Application of Tensor Analysis in Physics

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Homi Bhabha National Institute
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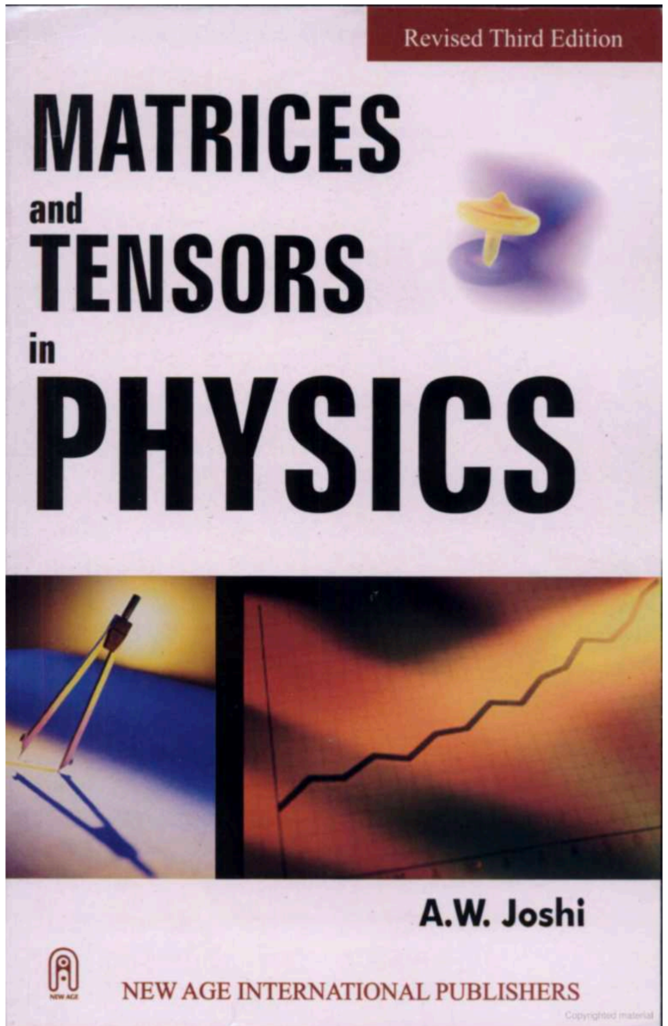
Teaching material

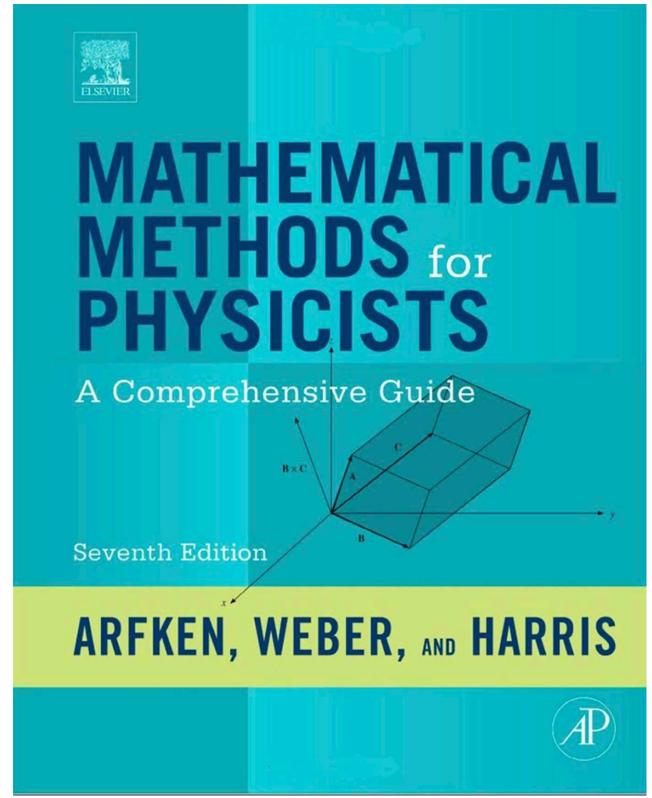
for

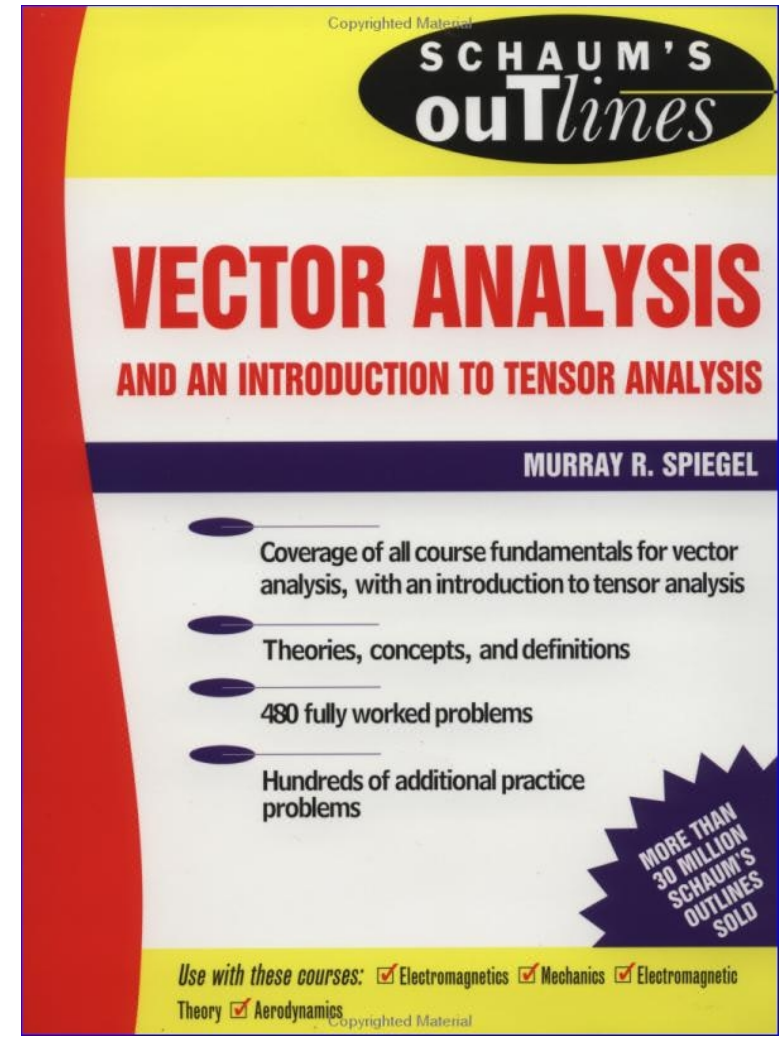
School of Basic Science,

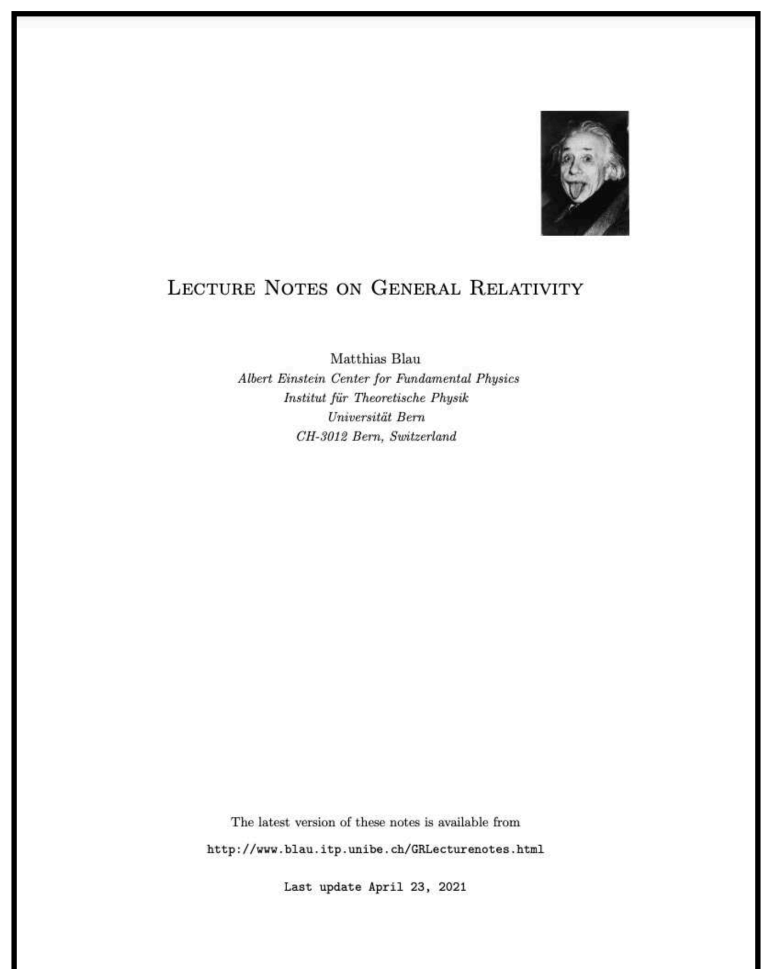
Indian Institute of Technology, Mandi, India

Useful references and study materials









Lecture Notes on General Relativity

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The latest version of these notes is available from

http://www.blau.itp.unibe.ch/GRLecturenotes.html

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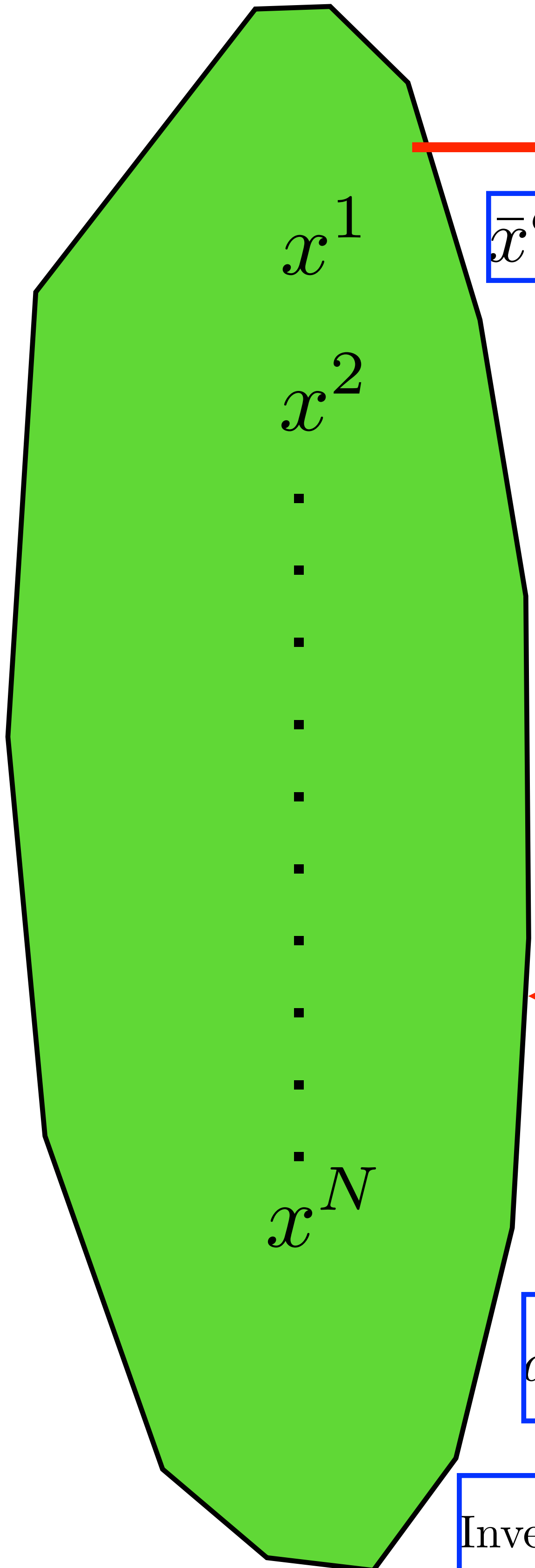
Lecture contents

- **Coordinate transformation**
- **Tensor of rank zero (scalar)**
- **Tensor of rank one (vector): Contra variant and Co variant**
- **Tensor of rank two: Contra variant, Co variant and Mixed**
- **Tensor of higher ranks: Contra variant, Co variant and Mixed**
- **Various examples: Metric tensor, reciprocal tensor, associated, relative, absolute, symmetric and antisymmetric, permutation tensor**
- **Tensor calculus and its applications in Physics**
- **Assignments for students**

Coordinate transformation



A

Forward transformation/mapping

$$\bar{x}^{\alpha} = \bar{x}^{\alpha}(x^1, x^2, \cdots, x^N) \qquad \forall \quad \alpha = 1, 2, \cdots, N$$

$$d\bar{x}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}}\, dx^{\beta} = J^{\alpha}_{\beta}(\bar{x}, x) dx^{\beta} \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$$

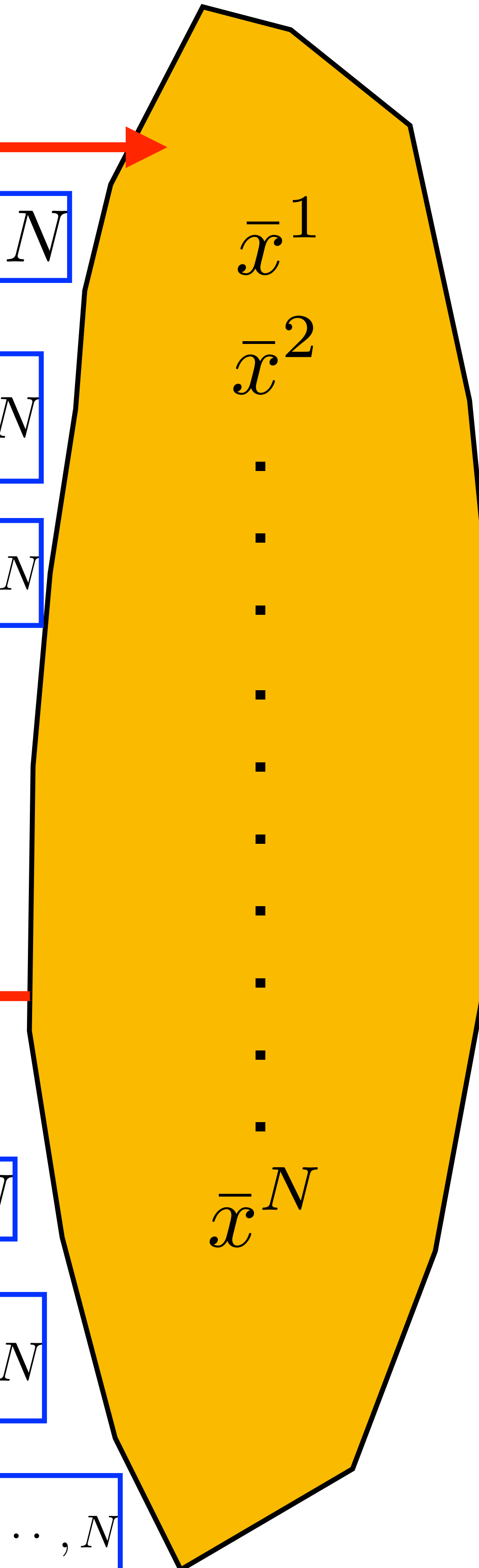
Transformation matrix : $\quad J^{\alpha}_{\beta}(\bar{x}, x) = \dfrac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$

Inverse transformation/mapping

$$x^{\beta} = x^{\beta}(\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^N) \qquad \forall \quad \beta = 1, 2, \cdots, N$$

$$dx^{\beta} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}}\, d\bar{x}^{\alpha} = \left[J^{-1}(\bar{x}, x)\right]^{\beta}_{\alpha} d\bar{x}^{\alpha} \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$$

Inverse transformation matrix : $\quad \left[J^{-1}(\bar{x}, x)\right]^{\beta}_{\alpha} = \dfrac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}} \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$



B

Tensor of rank zero (Scalar)

A $\phi = \phi(x^\beta)$ $\xrightarrow{\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \quad \forall \quad \alpha, \beta = 1, 2, \cdots, N}$ $\bar{\phi} = \bar{\phi}(\bar{x}^\alpha)$ B

Under the above mentioned coordinate transformation if we found

$$\phi = \bar{\phi}$$

Which means the function is invariant under the above mentioned coordinate transformation

=

Tensor of rank zero

=

Scalar

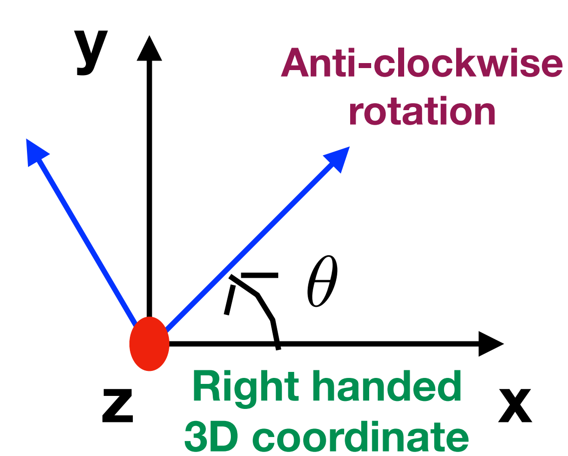
Simple example- Rotation about a specific axis/ arbitrary direction in 3D:

$$\bar{x}^\alpha = R^\alpha_\beta x^\beta \quad \forall \quad \alpha, \beta = 1, 2, 3 \qquad\qquad d\bar{x}^\alpha = R^\alpha_\beta dx^\beta \quad \forall \quad \alpha, \beta = 1, 2, 3$$

$$\text{Rotation matrix :} \quad R^\alpha_\beta = \frac{\partial \bar{x}^\alpha}{\partial x^\beta} \qquad \forall \quad \alpha, \beta = 1, 2, 3$$

R: Orthogonal Matrix

$$\left[R^T R\right]^\alpha_\gamma = \left[R^{-1} R\right]^\alpha_\gamma = \frac{\partial x^\alpha}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\gamma} = \frac{\partial x^\alpha}{\partial x^\gamma} = \delta^\alpha_\gamma \qquad \forall \quad \alpha, \beta = 1, 2, 3$$



$$\bar{x}_\alpha \bar{x}^\alpha = \left(R^\gamma_\alpha x_\gamma\right) \left(R^\alpha_\beta x^\beta\right) = \left(R^\gamma_\alpha R^\alpha_\beta\right) x_\gamma x^\beta = \delta^\gamma_\beta x_\gamma x^\beta = x_\beta x^\beta \qquad \forall \quad \alpha, \beta, \gamma = 1, 2, 3$$

Length of a vector is unchanged under rotation (Scalar)

Tensor of rank one (Vector): Contra variant & Co variant

Contra variant vector:

$$\text{A} \quad \begin{array}{c} A^{\beta}(x^{\beta}) \equiv A^{\beta} \\ \forall \;\; \beta = 1, 2, \cdots, N \end{array} \xrightarrow{\bar{x}^{\alpha} = \bar{x}^{\alpha}(x^{\beta}) \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N} \begin{array}{c} \bar{A}^{\alpha}(x^{\alpha}) \equiv \bar{A}^{\alpha} \\ \forall \;\; \alpha = 1, 2, \cdots, N \end{array} \quad \text{B}$$

$$\bar{A}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \; A^{\beta}$$

Transformation rule of a Contra variant vector or Contra variant tensor of rank one.

Simple example: Velocity transform as a contra variant vector.

$$\bar{v}^{\alpha} = \frac{d\bar{x}^{\alpha}}{dt} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \; \frac{dx^{\beta}}{dt} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} \; v^{\beta}$$

Co variant vector:

$$\text{A} \quad \begin{array}{c} A_{\beta}(x^{\beta}) \equiv A_{\beta} \\ \forall \;\; \beta = 1, 2, \cdots, N \end{array} \xrightarrow{\bar{x}^{\alpha} = \bar{x}^{\alpha}(x^{\beta}) \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N} \begin{array}{c} \bar{A}_{\alpha}(\bar{x}^{\alpha}) \equiv \bar{A}_{\alpha} \\ \forall \;\; \alpha = 1, 2, \cdots, N \end{array} \quad \text{B}$$

$$\bar{A}_{\alpha} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}} \; A_{\beta}$$

Transformation rule of a Co variant vector or Co variant tensor of rank one.

Simple example: Components of gradient of a scalar function transform as a co variant vector.

$$\left(\bar{\nabla}\phi\right)_i = \bar{\partial}_i \phi = \frac{\partial \phi}{\partial \bar{x}^i} = \frac{\partial x^j}{\partial \bar{x}^i} \; \frac{\partial \phi}{\partial x^j} = \frac{\partial x^j}{\partial \bar{x}^i} \; \partial_j \phi = \frac{\partial x^j}{\partial \bar{x}^i} \; \left(\nabla \phi\right)_j$$

Tensor of rank two: Contra variant , Co variant & Mixed

Contra variant tensor:

A $A^{\beta_1\beta_2}(x^\beta) \equiv A^{\beta_1\beta_2}$ $\forall\ \beta, \beta_1, \beta_2 = 1, 2, \cdots, N$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$$

B $\bar{A}^{\alpha_1\alpha_2}(x^\alpha) \equiv \bar{A}^{\alpha_1\alpha_2}$ $\forall\ \alpha, \alpha_1, \alpha_2 = 1, 2, \cdots, N$

$$\bar{A}^{\alpha_1\alpha_2} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\beta_1}} \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{\beta_2}} \; A^{\beta_1\beta_2}$$

Transformation rule of a Contra variant tensor of rank two.

Co variant tensor:

A $A_{\beta_1\beta_2}(x^\beta) \equiv A_{\beta_1\beta_2}$ $\forall\ \beta, \beta_1, \beta_2 = 1, 2, \cdots, N$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$$

B $\bar{A}_{\alpha_1\alpha_2}(x^\alpha) \equiv \bar{A}_{\alpha_1\alpha_2}$ $\forall\ \alpha, \alpha_1, \alpha_2 = 1, 2, \cdots, N$

$$\bar{A}_{\alpha_1\alpha_2} = \frac{\partial x^{\beta_1}}{\partial \bar{x}^{\alpha_1}} \frac{\partial x^{\beta_2}}{\partial \bar{x}^{\alpha_2}} \; A_{\beta_1\beta_2}$$

Transformation rule of a Co variant tensor of rank two.

Mixed tensor:

A $A^{\beta_1}_{\beta_2}$

$$\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \qquad \forall \quad \alpha, \beta = 1, 2, \cdots, N$$

B $\bar{A}^{\alpha_1}_{\alpha_2}$

$$\bar{A}^{\alpha_1}_{\alpha_2} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\beta_1}} \frac{\partial x^{\beta_2}}{\partial \bar{x}^{\alpha_2}} \; A^{\beta_1}_{\beta_2}$$

Transformation rule of a mixed tensor of rank two.

Simple example: Krönecker Delta function transform as a mixed tensor of rank two.

$$\bar{\delta}^j_k = \frac{\partial \bar{x}^j}{\partial \bar{x}^k} = \frac{\partial \bar{x}^j}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \; \frac{\partial x^m}{\partial x^n} = \frac{\partial \bar{x}^j}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^k} \; \delta^m_n$$

Tensor of higher ranks: Contra variant , Co variant & Mixed

Contra variant tensor:

$$\mathbf{A}\quad A^{\beta_1\beta_2\cdots\beta_k} \xrightarrow{\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \quad \forall \quad \alpha,\beta = 1,2,\cdots,N} \bar{A}^{\alpha_1\alpha_2\cdots\alpha_m}\quad \mathbf{B}$$

$$\bar{A}^{\alpha_1\alpha_2\cdots\alpha_m} = \frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\beta_1}}\frac{\partial \bar{x}^{\alpha_2}}{\partial x^{\beta_2}}\cdots\frac{\partial \bar{x}^{\alpha_m}}{\partial x^{\beta_k}} A^{\beta_1\beta_2\cdots\beta_k} = \prod_{j=1,2,\cdots,m}\prod_{p=1,2,\cdots,k}\frac{\partial \bar{x}^{\alpha_j}}{\partial x^{\beta_p}} A^{\beta_1\beta_2\cdots\beta_k}$$

Transformation rule of a Contra variant tensor higher rank.

Co variant tensor:

$$\mathbf{A}\quad A_{\beta_1\beta_2\cdots\beta_k} \xrightarrow{\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \quad \forall \quad \alpha,\beta = 1,2,\cdots,N} \bar{A}_{\alpha_1\alpha_2\cdots\alpha_m}\quad \mathbf{B}$$

$$\bar{A}_{\alpha_1\alpha_2\cdots\alpha_m} = \frac{\partial x^{\beta_1}}{\partial \bar{x}^{\alpha_1}}\frac{\partial x^{\beta_2}}{\partial \bar{x}^{\alpha_2}}\cdots\frac{\partial x^{\beta_k}}{\partial \bar{x}^{\alpha_m}} A_{\beta_1\beta_2\cdots\beta_k} = \prod_{j=1,2,\cdots,m}\prod_{p=1,2,\cdots,k}\frac{\partial x^{\beta_p}}{\partial \bar{x}^{\alpha_j}} A_{\beta_1\beta_2\cdots\beta_k}$$

Transformation rule of a Co variant tensor of rank two.

Mixed tensor:

$$\mathbf{A}\quad A^{\beta_1\beta_2\cdots\beta_k}_{\gamma_1\gamma_2\cdots\gamma_q} \xrightarrow{\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \quad \forall \quad \alpha,\beta = 1,2,\cdots,N} \bar{A}^{\alpha_1\alpha_2\cdots\alpha_m}_{\lambda_1\lambda_2\cdots\lambda_p}\quad \mathbf{B}$$

$$\bar{A}^{\alpha_1\alpha_2\cdots\alpha_m}_{\lambda_1\lambda_2\cdots\lambda_p} = \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\beta_1}}\frac{\partial \bar{x}^{\alpha_2}}{\partial x^{\beta_2}}\cdots\frac{\partial \bar{x}^{\alpha_m}}{\partial x^{\beta_k}}\right)\left(\frac{\partial x^{\gamma_1}}{\partial \bar{x}^{\lambda_1}}\frac{\partial x^{\gamma_2}}{\partial \bar{x}^{\lambda_2}}\cdots\frac{\partial x^{\gamma_q}}{\partial \bar{x}^{\lambda_p}}\right) A^{\beta_1\beta_2\cdots\beta_k}_{\gamma_1\gamma_2\cdots\gamma_q}$$

$$= \prod_{j=1,2,\cdots,m}\prod_{i=1,2,\cdots,k}\prod_{a=1,2,\cdots,q}\prod_{b=1,2,\cdots,p}\frac{\partial \bar{x}^{\alpha_j}}{\partial x^{\beta_i}}\frac{\partial x^{\gamma_a}}{\partial \bar{x}^{\lambda_b}} A^{\beta_1\beta_2\cdots\beta_k}_{\gamma_1\gamma_2\cdots\gamma_q}$$

Transformation rule of a mixed tensor of rank two.

Various examples

Metric tensor:

$$\text{A} \quad g_{\alpha\beta}, g^{\alpha\beta} \xrightarrow{\bar{x}^\alpha = \bar{x}^\alpha(x^\beta) \quad \forall \quad \alpha, \beta = 1, 2, \cdots, N} \bar{g}_{\gamma\lambda}, \bar{g}^{\gamma\lambda} \quad \text{B}$$

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = g^{\alpha\beta} dx_\alpha dx_\beta$$

$$d\bar{s}^2 = \bar{g}_{\gamma\lambda} d\bar{x}^\gamma d\bar{x}^\lambda = \bar{g}^{\gamma\lambda} d\bar{x}_\gamma d\bar{x}_\lambda$$

$$\boxed{ds^2 = d\bar{s}^2 \longrightarrow g_{\alpha\beta} = \bar{g}_{\gamma\lambda} J^\gamma_\alpha J^\lambda_\beta = \bar{g}_{\gamma\lambda} \frac{\partial \bar{x}^\gamma}{\partial x^\alpha} \frac{\partial \bar{x}^\lambda}{\partial x^\beta}}$$

$$\boxed{ds^2 = d\bar{s}^2 \longrightarrow g^{\alpha\beta} = \bar{g}^{\gamma\lambda} \left[J^{-1}\right]^\gamma_\alpha \left[J^{-1}\right]^\lambda_\beta = \bar{g}^{\gamma\lambda} \frac{\partial x^\alpha}{\partial \bar{x}^\gamma} \frac{\partial x^\beta}{\partial \bar{x}^\lambda}}$$

Receprocal tensor:

$$g^{\beta\lambda} = \frac{\text{Cofactor of } g_{\beta\lambda}}{g} \qquad \text{where} \quad g = \text{Det}(g_{\alpha\beta}) \neq 0 \qquad g_{\alpha\beta} g^{\beta\lambda} = \delta^\lambda_\alpha$$

Associated tensor:

$$A^\alpha_{*\beta} = g^{\gamma\alpha} A_{\gamma\beta} \qquad A^{\alpha\beta} = g^{\gamma\alpha} g^{\lambda\beta} A_{\gamma\lambda}$$

Relative and absolute tensor:

$$\bar{A}^{\alpha_1\alpha_2\cdots\alpha_m}_{\lambda_1\lambda_2\cdots\lambda_p} = \left|\frac{\partial x}{\partial \bar{x}}\right|^w \left(\frac{\partial \bar{x}^{\alpha_1}}{\partial x^{\beta_1}} \frac{\partial \bar{x}^{\alpha_2}}{\partial x^{\beta_2}} \cdots \frac{\partial \bar{x}^{\alpha_m}}{\partial x^{\beta_k}}\right) \left(\frac{\partial x^{\gamma_1}}{\partial \bar{x}^{\lambda_1}} \frac{\partial x^{\gamma_2}}{\partial \bar{x}^{\lambda_2}} \cdots \frac{\partial x^{\gamma_q}}{\partial \bar{x}^{\lambda_p}}\right) A^{\beta_1\beta_2\cdots\beta_k}_{\gamma_1\gamma_2\cdots\gamma_q}$$

$$= \left|\frac{\partial x}{\partial \bar{x}}\right|^w \prod_{j=1,2,\cdots,m} \prod_{i=1,2,\cdots,k} \prod_{a=1,2,\cdots,q} \prod_{b=1,2,\cdots,p} \frac{\partial \bar{x}^{\alpha_j}}{\partial x^{\beta_i}} \frac{\partial x^{\gamma_a}}{\partial \bar{x}^{\lambda_b}} A^{\beta_1\beta_2\cdots\beta_k}_{\gamma_1\gamma_2\cdots\gamma_q}$$

If the weight factor

$w = 0 \longrightarrow$ Absolute tensor

$w = 1 \longrightarrow$ (Relative) tensor density

Various examples

Symmetric tensor:

$$A_{\alpha\beta} = A_{\beta\alpha} \qquad \text{symmetric under index exchange} \;\; \alpha \longleftrightarrow \beta$$

Anti-symmetric tensor:

$$A_{\alpha\beta} = -A_{\beta\alpha} \qquad \text{anti} - \text{symmetric under index exchange} \;\; \alpha \longleftrightarrow \beta$$

Simple example: A rank two tensor can be expressed as a sum of an symmetric and anti-symmetric tensor

$$A_{\alpha\beta} = C_{\alpha\beta} + D_{\alpha\beta}$$

$$C_{\alpha\beta} = \frac{1}{2}\left(A_{\alpha\beta} + A_{\beta\alpha}\right) = \frac{1}{2}\left(A_{\beta\alpha} + A_{\alpha\beta}\right) = C_{\beta\alpha} \quad \text{symmetric under} \quad \alpha \leftrightarrow \beta$$

$$D_{\alpha\beta} = \frac{1}{2}\left(A_{\alpha\beta} - A_{\beta\alpha}\right) = -\frac{1}{2}\left(A_{\beta\alpha} - A_{\alpha\beta}\right) = -D_{\beta\alpha} \quad \text{anti} - \text{symmetric under} \quad \alpha \leftrightarrow \beta$$

Permutation tensor: Levi-Civita

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \begin{cases} +1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an even permutation of } (1, 2, 3, \ldots, n) \\ -1 & \text{if } (a_1, a_2, a_3, \ldots, a_n) \text{ is an odd permutation of } (1, 2, 3, \ldots, n) \\ 0 & \text{otherwise} \end{cases}$$

$$\varepsilon_{a_1 a_2 a_3 \ldots a_n} = \prod_{1 \le i < j \le n} \operatorname{sgn}(a_j - a_i)$$

$$\varepsilon_{i_1 \ldots i_n} \varepsilon^{j_1 \ldots j_n} = \delta^{j_1 \ldots j_n}_{i_1 \ldots i_n}$$

$$\varepsilon_{i_1 \ldots i_k\, i_{k+1} \ldots i_n} \varepsilon^{i_1 \ldots i_k\, j_{k+1} \ldots j_n} = \delta^{i_1 \ldots i_k\, j_{k+1} \ldots j_n}_{i_1 \ldots i_k\, i_{k+1} \ldots i_n} = k!\, \delta^{j_{k+1} \ldots j_n}_{i_{k+1} \ldots i_n}$$

$$\varepsilon_{i_1 \ldots i_n} \varepsilon^{i_1 \ldots i_n} = n!$$

$$\varepsilon_{i_1 i_2 \ldots i_n} \varepsilon_{j_1 j_2 \ldots j_n} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \ldots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \ldots & \delta_{i_2 j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \ldots & \delta_{i_n j_n} \end{vmatrix}.$$

$$\varepsilon_{ijk} \varepsilon^{imn} = \delta_j{}^m \delta_k{}^n - \delta_j{}^n \delta_k{}^m$$

$$\varepsilon_{jmn} \varepsilon^{imn} = 2\delta_j{}^i$$

$$\varepsilon_{ijk} \varepsilon^{ijk} = 6.$$

Useful for vector calculus

Tensor calculus and its application in Physics

Vector calculus:

$$\vec{A}.\vec{B} = A_i B^j \delta^i_j = A_i B^i = A_i B_i$$

In 3D Euclidean geometry

$$ds^2 = \delta^{ij} dx_i dx_j = dx_i dx_i = dx^2 + dy^2 + dz^2$$

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A^j B^k = \epsilon_{ijk} A_j B_k$$

$$(\vec{\nabla}\phi)_i = \partial_i \phi$$

$$\nabla^2 \phi = \partial_i \partial^i \phi = \partial_i \partial_i \phi$$

$$(\vec{\nabla} \times \vec{A})_i = \epsilon_{ijk} \partial_j A_k$$

$$(\nabla^2 \vec{A})_i = \partial_j \partial_j A_i$$

$$\vec{\nabla}\phi.\vec{\nabla}\psi = \partial_i \phi \; \partial_i \psi$$

$$\vec{\nabla}\phi.\vec{\nabla}\phi = \partial_i \phi \; \partial_i \phi = (\partial_i \phi)^2$$

$$A_\mu B^\mu = -A_0 B_0 + A_i B_i$$

$$A_\mu = (-A_0, \vec{A}), \quad B^\mu = (B_0, \vec{B})$$

In 3+1 D Minkowski flat geometry

$$ds^2 = \eta^{\alpha\beta} dx_\alpha dx_\beta = -dt^2 + dx^2 + dy^2 + dz^2 \quad (in\ c = 1\ unit)$$

$$\partial_\mu = \left(\partial_t, \vec{\nabla}\right), \quad \partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\partial_t, -\vec{\nabla}\right), \quad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$$

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1) = \eta^{\mu\nu}$$

$$\eta_{\mu\nu}\eta^{\mu\nu} = (-1)^2 + 3 \times (1)^2 = 4 \quad \text{in } 3+1 \text{ D Minkowski flat space}$$

$$\eta_{\mu\nu}\eta^{\nu\alpha} = \delta^\alpha_\mu$$

$$\partial_\mu \phi = (\partial_t, \vec{\nabla})\phi$$

$$\partial^\mu \phi = (\partial_t, -\vec{\nabla})\phi$$

$$\Box\phi = (\partial_t^2 - \nabla^2)\phi$$

$$\Box A^\mu = (\partial_t^2 - \nabla^2) A^\mu$$

$$\Box C^{\mu\nu} = (\partial_t^2 - \nabla^2) C^{\mu\nu}$$

$$(\partial_\mu \phi)(\partial^\mu \phi) = (\partial_t \phi)^2 - (\partial_i \phi)^2$$

$$(\partial_\mu \phi)(\partial^\mu \psi) = (\partial_t \phi)(\partial_t \psi) - (\partial_i \phi)(\partial_i \psi)$$

Tensor calculus and its application in Physics

Electromagnetic theory in Minkowski flat space:

$$F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad A_\mu = \left(-\Phi, \vec{A}\right)$$

$$F_{\mu\nu} = -F_{\nu\mu} \qquad \text{anti} - \text{symmetric under} \quad \mu \leftrightarrow \nu$$

$$\eta_{\mu\nu} = \text{diag}\,(-1,1,1,1) = \eta^{\mu\nu} \qquad F^{\mu\nu} = \partial^{[\mu} A^{\nu]} = \eta^{\mu\alpha}\eta^{\nu\beta} F_{\alpha\beta}$$

$$\eta_{\mu\nu}\eta^{\mu\nu} = (-1)^2 + 3 \times (1)^2 = 4 \quad \text{in } 3+1 \text{ D Minkowski flat space}$$

$$\eta_{\mu\nu}\eta^{\nu\alpha} = \delta^\alpha_\mu$$

$$A^\mu = \eta^{\mu\nu} A_\nu = \left(\Phi, \vec{A}\right) \qquad \Box F^{\mu\nu} = 0 \quad \longrightarrow \quad \text{EM wave eqns}$$

$$\partial_\mu = \left(\partial_t, \vec{\nabla}\right), \qquad \partial^\mu = \eta^{\mu\nu}\partial_\nu = \left(\partial_t, -\vec{\nabla}\right), \qquad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$$

$$\partial_\mu F^{\mu\nu} = 0 = \partial^\mu F_{\mu\nu} \quad \text{Gauss law of electric field \& Ampere's law in magnetism}$$

$$G_{\mu\nu} = \frac{1}{2!}\epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \qquad \text{Electromagnetic dual tensor}$$

$$\partial_\mu G^{\mu\nu} = 0 = \partial^\mu G_{\mu\nu} \quad \longrightarrow \quad \text{Bianchi identity}$$

$$\longrightarrow \quad \text{Faraday's law of EM induction}$$

$$\& \quad \text{Gauss law of magnetism}$$

Sourceless Maxwell's eqns

Assignments for students

Q1. Show that the elements of 3D rotation matrix (consider rotation about x,y,z and arbitrary axis) transform like a tensor. Apart from rotation matrix if we consider a general matrix then in that case is it possible to show that the elements transform like a tensor?

Q2. Show that the vector product is unique to 3-D space, that is, only in three dimensions can we establish a one-to-one correspondence between the components of an antisymmetric tensor (second-rank) and the components of a vector.

Q3. Show that if all the components of any tensor of any rank vanish in one particular coordinate system, they vanish in all coordinate systems.

Q4. Prove that a necessary and sufficient condition that a tensor of rank R become an invariant by repeated contraction in that R be even and that the number of contra variant and co variant indices be equal to R/2.

Thanks for your time.....