GENERAL PRINCIPLES OF QUANTUM FIELD THEORY

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GENERAL PRINCIPLES OF QUANTUM FIELD THEORY

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Contents

Preface

The majority of the "memorable" results of relativistic quantum theory were obtained within the framework of the local quantum field approach. The explanation of the basic principles of the local theory and its mathematical structure has left its mark on all modern activity in this area.

Originally, the axiomatic approach arose from attempts to give a mathematical meaning to the quantum field theory of strong interactions (of Yukawa type). The fields in such a theory are realized by operators in Hilbert space with a positive Poincaré-invariant scalar product. This "classical" part of the axiomatic approach attained its modern form as far back as the sixties. *

It has retained its importance even to this day, in spite of the fact that nowadays the main prospects for the description of the electro-weak and strong interactions are in connection with the theory of gauge fields. In fact, from the point of view of the quark model, the theory of strong interactions of Wightman type was obtained by restricting attention to just the "physical" local operators (such as hadronic fields consisting of "fundamental" quark fields) acting in a Hilbert space of physical states. In principle, there are enough such "physical" fields for a description of hadronic physics, although this means that one must reject the traditional local Lagrangian formalism. (The connection is restored in the approximation of low-energy "phenomenological" Lagrangians.) Therefore our desire to include in our discussion such "unobservable" (that is, gauge dependent) fundamental fields as 4-vector potentials and the local field of an electron or quark, which are used in practical calculations in perturbation theory, necessitates a certain broadening and modification of the Wightman scheme.

This monograph is devoted to a logical account of the principles of local quantum field theory, including the gauge theories with indefinite metric. Although the amount of space allotted to the gauge theories proper is relatively small, the entire make-up of this book gives due attention to their inclusion. We have laid great emphasis on the algebraic approach (by comparison with [B8]).

Along with the predominantly purely theoretical material, we have included in the last part applications of the techniques developed to the derivation of dispersion relations and an examination of the behaviour of the cross sections of the interaction of elementary particles at high energies.

* Even at that time it was the subject of specialist monographs [S16], [J3], [B8]. The present book was originally planned as a second edition of [B8]. However, the overall project grew considerably as it evolved, and this current new book is the result. (The content of the book is broader than that suggested by the title: the mathematical methods of quantum field theory are set forth alongside the principles.)

This book is intended for theoretical physicists and mathematicians interested in the problems of quantum field theory and mathematical physics. Although we have tried as far as possible to make the exposition independent of other sources, this book cannot be recommended for a first acquaintance with quantum theory. Apart from a familiarity with a regular course in quantum mechanics and the general notions of elementary particles and their interactions, it is helpful to have some ideas about the fundamentals of quantum field theory, for example, within the framework of [B11] (or the first four chapters of [B10]; see also [H3], [S5]). The ancillary mathematical apparatus that falls outside the scope of the compulsory courses in physics faculties (an account of functional analysis, the theory of generalized functions, the theory of analytic functions of several variables and the vocabulary of the theory of Lie groups and their applications) is set out in the text.

It is our pleasant duty to express our thanks to our colleagues of the Steklov Mathematical Institute, the Joint Institute of Nuclear Research (Dubna), the Institute for High Energy Physics (Serpukhov), the Institute of Nuclear Research of the Academy of Sciences of the USSR, and the Institute of Nuclear Research and Nuclear Energy of the Bulgarian Academy of Sciences for numerous helpful discussions. One of the authors (I.T. Todorov) expresses his gratitude to M.K. Polivanov for his hospitality at the Steklov Mathematical Institute during the final stages of the work on this book.

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Introduction

THE PLACE OF THE AXIOMATIC APPROACH IN QUANTUM FIELD THEORY

In physics as well as in mathematics, the role of the axiomatic method is twofold. On the one hand it clarifies the logical foundations of a given topic by showing what the independent premises are (as , for example, in Euclidean geometry or Newtonian mechanics), and so opening up new possibilities. On the other hand, by isolating their fundamental structures, it enables one to find relationships between branches of science that at first glance appear different. Such an approach is typical of modern mathematics where it has given rise to a number of new areas. The role of the axiomatic method is less in evidence in theoretical physics; but even here, it gained wide dissemination as far back as Newton's time, both as a method of systematizing known results and as a method of describing new phenomena by means of formal schemes developed earlier. At the basis of every axiomatic physical discipline there lie deep physical ideas that are expressible in a mathematically consistent form. It is remarkable that the formal schemes sometimes contain more than was originally invested in them (examples: the action principle, the canonical formalism, the Gibbs ensemble). As a result of this there is a reverse influence of the mathematical structures of theoretical physics on the formation of physical ideas.

In the thirties, the axiomatic method was successfully applied (in the work of Jordan, von Neumann and Wigner) to quantum mechanics (see [V6]; we can recommend the books [M1], [K5] as examples of the later development in this direction). The structural analysis of quantum mechanics has led to a remarkable synthesis of physical and mathematical ideas, which has become part of the generally accepted formalism of quantum theory and has influenced the development of mathematics. (Under the stimulus of quantum theory, new branches of functional analysis have come into being: the theory of operators in Hilbert space, operator algebras, unitary representations of groups, harmonic analysis.) The mathematical problems of quantum theory have in large measure determined the interests of modern mathematical physics.

In quantum field theory, the axiomatic approach relates to the activities of fifty years in connection with the successes and difficulties of the method of perturbations in Lagrangian quantum field theory. The apparatus of renormalizations developed in the perturbation method led to brilliant success in quantum electrodynamics where the parameter of the expansion (the coupling constant) is small, so that it was possible to restrict attention to the first terms of the perturbation-theory series for a comparison with experiment. This method, however, has proved to be unsuitable for a description of the strong interactions of elementary particles (where the effective coupling constant is greater than unity). The theory of renormalizations yields something better; this is a formal infinite series for the solutions of quantum equations in the class of physically interesting renormalizable Lagrangians. The axiomatic approach was called upon in the first instance to answer the question: what is hidden behind these formal infinite series? For this purpose, the creation of new principles of quantum theory were required that were different from the Lagrangian method with its perturbation theory.*

The first attempt to go beyond the framework of the Lagrangian approach goes back to Heisenberg (1943). In analysing what, in fact, is measured in the physics of elementary particles, Heisenberg came to the conclusion that the basic observable is the scattering matrix; he suggested that a theory should be constructed directly in terms of the elements of the S-matrix which would do away with the notion of a field, the adiabatic hypothesis of the exclusion of the interaction (which was at the basis of perturbation theory) and so on. It turned out, however, that the Heisenberg approach was too radical. The complete banishment of the local quantities of the theory deprives us of the possibility of considering the evolution of the system in space and time by taking the causality principle into account. Therefore the development of the axiomatic approach proceeded via the study of local quantities; and at the very beginning (in the 50's) at least three lines of approach took shape.

The Wightman formalism singles out as the basic objects the most regular quantities, namely, the quantized fields in the Heisenberg representation and the vacuum expectation values of their ordinary products (the Wightman functions are analogous to the correlation functions in statistical physics). In principle, the Wightman functions enable one to extract all the physical information contained within the theory. In particular, the asymptotic condition (which was originally stated by Haag as one of the postulates of the theory) and the scattering matrix are derived concepts. (Only the asymptotic completeness condition remains as an independent hypothesis.)

In the Lehmann-Symanzik-Zimmermann (LSZ) approach, the basic concepts are the chronological (or T-) products of the fields (also their vacuum expectation values — the Green's functions) and the asymptotic condition. In this connection it can be shown that it makes no sense to talk about an independent approach, since the Green's functions and the T-products are formally expressed in terms of the Wightman functions and the ordinary derivatives of the Heisenberg fields. In fact, this formal definition is not mathematically well defined since it contains the product of an (operator-valued) generalized function with discontinuous θ-functions (which leads to divergences of the same type as in perturbation theory); and this blocks the application of the alternative point of view, that the T-products (or, equivalently, the retarded products) are the primitive objects of the theory along with the Heisenberg fields and are defined only indirectly by means of a certain set of properties. This sort of approach is not the most economical one (since the scattering theory can, in principle, be developed without introducing the notion of T-product in advance; see Ch.12), but it is convenient in practice and brings us close to the traditional Lagrangian method.

The Bogolubov-Medvedev-Polivanov approach, in which the basic object is the

* It is appropriate here to recall the analogous situation that arose in probability theory: the axiom scheme of Kolmogorov (put forward at the end of the 20's) brought about a decisive restructuring of this discipline on completely mathematical foundations.

extended S-matrix (beyond the mass shell), is superficially closer to Heisenberg's original programme. More interestingly, it is closely related to the LSZ approach which occurred in parallel, since the extended S-matrix is, in essence, the generating functional of the T-products of the current operators. Whereas the derivation of the reduction formulae (which express the S-matrix elements in terms of Green's functions) are non-trivial in the Wightman or LSZ formalism, in the S-matrix method these formulae are derived automatically by formally taking the variational derivative of the S-operator (or the radiation operators expressed in terms of it) with respect to the asymptotic fields. The practical convenience of the S-matrix method is afforded by the effective calculation of the complicated combinatorics in operations of this kind. It is no accident that the dispersion relations were first proved by this means.

In the 60's quite a considerable number of relationships manifested themselves between the approaches which historically had independent origins. Disregarding certain mathematical niceties and technical differences, we can say that all three approaches are applicable with equal success to the class of quantum field theories with the asymptotic completeness condition. Since this condition is made essential use of only in the last two approaches, the Wightman formalism is somewhat more general.

Alongside this, there arose in the 60's an even more general axiomatic line of development. In the work of Haag, Araki and Kastler, the principles of local quantum theory were stated in the language of the algebraic approach originating from the work of von Neumann and Segal. The significance of this approach for the general statement of the problems of the physics of systems with an infinite number of degrees of freedom is on the increase. In particular, it provides a method of describing gauge theories, spontaneous breakdown of symmetries (and in statistical physics, phase transitions, although the latter are beyond the scope of this book).

Of the various routes along which modern elementary particle theory has moved, the axiomatic approach has occupied a relatively small place (especially if we judge from the number of publications). However, the short life of the phenomenological results of theories based on a number of special assumptions, makes the results arising from the fundamental principles of quantum theory all the more interesting; these principles are: relativistic invariance, the existence of a complete system of states with positive energy, and causality.

There is to date no complete answer to the fundamental question which resulted in the development of the axiomatic approach; namely, are the principles of relativistic local quantum theory (in four-dimensional space-time) compatible with the existence of a non-trivial scattering matrix ? Here, considerable advances have been made in the last decade. An essentially new area has emerged, namely, constructive quantum field theory (see [G8]); as a result of its development, non-trivial models have been constructed in two- and three-dimensional space-time. However, these models are super-renormalizable* and do not require an infinite renormalization of the charge. If one is to consider realistic renormalizable models in four-dimensional space-time, then essentially new methods are required. Even so, the successes obtained along this route, together with the discovery of renormalizability and asymptotic freedom

* That is, they have a finite number of primitively divergent (one-particle irreducible) diagrams (see, for example, the definition in [S5],§16).

of the non-abelian gauge fields in the traditional (formal) approach* have opened up further prospects for the development of local quantum field theory.

THE LAYOUT OF THIS BOOK

Part I is of a preliminary character: it contains various topics in functional analysis and the theory of functions required for the subsequent parts. In places the account is somewhat terse and is not, of course, a substitute for a systematic exposition of all the questions touched upon: some proofs are omitted (and replaced by detailed references to the literature), and not all of the definitions and statements are accompanied by covering comments. The systematic account begins in Part II. After an account of the fundamental ideas of the quantum phenomenology (given in algebraic language) we formulate those principles of relativistic quantum theory that do not require the introduction of local quantities: the invariance principle with respect to the Poincaré group (that is, the non-homogeneous Lorentz group) and the spectral condition (that is, the existence of a complete system of physical states with non-negative energy).

In Part III we deal, in the main, with the Wightman formulation of the theory of local quantized fields. Examples of free (and generalized free) fields are analysed in detail. A number of general results are given here: the CPT theorem, the theorem on the connection of spin with statistics and the theorems of Haag and Goldstone. One of the chapters is devoted to a generalization of the Wightman formalism for fields with indefinite metric (the importance of this class of theories is that the gauge theories in local covariant gauges come out of it). The chapter on two-dimensional explicitly soluble models serves as an illustration, as it were, of the Wightman formalism and its generalizations.

Part IV contains a survey of the Haag-Ruelle scattering theory, its connection with the LSZ theory, also the S-matrix approach.

In Part V, the apparatus developed in the earlier parts is applied to the analytic properties of the amplitudes of elementary processes. Notwithstanding its simplicity, the idea of analyticity has turned out to have had a very fruitful influence on the development of the theory of strong interactions. (In this connection, we note at least the dual resonance models.) In our account of this, we present the basic results deduced from the principles of quantum field theory; these are, in the first instance, the analyticity with respect to the cosine of the scattering angle, the dispersion relations, and the crossing. An extensive literature is devoted to the applications of the results of local quantum field theory to high-energy elementary particle processes (the reader will find detailed information concerning this in the surveys [G7]). Several typical examples of this sort are given in the final chapter.

Each part is preceded by a brief summary of the contents.

This book includes appendices which set out auxiliary material or questions of independent interest. The numerous exercises form an integral part of the text. They are referred to in the subsequent parts of the text and, as a rule, hints are provided. The appendices, exercises, proofs, also some remarks are printed in a smaller typeface.

The references to the literature are separated into two parts. In the first part are the textbooks and monographs; references to these are given in square brackets (for example, [A1]). The second half contains articles and journals, preprints and lectures at seminars and workshops. In the text, references to this part are given by

* See, for example, the collection [Q1] and the survey article by Crewther (1976).

the surnames of the authors (or just the first author plus "et al." if there are more than two authors) and the publication date (for example, Zwanziger, 1979b).

At the back of the book is a glossary of the notation of frequent occurrence in this book. It should be noted that in this book all the coordinates of the 4-vectors of Minkowski space-time $\mathbf{M}$ are real and the metric tensor in $\mathbf{M}$ is defined by the formulae

$$g^{00} = -g^{kk} = 1 \text{ for } k = 1,2,3, \quad (g^{\mu\nu} = 0 \text{ for } \mu \neq \nu, \quad \mu, \nu = 0,1,2,3).$$

The three-dimensional spatial part of the 4-vector p is denoted by bold-face type so that $p \equiv (p^0, \mathbf{p})$, $p^2 \equiv (p^0)^2 - \mathbf{p}^2$. Throughout, a system of units in which $c = \hbar = 1$ is used.

Part I

Elements of Functional Analysis and the Theory of Functions

Synopsis

The reader will undoubtedly have encountered in his study of quantum mechanics, the idea of a Hilbert space and the spectral analysis of self-adjoint operators in it. Further information on this topic appears in §§1.1,1.4. From the remaining material of Chapter 1, we single out the concept of a locally convex space and the Gel'fand-Naimark-Segal (GNS) representation of a C^*-algebra.

A non-negative function $p(u)$ on a vector space Ω is called a seminorm if it is positive homogeneous: $p(\lambda u) = |\lambda| p(u)$ (for any scalar λ) and satisfies the triangle inequality $p(u+v) \leq p(u) + p(v)$. Each system of seminorms $\{p_\alpha\}$ in a space Ω that satisfies the separation property ($p_\alpha(u) = 0$ for all α in the given index set $\mathcal{A}$ implies that $u = 0$) defines a (separated) locally convex topology on Ω. In §1.2.A we introduce a natural notion of equivalence (mutual subordination) of two systems of seminorms; two locally convex topologies on a given linear space Ω are the same if and only if the seminorms inducing them are equivalent. A complete (separated) locally convex space with a countable system of seminorms is called a Fréchet space. Every Fréchet space is metrizable and hence the Baire category theorem (Theorem 1.3 of §1.2.B) is applicable to it.

An involutive Banach algebra $\mathfrak{A}$ is called a C^*-algebra if $\|A^*A\| \equiv \|A\|^2$ (§1.5.A). This notion is abstracted from algebras of bounded operators in Hilbert space. A linear functional F over an involutive algebra $\mathfrak{A}$ is said to be positive if $F(A^*A) \geq 0$ for all A in $\mathfrak{A}$ (§1.5.C). The GNS construction enables one to construct from a given positive functional F over a C^*-algebra $\mathfrak{A}$ a (cyclic) representation π_F of $\mathfrak{A}$ in a Hilbert space $\mathcal{H}_F$ with cyclic vector Φ_F such that $F(A) = \langle \Phi_F, \pi_F(A)\Phi_F \rangle$ for all $A \in \mathfrak{A}$ (Theorem 1.25 of §1.5.D).

An important example of a Fréchet space is the space $\mathcal{S}(\mathbf{R}^n)$ of (complex) infinitely smooth (C^∞ for short) rapidly decreasing functions (§1.2.C); a topology can be defined on it by means of the increasing system of Hilbert norms

$$\|u\|_m^2 = \int \bar{u}(x)(|x|^2 - \Delta + 1)^m u(x) d^n x$$

(where $|x|^2 \equiv x_1^2 + \ldots + x_n^2$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$) (which is equivalent to the system (1.42)). Generalized functions are defined (§2.1.A) as continuous linear functionals over $\mathcal{S}(\mathbf{R}^n)$. (Sometimes the terminology "tempered generalized functions" (or tempered distributions) is used for these objects; but we shall reserve the terminology "distribution" of Schwartz for the more general notion of a linear functional over the space $\mathcal{D}(\mathcal{O})$ of C^∞-functions with compact support in a given open subset $\mathcal{O}$ of $\mathbf{R}^n$.) Generalized functions (or distributions) can be differentiated and multiplied by smooth functions of polynomial growth without going outside the space $\mathcal{S}'(\mathbf{R}^n)$ (or $\mathcal{D}'(\mathcal{O})$). A remarkable property of the space $\mathcal{S}(\mathbf{R}^n)$ and its dual is stability under the Fourier transformation; that is, if $u(x) \in \mathcal{S}(\mathbf{R}^n)$, then

$$\tilde{u}(p) = \int e^{ipx} u(x) d^n x \in \mathcal{S}(\mathbf{R}_n),$$

where $\mathcal{S}(\mathbf{R}_n)$ is the space of functions of p with measure $d_n p = d^n p/(2\pi)^n$ (Proposition 2.6). The Parseval identity

$$\int \overline{f(x)} u(x) d^n x = \int \overline{\tilde{f}(p)} \tilde{u}(p) d_n p$$

(which holds for arbitrary test functions u and f) is used to define the Fourier transform of a generalized function f. Within the framework of the theory of Fourier transforms of generalized functions (§2.5.B) one can obtain a justification of a formula of the type $\delta(x) = \int e^{-ipx} d_n p$.

In §2.7 we introduce the notions of vector- and operator-valued generalized functions. The first of these enables us to define generalized eigenvectors (corresponding to the continuous spectrum of a self-adjoint operator, see §2.7.C); the second is needed for the statement of the field axioms in Part III.

Chapter 3 is devoted to a study of the Lorentz-invariant and covariant generalized functions. By way of introduction, we give a classification of the finite-dimensional representations of the group $SL(2, C)$ (the Lorentz group of quantum mechanics). Proposition 3.2 (§3.2.B) is a precise statement of the intuitive idea that any even Lorentz-invariant function of a 4-vector p is a function of p^2. Also related to this are the invariant generalized functions concentrated at a point, which have the form $P(\Box)\delta(p)$ (see (3.89)). An arbitrary odd invariant function is defined, on its part, by a functional of type

$$(f, u) = \int f(p)u(p)d_4p = \int d\tau \psi_f(\tau) \int d_4 p \epsilon(p^0)\delta(\tau - p^2)u(p) \quad (u \in \mathcal{S}(\mathbf{M})),$$

where ψ_f is a generalized function in $\mathcal{S}'(\mathbf{R})$ that vanishes when $\tau < 0$ (§3.2.C).

In the study of the Lorentz-covariant (generalized) functions (§3.3) we shall use the formalism of homogeneous polynomials of the spinor variables ω ($\in \mathbf{C}^2$) and $\bar{\omega}$ (instead of working with spin-tensors). The function $f(p; \omega, \bar{\omega})$ is covariant if $f(\Lambda(\underset{\sim}{A})p; \underset{\sim}{A}\omega, \bar{\underset{\sim}{A}}\bar{\omega}) = f(p; \omega, \bar{\omega})$. A non-trivial covariant function f exists provided that the degree of homogeneity in ω is the same as that in $\bar{\omega}$. Denoting this common degree by n, we have $f_n(p; \omega, \bar{\omega}) = f_0(p)(\omega \underset{\sim}{p} \bar{\omega})^n$, where f_0 is a Lorentz-invariant (generalized) function (see (3.127)).

A generalized function $f_+(x)$ in $\mathcal{S}'(\mathbf{M})$ is called retarded if its support lies in the future cone $\overline{V}^+$; similarly, $f_-(x)$ is an advanced function if $\operatorname{supp} f_- \subset \overline{V}^-$. It is proved in Appendix B that the Fourier transform of a retarded (advanced) function is the limiting value of an analytic function that is holomorphic in the tube domain $T^\pm = \mathbf{R}^4 + iV^\pm$.

Let $h_+(p)$ and $h_-(p)$ be the Fourier transforms of retarded and advanced functions and let their difference $g(p) = h_+(p) - h_-(p)$ be concentrated in the domain $((-m, 0) + \overline{V}_M^+) \cup ((m, 0) + \overline{V}_M^-)$, where $\overline{V}_M^\pm = \{p \in \mathbf{R}^4 : \pm p^0 \geq \sqrt{M^2 + \mathbf{p}^2}\}$. Then we have the Jost-Lehmann representation

$$g(p) = \epsilon(p^0) \int_0^\infty d\tau \int d_3 q \delta(p_0^2 - (\mathbf{p} - \mathbf{q})^2)[\Phi_0(\mathbf{q}, \tau) + p^2 \Phi_1(\mathbf{q}, \tau)],$$

where Φ_0, Φ_1 are generalized functions in $\mathcal{S}'(\mathbf{R}_3 \times \overline{\mathbf{R}}_+)$ concentrated on the set

$$\{(\mathbf{q}, \tau) \in \mathbf{R}_3 \times \overline{\mathbf{R}}_+;\ |\mathbf{q}| \leq m, \sqrt{\tau} \geq M - \sqrt{m^2 - \mathbf{q}^2}\}.$$

Chapter 4 is devoted to a derivation of the Jost-Lehmann-Dyson (JLD) representation, which generalizes the formula to the case of a more general (asymmetric) region in which the function $g(p)$ vanishes.

The JLD representation enables one to extend by analytic continuation the single function $h(k)$ originally defined on the union of the domains $T^+ \cup T^- \cup \mathcal{O}$ (where $\mathcal{O}$ is the (real) region where h_+ and h_- coincide):

$$h(k) = \begin{cases} h_+(k) & \text{if } k \in T^+, \\ h_-(k) & \text{if } k \in T^-, \end{cases}$$

to a larger region of 4-dimensional complex space. This is the simplest example of the "edge of the wedge theorem" (Theorem 5.12). It illustrates an important fact which distinguishes the theory of analytic functions of several complex variables from the corresponding single-variable theory. Whereas any domain D of the complex plane $\mathbf{C}$ is the domain of holomorphy of an analytic function that does not admit an analytic continuation outside D, for the case of several variables there exist domains that are not domains of holomorphy of any analytic function. This enables one to introduce the notion of analytic continuation, domains of holomorphy and envelopes of holomorphy; Chapter 5 is devoted to the study of these.

CHAPTER 1

Preliminaries on Functional Analysis

1.1. Normed Spaces

A. LINEAR SPACES

In the study of infinite-dimensional linear spaces by the methods of functional analysis, emphasis is placed on topological properties. Leaving aside the topological details for the moment, we give a brief run-through here of the basic concepts relating to linear spaces and familiar from linear algebra.

A linear space is a set of elements of any kind for which the operations of addition and multiplication by a real (or complex) number are defined, the usual laws being satisfied for these operations. Examples of a linear space are vectors in n-dimensional Euclidean space, or the set of continuous (or integrable) functions defined on some set of points in finite-dimensional space, or functionals defined on some class of functions. The concrete nature of the elements is irrelevant for the abstract theory of linear spaces.

We now give the precise definitions.

A set Ω is called a *linear space* (over the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers) if Conditions I–III hold.

I. A commutative and associative law of addition is defined on Ω. This means that for any two elements u and v of Ω a third element $u+v$ is defined such that

Ia. $u+v=v+u$;

Ib. $u+(v+w)=(u+v)+w$;

Ic. For any u and v there exists an element x, depending on u and v, such that $u+x=v$ (this element being denoted by $v-u$).

II. Multiplication by numbers λ, μ, ... is defined on Ω, where

IIa. $1\cdot u=u$;

IIb. $\lambda(\mu u)=(\lambda\mu)u$ for any $u\in\Omega$.

III. The operations of addition and multiplication by a number are related by the distributive laws:

IIIa. $(\lambda+\mu)u=\lambda u+\mu u$;

IIIb. $\lambda(u+v)=\lambda u+\lambda v$.

The elements of a linear space are also referred to as points or vectors, and the linear space itself as a *vector space.* (It is called a real or complex (linear or vector) space depending on the choice of $\mathbf{R}$ or $\mathbf{C}$ as the field of scalars; in this book we shall be dealing, in the main, with complex spaces.)

Condition Ic is equivalent to the requirement that both the following conditions hold:

Ic$'$. There exists an element $0\in\Omega$ such that $u+0=u$ for all u;

Ic$''$. There exists for each $u \in \Omega$ a negative element $-u$ such that $u + (-u) = 0$.

Exercise 1.1. Prove that Condition Ic is equivalent to the pair of conditions Ic$'$ and Ic$''$.

Exercise 1.2. Deduce from the axioms that $0 \cdot u = 0$ and $(-1)u = -u$.

A subset X of Ω is called a *linear subspace* if it is itself a linear space with respect to the linear operations (addition and multiplication by scalars) inherited from X. If X is a linear subspace of Ω, then we can introduce an equivalence relation in Ω by defining two vectors $u, v \in \Omega$ to be equivalent if $u - v \in X$. Thus, the equivalence class of an element $u \in \Omega$ is the subset $\dot{u} = u + X \equiv \{u + v : v \in X\}$ of Ω. The set of equivalence classes is called the *quotient space* of Ω modulo the subspace X and is denoted by Ω/X. It is not difficult to see that this is a linear space if we define the linear operations thus: $(u+v)^{\cdot} = \dot{u} + \dot{v}$, $(\lambda u)^{\cdot} = \lambda \dot{u}$.

For a given set X in Ω there is a smallest subspace of Ω containing X; it is called the *linear span* of X (or the linear space spanned by X) and consists of all possible linear combinations of elements of X, that is, vectors of the form $\sum_{j=1}^{n} \lambda_j u_j$, where the λ_j are arbitrary scalars and $u_j \in X$. If the equality $\sum_{j=1}^{n} \lambda_j u_j = 0$ holds only when all the λ_j are equal to zero, then the vectors of X are said to be *linearly independent.* A space Ω in which any set of linearly independent vectors is finite is called finite-dimensional, and each such maximal set is called a *basis* in Ω. If a basis consists of n elements, then Ω is said to be n-dimensional.

The fundamental concepts of the theory of linear spaces are the linear functionals and linear operators. More generally, by a *functional* on a linear space Ω one means a (usually scalar-valued) function defined on Ω. A functional F is said to be *linear* if it takes values in the field of scalars and satisfies the linearity condition:

$$F(\lambda u + \mu v) = \lambda F(u) + \mu F(v), \tag{1.1}$$

where λ, μ are arbitrary scalars and $u, v \in \Omega$. (The value $F(u)$ of the linear functional at the element u is sometimes written in the form (F, u) or $\langle F, u \rangle$.)

Let Ω_1 and Ω_2 be two linear spaces over the same field of scalars; then by a *linear operator* from Ω_1 to Ω_2 we mean a function, say, $T : \Omega_1 \to \Omega_2$, defined on Ω_1 with values in Ω_2 and satisfying the linearity conditions of type (1.1). The value $T(u)$ of the linear operator at the element u is usually written as Tu. If $\Omega_1 = \Omega_2 = \Omega$, then T is called a linear operator in Ω. If $T(\Omega_1) = \Omega_2$ (that is, if the image of Ω_1 is the whole of Ω_2), then T is said to be an operator from Ω_1 onto Ω_2. A one-to-one linear map from Ω_1 onto Ω_2 is called an *isomorphism* of the linear spaces. (An isomorphism from Ω onto itself is called a *linear transformation* of Ω.)

A linear functional is a special case of an operator when Ω_2 is the field of scalars. Another important example of a linear operator is the *natural projection* $J : \Omega \to$ $\to \Omega/X$ that associates with the vector $u \in \Omega$ its equivalence class $\dot{u} = u + X$.

In the case when Ω_1 and Ω_2 are complex linear spaces, the notion of *anti-linear operator* is also commonly used; this is a function $T : \Omega_1 \to \Omega_2$ satisfying the anti-linearity condition

$$T(\lambda u + \mu v) = \bar{\lambda} T(u) + \bar{\mu} T(v) \tag{1.2}$$

for all $\lambda, \mu \in \mathbf{C}$, $u, v \in \Omega_1$. An example of an antilinear operator is the operation of complex conjugation in $\mathbf{C}^n : (Tu)_i = \bar{u}_i$.

There is an extension theorem for linear functionals according to which, each linear functional F_0 defined on a linear subspace X of Ω can be extended to a linear functional F on the whole of Ω.

The theorem is proved by means of Zorn's lemma, which is a special version of "transfinite induction". With later references to this in view, we give the lemma here in a form suitable for application. (The various formulations and a discussion of Zorn's lemma can be found in §1.5 of [K10]).

Lemma 1.1 (Zorn). *Let $\mathcal{A}$ be a family of subsets of some set A with the property that the union of any chain of subsets of the family $\mathcal{A}$ is contained in some subset of $\mathcal{A}$. Then the family $\mathcal{A}$ has a maximal subset.*

Here, a subfamily of $\mathcal{A}$ is called a *chain* if either $U \subset V$ or $V \subset U$ for any two elements U, V of it. A subset U of $\mathcal{A}$ is said to be *maximal* if for any V in $\mathcal{A}$, the condition $U \subset V$ implies $U = V$.

Exercise 1.3.(a) Prove the extension theorem for linear functionals. [Hint: Choose as the set A in Zorn's lemma the set $\Omega \times K$, where K is the field of scalars **R** or **C**; for the family $\mathcal{A}$ of subsets of A, choose the set of graphs of linear functionals F defined on linear subspaces of Ω containing X and coinciding with F_0 on X.]

(b) Let $u_1, \ldots, u_n$ be linearly independent vectors in Ω. Prove that there exist n linear functionals $f_1, \ldots, f_n$ on Ω such that $f_i(u_j) = \delta_{ij}$ (where δ_{ij} is the Kronecker delta). [Hint: Use the extension theorem.]

The *null space* (or *kernel*) is defined by $\ker T = \{u \in \Omega_1 : Tu = 0\}$; clearly this is a linear subspace of Ω_1.

Exercise 1.4. Suppose that the null space of the linear operator $T : \Omega_1 \to \Omega_2$ contains the linear subspace $X \subset \Omega_1$. Prove that there exists a unique linear operator $S : \Omega_1/X \to \Omega_2$ such that $T = Sp$, where $p : \Omega_1 \to \Omega_1/X$ is the natural projection. (In this case we say that the operator T is lowered by S from Ω_1 onto Ω_1/X.)

B. DIRECT SUM AND TENSOR PRODUCT OF LINEAR SPACES

Using the linear spaces at our disposal, one can construct new ones. Let $\{\Omega_\nu\}_{\nu\in N}$ be a family of linear spaces indexed by ν (running through an index set N). We consider the set Ω consisting of all possible families $\{u_\nu\}_{\nu\in N} \equiv u$, where the u_ν run through the elements of Ω_ν and for each such family u, only a finite number of the u_ν are non-zero. The element u_ν is called the νth projection (or component) of u. In particular Ω has a zero element all of whose projections are zero. It is easy to see that Ω becomes a linear space if we define the sum $u + v$ and product $\lambda \cdot u$ by a number λ in terms of the projections: $(u+v)_\nu = u_\nu + v_\nu$, $(\lambda u)_\nu = \lambda u_\nu$. It is called the (algebraic)* *direct sum* of the spaces Ω_ν.

Another method of constructing new spaces is tensor multiplication. Let $\Omega_1, \ldots$ $\ldots, \Omega_n$ be a finite family of linear spaces. We consider the set $\mathcal{F}$ consisting of all possible functions f of n variables $v_1 \in \Omega_1, \ldots, v_n \in \Omega_n$ taking values in the scalar field and such that $f(v_1, \ldots, v_n) = 0$ everywhere except at a finite number of points $(v_1, \ldots, v_n) \in \Omega_1 \times \ldots \times \Omega_n$. Clearly $\mathcal{F}$ becomes a linear space if we define the linear operations as follows:

$$(f+g)(v_1, \ldots, v_n) = f(v_1, \ldots, v_n) + g(v_1, \ldots, v_n),$$

* The algebraic direct sum (or algebraic tensor product) must be distinguished from the topological direct sum (or topological tensor product); see, for example, §1.1.E.

$$(\lambda f)(v_1, \dots, v_n) = \lambda \cdot f(v_1, \dots, v_n).$$

We associate with each fixed point $(u_1, \dots, u_n) \in \Omega_1 \times \dots \times \Omega_n$ the element $f_{u_1,\dots,u_n} \in$ $\in \mathcal{F}$ by setting

$$f_{u_1,\dots,u_n}(v_1, \dots, v_n) = \begin{cases} 0, & \text{if } (v_1, \dots, v_n) \neq (u_1, \dots, u_n), \\ 1, & \text{if } (v_1, \dots, v_n) = (u_1, \dots, u_n). \end{cases}$$

Then $\mathcal{F}$ is the linear span of all the $f_{u_1,\dots,u_n}$ (these elements form a basis in $\mathcal{F}$). We consider the linear subspace X of $\mathcal{F}$ spanned by vectors of the form

$$f_{u_1,\dots,u_\nu+v_\nu,\dots,u_n} - f_{u_1,\dots,u_\nu,\dots,u_n} - f_{u_1,\dots,v_\nu,\dots,u_n},$$

$$f_{u_1,\dots,\lambda u_\nu,\dots,u_n} - \lambda f_{u_1,\dots,u_\nu,\dots,u_n},$$

where the ν can take any values from 1 to n; $u_1, \dots, u_j, v_j, \dots, u_n$ are arbitrary vectors in the appropriate $\Omega_1, \dots, \Omega_n$, and λ is an arbitrary scalar. The quotient space $\mathcal{F}/X$ is called the (algebraic) *tensor product* of $\Omega_1 \dots, \Omega_n$. In particular, the equivalence class $\dot{f}_{u_1,\dots,u_n}$ of $f_{u_1,\dots,u_n}$ is called the tensor product of the elements $u_1, \dots, u_n$ and is denoted by $u_1 \otimes \dots \otimes u_n$.

Here are the main properties of the tensor product.

(a) The tensor product of $\Omega_1, \dots, \Omega_n$ is spanned by the elements $u_1 \otimes \dots \otimes u_n$.

(b) Multilinearity:

$$u_1 \otimes \dots \otimes (\lambda u_\nu + \mu v_\nu) \otimes \dots \otimes u_n =$$

$$= \lambda u_1 \otimes \dots \otimes u_\nu \otimes \dots \otimes u_n + \mu u_1 \otimes \dots \otimes v_\nu \otimes \dots \otimes u_n.$$

(c) If $v_1^{(1)}, \dots, v_\nu^{(m_\nu)}$ is a family of linearly independent vectors of Ω_ν, then the vectors $v_1^{(i_1)} \otimes \dots \otimes v_n^{(i_n)}$ (where the i_ν take all possible values from 1 to m_ν) are linearly independent.

Exercise 1.5. Prove the property (c) of the tensor product. [Hint: Suppose that

$$\sum_{i_1 \dots i_n} \lambda_{i_1 \dots i_n} v_1^{(i_1)} \otimes \dots \otimes v_n^{(i_n)} = 0. \tag{1.3}$$

To prove, for example, that $\lambda_{1 \dots 1} = 0$, define for each ν the linear functional h_ν on Ω_ν such that $h_\nu\left(u_\nu^{(1)}\right) = 1$ and $h_\nu(u_\nu^{(j)}) = 0$ for $j > 1$. Then define the linear functional H on the space $\mathcal{F}$ (defined above) as:

$$H(f) = \sum_{v_1 \in \Omega_1} \dots \sum_{v_n \in \Omega_n} f(v_1, \dots, v_n) h_1(v_1) \dots h_n(v_n).$$

Verify that this functional is lowered to a linear functional h on the quotient space $\mathcal{F}/X = \Omega_1 \otimes \dots$ $\otimes \Omega_n$, where $h\left(v_1^{(i_1)} \otimes \dots \otimes v_n^{(i_n)}\right)$ is equal to 1 for $i_1 = 1, \dots, i_n = 1$ and is equal to 0 for all other values of $i_1, \dots, i_n$. Finally, let h act on both sides of (1.3).]

Exercise 1.6. Suppose that each of the spaces Ω_ν is finite-dimensional and that the $\left\{v_\nu^{(j)}\right\}$ $(j =$ $= 1, \dots, m_\nu)$ forms a basis in Ω_ν. Prove that the vectors $v_1^{(i_1)} \otimes \dots \otimes v_n^{(i_n)}$ then form a basis in the tensor product of the $\Omega_1, \ldots, \Omega_n$, so that the dimension of the tensor product is equal to the product of the dimensions of the spaces $\Omega_1, \ldots, \Omega_n$.

C. NORMED SPACES

We now turn to the topological concepts. We begin with the simplest example of a linear topological space, namely, a normed space. A more general class of linear topological spaces (the class of locally convex spaces) will be introduced in §1.2.

A real function $p(u)$ defined on Ω is called a *norm* if it satisfies the following conditions:

(a) $p(\lambda u) = |\lambda| p(u)$ for any scalar λ (positive homogeneity);
(b) $p(u+v) \leq p(u) + p(v)$ (triangle inequality, or convexity of the norm);
(c) if $p(u) = 0$, then $u = 0$ (separation property).

It follows from (a) and (b) that the norm is non-negative:

$$0 = p(u-u) \leq p(u) + p(-u) = 2p(u).$$

Functions satisfying conditions (a) and (b) only (but not necessarily (c)) are called *seminorms*.

A linear space in which a norm is defined is called a *normed space*. The norm of an element u is sometimes denoted by $\|u\|$.

If we define the distance between two elements u, v as $d(u,v) = p(u-v)$, then the normed space becomes a *metric space* [that is, the set of Ω is endowed with a non-negative function $d(u,v)$ defined on $\Omega \times \Omega$, called a *metric*, which by definition must satisfy the conditions of symmetry ($d(u,v) \equiv d(v,u)$), the triangle inequality ($d(u,v) \leq d(u,w) + d(v,w)$ for all $u, v, w \in \Omega$), and separation ($d(u,v) = 0 \Rightarrow u = v$)]. We say that a sequence $\{u_n\}$ in Ω converges to u if the distance from u_n to u tends to zero as $n \to \infty$: $\lim\limits_{n\to\infty} d(u_n, u) = 0$. This type of convergence is called *convergence in norm* or *strong convergence*.

Exercise 1.7. (a) Prove that the norm is continuous, that is, $\|u_n\| \to \|u\|$ if $u_n \to u$ in norm as $n \to \infty$.

(b) Prove the continuity of the sum $u+v$ with respect to u, v and also the continuity of multiplication λu with respect to u, λ, that is, $u_n + v_n \to u + v$ if $u_n \to u$, $v_n \to v$ as $n \to \infty$, and $\lambda_n u_n \to \lambda u$ if $u_n \to u$, $\lambda_n \to \lambda$ as $n \to \infty$.

A sequence $\{u_n\}$ of elements of a normed (or metric) space Ω is called *fundamental* if for any $\epsilon > 0$, a number $N(\epsilon)$ can be found such that $d(u_n, u_m) < \epsilon$ for any $n > N(\epsilon)$ and $m > N(\epsilon)$ (that is, if $\lim\limits_{\min(m,n)\to\infty} d(u_n, u_m) = 0$). (A fundamental sequence is sometimes called convergent in itself or a Cauchy sequence.) It is not difficult to see that if a sequence $\{u_n\}$ converges to some element $u \in \Omega$ then it is fundamental. The converse does not always hold. A normed (or metric) space in which every fundamental sequence converges to some element of this space is called *complete*. A complete normed space is called a *Banach space*. There is a theorem that states that any normed space can be completed to form a Banach space (see, for example, [Y1], §1.10). A Banach space Ω is said to be *separable* if it contains a countable subset that is everywhere dense in Ω (that is, each element of Ω is the limit of a sequence of elements of this countable set).

A trivial example of a Banach space is n-dimensional real Euclidean space $\mathbf{R}^n$, in which the norm is defined by the formula

$$|x|^2 = \sum_{i=1}^{n} x_i^2. \tag{1.4}$$

Another example of a (this time complex) Banach space is the space $C([a,b])$ of continuous complex functions on the interval $[a,b]$ with norm

$$p(u) = \sup_{x\in[a,b]} |u(x)|. \tag{1.5}$$

Exercise 1.8. Prove that $C([a,b])$ is complete.

Our next example of a Banach space is the space $\mathcal{S}_{\lambda,m}(\mathbf{R}^n)$, which plays an important role in the theory of generalized functions. Here λ, m are arbitrary non-negative integers. The space $\mathcal{S}_{\lambda,m}(\mathbf{R}^n)$ consists of all complex functions of n real variables $x \equiv (x_1,\ldots,x_n) \in \mathbf{R}^n$ having continuous partial derivatives up to order m and decreasing at infinity at least as rapidly as $|x|^{-m}$. In other words, for each function u in $\mathcal{S}_{\lambda,m}(\mathbf{R}^n)$ all its derivatives of the form

$$x^\alpha D^\beta u(x) \equiv x_1^{\alpha_1}\ldots x_n^{\alpha_n} \frac{\partial^{\beta_1+\ldots+\beta_n} u(x)}{\partial x_1^{\beta_1}\ldots\partial x_n^{\beta_n}} \tag{1.6}$$

are bounded for all multi-indices α and β whose (integral) components satisfy

$$|\alpha| \equiv \alpha_1 + \ldots + \alpha_n \le \lambda; \quad |\beta| \equiv \beta_1 + \ldots + \beta_n \le m. \tag{1.7}$$

The norm in $\mathcal{S}_{\lambda,m}(\mathbf{R}^n)$ is defined by the equality

$$\|u\|_{\lambda,m} = \max_{\substack{|\alpha|\le\lambda \\ |\beta|\le m}} \sup_{x\in\mathbf{R}^n} |x^\alpha D^\beta u(x)|. \tag{1.8}$$

In the next subsection we shall be considering examples of Hilbert spaces, which are an important special case of Banach spaces.

Two normed spaces are said to be *isomorphic* * if there exists a one-to-one map between them that preserves the linear operations and the norms of the vectors (that is, a linear isometric map from one space onto the other).

Exercise 1.9. Let X be a closed linear subspace of the Banach space Ω.

(a) Prove that X is a Banach space in its own right (with the linear operations and norm inherited from Ω).

(b) Prove that the quotient space Ω/X is a Banach space with norm

$$\|\dot{u}\| = \inf_{v\in X} \|u+v\| \quad \text{for all } u \in \Omega. \tag{1.9}$$

D. Hilbert spaces

A function ω that associates with each pair u, v of elements of a complex linear space Ω a complex number $\omega(u,v)$ is called a *sesquilinear form* on Ω if $\omega(u,v)$ is linear in v and antilinear in u, that is, if

$$\omega(u, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1\omega(u,v_1) + \lambda_2\omega(u,v_2),$$

* More general is the notion of topological isomorphism of (normed) spaces (see §1.3.A).

$$\omega(\lambda_1 u_1 + \lambda_2 v_2, v) = \bar{\lambda}_1 \omega(u_1, v) + \bar{\lambda}_2 \omega(u_2, v)$$

for all $u, u_1, u_2, v, v_1, v_2 \in \Omega$, $\lambda_1, \lambda_2 \in \mathbf{C}$. If, moreover, the Hermitian condition

$$\omega(u, v) = \overline{\omega(v,\, u)}$$

holds, then ω is called a *Hermitian form.*

The expression $\omega(u, v)$ is called a *scalar product* of the vectors $u, v \in \Omega$ if this Hermitian form is *non-degenerate*, that is, if the condition $\omega(u, v) = 0$ for all $v \in \Omega$ implies that $u = 0$, in other words, if there exists for each element $u \neq 0$ an element $v \in \Omega$ such that $\omega(u, v) \neq 0$). The usual notation for the scalar product $\omega(u, v)$ is $\langle u, v\rangle$ (or $\langle u|v\rangle$).

It is clear that the "scalar square" $\langle u, u\rangle$ of any $u \in \Omega$ is a real number. If it can take all real values, then Ω is called a *space with an indefinite metric.*

For the present, we are interested in a different case. A Hermitian form ω is said to be *non-negative-definite* if the scalar square of any vector is non-negative:

$$\omega(u, u) \geq 0 \quad \text{for all } u \in \Omega, \tag{1.10}$$

and *positive-definite** if, in addition to (1.10) we have

$$\omega(u, u) = 0, \quad \text{only if} \quad u = 0. \tag{1.11}$$

It follows from the next exercise (more precisely from (1.12)) that a non-negative definite form ω is positive-definite if and only if it is non-degenerate, which means that the expression $\omega(u, v) \equiv \langle u, v\rangle$ can be called the scalar product of the vectors u, v. (It is not difficult to prove from this (by contradiction) that a space with an indefinite form contains a vector $u \neq 0$ for which $\langle u, u\rangle = 0$.)

Exercise 1.10. Let ω be a non-negative definite Hermitian form on Ω.

(a) Prove that the following inequality (called the Cauchy-Bunyakovsky-Schwarz inequality) holds for any vectors $u, v \in \Omega$:

$$|\omega(u, v)|^2 \leq \omega(u, u)\omega(v, v). \tag{1.12}$$

[Hint: The expression $\omega(\lambda u + v, \lambda u + v)$ is non-negative for all $\lambda \in \mathbf{C}$.]

(b) Prove that the expression

$$p(u) = \sqrt{\omega(u, u)} \tag{1.13}$$

is a seminorm on Ω; it is a norm if and only if ω is non-degenerate.

A space Ω with a positive-definite Hermitian form $\omega(u, v) = \langle u, v\rangle$ is called a (complex) *pre-Hilbert space.*

It follows from (1.10) that every pre-Hilbert space is a normed space with norm

$$\|u\| = \sqrt{\langle u, u\rangle}. \tag{1.14}$$

Furthermore, the Cauchy-Bunyakovsky-Schwarz inequality holds:

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|. \tag{1.15}$$

* Sometimes non-negative-definite and positive-definite forms are called positive-definite and strictly positive-definite respectively.

A complete pre-Hilbert space $\mathcal{H}$ is called a ***Hilbert space.*** *

Exercise 1.11. Prove that the scalar product is continuous in u, v, that is, $\langle u_n, v_n\rangle \to \langle u, v\rangle$ as $u_n \to u$, $v_n \to v$ in norm.

The notion of orthogonality can be defined in a Hilbert (and in a pre-Hilbert) space: two vectors Φ, Ψ are said to be ***orthogonal*** if their scalar product $\langle\Phi, \Psi\rangle$ is zero. A subset X of a Hilbert space $\mathcal{H}$ is said to be ***total*** if the linear span of X is everywhere-dense in $\mathcal{H}$, that is, the closure of this linear span is the whole of $\mathcal{H}$.

Exercise 1.12. (a) Prove the ***parallelogram law*** for the Hilbert space norm:

$$||\Phi + \Psi||^2 + ||\Phi - \Psi||^2 = 2||\Phi||^2 + 2||\Psi||^2. \tag{1.16}$$

(b) Prove the ***polarization identity***:

$$\langle\Phi, \Psi\rangle = \tfrac{1}{4}\sum_{\omega=\sqrt[4]{1}} \omega||\omega\Phi + \Psi||^2; \tag{1.17}$$

here ω runs through the four complex roots of unity (that is, $\omega = \pm 1, \pm i$).

The polarization identity shows that the scalar product in a Hilbert space is completely determined by the norm. Of interest is the question, what properties of the norm distinguish Hilbert spaces in the class of all Banach spaces. It turns out that the parallelogram law is such a characteristic property of Hilbert space (Jordan and von Neumann, 1935). In other words, if the norm in a Banach space $\mathcal{H}$ satisfies the parallelogram law, then $\mathcal{H}$ is a Hilbert space with scalar product (1.17). Prove this. Clearly, it suffices to verify the equations

$$\langle\Phi, \Psi_1 + \Psi_2\rangle = \langle\Phi, \Psi_1\rangle + \langle\Phi, \Psi_2\rangle \quad \text{for } \Phi, \Psi_1, \Psi_2 \in \mathcal{H}, \tag{1.18}$$

$$\langle\Phi, \lambda\Psi\rangle = \lambda\langle\Phi, \Psi\rangle \quad \text{for } \Phi, \Psi \in \mathcal{H}, \lambda \in \mathbf{C}. \tag{1.19}$$

It follows from (1.17) that

$$4\mathrm{Re}\langle\Phi, \Psi_1 + \Psi_2\rangle = ||\Phi + \Psi_1 + \Psi_2||^2 - ||\Phi - \Psi_1 - \Psi_2||^2.$$

We transform the right hand side by means of the parallelogram law:

$$= (2||\Phi + \Psi_1||^2 + 2||\Psi_2||^2 - ||\Phi + \Psi_1 - \Psi_2||^2) - ||\Phi - \Psi_1 - \Psi_2||^2 \equiv$$

$$\equiv ||\Phi + \Psi_1||^2 + ||\Phi + \Psi_1||^2 + 2||\Psi_2||^2 - (||\Phi + \Psi_1 - \Psi_2||^2 + ||\Phi - \Psi_1 - \Psi_2||^2 =$$

$$= ||\Phi + \Psi_1||^2 + (-||\Phi - \Psi_1||^2 + 2||\Phi||^2 + 2||\Psi_1||^2) + 2||\Psi_2||^2 -$$

$$-(2||\Phi - \Psi_2||^2 + 2||\Psi_1||^2) = (||\Phi + \Psi_1||^2 - ||\Phi - \Psi_1||^2) + (2||\Phi||^2 + 2||\Psi_2||^2 - 2||\Phi - \Psi_2||^2) =$$

$$= 4\mathrm{Re}\langle\Phi, \Psi_1\rangle + 4\mathrm{Re}\langle\Phi, \Psi_2\rangle.$$

Hence and from the identity $\mathrm{Im}\langle\Phi, \Psi\rangle \equiv \mathrm{Re}\langle i\Phi, \Psi\rangle$, (1.18) follows. For $\lambda = \sqrt[4]{1}$, (1.19) follows from (1.17); therefore it suffices to prove (1.19) for $\lambda > 0$. For integral $\lambda > 0$ and for rational $\lambda > 0$, (1.19) follows from (1.18); for the remaining $\lambda > 0$, it follows from the continuity of the scalar product.

Exercise 1.13. Let $\mathcal{H}_1$ be a closed linear subspace of $\mathcal{H}$, and Φ any vector in $\mathcal{H}$.

(a) Prove that there exists a vector $\Phi_1 \in \mathcal{H}_1$ such that

$$\inf_{\Psi\in\mathcal{H}_1} ||\Phi - \Psi|| = ||\Phi - \Phi_1||. \tag{1.20}$$

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* We deal here only with complex Hilbert spaces, these being of primary interest to us.

[Hint: Let $d = \inf_{\Psi \in \mathcal{H}_1} \|\Phi - \Psi\|$ and Ψ_n a sequence in $\mathcal{H}_1$ such that $\|\Phi - \Psi_n\| \to d$ as $n \to \infty$. Deduce from the parallelogram identity (1.16) applied to the vectors $\Phi - \Psi_m$, $\Phi - \Psi_n$, that $\|\Psi_m - \Psi_n\| \to 0$ as $\min(m,n) \to \infty$, that is, the sequence Ψ_n is fundamental.]

(b) Set $\Phi_2 = \Phi - \Phi_1$. Prove that the vector Φ_2 is orthogonal to $\mathcal{H}_1$. [Hint: it follows from (1.20) that $\|\Phi_2 + \lambda\Psi_1\|^2 \geq \|\Phi_2\|^2$ for all $\Psi_1 \in \mathcal{H}_1$, $\lambda \in \mathbf{C}$.]

It is clear that every closed linear subspace $\mathcal{H}_1$ of a Hilbert space $\mathcal{H}$ is itself a Hilbert space. The set

$$\mathcal{H}_1^\perp = \{\Phi \in \mathcal{H} : \langle \Phi, \Psi \rangle = 0 \quad \text{for all } \Psi \in \mathcal{H}_1\} \tag{1.21}$$

is called the *orthocomplement* (or *orthogonal complement*) of the subspace $\mathcal{H}_1$.

Exercise 1.14. Let $\mathcal{H}_1$ be a closed linear subspace of the Hilbert space $\mathcal{H}$.

(a) Prove that $\mathcal{H}_1^\perp$ is a closed linear subspace of $\mathcal{H}$ and that $\mathcal{H}_1 \cap \mathcal{H}_1^\perp = \{0\}$.

(b) Prove that any vector in $\mathcal{H}$ can be uniquely represented in the form $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 \in \mathcal{H}_1$, $\Phi_2 \in \mathcal{H}_1^\perp$. [Hint: Use the preceding exercise and part (a) of this exercise.]

The result of part (b) of Exercise 1.14 can also be stated as follows: the Hilbert space $\mathcal{H}$ can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$; this is written in the form

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp.$$

(The orthocomplement is sometimes called "orthogonal difference" and one uses the notation: $\mathcal{H}^\perp = \mathcal{H} \ominus \mathcal{H}_1$.) In the decomposition $\Phi = \Phi_1 + \Phi_2$, the vectors $\Phi_1 \in \mathcal{H}_1$, $\Phi_2 \in \mathcal{H}_1^\perp$ are called the (*orthogonal*) *projections* of the vector Φ onto the subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$.

Exercise 1.15. (a) Prove that a set $X \subset \mathcal{H}$ is total if and only if the set of vectors in $\mathcal{H}$ that are orthogonal to X (that is, orthogonal to all the vectors in X) consists solely of the zero vector. [Hint: Apply Exercise 1.14 to the closed linear span of X.]

(b) Prove that a Hilbert space $\mathcal{H}$ is separable if and only if there exists a finite or countable orthonormal (that is, orthogonal and normalized) family of vectors $\{e_n\}$, $\langle e_i, e_j\rangle = \delta_{ij}$ forming a total set in $\mathcal{H}$.* [Hint: Use the orthogonalization process for a sequence of vectors.]

A trivial example of a Hilbert space is n-dimensional complex Euclidean space $\mathbf{C}^n$ with scalar product

$$\langle \Phi, \Psi \rangle = \sum_{i=1}^{n} \bar{\Phi}_i \Psi_i. \tag{1.22}$$

The complex measurable functions of a real variable x with square-integrable modulus over the interval $[a,b]$ form a Hilbert space. More precisely, it is the space of equivalence classes of functions, where functions differing only on a set of Lebesgue measure zero are deemed to be equivalent. This space is usually denoted by $\mathcal{L}^2([a,b])$. The scalar product is defined in it by the formula

$$\langle \Phi, \Psi \rangle = \int_a^b \bar{\Phi}(x)\Psi(x)dx. \tag{1.23}$$

One often obtains Hilbert spaces by means of the following general construction. Suppose that we are given a non-negative definite Hermitian form ω on the complex

* We call such a family of vectors an *orthonormal* basis of the (finite- or countable-dimensional) Hilbert space.

linear space Ω. Then (1.13) defines a seminorm $p(u)$ on Ω. Let Ω_0 be the set of all vectors in Ω whose seminorms are zero, that is, $\Omega_0 = \{u \in \Omega : p(u) = 0\}$. Clearly Ω_0 is a linear space and it follows from (1.12) that Ω_0 is orthogonal to all the vectors of Ω. It is not difficult to see that the formula

$$\langle \dot{u}, \dot{v} \rangle = \omega(u, v) \quad \text{for all } u, v \in \Omega \tag{1.24}$$

well-defines a positive-definite Hermitian form (or scalar product) on the quotient space Ω/Ω_0, thus converting it into a pre-Hilbert space. On completing this, we obtain a Hilbert space $\mathcal{H}$ (in which Ω/Ω_0 is an everywhere-dense linear subspace).

E. DIRECT SUM AND TENSOR PRODUCT OF HILBERT SPACES

In §1.1.B we introduced the notion of algebraic direct sum and algebraic tensor product of linear spaces. In the case when the original spaces are Hilbert spaces, the resultant spaces are in general (incomplete) pre-Hilbert spaces which need to be completed so as to obtain Hilbert spaces. We consider these constructions in greater detail.

Suppose that we are given a family of Hilbert spaces $\{\mathcal{H}_\nu\}_{\nu \in N}$. We consider the family $\Phi = \{\Phi_\nu\}_{\nu \in N}$, where the Φ_ν are arbitrary vectors in $\mathcal{H}_\nu$ subject to the condition:

$$\|\Phi\|^2 \equiv \sum_\nu \|\Phi_\nu\|^2 < \infty. \tag{1.25}$$

It is clear that $\Phi_\nu \neq 0$ for at most a finite or countable set of indices ν. The set of all such families is denoted by $\bigoplus_{\nu \in N} \mathcal{H}_\nu$ and is called the *direct sum of Hilbert spaces.*

It is obvious that the direct sum is endowed with a linear-space structure, the linear operations being carried out componentwise. Furthermore, it becomes a Hilbert space if we define the scalar product of vectors Φ,Ψ by the formula

$$\langle \Phi, \Psi \rangle = \sum_{\nu \in N} \langle \Phi_\nu, \Psi_\nu \rangle. \tag{1.26}$$

We suggest that the reader verifies that the completeness condition for a Hilbert space does in fact hold.

Exercise 1.16. (a) Let Ω be the algebraic direct sum of the family of Hilbert spaces $\{\mathcal{H}_\nu\}$. Prove that Ω is an everywhere-dense subspace of $\bigoplus_\nu \mathcal{H}_\nu$.

(b) Suppose that all the $\mathcal{H}_\nu$ are separable and distinct from $\{0\}$. Prove that their direct sum $\bigoplus \mathcal{H}_\nu$ is separable if and only if the index set N is at most countable. [Hint: Use Exercise 1.15.]

In the preceding construction, the original spaces $\mathcal{H}_\nu$ appear somehow to be external to the direct sum. However, they can be naturally identified with definite subspaces of the direct sum. Thus we fix an index $\lambda \in N$ and associate with each $u \in \mathcal{H}_\lambda$ the vector Φ of the direct sum for which $\Phi_\lambda = u$, while all the remaining components Φ_ν of Φ are zero. As a result we obtain a map $\mathcal{H}_\lambda \to \bigoplus_\nu \mathcal{H}_\nu$ which is an isomorphism of the Hilbert space $\mathcal{H}_\lambda$ onto a closed linear subspace $\mathcal{H}'_\lambda \subset \bigoplus \mathcal{H}_\nu$ (consisting of all the vectors of the direct sum whose components Φ_ν are zero for $\nu \neq \lambda$). On identifying $\mathcal{H}_\lambda$ and $\mathcal{H}'_\lambda$ we find that $\bigoplus \mathcal{H}_\nu$ is the direct sum of the subspaces $\mathcal{H}_\nu$.

Exercise 1.17. Let $\{\mathcal{H}_\nu\}$ be a family of closed linear subspaces of the Hilbert space $\mathcal{H}$. Prove that $\mathcal{H}$ is the direct sum of these subspaces $\mathcal{H}_\nu$ if and only if they are pairwise orthogonal and there does not exist a non-zero vector in $\mathcal{H}$ orthogonal to all of the $\mathcal{H}_\nu$.

By choosing an orthonormal basis in a Hilbert space $\mathcal{H}$, we obtain an example of a direct-sum decomposition of $\mathcal{H}$ into one-dimensional subspaces. By an orthonormal basis we mean any family $\{e_\nu\}_{\nu\in N}$ of vectors in $\mathcal{H}$ with the properties: 1) $\langle e_\lambda, e_\mu\rangle = \delta_{\lambda\mu}$, 2) the set $\{e_\nu\}_{\nu\in N}$ is total in $\mathcal{H}$. It is easy to conclude from this that any vector $u\in\mathcal{H}$ can be uniquely represented in the form

$$\Phi = \sum_{\nu\in N} \lambda_\nu e_\nu, \tag{1.27a}$$

where

$$\lambda_\nu = \langle e_\nu, \Phi\rangle. \tag{1.27b}$$

The sum of the series (1.27a) is to be understood in the following sense: for any $\epsilon > 0$ there exists a finite index subset $M_0 \subset N$ such that $\left\|\Phi - \sum_{\nu\in M}\lambda_\nu e_\nu\right\| < \epsilon$ for every finite index subset $M \supset M_0$. The scalar product of two vectors $\Phi = \sum_\nu \lambda_\nu e_\nu$ and $\Psi = \sum_\nu \mu_\nu e_\nu$ is given by the formula

$$\langle\Phi,\Psi\rangle = \sum_{\nu\in N}\bar{\lambda}_\nu\mu_\nu.$$

Every Hilbert space has an orthonormal basis $\{e_\nu\}_{\nu\in N}$ and the cardinality of this basis (that is, the cardinality of the index set N) is called the dimension of the Hilbert space (see, for example, [Y1] Ch. III, §4).

We now turn to the tensor product. Let Ω be the algebraic tensor product of a finite family of Hilbert spaces $\mathcal{H}_1,\dots,\mathcal{H}_n$. An arbitrary pair of vectors Φ, Ψ in Ω can be represented in the form

$$\Phi = \sum_{i=1}^{M} u_1^{(i)}\otimes\ldots\otimes u_n^{(i)}, \quad \Psi = \sum_{j=1}^{N} v_1^{(j)}\otimes\ldots\otimes v_n^{(j)}, \tag{1.28}$$

where $u_\nu^{(i)}, v_\nu^{(j)} \in \mathcal{H}_\nu$. We define the scalar product of this pair of vectors by the formula

$$\langle\Phi,\,\Psi\rangle = \sum_{i=1}^{M}\sum_{j=1}^{N}\langle u_1^{(i)}, v_1^{(j)}\rangle\ldots\langle u_n^{(i)}, v_n^{(j)}\rangle. \tag{1.29}$$

However, since the representation (1.28) is not unique, it is not clear that the right hand side of (1.29) depends only on the vectors u, v and not on the specific representation (1.28). The following exercise shows that the scalar product is well defined by (1.29), that is, the right hand side does not depend on the choice of the representation (1.28).

Exercise 1.18. (a) We define a Hermitian form on the space $\mathcal{F}$ of §1.1.B by setting

$$\omega(f,g) = \sum_{u_1,v_1\in\Omega_1,\dots,u_n,v_n\in\Omega_n} \overline{f(u_1,\dots,u_n)}g(v_1,\dots,v_n)\langle u_1,v_1\rangle\ldots\langle u_n,v_n\rangle. \tag{1.30}$$

Prove that the form ω is non-negative definite.

(b) Prove that if $f \in X$ or $g \in X$, then $\omega(f,g) = 0$. Deduce that ω can be lowered onto $\mathcal{F}/X = \Omega$, that is, there exists a form $\tilde{\omega}$ on Ω such that $\tilde{\omega}(\dot{f},\dot{g}) = \omega(f,g)$ for all $f,g \in \mathcal{F}$. Verify that $\tilde{\omega}(u,v)$ is the same as the expression $\langle u,v\rangle$ defined by (1.29).

(c) Let the vector Φ be as in (1.28) and let $\{e_\nu^{(j)}\}_{j=1,\dots,m_\nu}$ be an orthonormal basis in the subspace of $\mathcal{H}_\nu$ spanned by the vectors $u_\nu^{(1)},\dots,u_\nu^{(M)}$. Then Φ can be expressed in the form

$$\Phi = \sum_{i_1,\dots,i_n} \lambda_{i_1\dots i_n} e_1^{(i_1)} \otimes \dots \otimes e_n^{(i_n)}.$$

Prove that

$$\|\Phi\|^2 = \sum_{i_1,\dots i_n} |\lambda_{i_1\dots i_n}|^2.$$

Deduce that the Hermitian form $\langle\Phi,\Psi\rangle$ is positive-definite.

Thus, according to Exercise 1.18, Ω is a pre-Hilbert space with scalar product (1.29). Its completion is a Hilbert space called the *tensor product of the Hilbert spaces* $\mathcal{H}_1,\dots,\mathcal{H}_n$, denoted by $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$.

Exercise 1.19. Let $\left\{e_\nu^{(\alpha)}\right\}_{\alpha\in\mathcal{A}_\nu}$ be an orthonormal basis in the Hilbert space $\mathcal{H}_\nu$. Prove that the vectors $e_1^{(\alpha_1)} \otimes \dots \otimes e_n^{(\alpha_n)}$ (where α_ν runs through $\mathcal{A}_\nu$) form an orthonormal basis in $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$.

F. Linear functionals and dual spaces

As is well known from analysis, the continuity of a scalar-valued functional on a normed space can be defined by two equivalent methods — in terms of sequences and in terms of "epsilons and deltas".

Exercise 1.20. Prove that a linear functional F on a normed Ω is continuous if and only if there exists a constant $c \geq 0$ (dependent on F) such that

$$|F(u)| \leq c\|u\| \quad \text{for all } u \in \Omega. \tag{1.31}$$

Thus for a continuous linear functional on a normed space we can define the quantity

$$p'(F) = \sup_{u\in\Omega,\|u\|\leq 1} |F(u)|. \tag{1.32}$$

Continuous linear functionals on a normed space are also called *bounded linear functionals.* (As a rule we only deal with bounded linear functionals, so that we shall sometimes simply call them linear functionals, for brevity.) The set of all continuous linear functionals on the normed space Ω is called the *dual space*, denoted by Ω' (sometimes the notation Ω^* is used). It is a normed space in which the norm (also denoted by $\|F\|$) is defined by (1.32).

Exercise 1.21. Prove that the dual of a normed space is a Banach space.

We now consider some examples of dual spaces.

1) The space $\mathcal{C}'([a,b])$ dual to the space $\mathcal{C}([a,b])$ defined in §1.1.C, consists of functionals of Stieltjes-integral form

$$F(u) = \int_a^b u(x)d\phi(x), \tag{1.33}$$

where $\phi(x)$ is a function of bounded variation (see [K2],Ch.VI,§3) (Riesz representation theorem). The norm of the functional is equal to the total variation of ϕ:

$$p'(F) = V_a^b \phi(x) \equiv \sup_k \sup_{a<x_1<\ldots<x_k<b} \sum_{j=1}^{k-1} |\phi(x_{j+1}) - \phi(x_j)|. \tag{1.34}$$

A linear functional $F \in \mathcal{C}'([a,b])$ is said to be positive if $F(u) \geq 0$ for any non-negative function $u \in \mathcal{C}([a,b])$. There is another theorem due to Riesz which states that any positive functional on $\mathcal{C}([a,b])$ can be represented in the form (1.33) with monotone increasing ϕ. In what follows we shall also be using the generalization of Riesz's theorem to several dimensions: let $\mathcal{C}(K)$ be the space of complex continuous functions on the compact subset $K \subset \mathbf{R}^n$; then the dual $\mathcal{C}'(K)$ consists of functionals of the form

$$F(u) = \int u(x) d\mu(x), \tag{1.35}$$

where μ is an arbitrary (complex Borel) measure on K (see, e.g., [K2],Ch. VI, §3).

2) The dual $\mathcal{H}'$ of a Hilbert space consists of functionals of the form

$$F(\Phi) = \langle \Psi, \Phi \rangle, \tag{1.36}$$

where $\Psi \in \mathcal{H}$ (Riesz's theorem; see, for example, [K2],Ch.V,§3). It is not difficult to see that the norm of the functional F is equal to that of the corresponding vector $\Psi : p'(F) = \|v\|$. Thus (1.36) establishes a one-to-one antilinear map from $\mathcal{H}$ onto $\mathcal{H}'$ that is norm-preserving.*

The Hahn-Banach extension theorem plays an important role in the theory of dual spaces.

Theorem 1.2 (Hahn-Banach). *Let Ω be a normed space with norm p, and F_0 a continuous linear functional defined on a linear subspace $\Omega_0 \subset \Omega$. Then there exists a continuous linear functional F defined on the whole of Ω such that $F(u) = F_0(u)$ for $u \in \Omega_0$ and $p'(F) = p'(F_0)$.*

For a proof (which makes essential use of Zorn's lemma) and various applications of this theorem see, for example, [K2],ChII,§4.

It follows from the Hahn-Banach theorem that there exists a non-trivial continuous linear functional (that is, not identically zero) on any normed space Ω (other than $\{0\}$). It suffices to assign an arbitrary non-zero value to the functional F at some element $u \in \Omega$, $u \neq 0$, say, $F(u) = \|u\|$; we can then define the linear functional F_0 on the one-dimensional linear subspace Ω_0 spanned by u ($F_0(\lambda u) = \lambda \|u\|$) and apply the extension theorem to extend F_0 (as a continuous linear functional) onto the whole of Ω (where F can be chosen so that $\|F\| = 1$).

In fact, far-reaching conclusions can be drawn from this argument. We say that a given set S of linear functionals on a linear space Ω *separates the points of* Ω if the condition $F(u_1) = F(u_2)$ for all $F \in S$, implies that $u_1 = u_2$

Exercise 1.22. Prove that the dual of a normed space Ω separates the points of Ω.

* In Dirac's notation, "*ket vectors*" (that is, elements of $\mathcal{H}$) are converted into "*bra vectors*" (elements of $\mathcal{H}'$).

The space $\Omega'' \equiv (\Omega')'$ dual to Ω' is called the *second dual* of the normed space Ω. It is a Banach space with norm p''. The space Ω can always be regarded as a linear subspace of Ω'' (with norm induced by Ω'').

Exercise 1.23. We associate with each element $u \in \Omega$ the functional $\sigma(u)$ on Ω' according to the formula $\sigma(u)(F) = F(u)$, $F \in \Omega'$. Prove that the map $\sigma : \Omega \to \Omega''$ is a linear isometric map.

Thus we can identify Ω with a subspace of Ω'' by means of the map σ (of Exercise 1.23). If $\Omega = \Omega''$, that is, if σ is an isomorphism, then Ω is said to be *reflexive*.

The space $C([a, b])$ of Example 1) is non-reflexive, whereas every Hilbert space is reflexive.

1.2. Locally Convex Spaces

A. Equivalent systems of seminorms. Structure of LCS's.

In addition to the normed spaces, an important role is played by the more general classes of linear spaces with convergence. Of these, the Fréchet spaces, or F-spaces, are the closest generalizations of normed spaces; the aim of this section is to acquaint the reader with such spaces.

Let Ω be a linear space (say for definiteness, complex). To estimate the degree of closeness of an arbitrary vector $u \in \Omega$, we take some fixed family of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ on Ω; here α is an index distinguishing the seminorms and taking values in some (finite or infinite) set $\mathcal{A}$. We say that the system of seminorms $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ on Ω is *subordinated* to the system of seminorms $\{q_\beta\}_{\beta \in \mathcal{B}}$ on Ω if there exists for any $\alpha \in \mathcal{A}$ a finite set of indices $\beta_1, \dots, \beta_k$ in $\mathcal{B}$ and a number $c \geq 0$ such that $p_\alpha \leq c \sup_{j=1,\dots,k} q_{\beta_j}$.

According to the next exercise, the quantity on the right hand side of this last inequality is a seminorm on Ω.

Exercise 1.24. Let $\{p_\alpha\}_{\alpha \in \mathcal{A}}$ be a system of seminorms on Ω such that the quantity $p(u) = \sup_{\alpha \in \mathcal{A}} p_\alpha(u)$ is finite for all $u \in \Omega$. Prove that p is a seminorm on Ω.

Two systems of seminorms on Ω are said to be *equivalent* if each system is subordinated to the other. A linear space Ω endowed with a system of seminorms $\{p_\alpha\}$ is called a *locally convex space* (LCS for short) where by definition, two different systems of seminorms define the same LCS structure on Ω if and only if they are equivalent.* A given system of seminorms on Ω defining an LCS structure is called a *defining system of seminorms*; we also use the more detailed notation $(\Omega, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ for an LCS; this notation contains a reference to the determining system of seminorms.

A determining system of seminorms on an LCS can, without any harm, be replaced by an equivalent system. Because of this, we can (and shall) henceforth assume that the determining system $\{p_\alpha\}$ of seminorms satisfies the condition: any finite subsystem is subordinated to some seminorm of the given system. (If this condition is not satisfied, then we replace the original system by the equivalent system of seminorms $p_{\alpha_1 \dots \alpha_k} = \sup_{j=1,\dots,k} p_{\alpha_j}$; here k runs through all the natural numbers and $\alpha_1, \dots, \alpha_k$ run through all the indices in $\mathcal{A}$.) The aim of the above understanding,

* It will be clear from what follows that LCS structures are the same if and only if the topologies defined by them are the same.

which does not sacrifice any generality, is to simplify a number of the statements. (For instance, relations of type (1.37), (1.43) (see below) would have to be replaced by something much more cumbersome.)

An LCS $(\Omega, \{p_\alpha\})$, is called *separated* if the condition $p_\alpha(u - v) = 0$ for all $\alpha \in \mathcal{A}$ implies that $u = v$, that is, if for any non-zero $u \in \Omega$ there exists an index $\alpha \in \mathcal{A}$ such that $p_\alpha(u) > 0$. (In particular, an LCS $(\Omega, \{p_\alpha\})$ is separated if at least one of the seminorms p_α is a norm.) We can define the notion of a limit in a separated LCS in similar fashion to that for normed spaces. Namely, a sequence u_k in Ω converges to an element $u \in \Omega$ ($u_k \to u$ or $u = \lim_{k\to\infty} u_k$) if $\lim_{k\to\infty} p_\alpha(u_k - u) = 0$ for any $\alpha \in \mathcal{A}$; in view of the above separatedness condition , the limit, if it exists, is unique.

From now on we shall suppose (without special mention) that all our LCS's are separated.

If X is a linear subspace of the LCS $(\Omega, \{p_\alpha\})$, then the restrictions of the seminorms p_α to X define an LCS structure on X called the *induced structure* (induced from Ω).

All the topological concepts which, undoubtedly, are well known to the reader in the context of Euclidean spaces, can easily be carried over to an arbitrary LCS. Thus, the analogues of open balls at the origin (or at the point $u \in \Omega$) are the sets

$$V_\alpha^\epsilon = \{v \in \Omega : p_\alpha(v) < \epsilon\} \tag{1.37}$$

(or $u + V_\alpha^\epsilon$) associated with each index $\alpha \in \mathcal{A}$ and number $\epsilon > 0$. A subset X of Ω is said to be *open* (in Ω) if for any point $u \in X$, a set of the form $u + V_\alpha^\epsilon$ is contained in X (for some $\alpha \in \mathcal{A}$, $\epsilon > 0$). The complement of an open set in Ω is called a *closed set* (in Ω). The *closure* of a subset X of Ω is the smallest closed set containing X; it is denoted by $\overline{X}$. By a *neighbourhood* of a point $u \in \Omega$ we mean an open set containing this point. We say that a set $X \subset \Omega$ is *dense* in the set $Y \subset \Omega$ if $X \subset Y$ and $\overline{X} \supset Y$. (The latter inclusion is equivalent to the property that any neighbourhood of a point of Y has a non-empty intersection with X.)

We can summarize the above discussion as follows: every LCS has a canonically defined topology.

B. FRÉCHET SPACES

It should be noted that for the case of normed (and in particular, Euclidean) spaces, the notions of closedness and closure can be restated in terms of sequences (that is, sequential properties). Thus, to say that a set X is closed in a normed space Ω means that X contains the limit of any sequence of points of X that converges in Ω, while the relation $\overline{X} = Y$ means that Y is the set of limits of all possible sequences of points of X that converge in Ω.

To enable us to go over to such a characterization in an LCS $(\Omega, \{p_\alpha\})$, we have to impose the condition: one can choose from the family V_α^ϵ of neighbourhoods of the origin, a family $\{V_{\alpha_k}^{\epsilon_k}\}_{k=1,2,\ldots}$ that forms a countable basis of neighbourhoods of zero. (By definition, such a basis has the property that each V_α^ϵ contains at least one set $V_{\alpha_k}^{\epsilon_k}$ of the chosen family.) As in the case of normed spaces, the existence of such a basis is decisive for formulating closedness and closure (also completeness and continuity) in terms of sequences. In particular, this condition turns out to be satisfied for all LCS's with a finite or countable defining system of seminorms. (The case of a finite number of seminorms can be regarded as a special case of a countable number, since the addition of seminorms subordinated to the original system leads to an equivalent system.)

Exercise 1.25. (a) In an LCS with a countable determining system of seminorms $\{p_\alpha\}_{\alpha\in\mathcal{A}}$ there exists a countable basis of neighbourhoods of zero. [Hint: consider a family of sets $V_\alpha^{\epsilon_k}$, where α runs through the countable index set $\mathcal{A}$ and the ϵ_k are a sequence of positive numbers converging to zero.]

(b) The topology of an LCS Ω with a countable system of seminorms $\{p_k\}_{k=1}^\infty$ is the same as the topology on Ω defined by the metric

$$d(u,v) = \sum_{k=1}^\infty k^{-2} \frac{p_k(u-v)}{1+p_k(u-v)}.$$

We now single out from the LCS's whose determining systems of seminorms can be chosen to be countable, a class of spaces of the greatest interest. A separated LCS with a countable system of seminorms is called *complete* if every Cauchy sequence in Ω converges in Ω. (By analogy with the case of normed spaces, a sequence u_k in Ω is, by definition, a Cauchy sequence if for any $\alpha \in \mathcal{A}$, $p_\alpha(u_n - u_m) \to 0$ as $\min(n,m) \to \infty$.) A separated complete LCS with a countable system of seminorms is called an *F-space* (or *Fréchet space*). We shall, in the main, be concentrating attention on such spaces.

It follows from the metrizability of F-spaces (see Exercise 1.25(b)) that the following theorem is applicable.

Theorem 1.3 (Baire category theorem). *The intersection of any countable family $\{\mathcal{M}_k\}_{k=1}^\infty$ of open dense subsets $\mathcal{M}_k$ of a complete metric space Ω is dense in Ω.*

■ It suffices to prove that for any point $u_0 \in \Omega$ and any $\epsilon_0 > 0$, the open ball $U_0 = \{u \in \Omega : d(u,u_0) < \epsilon_0\}$ with centre at u_0 and radius ϵ_0 has a non-empty intersection with $\bigcap \mathcal{M}_k$. It follows from the density of $\mathcal{M}_1$ in Ω that there exists $u_1 \in \mathcal{M}_1 \cap U_0$. Since $\mathcal{M}_1$ is open, there is a closed ball $\overline{U}_1$ with centre at u_1 and radius ϵ_1 $(0 < \epsilon_1 < \epsilon_0)$ contained in $\mathcal{M}_1 \cap U_0$. By continuing this process by induction, we can prove the existence of a sequence of points $u_k \in \Omega$ and a monotone decreasing sequence of numbers $\epsilon_k \to 0$ such that each $\mathcal{M}_k \cap U_{k-1}$ contains a closed ball $\overline{U}_k$ with centre at u_k and radius ϵ_k. By construction, u_k is a Cauchy sequence, that is, $d(u_m,u_n) \to 0$ as $\min(m,n) \to \infty$. Consequently it has a limit u_∞ belonging to $\overline{U}_{k+1}$ for each $k \geq 0$. Since by construction, $\overline{U}_{k+1} \subset \mathcal{M}_k \cap U_k$, u_∞ belongs to the intersection of U_0 with $\bigcap_{k=1}^\infty \mathcal{M}_k$. ■

C. Examples

As is clear, Banach spaces belong to the class of Fréchet spaces. We now mention a number of other (complex) Fréchet spaces that are encountered in applications.

1) The space $C(\mathcal{O})$. Let $\mathcal{O}$ be an open subset of $\mathbf{R}^n$. Then $C(\mathcal{O})$ denotes the space of all complex continuous functions in (or on) $\mathcal{O}$ endowed with the system of seminorms

$$\|u\|^K = \sup_{x\in K} |u(x)|, \tag{1.38}$$

where K runs through all the compacta * in $\mathcal{O}$ (or a countable family of compacta

* By a *compactum* in $\mathcal{O}$ we mean a set $K \subset \mathcal{O}$ such that any sequence of points of K has a subsequence converging to a limit in K. (For the special case under consideration (namely $\mathbf{R}^n$), a compactum K can be defined as a bounded subset of $\mathcal{O}$ that is closed in $\mathbf{R}^n$.)

whose interiors cover $\mathcal{O}$). In this example, $\mathcal{O}$ is in fact allowed to be an arbitrary locally compact subset of $\mathbf{R}^n$. *

2) The space $\mathcal{E}(\mathcal{O})$. Let $\mathcal{O}$ be an open subset of $\mathbf{R}^n$. We denote by $\mathcal{E}(\mathcal{O})$ the space of all complex infinitely differentiable ($\mathcal{C}^\infty$ for short) functions on $\mathcal{O}$. We use the following notation for the partial derivatives of a function $u \in \mathcal{E}(\mathcal{O})$:

$$D^\alpha u(x) = \frac{\partial^{\alpha_1+\ldots+\alpha_n}}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} u(x), \tag{1.39}$$

where $\alpha \equiv (\alpha_1, \ldots, \alpha_n)$ is an ordered set of n non-negative integers (called the *n-index* or *multi-index*); here $D^\alpha u(x) \equiv u(x)$ when $\alpha = (0, \ldots, 0)$. The order of the derivative D^α is denoted by

$$|\alpha| = \alpha_1 + \ldots + \alpha_n. \tag{1.40}$$

Next we endow $\mathcal{E}(\mathcal{O})$ with the system of seminorms

$$\|u\|_{l,0}^K = \max_{|\alpha|\le l} \sup_{x\in K} |D^\alpha u(x)|; \tag{1.41}$$

where l can take all possible non-negative values and K runs through all the compacta of $\mathcal{O}$ (or a countable family of compacta whose interiors cover $\mathcal{O}$).

In mathematical analysis ([S4],Ch. IV,§8) there is a $\mathcal{C}^\infty$ variant of the classical *Weierstrass polynomial approximation theorem*: for any $\mathcal{C}^\infty$ function $u(x)$ defined on an open set $\mathcal{O}$, any compactum $K \subset \mathcal{O}$ and any $l \in \overline{\mathbf{Z}}_+$,** $\epsilon > 0$, there exists a complex polynomial $P(x)$ defined on $\mathcal{O}$ (or, what is the same, on $\mathbf{R}^n$) such that $\|u - P\|_{0,l}^K < \epsilon$. In other words, the complex polynomials on $\mathcal{O}$ form an everywhere-dense linear subspace of $\mathcal{E}(\mathcal{O})$.

3) The space $\mathcal{S}(\mathbf{R}^n)$. Let $\mathcal{S}(\mathbf{R}^n)$ be the space of $\mathcal{C}^\infty$ functions on $\mathbf{R}^n$ for which the expressions

$$\|u\|_{l,m} = \max_{|\alpha|\le l} \sup_{x\in\mathbf{R}^n} (1+|x|)^m \cdot |D^\alpha u(x)| \tag{1.42}$$

are finite for all non-negative integers l, m. The space $\mathcal{S}(\mathbf{R}^n)$ endowed with the system of seminorms $\|u\|_{l,m}$ is called *the space of rapidly decreasing test functions* on $\mathbf{R}^n$. It will play an important role later on.

Exercise 1.26. Prove that the spaces $\mathcal{C}(\mathcal{O}), \mathcal{E}(\mathcal{O})$ and $\mathcal{S}(\mathbf{R}^n)$ are Fréchet spaces.

Exercise 1.27. Prove that (a) $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathcal{S}_{\lambda,m}(\mathbf{R}^n)$ for all λ, m (see, for example, §1.1.C); (b) $\mathcal{S}(\mathbf{R}^n)$ is dense in $\mathcal{E}(\mathbf{R}^n)$; (c) $\mathcal{E}(\mathcal{O})$ is dense in $\mathcal{C}(\mathcal{O})$.

We note that the LCS's in the above examples are not equivalent to normed spaces.

As an example of an LCS that is not an F-space, we consider the space $\mathcal{D}(\mathcal{O})$ of complex † $\mathcal{C}^\infty$ functions defined on an open subset $\mathcal{O} \subset \mathbf{R}^n$ vanishing outside a

* A subset Γ of Euclidean space is *locally compact* if for any point $\gamma \in \Gamma$ the set $\{y \in \Gamma : \|y-\gamma\| \le \rho\}$ is compact for some value of $\rho = \rho_0 > 0$ (and hence for all $\rho \in (0, \rho_0]$). In particular, open and closed subsets of Euclidean space are locally compact.

** We denote by $\mathbf{Z}$ (or $\mathbf{Z}_+$ or $\overline{\mathbf{Z}}_+$) the set of all integers (or positive integers, or non-negative integers).

† The subsets of real functions of $\mathcal{S}(\mathbf{R}^n)$ (or of $\mathcal{D}(\mathcal{O})$) are denoted by $\mathcal{S}_r(\mathbf{R}^n)$ (or $\mathcal{D}_r(\mathcal{O})$).

compactum $K \subset \mathcal{O}$ (depending on the function). The topology on $\mathcal{D}(\mathcal{O})$ is defined by an uncountable collection of seminorms. Each seminorm is defined by some locally finite sequence * of infinitely differentiable functions $\{f_\alpha\}$ in terms of the formula

$$p_{\{f_\alpha\}}(u) = \sup_{x\in\mathcal{O}} \sum_\alpha |f_\alpha(x) D^\alpha u(x)|.$$

This space will be used later on (in §2.1) in the context of the theory of generalized functions.

1.3. Linear Operators and Linear Functionals in Fréchet Spaces

A. Continuous maps of LCS's

There are, in general, two distinct notions of continuity for functions defined on an arbitrary LCS Ω, namely, topological and sequential. In the first case, the continuity of a map $T : \Omega_1 \to \Omega_2$ at a point $u \in \Omega_1$ means that the inverse image of a neighbourhood of the point $Tu \in \Omega_2$ contains a neighbourhood of the point u. (Continuity without mention of the point means continuity at all points.) From this we easily obtain:

Proposition 1.4. *A linear map T from an LCS $(\Omega_1, \{p_\alpha\})$ to the LCS $(\Omega_2, \{q_\beta\})$ is continuous if and only if for any $\alpha \in \mathcal{A}$ there exist $\beta \in \mathcal{B}$ and $c \geq 0$ such that*

$$q_\beta(Tu) \leq c p_\alpha(u) \quad \text{for all } u \in \Omega_1. \tag{1.43}$$

In particular, for normed spaces $(\Omega_1, \|\cdot\|)$ and $(\Omega_2, \|\cdot\|)$, a linear operator $T : \Omega_1 \to \Omega_2$ such that $\|Tu\| \leq c\|u\|$ for some $c \geq 0$ and all $u \in \Omega_1$ is said to be bounded; the number

$$\|T\| = \sup_{u\in\Omega_1} \frac{\|Tu\|}{\|u\|} \tag{1.44}$$

is called the norm of T. Thus it follows from Proposition 1.4 that the notions of continuity and boundedness are equivalent for operators in normed spaces.

Sequential continuity of an operator T means that for any convergent sequence $u_k \to u$ in Ω_1 we have $Tu_k \to Tu$ in Ω_2. It is not difficult to verify using the standard arguments of real analysis, that these two concepts are equivalent if we restrict our attention to F-spaces (which have a countable basis of neighbourhoods of zero). When we add the additional condition that the map T be linear, we easily arrive at the following statement (which, in particular, enables one to identify sequential and ordinary continuity in F-spaces).

Proposition 1.5. *For linear maps T from the F-space $(\Omega_1, \{p_\alpha\}_{\alpha\in\mathcal{A}})$ to the F-space $(\Omega_2, \{q_\beta\}_{\beta\in\mathcal{B}})$ the following properties are equivalent:*

1) *T is sequentially continuous;*
2) *T is sequentially continuous at zero;*
3) *T is continuous;*

* A sequence $\{f_\alpha\}$ is said to be *locally finite* if on any compactum $K \subset \mathcal{O}$, only a finite number of terms of this sequence are non-zero.

4) *T is continuous at zero;*

5) *for any index $\beta \in \mathcal{B}$ there exists an index $\alpha \in \mathcal{A}$ and a number $c \geq 0$ satisfying* (1.43).

If an LCS Ω_1 is contained in the LCS Ω_2 and the inclusion map (associating an element $u \in \Omega_1$ with the same element u and regarded as a map from Ω_1 to Ω_2) is continuous, then we say that the topology of Ω_1 *majorizes* the topology induced by Ω_2.

Exercise 1.28. Let $(\Omega_1, \{p_\alpha\}_{\alpha \in \mathcal{A}})$ and $(\Omega_2, \{q_\beta\}_{\alpha \in \mathcal{B}})$ be two F-spaces with $\Omega_1 \subset \Omega_2$. Prove that the topology of Ω_1 majorizes that induced from Ω_2 if and only if the restriction to Ω_1 of the system of seminorms $\{q_\beta\}$ is subordinated to the system of seminorms $\{p_\alpha\}$.

Exercise 1.29. Prove that a seminorm p on an LCS $(\Omega, \{p_\alpha\})$ is continuous if and only if it is subordinated to the determining system of seminorms $\{p_\alpha\}$.

Exercise 1.30. (a) Prove that the sum $u + v$ of a pair of vectors u, v of an LCS Ω is jointly (sequentially) continuous in u, v.

(b) Prove that the product λu (where $\lambda \in \mathbf{C}$, $u \in \Omega$) is jointly continuous in λ, u.

In the next exercise, we discuss the construction of the topology on the quotient space of an F-space (cf. the similar Exercise 1.9 for Banach spaces).

Exercise 1.31. Let X be a closed linear subspace of the F space $(\Omega, \{p_k\})$; we suppose for definiteness that the sequence of seminorms p_k on Ω is non-decreasing as k increases. Prove that the quotient space Ω/X is an F space with system of seminorms $\dot{p}_k$ defined by the equality

$$\dot{p}_k(\dot{u}) = \inf_{v \in X} p_k(u + v), \quad \dot{u} \in \Omega/X. \tag{1.45}$$

The system of seminorms $\{\dot{p}_k\}$ defined in this way is called the *LCS structure on Ω/X induced by the LCS structure on Ω*. It is not difficult to see that the natural projection $J : \Omega \to \Omega/X$ is continuous:* it is also an *open map* (that is, the image of any open subset of Ω is an open subset of Ω/X). This result can be strengthened to the following more general form.

Exercise 1.32. Let T be a continuous linear operator from the F space $(\Omega_1, \{p_k\})$ onto the LCS $(\Omega_2, \{q_k\})$. We define on Ω_2 the new system of seminorms $\{p_k^T\}$ (called the *inductive LCS structure on Ω_2 with respect to the map T*):

$$p_k^T(v) = \inf\{p_k(u) : u \in \Omega_1, Tu = v\}, \quad \text{where } v \in \Omega_2. \tag{1.46}$$

Prove that the map T is open if and only if the systems of seminorms $\{p_k^T\}$ and $\{q_k\}$ are equivalent.

A linear map $T : \Omega_1 \to \Omega_2$ from the LCS Ω_1 onto the LCS Ω_2 is called a *topological homomorphism* if it is continuous and open. (An example is the natural projection $J : \Omega \to \Omega/X$.)

The set of all continuous linear functionals on an LCS is called the *dual space* to Ω; we denote it by Ω' (or Ω^*).

Exercise 1.33. Let Ω_1 and Ω_2 be F-spaces, where Ω_1 is a dense subset of Ω_2 and the topology of Ω_1 majorizes that induced by Ω_2. Prove that the dual spaces satisfy the reverse inclusion $\Omega_2' \subset \Omega_1'$ (in the sense that by restricting a functional $F_2 \in \Omega_2'$ to Ω_1, we obtain a functional $F_1 \in \Omega_1'$, the map $F_2 \to F_1$ being an inclusion, that is, $F_1 = 0$ only if $F_2 = 0$).

---

* This property completely determines the F- space structure on the quotient space Ω/X. In fact, it follows from Theorem 1.14 (open mapping theorem) and §1.3.C that there exists a unique F-space structure on Ω/X (where Ω is an F-space and X is a closed subspace) for which the natural projection J is continuous.

By an *isomorphism* (more precisely, a topological isomorphism) from the LCS Ω_1 to the LCS Ω_2, we mean a one-to-one continuous map from Ω_1 onto Ω_2 whose inverse is also continuous. Two spaces are said to be isomorphic if there exists an isomorphism between them.

Exercise 1.34. Reformulate the notion of isomorphism of F-spaces in terms of seminorms.

It should be noted that the Hahn-Banach Theorem 1.2 carries over to arbitrary LCS's and, in particular, to F-spaces.

B. THE UNIFORM BOUNDEDNESS PRINCIPLE. THE WEAK AND WEAK* TOPOLOGIES

In §1.3.A we did not make use of the completeness of F-spaces. The stronger results that we now consider rest on completeness. The following (at first glance, unexpected) property forms the basis of our arguments.

Lemma 1.6. *An arbitrary seminorm p on an F-space $(\Omega, \{p_\alpha\})$ is continuous on an everywhere-dense linear manifold in Ω. In other words, there exist (for this seminorm p on Ω) a dense linear manifold $\mathcal{X}$ in Ω, $\alpha \in \mathcal{A}$ and $c \geq 0$ such that $p(u) \leq cp_\alpha(u)$ for all $u \in \mathcal{X}$.*

■ We may assume that the defining system of seminorms $\{p_\alpha\}$ on Ω is a monotone-decreasing sequence $\{p_k\}_{k=1}^{\infty}$ of seminorms. We now introduce the non-decreasing sequence of linear manifolds $\mathcal{X}_k \subset \Omega$:

$$\mathcal{X}_k = \{u \in \Omega : p(u) \leq kp_k(u)\}.$$

It is clear that each $u \in \Omega$ belongs to some $\mathcal{X}_k$ (since $kp_k(u) \to \infty$ as $k \to \infty$ when $u \neq 0$). It remains to prove that $\mathcal{X}_k$ is dense in Ω for some k. Suppose the contrary: $\overline{\mathcal{X}}_k$ (the closure of $\mathcal{X}_k$ in Ω) is not the whole of Ω for any k. Thus for each k, the set $\mathcal{M}_k = \Omega \backslash \overline{\mathcal{X}}_k$ is dense in Ω (in fact, any $u \in \Omega$ can be approximated by elements of the form $u + cv$, where v is a fixed element of $\mathcal{M}_k$ and $c \neq 0$ is an arbitrary number sufficiently small in modulus). It is clear that all the $\mathcal{M}_k$ are open and dense in Ω. But by Theorem 1.3, their intersection is non-empty; whence it follows that the union of all the $\overline{\mathcal{X}}_k$ is not the whole of Ω. This contradiction completes the proof. ■

From Lemma 1.6 we have the following important principle.

Theorem 1.7 (Uniform boundedness principle). *Suppose that an arbitrary system of continuous seminorms $\{q_\beta\}_{\beta \in \mathcal{B}}$ is given on the F-space $(\Omega, \{p_\alpha\})$such that $\sup_{\beta \in \mathcal{B}} q_\beta(u) < \infty$ for all $u \in \Omega$. Then the seminorm $q(u) = \sup_{\beta \in \mathcal{B}} q_\beta(u)$ is also continuous on Ω.*

■ It is clear that $q(u)$ is a seminorm (see Exercise 1.24). By Lemma 1.6, there exist a dense linear manifold $\mathcal{X} \subset \Omega$, an index α and a number $c \geq 0$ such that $q(u) \leq cp_\alpha(u)$ for all $u \in \mathcal{X}$. It follows that for all $\beta \in \mathcal{B}$

$$q_\beta(u) \leq cp_\alpha(u) \quad \text{for all } u \in \mathcal{X}. \tag{1.47}$$

Since q_β and p_α are continuous on Ω, equation (1.47) can be extended by continuity to the closure $\overline{\mathcal{X}}$ of $\mathcal{X}$, that is, to the whole of Ω. Thus we have

$$q_\beta(u) \leq cp_\alpha(u) \quad \text{for all } u \in \Omega,$$

and from this we obtain the required result: $q(u) \leq cp_\alpha(u)$ for all $u \in \Omega$. ■

Corollary 1.8. Let the sequence T_k of continuous linear operators from the F-space Ω_1 to the LCS space Ω_2 be such that the limit $\lim_{k\to\infty} T_k u$ in Ω_2 exists for each $u \in \Omega_1$. Then the formula $Tu = \lim_{k\to\infty} T_k u$ defines a continuous linear operator from Ω_1 to Ω_2.

Exercise 1.35. Deduce Corollary 1.8 from the uniform boundedness principle. [Hint: Let r be an arbitrary seminorm from the defining system of seminorms on Ω_2. Apply Theorem 1.7 to the sequence of seminorms q_k on Ω_1 defined by $q_k(u) = r(T_k u)$, $u \in \Omega_1$.]

We consider the important special case of Corollary 1.8 when $\Omega_2 = \mathbf{C}$.

Corollary 1.9. If the sequence F_k of continuous linear functionals on the F-space $(\Omega, \{p_\alpha\})$ is such that for each $u \in \Omega$, the limit $\lim(F_k, u)$ exists as $k \to \infty$, then the linear functional F_∞ on Ω equal to

$$(F_\infty, u) = \lim_{k\to\infty} (F_k, u) \quad \text{for all } u \in \Omega, \tag{1.48}$$

is continuous and there exist an index α and a number $c \geq 0$ such that

$$|(F_k, u)| \leq c p_\alpha(u) \quad \text{for all } u \in \Omega,\ k = 1, 2, \ldots, \infty. \tag{1.49}$$

Corollary 1.9 is a way of saying that the dual Ω' of an F-space Ω is sequentially complete. But prior to this, we note that Ω' provides us with the possibility of endowing Ω with an LCS topology that is different from the original one; it is called the *weak topology* (or $\sigma(\Omega, \Omega')$ topology) and is defined by the seminorms

$$p_{F_1 \ldots F_n}(u) = \sup_{j=1,\ldots,n} |F_j, u)| \tag{1.50}$$

where n is any natural number and the F_j run through Ω'. Similarly, we can endow Ω' with an LCS topology called the *weak* topology* (or $\sigma(\Omega', \Omega)$ topology) and defined by the seminorms

$$p^{u_1 \ldots u_n}(F) = \sup_{j=1,\ldots,n} |(F, u_j)| \tag{1.51}$$

for any n and any $u_j \in \Omega$. As a rule, by convergence in Ω we mean convergence in the original topology of Ω, whereas by convergence in Ω' we usually mean with respect to the weak* topology.

Thus Corollary 1.9 means that the dual Ω' of an F-space is sequentially complete (in the weak* topology).

Exercise 1.36. Prove that every reflexive Banach space is sequentially complete in the weak topology. [Hint: use the fact that Ω is the dual of Ω'.]

Corollary 1.10. If u_k is a sequence in the F-space Ω converging to u (in Ω) and F_k is a sequence in the dual Ω' that is (weak*) convergent to a functional F in Ω, then

$$(F_k, u_l) \to (F, u) \text{ as } \min(k, l) \to \infty.$$

Corollary 1.10 follows from Corollary 1.9 by a simple application of the identity

$$(F, u) - (F_k, u_l) = (F - F_k, u) + (F_k, u - u_l).$$

C. THE CLOSED GRAPH AND OPEN MAPPING THEOREMS

A remarkable property of F-spaces is that the closed graph theorem is applicable. By the *graph* of an operator $T : \Omega_1 \to \Omega_2$ we mean the appropriate subset of the direct product $\Omega_1 \times \Omega_2$; to say that the graph of T is closed then means that the relations

$u_k \to u$ in Ω_1 and $Tu_k \to v$ in Ω_2 together imply that $v = Tu$. The linearity of T enables us to simplify this definition.

Exercise 1.37. Prove that a linear operator T from the F-space Ω_1 to the F-space Ω_2 has a closed graph if and only if the relations $u_k \to 0$ in Ω_1 and $Tu_k \to v$ in Ω_2 together imply that $v = 0$.

Theorem 1.11 (Closed graph theorem). *A linear operator* T from an F-space Ω_1 to an F-space Ω_2 that has a closed graph is continuous.*

■ We can suppose without loss of generality that the countable defining systems of seminorms $\{p_k\}$ in Ω_1 and $\{q_k\}$ in Ω_2 are indexed by the natural numbers k and are non-decreasing as k increases. Since $q_k(Tu)$ is a seminorm on Ω_1, for each k there exists, by Lemma 1.6, a dense linear subspace $\mathcal{X}_k$ of Ω_1 and numbers j_k, c_k such that for all $u \in \mathcal{X}_k$

$$q_k(Tu) \le c_k p_{j_k}(u). \tag{1.52}$$

By increasing the numbers c_k, j_k if necessary, we can arrange matters so that c_k and j_k increase with increasing k and so that $j_k \to \infty$ as $k \to \infty$. We can now extend $\mathcal{X}_k$ by adding to it all the vectors $u \in \mathcal{X}_k$ satisfying (1.52); thus we may suppose that $\mathcal{X}_k \supset \mathcal{X}_l$ for $k < l$.

It remains to extend (1.52) to all the $u \in \Omega_1$. Let u be an arbitrary element of Ω_1. Since $\mathcal{X}_n$ is dense in Ω_1, there exists a sequence $u_n \in \mathcal{X}_n$ such that

$$c_n p_{j_n}(u_n - u) \to 0 \text{ as } n \to \infty. \tag{1.53}$$

Clearly $u_n \to u$ as $n \to \infty$. On the other hand, the sequence Tu_n is fundamental, since it follows from (1.52) and (1.53) that for all $k = 1, 2, \ldots$ and $m, n > k$,

$$q_k(T(u_m - u_n)) \le c_k p_{j_k}(u_m - u_n) \le c_m p_{j_m}(u_m - u) + c_n p_{j_n}(u_n - u) \to 0$$

as $\min(m,n) \to \infty$. It follows from this and the fact that T is closed that $Tu_n \to Tu$ as $n \to \infty$. By passing to the limit as $n \to \infty$ in the inequality $q_k(Tu_n) \le c_k p_{j_k}(u_n)$ we obtain $q_k(Tu) \le c_k p_{j_k}(u)$. Thus (1.52) holds for all $u \in \Omega_1$ and all $k = 1, 2, \ldots$. ■

The closed graph theorem is usually applied in the following context.

Corollary 1.12. Let $T : \Omega_1 \to X$ be a (sequentially) continuous linear operator from the F-space Ω_1 to the LCS X, ** where T takes values in the linear subspace $\Omega_2 \subset X$. Suppose that Ω_2 is itself an F-space with topology majorizing that induced by X on F. Then T, regarded as an operator from the F-space Ω_1 to the F-space Ω_2, is continuous.

Corollary 1.13. Suppose that two F-space structures with defining systems of seminorms $\{p_\alpha\}$ and $\{q_\beta\}$ are given on the linear space Ω and that the first system is subordinated to the second. Then both systems of seminorms are equivalent.

In fact, the identity map $J : (\Omega, \{p_\alpha\}) \to (\Omega, \{q_\beta\})$ is such that $Ju = u$ is continuous (by hypothesis). Hence it has a closed graph and this clearly is equivalent to the inverse map J^{-1} having a closed graph. It then follows from Theorem 1.11 that J^{-1} is a continuous operator, that is, the second system of seminorms is subordinated to the first.

Closely related to the closed graph theorem is another important theorem.

* We emphasize that the domain of the operator T is the whole of Ω_1. The fact is that not infrequently, one has to consider linear operators that are only defined on a linear proper subspace of Ω_1 (see the next section). Theorem 1.11 is not applicable to such operators.

** Here the supposition made in §1.2.A that the topology of our LCS is separated, is essential.

Theorem 1.14 (Open mapping theorem). *Any continuous linear map T from an F-space $(\Omega_1, \{p_\alpha\})$ onto (the whole) F-space $(\Omega_2, \{q_\beta\}$ is open.**

■ By Exercise 1.32, the system of seminorms $\{p_\alpha^T\}$ on Ω_2 defines an F space structure on Ω_2; since T is continuous, the system of seminorms $\{q_\beta\}$ is subordinated to the system $\{p_\alpha^T\}$. Hence by Corollary 1.13 we see that the systems $\{p_\alpha^T\}$ and $\{q_\beta\}$ are equivalent. ■

Exercise 1.38. Let T be a continuous linear operator from the F-space Ω_1 onto the F-space Ω_2. Prove that for any sequence v_k in Ω_1 that converges to zero, there exists a sequence u_k in Ω_2 convergent to zero such that $v_k = Tu_k$.

An important application of Theorem 1.14 is that it enables one to establish a relationship between Ω_2' and Ω_1'. We introduce the relevant concepts. If X is a linear manifold in an LCS Ω, then we call the set

$$X^\circ = \{F \in \Omega' : (F, u) = 0 \quad \text{for all } u \in X\} \tag{1.54}$$

the *orthogonal subspace* (or *polar*) to X in Ω'. We suppose X° to be endowed with the convergence induced from Ω' in the sense that $F_k \to F$ in X° is equivalent to the condition $(F_k, u) \to (F, u)$ for all $u \in \Omega$. We recall that $\ker T$ denotes the null space of $T : \Omega_1 \to \Omega_2$. (If T is continuous, then $\ker T$ is a closed linear subspace of Ω_1.)

Let T be a continuous linear operator from Ω_1 to Ω_2: by its *adjoint* we mean the operator T' that associates with a functional $G \in \Omega_2'$ the functional $T'G \in \Omega_1'$ according to the rule $(T'G, u) = (G, Tu)$ for all $u \in \Omega_1$. (As is easy to see, if $G_k \to G$ in Ω_2' then $T'G_k \to T'G$ in Ω_1'.) It is easily verified that the image of Ω_2' under the map T' is always a subset of $(\ker T)^\circ$. The stronger statement given in Exercise 1.39 is obtained by applying the result of the preceding exercise.

Exercise 1.39. Let T be a continuous linear operator from the F-space Ω_1 to the F-space Ω_2; then T' is an isomorphism (that is, a one-to-one linear map preserving convergence) from Ω_2' to $(\ker T)^\circ$. [Hint: For a given $F \in (\ker T)^\circ$ a functional $G \in \Omega_2'$ such that $F = T'G$ can be defined by the formula $(G, v) = (F, u)$, where $u \in \Omega_1$ is such that $Tu = v$.]

1.4. Operators in Hilbert Space

A. The notion of an (unbounded) self-adjoint operator

We shall be considering here linear operators A defined on certain dense linear manifolds D_A of a Hilbert space $\mathcal{H}$ and taking values in $\mathcal{H}$:

$$\mathcal{H} \supset D_A \ni \Phi \to A\Phi \in \mathcal{H}. \tag{1.55}$$

The operator A is said to be *bounded* (in D_A) if the square of the norm

$$\|A\Phi\|^2 = \langle A\Phi, A\Phi\rangle$$

is bounded when $\Phi \in D_A$ and $\|\Phi\| \leq 1$. The least upper bound of $\|A\Phi\|$ as Φ runs through the intersection of D_A and the unit sphere is called the *norm of the operator* A and is denoted by $\|A\|$:

$$\|A\| = \sup_{\substack{\|\Phi\|=1 \\ \Phi \in D_A}} \|A\Phi\|. \tag{1.56}$$

* According to Exercise 1.32 this means that the system of seminorms $\{q_\beta\}$ on Ω_2 is equivalent to the inductive system of seminorms $\{p_\alpha^T\}$ with respect to the map T.

If the right hand side of (1.56) is equal to infinity, then the operator A is said to be *unbounded.*

Any bounded operator A with domain $D_A \subset \mathcal{H}$ that is dense in H can be (canonically) extended to the whole of $\mathcal{H}$ so as to be linear and bounded and with the same norm $\|A\|$ (see [K2],ChIV,§1.1). We can therefore suppose without loss of generality that bounded operators are defined everywhere on $\mathcal{H}$ and we do not refer to their domain. But for unbounded operators, which are encountered a great deal in quantum theory, this is not the case. For these operators it is absolutely essential to indicate their domain.

It will be recalled that the graph $\Gamma(A)$ of an operator A is the set of all pairs $(\Phi, A\Phi)$, where $\Phi \in D_A$ and $A\Phi \in \mathcal{H}$. We then say that A is *closed* if its graph is a closed subset of $\mathcal{H} \oplus \mathcal{H}$. According to Theorem 1.11 (closed graph theorem) a closed linear operator defined on the whole of $\mathcal{H}$ (that is, $D_A = \mathcal{H}$) is bounded. Since we shall frequently be dealing with closed unbounded operators (or at any rate with operators admitting a closure), it is clear from the closed graph theorem that they cannot be defined on the whole of $\mathcal{H}$. The best that we can say is that the domain D_A of a closed unbounded operator A is everywhere dense in $\mathcal{H}$. We say that the operator B is an *extension* of A (written $A \subset B$) if the graph $\Gamma(A)$ is contained in $\Gamma(B)$ $(\Gamma(A) \subset \Gamma(B))$, that is, if $D_A \subset D_B$ and $A\Phi = B\Phi$ for all Φ in D_A.

The importance of Hilbert space in quantum theory is that the presence of the scalar product enables one to introduce the notion of a Hermitian operator corresponding to observable quantities, just as the notion of a unitary operator is used to describe the symmetry of a physical system. We now turn to the definition of these concepts.

Consider the bilinear form $\langle \Psi, A\Phi \rangle$ $(\Phi \in D_A, \Psi \in \mathcal{H})$. If for some $\Psi \in \mathcal{H}$

$$|\langle \Psi, A\Phi \rangle| \leq C(\Psi, A)\|\Phi\| \quad \text{for all } \Phi \in D(A), \tag{1.57}$$

where $C(\Psi, A)$ is a positive number that does not depend on Φ, then according to Riesz's theorem on the representation of continuous linear functionals on $\mathcal{H}$ (see §1.1.F), there exists $\Psi_1 \in \mathcal{H}$ such that

$$\langle \Psi, A\Phi \rangle = \langle \Psi_1, \Phi \rangle. \tag{1.58}$$

Since D_A is everywhere dense in $\mathcal{H}$, it follows that the vector Ψ_1 is uniquely defined by Ψ. We define on such Ψ the (linear) operator A^*, the (*Hermitian*) *adjoint* of A, by the formula $A^*\Psi = \Psi_1$. The domain of A^* consists of all $\Psi \in \mathcal{H}$ for which (1.57) holds.

In the physical literature, the adjoint operator is often defined by the equality

$$\langle \Psi, A\Phi \rangle = \langle A^*\Psi, \Phi \rangle \tag{1.59}$$

(which is a corollary of the last two formulae) without reference to the domains of A and A^*. This imprecision is allowable if A is a bounded operator, since, as we have already pointed out, we can suppose in this case that $D_A = \mathcal{H}$ and, furthermore, it is fairly easy to see that (1.57) holds for all $\Phi \in \mathcal{H}$ (with $C(\Psi, A) = \|\Psi\| \cdot \|A\|$), so that the adjoint operator does in fact exist and is defined on the whole of $\mathcal{H}$.

In the general case of unbounded operators, the adjoint operator does not always exist.* A necessary and sufficient condition for an operator A to possess an adjoint is that A should have a closure in $\mathcal{H}$ (see [A4],§44). This means that if $\{\Phi_n\}$ is a convergent sequence of vectors in D_A, then the sequence $\{A\Phi_n\}$ either converges or, more generally, has no accumulation point in $\mathcal{H}$. (In other words, the possibility that two subsequences of the sequence $\{A\Phi_n\}$ converge to different limits in $\mathcal{H}$ is excluded.) In this case the closure (that is, the smallest closed extension) of A is equal to A^{**}.

An operator A is said to be *symmetric* (*Hermitian* in the physical literature) if $A \subset A^*$, that is, if

$$\langle A\Psi, \Phi\rangle = \langle \Psi, A\Phi\rangle \quad \text{for } \Phi, \Psi \in D_A. \tag{1.60}$$

An important special case of a symmetric operator is given by operators A (with dense domain) satisfying the condition:

$$\langle \Phi, A\Phi\rangle \geq 0 \quad \text{for all } \Phi \in D_A.$$

Such operators are called *positive* (sometimes *positive-definite*). If $\langle \Phi, A\Phi\rangle > 0$ for all non-zero $\Phi \in D_A$, then A is called a *strictly positive* (or *strictly positive-definite*) operator.

Exercise 1.40. Prove that every positive operator is symmetric.

If $A = A^*$ (that is, if in addition to condition (1.60) we further stipulate that the domains D_A and D_{A^*} are equal) then we say that the operator is *self-adjoint* (or, in the terminology of von Neumann, hypermaximal symmetric). An operator is called *essentially self-adjoint* if its closure is a self-adjoint operator.

Exercise 1.41. Let $\mathcal{H} = \mathcal{L}^2([0,1])$ be the set of square-integrable functions on $[0,1]$. Let D be the set of all absolutely continuous functions on this interval (see [K8],Ch.VI,§4) whose derivatives belong to $\mathcal{H}$, the functions themselves satisfying the condition of periodicity $\psi(0) = \psi(1)$. Let D_0 be the subset of D consisting of all functions in D that vanish on the boundary of this interval $\psi(0) = \psi(1) = 0$. Prove that the operator

$$P = -i\frac{d}{dx}$$

with domain D is a self-adjoint operator, whereas the operator P_0 given by the same formula but with domain D_0 is only symmetric and not self-adjoint. Find the domain D_0^* of the adjoint operator P_0^*. Verify directly that P is unbounded.

Exercise 1.42. Consider the symmetric differential operator

$$T = Px^3 + x^3P \quad (P = -i\frac{d}{dx})$$

in the Hilbert space $\mathcal{L}^2(\mathbf{R})$ defined originally on the dense domain $\mathcal{S}(\mathbf{R})$. Show that the infinitely differentiable square-integrable function $\psi(x) = x^{-3/2}\exp(-1/4x^2)$ is an eigenfunction of the operator T^* with a purely imaginary eigenvalue ($T^*\psi = -i\psi$).

It can be shown that the operator T in this example does not have a self-adjoint extension in $\mathcal{H}$.

We now give a criterion for the essential self-adjointness of a symmetric operator (Nelson, 1959) which is useful in applications. First of all, we note that if $\Phi \in D_A$ and $A\Phi \in D_A$, then A^2 is defined at Φ by $A^2\Phi = A(A\Phi)$. By induction we can define

* If D_{A^*} is dense in $\mathcal{H}$, then we say that the adjoint operator A^* *exists*.

A^n for any natural number n. We say that the vector $\Phi \in D_A$ is an *analytic vector* of A if $\Phi \in D_{A^n}$ for all natural numbers n and the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} \|A^n \Phi\| \cdot z^n$$

has a non-zero radius of convergence.

Theorem 1.15 (on analytic vectors). *If A is a symmetric operator in a Hilbert space $\mathcal{H}$ and D_A contains a set of analytic vectors of A that is dense in $\mathcal{H}$, then A is an essentially self-adjoint operator.*

A proof of this theorem can be found in [R1], vol. 2 §X.6.

B. ISOMETRIC, UNITARY AND ANTI-UNITARY OPERATORS

A linear operator U defined on the whole of a Hilbert space $\mathcal{H}$ and with range coinciding with $\mathcal{H}$ is said to be *unitary* if it preserves the scalar product:

$$\langle U\Phi, U\Psi \rangle = \langle \Phi, \Psi \rangle \quad \text{for all } \Phi, \Psi \in \mathcal{H}. \tag{1.61}$$

The requirement that the range of U be the whole of $\mathcal{H}$ is essential. If we do not insist on this, then U is said to be *isometric*. An isometric operator is characterized by the equality $U^*U = 1$. This equality does not, in general, imply that $UU^* = 1$. If both equalities hold, then U is unitary. Thus a unitary operator U has an inverse U^{-1} which is also unitary.

Exercise 1.43. Show that the condition that a linear operator U in $\mathcal{H}$ be isometric is equivalent to the condition that $\|U\Phi\| = \|\Phi\|$ for all $\Phi \in \mathcal{H}$ (so that the norm of an isometric operator U is equal to 1). Verify that UU^* is a projector in $\mathcal{H}$.

In the above exercise we have used the notion of a projector. By a *projector* (or projection) (more precisely, an orthogonal or Hermitian projector) in the Hilbert space $\mathcal{H}$ we mean an operator E such that $E = E^* = E^2$. Clearly E behaves like the identity operator (multiplication by unity) in $E\mathcal{H}$ and like the zero operator (multiplication by zero) in $(1-E)\mathcal{H}$; also $E\mathcal{H}$ and $(1-E)\mathcal{H}$ are mutually orthogonal closed subspaces of $\mathcal{H}$.

Unitary operators are isomorphisms (or automorphisms) of the Hilbert space. More generally, by an *isomorphism of Hilbert spaces* (sometimes called a unitary operator) we mean a linear operator V from one Hilbert space $\mathcal{H}_1$ onto another Hilbert space $\mathcal{H}_2$, possessing an inverse and preserving the scalar product (or norm, which is the same, in view of the polarization formula). It is clear that V^{-1} is an isomorphism from $\mathcal{H}_2$ onto $\mathcal{H}_1$.

Exercise 1.44. (a) Let $A_1, \dots, A_n$ be bounded linear operators in the Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$. Prove that there exists a unique bounded linear operator $A \equiv A_1 \otimes \dots \otimes A_n$ in $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$ (called the *tensor product of the operators* $A_1, \dots, A_n$) such that

$$A(\psi_1 \otimes \dots \otimes \psi_n) = A_1\psi_1 \otimes \dots \otimes A_n\psi_n$$

for any $\psi_1 \in \mathcal{H}_1, \dots, \psi_n \in \mathcal{H}_n$; furthermore, $\|A_1 \otimes \dots \otimes A_n\| = \|A_1\| \cdots \|A_n\|$. [Hint: Consider the space $\mathcal{F}$ in §1.1.B and introduce the Hermitian form

$$\omega'(f,g) = \sum_{u_1, v_1 \in \Omega_1, \dots, u_n, v_n \in \Omega_n} \overline{f(u_1, \dots, u_n} g(v_1, \dots, v_n) < A_1u_1, A_1v_1 > \dots < A_nu_n, A_nv_n >$$

and prove that it is subordinated to the form $\omega(f,g)$ in (1.30) in the sense that

$$\omega'(f,f) \leq \|A_1\|^2 \dots \|A_n\|^2 \cdot \omega(f,f) \quad \text{for all } f \in \mathcal{F}.$$

Hence show that the operator

$$f \to a(f) = \sum f(u_1,\dots,u_n) A_1 u_1 \otimes \dots \otimes A_n u_n, \; f \in \mathcal{F}$$

on $\mathcal{F}$ is lowered onto $\mathcal{F}/X = \Omega$, that is, there exists a bounded linear operator $\tilde{A}$ on Ω such that $\tilde{A} f = a(f)$. The required operator A is now obtained by extending $\tilde{A}$ by continuity to the whole of $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n$.]

(b) Prove that if $A_1,\dots,A_n$ are Hermitian (or unitary), then $A_1 \otimes \dots \otimes A_n$ is Hermitian (or unitary).

In the foregoing discussions we have sometimes referred to linear operators simply as operators (and we shall continue to do so). The antilinear operators characterized by condition (1.2) are close to linear operators. For a continuous antilinear operator A in $\mathcal{H}$, the *Hermitian adjoint* operator A^* is not defined by (1.59) but by the formula

$$\langle \Psi, A\Phi \rangle = \overline{\langle A^*\Psi, \Phi \rangle} \equiv \langle \Phi, A^*\Psi \rangle. \tag{1.62}$$

The antilinear operators that are of importance to quantum theory are the *anti-unitary operators*; these are antilinear operators in $\mathcal{H}$ that define a one-to-one map from $\mathcal{H}$ onto itself (and hence have inverse operators) and satisfy the following analogue of (1.61):

$$\langle A\Phi, A\Psi \rangle = \overline{\langle \Phi, \Psi \rangle} \equiv \langle \Psi, \Phi \rangle, \quad \Phi, \Psi \in \mathcal{H}. \tag{1.63}$$

We note that the product of anti-unitary operators (by which, of course, we mean the composition of the two maps) is a unitary operator, while the product of a unitary and an anti-unitary operator is anti-unitary. It is not difficult to show that an antilinear operator A in $\mathcal{H}$ is anti-unitary if and only if it preserves the norm (that is, if $\|A\Phi\| = \|\Phi\|$ for all $\Phi \in \mathcal{H}$) and maps $\mathcal{H}$ onto itself.

Another equivalent characterization which singles out the anti-unitary operators from among the antilinear ones is the relation $A^*A = AA^* = 1$.

C. The spectral theory of self-adjoint and unitary operators.

We now give a brief discussion of the spectral decomposition of self-adjoint and unitary operators. (For a more detailed account of spectral theory, we refer the reader to Ch.VI in [A4] or Ch.VII and VIII in [R1], vol.1.)* It should be noted that this theory is not applicable to symmetric operators (see, for example, Ch.VI of [A4].)

In a finite-dimensional space, the spectral decomposition of an operator is usually interpreted as the problem of finding an orthonormal basis of eigenvectors of it. However, for infinite-dimensional spaces, this point of view has to be somewhat modified, since even the simplest examples show that a self-adjoint or unitary operator in $\mathcal{H}$ may have no eigenvectors whatsoever in $\mathcal{H}$. For example, in $\mathcal{L}^2(\mathbf{R})$ the operator of multiplication by x, the operator $P = -i\frac{d}{dx}$, and the unitary translation operator $\psi(x) \to \psi(x+a)$ (where $a \neq 0$) are operators of this type. (Although books on quantum mechanics usually call the functions e^{ipx} (where $p \in \mathbf{R}$) eigenvectors of the operator P, one must remember that these functions are not square-integrable on the x axis, so that they do not belong to the given Hilbert space. A similar observation can be made about the other examples.) In most

*The theory of commutative C^*-algebras provides another method of obtaining the spectral theorems (see, for example, [N2],§17.4 and §31.7; we shall be dealing with C^*-algebras in the next section).

mathematical textbooks, the spectral theory of a self-adjoint operator A in $\mathcal{H}$ is set out as a problem of realizing the Hilbert space $\mathcal{H}$ in terms of complex functions of some variable, so that under this realization, the action of A reduces to multiplication by a suitable function α. (In the finite-dimensional case, the role of this variable is played by the index k that distinguishes the eigenvectors of the operator A; the vectors of $\mathcal{H}$ are in one-to-one correspondence with their coordinates ξ_k in this basis and are regarded as functions of the variable k; similarly, the eigenvalues α_k of the operator A can be regarded as functions of k.)

A more detailed account of spectral theory can be given using the notion of a measure space (which we do not define here; see, for example, [D6],[K8]). Let $\mathcal{L}^2(X;\mu)$ be the Hilbert space of (the equivalence classes of) complex measurable square-integrable functions on the space X with measure μ. Each (real or complex) measurable function α on X gives rise to the operation of multiplication by α in $\mathcal{L}^2(X;\mu)$ which is defined on the set D_α of elements $u \in \mathcal{L}^2(X;\mu)$ for which $\alpha u \in \mathcal{L}^2(X;\mu)$. By a realization of the operator A in the Hilbert space $\mathcal{H}$ by an operator of multiplication in $\mathcal{L}^2(X;\mu)$ we mean an isomorphism V from $\mathcal{H}$ onto $\mathcal{L}^2(X;\mu)$ such that $A = V^{-1}\alpha V$, where α is a multiplication operator in $\mathcal{L}^2(X;\mu)$; here it is understood that $D_\alpha = VD_A$. What the spectral theorem states is that every self-adjoint or unitary operator A in the Hilbert space $\mathcal{H}$ can be realized by a multiplication operator in a suitable Hilbert space $\mathcal{L}^2(X;\mu)$.

The spectral theorem is often given a somewhat different form (which does not depend on the choice of X, μ) in terms of the so-called "resolution of the identity" in $\mathcal{H}$. For the sake of definiteness, we begin with the case of a self-adjoint operator. If the operator A is realized as the operation of multiplication by a real function in $\mathcal{L}^2(X;\mu)$, then it is possible to introduce a family of operations of multiplication in $\mathcal{L}^2(X;\mu)$ by functions e_λ that depend on the real parameter λ and are defined by the equalities

$$e_\lambda(x) = 1 \text{ for } \alpha(x) < \lambda \text{ and } e_\lambda(x) = 0 \text{ for } \alpha(x) \geq \lambda.$$

It is clear that the e_λ are projectors in $\mathcal{L}^2(X;\mu)$, therefore the formula $E_\lambda = V^{-1}e_\lambda V$ defines a family of (orthogonal) projectors in $\mathcal{H}$ which depend on the real parameter λ. The following properties are not too difficult to verify:

(a) $$E_\lambda E_\mu = E_\lambda \quad \text{for } \lambda \leq \mu,$$

(b) $$\lim_{\lambda \to -\infty} E_\lambda \Phi = 0, \quad \lim_{\lambda \to +\infty} E_\lambda \Phi = \Phi, \quad \lim_{\lambda \to \mu - 0} E_\lambda \Phi = E_\mu \Phi \quad \text{for all } \Phi \in \mathcal{H}, \mu \in \mathbf{R};$$

(c) for all $\Phi \in D_A$, the following integral representation holds (in the sense of Stieltjes integrals):

$$A\Phi = \int_{-\infty}^{\infty} \lambda dE_\lambda \Phi. \tag{1.64}$$

The family of (orthogonal) projectors E_λ in $\mathcal{H}$ satisfying properties (a)–(c) is called the *spectral decomposition* of the self-adjoint operator A. (By contrast with the realization of A by multiplication operators, it is unique.)

The commutativity condition $AB = BA$ for operators can be carried over to the language of spectral decompositions (provided that at least one of these operators is self-adjoint).

Proposition 1.16. *Let A and B be two bounded operators in a Hilbert space $\mathcal{H}$, where A is Hermitian and $\{E_\lambda\}$ is its spectral decomposition. Then the operators A and B commute if and only if the operators E_λ and B commute for all $\lambda \in \mathbf{R}$.*

The spectral decomposition for a unitary operator U is completely analogous; instead of (1.64) we have the decomposition (in the same sense)

$$U = \int e^{i\lambda} dE_\lambda, \tag{1.65}$$

where $E_\lambda = 0$ for $\lambda \leq 0$ and $E_\lambda = 1$ for $\lambda > 2\pi$.

In essence, we can use the same methods to solve the problem of the spectral decomposition of, say, a *unitary n-parameter abelian group* of operators in $\mathcal{H}$. By such a group we mean a family of unitary operators $U(a)$ in $\mathcal{H}$ that depend strongly continuously (or weakly continuously, which in the present case is the same) on the parameter $a \in \mathbf{R}^n$, where

$$U(a)U(b) = U(a+b). \tag{1.66}$$

Here strong (or weak) continuity means that $U(a)\Phi$ is a continuous vector-valued function of a for any fixed $\Phi \in \mathcal{H}$ (or $\langle \Psi, U(a)\Phi \rangle$ is a continuous complex-valued function of a for any $\Phi, \Psi \in \mathcal{H}$).

For the simultaneous reduction of the operators $U(a)$ to the form (1.65), we have to introduce the n-dimensional analogue of the family of projectors E_λ satisfying the above conditions (a), (b). Thus by a *spectral measure* on $\mathbf{R}^n$ (with values in the set of projectors in Hilbert space $\mathcal{H}$) we mean a correspondence that associates with each Borel subset $\Delta \subset \mathbf{R}^n$ an orthogonal projector $E(\Delta)$ in $\mathcal{H}$ with the properties:

(a') $$E(\Delta)E(\Delta') = E(\Delta \cap \Delta');$$

(b') $$E(\emptyset) = 0,\ E(\mathbf{R}^n) = 1,\ E(\cup_{k=1}^{\infty}\Delta_k) = \sum_{k=1}^{\infty} E(\Delta_k), \quad \text{if} \quad \Delta_j \cap \Delta_k = \emptyset \quad \text{for } j \neq k.$$

It is not difficult to see that for the case $n = 1$ these conditions become the conditions of (a) and (b) given above if we set $E_\lambda = E((-\infty, \lambda))$. It is clear that if $E(\Delta)$ is a spectral measure on $\mathbf{R}^n$, then for any vector $\Phi \in \mathbf{R}^n$, the correspondence $\Delta \to \langle \Phi, E(\Delta)\Phi \rangle$ is a finite positive Borel measure on $\mathbf{R}^n$. If now $\chi(p)$ is a continuous (or Borel) complex bounded function of $p \in \mathbf{R}^n$, then we can define the integral

$$\int \chi(p) dE(p) \equiv \int \chi(p) E(dp), \tag{1.67}$$

by setting

$$\left\langle \Phi, \int \chi(p) dE(p) \Phi \right\rangle = \int \chi(p) \langle \Phi, E(dp)\Phi \rangle \quad \text{for all } \Phi \in \mathcal{H}.$$

(In the case of unbounded $\chi(p)$ the integral (1.67) defines an operator which, in general, is not defined on the entire Hilbert space $\mathcal{H}$; more precisely the domain of this operator consists of all vectors $\Phi \in \mathcal{H}$ for which $\int |\chi(p)|^2 \langle \Phi, E(dp)\Phi \rangle < \infty$.)

Returning to the abelian group of unitary operators, we mention the following classical result.

Theorem 1.17 (Stone). *Every unitary n-parameter group of operators $U(a)$ in $\mathcal{H}$ has a (unique) representation of the form*

$$U(a) = \int e^{ipa} dE(p), \tag{1.68}$$

where $E(\Delta)$ is a spectral measure on $\mathbf{R}^n$.

We note that in the physical literature, instead of the spectral decomposition theory outlined above, the terminology of eigenvectors and eigenvalues, including the case of a continuous spectrum, is widely used. (These are, more strictly, generalized eigenvectors, since as we have already remarked, a self-adjoint or unitary operator may, in general, not have eigenvectors in the Hilbert space.) This terminology can be given a proper meaning by means of the notion of a rigged (or nested) Hilbert space (see [G6],Ch.1) or generalized vector-valued functions (see §2.7.C).

1.5. Algebras with Involution. C*-Algebras

A. Definition and Elementary Properties

By an *algebra* we mean a linear space over the field of complex numbers in which the bilinear operation of multiplication is defined. In other words, a set $\mathfrak{A}$ of elements

$A, B, \dots$ is called an algebra if commutative and associative addition and multiplication by complex numbers are defined on it, with conditions I–III of §1.1.A being fulfilled and if in addition a product AB is defined for any two elements satisfying the conditions:

$$\begin{aligned}(\lambda_1 A_1 + \lambda_2 A_2)B &= \lambda_1(A_1 B) + \lambda_2(A_2 B),\\ A(\lambda_1 B_1 + \lambda_2 B_2) &= \lambda_1(AB_1) + \lambda_2(AB_2).\end{aligned} \tag{1.69}$$

Here we shall be considering *associative algebras* for which it is further supposed that

$$A(BC) = (AB)C. \tag{1.70}$$

A subset $\mathfrak{B}$ of an algebra $\mathfrak{A}$ is called a *subalgebra* if it is closed with respect to the algebra operations of $\mathfrak{A}$ (that is, if $\mathfrak{B}$ is a linear subspace of $\mathfrak{A}$ and if $AB \in \mathfrak{B}$ whenever $A \in \mathfrak{B}$ and $B \in \mathfrak{B}$). A subalgebra $\mathfrak{B}$ is called a *left* (*or right*) *ideal* if the elements of $\mathfrak{A}$ behave on $\mathfrak{B}$ like linear operations of multiplication on the left (or right), that is, if $AB \in \mathfrak{B}$ (or $BA \in \mathfrak{B}$) for all $A \in \mathfrak{A}$, $B \in \mathfrak{B}$. If $\mathfrak{B}$ is simultaneously a left and a right ideal, then it is called a *two-sided ideal.* (In a *commutative algebra*, in which the identity $AB = BA$ always holds, all three types of ideal are the same.)

We say that $\mathfrak{A}$ is an *algebra with involution* (or a **-algebra*, or a *star algebra*) if to each element $A \in \mathfrak{A}$ there corresponds an *adjoint element* A^* such that the following conditions hold:

$$(\alpha A)^* = \bar{\alpha} A^*, \quad (A + B)^* = A^* + B^*, \quad (AB)^* = B^* A^*, \tag{1.71}$$

$$(A^*)^* = A. \tag{1.72}$$

A star algebra $\mathfrak{A}$ is called *normed* if a norm $\|A\|$ is defined on it such that, in addition to conditions (a)–(c) of §1.1.C, the following conditions hold:

$$\|AB\| \le \|A\| \cdot \|B\|, \quad \|A^*\| = \|A\|. \tag{1.73}$$

A complete normed algebra is called a *Banach algebra.*

Unless otherwise stated, we suppose that the algebra has an identity 1 such that $1 \cdot A = A \cdot 1 = A$ for all $A \in \mathfrak{A}$. It is not difficult to show that the identity element is unique and that $1^* = 1$, $\|1\| = 1$. Elements of an algebra of the form $\lambda \cdot 1$ where λ is a complex number are sometimes called scalars or complex numbers (and are simply denoted by λ instead of $\lambda \cdot 1$). We shall also suppose that the identity of a subalgebra is the same as the identity of the algebra.

An element A of an algebra $\mathfrak{A}$ with involution is called *Hermitian* (or self-adjoint) if it satisfies the relation $A^* = A$; similarly, an element $U \in \mathfrak{A}$ such that $U^*U = UU^* = 1$ is called *unitary*. A Hermitian element E with the property $E^2 = E$ is called a (Hermitian) *projector* (or projection).

An involutive Banach algebra $\mathfrak{A}$ is called a *C^*-algebra** if along with (1.73) the equality

$$\|A^* A\| = \|A\|^2 \tag{1.74}$$

* Naimark [N2] calls a C^*-algebra a Banach (or complete) fully regular symmetric ring. Some authors use the term B^*-algebra for an abstract C^*-algebra, preserving the name C^*-algebra for algebras of bounded operators in Hilbert space (see, for example, [R4]). We refer the reader to the books [N2] and [D5] for a detailed exposition of the questions touched upon in the present section.

holds for all $A \in \mathfrak{A}$.

Since every C^*-algebra $\mathfrak{A}$ is, in particular, a Banach space, we can consider the set of all continuous linear functionals on $\mathfrak{A}$ which also form a Banach space (the dual space $\mathfrak{A}'$). Accordingly, we can define the weak topology on $\mathfrak{A}$, while in $\mathfrak{A}'$ we use the weak* topology.

The choice of the axioms defining a C^*-algebra is to a considerable extent motivated by the fact that these axioms are satisfied by the algebra of all bounded linear operators in an arbitrary Hilbert space $\mathcal{H}$, which henceforth we shall denote by $\mathcal{B}(\mathcal{H})$. (Here, the norm of an operator is the operator norm, while the involution is the Hermitian adjoint; see Exercise 1.45 below). Therefore every subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the norm topology and is closed under the operation of forming the Hermitian adjoint, is a C^*-algebra. (More generally, by a *C^*-subalgebra* of a C^*-algebra $\mathfrak{A}$ we mean a subset of $\mathfrak{A}$ that is itself a C^*-algebra with respect to the algebraic operations, the involution and the norm inherited from $\mathfrak{A}$.) The C^*-subalgebras of $\mathcal{B}(\mathcal{H})$ are also known as *operator* (or concrete) *C^*-algebras*, in contrast to general C^*-algebras, which are sometimes called *abstract C^*-algebras*. There is a deep result due to Gel'fand and Naimark (see §1.5.D) which states that every (abstract) C^*-algebra can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some suitable choice of $\mathcal{H}$; in other words, every abstract C^*-algebra is isomorphic to some concrete C^*-algebra.

Exercise 1.45. Prove that the norm of a bounded operator satisfies (1.74). [Hint: It follows from the relations $||A\psi||^2 \equiv \langle \psi, A^*A\psi \rangle \leq ||\psi||^2||A^*A||$ that $||A^2|| \leq ||A^*A||$; then use the inequality $||A^*A|| \leq$ $\leq ||A||^2$ which follows from (1.73).]

More general than an isomorphism is a *homomorphism* of C^*-algebras (or a *C^*-homomorphism*, or simply *morphism*), by which we mean a linear map γ from the C^*-algebra $\mathfrak{A}_1$ to the C^*-algebra $\mathfrak{A}_2$ that preserves the operations of multiplication (that is, $\gamma(AB) \equiv \gamma(A)\gamma(B)$) and involution (that is, $\gamma(A^*) \equiv \gamma(A)^*$) and takes the identity to the identity (that is, $\gamma(1) \equiv 1$). A *C^*-isomorphism* (or *algebraic isomorphism*) from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ can now be defined as a C^*-homomorphism from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$ that is a one-to-one map from $\mathfrak{A}_1$ onto $\mathfrak{A}_2$. A C^*-isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}$ is called a *C^*-automorphism* of $\mathfrak{A}$. Note that every C^*-isomorphism is automatically an isometric map, that is, $||\gamma(A)|| = ||A||$ (see the next subsection).

B. THE SPECTRUM

Although we shall not be making any essential use of the notion of a spectrum in an abstract C^*-algebra, we give some results relating to it that are important in their own right.

An element A of a C^*-algebra $\mathfrak{A}$ is called *invertible* (or regular) if there exists an element $A^{-1} \in \mathfrak{A}$, called its *inverse*, such that

$$A^{-1}A = AA^{-1} = 1. \tag{1.75}$$

Exercise 1.46. Let $||A|| < 1$. Prove that the element $1 - A$ is invertible and that

$$||(1-A)^{-1} - 1 - A|| \leq \frac{||A||^2}{1 - ||A||}.$$

[Hint: Show that the series $1 + A + A^2 + \ldots$ converges in the norm topology and define $(1 - A)^{-1}$ as its sum.]

Let A be a fixed element of the C^*-algebra $\mathfrak{A}$. The set of all complex numbers λ for which $\lambda - A$ is invertible is called the *resolvent set* of A and the inverse element $(\lambda - A)^{-1}$ (as a function of the parameter λ) is called the *resolvent* of A. The complement of the resolvent set is called the *spectrum* of A, denoted by $\sigma(A)$). The quantity

$$\rho(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} \tag{1.76}$$

is called the *spectral radius* of A.

Proposition 1.18. *The spectrum $\sigma(A)$ of any element A of a C^*-algebra $\mathfrak{A}$ is a non-empty compact set whose spectral radius is given by the formula*

$$\rho(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \inf_n \|A^n\|^{1/n}. \tag{1.77}$$

The proof of this proposition can be found in [R4],§10.13.

In fact, we have not so far made use of the involution in an algebra and Proposition 1.18 holds for any Banach algebra (with an identity). The specific properties of C^*-algebras emerge in the following exercise.

*Exercise*1.47. Prove that for a C^*-algebra, the following formula for the norm holds:

$$\|A\| = \rho(A^*A)^{1/2}. \tag{1.78}$$

In particular,

$$\|A\| = \rho(A), \quad \text{if} \quad A = A^*. \tag{1.79}$$

[Hint: Prove (1.79) first of all, by using the first of the relations (1.77) in which n should be chosen in the form 2^m, $m \to \infty$: then prove by induction on m that $\|A^{(2^m)}\| = \|A\|^{(2^m)}$.]

Formula (1.78) provides us with an expression for the norm in a C^*-algebra in terms of algebraic concepts (and the involution). In particular, it leads to the following important result.

Exercise 1.48. Prove that each C^*-homomorphism γ satisfies the inequality $\|\gamma(A)\| \leq \|A\|$ for all A and that every C^*-isomorphism is norm-preserving. [Hint: Verify that if γ is a C^*-homomorphism, then $\sigma(\gamma(A)) \subset \sigma(A)$ and that if γ is a C^*-isomorphism, then $\sigma(\gamma(A)) = \sigma(A)$ for all A.]

The next proposition characterizes the *positive elements* of a C^*-algebra, that is, elements A that can be represented in the form $A = B^*B$. (Clearly such elements are Hermitian.)

Proposition 1.19. *The spectrum of a Hermitian element of a C^*-algebra is real. A Hermitian element A of a C^*-algebra is positive if and only if its spectrum is non-negative* (*that is,* $\sigma(A) \subset [0,\infty)$).

For the proof, see [R4],§§11.20,11.28.

We mention without proof, a further result on the mapping of spectra (see [R4],§§10.26–10.28).

Proposition 1.20. *Let f be a holomorphic function defined on an open set $\mathcal{O} \subset \mathbf{C}$, and let $\sigma(A) \subset \mathcal{O}$. Then we can define an element $f(A) \in \mathfrak{A}$ such that:* (a) *the map $f \to f(A)$ is a homomorphism of the algebra of holomorphic functions to the algebra $\mathfrak{A}$*; (b) $f(A) = A$ *if* $f(\lambda) \equiv \lambda$; (c) $f(A) = (A - \lambda_0)^{-1}$ *if* $f(\lambda) \equiv (\lambda - \lambda_0)^{-1}$; (d) $\sigma(f(A)) = f(\sigma(A))$.

For the case of Hermitian elements A, we only require that the function f be continuous.

Proposition 1.21. *Let $\mathcal{C}(K)$ be the C^*-algebra of continuous functions on a compact set $K \subset \mathbf{R}$ where the algebraic operations are defined pointwise, the involution is the complex conjugate and the norm is $\|f\| = \sup\{|f(\lambda)| : \lambda \in K\}$. Suppose further that A is a Hermitian element of the C^*-algebra $\mathfrak{A}$ with $\sigma(A) \subset K$. Then we can define the element $f(A) \in \mathfrak{A}$ such that:* (a) *the map $f \to f(A)$ is a C^*-homomorphism from the algebra $\mathcal{C}(K)$ to the algebra $\mathfrak{A}$*; (b) $f(A) = A$ *if* $f(\lambda) \equiv \lambda$; (c) $\sigma(f(A)) = f(\sigma(A))$.

For the proof see [D5],§1.5.1.

C. Positive functionals

A linear functional F defined on an involutive algebra $\mathfrak{A}$ is called a *positive functional* if

$$F(A^*A) \geq 0 \quad \text{for all } A \in \mathfrak{A}. \tag{1.80}$$

We give the simplest properties of positive functionals.

(a) A positive functional takes real values at the Hermitian element of the algebra.

(b) If F is a positive functional, then

$$|F(A^*B)|^2 \leq F(A^*A) \cdot F(B^*B) \quad \text{for all } A, B \in \mathfrak{A}. \tag{1.81}$$

(Inequality (1.81), which is a special case of (1.12), is also called the Cauchy-Bunyakovsky-Schwarz inequality.)

For the case of C^*-algebras, there is the further interesting property:

(c) Every positive functional F on a C^*-algebra is continuous (with respect to the norm) and its norm is equal to $F(1)$.

Exercise 1.49. Prove property (c). [Hint: By virtue of (1.81) we have $|F(A)|^2 \leq F(1)F(A^*A)$; it remains to prove that $|F(A)| \leq F(1)$ for all Hermitian elements A with norm $\|A\| < 1$. Use the fact that such elements can be represented in the form $A = 1 - B^2$, where B is the Hermitian element defined by the convergent (in norm) series

$$B = \sqrt{1-A} = \sum_{n=0}^{\infty}(-1)^n \binom{1/2}{n} A^n.]$$

Exercise 1.50. Prove that for a positive functional F on the C^*-algebra $\mathfrak{A}$ and for any $A, B \in \mathfrak{A}$ we have

$$|F(B^*AB)| \leq \|A\| \cdot F(B^*B). \tag{1.82}$$

[Hint: Apply property (c) to the positive functional $G(A) = F(B^*AB)$, where B is fixed.]

Exercise 1.51. Prove that the set

$$\mathfrak{I}_F = \{A \in \mathfrak{A} : F(A^*A) = 0\} \tag{1.83}$$

is a left ideal in $\mathfrak{A}$. [Hint: Use the Cauchy-Bunyakovsky-Schwarz inequality or (1.82).]

The positive functionals separate the elements of a C^*-algebra. Moreover, the following statement holds.

Proposition 1.22. *For each element A in the C^*-algebra $\mathfrak{A}$ there exists a positive functional $F \equiv F_A$ on $\mathfrak{A}$ such that*

$$\|F_A\| = 1 \quad \text{and} \quad F_A(A^*A) = \|A\|^2. \tag{1.84}$$

See [R4],§12.39 for a proof of this.

It is easy to see that if $F_1, \ldots, F_n$ are positive functionals on an involutive algebra and $\lambda_1, \ldots, \lambda_n \geq 0$, then the functional $F = \sum_{j=1}^{n} \lambda_j F_j$ is positive; we call it a *convex linear combination of the functionals* $F_1, \ldots, F_n$.

A positive functional F is said to be *indecomposable* if it cannot be represented as a sum of non-collinear positive functionals, that is, if the equality $F(A) = F_1(A) + F_2(A)$ for all $A \in \mathfrak{A}$ (where F_1 and F_2 are positive functionals) implies that $F_1 = \lambda F$ $(\lambda \geq 0)$.

A subset $[a, b]$ of a linear space Ω consisting of points x of the form $x = \lambda a + (1-\lambda)b$, $0 \leq \lambda \leq 1$, is called an *interval* (or *segment*). A point x_0 is called an *extreme point* of the subset K of the linear space Ω (where $x_0 \in K$) if it is not an internal point of any interval with distinct end points in K. We introduce the notation

$$\mathfrak{A}'_{1+} = \{F \in \mathfrak{A}' : F \geq 0, \|F\| = 1\} \tag{1.85}$$

for the set of all normalized positive functionals on the C^*-algebra $\mathfrak{A}$. It is not difficult to see that a non-zero positive functional F on a C^*-algebra $\mathfrak{A}$ is indecomposable if and only if $\|F\|^{-1}F$ is an extreme point of the set $\mathfrak{A}'_{1+}$.

Exercise 1.52. Let $F = F_1 + F_2$, where F_1, F_2 (and F) are positive functionals on the *-algebra $\mathfrak{A}$. Show that the left ideals corresponding to them (see (1.83)) satisfy the relation $\Im_F = \Im_{F_1} \cap \Im_{F_2}$.

The set $\mathfrak{A}'_{1+}$ is, in a certain sense, generated by its extreme points. The precise meaning of this assertion is revealed in the next two results.

Proposition 1.23. *The unit ball $\mathfrak{A}'_1$ in $\mathfrak{A}'$ is compact in the weak*-topology (therefore the set $\mathfrak{A}'_{1+}$ is also compact in this topology).*

This result is a special case of the Banach-Alaoglu theorem ([R4],§4.3).

Theorem 1.24 (Krein-Milman). *Every compact convex subset of an LCS is the closed convex hull of its extreme points.*

For a proof, see, for example, [R4],§3.21.

Examples of indecomposable positive functionals on the algebra $\mathcal{B}(\mathcal{H})$ are provided by the so-called *vector functionals of type F_Φ*:

$$F_\Phi(A) = \langle \Phi, A\Phi \rangle, \ A \in \mathcal{B}(\mathcal{H}), \tag{1.86}$$

where Φ is an arbitrary vector of $\mathcal{H}$. This example will be typical when we consider representations of *-algebras (see subsection D).

Exercise 1.53. Let F be a Hermitian linear functional on the C^*-algebra $\mathfrak{A}$ (where by Hermitian we mean that $F(A^*) = \overline{F(A)}$ for all $A \in \mathfrak{A}$). Prove that

$$\|F\| = \sup\{|F(A)| : A \in \mathfrak{A},\, A = A^*,\, \|A\| \leq 1\}. \tag{1.87}$$

[Hint: If $F(A) = a \geq 0$ for some $A \in \mathfrak{A}$ with $\|A\| \leq 1$, then by setting $B = \frac{1}{2}(A + A^*)$, we have $F(B) = a$, where $B = B^* \in \mathfrak{A}$ and $\|B\| \leq 1$.]

D. Representations

By a *representation* of a C^*-algebra $\mathfrak{A}$ in a Hilbert space $\mathcal{H}$ we mean a C^*-homomorphism π of $\mathfrak{A}$ into the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators in the Hilbert space $\mathcal{H}$. A representation π is called *irreducible* if each closed subspace of $\mathcal{H}$ that is invariant with respect to all the operators $\pi(A)$ ($A \in \mathfrak{A}$) is either $\{0\}$ or the whole of $\mathcal{H}$. A vector $\Phi \in \mathcal{H}$ is called *cyclic* for the representation π if all the vectors of the form $\pi(A)\Phi$ (where $A \in \mathfrak{A}$) form a total subset of $\mathcal{H}$. (A representation with a cyclic vector is called *cyclic*.) By an equivalence of two representations π_1 and π_2 of the C^*-algebra $\mathfrak{A}$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we mean as a rule a *unitary equivalence*,* which means an isomorphism $V : \mathcal{H}_1 \to \mathcal{H}_2$ of the Hilbert spaces with the property:

$$\pi_2(A) = V\pi_1(A)V^{-1} \quad \text{for all } A \in \mathfrak{A}. \tag{1.88}$$

Also of importance in applications are representations of involutive algebras that are not C^*-algebras; here, in general, the operators of the representations π are allowed to be unbounded operators

* The following concept is closely tied up with that of unitary equivalence. Two algebraically isomorphic concrete C^*-algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively) with $C*$-isomorphism $\pi : \mathfrak{A}_1 \to \mathfrak{A}_2$ are called *spatially isomorphic* if this C^*-isomorphism π, regarded as a representation of $\mathfrak{A}_1$ in $\mathcal{H}_2$, is unitarily equivalent to the identity representation $A \to A$ of $\mathfrak{A}_1$ in $\mathcal{H}_1$ (in other words, there exists a unitary operator $V : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi(A) = VAV^{-1}$ for all $A \in \mathfrak{A}_1$).

in Hilbert space. The usual requirement is that all the operators $\pi(A)$ are defined together with their adjoints on some dense domain D (linear subspace) of $\mathcal{H}$, where D is invariant with respect to the operators $\pi(A)$. The characteristic identities defining such representations are then meaningful on D:

$$\pi(\lambda A+\mu B)=\lambda\pi(A)+\mu\pi(B),\quad \pi(A^*)=\pi(A)^*,\quad \pi(AB)=\pi(A)\pi(B).$$

We shall be encountering such representations in Ch.8; many of the important results set out here for C^*-algebras can be extended to these more general representations. (These are results in which the topology of the C^*-algebra is inessential.)

Exercise 1.54. Prove the equivalence of the following three properties of a representation π of a C^*-algebra $\mathfrak{A}$ in $\mathcal{H}$:

(a) π is irreducible;

(b) every operator of $\mathcal{B}(\mathcal{H})$ that commutes with the whole of $\pi(\mathfrak{A})$ (that is, with all the operators $\pi(A)$, where $A\in\mathfrak{A}$) is a multiple of the identity;

(c) every vector in $\mathcal{H}$ is cyclic for π.

[Hint: Prove the equivalences (a)$\Leftrightarrow$ (b) and (a)$\Leftrightarrow$(c); for the proof of the implication (a)$\Rightarrow$ (b), consider to begin with, projectors and Hermitian operators that commute with $\pi(\mathfrak{A})$.]

The equivalence of (a) and (b) in Exercise 1.54 is usually called *Schur's lemma.*

Exercise 1.55. Let π_1 and π_2 be two irreducible unitary inequivalent representations of the C^*-algebra $\mathfrak{A}$ in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Prove that for each bounded linear operator $T:\mathcal{H}_1\to\mathcal{H}_2$, the condition $T\pi_1(A)=\pi_2(A)T$ for all $A\in\mathfrak{A}$ (that is, T intertwines π_1 and π_2) implies that $T=0$. [Hint: Consider the operator T^*T and use the previous exercise.]

If Φ is a vector in the space $\mathcal{H}$ of the representation π of the *-algebra $\mathfrak{A}$, then it induces the positive functional

$$F_\Phi=\langle\Phi,\pi(A)\Phi\rangle \tag{1.89}$$

on $\mathfrak{A}$; it is called the *vector functional* associated with the representation π and corresponding to the vector Φ.

Exercise 1.56. Let π_i $(i=1,2)$ be two cyclic representations of $\mathfrak{A}$ on the Hilbert spaces $\mathcal{H}_i$ with cyclic vectors Φ_i for which the vector functionals corresponding to these Φ_i are the same:

$$\langle\Phi_1,\pi_1(A)\Phi_1\rangle=\langle\Phi_2,\pi_2(A)\Phi_2\rangle,\ A\in\mathfrak{A}.$$

Prove that the representations π_i are unitarily equivalent. [Hint: The sets $\mathcal{L}_i=\pi_i(\mathfrak{A})\Phi_i$ are everywhere dense linear subspaces of $\mathcal{H}_i$, and

$$\langle\pi_1(A)\Phi_1,\pi_2(B)\Phi_1\rangle=\langle\pi_2(A)\Phi_2,\pi_2(B)\Phi_2\rangle\quad\text{for all } A,B\in\mathfrak{A}\,.$$

Deduce that the formula

$$V\pi_1(A)\Phi_1=\pi_2(A)\Phi_2,\quad A\in\mathfrak{A}, \tag{1.90}$$

well-defines the operator $V:\mathcal{L}_1\to\mathcal{L}_2$, which can be extended by continuity to an isomorphism of the Hilbert spaces $V:\mathcal{H}_1\to\mathcal{H}_2$. Finally, use (1.90) to prove (1.88).]

Let $\mathcal{H}_1$ be a closed linear subspace of the Hilbert space $\mathcal{H}$ on which a representation π of the C^*-algebra $\mathfrak{A}$ is given. Then this subspace is called *invariant* for π if $\pi(A)\mathcal{H}_1\subset\mathcal{H}_1$ for all $A\in\mathfrak{A}$ (that is, if $\pi(\mathfrak{A})\mathcal{H}_1\subset\mathcal{H}_1$). Clearly the restrictions $\pi_1(A)$ of the operators $\pi(A)$ to $\mathcal{H}_1$ form a representation π_1 of $\mathfrak{A}$; it is called a *subrepresentation* of π. It is not difficult to see that the orthocomplement $\mathcal{H}_2=\mathcal{H}\ominus\mathcal{H}_1$ is also invariant for π, so that a subrepresentation π_2 acts in it. Thus (when $\{0\}\neq\mathcal{H}_1\neq\mathcal{H}$) the representation decomposes into a direct sum of the two subrepresentations:

$$\pi(A)=\pi_1(A)\oplus\pi_2(A). \tag{1.91}$$

More generally, if we are given a (finite or infinite) family of Hilbert spaces $\{\mathcal{H}_\nu\}_{\nu\in N}$ and a corresponding family of representations $\{\pi_\nu\}_{\nu\in N}$ of the C^*-algebra $\mathfrak{A}$ in which π_ν acts in $\mathcal{H}_\nu$, then we can define on the direct sum of Hilbert spaces $\mathcal{H} = \bigoplus \mathcal{H}_\nu$ a new representation π of $\mathfrak{A}$ in terms of the formula

$$(\pi(A)\psi)_\nu = \pi_\nu(A)\psi_\nu; \tag{1.92}$$

this representation is called the *direct sum* of the family of representations $\{\pi_\nu\}$.

A representation π of a C^*-algebra $\mathfrak{A}$ is called *faithful* if $\pi(A) = \pi(B)$ implies $A = B$ (or equivalently, if zero is the only element of $\mathfrak{A}$ that is taken to the zero operator under the map π). Clearly, faithfulness of a representation π means that the map $A \to \pi(A)$ is an isomorphism from $\mathfrak{A}$ onto $\pi(\mathfrak{A})$.

The most important result of this section is the *Gel'fand-Naimark-Segal construction* (*GNS construction* for short) which enables one to interpret any positive functional on a C^*-algebra as a vector functional in a suitable representation of the algebra.

Theorem 1.25 (GNS construction). *Let F be a positive functional on the C^*-algebra $\mathfrak{A}$. Then a (cyclic) representation π_F of $\mathfrak{A}$ can be defined on a Hilbert space with a cyclic vector Φ_F such that*

$$F(A) = \langle \Phi_F, \pi_F(A)\Phi_F\rangle \quad \textit{for all} \quad A \in \mathfrak{A}. \tag{1.93}$$

The representation π_F is uniquely defined by these conditions to within a unitary equivalence (which associates the cyclic vectors of different representations with each other).

■ The formula

$$\omega(A,B) = F(A^*B) \tag{1.94}$$

defines on $\mathfrak{A}$ a non-negative definite Hermitian form ω, therefore the construction of the Hilbert space described in §1.1.D is applicable here. The vectors of zero seminorm $p(A) = F(A^*A)^{1/2}$ form a left ideal $\Im_F$ (see (1.83)) and the formula $\langle \dot{A}, \dot{B}\rangle = F(A^*B)$ defines a scalar product on the quotient space $\mathfrak{A}/\Im_F$. The completion of this pre-Hilbert space is, in fact, the Hilbert space $\mathcal{H}_F$ featuring in the GNS construction.

Denoting $\dot{A}$ by $\xi_F(A)$ we obtain a linear map $\xi_F : \mathfrak{A} \to \mathcal{H}_F$ from $\mathfrak{A}$ to a dense linear subspace $\mathfrak{A}/\Im_F = \xi_F(\mathfrak{A})$ of $\mathcal{H}$, where

$$\langle \xi_F(A), \xi_F(B)\rangle = F(A^*B) \quad \text{for } A, B \in \mathfrak{A}. \tag{1.95}$$

We now turn to the construction of the representation π_F. To begin with, we associate each element $A \in \mathfrak{A}$ with an operator $\pi_F(A)$ on $\xi_F(\mathfrak{A})$ by means of the formula

$$\pi_F(A)\xi_F(B) = \xi_F(AB). \tag{1.96}$$

This is well-defined (in the sense that the right hand side depends on B only inasmuch as it depends on $\xi_F(B)$). For suppose $\xi_F(B) = 0$; then $B \in \Im_F$ and (since $\Im_F$ is a left ideal) $AB \in \Im_F$, so that $\xi_F(AB) = 0$. It now follows from formula

$$\langle \xi_F(B), \pi_F(A)\xi_F(B)\rangle = F(B^*AB) \tag{1.97}$$

and from (1.82) that the operator $\pi_F(A)$ is bounded: $\|\pi_F(A)\| \le \|A\|$; it can therefore be extended by continuity to the whole of $\mathcal{H}$. We use the same notation $\pi_F(A)$ for this extension.

It follows from (1.96) that the map $A \to \pi_F(A)$ is linear, preserves multiplication (that is, $\pi_F(AB) = \pi_F(A)\pi_F(B)$) and takes the identity into the identity: $\pi_F(1) = 1$. We now verify that the adjoint is preserved:

$$\langle \xi_F(B), \pi_F(A)^*\xi_F(C)\rangle = \langle \xi_F(AB), \xi_F(C)\rangle = F((AB)^*C) =$$

$$= F(B^*A^*C) = \langle \xi_F(B), \xi_F(A^*C)\rangle = \langle \xi_F(B), \pi_F(A^*)\xi_F(C)\rangle;$$

hence $\pi_F(A)^* = \pi_F(A^*)$.

Thus the map π_F so constructed is a representation of the C^*-algebra in $\mathcal{H}_F$. We set $\Phi_F = \xi_F(1)$. It follows from (1.96) (with $B = 1$) that the vector Φ_F is cyclic for π_F, while (1.94) follows from (1.97) (with $B = 1$).

It remains to prove the uniqueness (to within unitary equivalence) of π_F in the sense made explicit in the statement of the theorem. Thus, let π be a representation of $\mathfrak{A}$ on $\mathcal{H}$ with cyclic vector Φ, where

$$F(A) = \langle \Phi, \pi(A)\Phi\rangle \quad \text{for all } A \in \mathfrak{A}. \tag{1.98}$$

Then the representations π and π_F are unitarily equivalent according to Exercise 1.56. ∎

The representation π_F in the GNS construction is called the *representation corresponding to the positive functional F*. It turns out that there is a close connection between irreducibility of π_F and indecomposability of F. As a preliminary to this, we prove the following lemma.

Lemma 1.26. *Let F and F_1 be two positive functionals on the C^*-algebra $\mathfrak{A}$, where F_1 is subordinated to F in the sense that there exists $\lambda \geq 0$ such that*

$$F_1(A) \leq \lambda F(A) \tag{1.99}$$

for all positive $A \in \mathfrak{A}$. Then there exists in the space $\mathcal{H}_F$ of the representation π_F (corresponding to F) a (unique) positive bounded operator T commuting with all the operators $\pi_F(A)$ and such that

$$F_1(A) = \langle \Phi_F, T\pi_F(A)\Phi_F\rangle, \quad A \in \mathfrak{A}. \tag{1.100}$$

∎ We give a sketch proof, referring the reader to [N2],§19.1 for further details. We consider the Hermitian form ω on $\mathfrak{A}$: $\omega(A, B) = F_1(A^*B)$. It follows from the Cauchy-Bunyakovsky-Schwarz inequality and from (1.99) that $|\omega(A, B)| \leq \lambda\|\xi_F(A)\| \cdot \|\xi_F(B)\|$. Hence $\omega(A, B)$ in fact depends on A and B only inasmuch as it depends on $\xi_F(A)$ and $\xi_F(B)$. As a result, we have constructed a Hermitian form ω on the vectors of the form $\xi_F(A)$ (that is, on $\pi_F(\mathfrak{A})\Phi_F$) such that

$$\dot{\omega}(\xi_F(A), \xi_F(B)) = F_1(A^*B),$$

where $|\dot{\omega}(\xi_F(A), \xi_F(B))| \leq \lambda\|\xi_F(A)\| \cdot \|\xi_F(B)\|$. It follows from Riesz's theorem (§1.1.F,Example 2) that there exists in $\mathcal{H}$ a (unique) linear operator T such that

$$F_1(A^*B) = \langle \xi_F(A), T\xi_F(B)\rangle, \quad A, B \in \mathfrak{A}. \tag{1.101}$$

It is clear that T is a positive operator. Finally it is not difficult to verify, using (1.101), that

$$\langle \xi_F(A), \pi_F(C)T\xi_F(B)\rangle = \langle \xi_F(A), T\pi_F(C)\xi_F(B)\rangle$$

for all A, B, $C \in \mathfrak{A}$, whence it follows that T commutes with $\pi_F(\mathfrak{A})$. ∎

We are now able to prove the following supplement to Theorem 1.25.

Proposition 1.27. *The representation π_F corresponding to a positive functional F is irreducible if and only if F is indecomposable.*

∎ Assume that π_F is irreducible. Then it suffices to prove that if F can be represented in the form $F = F_1 + F_2$, where F_1 and F_2 are positive functionals, then $F_1 = \lambda F$, where $\lambda \geq 0$. Clearly, F_1 is subordinated to F, therefore (by Lemma 1.26) it can be represented in the form (1.100). It follows from the irreducibility of π_F (by Schur's lemma, see Exercise 1.54) that T is a multiple of the identity: $T = \lambda$, so that $F_1 = \lambda F$ by (1.100).

Suppose, conversely, that F is indecomposable. By Schur's lemma, it suffices to show that any bounded operator T in $\mathcal{H}$ that commutes with all the $\pi_F(A)$ $(A \in \mathfrak{A})$ is a multiple of the identity. Clearly, it suffices to consider just the bounded Hermitian operators (since any bounded operator can be expressed in the form $T = T_1 + iT_2$, where the T_i are bounded Hermitian operators). Furthermore, it suffices, in view of Proposition 1.16, merely to consider the projectors T in $\mathcal{H}_F$ that commute with $\pi_F(\mathfrak{A})$. We claim that each such projector is either 0 or 1. For clearly, the functional F_1 on $\mathfrak{A}$ defined by (1.100) is subordinated to the functional F: $F_1(A^*A) \leq F(A^*A)$; consequently, F is the sum of the two positive functionals F_1 and $F - F_1$. Since F is indecomposable, $F_1 = \lambda F$ for some $\lambda \geq 0$. It now follows from the uniqueness of the representation of F_1 in the form (1.100) that $T = \lambda$. This completes the proof. ∎

From Theorem 1.25 there follows the fundamental result due to Gel'fand and Naimark.

Theorem 1.28. *Any ("abstract") C^*-algebra is isomorphic to a "concrete" C^*-algebra of operators in Hilbert space.*

∎ We associate with an arbitrary element $A \in \mathfrak{A}$ a positive functional F_A with the properties (1.84) and we let π_{F_A} be the representation of $\mathfrak{A}$ corresponding to it with cyclic vector Φ_{F_A}. It follows from (1.84) that $\|\Phi_{F_A}\| = 1$, $\|\pi_{F_A}(A)\Phi_{F_A}\| = \|A\|$, so that

$$\|\pi_{F_A}(A)\| \geq \|A\|. \tag{1.102}$$

We consider the direct sum of all such representations:

$$\pi = \bigoplus_{A \in \mathfrak{A}} \pi_{F_A}, \tag{1.103}$$

It now follows from (1.102) and Exercise 1.48 that

$$\|\pi(A)\| = \|A\| \quad \text{for all } A \in \mathfrak{A}, \tag{1.104}$$

so that the representation π is faithful and hence is a C^*-isomorphism of $\mathfrak{A}$ onto the operator algebra $\pi(\mathfrak{A})$. ∎

Corollary 1.29. The following formula holds for all elements of a C^*-algebra $\mathfrak{A}$:

$$\sup_{\pi} \|\pi(A)\| = \|A\|, \tag{1.105}$$

where the supremum is taken over all the irreducible representations of $\mathfrak{A}$.

In fact, the set $\mathfrak{A}'_{1+}$ of (1.85) is the closed convex hull of its extreme points, therefore in the proof of Theorem 1.28 we can suppose that the positive functionals F_A have been chosen to be indecomposable. The required formula (1.105) now follows from (1.102) (and from the fact that $\|\pi(A)\| \leq \|A\|$ for all A, by virtue of Exercise 1.48).

Given an automorphism γ of a C^*-algebra $\mathfrak{A}$, we say that a positive functional F on $\mathfrak{A}$ is *invariant* with respect to γ if

$$F(\gamma(A)) = F(A) \quad \text{for all } A \in \mathfrak{A}.$$

Proposition 1.30. *Let F be a positive functional on the C^*-algebra $\mathfrak{A}$ that is invariant with respect to the automorphism γ and let π_F, $\mathcal{H}_F$, Φ_F be the components in accordance with the GNS construction. Then there exists a unitary operator U_F in $\mathcal{H}_F$ such that*

$$\pi_F(\gamma(A)) = U_F \pi_F(A) U_F^{-1} \quad \textit{for all} \quad A \in \mathfrak{A}. \tag{1.106}$$

If it is further required that

$$U_F\Phi_F = \Phi_F, \tag{1.107}$$

then conditions (1.106) and (1.107) define U_F uniquely.

Exercise 1.57. Prove Proposition 1.30. [Hint: U_F can be defined by the formula

$$U_F\pi_F(A)\Phi_F = \pi_F(\gamma(A))\Phi_F \tag{1.108}$$

for all $A \in \mathfrak{A}$.]

Theorem 1.25 contains the construction of all the cyclic representations of an algebra $\mathfrak{A}$. On the other hand, every representation $\mathfrak{A}$ can be decomposed into a direct sum of cyclic representations.

Exercise 1.58. Let π be a representation of the C^*-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$ and let $\psi \in \mathcal{H}$. Then the closure $\mathcal{H}_1 = \overline{\pi(\mathfrak{A})\psi}$ is a (closed) invariant subspace of $\mathcal{H}$ and the vector ψ is cyclic for the restriction π_1 of π to $\mathcal{H}_1$.

The above statement on the decomposition of any representation of an algebra $\mathfrak{A}$ into cyclic ones follows from the result of Exercise 1.58 by using Zorn's lemma (see [N2], §17.2).

It is worth noting that the vector functionals of the form (1.86) do not exhaust all the indecomposable functionals on $\mathcal{B}(\mathcal{H})$ ([N2],§22). On the other hand, as we have seen, the GNS construction provides the possibility of associating with each indecomposable positive functional on $\mathcal{B}(\mathcal{H})$ a vector functional in (possibly) some other irreducible representation of $\mathcal{B}(\mathcal{H})$. This means that $\mathcal{B}(\mathcal{H})$ has unitarily inequivalent representations. However, for a given space $\mathcal{H}$ in quantum mechanics, we restrict ourselves to vector functionals of the form (1.86) and their convex linear combinations (possibly countable). We shall go into the physical grounds for this choice later in §6.4.A.

E. TRACE CLASS OPERATORS

We now give a method of representing vector functionals and their convex linear combinations by operators in $\mathcal{B}(\mathcal{H})$. For this we need the notion of the *trace* of an operator. In a finite dimensional $\mathcal{H}$, the trace of an operator A is the sum of the diagonal elements of the matrix of A in any basis of $\mathcal{H}$. By choosing an orthonormal basis $\{e_n\}$ in $\mathcal{H}$, we can express this definition in the form

$$\operatorname{tr} A = \sum_n \langle e_n | A e_n \rangle. \tag{1.109}$$

(It is not difficult to verify that the trace of an operator does not depend on the choice of the basis.) For an infinite-dimensional $\mathcal{H}$, the trace is not defined for an arbitrary operator in $\mathcal{B}(\mathcal{H})$, but only for the so-called *trace class operators* (or *nuclear operators*) for which the series on the right hand side of (1.109) converges under any choice of orthonormal basis $\{e_n\}$ in $\mathcal{H}$. The class of all such operators can be described as follows.

We associate with each pair of vectors $u, v \in \mathcal{H}$ an operator, which we write (in Dirac notation) as $|u\rangle\langle v|$ and which acts on $\Phi \in \mathcal{H}$ according to the formula

$$(|u\rangle\langle v|)\Phi = \langle v, \Phi\rangle \cdot u. \tag{1.110}$$

It is clear that $\big\| |u\rangle\langle v| \big\| = \|u\| \cdot \|v\|$. It is not difficult to verify that the set of finite linear combinations of operators of the form $|u\rangle\langle v|$ is precisely the class of *operators of finite rank* (characterized by the property that the range of such an operator A is finite-dimensional or equivalently, that the orthocomplement of the null space $\ker A$ is finite-dimensional). In particular, the operators of the form $|u\rangle\langle v|$ constitute the class of operators of rank ≤ 1. The trace class of operators can now be characterized as the set of all operators A in $\mathcal{H}$ that are representable in the form of a (finite or infinite) sum $A = \sum_k A_k$ of operators A_k of rank ≤ 1 for which the series $\sum_k \|A_k\|$ converges. For such operators A, the trace is well defined by (1.109) as the following exercise shows.

Exercise 1.59. Prove that

$$\operatorname{tr}\Big(\sum_k |u_k\rangle\langle v_k|\Big) = \sum_k \langle v_k, u_k\rangle$$

provided that $\sum_k \|u_k\| \cdot \|v_k\| < \infty$.

We denote the trace class of operators in $\mathcal{H}$ by $\mathcal{L}_1(\mathcal{H})$. It is clear that $\mathcal{L}_1(\mathcal{H})$ is an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ (without an identity when $\mathcal{H}$ is infinite-dimensional). Furthermore it is a two-sided ideal in $\mathcal{B}(\mathcal{H})$ as the next exercise states.

Exercise 1.60. If $A \in \mathcal{L}_1(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})$, then AB and BA belong to $\mathcal{L}_1(\mathcal{H})$.

Exercise 1.61. Prove the following properties of the trace:

(a) $\operatorname{tr} A^* = \overline{\operatorname{tr} A}$ for $A \in \mathcal{L}_1(\mathcal{H})$;

(b) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for $A \in \mathcal{L}_1(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})$;

(c) for any $A \in \mathcal{L}_1(\mathcal{H})$ the quantity

$$\|A\|_1 = \sup_{B\in\mathcal{B}(\mathcal{H})} \frac{|\operatorname{tr}(AB)|}{\|B\|} \tag{1.111}$$

is finite and enjoys the properties of a norm on $\mathcal{L}_1(\mathcal{H})$;

(d) if $A \in \mathcal{L}_1(\mathcal{H})$ is represented in the form $\sum_k |u_k\rangle\langle v_k|$, where each of the sequences $\{u_k\}$ and $\{v_k\}$ is orthogonal then $\|A\|_1 = \sum_k \|u_k\| \cdot \|v_k\|$.

The norm $\|A\|_1$ of (1.111) is called the *trace norm* in $\mathcal{L}_1(\mathcal{H})$. A comparison of (1.32) and (1.111) shows that there is an isometric embedding of $\mathcal{L}_1(\mathcal{H})$ into the space $\mathcal{B}'(\mathcal{H})$ of continuous linear functionals on $\mathcal{B}(\mathcal{H})$ under which the element $A \in \mathcal{L}_1(\mathcal{H})$ is associated with the functional $(F, B) = \operatorname{tr}(AB)$ over $B \in \mathcal{B}(\mathcal{H})$. In this sense, we can identify $\mathcal{L}_1(\mathcal{H})$ with a subspace of $\mathcal{B}'(\mathcal{H})$ and write $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{B}'(\mathcal{H})$. In similar fashion, (1.111) shows that we can embed $\mathcal{B}(\mathcal{H})$ isometrically in $\mathcal{L}_1'(\mathcal{H})$ by associating with an element $B \in \mathcal{B}(\mathcal{H})$ the functional $\operatorname{tr}(AB)$ over $A \in \mathcal{L}_1(\mathcal{H})$. It is worth noting that in this identification of $\mathcal{B}(\mathcal{H})$ with the subspace of $\mathcal{L}_1'(\mathcal{H})$ we obtain, in fact, the whole of $\mathcal{L}_1'(\mathcal{H})$, that is, $\mathcal{B}(\mathcal{H}) = \mathcal{L}_1'(\mathcal{H})$.

Exercise 1.62. Prove that $\mathcal{L}_1(\mathcal{H})$ is a Banach space with trace norm and that the operators of finite rank form a dense subset of $\mathcal{L}_1(\mathcal{H})$.

Exercise 1.63. Prove that an operator $A \in \mathcal{L}_1(\mathcal{H})$ is positive if and only if the functional F corresponding to it is positive over $\mathcal{B}(\mathcal{H})$.

The result of the last exercise enables us to describe the closure (in the norm) in $\mathcal{B}'(\mathcal{H})$ of the set of all convex linear combinations of vector functionals. It consists of all functionals of the form $B \to \operatorname{tr}(AB)$, where A is a positive operator in $\mathcal{L}_1(\mathcal{H})$. Using the spectral decomposition for A, which in the present case takes the form

$$A = \sum_n \lambda_n |u_n\rangle\langle u_n|, \tag{1.112}$$

where $\{u_n\}$ is an orthonormal basis in $\mathcal{H}, \lambda_n \geq 0$ and $\sum_n \lambda_n < \infty$, we see that such functionals can be represented as a countable sum of vector functionals over $\mathcal{B}(\mathcal{H})$.

F. VON NEUMANN ALGEBRAS

For operator C^*-algebras $\mathfrak{A} \subset \mathcal{B}(\mathcal{H})$ a number of special topologies can be introduced in addition to the norm topology and weak topology.

By the *weak operator topology* (or *W-topology*) we mean the LCS topology defined on $\mathcal{B}(\mathcal{H})$ by the seminorms

$$p_{\Psi_1,\ldots,\Psi_n}^{\Phi_1,\ldots,\Phi_n}(A) = \max_{j=1,\ldots,n} |(\Phi_j, A\Psi_j)|; \tag{1.113}$$

here n is an arbitrary natural number and the $\Phi_1, \ldots, \Phi_n$ and $\Psi_1, \ldots, \Psi_n$ are arbitrary vectors of $\mathcal{H}$. Similarly, the seminorms

$$p_{\Psi_1,\ldots,\Psi_n}(A) = \max_{j=1,\ldots,n} \|A\Psi_j\|$$

define the *strong operator topology* (or *S-topology*) on $\mathcal{B}(\mathcal{H})$. Also of importance is the *σ-weak operator topology* on $\mathcal{B}(\mathcal{H})$ with seminorms

$$p^{B_1,\ldots,B_n}(A) = \sup_{j=1,\ldots,n} |\operatorname{tr}(AB_j)|, \tag{1.114}$$

where $B_1, \ldots, B_n$ is any finite set of trace class operators. It is clear that the W-topology is weaker than the S-topology and the σ-weak topology, while all these topologies are weaker than the norm topology; therefore a subset of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the norm may not be closed in these other topologies.

We have said above that an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the norm topology is a C^*-algebra. Similarly, an involutive subalgebra of $\mathcal{B}(\mathcal{H})$ (with identity) that is closed in the W-topology is called a *von Neumann algebra* or *W^*-algebra*. Every von Neumann algebra is automatically a C^*-algebra. It is worthy of note that the closedness condition for an involutive algebra with respect to the W-topology in the definition of a von Neumann algebra is equivalent to closedness in the S-topology or in the σ-weak operator topology ([N2],§34.2 or [D4],p.41).

The reason that the notion of a von Neumann algebra is a natural one to use is that it is possible to give it a purely algebraic (non-topological) definition.

By the *commutant* of a subset $\mathfrak{M}$ of $\mathcal{B}(\mathcal{H})$ we mean the set $\mathfrak{M}^c$ of those operators $A \in \mathcal{B}(\mathcal{H})$ that commute with all the elements of $\mathfrak{M}$. It is clear that the commutant is an algebra. If the set $\mathfrak{M}$ is closed with respect to forming the adjoint (that is, $A^* \in \mathfrak{M}$ if $A \in \mathfrak{M}$), then $\mathfrak{M}^c$ is an involutive subalgebra of $\mathcal{B}(\mathcal{H})$.

Exercise 1.64. Prove that the commutant $\mathfrak{M}^c$ of any set $\mathfrak{M}$ that is closed with respect to forming the adjoint is closed in the weak (or σ-weak) operator topology and hence is a von Neumann algebra.

The algebraic characterization of von Neumann algebras is given in terms of the *bicommutant* (or double commutant) $\mathfrak{M}^{cc} \equiv (\mathfrak{M}^c)^c$.

Theorem 1.31 (on the bicommutant, or the von Neumann density theorem). *The weak operator closure of an arbitrary involutive subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ coincides with the bicommutant $\mathfrak{A}^{cc}$. In particular every von Neumann algebra coincides with its bicommutant: $\mathfrak{A}^{cc} = \mathfrak{A}$.*

For the proof see [N2],§34.2.

For each subset $\mathfrak{M}$ in $\mathcal{B}(\mathcal{H})$ there exists a minimal von Neumann algebra $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$ containing $\mathfrak{M}$; it is called the *von Neumann algebra generated by* $\mathfrak{M}$. It is not difficult to conclude from the bicommutant theorem that this algebra is the bicommutant of $\mathfrak{M} \cup \mathfrak{M}^*$:

$$\mathfrak{A} = (\mathfrak{M} \cup \mathfrak{M}^*)^{cc}. \tag{1.115}$$

Exercise 1.65. Let π be an irreducible representation of the C^*-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$. Then the weak (or σ-weak) closure of $\pi(\mathfrak{A})$ is the same as $\mathcal{B}(\mathcal{H})$.

There is also a useful strengthening of Theorem 1.31, which we state in terms of representations. We denote by $\mathfrak{A}_1$ the (norm-closed) unit ball in the C^*-algebra $\mathfrak{A}$ and by $\bar{\mathfrak{A}}$ the weak (or, equivalently, σ-weak) operator closure of the operator algebra $\mathfrak{A}$.

Proposition 1.32. *Let π be a representation of the C^*-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$. Then $\pi(\mathfrak{A}_1)$ is dense in $(\overline{\pi(\mathfrak{A})})_1$ in the weak (or σ-weak) operator topology.*

This proposition is a combination of the Kaplansky density theorem ([D4],p.43) and a property of homomorphisms given in [D5],§1.8.3.

Exercise 1.66. Let π be a representation of a C^*-algebra in the Hilbert space $\mathcal{H}$ and B a trace class operator in $\mathcal{H}$. Prove that

$$\sup\Big\{|\operatorname{tr}(B\cdot\pi(X))| : X \in \mathfrak{A},\ \|X\| \le 1\Big\} = \sup\Big\{|\operatorname{tr}(B\cdot Y)| : Y \in \overline{\pi(\mathfrak{A})},\ \|Y\| \le 1\Big\}. \tag{1.116}$$

In particular, if π is irreducible, then

$$\sup\Big\{|\operatorname{tr}(B\cdot\pi(X))| : X \in \mathfrak{A},\ \|X\| \le 1\Big\} = \|B\|_1. \tag{1.117}$$

[Hint: Use (1.111), Proposition 1.32 and Exercise 1.65.]

We give a further corollary of the bicommutant theorem.

Exercise 1.67. If the C^*-algebra $\mathfrak{A}$ of operators in $\mathcal{H}$ is commutative, then its weak operator closure is a commutative von Neumann algebra. [Hint: Apply the operation of taking the bicommutant to the relation $\mathfrak{A} \subset \mathfrak{A}^c$ and note that $(\mathfrak{A}^c)^{cc} = (\mathfrak{A}^{cc})^c = (\bar{\mathfrak{A}})^c$.]

An important role in the study and classification of von Neumann algebras is played by the notion of factor, which can be regarded as a sort of algebraic counterpart of irreducible representation. An algebra $\mathfrak{A} \subset \mathcal{B}$ is called a *factor* if its *centre* $\mathfrak{A} \cap \mathfrak{A}'$ only contains the multiples of the identity, that is, if $\mathfrak{A} \cap \mathfrak{A}' = \{\alpha 1\}$. An arbitrary von Neumann algebra can be represented as a generalized direct sum of factors.

The self-adjoint elements of a von Neumann algebra (and functions of them) are defined by the spectral decomposition properties (1.64) in terms of the system $E(\lambda)$ of projection operators. The classification of factors is based on the existence (and uniqueness to within a numerical factor) of a characteristic additive functional of the projectors, which can be regarded as a generalization of the notion of dimension. The next two theorems give the precise statements of this (see [N2],§36).

Theorem 1.33. *There exists a numerical function $D(E)$ defined on all the operators of the projection E belonging to a given factor F, which satisfies the following conditions.*

1) $D(E) \ge 0$, where $D(E) = 0$ only when $E = 0$.

2) If the operators E_1 and E_2 are equivalent with respect to the factor F, that is, if there exists $S \in F$ such that $E_2 = SE_1S^{-1}$, then $D(E_1) = D(E_2)$.

3) If $E_1E_2 = 0$, then $D(E_1 + E_2) = D(E_1) + D(E_2)$.

*The function $D(E)$ is uniquely defined to within a constant positive factor * from conditions* 1)–3) *(for a given factor F).*

Theorem 1.34. *The range of the relative dimension can be reduced under a suitable normalization to one of the following sets:*

(I_n) *the set of all integers k in the interval $0 \le k \le n$;*

* The function $D(E)$ is called the *relative dimension.*

(I_∞) *the set of all non-negative integers, including* ∞*;*
(II_1) *the interval* $[0,1]$;
(II_∞) *the interval* $[0,\infty]$;
(III) *the numbers 0 and* ∞.

One refers to factors of type I_n, I_∞, II_1, II_∞, and III, depending on which of these sets the function $D(E)$ runs through.

There are examples of factors of each of these classes. In the algebraic formulation of quantum theory we shall be dealing with the least familiar (and least studied) class of factors of type III.

CHAPTER 2

The Technique of Generalized Functions

2.1. The Concept of a Generalized Function

A. FUNCTIONAL DEFINITION

Both in physics and in mathematics itself, the classical notion of a function, characterized by assigning functional values at all possible values of the arguments, was long ago felt to be restrictive. Important notions of classical physics such as the density of a point mass or charge, are not built in to this ordinary notion of function. In quantum mechanics and quantum field theory, such "singular" or "improper" functions as $\delta(x)$ (the Dirac delta function), $D(x)$ (the commutation function), etc., have been systematically used; these arose, in particular, in the so-called commutation relations which have played an important role in quantum theory. In mathematics itself, the restrictiveness of the classical notion of function was felt, for example, in attempts to define the Green's function for equations of hyperbolic type (in particular, for the wave equation), in the normalization of eigenfunctions corresponding to the continuous spectrum of an operator, and in a number of other questions.

There are two ways in which physicists have justified their applications on such concepts as the delta function, which are in conflict with the classical definition of a function. Firstly it was said that singular functions could be multiplied by "good" functions and that a law of integration could be given for such products; the expressions so obtained then made sense. Secondly, singular functions were represented in the form of improper limits of sequences of ordinary smooth functions. Both these intuitive grounds for the application of singular functions have since found a precise mathematical formulation and can be used as a starting point for the theory of generalized functions.

In order to attach a precise meaning to the first definition, one first has to be given a certain class (a linear topological space) of "good" *test functions* $u(x)$ (for example, smooth functions that decrease to zero at infinity). One can then define a generalized function $f \equiv f(x)$ by "integrating " it with all the possible test functions:

$$(f, u) \equiv f(u) \equiv \int f(x)u(x)d^n x. \tag{2.1}$$

In the general case, this "integral" has a purely symbolic meaning: it merely indicates that the functional $f(u)$ depends linearly on u. (It is also supposed that $f(u)$ depends continuously on u; this is essential from the mathematical point of view, so as to ensure that the apparatus of functional analysis, which is well suited for applications, can in particular be adapted so as to maintain the analogy with an integral with respect to some measure; it is also essential from the point of view of the physical interpretation of such distributions as densities of masses, charges, etc.) In particular, if $f(x)$ is an ordinary continuous function (such that the integrals of type (2.1) exist for all test functions $u(x)$), equation (2.1) defines it as a generalized function f. If the class of test functions is chosen to be sufficiently wide, then the continuous function $f(x)$ is defined uniquely by the functional $f(u)$. (If we only assume that $f(x)$ is locally integrable, then the functional $f(u)$ can define it only almost everywhere, that is, to within values on a set of Lebesgue measure zero.)

The question then arises, what considerations make us choose some particular space of test functions ?

We note first of all, that a given singular function (in some physical context) can be treated as a generalized function by many methods that differ formally from one another, by varying the space of test functions. Thus we consider by way of example, a charge q and a dipole $\mathbf{d}$ concentrated at a point $\mathbf{a}$ in three-dimensional space $\mathbf{R}^3$. Their densities can be defined as the generalized functions $f_{q,\mathbf{a}}$ and $f_{\mathbf{d},\mathbf{a}}$, related to the Dirac δ-function by the formulae

$$\begin{aligned}(f_{q,\mathbf{a}}, u) &= q \cdot u(\mathbf{a}), & f_{q,\mathbf{a}} &= q\delta(\mathbf{x}-\mathbf{a}),\\ (f_{\mathbf{d},\mathbf{a}}, u) &= -\mathbf{d}\nabla u(\mathbf{a}), & f_{\mathbf{d},\mathbf{a}} &= \mathbf{d}\nabla\delta(\mathbf{x}-\mathbf{a}).\end{aligned}$$

In the first case, the test functions can be taken to be all functions continuous in a neighbourhood of $\mathbf{a}$, and in the second, all functions that are continuously differentiable in a neighbourhood of this point. But in either case, we could choose as the test functions, simply all the infinitely differentiable (C^∞ for short) functions in $\mathbf{R}^3$. It is clear at the intuitive level that in narrowing down the space of test functions, we gain (or lose) nothing in the sense of the physical information on the distribution of charge. (At the formal level, the equivalence of the two approaches can be justified by considerations of continuity of functionals and the possibility of their extension by continuity.) However, the choice of the set of test functions of class C^∞ is preferable because of its great universality: it is suitable not only for the two examples given above, but also for many other more singular distributions. Generally speaking, the greater the smoothness of the test functions, the greater the admissible singularity of the generalized functions. (Intuitive ideas on the degree of singularity can be borrowed from the theory of meromorphic functions; this concept is introduced into generalized functions, based on the above principle.) On the other hand, it is clear that the more rapid the decrease of the test functions as $|x| \to \infty$, the greater the allowable increase of the generalized functions at infinity.

In quantum field theory it is usually assumed that the matrix elements of the operators of the field $\phi(x)$ can increase at infinity with at most polynomial growth in x (where x is a point of the "coordinate" space) and that their Fourier transforms have similar properties in the "momentum" space. Most suitable for our purposes is the following definition.

By the *generalized functions* in $\mathbf{R}^n$ we mean the continuous linear functionals on the (complex) space $\mathcal{S}(\mathbf{R}^n)$ of rapidly decreasing (complex-valued) test functions (introduced in §1.2.C). They form the dual space $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{S}(\mathbf{R}^n)$ and are also called *tempered distributions*. Using Proposition 1.2, we can also say that a generalized function in $\mathbf{R}^n$ is a linear functional on $\mathcal{S}(\mathbf{R}^n)$ satisfying the condition: there exist natural numbers l, m and a constant $c \geq 0$ such that

$$|(f,u)| \leq c \cdot \|u\|_{l,m} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{2.2}$$

Here, the norms on the right hand side are defined by (1.42). (When we want to indicate the argument explicitly, we use the notation $(f(x), u(x))_x$ or $\int f(x)u(x)d^n x$ for (f,u).)

An important example of a generalized function in $\mathbf{R}^n$ is the Dirac δ-function referred to above and defined by the equality

$$(\delta, u) \equiv \int \delta(x)u(x)dx = u(0) \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{2.3}$$

It turns out that, in accordance with the intuitive idea that a generalized function is a singular function of polynomial growth with a singularity of finite degree over the whole space, one can write an arbitrary element $f \in \mathcal{S}'$ in the form *

$$(f,u) = \sum_{|\alpha| \leq N} \int f_\alpha(x) D^\alpha u(x) d^n x, \quad u \in \mathcal{S}(\mathbf{R}^n); \tag{2.4}$$

here the $f_\alpha(x)$ are continuous functions in $\mathbf{R}^n$ of polynomial growth.

A proof of this fact is easily obtained by combining the Hahn-Banach theorem and the Riesz-Markov theorem on the general form of continuous linear functionals on the space of complex continuous functions in $\mathbf{R}^n$ that decrease to zero at ∞ (see also [G5],§§II.4.2,II.4.3).

In the literature, the (*Schwartz*) *distributions* ** in an arbitrary open † subset $\mathcal{O} \subset \mathbf{R}^n$ are used; these are defined as continuous linear functionals on the space $\mathcal{D}(\mathcal{O})$ of infinitely smooth functions with compact support in $\mathcal{O}$ (§1.2.C). The space of Schwartz distributions in $\mathcal{O}$ is denoted by $\mathcal{D}'(\mathcal{O})$. There is no restriction on the growth at infinity of the Schwartz distributions in $\mathbf{R}^n$ (in contrast to the tempered growth of the generalized functions in $\mathcal{S}'$).

The functionals of the space $\mathcal{D}'(\mathcal{O})$ satisfy (by definition) the following sequential continuity condition. We say that a sequence of test functions $u_k \in \mathcal{D}(\mathcal{O})$ converges to the function $u \in \mathcal{D}(\mathcal{O})$ (in $\mathcal{D}(\mathcal{O})$ convergence) if there exists a compactum K in $\mathcal{O}$ outside which all the functions u_k and u are zero, and if u_k converges uniformly (in x) to u along with all its derivatives (that is, $D^\alpha u_k \to D^\alpha u$ as $k \to \infty$ uniformly in x, where the multiindex α is arbitrary). A functional f on $\mathcal{D}(\mathcal{O})$ is called *sequentially continuous* if $(f, u_k) \to (f,u)$ for any convergent sequence $u_k \to u$ in $\mathcal{D}(\mathcal{O})$.

We note that every generalized function in $\mathcal{S}'(\mathbf{R}^n)$ can be regarded as a distribution in $\mathcal{D}'(\mathbf{R}^n)$. To see this, it suffices to observe that $\mathcal{D}(\mathbf{R}^n)$ is a dense linear subspace ‡ of the space of test functions $\mathcal{S}(\mathbf{R}^n)$ and that convergence $u_k \to u$ in $\mathcal{D}(\mathbf{R}^n)$ implies

* It can be arranged that the number of terms on the right hand side of (2.4) is equal to 1; nevertheless, the general formula (2.4) proves to be preferable in those cases when we are interested in the properties of the supports of f and f_α.

** By analogy with the French *distributions* (Schwartz's terminology). Schwartz calls the generalized functions (that is, the members of $\mathcal{S}'$) *distributions tempérées*, while Gel'fand and Shilov [G5] call them generalized functions of temperate growth.

† A non-empty open connected subset of Euclidean space $\mathbf{R}^n$ is also called a *domain* in $\mathbf{R}^n$.

‡ This density property means that for each function $u \in \mathcal{S}(\mathbf{R}^n)$ there exists a sequence of functions $\{u_k\} \in \mathcal{D}(\mathbf{R}^n)$ converging to u. To see this it suffices to consider the sequence of functions $u_k(x) = \omega(x/k)u(x), k = 1, 2, \ldots$, where $\omega(x) \in \mathcal{D}$ and $\omega(x) = 1$ for $|x| \leq 1$.

convergence $u_k \to u$ in $\mathcal{S}(\mathbf{R}^n)$. It follows from the inclusion $\mathcal{D}(\mathbf{R}^n) \subset \mathcal{S}(\mathbf{R}^n)$ and from the definition of a continuous functional on a Fréchet space, that there exists for any $f \in \mathcal{S}'(\mathbf{R}^n)$, a number $c \geq 0$ and natural numbers l, m such that

$$|(f,u)| \leq c\|u\|_{l,m} \quad \text{for all } u \in \mathcal{D}(\mathbf{R}^n). \tag{2.5}$$

It is not difficult to see that this property characterizes the restrictions of functionals in $\mathcal{S}'$ to $\mathcal{D}$. In other words, the following statement holds.

Proposition 2.1. *Every distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ that satisfies (2.5) can be uniquely extended to a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ (in other words, it can be uniquely extended in linear and continuous fashion to $\mathcal{S}(\mathbf{R}^n)$).*

This proposition is proved by the standard procedure of extending functions (or functionals) by continuity.

A sequence f_k in $\mathcal{S}'(\mathbf{R}^n)$ is said to be weakly (or weak* in the terminology of §1.3.B) convergent* to $f \in \mathcal{S}'(\mathbf{R}^n)$ if $\lim_{k\to\infty}(f_k, u) = (f,u)$ for each test function $u \in \mathcal{S}(\mathbf{R}^n)$. According to Corollary 1.9, $\mathcal{S}'(\mathbf{R}^n)$ is weakly sequentially complete.

In similar fashion we can define weak convergence in the space $\mathcal{D}'(\mathcal{O})$ (which is also weakly sequentially complete); in connection with this, see the remark at the end of §2.1.C.

We note that in quantum field theory, other versions of extending a space of generalized functions have been put forward with the object of enabling one to consider the so-called unrenormalizable models of field theory (see, for example, Meiman, 1964; Jaffe, 1967).

B. Definition in terms of fundamental sequences

We now turn to another definition of generalized functions which is a mathematical statement of the intuitive view of generalized functions as some kind of limits of sequences of continuous functions. This definition does not make use of the apparatus of functional analysis and from this point of view is more elementary than the first definition.

We shall regard the space of generalized functions as an extension of the set of continuous functions (in the spirit of the Cantor definition of the set of real numbers as an extension of the set of rationals).

A sequence of continuous functions $\{f_\nu(x)\}$ in $\mathbf{R}^n$ is said to be *fundamental* if there exists a sequence of continuous functions $\{F_\nu(x)\}$ and a multi-index $\alpha = (\alpha_1, \dots \alpha_n)$ such that the functions F_ν have continuous partial derivatives of order $|\alpha|$ and satisfy the following conditions:

1) $D^\alpha F_\nu(x) = f_\nu(x)$ for $\nu = 1, 2, \dots$;

2) the sequence $F_\nu(x)$ converges to the continuous function $F(x)$, the convergence being uniform over each bounded set;

3) the functions $F_\nu(x)$ are bounded by a single polynomial, that is, there exist constants $A > 0$ and $k \geq 0$ not dependent on ν, such that for all $\nu = 1, 2, \dots$

$$|F_\nu(x)| \leq A[1 + (x_1^2 + \dots + x_n^2)^k]. \tag{2.6}$$

We say that the fundamental sequences $\{f_\nu(x)\}$ and $\{g_\nu(x)\}$ are equivalent (and we write $\{f_\nu\} \sim \sim \{g_\nu\}$) if the mixed sequence

$$f_1(x),\ g_1(x),\ f_2(x),\ g_2(x),\ \dots$$

is fundamental. This equivalence relation partitions the set of fundamental sequences into equivalence classes. The sequences $\{f_\nu(x)\}$ and $\{g_\nu(x)\}$ belong to the same equivalence class if and only if $\{f_\nu(x)\} \sim \{g_\nu(x)\}$. To assign some equivalence class, it suffices to give some fundamental sequence of this class. The equivalence classes of fundamental sequences are called generalized functions (in $\mathbf{R}^n$).

* The space $\mathcal{S}(\mathbf{R}^n)$ is reflexive, therefore there is no difference between weak and weak* convergence in $\mathcal{S}'(\mathbf{R}^n)$.

Every continuous function of polynomial growth $f(x)$ can be regarded as a generalized function by identifying it with the equivalence class containing the sequence

$$f(x), f(x), \ldots, f(x), \ldots$$

The linear operations of addition and multiplication by a scalar are defined in the obvious way on the set of generalized functions.

If the fundamental sequence $\{f_\nu\}$ defines the generalized function f, and λ is a scalar, then clearly, the sequence $\{\lambda f_\nu\}$ is also fundamental. The equivalence class defined by this sequence is called the product of λ by the generalized function f, denoted by λf.

Exercise 2.1. Prove that if $\{f_\nu\}$ and $\{g_\nu\}$ are fundamental sequences, then so is the sequence $\{f_\nu + g_\nu\}$.

The class defined by the sequence $\{f_\nu + g_\nu\}$ depends only on the classes $f \ni \{f_\nu\}$ and $g \ni \{g_\nu\}$. We call this class the sum of the generalized functions f and g, denoted by $f + g$.

We give an example of equivalent fundamental sequences of functions of one variable.

Exercise 2.2. Prove that the sequences

$$f_\nu(x) = \sqrt{\frac{\nu}{2\pi}} exp(-\frac{\nu x^2}{2}), \quad g_\nu(x) = \frac{1}{\pi}\frac{\nu}{\nu^2 x^2 + 1}, \quad h_\nu(x) = \frac{1}{\pi}\frac{sin \nu x}{x} \tag{2.7}$$

are fundamental and equivalent to one another.

At first glance it would seem that here and in §2.1.A we have given the same name, generalized function, to objects of a completely different nature. We will show that these two definitions are, in fact, equivalent; more precisely, the set of equivalence classes considered here is isomorphic to the space of linear functionals $\mathcal{S}'$.

Let $\{f_\nu(x)\}$ be a fundamental sequence and $F(x)$ the continuous function corresponding to it (in accordance with Condition 2 of the definition of fundamental sequence). We associate with the equivalence class to which f_ν belongs, the functional (f, u) $(u \in \mathcal{S})$, defined by the following formula of type (2.4):

$$(f, u) = (-1)^{|\alpha|} \int_{\mathbf{R}^n} F(x) D^\alpha u(x) d^n x \quad (|\alpha| \equiv \alpha_1 + \ldots + \alpha_n). \tag{2.8}$$

Exercise 2.3. Show that the functional (f, u) defined by (2.8) depends only on the equivalence class of the sequences and not on the specific choice of the sequence $\{f_\nu\}$, the set of integers α or the primitive functions $\{F_\nu\}$.

Exercise 2.4. Show that the sequence (2.7) defines the generalized function $\delta(x)$ in $\mathbf{R}$.

Exercise 2.5. Show that the algebraic operations are preserved under the above correspondence (for example, the sum of equivalence classes corresponds to the sum of the corresponding functionals).

We have already noted that an arbitrary functional in $\mathcal{S}'$ can be represented in the form (2.4). To complete the proof of the isomorphism of the space $\mathcal{S}'$ with the set of generalised functions defined as equivalence classes of fundamental sequences, it remains to show that every continuous function of polynomial growth can be approximated by smooth functions of polynomial growth in such a way that Conditions 2) and 3) of the definition of a fundamental sequence are satisfied.

To this end, we use the sequence of functions (see (2.7))

$$\phi_\nu(x) = \prod_{j=1}^{n} f_\nu(x_j) = \left(\frac{\nu}{2\pi}\right)^{n/2} exp\left[-\frac{\nu}{2}(x_1^2 + \ldots + x_n^2)\right]. \tag{2.9}$$

Any continuous function of polynomial growth can be approximated by the sequence of functions $\{F_\nu)(x)\}$, where

$$F_\nu(x) = \int_{\mathbf{R}^n} F(\xi)\phi_\nu(x - \xi) d^n\xi. \tag{2.10}$$

We invite the reader to prove the following assertions:

1) The functions $F_\nu(x)$ are infinitely differentiable;

2) $F_\nu(x) \to F(x)$ as $\nu \to \infty$, the convergence being uniform in x over each bounded subset of $\mathbf{R}^n$;

3) There exist constants $A > 0$ and $k \geq 0$ depending on $F(x)$ but not on ν, such that (2.6) holds.

In what follows, we shall be working with the functional definition of generalized functions; but we shall also make use of the construction (2.10) which enables one to approximate (in the sense of convergence in $\mathcal{S}'$) any new generalized function by smooth functions.

C. LOCAL PROPERTIES OF GENERALIZED FUNCTIONS

In contrast to ordinary functions which are defined by their values at each point of some set, generalized functions are defined in the large as functionals on a space of test functions. They do not, in general, have specified values at individual points. Even so, it is possible to talk about certain local properties of generalized functions.

In the study of the local properties of generalized functions and distributions, an important role is played by the notion of a (smooth) *partition of unity* subordinated to a covering $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ of a given open subset $\mathcal{O} \subset \mathbf{R}^n$ by a (finite or infinite) family of open subsets $\mathcal{O}_\lambda \subset \mathcal{O}$. By such a partition of unity we mean a family $\{e_\lambda\}$ of non-negative C^∞ functions $e_\lambda(x)$ on $\mathcal{O}$ with the following properties:

(a) the function e_λ vanishes outside $\mathcal{O}_\lambda$;

(b) only a finite number of the functions e_λ are non-zero on any given compactum $K \subset \mathcal{O}$;

(c) $\sum_{\lambda\in\Lambda} e_\lambda(x) = 1$ for all $x \in \mathcal{O}$.

We shall not go into the relatively simple proof here of the existence of a partition of unity subordinated to the covering $\{\mathcal{O}_\lambda\}_{\lambda\in\Lambda}$ of any given subset $\mathcal{O} \subset \mathbf{R}^n$ by open subsets. A detailed account of this can be found in [G4],Ch.1, Appendix 1.

By the *support* of a test function we mean the closure of the set of points x for which $u(x) \neq 0$. The support of u is usually denoted by $\operatorname{supp} u$. We say that the generalized functions f and g in $\mathcal{S}'(\mathbf{R}^n)$ *coincide* on (or in) the open subset $\mathcal{O} \subset \mathbf{R}^n$ if $(f,u) = (g,u)$ for all test functions $u(x) \in \mathcal{S}(\mathbf{R}^n)$ with support in $\mathcal{O}$. In particular, we say that $f(x) = 0$ (or that f *vanishes*) in an open set $\mathcal{O}$ if $(f,u) = 0$ for all $u \in \mathcal{S}(\mathbf{R}^n)$ with support in $\mathcal{O}$. The partition of unity enables one to prove the following remarkable fact.

Exercise 2.6. Let $\{\mathcal{O}_\alpha\}$ be a (finite or infinite) family of open subsets of $\mathbf{R}^n$ and $f \in \mathcal{S}'(\mathbf{R}^n)$ a generalized function vanishing on each of the $\mathcal{O}_\alpha$. Then f vanishes on their union $\mathcal{O} = \cup_\alpha \mathcal{O}_\alpha$. [Hint: The density considerations used in Proposition 2.1 reduce the problem to proving that $(f,u) = 0$ for any function $u(x) \in \mathcal{D}(\mathbf{R}^n)$ with support in $\mathcal{O}$. Now choose an open set $\mathcal{Q}$ containing $\operatorname{supp} u$ the closure $\bar{\mathcal{Q}}$ of which is a compactum contained in $\mathcal{O}$. Then use the property of compact sets to choose from the family $\{\mathcal{O}_\alpha\}$ a finite subfamily, say, $\{\mathcal{O}_{\alpha_j}\}_{j=1}^m$ covering $\bar{\mathcal{Q}}$. Finally, use a partition of unity $\{e_\nu\}$ subordinated to the covering of $\mathbf{R}^n$ by the sets $\mathbf{R}^n \backslash \bar{\mathcal{Q}}$ and $\{\mathcal{O}_{\alpha_j}\}_{j=1}^m$ to represent ϕ in the form $\phi = \sum_{j=1}^m \phi_j$, where $\operatorname{supp} \phi_j \subset \mathcal{O}_{\alpha_j}$.]

It follows from this exercise that for any generalized function f in $\mathcal{S}'(\mathbf{R}^n)$ there exists a maximal open subset $\mathcal{O} \subset \mathbf{R}^n$ (possibly empty) on which f vanishes. (Here, by maximal we mean that $\mathcal{O}$ contains every other open subset of $\mathbf{R}^n$ on which f vanishes.) The complement of this set with respect to $\mathbf{R}^n$ is called the *support of the generalized function*, denoted by $\operatorname{supp} f$. (Clearly, $\operatorname{supp} f$ is a closed subset of $\mathbf{R}^n$.) Thus, we have

$$(f,u) = 0, \quad \text{if} \quad \operatorname{supp} f \cap \operatorname{supp} u = \emptyset. \tag{2.11}$$

If $\operatorname{supp} f$ is contained in a closed set $S \subset \mathbf{R}^n$, then we also say that f is *concentrated on the set* S. The support of a distribution in $\mathcal{D}'(\mathbf{R}^n)$ (or, more generally, in $\mathcal{D}'(\mathcal{O})$) is defined in precisely the same way.

Exercise 2.7. Every distribution $f \in \mathcal{D}'(\mathbf{R}^n)$ with compact support is a generalized function (satisfying the estimate (2.5) with $m = 0$). [Hint: Extend T to $\mathcal{S}(\mathbf{R}^n)$ by means of the formula

$$(T,f) = (T,\omega f), \tag{2.12}$$

where ω is a fixed function in $\mathcal{D}(\mathbf{R}^n)$ equal to unity in a neighbourhood of supp T; verify that $(T, f_k) \to 0$ as $f_k \to 0$ in $\mathcal{S}(\mathbf{R}^n)$.]

Thus the distributions in $\mathbf{R}^n$ with compact supports can be identified with the corresponding generalized functions in $\mathbf{R}^n$. The following proposition deals with an important special case.

Proposition 2.2. *Every generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ with support at the origin has the form*

$$(f, u) = \sum_{|\alpha| \le N} c_\alpha D^\alpha u(0), \quad u \in \mathcal{S}(\mathbf{R}^n), \tag{2.13}$$

where the c_α are a fixed finite set of numbers.

Using the notion of partial derivative of a generalized function (see below §2.2.B), we could say that f is a finite linear combination of the δ-function and its partial derivatives.

■ According to Exercise 2.7, we have the following estimate for f:

$$|(f, u)| \le c \cdot \|u\|_{N,0}, \quad u \in \mathcal{S}(\mathbf{R}^n)$$

for some $c \ge 0$ and integer N. It suffices to show that

$$(f, u) = 0, \quad \text{if} \quad u \in \mathcal{S}(\mathbf{R}^n) \quad \text{and} \quad D^\alpha u(0) = 0 \quad \text{for } |\alpha| \le N. \tag{2.14}$$

In fact, let ω be a fixed function in $\mathcal{D}(\mathbf{R}^n)$ equal to unity in a neighbourhood of zero; then for any $u \in \mathcal{S}(\mathbf{R}^n)$, the function

$$v(x) = u(x) - \omega(x) \sum_{|\alpha| \le N} \frac{1}{\alpha!} x^\alpha D^\alpha u(0)$$

belongs to $\mathcal{S}(\mathbf{R}^n)$ and has zero partial derivatives of order $\le N$. It therefore follows from (2.14) that $(f, v) = 0$, and we arrive at the representation (2.13), where $c_\alpha = \frac{1}{\alpha!}(f, x^\alpha \omega)$.

It remains to prove (2.14). It is clear that our function $u(x)$ satisfies the estimate

$$\sup_x |x|^{-|\alpha|-1} |D^\alpha u(x)| < \infty \quad \text{for } |\alpha| \le N,$$

so that

$$|(f, u)| \equiv |(f, \omega(kx)u)| \le c \, \|\omega(kx)u(x)\|_{N,0} \to 0 \quad \text{as } k \to \infty.$$

This proves (2.14). ■

If $\mathcal{O}$ and $\mathcal{Q}$ are two open subsets of $\mathbf{R}^n$ with $\mathcal{Q} \subset \mathcal{O}$, then we can uniquely associate with each distribution $f \in \mathcal{D}'(\mathcal{O})$ the restriction $f|_\mathcal{Q} \in \mathcal{D}'(\mathcal{Q})$ of the distribution f to the (open) subset $\mathcal{Q}$:

$$(f|_\mathcal{Q}, u) = (f, u_1) \quad \text{for all } u \in \mathcal{D}(\mathcal{Q});$$

here, u_1 is a function in $\mathcal{D}(\mathcal{O})$ equal to $u(x)$ in $\mathcal{Q}$ and 0 in $\mathcal{O} \backslash \mathcal{Q}$. Using a partition of unity, it is not difficult to prove the following principle which provides an important practical means of defining distributions (see also Proposition 2.4 in §2.3.A).

Proposition 2.3 (The gluing principle for distributions). *Let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ be a family of open subsets of $\mathbf{R}^n$ and $\{f_\lambda\}_{\lambda \in \Lambda}$ a family of distributions $f_\lambda \in \mathcal{D}'(\mathcal{O}_\lambda)$ satisfying the compatibility condition*

$$f_\lambda|_{\mathcal{O}_\lambda \cap \mathcal{Q}_\mu} = f_\mu|_{\mathcal{O}_\lambda \cap \mathcal{Q}_\mu}$$

for all $\lambda, \mu \in \Lambda$. *Then there exists a unique distribution* $f \in \mathcal{D}'(\mathcal{O})$, *where* $\mathcal{O} = \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$ *coinciding with* f_λ *in* $\mathcal{O}_\lambda$ *for each* λ.

Remark. As we recalled in §1.2.C, the space $\mathcal{D}(\mathcal{O})$ is not an F-space; it belongs to the wider class of so-called strict inductive limits of F-spaces to which most of the results of §1.3 on F-spaces can be extended. We shall not give the notions relating to this. Instead, we confine ourselves to an intuitive explanation of how the results relating to questions of convergence of sequences in $\mathcal{D}(\mathcal{O})$ or $\mathcal{D}'(\mathcal{O})$ can be obtained from the corresponding results for $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$ on the basis of the principles of localization and gluing of distributions. In accordance with these principles, each distribution $f \in \mathcal{D}'(\mathcal{O})$ (or more generally, each linear functional on $\mathcal{D}(\mathcal{O})$) is uniquely defined by its restrictions $f|_K$ to the open subsets $K \subset \mathcal{O}$ with compact closures in $\mathcal{O}$. Instead of the family $\{f|_K\}_{K\subset\mathcal{O}}$, we can equally well choose a family $\{f^K\}_{K\subset\mathcal{O}}$ of generalized functions $f^K \in \mathcal{S}'(\mathbf{R}^n)$ coinciding with f in K. For f^K we can take the generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with compact support in $\mathcal{S}'(\mathbf{R}^n)$ defined by the formula $(f^K, u) = (f, \omega_K u|_K)$ where $u \in \mathcal{S}(\mathbf{R}^n)$ and ω_K is a function in $\mathcal{D}(\mathcal{O})$ equal to 1 in a neighbourhood of K. It is not difficult to see that the convergence of the sequence $f_j \to f$ in $\mathcal{D}'(\mathcal{O})$ is equivalent to the convergence $f_j^K \to f^K$ in $\mathcal{S}'(\mathbf{R}^n)$ for all $K \subset \mathcal{O}$ with compact closure in $\mathcal{O}$. Moreover, the definition of convergence in $\mathcal{D}(\mathcal{O})$ as well as the condition of continuity of a linear functional on $\mathcal{D}(\mathcal{O})$ can be stated in terms of sequences in $\mathcal{D}(\mathcal{O})$ whose supports are contained in some common subset $K \subset \mathcal{O}$ with compact closure in $\mathcal{O}$ (and for such a sequence, the condition $u_j \to u$ in $\mathcal{D}(\mathcal{O})$ is equivalent to the condition $\breve{u}_j \to \breve{u}$ in $\mathcal{S}(\mathbf{R}^n)$, where $\breve{u}$ is a function in $\mathcal{S}(\mathbf{R}^n)$ equal to u in $\mathcal{O}$ and vanishing outside $\mathcal{O}$). Thus, the questions of convergence in $\mathcal{D}(\mathcal{O})$ and $\mathcal{D}'(\mathcal{O})$ completely reduce to the corresponding questions of convergence in $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{S}'(\mathbf{R}^n)$.

2.2. Transformation of Arguments and Differentiation

A. Change of Variables in a Generalized Function

Many of the important operations on distributions are defined by means of an extension of the corresponding operations on ordinary smooth functions. We already know that the generalized functions form a linear space so that they can be added and multiplied by a scalar. It is important that such local operations as a (smooth) change of variables and differentiation can also be extended to generalized functions. The definitions below are obtained, so to speak, by carrying over these operations to the language of linear functionals.

Let $y = \phi(x)$ or, more precisely, let

$$y_1 = \phi_1(x), \dots, y_n = \phi_n(x), \quad x \equiv (x_1, \dots, x_n), \tag{2.15}$$

be a *diffeomorphism* of $\mathbf{R}^n$ onto itself (that is, a one-to-one C^∞-map from $\mathbf{R}^n$ onto itself whose inverse is also C^∞). Then, in particular, the Jacobian of the transformation (2.15) is non-zero:

$$J(\phi) = \begin{vmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_n}{\partial x_1} & \cdots & \frac{\partial\phi_n}{\partial x_n} \end{vmatrix} \neq 0. \tag{2.16}$$

The inverse transformation to (2.15) is denoted by $x = \phi^{-1}(y)$. Then if $f(x)$ is an ordinary (locally integrable) function and $u(x) \in \mathcal{D}$,

$$(f(\phi^{-1}(y)), u(y)) = \int_{\mathbf{R}^n} f(x)u(\phi(x))|J(\phi)|d^n x = (f(x), |J(\phi)|u(\phi(x))). \tag{2.17}$$

In the case when $f(x)$ is a distribution, this equality is taken as the definition of the distribution $f\circ\phi^{-1} \equiv f(\phi^{-1}(y))$. In what follows we shall usually be dealing only with transformations of arguments in distributions that leave the subspace S' invariant, that is, ones that associate with the generalized function $f(x)$ the generalized function $f(\phi^{-1}(y))$.

Exercise 2.8. Let ϕ be a diffeomorphism of $\mathbf{R}^n$ onto itself such that the partial derivatives of any order of the components $\phi_j(x)$ and $(\phi^{-1})_j(y)$ of the maps ϕ and ϕ^{-1} are of polynomial growth (in $|x|$ and $|y|$ respectively). Verify that for each generalized function $f(x) \in S'(\mathbf{R}^n)$, formula (2.17) defines the generalized function $f \circ \phi^{-1} \equiv f(\phi^{-1}(y)) \in S'(\mathbf{R}^n)$, where the map $f \to f \circ \phi^{-1}$ is an isomorphism from $S'(\mathbf{R}^n)$ onto itself.

B. Differentiation of Generalized Functions. Examples

In the preceding discussion we can choose as a special case the transformation of translation by one of the variables, for example,

$$y_1 = x_1 - \Delta x_1, \quad y_2 = x_2, \ldots, y_n = x_n;$$

we then obtain from (2.17) (with $\Delta x_1 \neq 0$)

$$\left(\frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x_1}, u(x)\right) =$$

$$= -\left(f(x), \frac{u(x_1, -\Delta x_1, x_2, \ldots, x_n) - u(x_1, \ldots, x_n)}{-\Delta x_1}\right). \tag{2.18}$$

Using the fact that the functional f is continuous, so that the right hand side of (2.17) tends to the limit $-(f, \partial u/\partial x_1)$ as $\Delta x_1 \to 0$, we can define this limit to be the partial derivative of the generalized function $f(x)$ with respect to x_1. Thus, in the general case, we have

$$\left(\frac{\partial f}{\partial x_k}, u\right) \equiv -\left(f, \frac{\partial u}{\partial x_k}\right). \tag{2.19}$$

Formula (2.19) can be obtained by integrating by parts when the function $f(x)$ has continuous partial derivatives of polynomial growth. This fact is usually taken to be the intuitive justification of the definition (2.19). The preceding discussions (based on formula (2.18)) show that this definition is the same as the usual definition of the derivative of a function.

Since the derivative of a generalized function is again a generalized function (in S'), it is clear that generalized functions have partial derivatives of any order. We note also that the differentiation operation is continuous in S'. In other words, if for any $u \in S$

$$\lim_{n\to\infty} (f_n, u) = (f, u),$$

the following relation holds for each $u \in S$:

$$\lim_{n\to\infty} \left(\frac{\partial f_n}{\partial x}, u\right) = \left(\frac{\partial f}{\partial x}, u\right). \tag{2.20}$$

We note also that the order of differentiation of generalized functions is immaterial:

$$\frac{\partial^2 f}{\partial x_i \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_i}, \tag{2.21}$$

since this equality holds for the test functions.

We now consider some examples of generalized functions of one variable that are derivatives of ordinary locally integrable functions.

1) The derivative of the discontinuous (locally integrable) function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0 \end{cases} \tag{2.22}$$

is equal to $\delta(x)$. For we have :

$$\left(\frac{d\theta}{dx}, u\right) = -\int_{-\infty}^{\infty} \theta(x)\frac{du}{dx}dx = -\int_{0}^{\infty} \frac{du}{dx}dx = u(0) = (\delta, u). \tag{2.23}$$

2) The functional $d \ln|x|/dx$ is the same as the principal value of $1/x$ (in the Cauchy sense); in other words,

$$\left(\frac{d \ln|x|}{dx}, u(x)\right) = \mathcal{P}\int_{-\infty}^{\infty} \frac{u(x)}{x}dx, \tag{2.24}$$

where $\mathcal{P}$ denotes the principal value of the integral. In fact, in view of definition (2.19) we have:

$$\left(\frac{d \ln|x|}{dx}, u(x)\right) = -\int_{-\infty}^{\infty} \ln|x| u'(x)dx =$$

$$= \lim_{\epsilon \to +0} \left\{\int_{-\infty}^{-\epsilon} \frac{u(x)}{x}dx + \int_{\epsilon}^{\infty} \frac{u(x)}{x}dx\right\} = \int_{0}^{\infty} \frac{u(x) - u(-x)}{x}dx = \mathcal{P}\int_{-\infty}^{\infty} \frac{u(x)}{x}dx$$

$$\left(u'(x) \equiv \frac{du}{dx}\right).$$

The generalized function $d \ln(x)/dx$ will be denoted by $1/x$ (and sometimes by $\mathcal{P}\frac{1}{x}$). Note that the ordinary function $1/x$ is not locally integrable (in a neighbourhood of the point $x = 0$) and is therefore not identified with the generalized function $1/x$.

In similar fashion, we define the generalized function $1/x^2$ as the derivative of the generalized function $-1/x$:

$$\left(\frac{1}{x^2}, u(x)\right) = \left(\frac{1}{x}, u'(x)\right) = \int_{0}^{\infty} \frac{u'(x) - u'(-x)}{x}dx = \int_{0}^{\infty} \frac{u(x) + u(-x) - 2u(0)}{x^2}dx. \tag{2.25}$$

More generally, we set by definition

$$\frac{1}{x^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^n}{dx^n} \ln|x|, \quad n = 1, 2, \dots, \tag{2.26}$$

where the derivative has to be understood in the sense of the definition of differentiation in $\mathcal{S}'$.

3) The ordinary locally integrable function $\ln(x \pm i0)$ is defined by the formula

$$\ln(x \pm i0) = \lim_{y \to +0} \ln(x \pm iy) = \ln|x| + i \lim_{y \to +0} \arg(x \pm iy) = \ln|x| \pm i\pi\theta(-x), \tag{2.27}$$

where $\theta(x)$ is defined by (2.23). The derivative of the generalized function (2.27) is denoted by $1/(x \pm i0)$. In view of (2.23), (2.26), (2.27), the *Sokhotskii formulae* hold:

$$\frac{1}{x \pm i0} \equiv \frac{d}{dx} \ln(x \pm i0) = \frac{1}{x} \mp i\pi\delta(x), \tag{2.28}$$

from which it follows that

$$\frac{1}{x - i0} - \frac{1}{x + i0} = 2\pi i\delta(x). \tag{2.29}$$

By successively differentiating (2.28) we obtain

$$\frac{1}{(x \pm i0)^n} = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^n}{dx^n} \ln(x \pm i0) = \frac{1}{x^n} \pm \frac{(-1)^n}{(n-1)!}i\pi\delta^{(n-1)}(x). \tag{2.30}$$

The derivatives of the delta-function on the right hand side of (2.30) are defined from (2.19) by the formula

$$(\delta^{(n)}, u) = (-1)^n u^{(n)}(0). \tag{2.31}$$

2.3. Multiplication of a Generalized Function by a Smooth Function

A. THE PROBLEM UNDERLYING MULTIPLICATION OF GENERALIZED FUNCTIONS. THE CONCEPT OF A MULTIPLICATOR

We have defined the set of generalized functions as a linear space, so that it is no surprise that the apparatus of generalized functions is well suited for discussing linear problems. We have already seen in the preceding section that generalized functions can be differentiated any number of times and that their mixed partial derivatives do not depend on the order of differentiation (which, in general, is not true for ordinary functions that are "not too good"). We shall see in the next section that the Fourier transform can be freely applied to generalized functions.

However, there is no natural definition of multiplication for an arbitrary pair of generalized functions. It is easy to see that it is impossible to define a product that is bilinear and associative and satisfies, for example, the following natural conditions:

$$\delta(x)\cdot x = 0, x\cdot\frac{1}{x} = 1.$$

In fact,*

$$\delta(x)\{x\cdot\frac{1}{x}\} = \delta(x) \neq \{\delta(x)\cdot x\}\frac{1}{x} = 0. \tag{2.32}$$

Nevertheless there is a wide class of functions for which it is possible to define the product with generalized functions in $\mathcal{S}'$ in a natural manner. This class is defined in the following way.

We say that a function $\phi(x)$ is a *multiplicator* in the test function space $\mathcal{S}$ if $u(x) \in \mathcal{S}$ implies that $\phi(x)u(x) \in \mathcal{S}$ as well. The space of all multiplicators will be denoted by Θ_M (or $\Theta_M(\mathbf{R}^n)$). It will be shown that for a function $\phi(x)$ to be a multiplicator, it is necessary and sufficient that it be infinitely differentiable and have polynomial growth along with its partial derivatives:

$$|D^\alpha\phi(x)| \leq c_\alpha(1+|x|)^{p_\alpha}. \tag{2.33}$$

If $\phi(x)$ is a multiplicator, then the product of $\phi(x)$ by a generalized function $f \in \mathcal{S}'$ is defined by the formula

$$(\phi(x)f, u(x)) = (f, \phi(x)u(x)). \tag{2.34}$$

This multiplication is, by definition, commutative: $\phi f = f\phi$.

In certain cases we can also define the product of two generalized functions . The theory of such "singular" products is less natural and more complex than the above case of multiplication of a generalized function by a multiplicator.

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* It should be noted that products of type $\delta(x)\{x\,f(x)\}$ occur extensively in quantum field theory (see, for example, §13.2.C and thereafter).

In §2.6.C below we set out one of the possible variants of the definition of the product of generalized functions. Another approach is set out in [R1],vol.2,§IX.10.

Exercise 2.9. Suppose that the support of the generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ is contained in the closed set $\Omega \subset \mathbf{R}^n$, and that for each $\epsilon > 0$, Ω^ϵ is the closure of the ϵ-neighbourhood of Ω (that is, Ω^ϵ is the set of points of $\mathbf{R}^n$ whose distance from Ω does not exceed ϵ). Prove that for any $\epsilon > 0$ there exist a constant c and natural numbers l, m (depending on ϵ) such that

$$|(f,u)| \leq c\|u\|_{l,m}^{\Omega^\epsilon} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n), \tag{2.35}$$

where by definition

$$\|u\|_{l,m}^{\Omega} = \max_{|\alpha|\leq l} \sup_{x\in\Omega}(1+|x|^m|D^\alpha u(x)|). \tag{2.36}$$

[Hint: Take a multiplicator ϕ equal to 1 in Ω^δ, where $0 < \delta < \epsilon$, and having support in Ω^ϵ; for example,

$$\phi(x) = \int_{\Omega^\delta} \omega(x-y)dy,$$

where ω is a function in $\mathcal{D}(\mathbf{R}^n)$ with integral equal to 1 and with support in a sufficiently small neighbourhood of zero. Then use the equality $(f,u) \equiv (f,\phi u)$; use (2.2) to estimate the right hand side.]

It should be noted that for "sufficiently good" sets Ω (and this will be made precise in Appendix A) the estimate (2.35) can be sharpened: there exist a constant c and natural numbers l, m such that

$$|(f,u)| \leq c \cdot \|u\|_{l,m}^{\Omega} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n).$$

By using a suitable partition of unity into multiplicators in $\mathbf{R}^n$, it is possible to extend the gluing principle for distributions (§2.1.C) to generalized functions of class $\mathcal{S}'$. To simplify the statement, we restrict ourselves to finite coverings of $\mathbf{R}^n$.

Proposition 2.4. *Let $\{\mathcal{O}_j\}_{j=1,\dots,m}$ be a finite family of open subsets of $\mathbf{R}^n$ covering $\mathbf{R}^n$ such that for each $j = 1,\dots,m$ the set*

$$Q_j = \mathbf{R}^n \setminus \bigcup_{i\neq j} \mathcal{O}_i$$

is closed and contained in $\mathcal{O}_j$. Suppose that the distance $d(x,\partial\mathcal{O}_j)$ from the point $x \in Q_j$ to the boundary $\partial\mathcal{O}_j$ of $\mathcal{O}_j$ satisfies the condition $d(x,\mathcal{O}_j) \geq A\cdot(1+|x|)^{-\delta}$ for all $x \in Q_j$; here the numbers $A > 0$ and $\delta \geq 0$ depend only on the given covering. Then:

(a) *there exists a partition of unity $\{e_j\}_{j=1}^m$, subordinated to the covering $\{\mathcal{O}_j\}$, where all the e_j are multiplicators in $\Theta_M(\mathbf{R}^n)$;*

(b) *for any family $\{f_j\}_{j=1,\dots,m}$ of generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ satisfying the compatibility condition*

$$(f_i - f_j)|_{\mathcal{O}_i\cap\mathcal{O}_j} = 0 \quad \textit{for all } i,j = 1,\dots,m,$$

there exists a unique generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ coinciding with f_j in $\mathcal{O}_j$ for all j.

Exercise 2.10. Prove Proposition 2.4. [Hint: For the case $\delta = 0$, the e_j can be defined as follows:

$$e_j(x) = \phi_j(x)\left(\sum_{i=1}^m \phi_j(x)\right)^{-1}, \quad \phi_j(x) = \int \omega\left(\frac{x-y}{A}\right)\chi_j(y)d^ny;$$

here ω is a non-negative function in $\mathcal{D}(\mathbf{R}^n)$ with support in the ball $|x| \le \frac{1}{2}$, where $\int \omega(x)d^nx = 1$, and χ_j is the characteristic function of the set $\{x \in \mathcal{O}_j : d(x, \partial\mathcal{O}_j) \ge A/2\} \supset Q_j$. For the case $\delta > 0$, one has to modify the definition of $\phi_j(x)$ by setting

$$\phi_j(x) = \sum_{k=1}^{\infty} \int \omega\left(\frac{x-y}{A_k}\right) \chi_{j,k}(y)d^ny,$$

where ω is the same function as before, $A_k = A(1+k)^{-\delta}$ and $\chi_{j,k}$ is the characteristic function of the set $\{x \in \mathcal{O}_j : k-1 \le |x| \le k,\ d(x, \partial\mathcal{O}_j) \ge \frac{1}{2}A_k\}$.]

B. THE DIVISION PROBLEM

We now come to the converse problem of division, that is, the study of the equation

$$\phi(x)f = g, \tag{2.37}$$

where $g \in \mathcal{S}'$ and $\phi(x) \in \Theta_M$ are given and f is an unknown generalized function. In the case when the function $\phi(x) \not\equiv 0$ for all x and does not tend to zero too rapidly as $x \to \infty$, more precisely, if the function $1/\phi(x)$ is also a multiplicator, then (2.37) is solved in elementary fashion. The problem becomes considerably more complex if $\phi(x)$ vanishes at certain points. We restrict ourselves to the case of a single independent variable and suppose that the function $\phi(x)$ occurring in (2.37) has only a discrete set of zeros of finite order. Under these assumptions, the division problem in $\mathcal{S}'(\mathbf{R})$ reduces essentially to the solution of a very simple equation of type (2.37), namely,

$$xf = g. \tag{2.38}$$

In view of (2.34), equation (2.38) can be written in the form

$$(g, u(x)) = (xf, u(x)) = (f, xu(x)),$$

where $u(x)$ is an arbitrary test function. Thus if a generalized function f satisfying (2.38) exists and

$$v(x) = xu_1(x), \quad u_1(x) \in \mathcal{S}, \tag{2.39}$$

then

$$(f, v) = (g, u_1). \tag{2.40}$$

Thus the functional f is uniquely defined on the subspace (of $\mathcal{S}$) of functions $v(x)$ for which $v(0) = 0$. For the complete definition of the functional f it suffices to define its action on some function $u_0(x)$ in $\mathcal{S}$ such that $u_0(0) = 1$. Any function $u(x) \in \mathcal{S}$ can be expressed in the form

$$u(x) = u(0)u_0(x) + v(x), \tag{2.41}$$

where $v(0) = 0$, so that the function $v(x)$ has the form (2.39). Applying the functional f to both sides of (2.71) and taking (2.40) into account, we obtain

$$(f, u(x)) = u(0)(f, u_0(x)) + (g, u_1(x)). \tag{2.42}$$

So far the discussion has been carried out under the assumption that equation (2.38) has a solution. It is not difficult to verify directly that for any choice of the constant C, the linear functional

$$(f, u) = Cu(0) + (g, u_1(x)) \tag{2.43}$$

is continuous in S and in fact satisfies (2.38). The arguments leading to formula (2.42) show that the functional (2.43) is the general solution of (2.38) (any two particular solutions of this equation differing only by the value of the constant C). The general solution of the homogeneous equation

$$xf_0 = 0 \tag{2.44}$$

has the form

$$(f_0, u) = Cu(0), \quad \text{that is} \quad f_0 = C\delta(x). \tag{2.45}$$

Once we have verified that (2.38) has a solution, it is not difficult to show that the more general equation

$$x^l f = g, \tag{2.46}$$

where $g \in \mathcal{S}'$ and l is any natural number, always has a solution for the unknown f in $\mathcal{S}'$. The arbitrariness of this solution is defined by the general solution of the homogeneous equation

$$x^l f_0 = 0. \tag{2.47}$$

Exercise 2.11. Prove that the general solution of (2.47) has the form

$$f_0 = \sum_{\nu=0}^{l-1} \frac{c_\nu}{\nu!}\delta^{(\nu)}(x), \tag{2.48}$$

where the c_ν are arbitrary constants. [Hint: Use the formula

$$x^k \frac{1}{\nu!}\delta^{(\nu)}(x) = \begin{cases} 0 & \text{for} \quad k > \nu, \\ \frac{(-1)^k}{(\nu-k)!}\delta^{(\nu-k)}(x) & \text{for} \quad k \le \nu \end{cases} \tag{2.49}$$

and proceed by induction on l.]

Formula (2.49) can be obtained by Leibnitz's rule for differentiating the product of two functions

$$\frac{d^n}{dx^n}[\phi(x)f] = \sum_{\nu=0}^{n} \binom{n}{\nu} \phi^{(n-\nu)}(x) f^{(\nu)}(x), \quad \phi \in \Theta_M, \quad f \in \mathcal{S}'. \tag{2.50}$$

It is a special case of the more general formula

$$u(x)\delta^{(n)}(x) = \sum_{\nu=0}^{n} (-1)^{(\nu)} \binom{n}{\nu} u^{(\nu)}(0)\delta^{(n-\nu)}(x). \tag{2.51}$$

Similarly in $\mathcal{S}'$ we can divide by the function $(x-a)^l$, where a is a real number and, more generally, by an arbitrary polynomial.

Exercise 2.12. Let $a_1 \ldots, a_r$ be the real roots of the polynomial $P(x)$ (generally speaking, $P(x)$ will also have complex roots), where l_i is the multiplicity of the root a_i. Show that the general solution of the equation

$$P(x)f_0 = 0 \tag{2.52}$$

has the form

$$f_0 = \sum_{i=1}^{r} \sum_{\nu_i=0}^{l_i-1} c_i^{\nu_i} \delta^{(\nu_i)}(x - a_i). \tag{2.53}$$

The above results generalize to the wider class of infinitely smooth functions that only have zeros of finite multiplicity on the real axis. For example, the general solution of the equation

$$\sin x \cdot f = 0$$

has the form

$$f = \sum_{n=-\infty}^{\infty} c_n \delta(x - n\pi),$$

where the coefficients c_n can increase no faster than some power of $|n|$ as $|n| \to \infty$, that is, there exist positive constants A and λ (not depending on n) such that

$$|c_n| \leq A(1 + |n|)^{\lambda}.$$

Apart from this, the constants c_n are arbitrary.

However, not every equation of type (2.37) has a solution in $\mathcal{S}'$ (even if we suppose that the multiplicator $\phi(x)$ vanishes only at one point). Thus, for example, it can be shown that the equation

$$\exp(-1/x^2)f = 1$$

has no solution in the class of generalized functions $\mathcal{S}'$, since the function $\exp(-1/x^2)$ has a zero of infinite multiplicity at the point $x = 0$ (which is an essential singularity of this function).

The results obtained in the division problem in $\mathcal{S}'(\mathbf{R})$ can also be applied to certain special problems on the division of generalized functions of several variables (although the general problem of the division of functionals in $\mathcal{S}'(\mathbf{R}^n)$ by an arbitrary polynomial in n variables is more complicated).

An example of this sort is the equation

$$(p^2 - m^2)f = 1 \tag{2.54}$$

for a generalized function in Minkowski space $\mathbf{M}$ (see below §3.1.A); here m is a positive parameter. A particular solution of this equation has the form

$$f_0 = \mathcal{P}\frac{1}{p^2 - m^2}.$$

It is well defined. For in the region $p^2 < m_1^2$ it is an ordinary regular function ($0 < m_1 < m$). It remains to consider the two regions

$$p^2 > m_2^2, \quad \pm p^0 > 0$$

(where $0 < m_2 < m_1$); in each of them we can go over from the variables $p \equiv (p^0, \mathbf{p})$ to the variables $(p^2, \mathbf{p})$, so that (according to the one variable case considered above), the formula defines distributions in these regions. It remains to apply the gluing principle for distributions (or generalized functions). It can be shown that the general solution of (2.54) in the class $\mathcal{S}'(\mathbf{M})$ has the form

$$f = \mathcal{P}\frac{1}{p^2 - m^2} + \theta(p^0)\delta(p^2 - m^2)f_1(\mathbf{p}) + \theta(-p^0)\delta(p^2 - m^2)f_2(\mathbf{p}), \tag{2.55}$$

where f_1 and f_2 are arbitrary generalized functions in $\mathcal{S}'(\mathbf{R}^3)$; here, the second term in (2.55) is defined as follows:

$$\int \theta(p^2)\delta(p^2 - m^2)f_1(\mathbf{p})u(p)dp = \int f_1(\mathbf{p})\frac{u(\sqrt{m^2 + \mathbf{p}^2}, \mathbf{p})}{2\sqrt{m^2 + \mathbf{p}^2}}d^3p;$$

the last term in (2.55) is defined similarly.

We recall the classical result due to Hörmander (1958) called the "Hörmander division theorem", which asserts that equation (2.37) (in the unknown $f \in \mathcal{S}'(\mathbf{R}^n)$ for a given $g \in \mathcal{S}'(\mathbf{R}^n)$ and ϕ) always has a solution if ϕ is a (complex) polynomial in $\mathbf{R}^n$ that does not vanish identically.

2.4. The Kernel Theorem. Tensor Products of Generalized Functions

A. Bilinear functionals on spaces of type $\mathcal{S}$

We now state a certain property of spaces of type $\mathcal{S}$ that essentially distinguishes them from a Hilbert (and more generally from a normed) space and which to some extent is responsible for the fact that the spaces $\mathcal{S}(\mathbf{R}^n)$ are "good", that is, convenient for applications. We can arrive at this property by two equivalent routes: by studying the general form of bilinear (continuous) functionals in $\mathcal{S}$ or by considering linear operators taking $\mathcal{S}$ into its adjoint $\mathcal{S}'$.

A functional $B(u,v)$ is called *bilinear* if it is linear in each of its arguments u and v when the other argument is kept fixed. If u and v are tensors of degree k and m respectively in a finite-dimensional space: $u = (u_{i_1 i_2 \dots i_k})$, $v = (v_{j_1 j_2 \dots j_m})$, then every bilinear functional $B(u,v)$ can be represented in the form

$$B(u,v) = \sum_{i_1 \dots i_k j_1 \dots j_m} b_{i_1 \dots i_k j_1 \dots j_m} u_{i_1 \dots i_k} v_{j_1 \dots j_m}, \tag{2.56}$$

where $b_{i_1 \dots j_m}$ is a tensor of degree $k+m$ in the same finite-dimensional space. This simple but important theorem has no analogue in Hilbert space. We illustrate this by the example of the space $\mathcal{L}^2(\mathbf{R}^k)$ of square-integrable functions of k variables. For this case, the analogue of (2.56) is the representation

$$B(u,v) = \int_{\mathbf{R}^{k+m}} \cdots \int dx_1 \dots dx_k dy_1 \dots dy_m u(x_1, \dots, x_k) \times \\ \times v(y_1, \dots, y_m) K(x_1, \dots, x_k; y_1, \dots, y_m), \tag{2.57}$$

where the kernel $K(x_1, \dots, x_k; y_1, \dots, y_m)$ is square-integrable over $\mathbf{R}^{k+m}$. In fact, not every bilinear functional in $\mathcal{L}^2(\mathbf{R}^k) \times \mathcal{L}^2(\mathbf{R}^m)$ can be expressed in the form (2.57). For example, the simplest continuous bilinear functional (with $k = m$)

$$B(u,v) = \int_{\mathbf{R}^k} \cdots \int dx_1 \dots dx_k u(x_1, \dots, x_k) v(x_1, \dots, x_k)$$

cannot be expressed in the form (2.57) (with kernel K an ordinary square-integrable function).

By contrast, the analogue of (2.57) holds for continuous linear functionals on the spaces $\mathcal{S}$. Here, the continuity condition can be slightly weakened, replacing it by separate continuity (in each argument). We add by way of explanation that continuity of a bilinear function $B(u,v)$ over $u \in \mathcal{S}(\mathbf{R}^k)$, $v \in \mathcal{S}(\mathbf{R}^m)$ means that $B(u_n, v_n) \to B(u,v)$ if $u_n \to u$ in $\mathcal{S}(\mathbf{R}^k)$ and $v_n \to v$ in $\mathcal{S}(\mathbf{R}^m)$ as $n \to \infty$. On the other hand, separate continuity means continuity of $B(u,v)$ in u for each fixed v (that is, $B(u_n, v) \to B(u,v)$ if $u_n \to u$) and continuity of $B(u,v)$ in v for each fixed u. It turns out that separate continuity automatically implies continuity. Furthermore, there is the following classical result.

Theorem 2.5 (Schwartz kernel theorem). *An arbitrary bilinear (separately) continuous functional $B(u,v)$ over $u \in \mathcal{S}(\mathbf{R}^k)$ and $v \in \mathcal{S}(\mathbf{R}^m)$ is uniquely representable in the form*

$$B(u,v) = (F(x,y), u(x) \cdot v(y)), \tag{2.58}$$

where $F(x,y)$ *is a generalized function in* $\mathcal{S}'(\mathbf{R}^k \times \mathbf{R}^m) \equiv \mathcal{S}'(\mathbf{R}^{k+m})$, *which depends on the variables* $x \in \mathbf{R}^k$, $y \in \mathbf{R}^m$.

We omit the proof of this theorem, referring the reader to the literature (see, for example, [G6],§1.1.3 or [R1],vol.1, Appendix 2,§V.3). It should be noted that by induction on the number of arguments, the kernel theorem has a natural generalization to multilinear continuous functionals $B(u_1, \ldots, u_n)$ (depending on n arguments $u_1, \ldots, u_n$ instead of just the two u, v). Furthermore, one can allow $B(u_1, \ldots, u_n)$ to take values in an LCS Ω. For example, in the case $n = 2$, the (separate) continuity of the functional $B(u,v)$ in $u \in \mathcal{S}(\mathbf{R}^k)$, $v \in \mathcal{S}(\mathbf{R}^m)$ implies that there exists a (unique) continuous linear operator $b : \mathcal{S}(\mathbf{R}^{k+m}) \to \Omega$ such that

$$B(u,v) = b(u \otimes v),$$

where $u \otimes v$ denotes the function $u(x)v(y) \in \mathcal{S}(\mathbf{R}^{k+m})$.

Formula (2.58) also enables one to obtain the general form of a continuous linear operator A mapping $\mathcal{S}(\mathbf{R}^n)$ into $\mathcal{S}'(\mathbf{R}^m)$. For this purpose it suffices to consider the (separately) continuous bilinear functional $B(u,v) = (Au, v)$, where $u \in \mathcal{S}(\mathbf{R}^k)$, $v \in \mathcal{S}(\mathbf{R}^m)$. As a result we have the following analogue of (2.58):

$$(Au, v) = (F(x,y),\ u(x)v(y)). \tag{2.59}$$

B. TENSOR PRODUCTS

It is easy to construct by means of the kernel theorem the tensor (or direct) product of generalized functions. By the tensor product of (ordinary) functions we mean the product of the functions depending on different independent variables; thus, the function $u(x) \cdot v(y)$ in $\mathbf{R}^{k+m}$ serves as the tensor product of the functions $u(x)$ in $\mathbf{R}^k$ and $v(y)$ in $\mathbf{R}^m$. Similarly, using the functional definition of generalized functions, by the *tensor product of the generalized functions* $f(x) \in \mathcal{S}'(\mathbf{R}^k)$ and $g(y) \in \mathcal{S}'(\mathbf{R}^m)$, we mean the generalized function in $\mathcal{S}'(\mathbf{R}^{k+m})$, denoted by $f(x)g(y)$ (or $f \otimes g$ if the arguments are omitted), satisfying the property

$$\int f(x)g(y)u(x)v(y)dxdy = \left(\int f(x)u(x)dx\right)\left(\int g(y)v(y)dy\right) \tag{2.60}$$

for all $u \in \mathcal{S}(\mathbf{R}^k)$, $v \in \mathcal{S}(\mathbf{R}^m)$. As is immediately clear from the kernel theorem, the generalized function $f \otimes g$ does indeed exist and is unique.

However, the construction of the tensor product of generalized functions can be carried out by a more elementary argument. Thus, the value of $f \otimes g$ at an arbitrary test function $w(x,y) \in \mathcal{S}(\mathbf{R}^{k+m})$ can be defined by either of the following two formulae:

$$\int f(x)g(y)w(x,y)dxdy = \int f(x)\left[\int g(y)w(x,y)dy\right]dx, \tag{2.61}$$

$$\int f(x)g(y)w(x,y)dxdy = \int g(y)\left[\int f(x)w(x,y)dx\right]dy. \tag{2.62}$$

Here the expression in the square brackets, for example, in (2.61), is to be taken as the value of the generalized function $g(y)$ at the test function $w(x,y)$ (as a function of y), which depends on x as a parameter. It is not too difficult to verify that this expression defines a test function, say, $W(x) \in \mathcal{S}(\mathbf{R}^k)$ and that convergence $w_\nu \to 0$ in $\mathcal{S}(\mathbf{R}^n)$ implies convergence $W_\nu \to 0$. This latter observation shows that formulae (2.61) and (2.62) do, in fact, define generalized functions.

The uniqueness of the tensor product $f \otimes g$ follows from the fact that the linear combinations of functions of the form $u(x)v(y)$ with arbitrary $u(x) \in \mathcal{D}(\mathbf{R}^k)$, $v(y) \in \mathcal{D}(\mathbf{R}^m)$ form a dense subset of $\mathcal{S}(\mathbf{R}^{k+m})$. This is easily obtained from the denseness of $\mathcal{D}(\mathbf{R}^{k+m})$ in $\mathcal{S}(\mathbf{R}^{k+m})$ and from the Weierstrass theorem on approximation by polynomials (§1.2.C). In particular, it follows that the right hand sides of (2.61) and (2.62) are the same.

We note that the tensor product of generalized functions has a natural generalization to any (finite) number of generalized functions $f_1(x_1), \dots, f_n(x_n)$ of distinct independent variables $x_1 \in \mathbf{R}^{k_1}, \dots, x_n \in \mathbf{R}^{k_n}$.

Exercise 2.13. Let $f(x) \in \mathcal{S}'(\mathbf{R}^k)$, $g(y) \in \mathcal{S}'(\mathbf{R}^m)$. Prove that

$$\operatorname{supp} f(x) \cdot g(y) \subset \operatorname{supp} f(x) \times \operatorname{supp} g(y), \tag{2.63}$$

$$D_x^\alpha \cdot D_y^\beta f(x) \cdot f(y) = (D_x^\alpha f(x)) \cdot (D_y^\beta f(y)); \tag{2.64}$$

here D_x^α and D_y^β denote differential monomials of degree $|\alpha|$ and $|\beta|$ in the variables x and y respectively.

In the following exercise, it is convenient to use the Weierstrass polynomial approximation theorem mentioned above.

Exercise 2.14. Let $f(x,y) \in \mathcal{S}'(\mathbf{R}^{k+m})$ be a generalized function with support contained in the set $\{a\} \times S$, where a is a point in $\mathbf{R}^k$ and S is a closed subset of $\mathbf{R}^m$. Prove that f is uniquely expressible as a finite sum:

$$f(x,y) = \sum_\alpha D_x^\alpha \delta(x) h_\alpha(y), \tag{2.65}$$

where the $h_\alpha(y)$ are generalized functions in $\mathcal{S}'(\mathbf{R}^m)$ with supports in S. [Hint: Let $u(x) \in \mathcal{S}(\mathbf{R}^k)$, $v(y) \in \mathcal{S}(\mathbf{R}^m)$; using Proposition 2.2, for fixed v write down the representation

$$(f, u \otimes v) = \sum_{|\alpha| \le N} c_\alpha(v) D^\alpha u(0);$$

use a continuity argument to show that the number N in this representation can be chosen to be independent of v. Finally, use the fact that linear combinations of functions of the form $u \otimes v$ are dense in $\mathcal{S}(\mathbf{R}^{k+m})$.]

2.5. Fourier Transform and Convolution

A. FOURIER TRANSFORM OF TEST FUNCTIONS

One of the main advantages of the space $\mathcal{S}'(\mathbf{R}^n)$ of slowly increasing distributions over the space $\mathcal{D}'(\mathbf{R}^n)$ of all Schwartz distributions is that it is possible to define the Fourier transform as a map from $\mathcal{S}'(\mathbf{R}^n)$ to a space that is isomorphic to it.

We define the Fourier transform and its inverse for square-integrable functions (in particular, for test functions) by the formulae

$$\tilde{u}(p) = \mathcal{F}u \equiv \mathcal{F}_{x \to p}[u(x)] = \int u(x) e^{ipx} d^n x, \tag{2.66}$$

$$u(x) = \mathcal{F}^{-1}\tilde{u} \equiv \mathcal{F}^{-1}_{p \to x}[\tilde{u}(p)] = \int \tilde{u}(p) e^{-ipx} d_n p, \tag{2.67}$$

where

$$d_n p = (2\pi)^{-n} d^n p. \tag{2.68}$$

Here px is a fixed non-degenerate bilinear form in p and x. In the mathematical literature, xp is usually chosen to be the Euclidean scalar product. But with the application in view to functions of (a finite number of) vectors in Minkowski space, we shall set

$$xp \equiv x \cdot p = \sum_{j=1}^{n} \epsilon_j x_j p_j, \ \epsilon_j = \pm 1, \ j = 1, \dots, n \tag{2.69}$$

(where the ϵ_j are fixed).

From now on, we shall make a distinction between the n-dimensional x-space endowed with Lebesgue measure $d^n x$, previously denoted by $\mathbf{R}^n$, and the n-dimensional p-space endowed with the measure $d_n p$ (2.68), which we denote by $\mathbf{R}_n$. Similarly, we shall draw a distinction between the isomorphic spaces of test (and generalized) functions on $\mathbf{R}^n$ and $\mathbf{R}_n$, but with the following understanding. In the first case, when we convert an ordinary function $f(x)$ in $\mathbf{R}^n$ to a generalized function (also when we write down the result of smoothing a generalized function) $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ with a test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$), we shall use (2.1) as before. On the other hand, the conversion of an ordinary function $f(p)$ in $\mathbf{R}_n$ to a generalized function and the smoothing of $f(p) \in \mathcal{S}'(\mathbf{R}_n)$ with $u(p) \in \mathcal{S}(\mathbf{R}_n)$, will now be carried out in terms of the measure $d_n p$:

$$\int f(p) u(p) d_n p. \tag{2.70}$$

From the physical point of view, it is advisable to preserve the individualities of these function spaces and their Fourier transforms since these spaces have different interpretations (say, $x \in \mathbf{R}^n$ as "coordinates" and $p \in \mathbf{R}_n$ as "momenta").

Proposition 2.6. *The operator $\mathcal{F}$ (2.66) defines a (topological) isomorphism from $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}(\mathbf{R}_n)$. The inverse of $\mathcal{F}$ is the operator $\mathcal{F}^{-1}$ of (2.67).*

Exercise 2.15. Prove Proposition 2.5 by establishing en route the relations

$$D_p^\alpha \tilde{u}(p) = \int (i\epsilon x)^\alpha u(x) e^{ipx} d^n x, \tag{2.71}$$

$$p^\alpha \tilde{u}(p) = \int [(-i\epsilon \partial_x)^\alpha u(x)] e^{ipx} d^n x, \tag{2.72}$$

where, for example,

$$(i\epsilon x)^\alpha = \prod_{j=1}^{n} (i\epsilon_j x_j)^{\alpha_j}. \tag{2.73}$$

[Hint: In the proof of the fact that the operators (2.66) and (2.67) are inverses of each other, change the order of integration on the right hand side of the equality

$$\mathcal{F}^{-1}_{p\to x} \mathcal{F}_{y\to p}[u(y)] = \lim_{\epsilon\to +0} \int \exp\Big(-\frac{\epsilon}{2}|p|^2 - ixp\Big) \Big(\int e^{iyp} u(y) dy\Big) d_n p,$$

and then use the equality

$$\int \exp\Big(-\frac{\epsilon}{2}|p|^2 - ixp\Big) d_n p = (2\pi)^{-n/2} \exp\Big(-\frac{1}{2\epsilon}|x|^2\Big)$$

and Exercise 2.4.]

In particular, the Parseval identity follows from (2.66), (2.67):

$$\int \overline{u(x)} v(x) d^n x = \int \overline{(\mathcal{F}u)(p)} (\mathcal{F}v)(p) d_n p. \tag{2.74}$$

B. FOURIER TRANSFORM OF GENERALIZED FUNCTIONS

As the starting point for the definition of the Fourier transform of generalized functions, we apply the formula

$$\int f(x)u(x)d^n x = \int \tilde{f}(p)(\tilde{u})(-p)d_n p, \tag{2.75}$$

which is valid for any absolutely integrable function $f(x)$ and $u(x) \in \mathcal{S}(\mathbf{R}^n)$. We define the Fourier transform of a generalized function $f \in \mathcal{S}'(\mathbf{R}_n)$, denoted by $\tilde{f} = \mathcal{F}f \in \mathcal{S}'(\mathbf{R}_n)$, as the linear functional on the space of test functions $\tilde{u}(p) \in \mathcal{S}(\mathbf{R}_n)$, defined by (2.75).

Clearly, this definition is an extension of the definition (2.66) of the Fourier transform of test functions. Therefore we shall often use the notation for $\mathcal{F}f$ in the form of the symbolic integral

$$\tilde{f}(p) = \mathcal{F}_{x\to p}[f(x)] = \int f(x)e^{ipx}d^n x \tag{2.76}$$

and similarly,

$$\mathcal{F}^{-1}g \equiv \mathcal{F}^{-1}_{p\to x}[g(p)] = \int g(p)e^{-ipx}d_n p. \tag{2.77}$$

Exercise 2.16. Prove that the definition of $\mathcal{F}f$ via the Parseval formula

$$\int \mathcal{F}f \cdot \overline{\mathcal{F}} u d_n p = \int f(x)\bar{u}(x)d^n x \tag{2.78}$$

is equivalent to the above definition.

Proposition 2.7. *The Fourier transformation $\mathcal{F}$ of generalized functions defined by* (2.75) *or* (2.78), *defines an isomorphism from $\mathcal{S}'(\mathbf{R}^n)$ onto $\mathcal{S}'(\mathbf{R}_n)$. The operator $\mathcal{F}^{-1}$* (2.67) *is the inverse of $\mathcal{F}$.*

Exercise 2.17. Prove Proposition 2.7 using Proposition 2.6. Show that the formulae of type (2.71), (2.72) remain valid for generalized functions.

We now give some examples of Fourier transforms of generalized functions. *

The Fourier transform of the function $\theta(x)$ of (2.22) can be calculated as the limit of the ordinary Fourier transform of the function $\theta(x)e^{-\epsilon x}$ as $\epsilon \to +0$:

$$[\mathcal{F}\theta](p) = \lim_{\epsilon\to+0}\int_0^\infty e^{ipx-\epsilon x}dx = \frac{i}{p+i0}. \tag{2.79}$$

The Fourier transform of the n-dimensional δ-function (see (2.3)) can be obtained as the value of the functional δ at the (smooth) test function e^{ipx}:

$$\int \delta(x-a)e^{ipx}d^n x = e^{ipa}; \tag{2.80}$$

while the inversion formula gives

$$\delta(x) = \int e^{-ipx}d_n p. \tag{2.81}$$

* A table of frequently encountered transforms of generalized functions is given in [G4],§II.2.5.

Below (in Appendix F) we also give the Fourier transforms of certain Lorentz-invariant generalized functions in Minkowski space. For the moment we merely note that the property of Lorentz-invariance is preserved under the Fourier transformation (where $p \cdot x$ denotes the Minkowski scalar product).

Exercise 2.18. Let A be a non-linear transformation in n-dimensional space $\mathbf{R}^n$ endowed with the bilinear form (2.69), and A' the transformation related to A by the formula $(Ax) \cdot p = x \cdot A'p$. Prove that

$$\int f(A^{-1}x)e^{ipx}d^n x = |detA| \int f(x)e^{ixA'p}d^n x.$$

[Hint: Prove first of all that this formula holds for test functions, and then make its meaning precise and establish its validity for generalized functions.]

We note finally that the Fourier transformation diagonalizes the translation operator both for test functions and generalized functions:

$$\mathcal{F}_{x\to p}[f(x-a)] = \int f(x-a)e^{ipx}d^n x = e^{ipa}\mathcal{F}_{x\to p}[f(x)].$$

C. CONVOLUTES

In classical analysis one often uses the operation of convolution of two functions $f(x)$ and $g(x)$ defined by the formula

$$f(x) * g(x) = \int_{R^n} f(x-y)g(y)d^n y = \int_{R^n} f(y)g(x-y)d^n y. \tag{2.82}$$

This operation plays an even more important role in the analysis of generalized functions. For the sake of definiteness, we restrict ourselves to the "coordinate" space $\mathbf{R}^n$ with measure $d^n x$ (in the "momentum" space the convolution is defined in terms of the measure $d_n p$).

Exercise 2.19. Prove that the convolution of test functions in $\mathcal{S}(\mathbf{R}^n)$ is commutative (that is, $f * g = g * f$) and associative (that is, $f * (g * h) = (f * g) * h$). Also verify the following formulae (where $f, g, u \in \mathcal{S}(\mathbf{R}^n)$):

$$(f * g, u) = \int\int f(x) \cdot g(y)u(x+y)d^n x d^n y, \tag{2.83}$$

$$\mathcal{F}_{x\to p}[f * u(x)] = \mathcal{F}_{x\to p}[f(x)] \cdot \mathcal{F}_{x\to p}[u(x)]. \tag{2.84}$$

As with the operation of multiplication, the convolution is not defined for arbitrary pairs of generalized functions. The convolution of a generalized function f with a test function $u(x)$ is easily defined using the second of equations (2.82):

$$f * u(x) = (f(y), u(x-y))_y = \int_{R^n} f(y)u(x-y)d^n y. \tag{2.85}$$

The first of the equalities (2.85) is the definition of the convolution as the action of the functional $f(y)$ on the test function $u(x-y)$ regarded as a function of y for fixed x. It is not difficult to see that the convolution (2.85) is an infinitely differentiable function of x of polynomial growth (as are all its derivatives). In other words, if $f \in \mathcal{S}'$ and $u \in \mathcal{S}$, then $f * u(p) \in \Theta_M$, that is, the convolution (2.85) is a multiplicator. In general, however, the function $f * u$ does not belong to the space of test functions $\mathcal{S}$. For example, if $f(y)$ is a polynomial, then the convolution (2.85) is also a polynomial and consequently does not, in general, decrease as $x \to \infty$.

We say that the generalized function $g(x)$ is a *convolute* (or *convolution functional*) in $\mathcal{S}$ if for any $u(x) \in \mathcal{S}$, $f * u$ also belongs to $\mathcal{S}$. We denote the space of convolutes by $\Theta_c(\mathbf{R}^n)$ (or simply Θ_c).

If g is a convolute, then the convolution of g with an arbitrary generalized function $f \in \mathcal{S}'$ can be defined by the formula

$$(f * g, u) = (f(y), (g(x), u(x+y))_x)_y = (f(x), g(-x) * u(x)). \tag{2.86}$$

Here is an example of a convolute. The generalized function $D^\alpha \delta(x)$ is a convolute for any $\alpha = (\alpha_1, \ldots, \alpha_n)$. (More generally, if a generalized function f has bounded support, then it is a convolute.)

Test functions are convolutes and it is not difficult to see that the two definitions of the convolution $f * u$ of a generalized function $f(x)$ with a test function $u(x)$ given by formulae (2.85) and (2.86) are equivalent.

The notion of a convolute is related to that of a multiplicator.

Exercise 2.20. Prove that (2.84) still holds if f is an arbitrary generalized function and u is a test function. Verify by means of this formula that a generalized function f is a convolute if and only if its Fourier transform $\tilde{f}$ is a multiplicator.

Exercise 2.21. Prove that the Fourier transform of the product of a generalized function $h(p) \in \mathcal{S}'(\mathbf{R}_n)$ with a test function $\phi(p) \in \mathcal{S}(\mathbf{R}_n)$ is the following function:

$$\mathcal{F}^{-1}_{p \to x}[h(p)\phi(p)] = \int h(p) e^{-ipx} \phi(p) d_n p; \tag{2.87}$$

here, the right hand side is to be taken in the sense of the value of the generalized function $h(p)$ at the test function $e^{-ipx}\phi(p)$. [Hint: Use the formula

$$\mathcal{F}^{-1}(h(p)\phi(p)) = (\mathcal{F}^{-1}h) * (\mathcal{F}^{-1}\phi). \tag{2.88}$$

obtained from (2.85) by substituting $h = \mathcal{F}f, \phi = \mathcal{F}u$.)

The space of convolutes Θ_c consists of the rapidly decreasing generalized functions. In other words, the following assertion holds:

Proposition 2.8. *A necessary and sufficient condition for a generalized function f to be a convolute is that for each natural number N, f can be represented as a finite sum of derivatives of continuous functions $F_{kN}(p)$, each of which satisfies the inequality*

$$|F_{kN}(p)| \leq C_{kN}/[1 + |p|^2]^N.$$

We omit the proof of this proposition (see [S3],vol.II,Ch.7,§5,Theorem 9).

Exercise 2.22. Prove that for all $f \in \mathcal{S}'(\mathbf{R}^n)$ and $u \in \mathcal{S}(\mathbf{R}^n)$, $fu \in \Theta_c$. [Hint: Use (2.84).]

Exercise 2.23. Prove that every generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ with compact support is a convolute. [Hint: f can be expressed in the form $f = f \cdot \omega$ for a suitable function $\omega \in \mathcal{D}(\mathbf{R}^n)$.]

Exercise 2.24. Let $f \in \mathcal{S}'(\mathbf{R}^n)$ and g_1 and g_2 convolutes. Prove that

$$(f * g_1) * g_2 = f * (g_1 * g_2). \tag{2.89}$$

[Hint: Use Exercise 2.19.]

D. Generalized functions of integrable type

There is a class of generalized functions $f \in \mathcal{S}'(\mathbf{R}^n)$, called *generalized functions of integral type*, which is broader than $\Theta_c(\mathbf{R}^n)$; these are characterized by the condition:

for any function $u \in \mathcal{S}(\mathbf{R}^n)$, the convolution $f*u$ is an (absolutely) integrable function in $\mathbf{R}^n$ (with respect to Lebesgue measure in $\mathbf{R}^n$). We define the *integral* $\int f(x)d^nx$ of such generalized functions by the formula

$$\left(\int f(x)d^nx\right)\left(\int u(x)d^nx\right) = \int (f*u)(x)d^nx, \tag{2.90}$$

where u is an arbitrary function in $\mathcal{S}(\mathbf{R}^n)$.

This definition is natural since (2.90) holds identically for arbitrary absolutely integrable functions f and u in $\mathbf{R}^n$. The integral is well defined in the sense that there exists a number $I(f)$ (depending only on the generalized function of integrable type f) such that

$$\int (f*u)dx = I(f)\int u(x)dx \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n).$$

Exercise 2.25. Prove that such a number $I(f)$ exists. [Hint: Integrate the identity $(f*u)*v = (f*v)*u$, which holds for all $u, v \in \mathcal{S}(\mathbf{R}^n)$; see (2.89).]

Exercises 2.28, 2.29 also illustrate the naturalness of the idea of the integral of a generalized function.

Exercise 2.26. Prove that $f \in \mathcal{S}'(\mathbf{R}^n)$ is a generalized function of integrable type if and only if

$$|(f, u*v)| \le c(u)\sup_x |v(x)| \quad \text{for all } u, v \in \mathcal{S}(\mathbf{R}^n), \tag{2.91}$$

where $c(u)$ is some functional of u. [Hint: Use (2.86) and the fact that any continuous function $h(x)$ of polynomial growth in $\mathbf{R}^n$ is absolutely integrable if and only if an estimate of the following type holds:

$$|\int h(x)v(x)dx| \le c\sup_x |v(x)| \quad \text{for all } v \in \mathcal{S}(\mathbf{R}^n).]$$

Exercise 2.27. Prove that the property of being a generalized function of integrable type does not depend on the behaviour of the generalized function in any fixed finite region. [Hint: Use Exercise 2.23.]

Exercise 2.28. (a) If f is a generalized function of integrable type in $\mathbf{R}^n$, then its Fourier transform $\tilde{f}(p) = \int f(x)e^{ipx}dx$ is a continuous function of p that converges to 0 as $|p| \to \infty$ and

$$\int f(x)dx = \tilde{f}(0). \tag{2.92}$$

[Hint: Use Exercise 2.20.]

(b) If f is a generalized function of integrable type in $\mathbf{R}^n$, then so is the generalized function $D^\alpha f$, where $\int D^\alpha f d^nx = 0$ for $|\alpha| \ne 0$. [Hint: Use the formula

$$D^\alpha(f*u) = (D^\alpha f)*u = f*D^\alpha u \tag{2.93}$$

where $f \in \mathcal{S}'(\mathbf{R}^n)$, $u \in \mathcal{S}(\mathbf{R}^n)$.]

Exercise 2.29. Prove that for any $f \in \mathcal{S}'(\mathbf{R}^n)$, $u \in \mathcal{S}(\mathbf{R}^n)$, the product fu is of integrable type, where

$$\int (fu)dx = (f, u). \tag{2.94}$$

[Hint: Use Exercises 2.22, 2.28 and 2.21.]

Closely linked to the notion of integral is that of partial integral. We suppose that the independent variables are divided into two groups, $x \in \mathbf{R}^n$ and $y \in \mathbf{R}^m$. Then we can associate with each generalized function $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ and any test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$, the generalized function $\int f(x,y)u(x)d^n x \in \mathcal{S}'(\mathbf{R}^m)$ by means of the rule

$$\int \left(\int f(x,y)u(x)d^n x \right) v(y)d^m y = (f, u \otimes v) \quad \text{for all } v \in \mathcal{S}(\mathbf{R}^m). \tag{2.95}$$

We call $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ a *generalized function of integrable type with respect to* y (or a convolute with respect to y) if the generalized function $\int f(x,y)u(x)dx$ in $\mathcal{S}'(\mathbf{R}^m)$ is of integrable type in $\mathbf{R}^m$ (or a convolute in $\Theta_c(\mathbf{R}^m)$) for any $u \in \mathcal{S}(\mathbf{R}^n)$. In this case, the *partial integral* of the generalized function $f(x,y)$ with respect to y is defined: this is the generalized function $\int f(x,y)dy$, defined by the formula

$$\left(\int f(x,y)dy, u(x) \right)_x = \int \left(\int f(x,y)u(x)dx \right) dy \tag{2.96}$$

for all $u \in \mathcal{S}(\mathbf{R}^n)$.

It is clear that the partial integral is a linear functional on $\mathcal{S}(\mathbf{R}^n)$; its continuity is established in the following exercise.

Exercise 2.30. Prove that for a generalized function $f(x,y)$ of integrable type with respect to y, the right hand side of (2.96) is a continuous functional of $u \in \mathcal{S}(\mathbf{R}^n)$. [Hint: Take a fixed function $v(y) \in \mathcal{S}(\mathbf{R}^m)$ such that $\int v(y)dy = 1$. Then by definition, the right hand side of (2.96) is $\int h(y)dy$, where $h(y) = (f(x,y-z),\ u(x)v(z))_{x,z}$ is a function in the space $\mathcal{L}^1(\mathbf{R}^m)$.* Verify that the map $u \to h$ has a closed graph and is therefore continuous.]

We now give tests for a generalized function to be a convolute with respect to y.

Exercise 2.31. Suppose that for the generalized function $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ there exists a function $\chi(x,y) \in \Theta_M(\mathbf{R}^{n+m})$ with the following properties: firstly, $\chi(x,y)f(x,y) = f(x,y)$: secondly, $\chi(x,y)u(x) \in \mathcal{S}(\mathbf{R}^{n+m})$ for any function $u(x) \in \mathcal{S}(\mathbf{R}^n)$. Prove that $f(x,y)$ is a convolute with respect to y. [Hint: It suffices to verify that for any $u \in \mathcal{S}(\mathbf{R}^n)$, $v \in \mathcal{S}(\mathbf{R}^m)$ the function $h(z) = (f(x,y), u(x)v(y-z)) \equiv (f(x,y), \chi(x,y)u(x)v(y-z))$ with respect to z belongs to $\mathcal{S}(\mathbf{R}^m)$, and hence that $\chi(x,y)u(x)v(y-z) \in \mathcal{S}(\mathbf{R}^{n+2m})$.]

Exercise 2.32. Prove that every generalized function $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ whose support lies in the set

$$\{(x,y) \in \mathbf{R}^{n+m} : |y| \leq A(1+|x|)^\delta\}$$

(where A, δ are positive numbers) is a convolute with respect to y. [Hint: Let $\chi(x,y) = \omega(|y|^2 A^{-2} \times \times (2(1+|x|^2))^{-\delta})$, where $\omega(t)$ is a function in $\mathcal{D}(\mathbf{R})$ that is equal to 1 when $|t| \leq 2$. Verify that $\chi(x,y)$ satisfies the conditions of the preceding exercise.]

Exercise 2.33. Prove that a generalized function $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$, satisfying the equation

$$(|x|^2 - |y|^2 + \lambda)f(x,y) = 0 \tag{2.98}$$

(where λ is real) is a convolute with respect to y. [Hint: Apply Exercise 2.32.]

Certain properties of generalized functions of integrable type with respect to y are considered in Exercises 2.34 and 2.35.

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* $\mathcal{L}^1(\mathbf{R}^n)$ is the Banach space of (equivalence classes of) functions in $\mathbf{R}^n$ that are absolutely integrable with respect to Lebesgue measure; the norm of such a function $f(x)$ is given by

$$\|f\|_1 = \int |f(x)|dx. \tag{2.97}$$

Exercise 2.34. Let $f(x,y) \in S'(\mathbf{R}^{n+m})$ be a generalized function of integrable type with respect to y (or a convolute with respect to y). Prove that for any $w(x,y) \in S(\mathbf{R}^{n+m})$, the function

$$(f(x,y),\ w(x,y-z))_{x,y} \tag{2.99}$$

of $z \in \mathbf{R}^m$ is absolutely integrable (or a test function in $S(\mathbf{R}^m)$) that is continuously dependent on w (in the respective topology of the spaces $\mathcal{L}^1(\mathbf{R}^m)$ and $S(\mathbf{R}^m)$), where

$$\int (f(x,y),\ w(x,y-z))_{x,y} dz = \left(\int f(x,y)dy,\ \int w(x,z)dz\right)_x . \tag{2.100}$$

[Hint: For functions of the form $w(x,y) = u(x)v(y)$ with arbitrary $u \in S(\mathbf{R}^n)$, $v \in S(\mathbf{R}^m)$, the statement is obvious. For such $u(x,y)$, the expression (2.98) defines a bilinear map from $S(\mathbf{R}^n) \times$ $\times\ S(\mathbf{R}^m)$ to $\mathcal{L}^1(\mathbf{R}^n)$ (or to $S(\mathbf{R}^m)$). Using the closed graph theorem, prove that it is separately continuous in u and v; then use the remark following Theorem 2.5.]

Exercise 2.35. Prove that the generalized function $f(x,y) \in S'(\mathbf{R}^{n+m})$ is of integrable type with respect to y if and only if for each test function $u(x) \in S(\mathbf{R}^n)$ the generalized function $u(x)f(x,y)$ is of integrable type with respect to x, y, where

$$\left(\int f(x,y)dy, u(x)\right)_x = \int u(x)f(x,y)dxdy. \tag{2.101}$$

[Hint: If $f(x,y)$ is of integrable type with respect to y, then it is required to prove that for any $u(x) \in S(\mathbf{R}^n)$ and $w(x,y) \in S(\mathbf{R}^{n+m})$, the function

$$(f(x,y), u(x), w(x-\xi,\ y-z)) \tag{2.102}$$

is a member of $\mathcal{L}^1(\mathbf{R}^{n+m})$ with respect to $\xi \in \mathbf{R}^n$, $z \in \mathbf{R}^m$. For this purpose, define $w_\xi(x,y) =$ $= u(x)w(x-\xi,y)$ and verify that it is a test function with respect to x,y, that is continuously dependent on the parameter ξ and that $w_\xi \to 0$ faster than any negative power of $|\xi|$ as $|\xi| \to \infty$. Then use Exercise 2.34 where w_ξ takes on the role of w. Conversely, if the function (2.102) is a member of $\mathcal{L}^1(\mathbf{R}^{n+m})$ for any $u \in S(\mathbf{R}^n)$ and $w(x,y) \in S(\mathbf{R}^{n+m})$, then it suffices to set $w(x,y) = \alpha(x)v(y)$, where $\alpha(x)$ is a fixed function of $S(\mathbf{R}^n)$ with integral equal to one and $v(y)$ is an arbitrary function of $S(\mathbf{R}^m)$. Then by Fubini's theorem applied to the function (2.102), the expression

$$\int (f(x,y), u(x)\alpha(x-\xi)v(y-z))_{x,y} d\xi \equiv (f(x,y)\cdot 1(\xi), u(x)\alpha(x-\xi)v(y-z))_{x,y,\xi}$$

is an integrable function of z; the right hand side of this identity can be converted to the form $(f(x,y), u(x)v(y-z))_{x,y}$, whence the result follows.]

Exercise 2.36. Let $f(x,y) \in S'(\mathbf{R}^{n+m})$ be a generalized function of integrable type with respect to y (or a convolute with respect to y). Prove that the generalized function $f(x, y+Ax)$ is of the same type, where $A : \mathbf{R}^n \to \mathbf{R}^m$ is an arbitrary linear transformation and

$$\int f(x,y+Ax)dy = \int f(x,y)dy. \tag{2.103}$$

[Hint: Apply Exercise 2.34 to the case $w(x,y) = u(x)v(y-Ax)$, where $u \in S(\mathbf{R}^n)$, $v \in S(\mathbf{R}^m)$.]

E. Convolution of Generalized Functions

We now give a definition of convolution of generalized functions which applies in a considerably more general situation than the previous definition (2.86), where $g(x)$ was assumed to be a convolute in $S(\mathbf{R}^n)$.

We say that the convolution of the generalized functions $f(x)$ and $g(x)$ in $S'(\mathbf{R}^n)$ is "canonically" defined if the generalized function $f(x-y)g(y)$ in $S'(\mathbf{R}^{2n})$ is of integrable type with respect to y; in this case, by the *convolution* of the generalized functions f and g, we mean the generalized function

$$(f * g)(x) = \int f(x-y)g(y)dy \tag{2.104}$$

in $\mathcal{S}'(\mathbf{R}^n)$.

The next two exercises demonstrate a number of natural properties of the convolution of generalized functions (in particular, *commutativity of convolution*).

Exercise 2.37. Let f and g be generalized functions in $\mathcal{S}'(\mathbf{R}^n)$. Prove that $f(x-y)g(y)$ is a generalized function of integrable type with respect to y if and only if $g(x-y)f(y)$ is; under these conditions, we have

$$\int f(x-y)g(y)dy = \int g(x-y)f(y)dy,$$

that is

$$f * g = g * f. \tag{2.105}$$

[Hint: Use the fact that $f(x+y)g(-y)$ is obtained from $g(x-y)f(y)$ by the change of variables $(x, y) \to (x, x+y)$ and apply Exercise 2.36.]

Exercise 2.38. Prove that the convolution of the generalized functions $f, g \in \mathcal{S}'(\mathbf{R}^n)$ is "canonically" defined if and only if for any test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$, the generalized function $f(x)g(y)u(x+y)$ is of integrable type with respect to x, y, and that the following equality then holds:

$$(f * g, u) = \int f(x)g(y)u(x+y)dxdy. \tag{2.106}$$

[Hint: Use Exercise 2.35.]

Exercise 2.39. Suppose that the convolution of f and $g \in \mathcal{S}'(\mathbf{R}^n)$ is "canonically" defined.

(a) Prove that the convolution of $D^\alpha f$ and g (and of f and $D^\alpha g$) exists for any $\alpha \in \overline{\mathbf{Z}}^n_+$ and that the following rule holds for differentiation of a convolution:

$$D^\alpha(f * g) = (D^\alpha f) * g = f * (D^\alpha g). \tag{2.107}$$

[Hint: Use (2.93).]

(b) Prove the following property for the support of a convolution:

$$\operatorname{supp} f * g \text{ is contained in the closure of } \operatorname{supp} f + \operatorname{supp} g. \tag{2.108}$$

(In the above formula, by the algebraic sum of two sets A and B in $\mathbf{R}^n$ we mean the set of points of the form $a + b$ with $a \in A$, $b \in B$.)

It is possible to deduce the existence of the convolution of generalized functions from specified properties of their supports.

Exercise 2.40. Let Q_1 and Q_2 be two closed subsets of $\mathbf{R}^n$ such that

$$|x| + |y| \le A(1 + |x+y|)^\delta \quad \text{for all } x \in Q_1,\ y \in Q_2, \tag{2.109}$$

where A, δ are positive numbers. Prove that the convolution $f_1 * f_2$ is "canonically" defined if for $j = 1, 2$, f_j is a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with support in Q_j. [Hint: Use Exercise 2.32 and the trivial inequality $|y| \le |x| + |x-y|$ to prove that the generalized function $f_1(x-y)f_2(y)$ is a convolute with respect to y.]

We now consider the case of generalized functions with supports in cones. Here, by a *cone* in $\mathbf{R}^n$ with vertex at the origin, we mean a set $K \subset \mathbf{R}^n$ such that $\lambda x \in K$ for all $\lambda > 0$ and $x \in K$; similarly, the set $a + K$ (where a is a fixed vector in $\mathbf{R}^n$) is called a cone with vertex at the point a. Usually by a cone (without mentioning the vertex), we mean a cone with vertex at the origin. In particular, a set of the form

$$\{x \in \mathbf{R}^n : b^{(\nu)}x \geq 0 \quad \text{for } \nu = 1, \ldots, N, \tag{2.110}$$

where $b^{(1)}, \ldots, b^{(N)}$ is a finite set of vectors in $\mathbf{R}^n$, is called a (closed convex) *polyhedral cone* in $\mathbf{R}^n$; if $N = n$ and $b^{(1)}, \ldots, b^{(n)}$ is a linear basis in $\mathbf{R}^n$, then the cone (2.110) is called a (closed) *simplicial* cone. A cone $K \subset \mathbf{R}^n$ is called *pointed* if it is contained in some simplicial cone.

Exercise 2.41. Let K_1 and K_2 be two closed cones in $\mathbf{R}^n$ (with vertex at the origin) such that $K_1 \cap (-K_2) = \{0\}$, and A_1, A_2 two compact subsets of $\mathbf{R}^n$. Prove that the convolution $f_1 * f_2$ is "canonically" defined if f_j is a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with support in $A_j + K_j$ ($j = 1, 2$). [Hint: Use the preceding exercise.]

Exercise 2.42. * Let K be a closed convex pointed cone in $\mathbf{R}^n$ with non-empty interior. Prove that the generalized functions $f \in \mathcal{S}'(\mathbf{R}^n)$ for which $\operatorname{supp} f \subset a + K$ (where the point a is allowed to depend on f) form a (commutative and associative) convolution algebra (and hence, the convolution of any such generalized functions f_1 and f_2 is defined and enjoys the above support property). [Hint: Apply the preceding exercise.]

2.6. Generalized Functions Dependent on a Parameter

A. General information

The questions of convergence set forth in §§1.2 and 1.3 are mainly of interest to us in the context of test and generalized functions depending on a parameter (which runs through some locally compact subset Λ of Euclidean space $\mathbf{R}^m$). The notion of continuity of such a family (in terms of sequences or in the "$\epsilon - \delta$" language) is reasonably clear.** It should be noted, however, that in this connection we use the usual topology of $\mathcal{S}(\mathbf{R}^n)$ for the test functions but the weak topology for the generalized functions. Similar remarks apply (in the case when Λ is an open subset of $\mathbf{R}^m$) to the notions of differentiability or infinite differentiability (that is, class C^∞ dependence) of test functions and generalized functions with respect to the parameter, also (when Λ is an open subset of $\mathbf{C}^k$) holomorphy (or, what is the same, analyticity) of the dependence on the parameter.

Exercise 2.43. Let $\{f_\lambda\}$ be a family of generalized functions in $\mathbf{R}^n$ that depends continuously on the parameter λ which runs through the locally compact set Λ. Prove that for any compactum $K \subset \Lambda$, there exist numbers c, l, m (depending on K) such that

$$|(f_\lambda, u)| \leq c\|u\|_{l,m} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n),\ \lambda \in K. \tag{2.111}$$

[Hint: By regarding the map $u \to (f_\lambda, u)$ as a linear operator from $\mathcal{S}(\mathbf{R}^n)$ to the space $\mathcal{C}(\Gamma)$ of §1.2.C, verify that this operator has a closed graph and hence is continuous by Theorem 1.11.]

It is not obvious (but true) that the function $\lambda \to (f_\lambda, u_\lambda)$ is continuous if the families $\{f_\lambda\}$ in $\mathcal{S}'$ and $\{u_\lambda\}$ in $\mathcal{S}$ depend continuously on the parameter λ. (The case when the families $\{f_\lambda\}$ and $\{u_{\lambda'}\}$ depend on different parameters $\lambda \in \Lambda$ and $\lambda' \in \Lambda'$ would appear to be more general; however, it reduces to the previous case if we regard (λ, λ') as a single parameter running through $\Lambda \times \Lambda'$.)

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* See [V5],§5.6.

** At least in the present context of a locally compact set Λ. (In Appendix B, we shall encounter examples going beyond this framework.)

Proposition 2.9. *Let $\{u_\lambda\}$ and $\{f_\lambda\}$ be families of test functions in $\mathcal{S}$ and generalized functions in $\mathcal{S}'$ respectively, which depend continuously (or continuously differentiably, or C^∞-, or holomorphically)on the parameter $\lambda \in \Lambda$. Then the function $\lambda \to (f_\lambda, u_\lambda)$ is continuous (resp. continuously differentiable, or C^∞-, or holomorphic)on Λ. The differentiation-type formulae*

$$\frac{\partial}{\partial \lambda}(f_\lambda, u_\lambda) = (\frac{\partial}{\partial \lambda} f_\lambda, u_\lambda) + (f_\lambda, \frac{\partial}{\partial \lambda} u_\lambda) \tag{2.112}$$

are valid (provided that they are meaningful).

Exercise 2.44. Prove Proposition 2.9. [Hint: Use Corollary 1.10.]

We give a further result relating to families of generalized functions that depend holomorphically on a parameter λ. As is well known, complex holomorphic functions defined in a complex domain $\Lambda \subset \mathbf{C}^k$ can have a holomorphic continuation to a larger domain $\Lambda' \subset \mathbf{C}^k$, where for a fixed domain Λ' this continuation is unique. (For further details see §5.1.C.) It can therefore happen that a given family $\{f_\lambda\}$ of generalized functions that is holomorphically dependent on a parameter λ in a domain $\Lambda \subset \mathbf{C}^k$ is such that for all $u \in \mathcal{S}(\mathbf{R}^n)$, the functions $\lambda \to (f_\lambda, u)$ can be holomorphically continued to the domain Λ' ($\Lambda \subset \Lambda' \subset \mathbf{C}^k$). However, it is not clear in advance that the linear functional f_λ on $\mathcal{S}(\mathbf{R}^n)$ is continuous for all $\lambda \in \Lambda'$ (and therefore belongs to $\mathcal{S}'(\mathbf{R}^n)$). It is not difficult to see that the space $\mathcal{H}(\Lambda')$ of all holomorphic functions in Λ' is a Fréchet space (see Proposition 5.3 concerning this). It is also easily verified that the map that associates with each $u \in \mathcal{S}(\mathbf{R}^n)$ the function (f_λ, u) in $\mathcal{H}(\Lambda')$ (which is a function of the variable λ), is a linear operator with a closed graph and is therefore continuous. Thus we have the following result.

Proposition 2.10. *Let $\{f_\lambda\}$ be a family of generalized functions analytically dependent on the parameter λ in the domain $\Lambda \subset \mathbf{C}^k$, and suppose that for each test function $u \in \mathcal{S}(\mathbf{R}^n)$ the function $\lambda \to (f_\lambda, u)$ has an analytic continuation to a domain $\Lambda' \subset \mathbf{C}^k$ containing Λ. Then there exists a unique family $\{f_\lambda\}_{\lambda \in \Lambda'}$ in $\mathcal{S}'(\mathbf{R}^n)$ that continues the original family analytically in λ.*

An example of a family of generalized functions analytically dependent on a parameter is provided by the homogeneous generalized functions of one variable (Appendix C.2).

In the following exercises we give a number of simple criteria for the continuity or infinite differentiability or holomorphy of a given family $\{u_y\}$ of test functions depending on a parameter, in terms of the function $u(x, y)$ of the two variables x and y:

$$u(x, y) \equiv u_y(x). \tag{2.113}$$

For greater generality, we shall state the results for the spaces of test functions $\mathcal{S}(\Omega)$ defined in Appendix A.2 (here Ω is a canonically closed regular subset of $\mathbf{R}^n$; the reader can take Ω to be $\mathbf{R}^n$).

Exercise 2.45. Let $u(x, y)$ be a complex function on $\Omega \times S$, where S is a locally compact subset of $\mathbf{R}^m$. Suppose that all the partial derivatives (of any order) in x $D_x^\alpha f(x, y)$ are defined in $\operatorname{int}\Omega \times S$ and can be continuously extended onto $\Omega \times S$, where for each $l \in \mathbf{Z}_+$, $\alpha \in \mathbf{Z}_+^n$, the function $|x|^l D_x^\alpha f(x, y)$ is bounded on each subset of the form $\Omega \times K$, where K is a compactum in S. Prove that (2.113) defines a family $\{u_y\}$ of test functions in $\mathcal{S}(\Omega)$ that is continuously dependent on the parameter $y \in S$.

Exercise 2.46. Let $u(x, y)$ be a complex function on $\Omega \times S$, where S is a locally compact subset of $\mathbf{R}^m$ whose interior is dense in S. Suppose that all the partial derivatives $D_x^\alpha D_y^\beta u(x, y)$ are defined in

$\operatorname{int}\Omega \times \operatorname{int} S$ and can be continuously extended to $\Omega \times S$, where for each $l \in \mathbf{Z}_+$, $\alpha \in \mathbf{Z}_+^n$, $\beta \in \mathbf{Z}_+^m$, the function $|x|^l D_x^\alpha D_y^\beta u(x,y)$ is bounded on any subset of the form $\Omega \times K$, K being an arbitrary compactum in S. Prove that (2.113) defines a family $\{u_y\}$ in $\mathcal{S}(\Omega)$ that is C^∞-dependent on the parameter $y \in S$ and that $(D_y^\alpha u_y)(x) = D_y^\alpha u(x,y)$.

Exercise 2.47. Let S be an open subset of $\mathbf{C}^k$ and $u(x,\zeta)$ a complex C^∞-function on $\operatorname{int}\Omega \times S$ that is holomorphic with respect to ζ in S for each $x \in \operatorname{int}\Omega$ and is such that for each $l \in \mathbf{Z}_+$, $\alpha \in \mathbf{Z}_+^m$ and any compactum $K \subset S$, the function $|x|^l D_x^\alpha u(x,\zeta)$ can be continuously extended onto $\Omega \times S$ and is bounded on $\Omega \times K$. Then the formula $u_\zeta(x) = u(x,\zeta)$ defines a family $\{u_\zeta\}$ in $\mathcal{S}(\Omega)$ that is holomorphically dependent on the parameter $\zeta \in S$. [Hint: It suffices to verify that $u(x,\zeta)$ satisfies the conditions of the preceding exercise, where the role of y is now taken on by $(\xi,\eta) \equiv (\operatorname{Re}\zeta, \operatorname{Im}\zeta)$. Use the Cauchy integral representation:

$$u(x,\zeta) = \int u(x,\zeta') \prod_{j=1}^{k} \frac{d\zeta_j'}{2\pi i(\zeta_j' - \zeta_j)}; \tag{2.114}$$

here $x \in \operatorname{int}\Omega$ and ζ runs through an arbitrary fixed polydisc K with closure in S of the form $K = \{\zeta \in S : |\zeta_j - b_j| < \epsilon,\ j = 1,\dots,k\}$; the integration is carried out over the set defined by the equations $|\zeta_j' - b_j| = \epsilon$, $j = 1,\dots,k$.]

B. Restriction of Generalized Functions

We were interested above in the problem of interpreting a function of two variables as a family of test functions of one variable depending on the other variable as a parameter. A similar point of view is also of interest with regard to generalized functions.

Let $f \equiv f(x,y)$ be a generalized function in $\mathcal{S}'(\mathbf{R}^{n+m})$ (the argument of which is represented as a pair of points $x \in \mathbf{R}^n$, $y \in \mathbf{R}^m$), and S an open subset of $\mathbf{R}^n$. We say that $f(x,y)$ (where $(x,y) \in \mathbf{R}^n \times S$) is a *generalized function* in x that is continuously (or C^∞-, or (in the case when $m = 2k$ and $\mathbf{R}^m$ is identified with $\mathbf{C}^k$) holomorphically) *dependent on $y \in S$ as a parameter*, if there exists a family $\{f_y\}_{y \in S}$ of generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ that is continuously (or $\mathbf{C}^\infty$-, or holomorphically) dependent on the parameter y and is such that

$$(f, u \otimes v) = \int_S (f_y, u) v(y)dy \tag{2.115}$$

for all $u \in \mathcal{S}(\mathbf{R}^n)$, $v \in D(\mathbf{R}^m)$ with $\operatorname{supp} v \subset S$.

In connection with this definition, it is natural to associate with an arbitrary $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$, $u(x) \in \mathcal{S}(\mathbf{R}^n)$ the generalized function $\int f(x,y)u(x)dx$ in $\mathcal{S}'(\mathbf{R}^m)$ according to the rule

$$\int \Big(\int f(x,y)u(x)dx\Big) v(y)d^m y = (f, u \otimes v),\ v \in \mathcal{S}(\mathbf{R}^m). \tag{2.116}$$

We can now give another statement of the above concept.

Proposition 2.11. *Let $f(x,y)$ be a generalized function in $\mathcal{S}'(\mathbf{R}^{n+m})$ such that for all $u \in \mathcal{S}(\mathbf{R}^n)$, the generalized function $\int f(x,y)u(x)d^n x$ coincides with the continuous (or C^∞-, or holomorphic) function of y on an open subset $S \subset \mathbf{R}^m$ (or $\mathbf{C}^k$ where $m = 2k$). Then over $\mathbf{R}^n \times S$, $f(x,y)$ is a generalized function in x which is continuously (or C^∞-, or holomorphically) dependent on y as a parameter.*

■ By hypothesis, the expression

$$F(u;y) = \int f(x,y)u(x)dx\big|_S$$

defines a continuous function of $y \in S$ for each $u \in \mathcal{S}(\mathbf{R}^n)$. It suffices to show that for any fixed $y \in S$, this expression is a continuous functional of $u \in \mathcal{S}(\mathbf{R}^n)$; it will then define the required family of generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ which is continuously (or $\mathbf{C}^\infty$-, or holomorphically) dependent on the parameter $y \in S$. To this end we note that the linear map $u \to F(u;y)$ from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{D}'(S)$ is (sequentially) continuous. On the other hand, by hypothesis, the image of $\mathcal{S}(\mathbf{R}^n)$ under this map lies in the F-space $\mathcal{C}(S)$ (see §1.2.C), the topology of which is stronger than the weak topology of $\mathcal{D}'(S)$. The continuity of the map $u \to F(u;y)$ from $\mathcal{S}(\mathbf{R}^n)$ to $\mathcal{C}(S)$ now follows from Corollary 1.12. ■

An interesting illustration of the above definition is provided by the partially holomorphic generalized functions (we return to this in §5.1.E).

Exercise 2.48. (a) Let $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ be a generalized function in x that is continuously (or $\mathcal{C}^\infty$-) dependent on y as a parameter. Prove that the product $f(x,y)\phi(x,y)$ has the same property, where $\phi(x,y) \in \mathcal{O}_M(\mathbf{R}^{n+m})$. [Hint: Let $u(x) \in \mathcal{S}(\mathbf{R}^n)$, $v(y) \in \mathcal{D}(\mathbf{R}^m)$; then by formula (2.115), $f \cdot \phi, u \otimes v) \equiv (f, \phi \cdot (u \otimes v)) = \int (f_y(x), \phi(x,y)u(x))_x v(y) d^m y$; then use the fact that according to Proposition 2.9, $(f_y(x), \phi(x,y)u(x))_x$ depends continuously (or $\mathcal{C}^\infty$) on y.]

(b) Suppose that the generalized function $h(p,q) \in \mathcal{S}'(\mathbf{R}_{n+m})$ is of integrable type with respect to q (or a convolute with respect to q). Prove that the Fourier transform

$$f(x,y) = \int h(p,q)e^{i(px+qy)}d_n p\, d_m q$$

is a generalized function in x which is continuously (or $\mathcal{C}^\infty$-) dependent on y as a parameter. State the analogous assertion for the partial Fourier transform

$$F(p,y) = \int h(p,q)e^{iqy}d_m q. \tag{2.117}$$

[Hint: The Fourier transform of a function in $\mathcal{L}^1(\mathbf{R}_m)$ (or in $\mathcal{S}(\mathbf{R}_m)$) is a continuous (or a test) function.]

Exercise 2.49. Prove that every solution $f(x,y)$ of the differential equation (where λ is a real parameter)

$$\left(\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} - \sum_{j=1}^{m}\frac{\partial^2}{\partial y_j^2} - \lambda\right) f(x,y) = 0 \tag{2.118}$$

in the space of generalized functions $\mathcal{S}'(\mathbf{R}^{n+m})$ is a generalized function in x that is $\mathcal{C}^\infty$-dependent on y as a parameter. [Hint: Combine the results of Exercises 2.33 and 2.48(b).]

We now consider an important example. Let us agree to denote by $1(y)$ the generalized function in $\mathcal{S}'(\mathbf{R}^m)$ (or the distribution in $\mathcal{D}'(\mathcal{O})$, $\mathcal{O} \subset \mathbf{R}^m$) the respective functions identically equal to one. Then each generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ can be regarded as a generalized function of a larger number (say, $n+m$) variables by letting it correspond to the generalized function $f(x) \cdot 1(y) \in \mathcal{S}'(\mathbf{R}^{n+m})$. It is clear that $f(x) \cdot 1(y)$ is a generalized function in x that is $\mathcal{C}^\infty$-dependent on $y \in \mathbf{R}^m$. (Generalized functions of the form $f(x) \cdot 1(y)$ are called generalized functions that are independent of y.) In Exercise 2.50 we give a natural property of such generalized functions; Exercise 2.51 gives another characterization of them.

Exercise 2.50. Let $f(x) \in \mathcal{S}'(\mathbf{R}^n)$, $A : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Prove that the distribution $f(x) \cdot 1(y + Ax)$ (obtained from $f(x) \cdot 1(y)$ by a linear change of variables) coincides with $f(x) \cdot 1(y)$. [Hint: Apply (2.61) to the expression $(f(x) \cdot 1(y), w(x, y - Ax))$, $w \in \mathcal{S}(\mathbf{R}^{n+m})$.]

Exercise 2.51. (a) Prove that every generalized function $f(x) \in \mathcal{S}'(\mathbf{R})$, satisfying the equation

$$\frac{d}{dx}f(x) = 0, \tag{2.119}$$

has the form $f(x) = c \cdot 1(x)$, where c is a complex number. [Hint: The above equation implies that $(f, u) = 0$ for all u in the subspace $\frac{d}{dx}\mathcal{S}(\mathbf{R})$ in $\mathcal{S}(\mathbf{R})$; this subspace in turn is defined by the equation $\int f(x)dx \equiv (1(x), u(x)) = 0$.]

(b) Prove that the general solution of the system of equations

$$\frac{\partial}{\partial y_j} F(x,y) = 0, \; j = 1, \ldots, m,$$

in the class of generalized functions $F(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ has the form

$$F(x,y) = f(x) \cdot 1(y),$$

where $f(x) \in \mathcal{S}'(\mathbf{R}^n)$. [Hint: For the case $m = 1$, use the first part of the exercise and show that $F(x,y)$ is a generalized function in x that is C^∞-dependent on the parameter y; then prove the general case $m \geq 1$ by induction.]

(c) Let $\mathcal{O}$ be an open subset of $\mathbf{R}^n$. Prove that the system of equations

$$\frac{\partial}{\partial x_j} f(x) = 0, \; j = 1, \ldots, n. \tag{2.120}$$

describes complex functions in $\mathcal{O}$ that are constant on each component* of $\mathcal{O}$.

Exercise 2.52. Let $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ (defined on the open set $\mathbf{R}^n \times S$) be a generalized function in x that is continuously (or C^∞-) dependent on y as a parameter.

(a) Prove that the map associating the function $u \in \mathcal{S}(\mathbf{R}^n)$ with the continuous function $h(y) = (f(x,y), u(x))_x$ on S is a continuous linear operator from $\mathcal{S}(\mathbf{R}^n)$ into $C(S)$. [Hint: See Exercise 2.43.]

(b) Prove that the following generalization of (2.115):

$$(f(x,y)h(y), w(x,y)) = \int (f_y(x), w(x,y))_x h(y) dy \tag{2.121}$$

holds for all $w \in \mathcal{S}(\mathbf{R}^{n+m})$, $h \in \mathcal{D}(S)$. [Hint: By Proposition 2.9, the right hand side of (2.121) as a functional of w, is a distribution in $\mathcal{S}'(\mathbf{R}^{n+m})$ which is the same as $f(x,y)h(y)$ for linear combinations of functions of the form $u \otimes v$, where $u \in \mathcal{S}(\mathbf{R}^n)$, $v \in \mathcal{S}(\mathbf{R}^m)$.]

(c) Prove that for any linear transformation $A : \mathbf{R}^m \to \mathbf{R}^n$, $f(x+Ay, y)$ is a generalized function in x that is continuously (or C^∞-)dependent on y as a parameter. [Hint: Use (2.121).]

We say that the generalized function $f(x,y) \in \mathcal{S}'(\mathbf{R}^{n+m})$ has a restriction to the plane $y = b$ (where b is a fixed point of $\mathbf{R}^m$) if there exists a neighbourhood S of b in $\mathbf{R}^m$ such that over $\mathbf{R}^n \times S$, $f(x,y)$ is a generalized function of x which is continuously dependent on y as a parameter. If $\{f_y\}$ is the corresponding family in $\mathcal{S}'(\mathbf{R}^n)$ (see (2.115)), then the distribution $f_b \in \mathcal{S}'(\mathbf{R}^n)$ is called the *restriction of the generalized function* $f(x,y)$ to the plane $y = b$, denoted by $f(x,y)|_{y=b}$.

By a suitable change of variables, it is possible to define the notion of the restriction of a generalized function to a "sufficiently good" submanifold.

We note that the above definition of restriction is not the only possible one. For example, in Ch.13 we shall use an alternative definition (in terms of limits) of the restriction to the mass shell of a generalized function in Minkowski space.

C. MORE ON THE MULTIPLICATION OF GENERALIZED FUNCTIONS

We now apply the notion of restriction to the problem of multiplication of generalized functions. If $f(x)$ and $g(x)$ are two generalized functions in $\mathcal{S}'(\mathbf{R}^n)$, then we say that their product is "canonically" defined if the generalized function $f(x)g(x+y) \in$

* Every open subset $\mathcal{O}$ of $\mathbf{R}^n$ is the union of a family of disjoint regions (i.e., open connected sets), called the (*connected*) *components* of $\mathcal{O}$.

$\in \mathcal{S}'(\mathbf{R}^{2n})$ (obtained from $f(x)g(y)$ by a change of variables) has a restriction to the plane $y = 0$; in this case, the generalized function

$$f(x)g(x) \equiv f(x)g(x+y)|_{y=0} \tag{2.122}$$

in $\mathcal{S}'(\mathbf{R}^n)$ is called *the product of the generalized functions* $f(x)$ and $g(x)$.

In the case when one of the generalized functions f or g is a multiplicator, the new definition of product coincides with the old one (§2.3.A).

In the next exercise, it is asserted that the product of generalized functions is in fact symmetric in f and g and is therefore commutative (but non-associative, as Example (2.32) shows).

Exercise 2.53. Let $f(x)$ and $g(x)$ be two generalized functions in $\mathcal{S}'(\mathbf{R}^n)$. Prove that the following three conditions are equivalent:

1) the generalized function $f(x)g(x+y) \in \mathcal{S}'(\mathbf{R}^{2n})$ has a restriction to the plane $y = 0$;

2) the generalized function $f(x+y)g(x+z) \in \mathcal{S}'(\mathbf{R}^{3n})$ (obtained from $f(x)g(y)\cdot 1(z)$ by a linear change of variables) has a restriction to the plane $y = z = 0$;

3) the generalized function $f(x+y)g(x) \in \mathcal{S}'(\mathbf{R}^{2n})$ has a restriction to the plane $y = 0$.

Under these conditions, the following equation holds:

$$f(x)g(x+y)|_{y=0} = f(x+y)g(x+z)|_{y=z=0} = f(x+y)g(y)|_{y=0}. \tag{2.123}$$

[Hint: According to Exercise 2.52(c), condition 2) implies that $f(x)g(x-y+z)\cdot 1(z)$ has a restriction to the plane $y = z = 0$; next, this generalized function is converted to $f(x)g(x+y)\cdot 1(z)$ after the change of variables $(x,y,z) \to (x,z,y+z)$.]

Exercise 2.54. Let $f(x)$ and $g(x)$ be generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ such that $f(x)g(x+y)$ is a generalized function in x that is C^∞-dependent on y as a parameter. Prove that for any multi-indices $\beta, \gamma \in \overline{\mathbf{Z}}^n_+$ is the product of the generalized functions $D^\beta f(x)$ and $D^\gamma g(x)$ defined "in canonical fashion", and that the following Leibnitz differentiation formula holds:

$$D^\alpha(f(x)g(x)) = \sum_{\beta+\gamma=\alpha} \prod_{j=1}^{n} \frac{\alpha_j!}{\beta_j!\gamma_j!} D^\beta f(x)\cdot D^\gamma g(x). \tag{2.124}$$

[Hint: Here the same considerations apply as in the previous exercise in the proof of the implication 1) $\Rightarrow$ 2).]

Exercise 2.55. Prove that the support of a product of generalized functions is contained in the intersection of their supports.

The relationship between product and convolution pointed out in Exercises 2.19 and 2.21 can be further extended.

Exercise 2.56. Suppose that the convolution of the generalized functions $h_1(p)$, $h_2(p) \in \mathcal{S}'(\mathbf{R}_n)$ is "canonically" defined (in the sense of §2.5.E). Prove that the product of their Fourier transforms

$$f_j(x) = \int h_j(p)e^{ipx} d_n p, \ j = 1, 2, \tag{2.125}$$

is "canonically" defined, where

$$f_1(x)f_2(x) = \int (h_1 * h_2)(p)e^{ipx} d_n p. \tag{2.126}$$

[Hint: Use Exercise 2.48.]

Exercise 2.56 shows how the tests given in §2.5.E for the existence of a convolution can be adapted for products of generalized functions.

For example, the product of generalized functions $f_1(x)$, $f_2(x) \in \mathcal{S}'(\mathbf{R}^n)$ is defined if their Fourier transforms have supports in the sets $A_1 + K_1$ and $A_2 + K_2$ respectively, where A_1 and A_2 are compacta in $\mathbf{R}^n$ and K_1 and K_2 are two closed cones in $\mathbf{R}^n$ (with vertex at the origin) such that $K_1 \cap (-K_2) = \{0\}$.

Example. Consider, for example, generalized functions $f \in \mathcal{S}'(\mathbf{R}^n)$ whose Fourier transforms are concentrated on sets of the form $a + K$, where a is a point in $\mathbf{R}^n$ depending on f, and K is a fixed closed convex pointed cone in $\mathbf{R}^n$ with non-empty interior. It follows from Exercise 2.42 (combined with Exercise 2.56) that such generalized functions form a (commutative and associative) algebra with respect to multiplication.*

We now give an application of the notion of product of generalized functions. In practice, one is required from time to time to regard a given generalized function (or a given distribution) as a functional on a wider class of functions or generalized functions than the original space of test functions. (We have already dealt with extension "by continuity" in Proposition 2.1.) The notion of the product and integral of generalized functions is a natural one for this purpose. We say that a generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ is *integrable in the generalized sense with the generalized function* $g(x) \in \mathcal{S}'(\mathbf{R}^n)$ (or that $g(x)$ is integrable with $f(x)$) if their product fg is "canonically" defined and is a generalized function of integrable type. We can regard g as a linear functional on the set of generalized functions and, in particular, the set of functions f that are integrable (in the generalized sense) with g, by setting

$$(g, f) = \int f(x)g(x)dx; \tag{2.127}$$

the functional g defined in this manner is called the *canonical extension of the generalized function g.*

We have already used this kind of extension in the section on convolutions. In fact, according to Exercise 2.38, the convolution of the generalized functions $f(x), g(x) \in \mathcal{S}'(\mathbf{R}^n)$ is "canonically" defined if and only if for each test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$, the generalized function $f(x)g(y) \in \mathcal{S}'(\mathbf{R}^{2n})$ is integrable in the generalized sense with the function $u(x+y)$; furthermore, (2.106) holds.

Exercise 2.57. Prove that the generalized function $(x + i0)^{-1}$ is integrable in the generalized sense with the generalized functions $(x + a + i0)^{-1}$ for each $a \in \mathbf{R}$, where

$$\int (x + i0)^{-1}(x + a + i0)^{-1}dx = 0.$$

In this chapter we have only touched upon those situations when the product and convolution of generalized functions can be defined "in canonical fashion". However, in quantum field theory, the need also arises to multiply and form convolution of generalized functions in those ("non-regular") cases when these operations cannot be defined in a unique manner (or when the products and convolutions "do not exist" according to our definition). In these situations as a rule, it is a question of extending the definition by including arbitrary parameters ("renormalization constants" in the quantum-field context). The procedure corresponding to this is the most important part of the so-called theory of renormalizations (see Bogolubov and Parasiuk (1957), [B10],[Z1],[H4]). Although this theory is not required here, later we shall repeatedly encounter similar situations (see, for example §§A.3, B.5, 4.3.D, 15.2.E).

2.7. Vector- and Operator-Valued Generalized Functions

A. Generalized Functions with Values in Hilbert Space

In the previous sections we have considered complex generalized functions, and the

* In the book [V5],§12, an approach to this algebra of generalized functions based on the technique of the Laplace transform is set out in detail.

results of smoothing these with test functions are expressed by (complex) numbers. If instead of numbers we allow vectors in some linear space $\mathcal{H}$, then we arrive at the notion of a vector-valued generalized function.

Thus, a *vector-valued generalized function* in $\mathbf{R}_n$ (or a generalized function in $\mathbf{R}_n$ with values in a complex* locally convex space $\mathcal{H}$) is a continuous linear map f from $\mathcal{S}(\mathbf{R}_n)$ into $\mathcal{H}$ that associates an arbitrary test function $u \in \mathcal{S}(\mathbf{R}_n)$ with a vector $(f,u) \equiv f(u) \in \mathcal{H}$, which we write in the form of a formal integral

$$f(u) = \int f(p)u(p)d_np. \tag{2.128}$$

With applications to quantum theory in view, we shall restrict ourselves to the case when $\mathcal{H}$ is a Hilbert space. We can then use a weaker continuity requirement in our definition of a vector-valued generalized function.

Proposition 2.12. *Let $f : u \to (f,u)$ be a given linear map from $\mathcal{S}(\mathbf{R}_n)$ into $\mathcal{H}$ such that for each vector Φ in some dense linear variety $\mathfrak{M}$ in $\mathcal{H}$, the scalar product $\langle \Phi, (f,u)\rangle$ is continuous in $u \in \mathcal{S}(\mathbf{R}_n)$ and hence defines a (complex) generalized function, denoted by $\langle \Phi, f(p)\rangle$. Then f is a vector-valued generalized function.*

■ Along with the strong (norm) topology on the Hilbert space $\mathcal{H}$, we define the so-called weak topology by the seminorms $p_{\Phi_1,\ldots,\Phi_n}(\Psi) = \max_{j=1,\ldots,n} |\langle \Phi_j, \Psi\rangle|$, where $\Psi \in \mathcal{H}$, and $\Phi_1,\ldots,\Phi_n$ is an arbitrary finite set of vectors in $\mathfrak{M}$. Then the map $u \to (f,u)$ from $\mathcal{S}(\mathbf{R}_n)$ to $\mathcal{H}$ is, by hypothesis, continuous in the weak topology on $\mathcal{H}$. Hence by Corollary 1.10, the map $u \to (f,u)$ is continuous from $\mathcal{S}(\mathbf{R}_n)$ to $\mathcal{H}$ in the norm topology on $\mathcal{H}$. ■

To some extent, Proposition 2.12 reduces vector-valued generalized functions to complex ones. We can carry over most of the operations defined above on generalized functions to vector-valued generalized functions (either directly from the definition or via Proposition 2.12); these operations include addition, multiplication by a smooth function, transformation of arguments, differentiation, the Fourier transformation, etc. We can also talk about local properties and the support of a vector-valued generalized function.

We now give two operations that are specific to vector-valued generalized functions. Let $f(p)$ be a vector-valued generalized function (with values in $\mathcal{H}$) and A a closed operator in $\mathcal{H}$ whose domain D_A contains all the vectors of the form (2.128) (for any $u \in \mathcal{S}(\mathbf{R}_n)$). Then there exists a vector-valued function $Af(p)$ such that

$$Af(u) = \int Af(p)u(p)d_np \tag{2.129}$$

for all $u \in \mathcal{S}(\mathbf{R}_n)$.

The crucial point of this assertion is the fact that the map $u \to A(f,u)$ is continuous, this being easily established from Proposition 2.12. To this end we choose $\mathfrak{M}$ to be D_{A^*} (which is a dense linear variety in $\mathcal{H}$ by virtue of the closure of A). We then have $\langle \Phi, A(f,u)\rangle = \langle A^*\Phi, (f,u)\rangle$ for all $\Phi \in D_{A^*}$, from which it follows that both sides of this equality are continuous in u.

If $f(p)$ and $g(q)$ are generalized functions in $\mathbf{R}_n$ and $\mathbf{R}_m$ respectively, with values in $\mathcal{H}$, then we can uniquely define their scalar product $\langle f(p), g(q)\rangle \in \mathcal{S}'(\mathbf{R}_{n+m})$ by the following formula (for any $u(p) \in \mathcal{S}(\mathbf{R}_n)$, $v(q) \in \mathcal{S}(\mathbf{R}_m)$)

$$\int \langle f(p), g(q)\rangle u(p)v(q)d_npd_mq = \langle (f,\bar{u}),\ (g,v)\rangle. \tag{2.130}$$

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* In the case of a real space $\mathcal{H}$, one has to replace $\mathcal{S}(\mathbf{R}_n)$ by $\mathcal{S}_r(\mathbf{R}_n)$, which is the set of all real-valued functions in $\mathcal{S}(\mathbf{R}_n)$.

In fact, the right hand side of this equation determines a bilinear functional in $u \in \mathcal{S}(\mathbf{R}_n)$ and $v \in \mathcal{S}(\mathbf{R}_m)$ which is continuous in each of the arguments u and v. The existence and uniqueness of the generalized function $\langle f(p), g(q)\rangle$ now follows from Theorem 2.5 (Kernel Theorem). We note that $\langle f(p), g(q)\rangle$ appears here in the role of an analogue of the tensor product $f \otimes g$ defined in §2.4.B for the complex generalized functions $f(p)$ and $g(q)$.

B. Operator-valued generalized functions

Let D be an everywhere-dense linear submanifold of the Hilbert space $\mathcal{H}$, and suppose that there is associated with each test function $u(p) \in \mathcal{S}(\mathbf{R}_n)$ a closable linear operator $(A, u) \equiv A(u)$ defined on D (and having values in $\mathcal{H}$). We then say that $u \to A(u)$ is an *operator-valued generalized function* in $\mathcal{H}$ if the expression $A(u)\Phi$ is continuous in $u \in \mathcal{S}(\mathbf{R}_n)$ for each vector $\Phi \in D$; in other words, if for all $\Phi \in D$, the correspondence $u \to A(u)\Phi$ is a vector-valued generalized function. In this case, we write

$$A(u) = \int A(p)u(p)d_np \tag{2.131}$$

(the symbol $A(p)$ occurring here is also called an operator-valued generalized function).

In the definition of an operator-valued generalized function, the continuity condition can be weakened without making any essential change; namely, it is enough to require that for any $\Phi, \Psi \in D$, the matrix element $\langle \Psi, A(u)\Phi\rangle$ be continuously dependent on $u \in \mathcal{S}(\mathbf{R}_n)$ (that is, a complex generalized function); we use the following symbolic integral notation for this:

$$\langle \Psi, A(u)\Phi\rangle = \int \langle \Psi, A(p)\Phi\rangle u(p)d_np. \tag{2.132}$$

The fact that this weakened continuity condition is equivalent to the original one is an immediate corollary of Proposition 2.12. A second proof of the equivalence of the two definitions can be obtained by means of Theorem 2.5 (from which it follows that every separately continuous bilinear functional on $\mathcal{S}(\mathbf{R}_n)$ is continuous). We fix an arbitrary vector $\Phi \in D$ and consider the bilinear functional $B(u, v) = \langle A(\bar{v})\Phi,\ A(u)\Phi\rangle$, where $u, v \in \mathcal{S}(\mathbf{R}^n)$. Supposing that $A(u)$ satisfies the weaker continuity condition, we find that $B(u, v)$ is separately continuous and therefore continuous. It follows in particular, that if $u_k \to 0$ as $k \to \infty$, then $\|A(u_k)\Phi\|^2 = B(u_k, \bar{u}_k) \to 0$. This means that $A(u)$ satisfies the original continuity condition.

We now discuss two operations that are specific for operator-valued generalized functions. We suppose that the operator-valued generalized function $A(p)$ is such that the adjoint operators $A(u)^*$ are defined on D for all $u \in \mathcal{S}(\mathbf{R}_n)$. Then the correspondence $u \to A(\bar{u})^*|_D$ is also an operator-valued generalized function; we use the following notation for this:

$$A(u)^*|_D = \int A(p)^*\bar{u}(p)d_np. \tag{2.133}$$

The (weak and hence, ordinary) continuity of the functional $A(u)^*|_D$ with respect to u is a corollary of the identity

$$\langle \Psi, A(u)^*\Phi\rangle \equiv \overline{\langle \Phi, A(u)\Psi\rangle}.$$

Let $A(p)$ and $B(q)$ be two operator-valued generalized functions in the Hilbert space $\mathcal{H}$ defined on the test-function spaces $u(p) \in \mathcal{S}(\mathbf{R}_n)$ and $v(q) \in \mathcal{S}(\mathbf{R}_m)$ respectively. In addition to the hypothesis that the operators $A(u)$ and $B(v)$ be continuous on their common linear subspace D (which is dense in $\mathcal{H}$), we suppose further that the operators $B(v)$ leave D invariant:

$$B(v)D \subset D. \tag{2.134}$$

There then exists a (unique) operator-valued generalized function, which we denote by $A(p)B(q)$, on the space of test functions $\mathcal{S}(\mathbf{R}_{n+m})$, having D as its domain and such that

$$A(u)B(v) = \int A(p)B(q)u(p)v(q)d_npd_mq \tag{2.135}$$

for all $u \in \mathcal{S}(\mathbf{R}_n)$, $v \in \mathcal{S}(\mathbf{R}_m)$.

The proof is based on the scalar or vector version of the Kernel Theorem (see Theorem 2.5 and the remark following it). We shall give a more instructive argument based on the scalar version. For this purpose we note that the linear combinations of functions $u(x)v(y)$ with $u \in \mathcal{S}(\mathbf{R}_n)$, $v \in \mathcal{S}(\mathbf{R}_m)$ form an everywhere-dense linear subspace $\mathcal{M}$ of $\mathcal{S}(\mathbf{R}_{n+m})$ ($\mathcal{M}$ is the algebraic tensor product of $\mathcal{S}(\mathbf{R}_n)$ and $\mathcal{S}(\mathbf{R}_m)$). Since for each $\Phi \in D$, the expression $A(u)B(v)\Phi$ is a bilinear functional of u, v, there exists a unique linear functional C on $\mathcal{M}$ such that $C(u \otimes v) = A(u)B(v)\Phi$. It suffices to verify that C is continuous, that is, that $\|C(w_k)\| \to 0$ if $w_k \in \mathcal{M}$ and $w_k \to 0$ as $k \to \infty$ in the topology of $\mathcal{S}(\mathbf{R}_{n+m})$. For this we use the kernel theorem (scalar version) according to which there exists a generalized function in $\mathcal{S}'(\mathbf{R}_{2n+2m})$, which we denote by $\langle A(p')B(q')\Phi,\ A(p)B(q)\Phi\rangle$, such that

$$\langle A(u')B(v')\Phi,\ A(u)B(v)\Phi\rangle = \int \langle A(p')B(q')\Phi,\ A(p)B(q)\Phi\rangle \times$$
$$\times\ \overline{u'(p')v'(q')}u(p)v(q)d_npd_np'd_mqd_mq'.$$

From this it follows that (for $w_k \in \mathcal{M}$)

$$\|C(w_k)\|^2 = \int \langle A(p')B(q')\Phi, A(p)B(q)\Phi\rangle \overline{w_k(p',q')}w_k(p,q)d_npd_np'd_mqd_mq'.$$

If now $w_k \to 0$ in the $\mathcal{S}(\mathbf{R}_{n+m})$ topology as $k \to +\infty$, then $\overline{w_k(p',q')}w_k(p,q) \to 0$ in the $\mathcal{S}(\mathbf{R}_{2n+2m})$ topology; it therefore follows from the above representation that $\|C(w_k)\| \to 0$ as $k \to \infty$. This completes the proof.

The above result extends to the case of any number of operator-valued "factors", which we now state.

Proposition 2.13. *Let $A_j(p_j)$ $(j = 1, \dots, k)$ be a family of operator-valued generalized functions in the Hilbert space $\mathcal{H}$ over the respective test function spaces $u_j(p_j) \in \mathcal{S}(\mathbf{R}_{n_j})$. Suppose that all the operators $A_j(u_j)$ are defined on a dense linear subspace D of $\mathcal{H}$ and leave this subspace invariant. Then there exists a (unique) operator-valued generalized function, denoted by $A_1(p_1)\dots A_k(p_k)$, (defined on the test function space $\mathcal{S}(\mathbf{R}_{n_1+\dots+n_k})$ and having D as its domain) such that*

$$A_1(u_1)\dots A_k(u_k) = \int A_1(p_1)\dots A_k(p_k)u_1(p_1)\dots u_k(p_k)d_{n_1}p_1\dots d_{n_k}p_k \tag{2.136}$$

for all $u_j \in \mathcal{S}(\mathbf{R}_{n_j})$.

Remark. Whereas the operators (2.136) leave D invariant (by hypothesis), operators of the form

$$\int A_1(p_1)\dots A_k(p_k)u(p_1,\dots,p_k)d_{n_1}p_1\dots d_{n_k}p_k \tag{2.137}$$

($u \in \mathcal{S}(\mathbf{R}_{n_1+\ldots+n_k})$ being arbitrary) may in general take vectors in D outside D. It is not difficult to see, however, that under the action of operators (2.137) on D, the vectors so obtained lie in the domain of the closures of the operators $A_j(u_j)$ ($u_j \in \in \mathcal{S}(\mathbf{R}_{n_j})$) and, more generally, in the domain of operators of type (2.137). If therefore we suitably extend the domain D (for example, by replacing D by the linear span of the subspaces D and XD, where the X run through operators that are the results of smoothing $A_{i_1}(p_1)\ldots A_{i_r}(p_r)$, where $r = 1, 2, \ldots$, by arbitrary test functions), we can arrange matters so that the new domain D is invariant with respect to all operators of the form (2.137).

C. The notion of a generalized eigenvector

Let A be a closed operator in $\mathcal{H}$ (with domain D_A). Then a generalized function $f(p)$ in $\mathbf{R}_n$ with values in $\mathcal{H}$ is called a *generalized eigenvector* if

$$Af(p) = \lambda(p)f(p), \tag{2.138}$$

where $\lambda(p)$ is a multiplicator* in $\mathcal{O}_M(\mathbf{R}_n)$. The restriction $\lambda|_{\mathrm{supp}\, f}$ is called the *generalized eigenvalue* (or system of generalized eigenvalues) of A corresponding to the generalized eigenvector $f(p)$. Similarly, if $U(a)$ is a (continuous) unitary n-parameter abelian group of operators in $\mathcal{H}$, then a generalized function $f(p)$ in $\mathbf{R}_n$ with values in $\mathcal{H}$ is called a generalized eigenvector for this subgroup if

$$U(a)f(p) = e^{ia\mu(p)}f(p) \tag{2.139}$$

for all $a \in \mathbf{R}^n$, where $a\mu(x) \equiv \sum_{j=1}^{n} \epsilon_j a_j \mu_j(x)$ and the $\mu_j(p)$ are real multiplicators in $\mathcal{O}_M(\mathbf{R}_n)$.

We now consider the example of the operator $P = -i\frac{d}{dx}$ and the (abelian) translation group $U(a)$ in $\mathcal{L}^2(\mathbf{R})$, where $(U(a)f)(x) = f(x-a)$. We define the generalized function f_p over $\mathcal{S}(\mathbf{R}_1)$ with values in $\mathcal{L}^2(\mathbf{R})$ by the formula

$$\left(\int f_p u(p) d_1 p\right)(x) = \int e^{ipx} u(p) d_1 p;$$

in other words, we choose as the kernel $f_p(x)$ of the map $f: \mathcal{S}(\mathbf{R}_1) \to \mathcal{L}^2(\mathbf{R})$, the "plane wave" $f_p(x) = e^{ipx}$. It is easily verified that f_p is a generalized function for P and for $U(a)$:

$$Pf_p = pf_p, \; U(a)f_p = e^{-ipa}f_p.$$

Furthermore, it satisfies an orthogonality relation in the sense that

$$\langle f_q | f_p \rangle = 2\pi\delta(p-q). \tag{2.140}$$

In this example, the generalized eigenvectors f_p of P form a complete orthonormal system of eigenvectors in $\mathcal{L}^2(\mathbf{R})$ in the sense of the following definition (which is more suitably stated in a more general context).

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* If we invoke the notion of the product of a scalar and a vector generalized function (by analogy with the product of scalar generalized functions in §2.6.C), then less smooth functions can be allowed for $\lambda(p)$ in our definition.

Let $f^{(1)}(p^{(1)}),\ldots,f^{(j)}(p^{(j)}),\ldots$ be a sequence of vector-valued generalized functions depending on the variables $p^{(j)} \in \mathbf{R}_{n_j}$ (where n_j can depend on j) and taking values in the Hilbert space $\mathcal{H}$. Then we say that this sequence forms a *complete system* of generalized eigenvectors in $\mathcal{H}$ if the linear combinations of vectors of the form

$$\int f^{(j)}(p^{(j)})u^{(j)}(p^{(j)})d_{n_j}p^{(j)} \tag{2.141}$$

(with arbitrary $u^{(j)} \in \mathcal{S}(\mathbf{R}_{n_j})$) form a dense subset of $\mathcal{H}$. If in addition, we have $\langle f^{(j)}(p^{(j)}),\ f^{(k)}(p^{(k)})\rangle = 0$, when either $j \neq k$ or $j = k$ but $p^{(j)} \neq p^{(k)}$, then we have an *orthogonal system* of generalized vectors in $\mathcal{H}$. Now by an expansion of the vectors of $\mathcal{H}$ in a complete orthogonal system of generalized vectors $\{f^{(j)}(p^{(j)})\}$, we simply mean that the linear combinations of vectors of the form (2.141) are dense in $\mathcal{H}$. For instance, in the above example of a system of plane waves f_p, the vector $\int f_p(x)u(p)d_1p$ (where $u \in \mathcal{S}(\mathbf{R}_1)$) is simply the Fourier transform of u and hence (by Proposition 2.7) such vectors form a dense subset $\mathcal{S}(\mathbf{R})$ of $\mathcal{L}^2(\mathbf{R})$. In this instance, the orthogonality relation (2.140) is another form of writing down the Parseval equality (2.74).

Second quantization provides an example of a complete orthogonal system of generalized vectors in Fock space (see §7.3.A). We shall encounter another important example of a complete (but this time, non-orthogonal) system of generalized vectors in the study of quantized fields (Ch.8). .

Appendix A. Generalized Functions on Subsets of $\mathbf{R}^n$

A.1. Generalized functions on an open subset

So far, we have only dealt with generalized functions defined on the whole of $\mathbf{R}^n$. Not infrequently, however, it proves to be convenient (and even necesssary) to consider generalized functions on subsets of $\mathbf{R}^n$. Here we shall restrict ourselves to the most important cases when the given set is either open or belongs to a special class of closed sets.

The case of open sets is very simple. For an open subset $\mathcal{O}$ of $\mathbf{R}^n$, we define $\mathcal{S}(\mathcal{O})$ as the set of all functions in $\mathcal{S}(\mathbf{R}^n)$ that vanish in $\mathbf{R}^n\backslash\mathcal{O}$ (the complement of $\mathcal{O}$). If we endow $\mathcal{S}(\mathcal{O})$ with the system of seminorms (1.42), then $\mathcal{S}(\mathcal{O})$ becomes a closed subspace of $\mathcal{S}(\mathbf{R}^n)$ and hence a Fréchet space. We note an essential detail: the functions of $\mathcal{S}(\mathcal{O})$ and any of their derivatives vanish on the boundary of $\mathcal{O}$ as well. Since the functions in $\mathcal{S}(\mathcal{O})$ are determined by their values on $\mathcal{O}$, we can identify each function in $\mathcal{S}(\mathcal{O})$ with its restriction to $\mathcal{O}$. We can therefore call $\mathcal{S}(\mathcal{O})$ the space of rapidly decreasing test functions on $\mathcal{O}$.

The space $\mathcal{S}'(\mathcal{O})$ of continuous linear functionals on $\mathcal{S}(\mathcal{O})$ is called the space of generalized functions on $\mathcal{O}$. As before, we write the value of $f \in \mathcal{S}'(\mathcal{O})$ at $u \in \mathcal{S}(\mathcal{O})$ in the symbolic integral form (2.1). As for the case of $\mathbf{R}^n$, we can also define the various operations on $\mathcal{S}(\mathcal{O})$ such as differentiation, multiplication by a smooth function and change of variables by a diffeomorphism, in terms of the previous formulae (2.19), (2.34), (2.17). To motivate formula (2.19), we point out that in the case of a "sufficiently good" function f (more precisely, when f has continuous partial derivatives of polynomial growth in $\mathbf{R}^n$) , formula (2.19) is the same as integration by parts in classical analysis (since, according to our definition, we can regard the test functions $u \in \mathcal{S}(\mathcal{O})$ as functions in $\mathcal{S}(\mathbf{R}^n)$). We have to impose on the function in formula (2.34) the condition that $\phi(x)\cdot u(x) \in \mathcal{S}(\mathcal{O})$ for all $u(x) \in \mathcal{S}(\mathcal{O})$. We call such functions *multiplicators* in $\mathcal{S}(\mathcal{O})$ and denote the set of them by $\Theta_M(\mathcal{O})$. Finally, the ϕ in formula (2.17) is a diffeomorphism from $\mathcal{O}$ onto an open subset $\mathcal{Q} \subset \mathbf{R}^n$ such that the map $u \to |J(\phi)|\cdot u\circ\phi$ is an isomorphism from $\mathcal{S}(\mathcal{Q})$ onto $\mathcal{S}(\mathcal{O})$; definition (2.17) then makes sense for all $f \in \mathcal{S}'(\mathcal{Q}),\ u \in \mathcal{S}(\mathcal{Q})$.

The practical value of the spaces $\mathcal{S}'(\mathcal{O})$ (for open sets $\mathcal{O}$) is essentially that they provide the possibility of studying generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ by local methods. Namely, if $f \in \mathcal{S}'(\mathbf{R}^n)$ then the restriction of f (as a linear functional) to the subspace $\mathcal{S}(\mathcal{O}) \subset \mathcal{S}(\mathbf{R}^n)$ gives us a generalized function in $\mathcal{S}'(\mathcal{O})$, denoted by $f|_{\mathcal{O}}$ and called the *restriction* of f to $\mathcal{O}$. By the Hahn-Banach theorem, all the generalized functions on $\mathcal{O}$ are obtained via restrictions of the generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ to $\mathcal{O}$.

Conversely, the gluing principle enables us to construct the generalized functions in $\mathcal{S}(\mathbf{R}^n)$ by means of a suitable collection of generalized functions on open subsets. More precisely, suppose that we are given a finite collection $\{\mathcal{O}_j\}_{\nu=1}^N$ of open subsets of $\mathbf{R}^n$ covering $\mathbf{R}^n$. We further suppose that there is a partition of unity in $\Theta_M(\mathbf{R}^n)$ subordinated to this covering. Here, by a partition we mean a set $\{e_\nu\}$ of multiplicators in $\mathcal{S}(\mathbf{R}^n)$ with the properties (1), (2), (3) mentioned in §2.1.C. If there is associated with each subset $\mathcal{O}_\nu$ of the given covering a generalized function $f_\nu \subset \mathcal{S}'(\mathcal{O}_\nu)$, such that the compatibility condition $f_\nu(x) = f_{\nu'}(x)$ on $\mathcal{O}_\nu \cap \mathcal{O}_{\nu'}$ holds for all ν, ν', then there exists a unique generalized function f in $\mathcal{S}'(\mathbf{R}^n)$ coinciding with $f_\nu(x)$ on $\mathcal{O}_\nu$ for each ν.

A.2. Generalized functions on canonically closed regular subsets

We now go over to the construction of spaces of test functions and generalized functions on a closed subset $\Omega \subset \mathbf{R}^n$. So as not to complicate matters, we restrict ourselves to the class of so-called canonically closed regular subsets Ω of $\mathbf{R}^n$. A set Ω is said to be ***canonically closed*** if it is the closure of its interior (or, what is the same, the closure of some open set) in $\mathbf{R}^n$. By a ***regular set*** we mean a set $\Omega \subset \mathbf{R}^n$ with the following property: any pair of points $x, y \in \Omega$ with distance $|x - y| \le \rho$ can be joined by a rectifiable curve in Ω of length $\le A(1 + |x| + |y|)^p |x - y|^\lambda$; here ρ, A, p, λ are positive numbers (depending on Ω).

Exercise A.1. Prove that every closed convex subset of $\mathbf{R}^n$ with a non-empty interior is canonically closed and regular.

Below (and throughout this appendix), by Ω we mean a fixed canonically closed regular set.

The standard definition of the partial derivatives of a function defined on a subset of $\mathbf{R}^n$ presupposes that the given set is open. In our case, we say that a complex function u on Ω is infinitely differentiable (or of class C^∞) if it is the restriction to Ω of some infinitely differentiable function, say, $\hat{u}$ on $\mathbf{R}^n$; similarly, the partial derivatives $D^\alpha u$ are defined as the restrictions to Ω of the partial derivatives $D^\alpha \hat{u}$. These derivatives are well defined, that is, the definition does not depend on the choice of the extension to the function u outside Ω, and this is due to the canonical regularity of Ω. We denote by $\mathcal{S}(\mathcal{O})$ the space of all complex C^∞-functions u on Ω for which all the norms (for any non-negative integral l, m) are finite:

$$\|u\|_{l,m}^{\Omega} = \max_{|\alpha|\le l} \sup_{x\in\Omega} |(1 + |x|)^m D^\alpha u(x)|. \tag{A.1}$$

Omitting the proof (based on Whitney's extension theorem [M3]), we note the following important fact: because of the regularity of Ω, $\mathcal{S}(\Omega)$ is a complete space and hence is also a Fréchet space. Furthermore, each function in $\mathcal{S}(\Omega)$ is obtained as the result of restricting some function in $\mathcal{S}(\mathbf{R}^n)$ to Ω.

As usual, by the space of generalized functions on Ω we mean the dual space $\mathcal{S}'(\Omega)$ to $\mathcal{S}(\Omega)$.

As before, we use the notation (2.1) for the value of $f \in \mathcal{S}'(\Omega)$ at $u \in \mathcal{S}(\Omega)$.

As might be expected, by the regular generalized functions on Ω we mean generalized functions f representable in the form (2.1), where the right hand side is now taken in the sense of the Lebesgue integral over Ω of the product of a locally integrable $f(x)$ on Ω with $u \in \mathcal{S}(\Omega)$. Here in order that the integral be absolutely convergent, we have to restrict the behaviour of the function $f(x)$ at infinity in the following way: for some $A,\ \delta > 0$,

$$\int_{\Omega\cap\sigma_0(r)} |f(x)| d^n x \le A(1 + r)^\delta \quad \text{for all } r > 0,$$

where $\sigma_0(r)$ is a ball of radius r with centre at the origin.

In particular, this condition is satisfied by the continuous functions $f(x)$ of polynomial growth in Ω; thus we can identify them with regular generalized functions in Ω.

The operations of multiplication by a smooth function and change of variables in $\mathcal{S}'(\Omega)$ are completely analogous to the corresponding operations in $\mathcal{S}'(\mathbf{R}^n)$. For example, by a multiplicator in $\mathcal{S}(\Omega)$ we mean a C^∞-function ϕ on Ω for which any derivative $D^\alpha\phi$ is of polynomial growth on Ω, that is, there exist $A,\ \delta \ge 0$ (depending on ϕ and α) such that $|D^\alpha\phi(x)| \le A(1 + |x|)^\delta$ in Ω.

The operation of differentiation in $\mathcal{S}'(\Omega)$ is given by the previous formula:

$$(D^\alpha f, u) = (-1)^{|\alpha|}(f, D^\alpha u),\ f \in \mathcal{S}'(\Omega),\ u \in \mathcal{S}(\Omega). \tag{A.2}$$

However, this formula is in a certain sense unnatural and one should bear in mind the following warning when differentiating generalized functions. In contrast to the situation for $\mathcal{S}'(\mathbf{R}^n)$, when $\Omega \neq \mathbf{R}^n$ the generalized derivative (in the sense of (A.2)) of a smooth function f (for example, a multiplicator in $\mathcal{S}(\Omega)$) is not, generally speaking, the same as the ordinary (or "classical") derivative due to the presence of boundary terms in the formula for integration by parts (assuming it holds for Ω). For example, in the case $\Omega = [0,1] \subset \mathbf{R}$, the generalized derivative in $\mathcal{S}'([0,1])$ of the function $h(x) \equiv 1$ on Ω is not 0, but $\delta(x) - \delta(1-x)$, that is,

$$\left(\frac{d}{dx}h(x), u(x)\right) = u(0) - u(1), \; u \in \mathcal{S}([0,1]).$$

Proposition A.1 given below justifies to a considerable extent this "strangeness" of generalized differentiation in Ω. Namely, if we identify generalized functions on Ω with generalized functions on $\mathbf{R}^n$ with supports in Ω, then it turns out that in differentiating a smooth function f in the $\mathcal{S}'(\Omega)$ sense, we are in fact differentiating the function $\chi_\Omega f$ in the $\mathcal{S}'(\mathbf{R}^n)$ sense, where χ_Ω is the characteristic function of Ω. It is the derivatives of χ_Ω that are responsible for the appearance of the "boundary terms" in $D^\alpha(\chi_\Omega f)$.

It is very interesting to note that for generalized functions in $\mathcal{S}'(\Omega)$ we can define a "canonical extension" in $\mathbf{R}^n$. Namely, if $f \in \mathcal{S}'(\Omega)$, then we can associate the generalized function $F \in \mathcal{S}'(\mathbf{R}^n)$ with it according to the formula

$$(F, u) = (f, u|_\Omega) \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{A.3}$$

Let $j : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\Omega)$ be the restriction operator that associates with the function $u \in \mathcal{S}(\mathbf{R}^n)$ its restriction to Ω:

$$ju = u|_\Omega. \tag{A.4}$$

(According to the above characterization of $\mathcal{S}(\Omega)$, j maps $\mathcal{S}(\mathbf{R}^n)$ onto the whole of $\mathcal{S}(\Omega)$.)

Let $j' : \mathcal{S}'(\Omega) \to \mathcal{S}'(\mathbf{R}^n)$ be the adjoint operator (see §1.3.8). Then formula (A.3) means that $F = j'f$. It is not difficult to see that the generalized function F has support in Ω. We denote by $\mathcal{S}'(\mathbf{R}^n|\Omega)$ the set of all generalized functions in $\mathbf{R}^n$ with supports in Ω. It turns out that as f runs through $\mathcal{S}'(\Omega)$, $j'f$ runs through $\mathcal{S}'(\mathbf{R}^n|\Omega)$. In other words, j' is an isomorphism (in the sense of a one-to-one linear correspondence that preserves convergence) from $\mathcal{S}'(\Omega)$ to $\mathcal{S}'(\mathbf{R}^n|\Omega)$.

Proposition A.1. *Let Ω be a canonically closed regular subset of $\mathbf{R}^n$. Then the restriction operator j of (A.4) maps $\mathcal{S}(\mathbf{R}^n)$ continuously onto $\mathcal{S}(\Omega)$, while its adjoint j' maps $\mathcal{S}'(\Omega)$ isomorphically onto the subspace $\mathcal{S}'(\mathbf{R}^n|\Omega)$ of all generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ with supports in Ω.*

The isomorphism constructed in Proposition A.1 is called the *canonical isomorphism* between $\mathcal{S}'(\Omega)$ and $\mathcal{S}'(\mathbf{R}^n|\Omega)$.

Corollary A.2. For any generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ with support in the canonically closed regular set $\Omega \subset \mathbf{R}^n$, there exist numbers $c \geq 0$ and natural numbers l, m such that

$$|(f,u)| \leq c\|u\|_{l,m}^{\Omega} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{A.5}$$

Furthermore, $f(x)$ can be represented as a finite sum:

$$f(x) = \sum_{|\alpha| \leq N} D^\alpha f_\alpha(x), \tag{A.6}$$

where $f_\alpha(x)$ is a measure of power growth on $\mathbf{R}^n$ with support in Ω.

Formula (A.6) follows from the estimate (A.5) in the same way that (2.4) follows from (2.2).

Exercise A.2. Prove Proposition A.1 and Corollary A.2. [Hint: According to Exercise 1.39, the proof of Proposition A.1 reduces to establishing the equality $\mathcal{S}'(\mathbf{R}^n|\Omega) = (\ker j)^\circ$, which means that the generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ has support in Ω if and only if $(f, u) = 0$ for all functions $u \in \mathcal{S}(\Omega)$ that vanish on Ω. Use the fact that every function $u \in \mathcal{S}(\Omega)$ that vanishes on Ω can be expressed as the limit in $\mathcal{S}(\mathbf{R}^n)$ of a sequence of functions $u_k \in \mathcal{S}(\mathbf{R}^n)$ with supports in $\mathbf{R}^n \backslash \Omega$, that is, functions that vanish on a neighbourhood of Ω.]

Proposition A.1 and Corollary A.2 shed considerable light on the role of the spaces $\mathcal{S}'(\Omega)$. These spaces prove to be extremely useful in the study of generalized functions with support in the (canonically closed regular) set Ω. If it follows from the definition of the support of the generalized function f that the value (f,u) of $f \in \mathcal{S}'(\mathbf{R}^n|\Omega)$ at the test function $u \in \mathcal{S}(\mathbf{R}^n)$ depends only on the behaviour of u in (a neighbourhood of) Ω, then we now have the quantitative estimate A.5 for this dependence.

The spaces $\mathcal{S}'(\Omega)$ find another typical application in the so-called weak integral representations. Suppose that associated with each value y of the canonically closed regular subset Ω of $\mathbf{R}^n$ is a generalized function $K(x;y)$ in $\mathcal{S}'(\mathbf{R}^m)$ (with respect to x), where for each test function $u(x) \in \mathcal{S}(\mathbf{R}^m)$, the function $\int K(x,y)u(x)d^mx$ belongs to $\mathcal{S}(\Omega)$. Then the map $u \to \int K(x,y)u(x)d^mx$ is a continuous linear operator from $\mathcal{S}(\mathbf{R}^m)$ to $\mathcal{S}(\Omega)$ (the reasoning is the same as in Exercise 2.43 or Proposition 2.11). Therefore for the generalized function $h(y)$ in $\mathcal{S}'(\Omega)$, the formula

$$(f,u) = (h(y), \int K(x,y)u(x)d^mx)_y \tag{A.7}$$

defines a generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^m)$. This $f(x)$, called the *weak integral representation** for the generalized function $f(x)$:

$$f(x) = \int K(x,y)h(y)d^ny; \tag{A.8}$$

here $K(x,y)$ and $h(y)$ are called the *kernel* and *generalized density*, respectively, of the weak integral representation.

We give one further useful result on the decomposition of generalized functions into a (finite) sum of generalized functions with smaller supports. We confine ourselves to the case of two variables.

Proposition A.3. *Let $f(x)$ be a generalized function in $\mathbf{R}^n$ whose support is a canonically closed regular set $\Omega \subset \mathbf{R}^n$ that is the union of two canonically closed regular subsets Ω_1 and Ω_2 of $\mathbf{R}^n$. Then $f(x)$ can be expressed in the form*

$$f(x) = f_1(x) + f_2(x), \tag{A.9}$$

where f_j is a generalized function in $\mathbf{R}^n$ with support in Ω_j, $j = 1, 2$.

■ Consider the direct sum $\mathcal{X} = \mathcal{S}(\Omega_1) \oplus \mathcal{S}(\Omega_2)$. The elements of $\mathcal{X}$ are all the pairs (u_1, u_2) with $u_j \in \mathcal{S}(\Omega_j)$. We make $\mathcal{X}$ into a Fréchet space by defining for any norms p_1 and p_2 of the form (A.1) on $\mathcal{S}(\Omega_1)$ and $\mathcal{S}(\Omega_2)$, the norm p on $\mathcal{X}$ by setting $p(u_1,u_2) = p_1(u_1) + p_2(u_2)$. $\mathcal{X}$ contains the (closed) subspace $\mathcal{Y}$ consisting of elements of the form $(u|_{\Omega_1}, u|_{\Omega_2})$, where u runs through $\mathcal{S}(\Omega)$. It is easy to see that the map $u \to (u|_{\Omega_1}, u|_{\Omega_2})$ is an isomorphism from $\mathcal{S}(\Omega)$ onto $\mathcal{Y}$.

We now turn our attention to the generalized function $f(x)$. Let $h(x)$ be the generalized function in $\mathcal{S}'(\Omega)$ corresponding to $f(x)$ under the canonical isomorphism from $\mathcal{S}'(\Omega)$ onto $\mathcal{S}'(\mathbf{R}^n|\Omega)$. Then h defines a continuous linear functional h' on $\mathcal{Y}$ by means of the formula

$$(h', (u|_{\Omega_1}, u|_{\Omega_2})) = (h, u). \tag{A.10}$$

By the Hahn-Banach theorem (Theorem 1.1) and §1.3.A), h' can be extended to a continuous functional h'' on $\mathcal{X}$. It is clear that each such functional h'' has the form $(h'', (u_1, u_2)) = (h_1, u_1) + (h_2, u_2)$, where $h_j \in \mathcal{S}(\Omega_j)$. In particular, we find by taking (A.10) into account that for all $u \in \mathcal{S}(\mathbf{R}^n)$,

$$(f,u) = (h, u|_\Omega) = (h_1, u|_{\Omega_1}) + (h_2, u|_{\Omega_2}). \tag{A.11}$$

Suppose that under the canonical isomorphism from $\mathcal{S}'(\Omega_j)$ onto $\mathcal{S}'(\mathbf{R}^n|\Omega_j)$, h_j is taken to the generalized function $f_j(x)$ in $\mathbf{R}^n$ with support in Ω_j. Then (A.11) is clearly equivalent to the required decomposition (A.9). ■

A.3. Application: generalized functions on the compactified sets $[a,\infty]$, $\mathbf{R}_\infty$, $[-\infty,+\infty]$

The extension of the definition to divergent integrals (or "removal of divergences") is a typical problem of quantum field theory, which we illustrate with a typical example (essential for us later).

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* It is also called the integral representation for generalized functions (or "in the weak sense").

Suppose that it is required to define the convolution in $\mathcal{S}'(\mathbf{R})$

$$\frac{1}{\lambda} * f(\lambda) = \int \frac{1}{\lambda - \lambda'} f(\lambda')d\lambda', \tag{A.12}$$

where $f(\lambda)$ is a generalized function in $\mathcal{S}'(\mathbf{R})$. It is clear that, in general, this convolution is not defined uniquely (in other words, it "does not exist", in the sense of §2.5). Thus, if we start from the formula of type (2.86)

$$\left(\frac{1}{\lambda} * f, u\right) = -\left(f, \frac{1}{\lambda} * u\right), \; u \in \mathcal{S}(\mathbf{R}), \tag{A.13}$$

then the right hand side is ill-defined here since, in general, $\frac{1}{\lambda} * u$ does not belong to $\mathcal{S}(\mathbf{R})$. To solve this problem, we avail ourselves of the larger space $\mathcal{S}(\mathbf{R}_\infty)$ of functions on the extended real line * $\mathbf{R}_\infty = \mathbf{R} \cup \{\infty\}$ containing functions of the form $\frac{1}{\lambda} * u$ (where $u \in \mathcal{S}(\mathbf{R})$). If we now extend f from $\mathcal{S}(\mathbf{R})$ onto $\mathcal{S}(\mathbf{R}_\infty)$, then the right hand side of (A.13) becomes meaningful for all $u \in \mathcal{S}(\mathbf{R})$. Thus we see that the convolution (A.12) is naturally defined for $f \in \mathcal{S}'(\mathbf{R}_\infty)$ rather than for an arbitrary f in $\mathcal{S}'(\mathbf{R})$. The procedure for extending a generalized function in $\mathcal{S}'(\mathbf{R})$ to a generalized function in $\mathcal{S}'(\mathbf{R}_\infty)$ contains certain arbitrary parameters (by analogy with what in quantum field theory are called the "renormalization constants"). We note that the problem of multiplying a generalized function $\tilde{f}(t)$ in $\mathcal{S}'(\mathbf{R})$ by $\epsilon(t)$ (or by $\theta(t)$) reduces to the problem of defining the convolution (A.12).

We now turn to the construction of the space $\mathcal{S}(\mathbf{R}_\infty)$, as well as the analogous spaces $\mathcal{S}([a,\infty])$ (where a is an arbitrary real number) and $\mathcal{S}([-\infty, +\infty])$.

We go into the details for the case $\mathcal{S}([1,\infty])$. Using the fact that the map $\xi = -1/\lambda$ maps $[1,\infty]$ onto the interval $[-1,0]$, we can define $\mathcal{S}([1,\infty])$ as the set of functions on $[1,\infty]$ of the form

$$u(\lambda) = v(\xi), \quad \text{where } \xi = -1/\lambda \tag{A.14}$$

where $v \in \mathcal{S}([-1,0])$ is arbitrary. In accordance with §A.2, the topology on $\mathcal{S}([-1,0])$ is defined by the norms

$$q_n(u) = \max_{k=0,\dots,n} \sup_{\xi\in[-1,0]} |v^{(k)}(\xi)|, \; n \in \overline{\mathbf{Z}}_+, \tag{A.15}$$

where $v^{(k)}(\xi) \equiv \frac{d^k v(\xi)}{d\xi^k}$. We endow $\mathcal{S}([1,\infty])$ with the inductive topology relative to the map $v \to u$ (A.14) (so that this map becomes a topological isomorphism). It is not difficult to see that this topology is defined by the norms

$$p_n(u) = \max_{k=0,\dots,n} \sup_{\lambda\geq 1} |\Lambda^k u(\lambda)|, \; n \in \overline{\mathbf{Z}}_+, \tag{A.16}$$

where $\Lambda = \lambda^2 \frac{\partial}{\partial\lambda}$. It is clear that $\mathcal{S}([1,\infty))$ is the subspace of $\mathcal{S}([1,\infty])$ consisting of functions u such that $\Lambda^n u(\lambda) \to 0$ as $\lambda \to \infty$ for each $n \in \overline{\mathbf{Z}}_+$. It follows from Exercise A.3 that it is also a subspace in the topological sense, that is, the natural topology on $\mathcal{S}([1,\infty))$ is the same as that induced by $\mathcal{S}([1,\infty])$.

Exercise A.3. (a) Prove that $\mathcal{S}([1,\infty))$ is the image of $\mathcal{S}_0([-1,0])$ under the map $v \to u$ (A.14); here $\mathcal{S}_0([-1,0])$ is the subspace of all functions in $\mathcal{S}([-1,0])$ such that $v^{(n)}(0) = 0$ for all $n \in \overline{\mathbf{Z}}_+$. Prove also that this isomorphism $\mathcal{S}_0([-1,0]) \to \mathcal{S}([1,\infty))$ is topological if we take $\mathcal{S}([1,\infty))$ with its natural topology (see §A.2), while the topology of $\mathcal{S}_0([-1,0])$ is defined by the norms

$$q_{m,n}(v) = \max_{k=0,\dots,n} \sup_{-1\leq\xi\leq 0} |\xi^{-m} v^{(k)}(\xi)|, \quad m,n \in \overline{\mathbf{Z}}_+. \tag{A.17}$$

(b) Prove that the systems of norms $\{q_{m,n}\}$ and $\{q_n\}$ are equivalent on $\mathcal{S}_0([-1,0])$. [Hint: Use the equality $\xi^{-1}v^{(k)}(\xi) = \int_{-1}^{0} v^{(k+1)}(t\xi)dt$ to verify that $q_{m,n}(v) \leq q_{m-1,n+1}(v)$, from which it follows by induction on m that $q_{mn}(v) \leq q_{m+n}(v)$.]

* This is the so-called one-point compactification of the real line; we also use the two-point compactification $[-\infty, +\infty] = \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ (obtained by adjoining to $\mathbf{R}$ the two points at infinity $-\infty$ and $+\infty$).

As usual, the elements of the dual $\mathcal{S}'([1,\infty])$ of $\mathcal{S}([1,\infty])$ are called generalized functions on $[1,\infty]$; we also use the symbolic integral notation $\int f(\lambda)u(\lambda)d\lambda$ for (f,u). In particular, the integral $(f(\lambda),1(\lambda)) \equiv \int f(\lambda)d\lambda$ is defined for all $f \in \mathcal{S}'([1,\infty])$.

Exercise A.4. Prove that the generalized functions l_k on $[1,\infty]$ defined by the asymptotic expansion

$$u(\lambda) = \sum_{k=0}^{n} \lambda^{-k}(l_k,u) + o(\lambda^{-n}) \quad \text{for } \lambda \to \infty,\ n \in \overline{\mathbf{Z}}_+, \tag{A.18}$$

form a basis in the subspace of generalized functions of $\mathcal{S}'([1,\infty])$ with support at the point $+\infty$. [Hint: Use the isomorphism between $\mathcal{S}'([1,\infty])$ and $\mathcal{S}'([-1,0])$ and Proposition 2.2.]

By the Hahn-Banach theorem, every continuous linear functional on $\mathcal{S}([1,\infty))$ can be extended to a continuous linear functional on $\mathcal{S}([1,\infty])$. This extension can be effected by the following constructive method.

Exercise A.5. Let $f_0 \in \mathcal{S}'([1,\infty))$, so that f_0 can be regarded as a continuous linear functional on $\mathcal{S}_{\lambda,m}([1,\infty))$ (the space of restrictions to $[1,\infty)$ of functions in $\mathcal{S}_{\lambda,m}(\mathbf{R})$, see §1.1.C). Prove that the general form of an extension of f_0 to a generalized function in $\mathcal{S}'([1,\infty])$ has the form

$$(f,u) = \left(f_0(\lambda), u(\lambda) - \sum_{k=0}^{N} \omega(\lambda)\lambda^{-k}(l_k,u)\right) + \sum_{n=0}^{\infty} c_n(l_n,u), \tag{A.19}$$

where $n \in \mathbf{Z}_+$ is a sufficiently large fixed number; c_n is an arbitrary numerical sequence with a finite number of non-zero terms.

The spaces $\mathcal{S}([-\infty,\ +\infty])$, $\mathcal{S}(\mathbf{R}_\infty)$ and their duals $\mathcal{S}'([-\infty,\ +\infty])$ and $(\mathcal{S}'(\mathbf{R}_\infty)$ are defined similarly. Thus, $\mathcal{S}([-\infty,\infty])$ (or $\mathcal{S}(\mathbf{R}_\infty)$) is the space of functions $u(\lambda)$ on $[-\infty,\ +\infty]$ (or on $\mathbf{R}_\infty$) of the form

$$u(\lambda) = w(\theta), \quad \text{where } \theta = 2\tan^{-1}\lambda, \tag{A.20}$$

for arbitrary (or arbitrary with period 2π) $w(\theta)$ in $\mathcal{S}([-\pi,\ \pi])$ (or in $\mathcal{E}(\mathbf{R})$). The topology on these spaces is defined in similar fashion to that given above for $\mathcal{S}([1,\infty])$. We note that for each $a \in \mathbf{R}$, $\mathcal{S}'([a,\infty])$ can be identified with the subspace of $\mathcal{S}'(\mathbf{R}_\infty)$ (or of $\mathcal{S}'([-\infty,\ +\infty])$) consisting of generalized functions on $\mathbf{R}_\infty$ (or on $[-\infty,\ +\infty]$) with supports in $[a,\infty]$.

Exercise A.6. Prove that the map $u \to 1/\lambda * u$ is a continuous linear operator from $\mathcal{S}(\mathbf{R})$ to $\mathcal{S}(\mathbf{R}_\infty)$ (or to $\mathcal{S}([-\infty,\ +\infty])$). [Hint: It suffices to consider the case $\operatorname{supp} u \subset (\infty,\ -1] \cup [1,\ +\infty)$; use the change of variables (A.14).]

This exercise enables one to define the convolutions $\frac{1}{\lambda} * f$, $\frac{1}{\lambda+i0} * f$ as elements of the space $\mathcal{S}'(\mathbf{R})$ for any generalized function $f \in \mathcal{S}'(\mathbf{R}_\infty)$ (or in $\mathcal{S}'([-\infty,\ +\infty])$) and, in particular, for $f \in \mathcal{S}'([0,\infty])$.

Exercise A.7.(a) Prove the formula

$$\frac{1}{\lambda \pm i0} * l_{n+1}(\lambda) = -\lambda^n,\ n \in \overline{\mathbf{Z}}_+, \tag{A.21}$$

where the $l_{n+1}(\lambda)$ are the generalized functions in $\mathcal{S}'([0,\infty])$ defined by (A.18).

(b) Prove that every complex polynomial $P(\lambda)$ in $\lambda \in \mathbf{R}$ can be uniquely represented as a convolution

$$P(\lambda) = \int \frac{1}{\lambda - \lambda' \pm i0} f(\lambda')d\lambda', \tag{A.22}$$

where $f(\lambda)$ is a generalized function in $\mathcal{S}'([0,\infty])$ with support at the point ∞ and with integral zero. [Hint: Use (A.21).]

To define generalized functions in $\mathcal{S}'(\mathbf{R}_\infty)$, we can use the following particular case of the gluing principle (which, like Proposition 2.4, is proved by means of a suitable partition of unity on $\mathbf{R}_\infty$).

Proposition A.4. *Let $f_1(\lambda)$ and $f_2(\xi)$ be a pair of generalized functions in $\mathcal{S}'(\mathbf{R})$ satisfying the compatibility condition*

$$f_1(\lambda) = \lambda^{-2} f_2\left(\frac{-1}{\lambda}\right) \quad \text{for } \lambda \neq 0. \tag{A.23}$$

Then there exists a (unique) generalized function $f(\lambda) \in \mathcal{S}'(\mathbf{R}_\infty)$ *such that*

$$(f,u) = \int f_1(\lambda)u(\lambda)d\lambda, \qquad \textit{if } u \in \mathcal{S}(\mathbf{R}_\infty) \textit{ and } \operatorname{supp} u \subset \mathbf{R}, \tag{A.24a}$$

$$(f,u) = \int f_2(\xi)u\Big(-\frac{1}{\xi}\Big)d\xi, \quad \textit{if } u \in \mathcal{S}(\mathbf{R}_\infty) \textit{ and } \operatorname{supp} u \subset \mathbf{R}_\infty \backslash \{0\}. \tag{A.24b}$$

Appendix B. The Laplace Transform of Generalized Functions

B.1. The Laplace transform as an analytic function in the complex plane

The classical Laplace transform of a locally integrable function $f(x)$ of the single variable x is defined by the formula

$$\int_0^\infty e^{-kx} f(x)dx \tag{B.1}$$

The essential difference between this and the Fourier transform is that the integral (B.1) is regarded as an analytic function in the complex plane. Because, in the present case, the transformation is one-sided (the integral on the right hand side extending only along the positive semi-axis), the function (B.1) is analytic in the right half-plane $\operatorname{Re} k > 0$, provided that the integral is absolutely convergent for such k. (For this it is enough to suppose that $f(x)$ is bounded in modulus by expressions $c_\epsilon e^{\epsilon x}$ for all $\epsilon > 0$, where c_ϵ may depend on ϵ.)

Here we shall use a slightly different (more modern) definition in which the Laplace transform is written down in a form identical to the Fourier transform. Thus by the Laplace transform of the locally integrable function $f(x)$ in $\mathbf{R}^n$ we mean the function

$$\tilde{f}(k) = \int e^{ikx} f(x)d^n x, \tag{B.2}$$

defined for all those $k \in \mathbf{C}^n$ for which $e^{ikx}f(x)$ is an absolutely integrable function with respect to x. (Since k is complex, the absence of the imaginary unit in the exponent in (B.1) is merely a question of notation.) As in (2.69), the expression kx here stands for

$$kx = \sum_{j=1}^n \epsilon_j k_j x_j \tag{B.3}$$

(where the $\epsilon_j = \pm 1$ are fixed). It is obvious that if $n = 1$ and $f(x)$ vanishes when $x < 0$, then under the substitution of ik by $-k$, the expression (B.2) reverts to the classical Laplace transform of f.

Throughout this appendix we shall write $p = \operatorname{Re} k \in \mathbf{R}^n$ and $q = \operatorname{Im} k \in \mathbf{R}^n$ to denote the real and imaginary parts of the complex vector $k \in \mathbf{C}^n$, so that $k = p + iq$.

It is appropriate to use the complex vector k rather than the pair of real vectors p and q because of the analyticity property of the Laplace transform which we shall be discussing below. Here we shall be requiring only the most elementary facts concerning analytic (or holomorphic) functions of several complex variables. Here is the main definition. By a ***holomorphic function*** in a domain (or, more generally, an open set) $\Omega \subset \mathbf{C}^n$, we mean a complex continuously differentiable function $h(k)$ of the complex vector $k \in \Omega$ that satisfies the *Cauchy-Riemann equations*

$$\frac{\partial}{\partial \bar{k}_j} h(k) \equiv \frac{1}{2}\Big(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial p_j}\Big)h(k) = 0, \quad j = 1, \dots, n \tag{B.4}$$

(which in view of the continuous differentiability means that $h(k)$ is an analytic function in each component k_j for any admissible values of the remaining components). By a *complex analytic** (or

* There is also the notion of ***real analytic function*** which comprises the (real or complex) functions $f(x)$ of a real vector x that can be expanded in a convergent Taylor series in a neighbourhood of each point of its domain of definition.

simply *analytic*) function $h(k)$, we mean a complex function $h(k)$ defined in the domain Ω that can be expanded in a Taylor series which converges in a neighbourhood of each point of the domain. In fact (as we shall see in §5.1.A) the notions of holomorphy and analyticity are equivalent.

It is clear that (B.2) can be interpreted as the Fourier transform $\mathcal{F}_{x\to p}$ of the absolutely integrable function $e^{-qx}f(x)$ depending on q as a parameter. Such an interpretation enables us to extend the Laplace transform to distributions in natural fashion.

Let $f(x)$ be an arbitrary distribution in $\mathcal{D}'(\mathbf{R}^n)$. We associate with it the set * $\Gamma(f)$ of all vectors $q \in \mathbf{R}^n$ such that $e^{-qx}f(x)$ is a generalized function (that is, a tempered distribution) with respect to x:

$$\Gamma(f) = \{q \in \mathbf{R}^n : \ e^{-qx}f(x) \in \mathcal{S}'(\mathbf{R}^n) \text{ with respect to } x\} \tag{B.5}$$

We denote by $\operatorname{int}\Gamma(f) \equiv \mathring{\Gamma}(f)$ the interior of $\Gamma(f)$. By the *Laplace transform*** of the distribution f, we mean the generalized function $\tilde{f}(p+iq)$ with respect to p and depending on the parameter $q \in \Gamma(f)$, defined by the formula $\tilde{f}(p+iq) = \mathcal{F}_{x\to p}[e^{-qx}f(x)]$. The Laplace transform of f is also denoted by $\mathcal{L}_{x\to p+iq}[f(x)]$, or $(\mathcal{L}f)(p+iq)$, or $\mathcal{L}f$. Thus,

$$\mathcal{L}_{x\to p+iq}[f(x)] = \mathcal{F}_{x\to p}[e^{-qx}f(x)] \tag{B.6}$$

or (in accordance with the symbolic integrals (2.76), (2.77))

$$\tilde{f}(k) = \int e^{ikx}f(x)dx. \tag{B.7}$$

As in the case of the Fourier transform, the transforms obtained from (B.7) by changing the sign in the exponent or multiplying the right hand side by $(2\pi)^{-n}$ are also known as the Laplace transform. (The definition of $\Gamma(f)$ is then suitably modified.) For example, in accordance with the definition (2.67) of the Fourier transform of generalized functions in p-space, the Laplace transform of the distribution $g(p) \in \mathcal{D}'(\mathbf{R}_n)$ (in p-space) is defined by the formula

$$(\mathcal{L}^*g)(z) \equiv \mathcal{L}^*_{p\to x+iy}[g(p)] = \int e^{-ipz}g(p)d_np, \tag{B.8}$$

and is to be understood in the same sense as (B.7).

The Laplace transform of distributions is a generalization of that of functions and preserves its main properties.

Exercise B.1. Prove the following properties of the Laplace transform.

(a) Linearity: if $f = \alpha_1 f_1 + \alpha_2 f_2$, where $\alpha_1, \alpha_2 \in \mathbf{C}$, $f_1, f_2 \in \mathcal{D}'(\mathbf{R}^n)$, then $\Gamma(f) \supset \Gamma(f_1)\cap\Gamma(f_2)$ and $\mathcal{L}f = \alpha_1\mathcal{L}f_1 + \alpha_2\mathcal{L}f_2$ for $q \in \Gamma(f_1)\cap\Gamma(f_2)$.

(b) The rule for multiplication by a polynomial and differentiation:

$$\Gamma(x^\alpha f) \supset \Gamma(f), \quad \Gamma(D^\alpha f) \supset \Gamma(f)$$

$$\mathcal{L}_{x\to k}[x^\alpha f(x)] = (-i\epsilon D_p)^\alpha \mathcal{L}_{x\to k}[f(x)], \tag{B.9}$$

$$\mathcal{L}_{x\to k}[D^\alpha f(x)] = (-i\epsilon k)^\alpha \mathcal{L}_{x\to k}[f(x)]; \tag{B.10}$$

here the notation is the same as in Exercise 2.15.

(c) Behaviour under inhomogeneous linear transformations of the arguments:

$$\mathcal{L}_{x\to k}f(x-a) = e^{ika}\mathcal{L}_{x\to k}f(x), \ a \in \mathbf{R}^n, \tag{B.11}$$

$$\mathcal{L}_{x\to k}[e^{iax}f(x)] = \mathcal{L}_{x\to k+a}f(x), \ a \in \mathbf{R}^n, \tag{B.12}$$

$$\mathcal{L}_{x\to k}f(A^{-1}x) = |\det A| \cdot \mathcal{L}_{x\to A'k}f(x), \quad \text{where } \det A \neq 0 \tag{B.13}$$

(the notation of Exercise 2.18 is used in the last formula).

* Not excluding the possibility that $\Gamma(f)$ is the empty set.

** Sometimes called the *Fourier-Laplace transform*.

Exercise B.2. Prove the inversion formula for the Laplace transform: if

$$\tilde{f}(p+iq) = \mathcal{L}_{x\to p+iq} f(x), \tag{B.14a}$$

then

$$f(x) = e^{qx} \mathcal{F}^{-1}_{p\to x} \tilde{f}(p+iq). \tag{B.14b}$$

The sets $\Gamma(f)$ have the following important property.

Proposition B.1. *For each distribution $f \in \mathcal{D}'(\mathbf{R}^n)$, the set $\Gamma(f)$ is convex.*

■ Let $a, b \in \Gamma(f)$ and $c = ta + (1-t)b$ for $0 \le t \le 1$. Then

$$e^{-cxf}(x) = u_c(x)[e^{-ax} f(x) + e^{-bx} f(x)],$$

where

$$u_c(x) = (e^{-ax} + e^{-bx})^{-1} e^{-cx}.$$

If we show that u_c is a multiplicator in $\mathcal{S}(\mathbf{R}^n)$, then it will follow from the above relation that $e^{-cx} f(x) \in \mathcal{S}'(\mathbf{R}^n)$ and hence $c \in \Gamma(f)$. Clearly, $|u_c(x)| \le 1$. Furthermore, the first derivatives of $u_c(x)$ are polynomials of functions of the same form as u_c:

$$\frac{\partial}{\partial x_j} u_c(x) = \epsilon_j u_c(x)\{-c_j + a_j u_a(x) + b_j u_b(x)\};$$

consequently they are also bounded. It is now easily verified by induction on the order of the derivative that any derivative of u_c is bounded and hence u_c is a multiplicator in $\mathcal{S}(\mathbf{R}^n)$. ■

An important property of the Laplace transform is its analyticity. Namely, $\mathcal{L}_{x\to k} f(x)$ is an analytic function of k in the open set $\mathbf{R}^n + i\mathring{\Gamma}(f)$. A subset of $\mathbf{C}^n$ of the form

$$\mathcal{T}^K \equiv \mathbf{R}^n + iK = \{p + iq \in \mathbf{C}^n : \; p \in \mathbf{R}^n, \; q \in K\}, \tag{B.15}$$

where K is a fixed set (or, more particularly, a domain) in $\mathbf{R}^n$, is called a *tube* (or a *tube domain*) in $\mathbf{C}^n$; we call K the *base* of the tube $\mathcal{T}^K$, also denoted by $\operatorname{Im} \mathcal{T}^K$. Thus the above result means that if $\mathring{\Gamma}(f)$ is non-empty, then $\mathcal{L}f$ is analytic in the (convex) tube domain $\mathring{\Gamma}(f)$.

The description of the analytic properties of the Laplace transform given below in Theorem B.2 is complete in the sense that there is a converse Theorem B.4 giving criteria for the representability of an analytic function in a tube domain as the Laplace transform of some distribution.

Theorem B.2. *Let $f(x)$ be a distribution in $\mathcal{D}'(\mathbf{R}^n)$ such that $\mathring{\Gamma}(f)$ is non-empty. Then the Laplace transform $\tilde{f}(k)$ is infinitely differentiable and, furthermore, is a (complex) analytic function of k in the tube domain $\mathbf{R}^n + i\mathring{\Gamma}(f)$, where for each compactum $Q \subset \mathring{\Gamma}(f)$, there exist numbers A, $m > 0$ (depending on Q and, of course, on f) such that*

$$|\tilde{f}(p+iq)| \le A(1+|p|)^m \quad \textit{for all}\; p \in \mathbf{R}^n,\; q \in Q. \tag{B.16}$$

■ Let a be an arbitrary fixed point of $\mathring{\Gamma}(f)$. Since $\mathring{\Gamma}(f)$ is an open set, there exists $\rho > 0$ such that the cube P' in $\mathbf{R}^n$ defined by the inequalities $|q_j - a_j| \le \rho$ $(j = 1, \dots, n)$, belongs to $\mathring{\Gamma}(f)$. Let P be the smaller cube defined by the inequalities $|q_j - a_j| < \rho/2$ $(j = 1, \dots, n)$. We claim that $\mathcal{L}f$ is analytic in $\mathbf{R}^n + iP$.

For we have the identity

$$e^{-qx} = u_q(x)\left[e^{-ax} \prod_{j=1}^{n} (e^{-\rho x_j} + e^{\rho x_j}\right], \tag{B.17}$$

where

$$u_q(x) = \prod_{j=1}^{n} \left[(e^{-\rho x_j} + e^{\rho x_j})^{-1} e^{-\epsilon_j (q_j - a_j) x_j}\right]. \tag{B.18}$$

If $q \in P$, then $|\epsilon_j(q_j - a_j)| < \rho$ and, clearly, $u_q \in \mathcal{S}(\mathbf{R}^n)$. The terms in the expression in square brackets in (B.17) can be expressed in the form e^{-bx} with $b \in P'$ and hence its product with $f(x)$ is a tempered distribution which we denote by $f_0(x)$. Thus for $q \in P'$, we have $e^{-qx}f(x) = u_q(x)f_0(x)$. We apply the equality for the Fourier transform operator $\mathcal{F}_{x\to p}$ and use formula (2.87); we then obtain

$$\tilde{f}(p+iq) = \mathcal{F}_{x\to p}[e^{-qx}f(x)] = \mathcal{F}_{x\to p}[u_q(x)f_0(x)] = (f_0(x), e^{ipx}u_q(x)) = (f_0(x), v_{p+iq}(x)),$$

where

$$v_k(x) = \prod_{j=1}^{n}(e^{-\rho x_j} + e^{\rho x_j})^{-1}e^{i\epsilon_j k_j x_j}.$$

It is clear that $v_k(x)$ is a function of x in $\mathcal{S}(\mathbf{R}^n)$ for $k \in \mathcal{T}^{P'}$ that is infinitely differentiably dependent on the parameter k; hence, $\tilde{f}(k)$ is an infintely differentiable function of k. Furthermore $\frac{\partial v_k(x)}{\partial \bar{k}_j} \equiv$ $\equiv \frac{1}{2}\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q_j}\right)v_{p+iq}(x) = 0$ for $j = 1, \ldots, n$. It follows that $\tilde{f}(k)$ is a holomorphic function of k in the tube $\mathcal{T}^P$ and hence throughout $\mathbf{R}^n + i\mathring{\Gamma}(f)$ (since the point a is arbitrary).

It remains to obtain the estimate for $\tilde{f}(k)$. Let c, l, m be such that

$$|(f_0, u) \le c\|u\|_{l,m} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n).$$

Then for $k \in \mathbf{R}^n + iP$ we have

$$|\tilde{f}(k)| \le c\|v_k\|_{l,m} \le A(1+|p|)^m \quad \text{for } k \in \mathcal{T}^P$$

(where A, m depend on P). It is clear that this estimate holds in any tube $\mathcal{T}^K$, where any compactum K in $\mathring{\Gamma}(f)$ can be used as the base, since each such compactum can be covered by a finite number of open cubes of the above form. ■

Combining Theorem B.2 and Proposition B.1, we obtain the following important corollary.

Corollary B.3. Let S_1 and S_2 be two domains in $\mathbf{R}^n$ and h_1 and h_2 two holomorphic functions in the tube domains $\mathcal{T}^{S_1}$ and $\mathcal{T}^{S_2}$ respectively, and suppose that in these domains they are Laplace transforms of the same distribution $f \in \mathcal{D}'(\mathbf{R}^n)$. Then there exists a holomorphic function in $\mathcal{T}^S$, where S is the convex hull of $S_1 \cup S_2$, such that h coincides with h_j in $\mathcal{T}^{S_j}$ $(j = 1, 2)$.

As noted above, the following theorem can be regarded as the converse of Theorem B.2.

Theorem B.4. *Let h be a holomorphic function in the tube domain $\mathcal{T}^S$ in $\mathbf{C}^n$, where the following estimate holds for any compactum $Q \subset S$:*

$$|h(p+iq)| \le A(1+|p|)^m \textit{ for all } p \in \mathbf{R}^n,\ q \in Q \tag{B.19}$$

(*where the numbers A, m depend on Q*). *Then there exists a distribution $f \in \mathcal{D}'(\mathbf{R}^n)$* (*which is clearly unique*)*such that $\Gamma(f) \supset S$ and $h(k) = (\mathcal{L}f)(k)$ in $\mathcal{T}^S$.*

■ The crucial point of the proof is that $h(p+iq)$ can be regarded as a distribution in $\mathcal{S}'(\mathbf{R}^n)$ with respect to p that is continuously differentiably (and even $\mathcal{C}^\infty$-) dependent on the parameter $q \in S$, where the generalized derivatives with respect to the argument p and the derivatives with respect to the parameter q (in $\mathcal{S}'(\mathbf{R}^n)$) are the same as the corresponding ordinary derivatives.

The fact that $h(p+iq)$ is a tempered distribution with respect to p for each fixed $q \in S$ follows from the fact that h is of polynomial growth in p. As in the proof of Theorem B.2, we now construct for each fixed point $a \in S$, two cubes with centre at a and contained in S, namely P' (closed cube with edge of length ρ) and P (open cube with edge of length $\rho/2$). By hypothesis, we have the estimate

$$|h(p+iq)| \le A(1+|p|)^m \quad \text{for } p+iq \in \mathcal{T}^{P'},$$

from which (by Cauchy's theorem) we obtain the representation

$$h(k) = \frac{1}{2\pi i}\int_C \left(\frac{k_j - i(a_j+2\rho)}{\zeta_j - i(a_j+2\rho)}\right)^{m+2} \frac{h(k_1, \ldots, \zeta_j, \ldots, k_n)}{\zeta_j - k_j} d\zeta_j$$

for $k \in \mathcal{T}^P$, where C is the contour, oriented in usual fashion, in $\mathbf{C}$ consisting of the two straight lines forming the boundary of the strip $a_j - \rho < \operatorname{Im}\zeta_j < a_j + \rho$. If we differentiate this representation with respect to p_j and q_j, we see that $\frac{\partial}{\partial p_j}h$ and $\frac{\partial}{\partial q_j}h$ are of polynomial growth in $\mathcal{T}^P$; therefore the generalized derivative with respect to p and the derivative with respect to the parameter q in $\mathcal{S}'(\mathbf{R}^n)$ of $h(p+iq)$ coincides with the ordinary derivatives with respect to p and q respectively. It now follows from the holomorphy of $h(k)$ that

$$\left(\frac{\partial}{\partial p_j} - i\frac{\partial}{\partial q_j}\right)h(p+iq) = 0 \quad \text{in} \quad \mathcal{S}'(\mathbf{R}^n)$$

for $q \in P$ and hence for all $q \in S$.

Applying the Fourier operator $\mathcal{F}^{-1}_{p\to x}$, we obtain

$$\left(x_j + \frac{\partial}{\partial q_j}\right)\mathcal{F}^{-1}_{p\to x}[h(p+iq)] = 0. \tag{B.20}$$

We now introduce the distribution $e^{qx}\mathcal{F}^{-1}_{p\to x}[h(p+iq)]$ in $\mathcal{D}'(\mathbf{R}^n)$ with respect to x, which is C^∞-dependent on the parameter $q \in S$. Equality (B.20) means that all the first partial derivatives of this distribution with respect to the parameter q vanish and hence (since S is connected), this distribution does not, in fact, depend on the parameter q. We have proved that there exists a distribution $f(x) \in \mathcal{D}'(\mathbf{R}^n)$ such that

$$e^{qx}\mathcal{F}^{-1}_{p\to x}[h(p+iq)] = f(x) \quad \text{for all } q \in S,$$

that is,

$$\mathcal{F}^{-1}_{p\to x}[h(p+iq)] = e^{-qx}f(x) \quad \text{for all } q \in S.$$

This completes the proof. ∎

In the next exercise we give alternative statements of the condition of Theorem B.4 (for representing a holomorphic function in a tube as the Laplace transform of a distribution).

Exercise B.3. Let $h(p+iq)$ be a holomorphic function in the tube $\mathcal{T}^S$, where S is a domain in $\mathbf{R}^n$. Prove the equivalence of the following conditions:

(1) the expression $h(p+iq)$ defines a family of tempered distributions * in $\mathcal{S}'(\mathbf{R}^n)$ with respect to the variable p, that are continuously dependent on the parameter $q \in S$;

(2) for each compactum $Q \subset S$, there exist numbers $c \geq 0, l, m \in \overline{\mathbf{Z}}_+$ (depending on Q) such that

$$\left|\int h(p+iq)u(p)dp\right| \leq c\|u\|_{l,m} \quad \text{for all } u \in D(\mathbf{R}^n),\ q \in Q_j; \tag{B.21}$$

(3) the estimate (B.19) holds for each compactum K in S.

[Hint: The implications (1) ⇒ (2) and (3) ⇒ (1) follow respectively from Exercise 2.43 and the proof of Theorem B.4; for the proof of the implication (2) ⇒ (3), we use the mean value formula for the holomorphic function:

$$h(z) = \int h(z+p+iq)\omega(|p+iq|^2)d^npd^nq; \tag{B.22}$$

here $\omega(x)$ is a function in $\mathcal{D}(\mathbf{R})$ such that

$$\int \omega(|p+iq|^2)d^npd^nq = 1; \tag{B.23}$$

for (B.22) to hold, we have to assume, of course, that $p+iq \in \mathcal{T}^S$ as $p+iq$ runs through the domain of integrability.]

* Here we are not supposing in advance that $h(p+iq)$ is of polynomial growth in p for each fixed $q \in S$, although this follows from the equivalence of conditions (1) and (3). In condition (1), we have in mind that the expression $h(p+iq)$, initially regarded as a distribution in $\mathcal{D}'(\mathbf{R}^n)$ with respect to p, admits a continuous extension to a tempered distribution (for each fixed $q \in S$).

We now give another special version of Theorems B.2 and B.4 for the Laplace transform of tempered distributions f (that is, generalized functions in $\mathcal{S}'(\mathbf{R}^n)$). If the point $q = 0$ is contained in $\mathring{\Gamma}(f)$ (or in the set S of Theorem 3.3), then, of course, this case is already covered by Theorems 3.2 and 3.3 (since the distribution f is then automatically tempered). Therefore the specific features only arise when 0 is a boundary point of $\Gamma(f)$ (or of S). It turns out that the property that the distribution f be tempered is equivalent to some specific restriction on the growth of $(\mathcal{L}f)(p+iq)$ as $q \to 0$ in $\mathring{\Gamma}(f)$.

By a *truncated cone* in $\mathbf{R}^n$, we mean a set of the form

$$K^r = \{x \in K : |x| \leq r\}, \tag{B.24}$$

where K is a cone in $\mathbf{R}^n$ with vertex at the origin and r is a positive number.

Theorem B.5. *Let S be a convex domain in $\mathbf{R}^n$ for which 0 is a boundary point. Then a holomorphic function h in the tube domain T^S is equal to the Laplace transform of some generalized function $f \in \mathcal{S}'(\mathbf{R}^n)$ (with $\Gamma(f) \supset S$) if and only if the following estimate holds for each compact truncated cone $K^r \subset S \cup \{0\}$:*

$$|h(p+iq)| \leq A\frac{(1+|p|)^m}{|q|^l} \quad \textit{for all } p \in \mathbf{R}^n,\ q \in K^r \backslash \{0\} \tag{B.25}$$

(*where the numbers A, m, l depend on K^r*).

■ We begin by supposing that $h = \mathcal{L}f$ in T^Q, where f is a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ (with $\Gamma(f) \supset S$). We derive (B.25). In fact, instead of the truncated cones K^r with compact closure in $S \cup \{0\}$, it suffices to consider the open parallelepipeds $Q \subset S$ contained in the convex hulls of the $(n+1)$-point sets $\{0, a_1, \ldots, a_n\}$, where $a_1, \ldots, a_n$ are arbitrary linearly independent points of S:

$$Q = \Big\{x \in \mathbf{R}^n : x = \sum_{j=1}^{n} \xi_j a_j, \quad \text{where } 0 < \xi_j < \frac{1}{n}(j = 1, \ldots, n)\Big\}. \tag{B.26}$$

The fact that Q is contained in S follows from the fact that S is a convex open set whose boundary contains 0. We can confine our attention to the parallelepipeds Q because each compact truncated cone $K \subset S \cup \{0\}$ can be covered by a finite number of such parallelepipeds Q. In fact the intersection of $\operatorname{clos} K^r$ with the sphere $|x| = r$ is a compactum contained in S and since the Q in totality cover K, we can apply the Heine-Borel theorem to choose a finite family covering the given compactum; this family also covers K^r, since $\rho Q \subset Q$ for all $0 < \rho < 1$.

The subsequent argument is almost a repetition of the proof of Theorem B.2. We shall characterize an arbitrary point $q \in Q$ by its coordinates $\eta_1, \ldots, \eta_n$ with respect to the basis $a_1, \ldots, a_n$ (as in (B.26)). We have the identity of type (B.17):

$$e^{-qx} = u_q(x) \prod_{j=1}^{n} \left(1 + e^{-\frac{1}{n}(a_j x)}\right),$$

where

$$u_q(x) = \prod_{j=1}^{n} e^{-\eta_j (a_j x)} \left(1 + e^{-\frac{1}{n}(a_j x)}\right)^{-1}.$$

Next, as in the proof of Theorem B.2, $\mathcal{L}f$ in T^Q has a representation of the form

$$(\mathcal{L}f)(p+iq) = (g(x),\ v_{p+iq}(x)),$$

where g is a fixed distribution in $\mathcal{S}'(\mathbf{R}^n)$ and

$$v_{p+iq} = \prod_{j=1}^{n} e^{i(\xi_j + i\eta_j)(a_j x)} \left(1 + e^{-\frac{1}{n}(a_j x)}\right)^{-1}$$

is a function in x in $\mathcal{S}(\mathbf{R}^n)$ which is holomorphically dependent on the parameter $p+iq$. (Here the ξ_j are the coordinates of p in the basis $a_1, \ldots, a_n$.) From this we easily obtain an estimate of type

$$|(\mathcal{L}f)(p+iq)| \leq B \frac{(1+|p|)^m}{(\eta_1 \cdots \eta_n)^s} \quad \text{for } p+iq \in \mathcal{T}^Q. \tag{B.27}$$

If now we choose instead of Q the somewhat larger parallelepiped Q' with compact closure in $Q' \cup \{0\}$, then we have the estimate $\eta_j' \geq \alpha|q|$ (for some $\alpha > 0$ depending on Q and Q') for the coordinates η_j' of the points $q \in Q$ in the new basis $a_1', \ldots, a_n'$. Since an estimate of type (B.27) (with η_j' instead of η_j) holds in the tube $\mathcal{T}^{Q'}$, we obtain the required estimate of type (B.25) in $\mathcal{T}^Q$.

Conversely, let h be a given holomorphic function in the tube $\mathcal{T}^S$ satisfying an estimate of type (B.25) for each compact truncated cone $K^r \subset S \cup \{0\}$. Then by Theorem B.4, h coincides in $\mathcal{T}^S$ with the Laplace transform of a distribution f in $\mathbf{R}^n$ with $\Gamma(f) \supset S$, so that the problem reduces to proving that f is tempered.

To this end, we estimate (f,u) for arbitrary $u \in \mathcal{D}(\mathbf{R}^n)$. Taking a to be a fixed point of S, we write down the formula of type (B.14b):

$$f(x) = e^{tax} \mathcal{F}^{-1}_{p \to x}[h(p+ita)] \quad \text{for all } 0 < t \leq 1, \tag{B.28}$$

from which it follows that $(f,u) = \int h(p+ita) \mathcal{T}^{-1}_{p \to x}[e^{tax}u(x)] d^n p$. This together with the estimate (B.25) (for the case when K^r is the interval joining 0 and a) gives:

$$|(f,u)| \leq ct^{-L} \sup_p \left|(1+|p|^2)^M \mathcal{F}^{-1}_{p\to x}[e^{tax}u(x)]\right| = ct^{-L} \sup_p \left|\mathcal{F}^{-1}_{p\to x}[(1-\Delta)^M (e^{tax}u(x))]\right|$$

for all $u \in \mathcal{D}(\mathbf{R}^n)$, $0 < t \leq 1$; here $c \geq 0$ and $L, M \in \overline{\mathbf{Z}}_+$ are certain numbers (depending on f and a); we denote by Δ the Laplacian

$$\Delta = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j}\right)^2. \tag{B.29}$$

This estimate can clearly be rewritten in the form

$$|(f,u)| \leq c' t^{-L'} \sup_{|\alpha| \leq 2M} \sup_x |(1+|x|)^{n+1} e^{tax} D^\alpha u(x)| \tag{B.30}$$

for all $u \in \mathcal{D}(\mathbf{R}^n)$, $0 < t \leq 1$.

We now consider functions $u \in \mathcal{D}(\mathbf{R}^n)$ concentrated on a set of the form $k-2 \leq |x| \leq k$ (for some $k = 1, 2, \ldots$). If we set $t = 1/k$, then (B.30) clearly leads to the estimate

$$|(f,u)| \leq B\|u\|_{\lambda,\mu} \text{ for all } u \in \mathcal{D}(\mathbf{R}^n) \text{ with supports in } k-2 \leq |x| \leq k; \tag{B.31}$$

here $B \geq 0$ and $\lambda, \mu \in \overline{\mathbf{Z}}_+$ depend only on f and not on k. To obtain such an estimate for all $u \in \mathcal{D}(\mathbf{R}^n)$, it suffices to take a partition of unity $\{e_k\}_{k=1}^\infty$ in $\mathbf{R}^n$ subordinated to the covering of $\mathbf{R}^n$ by sets of the form $k-2 < |x| < k$ ($k = 1, 2, \ldots$) such that the $\|e_k\|_{\lambda,\mu}$ are of polynomial growth in k. If we now express the derivative $u \in \mathcal{D}(\mathbf{R}^n)$ in the form $u = \sum_k e_k u$ and use (B.31), we obtain an estimate of type $|(f,u)| \leq B'\|u\|_{\lambda',\mu'}$ for all $u \in \mathcal{D}(\mathbf{R}^n)$. By Proposition 2.1, this implies that $f \in \mathcal{S}'(\mathbf{R}^n)$. ∎

The next exercise (which is similar to Exercise B.3) contains other reformulations of the condition in Theorem B.5 for the representability of a holomorphic function in a tube as the Laplace transform of a tempered distribution.

Exercise B.4. Let $h(p+iq)$ be a holomorphic function in the tube $\mathcal{T}^S$, where S is a convex domain in $\mathbf{R}^n$ having 0 as a boundary point. Prove the equivalence of the following conditions:

(1) for each compact truncated cone $K^r \subset S \cup \{0\}$, there exist numbers $c \geq 0$, $l, m \in \overline{\mathbf{Z}}_+$ (depending on K^r) such that an estimate of type (B.21) holds with $q \in K^r \backslash \{0\}$;

(2) for each compact truncated cone $K^r \subset S \cup \{0\}$, an estimate of type (B.25) holds.

[Hint: For the proof of the implication (1) $\Rightarrow$ (2), use the formula of type (B.22) where $\omega(|p+iq|^2)$ is replaced by $|q|^{-2n}\omega\left(\frac{1}{|q|^2}|p+iq|^2\right)$ under a suitable choice of ω.]

Remark. In the preceding theorems we have been dealing with distributions f in $\mathbf{R}^n$ for which the set $\Gamma(f)$ has a non-empty interior. The general case (when $\Gamma(f)$ is non-empty but has an empty interior) reduces to the special case by means of the so-called partial Laplace transform (that is, the Laplace transform with respect to some of the independent coordinate variables in $\mathbf{R}^n$). We shall not develop the more general point of view here, but restrict ourselves to extending the previous theorem for a special case (which allows $\Gamma(f)$ to have an empty interior).

Proposition B.6. *Let $h(p+iq)$ be an infinitely differentiable function (with respect to the variables $p_1, \dots, p_n, q_1, \dots, q_n$) in the tube $\mathcal{T}^\Omega$, the base of which is*

$$\Omega = \{q \in \mathbf{R}^n : 0 < q_j < 1 \ for\ j = 1, \dots, \nu; \quad q_j = 0 \ for\ j = \nu+1, \dots, n\} \tag{B.32}$$

(where $\nu \in \mathbf{Z}_+$ is fixed, $1 \le \nu \le n$), $h(p+iq)$ being holomorphic with respect to the variables $p_1 + iq_1, \dots, p_\nu + iq_\nu$ and satisfying the estimate (for some A, m, l)

$$|h(p+iq)| \le A(1+|p|)^m |q|^{-l}. \tag{B.33}$$

Then there exists $f \in \mathcal{S}'(\mathbf{R}^n)$ such that $\Gamma(f) \supset \Omega$ and $h(p+iq) = (\mathcal{L}f)(p+iq)$ in $\mathcal{T}^\Omega$.

■ The existence of $f \in \mathcal{D}'(\mathbf{R}^n)$ such that $\Gamma(f) \supset \Omega$ and $h = \mathcal{L}f$, is established by the same arguments as in Theorem B.4; the fact that $f \in \mathcal{S}'(\mathbf{R}^n)$ is proved in exactly the same way as at the corresponding place in Theorem B.5. ■

B.2. THE CASE OF A GENERALIZED FUNCTION WITH SUPPORT IN A POINTED CONE

We recall that the existence (in the sense that its domain of analyticity is non-empty) of the classical Laplace transform (B.1) of, say, a function f of polynomial growth is guaranteed by the one-sidedness of this transform (that is, that the range of integration in (B.1) is a half-line). The n-dimensional analogue of the half-line is obtained by considering closed simplicial cones or, more generally, closed pointed cones in $\mathbf{R}^n$ (see §2.5.E). For the cone $K \subset \mathbf{R}^n$ we define the *dual cone* (with respect to the bilinear form $xy \equiv \sum_j \epsilon_j x_j y_j$) K^* by the formula

$$K^* = \{q \in \mathbf{R}^n : \ xq \ge 0 \quad \text{for all } x \in K\}. \tag{B.34}$$

Exercise B.5. Let K be a non-empty cone in $\mathbf{R}^n$. Prove the following:

a) The cone K has the same dual cone as that of its closed convex hull. (Here, by the closed convex hull we mean, more precisely, the closure of the convex hull of K.)

b) K^{**} is the closed convex hull of K. [Hint: It follows from part (a) that $K^{**} = (\hat{K})^{**}$, where $\hat{K}$ is the closed convex hull of K. For the proof that $\hat{K}^{**} = \hat{K}$, use the geometric theorem ([S2],p.65) which states that every non-empty closed convex subset of Euclidean space is the intersection of all closed linear subspaces containing it.]

c) The cone K^* has a non-empty interior if and only if K is a pointed cone.

Exercise B.6. a) Let K_1 and K_2 be closed convex polyhedral cones. Prove that the closed convex hull $K_1 + K_2$ of $K_1 \cup K_2$ is also a convex polyhedral cone. [Hint: K_1 (and K_2) is the linear span of a finite number of elements of it.]

(b) Prove that the dual of a closed convex polyhedral cone is a closed convex polyhedral cone. [The hint is the same as for part (a).]

It turns out that the Laplace transform of an arbitrary generalized function $f(x)$ with support in a pointed cone K is analytic in a tube with base $\operatorname{int} K^*$ (the interior of the dual cone to K). The role of this condition shows up very well in the following example. Let $f(x)$ be a continuous function of polynomial growth in $\mathbf{R}^n$ that vanishes outside a pointed cone K and therefore defines a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with support in K. Then for each $q \in K^*$, the exponential e^{-qx} is bounded on K, therefore $e^{-qx} f(x)$ is also a continuous function of x of polynomial growth. Hence its Laplace transform is defined for all $q \in K^*$, that is, $\Gamma(f) \supset K^*$. In this example, it can be seen directly that $\mathcal{L}f$ is analytic in the tube with base $\operatorname{int} K^*$.

We now turn to a characterization of the Laplace transform of generalized functions with supports in a closed pointed cone K. Prior to this we note that (by virtue of Exercise B.5) the cone K^* is, in essence, defined by just the closed convex hull of K, therefore we may suppose that K is a closed convex pointed cone. We may further suppose that K has a non-empty interior and hence (by Exercise A.1) is a canonically closed regular subset of $\mathbf{R}^n$. This suffices for our purposes (although the more general case (without the latter hypothesis) can easily be reduced to this special case).

Theorem B.7. *If $f(x)$ is a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with support in the closed convex pointed cone $K \subset \mathbf{R}^n$ with non-empty interior, then $\Gamma(f) \supset K^*$ and the Laplace transform is a holomorphic function in the tube domain with base $\operatorname{int} K^*$ and satisfies the estimate*

$$|\mathcal{L}(f)(p+iq)| \leq A\frac{(1+|p+iq|)^m}{d(q,\partial K^*)^l} \quad \textit{for} \quad p \in \mathbf{R}^n,\ q \in \operatorname{int} K^*; \tag{B.35}$$

here A, l, m are numbers that depend only on f and $d(q, \partial K^*)$ is the distance from q to the boundary of K^*. Furthermore the Fourier transform $\mathcal{F}f$ of f is the (generalized) boundary value (as $q \to 0$, $q \in \operatorname{int} K^*$) of the Laplace transform in the following sense:

$$\lim_{q\to 0, q\in \operatorname{int} K^*} (\mathcal{L}f)(p+iq) = (\mathcal{F}f)(p) \textit{ in } \mathcal{S}'(\mathbf{R}^n). \tag{B.36}$$

■ Since K is a canonically closed regular set, we can (by Proposition A.1) associate with the given generalized function f the generalized function $F(x) \in \mathcal{S}'(K)$ such that

$$(f,u) = (F, u|_K), \quad u \in \mathcal{S}(\mathbf{R}^n). \tag{B.37}$$

Let $q \in K^*$. Then the function e^{-qx} is a multiplicator in $\mathcal{S}(K)$, therefore the generalized function $e^{-qx}F(x) \in \mathcal{S}'(K)$ is defined with respect to x. It is now obvious from (B.37) that $e^{-qx}u(x)$ is also a generalized function in $\mathbf{R}^n$ such that

$$(e^{-qx}f(x), u(x)) = (F(x), e^{-qx}u|_K(x)), \quad u \in \mathcal{S}(\mathbf{R}^n),\ q \in K^*. \tag{B.38}$$

Next, let $q \in \operatorname{int} K^*$. We set $r = d(q, \partial K^*)$. Then $(q-y)x \geq 0$ for all $x \in K$ and $y \in \mathbf{R}^n$ with $|y| \leq r$; hence

$$qx \geq \sup_{y:|y|\leq r} yx = r|x| \quad \text{for all } x \in K. \tag{B.39}$$

This estimate implies that e^{-qx} is a test function in $\mathcal{S}(K)$ with respect to x (for $q \in \operatorname{int} K^*$) and hence (by the characterization of $\mathcal{S}(\Omega)$ in Appendix A) there exists a function of x in $\mathcal{S}(\mathbf{R}^n)$, say, $u_q(x)$ that is equal to e^{-qx} when $x \in K$. It follows from (B.38) that $e^{-qx}f(x) = u_q(x)f(x)$, whence (and from Exercise 2.21) it follows that the Fourier transform of $e^{-qx}f(x)$ is the following (ordinary) function:

$$\mathcal{F}_{x\to p}[e^{-qx}f(x)] = \mathcal{F}_{x\to p}[u_q(x)f(x)] = (f(x),\ e^{ipx}u_q(x)).$$

This, together with (B.37) gives us a formula for the Laplace transform of f:

$$\mathcal{L}_{x\to k}[f(x)] = (F(x), e^{ikx}|_K), \quad q \in \operatorname{int} K^*. \tag{B.40}$$

The remaining assertions of Theorem B.7 now follow easily. In fact since for $q \in \operatorname{int} K^*$, e^{ikx} is a test function of x in $\mathcal{S}(K)$ which is analytically dependent on the parameter k, it follows from the continuity of the functional F over $\mathcal{S}(K)$ that the right hand side of (B.40) is also analytic in k. In fact, the continuity of F over $\mathcal{S}(K)$ is in accordance with the estimate

$$|(F(x),\ u(x))| \leq A'\|u\|^K_{l',m'} \quad \text{for all } u \in \mathcal{S}(K);$$

which along with (B.40) gives the following estimate for $\mathcal{L}f$:

$$|(\mathcal{L}f)(K)| \leq A'\|e^{ikx}\|^K_{l',m'},\ q \in \operatorname{int} K^*. \tag{B.41}$$

Using (B.39), we now obtain an estimate of type (B.35). Finally, it follows from (B.38) that $e^{-qx}f(x) \to$ $\to f(x)$ in $\mathcal{S}'(\mathbf{R}^n)$ as $q \to 0$ in $\operatorname{int} K^*$. This is equivalent to (B.36). ■

It should be noted that Theorem B.7 gives a complete characterization of the Laplace transform of generalized functions with support in K, since the following converse theorem holds.

Theorem B.8. *Let K be a non-empty closed pointed cone in $\mathbf{R}^n$ (with non-empty interior), and h an analytic function in the tube domain with base* $\operatorname{int} K^*$, *satisfying an estimate of type* (B.35). *Then there exists a (unique) generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ with support in K such that $\Gamma(f) \supset K^*$ and $h = \mathcal{L}f$.*

■ The fact that (in the tube with base $\operatorname{int} K^*$) f coincides with the Laplace transform of a tempered distribution f is a direct consequence of Theorems B.4 and B.5. It merely remains to prove that f has support in K.

We need to check that for an arbitrary fixed point x_0 in $\mathbf{R}^n \backslash K$, there exists a neighbourhood $\mathcal{N}$ of x_0 such that $(f, u) = 0$ for all $u \in \mathcal{D}(\mathbf{R}^n)$ with support in $\mathcal{N}$. Using the fact that $K = K^{**}$ (see Exercise B.5), we can fix $a \in \operatorname{int} K^*$ such that $ax_0 = -2$. We then take the half-space $ax \leq -1$ as our neighbourhood $\mathcal{N}$ of x_0. We now use (B.28), only this time we suppose that $t \geq 1$. Using an estimate of type (B.35) and carrying out the subsequent estimates as in the proof of Theorem B.5, we arrive at the following estimate of type (B.30):

$$|(f,u)| \leq ct^s \sup_{|\alpha| \leq 2M} \sup_x |(1+|x|)^{n+1} e^{tax} D^\alpha u(x)| \tag{B.42}$$

for all $u \in \mathcal{D}(\mathbf{R}^n)$, $t \geq 1$. If the function $u \in \mathcal{D}(\mathbf{R}^n)$ has support in $\mathcal{N}$, then we obtain the estimate

$$|(f,u)| \leq c' t^s e^{-t} \|u\|_{n+1,2M} \quad \text{for all } t \geq 1.$$

By taking the lower bound with respect to t for $t \geq 1$ in this inequality, we obtain $(f, u) = 0$. ■

B.3. Example: Generalized functions of retarded type

The following example will be of interest in the later chapters. We consider the space $\mathbf{R}^{4n}$ the points of which are written as sets $x \equiv (x_1, \dots, x_n)$ of vectors x_j in $\mathbf{R}^4$ with coordinates x_j^μ ($j = 1, \dots, n$; $\mu = 0, 1, 2, 3$). Along with the ordinary Euclidean scalar product

$$(x, y) = \sum_{j,\mu} x_j^\mu y_j^\mu \tag{B.43}$$

in $\mathbf{R}^{4n}$ we also introduce the pseudo-Euclidean scalar product

$$xy = \sum_{j,\mu} g_{\mu\mu} x_j^\mu y_j^\mu, \tag{B.44}$$

where $g_{\mu\mu} = 1$ for $\mu = 0$ and $g_{\mu\mu} = -1$ for $\mu = 1, 2, 3$. Thus $\mathbf{R}^{4n}$ is in fact the direct product $\mathbf{M}^n$ of n copies of Minkowski space (see §3.1.A for details). In Minkowski space $\mathbf{M}$, a distinguished role is played by the open (V^+) or closed ($\overline{V}^+$) upper light cone:

$$V^+ = \Big\{ x \in \mathbf{M} : \ x^0 > |\boldsymbol{x}| \equiv \Big(\sum_{\mu=1}^{3} (x^\mu)^2 \Big)^{1/2} \Big\}, \tag{B.45}$$

$$\overline{V}^+ = \{ x \in \mathbf{M} : \ x^0 \geq |\boldsymbol{x}| \}. \tag{B.46}$$

Similarly for $\mathbf{M}^n$, we introduce the direct product of n copies of V^+ or $\overline{V}^+$:

$$(V^+)^n = \{ (x_1, \dots, x_n) \in \mathbf{M}^n : \ x_j \in V^+ \quad \text{for } j = 1, \dots, n \}, \tag{B.47}$$

$$(\overline{V}^+)^n = \{ (x_1, \dots, x_n) \in \mathbf{M}^n : \ x_j \in \overline{V}^+ \quad \text{for } j = 1, \dots, n \} \tag{B.48}$$

Exercise B.7.(a) Prove that $(V^+)^n$ is an open convex pointed cone in $\mathbf{M}^n$ and that its dual with respect to either the Euclidean (B.43) or the pseudo-Euclidean (B.44) scalar product is the cone $(\overline{V}^+)^n$ (which, of course, is a closed convex pointed cone in $\mathbf{M}^n$ with non-empty interior).

(b) Prove that the (Euclidean) distance from a point q of the cone $(\overline{V}^+)^n$ to its boundary is given by

$$d(q, \partial(\overline{V}^+)^n) = \frac{1}{\sqrt{2}} \min_j (q_j^0 - |q_j|).$$

(c) Prove the following inequality for any point $q_j \in \overline{V}^+$:

$$\frac{1}{\sqrt{2}|q_j|} q_j^2 \leq q_j^0 - |q_j| \leq (q_j^2)^{1/2},$$

where q_j^2 is the scalar square of q_j in the Minkowski metric.

We can replace $\mathbf{R}^n$ by $\mathbf{M}^n$ and K by $(\overline{V}^+)^n$ in Theorems B.7 and B.8 to obtain the following special result.

Corollary B.9. The Laplace transform of $f(x)$ in $\mathcal{S}'(\mathbf{M}^n)$ with support * in $(\overline{V}^+)^n$ is a holomorphic function in the future tube $T_n^+ = \mathbf{M}^n + i(V^+)^n$, satisfying the estimate

$$|(\mathcal{L}f)(k)| \leq A \frac{(1+|k|)^m}{\min_j (q_j^2)^l}, \tag{B.49}$$

where A, m, l are non-negative numbers depending on f. (In (B.49), q_j^2 is the scalar square of the vector $q_j = \operatorname{Im} k_j$ in the Minkowski metric.) Conversely, every holomorphic function in the tube T_n^+ having an estimate of type (B.49) is the Laplace transform of a generalized function in $\mathcal{S}'(\mathbf{M}^n)$ with support in $(\overline{V}^+)^n$.

B.4. Boundary values of the Laplace transform

For a generalized function $f(x) \in \mathcal{S}'(\mathbf{R}^n)$ such that the point $q = 0$ belongs to the boundary of $\mathring{\Gamma}(f)$, an interesting question is whether the limit exists in the class $\mathcal{S}'$ (or generalized boundary value) of the expression $(\mathcal{L}f)(p+iq)$, regarded as a generalized function with respect to p that depends on the parameter q as $q \to 0$ in $\mathring{\Gamma}(f)$. For this purpose, we introduce the following notion of limit** (the usefulness of which will emerge in Theorem B.11).

Let S be a convex domain in $\mathbf{R}^n$ such that $0 \in \partial S$, and $h(p+iq)$ a holomorphic function in the tube T^S. We suppose that the expression $h(p+iq)$ defines a family of generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ with respect to p that is continuously dependent on the parameter q (see Exercise B.3 in this connection). We say that h has the *generalized boundary value* (or simply boundary value) $\lim_{q\to 0, q\in S} h(p+iq) = h_0(p)$ in $\mathcal{S}'(\mathbf{R}^n)$ (and also write $h(p+iq) \to h_0(p)$ in $\mathcal{S}'(\mathbf{R}^n)$ as $q \to 0$ in S), if for any compact truncated cone $K^r \subset S \cup \{0\}$, the inequality

$$\lim_{|q|\to 0, q\in K^r\setminus\{0\}} (h(p+iq), u(p))_p = (h_0(p), u(p))_p \tag{B.50}$$

holds for all $u \in \mathcal{S}(\mathbf{R}^n)$.

In similar fashion, we can define the notion of boundary value of a holomorphic function $h(p+iq)$ in $\mathcal{D}'(\mathcal{O})$ as $q \to 0$ in S, where $\mathcal{O}$ and S are two domains in $\mathbf{R}^n$, S being convex and $0 \in \partial S$. Since q tends to 0, it is not necessary to assume that h is defined throughout the domain $\mathcal{O} + iS$.

It suffices to suppose that f is defined and holomorphic in some domain $D \subset \mathbf{C}^n$ that is *adjacent to the domain* $\mathcal{O} \subset \mathbf{R}^n$ from the side iS in the sense that for any compactum $Q \subset \mathcal{O}$ and any truncated cone $K^r \subset S$ with compact closure in $S \cup \{0\}$, there exists $\rho \in (0, r)$ such that $D \supset Q + iK^\rho$. We say that h has a (generalized) boundary value

$$\lim_{q\to 0, q\in S} h(p+iq) = h_0(p) \text{ in } \mathcal{D}'(\mathcal{O}), \tag{B.51}$$

if (B.50) holds for any $u \in \mathcal{D}(\mathcal{O})$ and for any compact truncated cone $K^r \subset S \cup \{0\}$.

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* Such generalized functions are called *generalized functions of retarded type*.

** In §2.6.A, the notion of limit (say, as $\gamma \to b$) of a family of functions or distributions depending on the parameter $\gamma \in \Gamma$ was used in application to the case when the set $\Gamma \cup \{b\}$ was locally compact.

These definitions generalize the following classical notion of boundary value of a holomorphic function. If the function h is holomorphic in some domain D adjacent to $\mathcal{O}$ from the side iS and if $h(p+iq) \to h_0(p)$ as $q \to 0$ in $K^r\backslash\{0\}$ uniformly in p over the compacta of $\mathcal{O}$ (where $h_0(p)$ is a continuous function in $\mathcal{O}$ and K^r is any compact truncated cone in $S \cup \{0\}$), then we say that h has a boundary value in the class $C(\mathcal{O})$. We can also talk about boundary values in the class $\mathcal{E}(\mathcal{O})$ if the relation $\lim_{q\to 0, q\in S} D_p^\alpha h(p+iq) = D^\alpha h_0(p)$ holds in the class $C(\mathcal{O})$ for all multi-indices α. (It is clear that these notions are in complete accord with the definitions of $C(\mathcal{O})$ and $\mathcal{E}(\mathcal{O})$ given in §1.2.C.)

The broadest of these concepts is the (generalized) boundary value in the class $\mathcal{D}'(\mathcal{O})$. Thus if h_0 is a boundary value of f in the class $C(\mathcal{O})$ (or $\mathcal{S}'(\mathbf{R}^n)$), then it is a boundary value in the class $\mathcal{D}'(\mathcal{O})$ (or $\mathcal{D}'(\mathbf{R}^n)$).

Exercise B.8. Let h be a holomorphic function in the domain $D \subset \mathbf{C}^n$ adjacent to the domain $\mathcal{O} \subset \mathbf{R}^n$ from the side iS (where S is a convex domain in $\mathbf{R}^n$ such that $0 \in \partial S$). Prove that if $\lim_{q\to 0, q\in S} h(p+iq) = h_0(p)$ in $\mathcal{D}'(\mathcal{O})$, then for any function $a(k)$ that is holomorphic in a complex neighbourhood (in $\mathbf{C}^n$) of $\mathcal{O}$, we have $\lim_{q\to 0, q\in S} a(p+iq)h(p+iq) = a(p)h_0(p)$ in $\mathcal{D}'(\mathcal{O})$. [Hint: The proof that

$$\lim_{|q|\to 0, q\in S} (h(p+iq), a(p+iq)u(p))_p = (h_0(p), a(p)u(p))_p$$

uses the same arguments as in Proposition 2.9.]

The boundary value of a holomorphic function uniquely defines the function itself. This implies the following generalization of the classical uniqueness theorem.*

Theorem B.10 (generalized uniqueness theorem). *Let h be a function that is holomorphic in the domain $D \subset \mathbf{C}^n$ containing a set of the form $\mathcal{O} + iS$, where $\mathcal{O}$ and S are domains in $\mathbf{R}^n$, S being convex with $0 \in \partial S$. If $h(p+iq)$ has 0 as its generalized boundary value in $\mathcal{D}'(\mathcal{O})$ as $q \to 0$ in S, then $h \equiv 0$.*

■ Acccording to the classical uniqueness theorem, it suffices to show that $f = 0$ in $\mathcal{O} + iS$ or, equivalently, that

$$(h(p+iq), u(p))_p = 0 \tag{B.52}$$

for any $u \in \mathcal{D}(\mathcal{O})$ and any $q \in S$. We fix for given u and q a positive number ϵ such that $\epsilon < \frac{1}{|q|} \times$ $\times\ d(\operatorname{supp} u, \partial\mathcal{O})$ and $\epsilon < \frac{1}{|q|} d(q, \partial\Omega)$. Then (by the property of integrals depending on a parameter) the function

$$g(\zeta) = (h(p+\zeta q), u(p))_p$$

is defined and holomorphic with respect to ζ in the strip

$$\mathcal{P} = \{\zeta \equiv (\xi + i\eta) \in \mathbf{C} : \ |\xi| < \epsilon,\ 0 < \eta < 1 + \epsilon\}.$$

Since the left hand side of (B.52) is $g(i)$, it suffices to show that $g \equiv 0$. First we show that g can be continuously extended onto the strip $\mathcal{P}_1 = \{\zeta \in \mathbf{C} : \ |\zeta| < \epsilon,\ 0 \le \eta < 1 + \epsilon\}$. To this end we rewrite $g(\zeta)$ in the form

$$g(\zeta) = (h_\eta(p), u_\xi(p))_p, \tag{B.53}$$

where $h_\eta(p) = h(p + i\eta q)$ for $0 < \eta < 1+\epsilon$, $h_0(p) = 0$ and $\{u_\xi\}$ is a family of functions in $\mathcal{D}(\mathcal{O})$ that is continuously dependent on the parameter $\xi \in (-\epsilon, \epsilon)$ and defined by the equalities: $u_\xi(p) = u(p - \xi q)$ for $p \in \mathcal{O} \cap (\xi q + \mathcal{O})$ and $u_\xi(p) = 0$ for $p \notin \xi q + \mathcal{O}$.

By hypothesis, $h_\eta(p) \to 0$ in $\mathcal{D}'(\mathcal{O})$ as $\eta \to +0$; hence $\{h_\eta\}$ can be regarded as a family of distributions in $\mathcal{D}'(\mathcal{O})$ which depends continuously on the parameter η for $0 \le \eta < 1+\epsilon$. In this case, the expression (B.53) is, by Proposition 2.9, a continuous function of ζ in the strip $\mathcal{P}_1$.

It remains to apply the Cauchy integral formula to g over an upper semicircular contour of radius $< \epsilon$ and centre at 0. Since in this formula, the integration is carried out only along the semicircle (since g vanishes on the interval $(-\epsilon, \epsilon)$), g can be analytically continued onto the real interval $(-\epsilon, \epsilon)$; the fact that $g = 0$ on $(-\epsilon, \epsilon)$ now implies that $g = 0$ in $\mathcal{P}_1$. ■

We now give the main theorem of this subsection, which states that the generalized boundary value (as $q \to 0$) of the Laplace transform of a tempered distribution f is the Fourier transform of f.

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*This theorem for the case of a single complex variable is assumed to be well known to the reader; the case of several complex variables will be set forth in §5.1.C.

Theorem B.11. *Let $f \in \mathcal{S}'(\mathbf{R}^n)$, $\mathring{\Gamma}(f)$ non-empty and $0 \in \partial\mathring{\Gamma}(f)$. Then* $\lim_{q \to 0, q \in \mathring{\Gamma}(f)} (\mathcal{L}f)(p+iq)$ *exists in $\mathcal{S}'(\mathbf{R}^n)$ and is equal to $\mathcal{F}_{x\to p}[f(x)]$.*

■ It suffices to show that

$$\lim_{|q|\to 0, q\in K^r} ((\mathcal{L}f)(p+iq), u(p)) = (\mathcal{F}f, u)$$

for any $u \in \mathcal{S}(\mathbf{R}^n)$ and any compact truncated cone $K^r \subset \mathring{\Gamma}(f) \cup \{0\}$. But as in the proof of Theorem B.5, instead of considering truncated cones K^r we can confine ourselves to parallelepipeds Q of the form (B.26) associated with an arbitrary set $\{a_1, \ldots, a_n\}$ of linearly independent vectors of $\mathring{\Gamma}(f)$. On replacing u by $u = \mathcal{F}^{-1}v$, the proof reduces to showing that

$$\lim_{|q|\to 0, q\in Q} (e^{-qx} f(x),\ v(x)) = (f(x), v(x)) \tag{B.54}$$

for all $v \in \mathcal{S}(\mathbf{R}^n)$. As in the proof of Theorem B.5, we can rewrite this equality in the form

$$\lim_{|q|\to 0, q\in Q} (g(x), \phi_q(x)v(x)) = (g(x), \phi_0(x)v(x)), \tag{B.55}$$

where g is a fixed distribution in $\mathcal{S}'(\mathbf{R}^n)$ and $\phi_q(x)$ is a multiplicator in $\mathcal{S}(\mathbf{R}^n)$ depending on the parameter q such that $\phi_q v \to \phi_0 v$ in $\mathcal{S}(\mathbf{R}^n)$ as $q \to 0$ in Q. This is sufficient to establish the claim that (B.55) is indeed true. ■

Remark. In certain cases, the boundary value of the Laplace transform of the generalized function f as $q \to 0$ in S (where S is a domain in $\Gamma(f)$ such that $0 \in \partial S$) can be understood in a more natural manner, that is, without using truncated cones (just as the limit on a locally compact set of parameters was considered in §2.6.A); more precisely, the equation

$$\lim_{|q|\to 0, q\in S} ((\mathcal{L}f)(p+iq), u(p)) = ((\mathcal{F}f)(p), u(p))$$

holds for all $u \in \mathcal{S}(\mathbf{R}^n)$.

We mention two such cases.

(1) There exists a closed convex polyhedron (or a closed convex polyhedral cone) Q in $\mathbf{R}^n$ such that $S \subset Q \subset \Gamma(f)$;

(2) $\operatorname{supp} f \subset K$ and $S \subset K^*$, where K is a convex pointed cone.

In case (1), the proof that $e^{-qx}f(x) \to f(x)$ in $\mathcal{S}'(\mathbf{R}^n)$ as $|q| \to 0$, $q \in S$, is a virtual repetition of the argument of Theorem B.11; in case (2) (as in Theorem B.7) we have to use Proposition A.1 and the fact that $e^{-qx}u(x) \to u(x)$ in $\mathcal{S}(K)$ as $|q| \to 0$, $q \in K^*$.

An important property of the operation of passing to the boundary value of a holomorphic function is that it commutes with a change of variables under a conformal map (say, Φ under natural hypotheses concerning Φ). We confine ourselves to the statement and proof of a special case used in the proof of the "edge of the wedge" theorem (§5.1.D).

We suppose that $h(p+iq)$ is a C^∞- partially holomorphic function (more precisely, holomorphic in $p_1+iq_1, \ldots, p_\nu + iq_\nu$) in the tube $\mathcal{T}^\Omega$ with base (B.32). We fix a conformal map ϕ taking the disc

$$U = \{\zeta \in \mathbf{C} :\ |\zeta| < \rho\} \tag{B.56}$$

to the strip $\{z \in \mathbf{C} :\ |\operatorname{Im} z| < 1\}$, the upper semidisc (where $\operatorname{Im}\zeta > 0$) to the upper half-strip (where $\operatorname{Im} z > 0$), and the interval $(-\rho, \rho)$ to $\mathbf{R}$. Next we define the n-dimensional conformal map Φ:

$$\Phi(\zeta) \equiv \Phi(\zeta_1, \ldots, \zeta_n) = (\phi(\zeta_1), \ldots, \phi(\zeta_n)), \tag{B.57}$$

where $\zeta_1, \ldots, \zeta_n \in U$. It is then clear that the function $h(\Phi(\zeta))$ is defined and is C^∞- and partially holomorphic on the set

$$G = \{\zeta \equiv \zeta + i\eta \in \mathbf{C}^n :\ |\zeta_\alpha| < \rho \quad \text{for } \alpha = 1, \ldots, n; \quad \eta_\beta > 0 \quad \text{for } \beta = 1, \ldots, \nu; \\ \eta_\beta = 0 \quad \text{for } \beta = \nu+1, \ldots, n\}. \tag{B.58}$$

Proposition B.12. *Suppose that the function $h(k) \equiv h(k^{(1)}, \dots, k^{(n)})$ is defined in the tube T^Ω with base* (B.32) *and satisfies the other conditions of Proposition* B.6. *Let Φ be the conformal map defined by* B.57. *Then the functions $h(k)$ and $h(\Phi(\zeta))$ (defined respectively in T^Ω and G) have the generalized boundary values*

$$\lim_{q\to 0, q\in\Omega} h(p+iq) = h_0(p) \text{ in } \mathcal{S}'(\mathbf{R}^n), \tag{B.59a}$$

$$\lim_{\eta\to 0, \eta\in\Omega} h(\Phi(\xi+i\eta)) = g(\xi) \text{ in } \mathcal{D}'(\mathcal{O}), \tag{B.59b}$$

which are related by

$$g(\xi) = h_0(\Phi(\xi)) \text{ in } \mathcal{D}'(\mathcal{O}); \tag{B.59c}$$

here

$$\mathcal{O} = \{x \in \mathbf{R}^n : \ -\rho < x_j < \rho; \ j = 1, \dots, n\}. \tag{B.60}$$

■ First we show that h can be represented (for some $\Lambda \in \mathbf{Z}_+$) in the form

$$h(k) = \prod_{j=1}^{\nu} \left(\frac{\partial}{\partial k_j}\right)^\Lambda H(k), \tag{B.61}$$

where H has the property mentioned above for h and furthermore, has an extension as a continuous function to the tube T^{Ω_1}, where

$$\Omega_1 = \{q \in \mathbf{R}^n : \ 0 \le q_j < 1 \quad \text{for } \ j = 1, \dots, \nu; \ q_j = 0 \quad \text{for } \ j = \nu+1, \dots, n\}. \tag{B.62}$$

To this end, we choose $\Lambda \in \mathbf{Z}_+$ such that

$$|h(p+iq)| \le A_0(1+|p|)^m \prod_{j=1}^{\nu} (q_j)^{-\Lambda+3/2},$$

and we define the functions $H_0, H_1, \dots, H_\Lambda$ by the recurrence relations:

$$H_0 = h, H_\lambda(k) = \int_{i/2}^{k_1} dz_1 \dots \int_{i/2}^{k_\nu} dz_\nu H_{\lambda-1}(z_1, \dots, z_\nu, k_{\nu+1}, \dots, k_n).$$

The function H_λ satisfies an estimate of type

$$|H_\lambda(p+iq)| \le A_\lambda(1+|p|)^{m+\nu\lambda} \prod_{j=1}^{\nu} (q_j)^{-\Lambda+\lambda+3/2}, \tag{B.63}$$

which is easily obtained by induction on λ:

$$|H_\lambda(k)| = \left|\prod_{j=1}^{\nu}\left(k_j - \frac{i}{2}\right) \int_0^1 dt_1 \dots \int_0^1 dt_\nu H_{\lambda-1}\left(\frac{i}{2} + t_1\left(k_1 - \frac{i}{2}\right), \dots\right.\right.$$

$$\left.\left. \dots, \frac{i}{2} + t_\nu\left(k_\nu + \frac{i}{2}\right), k_{\nu+1}, \dots, k_n\right)\right| \le A'_{\lambda-1}(1+|p|)^{m+\nu(\lambda-1)} \prod_{j=1}^{\nu} \left(\left|k_j - \frac{i}{2}\right| \times\right.$$

$$\left.\times \int_0^1 \left[\frac{1}{2} + t_j\left(q_j - \frac{1}{2}\right)\right]^{-\Lambda+\lambda+1/2} dt_j\right) \le A_\lambda(1+|p|)^{m+\nu\lambda} \prod_{j=1}^{\nu} (q_j)^{-\Lambda+\lambda+3/2}.$$

It follows from (B.63) that the function $H_{\Lambda-1}(p+iq)$ is uniformly bounded as p varies over a compactum in $\mathbf{R}^n$ and $0 < q_j \le \frac{1}{2}$ for $j = 1, \dots, \nu$. Consequently, $H_\Lambda(p+iq)$ is a continuous function of p, q in the tube with base defined by $0 \le q_j \le \frac{1}{2}$ for $j = 1, \dots, \nu$ and $q_k = 0$ for $k = \nu+1, \dots, n$. Setting $H_\Lambda = H$, we obtain the required representation (B.61).

The question of the boundary values in Proposition B.12 now presents no difficulties. Setting $H_o(p) = H(p+iq)|_{q=0}$, we find that $H_0(p)$ is the boundary value of $H(p+iq)$ as $q \to 0,\ q \in \Omega$ in the class $\mathcal{C}(\mathbf{R}^n)$ of continuous functions. In such a case it follows from (B.61) that $h(p+iq)$ has the generalized boundary value $h_0(p)$ in $\mathcal{S}'(\mathbf{R}^n)$ as $y \to 0,\ y \in \Omega$, where

$$h_0(p) = \prod_{j=1}^{\nu} (\frac{\partial}{\partial p_j})^{\Lambda} H_0(p). \tag{B.64}$$

Now by construction, the function $H(\Phi(\xi + i\eta))$ in G has the boundary value $H_0(\Phi(\xi))$ (as $\eta \to 0,\ \eta \in \Omega$) in the class $\mathcal{C}(\mathcal{O})$ of continuous functions and hence in the class $\mathcal{D}'(\mathcal{O})$. In this case, $h(\Phi(\zeta))$ also has a boundary value in $\mathcal{D}'(\mathcal{O})$ as $\eta \to 0,\ \eta \in \Omega$. This obviously follows from the fact that by virtue of (B.61) and (B.64) we can write the following representations for the expressions $h(\Phi(\zeta))$ and $h_0(\Phi(\xi))$, which are completely identical (apart from the substitution $k \to \zeta = \Phi^{-1}(k)$)

$$h(\Phi(\zeta)) = \sum_{\alpha} b_{\alpha_1 \ldots \alpha_\nu}(\zeta) \prod_{j=1}^{\nu} \left(\frac{\partial}{\partial \zeta_j}\right)^{\alpha_j} H(\Phi(\zeta)),$$

$$h_0(\Phi(\xi)) = \sum_{\alpha} b_{\alpha_1 \ldots \alpha_\nu}(\xi) \prod_{j=1}^{\nu} \left(\frac{\partial}{\partial \xi_j}\right)^{\alpha_j} H_0(\Phi(\xi)),$$

where α runs through a finite set of multiplicators in $\overline{\mathbf{Z}}_+^\nu$ and the $b_{\alpha_1 \ldots \alpha_\nu}(\zeta)$ are holomorphic functions for $|\zeta_j| < \rho$. By Exercise B.8, it follows from these equalities, as well as from the fact that $H(\Phi(\xi + i\eta))$ has $H_0(\Phi(\xi))$ as its boundary value as $\eta \to 0,\ \eta \in \Omega$, that our assertion (and in particular, B.59c)) holds. ■

The following two exercises contain criteria for the existence of a generalized boundary value for a holomorphic function in the class $\mathcal{S}'(\mathbf{R}^n)$ and $\mathcal{D}'(\mathcal{O})$.

Exercise B.9.(a) Let S be a convex domain in $\mathbf{R}^n$ for which 0 is a boundary point, and $h(k)$ a holomorphic function in the tube $\mathcal{T}^S$, where the expression $h(p+iq)$ defines a family of distributions in $\mathcal{S}'(\mathbf{R}^n)$ with respect to p that depends continuously on the parameter $q \in S$ (see Exercise B.3 with regard to this). Prove the equivalence of the following three conditions:

(1) $h(p+iq)$ has a generalized boundary value in $\mathcal{S}'(\mathbf{R}^n)$ as $q \to 0,\ q \in S$;

(2) $h(p+iq)$ coincides in $\mathcal{T}^S$ with the Laplace transform of a tempered distribution;

(3) an estimate of type (B.25) holds for any compact truncated cone $K^r \subset S \cup \{0\}$.

[Hint: The implications (2) $\Rightarrow$ (1) and (2) $\Leftrightarrow$ (3) follow from Theorems B.11 and B.5; for the proof of the implication (1) $\Rightarrow$ (3), use Exercise 2.43 with $\Gamma = K = K^r$ and Theorem B.4.]

(b) Let $h(k)$ be a holomorphic function in the domain $D \subset \mathbf{C}^n$ adjacent to the domain $\mathcal{O} \subset \mathbf{R}^n$ from the side iS where S is a convex domain in $\mathbf{R}^n$ having 0 as a boundary point. Prove that in order that $h(p+iq)$ have a boundary value in $\mathcal{D}'(\mathcal{O})$ as $q \to 0,\ q \in S$, it is necessary and sufficient that for a compact set $Q \subset \mathcal{O}$ and any compact truncated cone $K^r \subset S \cup \{0\}$ such that $Q + iK^r \subset D$, there exist numbers A, l (depending on Q and K^r) providing an estimate

$$|h(p+iq)| \le A|q|^{-l} \quad \text{for } p \in Q,\ q \in K^r \backslash \{0\}. \tag{B.65}$$

[Hint: For the proof of the necessity, use Exercise 2.43 with $\Gamma = K = K^r$ in order to obtain for any compact subset $Q \subset \mathcal{O}$ and any compact truncated cone $K^r \subset S \cup \{0\}$ an estimate of type (B.21) for all $u \in \mathcal{S}(\mathbf{R}^n)$ with support in Q. Clearly such functions u can be regarded as elements of $\mathcal{D}(\mathcal{O})$; the subsequent argument for the derivation of (B.65) is the same as in Exercise B.4 in the proof of the implication (1) $\Rightarrow$ (2); the proof of the sufficiency is established via arguments used in the proof of Proposition B.12.]

B.5. Example: the "mathematics" of dispersion relations

We consider the problem of expressing an analytic function $h(z)$ of the complex variable $z \equiv x + iy$ with $\operatorname{Im} z \neq 0$ in the form of the so-called *dispersion integral*

$$h(z) = \frac{1}{2\pi i} \int \frac{\rho(\xi)}{\xi - z} d\xi, \ \operatorname{Im} z \neq 0, \tag{B.66}$$

where $\rho(x)$ is the jump of the function $h(z)$ on the real axis (as we shall see, $\rho(x)$ can in fact be naturally taken to be given on the extended real line $\mathbf{R}_\infty$). For the existence of generalized boundary values as $\operatorname{Im} z \to \pm 0$ in the class $\mathcal{S}'(\mathbf{R})$, it suffices to suppose that there is an estimate

$$|h(z)| \le A(1+|z|)^m |\operatorname{Im} z|^{-l}, \tag{B.67}$$

where A, m, l are real numbers.

We begin by considering the case $\operatorname{Im} z > 0$. Then (as we know from Theorem B.8) $h(z)$ is the Laplace transform of a generalized function in $\mathcal{S}'(\mathbf{R})$ with support* in $\overline{\mathbf{R}}_+$, so that (by Theorem B.11) it has a boundary value $f_1(x)$ in the class $\mathcal{S}'(\mathbf{R})$:

$$f_1(x) = \lim_{y\to 0} h(z). \tag{B.68}$$

Similarly the function $z^{-2}h(-1/z)$ is analytic for $\operatorname{Im} z > 0$ and satisfies an estimate of the same type, therefore it also has a boundary value in $\mathcal{S}'(\mathbf{R})$:

$$f_2(x) = \lim_{y\to 0} z^{-2}h(-1/z), \tag{B.69}$$

where

$$f_1(x) = x^{-2} f_2(-1/x) \quad \text{for } x \ne 0 \tag{B.70}$$

(for the proof of the latter equality we apply the arguments used in Proposition B.12). Formula (B.70) is the compatibility condition which (by Proposition A.4) ensures the existence of a generalized function $h_+(x)$ in $\mathcal{S}'(\mathbf{R}_\infty)$ related to f_1 and f_2 by formulae of type (A.24). We call $h_+(x)$ the *(generalized) boundary value* of $h(z)$ as $\operatorname{Im} z \to +0$ in the class $\mathcal{S}'(\mathbf{R}_\infty)$.

Lemma B.13. *An arbitrary function $h(z)$ that is analytic in the upper half-plane and satisfies* (B.67), *can be recovered from its generalized boundary value $h_+(x) \in \mathcal{S}'(\mathbf{R}_\infty)$ by means of the following formula* (*of Cauchy type*):

$$\frac{1}{2\pi i}\int \frac{h_+(\xi)}{\xi - z}\,d\xi = \begin{cases} h(z) & \text{for } \operatorname{Im} z > 0, \\ 0 & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{B.71}$$

■ We give an outline of the proof. We shall consider the particular case when $h(z)$ is an analytic function for $\operatorname{Im} z > 0$ of polynomial growth for $\operatorname{Im} z \ge 0$. For $\alpha > 0$ let $\beta \equiv \beta(\alpha)$ be the imaginary part of the points of intersection of the circle $|z| = 1$ and $\operatorname{Im}(-1/z) = \alpha$.

We apply the usual Cauchy formula to the domain defined by the inequalities $\operatorname{Im}(-1/z) >$ $> \alpha$, $\operatorname{Im} z > \beta$:

$$\frac{1}{2\pi i}\int_{\substack{\operatorname{Im}\zeta=\beta\\ |\zeta|<1}} \frac{h(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\int_{\substack{\operatorname{Im} w=\alpha\\ |w|<1}} \frac{w^{-1}h(-1/w)}{1 - zw}\,dw = \begin{cases} h(z), & \text{for } \operatorname{Im} z > 0, \\ 0, & \text{for } \operatorname{Im} z < 0. \end{cases} \tag{B.72}$$

We have made the substitution $w = -1/\zeta$ in the second integral. Also we suppose that α is sufficiently small. Since $h(\zeta)$ is continuous for $\operatorname{Im}\zeta \ge 0$ and the limit (B.69) exists, we can pass to the limit as $\alpha \to +0$ in (B.72) to obtain (B.71).

For the proof of the general case, we note that $h(z)$ (for $\operatorname{Im} z > 0$) is a sufficiently high order derivative of an analytic function in the upper half-plane of polynomial growth and continuous when $\operatorname{Im} z \ge 0$. (This fact was established at the beginning of the proof of Proposition B.12.) Therefore it is enough to prove that if (B.71) holds for $h(z)$, then it also holds for its derivative $h'(z)$ (which also satisfies an estimate of type (B.67) since it is also the Laplace transform of a generalized function in $\mathcal{S}'(\mathbf{R})$ with support in $\overline{\mathbf{R}}_+$). We leave the proof of this as an exercise. [Hint: In the integral (B.71) one needs to apply a partition of unity on $\mathbf{R}_\infty$ of the form $u(\xi) + v(-1/\xi) = 1$, where $u, v \in \mathcal{D}(\mathbf{R})$ and differentiate (B.71) with respect to z.] ■

Exercise B.10. Suppose that the conditions of Lemma B.13 hold. Prove that the generalized boundary value of $h(z)$ as $\operatorname{Im} z \to 0$ in the class $\mathcal{S}'(\mathbf{R}_\infty)$ has zero integral. [Hint: The integral of $h(z)$ along the contour used in the proof of Lemma B.13 is zero; then proceed as in the proof of that lemma.]

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* Here and in what follows $\mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$, $\overline{\mathbf{R}}_+ = \{x \in \mathbf{R} : x \ge 0\}$.

We can associate with a function $h(z)$ that is analytic when $\operatorname{Im} z \neq 0$ and satisfies (B.67) two boundary values $h_{\pm}(x)$ in the class $\mathcal{S}'(\mathbf{R}_\infty)$ corresponding to $\operatorname{Im} z \to \pm 0$. The difference

$$\rho(x) = h_+(x) - h_-(x) \tag{B.73}$$

is called the *jump* or *saltus* of class $\mathcal{S}'(\mathbf{R}_\infty)$ (or on $\mathbf{R}_\infty$) of $h(z)$.

Proposition B.14. *An arbitrary function $h(z)$ that is analytic for* $\operatorname{Im} z \neq 0$ *and satisfies* (B.67) *can be recovered from its jump $\rho(x)$ of class $\mathcal{S}'(\mathbf{R}_\infty)$ by means of the dispersion relation* (B.66).

Exercise B.11. Prove Proposition B.14. [Hint: Apply (B.71) and the corresponding formula for h_-.]

Exercise B.12. Prove that the generalized boundary values in the class $\mathcal{S}'(\mathbf{R})$ of a function $h(z)$ that is analytic for $\operatorname{Im} z \neq 0$ and satisfies (B.67) can be recovered from the jump $\rho(x)$ of class $\mathcal{S}'(\mathbf{R}_\infty)$ by means of the following dispersion relations:

$$\lim_{y \to \pm 0} h(x \pm iy) = \frac{1}{2\pi i} \int \frac{\rho(\xi)}{\xi - x \mp i0} d\xi. \tag{B.74}$$

[Hint: See §A.3 regarding the definition of the convolutions in (B.74)]

Exercise B.13. Prove that formula (B.66) establishes an isomorphism between the analytic functions $h(z)$ (for $\operatorname{Im} z \neq 0$) satisfying an estimate of type (B.67), and the generalized functions $\rho(x)$ in $\mathcal{S}'(\mathbf{R}_\infty)$ with zero integral. [Hint: If for some generalized function $\rho(x)$ in $\mathcal{S}'(\mathbf{R}_\infty)$ with zero integral, the associated function $h(z)$ is equal to zero, then it follows from (2.29) that $\rho(x) = 0$ in $\mathbf{R} \subset \mathbf{R}_\infty$, so that ρ has support at the point ∞; then use part (b) of Exercise A.7.]

We draw attention to the fact that the jump in a neighbourhood of the point ∞ is defined by the function $z^{-2}h(-1/z)$. Therefore the function $h(z) = \text{const} \neq 0$ in the complex plane has a non-zero jump at infinity (otherwise it would be impossible to understand how it could have a representation in the form of a dispersion integral). We note also that we have already encountered dispersion relations for polynomials in Exercise A.7(b).

B.6. Restriction of the Laplace transform

Theorem B.8 is an effective criterion enabling one to determine from the properties of a function $h(k)$ in the tube $\mathbf{R}^n + i \operatorname{int} K^*$ whether it is the Laplace transform of a generalized function in $\mathcal{S}'(\mathbf{R}^n)$ with support in K. If in fact $h(k)$ is known only on some subset (for example, a non-open one) in the tube $\mathbf{R}^n + i \operatorname{int} K^*$, then we can naturally expect that the corresponding criterion will be more complex and less effective. We consider the simplest situation (which we shall come across in §9.5) where the cone K is given as the direct product $\overline{\mathbf{R}}^n_+$ of n copies of the set $\overline{\mathbf{R}}_+$ of non-negative numbers. In this case, $K^* = K$ (with respect to the Euclidean scalar product in $\mathbf{R}^n$). It is clear that $i\mathbf{R}^n_+$ is not an open subset of $\mathbf{R}^n + i \operatorname{int} K$; nevertheless, this is a set of uniqueness for the holomorphic functions in the tube $\mathbf{R}^n + i \operatorname{int} K$ (that is, every holomorphic function in the tube is uniquely defined by its restriction to the subset of purely imaginary points $i\mathbf{R}^n_+$).

It is of interest to give a criterion enabling one to judge whether a given complex function $h(k)$ on $i\mathbf{R}^n_+$ is the restriction to $i\mathbf{R}^n_+$ of the Laplace transform of a generalized function $f(x)$ in $\mathcal{S}'(\mathbf{R}^n)$ with support in $\overline{\mathbf{R}}^n_+$.

If $h(k)$ is such a function, then the formula

$$(h, u) = \int_{\mathbf{R}^n_+} h(iq) u(q) d_n q, \ u \in \mathcal{S}(\mathbf{R}^n_+), \tag{B.75}$$

clearly defines a continuous linear functional on $\mathcal{S}(\mathbf{R}^n_+)$. It turns out that this functional is also continuous in a certain weaker topology on $\mathcal{S}(\mathbf{R}^n)$. To see this, we represent $f(x)$ as a finite sum

$$f(x) = \sum_\alpha \partial_x^\alpha f_\alpha(x),$$

where the $f_\alpha(x)$ are measures of power growth on $\mathbf{R}^n$ with supports in $\overline{\mathbf{R}}^n_+$ (see Corollary A.2). Then

$$h(iq) = \sum_\alpha (-\partial_x)^\alpha \int e^{-xq} f_\alpha(x) d^n x,$$

whence (by Fubini's theorem) we have

$$(h,u) = \sum_{\alpha} \int_{\overline{\mathbf{R}}^n_+} \left((-\partial_x)^\alpha \int e^{-xq} u(q)dq\right) f_\alpha(x) d^n x,$$

that is,

$$(h,u) = (f,\hat{u}). \tag{B.76}$$

Here we have introduced the function of $x \in \overline{\mathbf{R}}^n_+$:

$$\hat{u}(x) = \int_{\mathbf{R}^n_+} e^{-xq} u(q) d_n q, \ x \in \overline{\mathbf{R}}^n_+. \tag{B.77}$$

Exercise B.14. (a) Prove that the map $u \to \hat{u}$ is a continuous linear operator from $\mathcal{S}(\mathbf{R}^n_+)$ to $\mathcal{S}(\overline{\mathbf{R}}^n_+)$ with zero null space. [Hint: $\hat{u}(x)$ is the restriction onto the set of purely imaginary points of a holomorphic function in the tube, so that $\hat{u}(x) \equiv 0$ implies that $u(q) \equiv 0$.]

(b) Prove that the iamge of $\mathcal{S}(\mathbf{R}^n_+)$ under the map $u \to \hat{u}$ is an everywhere dense linear subspace of $\mathcal{S}(\overline{\mathbf{R}}^n_+)$. [Hint: It suffices to prove that if $f \in \mathcal{S}'(\overline{\mathbf{R}}^n_+)$ is such that $(f,\hat{u}) = 0$ for all $u \in \mathcal{S}(\mathbf{R}^n_+)$, then $f = 0$. For this we introduce the Laplace transform $h(k)$ of the generalized function f with the help of (B.76), from which it follows that $h(k)$ is equal to zero for all purely imaginary k, so that $h(k) \equiv 0$.]

From (B.76) we have the estimate

$$|(h,u)| \le c\|\hat{u}\|^{\overline{\mathbf{R}}^n_+}_{l,m} \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n_+). \tag{B.78}$$

According to Exercise B.10, the collection of norms

$$\|u\|^{\hat{}}_{l,m} = \|\hat{u}\|^{\overline{\mathbf{R}}^n_+}_{l,m} \ (l,m = 0,1,\ldots)$$

for $u \in \mathcal{S}(\mathbf{R}^n_+)$ defines a new topology on $\mathcal{S}(\mathbf{R}^n_+)$ which is weaker than the natural topology on $\mathcal{S}(\mathbf{R}^n_+)$; we shall refer to this as the weakened topology.

The above estimate (B.78) is a characteristic property of the Laplace transform.

Proposition B.15. *Formulae* (B.75), (B.76) *realize a one-to-one correspondence between the restrictions to the set of purely imaginary points of Laplace transforms of generalized functions in* $\mathcal{S}'(\mathbf{R}^n)$ *with supports in* $\overline{\mathbf{R}}^n_+$*, and the linear functionals on* $\mathcal{S}(\mathbf{R}^n_+)$ *that are continuous in the weakened topology on* $\mathcal{S}(\mathbf{R}^n_+)$.

■ It remains to verify that each linear functional (h,u) on $\mathcal{S}(\mathbf{R}^n_+)$ that is continuous in the weakened topology is obtained by (B.75) from the Laplace transform $h(k)$ of some generalized function f with support in $\overline{\mathbf{R}}^n_+$. In fact we define the linear functional f according to (B.76) first on functions $\hat{u}$ of the form (B.77). By hypothesis, this functional is continuous and its domain (by Exercise B.14) is everywhere dense in $\mathcal{S}(\overline{\mathbf{R}}^n_+)$. Consequently, f is a generalized function in the space $\mathcal{S}'(\overline{\mathbf{R}}^n_+) \approx \mathcal{S}'(\mathbf{R}^n|\overline{\mathbf{R}}^n_+)$. Now let $h(k)$ be the Laplace transform of the generalized function $f(x)$. Then the functional (B.75) defined by it on $\mathcal{S}(\mathbf{R}^n_+)$, clearly coincides with the given functional (h,u). ■

Appendix C. Homogeneous Generalized Functions

C.1. Homogeneous generalized functions in $\mathring{\mathbf{R}}^n$

Homogeneous generalized functions (HGF's) are widely used in quantum field theory (and, more generally, in mathematical physics). To define an HGF in $\mathbf{R}^n$ often reduces to the following problem: to extend (if possible) a given HGF in $\mathring{\mathbf{R}}^n$, where

$$\mathring{\mathbf{R}}^n = \mathbf{R}^n \backslash \{0\} \tag{C.1}$$

(that is, the space $\mathbf{R}^n$ punctured at the origin), to an HGF in $\mathbf{R}^n$. The fact is that HGF's in $\mathring{\mathbf{R}}^n$ enjoy a much simpler structure than those in $\mathbf{R}^n$, and are essentially described by generalized functions of

$n-1$ variables. In this Appendix we set forth questions relating to HGF's in a form that is invariant with respect to the group $GL(n,R)$ of all linear transformations of $\mathbf{R}^n$. (In this connection, an important role is played by the HGF's of one variable.)

Let Ω be one of the following manifolds: $\mathring{\mathbf{R}}^n$, $\mathbf{R}^n$, $\mathbf{R}_+$, $\overline{\mathbf{R}}_+$. A function or generalized function $f(x)$ in Ω is said to be *homogeneous of degree* λ ($\in \mathbf{C}$), if

$$f(\rho x) = \rho^\lambda f(x) \quad \text{for all } \rho > 0. \tag{C.2}$$

Differentiating (C.2) with respect to the parameter ρ at the point $\rho = 1$, we obtain the Euler equation

$$\left(x\frac{\partial}{\partial x}\right) f(x) = 0, \tag{C.3}$$

which (as is easy to see) is equivalent to (C.2). We denote by $\mathfrak{D}_\lambda(\Omega)$ and $\mathfrak{d}_\lambda(\Omega)$ the spaces of test functions $f(x) \in \mathcal{E}(\Omega)$ and generalized functions $f(x) \in \mathcal{S}'(\Omega)$ respectively, of degree $\lambda - n/2$.

Exercise C.1. Prove that the space $\mathfrak{d}_\lambda(\mathbf{R}_+)$ is one-dimensional and is generated by the smooth function $\rho^{\lambda-1/2}$.

We begin with HGF's in $\mathring{\mathbf{R}}^n$. For this purpose we fix (arbitrarily) a test function $h(x)$ in $\mathcal{D}(\mathring{\mathbf{R}}^n)$ (or $\mathcal{S}(\mathring{\mathbf{R}}^n)$) such that

$$\int h(\rho x)\frac{d\rho}{\rho} = 1 \text{ in } \mathring{\mathbf{R}}^n. \tag{C.4}$$

(For example, $h(x)$ can be taken to be $h(x) = u(|x|)$, where $u \in \mathcal{D}(\mathbf{R}_+)$ and $\int u(\rho)\rho^{-1}d\rho = 1$.) By the integral with respect to the measure

$$[d^n x] = h(x)d^n x \tag{C.5}$$

of the generalized function $f(x) \in \mathfrak{d}_{-n/2}(\mathring{\mathbf{R}}^n)$, we mean the expression

$$\int f(x)[d^n x] = (f,h). \tag{C.6}$$

It turns out that (C.6) does not depend on the choice of the function h satisfying (C.4). We shall shortly prove this fact, which is based on Proposition C.1. As a preparation for this, we define the operator $\mathfrak{Z}_\lambda : \mathcal{S}(\mathbf{R}^n) \to \mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$ by the formula

$$(\mathfrak{Z}_\lambda u)(x) = \int_0^\infty \rho^{-\lambda+n/2}u(\rho x)\frac{d\rho}{\rho} \quad \text{for } u \in \mathcal{S}(\mathring{\mathbf{R}}^n),\ x \in \mathbf{R}^n. \tag{C.7}$$

It is clear that $u(\rho x)$ is an element of $\mathcal{S}(\mathbf{R}_+)$ with respect to ρ which depends in C^∞ fashion on $x \in \mathring{\mathbf{R}}^n$ as a parameter; therefore (C.7) defines a continuous operator from $\mathcal{S}(\mathring{\mathbf{R}}^n)$ to $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$.

Proposition C.1. *The following formula holds for any* $f \in \mathfrak{d}_\lambda(\mathring{\mathbf{R}}^n)$, $u \in \mathcal{S}(\mathring{\mathbf{R}}^n)$:

$$(f,u) = \int f(x)(\mathfrak{Z}_{-\lambda}u)(x)[d^n x]. \tag{C.8}$$

■ It follows from the homogeneity of f that

$$(f(x), h(\rho^{-1}x)u(x))_x = \rho^{\lambda+n/2}(f(x), h(x)u(\rho x))_x \tag{C.9}$$

for all $\rho > 0$. We integrate with respect to the measure $\rho^{-1}d\rho$ on $\mathbf{R}_+$. The left hand side of this relation can be regarded as the value of the generalized function $f(x)\rho^{-1} \in \mathcal{S}'(\mathring{\mathbf{R}}^n \times \mathbf{R}_+)$ at the test function $h(\rho^{-1}x)u(x)$, which (by virtue of the "commutativity" of the tensor product of generalized functions, see (2.61), (2.62)) can be written in the form $(f(x), (\rho^{-1}, h(\rho^{-1}x)u(x))_\rho)_x = (f,u)$. Similar reasoning enables one to write the right hand side in the form $(f, h(\mathfrak{Z}_{-\lambda}u))$, and this completes the proof of (C.8). ■

Corollary C.2. The integral $\int f(x)[d^n x]$ of an HGF $f(x)$ of degree $-n$ in $\mathring{\mathbf{R}}^n$ does not depend on the choice of the function $h(x) \in \mathcal{S}(\mathring{\mathbf{R}}^n)$ satisfying (C.4) and entering into the definition (C.5) of the measure $[d^n x]$.

In fact, if $f \in \mathfrak{h}_{-n/2}(\mathring{\mathbf{R}}^n)$ and $u(x)$ is an arbitrary function of $\mathcal{S}(\mathring{\mathbf{R}}^n)$ satisfying (C.4), then

$$(f,u) = \int f(x)(\mathfrak{J}_{n/2}u)(x)[d^n x] = \int f(x)[d^n x] = (f,h).$$

Exercise C.2. Prove that

$$\int f(Ax)[d^n x] = |det A^{-1}| \int f(x)[d^n x], \tag{C.10}$$

where $f \in \mathfrak{h}_{-n/2}(\mathring{\mathbf{R}}^n)$, $A \in GL(n,R)$.

Exercise C.3. Prove the following analogue of the "integration by parts" formula:

$$\int [D^\alpha f(x)]\, v(x)[d^n x] = (-1)^{|\alpha|} \int f(x) D^\alpha v(x)[d^n x], \tag{C.11}$$

where $f \in \mathfrak{h}_\lambda(\mathring{\mathbf{R}}^n)$, $v \in \mathfrak{D}_\mu(\mathring{\mathbf{R}}^n)$, $\lambda + \mu = |\alpha|$.

The next Proposition C.3 reveals the topological properties of the operators $\mathfrak{J}_\lambda$ and their adjoints $\mathfrak{J}'_\lambda$. Here we shall suppose that $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$ is endowed with the Fréchet space structure induced by the LCS structure on $\mathcal{E}(\mathring{\mathbf{R}}^n)$. For definiteness, we consider the dual spaces $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)'$ and $\mathcal{S}'(\mathring{\mathbf{R}}^n)$ in their weak topologies.

Proposition C.3. (a) *The operator $\mathfrak{J}_\lambda$ is a topological homomorphism from $\mathcal{S}(\mathring{\mathbf{R}}^n)$ onto $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$.*

(b) *The operator $\mathfrak{J}'_\lambda : \mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)' \to \mathcal{S}'(\mathring{\mathbf{R}}^n)$ maps $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)'$ topologically isomorphically onto the subspace $\mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n) \subset \mathcal{S}'(\mathring{\mathbf{R}}^n)$; it associates with the functional $\Phi \in D_\lambda(\mathring{\mathbf{R}}^n)'$ an HGF $f \in \mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n)$ such that*

$$(f,u) = \Phi(\mathfrak{J}'_\lambda u) \quad for \quad u \in \mathcal{S}(\mathring{\mathbf{R}}^n), \tag{C.12}$$

$$\Phi(v) = \int f(x)v(x)[d^n x] \quad for \quad v \in \mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n). \tag{C.13}$$

■ We give an outline of the proof. Part (a) is a direct consequence of the fact that $\mathfrak{J}_\lambda$ has a continuous right inverse $\mathfrak{R}_\lambda : \mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n) \to \mathcal{S}(\mathring{\mathbf{R}}^n)$, that is, an operator such that $\mathfrak{J}_\lambda \mathfrak{R}_\lambda$ is the identity operator in $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$. Thus it is not difficult to see that we can choose for $\mathfrak{R}_\lambda$ the operator of multiplication by a fixed function h satisfying (C.4). The corresponding statement for the adjoint follows from (a): $\mathfrak{J}'_\lambda$ maps $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)'$ isomorphically onto the subspace $\mathcal{M}_\lambda^\perp \subset \mathcal{S}'(\mathring{\mathbf{R}}^n)$ orthogonal to the null space $\mathcal{M}_\lambda \equiv \mathfrak{J}_\lambda^{-1}\{0\}$ of $\mathfrak{J}_\lambda$ (where $\mathcal{M}_\lambda^\perp$ is defined as the set of generalized functions f in $\mathcal{S}'(\mathring{\mathbf{R}}^n)$ such that $(f,u) = 0$ for all $u \in \mathcal{M}_\lambda$). It remains to prove the equality $\mathcal{M}_\lambda^\perp = \mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n)$. Since $\mathfrak{J}'_\lambda$ maps $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$ onto $\mathcal{M}_\lambda^\perp$, each generalized function $f \in \mathcal{M}_\lambda^\perp$ has the form $f = \mathfrak{J}'_\lambda \Phi$ for $\Phi \in \mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)'$, that is, f is defined by (C.12). It follows that $f \in \mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n)$, which proves the relation $\mathcal{M}_\lambda^\perp \subset \mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n)$. The reverse inclusion $\mathfrak{h}_{-\lambda}(\mathring{\mathbf{R}}^n) \subset \mathcal{M}_\lambda^\perp$ follows from (C.8). ■

Proposition C.3 finds its most important applications in problems related to multilinear functionals on the spaces $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$. Let Φ be a (separately) continuous multilinear functional on $\mathfrak{D}_{\lambda_1}(\mathring{\mathbf{R}}^n) \times \ldots \times \mathfrak{D}_{\lambda_k}(\mathring{\mathbf{R}}^n)$. Then the formula

$$(K, u_1 \otimes \ldots \otimes u_k) = \Phi(\mathfrak{J}_{\lambda_1}(u_1), \ldots, \mathfrak{J}_{\lambda_k}(u_k)) \quad \text{for } u_1, \ldots, u_k \in \mathcal{S}(\mathring{\mathbf{R}}^n) \tag{C.14}$$

associates with it the generalized function $K(x_1, \ldots, x_k) \in \mathcal{S}'((\mathring{\mathbf{R}}^n)^k)$, which satisfies the following condition of *separate homogeneity* with respect to $x_1, \ldots, x_k$ (of degree $-\lambda_j - n/2$ in x_j $j = 1, \ldots, k$):

$$K(\rho_1 x_1, \ldots, \rho_k x_k) = \prod_{j=1}^{k} \rho_j^{-\lambda_j - n/2} K(x_1, \ldots, x_k) \quad \text{for } \rho_1, \ldots, \rho_k > 0. \tag{C.15}$$

We call $K(x_1, \dots, x_k)$ *the kernel of the multilinear functional* Φ. Here, Φ can be recovered from the kernel via the formula

$$\Phi(v_1, \dots, v_k) = \int K(x_1, \dots, x_k) v_1(x_1) \dots v_k(x_k) [d^n x_1] \dots [d^n x_k] \tag{C.16}$$

for $v_j \in \mathfrak{D}_{\lambda_j}(\mathring{\mathbf{R}}^n)$ $j = 1, \dots, k$. It is not difficult to see from Proposition C.3 (and from Schwartz's kernel theorem) that the map $\Phi \to K$ defined by (C.14) sets up an isomorphism between the space of all continuous multilinear functions over $\mathfrak{D}_{\lambda_1}(\mathring{\mathbf{R}}^n) \times \dots \times \mathfrak{D}_{\lambda_k}(\mathring{\mathbf{R}}^n)$ and the space of all generalized functions $K(x_1, \dots, x_k) \in \mathcal{S}'((\mathring{\mathbf{R}}^n)^k)$ with the separate homogeneity property (C.15).

Of special interest for applications are multilinear functionals Φ with the additional condition of invariance with respect to some subgroup G of the group of special linear transformations on $\mathbf{R}^n$. In view of the obvious invariance of the isomorphism $\Phi \to K$, this condition easily carries over to the language of generalized functions as the condition that the kernel $K(x_1, \dots, x_n)$ be G-invariant. Thus the G-invariant multilinear functionals over $\mathfrak{D}_{\lambda_1}(\mathring{\mathbf{R}}^n) \times \dots \times \mathfrak{D}_{\lambda_k}(\mathring{\mathbf{R}}^n)$ are in one-to-one correspondence with the G-invariant separately homogeneous generalized functions $K(x_1, \dots, x_k)$.

C.2. THE SINGLE REAL VARIABLE CASE

Before we turn to the problem of the extension of an HGF, we consider HGF's of a single real variable. We set

$$j_\lambda(r) \equiv \frac{r_+^\lambda}{\Gamma(\lambda+1)} = \theta(r) \frac{r^\lambda}{\Gamma(\lambda+1)} \tag{C.17}$$

for $r \in \mathbf{R}$, $\lambda \in \mathbf{C}$, $\operatorname{Re}\lambda > -1$. This is a locally integrable function that defines a generalized function $j_\lambda(r)$ in $\mathcal{S}'(\mathbf{R})$ with support in $\overline{\mathbf{R}}_+$ that is homogeneous of degree λ. The family of generalized functions (C.17) is clearly analytically dependent on the parameter λ. Furthermore, it has an analytic continuation in λ to the entire complex plane. For the identity

$$j_\lambda(r) = \frac{d}{dr} j_{\lambda+1}(r) \tag{C.18}$$

allows us successively to extend the left hand side to the domains $\operatorname{Re}\lambda > -2$, $\operatorname{Re}\lambda > -3$, and so on. It is clear that the condition of homogeneity of degree λ now holds for all complex λ. In particular, since $j_0(r) = \theta(r)$, it follows from the differentiation formula (C.18) that

$$j_{-l-1}(r) = \delta^{(l)}(r) \quad \text{for } l \in \overline{\mathbf{Z}}_+. \tag{C.19}$$

It turns out that every generalized function in $\mathcal{S}'(\mathbf{R})$ with support in $\overline{\mathbf{R}}_+$ that is homogeneous of degree λ, is proportional to $j_\lambda(r)$ (see Exercise C.8 below).

Exercise C.4. Prove the formula

$$j_\lambda(r) * j_\mu(r) = j_{\lambda+\mu}(r). \tag{C.20}$$

The following limits exist in $\mathcal{S}'(\mathbf{R})$ for all complex λ:

$$\lim_{\alpha \to +0} (\alpha \pm ix)^\lambda = (0 \pm ix)^\lambda, \tag{C.21}$$

which define generalized functions that are homogeneous of degree λ and are entire functions of the parameter λ.

Exercise C.5. Prove that the limit

$$\lim_{\alpha \to +0} (\alpha - ix)^\lambda = (0 - ix)^\lambda$$

exists in $\mathcal{S}'(\mathbf{R})$ for all $\lambda \in \mathbf{C}$ and that the Fourier transform formula holds for it:*

$$\int j_{-\lambda-1}(r) e^{ixr} dr = (0 - ix)^\lambda. \tag{C.22}$$

* In this exercise we touch upon the Laplace transform technique (set out in Appendix B).

We define for any complex λ the generalized function r_+^λ in $\mathcal{S}'(\mathbf{R})$ with support in $\overline{\mathbf{R}}_+$ by setting

$$r_+^\lambda = \begin{cases} \Gamma(\lambda+1) j_\lambda(r) & \text{for } \lambda \neq -1, -2, \ldots, \\ \frac{(-1)^l}{l!} \frac{\partial}{\partial \lambda} j_\lambda(r) & \text{for } \lambda = -1 - l, \ l \in \overline{\mathbf{Z}}_+. \end{cases} \tag{C.23}$$

It is clear that r_+^λ is equal to the function $\theta(r) r^\lambda$ when $r \neq 0$ (and for this reason is called a regularization of this function). It follows from the definition that when $-(1+\lambda) \notin \overline{\mathbf{Z}}_+$, r_+^λ is an HGF of degree λ while for $-(1+\lambda) = l \in \overline{\mathbf{Z}}_+$ it is a so-called *associated* HGF of the first kind:

$$(\rho r)_+^{-l-1} = \rho^{-l-1}\{r_+^{-l-1} + \frac{(-1)^l}{l!} \ln \rho \cdot j_{-l-1}(r)\} \quad \text{for } \rho > 0, \ l \in \overline{\mathbf{Z}}_+. \tag{C.24}$$

Exercise C.6. Prove that r_+^{-1} is not a positive generalized function. [Hint: If r_+^{-1} were positive, then $\rho(\rho r)_+^{-1}$ would also be a positive generalized function in contradiction with (C.24).]

In the next exercise we examine other methods of defining r_+^{-l-1} for $l \in \overline{\mathbf{Z}}_+$.

Exercise C.7. (a) Prove the relations

$$r_+^{-l-1} = \frac{(-1)^l}{l!} \left(\frac{\partial}{\partial r} \right)^l r_+^{-1} \quad \text{for } l \in \overline{\mathbf{Z}}_+, \tag{C.25}$$

$$r_+^{-1} = \frac{\partial}{\partial r} [\theta(r) \ln (e^\gamma r)], \tag{C.26}$$

where γ is the Euler-Masceroni constant:

$$\gamma = -\Gamma'(1). \tag{C.27}$$

(b) Prove that $\Gamma(\lambda) j_{\lambda-1}(r)$ has a (Laurent) expansion in λ of the form*

$$\Gamma(\lambda) j_{\lambda-1}(r) = \frac{\delta(r)}{\lambda} + [r_+^{-1} - \gamma\delta(r)] + O(\lambda) \quad \text{for } \lambda \to 0. \tag{C.28}$$

(c) Prove the formula for the Fourier transform:

$$\int r_+^{-l-1} e^{ixr} dr = \frac{(ix)^l}{l!} \ln(0 - ix). \tag{C.29}$$

Exercise C.8. Prove that every HGF $f(r)$ of degree λ in $\mathcal{S}'(\mathbf{R})$ with support in $\overline{\mathbf{R}}_+$ is proportional to $j_\lambda(r)$. [Hint: It follows from Exercise C.1 that $f(r) = A\theta(r) r^\lambda$ for $r \neq 0$. If $-(1+\lambda) \notin \overline{\mathbf{Z}}_+$ then this equality implies that $f(r) - c j_\lambda(r)$, where $c \in \mathbf{C}$ is a parameter, is an HGF of degree λ with support at the point $r = 0$; deduce that $f(r) = c j_\lambda(r)$. If $-(1+\lambda) = l \in \overline{\mathbf{Z}}_+$, then $f(r) = A r_+^{-l-1} + g(r)$, where $g(r)$ is a generalized function with support at the origin. Conclude from the homogeneity condition that $A = 0$ and $g(r) = c\delta^{(l)}(r)$.]

Exercise C.9. Prove that the space of homogeneous generalized functions of degree λ in $\mathcal{S}'(\mathbf{R})$ is two-dimensional. For $\lambda \notin \overline{\mathbf{Z}}_+$, the HGF's $(0+ix)^\lambda$ and $(0-ix)^\lambda$ form a basis in it, while for $\lambda = l \in \overline{\mathbf{Z}}_+$, the HGF's x^l and $x^l \operatorname{sgn} x$ form a basis. [Hint: The general reasoning is the same as in the previous exercise.]

C.3. Extension of homogeneous generalized functions

We define for any complex λ the linear operators $\mathcal{J}_\lambda$ and $\mathfrak{J}_\lambda$ in $\mathcal{S}(\mathbf{R}^n)$ and $\mathcal{E}(\mathring{\mathbf{R}}^n)$ by the formulae

$$(\mathcal{J}_\lambda u)(x) = \langle j_{\lambda'}(r), u(rx) \rangle_r \quad \text{for } u \in \mathcal{S}(\mathbf{R}^n), \ x \in \mathring{\mathbf{R}}^n, \tag{C.30}$$

* In [G4] the generalized function r_+^{-1} is defined from the decomposition $\Gamma(\lambda) j_{\lambda-1}(r) = \delta(r)\lambda^{-1} + r_+^{-1} + O(\lambda)$ as $\lambda \to 0$; therefore Gel'fand and Shilov's definition of r_+^{-1} differs from ours by the term $-\gamma\delta(r)$ concentrated at the origin. (A similar remark applies to the generalized functions r_+^{-l-1} for $l \in \mathbf{Z}_+$.)

$$(\hat{\mathfrak{Z}}_\lambda u)(x) \quad = \langle r_+^{\lambda'}, u(rx)\rangle_r \quad \text{for } u \in \mathcal{S}(\mathbf{R}^n),\ x \in \mathring{\mathbf{R}}^n, \tag{C.31}$$

where $\lambda' = -\lambda - 1 + n/2$. These operators are continuous for the same reason that $\mathfrak{Z}_\lambda$ is continuous. Furthermore, the analyticity of j_λ with respect to λ implies that of the operators $\mathcal{J}_\lambda$: thus for any $u \in \mathcal{S}(\mathbf{R}^n)$, $\mathcal{J}_\lambda u$ is a vector-valued analytic function in $\lambda \in \mathbf{C}$ with values in $\mathcal{E}(\mathring{\mathbf{R}}^n)$.

We indicate the main properties of the operators $\mathcal{J}_\lambda$ and $\hat{\mathfrak{Z}}_\lambda$.

Firstly, for all $u \in \mathcal{S}(\mathring{\mathbf{R}}^n)$ and $A \in GL(n, \mathbf{R})$

$$\mathcal{J}_\lambda(u_A) = (\mathcal{J}_\lambda u)_A, \quad (\hat{\mathfrak{Z}}_\lambda u_A) = (\hat{\mathfrak{Z}}_\lambda u)_A, \quad \text{where } u_A(x) \equiv u(Ax). \tag{C.32}$$

Secondly, $\mathcal{J}_\lambda$ maps $\mathcal{S}(\mathbf{R}^n)$ into $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$ (in view of the homogeneity of $j_{\lambda'}(r)$). In fact for $\lambda - n/2 \notin \overline{\mathbf{Z}}_+$, the image of $\mathcal{J}_\lambda$ is the whole of the space $D_\lambda(\mathring{\mathbf{R}}_n)$ (since on $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{J}_\lambda$ is proportional to the operator $\mathfrak{Z}_\lambda$ which, according to Proposition C.3, maps $\mathcal{S}(\mathring{\mathbf{R}}^n)$ onto $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$). For $\lambda - n/2 = l \in \overline{\mathbf{Z}}_+$, the image of $\mathcal{S}(\mathbf{R}^n)$ under the map $\mathcal{J}_\lambda$ is the space $\mathfrak{p}^{[l]}(\mathring{\mathbf{R}}^n)$ of polynomial functions in $\mathcal{E}(\mathring{\mathbf{R}}^n)$ that are homogeneous of degree l. This follows from the explicit form of $\mathcal{J}_\lambda$ when $\lambda - n/2 = l \in \overline{\mathbf{Z}}_+$:

$$(\mathcal{J}_\lambda u)(x) = \left(-x\frac{\partial}{\partial y}\right)^l u(y)\Big|_{y=0} = \left(\left(x\frac{\partial}{\partial y}\right)^l \delta(y), u(y)\right)_y. \tag{C.33}$$

Thirdly, by regarding $\mathcal{S}(\mathring{\mathbf{R}}^n)$ as a subspace of $\mathcal{S}(\mathbf{R}^n)$ we see that $\hat{\mathfrak{Z}}_\lambda$ is an extension of the operator $\mathfrak{Z}_\lambda : \mathcal{S}(\mathring{\mathbf{R}}^n) \to \mathcal{E}(\mathring{\mathbf{R}}^n)$. For $\lambda - n/2 \notin \overline{\mathbf{Z}}_+$, $\hat{\mathfrak{Z}}_\lambda$ differs from $\mathcal{J}_\lambda$ by a non-zero factor, therefore it also maps $\mathcal{S}(\mathbf{R}^n)$ onto $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$. However, when $\lambda - n/2 = l \in \overline{\mathbf{Z}}_+$, the image of $\mathcal{S}(\mathbf{R}^n)$ under the map $\hat{\mathfrak{Z}}_\lambda$ contains the functions of $\mathfrak{D}_\lambda(\mathring{\mathbf{R}}^n)$ along with the associated homogeneous functions of the first kind; namely, for $u \in \mathcal{S}(\mathbf{R}^n)$ and $\rho > 0$

$$(\hat{\mathfrak{Z}}_\lambda u)(\rho x) = \rho^l\left\{(\hat{\mathfrak{Z}}_\lambda u)(x) + \frac{(-1)^l}{l!}\ln\rho(\mathcal{J}_\lambda u)(x)\right\}. \tag{C.34}$$

We give the solution of the problem of extending an HGF in $\mathring{\mathbf{R}}^n$ to an HGF in $\mathbf{R}^n$ in terms of the above operators. Suppose that we are given $f \in \mathfrak{d}_\lambda(\mathring{\mathbf{R}}^n)$; then it follows from (C.8) and the indicated properties of the operator $\hat{\mathfrak{Z}}_{-\lambda}$ that the generalized function $\hat{f}(x) \in \mathcal{S}'(\mathbf{R}^n)$ defined by the formula

$$(\hat{f}, u) = \int f(x)(\hat{\mathfrak{Z}}_{-\lambda} u)(x)[d^n x] \quad \text{for } u \in \mathcal{S}(\mathbf{R}^n), \tag{C.35}$$

is equal to $f(x)$ in $\mathring{\mathbf{R}}^n$. It is clear that the general extension of f to a generalized function in $\mathbf{R}^n$ has the form $\hat{f} + f_0$, where $f_0 \in \mathcal{S}'(\mathbf{R}^n)$ and $\operatorname{supp} f_0 = \{0\}$. For the case $-(\lambda + n/2) \notin \overline{\mathbf{Z}}_+$, it follows from the properties (C.32) (with $A = \rho \cdot 1, \rho > 0$) and the equality $\hat{\mathfrak{Z}}_{-\lambda}\mathcal{S}(\mathbf{R}^n) = \mathfrak{D}_{-\lambda}(\mathbf{R}^n)$, that $\hat{f} \in \mathfrak{d}_{-\lambda}(\mathbf{R}^n)$; in fact $\hat{f}$ is the unique extension of such an f. In the case when $-(\lambda + n/2) \in \overline{\mathbf{Z}}_+$, $\hat{f}$ may not in general be homogeneous. In fact it follows from (C.32) (with $A = \rho \cdot 1,\ \rho > 0$) and (C.34), that $\hat{f}$ is an HGF (of degree $\lambda - n/2$) if and only if f satisfies the condition

$$\int f(x)(\mathcal{J}_{-\lambda} u)(x)[d^n x] = 0 \quad \text{for all } u \in \mathcal{S}(\mathbf{R}^n). \tag{C.36}$$

If this condition holds, then the homogeneous extension of f is clearly non-unique (we can add to $\hat{f}$ an arbitrary $f_0 \in \mathfrak{d}_{-\lambda}(\mathbf{R}^n)$ with support at the origin). On the other hand, if (C.36) does not hold, then the lack of homogeneity of $\hat{f}$ cannot be made good by adding some generalized function $f_0 \in \mathcal{S}'(\mathbf{R}^n)$ with support at the origin.

We have obtained the following result.

Theorem C.4. *Let $f(x)$ be an HGF of degree $\lambda - n/2$ in $\mathcal{S}'(\mathring{\mathbf{R}}^n)$.*

(a) *If $-(\lambda + n/2) \notin \overline{\mathbf{Z}}_+$, then f has a unique extension to an HGF of degree $\lambda - n/2$ in $\mathcal{S}'(\mathbf{R}^n)$; this extension is given by* (C.35).

(b) *If* $-(\lambda + n/2) = l \in \overline{\mathbf{Z}}_+$, *then* f *has an extension to an HGF of degree* $\lambda - n/2$ *in* $\mathcal{S}'(\mathbf{R}^n)$ *if and only if it satisfies the condition*

$$\int f(x)P(x)[d^n x] = 0 \quad \textit{for all} \quad P \in \mathfrak{p}^{[l]}(\mathring{\mathbf{R}}^n). \tag{C.37}$$

If this condition holds, then the general form of such an extension is $\hat{f} + f_0$, *where* $\hat{f}$ *is defined by* (C.35) *and* f_0 *is an HGF of degree* $-(l+n)$ *in* $\mathcal{S}'(\mathbf{R}^n)$ *with support at the origin.*

Corollary C.5. An HGF $f(x)$ of degree $-(l+n)$ in $\mathring{\mathbf{R}}^n$ with $l \in \overline{\mathbf{Z}}_+$ admits a homogeneous extension in $\mathbf{R}^n$ if and only if for all $P \in \mathfrak{p}^{[l]}(\mathring{\mathbf{R}}^n)$, the HGF $P(x)f(x)$ of degree $-n$ in $\mathring{\mathbf{R}}^n$ has a homogeneous extension in $\mathbf{R}^n$.

In the case when the homogeneous extension of f exists and is unique, as in part (a) of Theorem C.4, we talk about a *forced extension.* The forced extension can also involve cases in part (b) if we confine ourselves to HGF's with parity σ such that

$$\sigma(-1)^l = -1. \tag{C.38}$$

Here we say that the HGF $f(x)$ has *parity* $\sigma = \pm$ if

$$f(-x) = \sigma f(x). \tag{C.39}$$

Exercise C.10. Let $l \in \overline{\mathbf{Z}}_+$. Prove that every HGF $f(x)$ of degree $-(l+n)$ in $\mathring{\mathbf{R}}^n$ with parity σ (C.38) has a unique extension to an HGF in $\mathbf{R}^n$ of the same degree and the same parity; this extension is defined by (C.35) where we now further suppose that the function h defining the measure $[d^n x]$ is even.

In contrast to what happens in Corollary C.2, the homogeneous extension $\hat{f}$ given by (C.35) in the case when $-(\lambda + n/2) \in \overline{\mathbf{Z}}_+$, subject, of course, to condition (C.37), depends in general on the function $h(x)$ satisfying (C.4) and featuring in the definition of the measure $[d^n x]$. Of interest in this case is a variant of the method of analytic continuation with respect to the parameter that enables one to construct homogeneous extensions by a method that does not depend on h. Suppose that we are given a family $\{f_\lambda\} \subset \mathcal{S}'(\mathring{\mathbf{R}}^n)$ of HGF's in $\mathbf{R}^n$, where f_λ depends analytically on the parameter $\lambda \in \mathbf{C}$ and satisfies the condition $f_\lambda \in \mathfrak{d}_\lambda(\mathring{\mathbf{R}}^n)$. We associate with it the family $\{F_\lambda\} \in \mathcal{S}'(\mathbf{R}^n)$ of HGF's in $\mathbf{R}^n$, where F_λ is defined by the formula

$$(F_\lambda, u) = \int f_\lambda(x)(\mathcal{J}_{-\lambda}u)(x)[d^n x] \tag{C.40}$$

and depends analytically on $\lambda \in \mathbf{C}$. It is clear that F_λ belongs to $\mathfrak{d}_\lambda(\mathbf{R}^n)$ and is independent of the choice of h. It is now fairly easy to see that if for some fixed $-(\lambda + n/2) = l \in \overline{\mathbf{Z}}_+$, f_λ has a homogeneous extension in $\mathbf{R}^n$, that is, if $F_\lambda = 0$ for this λ, then

$$\hat{f}_\lambda = \frac{(-1)^l}{l!}\frac{\partial}{\partial \lambda}F_\lambda \tag{C.41}$$

provides us with one of these extensions (defined independently of the choice of h). More generally, we can differentiate F_λ with respect to the parameter to obtain non-trivial HGF's in $\mathfrak{d}_\lambda(\mathbf{R}^n)$ for those values of λ at which $F_\lambda = 0$; it suffices to take a non-vanishing lower derivative of F_λ with respect to λ.

The method of differentiating with respect to the parameter with the aim of obtaining new HGF's can be generalized to separately homogeneous generalized functions. Let $\lambda \equiv (\lambda_1, \ldots, \lambda_k) \in \mathbf{C}^k$, and $\mathfrak{d}_\lambda$ the subspace of $\mathcal{S}'(\Omega_1 \times \ldots \times \Omega_k)$ (where Ω_j is $\mathring{\mathbf{R}}^{n_j}$ or $\mathbf{R}^{n_j}$) of homogeneous generalized functions $f(x_1, \ldots, x_k)$ of degree $\lambda_j - n_j/2$ in x_j ($j = 1, \ldots, k$). Finally, let $\{F_\lambda\}$ be a family of generalized functions that are analytically dependent on the parameter λ and are such that $F_\lambda \in \mathfrak{d}_\lambda$. Then it is not difficult to see that $P\left(\frac{\partial}{\partial\lambda}\right) F_\lambda \equiv P\left(\frac{\partial}{\partial\lambda_1}, \ldots, \frac{\partial}{\partial\lambda_k}\right) F_\lambda$ belongs to $\mathfrak{d}_\lambda$ (where $P(z)$ is a complex polynomial in the k variables $z \equiv (z_1, \dots, z_k)$) if and only if $P^{(\alpha)}\left(\frac{\partial}{\partial \lambda}\right) F_\lambda = 0$ for any non-zero multi-indices $\alpha \equiv (\alpha_1, \dots, \alpha_k) \in \overline{\mathbf{Z}}_+^k \setminus \{0\}$; here $P^{(\alpha)}$ denotes the polynomial $P^{(\alpha)}(z) = \prod_{j=1}^k \left(\frac{\partial}{\partial z_j}\right)^{\alpha_j} P(z)$. We say that each HGF in $\mathfrak{h}_\lambda$ of the form $P\left(\frac{\partial}{\partial \lambda}\right) F_\lambda$ is *associated* with the family $\{F_\lambda\}$ (many of the HGF's encountered in physics are of this sort, for example, the HGF's associated with quadratic forms [G4]).

C.4. Application to covariant homogeneous generalized functions*

We give a version of Theorem C.4 relating to the extension of HGF's that are covariant (in particular, invariant) with respect to a Lie group G acting linearly in $\mathbf{R}^n$. We suppose that we are given an L-dimensional complex matrix representation $T(g) \equiv (T^\alpha_\beta(g))$ of G; $\widetilde{T}(g)$ is the contragradient representation.

Theorem C.6. *Let $f(x) \equiv (f_\alpha(x))_{\alpha=1,\dots,L}$ be a G-covariant HGF of degree $\lambda - n/2$ in $\mathring{\mathbf{R}}^n$ that is transformed according to the representation $\widetilde{T}$ of the group G.*

(a) *Let $-(\lambda+n/2) \notin \overline{\mathbf{Z}}_+$; then f has a unique extension to a G-covariant HGF of degree $\lambda - n/2$ in $\mathbf{R}^n$ that is transformed under the representation $\widetilde{T}$; it can be defined by the formula*

$$(\hat{f}, u) = \int \sum_{\alpha=1}^{L} f_\alpha(x)(\mathfrak{S}_{-\lambda} u^\alpha)(x)[d^n x] \ \text{for}\ u^\alpha \in \mathcal{S}(\mathbf{R}^n). \tag{C.42}$$

(b) *Let $-(\lambda + n/2) = l \in \overline{\mathbf{Z}}_+$ and suppose that G has the property that all the reducible complex finite-dimensional representations of G are completely reducible.*** *The HGF f has a G-covariant homogeneous extension (of degree $\lambda - n/2$) in $\mathbf{R}^n$ (which transforms according to the representation $\widetilde{T}$) if and only if for any G-covariant polynomial $p(x) \equiv (p^\alpha(x))_{\alpha=1,\dots,L}$ in $\mathbf{R}^n$ that is transformed according to T and is homogeneous of degree l, the G-invariant HGF $\sum_\alpha f_\alpha(r)p^\alpha(x)$ in $\mathfrak{h}_{-n/2}(\mathbf{R}^n)$ has a homogeneous extension in $\mathbf{R}^n$, that is,*

$$\int \sum_\alpha f_\alpha(x)p^\alpha(x)[d^n x] = 0. \tag{C.43}$$

The non-uniqueness of this extension is clearly characterized by the dimension of the space of G-covariant HGF's in $\mathbf{R}^n$ that are transformed under $\widetilde{T}$, are homogeneous of degree $\lambda - n/2$ and have support at the origin.

■ We restrict ourselves to an outline of the proof. Part (a) is a consequence of the forced nature of the homogeneous extension in part (a) of Theorem C.4. The necessity of condition (C.43) in case (b) follows from Corollary C.5. For the proof of its sufficiency, we introduce the finite-dimensional space $\mathcal{P}$ of vector-valued functions $p(x) \equiv (p^\alpha(x))_{\alpha=1,\dots,L}$ whose components $p^\alpha(x)$ are complex polynomial functions in $\mathring{\mathbf{R}}^n$ that are homogeneous of degree l. The representation τ of the group G acts in $\mathcal{P}$: $(\tau(g)p)^\alpha(x) = \sum_\beta T^\alpha_\beta(g)p^\beta(g^{-1}x)$. The left hand side of (C.43) defines a G-invariant linear functional $\Phi(p)$ on $\mathcal{P}$ which vanishes at those elements $p \in \mathcal{P}$ for which $\tau(g)p = p$ for all $g \in G$. We claim that $\Phi(p) = 0$ on $\mathcal{P}$; thus we will prove that according to Corollary C.5, f has a homogeneous extension of degree $-(l+n)$ in $\mathbf{R}^n$. To this end, we decompose $\mathcal{P}$ into its invariant irreducible subspaces $\mathcal{P}_i$. There are three possibilities. The first is that $\mathcal{P}_i$ is one-dimensional and is transformed according to the trivial representation of G; then $\Phi(p) = 0$ on $\mathcal{P}_i$ by hypothesis. The second possibility is that $\mathcal{P}_i$ is one-dimensional and is transformed according to a non-trivial representation; it then follows from the invariance of f that $\Phi(p) = \tau_i(g)\Phi(p)$ for $p \in \mathcal{P}_i$, so that $\Phi(p) = 0$ on $\mathcal{P}_i$. Finally, if the dimension of $\mathcal{P}_i$ exceeds unity, then Φ is also equal to zero on $\mathcal{P}_i$ (otherwise the null space of the functional Φ restricted to $\mathcal{P}_i$ would be a proper invariant subspace of $\mathcal{P}_i$). Thus we have proved that f has a

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* The concepts used in this subsection will be discussed in greater detail in §3.4.B.

** All compact and all connected semisimple Lie groups have this property. It can be deduced from this property that the modulus of the determinant of any finite-dimensional representation of G is identically equal to unity. (Otherwise it would define a homomorphism from G onto the group $\mathbf{R}_+$, which does not enjoy such a property of complete reducibility of the representations; in this case G would not have this property either.)

one-dimensional extension $\hat{f}$ in $\mathbf{R}^n$. It remains to prove that such an extension can be chosen to be G-covariant. For this we note that the homogeneity property is preserved if we replace $\hat{f}$ by $\hat{f} + h$, where $h \equiv (h_\alpha(x))_{\alpha=1,\ldots,L}$ is an arbitrary vector-valued generalized function whose components $h_\alpha(x)$ are HGF's of degree $-(l+n)$ in $\mathbf{R}^n$ with support at the origin. We denote the finite-dimensional complex space of all such vector-valued HGF's h by Q. We can now use this arbitrariness to arrange for $\hat{f} + h$ to be a G-covariant HGF. (At this final stage of the proof, it is necessary to invoke the fact ([K6],§8.1) that the property of complete reducibility of all reducible complex finite-dimensional representations of a group G is equivalent to the triviality of the one-dimensional cohomology groups $H^1(G, V)$ associated with arbitrary representations of G in complex finite-dimensional vector spaces V. We obtain the required result by applying this remark to the case $V = Q$.) ∎

Part (b) of Theorem C.6 gives no indication how to construct covariant homogeneous extensions. If in case (b) we further suppose that G is compact, then at least one of the desired extensions (if it exists) is given by (C.42), where it is now supposed that the measure $[d^n x]$ is constructed from a G-invariant function $h \in \mathcal{S}(\mathbf{R}^n)$ satisfying (C.4).

In the next two exercises, situations with forced extensions are considered.

Exercise C.11. Prove that the result stated in Exercise C.10 still holds if it is further required that $f(x)$ and its extension be G-covariant.

Exercise C.12. Suppose that $l \in \mathbf{Z}_+$ and that the group G is such that its reducible complex finite-dimensional representations are completely reducible. Prove that if the space of G-covariant polynomial functions in $\mathbf{R}^n$ that are transformed according to the representation T and are homogeneous of degree l consists of just the zero function, then every G-covariant HGF in $\mathring{\mathbf{R}}^n$ that is transformed according to $\widetilde{T}$ and is homogeneous of degree $-(l+n)$ has a unique G-covariant extension in $\mathbf{R}^n$ that is transformed according to $\widetilde{T}$ and is homogeneous of degree $-(l+n)$.

The next example shows that the conditions on the group G in part (b) of Theorem C.6 are essential.

Exercise C.13. Let the group $G = \mathbf{R}$ act in $\mathbf{R}^2$ by the transformation $(x_1, x_2) \to (x_1 + tx_2, x_2)$, $t \in \mathbf{R}$ and let $f \in \mathcal{S}'(\mathring{\mathbf{R}}^2)$ be defined by the equality $f(x) = \theta(x_1)\delta'(x_2)$. Prove that f is homogeneous and G-invariant and has a homogeneous but not G-invariant extension in $\mathbf{R}^2$. [Hint: The infinitesimal version of the G-invariance condition has the form $x_2 \frac{\partial}{\partial x_1} f = 0$.]

C.5. HOMOGENEOUS GENERALIZED FUNCTIONS IN THE COMPLEX PLANE

Let $\Omega \subset \mathbf{C}^n$ be an arbitrary open set. Then we define the spaces of test and generalized functions $\mathcal{S}(\Omega)$ and $\mathcal{S}'(\Omega)$ by regarding the test (or generalized) function $f(z)$ of the vector $z \equiv x + iy \in \Omega$ as the test (or generalized) function of $(x, y) \in \mathbf{R}^{2n}$. Again we use the following symbolic integral notation for the value of the generalized function $f(z) \in \mathcal{S}'(\Omega)$ at $u(z) \in \mathcal{S}(\Omega)$:

$$(f, u) = \int f(z)u(z)|d^n z d^n \bar{z}|, \tag{C.44}$$

where

$$2^{-n}|d^n z d^n \bar{z}| \equiv d^n x d^n y \tag{C.45}$$

is Lebesgue measure on $\mathbf{C}^n$. The Schwartz spaces of test functions $\mathcal{D}(\Omega)$ and distributions $\mathcal{D}'(\Omega)$ are defined in similar fashion. In particular, the delta-function $\delta(z) \in \mathcal{S}'(\mathbf{C}^n)$ is defined by the formula

$$\int \delta(z)u(z)|d^n z d^n \bar{z}| = u(0). \tag{C.46}$$

By the *index of homogeneity* we mean the ordered pair $\chi \equiv (\lambda, \mu)$ of complex numbers such that $\lambda - \mu \in \mathbf{Z}$. We denote the set of all such indices by $\mathcal{X}$, which is a complex manifold consisting of a countable number of copies of the complex plane $\mathbf{C}$ with the variable $\frac{1}{2}(\lambda + \mu)$ as a local coordinate.

Let Ω be either $\mathbf{C}^n$ or $\mathring{\mathbf{C}}^n$, where

$$\mathring{\mathbf{C}}^n = \mathbf{C}^n \setminus \{0\} \tag{C.47}$$

is the complex space $\mathbf{C}^n$ punctured at the origin. We say that the function (or generalized function) $f(z)$ in Ω is *homogeneous of index* χ (or of bidegree $(\lambda - n/2, \mu - n/2)$), if

$$f(az) = \phi_\chi^{[n]}(a)f(z) \quad \text{for all } a \in \mathring{\mathbf{C}}, \tag{C.48}$$

where

$$\phi_\chi^{[n]}(a) = |a|^{\lambda+\mu-n}\exp(i(\lambda-\mu)\arg a) \quad \text{for } a \in \overset{\circ}{\mathbf{C}}. \tag{C.49}$$

We denote by $\mathfrak{D}_\chi(\Omega)$ and $\mathfrak{d}_\chi(\Omega)$ the subspaces of $\mathcal{E}(\Omega)$ and $\mathcal{S}'(\Omega)$ respectively, of smooth (or generalized) functions in Ω that are homogeneous of index χ. A special role is played by the indices in the sets

$$\Xi_+^{[n]} = \{\chi \equiv (\lambda,\mu) \in \mathcal{X} : \lambda - n/2 \in \overline{\mathbf{Z}}_+, \mu - n/2 \in \overline{\mathbf{Z}}_+\}, \tag{C.50}$$

$$\Xi_-^{[n]} = \{-\chi \equiv (-\lambda,-\mu) : \chi \in \Xi_+^{[n]}\}. \tag{C.51}$$

In the first case, $\mathfrak{D}_\chi(\overset{\circ}{\mathbf{C}}{}^n)$ contains the (finite-dimensional) subspace $\mathfrak{p}_\chi(\overset{\circ}{\mathbf{C}}{}^n)$ of polynomials in $z, \bar{z}$ that are homogeneous of degree $\lambda - n/2$ in z and of degree $\mu - n/2$ in $\bar{z}$; in the second case, $\mathfrak{d}_\chi(\mathbf{C}^n)$ contains the (finite-dimensional) subspace $\mathcal{Q}_\chi(\mathbf{C}^n)$ of HGF's of index χ with support at the origin.

For HGF's in a complex domain there are results that are completely analogous to those obtained in the real case. We therefore confine ourselves to their statements without proofs.

In fact the complex case reduces to the real case. To see this, we rewrite (C.48) in the form of the two conditions:

$$f(\rho z) = \rho^{\lambda+\mu-n} f(z) \quad \text{for } \rho > 0, \tag{C.52a}$$

$$f(e^{i\phi} z) = e^{i(\lambda-\mu)\phi} f(z) \quad \text{for } \phi \in \mathbf{R}; \tag{C.52b}$$

(C.52a) is the condition of homogeneity of degree $\lambda + \mu - n$ in the variables $(x, y) \in \mathbf{R}^{2n}$, whereas (C.52b) can be interpreted as the condition of covariance of $f(z)$ with respect to the group $U(1) \equiv$ $\equiv \{a \in \mathbf{C} : |a| = 1\}$.

For each index χ, the spaces $\mathfrak{d}_\chi(\overset{\circ}{\mathbf{C}})$ and $\mathfrak{d}_\chi(\mathbf{C})$ are one-dimensional and are generated by the function $\phi_\chi^{[1]}(z)$ and the HGF $\psi_\chi(z)$ respectively, which are defined as follows. We set

$$\lambda \vee \mu = 1/2(\lambda + \mu + |\lambda - \mu|), \quad \lambda \wedge \mu = 1/2(\lambda + \mu - |\lambda - \mu|). \tag{C.53}$$

We define $\psi_\chi(z)$ for $\mathrm{Re}(\lambda + \mu) > 1$ as the following continuous function in $\mathbf{C}$:

$$\psi_\chi(z) = \frac{1}{\Gamma((\lambda \vee \mu) + 1/2)} |z|^{\lambda+\mu-1} \exp(i(\lambda - \mu)\arg z)). \tag{C.54}$$

Clearly ψ_χ is a generalized function in $\mathcal{S}'(\mathbf{C})$ which is analytically dependent on the parameter χ (in the sense of of analyticity in $\frac{1}{2}(\lambda + \mu)$ for fixed $\lambda - \mu \in \mathbf{Z}$). Then ψ_χ has a (unique) analytic continuation in χ to the whole of the index manifold $\mathcal{X}$; this continuation can be carried out (by induction on the integer part of $-\mathrm{Re}(\lambda + \mu)$) by means of the identities in $\mathcal{S}'(\mathbf{C})$:

$$\psi_\chi(z) = \frac{\partial}{\partial z}\psi_{(\lambda+1,\mu)}(z) \quad \text{for } \lambda - \mu \in \overline{\mathbf{Z}}_+, \tag{C.55a}$$

$$\psi_\chi(z) = \frac{\partial}{\partial \bar{z}}\psi_{(\lambda,\mu+1)}(z) \quad \text{for } \mu - \lambda \in \overline{\mathbf{Z}}_+. \tag{C.55b}$$

Exercise C.14. Prove that for $\chi \in \Xi_-^{[1]}$

$$\psi_\chi(z) = \frac{(-1)^{-(\lambda\wedge\mu)-1/2}}{\Gamma(-(\lambda \wedge \mu) + 1/2)} \left(\frac{\partial}{\partial z}\right)^{-\lambda-1/2} \left(\frac{\partial}{\partial \bar{z}}\right)^{-\mu-1/2} [2\pi\delta(z)] = \tag{C.56a}$$

$$= (-1)^{-(\lambda\wedge\mu)-1/2} \psi_{-\chi}\left(\frac{\partial}{\partial z}\right) [2\pi\delta(z)]; \tag{C.56b}$$

where $\psi_{-\chi}\left(\frac{\partial}{\partial z}\right)$ is a differential polynomial of order $-(\lambda + \mu + 1)$.

It is not difficult to see that

$$\psi_\chi(z) = \frac{1}{\Gamma((\lambda \vee \mu) + 1/2)} \phi_\chi^{[1]}(z) \text{ in the domain } z \in \overset{\circ}{\mathbf{C}}. \tag{C.57}$$

It follows from (C.56) that the support of ψ_χ is $\mathbf{C}$ when $\chi \notin \Xi_-^{[1]}$ and the support of ψ_χ is $\{0\}$ when $\chi \in \Xi_-^{[1]}$.

Another useful family of generalized functions in $\mathbf{C}$ is formed by the continuations of $\phi_\chi^{[1]}$ from $\overset{\circ}{\mathbf{C}}$ to $\mathbf{C}$:

$$\hat{\phi}_\chi = \Gamma((\lambda \vee \mu) + 1/2)\psi_\chi \quad \text{for } \chi \notin \Xi_-^{[1]}, \tag{C.58a}$$

$$\hat{\phi}_\chi = \frac{(-1)^{-(\lambda\vee\mu)-1/2}}{\Gamma(-(\lambda \vee \mu) + 1/2)} \frac{\partial}{\partial \chi}\psi_\chi \quad \text{for } \chi \in \Xi_-^{[1]} \tag{C.58b}$$

(where $\frac{\partial}{\partial \chi}$ is the derivative with respect to the parameter $\frac{1}{2}(\lambda + \mu)$ for fixed $\lambda - \mu$). It is not difficult to see that $\hat{\phi}_\chi$ is the continuation of ϕ_χ from $\overset{\circ}{\mathbf{C}}$ to $\mathbf{C}$. For $\chi \notin \Xi_-^{[1]}$, this continuation is homogeneous, whereas for $\chi \in \Xi_-^{[1]}$, it is an associated HGF of the first kind:

$$\hat{\phi}_\chi(az) = \phi_\chi^{[1]}(a)\left\{\hat{\phi}_\chi(z) + \ln|a| \frac{2(-1)^{-(\lambda\vee\mu)-1/2}}{\Gamma(-(\lambda \vee \mu) + 1/2)}\psi_\chi(z)\right\} \quad \text{for } a \in \overset{\circ}{\mathbf{C}}. \tag{C.59}$$

We now go over to the case of several complex variables. We set

$$[d^n z d^n \bar{z}] = H(z)|d^n z d^n \bar{z}|, \tag{C.60}$$

where $H(z)$ is a fixed function in $\mathcal{D}(\overset{\circ}{\mathbf{C}}{}^n)$ such that

$$\int_{\overset{\circ}{\mathbf{C}}} H(az)\frac{|da d\bar{a}|}{|a|^2} = 1 \quad \text{for all } z \in \overset{\circ}{\mathbf{C}}{}^n. \tag{C.61}$$

The integral of the HGF $f(z) \in \mathfrak{d}_{(-n/2,-n/2)}(\overset{\circ}{\mathbf{C}}{}^n)$ with respect to the measure $[d^n z d^n \bar{z}]$

$$\int f(z)[d^n z d^n \bar{z}] = (f, H) \tag{C.62}$$

does not depend on the choice of the function H satisfying (C.61). Proposition C.7 generalizes this result.

Proposition C.7. *For any $f \in \mathfrak{d}_\chi(\overset{\circ}{\mathbf{C}}{}^n)$, $u \in \mathcal{S}(\overset{\circ}{\mathbf{C}}{}^n)$ we have*

$$(f, u) = \int f(z)(\Phi_{-\chi}u)(z)[d^n z d^n \bar{z}]; \tag{C.63}$$

where the continuous operator $\Phi_\chi : \mathcal{S}(\overset{\circ}{\mathbf{C}}{}^n) \to \mathfrak{D}_\chi(\overset{\circ}{\mathbf{C}}{}^n)$ is defined by the formula

$$(\Phi_\chi u)(z) = \int_{\overset{\circ}{\mathbf{C}}} \phi_\chi^{[n]}(a^{-1})u(az)\frac{|da d\bar{a}|}{|a|^2}. \tag{C.64}$$

The next result is an analogue of Proposition C.3.

Proposition C.8. (a) *The operator Φ_χ is a topological isomorphism from $\mathcal{S}(\overset{\circ}{\mathbf{C}}{}^n)$ onto $\mathfrak{D}_\chi(\overset{\circ}{\mathbf{C}}{}^n)$.*

(b) *The adjoint $\Phi'_\chi : \mathfrak{D}_\chi(\overset{\circ}{\mathbf{C}}{}^n)' \to \mathcal{S}'(\overset{\circ}{\mathbf{C}}{}^n)$ maps $\mathfrak{D}_\chi(\overset{\circ}{\mathbf{C}}{}^n)'$ topologically isomorphically onto the subspace $\mathfrak{d}_{-\chi}(\overset{\circ}{\mathbf{C}}{}^n)$.*

In particular, this proposition enables us to associate with each (separately) continuous multilinear functional $\mathcal{K}$ on $\mathfrak{D}_{\chi_1}(\overset{\circ}{\mathbf{C}}{}^n) \times \ldots \times \mathfrak{D}_{\chi_k}(\overset{\circ}{\mathbf{C}}{}^n)$ the *kernel of the multilinear functional*, that is, the generalized function $K(z_1, \ldots, z_k) \in \mathcal{S}'((\overset{\circ}{\mathbf{C}}{}^n)^k)$ related to $\mathcal{K}$ by the formula

$$(K, u_1 \otimes \ldots \otimes u_k) = \mathcal{K}(\Phi)_{\chi_1}(u_1), \ldots, \Phi_{\chi_k}(u_k)) \quad \text{for } u_j \in \mathcal{S}(\overset{\circ}{\mathbf{C}}{}^n). \tag{C.65}$$

The kernel K satisfies the condition of *separate homogeneity* in $z_1, \ldots, z_k$ (of index $-\chi_j$ in z_j; $j = 1, \ldots, n$):

$$K(a_1 z_1, \ldots, a_k z_k) = \prod_{j=1}^{k} \phi^{[n]}_{-\chi_j}(a_j) K(z_1, \ldots, z_k) \quad \text{for } a_1, \ldots, a_k \in \mathring{\mathbf{C}}. \tag{C.66}$$

By means of formula (C.65) and its inverse

$$\mathcal{K}(v_1, \ldots, v_k) = \int K(z_1, \ldots, z_k) v_1(z_1) \ldots v_k(z_k) [d^n z d^n \bar{z}_1] \ldots [d^n z_k d^n \bar{z}_k] \tag{C.67}$$

(where $v_j \in \mathfrak{D}_{\chi_j}(\mathring{\mathbf{C}}^n)$, $j = 1, \ldots, k$), we can set up a one-to-one correspondence $\mathcal{K} \leftrightarrow K$ between all the (separately) continuous multilinear functionals $\mathcal{K}$ on $\mathfrak{D}_{\chi_1}(\mathring{\mathbf{C}}^n) \times \ldots \times \mathfrak{D}_{\chi_k}(\mathring{\mathbf{C}}^n)$ and the space of all generalized functions $K(z_1, \ldots, z_k) \in \mathcal{S}'((\mathring{\mathbf{C}}^n)^k)$ satisfying the condition of separate homogeneity (C.66).

In conclusion we consider the question of extending an HGF from $\mathring{\mathbf{C}}^n$ to $\mathbf{C}^n$. We define the continuous operators Ψ_χ and $\hat{\Phi}_\chi$ from $\mathcal{S}(\mathbf{C}^n)$ to $\mathcal{E}(\mathring{\mathbf{C}}^n)$ by setting

$$(\Psi_\chi u)(z) = (\psi_{\chi'}(a),\ u(az))_a, \tag{C.68}$$

$$(\hat{\Phi}_\chi u)(z) = (\hat{\phi}_{\chi'}(a),\ u(az))_a, \tag{C.69}$$

where $u \in \mathcal{S}(\mathbf{C}^n)$, $z \in \mathring{\mathbf{C}}^n$, $\chi' = \left(-\lambda + \frac{n-1}{2}, -\mu + \frac{n-1}{2}\right)$; here the right hand sides are to be understood in the sense of the values of generalized functions in $\mathcal{S}'(\mathbf{C})$ with respect to the variable a at the test functions $a \to u(az)$ in $\mathcal{S}(\mathbf{C})$, depending in C^∞ fashion on the parameter $z \in \mathring{\mathbf{C}}^n$.

Exercise C.15. Prove that for $\chi \in \Xi^{[1]}_+$, we have the following explicit expression for Ψ_χ:

$$(\Psi_\chi u)(z) = (-1)^{(\lambda \vee \mu) - n/2} \left(\psi_{-\chi'}\left(z \frac{\partial}{\partial w}\right) \delta(w),\ u(w)\right)_w. \tag{C.70}$$

For $\chi \notin \Xi^{[n]}_+$, the operators Ψ_λ and $\hat{\Phi}_\chi$ map $\mathcal{S}(\mathbf{C}^n)$ onto $\mathfrak{D}_\chi(\mathring{\mathbf{C}}^n)$. When $\chi \in \Xi^{[n]}_+$, Ψ_χ maps $\mathcal{S}(\mathbf{C}^n)$ onto the (finite-dimensional) subspace of $\mathfrak{D}_\chi(\mathring{\mathbf{C}}^n)$ consisting of polynomial functions in $z, \bar{z}$, that are homogeneous of index χ, whereas the image of $\mathcal{S}(\mathbf{C}^n)$ under the map $\tilde{\Phi}_\chi$ for such χ contains the homogeneous functions in $\mathfrak{D}_\chi(\mathring{\mathbf{C}}^n)$ along with the associated homogeneous functions of the first kind. In fact it follows from (C.59) that for $\chi \in \Xi^{[n]}_+$

$$(\hat{\Phi}_\chi u)(az) = \phi^{[n]}_\chi(a) \left\{(\hat{\Phi}_\chi u)(z) + \ln|a| \frac{2(-1)^{\lambda \wedge \mu) - n/2}}{\Gamma((\lambda \wedge \mu) - n/2 + 1)} (\Psi_\chi u)(z)\right\} \text{ for } a \in \mathring{\mathbf{C}}. \tag{C.71}$$

Using the above properties of the operators $\hat{\Phi}_\chi$ and $\Psi\chi$, we establish the following result on the extension of HGF's of several complex variables.

Theorem C.9. *Let $f(z)$ be an HGF in $\mathcal{S}'(\mathring{\mathbf{C}}^n)$ that is homogeneous of index χ.*

(a) *If $\chi \notin \Xi^{[n]}_-$, then f has a unique extension to an HGF of index χ in $\mathcal{S}'(\mathbf{C}^n)$; this extension $\hat{f}$ is defined by the formula*

$$(\hat{f}, u) = \int f(z) (\hat{\Phi}_{-\chi} u)(z) [d^n z d^n \bar{z}]. \tag{C.72}$$

(b) *If $\chi \in \Xi^{[n]}_-$, then f has a homogeneous extension of index ξ in $\mathbf{C}^n$ if and only if*

$$\int f(z) P(z) [d^n z d^n \bar{z}] = 0 \tag{C.73}$$

for all functions $P(z)$ in $\mathfrak{D}_{-\chi}(\mathring{\mathbf{C}}^n)$, that are polynomial in $z, \bar{z}$. When this condition holds, the general form of such an extension is the sum of the HGF $\hat{f}$ defined by (C.72) and an arbitrary HGF in $\mathfrak{d}_\chi(\mathbf{C}^n)$ with support at the origin

CHAPTER 3

Lorentz-Covariant Generalized Functions

3.1 The Lorentz Group

A. THE GEOMETRY OF MINKOWSKI SPACE

By *Minkowski space* $\mathbf{M}$ we mean the four-dimensional real space $\mathbf{R}^4$ endowed with the pseudo-Euclidean scalar product

$$pq = p^0 q^0 - \mathbf{pq} = g_{\lambda\mu} p^\lambda q^\mu = p_\lambda q^\lambda. \tag{3.1}$$

We shall agree to write an arbitrary 4-vector in $\mathbf{M}$ in the form $p \equiv (p^0, p^1, p^2, p^3) \equiv$ $\equiv (p^0, \mathbf{p})$, where p^0 is the time coordinate and $\mathbf{p}$ is the spatial part of p. We denote by $g \equiv (g_{\lambda\mu})$ the metric tensor with components

$$g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \ g_{\lambda\mu} = 0 \text{ for } \lambda \neq \mu. \tag{3.2}$$

Where there are repeated upper and lower indices (in formula (3.1) and hereafter), it is understood that these indices are to be summed from 0 to 3 ("Einstein summation rule"). Raising and lowering of indices is achieved by means of the metric tensor g and its inverse $g^{-1} = (g^{\lambda\mu}) = (g_{\lambda\mu})$:

$$p_\lambda = g_{\lambda\mu} p^\mu, \ p^\lambda = g^{\lambda\mu} p_\mu. \tag{3.3}$$

We denote the scalar square* of a vector by p^2:

$$p^2 = p \cdot p. \tag{3.4}$$

The basis

$$e_0 = (1,0,0,0), \ e_1 = (0,1,0,0), \ e_2 = (0,0,1,0), \ e_3 = (0,0,0,1) \tag{3.5}$$

is called the *standard Lorentz system* (or standard frame) in $\mathbf{M}$; here e_0 is called the time basis vector and e_1, e_2, e_3 the spatial basis vectors (of the given system).

Along with the Minkowski vector space, we also introduce the *affine* (coordinate) *Minkowski space* in which the invariant object is not the scalar product but the pseudo-Euclidean distance, the square of which is equal to $(x-y)^2 \equiv (x^0 - y^0)^2 - (\mathbf{x} - \mathbf{y})^2$. In the physical context, coordinate Minkowski space, denoted sometimes by $\mathbf{M}_x$, serves as the space-time of the special theory of relativity; the natural element of volume in $\mathbf{M}_x$ is the Lebesgue measure d^4x. The Minkowski vector space then

* This will lead to no misunderstanding; for in those rare cases when p^2 denotes the second component of the vector p we shall make specific mention of it.

features either as the tangent space to $\mathbf{M}_x$ (that is, the space of distances $x - y$, where $x, y \in \mathbf{M}_x$) or as the dual to the tangent space, when it then plays the role of *Minkowski momentum space* and is sometimes denoted by $\mathbf{M}_p$. The natural element of volume in $\mathbf{M}_p$ is

$$d_4p = d^4p/(2\pi)^4.$$

Whereas the group of symmetries of the Minkowski vector (or momentum) space is the Lorentz group (which we shall be talking about below), the group of symmetries (or motions) of Minkowski affine (or coordinate) space is the Poincaré group (§7.1). In what follows, we shall usually use $\mathbf{M}$ to denote either the Minkowski coordinate or the Minkowski momentum space. (It will generally be clear from the context which space we are referring to.)

A vector $p \in \mathbf{M}$ is said to be *positive* (or *negative*) *time-like* if its square p^2 is positive and the zeroth component p^0 is positive (or negative). If, on the other hand, instead of $p^2 > 0$ we have $p^2 = 0$ but $p \neq 0$, then p is called a *positive* (or *negative*) *isotropic* vector. If $p^2 < 0$, the vector p is called *space-like*. In accordance with the classification of vectors in $\mathbf{M}$ in terms of the quantity p^2 and the sign of p^0 (when $p^2 \geq 0$), we introduce the following frequently encountered sets:

$$V = \{p \in \mathbf{M} : p^2 > 0\},\ \overline{V} = \{p \in \mathbf{M} : p^2 \geq 0\}, \tag{3.6}$$

$$V^+ = \{p \in \mathbf{M} : p^2 > 0, p^0 > 0\},\ \overline{V}^+ = \{p \in \mathbf{M} : p^2 \geq 0, p^0 \geq 0\}, \tag{3.7}$$

$$V^- = -V^+,\ \overline{V}^- = -\overline{V}^+, \tag{3.8}$$

$$\overline{V}^+_m = \{p \in \mathbf{M} : p^2 \geq m^2, p^0 \geq 0\} \quad \text{for } m \geq 0, \tag{3.9}$$

$$\Gamma^+_m = \{p \in \mathbf{M} : p^2 = m^2, p^0 > 0\} \quad \text{for } m \geq 0. \tag{3.10}$$

B. Definition of the General Lorentz Group and Its Connected Components

By a *Lorentz transformation* we mean any linear transformation Λ of Minkowski space that leaves the Minkowski scalar product invariant. The set of all such transformations forms the (*full*) *Lorentz group*, denoted by L. Each Lorentz transformation $\Lambda \in L$ is defined by the four-dimensional matrix

$$y^\mu = (\Lambda x)^\mu = \Lambda^\mu_\nu x^\nu. \tag{3.11}$$

The condition that the form (3.1) be invariant is equivalent to the condition

$$\Lambda^\chi_\mu g_{\chi\lambda} \Lambda^\lambda_\nu = g_{\mu\nu} \tag{3.12}$$

or, in matrix form,

$$\Lambda^T g \Lambda = g, \quad \text{that is,} \quad \Lambda^{-1} = g\Lambda^T g, \tag{3.13}$$

where Λ^T is the transpose of Λ: $(\Lambda^T)^\mu_\nu = \Lambda^\nu_\mu$. If we take the determinant of both sides of (3.13), we see that

$$(\det \Lambda)^2 = 1, \text{ that is } \det \Lambda = \pm 1. \tag{3.14}$$

If $\det \Lambda = 1$, then Λ is called *special*.

An example of a Lorentz transformation with $\det \Lambda = 1$ is the identity transformation $\Lambda = 1$ (which is the identity of the group), also reflections in all four axes

$\Lambda = -1$. (An example of an improper transformation is spatial reflection; see (3.19) below.) Formula (3.12) with $\mu = \nu = 0$ gives

$$(\Lambda^0_0)^2 - \sum_{j=1}^{3} (\Lambda^j_0)^2 = 1,$$

and hence there are two possible domains of variation of the parameter Λ^0_0:

$$\Lambda^0_0 \geq 1 \text{ and } \Lambda^0_0 \leq -1. \tag{3.15}$$

If $\Lambda^0_0 \geq 1$, then we say that Λ is *orthochronous* (does not change the direction of time); a special orthochronous Lorentz transformation is called *proper*.

Exercise 3.1. Verify that the Lorentz group is a six-dimensional Lie group and that local coordinates can be introduced in it by means of the real antisymmetric 4×4-matrices $\theta \equiv (\theta_{\lambda\mu}) = (-\theta_{\mu\lambda})$ according to the formula

$$\Lambda = \exp(1/2 l^{\lambda\mu}\theta_{\lambda\mu}); \tag{3.16}$$

here $l^{\lambda\mu} \equiv -l^{\lambda\mu}$ are the following linear operators in Minkowski space, called *infinitesimal Lorentz rotation operators*:

$$(l^{\lambda\mu})^\alpha_\beta = -g^{\lambda\alpha}\delta^\mu_\beta + g^{\mu\alpha}\delta^\lambda_\beta. \tag{3.17}$$

Verify that these operators satisfy the following commutation relations:

$$[l^{\lambda\mu}, l^{\rho\sigma}] = g^{\lambda\rho}l^{\mu\sigma} - g^{\mu\rho}l^{\lambda\sigma} + g^{\mu\sigma}l^{\lambda\rho} - g^{\lambda\sigma}l^{\mu\rho}. \tag{3.18}$$

The real linear combinations of the operators $l^{\lambda\mu}$ form a (real six-dimensional) Lie algebra $\mathcal{L}$ with commutation relations defined by (3.18). The algebra $\mathcal{L}$ is called the *Lie algebra of the Lorentz group*.

Exercise 3.2. (a) A quadruple of vectors Y_λ ($\lambda = 0, \ldots, 3$) in M is called a Lorentz basis if $Y_\lambda Y_\mu = g_{\lambda\mu}$, $Y^0_0 > 0$ and $\det(Y^\lambda_\mu) = 1$. Prove that the Lorentz bases are in one-to-one correspondence with the proper transformations of the Lorentz group $\Lambda \leftrightarrow \{Y_\lambda = \Lambda e_\lambda\}$.

(b) Prove that each Lorentz basis $\{Y_\lambda\}$ can be obtained from the standard basis (3.5) by means of two Lorentz transformations: $Y_\lambda = R\Lambda e_\lambda$, where Λ is a pure Lorentz rotation (say, in the 0,1 plane) and R is a three-dimensional rotation.

(c) Use part (b) of this exercise to prove that the group of proper Lorentz transformations is connected.

The Lorentz group L is disconnected. It consists of four connected components denoted respectively by $L^\uparrow_+$ (with the conditions $\det\Lambda = 1$ and $\Lambda^0_0 \geq 1$), $L^\downarrow_+$ ($\det\Lambda = 1$ and $\Lambda^0_0 \leq -1$), $L^\uparrow_-$ ($\det\Lambda = -1$ and $\Lambda^0_0 \geq 1$), $L^\downarrow_-$ ($\det\Lambda = -1$ and $\Lambda^0_0 \leq -1$). Of these, only $L^\uparrow_+$ is a subgroup (clearly it contains the identity of L). We call this subgroup the *proper Lorentz group* or the *Lorentz rotation group*. This group plays a special role, since at the present moment it is considered the correct symmetry group in relativistic field theory and elementary particle theory. The following subgroups can be formed from the connected components of L: $L^\uparrow_0 = L^\uparrow_+ \cup L^\uparrow_-$ (*orthochronous Lorentz group*), $L_+ = L^\uparrow_+ \cup L^\downarrow_+$ (*special Lorentz group*) and $L^\uparrow_+ \cup L^\downarrow_-$. (We note that although this last group does not have a name of its own, there is a wide range of physical phenomena that are invariant with respect to this group, which contains the Lorentz rotations and time reversal.*) Here the group $L^\uparrow$ is generated by the

* In fact up to 1964, physicists believed in the invariance with respect to time reversal (T-invariance, for short) of all the elementary interactions. It was only the experiment of Fitch and Cronin, in which the decay of a long-lived neutral K-meson into two π-mesons was observed (see, for example, [K1]), that shook this belief. In fact, if we accept the validity of the CPT theorem (for which there are impressive grounds both at the theoretical (see Ch.9) and experimental levels), we can be satisfied that this experiment testifies to the violation of T-invariance. According to the currently accepted view of Wolfenstein, only the so-called superweak interaction violates T-invariance.

subgroup $L_+^\uparrow$ and one element of $L_-^\uparrow$, say, the *spatial reflection* I_s:

$$I_s(x^0, \mathbf{x}) = (x^0, -\mathbf{x}). \tag{3.19}$$

Similarly, $L_+^\uparrow \cup L_-^\downarrow$ is generated by $L_+^\uparrow$ and the *time reversal* I_t:

$$I_t(x^0, \mathbf{x}) = (-x^0, \mathbf{x}), \tag{3.20}$$

while L_+ is generated by $L_+^\uparrow$ and the *full reflection* I_{st}:

$$I_{st}x = -x. \tag{3.21}$$

We can regard Minkowski space $\mathbf{M}$ as a G-space for the proper Lorentz group $L_+^\uparrow$ (see Appendix D.1 for the general definition). Under the action of $L_+^\uparrow$, the space $\mathbf{M}$ is partitioned into the following orbits: the hyperboloids $x^2 = s$ for $s < 0$, the sets Γ_m^+ and $\Gamma_m^- = -\Gamma_m^+$ for $m \geq 0$ (see (3.10)) and the point $x = 0$. The orbits of $L^\uparrow$ are the same. It is not difficult to write down the orbits of L_+ (or L) in $\mathbf{M}$; they are the hyperboloids $x^2 = s$ for $s < 0$, the sets $\Gamma_m^+ \cup \Gamma_m^-$ for $m \geq 0$ and the point $x = 0$.

C. THE UNIVERSAL COVERING OF THE GROUP $L_+^\uparrow$

Representations* of connected Lie groups can be studied by algebraic methods. Thus if T is a representation of $L_+^\uparrow$ in the space $\mathcal{X}$, then in some neighbourhood of the identity of the group (where the elements of the group have the form (3.16)), $T(\Lambda)$ can be written in the form

$$T(\Lambda) = \exp\left(\frac{i}{2} X^{\lambda\mu}\theta_{\lambda\mu}\right); \tag{3.22}$$

here, the $X^{\lambda\mu} = -X^{\mu\lambda}$ are linear operators in $\mathcal{X}$ called the generators of the representation T; they satisfy the commutation relations of a Lie algebra $\mathcal{L}$:

$$\left[X^{\lambda\mu}, X^{\rho\sigma}\right] = -i\left(g^{\lambda\rho}X^{\mu\sigma} - g^{\mu\rho}X^{\lambda\sigma} + g^{\mu\sigma}X^{\lambda\rho} - g^{\lambda\sigma}X^{\mu\rho}\right). \tag{3.23}$$

Sometimes the operators $iX^{\lambda\mu}$ are called the generators of the representation. We shall follow the convention adopted in theoretical physics which ensures that the generators for unitary representations are Hermitian. (We note that the Lorentz group has no finite-dimensional unitary representations apart from the trivial identity representation $T(\Lambda) \equiv 1$.)

Thus every representation of $L_+^\uparrow$ defines a representation of its Lie algebra $\mathcal{L}$. Therefore the construction of the representations of $L_+^\uparrow$ would be completely reduced to an algebraic problem if there corresponded to each representation of the Lie algebra $\mathcal{L}$ a representation of $L_+^\uparrow$. However, the situation is more complicated. The fact is that if for given operators $X^{\alpha\beta}$ satisfying (3.23) we construct the operators $T(\Lambda)$ for Λ in a neighbourhood of the identity as in (3.22), then we obtain a so-called *local representation of the group* $L_+^\uparrow$. It is natural to try to extend this to the whole group using the formula $T(\Lambda_1\Lambda_2) = T(\Lambda_1)T(\Lambda_2)$ as the definition. In some cases, we do in fact obtain a representation of $L_+^\uparrow$. But in other cases, $T(\Lambda)$ turns out not to be single-valued. (In the case of an irreducible representation, the Lie algebras are said to be non-integrable to a representation of the whole group.) In these cases we do not obtain a true representation, but a so-called two-valued representation

* In this subsection and throughout, our representations will always be finite-dimensional linear representations.

of $L_+^\uparrow$. (The situation is entirely similar for the rotation group in three-dimensional space.) In relativistic quantum theory they play just as important a role as the "true" representations; therefore it is advisable to have a more logical treatment of them.

The difficulties involving non-integrable representations of Lie algebras are fully analysed in the theory of Lie groups (see [P1]). The cause of these difficulties is the fact that $L_+^\uparrow$ is not simply connected; more precisely, it is 2-connected. The difficulty would not arise if from the very beginning we were to replace $L_+^\uparrow$ by a simply connected group whose structure in a neighbourhood of the identity is the same as $L_+^\uparrow$, that is, one that is locally isomorphic to $L_+^\uparrow$. (By a *local isomorphism* of two groups G and G' we mean a one-to-one correspondence $\phi : N \to N'$ between some pair of neighbourhoods of the identity $N \subset G$ and $N' \subset G'$ that preserves multiplication: $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$.) It turns out that for any connected Lie group G there exists a (unique to within isomorphism) connected simply connected Lie group $\tilde{G}$ that is locally isomorphic to G. This group G is called the *universal covering* of G.

In a certain sense the group $\tilde{G}$ is larger than G. However, the relation between them is more complex than a simple inclusion $G \subset \tilde{G}$ (apart from the trivial case $G = \tilde{G}$). The group $\tilde{G}$ covers G several times. This means that there is a homomorphism ϕ from $\tilde{G}$ onto G which is at the same time a local isomorphism. (This homomorphism is called a *covering homomorphism.*) The inverse image of an element $g \in G$ under ϕ consists of a certain (finite or infinite) number of isolated points $\tilde{g} \in \tilde{G}$. The number of such points does not depend on g; and it is this number that determines how many times $\tilde{G}$ covers G.

The universal covering of $L_+^\uparrow$ can be constructed as follows. The group $L_+^\uparrow$ has a two-valued complex two-dimensional representation. The matrices of this "representation" form the group* $SL(2, C)$ consisting of all complex 2×2 matrices with determinant 1.

The group $SL(2, C)$ is a complex Lie group of complex dimension 3 (and hence of real dimension 6) and we can introduce local coordinates by means of the complex three-dimensional vector $\phi \in \mathbf{C}^3$ in the following way:

$$\underset{\sim}{A} = e^{\frac{1}{2}i\tau\phi} \equiv e^{\frac{1}{2}i\tau_j\phi_j}. \tag{3.24}$$

Here the τ_j are the Pauli matrices:

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.25}$$

satisfying the commutation relations

$$[\tau_j, \tau_k] = 2i\epsilon_{jkl}\tau_l; \tag{3.26}$$

ϵ_{jkl} is the completely antisymmetric tensor in three-dimensional space with $\epsilon_{123} = 1$:

$$\epsilon_{jkl} = \epsilon^{jkl}; \ \epsilon_{123} = 1. \tag{3.27}$$

The matrices $\tau_j/2$ are called the *complex generators* of $SL(2, C)$. The complex linear combinations of the matrices $i\tau_j$ form a three-dimensional Lie algebra, called the complex Lie algebra of the group $SL(2, C)$.

It is easy to see from the defining equation $\det A = 1$ of the group $SL(2, C)$, that this group is a connected simply connected manifold in $\mathbf{C}^4$. We now construct a local

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* More generally, the adopted notation for the general linear group of complex non-singular $n \times n$ matrices is $GL(n, C)$, while $SL(n, C)$ denotes the special linear group, that is, the subgroup of $GL(n, C)$ consisting of matrices with determinant 1.

isomorphism between the groups $SL(2,C)$ and $L_+^\uparrow$. It will be proved that $SL(2,C)$ is the universal covering of $L_+^\uparrow$.

The group $SL(2,C)$ acts in standard fashion on $\mathbf{C}^2$: for $z \equiv (z^1, z^2) \in \mathbf{C}^2$ and $\underset{\sim}{\Lambda} \in SL(2,C)$ we have

$$(\underset{\sim}{\Lambda} z)^a = \underset{\sim}{\Lambda}{}^a_b z^b, \ a = 1,2 \tag{3.28}$$

(here the sum is with respect to b from 1 to 2). It leaves invariant the skew symmetric bilinear form

$$z\epsilon w \equiv z^1 w^2 - z^2 w^1, \tag{3.29}$$

corresponding to the matrix

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.30}$$

Let $x \to \underset{\sim}{x}$ be a fixed linear isomorphism from $\mathbf{M}$ onto the space of all Hermitian 2×2 matrices such that

$$x^2 \equiv x \cdot x = \det \underset{\sim}{x}, \tag{3.31}$$

and $\underset{\sim}{x}$ is a positive definite matrix for $x \in V^+$. We set

$$\tilde{x} = \epsilon \underset{\sim}{x}^T \epsilon^{-1}, \tag{3.32}$$

where $\underset{\sim}{x}^T$ is the transpose of $\underset{\sim}{x}$. For definiteness we shall use the following special realization of the matrices $\underset{\sim}{x}$ and $\tilde{x}$:

$$\underset{\sim}{x} = x^\mu \underset{\sim}{e}_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{3.33a}$$

$$\tilde{x} = x^\mu \tilde{e}_\mu = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix}, \tag{3.33b}$$

where x^2 is the second component of x, $\underset{\sim}{e}_0 = \tilde{e}_0$ is the identity matrix and $\underset{\sim}{e}_j = -\tilde{e}_j$ are the Pauli matrices*

$$\underset{\sim}{e}_0 = \tilde{e}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underset{\sim}{e}_1 = -\tilde{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\underset{\sim}{e}_2 = -\tilde{e}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \underset{\sim}{e}_3 = -\tilde{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.34}$$

The matrices $\underset{\sim}{e}_\lambda$ and $\tilde{e}_\lambda$ can be regarded as the result of applying the maps $x \to \underset{\sim}{x}$ and $x \to \tilde{x}$ to the basis vectors (3.5). We shall also use the matrices $\underset{\sim}{e}^\lambda = g^{\lambda\mu} \underset{\sim}{e}_\mu$ and $\tilde{e}^\lambda = g^{\lambda\mu} \tilde{e}_\mu$.

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* We draw a distinction between the Pauli matrices in the formulae (3.24) and (3.34), since they relate to objects with different transformation properties (with respect to the group $SL(2,C)$); the fact that these matrices are numerically the same is due to the special choice of the basis.

Exercise 3.3. Verify the following relations:

$$\underset{\sim}{e}_\lambda \tilde{e}_\mu + \underset{\sim}{e}_\mu \tilde{e}_\lambda = 2g_{\lambda\mu} \cdot \mathbf{1}, \tag{3.35}$$

$$\underset{\sim}{e}_\lambda \tilde{e}_\mu - \underset{\sim}{e}_\mu \tilde{e}_\lambda = \frac{i}{2}\epsilon_{\lambda\mu\rho\sigma}(\underset{\sim}{e}^\sigma \tilde{e}^\rho - \underset{\sim}{e}^\rho \tilde{e}^\sigma), \tag{3.36}$$

$$\operatorname{tr}(\underset{\sim}{a}\tilde{b}) = \operatorname{tr}(\tilde{a}\underset{\sim}{b}) = 2ab, \; a, b \in \mathbf{M}. \tag{3.37}$$

Here $\epsilon_{\lambda\mu\nu\rho}$ denotes the absolute antisymmetric tensor: $\epsilon_{\lambda\mu\nu\rho} = 1$ of $\lambda\mu\nu\rho$ is an even permutation of 0,1,2,3 and $\epsilon_{\lambda\mu\nu\rho} = -1$ if $\lambda\mu\nu\rho$ is an odd permutation; $\epsilon_{\lambda\mu\nu\rho} = 0$ if at least two of the indices are the same. In what follows we shall also use the absolute antisymmetric tensor with upper indices

$$\epsilon^{\lambda\mu\nu\rho} = -\epsilon_{\lambda\mu\nu\rho} \quad (\text{so that} \quad \epsilon_{0123} = -\epsilon^{0123} = 1). \tag{3.38}$$

We now associate with element $\underset{\sim}{\Lambda} \in SL(2, C)$ the element $\Lambda \in L_+^\uparrow$ by means of either of the following two formulae:

$$\underset{\sim}{\Lambda x} = \underset{\sim}{\Lambda}\underset{\sim}{x}\underset{\sim}{\Lambda}^*, \widetilde{\Lambda x} = \underset{\sim}{\Lambda}^{*-1}\tilde{x}\underset{\sim}{\Lambda}^{-1} \tag{3.39}$$

for all $x \in \mathbf{M}$. It is not difficult to see that the map $\underset{\sim}{\Lambda} \to \Lambda(\underset{\sim}{\Lambda})$ is a homomorphism from $SL(2, C)$ to $L_+^\uparrow$ (that is, it is a representation of $SL(2, C)$ in Minkowski space). The elements of the matrix $\Lambda(\underset{\sim}{\Lambda})$ can be defined by the formula

$$\Lambda^\alpha_\beta = \frac{1}{2}\operatorname{tr}(\tilde{e}^\alpha \underset{\sim}{\Lambda e_\beta}) = \frac{1}{2}\operatorname{tr}(\tilde{e}^\alpha \underset{\sim}{\Lambda}\underset{\sim}{e}_\beta\underset{\sim}{\Lambda}^*). \tag{3.40}$$

Exercise 3.4. (a) Prove that the homomorphism $\underset{\sim}{\Lambda} \to \Lambda(\underset{\sim}{\Lambda})$ realizes a local isomorphism between $SL(2, C)$ and $L_+^\uparrow$. [Hint: In a neighbourhood of the identity of $SL(2, C)$ we can choose a coordinate system of real and imaginary parts of the vector $\phi \in \mathbf{C}^3$ in (3.24). Also verify by means of (3.40) that the rank of the map $\underset{\sim}{\Lambda} \to \Lambda(\underset{\sim}{\Lambda})$ at the identity $\Lambda = 1$ of $SL(2, C)$ is equal to 6. Deduce by means of the open mapping theorem ([G12]) vol.II,Ch.IV,Theorem 5.2) that this map is a diffeomorphism of a neighbourhood of the identity of $SL(2, C)$.]

(b) Prove that the kernel of the homomorphism $\underset{\sim}{\Lambda} \to \Lambda(\underset{\sim}{\Lambda})$ consists of just the two elements $\underset{\sim}{\Lambda} = 1$ and $\underset{\sim}{\Lambda} = -1$.

It follows from this exercise (and from Exercise D.1 in Appendix D) that $SL(2, C)$ is the universal covering for $L_+^\uparrow$, it being a double covering of this group.

Exercise 3.5. (a) Prove that under the homomorphism $\underset{\sim}{\Lambda} \to \Lambda$, the subgroup $SU(2) \subset SL(2, C)$ of unitary 2×2 matrices is mapped onto the group $O_+(3)$ of three-dimensional rotations regarded as elements of $L_+^\uparrow$ with $\Lambda^{0j} = 0$, $j = 1, 2, 3$. (This means that $SU(2)$ is a double covering of $O_+(3)$.) Prove in particular that the unitary matrix

$$V(\theta) = \cos\frac{\theta}{2} \cdot \mathbf{1} - i\sin\frac{\theta}{2} \cdot \tau_3 = \exp\left(-\frac{i\theta}{2}\tau_3\right) \tag{3.41}$$

corresponds to a rotation $R(\theta)$ in the (x^1, x^2)-plane through an angle θ:

$$(R(\theta)x)^1 = \cos\theta \cdot x^1 - \sin\theta \cdot x^2, \; (R(\theta)x)^2 = \sin\theta \cdot x^1 + \cos\theta \cdot x^2. \tag{3.42}$$

(b) Prove that under the homomorphism $\underset{\sim}{\Lambda} \to \Lambda$ the positive definite Hermitian matrix in $SL(2, C)$ of the form

$$H(n, \alpha) = \cosh\frac{\alpha}{2} \cdot 1 + \sinh\frac{\alpha}{2} \cdot \tau\mathbf{n} = \exp\left(\frac{\alpha}{2}\tau\mathbf{n}\right) \tag{3.43}$$

(where $\mathbf{n}$ is a unit vector in $\mathbf{R}^3$, $\alpha \in \mathbf{R}$) corresponds to a pure Lorentz transformation ("hyperbolic rotation") $x \to y$, where

$$\begin{aligned} y^0 &= x^0 \cosh\alpha + (\mathbf{x}\mathbf{n})\sinh\alpha, \\ \mathbf{y} &= \mathbf{x} - (\mathbf{x}\mathbf{n})\mathbf{n} + [(\mathbf{x}\mathbf{n})\cosh\alpha + x^0\sinh\alpha]\mathbf{n}. \end{aligned} \tag{3.44}$$

In particular, the diagonal matrix

$$h(\alpha) = \begin{pmatrix} e^{\alpha/2} & 0 \\ 0 & e^{-\alpha/2} \end{pmatrix} \tag{3.45}$$

corresponds to the hyperbolic rotation in the (x^0, x^3)-plane:

$$y^0 = x^0\cosh\alpha + x^3\sinh\alpha, \; y^3 = x^0\sinh\alpha + x^3\cosh\alpha. \tag{3.46}$$

(c) Show that any proper Lorentz transformation can be represented as the successive application of a hyperbolic rotation of type (3.43) and a three-dimensional Euclidean rotation. [Hint: Every matrix $B \in SL(2,C)$ can be represented in the form $B = VH$, where $H = \sqrt{B^*B}$ is a positive definite matrix and $V = BH^{-1}$ is a unitary matrix.]

(d) Verify the identity

$$\Lambda[\exp(-\frac{1}{8}(\underset{\sim}{e}^{\lambda}\tilde{e}^{\mu} - \underset{\sim}{e}^{\mu}\tilde{e}^{\lambda})\theta_{\lambda\mu}] = \exp(\frac{1}{2}l^{\lambda\mu}\theta_{\lambda\mu}), \tag{3.47}$$

where the operators $l^{\lambda\mu}$ in Minkowski space are defined by (3.17) and $\theta_{\lambda\mu}$ is an arbitrary real antisymmetric 4×4 matrix. [Hint: It suffices to verify that the linear parts in the expansion in $\theta_{\lambda\mu}$ on both sides of (3.47) are the same; then use (3.40) to reduce the problem to proving the relation

$$-\frac{1}{8}\operatorname{tr}\{\tilde{e}^{\alpha}(\underset{\sim}{e}^{\lambda}\tilde{e}^{\mu} - \underset{\sim}{e}^{\mu}\tilde{e}^{\lambda})\underset{\sim}{e}_{\beta} + \tilde{e}^{\alpha}\underset{\sim}{e}_{\beta}(\tilde{e}^{\mu}\underset{\sim}{e}^{\lambda} - \tilde{e}^{\lambda}\underset{\sim}{e}^{\mu})\} = (l^{\lambda\mu})^{\alpha}_{\beta};$$

to do this, use (3.35) and (3.37).]

We note that we can form from the two spinor vectors w, $\omega \in \mathbf{C}^2$ and the 4-vector x, the following algebraic combination, which is invariant under the simultaneous transformations $w \to \underset{\sim}{\Lambda}w$, $\omega \to \underset{\sim}{\Lambda}\omega$, $x \to \Lambda(\underset{\sim}{\Lambda})x$:

$$\bar{\omega}\tilde{x}w \equiv \bar{\omega}^1\tilde{x}_{11}w^1 + \bar{\omega}^1\tilde{x}_{12}w^2 + \bar{\omega}^2\tilde{x}_{21}w^1 + \bar{\omega}^2\tilde{x}_{22}w^2. \tag{3.48}$$

Clearly, in the left hand side of this equation $\bar{\omega}$ is to be interpreted as a 1×2-matrix (row) and w as a 2×1-matrix (column), so that the matrix multiplication yields a scalar. More complicated expressions (including invariant differential operators) are similarly interpreted; for example,

$$w\epsilon\underset{\sim}{x}\frac{\partial}{\partial\bar{\omega}}, \; \frac{\partial}{\partial\omega}\epsilon^{-1}\frac{\partial}{\partial w}, \bar{\omega}\frac{\tilde{\partial}}{\partial x}w, \; \frac{\partial}{\partial w}\underset{\sim}{x}\frac{\partial}{\partial\bar{\omega}} \tag{3.49}$$

and so on.

D. Finite-dimensional representations of the group $SL(2,C)$

According to what we said above, the construction of the representations of $L^{\uparrow}_{+}$ reduces to the analogous problem for $SL(2,C)$. If we have constructed some representation $\underset{\sim}{\Lambda} \to T(\underset{\sim}{\Lambda})$ for $SL(2,C)$, then by setting

$$T_1(\Lambda) = T(\underset{\sim}{\Lambda}) \quad \text{for } \Lambda = \Lambda(\underset{\sim}{\Lambda}),$$

we obtain a single-valued or two-valued representation T_1 for $L_+^\uparrow$ (and this method gives all the representations of $L_+^\uparrow$). A representation of $L_+^\uparrow$ produced in this way is single-valued if and only if $T(-1) = 1$.

We begin by giving results relating to the complex (finite-dimensional) representations of $SL(2,C)$. For the group $SL(2,C)$ (and for every semisimple Lie group) the irreducibility condition for complex representations (meaning the absence of non-trivial invariant subspaces other than $\{0\}$ and the whole space of the representation) is equivalent to the condition of operator irreducibility in the sense of Schur (which means that every operator that commutes with the operators of the representation is a multiple of the identity operator). An arbitrary representation is a direct sum of irreducible representations. (In other words, an arbitrary representation is *completely reducible.*) Therefore to classify the representations it suffices to describe the irreducible ones (to within isomorphism).

Each irreducible complex representation of $SL(2,C)$ is characterized by an ordered pair (j,k), where j and k are integral or half-integral* non-negative numbers. The representation $(j,0)$ can be regarded as the symmetric tensor product of $2j$ copies of the self representation $\underset{\sim}{\Lambda} \to \underset{\sim}{\Lambda}$ of $SL(2,C)$. (Here we can equally take the equivalent contragradient representation $\underset{\sim}{\Lambda} \to (\underset{\sim}{\Lambda}^T)^{-1}$, and we use this possibility below.) The representation $(0,j)$ is the complex conjugate of $(j,0)$. An arbitrary irreducible representation (j,k) can be defined as the tensor product of the representations $(j,0)$ and $(0,k)$. Thus the representation (j,k) can be realized in the space of spin tensors $\Psi^{\alpha_1\ldots\alpha_{2j};\,\beta_1'\ldots\beta_{2k}'}$ which depend on two groups of indices: $2j$ "unprimed" indices $\alpha_1,\ldots,\alpha_{2j}$ and $2k$ "primed" indices $\beta_1',\ldots,\beta_{2k}'$; all the indices run through the values from 1 to 2. Apart from this, the spin tensors are subject to the condition of symmetry with respect to the indices of each group (that is, with respect to the primed and unprimed indices separately).

The representation $\mathfrak{D}^{(j,k)}$ in the space of spin tensors is defined by the formula

$$(\mathfrak{D}^{(j,k)}(\underset{\sim}{\Lambda})\psi)^{\alpha_1\ldots\alpha_{2j};\beta_1'\ldots\beta_{2k}'} =$$
$$= \sum_{\gamma_1,\ldots,\gamma_{2j};\delta_1',\ldots,\delta_{2k}'} \left(\prod_{s=1}^{2j} \underset{\sim}{\Lambda}^{\alpha_s}_{\gamma_s}\right)\left(\prod_{t=1}^{2k} \bar{\underset{\sim}{\Lambda}}^{\beta_t'}_{\sigma_t'}\right) \psi^{\gamma_1\ldots\gamma_{2j}\sigma_1'\ldots\sigma_{2k}'}. \tag{3.50}$$

We could equally have used spin tensors with lower ("contragradient") indices $\alpha_1,\ldots$ $\ldots,\alpha_{2j};\beta_1',\ldots,\beta_{2k}'$; in this case the matrices $\underset{\sim}{\Lambda}$ and $\bar{\underset{\sim}{\Lambda}}$ on the right hand side of (3.50) must be replaced by $(\underset{\sim}{\Lambda}^T)^{-1}$ and $\underset{\sim}{\Lambda}^{*-1}$ respectively.

The spin tensors are conveniently characterized by arbitrary polynomials in the variables ω and $\bar\omega$:

$$\psi(\omega,\bar\omega) = \sum_{\alpha_1,\ldots,\alpha_{2j};\beta_1',\ldots,\beta_{2k}'} \psi_{\alpha_1\ldots\alpha_{2j};\beta_1'\ldots\beta_{2k}'}\omega^{\alpha_1}\ldots\omega^{\alpha_{2j}}\bar\omega^{\beta_1'}\ldots\bar\omega^{\beta_{2k}'}. \tag{3.51}$$

We may therefore suppose that the representation (j,k) is realized in the space $\mathfrak{p}^{(j,k)}$ of all complex polynomials $\psi(\omega,\bar\omega)$ in $\omega \equiv (\omega^1,\omega^2)$ and $\bar\omega \equiv (\bar\omega^1,\bar\omega^2)$, that are homogeneous of degree $2j$ in ω and homogeneous of degree $2k$ in $\bar\omega$:

$$\psi(\lambda\omega,\bar\lambda\bar\omega) = \lambda^{2j}(\bar\lambda)^{2k}\psi(\omega,\bar\omega) \quad \text{for all } \lambda \in \mathbf{C}. \tag{3.52}$$

* By a half-integral number we mean one of the form $n + 1/2$ where n is an integer.

The action $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ in $\mathfrak{p}^{(j,k)}$ has the form

$$(\mathfrak{D}^{(j,k)}(\underset{\sim}{\Lambda})\psi)(\omega,\bar{\omega}) = \psi(\underset{\sim}{\Lambda}^{-1}\omega, \bar{\underset{\sim}{\Lambda}}^{-1}\bar{\omega}). \tag{3.53}$$

Exercise 3.6. Prove that the map associating each pair of polynomials $\phi, \psi \in \mathfrak{p}^{(j,k)}$ with the number

$$\phi\left(\epsilon^{-1}\frac{\partial}{\partial\omega}, \epsilon^{-1}\frac{\partial}{\partial\bar{\omega}}\right)\psi(\omega,\bar{\omega}) \equiv$$

$$\equiv \frac{1}{((2j)!\,(2k)!)^2}\left(\frac{\partial}{\partial w}\epsilon^{-1}\frac{\partial}{\partial\omega}\right)^{2j}\left(\frac{\partial}{\partial\bar{w}}\epsilon^{-1}\frac{\partial}{\partial\bar{\omega}}\right)^{2k}\phi(w,\bar{w})\psi(\omega,\bar{\omega}), \tag{3.54}$$

is a non-degenerate $SL(2,C)$-invariant bilinear form on $\mathfrak{p}^{(j,k)}$. (This form is generated by the bilinear form (3.29) on $\mathbf{C}^2$.)

In the next exercise, we give the form of the generators of the representation $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$. According to formulae (3.22) and (3.47), the generators $X^{\lambda\mu}$ of $\mathfrak{D}^{(j,k)}$ can be defined by the relations

$$\mathfrak{D}^{(j,k)}(\exp(-\frac{1}{8}(\underset{\sim}{e}^{\lambda}\tilde{e}^{\mu} - \underset{\sim}{e}^{\mu}\tilde{e}^{\lambda})\theta_{\lambda\mu})) = \exp\left(\frac{i}{2}X^{\lambda\mu}\theta_{\lambda\mu}\right) \tag{3.55}$$

for all real antisymmetric 4×4 matrices θ.

Exercise 3.7. Prove the following formulae for the generators $X^{\lambda\mu}$ of the representation $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ in the space $\mathfrak{p}^{(j,k)}$:

$$X^{\lambda\mu} = -\frac{i}{4}z\epsilon(\underset{\sim}{e}^{\lambda}\tilde{e}^{\mu} - \underset{\sim}{e}^{\mu}\tilde{e}^{\lambda})\epsilon^{-1}\frac{\partial}{\partial z} - \frac{i}{4}\bar{z}(\tilde{e}^{\lambda}\underset{\sim}{e}^{\mu} - \tilde{e}^{\mu}\underset{\sim}{e}^{\lambda})\frac{\partial}{\partial\bar{z}}, \tag{3.56}$$

$$\frac{1}{2}\epsilon_{\lambda\mu\rho\sigma}X^{\rho\sigma} = -\frac{1}{4}z\epsilon(\underset{\sim}{e}_{\lambda}\tilde{e}_{\mu} - \underset{\sim}{e}_{\mu}\tilde{e}_{\lambda})\epsilon^{-1}\frac{\partial}{\partial z} + \frac{1}{4}\bar{z}(\tilde{e}_{\lambda}\underset{\sim}{e}_{\mu} - \tilde{e}_{\mu}\underset{\sim}{e}_{\lambda})\frac{\partial}{\partial\bar{z}}. \tag{3.57}$$

[Hint: For the derivation of (3.56) it suffices to substitute $\underset{\sim}{\Lambda} = \exp[-\frac{1}{8}(\underset{\sim}{e}^{\lambda}\tilde{e}^{\mu} - \underset{\sim}{e}^{\mu}\tilde{e}^{\lambda})\theta_{\lambda\mu}]$ in (3.53) and extract the linear part in $\theta_{\lambda\mu}$; (3.57) follows from (3.56) and (3.36).]

We note that $\mathfrak{D}^{(j,k)}$ forms a single-valued representation of $L_+^{\uparrow}$ if and only if $\mathfrak{D}^{(j,k)}(-1) = 1$, that is, when $j+k$ is an integer. More generally, we call the quantity $\mathfrak{D}^{(j,k)}(-1)$ the *valency*; if it is equal to 1 (or -1), then we say that the representation has *even* (or *odd*) *valency.*

It is worth while discussing the physical meaning of the notion of valency. We consider some one-parameter subgroup of the rotations in a two-dimensional space-like plane of Minkowski space, say, the subgroup $V(\theta) = \exp(\frac{1}{2}i\tau_3\theta)$ (in the notation of (3.41), where the real parameter denotes the angle of rotation around the vector e_3 in three-dimensional space). Then we can interpret the element -1 of $SL(2,C)$ as a rotation in three-dimensional space through an angle 2π. However, this is physically the identity transformation, so that observable physical quantities cannot change under such a transformation. In field theory, observable quantities are constructed from fields that transform under a representation of the Lorentz group (or $SL(2,C)$) as spin tensors. Then the requirement that observables do not change under a spatial rotation through an angle of 2π essentially comes down to stipulating that the dependence of the observables of the fields can only occur as combinations containing products of an even number of fields with odd valency.

The results given above on the irreducible representations of $SL(2,C)$ follow easily from the theory of angular momentum in three-dimensional space. Suppose that a neighbourhood of the identity in $SL(2,C)$ is parametrized by a three-dimensional complex vector $\phi \in \mathbf{C}^3$ as in (3.24). According to general results of the theory of representations of complex Lie groups (see, for example, [Z2],§43), every irreducible representation T of $SL(2,C)$ can be expressed in the form

$$T(\underset{\sim}{\Lambda}) = T_1(\underset{\sim}{\Lambda}) \otimes T_2(\underset{\sim}{\Lambda}), \tag{3.58}$$

where $T_1(\underset{\sim}{\Lambda})$ is a complex analytic representation of $SL(2,C)$ and $T_2(\underset{\sim}{\Lambda})$ is an anti-analytic representation (that is, the complex conjugate of an analytic representation). Therefore it is enough for us to confine ourselves to complex analytic representations T. This means that in the parametrization (3.24), $T(\underset{\sim}{\Lambda})$ can be expressed in the form

$$T(\underset{\sim}{\Lambda}) = \exp(iJ\phi), \tag{3.59}$$

where the J_j are operators in the representation space satisfying commutation relations of type (3.26):

$$[J_i, J_j] = i\epsilon_{ijk}J_k. \tag{3.60}$$

In the theory of angular momentum, all the representations of these commutation relations are well known. Each irreducible (finite-dimensional) representation is characterized (to within equivalence) by a single number, namely, the spin s, which takes integral or half-integral non-negative values in accordance with the equality

$$J^2 = s(s+1), \tag{3.61}$$

The corresponding representation is a symmetric tensor product of $2s$ copies of the self representation $\underset{\sim}{\Lambda} \to \underset{\sim}{\Lambda}$ and hence is $(s,0)$ in the above classification. An arbitrary irreducible representation has the form (3.58), that is, it has to be equivalent to one of the representations (j,k).

E. Simply reducible finite-dimensional representations of $SL(2,C)$. Spatial reflection

The realization of representations of $SL(2,C)$ in the space of (generally inhomogeneous) polynomials in ω and $\bar{\omega}$ is also applicable to the important class of reducible representations of $SL(2,C)$, namely, the class of *simply reducible representations*. Such a representation is a direct sum of a finite number of pairwise inequivalent irreducible representations. It is defined to within isomorphism by a finite set S of ordered pairs (j,k) of integral or half-integral non-negative numbers. It can be realized in the space $\mathfrak{p}_S$ of polynomials in ω and $\bar{\omega}$ of the form

$$\psi(\omega,\bar{\omega}) = \sum_{(j,k)\in S} \psi_{(i,j)}(\omega,\bar{\omega}), \quad \text{where } \psi_{(i,j)} \in \mathfrak{p}^{(i,j)}, \tag{3.62a}$$

so that

$$\mathfrak{p}_S = \bigoplus_{(j,k)\in S} \mathfrak{p}^{(j,k)}. \tag{3.62b}$$

As before, the action of $SL(2,C)$ is given by a formula of type (3.53):

$$(\mathfrak{D}(\underset{\sim}{\Lambda})\psi)(\omega,\bar{\omega}) = \psi(\underset{\sim}{\Lambda}^{-1}\omega, \bar{\underset{\sim}{\Lambda}}^{-1}\bar{\omega}). \tag{3.63}$$

Exercise 3.8. (a) Suppose that the tensor product of the representations $\mathfrak{D}^{(s_1)}$ and $\mathfrak{D}^{(s_2)}$ of $SU(2)$ is realized in the space $\mathfrak{p}^{(s_1,0)} \otimes \mathfrak{p}^{(s_2,0)}$ of polynomials $P(\omega, w)$ that are homogeneous of degree $2s_1$ in $\omega \in \mathbf{C}^2$ and homogeneous of degree $2s_2$ in $w \in \mathbf{C}^2$. Prove the following equivalence relation for the representations:

$$\mathfrak{D}^{(s_1)} \otimes \mathfrak{D}^{(s_2)} \approx \bigoplus_{\substack{|s_1-s_2|\le s\le s_1+s_2 \\ s\in|s_1-s_2|+\mathbf{Z}_+}} \mathfrak{D}^{(s)}. \tag{3.64}$$

To do this, verify that an $SU(2)$-invariant isomorphism from $\mathfrak{p}^{(s_1,0)} \otimes \mathfrak{p}^{(s_2,0)}$ to $\bigoplus_s \mathfrak{p}^{(s)}$, realizing the equivalence of the representations can be defined as follows:

$$\psi(\omega, w) \to \Psi(\zeta) =$$

$$= \sum_s \frac{\left(\zeta \frac{\partial}{\partial \omega}\right)^{s+s_1-s_2}}{(s+s_1-s_2)!} \frac{\left(\zeta \frac{\partial}{\partial w}\right)^{s-s_1+s_2}}{(s-s_1+s_2)!} \frac{\left(\frac{\partial}{\partial \omega} \epsilon^{-1} \frac{\partial}{\partial w}\right)^{s_1+s_2-s}}{(s_1+s_2-s)!} \psi(\omega, w)\Big|_{\omega=w=0}; \tag{3.65a}$$

the inverse map has the form

$$\Psi(\zeta) \to \psi(\omega, w) = \sum_s c(s, s_1, s_2) \frac{\left(\omega \frac{\partial}{\partial \zeta}\right)^{s+s_1-s_2}}{(s+s_1-s_2)!} \frac{\left(w \frac{\partial}{\partial \zeta}\right)^{s-s_1+s_2}}{(s-s_1+s_2)!} \times$$

$$\times \frac{(\omega \epsilon w)^{s_1+s_2-s}}{(s_1+s_2-s)!} \Psi(\zeta)|_{\zeta=0}; \tag{3.65b}$$

here the $c(s, s_1, s_2) \neq 0$ are numbers. [Hint: Substitute the $\psi(\omega, w)$ in (3.65b) into (3.65a).]

(b) Derive the analogous formulae for the decomposition of the tensor product of two irreducible representations of $SL(2, C)$ into irreducible representations:

$$\mathfrak{D}^{(j_1,k_1)} \otimes \mathfrak{D}^{(j_2,k_2)} \cong \bigoplus_{\substack{|j_1-j_2| \le j \le j_1+j_2 \\ j \in |j_1-j_2| + \mathbb{Z}_+}} \bigoplus_{\substack{|k_1-k_2| \le k \le k_1+k_2 \\ k \in |k_1-k_2| + \mathbb{Z}_+}} \mathfrak{D}^{(j,k)}. \tag{3.66}$$

We now give some simple examples.

We realize the representation $\mathfrak{D}^{(1/2,0)}$ in the space $\mathfrak{p}^{(1/2,0)}$ of linear forms on $\mathbf{C}^2$,

$$\psi(\omega) = \sum_{\alpha=1,2} a_\alpha \omega^\alpha, \tag{3.67}$$

so that the representation $\mathfrak{D}^{(1/2,0)}$ is contragradient to the self representation $\underset{\sim}{\Lambda} \to \underset{\sim}{\Lambda}$ in $\mathbf{C}^2$ and is equivalent to it. Similarly, $\mathfrak{D}^{(0,1/2)}$ acts in the space $\mathfrak{p}^{(0,1/2)}$ of antilinear forms on $\mathbf{C}^2$:

$$\psi(\bar{\omega}) = \sum_{\beta'=1,2} b_{\beta'} \bar{\omega}'. \tag{3.68}$$

These two representations form the simply reducible representation

$$\mathfrak{D}^{(1/2,0)} \oplus \mathfrak{D}^{(0,1/2)}, \tag{3.69}$$

called the *Dirac representation.* It acts in the space of Dirac spinors ψ, which are usually regarded as columns

$$\psi = \begin{pmatrix} a_\alpha \\ b_{\beta'} \end{pmatrix}, \quad \text{where } \alpha = 1, 2, \ \beta' = 1, 2; \tag{3.70}$$

here the spinors a_α and $b_{\beta'}$ are transformed by the representations $\mathfrak{D}^{(1/2,0)}$ and $\mathfrak{D}^{(0,1/2)}$ respectively. The spinor ψ can, of course, be treated as a polynomial in ω and $\bar{\omega}$:

$$\psi(\omega, \bar{\omega}) = \sum_{\alpha=1,2} a_\alpha \omega^\alpha + \sum_{\beta'=1,2} b_{\beta'} \bar{\omega}^{\beta'}, \tag{3.71}$$

and then the transformation law is expressed by (3.63).

We consider one more representation $(1/2, 1/2)$. Since it is the tensor product of the representations $(1/2, 0)$ and $(0, 1/2)$, it can be realized in the space of complex 2×2 matrices $\underset{\sim}{x} \equiv (x^{\alpha\beta'})$ by the representations

$$\mathfrak{D}^{(1/2,1/2)}(\underset{\sim}{\Lambda}) : \underset{\sim}{x} \to \underset{\sim}{\Lambda} \underset{\sim}{x} \underset{\sim}{\Lambda}^*. \tag{3.72}$$

If we compare (3.72) with (3.39), we see that this transformation is the same as $\Lambda(\underset{\sim}{A})$ (but now acting in complex Minkowski space). Thus in the language of the group $L_+^\uparrow$, the representation $\mathfrak{D}^{(1/2,1/2)}$ is the standard action of the Lorentz group in the space of 4-vectors.

So far we have dealt with complex representations; real representations can be constructed from complex ones. A complex representation T of a group G in the complex linear space $\mathcal{X}$ is called a *complexification* of a real representation T_R of G if a linear basis (over the field $\mathbf{C}$) can be chosen in $\mathcal{X}$ in which the matrix elements of T are real. Then the real linear span of this basis is a G-invariant subspace of $\mathcal{X}$ over the field $\mathbf{R}$ (the real dimension being equal to the complex dimension of $\mathcal{X}$) in which the real representation T_R of G is defined. Since T_R carries all the information of T (since the complex linear span of $\mathcal{X}_R$ is $\mathcal{X}$) we do not usually draw a distinction between the real representation T_R and its complexification T.

The condition that T be the complexification of a real representation can be conveniently expressed in basis-free form as the condition of the existence of an *involution* K (that is, an antilinear operator whose square is 1) in $\mathcal{X}$ that commutes with the operators of the representation. In fact this operator can be defined by the formula $K(x+iy) = x - iy$ for any $x, y \in \mathcal{X}_R$. In its turn, $\mathcal{X}$ is defined in terms of K, namely, $\mathcal{X}_R$ consists of all vectors $x \in \mathcal{X}$ that are *real with respect to* K (that is, such that $Kx = x$).

It is easy to see that the real irreducible representations $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ (that is, the complexifications of real representations) consist of those representations for which $j = k$. An invariant involution in $\rho^{(j,k)}$ can be defined by setting

$$(K\psi)(\omega,\bar{\omega}) = \overline{\psi(\omega,\bar{\omega})}. \tag{3.73}$$

In the case $j \neq k$, we can construct a real representation from the two representations $\mathfrak{D}^{(j,k)}$ and $\mathfrak{D}^{(k,j)}$, namely, $\mathfrak{D}^{(j,k)} \oplus \mathfrak{D}^{(k,j)}$. Once again, an invariant involution K in $\rho^{(j,k)} \oplus \rho^{(k,j)}$ can be defined by (3.73).

We give two useful realizations of the representations $\mathfrak{D}^{(j,j)}$.

Exercise 3.9. Let $H^{(n)}$ be the space of all complex pseudo-harmonic polynomial functions of the (real or complex) 4-vector r that are homogeneous of degree n $(= 0, 1, 2\ldots)$. Here the condition that a polynomial $q(r)$ be pseudo-harmonic means that

$$\Box q(r) \equiv \frac{\partial}{\partial r_\mu}\frac{\partial}{\partial r^\mu} q(r) = 0. \tag{3.74}$$

Prove that the action of $SL(2,C)$ in $H^{(n)}$ that associates the element $\underset{\sim}{A} \in SL(2,C)$ with the map $q(r) \to q(\Lambda(\underset{\sim}{A}^{-1})r)$ is equivalent to the representation $\mathfrak{D}^{(n/2,n/2)}$. [Hint: Consider the maps

$$\psi(\omega,\bar{\omega}) \to q(r) = \frac{1}{(n!)^2}\Big(\frac{\partial}{\partial\omega}\underset{\sim}{r}\frac{\partial}{\partial\bar{\omega}}\Big)^n \psi(\omega,\bar{\omega}), \tag{3.75a}$$

$$q(r) \to \psi(\omega,\bar{\omega}) = \frac{1}{n!}(\bar{\omega}\bar{\partial}_r\omega)^n q(r) \tag{3.75b}$$

and verify that they are inverses of each other.]

Exercise 3.10. Let $Q^{(n)}$ be the space of all complex polynomial functions on the light cone*

$$\Gamma_0 = \{\xi \in \mathbf{M} : \xi^2 = 0\}, \tag{3.76}$$

that are homogeneous of degree $n = 0, 1, 2, \ldots$. Prove that the action $Q(\xi) \to Q(\Lambda(\underset{\sim}{A}^{-1})\xi)$ of $SL(2,C)$ in $Q^{(n)}$ is equivalent to a representation in $\rho^{(n/2,n/2)}$. [Hint: Each vector on the upper light cone Γ_0^+

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* By a polynomial function on Γ_0 we mean the restriction to Γ_0 of a polynomial function in $\mathbf{M}$.

can be represented in the form

$$\xi^{\mu}(\omega) = -\omega\epsilon\underline{\sigma}^{\mu}\epsilon\bar{\omega} \tag{3.77}$$

for some $\omega \in \mathbf{C}^2$. Consider the maps

$$Q(\xi) \to \psi(\omega,\bar{\omega}) = Q(\xi(\omega)), \tag{3.78a}$$

$$\psi(\omega,\bar{\omega}) \to Q(\xi) = \frac{1}{(n!)^2}\left(\frac{\partial}{\partial\omega}\xi\frac{\partial}{\partial\bar{\omega}}\right)^n \psi(\omega,\bar{\omega}) \tag{3.78b}$$

and verify that they are inverses of each other.]

The value of real representations of $L_+^\uparrow$ is in large measure due to the fact that such representations can be extended to representations (in the same space) of $L^\uparrow$ which contains the spatial reflection* I_s. A discussion of this question is complicated by the non-single-valuedness of the representation arising from the representations of $SL(2,C)$ with odd valency. We shall be considering this question in the following chapters. Here we restrict ourselves to representations $\mathfrak{D}^{(n/2,n/2)}$ with even valency. (This will be sufficient for the questions discussed in §3.3 of this chapter.)

Exercise 3.11. Let T be a (single-valued) representation of $L_+^\uparrow$, and $T(I_s)$ an operator in the space of this representation, such that

$$T(I_s)T(\Lambda)T(I_s)^{-1} = T(I_s\Lambda I_s^{-1}) \quad \text{for all } \Lambda \in L_+^\uparrow, \tag{3.79a}$$

$$T(I_s)^2 = 1. \tag{3.79b}$$

Then there exists a unique representation of $L^\uparrow$ that coincides with $T(\Lambda)$ for $\Lambda \in L_+^\uparrow$ and for $\Lambda = I_s$.

Let $\mathfrak{D}^{(n/2,n/2)}$ be a representation of $L_+^\uparrow$ that is realized in the space $\mathcal{Q}^{(n)}$ (see Exercise 3.10). Using Exercise 3.10, it is easily shown that there exist exactly two extensions of this representation to a representation of $L^\uparrow$. We denote these representations by $\mathfrak{D}_\eta^{(n/2,n/2)}$, where the parameter $\eta \equiv \eta_s$ takes two values and is called the *spatial parity* of the representation. To define this representation, it is sufficient to indicate the action of I_s in $\mathcal{Q}^{(n)}$:

$$(\mathfrak{D}_\eta^{(n/2,n/2)}(I_s)Q)(\xi) = \eta_s Q(I_s\xi). \tag{3.80}$$

3.2. Lorentz-Invariant Generalized Functions in Minkowski Space

A. Definition

A generalized function $f(p_1,\dots,p_n) \in \mathcal{S}'(\mathbf{M}^n)$ depending on n 4-vectors $p_1,\dots,p_n \in$ $\in \mathbf{M}$ is said to be *Lorentz-invariant* (or $L_+^\uparrow$*-invariant*) if

$$f(\Lambda p_1,\dots,\Lambda p_n) = f(p_1,\dots,p_n) \quad \text{for } \Lambda \in L_+^\uparrow. \tag{3.81}$$

(If we replace $L_+^\uparrow$ here by the other subgroups $G \subset L$, we obtain the G*-invariant generalized functions.*)

The Lorentz-invariant functions play an important role in relativistic quantum physics. One uses for these, explicit invariant representations in the form of generalized functions of invariant algebraic combinations which can consist of 4-vectors

* We shall not deal here with the other subgroups of the full Lorentz group containing the time reversal I_t and the full reflection I_{st}. As will be clear in the later chapters, it is only realizations of the operations I_t and I_{st} by antilinear operators that are of physical interest.

$p_1, \ldots, p_n$ (and, possibly certain other variables of type $\operatorname{sgn} p_j^0$ for $p_j^2 > 0$ and taking discrete values). Such representations are particularly effective in the case when n is not large ($n \le 4$).

Here we shall be considering invariant generalized functions $f(p) \in \mathcal{S}'(\mathbf{M})$ of a single 4-vector p:

$$f(\Lambda p) = f(p) \quad \text{for } \Lambda \in L_+^\uparrow. \tag{3.82}$$

We denote the collection of all such generalized functions by $\mathcal{L}$.

This description enables us to obtain invariant representations for the so-called two-point (Wightman and Green) functions of scalar fields. Here p is interpreted either as a 4-momentum or as the difference of the coordinates $x_1 - x_2$ of two points in $\mathbf{M}_x$.

Every Lorentz-invariant function $f(p) \in \mathcal{S}'(\mathbf{M})$ turns out to be automatically $L^\uparrow$-invariant, that is, it satisfies the relation

$$f(I_s p) = f(p). \tag{3.83}$$

Exercise 3.12. Prove (3.83). [Hint: Write (3.82) in the form

$$(f(p), u(p)) = (f(p), u(\Lambda p)) \quad \text{for } \Lambda \in L_+^\uparrow, \tag{3.84}$$

where $u(p)$ is an arbitrary test function, and then average it over the rotation group $O_+(3) \subset L_+^\uparrow$:

$$(f(p), u(p)) = \left(f(p), \int_{R \subset O_+(3)} u(Rp)dR \right). \tag{3.85}$$

Then observe that the averaged function on the right hand side of this equation depends only on p^2 and p^0.]

B. Even Invariant Generalized Functions. Invariant Generalized Functions with Support at a Point

A Lorentz-invariant generalized function may not be invariant with respect to the full Lorentz group (and in particular, with respect to the transformations of time reversal I_t and full reflection I_{st}). We represent it in the form

$$f(p) = f_+(p) + f_-(p), \tag{3.86}$$

where

$$f_\pm(p) = \frac{1}{2}(f(p) \pm f(-p)) = \frac{1}{2}(f(p) \pm f(I_t p)). \tag{3.87}$$

It is clear that for any Lorentz transformation Λ, the following relation holds:

$$f_+(\Lambda p) = f(p), \quad f_-(\Lambda p) = \epsilon(\Lambda) f(p), \tag{3.88}$$

where $\epsilon(\Lambda) = 1$ if Λ does not reverse the direction of time (orthochronous transformation), otherwise $\epsilon(\Lambda) = -1$.

Lorentz-invariant generalized functions that transform in the same way as $f_+(p)$ under a full reflection are called *even* generalized functions. We denote their totality by $\mathcal{L}_+$. Similarly, we denote the family of generalized functions of type f_- (called *odd*) by $\mathcal{L}_-$. Thus the set $\mathcal{L}$ of invariant generalized functions splits up into a direct sum of subspaces $\mathcal{L}_+$ and $\mathcal{L}_-$ each of which consists of eigen-"functions" of the time reversal operator.

We consider the classes $\mathcal{L}_+$ and $\mathcal{L}_-$ separately.

It is not difficult to prove (using Proposition 2.2) that every Lorentz-invariant generalized function f in $\mathbf{M}$ that is concentrated at the point $p = 0$, can be (uniquely) expressed in the form

$$f(p) = P(\Box)\delta(p), \tag{3.89}$$

where P is an arbitrary complex polynomial (of one variable) and $\Box \equiv \Box_p$, the d'Alembertian, is defined by the formula

$$\Box_p = \frac{\partial}{\partial p^\mu}\frac{\partial}{\partial p_\mu} \equiv g_{\lambda\mu}\frac{\partial}{\partial p_\lambda}\frac{\partial}{\partial p_\mu}. \tag{3.90}$$

We can make the natural (but so far, somewhat imprecise) suggestion that any generalized function f in $\mathcal{L}_+$ depends only on the scalar square of p and has the form

$$f(p) = \phi(p^2). \tag{3.91}$$

Although generalized functions in $\mathcal{L}_+$ with support at the origin look as though they are exceptions to the rule, it turns out that if suitably interpreted, (3.91) does in fact describe all the generalized functions of $\mathcal{L}_+$.

In particular, the possibility of representing generalized functions in $\mathcal{L}_+$ that are concentrated at the origin in the form (3.91) becomes less surprising if we note that $\delta(p)$ is the limit of ordinary functions of p^2.

Exercise 3.13. Prove that

$$-\lim_{\nu\to\infty}\frac{\nu^2}{\pi^2}\sin(\nu p^2) = \delta(p) \quad \text{in} \quad \mathcal{S}'(\mathbf{M}). \tag{3.92}$$

In order to attach a precise meaning to (3.91), we rewrite it as follows:

$$\int f(p)u(p)d_4p = \int \phi(\tau)(\mathfrak{m}_+u)(\tau)d\tau, \tag{3.93}$$

where $\mathfrak{m}_+$ is the operator that associates with an arbitrary test function $u \in \mathcal{S}(\mathbf{M})$ the function $\mathfrak{m}_+u$ of the single real variable (τ):

$$(\mathfrak{m}_+u)(\tau) = \int \delta(\tau - p^2)u(p)d_4p. \tag{3.94}$$

We denote by $\mathcal{M}_+(\mathbf{R})$ the space of functions $h(\tau)$ in $\mathbf{R}$ that can be represented in the form $h = \mathfrak{m}_+u$ for some $u \in \mathcal{S}(\mathbf{M})$. The right hand side of (3.93) is then to be understood as the value of the linear functional ϕ at $\mathfrak{m}_+u$ (where ϕ has to be continuous in a suitable topology on $\mathcal{M}_+(\mathbf{R})$).

It turns out that $\mathcal{M}_+(\mathbf{R})$ is the space of all complex functions $h(\tau)$ in $\mathbf{R}$ of the form

$$h(\tau) = h_1(\tau) + h_2(\tau)\ln\frac{1}{|\tau|}, \tag{3.95}$$

where h_1 and h_2 are arbitrary functions in $\mathcal{S}(\mathbf{R})$ with $h_2(0) = 0$.

We give an outline of the argument showing that the functions (3.94) can in fact be represented in the form (3.95). (See Methée(1954) for a detailed proof.) If in the integral in (3.94) we introduce the spherical coordinates ρ, e of the vector p and carry out the integration with respect to p^0 and the angles (that is, with respect to the unit vector e), we obtain

$$h(\tau) = (\mathfrak{m}_+u)(\tau) = \frac{1}{(2\pi)^4}\int_{\theta(-\tau)\sqrt{-\tau}}^{\infty}\rho^2 d\rho \int d\Omega_e \frac{u(\sqrt{\tau+\rho^2}, \rho e) + u(-\sqrt{\tau+\rho^2}, \rho e)}{2\sqrt{\tau+\rho^2}} =$$

$$= \int_{\theta(-\tau)|\tau|}^{\infty} v(\tau+\sigma,\sigma)\sqrt{\frac{\sigma}{\tau+\sigma}}d\sigma,$$

where θ is the step function (2.22), and

$$v(\omega,\sigma) = \frac{1}{4(2\pi)^4}\int (u(\sqrt{\omega},\sqrt{\sigma}e + u(-\sqrt{\omega},\sqrt{\sigma}e))d\Omega_e$$

is a function of ω, σ in $\mathcal{S}(\bar{\mathbf{R}}_+ \times \bar{\mathbf{R}}_+)$. By setting

$$w(\xi,\sigma) = v(\xi-\sigma,\sigma)$$

(of class $\mathcal{S}$ in the variables $\sigma \geq 0, \xi \geq \sigma$), $h(\tau)$ can be written in the form

$$h(\tau) = \int_{\theta(-\tau)|\tau|}^{\infty} w(\tau,\sigma)\sqrt{\frac{\sigma}{\tau+\sigma}}d\sigma.$$

It is clear that $h(\tau)$ is of class $\mathcal{S}$ on any interval $(-\infty,-\epsilon]$ or $[\epsilon,\infty)$ (for $\epsilon > 0$). A possible unsmoothness of $h(\tau)$ at $\tau = 0$ can only arise as a result of a singularity at the lower limit of integration with respect to σ:

$$\int_{\theta(-\tau)|\tau|}^{A} w(\tau,\sigma)\sqrt{\frac{\sigma}{\tau+\sigma}}d\sigma \quad \text{(where } 0 < A < \infty).$$

From this we can easily infer that $h(\tau)$ becomes n times differentiably continuous with respect to τ in a neighbourhood of $\tau = 0$ after subtracting an expression of the form $\sum_{k=1}^{n+1} l_k \frac{\tau^k}{k!}\ln|\tau|$; this singularity arises in the integration of the finite sums of the Taylor series of the function $w(\tau,\sigma)$ with respect to σ in a neighbourhood of the point $\sigma = 0$. Thus for any natural number n we have

$$h(\tau) = h_n(\tau) + \sum_{k=1}^{n+1} l_k \frac{\tau^k}{k!}\ln\frac{1}{|\tau|}, \tag{3.96}$$

where $h_n(\tau)$ is an n times continuously differentiable function. The representation (3.95) follows from this. We note that the coefficients l_k depend only on the behaviour of $u(p)$ in a neighbourhood of $p = 0$. It is not difficult to conclude from dimensional considerations and Lorentz-invariance that $l_k = c_k\Box^{k-1}u(p)|_{p=0}$, where the c_k are coefficients (the numerical value of which is given in Exercise 3.14).

Exercise 3.14. Prove the following formula for the coefficients l_k in the expansion (3.96):

$$l_k = \frac{1}{2^{2k-1}(2\pi)^3(k-1)!}\Box^{k-1}u(p)|_{p=0}. \tag{3.97}$$

[Hint: First derive the formula for l_1; for the calculation of c_k for arbitrary k, it suffices to consider functions $u(p)$ of the special form: $u(p) = (p^2)^{k-1}u_0(p)$, where $u_0(p) \in \mathcal{S}(\mathbf{M})$.]

We call $\mathcal{M}_+(\mathbf{R})$ the *Methée space* of test functions in $\mathbf{R}$. The topology on $\mathcal{M}_+(\mathbf{R})$ is defined as follows. We associate with each natural number $k = 1, 2, \ldots$ the linear functional L_k on the functions $h \in \mathcal{M}_+(\mathbf{R})$ (represented in the form (3.95)) defined by

$$(L_k, h) = \frac{d^k}{d\tau^k}h_2(\tau)|_{\tau=0} \tag{3.98}$$

and the linear operator

$$(Q_k h)(\tau) = \sum_{\nu=1}^{k+1}(L_\nu, h)\frac{\tau^\nu}{\nu!}\ln\frac{1}{|\tau|}. \tag{3.99}$$

We now define on $\mathcal{M}_+(\mathbf{R})$ the countable system of norms:

$$p_{k,m}(h) = \|h - \omega Q_k h\|_{k,m} + \sum_{\nu=1}^{k+1} |(L_\nu, h)|, \tag{3.100}$$

where the $\|\cdot\|_{k,m}$ are defined by (1.42) (for $n = 1$) and $\omega(\tau)$ is a fixed function in $\mathcal{D}(\mathbf{R})$ that is equal to unity in a neighbourhood of the points $\tau = 0$ (the resultant topology does not depend on the choice of ω); $\mathcal{M}_+(\mathbf{R})$ endowed with this topology is a Fréchet space. Its dual $\mathcal{M}'_+(\mathbf{R})$ (the continuous linear functionals on $\mathcal{M}_+(\mathbf{R})$) is called the *Methée space of generalized functions* on $\mathbf{R}$.

The properties of the operator $\mathfrak{m}_+$ can be summarized as follows.

Lemma 3.1. *Formula* (3.94) *defines a continuous linear map* $\mathfrak{m}_+$ *from* $\mathcal{S}(\mathbf{M})$ *onto* $\mathcal{M}_+(\mathbf{R})$.

Our problem is to prove that every generalized function $f \in \mathcal{L}_+$ can be uniquely represented in the form (3.91) for some Methée generalized function $\phi(\tau) \in \mathcal{M}'_+(\mathbf{R})$, that is,

$$f = \mathfrak{m}'_+ \phi, \tag{3.101}$$

where $\mathfrak{m}'_+ : \mathcal{M}'_+(\mathbf{R}) \to \mathcal{S}'(\mathbf{M})$ is the adjoint of $\mathfrak{m}_+$.

Proposition 3.2. *The operator* $\mathfrak{m}'_+ : \mathcal{M}'_+(\mathbf{R} \to \mathcal{S}'(\mathbf{M})$ *maps* $\mathcal{M}'_+(\mathbf{R})$ *isomorphically onto* $\mathcal{L}_+$.

Proposition 3.2 is based on the results of Exercises 3.14 and 3.15.

Exercise 3.15. Prove that for any generalized function $f \in \mathcal{L}$ there exists a pair of generalized functions $\phi_+(\tau)$ and $\phi_-(\tau)$ in $\mathcal{S}'(\mathbf{R})$ that coincide when $\tau < 0$ and are such that

$$f(p) = \psi_+(p^2) \quad \text{for } p \notin -\bar{V}_+, \tag{3.102a}$$

$$f(p) = \psi_-(p^2) \quad \text{for } p \notin \bar{V}_+. \tag{3.102b}$$

Furthermore, if $f \in \mathcal{L}_+$ then $\psi_+ = \psi_-$. [Hint: Differentiation of (3.82) with respect to the parameters of the Lorentz group in a neighbourhood of the identity element gives a system of differential equations:

$$(p_\lambda \partial_\mu - p_\mu \partial_\lambda) f(p) = 0; \quad \lambda, \mu = 0, 1, 2, 3. \tag{3.103}$$

In each half-space $p^0 > 0$ and $p^0 < 0$ one can go over (via a diffeomorphism) to coordinates $\tau = p^2$ and $\mathbf{p}$, so that $f(p) = \hat{f}(\tau, \mathbf{p})$. The system (3.103) can then be written in the form $\partial_j \hat{f}(\tau, \mathbf{p}) = 0$, $j = 1, 2, 3$.]

We now deduce Proposition 3.2 from the last two exercises. The fact that $\mathfrak{m}'_+\phi$ belongs to $\mathcal{L}_+$ for any $\phi \in \mathcal{M}'_+(\mathbf{R})$ is fairly obvious. Conversely, let $f \in \mathcal{L}$. By Exercise 3.15, it corresponds to a generalized function $\psi \in \mathcal{S}'(\mathbf{R})$ such that

$$f(p) = \psi(p^2) \quad \text{for } p \neq 0. \tag{3.104}$$

By the Hahn-Banach theorem, ψ can be extended to a continuous linear functional, say, ϕ_1, on $\mathcal{M}_+(\mathbf{R})$. It then follows from (3.104) that $f - \mathfrak{m}'_+\phi_1$ is a generalized function in $\mathcal{L}_+$ with support at the origin; hence (by Exercise 3.14) it can be represented in the form $\mathfrak{m}'_+\phi_2$, where ϕ_2 is some linear combination of functionals L_k. By setting $\phi = \phi_1 + \phi_2$, we now obtain the representation (3.101) for f. It is clearly unique (that is, $f = 0$ implies that $\phi = 0$, which is obvious from the definition of the adjoint operator and the fact that $\mathfrak{m}_+\mathcal{S}(\mathbf{M}) = \mathcal{M}_+(\mathbf{R})$).

C. ODD INVARIANT GENERALIZED FUNCTIONS

The description of the odd Lorentz-invariant functions is somewhat simpler. Any generalized function $f(x) \in \mathcal{L}_-$ can be uniquely expressed in the form

$$f(p) = \epsilon(p^0)\psi(p^2), \tag{3.105}$$

where $\psi(\tau)$ is a generalized function in $\mathcal{S}'(\mathbf{R})$ that vanishes for $\tau < 0$ or equivalently, (in terms of the canonical isomorphism introduced in Appendix A) $\psi \in \mathcal{S}'(\bar{\mathbf{R}}_+)$. For the precise definition of this representation, we introduce the operator $\mathfrak{m}_-$:

$$(\mathfrak{m}_- u)(\tau) = \int \epsilon(p^0)\delta(\tau - p^2)d_4p, \tag{3.106}$$

which continuously maps functions $u \in \mathcal{S}(\mathbf{M})$ to functions $\mathfrak{m}_- u \in \mathcal{S}(\bar{\mathbf{R}}_+)$. (In fact, $\mathfrak{m}_-$ maps $\mathcal{S}(\mathbf{M})$ onto the whole of $\mathcal{S}(\bar{\mathbf{R}}_+)$.) Equation (3.105) now implies that

$$\int f(p)u(p)d_4p = \int \psi(\tau)(\mathfrak{m}_- u)(\tau)d\tau, \tag{3.107}$$

that is,

$$f = \mathfrak{m}'_- \psi. \tag{3.108}$$

Proposition 3.3. *The operator* $\mathfrak{m}'_- : \mathcal{S}'(\bar{\mathbf{R}}_+) \to \mathcal{S}'(\mathbf{M})$ *maps* $\mathcal{S}'(\bar{\mathbf{R}}_+)$ *isomorphically onto* $\mathcal{L}_-$.

Exercise 3.16. Prove Proposition 3.3. [Hint: Use Exercise 3.15.]

We note that it follows from (3.105) that each generalized function $f(x) \in \mathcal{L}_-$ automatically vanishes for space-like p.

As we have already remarked, in view of the decomposition $\mathcal{L} = \mathcal{L}_+ + \mathcal{L}_-$, we now have, in fact, a description of all the Lorentz-invariant functions in $\mathbf{M}$. We now introduce the manifold $\mathfrak{R}$, which is customarily called "the real line doubled along the positive semi-axis". An arbitrary point of $\mathfrak{R}$ has the form τ, where $\tau < 0$ or $\tau + i0$ or $\tau - i0$, where $\tau \geq 0$. The functions $h_+(\tau)$ in the Methée class $\mathcal{M}_+$ can be regarded as being defined on $\mathfrak{R}$ by setting

$$h(\tau) = h_+(\tau) \text{ for } \tau < 0, \quad h(\tau \pm i0) = h_+(\tau) \text{ for } \tau \geq 0. \tag{3.109a}$$

Similarly, the functions $h_-(\tau)$ in the class $\mathcal{M}_- = \mathcal{S}(\bar{\mathbf{R}}_+)$ can be regarded as being defined on $\mathfrak{R}$ by setting

$$h(\tau) = 0 \text{ for } \tau < 0, \quad h(\tau \pm i0) = \pm h_-(\tau) \text{ for } \tau \geq 0. \tag{3.109b}$$

The Methée space $\mathcal{M}$ is now defined as the direct sum $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-$; its elements are given by functions on $\mathfrak{R}$ that are expressible as the sum of a pair of functions of the form (3.109a) and (3.109b). The Methée operator $\mathfrak{m} : \mathcal{S}(\mathbf{M}) \to \mathcal{M}$ is defined as the sum of the operators $\mathfrak{m}_\pm$: $\mathfrak{m} = \mathfrak{m}_+ + \mathfrak{m}_-$. This implies that

$$(\mathfrak{m}u)(\tau) = \int \delta(\tau - p^2)u(p)d_4p \quad \text{for} \quad \tau < 0, \tag{3.110a}$$

$$(\mathfrak{m}u)(\tau \pm i0) = \int 2\theta(\pm p^0)\delta(\tau - p^2)u(p)d_4p \quad \text{for} \quad \tau \geq 0. \tag{3.110b}$$

It is clear that $\mathfrak{m}$ maps $\mathcal{S}(\mathbf{M})$ linearly and continuously onto $\mathcal{M}$.

Let $\mathcal{M}'$ be the dual of $\mathcal{M}$. We shall also write the value (ϕ, h) of the functional $\phi \in \mathcal{M}'$ at the element $h \in \mathcal{M}$ in the symbolic integral form:

$$(\phi, h) = \int_{\Re} \phi(s)h(s)[ds] = \int_{\tau<0} \phi(\tau)h(\tau)d\tau + 1/2 \sum_{\epsilon=\pm} \int_{\tau\geq 0} \phi(\tau + \epsilon i0)h(\tau + \epsilon i0)d\tau, \tag{3.111}$$

where $[ds]$ is the natural measure on $\Re$ which (taking the doubling of the positive semi-axis into account) is equal to

$$[ds] = d\tau \quad \text{for } \tau < 0, \quad [ds] = 1/2 d\tau \quad \text{for } s = \tau \pm i0, \ \tau \geq 0. \tag{3.112}$$

From Propositions 3.2 and 3.3 we have the following characterization of $\mathcal{L}$.

Proposition 3.4. *The operator* $\mathfrak{m}' : \mathcal{M}' \to \mathcal{S}'(\mathbf{M})$ *maps* $\mathcal{M}'$ *(the Methée space of generalized functions on* $\Re$*) onto the space* $\mathcal{L}$ *(of Lorentz-invariant generalized functions on* $\mathbf{M}$*).*

By analogy with the representation (3.91) for $\mathcal{L}_+$, we can now write an arbitrary Lorentz-invariant generalized function $f(p)$ on $\mathbf{M}$ symbolically in the form

$$f(p) = \begin{cases} \phi(p_2) & \text{for } p^2 < 0, \\ \phi(p^2 + i0 \cdot \epsilon(p^0)) & \text{for } p^2 \geq 0, \end{cases} \tag{3.113}$$

where $\phi \in \mathcal{M}'$ which is another notation for the relation

$$f = \mathfrak{m}'\phi. \tag{3.114}$$

We now give another result on the extension of Lorentz-invariant generalized functions $f(p)$ defined away from the origin in $\mathbf{M}$ (the definition of the appropriate space $\mathcal{S}'(\mathbf{M}\backslash\{0\})$ was given in Appendix A) to generalized functions in $\mathbf{M}$. The statement of Exercise 3.14 still holds for such f, so that we have the following result, which was implicit in the discussion leading up to Propositions 3.2 and 3.3.

Proposition 3.5. Every Lorentz-invariant generalized function $f(p) \in \mathcal{S}'(\mathbf{M}\backslash\{0\})$ can be extended to a Lorentz-invariant generalized function in $\mathbf{M}$. Such an extension is not unique: if $f_0(p)$ is a particular extension, then the general form of the extensions is $f_0(p) + P(\Box)\delta(p)$, where P is an arbitrary polynomial.

The descriptions simplify for non-negative generalized functions (which we shall be encountering later on; see, for example, the Källén-Lehmann representation in §8.3.B).

A generalized function f is said to be *non-negative* if for any non-negative test function u,

$$(f, u) \geq 0. \tag{3.115}$$

It follows from Riesz's theorem (see §1.1.F,Example 1) that any non-negative generalized function $f \in \mathcal{S}(\mathbf{R})$ is associated with a non-negative measure μ of power growth via the Stieltjes integral

$$(f, u) = \int u(p)d\mu(p). \tag{3.116}$$

Here we say that a (generally complex-valued) measure μ in $\mathbf{R}^n$ is of power growth if

$$\int_{|p|<r} |d\mu(p)| \tag{3.117}$$

is a function of r that is bounded in modulus by a polynomial or, equivalently, if for some l we have

$$\int (1+|p|)^{-l}|d\mu(p)| < \infty. \tag{3.118}$$

Measures are a rather special case of generalized functions. If $d\mu(p) = f(p)d_4p$, then $f(p)$ may contain singularities of δ-function type, but cannot be derivatives of a δ-function. This enables us to obtain a representation for invariant measures that is closer to the intuitive one.

An even Lorentz-invariant non-negative measure in $\mathcal{S}'(\mathbf{M})$ has the general form

$$d\mu(p) = \left\{ \int_{-\infty}^{\infty} \delta(\tau - p^2)d\rho(\tau) + c(2\pi)^4\delta(p) \right\} d_4p, \tag{3.119}$$

where $\rho(\tau)$ is a monotone increasing function of polynomial growth on the real axis and $c \geq 0$. There is the following analogous representation for a Lorentz-invariant non-negative measure in $\mathcal{S}'(\mathbf{M})$ with support in $\overline{V}^+$:

$$d\mu(p) = \left\{ \int_{0}^{\infty} \theta(p^0)\delta(\tau - p^2)d\rho(\tau) + c(2\pi)^4\delta(p) \right\} d_4p, \tag{3.120}$$

where $\rho(\tau)$ is a monotone decreasing function of polynomial growth on $\overline{\mathbf{R}}_+$ and $c \geq 0$.

Exercise 3.17. Prove the representations (3.119) and (3.120) by using the general representations (3.91) and (3.113) (or (3.101) and (3.114)) for $\mathcal{L}_+$ and $\mathcal{L}_-$.

3.3. Lorentz-Covariant Generalized Functions in Minkowski Space

A. DEFINITION

Let $f(p_1, \ldots, p_n)$ be a vector-valued generalized function depending on n 4-vectors $p_1, \ldots, p_n \in \mathbf{M}$ and taking values in a complex finite-dimensional vector space $\mathcal{X}$ in which a representation T of the group $L_+^\uparrow$ (or $SL(2,C)$) is given. We denote the space of such generalized functions by $\mathcal{S}'(\mathbf{M}^n; \mathcal{X})$. We distinguish the *Lorentz-covariant* (or $L_+^\uparrow$*-covariant*) generalized functions by the condition

$$f(\Lambda p_1, \ldots, \Lambda p_n) = T(\Lambda) f(p_1, \ldots, p_n) \quad \text{for } \Lambda \in L_+^\uparrow. \tag{3.121}$$

(The more general notion of G-covariant generalized functions is similarly defined.) By choosing a basis $\{b_\alpha\}$ ($\alpha = 1, \ldots, \dim \mathcal{X}$) and introducing the components of f in this basis

$$f(p_1, \ldots, p_n) = \sum_\alpha b_\alpha f^\alpha(p_1, \ldots, p_n),$$

we can rewrite (3.121) in the component form

$$f^\alpha(\Lambda p_1, \ldots, \Lambda p_n) = \sum_\beta T^\alpha_\beta(\Lambda) f^\beta(p_1, \ldots, p_n). \tag{3.122}$$

In terms of the group $SL(2,C)$, condition (3.121) can be rewritten in the form

$$f(\Lambda(\underset{\sim}{A})p_1, \ldots, \Lambda(\underset{\sim}{A})p_n) = T(\underset{\sim}{A}) f(p_1, \ldots, p_n) \quad \text{for } \underset{\sim}{A} \in SL(2,C), \tag{3.123}$$

where T is the representation of $SL(2,C)$.

Here it is only meaningful to consider representations of $SL(2,C)$ with even valency (that is, representations corresponding to single-valued representations of $L_+^\uparrow$). In fact, if T is a representation with odd valency, so that $T(-1) = -1$, then (3.123) automatically implies that $f \equiv 0$.

As in the case of the Lorentz-invariant generalized functions, of interest are the representations that explicitly take into account the covariance properties. In the first instance these refer to representations of the form

$$f(p_1, \ldots, p_n) = \sum_{\rho=1}^{N} Q_\rho(p_1, \ldots, p_n) f_\rho(p_1, \ldots, p_n); \tag{3.124}$$

here the $Q_\rho(p_1, \ldots, p_n)$ are Lorentz-covariant polynomial functions of $p_1, \ldots, p_n$ and the f_ρ are Lorentz-invariant generalized functions. For the Q_ρ, one generally fixes some finite family of Lorentz-covariant polynomials (called *standard covariants* for the given representation T) defined by the following algebraic condition: for all Lorentz-covariant polynomials f that are transformed according to the representation T, there is a representation (3.124) with Lorentz-invariant polynomials f_ρ. If in addition, $f \equiv 0$ always implies that $f_\rho \equiv 0$ (that is, if the covariant decomposition is unique in the class of polynomials), then the family $\{Q_\rho\}$ is called a *polynomial basis of standard covariants* that transform according to the representation T.

The representation (3.124) is of practical significance only for smallish numbers n (mainly $n < 4$); otherwise the number of standard covariants may turn out to be larger than the number ($\dim \mathcal{X}$) of components of f.

B. Structure of Covariant Generalized Functions

We describe the structure of Lorentz-covariant generalized functions $f(p) \in \mathcal{S}'(\mathrm{M}; \mathcal{X})$ of a single 4-vector p:

$$f(\Lambda p) = T(\Lambda) f(p) \quad \text{for } \Lambda \in L_+^\uparrow. \tag{3.125}$$

Because of the complete reducibility of the finite-dimensional representations of $L_+^\uparrow$, we can of course, in principle, restrict ourselves to the irreducible representations $\mathfrak{D}^{(j,k)}$. If this representation is realized in the space $\mathfrak{p}^{(j,k)}$ of homogeneous polynomials in ω and $\bar{\omega}$ (of degree $2j$ and $2k$ respectively), then the vector-valued generalized function $f(p)$ becomes the complex-valued generalized function $f(p; \omega, \bar{\omega})$ which depends both on p and on the spinor variables $\omega \in \mathbf{C}^2$ and $\bar{\omega}$ (as homogeneous polynomials of degree $2j$ and $2k$ respectively). In this case, (3.125) assumes the form

$$f(\Lambda(\underset{\sim}{\Lambda})p;\ \underset{\sim}{\Lambda}\omega,\ \bar{\underset{\sim}{\Lambda}}\bar{\omega} = f(p, \omega, \bar{\omega}) \quad \text{for all } \underset{\sim}{\Lambda} \in SL(2, C), \tag{3.126}$$

that is, the condition of Lorentz-covariance reduces to one of Lorentz-invariance (with respect to a larger number of variables).

It turns out that polynomial covariants of a single 4-vector p that transform according to the representation $\mathfrak{D}^{(j,k)}$, exist only for $j = k = n/2$ (where n is a non-negative integer) and in this case they form (according to (3.124) with the polynomial f_ρ, $\rho = 1$) the single standard covariant

$$Q(p; \omega, \bar{\omega}) = (\bar{\omega} \tilde{p} \omega)^n. \tag{3.127a}$$

polynomial functions (of degree n) on the cone Γ_0 (see Exercise 3.10), the covariant (3.127a) takes on the form

$$Q(p;\xi) = (p\xi)^n. \tag{3.127b}$$

From this it is clear what the general form of Lorentz-covariant generalized functions must be.

Proposition 3.6. *An arbitrary Lorentz-covariant generalized function $f(p)$ in* M *that transforms according to the representation $\mathfrak{D}^{(j,k)}$, is non-zero only for $j = k = n/2$, and in this case it can be represented in the form*

$$f(p;\omega,\bar\omega) = (\bar\omega\tilde p\omega)^n F(p) = \tag{3.128a}$$

$$= (i\bar\omega\tilde\partial_p\omega)^n H(p), \tag{3.128b}$$

where $F(p)$ and $H(p)$ are Lorentz-invariant generalized functions in M *that are (for the given generalized function f) defined to within n arbitrary constants. More precisely, if $F_0(x)$ and $H_0(x)$ are fixed solutions of (3.128) (regarded as equations in $F(x)$ and $H(x)$), then the general solution has the form*

$$F(p) = F_0(p) + \sum_{l=0}^{n-1} a_l \Box^l \delta(p), \tag{3.129a}$$

$$H(p) = H_0(p) + \sum_{l=0}^{n-1} b_l \cdot (p^2)^l. \tag{3.129b}$$

Proposition 3.6 is based on the following two lemmas.

Lemma 3.7. *Let $f(p)$ be a Lorentz-covariant generalized function in* M *that transforms according to the representation $\mathfrak{D}^{(j,k)}$. Then there exists a pair of generalized functions (which is unique in the non-trivial case $j = k$) $\phi_\pm(\tau) \in \mathcal{S}'(\mathbf{R})$, that coincide for $\tau < 0$ and are such that*

$$f(p;\omega,\bar\omega) = \delta_{jk}(\bar\omega\tilde p\omega)^{2j}\phi_\pm(p^2) \quad \text{for } p \in \mathcal{V}_\pm, \tag{3.130}$$

where

$$\mathcal{V}_+ = \mathbf{M}\backslash\overline{V^-},\ \mathcal{V}_- = -\mathcal{V}_+. \tag{3.131}$$

■ We begin by proving (3.130) on $\mathcal{V}_+$. We rewrite the covariance condition (3.126) in infinitesimal form. To this end we parametrize the complex 2×2 matrix $\underset{\sim}{\Lambda} = (\underset{\sim}{\Lambda}{}^\alpha_\beta)$ in $SL(2,C)$ in a neighbourhood of the identity using $\underset{\sim}{\Lambda}{}^1_1$, $\underset{\sim}{\Lambda}{}^1_2$, $\underset{\sim}{\Lambda}{}^2_1$ as the parameters; we identify an arbitrary 4-vector p with the Hermitian matrix $\tilde p \equiv (p_{\alpha'\beta})$. We regard the left hand side of the covariance condition

$$f(\underset{\sim}{\Lambda}^{*-1}\tilde p\underset{\sim}{\Lambda}^{-1},\ \underset{\sim}{\Lambda}\omega,\ \bar{\underset{\sim}{\Lambda}}\bar\omega) = f(\tilde p,\omega,\bar\omega) \tag{3.132}$$

as a generalized function in $\mathcal{S}'(\mathbf{M}\times\mathbf{C}^2)$, depending on $\underset{\sim}{\Lambda}{}^1_1$, $\underset{\sim}{\Lambda}{}^1_2$, $\underset{\sim}{\Lambda}{}^2_1$ as parameters; we differentiate (3.126) with respect to these parameters and their complex conjugates. As a result we obtain six differential equations. We write down just four of these (the simplest ones), namely those obtained by differentiating with respect to $\underset{\sim}{\Lambda}{}^1_2$, $\bar{\underset{\sim}{\Lambda}}{}^1_2$, $\underset{\sim}{\Lambda}{}^2_1$, $\bar{\underset{\sim}{\Lambda}}{}^2_1$, since the remaining two are corollaries of these four:

$$\left(p_{11}\frac{\partial}{\partial p_{12}} + p_{21}\frac{\partial}{\partial p_{22}} - \omega^2\frac{\partial}{\partial\omega^1}\right)f = 0, \quad \left(p_{11}\frac{\partial}{\partial p_{21}} + p_{12}\frac{\partial}{\partial p_{22}} - \bar\omega\frac{\partial}{\partial\bar\omega^1}\right)f = 0, \tag{3.133a}$$

$$\left(p_{12}\frac{\partial}{\partial p_{11}} + p_{22}\frac{\partial}{\partial p_{21}} - \omega^1\frac{\partial}{\partial\omega^2}\right)f = 0, \quad \left(p_{21}\frac{\partial}{\partial p_{11}} + p_{22}\frac{\partial}{\partial p_{12}} - \bar\omega^1\frac{\partial}{\partial\bar\omega^2}\right)f = 0. \tag{3.133b}$$

We now observe that to prove (3.130) in the region $\mathcal{V}_+ \times \mathbf{C}^2$, it suffices to prove it for the region

$$Q = \{(p,z) \in \mathcal{V}_+ \times \mathbf{C}^2 : p_{11} > 0,\ \omega \underset{\sim}{p} \bar{\omega} > 0,\ \mathrm{Im}\left(\omega^1 + \frac{p_{12}}{p_{11}}\omega^2\right) > 0\}.$$

In fact, if (3.130) holds in Q, then since both sides of the equality are polynomials in $\omega, \bar{\omega}$, (3.130) holds in the larger region $Q' = \{(p,\omega) \in \mathcal{V}_+ \times \mathbf{C}^2 : p_{11} > 0\}$. But under the action of the transformations of $SL(2,C)$ on Q', we obtain the entire region $\mathcal{V}_+ \times \mathbf{C}^2$, therefore it follows from (3.126) and the validity of (3.130) in Q that (3.130) holds in $\mathcal{V}_+ \times \mathbf{C}^2$.

Thus we have to prove that (3.130) holds in Q. We carry out the regular change of variables in this region:

$$(p,\omega) \to (\tau = (p)^2, \quad u = \bar{\omega}\tilde{p}\omega, \quad p_{11}, p_{12}, \omega^2, \quad \theta = \arg(\omega^1 + p_{12}\omega^2/p_{11})), \tag{3.134}$$

which maps Q onto

$$\mathcal{R} = \mathbf{R} \times \mathbf{R}_+ \times \mathbf{R}_+ \times \mathbf{C} \times \mathbf{C} \times (0,\pi). \tag{3.135}$$

It is not difficult to see that the map (3.134) is regular by representing it as the composition of the two regular maps

$$(p_{11}, p_{22}, p_{12}, \omega^1, \omega^2) \to (p_{11}, \tau = p_{11}p_{22} - |p_{21}|^2,\ p_{22}, \lambda = p_{11}\omega^1 + p_{12}\omega^2, \omega^2);$$
$$(p_{11}, \tau, p_{12}, \lambda, \omega^2) \to (p_{11}, \tau, p_{12}, u = \frac{|\lambda|^2 + \tau|\omega^2|^2}{p_{11}}, \quad \theta = \arg\lambda, \omega^2).$$

Consequently we can make the following change of variables in $f(\underset{\sim}{p}; z) \in \mathcal{S}'(Q)$:

$$f(\underset{\sim}{p};\omega) = \psi(\tau, u, p_{11}, p_{12}, z^2, \theta), \tag{3.136}$$

where $\psi \in \mathcal{S}'(\mathcal{R})$. Upon substituting (3.136) into (3.133) we obtain

$$p_{11}\frac{\partial}{\partial p_{12}}\psi = 0,\ p_{11}\frac{\partial}{\partial p_{21}}\psi = 0, \tag{3.137a}$$

$$\left(p_{12}\frac{\partial}{\partial p_{11}} + p_{22}\frac{\partial}{\partial p_{21}} + \frac{p_{12}\omega^2 - \lambda}{p_{11}}\frac{\partial}{\partial \omega^2} - \frac{p_{12}\bar{\lambda} + \tau\bar{\omega}^2}{2ip_{11}\bar{\lambda}}\frac{\partial}{\partial\theta}\right)\psi = 0, \tag{3.137b}$$

$$\left(p_{21}\frac{\partial}{\partial p_{11}} + p_{22}\frac{\partial}{\partial p_{12}} + \frac{p_{21}\bar{\omega}^2 - \bar{\lambda}}{p_{11}}\frac{\partial}{\partial \bar{\omega}^2} + \frac{p_{21}\lambda + \tau\omega^2}{2ip_{11}\lambda}\frac{\partial}{\partial\theta}\right)\psi = 0. \tag{3.137c}$$

Here λ is regarded as a function of $\tau, u, p_{11}, z_2, \theta$ (not dependent on p_{12}). Now according to (3.137a), $\frac{\partial}{\partial p_{12}}\psi = \frac{\partial}{\partial p_{21}}\psi = 0$; therefore in (3.137b) and (3.137c) we can equate to zero the coefficients in front of the various powers of p_{12} and p_{21} $(= \bar{p}_{12})$. As a result, we have

$$\frac{\partial}{\partial p_{12}}\psi = \frac{\partial}{\partial \bar{p}_{12}}\psi = \frac{\partial}{\partial p_{11}}\psi = \frac{\partial}{\partial \omega^2}\psi = \frac{\partial}{\partial \bar{\omega}^2}\psi = \frac{\partial\psi}{\partial\theta} = 0. \tag{3.138}$$

Since any section $(\tau, u) = \text{const}$ of $\mathcal{R}$ is constant in the variables $p_{11}, p_{12}, \omega^2, \theta$, it follows from (3.138) that there exists $\Phi(\tau, u) \in \mathcal{S}'(\mathbf{R} \times \bar{\mathbf{R}}_+)$ such that

$$\psi(\tau, u, p_{11}, p_{12}, \omega^2, \theta) = \Phi(\tau, u),$$

that is

$$f(p,\omega) = \Phi(p^2, \bar{\omega}\tilde{p}\omega) \text{ in } Q. \tag{3.139}$$

It follows from the homogeneity condition $f(p, a\omega) = a^{2j}(\bar{a})^{2k}f(p,\omega)$, which holds for any complex a, that $\Phi(\tau,u) = 0$ for $j \neq k$ and $\Phi(\tau,u) = u^{2j}\phi_+(\tau)$ for $j = k$. Thus the representation (3.130) is proved for $\mathcal{V}_+$. Similarly, it is proved for $\mathcal{V}_-$. It now follows from the fact that (3.130) holds in $\mathcal{V}_+ \cap \mathcal{V}_-$, that $\phi_+(\tau) = \phi_-(\tau)$ for $\tau < 0$. ∎

As a corollary we have

$$f(p,\omega) = \delta_{jk}(\bar{\omega}\tilde{p}\omega)^{2j}F(p) \quad \text{for } p \neq 0, \tag{3.140}$$

where $F(p) \in \mathcal{S}'(\mathrm{M}\setminus\{0\})$ and $F(p) = \phi_\pm(p^2)$ for $p \in \mathcal{V}_\pm$.

Our next step is to determine the covariant generalized functions that are concentrated at $p = 0$.

Lemma 3.8. *Let $f(p)$ be a Lorentz-invariant generalized function in* M *that transforms according to the representation $\mathfrak{D}^{(j,k)}$ and is concentrated at $p = 0$. Then it has the form*

$$f(p;\omega,\bar{\omega}) = \delta_{jk}(\bar{\omega}\tilde{p}\omega)^{2j}\sum_{l=0}^{N} a_l(\Box_p)^{l+2j}\delta(p) = \tag{3.141a}$$

$$= \delta_{jk}(i\bar{\omega}\tilde{\partial}_p\omega)^{2j}\sum_{l=0}^{N} b_l(\Box_p)^{l}\delta(p), \tag{3.141b}$$

where a_l and b_l are arbitrary complex numbers.

■ Since $f(p;\omega,\bar{\omega})$ is concentrated at $p = 0$, its Fourier transform $\tilde{f}(x:\omega,\bar{\omega})$ with respect to $\bar{p}$ is a Lorentz-covariant polynomial in $x,\omega,\bar{\omega}$. If now we apply Lemma 3.7 (or the representation (3.140)) to $\tilde{f}(x;\omega,\bar{\omega})$, we find that for $x \in \mathcal{V}_+$

$$\tilde{f}(x,\omega,\bar{\omega}) = \delta_{jk}(\bar{\omega}\tilde{x}\omega)^{2j}\chi(x^2). \tag{3.142}$$

It is clear that we can henceforth restrict ourselves to the case $j = k = n/2$. Since $\tilde{f}(x,\omega,\bar{\omega})$ is a polynomial in x, there exists an integer m such that $(\bar{\omega}\tilde{\partial}_x\omega)^m\tilde{f}(x;\omega,\bar{\omega}) = 0$. Hence by substituting in (3.142) and using the identity

$$(\bar{\omega}\tilde{\partial}\omega)(\bar{\omega}\tilde{x}\omega) = 0,\ (\bar{\omega}\tilde{\partial}_x\omega)(x^2) = 2(\bar{\omega}\tilde{x}\omega), \tag{3.143}$$

we find that $(\bar{\omega}\tilde{x}\omega)^{m+n}\chi^{(m)}(x^2) = 0$, that is $(d/d\tau)^m\chi(\tau) = 0$, and hence, $\chi(\tau)$ is a polynomial. But in this case, (3.141b) follows from (3.142). We then see from the identities (3.143) that

$$(\bar{\omega}\tilde{\partial}_x\omega)^n(x^2)^l = 0 \quad \text{for } l < n, \tag{3.144a}$$

$$(\bar{\omega}\tilde{\partial}_x\omega)^n(x^2)^l = \frac{l!}{n!(n-l)!}\cdot 2^n(\bar{\omega}\tilde{x}\omega)^n(x^2)^{l-n} \quad \text{for } l \geq n, \tag{3.144b}$$

so that

$$(\bar{\omega}\tilde{x}\omega)^n(x^2)^l = \frac{l!n!}{(n+l)!}\cdot 2^{-n}(\bar{\omega}\tilde{\partial}_x\omega)^n(x^2)^{l+n}. \tag{3.145}$$

The last identity enables us to rewrite (3.142) in the form

$$\tilde{f}(x;\bar{\omega},\omega) = \delta_{jk}\cdot(\bar{\omega}\tilde{\partial}_x\omega)^n\sum_{l=0}^{N} c_l(x^2)^{l+n},$$

which is equivalent to (3.141a). ■

We now give the argument for Proposition 3.6. If $j \neq k$, then it follows from Lemmas 3.7 and 3.8 that $f(p;\omega,\bar{\omega}) = 0$. Let $j = k = n/2$ (where n is a non-negative integer); then for $p \neq 0$, $f(p;\omega,\bar{\omega})$ can be represented in the form (3.140). Extending $F(p)$ to a Lorentz-invariant generalized function$F_1(p) \in$ $\in \mathcal{S}'(\mathrm{M})$ (which is possible according to Proposition 3.5), we obtain a covariant generalized function $f_1(p,\omega,\bar{\omega})$. The difference $f - f_1$ is concentrated at $p = 0$ and therefore has the form (3.141a). Thus the representation (3.128a) is proved. The representation (3.128b) follows from (3.128a) applied to the Fourier transform $\tilde{f}(x;\omega,\bar{\omega})$ of $f(p,\omega,\bar{\omega})$ with respect to p. Finally, the arbitrariness in the choice of the functions $F(p)$ and $H(p)$ in (3.128) easily follows from the identities (3.144). This completes the proof of Proposition 3.6.

In the next exercise, a property of the standard covariant (3.127) is given that uniquely characterizes it to within a numerical factor.

Exercise 3.18. Prove that in the space of Lorentz-covariant polynomials $Q(p)$ in $p \in \mathrm{M}$ that transform according to the representation $\mathfrak{D}^{(n/2,n/2)}$, the condition of pseudo-harmonicity with respect to p

$$\Box_p Q(p) = 0 \tag{3.146}$$

singles out the one-dimensional subspace spanned by the standard covariant (3.127). [Hint: It suffices to verify that the condition $\Box_p Q(p,\xi) = 0$ (where $Q(p,\xi) = (p\xi)^n P(p^2)$, $\xi^2 = 0$ and $P(z)$ is a polynomial in z) implies that $P(z) = \text{const.}$]

We can easily obtain from Proposition 3.6 the structure of covariant functions $f(p)$ in $\mathbf{M}$ that transform according to an arbitrary finite-dimensional representation T of $SL(2,C)$. Each such representation T decomposes into a direct sum of representations $\mathfrak{D}^{(i,j)}$ entering into the direct sum with multiplicities $k_{ij} = 0, 1, 2, \ldots$ (where i, j are arbitrary integral or half-integral non-negative numbers) such that $\sum_{i,j} k_{ij} < \infty$. We write this representation symbolically in the form $\bigoplus_{i,j} k_{ij}\mathfrak{D}^{(i,j)}$.

Exercise 3.19. Let $T = \bigoplus_{i,j} k_{ij}\mathfrak{D}^{(i,j)}$ be an arbitrary finite-dimensional representation of the proper Lorentz group $L_+^\uparrow$ (or $SL(2,C)$)and let $k = \sum_j k_{jj}$.

(a) Prove that the pseudo-harmonic (with respect to p) Lorentz-covariant polynomials $Q(p)$ in $p \in \mathbf{M}$ that transform according to the representation T, form a k-dimensional linear subspace and that a linear basis $\{Q_\rho(p)\}$ in this space is at the same time a polynomial basis of standard covariants in $\mathbf{M}$ that transform according to T. [Hint: Use the preceding exercise.]

(b) Suppose that the standard covariants $Q_\rho(p)$ are chosen to be homogeneous polynomials in p of degree n_ρ (it can be easily verified that there is no loss of generality in making this assumption). Prove that each Lorentz-covariant generalized function $f(p)$ in $\mathbf{M}$ that transforms according to T, has the form

$$f(p) = \sum_\rho Q_\rho(p) F_\rho(p) = \tag{3.146a}$$

$$= \sum_\rho Q_\rho(i\partial_p) H_\rho(p) \tag{3.146b}$$

for certain Lorentz-invariant generalized functions $F_\rho(p)$ and $H_\rho(p)$ which are defined by these representations to within the additive terms $a_\rho(\Box)\delta(p)$ and $b_\rho(p^2)$ respectively, where $a_\rho(z)$ and $b_\rho(z)$ are arbitrary polynomials in z of degree n_ρ.

Examples. 1) Let T be the 4-vector representation $\mathfrak{D}^{(1/2,1/2)}$. Then there is one (independent) standard covariant in $\mathbf{M}$:

$$Q(p;\omega,\bar\omega) = \bar\omega\tilde{p}\omega. \tag{3.147}$$

Hence every 4-vector-valued Lorentz-covariant generalized function $F_\mu(p)$ in $\mathcal{S}'(\mathbf{M})$ is representable in the form

$$F_\mu(p) = p_\mu F(p), \tag{3.148}$$

where $F(p)$ is a Lorentz-invariant generalized function in $\mathcal{S}'(\mathbf{M})$.

2) Let T be the tensor product of two copies of the Dirac representation $\mathfrak{D}^{(1/2,0)} \oplus \oplus\mathfrak{D}^{(0,1/2)}$. In this case there are two (independent) standard covariants

$$\begin{aligned} Q_1(p;\omega,\bar\omega;w,\bar w) &= \bar w\tilde{p}\omega, \\ Q_2(p;\omega,\bar\omega,w,\bar w) &= \bar\omega\tilde{p}w. \end{aligned} \tag{3.149}$$

3.4. The Case of Several Vector Variables

A. Generalized Functions That Are Invariant With Respect to a Compact Group

We have seen in §3.2 that even for a single 4-vector, the description of the Lorentz-invariant generalized functions is fairly complicated, the reason for this being the non-compactness of the Lorentz group. However, in a situation (typical of quantum field theory) when at least one of the 4-vectors is time-like, there is a considerable simplification due to the fact that the invariance (or covariance) group reduces to a (compact) rotation group. Before turning our attention to this case, we discuss some auxiliary material on generalized functions that are invariant or covariant with respect to a compact group.*

We start with a compact Lie group G acting linearly on $\mathbf{R}^n$. We denote by gx the point in $\mathbf{R}^n$ obtained by the action of $g \in G$ on $x \in \mathbf{R}^n$. (We are assuming here that gx is continuous and even infinitely differentiable with respect to g.) We denote by $\mathcal{S}(\mathbf{R}^n)^G$ and $\mathcal{S}'(\mathbf{R}^n)^G$ respectively, the set of all G-invariant test and generalized functions in $\mathbf{R}^n$. The topology on these spaces is induced respectively by the topology on $\mathcal{S}(\mathbf{R}^n)$ and the weak topology on $\mathcal{S}'(\mathbf{R}^n)$.

Our main example will be the group $O(d)$ of all orthogonal transformations of the real Euclidean space $\mathbf{R}^d$ ($d \geq 2$) endowed with the Euclidean scalar product

$$\mathbf{xy} = \sum_{\delta=1}^{d} x^\delta y^\delta \tag{3.150}$$

(where x^δ and y^δ are the components of the vectors $\mathbf{x}, \mathbf{y} \in \mathbf{R}^d$). For each k ($= 1, 2, \ldots$) we identify the Euclidean space $\mathbf{R}^{dk}$ with the Cartesian product $\mathbf{R}^d \times \ldots \times \mathbf{R}^d$ of k copies of $\mathbf{R}^d$; a typical point of $\mathbf{R}^{dk}$ will be denoted by $x \equiv (\mathbf{x}_1, \ldots, \mathbf{x}_k)$, where $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are arbitrary vectors in $\mathbf{R}^d$. The group $O(d)$ now acts in natural fashion in $\mathbf{R}^{dk}$:

$$R(\mathbf{x}_1, \ldots, \mathbf{x}_k) \equiv (R\mathbf{x}_1, \ldots R\mathbf{x}_k), \;\; R \in O(d). \tag{3.151}$$

It should be noted that $O(d)$ is not connected: it consists of two components corresponding to $\det R = \pm 1$. The component of $O(d)$ containing the identity is the group $O_+(d)$ of all *proper orthogonal transformations* (or *Euclidean rotations*).

It follows from the compactness assumption on G that $\mathcal{S}'(\mathbf{R}^n)^G$ can be naturally identified with $(\mathcal{S}(\mathbf{R}^n)^G)'$, the space of all continuous linear functionals on $\mathcal{S}'(\mathbf{R}^n)^G$. More precisely, we have:

Lemma 3.9. *Let G be a compact Lie group acting linearly on $\mathbf{R}^n$. Then the operator associating with each generalized function in $\mathcal{S}'(\mathbf{R}^n)^G$ (as a functional on $\mathcal{S}(\mathbf{R}^n)$) its restriction to the subspace $\mathcal{S}(\mathbf{R}^n)^G$ is an isomorphism from $\mathcal{S}'(\mathbf{R}^n)^G$ onto $(\mathcal{S}(\mathbf{R}^n)^G)'$.*

This result is based on the existence of an averaging over G. It is well known (see, for example, [Z2],§7) that there exists on any compact Lie group G a (unique) invariant normalized integral that associates with each complex continuous function $h(g)$ on G a complex number $\int h(g)dg$ (the integral being over G) such that the conditions of invariance

$$\int h(gg')dg = \int h(g'g)dg = \int h(g)dg \quad \text{for all } g' \in G, \tag{3.152}$$

normalization

$$\int dg = 1 \tag{3.153}$$

and continuity all hold. (Here, by continuity we mean that if $h_k(g) \to 0$ uniformly in g as $k \to \infty$, then $\int h_k dg \to 0$.) This enables us to introduce the continuous linear operator E (called the averaging over G) that operates from $\mathcal{S}(\mathbf{R}^n)$ onto $\mathcal{S}(\mathbf{R}^n)^G$ according to the formula

$$(Eu)(x) = \int u(gx)dg, \;\; u \in \mathcal{S}(\mathbf{R}^n). \tag{3.154}$$

* See the cited references for the proofs of the results in §§3.4.A and 3.4.B

Using the operator E, a G-invariant generalized function $f(x)$ can be recovered from its restriction to $\mathcal{S}(\mathbf{R}^n)^G$ by means of the formula

$$(f, u) = (f, Eu). \tag{3.155}$$

Exercise 3.20. Let G and H be compact groups acting linearly in $\mathbf{R}^n$ and having identical orbits, that is, $Gx = Hx$ for all $x \in \mathbf{R}^n$. Prove that $\mathcal{S}'(\mathbf{R}^n)^G = \mathcal{S}'(\mathbf{R}^n)^H$. [Hint: the G-invariance of the function means that it is constant on the G-orbits. Deduce that $\mathcal{S}(\mathbf{R}^n)^G = \mathcal{S}(\mathbf{R}^n)^H$; then use Lemma 3.9.]

Exercise 3.21. Prove that for $k < d$, the condition that a generalized function in $\mathbf{R}^{dk}$ be $O_+(d)$-invariant is equivalent to $O(d)$-invariance. [Hint: Use the previous exercise, having first proved that there exists for each $x \in \mathbf{R}^{dk}$ with $k < d$, an element $R \in O(d)$ with $\det R = -1$ such that $Rx = x$; for example, R can be taken to be the reflection in a hyperplane in $\mathbf{R}^d$ containing $0, x_1, \dots, x_k$.]

Most significant of the representations used in theoretical physics for invariant (generalized) functions is the representation of the form

$$f(x) = \phi(I_1(x), \dots, I_\nu(x)) \equiv \phi \circ I, \tag{3.156}$$

where $I_1(x), \dots, I_\nu(x)$ is a fixed collection of real G-invariant polynomial functions in $\mathbf{R}^n$ defining the map

$$I : \mathbf{R}^n \ni x \to (I_1(x), \dots, I_\nu(x)) \in \mathbf{R}^\nu, \tag{3.157}$$

and ϕ is a function (or some specially understood "generalized function") on the set

$$\Omega = I(\mathbf{R}^n). \tag{3.158}$$

With regard to the polynomials $I_1, \dots, I_\nu$, it is supposed that they are generators in the rings of all (say, complex) G-invariant polynomials in $\mathbf{R}^n$, that is, each G-invariant polynomial $P(x)$ in $\mathbf{R}^n$ can be represented in the form

$$P = p \circ I \tag{3.159}$$

for some polynomial p in $\mathbf{R}^\nu$. In this case we say that $I_1, \dots, I_\nu$ are *standard (polynomial) invariants.*

In our presentation we shall further suppose that the *condition of algebraic independence* of the standard invariants is satisfied, which means that the standard invariants are not connected by any algebraic relations (or, equivalently, that the equality $p \circ I \equiv 0$ for some polynomial p in $\mathbf{R}^\nu$ implies that $p \equiv 0$). This condition is satisfied if and only if * the set Ω (3.158) (the image of $\mathbf{R}^n$ with respect to I) has a non-empty interior in $\mathbf{R}^\nu$. (Otherwise Ω lies on an algebraic surface in $\mathbf{R}^\nu$.) Whereas in the case under discussion of a compact group G, a set of standard polynomial invariants always exists ([Z2],§98), the condition of algebraic independence by no means holds in all cases (for example, for not too large k in the case of the group $O_{(+)}(d)$). On the other hand, when this condition does hold, the representation (3.156) has a simple direct interpretation.

Let G be the group $O(d)$ acting in $\mathbf{R}^{dk}$. It is not difficult to show ([W3],Ch.II,§§11,17), that the scalar products

$$\mathbf{x}_i \mathbf{x}_j \equiv I_{ij}(x) \tag{3.160}$$

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* Another equivalent condition for the algebraic independence of the standard invariants is the following: the rank of the Jacobian matrix $DI(x)$ of the map $I(x)$ is equal to ν at least at one point $x \in \mathbf{R}^n$.

$(i, j = 1, \dots, k)$ are generators in the space of $O(d)$-invariant polynomials. The obvious relation $I_{ij} = I_{ji}$ holds among these invariants, which allows us to restrict ourselves to the subsystem of $\nu = \frac{1}{2}k(k+1)$ invariants I_{ij} with $i \leq j$. This means that for $k \leq d$, we have algebraically independent standard invariants. For $k > d$, the condition of algebraic independence of the standard invariants breaks down, that is to say, there exist algebraic relations among the invariants I_{ij} (with $i \leq j$) (which reduce to the vanishing of all the minors of dimension $(d+1) \times (d+1)$ of the matrix (I_{ij})).

It is fairly easy to characterize the set $\Omega = I(\mathbf{R}^{dk})$ for $k \leq d$. We identify $\mathbf{R}^{\nu}$ with the set of all real symmetric matrices $y \equiv (y_{ij})$ of dimension $k \times k$; then Ω is the set of all real symmetric non-negative definite matrices (y_{ij}):

$$\Omega = \{y \equiv \{y_{ij}\}_{i,j=1,\dots,k}; y = y^T \text{ and } \det(yij)_{i,j=1,\dots,l} \geq 0 \quad \text{for } l = 1, \dots, k\}. \tag{3.161}$$

(This is a direct consequence of the theorem on the reduction of a quadratic form to canonical form, according to which a real symmetric $k \times k$ matrix y is positive-definite if and only if (for any $d \geq k$) it can be represented in the form $y = x^T x$, where x is a real $d \times k$ matrix and x^T is its transpose.)

For the group $O_+(d)$ acting in $\mathbf{R}^{dk}$ with $k < d$, the above polynomials, as before, are algebraically independent standard invariants (furthermore, according to Exercise 3.21, $O_+(d)$ can in this case be replaced by $O(d)$). However, for $k \geq d$, invariants of a new type appear, namely, $\det(x^i_{\alpha_j})$, where $i, j = 1, \dots, d$, and $\{\alpha_1, \dots, \alpha_d\}$ is an arbitrary set of distinct numbers in $\{1, \dots, k\}$ (in increasing order). In the case when $k = d$, there is the additional invariant

$$\det(x^i_j)_{i,j=1,\dots,d}. \tag{3.162}$$

For $k \geq d$, the standard invariants of $O_+(d)$ are algebraically dependent; for example, the invariant (3.162) and the invariant (3.160) satisfy the relation

$$\det{}^2(x^i_j) = \det(I_{ij}). \tag{3.163}$$

According to Lemma 3.9, the first stage in the description of the G-invariant generalized functions for the case of a compact group G must be to determine the structure of the G-invariant test functions. As is to be expected, the representation of type (3.156) which was required for the G-invariant polynomials, has a natural extension to G-invariant test functions.

Proposition 3.10. *Let G be a compact Lie group acting linearly in $\mathbf{R}^n$, for which there exists a set of algebraically independent standard polynomial invariants $I_1, \dots$ $\dots, I_\nu$. Then the set $\Omega = I(\mathbf{R}^n)$ is a canonically closed regular subset of $\mathbf{R}^\nu$ and the correspondence*

$$\mathcal{S}(\Omega) \ni v \to u = v \circ I \in \mathcal{S}(\mathbf{R}^n)^G \tag{3.164}$$

*realizes a (topological) isomorphism between $\mathcal{S}(\Omega)$ and $\mathcal{S}(\mathbf{R}^n)^G$.**

We now extend the representation (3.156) to the G-invariant generalized functions. We interpret the right hand side of (3.156) in the spirit of the weak integral representation (see Appendix A.2):

$$(\phi \circ I)(x) \equiv \int \delta(y - I(x))\phi(y) d^\nu y \tag{3.165}$$

or, equivalently,

$$(\phi \circ I, u) = \left(\phi(y), \int \delta(y - I(x)) u(x) d^n x\right) \tag{3.166}$$

* The space $\mathcal{S}(\Omega)$ was defined in Appendix A.2.

for all $u \in \mathcal{S}(\mathbf{R}^n)$. This formula shows that we must introduce the set of all functions on Ω of the form

$$\omega(y) = \int \delta(y - I(x))u(x)d^n x, \tag{3.167}$$

define a suitable LCS structure on it and then regard $\phi(y)$ as a continuous linear functional on this space. After this we can apply (3.166) as the correct definition of the composition $\phi \circ I$ of the map I and the functional ϕ.

In the general case we must allow the possibility that the function $\omega(y)$ is not defined by (3.167) on the whole of Ω, but only on some everywhere-dense subset. In fact, the expression $\delta(y - I(x))$ is a well-defined generalized function with respect to x (depending in $\mathcal{C}^\infty$ fashion on y as a parameter), at any rate if the following condition holds: y is a regular value of the map I, that is, the rank of the Jacobian matrix $D\,I(x)$ is equal to ν for all x on the variety $I(x) = y$. (This regularity condition enables us to treat the invariants $I_1(x), \ldots, I_\nu(x)$ as the first ν local coordinates in a neighbourhood of such a point x.) We introduce the set

$$\mathcal{R} = \{x \in \mathbf{R}^n : \operatorname{rank} D\,I(x) = \nu\}. \tag{3.168}$$

Clearly, it is open, G-invariant and (by virtue of the fact that the G-invariant polynomials and hence the map I separate the G-orbits, [Z2],§98) I-saturated, that is, $\mathcal{R} = I^{-1}(I(\mathcal{R}))$. Therefore $I(\mathcal{R})$ is open (by the implicit function theorem) and consists of regular values of the map I. Furthermore, $\mathcal{R}$ is clearly dense in $\mathbf{R}^n$, which means that $I(\mathcal{R})$ is dense in Ω. It follows that (3.167) defines for each $u \in \mathcal{S}(\mathbf{R}^n)$ a $\mathcal{C}^\infty$-function $\omega(y)$ of the variable y which runs through the subset $I(\mathcal{R})$, which is open in R^ν and dense in Ω. We shall understand this formula in this sense.

The subsequent characterization of the set of all functions $\omega(y)$ of the form (3.167) is essentially simplified by the fact that in (3.167) it suffices to restrict attention to just the G-invariant functions $u(x)$ (since according to (3.155), the right hand side of (3.167) is not altered if we replace an arbitrary $u \in \mathcal{S}(\mathbf{R}^n)$ by $Eu \in \mathcal{S}(\mathbf{R}^n)^G$). We now make use of Proposition 3.10, according to which every G-invariant test function has the form $u = v \circ I$, where $v \in \mathcal{S}(\Omega)$. Hence we obtain the general form of the functions ω of interest to us:

$$\omega(y) = \chi(y)v(y), \tag{3.169}$$

where $v(y)$ is an arbitrary function in $\mathcal{S}(\Omega)$ and $\chi(y)$ is a fixed $\mathcal{C}^\infty$-function on $I(\mathcal{R})$ equal to

$$\chi(y) = \int \delta(y - I(x))d^n x. \tag{3.170}$$

We have reached the conclusion that the set of test functions ω of interest to us are given by (3.169), that is multiplication of $\mathcal{S}(\Omega)$ by a fixed function χ. We denote the space of all such test functions ω by $\mathcal{S}_\chi(\Omega)$. The natural LCS structure is defined on $\mathcal{S}_\chi(\Omega)$ in such a way that the operator of multiplication by χ ($v \to \chi \cdot v$) from $\mathcal{S}(\Omega)$ onto $\mathcal{S}_\chi(\Omega)$ is a (topological) isomorphism. In terms of the defining system of seminorms, the LCS structure on $\mathcal{S}_\chi(\Omega)$ is given by the seminorms $p_{l,m}$

$$p_{l,m}(\chi v) = \|v\|^{\Omega}_{l,m} \quad \text{for all } v \in \mathcal{S}(\Omega), \tag{3.171}$$

where $l, m \in \overline{\mathbf{Z}}_+$. As usual, $(\mathcal{S}_\chi(\Omega))'$ denotes the space of all continuous linear functionals on $\mathcal{S}_\chi(\Omega)$. Above, we have regarded $\mathcal{S}_\chi(\Omega)$ as the space obtained from $\mathcal{S}(\Omega)$ by multiplication by χ; in like manner, we can regard $(\mathcal{S}_\chi(\Omega))'$ as the space obtained from $\mathcal{S}'(\Omega)$ by multiplication by $1/\chi$ and denote it by $\mathcal{S}'_{1/\chi}(\Omega)$ by means of the formula

$$\left(\frac{1}{\chi}\psi,\, \chi v\right) = (\psi, v) \quad \text{for all } \psi \in \mathcal{S}'(\Omega),\ v \in \mathcal{S}(\Omega) \tag{3.172}$$

or (more graphically) by means of the symbolic integrals

$$\int \left(\frac{1}{\chi}\psi\right)(\chi v)dy = \int \psi v dy. \tag{3.173}$$

It is easy to see that the operator of multiplication by $1/\chi$ from $\mathcal{S}'(\Omega)$ onto $\mathcal{S}'_{1/\chi}(\Omega)$ is a (topological) isomorphism (since it is the adjoint of the operator of multiplication by $1/\chi$ from $\mathcal{S}_\chi(\Omega)$ onto $\mathcal{S}(\Omega)$, which by construction is an isomorphism).

As a result we arrive at the following definition of the composition $\phi \circ I$ in (3.156), namely: for arbitrary ϕ in $\mathcal{S}'_{1/\chi}(\Omega)$, formula (3.166) defines $\phi \circ I \equiv \phi(I(x))$ as a G-invariant generalized function in $\mathcal{S}'(\mathbf{R}^n)$. This enables us to give a description of all the G-invariant generalized functions in $\mathcal{S}'(\mathbf{R}^n)$.

Proposition 3.11. *Let G be a compact Lie group acting linearly in $\mathbf{R}^n$ and suppose that there exist algebraically independent standard polynomial invariants $I_1, \dots, I_\nu$. Then the correspondence $\phi \to \phi \circ I$ is a (topological) isomorphism from $\mathcal{S}'_{1/\chi}(\Omega)$ onto $\mathcal{S}'(\mathbf{R}^n)^G$.*

Examples. 1) Consider the action of $O(d)$ on $\mathbf{R}^d$. The role of the standard invariant is played by $I(x) = |x|^2$, while the set Ω is the same as the non-negative real semi-axis $\overline{R}_+ = [0, \infty)$. In the given instance, $\chi(y)$ is proportional to $y^{d/2-1}$. Hence each $O(d)$-invariant generalized function f in $\mathcal{S}'(\mathbf{R}^d)$ is uniquely representable in the form

$$f(x) = \phi(|x|^2), \tag{3.174}$$

where $\phi \equiv \phi(y) \in \mathcal{S}'_{y^{-d/2+1}}(\overline{\mathbf{R}}_+)$.

2) Let $O(3)$ act in $\mathbf{R}^{3\cdot 2}$. In this case we have three standard invariants

$$I_1(x)\ (\equiv I_{11}(x)) = |\mathbf{x}_1|^2,\ I_2(x)\ (\equiv I_{12}(x)) = \mathbf{x}_1\mathbf{x}_2,\ I_3(x)\ (\equiv I_{22}(x)) = |\mathbf{x}_2|^2. \tag{3.175}$$

The set Ω is the cone in $\mathbf{R}^3$:

$$y_1 y_3 - y_2^2 \geq 0,\ y_1 \geq 0, \tag{3.176}$$

and the function $\chi(y)$ is a positive constant. Consequently we have the following representation for the $O(3)$-invariant generalized functions in $\mathcal{S}'(\mathbf{R}^{3\cdot 2})$:

$$f(\mathbf{x}_1, \mathbf{x}_2) = \phi(\mathbf{x}_1^2, \mathbf{x}_1\mathbf{x}_2, \mathbf{x}_2^2), \tag{3.177}$$

where $\phi \in \mathcal{S}'(\Omega)$.

3) Consider the $O(3)$-invariant generalized functions in $\mathcal{S}'(\mathbf{R}^{3\cdot 3})$. The standard invariants

$$I_{ij}(x) = \mathbf{x}_i \mathbf{x}_j \tag{3.178}$$

run through the set Ω:

$$y_{11} \geq 0,\ y_{11}y_{22} - y_{12}^2 \geq 0,\ \det(y_{ij}) \geq 0. \tag{3.179}$$

The function $\chi(y)$ is equal to

$$\chi(y) = \pi^2(\det(y_{ij}))^{-1/2}. \tag{3.180}$$

Thus we have the following representation for the $O(3)$-invariant generalized functions in $\mathbf{R}^{3\cdot 3}$

$$f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \phi(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2, \mathbf{x}_1\mathbf{x}_2, \mathbf{x}_2\mathbf{x}_3, \mathbf{x}_1\mathbf{x}_3), \tag{3.181}$$

where $\phi \in \mathcal{S}'_{1/\chi}(\Omega)$.

4) The $O_+(3)$-invariant generalized functions in $\mathbf{R}^{3\cdot 3}$ go beyond the framework of the representation (3.11), since the extra invariant $\det(x_i^j)$ is connected to (3.178) by an algebraic relation.

Clearly it suffices to consider the even and odd generalized functions separately; these are characterized respectively by the condition

$$f(-\mathbf{x}_1, -\mathbf{x}_2, -\mathbf{x}_3) = \pm f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3). \tag{3.182}$$

In the first case, we in fact obtain $O(3)$-invariant generalized functions for which (3.181) holds. It can be shown (see Exercise 3.27 below) that the odd $O_+(3)$-invariant generalized functions in $\mathbf{R}^{3\cdot 3}$ have the form

$$f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \det(x_i^j) \cdot f_0(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \tag{3.183}$$

where f_0 is an $O(3)$-invariant generalized function in $\mathbf{R}^{3\cdot 3}$ (admitting a representation of type (3.181)).

B. GENERALIZED FUNCTIONS THAT ARE COVARIANT WITH RESPECT TO A COMPACT GROUP

As before, we let G be a compact Lie group acting linearly in $\mathbf{R}^n$. We suppose further that $T(g)$ is a given representation of G in a finite-dimensional complex vector space $\mathcal{X}$. Then the contragradient representation acts in the dual $\mathcal{X}'$

$$\tilde{T}(g) = T(g^{-1})'. \tag{3.184}$$

We denote by $\mathcal{S}(\mathbf{R}^n; \mathcal{X})$ and $\mathcal{S}'(\mathbf{R}^n; \mathcal{X}')$ the spaces of $\mathcal{X}$-valued test functions and $\mathcal{X}'$-valued generalized functions, respectively, in $\mathbf{R}^n$. If $b_1, \ldots, b_L$ is a basis in $\mathcal{X}$, then an arbitrary element of $\mathcal{S}(\mathbf{R}^n; \mathcal{X})$ has the form $u(x) = \sum_{\alpha=1}^{L} b_\alpha u^\alpha(x)$, where the $u^\alpha(x)$ are arbitrary functions in $\mathcal{S}(\mathbf{R}^n)$. Similarly, we can take the dual basis $b'^1, \ldots, b'^L$ in $\mathcal{X}'$ and write an arbitrary element in $\mathcal{S}'(\mathbf{R}^n; \mathcal{X}')$ in the form $f(x) = \sum_{\alpha=1}^{L} b'^\alpha f_\alpha(x)$, where the $f_\alpha(x)$ are arbitrary generalized functions in $\mathcal{S}'(\mathbf{R}^n)$. We can regard $\mathcal{S}'(\mathbf{R}^n; \mathcal{X}')$ as the natural dual of $\mathcal{S}(\mathbf{R}^n; \mathcal{X})$ by setting

$$(f, u) = \sum_{\alpha=1}^{L} \int f_\alpha(x) u^\alpha(x) dx. \tag{3.185}$$

The *covariance condition*

$$T(g^{-1}) u(gx) = u(x) \quad \text{for all } g \in G \tag{3.186a}$$

distinguishes the subspace $\mathcal{S}(\mathbf{R}^n; \mathcal{X})^G$ of G-covariant test functions. In component notation, this condition has the form

$$\sum_\beta T^\alpha_\beta(g^{-1}) u^\beta(gx) = u^\alpha(x), \ g \in G. \tag{3.186b}$$

The subspace $\mathcal{S}'(\mathbf{R}^n; \mathcal{X}')^G \subset \mathcal{S}'(\mathbf{R}^n, \mathcal{X}')$ of G-covariant generalized functions is similarly defined.

There is a formula (analogous to (3.155)) that enables one to recover a G-covariant generalized function from its restriction to $\mathcal{S}(\mathbf{R}^n; \mathcal{X})^G$:

$$(f, u) = (f, Eu), \ f \in \mathcal{S}'(\mathbf{R}^n; \mathcal{X}')^G, \ u \in \mathcal{S}(\mathbf{R}^n; \mathcal{X}); \tag{3.187}$$

here

$$(Eu)(x) = \int T(g^{-1}) u(gx) dg, \ u \in \mathcal{S}(\mathbf{R}^n; \mathcal{X}). \tag{3.188}$$

Therefore (as in Lemma 3.9) we can identify $\mathcal{S}'(\mathbf{R}^n;\mathcal{X}')^G$ with the space of continuous linear functionals on $\mathcal{S}(\mathbf{R}^n;\mathcal{X})^G$, that is, $(\mathcal{S}(\mathbf{R}^n;\mathcal{X})^G)'$.

For the covariant test and generalized functions, extensive use is made of the so-called covariant decompositions

$$u(x) = \sum_{\sigma=1}^{N} Q_\sigma(x)v_\sigma(x), \tag{3.189}$$

$$f(x) = \sum_{\rho=1}^{N} Q'_\rho(x)h_\rho(x), \tag{3.190}$$

since they take into account the covariance condition in explicit fashion. Here $\{Q_\sigma\}$ and $\{Q'_\rho\}$ are fixed sets of covariant functions (covariants), and v_σ (and h_ρ) are invariant functions (and generalized functions), sometimes called invariant amplitudes. Of particular importance are the polynomial covariants.

Let $\mathcal{P}(\mathbf{R}^n;\mathcal{X})^G$ be the set of all $\mathcal{X}$-valued G-covariant polynomial functions in $\mathbf{R}^n$. The family $\{Q_\sigma\}_{\sigma=0}^N$ of elements in $\mathcal{P}(\mathbf{R}^n;\mathcal{X})^G$ is called a family of *standard polynomial covariants* in $\mathbf{R}^n$ if an arbitrary element $P \in \mathcal{P}(\mathbf{R}^n;\mathcal{X})^G$ is representable in the form

$$P(x) = \sum_{\sigma=1}^{N} Q_\sigma(x)p_\sigma(x) \tag{3.191}$$

for some G-invariant polynomials $p_\sigma(x)$. If the expansion (3.191) is unique, that is, if $P \equiv 0$ implies that $p_\sigma \equiv 0$ for all σ, then the family $\{Q_\sigma\}$ is called a *polynomial basis of standard covariants* in $\mathbf{R}^n$ which transform according to the representation T of G. (In particular, the above condition excludes the case, of little practical use, when the number N of invariant amplitudes exceeds the number $L = \dim \mathcal{X}$ of components of the $\mathcal{X}$-valued functions.) It can be shown that the condition of uniqueness of the expansion in the class $\mathcal{P}(\mathbf{R}^n;\mathcal{X})^G$ into standard polynomial covariants Q_σ holds if and only if the values of the standard covariants $Q_1(x),\dots,Q_N(x)$ at a fixed point x, regarded as vectors in $\mathcal{X}$, are linearly independent for at least one value of x (or for all x in some dense open subset of $\mathbf{R}^n$, which is the same in view of the polynomial dependence of the covariants on x). This condition does not always hold (for example, for smallish k in the case of the group $O_+(d)$ acting in $\mathbf{R}^{dk}$).

We note that the condition of the existence of a polynomial basis of standard covariants is symmetric with respect to $\mathcal{X}$ and $\mathcal{X}'$. If $\{Q_\sigma\}$ is such a basis in $\mathcal{P}(\mathbf{R}^n;\mathcal{X})$, then there exists a polynomial basis of standard covariants $\{Q'_\rho\}_{\rho=1}^N$ for $\mathcal{P}(\mathbf{R}^n;\mathcal{X}')$; for example, we can set

$$Q'_\sigma(x) = \theta Q_\sigma(x), \tag{3.192}$$

where θ is a G-invariant antilinear isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$.

Exercise 3.22. Construct a G-invariant antilinear isomorphism from $\mathcal{X}$ onto $\mathcal{X}'$. [Hint: There exists on $\mathcal{X}$ a G-invariant scalar product $\langle X_1, X_2\rangle$; define θX_1 by setting $(\theta X_1, X_2) = \langle X_1, X_2\rangle$.]

We define the $N \times N$-matrix $q(x) \equiv (q_{\rho\sigma}(x))$, the elements of which are the G-invariant polynomials:

$$q_{\rho\sigma}(x) = \sum_{\alpha=1}^{L} Q'^{\alpha}_{\rho}(x)Q_{\sigma,\alpha}(x). \tag{3.193}$$

The matrix-valued function $q(x)$ plays an important role for covariant expansions of generalized functions; this is similar to the role played by the function χ (3.170) for invariant representations.

Exercise 3.23. Prove that $\det q(x) \not\equiv 0$. [Hint: It suffices to consider the case when the covariants $Q'_\sigma(x)$ have the form (3.192); in this case, $q_{\rho\sigma}(x) = \langle Q_\rho(x), Q_\sigma(x)\rangle$ is a positive-definite matrix at all points x where the covariants $Q_\sigma(x)$ are linearly independent.]

We now go back to the examples of the groups $SU(2)$ and $O(3)$. As has already been noted (see Exercise 3.5(a)), the homomorphism $\underset{\sim}{A} \to \Lambda(\underset{\sim}{A})$ (3.39) restricted to the subgroup $SU(2) \subset SL(2,C)$ is a double covering of $O_+(3)$, so that

$$\Lambda(U) \in O_+(3) \quad \text{for } U \in SU(2). \tag{3.194}$$

It is well known from the theory of angular momentum that the irreducible representations of $SU(2)$ are parametrized by integral or half-integral non-negative numbers s, which is the ***spin*** of the representation. The representation $\mathfrak{D}^{(s)}$ with spin s can be realized in the space $\mathfrak{p}^{(s,0)}$ of homogeneous polynomials in $\omega \in \mathbf{C}^2$ of degree $2s$. In fact, $\mathfrak{D}^{(s)}$ is the restriction of $\mathfrak{D}^{(s,0)}$ (see (3.53)) of $SL(2,C)$ to the subgroup $SU(2)$.

Here we are only interested in the representation with integral spin. Such representations induce (unique) representations of the group $O_+(3)$. It is convenient to go over from the variable $\omega \in \mathbf{C}^2$ to the new variable $\zeta \equiv \zeta(\omega) \in \mathbf{C}^3$:

$$\zeta^1 = (\omega^2)^2 - (\omega^1)^2, \zeta^2 = -i((\omega^1)^2 - (\omega^2)^2);\ \zeta^3 = 2\omega^1\omega^2. \tag{3.195}$$

Here we have the equality

$$-\zeta\mathbf{x} = \omega\epsilon\underset{\sim}{x}\tilde{e}_0\omega \tag{3.196}$$

for all $x \in M$ with $x^0 = 0$. It is not difficult to see that for $\omega \in \mathbf{C}^2$, $\zeta(\omega)$ varies over the following quadric in $\mathbf{C}^3$:

$$\zeta\zeta \equiv (\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2 = 0. \tag{3.197}$$

The quantity ζ is convenient because it transforms under rotations like a 3-vector:

$$\zeta(U\omega) = \Lambda(U)\zeta(\omega) \quad \text{for all } U \in SU(2). \tag{3.198}$$

It is clear that when s is an integer, every homogeneous polynomial $\psi(\omega)$ of degree $2s$ in ω can naturally be expressed as a homogeneous polynomial $\hat{\psi}(\zeta)$ of degree s in the variable $\zeta \equiv \zeta(\omega)$ on the quadric (3.197):

$$\psi(\omega) \equiv \hat{\psi}(\zeta(\omega)). \tag{3.199}$$

Let $\hat{\mathfrak{p}}^{(s,0)}$ be the space of all such functions $\hat{\psi}(q)$. Then the representation $\mathfrak{D}^{(s)}$ of the group $O_+(3)$ in $\hat{\mathfrak{p}}^{(s,0)}$ has the form

$$(\mathfrak{D}^{(s,0)}(R)\hat{\psi})(\zeta) = \hat{\psi}(R^{-1}\zeta) \quad \text{for } R \in O_+(3). \tag{3.200}$$

With regard to the representations of $O(3)$, these are easily constructed by noting that $O(3)$ is a direct product of the subgroups $O_+(3)$ and $\mathbf{Z}_2 = \{1,-1\}$. We see from this that every irreducible representation of $O(3)$ is characterized to within equivalence by the (negative integral) spin s and sign $\eta = \pm 1$ (the spatial parity of the representation); we have for the corresponding representation $\mathfrak{D}^{(s)}_\eta$:

$$\mathfrak{D}^{(s)}_\eta(R) = \mathfrak{D}^{(s,0)}(R), \quad \mathfrak{D}^{(s)}_\eta(-R) = \eta \cdot i^{2s}\mathfrak{D}^{(s,0)}(R) \quad \text{for } R \in O_+(3). \tag{3.201}$$

In terms of the space $\hat{\mathfrak{p}}^{(s,0)}$, the representation (3.201) takes the form

$$(\mathfrak{D}^{(s)}_\eta(R)\hat{\psi})(\zeta) = (\det R)^{(1-\eta)/2}\hat{\psi}(R^{-1}\zeta) \quad \text{for all } R \in O(3). \tag{3.202}$$

Exercise 3.24. Let $\mathfrak{h}^{(s)}$ be the space of all complex harmonic polynomial functions $\phi(l)$ of the (real or complex) 3-vector l, that are homogeneous of degree s $(= 0, 1, 2, \ldots)$. Prove that the action of $O(3)$ that associates with an element $R \in O(3)$ the map

$$\phi(l) \to \phi'(l) = (\det R)^{(1-\eta)/2}\phi(R^{-1}l) \quad \text{for all } R \in O(3), \tag{3.203}$$

is equivalent to the representation $\mathfrak{D}_\eta^{(s)}$ of $O(3)$ in $\hat{\mathfrak{p}}^{(s,0)}$. (Hint: verify that the maps

$$\phi \to \hat{\psi}, \quad \text{where } \hat{\psi}(\zeta) = \phi(\zeta), \tag{3.204a}$$

$$\hat{\psi} \to \phi, \quad \text{where } \phi(l) = \frac{1}{4^s (s!)^2} \left(-\frac{\partial}{\partial \omega} \underset{\sim}{\epsilon}_0 \tilde{l} \epsilon^{-1} \frac{\partial}{\partial \omega} \right)^s \hat{\psi}(\zeta(\omega)), \tag{3.204b}$$

are inverses of each other; in formula (3.204b) l^0 is taken to be 0.)

We consider the structure of the standard polynomial covariants in the case in question of $O_+(3)$ (or $O(3)$) acting linearly in $\mathbf{R}^{3k}$. According to the above, we can suppose that the (irreducible) representation $\mathfrak{D}^{(s)}$ of $O_+(3)$ (or the representation $\mathfrak{D}_\eta^{(s)}$ of $O(3)$) is realized in the space $\hat{\mathfrak{p}}^{(s,0)}$.Then every function $f(\mathbf{x}_1, \cdots \mathbf{x}_k; \zeta)$ of the variables $\mathbf{x}_1, \cdots \mathbf{x}_k \in \mathbf{R}^3$ and dependent on the parameter ζ on the quadric (3.197) is, as a homogeneous polynomial function of degree s for which

$$f(R\mathbf{x}_1, \ldots, R\mathbf{x}_k; R\zeta) = f(\mathbf{x}_1, \ldots, \mathbf{x}_k; \zeta) \quad \text{for all } R \in O_+(3), \tag{3.205}$$

an $O_+(3)$-covariant function in $\mathbf{R}^{3k}$ that transforms according to the representation $\mathfrak{D}^{(s)}$.We see that the covariance condition becomes, in essence, an invariance condition if we regard ζ as just an additional variable.* There is an analogous characterization for the $O(3)$-covariant distributions in $\mathbf{R}^{3k}$, which transform according to the representation $\mathfrak{D}_\eta^{(s)}$.

The standard $O_+(3)$-covariants are constructed as suitable $O_+(3)$-invariant polynomial functions of $\mathbf{x}_1, \ldots, \mathbf{x}_k; \zeta$. A polynomial basis of standard covariants can only be constructed for all s for the case $k \le 2$. For $k = 1$, there is one standard covariant

$$(\mathbf{x}\zeta)^s. \tag{3.206}$$

For $k = 2$, we can choose as a polynomial basis of standard covariants, the sufficient family of polynomials of degree s of the three combinations: $\mathbf{x}_1\zeta$, $\mathbf{x}_2\zeta$ and $\det(\mathbf{x}_1, \mathbf{x}_2, \zeta)$ (where the last expression is the determinant consisting of the components of the vectors $\mathbf{x}_1, \mathbf{x}_2, \zeta$). Since $\zeta^2 = 0$, $\det^2(\mathbf{x}_1, \mathbf{x}_2, \zeta)$ can be expressed in terms of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_1\zeta$ and $\mathbf{x}_2\zeta$; hence it suffices to restrict attention to the zeroth and first powers of the combination $\det(\mathbf{x}_1, \mathbf{x}_2, \zeta)$. Thus we obtain the polynomial basis of $2s + 1$ standard covariants

$$\det(\mathbf{x}_1, \mathbf{x}_2, \zeta)^\sigma (\mathbf{x}_1\zeta)^a (\mathbf{x}_2\zeta)^b, \tag{3.207}$$

where

$$\sigma = 0 \text{ or } 1; \quad a, b \in \overline{\mathbf{Z}}_+, \quad \sigma + a + b = s. \tag{3.208}$$

Exercise 3.25. (a) Let $k = 1$. Prove that (3.206) defines a polynomial basis of standard covariants of $O_+(3)$ which transform according to the representation $\mathfrak{D}^{(s)}$. [Hint: Here it is convenient to use the isomorphism of the spaces $\hat{\mathfrak{p}}^{(s,0)}$ and $\mathfrak{h}^{(s)}$, described in Exercise 3.24. In this case, the covariant polynomials in $\mathbf{R}^3$ can be identified with the invariant polynomials $\phi(\mathbf{x}, l)$ of the vectors $\mathbf{x}, l \in \mathbf{R}^3$ that are harmonic and homogeneous of degree s in l. According to §3.4.A such polynomials can be written in the form $P(\mathbf{x}^2, \mathbf{x}l, l^2)$, where P is a polynomial of three variables. Hence by reverting to the variable ζ as in (3.204a), show that $\hat{\psi}(\mathbf{x}, \zeta) = p(\mathbf{x}^2)(\mathbf{x}\zeta)^s$, where p is a polynomial of one variable.]

(b) Let $k = 2$. Prove that the expressions (3.207) define a polynomial basis of standard covariants of the two vectors $\mathbf{x}_1, \mathbf{x}_2$, the representation of $O_+(3)$ being $\mathfrak{D}^{(s)}$. [Hint: With regard to proving that the expressions (3.207) can be taken as the standard polynomial covariants, see part (a) of this exercise. In fact, a polynomial basis of standard covariants is obtained: this follows from the fact that the covariants (3.207) regarded as functions of ζ on the quadric (3.197) are linearly independent for any fixed linearly independent $\mathbf{x}_1$ and $\mathbf{x}_2$.]

Exercise 3.26. Verify that under the realization of the representation $\mathfrak{D}^{(s)}$ (with integral s) of $O_+(3)$ in the space $\mathfrak{p}^{(s,0)}$, one can take the following as the standard covariants for $k = 1$ or 2: for $k = 1$, the single standard covariant

$$(\omega \epsilon \underset{\sim}{x} \omega)^s, \tag{3.209}$$

* This is the advantage of the explicitly covariant formalism which we adhere to in these examples (also in §3.3.B)

while for $k = 2$, the $2s+1$ covariants

$$(\omega\epsilon\underset{\sim}{x}_1\underset{\sim}{x}_2\omega)^\sigma (\omega\epsilon\underset{\sim}{x}_1\omega)^a (\omega\epsilon\underset{\sim}{x}_2\omega)^b, \tag{3.210}$$

where σ, a, b run through the values indicated in (3.208).

Similarly, the $O(3)$-covariance condition for (generalized) functions in $\mathbf{R}^{3k}$ that transform according to the representation $\mathfrak{D}_\eta^{(s)}$ (with integral spin s and parity $\eta = \pm 1$) is expressed by (3.205) together with the single relation:

$$f(-\mathbf{x}_1, \dots, -\mathbf{x}_k; -\zeta) = \eta f(\mathbf{x}_1, \dots, \mathbf{x}_k; \zeta). \tag{3.211}$$

If $k = 1$, then for $\eta = +1$ there is the single standard covariant (3.206), while for $\eta = -1$, there are, in general, no non-zero covariant functions. If $k = 2$, then for $\eta = +1$, we can take as a polynomial basis the $s+1$ standard covariants

$$(\mathbf{x}_1\zeta)^a (\mathbf{x}_2\zeta)^{s-a}, \quad \text{where } a = 0, 1, \dots, s, \tag{3.212}$$

while for $\eta = -1$ we can choose the following s standard covariants (for $s \geq 1$ of course):

$$\det(\mathbf{x}_1, \mathbf{x}_2, \zeta)\, (\mathbf{x}_1\zeta)^a (\mathbf{x}_2\zeta)^{s-a-1}, \quad \text{where } a = 0, 1, \dots, s-1. \tag{3.213}$$

The expansion (3.191) can be extended to covariant generalized functions.

Proposition 3.12. *Let G be a compact Lie group acting linearly in $\mathbf{R}^n$ and suppose that there exists a polynomial basis of standard covariants in $\mathbf{R}^n$ that transform according to the representation T of G in $\mathcal{X}$. Then each covariant test function $u(x) \in \mathcal{S}(\mathbf{R}^n; \mathcal{X})^G$ can be uniquely represented in the form (3.189), where the $v_\sigma(x)$ are G-invariant test functions; formula (3.189) defines a topological isomorphism from the direct sum $\bigoplus^N \mathcal{S}(\mathbf{R}^n)^G$ (k copies of $\mathcal{S}(\mathbf{R}^n)^G$) onto $\mathcal{S}(\mathbf{R}^n; \mathcal{X})^G$.*

Going over to an expansion of type (3.190) for covariant generalized functions, we first establish the meaning of invariant amplitudes. The result of smoothing (3.189) with an arbitrary test function $u \in \mathcal{S}(\mathbf{R}^n; \mathcal{X})$ has the form

$$(f, u) = \sum_{\rho=1}^{N} \Big(h_\rho(x), \sum_{\alpha=1}^{L} Q_\rho'^{\alpha}(x) u_\alpha(x)\Big). \tag{3.214}$$

According to (3.187), the left hand side of this equality is not altered on replacing u by Eu, therefore it suffices to restrict attention to G-covariant test functions. Using (3.189), we see that (3.214) takes on the form

$$(f, u) = \sum_{\rho=1}^{N} \Big(h_\rho(x), \sum_{\sigma=1}^{N} q_{\rho\sigma}(x) v_\sigma(x)\Big), \tag{3.215}$$

where $v \equiv (v_\sigma(x)) \in \bigoplus^N \mathcal{S}(\mathbf{R}^n)^G$. Here we shall regard the matrix q as an operator in $\bigoplus^N \mathcal{S}(\mathbf{R}^n)$, which associates with the element $v \equiv \{v_\sigma(x)\}_{\sigma=1}^N \in \bigoplus^N \mathcal{S}(\mathbf{R}^n)$ the element qv with components

$$(qv)_\rho(x) = \sum_{\sigma=1}^{N} q_{\rho\sigma}(x) v_\sigma(x). \tag{3.216}$$

It can be shown that the operator q is, in fact, a topological isomorphism from $\bigoplus^N \mathcal{S}(\mathbf{R}^n)$ to a closed linear subspace of $\bigoplus^N \mathcal{S}(\mathbf{R}^n)$, which we denote by $q \bigoplus^N \mathcal{S}(\mathbf{R}^n)$.

Formula (3.215) suggests the following interpretation of the invariant amplitudes $h_\rho(x)$ in the covariant expansion (3.190): the family $h \equiv \{h_\rho(x)\}$ is a continuous linear functional on $q\bigoplus\limits^{N} \mathcal{S}(\mathbf{R}^n)^G$ which (as in Lemma 3.9) can be identified with a G-invariant linear functional on $q\bigoplus\limits^{N} \mathcal{S}(\mathbf{R}^n)$.

In the same way that we introduced the operator $1/\chi$ in the preceding subsection (see (3.172)) we introduce the analogous isomorphism q'^{-1} from $\bigoplus \mathcal{S}'(\mathbf{R}^n)$ in the space $(q\bigoplus \mathcal{S}(\mathbf{R}^n))'$ of continuous linear functionals onto $q\bigoplus \mathcal{S}(\mathbf{R}^n)$; by definition,

$$(q'^{-1}H, qv) = (H, v) \tag{3.217}$$

for all $H \equiv \{H_\rho(x)\} \in \bigoplus \mathcal{S}'(\mathbf{R}^n)$, $v \in \{v_0(x)\} \in \bigoplus \mathcal{S}(\mathbf{R}^n)$.

Using this construction, we arrive at the following characterization of G-covariant generalized functions.

Proposition 3.13. *Suppose that the conditions of Proposition* 3.12 *hold and that* $\{Q'_\rho\}$ *is a standard basis of polynomial covariants in* $\mathbf{R}^n$ *which transform according to the contragradient representation. Then each* G*-covariant generalized function* $f(x) \in \mathcal{S}'(\mathbf{R}^n; \mathcal{X}')^G$ *can be represented in the form* (3.190), *where* $h \equiv (h_\rho(x))$ *is an element of the space* $q'^{-1}\bigoplus \mathcal{S}'(\mathbf{R}^n)^G$*; according to this formula, the value of* f *at the test function* $u \in \mathcal{S}(\mathbf{R}^n; \mathcal{X})$ *can be written in the form*

$$(f, u) = (h, qEu) \tag{3.218}$$

(*where* E *and* q *are the operators* (3.188) *and* (3.216)). *Therefore the formula* (3.190) *defines an isomorphism between* $\mathcal{S}'(\mathbf{R}^n; \mathcal{X}')^G$ *and* $q'^{-1}\bigoplus \mathcal{S}'(\mathbf{R}^n)^G$.

Combining Propositions (3.13) and (3.11) we can now express the invariant amplitudes h_ρ in the covariant expansion (3.190) in terms of the invariants $I_1, \dots, I_\nu$.

Proposition 3.14. *Let* G *be a compact Lie group acting in* $\mathbf{R}^n$, *and suppose that there exist algebraically independent standard polynomial invariants* $I_1, \dots, I_\nu$ *and a polynomial basis of standard covariants in* $\mathbf{R}^n$ *that transform according to the representation* T *of* G *in* $\mathcal{X}$ (*and according to the representation* $\widetilde{T}$ *in* $\mathcal{X}'$). *Then each* G*-covariant generalized function* $f(x) \in \mathcal{S}'(\mathbf{R}^n; \mathcal{X}')^G$ *can be represented in the form*

$$f(x) = \sum_\rho Q'_\rho(x) H_\rho(I_1(x), \dots, I_\nu(x)), \tag{3.219}$$

where $H \equiv \{H_\rho(y)\}$ *is an element of* $X'^{-1}\bigoplus\limits^{N} \mathcal{S}'_{1/\chi}(\Omega)$.

Here we are supposing that the matrix of invariant polynomials $q(x) \equiv (q_{\rho\sigma}(x))$ is expressed in terms of $I_1, \dots, I_\nu$:

$$q_{\rho\sigma}(x) = X_{\rho\sigma}(I(x)) \tag{3.220}$$

(the $X_{\rho\sigma}(y)$ being polynomials in $\mathbf{R}^\nu$), and that the corresponding operators X and X'^{-1} are defined by formulae analogous to (3.216), (3.217) in the spaces of test and generalized functions of the variable $y \in \mathbf{R}^\nu$. The expansion (3.219), written in the form

$$f(x) = \sum_\rho \int Q'_\rho(x)\delta(y - I(x)) H_\rho(y) d^\nu y, \tag{3.221}$$

belongs to a type of representation that is a natural generalization of the notion of weak integral representation (§A.2) to the case of vector-valued generalized functions.

Remark. We have seen that uniqueness considerations for the covariant expansion (3.190) led to the fact that the invariant amplitudes $h_\rho(x)$ are not the usual generalized functions. However, at the expense of losing the uniqueness of the expansion, we can allow G-invariant generalized functions in $\mathcal{S}'(\mathbf{R}^n)$ to serve as invariant amplitudes $h_\rho(x)$. For it was noted above that h is in fact a continuous linear functional on the subspace $q\bigoplus^N \mathcal{S}(\mathbf{R}^n)^G$ of $\bigoplus^N \mathcal{S}(\mathbf{R}^n)^G$. By the Hahn-Banach theorem, we can extend this continuously to a linear functional on $\bigoplus^N \mathcal{S}(\mathbf{R}^n)^G$ and such a functional can (by Lemma 3.9) be identified with an element of $\bigoplus^N \mathcal{S}'(\mathbf{R}^n)^G$, that is, with a family of G-invariant generalized functions. As a result , we arrive at the representation (3.190), where this time the $h_\rho(x)$ are G-invariant generalized functions. As we have said, such a decomposition is not, in general, unique. Even so, it uniquely defines the generalized functions h_ρ on an everywhere-dense subset of $\mathbf{R}^n$, where the standard covariants $Q_1(x), \dots, Q_N(x)$, regarded as vectors in $\mathcal{X}$, are linearly independent; this means that the non-uniqueness of the covariant expansion (3.190) in the class of generalized functions is confined to the complement of this set. For example, the $O_+(3)$-covariant vector-valued generalized function $\mathbf{f}(\mathbf{x})$ of the single-vector $\mathbf{x}$ can be represented in the form $\mathbf{f}(\mathbf{x}) = \mathbf{x}h(\mathbf{x})$, where $h(\mathbf{x})$ is an $O(3)$-invariant generalized function of $\mathbf{x}$ which is defined by this representation to within a term $c\delta(\mathbf{x})$ (where c is an arbitrary number).

Exercise 3.27. Prove the representation (3.183) for the odd $O_+(3)$-invariant generalized functions $f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ in $\mathbf{R}^{3\cdot 3}$. [Hint: Consider the one-dimensional representation $\mathfrak{D}^{(0)}_{-1}$ of $O(3)$: $\mathfrak{D}^{(0)}_{-1}(R) =$ $= \det R$. Then our generalized functions $f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ can be regarded as $O(3)$-covariant generalized functions. Now use the results set out in §3.4.A to verify that the expression $\det(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ is a polynomial basis of standard $O(3)$-covariants in $\mathbf{R}^{3\cdot 3}$ which transform according to the representation $\mathfrak{D}^{(0)}_{-1}$.]

C. Applications to Lorentz-invariant and Lorentz-covariant generalized functions

There is an important class of Lorentz-covariant (in particular, Lorentz-invariant)* generalized functions in $\mathbf{M}^n$ with a specific property of the support, a consideration of which reduces to $O_{(+)}(3)$-covariant (in particular, $O_{(+)}(3)$-invariant) generalized functions. This is the class of generalized functions $f(p_1, \dots, p_n)$ whose supports in one of the variables $p_1, \dots, p_n$, say p_n, are contained in the $L^\uparrow$-invariant set $\overline{V}^+_m$ for some $m > 0$, that is,

$$\operatorname{supp} f(p_1, \dots, p_n) \subset \{(p_1, \dots, p_n) \in \mathbf{M}^n : p_n \in \overline{V}^+_m\} \equiv \mathbf{M}^{n-1} \times \overline{V}^+_m. \tag{3.222}$$

According to Proposition A.1, such generalized functions f can be regarded as elements of $\mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}^+_m)$.

The standard method of regarding ($\mathcal{X}$-valued) Lorentz-covariant generalized functions in $\mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}^+_m; \mathcal{X})$ consists in going over to a frame of reference associated with the vector p_n or, to be more precise, it consists in fixing the direction of p_n, for example, along the zeroth basis vector $e_0 = (1, 0, 0, 0)$. This is possible because each Lorentz-covariant generalized function in $\mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}^+_m; \mathcal{X})$ can be regarded as a generalized function in some of the variables that is continuously (and even C^∞) dependent on the remaining variables; for the latter we must take quantities defining

* The invariant generalized functions can be regarded as a special case of the covariant ones (in this case, the role of T is played by the identity representation $T(g) \equiv 1$ in C).

the direction of p_n. We characterize the direction of an arbitrary positive time-like vector $p \in \mathbf{M}$ by means of the 4-velocity $v(p)$, defined by the formula

$$v(p) = p/\sqrt{p^2}. \tag{3.223}$$

It is easily seen that $v(p)$ takes values in Γ_1^+.

It is clear that each point v in Γ_1^+ is defined by its spatial part $v \in \mathbf{R}^3$, which can be regarded as an independent variable in Γ_1^+. This isomorphism between Γ_1^+ and $\mathbf{R}^3$ enables us to use the concept of a C^∞-function (or generalized function) $f(v)$ of v on Γ_1^+ (and, possibly, of other variables) simply in the sense of a C^∞-function (or generalized function) $f(v)$ of v on $\mathbf{R}^3$.

The map

$$\overline{V}_m^+ \ni p \to \left(\sqrt{p^2}, v(p)\right) \in [m, \infty) \times \Gamma_1^+ \tag{3.224}$$

is a diffeomorphism; in accordance with the definition of change of variables (§2.2.A), each generalized function $f \in \mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}_m^+; \mathcal{X})$ can be uniquely represented in the form

$$f(p_1, \ldots, p_n) = F\left(p_1, \ldots, p_{n-1}, \sqrt{p_n^2}, v(p_n)\right), \tag{3.225}$$

where $F(p_1, \ldots, p_{n-1}, \lambda, v)$ is a generalized function in $\mathcal{S}'(\mathbf{M}^{n-1} \times [m, \infty) \times \Gamma_1^+; \mathcal{X})$. The following result will be of interest to us.

Proposition 3.15. *Every $L_{(+)}^\uparrow$-covariant generalized function $f(p_1, \ldots, p_n) \in \mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}_m^+; \mathcal{X})$ can be uniquely represented in the form* (3.225)*, where $F(p_1, \ldots, p_{n-1}, \lambda, v)$ is a generalized function in $\mathcal{S}'(\mathbf{M}^{n-1} \times [m, \infty); \mathcal{X})$ with respect to $p_1, \ldots, p_{n-1}, \lambda$, which is C^∞-dependent on $v \in \Gamma_1^+$ as a parameter.*

The proof of this proposition is based on the following general argument. Let the connected Lie group G act in C^∞ fashion on the manifolds X and Y, its action on Y being transitive; suppose further that T is a fixed linear representation of G in some finite-dimensional vector space $\mathcal{X}$. Then each G-covariant $\mathcal{X}$-valued distribution $f(x, y)$ (say, of class $\mathcal{D}'(X \times Y)$) is a distribution with respect to x that is C^∞-dependent on y as a parameter. (This argument is also used below in the remark at the end of this subsection.)

We shall regard $O_{(+)}(3)$ as a subgroup of $L_{(+)}^\uparrow$ (consisting of all the transformations in $L_{(+)}^\uparrow$ that leave the vector $e_0 = (1, 0, 0, 0)$ fixed; this means that a representation of $O_{(+)}(3)$ is defined in $\mathcal{X}$ (as the restriction of the representation T of $L_{(+)}^\uparrow$ to the subgroup). (Note that it is, in general, reducible, even if T is irreducible.) Proposition 3.15 allows us to associate with each $L_{(+)}^\uparrow$-covariant generalized function $f(p_1, \ldots, p_n) \in \mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}_m^+; \mathcal{X})$, the generalized function $F(p_1, \ldots, p_{n-1}, \lambda, v)|_{v=e_0}$ in $\mathcal{S}'(\mathbf{M}^{n-1} \times [m, \infty); \mathcal{X})$, which is naturally called the *restriction of f to the rest frame of the vector p_n* and is denoted by

$$f(x_1, \ldots, x_{n-1}, p_n^0 e_0) \equiv F(p_1, \ldots, p_{n-1}, p_n^0, v)|_{v=e_0}. \tag{3.226}$$

From this we easily obtain the following result.

Corollary 3.16. The correspondence $f(p_1, \ldots, p_n) \to f(p_1, \ldots, p_{n-1}, p_n^0 e_0)$, defined by (3.223) and (3.224) is a topological isomorphism between the $L_{(+)}^\uparrow$-covariant generalized functions in $\mathcal{S}'(\mathbf{M}^{n-1} \times \overline{V}_m^+; \mathcal{X})$ and the $O_{(+)}(3)$-covariant generalized functions * in $\mathcal{S}'(\mathbf{M}^{n-1} \times [m, \infty); \mathcal{X})$.

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* Here we suppose that the representation $O_{(+)}(3)$ in $\mathcal{X}$ is obtained as the restriction of a fixed representation T of $L_{(+)}^\uparrow$ in $\mathcal{X}$.

The convenience of the device of restricting f to the rest frame of the vector p_n is clear, since it reduces the Lorentz-covariant (in particular, the Lorentz-invariant) generalized functions to $O_{(+)}(3)$-covariant (in particular, $O_{(+)}(3)$-invariant) generalized functions of a smaller number of variables. Thus we can adapt the results obtained earlier for compact groups (in particular, for $O_{(+)}(3)$) to the derivation of representations of type (3.156) (or (3.190)) for $L^{\uparrow}_{(+)}$-invariant (or $L^{\uparrow}_{(+)}$-covariant) generalized functions $f(p_1,\ldots,p_n)$ of class $\mathcal{S}'(\mathbf{M}^{n-1}\times\overline{V}^+_m)$. It is clear what the successive stages of this derivation must be. To begin with we write $f(p_1,\ldots,p_{n-1},p_n^0e_0)$ in terms of standard $O_{(+)}(3)$-invariants (or $O_{(+)}(3)$-covariants), then we represent the latter as restrictions to the rest frame of p_n of standard $L^{\uparrow}_{(+)}$-invariants (or $L^{\uparrow}_{(+)}$-covariants), which possibly additionally contain as a factor, some power of the invariant p_n^2, which is permissible since $p_n^2 \geq m^2 > 0$; for example,

$$\mathbf{p}_i\mathbf{p}_j = -p_ip_j + \frac{1}{p_n^2}(p_ip_n)(p_jp_n)\Big|_{v(p_n)=e_0}. \tag{3.227}$$

Finally, we write $f(p_1,\ldots,p_{n-1},p_n^0e_0)$ as the result of the restriction to the rest frame of p_n of some $L^{\uparrow}_{(+)}$-invariant (or $L^{\uparrow}_{(+)}$-covariant) generalized function $h(p_1,\ldots,p_n)$, written in the form (3.156) (or (3.190)). According to Corollary 3.16, f is the same as h. And so we arrive at the required representation for f.

The following exercises serve as an illustration of this argument (with $m > 0$ throughout).

Exercise 3.28. Let I be the map from $\mathbf{M}^2\times\overline{V}^+_m$ to $\mathbf{R}^6$:

$$I(p_1,p_2,p_3) = (p_1^2,p_2^2,p_3^2,p_1p_2,p_1p_3,p_2p_3); \tag{3.228}$$

set

$$\Omega = I(\mathbf{M}^2\times\overline{V}^+_m). \tag{3.229}$$

Prove that Ω is a canonically closed regular subset of $\mathbf{R}^6$ and that each $L^{\uparrow}_{(+)}$-invariant generalized function $f(p_1,p_2,p_3)\in\mathcal{S}'(\mathbf{M}^2\times\overline{V}^+_m)$ can be (uniquely) represented in the form

$$f(p_1,p_2,p_3) = \phi(p_1^2,p_2^2,p_3^2,p_1p_2,p_1p_3,p_2p_3) \tag{3.230}$$

for some generalized function $\phi\in\mathcal{S}'(\Omega)$. Here (3.228) is taken in the sense of the weak integral representation (see (3.165)):

$$f(p_1,p_2,p_3) = \int \phi(y_{11},y_{22},y_{33},y_{12},y_{13},y_{23})\prod_{i\leq j}\delta(y_{ij}-p_ip_j)dy_{ij}. \tag{3.231}$$

Show further that the correspondence $\phi\to f$ is an isomorphism of all the $L^{\uparrow}_{(+)}$-invariant distributions in $\mathcal{S}'(\mathbf{M}^2\times\overline{V}_m)$ onto $\mathcal{S}'(\Omega)$. [Hint: Go over to the rest frame of the vector p_3 in determining the set Ω and establishing (3.220). Then use (3.177)]

Exercise 3.29. Prove that the formula

$$f(p_1,\ldots,p_n;\omega,\bar{\omega}) \to g(p_1,\ldots,p_n;\omega,w) = \frac{1}{(2k)!}\left(w\epsilon \underset{\sim}{p}_n\frac{\partial}{\partial\underset{\sim}{\bar{\omega}}}\right)^{2k} f(p_1,\ldots,p_n;\omega,\bar{\omega}) \tag{3.232}$$

defines an isomorphism between the $L^{\uparrow}_{(+)}$-covariant generalized functions $f\in\mathcal{S}'(\mathbf{M}^{n-1}\times\overline{V}^+_m;\mathfrak{p}^{(j,k)})$ and the $L^{\uparrow}_{(+)}$-covariant generalized functions $g\in\mathcal{S}'(\mathbf{M}^{n-1}\times\overline{V}^+_m;\mathfrak{p}^{(j,0)}\otimes\mathfrak{p}^{(k,0)})$, which transform according to a representation isomorphic to $\bigoplus_s\mathfrak{D}^{(s,0)}$, where s takes the values

$$|j-k|\leq s\leq j+k,\; s-|j-k|\in\overline{\mathbf{Z}}_+. \tag{3.233}$$

Show that the inverse map has the form

$$f(p_1,\ldots,p_n;\omega,\bar{\omega}) = \frac{1}{(2k)!}(p_n^2)^{-k}\left(\frac{\partial}{\partial\underset{\sim}{w}}\underset{\sim}{p}_n\epsilon\bar{\omega}\right)^{2k} g(p_1,\ldots,p_n;\omega,w). \tag{3.234}$$

[Hint: Use the identity $p\tilde{p} = p^2 \cdot 1$.]

Exercise 3.30. Show that for $n = 1,2,3$ (and positive integral s), the $L_+^\uparrow$-covariant generalized functions $T(p_1, \dots, p_n; \omega) \in \mathcal{S}'(\mathrm{M}^{n-1} \times \overline{V}_m^+; \mathcal{P}^{(s,0)})$ have a decomposition of type (3.190), (3.219) with the following standard covariants. For $n = 1$, there are no covariant (generalized) functions. For $n = 2$, there is the single standard covariant

$$(\omega\epsilon\underset{\sim}{p}_1\tilde{p}_2\omega)^s; \tag{3.235}$$

for $n = 3$, there are $2s + 1$ standard covariants

$$(\omega\epsilon\underset{\sim}{p}_1\tilde{p}_2\omega)^\sigma(\omega\epsilon\underset{\sim}{p}_1\tilde{p}_3\omega)^a(\omega\epsilon\underset{\sim}{p}_2\tilde{p}_3\omega)^b, \tag{3.236}$$

which form a linear basis in $\mathcal{P}^{(s,0)}$ at each point $(p_1, p_2, p_3) \in \mathrm{M}^3$ of rank 3;* here σ, a, b run through the same values as in (3.208). [Hint: Use Exercise 3.26.]

Exercise 3.31. Prove that for $n = 1,2,3$ the $L_+^\uparrow$-covariant generalized functions $f(p_1, \dots, p_n) \in \mathcal{S}'(\mathrm{M}^{n-1} \times \overline{V}_m^+; \mathcal{X})$, that transform according to an arbitrary (finite-dimensional) representation of $L_+^\uparrow$ have a decomposition of type (3.190) or (3.219) in standard covariants. In particular, in the case $n = 3$, the number of elements of a basis of standard covariants $Q'_\rho(p_1, p_2, p_3)$ is $N = \dim \mathcal{X}$ (so that they form a linear basis in $\mathcal{X}$ at each point (p_1, p_2, p_3) of rank 3) and the (unique) covariant decomposition of type (3.219) has the form

$$f(p_1, p_2, p_3) = \sum_{\rho=1}^{N} Q'_\rho(p_1, p_2, p_3) H_\rho(p_1^2, p_2^2, p_3^2, p_1p_2, p_1p_3, p_2p_3), \tag{3.237}$$

where $H \equiv \{H_\rho(y)\}$ is an element of $X'^{-1} \bigoplus^{N} \mathcal{S}'(\Omega)$; Ω is defined in (3.229) and

$$X_{\rho\sigma}(I(p)) = \sum_{\alpha=1}^{N} Q'^{\alpha}_{\rho}(p_1, p_2, p_3) Q_{\sigma,\alpha}(p_1, p_2, p_3). \tag{3.238}$$

[Hint: Deal only with the case of an irreducible representation and use the two preceding exercises.]

Remark. In the foregoing discussion, we have fixed the direction of just one of the vector variables $p_1, \dots, p_n$ in the covariant generalized function $f(p_1, \dots, p_n)$, namely, the direction of the time basis vector e_0. We could have extended the procedure for constructing a Lorentz frame of vector variables. For example, let $f(p_1, \dots, p_n)$ be a covariant generalized function in $\mathcal{S}'(\mathcal{O})$ for $n \geq 2$, where

$$\mathcal{O} = \{(p_1, \dots, p_n) \in \mathrm{M}^n : p_n \in V_m^+;\ p_{n-1} \text{ and } p_n \text{ are linearly independent}\} \tag{3.239}$$

$(m > 0)$. We can construct from p_{n-1} and p_n the two basis vectors v_0 and v_1 of the Lorentz frame:

$$v_0 = \frac{p_n}{\sqrt{p_n^2}}, \quad v_1 = \frac{(p_n^2)p_{n-1} - (p_{n-1}p_n) \cdot p_n}{\sqrt{p_n^2((p_{n-1}p_n)^2 - p_n^2p_{n-1}^2)}}. \tag{3.240}$$

As in Proposition 3.15, we can now go over from p_{n-1}, p_n to the new variables $p_{n-1}^2, p_{n-1}p_n, p_n^2, v_0, v_1$ and verify that each covariant generalized function $f \in \mathcal{S}'(\mathcal{O})$ in the new variables is C^∞-dependent on v_0, v_1. This in turn enables us to fix v_0, v_1 for example: $v_0 = e_0 \equiv (1,0,0,0)$, $v_1 = e_1 \equiv (0,1,0,0)$. The analogue of Corollary 3.16 now asserts that there is an isomorphism (defined by the restriction $v_0 = e_0$, $v_1 = e_1$) between the $L_{(+)}^\uparrow$-covariant generalized functions$f_1(p_1, \dots, p_n)$ in $\mathcal{O}$ and the $O_{(+)}(2)$-covariant generalized functions of the vectors $p_1, \dots, p_{n-2}$ and the invariant variables $p_{n-1}^2, p_{n-1}p_n, p_n^2$.

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* A point $(p_1, p_2, p_3) \in \mathrm{M}^3$ is called a *point of rank* 3 if the Gramm determinant $\det(p_ip_j)_{i,j=1,2,3}$ is non-zero.

In exactly the same way, for $n \geq 3$ we can reduce the $L^{\uparrow}_{(+)}$-covariant generalized functions $f(p_1, \dots, p_n)$ in the region defined by the conditions

$$p_n \in \overline{V}^+_m;\ p_{n-2},\ p_{n-1} \text{ are linearly independent,} \tag{3.241}$$

to ($O_{(+)}(1)$-covariant) generalized functions of the vectors $p_1, \dots, p_{n-3}$ and scalar products of the vectors p_{n-2}, p_{n-1}, p_n. An alternative possibility is to fix the Lorentz frame constructed from the vectors p_{n-2}, p_{n-1}, p_n.

Appendix D. Vocabulary of Lie Groups and their Representations*

D.1. ABSTRACT GROUPS. ALGEBRAIC PROPERTIES

A set G is called a *group* if an associative operation of multiplication is defined in it that associates with each pair g_1, g_2 of elements of G an element $g_1 g_2 \in G$, and if there exists an *identity* in G (that is, an element e such that $eg = ge = g$ for all $g \in G$) and an *inverse element* g^{-1} of each element g (such that $g^{-1}g = gg^{-1} = e$).

A group G is called *abelian* (or commutative) if the factors in a product can be interchanged, that is, if the identity $g_1 g_2 = g_2 g_1$ holds for any g_1, $g_2 \in G$.

A subset H of a group G is called a *subgroup* if H is a group with respect to the same operation $h_1 h_2$ defined on G. Each group has two *trivial* (or improper) subgroups: the group G itself and the subgroup $\{e\}$ consisting of the identity element. A subgroup N of G is called a *normal subgroup* (or *invariant subgroup*) if $gng^{-1} \in N$ for all $n \in N$ and $g \in G$ (that is, if $gNg^{-1} = N$).

An example of a normal subgroup is the *centre* C of the group G, defined as the set of all elements of G that commute with any element of G. Another example of a normal subgroup is the *derived group* or *commutator subgroup*) G' of G, defined as the smallest subgroup of G containing all elements of the form $g_1 g_2 g_1^{-1} g_2^{-1}$, where $g_1, g_2 \in G$. The fact that G' is normal follows from the relation $hg_1 g_2 g_1^{-1} g_2^{-1} h^{-1} = \tilde{g}_1 \tilde{g}_2 \tilde{g}_1^{-1} \tilde{g}_2^{-1}$, where $\tilde{g}_i = hg_i h^{-1}$.

A map ϕ from a group G to a group H is called a *homomorphism* if

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad \text{for } g_1, g_2 \in G;$$

here, the set $\ker\phi = \{g \in G :\ \phi(g) = e\}$ is called the *kernel* of the homomorphism ϕ, and the set $\operatorname{im}\phi = \{\phi(g) : g \in G\}$ is called its *image*.

A sequence of groups and homomorphisms $\dots \xrightarrow{\phi_{n-1}} G_n \xrightarrow{\phi_n} G_{n+1} \xrightarrow{\phi_{n+1}}, \dots$ is said to be *exact* if the image of each homomorphism coincides with the kernel of the next one. For example, the exactness of the sequences $1 \longrightarrow G \xrightarrow{\phi} H$, $g \xrightarrow{\phi} H \longrightarrow 1$, $1 \longrightarrow G \xrightarrow{\phi} H \longrightarrow 1$ (where 1 denotes the group $\{e\}$ consisting of just the identity element) means that the map ϕ is a *monomorphism* ($\ker\phi = 1$), an *epimorphism* ($\operatorname{im}\phi = H$) or an *isomorphism*, respectively. (In the latter case we write $G \approx H$.) An isomorphism $G \xrightarrow{\phi} G$ (of G onto itself) is called an *automorphism*. An example of an automorphism of a group is given by the transformation $g \to g_0^{-1} g g_0$, where g_0 is a fixed element of G. An automorphism of this type is called *inner*.

The groups that we shall be dealing with (the Lorentz and Poincaré groups, the Euclidean group, etc.) turn up mostly in the role of transformation groups. We give a description of what occurs in this situation and give the requisite terminology in the general case.

Let G be a group and $\mathcal{X}$ a set (or a manifold). We say that the group G *acts* on $\mathcal{X}$ (or $\mathcal{X}$ is a *G-space*) if each element $g \in G$ is put in (one-to-one) correspondence with a transformation $\tau(g) : x \to \tau(g)(x) \equiv gx$ of $\mathcal{X}$ onto itself,** where $g_1(g_2 x) \equiv (g_1 g_2)x$ and $ex \equiv x$ (e being the identity

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* In this "vocabulary" we give in logical (rather than alphabetical) order, the main terminology and the various facts in the theory of groups and their representations that we shall be using in the text.

** The above definition of action and G-space is called *left*. One also uses *right action* and *right G-space*, for which $\tau(g)(x)$ is written in the form xg, and the defining property now becomes: $x(g_1 g_2) \equiv \equiv (xg_1)g_2$, $xe \equiv x$.

in G). In particular, if $\mathcal{X}$ is a (real or complex) vector space and all the transformations $\tau(g)$ are linear, then we say that a (real or complex) *linear representation* τ of G is defined in $\mathcal{X}$.

The representations τ and σ in the linear spaces $\mathcal{X}$ and $\mathcal{Y}$ are said to be *equivalent* if there exists a linear isomorphism $V : \mathcal{X} \to \mathcal{Y}$ such that $\tau(g) = V\sigma(g)V^{-1}$ for all $g \in G$.

The action on $\mathcal{X}$ is called *effective* if the condition $gx = x$ for all $x \in \mathcal{X}$ implies that $g = e$.

An important characteristic of the action of G on a manifold is the structure of the orbits: by a *G-orbit* (or *group orbit*) we mean a subset of $\mathcal{X}$ of the form $G_a \equiv \{x = ga : g \in G\}$, where a is a fixed element of $\mathcal{X}$. It is easy to see that different orbits are non-intersecting and that the totality of all the orbits gives a partitioning of $\mathcal{X}$ (where two points $x, y \in \mathcal{X}$ belong to the same orbit if and only if there exists an element g of the group such that $y = gx$). The importance of the notion of a G-orbit is clear, for example, from the following characterization of G-invariant functions on $\mathcal{X}$. A function f on $\mathcal{X}$ is *G-invariant* (that is, $f(gx) \equiv f(x)$) if and only if it is constant on each orbit. Thus, to say that a G-invariant function is defined on $\mathcal{X}$ is the same as saying that a function is defined on the set of G-orbits in $\mathcal{X}$.

If the G-space $\mathcal{X}$ has only one G-orbit, (namely, $\mathcal{X}$ itself) then G is said to act *transitively* on $\mathcal{X}$ or that $\mathcal{X}$ is a *homogeneous G-space*; in this case, any point of $\mathcal{X}$ is obtained from an arbitrary fixed point $x \in \mathcal{X}$ by the action of some element of G.

The set G_x of elements of G that leave a given point $x \in \mathcal{X}$ fixed, forms a subgroup of G. It is called the *stationary subgroup* or *stabilizer* (or even *small group*) of x. Let g be a fixed element of G; the set gG_x of all elements of the form gh, where h runs through the subgroup G_x is called a *left coset* of G with respect to G_x. It can be shown that a homogeneous space $\mathcal{X}$ is isomorphic to a *quotient space* G/G_x, the elements of which are the left cosets. Conversely, if H is an arbitrary subgroup of G, then the quotient space G/H is a homogeneous G-space with respect to the natural action of G:

$$g : g_1H \to gg_1H.$$

The space of *right cosets* $H\backslash G$ is similarly defined. If N is a normal subgroup, then the spaces $N\backslash G$ and G/N are the same. In this case, the quotient space $K = G/N$ can be given a group structure with multiplication $g_1N \cdot g_2N = g_1g_2N$; such a group K is called a *quotient group* while the group G itself is called an *extension of the group K* ($\approx G/N$) by the group N. In particular, if the normal subgroup N is contained in the centre C of G, then G is called a *central extension* (see[Z4],Ch.1,§3.9).

Examples. 1) By a *trivial extension* one means a group G isomorphic to the *direct product* $N \times K$ of the groups. Here, multiplication of elements of $N \times K$ represented as ordered pairs (n, k), where $n \in N$, $k \in K$, is defined "componentwise": $(n_1, k_1)(n_2, k_2) = (n_1n_2, k_1k_2)$. The groups N and K can be identified with the normal subgroups N_1 and K_1 in G corresponding to the subgroups $N \times \{e_2\}$ and $\{e_1\} \times K$ in $N \times K$ (where e_1 and e_2 are the identities in N and K); in this case, G is called the direct product of the subgroups N_1 and K_1.

2) A group G is called a *semidirect product* $N_1 \circ K_1$ of the normal subgroup $N_1 \subset G$ and the subgroup $K_1 \subset G$ if any element of G can be uniquely expressed as a product $g = nk$, where $n \in N_1, k \in K_1$. In this case, G is an extension of K_1 by N_1. In the definition of a semidirect product, the "factors" can also be taken to be external with respect to G; we then say that G is the semidirect product $N \circ K$ of the groups N and K if it is a semidirect product of the normal subgroup $N_1 \subset G$ and the subgroup $K_1 \subset G$, where $N_1 \approx N$, $K_1 \approx K$.

D.2. Lie groups

A topological space is said to be *Hausdorff* if any two distinct points have non-intersecting neighbourhoods. A group G is said to be a *topological group* if it is a Hausdorff (topological) space and if the map $G \times G \xrightarrow{\phi} G$ defined by the formula

$$\phi(g_1, g_2) = g_1g_2^{-1}, \tag{D.1}$$

is continuous. (From this it follows as a special case that the product and the operation $g \to g^{-1}$ are continuous maps in G.)

All properties of topological spaces automatically go over to topological groups. In particular, a group is (*arcwise*) *connected** if any element of it can be joined to the identity element by a continuous curve. A connected group is said to be *simply connected* if any closed curve in G can be continuously deformed to a point.

This means the following. Let $\overline{S}$ be the closed disc in the plane and ∂S the circle forming its boundary. Then an arcwise connected group G (more generally, an arcwise-connected topological space G) is said to be simply connected if any continuous map $\partial S \to G$ can be extended to a continuous map $\overline{S} \to G$.

A group G is *compact* if any covering of it by open sets has a finite subcovering. The group G is said to be *locally compact* if there exists a neighbourhood of the identity (and hence of any other) element of G whose closure is compact.

We give several examples. The group $O(3)$ of (real) orthogonal 3×3-matrices is compact but not connected [since the continuous function $\det R$ takes two values on $O(3)$: 1 (say for $R = e = 1$) and -1 (for example, for $R = -1$)]. The connected component of the identity element of $O(3)$ is the group $O_+(3)$ of proper Euclidean rotations R (such that $\det R = 1$). The group $O_+(3)$ is connected but not simply connected. Its double covering group $SU(2)$ (of unitary 2×2-matrices with determinant 1) is connected and simply connected, since it is homeomorphic as a topological space to the unit sphere S^3 in four-dimensional space. (To verify this last assertion, it suffices to use the quaternionic notation $V = v^0\sigma_0 + iv^j\sigma_j$ for an arbitrary matrix $\checkmark \in SU(2)$, where $v^\mu \in \mathbf{R}$ and $\sum_{\mu=0}^{3}(v^\mu)^2 = 1$.) The group $O_+(3)$ can be defined as a topological space by the same sphere S^3 with diametrically opposite points identified.

Exercise D.1. Verify that the groups $O(3)$ and $SU(2)$ are respectively trivial and non-trivial central extensions of $O_+(3)$ by the group $\mathbf{Z}_2 = \{+1, -1\}$.

The Lorentz group $L = O(1,3)$ is an example of a locally compact, non-compact group. Its non-compactness is evident, for example, from the unboundedness of the matrix elements of the one-parameter subgroup of Lorentz (hyperbolic) rotations

$$\begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We shall only be dealing with a special class of topological groups, namely Lie groups. A topological group G is called a *Lie group* of real dimension n if the following two conditions are satisfied. Firstly, there exists a neighbourhood, say $\mathcal{N}$, of the identity element of G that is homeomorphic to an open subset, say U, of $\mathbf{R}^n$. In other words, there exists a sufficiently small neighbourhood $\mathcal{N}$ of e that can be mapped in one-to-one fashion and bicontinuously into the interior of an n-dimensional open set $U \subset \mathbf{R}^n$. The coordinates of a point s in U are called *local coordinates* of the group. Thus in a sufficiently small neighbourhood of the identity e, the group multiplication $g = g_1g_2$ can be written in a form showing its dependence on the local coordinates s of the element g as a function of the local coordinates s_1, s_2 of the elements g_1, g_2:

$$s = \Phi(s_1, s_2), \; s_1, s_2 \in U_1 \subset U. \tag{D.2}$$

We note that the associativity of the group multiplication implies that

$$\Phi(\Phi(s_1, s_2), s_3) = \Phi(s_1, \Phi(s_2, s_3)), \quad s_1, s_2, s_3 \in U_2 \subset U \tag{D.3}$$

(at those points where this equality makes sense), while the existence of the inverse implies that for fixed s_2, the map $s_1 \to s = \Phi(s_1, s_2)$ must be invertible. The second requirement entering into the definition of a Lie group can be stated as a condition of sufficient smoothness (for definiteness, infinite differentiability) of the function Φ in (D.2) for some choice of local coordinates. We note that any two

* Apart from the notion of arcwise connectedness, there is the following more general definition: a topological space is said to be *connected* if it cannot be split up into two non-empty open subsets. For differentiable manifolds (in particular, for Lie groups) the notions of arcwise connectedness and connectedness are the same.

such systems of (differentiable) local coordinates are related by an infinitely differentiable transformation. Under the above-stated conditions, a Lie group is naturally endowed with the structure of an infinitely differentiable manifold, therefore the differential calculus can be applied to a Lie group. Furthermore, according to the Gleason-Montgomery-Zippin theorem (see, for example, Glushkov, 1957) it is always possible to choose local coordinates such that the law of multiplication (D.2) is expressed by a real analytic function of the coordinates. This means in effect, that every Lie group has the structure of a real analytic manifold.

Exercise D.2. Prove that a connected Lie group G is generated by any neighbourhood $\mathcal{O}$ of e in the sense that an arbitrary element $g \in G$ can be written in the form $g_1 \dots g_n$, where $g_j \in \mathcal{O}$ $(j = 1, \dots, n)$.

A Lie group H is said to be a *covering* of the Lie group G if there exists a homomorphism from H onto G that maps some neighbourhood of the identity in H homeomorphically onto a neighbourhood of the identity in G. Every connected Lie group G has a unique (to within isomorphism) connected, simply-connected covering group $\tilde{G}$; it is called the ***universal covering*** of G. Here, the kernel of the ***covering homomorphism*** $\phi : \tilde{G} \to G$ is a discrete subgroup in the centre of $\tilde{G}$ (so that $\tilde{G}$ is a central extension of G by $\ker \phi$). For example, $SU(2)$ is a universal covering of $O_+(3)$. It can happen that a universal covering of a compact group is non-compact. Thus the universal covering of the (compact) group of rotations of the plane $O_+(2) \approx U(1)$ is the (non-compact) additive group $\mathbf{R}$ of the real numbers. (The notion of a universal covering is discussed in greater detail in §3.1.C based on the example of the Lorentz group.)

A Lie group of real dimension $2n$ is called a *complex Lie group* of complex dimension n if some neighbourhood of the identity can be parametrized by n complex variables in such a way that the law of multiplication (D.2) is expressed by a complex analytic (that is, holomorphic) function of the coordinates. A complex Lie group is endowed with a complex manifold structure.

D.3. Lie algebras

First we give the axiomatic definition and list the properties of Lie algebras; after this we shall indicate their relationship to Lie groups.

By a (*real*) *Lie algebra* we mean a linear space X (over $\mathbf{R}$) endowed with a so-called *commutation* operation $[x, y]$, satisfying the following properties:

a) *bilinearity*;

b) *antisymmetry*: $[x, y] = -[y, x]$ for all $x, y \in X$;

c) *Jacobi identity*; $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for all $x, y, z \in X$.

(The notion of complex Lie algebra is similarly defined.)

It suffices for the definition of a Lie algebra structure on a (finite-dimensional) space X, to define the pairwise commutators of basis vectors $I_1, \dots, I_n$, that is, the *structure constants* c_{ij}^k in the expressions

$$[I_i, I_j] = c_{ij}^k I_k \tag{D.4}$$

(where k is to be summed from 1 to n). Here the coefficients c_{ij}^k must be antisymmetric in the lower indices and must satisfy the Jacobi identity (see (D.22), (D.23)).

Any linear space with zero commutator is a Lie algebra, which in this case is called *abelian*.

By a *derivation* of a (not necessarily associative) algebra $\mathfrak{A}$ we mean a linear map $D : \mathfrak{A} \to \mathfrak{A}$ with the property

$$D(xy) = (Dx)y + x(Dy). \tag{D.5}$$

Exercise D.3. (a) Let $\mathfrak{A}$ be a (real) associative algebra. Set $[x, y] = xy - yx$ for $x, y \in \mathfrak{A}$. Prove that $\mathfrak{A}$ with this commutation operation is a Lie algebra.

(b) Prove that the space of all derivations of an algebra $\mathfrak{A}$ is a Lie algebra with respect to the operation $[D_1, D_2] = D_1 D_2 - D_2 D_1$.

A subspace Y of a Lie algebra X is called a *subalgebra* (or an *ideal*) if $[Y, Y] \subset Y$ (or $[X, Y] \subset Y$). (The notation $[X, Y]$ here denotes the linear span of the vectors of the form $[x, y]$, $x \in X$, $y \in Y$.) A special case of an ideal is the *centre* of a Lie algebra X which is defined as the set of all elements $x \in X$ such that $[x, y] = 0$ for any $y \in X$. A subalgebra (or ideal) Y of X is said to be *trivial* if $Y = X$ or $Y = \{0\}$.

In every Lie algebra X we can define the sequences of subspaces:

$$X_1 = X \supset X_2 = [X,X] \supset X_3 = [X,X_2] \supset \ldots \supset X_{n+1} = [X,X_n] \supset \ldots, \tag{D.6a}$$
$$X^1 = [X,X] \supset X^2 = [X^1,X^1] \supset \ldots \supset X^{n+1} = [X^n,X^n] \supset \ldots. \tag{D.6b}$$

Exercise D.4. Prove that the subspaces X_n and X^n are ideals in X and that the quotient algebras X_n/X_{n+1} and X^n/X^{n+1} are abelian.

If $\dim X < \infty$, the sequences $\{X_n\}$ and $\{X^n\}$ stabilise: starting from some n, $X_n = X_{n+1} = \ldots = X_\infty$, $X^n = X^{n+1} = \ldots = X^\infty$. A Lie algebra X is called *soluble* (or *nilpotent*) if $X^\infty = \{0\}$ (or $X_\infty = \{0\}$). A Lie algebra is said to be *simple* (or *semisimple* if its dimension is greater than one and it contains no non-trivial ideals (or if it contains no soluble ideals other than the null ideal).

Theorem D.1 (E. Cartan-Levi-Mal'tsev). *All the soluble ideals of an arbitrary Lie algebra X are contained in a maximal soluble ideal $\mathfrak{R}$ (called the radical of X). There exists a semisimple subalgebra $Y \subset X$ such that $X = Y \oplus \mathfrak{R}$ (direct sum of linear spaces). Any two subalgebras Y with this property can be transformed into each other by an algebra automorphism of X (that preserves $\mathfrak{R}$).*

D.4. Relation between Lie groups and Lie algebras

There are many methods of constructing a Lie algebra from a Lie group. We give two of them. (See also [K6],§6.3, where four other properties and their interrelations are discussed.)

(a) We choose a parametrization of a neighbourhood of the identity e in the Lie group G for which the origin corresponds to e. Let $s = (s^1, \ldots, s^n)$, $t = (t^n, \ldots, t^n)$ ($n = \dim G$) be two points in the parameter space. Then the law of multiplication on the group defines a function $r = \Phi(s,t)$ (see (D.2)). Since $g(0) = e$, it follows that $\Phi(0,s) = \Phi(s,0) = s$. Consequently, the Taylor expansion of this function has the form

$$\Phi(s,t) = s + t + B(s,t) + \ldots, \tag{D.7}$$

where $B(s,t)$ is a bilinear vector-valued function, the dots denoting terms of third and higher orders. We now define a commutator in $\mathbf{R}^n$ by the formula

$$[s,t] = B(s,t) - B(t,s). \tag{D.8}$$

Exercise D.5. Prove that the commutator defined by (D.7), (D.8) satisfies the Jacobi identity. [Hint: Using the associativity of group multiplication, show that

$$B(B(r,s),t) = B(r,B(s,t)). \tag{D.9}$$

Thus $\mathbf{R}^n$ is endowed with a Lie algebra structure. It is called the *Lie algebra of the group* G and is denoted by $\mathfrak{g}$.

(b) The second method introduces the Lie algebra structure in the tangent space T_eG to the group G at the point e. Let $g(\tau)$ and $h(\tau)$ be two smooth curves on the group starting at the point e (that is, such that $g(0) = h(0) = e$). Let x and y be the tangent vectors at e corresponding to them. (If F is any real smooth function on the group ($F \subset C^\infty(G)$, then the vector x is defined as the functional $x(F) = \frac{d}{d\tau}F(g(\tau))\big|_{\tau=0}$.) We consider the (smooth) curve

$$q(\tau) = g(\sigma)h(\sigma)g(\sigma)^{-1}h(\sigma)^{-1}, \ \sigma = \sqrt{\tau}; \ \tau \geq 0; \tag{D.10}$$

let z be the tangent vector to it at $\tau = 0$ (that is, at $g = e$). Then the space T_eG can be identified with the Lie algebra $\mathfrak{g}$ if we define the commutator of x and y by the formula $[x,y] = z$. If G is a group of matrices, then there is a matrix realization of the Lie algebra in which $x = \frac{dg}{d\tau}\Big|_{\tau=0} \equiv g'(0)$. We also use the symbolic equality $x = g'(0)$ in the general case.

Exercise D.6. Verify the equivalence of the two definitions of the Lie algebra $\mathfrak{g}$ of the group G. [Hint: Introduce parameters $s = (s^1, \ldots, s^n)$ in a neighbourhood of the identity and use the fact that if s parametrizes the element g, then the parameters of g^{-1} are given by the expression $-s + B(s,s) + \ldots$ arising from (D.7); prove that if $g(\tau) = a(\tau) + o(\tau), h(\tau) = b\tau + o(\tau)$, then $q(\tau) = \tau B(a,b) - \tau B(b,a) + o(\tau)$.]

Conversely, corresponding to every Lie algebra X (with $\dim X < \infty$) there is a unique (to within isomorphism) connected simply connected Lie group G with Lie algebra $\mathfrak{g} \approx X$. All connected Lie groups with this property have the form G/H, where H is a discrete normal subgroup situated in the centre of G. (See [K6],§6.3,Theorem 2.)

The relation between one-parameter subgroups and their generators in a Lie algebra has an especially simple appearance. The parameter t in a one-dimensional subgroup can be chosen so that the following relations hold:

$$g(0) = e, \quad g(\tau)g(\sigma) = g(\tau + \sigma). \tag{D.11}$$

Since $g(\tau)$ is a smooth curve, its tangent vector can be defined:

$$x = g'(0) \in \mathfrak{g}. \tag{D.12}$$

The vector x is called a *generator* (or *infinitesimal generator*) of the one-parameter group (D.11).

Conversely, for each element x of a Lie algebra $\mathfrak{g}$, there exists a unique smooth curve $g(t)$ satisfying the conditions (D.11), (D.12) (and defining a one-parameter subgroup of G). (See [K6],§6.4, Theorem 1.) The element $g(t)$ is written in the form $\exp(tx)$ or e^{tx} and the map $x \to \exp x$ from $\mathfrak{g}$ to G is called the *exponential map*. It maps a neighbourhood of the origin in $\mathfrak{g}$ onto a neighbourhood of the identity of G.

Exercise D.7. Prove that any element of a connected Lie group can be represented in the form $g_1^2 \dots g_n^2$, where $g_j \in G$ $(j = 1, \dots, n)$. [Hint: Prove the identity $\exp x = (\exp x/2)^2$, $x \in \mathfrak{g}$; then use Exercise D.2.]

Exercise D.8. Let $SL(n, C)$ be the group of all complex $n \times n$-matrices with determinant 1. Prove that its Lie algebra consists of all $n \times n$-matrices with zero trace. [Hint: First derive the equality

$$\det e^x = e^{\operatorname{tr} x},$$

by using the reduction of x to Jordan normal form.]

Exercise D.9. Let $O_+(n)$ be the group of real orthogonal $n \times n$-matrices with determinant 1 $(\Lambda \in O_+(n) \Rightarrow \Lambda^T\Lambda = 1,\ \det\Lambda = 1)$. Show that its Lie algebra consists of all real antisymmetric $n \times n$-matrices.

A connected Lie group G is called *(semi)simple* if its Lie algebra is (semi)simple.

D.5. Local Lie groups. Canonical parametrization. Lie's theorems

In constructing the Lie algebra of a Lie group, we have only used the knowledge of the law of multiplication (D.5) in a suitably small neighbourhood of the identity. This leads to the following definition.

By a *local Lie group* we mean a pair (B, Φ), where $B \subset \mathbf{R}^n$ is the interior of an n-dimensional ball (of fixed radius) with centre at the origin, and Φ is the law of composition (D.7), that is, an analytic map from $B \times B$ to $\mathbf{R}^n$ satisfying the associative law (D.3) and the "initial conditions"

$$\Phi(s, 0) = \Phi(0, s) = s \tag{D.13}$$

(which follow from (D.7)).

A continuous curve $s = s(\tau)$, $-\rho < \tau < \rho$, is called a *local one-parameter subgroup* if

$$\Phi(s(\tau_1), s(\tau_2)) = s(\tau_1 + \tau_2).$$

The parameters s are said to be *canonical* if every curve with equation $s(\tau) = a\tau$, where a is a constant vector (or, in components, $s^\alpha(\tau) = a^\alpha\tau$), is a local one-parameter subgroup (for sufficiently small $|\tau|$).

Exercise D.10. Prove that if s and t are canonical parameters and $\Phi(s, t)$ is the law of composition, then $\Phi(s, -s) = 0$ and the expansion (D.7) takes the form

$$\Phi^i(s, t) = s^i + t^i + \frac{1}{2}c^i_{jk}s^j t^k + \dots, \tag{D.14}$$

where the c^i_{jk} are the structure constants of the Lie algebra $\mathfrak{g}$.

It can be shown that in canonical coordinates, the functions Φ^i are real analytic functions in a sufficiently small neighbourhood of the origin, so that the power series on the right hand side of (D.14) has a non-zero radius of convergence.

Proposition D.2 (first Lie theorem). *The functions $\Phi^i(s,t)$ satisfy the system of partial differential equations in t for fixed s*

$$\frac{\partial}{\partial t^j}\Phi^i(s,t)v^j_k(t) = v^i_k(\Phi(s,t)), \tag{D.15}$$

where

$$v^i_j(s) = \frac{\partial \Phi^i}{\partial t^j}(s,t)\Big|_{t=0}. \tag{D.16}$$

If for given (smooth) functions $v^i_j(s)$ satisfying the condition

$$v^i_j(0) = \delta^i_j, \tag{D.17}$$

the system (D.15) *has a solution, then it is uniquely defined by this system and the initial condition* (D.13).

The proof of this proposition is fairly simple. Equation (D.15) is obtained from the associative law (D.12) by differentiating with respect to t at $t = 0$ (and changing the notation for the arguments). To establish uniqueness, it is necessary to use the invertibility of the matrix $v(t)$ at the origin, which follows from (D.17) and from which it follows that (D.16) can be given the form

$$\frac{\partial \Phi^i}{\partial t^k}(s,t) = v^i_j(\Phi)u^j_k(t) \quad (v^i_j(t)u^j_k(t) \equiv \delta^i_k). \tag{D.18}$$

Proposition D.3 (second Lie theorem). *In order that the system* (D.15) (*or* (D.18)) *with initial condition* (D.13) *have a solution, it is necessary and sufficient that the functions $v^i_j(t)$ satisfy the system of differential equations*

$$\frac{\partial v^i_j}{\partial t^l}v^l_k - \frac{\partial v^i_k}{\partial t^l}v^l_j + v^i_l c^l_{jk} = 0 \tag{D.19}$$

and the initial condition (D.17). *Then the inverse matrix $u(t)$ must satisfy the system*

$$\frac{\partial}{\partial t^j}u^i_k - \frac{\partial,}{\partial t^k}u^i_j + c^i_{lm}u^l_j u^m_k = 0, \quad u^i_j(0) = \delta^i_j. \tag{D.20}$$

Equations (D.19) (and (D.20) are obtained by differentiating (D.18) with respect to t and comparing the mixed second partial derivatives (with respect to t_j, t_k) of Φ. Here the structure constants are given by the formula

$$c^i_{jk} = \left(\frac{\partial v^i_k}{\partial t^j} - \frac{\partial v^i_j}{\partial t^k}\right)\Big|_{t=0}. \tag{D.21}$$

Proposition D.4 (third Lie theorem). *In order that the systems* (D.19) (*or* (D.20)) *be locally soluble, it is necessary and sufficient that the constants* (D.21) *be antisymmetric in the lower indices and satisfy the Jacobi identity*:

$$c^i_{jk} = -c^i_{kj}, \tag{D.22}$$

$$c^i_{jk}c^k_{lm} + c^i_{mk}c^k_{jl} + c^i_{lk}c^k_{mj} = 0. \tag{D.23}$$

Exercise D.11. Denote by cs the matrix

$$(cs)^i_j = c^i_{jk}s^k.$$

Prove that the solutions of the systems (D.19) (D.17) and (D.20) are given in canonical coordinates by the equalities

$$u(s) = \frac{e^{cs}-1}{cs} = 1 + \frac{1}{2!}cs + \frac{1}{3!}(cs)^2 + \dots, \tag{D.24}$$

$$v(s) = \frac{cs}{e^{cs}-1} = \sum_{n=0}^{\infty} B_n \frac{(cs)^n}{n!}, \tag{D.25}$$

where the B_n are the Bernoulli numbers $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_3 = B_5 = = B_{2n+1} = 0, \sum_{r=0}^{n+1} \binom{n}{r} B_r = 0$).

D.6. LINEAR REPRESENTATIONS

In the case when G is a topological group (in particular, a Lie group) and $\mathcal{X}$ is a linear topological space, we shall suppose (usually without special mention) that the group representation T in question is *continuous*, that is, that the map $(g, \Phi) \to T(g)\Phi$ is jointly continuous in all the variables.

If the space $\mathcal{X}$ of the (continuous) representation T has a closed subspace $\mathcal{Y}$ that is invariant with respect to the operators $T(g)$, $g \in G$, then the restriction T_1 of T to $\mathcal{Y}$ is called a (*topological*) *subrepresentation* of T; the representation T_2 induced in the quotient space $\mathcal{Z} = \mathcal{X}/\mathcal{Y}$ is called the (*topological*) *factor representation* of T.

A representation T is said to be (*topologically*) *irreducible* if it has no non-trivial subrepresentations. We say that a representation T is (*topologically*) *decomposable* if $\mathcal{X}$ has (closed) non-trivial invariant subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$ such that $\mathcal{X}$ is isomorphic to the direct sum $\mathcal{X}_1 \oplus \mathcal{X}_2$. In this case, we write $T = T_1 \oplus T_2$, where the T_i are the restrictions of T to $\mathcal{X}_i$. A representation T is said to be *completely reducible* if each of its invariant subspaces has an invariant complement.

As an example, we consider the group $E_+(2)$ of Euclidean motions of the complex plane: $z \to e^{i\phi}z + \zeta$.

Exercise D.12. Verify that the two-dimensional representation

$$T(g) = \begin{pmatrix} e^{i\phi} & \zeta \\ 0 & 1 \end{pmatrix}$$

of $E_+(2)$ is reducible but not decomposable.

We can obtain from each (continuous) linear representation T of a Lie group G a linear representation of its Lie algebra by differentiating the operators $T(g)$ at the point $g = e$.

A representation T in Hilbert space $\mathcal{H}$ is called *unitary* if all the operators $T(g)$ are unitary.

Exercise D.13. Prove that an invariant subspace of a unitary representation of a group has an invariant (orthogonal) complement.

Theorem D.5 (Schur's lemma). (a) *Suppose that the representation T of the group G in the (complex) Hilbert space $\mathcal{H}$ is self-adjoint in the sense that there exists for each $g \in G$ an element $g' \in G$ such that $T(g)^* = T(g')$ (in the case of a unitary representation, $g' = g^{-1}$). Then this representation is irreducible if and only if any bounded linear operator B commuting with all the operators of the representation is a multiple of the identity operator.*

(b) *Let $T(g)$ and $S(g)$ be two self-adjoint irreducible representations of G in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then if A is a closed (linear) operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ such that $AT(g) = S(g)A$, then either $A = 0$, or there exists an inverse operator A^{-1} and the representations T and S are equivalent.* (This infinite-dimensional generalization of the well-known Schur's lemma is easily derived from results of the monograph [N2]; the first part of the theorem is proved in §17.6, and the second part is contained in Corollary 1 of §21.2.)

We note that the condition that the Hilbert space of the representation be complex is essential. The above version of Schur's lemma is invalid for real representations. We can see this from the example of the two-dimensional representation of $O_+(2)$ by the matrices

$$T(g) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

This representation is irreducible over $\mathbf{R}$ but at the same time, each of the matrices $T(g)$ commutes with all the rest, even though it is not a multiple of the identity matrix (when $\phi \neq n\pi$, $n \in \mathbf{Z}$). This representation is soluble over $\mathbf{C}$: there exists a non-singular matrix V such that

$$VT(g)V^{-1} = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix};$$

for V we can we can set $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$.

It should be noted that in additon to (linear) representations of groups, one can introduce the notion of a (linear) *Lie algebra representation* of the Lie algebra $\mathfrak{g}$, which means a homomorphism from $\mathfrak{g}$ to a Lie algebra of linear operators acting in some linear space $\mathcal{X}$.

D.7. ADJOINT AND CO-ADJOINT REPRESENTATIONS. KILLING FORMS

Let G be a Lie group and $A(g)$ the (inner) automorphism

$$A(g) : h \to ghg^{-1}, \; h, g \in G. \tag{D.26}$$

The point e is a fixed point for all such automorphisms $A(g)$. The derived map $A(g)_* : \mathfrak{g} \to \mathfrak{g}$, ordinarily denoted by Ad_g, is a homomorphism from G into the group $\mathrm{Aut}\,\mathfrak{g}$ of automorphisms of $\mathfrak{g}$. This homomorphism is called the *adjoint representation* of G.

If G is a group of matrices and $\mathfrak{g}$ is a matrix realization of its Lie algebra (see, for example, Exercises D.8, D.9), the representation Ad_g is defined by a formula similar to (D.26):

$$\mathrm{Ad}_g x = gxg^{-1}, \; g \in G, \; x \in \mathfrak{g}. \tag{D.27}$$

The derived map at the point $y = e$, denoted by ad_y is a Lie algebra derivation:

$$\mathrm{ad}_y\, x = [y, x], \; x, y \in \mathfrak{g}. \tag{D.28}$$

The correspondence $y \to \mathrm{ad}_y$ is called the *adjoint representation of the Lie algebra* $\mathfrak{g}$.

Let $\mathfrak{g}'$ be the (dual) space of all (real) linear functionals on $\mathfrak{g}$. Let (F, x) be the value of the linear functional F at the element $x \in \mathfrak{g}$. The representation K of the group G in $\mathfrak{g}'$ defined by the formula

$$(K_g F, x) = (F, \mathrm{Ad}_{g^{-1}}\, x), \tag{D.29}$$

is called the *co-adjoint representation of the Lie group* G.

If $\dim G (= \dim \mathfrak{g}) = n$, then the adjoint representation is an n-dimensional matrix representation. In particular, the operators ad_x are (real) $n \times n$-matrices. A basis of infinitesimal generators I_i of this representation is defined in terms of the structure constants c_{ij}^k of $\mathfrak{g}$ according to the formula

$$(I_i)_j^k = c_{ij}^k. \tag{D.30}$$

Exercise D.14. Prove that the matrices (D.30) satisfy the commutation relations (D.4). [Hint: Use the Jacobi identity.]

We can define the following (real) symmetric bilinear form on $\mathfrak{g}$ in terms of the matrices of the adjoint representation:

$$(x, y) = \mathrm{tr}(\mathrm{ad}_x\, \mathrm{ad}_y); \tag{D.31}$$

this is called the *Killing form*.

If $x = x^i I_i, \; y = y^i I_i$, where the I_i are the generators (D.30) of the adjoint representation, then

$$(x, y) = g_{jk} x^j x^k, \quad \text{where } g_{jk} = \mathrm{tr}\, I_j I_k = c_{ji}^s c_{ks}^i. \tag{D.32}$$

As an example, we consider the Lie algebra of the group $O_+(3)$ of three-dimensional rotations. It can be identified with the algebra of vectors in three-dimensional Euclidean space in which the "commutator" is defined by the vector product

$$[\mathbf{x}, \mathbf{y}] = \mathbf{x} \wedge \mathbf{y}; \tag{D.33}$$

or in coordinates (where $\mathbf{x} = x^i e_i, \; \mathbf{y} = y^i e_i$),

$$[\mathbf{x}, \; \mathbf{y}]^i = \epsilon_{ijk} x^i y^k, \tag{D.34}$$

where ϵ_{ijk} is the completely antisymmetric unit tensor ($\epsilon_{123} = 1$).

Exercise D.15. Verify that the metric tensor g_{jk} of (D.33) in this case is given by $g_{jk} = -2\delta_{jk}$.

Exercise D.16. Prove that the Killing form has the properties: (a) the operator ad_x is skew symmetric:

$$(\mathrm{ad}_x\, y, z) + (y, \mathrm{ad}_x\, z) = ([x, y], z) + (y, [x, z]) = 0; \tag{D.35}$$

(b) If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, then $(x, y)_{\mathfrak{h}} = (x, y)_{\mathfrak{g}}$ for all $x, y \in \mathfrak{h}$.

The Killing form enables us to characterize a Lie algebra.

Proposition D.6. (a) *The Killing form is invariant with respect to all Lie algebra automorphisms.*

(b) *The Killing form of a nilpotent Lie algebra X is identically zero.*

(c) *A Lie algebra X is soluble if and only if $(X, X^1) = 0$, where $X^1 = X_2 = [X, X]$* (see (D.6)) *is the commutator of the Lie algebra X.*

(d) *A Lie algebra X is semisimple if and only if its Killing form is non-degenerate (that is, if $(x, y) = 0$ for all $x \in X$ implies that $y = 0$).*

(e) *The Killing form of the Lie algebra of a compact semisimple Lie group is negative-definite.*

(See [Z2],§§87,90 and [K6]§6.2, Problems 12–14 for the proofs of these statements.)

Exercise D.17. Prove that for a semisimple Lie algebra X the co-adjoint representation is equivalent to the adjoint one. [Hint: Use Proposition D.3 (a, d) and the fact that $\dim X' = \dim X$.]

As an example of a Lie algebra for which the co-adjoint representation is different from the adjoint one, we consider the Lie algebra $\mathfrak{e}(2)$ of the group of motions $E_+(2)$ of the Euclidean plane (see Exercise D.12).

Each element g of $E_+(2)$ is defined by three parameters: the angle of rotation ϕ and the translation vector $\mathbf{a} \equiv (a_1, a_2)$. We consider the realization of this group by the 3×3-matrices

$$T(g) = \begin{pmatrix} \cos\phi & -\sin\phi & a_1 \\ \sin\phi & \cos\phi & a_2 \\ 0 & 0 & 1 \end{pmatrix}, \; T(g^{-1}) = \begin{pmatrix} \cos\phi & \sin\phi & -(\cos\phi \cdot a_1 + \sin\phi \cdot a_2) \\ -\sin\phi & \cos\phi & \sin\phi \cdot a_1 - \cos\phi \cdot a_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{D.36}$$

The general form of an element of the Lie algebra representation T is given by the matrix

$$x = \begin{pmatrix} 0 & -x_3 & x_1 \\ x_3 & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.37}$$

(From now on we shall identify the set of matrices of the form (D.37) with $\mathfrak{e}(2)$.) The components x_i of the vector x are transformed according to the adjoint representation Ad_g of $E_+(2)$ by the formula

$$\mathrm{Ad}_g x = T(g) x T(g)^{-1} = \begin{pmatrix} 0 & -x_3 & a_2 x_3 + \cos\phi \cdot x_1 - \sin\phi \cdot x_2 \\ x_3 & 0 & -a_1 x_3 + \sin\phi \cdot x_1 + \cos\phi \cdot x_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{D.38}$$

The space $\mathfrak{e}'(2)$ of the co-adjoint representation can be characterized as the set $\mathrm{Mat}(3, R)$ of all real 3×3-matrices F factored by the (six-dimensional) subspace $\mathfrak{e}(2)^\circ$ of matrices A satisfying the orthogonality condition

$$\mathrm{tr}\, Ax = 0 \quad \text{for all } x \in \mathfrak{e}(2).$$

Exercise D.18. Show that the space $\mathfrak{e}'(2)$ can be identified with the set of matrices of the form

$$F = \begin{pmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ t_1 & t_2 & 0 \end{pmatrix}, \tag{D.39}$$

while

$$\mathfrak{e}(2)^\circ = \left\{ \begin{pmatrix} s_{11} & s_{12} & s_1 \\ s_{12} & s_{22} & s_2 \\ 0 & 0 & s_3 \end{pmatrix} : s_i \text{ and } s_{ab} \in \mathbf{R} \right\}. \tag{D.40}$$

Let Π be the projection operator onto $\mathfrak{e}'(2)$ (in $\mathrm{Mat}(3,R)$) parallel to the subspace $\mathfrak{e}(2)^{\circ}$. Then the co-adjoint representation K is defined by the formula

$$K(g)F = \Pi(T(g)FT(g)^{-1}). \tag{D.41}$$

We leave the reader to verify that the parameters r, t_1 and t_2 in (D.39) are transformed under the action of $K(g)$ according to the law

$$\begin{aligned} r \to r' &= r + \frac{1}{2}(t_1 a_2^{-\phi} - t_2 a_1^{-\phi}) \\ (a_1^{-\phi} \equiv \cos\phi \cdot a_1 + \sin\phi \cdot a_2,\ & a_2^{-\phi} \equiv -\sin\phi \cdot a_1 + \cos\phi \cdot a_2), \\ t_1 \to t_1' &= \cos\phi \cdot t_1 - \sin\phi \cdot t_2, \\ t_2 \to t_2' &= \sin\phi \cdot t_1 + \cos\phi \cdot t_2 \quad (\text{or } \mathbf{t}' = \mathbf{t}^{\phi}). \end{aligned} \tag{D.42}$$

A vector with components I_j (where the I_j are basic infinitesimal generators of the Lie algebra $\mathfrak{g}$) transforms according to the K-representation of G.

Exercise D.19. Verify that the infinitesimal operators

$$R = \left.\frac{\partial T(g)}{\partial \phi}\right|_{g=e},\ T_i = \left.\frac{\partial T(g)}{\partial a_i}\right|_{g=e},\ i = 1, 2, \tag{D.43}$$

of the representation (D.36) of the group $E_+(2)$ are transformed under the inner automorphism (D.42) according to the formulae

$$\begin{aligned} &R \to R' = R + \frac{1}{2}\mathbf{T} \wedge \mathbf{a}^{-\phi}\ (\mathbf{T} \wedge \mathbf{b} \equiv T_1 b_2 - T_2 b_1), \\ &\mathbf{T} \to \mathbf{T}' = \mathbf{T}^{\phi}. \end{aligned}$$

The *universal enveloping algebra* $\mathfrak{A} = \mathfrak{A}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ can be defined as the quotient algebra $\mathfrak{I}/J$ of the (associative) algebra $\mathfrak{I}$ of all possible (ordered) tensor products of elements $\mathfrak{g}$ by the two-sided ideal J generated by the "tensors" of the form $x \otimes y - y \otimes x - [x, y]$. $\mathfrak{A}$ is an associative algebra with the natural embedding $i : \mathfrak{g} \to \mathfrak{A}$, which has the property

$$i([x, y]) = i(x)i(y) - i(y)i(x).$$

The automorphisms of a Lie algebra $\mathfrak{g}$ induce automorphisms of the enveloping algebra $\mathfrak{A}$. The elements of the centre $\mathfrak{Z}$ of $\mathfrak{A}$ (commuting with all the elements of this algebra) are invariants of the representations of $\mathfrak{g}$; they are called the *Casimir elements* of $\mathfrak{g}$.

Exercise D.20. Prove that the Lie algebra $\mathfrak{e}(2)$ has just one algebraically independent Casimir operator $\mathbf{T}^2 \equiv T_1^2 + T_2^2$.

In the case of a (semi)simple Lie algebra, there exists a general algorithm for constructing a quadratic Casimir element.

Exercise D.21. (a) Show that in the case of a (semi)simple group, the matrix (g_{ik}) defined by (D.32) is invertible. [Hint: Use Proposition D.6(d).]

(b) Let g^{ik} be the inverse tensor to g_{ik} (so that $g^{ik}g_{kj} = \delta^i_j$). Prove that the operator

$$C_2 = g^{jk} I_j I_k \tag{D.44}$$

is a Casimir element of the semisimple Lie algebra $\mathfrak{g}$.

CHAPTER 4

The Jost-Lehmann-Dyson Representation

4.1. Relation between the JLD Representation and the Wave Equation

A. Preliminary remarks

An examination of the analytic properties of the Green's functions in quantum theory leads to the following typical problem in the theory of functions of several complex variables.

Suppose that we are given two holomorphic functions $h_+(k)$ and $h_-(k)$ in the tubes $\mathbf{R}_n + iK$ and $\mathbf{R}_n - iK$ in $\mathbf{C}_n$ respectively, where K is a (non-empty) closed convex pointed cone in $\mathbf{R}_n$. We suppose for definiteness that $h_+(k)$ and $h_-(k)$ are the Laplace transforms of the generalized functions $f_+(x)$ and $f_-(x)$ in $\mathcal{S}'(\mathbf{R}^n)$ with supports in the dual cones K^* and $-K^*$ of K and $-K$ respectively (see [B.34]). According to Theorem B.7, the functions $h_\pm(k)$ have the generalized boundary values in $\mathcal{S}'(\mathbf{R}_n)$:

$$h_\pm(p) = \lim_{q\to 0, q\in\pm K} h(p+iq). \tag{4.1}$$

The case when these boundary values $h_+(p)$ and $h_-(p)$ coincide is trivial: the function $h_+(k) = h_-(k)$ is then a polynomial in k. (In fact, it follows from the equality $h_+(p) = h_-(p)$ that $f_+(x) = f_-(x)$, so that $f_+(x) = f_-(x)$ is a generalized function with support at the origin, and our assertion follows from this.) Of greater interest is the case when the generalized boundary values (4.1) coincide in some domain $\mathcal{O} \subset \mathbf{R}_n$. The functions $h_+(k)$ and $h_-(k)$ then have a common analytic continuation $h(k)$ in some complex neighbourhood $\mathcal{N}$ of $\mathcal{O}$ and we can talk about the single analytic function $h(k)$ in the domain $(\mathbf{R}_n + iK) \cup (\mathbf{R}_n - iK) \cup \mathcal{N}$. (This statement is a particular case of the "edge of the wedge theorem" in §5.1.D.)

It turns out that the functions $h_\pm(k)$ (and hence $h(k)$) are "in essence" defined by the difference

$$(\operatorname{disc} h)(p) = h_+(p) - h_-(p), \tag{4.2}$$

which is naturally called the jump of the function $h(k)$ in going from $\mathbf{R}_n + iK$ to $\mathbf{R}_n - iK$. More precisely, if $h'_\pm(k)$ is another pair of similar functions with common analytic continuation h', where $\operatorname{disc} h' = \operatorname{disc} h$, then $h' - h$ is a polynomial. (Here we apply the above argument.) There arises the natural problem: to find an effective expression (of integral representation type) defining the function $h(k)$ to within a polynomial, in terms of its jumps $\operatorname{disc} h$ and explicitly taking into account the above properties of analyticity and boundedness of type (B.35) on the growth of $h(k)$ in $(\mathbf{R}_n + iK) \cup (\mathbf{R}_n - iK)$. In view of the fact that $\operatorname{disc} h$ "almost" defines $h(k)$, it is natural to concentrate on the properties of the jump with regard to this problem. If we set

$$g(p) = (\operatorname{disc} h)(p), \tag{4.3}$$

then we are led to a class of generalized functions in $\mathcal{S}'(\mathbf{R}_n)$ with the properties: firstly, the Fourier transform

$$\tilde{g}(x) = \int e^{-ipx} g(p) d_n p \tag{4.4}$$

has support in the cone $\mathcal{K} = K^* \cup (-K^*)$:

$$\operatorname{supp} \tilde{g}(x) \subset \mathcal{K}; \tag{4.5a}$$

secondly,

$$\operatorname{supp} g(p) \subset \mathcal{R}, \quad \text{where } \mathcal{R} = \mathbf{R}^n \backslash \mathcal{O}. \tag{4.5b}$$

We denote the set of such functions $g(p)$ by $\sigma(\mathcal{K}; \mathcal{R})$.

Thus there arises the problem in the theory of generalized functions: to find an effective expression (of integral representation type) for generalized functions of class $\sigma(\mathcal{K}; \mathcal{R})$ that explicitly reflects the properties (4.5). After this we can return to the "problem of the adjoint" (or the determination of a holomorphic function $h(k)$ in terms of its jump $g(p)$).

This problem has been studied in detail by Jost, Lehmann and Dyson for the special case when $\mathbf{R}_n$ is taken to be Minkowski space $\mathbf{M}$ and the role of K is assumed by the upper light cone V^+, so that the closed light cone

$$\overline{V} = \{p \in \mathbf{M} : \ p^2 \geq 0\} \tag{4.6}$$

now serves as the cone $\mathcal{K}$. Certain extra conditions (which we shall discuss below in §4.3.B) are also imposed on $\mathcal{O}$. Taking into account the fact that the properties of the supports of the generalized functions $f_\pm(x)$ now assume the form $f_\pm(x) \subset \overline{V}^\pm$, we call f_+ and f_- *generalized functions of retarded* and *advanced type* respectively.

B. Outline of the derivation

Alongside $\sigma(\overline{V}; \mathcal{R})$ we introduce the wider class $\sigma(\overline{V})$ of all generalized functions satisfying (4.5a) with $\mathcal{K} = \overline{V}$. The essence of the method proposed by Dyson consists in the one-to-one correspondence between the generalized functions $g(p) \in \sigma(\overline{V})$ and a certain class of solutions $G(p, \eta) \in \mathcal{S}'(\mathbf{R}_6)$ of the six-dimensional wave equation

$$\square_{(6)} G(p, \eta) \equiv \left[\left(\frac{\partial}{\partial p^0} \right)^2 - \sum_{j=1}^{3} \left(\frac{\partial}{\partial p^j} \right)^2 - \sum_{\alpha=1}^{2} \left(\frac{\partial}{\partial \eta^\alpha} \right)^2 \right] G(p, \eta) = 0. \tag{4.7}$$

(Here and later in §4.3, we denote arbitrary points of $\mathbf{M}$ by x and p; $\xi \equiv (\xi^1, \xi^2)$ and $\eta \equiv (\eta^1, \eta^2)$ are arbitrary points of $\mathbf{R}^2$, where $\xi\eta \equiv \xi^1\eta^1 + \xi^2\eta^2$).

We give a heuristic argument which leads to this isomorphism. Consider the identity

$$\theta(x^2) = \frac{1}{\pi} \int \delta(x^2 - |\xi|^2) d^2\xi, \tag{4.8}$$

where $x \in \mathbf{M}$, $\xi \in \mathbf{R}^2$. Assuming that $g \in \sigma(\overline{V})$, we multiply both sides of (4.8) by $\tilde{g}(x)$. Since $\operatorname{supp} \tilde{g} \in \overline{V}$, we obtain the following representation:

$$\tilde{g}(x) = \int \tilde{G}(x, \xi) d^2\xi, \tag{4.9}$$

where we have introduced the generalized function $\tilde{G}(x, \xi) \in \mathcal{S}'(\mathbf{R}^6)$:

$$\tilde{G}(x, \xi) = \frac{1}{\pi} \tilde{g}(x) \delta(x^2 - |\xi|^2). \tag{4.10}$$

It turns out that (4.9) and (4.10) establish a one-to-one correspondence between $\sigma(\overline{V})$ and the generalized functions in $\mathcal{S}'(\mathbf{R}^6)$ with the properties:

$$(x^2 - |\xi|^2)\tilde{G}(x, \xi) = 0 \tag{4.11a}$$

$$\tilde{G}(x, R\xi) = \tilde{G}(x, \xi) \quad \text{for all } R \in O(2). \tag{4.11b}$$

We reformulate the construction in terms of Fourier transforms. We introduce the generalized function

$$G(p, \eta) = \int \tilde{G}(x, \xi) e^{i(px - \eta\xi)} d^4x \, d^2\xi, \tag{4.12}$$

in terms of which, (4.11) takes on the form

$$\Box_{(6)} G(p, \eta) = 0, \tag{4.13a}$$

$$G(p, R\eta) = G(p, \eta) \quad \text{for all } R \in O(2). \tag{4.13b}$$

Formula (4.9) now leads to the relation between $g(p)$ and $G(p, \eta)$ mentioned at the beginning of this subsection:

$$g(p) = G(p, \eta)|_{\eta=0}. \tag{4.14}$$

This formula can serve as the starting point for the integral representations of $g(p)$ which take (4.5a) into account in explicit fashion. It is well known that each solution G of the wave equation (4.7) can be recovered from the "values" of G and its normal derivative on an arbitrary fixed spacelike surface Σ in $\mathbf{R}^6$ by means of a formula of type

$$G(p, \eta) = \int D_{(6)}(p - p', \eta - \eta') H_\Sigma(p', \eta') d^4p' \, d^2\eta', \tag{4.15}$$

where $D_{(6)}(p, \eta)$ is the so-called fundamental solution of the Cauchy problem for the wave equation, and the generalized function H_Σ constructed from G has support on the surface Σ. In turn, $D_{(6)}(p - p', \eta)$ is a superposition (in the generalized sense) of generalized functions $K(p; p', \lambda)$ in the variable $p \in \mathbf{M}$ of the form

$$K(p; p', \lambda) = \epsilon(p^0 - p'^0)\delta((p - p')^2 - \lambda), \tag{4.16}$$

where $p' \in \mathbf{M}$ and $\lambda \geq 0$ are parameters. This reasoning together with (4.14) and (4.15) enables us to conclude that $g(p)$ can be represented as a superposition of generalized functions $K(p;p',\lambda)$ in the sense of the weak integral representation; it is called the ***Jost-Lehmann-Dyson*** (JLD) ***representation***:

$$g(p)=\int K(p;p',\lambda)\Psi(p',\lambda)d_4p'd\lambda. \tag{4.17}$$

The generalized function $\Psi(p',\lambda)$ occurring here is called the ***Jost-Lehmann-Dyson*** (JLD) ***spectral function.**** The Fourier transform of $K(p;p',\lambda)$ is $e^{ip'x}D_{\sqrt{\lambda}}(x)$ (to within a factor depending on λ), where $D_m(x)$ is the so-called Pauli-Jordan commutation function of the scalar field of mass m, which vanishes for $x^2<0$ (as does every odd $L_+^\uparrow$-invariant distribution in $\mathcal{S}'(\mathbf{M})$). Therefore the Fourier transform of $K(p;p',\lambda)$ also vanishes for $x^2<0$. Thus the JLD representation explicitly takes property (4.5a) into account.

We now turn our attention to the fact that the JLD spectral function $\Psi(p',\lambda)$ is not uniquely defined by the distribution $g(p)$ (which must be assumed to be the case, since Ψ depends on five real variables rather than four, as g does). Thus, if in (4.15) we choose different spacelike surfaces Σ we obtain different JLD spectral functions. We can try to use this freedom in order to obtain a representation (4.17) that also explicitly satisfies (4.5b) for $g\in\sigma(\overline{V};\mathcal{R})$. By this we mean the possibility of choosing $\Psi(p',\lambda)$ so that the supports of the generalized functions $K(p;p',\lambda)$ do not intersect with $\mathcal{O}$ when p',λ range over the set of integration in (4.17), that is,

$$\operatorname{supp} K(\cdot;p',\lambda)\cap\mathcal{O}=\emptyset \quad \text{for all } (p',\lambda)\in\operatorname{supp}\Psi. \tag{4.18}$$

It is not possible to achieve this for arbitrary domains $\mathcal{O}$. Dyson has indicated a fairly broad class of domains $\mathcal{O}$ for which there is a representation (4.17) for all distributions in $\sigma(\overline{V};\mathcal{O})$ that explicitly take into account (4.5) (which, in view of what we have said, reduces to (4.18)). This class of domains $\mathcal{O}$ is sufficient for applications to field theory. (Appropriate examples are given in §4.3.C.)

C. Departure into six-dimensional space

In the preceding subsection, we have given a heuristic argument for obtaining the JLD representation. Here we give a proof of the relation, mentioned above, between the given problem and the wave equation in $\mathbf{R}^6$. Since the further derivation requires an intensive use of the properties of generalized solutions of the wave equation, we set aside the next section for this theme. The derivation of the JLD representation will be completed in §4.3.

Our immediate aim is to establish the fact that formula (4.10) defines an isomorphism between the generalized functions $\tilde{g}(x)\in\mathcal{S}'(\mathbf{M})$ with supports in $\overline{V}$ and the generalized functions $\tilde{G}(x,\xi)\in\mathcal{S}'(\mathbf{R}^6)$, satisfying (4.11). We begin by verifying that (4.10) is well defined.

We note that $\overline{V}$ is a canonically closed regular subset of $\mathbf{M}$, therefore, (by Proposition A.1) there is an isomorphism j' from $\mathcal{S}'(\overline{V})$ onto the space of generalized functions in $\mathcal{S}'(\mathbf{M})$ with supports in $\overline{V}$; here the equality $\tilde{g}=j'\gamma$ (where $\gamma\in\mathcal{S}'(\overline{V})$) means that

$$(\tilde{g},u)=(\gamma,u|_{\overline{V}}) \quad \text{for all } u\in\mathcal{S}(\mathbf{M}). \tag{4.19}$$

* The fact that the JLD spectral function is a generalized function in the Schwartz space $\mathcal{S}'$ is far from obvious; we give a proof of this fact (under a certain additional assumption with regard to the domain $\mathcal{O}$) below in §4.3.A.

In what follows we shall always suppose that the generalized functions $\tilde{g}$ and γ are related in this way.

We can now assign a meaning to (4.10) by setting

$$(\tilde{G}(x,\xi), v(x,\xi)) = \big(\gamma(x), \frac{1}{\pi}\int \delta(x^2 - |\xi|^2)v(x,\xi)d^2\xi|_{\overline{V}}\big) \tag{4.20}$$

for all $v \in \mathcal{S}(\mathbf{R}^6)$ or, equivalently,

$$(\tilde{G}(x,\xi), v(x,\xi)) = (\gamma(x), w(x,x^2)|_{\overline{V}}), \tag{4.21}$$

where

$$w(x,t) = \frac{1}{\pi}\int \delta(t - |\xi|^2)v(x,\xi)d^2\xi \tag{4.22}$$

is a function in $\mathcal{S}(\mathbf{M} \times \overline{\mathbf{R}}_+)$, the map $v \to w$ being a continuous linear operator. (With regard to this, see Propositions 3.10 and 3.11 in §3.4.A, also the first example after Proposition 3.11.) Hence it is clear that the distribution $\tilde{G}(x,\xi)$ is well defined by (4.20) and satisfies (4.11).

We have constructed $\tilde{G}$ in terms of $\tilde{g}$. Conversely, $\tilde{g}$ can be recovered from $\tilde{G}$ by means of (4.9). To this end, we note that (4.11a) implies that $\tilde{G}(x,\xi)$ has support on the set $x^2 - |\xi|^2 = 0$, whence (by Exercise 2.33) it follows that $\tilde{G}(x,\xi)$ is a convolute with respect to ξ. Hence the partial integral with respect to ξ on the right hand side of (4.9) is well defined (see §2.5.D). Furthermore, by substituting (4.10) into the right hand side of (4.8), the validity of (4.9) is easily verified.

For the proof of (4.9) we can use the following equality, which is valid thanks to the property of the support of $\tilde{G}(x,\xi)$:

$$\tilde{G}(x,\xi) = \tilde{G}(x,\xi)\rho(|\xi|^2 - x^2), \tag{4.23}$$

where $\rho(t)$ is a function in $\mathcal{D}(\mathbf{R})$ which is equal to unity in a neighbourhood of $t = 0$. The right hand side can now be calculated as follows: for all $u \in \mathcal{S}(\mathbf{M})$

$$\Big(\int \tilde{G}(x,\xi)d^2\xi, u(x)\Big)_x = (\tilde{G}(x,\xi), u(x)\rho(|\xi|^2 - x^2)); \tag{4.24}$$

by (4.20), the last expression reduces to $(\tilde{g}(x), u(x))$.

We now state the result of interest to us.

Lemma 4.1. *Formula* (4.10) *realizes an isomorphism from the space of all generalized functions* $\tilde{g}(x) \in \mathcal{S}'(\mathbf{M})$ *with supports in* $\overline{V}$ *onto the space of all generalized functions* $\tilde{G}(x,\xi) \in \mathcal{S}'(\mathbf{R}^6)$ *satisfying the condition* (4.11), *where the inverse map is realized by* (4.9).

■ The arguments leading up to this lemma provide the main substance of the proof. It only remains to show that each distribution $\tilde{G}(x,\xi) \in \mathcal{S}'(\mathbf{R}^6)$ that satisfies the conditions (4.11) can be represented in the form (4.10) for some $\tilde{g} \in \mathcal{S}'(\mathbf{M})$ with support in $\overline{V}$. To see this, we define $\tilde{g}$ by means of (4.9). (This is meaningful since $\tilde{G}(x,\xi)$ is a convolute with respect to ξ because the support of $h(x,\xi)$ lies on the set $x^2 - |\xi|^2 = 0$.) Since the support of $\tilde{G}(x,\xi)$ lies on the set $\{(x,\xi) : x^2 \geq 0\}$, the support of $\tilde{g}$ also lies in $\overline{V}$. We can now define $\tilde{G}_1(x,\xi) \in \mathcal{S}'(\mathbf{R}^6)$ by the formula of type (4.10):

$$\tilde{G}_1(x,\xi) = \frac{1}{\pi}\delta(x^2 - |\xi|^2)\tilde{g}(x),$$

and it remains to prove that $\tilde{G}_1 = \tilde{G}$. To this end, we note that if $w(x,t)$ is defined (for arbitrary $v \in \mathcal{S}(\mathbf{R}^6)$) by (4.22), then

$$(\tilde{G}_1(x,\xi), v(x,\xi)) = (\gamma(x), w(x,x^2)|_{\overline{V}}).$$

Using (4.24) we can express the right hand side in terms of $\tilde{G}$

$$(\tilde{G}_1(x,\xi), v(x,\xi)) = (\tilde{G}(x,\xi), \rho(|\xi|^2 - x^2)w(x,x^2)). \tag{4.25}$$

It then follows from the $O(2)$-invariance of $\tilde{G}(x,\xi)$ in ξ that:

$$(\tilde{G}(x,\xi), v(x,\xi)) = \Big(\tilde{G}(x,\xi), \int_{O(2)} v(x, R\xi)dR\Big) = (\tilde{G}(x,\xi), w(x,|\xi|^2)),$$

which (using the property of the support of $\tilde{G}$) can be written in the form

$$(\tilde{G}(x,\xi), v(x,\xi)) = (\tilde{G}(x,\xi), \rho(|\xi|^2 - x^2)w(x,|\xi|^2)). \tag{4.26}$$

(Here, as in (4.25), $\rho(t)$ is a function in $\mathcal{D}(\mathbf{R})$ that is equal to unity in a neighbourhood of $t = 0$.) We introduce the function

$$w_1(x,\xi) = \rho(|\xi|^2 - x^2)\frac{w(x,|\xi|^2) - w(x,x^2)}{|\xi|^2 - x^2},$$

which clearly is a member of $\mathcal{S}(\mathbf{R}^6)$. It then follows from (4.25) and (4.26) that

$$(\tilde{G}_1(x,\xi) - \tilde{G}(x,\xi), v(x,\xi)) = ((|\xi|^2 - x^2)\tilde{G}(x,\xi), w_1(x,\xi)).$$

But this is zero in view of the property (4.11a). ∎

We now express the isomorphism, established above, between $\tilde{g}$ and $\tilde{G}$ in terms of their Fourier transforms $g(p)$ and $G(p,\eta)$ (see (4.4) and (4.12)). It is clear that the property $\operatorname{supp}\tilde{g} \subset \overline{V}$ just means that $g(p) \in \sigma(\overline{V})$. It is equally obvious that $\tilde{G}(x,\xi)$ satisfies the conditions (4.11) if and only if $G(p,\eta)$ satisfies conditions (4.13). We can then regard $G(p,\eta)$ as a generalized function with respect to p that is C^∞-dependent on η as a parameter (see Exercise 2.48(b)); consequently there exists the restriction $G(p,\eta)$ to the plane $\eta = 0$, given by

$$G(p,\eta)|_{\eta=0} = \int e^{ipx}\Big(\int \tilde{G}(x,\xi)d^2\xi\Big)d^4x \tag{4.27}$$

(see Exercise 2.48(b), also formula (2.92) in Exercise 2.28). Substituting (4.9) and (4.12) into this we now find that G and g are related by formula (4.14).

The isomorphism in Lemma 4.1 can now be restated as follows.

Proposition 4.2. *Formula* (4.14) *realizes an isomorphism between the space of all generalized functions* $G(p,\eta) \in \mathcal{S}'(\mathbf{R}^6)$ *satisfying conditions* (4.13), *and the space* $\sigma(\overline{V})$ *of all generalized functions* $g(p) \in \mathcal{S}'(\mathbf{M})$ *satisfying the condition* $\operatorname{supp}\tilde{g} \subset \overline{V}$.

4.2. Properties of Solutions of the d'Alembert Equation in $\mathcal{S}'$

A. Notation

In this section we make an excursion into the theory of generalized solutions of the n-dimensional d'Alembert wave equation. We denote the coordinates of an arbitrary point $x \in \mathbf{R}^n$ by $x_0, x_1, \ldots, x_{n-1}$; here, x_0 is called the time coordinate, while the vector $\mathbf{x} \equiv (x_1 \ldots, x_{n-1}) \in \mathbf{R}^{n-1}$ is called the spatial part of the vector x. We denote by xy the pseudo-Euclidean scalar product

$$xy \equiv x_0y_0 - \mathbf{x}\mathbf{y} \equiv x_0y_0 - \sum_{j=1}^{n-1} x_jy_j$$

(in particular, $x^2 \equiv xx$) and $\Box_{(n)}$ is the d'Alembertian:

$$\Box_{(n)} = \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

We introduce the accompanying notation (used in this section without further explanation):

$$V^+_{(n)} = \{x \in \mathbf{R}^n : x_0 > |\mathbf{x}|\},\ V^-_{(n)} = -V^+_{(n)},\ V_{(n)} = \{x \in \mathbf{R}^n : xx > 0\}.$$

These are respectively the *upper light, lower light* and (simply) *light* open *cone* in $\mathbf{R}^n$; we denote their closures by $\overline{V}^+_{(n)}$, $\overline{V}^-_{(n)}$, $\overline{V}_{(n)}$. The set

$$\Gamma = \{x \in \mathbf{R}^n : xx = 0\}$$

is a surface of the light cone. We also introduce the set (where $a, b \in \mathbf{R}^n$ are arbitrary):

$$\mathcal{Q}_{(n)}(a, b) = (a + V^+_{(n)}) \cap (b - V^-_{(n)}),$$

called an (open) diamond with vertices at the points a, b. It is clear that the diamond $\mathcal{Q}_{(n)}(a, b)$ is non-empty if and only if $b - a \in V^+_{(n)}$. For $b - a \in V^+_{(n)}$, an interval of a line in $\mathbf{R}^n$ of the form

$$(a, b) = \{a + bt \in \mathbf{R}^n : 0 < t < 1\}$$

is called a *timelike interval* (with end-points a and b).

B. FUNDAMENTAL SOLUTION OF THE CAUCHY PROBLEM

We now turn to a consideration of generalized functions $F(x) \in \mathcal{S}'(\mathbf{R}^n)$, satisfying the d'Alembert wave equation

$$\Box_{(n)} F(x) = 0. \tag{4.28}$$

Each solution is a generalized function in $\mathbf{x}$ that is C^∞-dependent on x_0 as a parameter (see Exercise 2.49; $F(x)$ can also be regarded as a generalized function in x_0 that is C^∞-dependent on $\mathbf{x}$ as a parameter). Consequently we can associate with the generalized function $F(x)$ the family of generalized functions $\{F_t(\mathbf{x})\}_{t\in R}$ in $\mathcal{S}'(\mathbf{R}^{n-1})$:

$$F_t(\mathbf{x}) = F(x)|_{x_0=t}, \tag{4.29}$$

which are C^∞-dependent on the parameter t and satisfy the following differential equation in $\mathcal{S}'(\mathbf{R}^{n-1})$:

$$\frac{\partial^2}{\partial t^2} F_t(\mathbf{x}) = \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2} F_t(\mathbf{x}). \tag{4.30}$$

One of the natural problems for the wave equation is the Cauchy problem: to determine the solution $F(x) \in \mathcal{S}'(\mathbf{R}^n)$ of equation (4.28) from the initial values $F_t(\mathbf{x})$ and $\frac{\partial}{\partial t} F_t(\mathbf{x})$ (in $\mathcal{S}'(\mathbf{R}^{n-1})$) at some fixed value of t, say, at $t = 0$. We call the pair of generalized functions

$$F_t(\mathbf{x})|_{t=0} = u_0(\mathbf{x}), \quad \frac{\partial}{\partial t} F_t(\mathbf{x})|_{t=0} = u_1(\mathbf{x}) \tag{4.31}$$

the *initial data of the problem* (at time $t = 0$). It is not difficult to see that for any initial data (in $\mathcal{S}'(\mathbf{R}^{n-1})$) the Cauchy problem is uniquely soluble (in $\mathcal{S}'(\mathbf{R}^n)$).

The uniqueness of the solution of the Cauchy problem can be proved, for example, by the following argument. Suppose that the initial data (4.31) is equal to zero. Then (by induction on m) it follows from (4.30) that the derivatives of any order m of $F_t(\mathbf{x})$ with respect to t vanish at $t = 0$. To see that $F_t(\mathbf{x}) = 0$ for all t, it now suffices to note that $(F_t(\mathbf{x}), u(\mathbf{x}))$ is an analytic function in t for all functions u ranging through some dense subset of $\mathcal{S}(\mathbf{R}^{n-1})$, say, $\mathcal{F}[\mathcal{D}(\mathbf{R}_{n-1})]$, the Fourier transform of the set of test functions $\mathcal{D}(\mathbf{R}_{n-1})$. In fact, it follows from the properties of the support of $\tilde{F}(p)$ that for all $\tilde{u} \in \mathcal{D}(\mathbf{R}_{n-1})$, the distribution $\int \tilde{F}(p_0, \mathbf{p})\tilde{u}(-\mathbf{p})d_{n-1}\mathbf{p}$ with respect to p_0 has compact support and hence its Fourier transform, which is equal to $\int F(x_0, \mathbf{x})u(\mathbf{x})d^{n-1}\mathbf{x}$, is an analytic function in x_0, as claimed.

To construct the solution of the Cauchy problem, we introduce the Fourier transform of $F_t(\mathbf{x})$ in the variable $\mathbf{x}$, which we denote by $\underset{\sim}{F}_t(\mathbf{p})$ (so as to distinguish it from $\tilde{F}(p)$):

$$\underset{\sim}{F}_t(\mathbf{p}) = \int e^{-ipx} F_t(\mathbf{x}) d^{n-1}\mathbf{x}. \tag{4.32}$$

The Cauchy problem now takes on the very simple form:

$$\frac{\partial^2}{\partial t^2}\underset{\sim}{F}_t(\mathbf{p}) = -|\mathbf{p}|^2 \underset{\sim}{F}_t(\mathbf{p}),$$

$$\underset{\sim}{F}_t(\mathbf{p})|_{t=0} = \underset{\sim}{u}_0(\mathbf{p}), \quad \frac{\partial}{\partial t}\underset{\sim}{F}_t(\mathbf{p})|_{t=0} = \underset{\sim}{u}_1(\mathbf{p}).$$

It is easy to see that these equations are satisfied by the expression

$$\underset{\sim}{F}_t(\mathbf{p}) = \cos(|\mathbf{p}|t)\underset{\sim}{u}_0(\mathbf{p}) + \frac{\sin(|\mathbf{p}|t)}{|\mathbf{p}|}\underset{\sim}{u}_1(\mathbf{p}), \tag{4.33}$$

which (by uniqueness) is the required solution. It should be added that the functions $\cos(|\mathbf{p}|)t)$ and $\frac{\sin(|\mathbf{p}|t)}{|\mathbf{p}|}$ entering into (4.33) are multiplicators in $\mathcal{O}_{\mathbf{M}}(\mathbf{R}^n)$ (in all the variables t and $\mathbf{p}$), so that the right hand side of (4.33) is well defined (as a family of generalized functions with respect to $\mathbf{p}$, that are C^∞-dependent on the parameter t, also as generalized functions in t and $\mathbf{p}$).

It is clear that in the x-space, (4.33) can be written in terms of $(n-1)$-dimensional convolutions:

$$F_t(\mathbf{x}) = \frac{\partial}{\partial t} D_{(n)}(t, \mathbf{x}) * u_0(\mathbf{x}) + D_{(n)}(t, \mathbf{x}) * u_1(\mathbf{x}). \tag{4.34}$$

Here we have introduced the *fundamental solution of the Cauchy problem* for the wave equation

$$D_{(n)}(x) = \int e^{-ipx} \frac{\sin(|\mathbf{p}|x_0)}{|\mathbf{p}|} d_{n-1}\mathbf{p}, \tag{4.35}$$

or, equivalently,

$$D_{(n)}(x) = \int 2\pi i \epsilon(p_0)\delta(p^2)e^{ipx} d_n p. \tag{4.36}$$

This is the solution of the wave equation $\Box_{(n)} D_{(n)}(x) = 0$ in $\mathcal{S}'(\mathbf{R}^n)$ with the following initial data (which follow directly from (4.35)):

$$D_{(n)}(t, \mathbf{x})|_{t=0} = 0, \ \frac{\partial}{\partial t} D_{(n)}(t, \mathbf{x})|_{t=0} = \delta(\mathbf{x}). \tag{4.37}$$

We shall only need the explicit form of $D_{(n)}(x)$ for even* $n = 2k \geq 4$:

$$D_{(n)}(x) = \frac{1}{2\pi^{k-1}}\epsilon(x_0)\delta^{(k-2)}(x^2) \text{ for even } n = 2k \geq 4. \tag{4.38}$$

As usual, the $\epsilon(t)$ in (4.36) and (4.38) denotes the sign function:

$$\epsilon(t) = \begin{cases} 1 & \text{for } t \geq 0; \\ -1 & \text{for } t < 0. \end{cases} \tag{4.39}$$

We elucidate formula (4.38) and outline its derivation. To give the form of $D_{(n)}(x)$ to within a factor, it is not necessary to carry out the Fourier transformation in (4.36); it is sufficient to give a general argument. It is clear from (4.36) that $D_{(n)}(x)$ is an odd generalized function in $\mathcal{S}'(\mathbf{R}^n)$ (with respect to the changes $x_0 \to -x_0$ and $x \to -x$), which is invariant with respect to the connected component G of the identity of the group of all linear pseudo-Euclidean transformations of $\mathbf{R}^n$. (As with the Lorentz transformations in Minkowski space $\mathbf{M} = \mathbf{R}^4$, such transformations are characterized by the condition that the pseudo-Euclidean scalar product xy in $\mathbf{R}^n$ is preserved.) The dimension n is unimportant for the derivation of the general form of these odd G-invariant generalized functions. We therefore have the general representation for the odd G-invariant generalized functions $f(x) \in \mathcal{S}'(\mathbf{R}^n)$, as in the case $n = 4$ (see (3.105)):

$$f(x) = \epsilon(x_0)h(x^2), \quad \text{where } h \in \mathcal{S}'(\overline{\mathbf{R}}_+). \tag{4.40}$$

We interpret this formula in the following way. Since $f(x)$ is an odd function of x_0, we have for any function $u(x) \in \mathcal{S}(\mathbf{R}^n)$:

$$\int f(x)u(x)dx = \tfrac{1}{2}\int f(x)[u(x_0, \mathbf{x}) - u(-x_0, \mathbf{x})]dx. \tag{4.41}$$

In turn, we can introduce $v(\sigma, x) \in \mathcal{S}(\overline{\mathbf{R}}_+ \times \mathbf{R}^{n-1})$ by means of the equality

$$\tfrac{1}{2}(u(x_0, \mathbf{x}) - u(-x_0, \mathbf{x})) = x_0 v(x_0^2, \mathbf{x}), \tag{4.42}$$

so that formal manipulations with (4.40) yield:

$$\int f(x)u(x)dx = \int h(x^2)|x_0|v(x_0^2, \mathbf{x})dx = \int h(\sigma - |\mathbf{x}|^2)v(\sigma, \mathbf{x})d\sigma\, d\mathbf{x}.$$

(Here we have replaced the variable x_0 by $x_0^2 = \sigma$ in the formal integral and have taken into account the fact that σ runs twice over $\overline{\mathbf{R}}_+$ as x_0 runs through $\mathbf{R}$.)

We arrive at a formula which can, in fact, be taken to be the correct definition of the representation (4.40):

$$(f(x), u(x)) = \left(h(\tau), \int v(\tau + |\mathbf{x}|^2, \mathbf{x})d\mathbf{x}\right)_\tau; \tag{4.43}$$

here v is taken to be related to u by (4.42) as before.

The subsequent argument leading to (4.38) is based on the property of homogeneity. It is clear that the generalized function $\epsilon(p_0)\delta(p^2)$, in (4.36) is homogeneous of degree -2, from which it follows that $D_{(n)}(x)$ is homogeneous of degree $-n+2$, that is,

$$D_{(n)}(\lambda x) = \lambda^{-n+2}D_{(n)}(x) \quad \text{for all } \lambda > 0.$$

Hence the generalized function $h(\tau) \in \mathcal{S}'(\overline{\mathbf{R}}_+)$ occurring in the representation (4.40) for $D_{(n)}(x)$ must be homogeneous of degree $\lambda^{-n/2+1}$. For even $n = 2k \geq 4$, each distribution $h(\tau)$ in $\mathcal{S}'(\overline{\mathbf{R}}_+)$ that is

* For the general case, see the book [V4],§30 (the case where n is odd reduces to the even case, see Exercise 4.2 below).

homogeneous of degree λ^{-k+1} has the form $c\delta^{(k-2)}(\tau)$ (see Exercise C.8). Thus we have found the form of $D_{(n)}(x)$ to within a factor:

$$D_{(n)}(x) = c\epsilon(x_0)\delta^{(k-2)}(x^2). \tag{4.44}$$

Finally (see Exercise 4.1 below), we can find the constant c by using the fact that (by (4.35)), $\underset{\sim}{D}_{(n)}(t,\mathbf{p}) = \frac{\sin(|\mathbf{p}|t)}{|\mathbf{p}|}$, therefore

$$\int D_{(n)}(t,\mathbf{x})d^{n-1}\mathbf{x} \equiv \underset{\sim}{D}_n(t,\mathbf{p})|_{\mathbf{p}=0} = t. \tag{4.45}$$

Exercise 4.1. Use (4.45) to verify that the constant c in (4.44) is equal to $1/2\pi^{k-1}$. [Hint: For arbitrary $u(t) \in \mathcal{S}(\mathbf{R})$ set $v(t^2) = \frac{1}{t}(u(t)-u(-t))$. Then $(t,u(t)) = 2\int_0^\infty v(t^2)t^2dt$. On the other hand, it follows from (4.40), (4.43) that

$$\left(\int \epsilon(x_0)\delta^{(k-2)}(x^2)d^{n-1}\mathbf{x}, u(x_0)\right) = s_{n-1}\int_0^\infty \left(-\frac{d}{dr^2}\right)^{k-2} v(r^2)r^{2k-2}dr,$$

where s_{n-1} is the area of the sphere of radius 1 in $\mathbf{R}^{n-1}$.]

Exercise 4.2. Prove that

$$D_{(n)}(x) = \int D_{(n+1)}(x,x_n)dx_n. \tag{4.46}$$

[Hint: This equality implies that $\underset{\sim}{D}_{(n)}(t,\mathbf{p}) = \int \underset{\sim}{D}_{(n+1)}(t,\mathbf{p},p_n)|_{p_n=0}$.]

The solution (4.34) of the Cauchy problem can be written in the form of an n-dimensional convolution:

$$F(x) = D_{(n)}(x) * \left\{\frac{\partial}{\partial x_0}(\delta(x_0)u_0(\mathbf{x})) + \delta(x_0)u_1(\mathbf{x})\right\}. \tag{4.47}$$

This convolution is "canonically" defined as a result of the properties of the supports of the generalized functions occurring here (see the existence criteria for the convolution in Exercise 2.41). In fact the support of $D_{(n)}$ is contained in the cone $\overline{V}_{(n)}$ (for even $n \geq 4$ this is clear from (4.38), while for the remaining n, it follows from (4.46)), while the support of the other "convolution factor" in (4.47) is concentrated on the plane $x_0 = 0$.

We state the result that we have obtained.

Proposition 4.3. *Each solution of the wave equation* (4.28) *in* $\mathcal{S}'(\mathbf{R}^n)$ *is a generalized function with respect to* $\mathbf{x}$ *that is* C^∞*-dependent on* x_0. *For any initial data in* $\mathcal{S}'(\mathbf{R}^{n-1})$ *(say, at time* $x_0 = t$*) the Cauchy problem is uniquely soluble in* $\mathcal{S}'(\mathbf{R}^n)$ *and its solution has the form*

$$F(x) = D_{(n)}(x) * \left\{\delta(x_0-t)\frac{\partial}{\partial\tau}F(\tau,\mathbf{x})|_{\tau=t} + \frac{\partial}{\partial x_0}\delta(x_0-t)F(\tau,\mathbf{x})|_{\tau=t}\right\}. \tag{4.48}$$

C. Cauchy problem on a spacelike hypersurface; Huygens' principle

Formula (4.48) naturally carries over to the case when the initial data are chosen on an arbitrary strictly spacelike hypersurface, rather than the surface x_0 =const. We say that the hypersurface

$$\Sigma = \{x \in \mathbf{R}^n : \ x_0 = \kappa(\mathbf{x})\} \tag{4.49}$$

(where $\kappa(\mathbf{x})$ is a smooth function in $\mathbf{R}^{n-1}$) is *spacelike* if $|\operatorname{grad}\kappa(\mathbf{x})| < 1$ for all $\mathbf{x} \in \mathbf{R}^{n-1}$ and *strictly spacelike* if $\kappa(\mathbf{x})$ is a multiplicator in $\mathcal{S}(\mathbf{R}^{n-1})$ and $|\operatorname{grad}\kappa(\mathbf{x})| < \rho$ for some $\rho < 1$ and all $\mathbf{x} \in \mathbf{R}^{n-1}$. For a spacelike hypersurface Σ we define the generalized functions in $\mathcal{S}'(\mathbf{R}^n)$:

$$\vartheta_\Sigma(x) = \theta(x_0 - \kappa(\mathbf{x})),\ Y_\Sigma^\mu = \eta_\mu \frac{\partial}{\partial x_\mu}\vartheta_\Sigma(x) \quad \text{for } \mu = 0, 1, \ldots, n-1, \tag{4.50}$$

where $\eta_0 = 1$ and $\eta_j = -1$ for $j = 1, \ldots, n-1$. It is clear that ϑ_Σ is locally integrable and that $\operatorname{supp} Y_\Sigma^\mu \subset \Sigma$.

It turns out that the generalization of (4.48) to the case of an arbitrary strictly spacelike hypersurface Σ has the form:

$$F(x) = D_{(n)}(x) * \left\{ \sum_{\mu=0}^{n-1} \left[Y_\Sigma^\mu(x) \frac{\partial F(x)}{\partial x_\mu} + \frac{\partial}{\partial x_\mu}(Y_\Sigma^\mu(x)F(x)) \right] \right\}, \tag{4.51}$$

or, equivalently,

$$F(x) = D_{(n)}(x) * \{\Box_{(n)} [\vartheta_\Sigma(x)F(x)]\}. \tag{4.52}$$

For the derivation of these formulae we prove first that the right hand sides are well defined. We define the products of the generalized functions in the square brackets in these formulae by using the notion of product of generalized functions (§2.6.C). The existence of the convolutions will then follow from the properties of the supports of the "convolution factors".

Lemma 4.4. *Let $F(x) \in \mathcal{S}'(\mathbf{R}^n)$ be a solution of the wave equation* (4.28) *and Σ the strictly spacelike hypersurface* (4.49). *Then the generalized function $F(x+y)\vartheta_\Sigma(x) \in \mathcal{S}'(\mathbf{R}^{2n})$ is a generalized function with respect to x that is C^∞-dependent on y as a parameter. For any multi-indices $\beta, \gamma \in \overline{\mathbf{Z}}_+^n$, the products of the generalized functions $D^\beta F(x)$ and $D^\gamma\vartheta_\Sigma(x)$ are "canonically" defined and have support in the set* $\operatorname{supp} F \cap \operatorname{supp} D^\gamma\vartheta_\Sigma$. *The Leibnitz differentiation rule is applicable to the products defined in this way.*

■ We claim that $F(x+y)\vartheta_\Sigma(x)$ is of class C^∞ with respect to y. (The remaining assertions of the lemma will then follow from Exercise 2.54.) Our claim means that for any function $u(x) \in \mathcal{S}(\mathbf{R}^n)$, the generalized function $(F(x+y)\vartheta_\Sigma(x), u(x))_x$ is of class $\mathcal{O}_M(\mathbf{R}^n)$ with respect to y. Since $\vartheta_\Sigma(x)u(x)$ is a convolute (Exercise 2.22), this means (after replacing $F(x)$ by $F(-x)$) that $F*(\vartheta_\Sigma u) \in \mathcal{O}_M(\mathbf{R}^n)$. In terms of the Fourier transform (Exercise 2.20) we have to prove that the following condition holds for any test function $u(x) \in \mathcal{S}(\mathbf{R}^n)$:

$$\tilde{F}(p)U(p) \in \mathcal{O}_c(\mathbf{R}^n); \tag{4.53}$$

here $\tilde{F}(p) \in \mathcal{S}'(\mathbf{R}^n)$ and $U(p) \in \mathcal{O}_M(\mathbf{R}^n)$ are the Fourier transforms of the generalized functions F and $\vartheta_\Sigma u$:

$$\tilde{F}(p) = \int F(x)e^{-ipx}d^n x, \quad U(p) = \int \vartheta_\Sigma(x)u(x)e^{-ipx}d^n x. \tag{4.54}$$

Since the support of $\tilde{F}(p)$ is concentrated on the surface of the light cone, using a suitable partition of unity in p-space, we can decompose $F(x)$ into a sum of three solutions of the wave equation

$$F(x) = F_+(x) + F_-(x) + F_0(x)$$

in such a way that $\operatorname{supp}\tilde{F}_0$ is compact and $\operatorname{supp}\tilde{F}_\pm$ is contained in the corresponding set $\pm p_0 = |\mathbf{p}| \geq 1$. Upon replacing $\tilde{F}$ by $\tilde{F}_0$, (4.53) becomes trivial (Exercise 2.23) and since the case $\tilde{F}_-$ is similar to $\tilde{F}_+$, it suffices to prove instead of (4.53), that for all $u \in \mathcal{S}(\mathbf{R}^n)$

$$\tilde{F}_+(p)U(p) \in \mathcal{O}_c(\mathbf{R}^n). \tag{4.55}$$

Since $|\mathbf{p}|$ is a smooth function of $\mathbf{p}$ when $|\mathbf{p}| \geq 1$, it follows from the wave equation $(p \cdot p)\tilde{F}_+(p) = 0$ and the condition on the support of $\tilde{F}_+$ that it is possible to represent $\tilde{F}_+$ in the form

$$\tilde{F}_+(p) = \delta(p_0 - |\mathbf{p}|)a(\mathbf{p}), \tag{4.56}$$

where $a(\mathbf{p})$ is a generalized function in $\mathcal{S}'(\mathbf{R}_{n-1})$ with support in $|\mathbf{p}| \geq 1$. Since the function

$$U(p) = \int e^{-i[p_0 t + p_0 \kappa(\mathbf{x}) - \mathbf{p}\mathbf{x}]}\theta(t)u(t + \kappa(\mathbf{x}), \mathbf{x})dt\, d\mathbf{x} \tag{4.57}$$

is what concerns us in the product with the generalized function $\tilde{F}_+(p)$, we can replace $p_0\kappa(\mathbf{x})$ in (4.57) by $|\mathbf{p}|\kappa(\mathbf{x})$. As a result we obtain

$$\tilde{F}_+(p)U(p) = \tilde{F}_+(p)U_1(p), \tag{4.58}$$

by setting

$$U_1(p) = \alpha(\mathbf{p})V\left(p_0, |\mathbf{p}|, \frac{\mathbf{p}}{|\mathbf{p}|}\right); \tag{4.59}$$

here $\alpha(\mathbf{p})$ is a multiplicator in $\mathcal{O}_M(\mathbf{R}_{n-1})$, which vanishes in a neighbourhood of the origin and is equal to unity for $|\mathbf{p}| \geq 1$; $V(p_0, \rho, \mathbf{r})$ is a function of the real variables p_0, ρ and the vector $\mathbf{r}$ in the spherical shell

$$1 - \delta/3 \leq |\mathbf{r}| \leq 1 + \delta/3 \tag{4.60}$$

in $\mathbf{R}^{n-1}$, where δ is the positive number defined from the condition that the hypersurface Σ be strictly spacelike:

$$|\text{grad}\, \kappa(\mathbf{x})| < 1 - \delta. \tag{4.61}$$

The function $V(p_o, \rho, \mathbf{r})$ is defined by a formula similar to (4.57):

$$V(p_0, \rho, \mathbf{r}) = \int e^{-i[p_0 t + \rho(\kappa(x) - \mathbf{r}\mathbf{x})]}\theta(t)u_1(t, \mathbf{x})dt\, d\mathbf{x}, \tag{4.62}$$

where $u_1(t, \mathbf{x}) \equiv u(t + \kappa(\mathbf{x}), \mathbf{x}) \in \mathcal{S}(\mathbf{R}^n)$.

It remains to verify that the function $U_1(p)$ given by (4.59) is of class $\mathcal{S}(\mathbf{R}^n)$, which means that $V(p_0, \rho, \mathbf{r})$ must be "sufficiently good"; in other words, $V(p_0, \rho, \mathbf{r})$ is a C^∞-function (in its domain of definition) such that any of its derivatives (of any order) is, for any $N > 0$, majorized in modulus by an expression $c_N(1 + |\rho|)^{-N}$ (for some c_N). To prove this, we suppose that $\mathbf{r}$ varies in the part of the spherical shell (4.60) defined by the extra condition

$$r_1 > 1 - \delta/2. \tag{4.63}$$

Since the entire shell is covered by a finite number of neighbourhoods obtained from (4.63) by rotations, it is clear that we can restrict attention to a single neighbourhood (4.63). We now make the substitution $(t, x_1, x_2, \ldots, x_{n-1}) \to (t, \tau = \kappa(\mathbf{x}) - \mathbf{r}\mathbf{x}, x_2, \ldots, x_{n-1})$ in (4.62):

$$V(p_0, \rho, \mathbf{r}) = \int e^{-i(p_0 t + \rho\tau)}\theta(t)\psi(t, \tau, x_2, \ldots, x_{n-1}; \mathbf{r})dt\, d\tau\, dx_2 \ldots dx_{n-1}.$$

It follows from conditions (4.60), (4.61), (4.63) that $a < |\frac{\partial\tau}{\partial x_1}| < b$ for some positive constants a and b. Hence $\psi(t, \tau, x_2, \ldots, x_{n-1}; \mathbf{r})$ is a member of $\mathcal{S}(\mathbf{R}^n)$ as a function of $t, \tau, x_2, \ldots, x_{n-1}$ and is C^∞-dependent on the parameter $\mathbf{r}$.

The required property of the function $V(p_0, \rho, \mathbf{r})$ now follows from the properties of the classical Fourier transform of integrable functions, and this in turn enables us to conclude that $U_1 \in$ $\in \mathcal{S}(\mathbf{R}^n)$. ■

Thus we have determined the expressions in the square brackets in formulae (4.51) and (4.52); their equality follows from the differentiation rule and the wave equation for F. It is now fairly easy to see that the convolutions in these formulae are "canonically" defined. For the support of $D_{(n)}$ lies in $\overline{V}_{(n)}$, while the supports of the distributions in the curly brackets are contained in Σ, and are therefore contained in a set of the form $b + \mathcal{K}$, where b is a fixed point of Σ and $\mathcal{K}$ is the cone:

$$\mathcal{K} = \{x \in \mathbf{R}^n : |x_0| \le \rho|\mathbf{x}|\} \tag{4.64}$$

($\rho < 1$ is defined from the condition $|\operatorname{grad}\kappa(\mathbf{x})| < \rho$). It now follows from the criterion for the existence of a convolution (see Exercise 2.41) that the right hand sides of (4.51) and (4.52) are well defined and are equal. It remains to prove that one of these formulae holds.

Proposition 4.5. *Let F be the solution in $\mathcal{S}'(\mathbf{R}^n)$ of the wave equation, and Σ the strictly spacelike hypersurface* (4.49). *Then the* (*equivalent*) *representations* (4.51) *and* (4.52) *hold* (*where $D_{(n)}$ is the fundamental solution of the wave equation and ϑ_Σ and X^μ_Σ are defined in* (4.50)).

■ Let b be a point in Σ, $b_0 = \tau$ and $\Sigma' = \{x \in \mathbf{R}^n : x_0 = \tau\}$. It is clear that if Σ is replaced by Σ', then (4.51) becomes the formula (4.48), which has already been proved. It therefore suffices to prove that the difference of the right hand sides of (4.51) and (4.48) is zero or, what is the same (in view of the equivalence of (4.51) and (4.52)):

$$D_{(n)} * (\Box_{(n)} f) = 0, \tag{4.65}$$

where $f(x) = F(x)(\vartheta_\Sigma(x) - \vartheta_{\Sigma'}(x))$. Since $\operatorname{supp}(\vartheta_\Sigma - \vartheta_{\Sigma'}) \subset b + \mathcal{K}$, where the cone $\mathcal{K}$ is defined in (4.64), it follows that $\operatorname{supp} f \subset b + \mathcal{K}$ and (by the existence criterion of Exercise 2.41), the convolution $D_{(n)} * f$ exists in $\mathcal{S}'(\mathbf{R}^n)$. Using the differentiation formula for the convolution (see Exercise 2.39) we obtain: $D_{(n)} * (\Box_{(n)} f) = (\Box_{(n)} D_{(n)}) * f$. But this expression is equal to zero by definition of $D_{(n)}$, and this completes the proof of (4.65). ■

From Proposition 4.5, we have the following property, which is of quasi-analytic character.

Corollary 4.6 (Huygens' principle). Let Σ be a (strictly) spacelike surface in $\mathbf{R}^n$, and $\mathcal{M}$ the subset of the form $\mathcal{M} = \{x \in \Sigma : \mathbf{x} \in D\}$, where D is a domain in $\mathbf{R}^{n-1}$. If the solution F of the d'Alembert equation vanishes in some neighbourhood of $\mathcal{M}$, then F vanishes in the causal hull of $\mathcal{M}$.*

Exercise 4.3. Prove this corollary. [Hint: Use the property of the support of a convolution of generalized functions; see Exercise 2.39.]

In the next exercise we give a "smoothed" version of the representation (4.52). (The arguments used in the proof of Proposition 4.5 are also applicable here.)

Exercise 4.4. Let $\Sigma_\alpha = \{x \in \mathbf{R}^n : x_0 = \kappa_\alpha(\mathbf{x})\}$ ($\alpha = 1, 2$) be two given strictly spacelike hypersurfaces and let $\omega(x)$ be a multiplicator in $\mathcal{O}_M(\mathbf{R}^n)$ such that

$$\omega(x) = 1 \text{ for } x_0 > \kappa_1(\mathbf{x}) \quad \text{and } \omega(x) = 0 \text{ for } x_0 < \kappa_2(\mathbf{x}). \tag{4.66}$$

Prove that the following representation holds for the solutions of the wave equation:

$$F(x) = D_{(n)}(x) * \{\Box_{(n)}[\omega(x)F(x)]\}. \tag{4.67}$$

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* By the *future* (or *past*) *causal shadow* of a subset $\mathcal{M}$ of $\mathbf{R}^n$, we mean the set of points $x \in \mathbf{R}^n$ such that the ray $x + l$ intersects $\mathcal{M}$, where l is an arbitrary ray in the cone $\overline{V}^-_{(n)}$ (or in $\overline{V}^+_{(n)}$) starting at the origin. The *causal hull* of the set $\mathcal{M}$ is the union of the future and past causal shadows.

D. THE ASGEIRSSON FORMULA AND ITS APPLICATIONS

It turns out that there is a remarkable similarity between the spacelike and timelike coordinates in the wave equation. We have already noted that the solutions of the wave equation in $\mathcal{S}'(\mathbf{R}^n)$ are generalized functions in $\mathbf{x}$ of class $\mathcal{C}^\infty$ with respect to x_0 and equally, generalized functions in x_0 of class $\mathcal{C}^\infty$ with respect to $\mathbf{x}$. Consequently, there are not only the restrictions $F(x)|_{x_0=\text{const}}$ but also the restrictions $F(\mathbf{x})|_{\mathbf{x}=\text{const}} \in \mathcal{S}'(\mathbf{R})$. Furthermore, there exists a restriction on each timelike line l (defined by an equation of type $\mathbf{x} = \mathbf{a} + \mathbf{b}x_0$ for $\mathbf{b} \in \mathbf{R}^{n-1}$, $|\mathbf{b}| < 1$), that is, there exists $f(x_0) = F(x)|_l \in \mathcal{S}'(\mathbf{R})$ and hence there exists a restriction of $F(x)$ on any timelike interval (a, b) as well. In fact, by means of a suitable linear transformation in $\mathbf{R}^n$ (namely, a pseudo-Euclidean transformation that leaves the wave equation invariant), a timelike line can be converted to a line along the time axis (and a restriction to this line exists by what we have just said).

It is interesting to note that the property of quasi-analyticity, analogous to the Huygens' principle and illustrating the symmetry between x_0 and $\mathbf{x}$, is also valid. For the derivation of this we can use the so-called *Asgeirsson formula* "for the mean value", see [C6], Ch.IV,§18.1) which we give in the following "partially smoothed" form:

$$F(x) * \{\delta(x_0)\psi(|\mathbf{x}|^2)\} = F(x) * \{\check{\psi}(|x_0|^2)\delta(\mathbf{x})\}. \tag{4.68}$$

Here $F(x) \in \mathcal{S}'(\mathbf{R}^n)$ is an arbitrary solution of the wave equation (4.28); $\psi(\rho)$ is a function in $\mathcal{D}(\overline{\mathbf{R}}_+)$, while $\check{\psi}$ is also a function in $\mathcal{D}(\overline{\mathbf{R}}_+)$, formed from ψ as follows:

$$\begin{aligned} &\check{\psi}(\tau) = \tfrac{1}{2}s_{n-2}\int_0^\infty \psi(\tau+\rho)\rho^{\frac{n-3}{2}}d\rho, \quad \text{if } n > 2, \\ &\check{\psi}(\tau) = \psi(\tau), \quad \text{if } n = 2 \end{aligned} \tag{4.69}$$

(s_{n-2} is the area of the unit sphere in $\mathbf{R}^{n-2}$).

The convolutions in (4.68) exist, since the generalized functions in the curly brackets have compact supports. The validity of (4.68) can be checked via the Fourier transform. We set

$$\phi(\mathbf{p}) = \int e^{i\mathbf{p}\mathbf{x}}\psi(|\mathbf{x}|^2)dx, \quad \phi_1(p_0) = \int e^{-ip_0x_0}\check{\psi}(|x_0|^2)dx_0;$$

formula (4.68) then implies that

$$\tilde{F}(p)(\phi(\mathbf{p}) - \phi_1(p_0)) = 0. \tag{4.70}$$

In view of the fact that $\phi(\mathbf{p})$ is a rotation-invariant function in $\mathcal{S}(\mathbf{R}^{n-1})$, it can be represented in the form $\phi(\mathbf{p}) = \Phi(|\mathbf{p}|^2)$ for some $\Phi(\sigma) \in \mathcal{S}(\overline{\mathbf{R}}_+)$ (see Proposition 3.10). Similarly, $\phi_1(p_0)$ is an even function in $\mathcal{S}(\mathbf{R})$, therefore it can be represented in the form $\phi_1(p_0) = \Phi_1(p_0^2)$, where $\Phi_1(\sigma)$ is a function in $\mathcal{S}(\overline{\mathbf{R}}_+)$. Direct verification shows that $\phi(s, 0, \ldots, 0) = \phi_1(s)$ for all $s \in \mathbf{R}$, hence $\Phi_1 = \Phi$. We introduce the function

$$\chi(\sigma_1, \sigma_2) = (\Phi(\sigma_1) - \Phi(\sigma_2))/(\sigma_1 - \sigma_2);$$

it is clear that this is a $\mathcal{C}^\infty$-function on $\overline{\mathbf{R}}_+ \times \overline{\mathbf{R}}_+$. In such a case $\chi(|\mathbf{p}|^2, |p_0|^2)$ is a multiplicator in $\mathcal{S}(\mathbf{R}^n)$ and we can rewrite the left hand side of (4.70) in the form

$$\tilde{F}(p)\{\Phi(|\mathbf{p}|^2) - \Phi(|p_0|^2)\} = (|\mathbf{p}|^2 - |p_0|^2)\tilde{F}(p)\chi(|\mathbf{p}|^2, |p_0|^2).$$

This is zero since $(p \cdot p)\tilde{F}(p) = 0$. Formula (4.70) is now proved.

We now go over to the quasi-analyticity property mentioned above.

Proposition 4.7. *Suppose that the solution $F(x) \in S'(\mathbf{R}^n)$ of the d'Alembert equation satisfies the condition: the restrictions of F and of all its partial derivatives $D^\alpha F(x)$ (of all orders) on some timelike interval (a, b) are zero. Then F vanishes in the diamond $Q_{(n)}(a, b)$.*

■ The interval (a, b) can be reduced to an interval on the time axis of the form $\Im = \{x \in \in \mathbf{R}^n : \mathbf{x} = 0, \ -l < x_0 < l\}$ via a translation of the origin and a subsequent linear pseudo-Euclidean transformation of $\mathbf{R}^n$; consequently, the set $Q_{(n)}(a, b)$ is taken to the set

$$Q = \{(t, x) \in \mathbf{R}^n : \ |t| < l, \ \mathbf{x} \in S_{r(t)}\},$$

where $S_{r(t)}$ is the ball in $\mathbf{R}^n$ with centre at the origin and radius $r(t) = l - |t|$.

Thus we may suppose that for each multi-index α, $D^\alpha F(\mathbf{x})|_{\mathbf{x}=0} = 0$ for $x_0 \in (-l, l)$. We are going to prove that $F(x) = 0$ in Q or, equivalently, that for any $t \in (-l, l)$, $F(\mathbf{x})|_{x_0=t} = 0$ for $\mathbf{x} \in S_{r(t)}$, that is

$$(F_t(\mathbf{x}), u(\mathbf{x}))_{\mathbf{x}} = 0 \quad \text{for all } t \in (-l, l), \ u \in \mathcal{D}(S_{r(t)}). \tag{4.71}$$

We use Asgeirsson's formula, which can be rewritten in the form

$$\int F(s, \mathbf{y})|_{s=t} \psi(|\mathbf{x} - \mathbf{y}|^2) d^{n-1}\mathbf{y} = \int F(s, \mathbf{y})|_{\mathbf{y}=\mathbf{x}} \check{\psi}(|s - t|^2) ds.$$

Clearly, both sides of this formula are C^∞-functions in $\mathbf{R}^n$. By setting $|t| < l$ and $\mathbf{x} = 0$, we obtain

$$\int F(s, \mathbf{x})|_{s=t} \psi(|\mathbf{x}|^2) d\mathbf{x} = \int F(s, \mathbf{x})|_{\mathbf{x}=0} \check{\psi}(|t - s|^2) ds. \tag{4.72}$$

We suppose in addition that $\psi(\rho)$ vanishes for $\rho \geq r^2(t)$; then $\check{\psi}(\tau)$ also vanishes for $\tau \geq r^2(t)$ and hence $\check{\psi}(|t - s|^2) = 0$ for $|s| \geq l$. (As before, we suppose that $|t| < l$.) By hypothesis, the support of $F(s, \mathbf{x})|_{\mathbf{x}=0}$ lies in the set $|s| \geq l$ which is exactly where $\check{\psi}(|t - s|^2) = 0$. Hence the right hand side of (4.72) is equal to zero and we conclude that $(F_t(\mathbf{x}), \psi(|\mathbf{x}|^2))_{\mathbf{x}} = 0$ for all $t \in (-l, l)$ and for all $\psi(\rho) \in \mathcal{D}(\overline{\mathbf{R}}_+)$ that vanish when $\rho \geq r^2(t)$.

Thus we have partially proved (4.71), namely, for all rotation-invariant functions $u(\mathbf{x})$. We now show that (as before, for $|t| < l$) for any polynomial $P(\mathbf{x})$ in $\mathbf{R}^{n-1}$ we have

$$(F_t(\mathbf{x}), P(\mathbf{x}) \psi(|\mathbf{x}|^2)) = 0 \tag{4.73}$$

for all rotation-invariant functions $\psi(|\mathbf{x}|^2)$ in $\mathcal{D}(S_{r(t)})$. It is clear that our assertion (4.71) follows from this, since by Weierstrass's theorem (§1.2.C), the set of all functions of the form $P(\mathbf{x})\psi(|\mathbf{x}|^2)$ (where $P(\mathbf{x})$ is a polynomial and $\psi(|\mathbf{x}|^2)$ is a rotation-invariant function in $\mathcal{D}(S_{r(t)})$) is dense in $\mathcal{D}(S_{r(t)})$. Formula (4.73) is proved by induction on the degree N of the polynomial $P(\mathbf{x})$. For $N = 0$, it has already been proved. We suppose that it has been established for all solutions $F(x)$ of the wave equation, for all rotation-invariant functions $\psi(|\mathbf{x}|^2)$ in $\mathcal{D}(S_{r(t)})$ and for all polynomials $P(\mathbf{x})$ of degree $\leq N$. We now claim that for any $j = 1, \ldots, n-1$

$$(F_t(\mathbf{x}), x_j P(\mathbf{x}) \psi(|\mathbf{x}|^2)) = 0. \tag{4.74}$$

To see this, we introduce the function $\Psi(t) = -\frac{1}{2}\int_t^\infty \psi(s) ds$, which, clearly, is a member of $\mathcal{D}(\overline{\mathbf{R}}_+)$. We then have for the left hand side of (4.74):

$$\begin{aligned}(F_t(\mathbf{x}), x_j \cdot P(\mathbf{x})\psi(|\mathbf{x}|^2)) &= (F_t(\mathbf{x}), P(\mathbf{x}) \frac{\partial}{\partial x_j} \Psi(|\mathbf{x}|^2)) = \\ &= -\left(\frac{\partial}{\partial x_j} F(x)|_{x_0=t}, P(\mathbf{x})\Psi(|\mathbf{x}|^2)\right) - \left(F_t(\mathbf{x}), \frac{\partial P(\mathbf{x})}{\partial x_j} \Psi(|\mathbf{x}|^2)\right).\end{aligned}$$

The last part of this equality is equal to zero by the induction hypothesis.

Thus (4.73) and (4.71) are proved; so also is Propositon 4.7. ■

4.3. Derivation of the Jost-Lehmann-Dyson Formula

A. CONSTRUCTION OF THE SPECTRAL FUNCTION

In §4.1.A we associated each generalized function $g(p) \in \sigma(\overline{V})$ with a solution $G(p, \eta)$ of the wave equation in six-dimensional space (Proposition 4.2). We now go over to the second stage of the derivation of the JLD representation (4.17) for generalized functions $g(p) \in \sigma(V; \mathcal{R})$. So as to simplify the description of the space that the JLD spectral functions $\Psi(p', \lambda)$ belong to, we make a "technical" assumption concerning the domain $\mathcal{O}$, namely: $\mathcal{R}$ (the complement of $\mathcal{O}$ in $\mathbf{M}$) is contained in a set of the form $K + \overline{V}$, where K is a compactum in $\mathbf{M}$. In other words, we suppose that *

$$\mathcal{R} \subset Q[-a, a] + \overline{V} \text{ for some } a \in V^+. \tag{4.75}$$

Here and in what follows (with $a, b \in \mathbf{M}$), $Q(a, b)$ and $Q[a, b]$ denote the (open and closed) *diamonds* with vertices at the points a, b in $\mathbf{M}$;

$$Q(a, b) = (a + V^+) \cap (b + V^-), \; Q[a, b] = (a + \overline{V}^+) \cap (b + \overline{V}^-) \tag{4.76}$$

($Q(a, b)$ is non-empty only when $b - a \in V^+$; compare the notation at the beginning of the previous section). Condition (4.75) can also be written in the form

$$\operatorname{supp} g(p) \subset (-a + \overline{V}^+) \cup (a + \overline{V}^-) \text{ for some } a \in V^+.$$

Theorem 4.8. *Let $\mathcal{R}$ be a closed subset of Minkowski space $\mathbf{M}$ satisfying condition (4.75), and $g(p)$ an arbitrary generalized function in $\sigma(\overline{V}; \mathcal{R})$ (that is, $g \in \in \mathcal{S}'(\mathbf{M})$, $\operatorname{supp} \tilde{g} \subset \overline{V}$, $\operatorname{supp} g \subset \mathcal{R}$). Then $g(p)$ has the (weak integral) JLD representation*

$$g(p) = \int \epsilon(p^0 - p^{0\prime})\delta((p - p')^2 - \lambda)\Psi(p', \lambda) d_4p' d\lambda \tag{4.77}$$

for some generalized function $\Psi(p', \lambda) \in \mathcal{S}'(Q[-a, a] \times \overline{\mathbf{R}}_+)$.

■ We begin by proving that for any test function $\phi(p) \in \mathcal{S}(\mathbf{M})$, the function $\psi(p', \lambda) = = \int K(p; p', \lambda)\phi(p) d_4p$ (where $K(p; p', \lambda)$ is defined by (4.16)) is a member of the space $\mathcal{S}(Q[-a, a] \times \times \overline{\mathbf{R}}_+)$. Let $\mathfrak{m}_-$ be the continuous linear operator mapping $\mathcal{S}(\mathbf{M})$ into $\mathcal{S}(\overline{\mathbf{R}}_+)$ according to the formula $(\mathfrak{m}_- \phi)(\lambda) = \int \epsilon(p^0)\delta(p^2 - \lambda)\phi(p) d_4p$. (We met this operator earlier in §3.2.C.) Then the definition of $\psi(p', \lambda)$ can be rewritten in the form $\psi(p', \lambda) = (\mathfrak{m}_- \phi_{p'})(\lambda)$, where $\phi_{p'}(p) = \phi(p + p')$ is an element of $\mathcal{S}(\mathbf{M})$ with respect to p that is C^∞-dependent on the parameter p'. It follows that $(\mathfrak{m}_- \phi_{p'})(\lambda)$ is an element of $\mathcal{S}(\overline{\mathbf{R}}_+)$ with respect to λ that is C^∞-dependent on $p' \in \mathbf{M}$. We have therefore proved that $\psi(p', \lambda) \in \mathcal{S}(Q[-a, a] \times \overline{\mathbf{R}}_+)$. Hence according to the definition given in §A.2, $K(p; p', \lambda)$ can be regarded as the kernel of an integral representation and for any $\Psi(p', \lambda) \in \mathcal{S}'(Q[-a, a] \times \overline{\mathbf{R}}_+)$, formula (4.77) makes sense and defines a generalized function in $\mathcal{S}'(\mathbf{M})$ which, as is easy to see, actually belongs to the class $\sigma(\overline{V}; Q[-a, a] + \overline{V})$.

We now prove that every generalized function $g(p) \in \sigma(\overline{V}; \mathcal{R})$ admits a representation (4.77). We use Proposition 4.2 and write $g(p)$ in the form (4.14) with the generalized function $G(p, \eta) \in \mathcal{S}'(\mathbf{R}^6)$ satisfying the conditions (4.13). Let $\delta_n(\eta)$ be a δ-sequence of test functions in $\mathcal{D}(\mathbf{R}^2)$. Then for any $\phi(p) \in \mathcal{S}(\mathbf{M})$ we have

$$\int g(p)\phi(p) d^4p = \int G(p, \eta)|_{\eta=0}\phi(p) d^4p = \lim_{n \to \infty} \int G(p, \eta)\phi(p)\delta_n(\eta) d^4p \, d^2\eta. \tag{4.78}$$

By hypothesis, $g(p)$ vanishes outside $Q[-a, a] + \overline{V}$. In this case it follows from Lemma 4.9 proved below, that $G(p, \eta) = 0$ outside $(Q[-a, a] + \overline{V}) \times \mathbf{R}^2$. Let Σ be an arbitrary strictly spacelike surface

* This condition is satisfied in the applications of interest to us.

in $\mathbf{R}^6$ such that its intersection with $(Q[-a,a]+\overline{V})\times\mathbf{R}^2$ is contained in $Q[-a,a]\times\mathbf{R}^2$. For example, we can choose for Σ the hypersurface $\{(p,\eta)\in\mathbf{R}^6 : ap=0\}$. Applying the representation (4.52) to $G(p,\eta)$, we obtain the formula

$$G(p,\eta)=D_{(6)}(p,\eta)*H_\Sigma(p,\eta), \tag{4.79}$$

where

$$H_\Sigma(p,\eta)=\square_{(6)}\{G(p,\eta)\vartheta_\Sigma(p,\eta)\} \tag{4.80}$$

is a generalized function with support in $Q[-a,a]\times\mathbf{R}^2$.

It now follows from (4.78) and (4.79) that *

$$\int g(p)\phi(p)d^4p=\lim_{n\to\infty}\langle 1(p,p',\eta,\eta'),\ D_{(6)}(p,\eta)H_\Sigma(p',\eta')\phi(p+p')\delta_n(\eta+\eta')\rangle=$$

$$=\lim_{n\to\infty}\langle 1(p',\eta),\ H_\Sigma(p',\eta)\langle 1(\eta'),\ \delta_n(\eta+\eta')\langle 1(p),\ D_{(6)}(p,\eta')\phi_{p'}(p)\rangle_p\rangle_{\eta'}\rangle_{p',\eta}.$$

Upon using the explicit form of $D_{(6)}(p,\eta')$:

$$D_{(6)}(p,\eta')=\frac{1}{2\pi^2}\epsilon(p^0)\delta'(p^2-|\eta'|^2),$$

we obtain

$$\int D_{(6)}(p,\eta')\phi_{p'}(p)d^4p=\frac{1}{2\pi^2}\frac{\partial}{\partial\lambda}({}_{m-}\phi_{p'})(\lambda)\big|_{\lambda=|\eta|^2};$$

this is clearly a function in $\mathcal{S}(Q[-a,a]\times\mathbf{R}^2)$ with respect to the variables p',η'. Then

$$\left\langle\delta_n(\eta+\eta'),\ \frac{\partial}{\partial\lambda}({}_{m-}\phi_{p'})(\lambda)\big|_{\lambda=|\eta'|^2}\right\rangle_{\eta'}$$

tends to $\frac{\partial}{\partial\lambda}({}_{m-}\phi_{p'})(\lambda)\Big|_{\lambda=|\eta|^2}$ in $\mathcal{S}(Q[-a,a]\times\mathbf{R}^2)$ as $n\to\infty$, and as a result $\int g(p)\phi(p)d^4p$ becomes

$$\int g(p)\phi(p)d^4p=\frac{1}{2\pi^2}\int H_\Sigma(p',\eta)\frac{\partial}{\partial\lambda}({}_{m-}\phi_{p'})(\lambda)\Big|_{\lambda=|\eta|^2}d^4p'd\eta.$$

We introduce the functional $\Psi(p',\lambda)\in\mathcal{S}'(\overline{Q}(-a,a)\times\overline{\mathbf{R}}_+)$ which acts on the functions $h(p',\lambda)\in\in\mathcal{S}(Q[-a,a]\times\overline{\mathbf{R}}_+)$ according to the formula

$$\int\Psi(p',\lambda)h(p',\lambda)d_4p'd\lambda=\frac{1}{2\pi^2}\int H_\Sigma(p',\eta)\frac{\partial}{\partial\lambda}h(p',\lambda)\Big|_{\lambda=|\eta|^2}d^4p'd\eta. \tag{4.82}$$

by means of which we obtain the required representation:

$$\int g(p)\phi(p)d^4p=\int\Psi(p';\lambda)({}_{m-}\phi_{p'})(\lambda)d_4p'd\lambda, \tag{4.83}$$

which completes the derivation of (4.77). ■

In the proof we used the following lemma which is also required for subsequent refinements of the properties of the support of $\Psi(p',\lambda)$. Here we shall say that a set

* Here we have used the formula

$$\langle 1(x,y),f(x,y)\rangle_{x,y}=\langle 1(y),\langle 1(x),f(x,y)\rangle_x\rangle_y \tag{4.81}$$

for integrals (in the sense of §2.5.D) of generalized functions of integrable type; this reduces double integrals to iterated integrals.

$\mathcal{O} \subset \mathbf{M}$ is *timelike convex* if for any $a, b \in \mathcal{O}$ such that $b - a \in V^+$, the interval (a, b) also belongs to $\mathcal{O}$.

Lemma 4.9. *Let $g(p) \in \sigma(\overline{V}, \mathcal{R})$, where $\mathcal{R} = \mathbf{M} \backslash \mathcal{O}$ and $\mathcal{O}$ is a timelike convex open subset of $\mathbf{M}$. Then the solution of the wave equation $G(p, \eta)$ corresponding to $g(p)$ according to Proposition 4.2, vanishes in the region*

$$\mathcal{B}(\mathcal{O}) = \bigcup_{\substack{a,b \in \mathcal{O} \\ b-a \in V^+}} \mathcal{Q}_{(6)}(\hat{a}, \hat{b}); \tag{4.84}$$

here $\mathcal{Q}_{(6)}(\hat{a}, \hat{b})$ is a diamond in $R^6 \equiv \mathbf{M} \times \mathbf{R}^2$ with vertices at the points $\hat{a} \equiv (a, 0)$ and $\hat{b} \equiv (b, 0)$.

■ It is clear that if (a, b) is a timelike interval in $\mathbf{M}$, then $(\hat{a}, \hat{b})$ is a timelike interval in $\mathbf{R}^6$. Taking the quasi-analyticity property into account (Proposition 4.7), we can confine ourselves to proving that the restrictions of all the partial derivatives (of any order) of $G(p, \eta)$ to the plane $\eta = 0$ vanishes in $\mathcal{O}$. Since for any differential operator $P(\partial/\partial p)$, the generalized function $P(\partial/\partial p)G(p, \eta)$ is a solution of the wave equation corresponding to the generalized function $P(\partial/\partial p)g(p)$, it is clear that it suffices to consider only the partial derivatives of $G(p, \eta)$ with respect to η. We claim that for any vector $\xi \in \mathbf{R}^2$ and natural number n, the generalized function $T_n(p, \xi) = \left(\xi_1 \frac{\partial}{\partial \eta_1} + \xi_2 \frac{\partial}{\partial \eta_2}\right)^n G(p, \eta)\big|_{\eta=0}$ with respect to p vanishes in $\mathcal{O}$ (ξ being regarded as a parameter). For by definition, $T_n(p, \xi)$ is a homogeneous polynomial in ξ and it follows from the $O(2)$-invariance of $G(p, \eta)$ in η that $T_n(p, \xi)$ is $O(2)$-invariant with respect to ξ. Consequently when n is odd, $T_n(p, \xi)$ is identically zero and $T_n(p, \eta) = (|\xi|^2)^k \tau_k(p)$ for any $n = 2k$ ($k = 1, 2, \ldots$). It remains to show that all the $\tau_k(p)$ vanish on $\mathcal{O}$. We have

$$\tau_k(p) = \alpha_k \left(\left(\frac{\partial}{\partial \xi_1} \right)^2 + \left(\frac{\partial}{\partial \xi_2} \right)^2 \right)^k (|\xi|^2)^k \tau_k(p) = \alpha_k \left(\left(\frac{\partial}{\partial \xi_1} \right)^2 + \left(\frac{\partial}{\partial \xi_2} \right)^2 \right)^k T_n(p, \xi) =$$

$$= \beta_k \left(\left(\frac{\partial}{\partial \eta_1} \right)^2 + \left(\frac{\partial}{\partial \eta_2} \right)^2 \right)^k G(p, \eta)|_{\eta=0} = \beta_k ((\Box_p)^k G(p, \eta)|_{\eta=0} = \beta_k (\Box_p)^k g(p),$$

where $\alpha_k = 4^{-k}(k!)^{-2}$, $\beta_k = (2k)!\alpha_k$. (In the penultimate equality, we have used the wave equation for $G(p, \eta)$.) It is now clear that all the τ_k vanish in $\mathcal{O}$. ■

Remark. It is fairly easy to see that $\mathcal{B}(\mathcal{O})$ (where $\mathcal{O}$ is an open timelike convex subset of $\mathbf{M}$) is the set of all points $(u, \eta) \in \mathbf{R}^6$ such that both sheets of the hyperboloid * $u + \Gamma_{|\eta|}$ intersect $\mathcal{O}$; here

$$\Gamma_{\sqrt{\lambda}} = \{p \in \mathbf{M} : p^2 = \lambda\}. \tag{4.85}$$

In fact, in view of the fact that $\mathcal{O}$ is open and timelike convex, the above set of points is also the set of points (p', η) such that both sheets of the solid hyperboloid $\{p \in \mathbf{M} : (p - p')^2 > |\eta|^2\}$ intersect $\mathcal{O}$. In turn, the latter condition means that there exist $a, b \in \mathcal{O}$ such that $a^0 < p'^0 - \sqrt{(\mathbf{p}' - \mathbf{a})^2 + |\eta|^2}$ and $b^0 > p'^0 + \sqrt{(\mathbf{p}' - \mathbf{b})^2 + |\eta|^2}$ or, equivalently, such that $b - a \in V^+$ and $(p', \eta) \in \mathcal{Q}_{(6)}(\hat{a}, \hat{b})$. But this means precisely that $(p', \eta) \in \mathcal{B}(\mathcal{O})$.

Exercise 4.5. Prove that each generalized function $g(p)$ in the class $\sigma(\overline{V}; \overline{V})$ (that is, such that $\operatorname{supp} \tilde{g} \subset \overline{V}$ and $\operatorname{supp} g \subset \overline{V}$) has any of the following two representations (with $N < \infty$):

$$g(p) = \sum_{n=0}^{N} \sum_{\mu_1, \ldots, \mu_n = 0, \ldots, 3} \partial^{\mu_1} \ldots \partial^{\mu_n} h_{\mu_1 \ldots \mu_n}(p) =$$

* When $\lambda = 0$, the hyperboloid $\Gamma_{\sqrt{\lambda}}$ degenerates into a (light) cone.

$$= \sum_{n=0}^{N} \sum_{\mu_1,\ldots,\mu_n=0,\ldots,3} p^{\mu_1}\ldots p^{\mu_n} t_{\mu_1\ldots\mu_n}(p), \tag{4.86}$$

where $h_{\mu_1\ldots\mu_n}(p)$ and $t_{\mu_1\ldots\mu_n}(p)$ are odd Lorentz-invariant generalized functions in $\mathcal{S}'(\mathbf{M})$. [Hint: It is sufficient to prove the first representation, since the second representation is obtained, for example, as a corollary of the invariance of the class $\sigma(\overline{V};\overline{V})$ under a Fourier transformation. Use Lemma 4.9 to verify that the solution $G(p,\eta)$ of the wave equation corresponding to $g(p)$ has support in $\overline{V}\times\mathbf{R}^2$. Now argue as in the proof of Proposition 4.8 to deduce that the spectral $\Psi(p',\lambda)$ in the representation (4.77) can for a given $g(p)$ be chosen with support at $p'=0$.]

Exercise 4.6. Suppose that the generalized function $g(p)$ has support in the set $(-a+\overline{V}^+)\cup(a+\overline{V}^-)$ (where $a\in V^+$) and that its Fourier transform $\tilde{g}(x)$ has support in the cone $\overline{V}$. Prove that the same property is enjoyed by the generalized function $g_1(p)$ defined by the equality $\tilde{g}_1(x)=F(x^2)\tilde{g}(x)$, where $F(t)$ is an arbitrary function in $\mathcal{O}_{\mathbf{M}}(\mathbf{R})$. [Hint: For the proof that $g_1(p)$ has support in the set $(-a+\overline{V}^+)\cup(a+\overline{V}^-)$, take an arbitrary function $\phi(p)\in\mathcal{S}(\mathbf{M})$ with support outside this set and verify that

$$(g_1(p),\phi(p))\equiv(g(p),\ G(p)*\phi(p))=0,$$

where the convolute $G(p)$ is the Fourier transform of $F(x^2)$. On the basis of the representation (4.77), it is sufficient to prove that the function $u(p',\lambda)=\int\epsilon(p^0-p'^0)\delta((p-p')^2-\lambda)(G(p)*\phi(p))d_4p$ vanishes for $p'\in Q[-a,a]$; to show this, use the fact that $u(p,\lambda)$ as a function of p is the convolution of $\phi(p)$ with some odd Lorentz-invariant generalized function and this vanishes when $p\in Q[-a,a]$ by the property of the support of a convolution.]

B. Further properties of the support of the spectral function

In §4.1.B, we also posed the problem: when can the JLD spectral density be chosen so that its support is contained in the set of points (p',λ) with the property that the hyperboloid $p'+\Gamma_{\sqrt{\lambda}}$ is entirely contained in $\mathcal{R}$? We now turn to a class of domains $\mathcal{O}$ for which this problem has an affirmative solution.

Let $\chi_+(\mathbf{p})$ and $\chi_-(\mathbf{p})$ be a pair of real functions on $\mathbf{R}_3$ satisfying the condition

$$(\chi_\sigma(\mathbf{p})-\chi_\sigma(\mathbf{q}))^2-(\mathbf{p}-\mathbf{q})^2\le 0 \quad \text{for all } \mathbf{p},\ \mathbf{q}\in\mathbf{R}_3,\ \sigma=\pm \tag{4.87}$$

(from which follows the continuity of these functions; this condition is satisfied, for example, by every function defining a spacelike hypersurface in Minkowski space). We suppose further that there exists a vector $a\in V^+$ such that

$$\chi_+(\mathbf{p})\ge -a^0+|\mathbf{p}+\mathbf{a}|,\ \ \chi_-(\mathbf{p})\le a^0-|\mathbf{p}-\mathbf{a}| \quad \text{for all } \mathbf{p}\in\mathbf{R}_3. \tag{4.88}$$

The closed set $\mathcal{R}$ in Minkowski space will be assumed to be of the form

$$\mathcal{R}=\mathcal{R}_+\cup\mathcal{R}_-, \tag{4.89}$$

where

$$\mathcal{R}_+=\{p\in\mathbf{M}:p^0\ge\chi_+(\mathbf{p})\},\ \ \mathcal{R}_-=\{p\in\mathbf{M}:p^0\le\chi_-(\mathbf{p})\}. \tag{4.90}$$

As before, the open set $\mathcal{O}$ is the complement of $\mathcal{R}$ in $\mathbf{M}$:

$$\mathcal{O}=\mathbf{M}\backslash\mathcal{R}. \tag{4.91}$$

Exercise 4.7. (a) Prove that the above sets $\mathcal{R}_\pm$ satisfy the conditions

$$\mathcal{R}_+ +\overline{V}_+\subset\mathcal{R}_+,\ \ \mathcal{R}_- +\overline{V}_-\subset\mathcal{R}_-, \tag{4.92}$$

and that $\mathcal{R}$ satisfies (4.75).

(b) Prove that the set $\mathcal{O}$ (defined by (4.89)–(4.91)) is timelike convex. [Hint: Use (4.92) and argue by contradiction.]

(c) Prove that the function $\chi(\mathbf{p}) = \frac{1}{2}(\chi_+(\mathbf{p}) + \chi_-(\mathbf{p}))$ is bounded in modulus.

In the case under discussion, the set $\mathcal{B}(\mathcal{O})$ defined by (4.84) (in which the solutions of the wave equation corresponding to generalized functions in $\sigma(\overline{V};\mathcal{R})$ vanish) has the comparatively simple form:

$$\mathcal{B}(\mathcal{O}) = \{(p',\eta) \in \mathbf{R}^6 : \ X_-(\mathbf{p}',\eta) < p'^0 < X_+(\mathbf{p}',\eta)\}, \tag{4.93}$$

where

$$X_+(\mathbf{p}',\eta) = \sup_{\mathbf{p}}\{\chi_+(\mathbf{p}) - \sqrt{(\mathbf{p}-\mathbf{p}')^2 + |\eta|^2}\}, \tag{4.94a}$$

$$X_-(\mathbf{p}',\eta) = \inf_{\mathbf{p}}\{\chi_-(\mathbf{p}) + \sqrt{(\mathbf{p}-\mathbf{p}')^2 + |\eta|^2}\} \tag{4.94b}$$

(see Exercise 4.8 concerning this). The fact that the expressions (4.94) define real functions follows from the fact that according to (4.87), $\chi_+(\mathbf{p}) \le \chi_+(0) + |\mathbf{p}|$ and $\chi_-(\mathbf{p}) \ge \chi_-(0) - |\mathbf{p}|$. The functions $X_\pm(\mathbf{p}',\eta)$ possess the analogous version of property (4.87) enjoyed by the functions $\chi_\pm(\mathbf{p})$. For it follows from the triangle inequality

$$-\sqrt{(\mathbf{p}'-\mathbf{p}'')^2 + |\eta-\eta'|^2} \le \sqrt{(\mathbf{p}-\mathbf{p}')^2 + |\eta|^2} - \sqrt{(\mathbf{p}-\mathbf{p}'')^2 + |\eta'|^2} \le$$
$$\le \sqrt{(\mathbf{p}'-\mathbf{p}'')^2 + |\eta-\eta'|^2}$$

that

$$(X_\sigma(\mathbf{p}',\eta) - X_\sigma(\mathbf{p}'',\eta'))^2 - (\mathbf{p}'-\mathbf{p}'')^2 - |\eta-\eta'|^2 \le 0$$
$$\text{for all } \mathbf{p}',\mathbf{p}'' \in \mathbf{R}_3,\ \eta,\eta' \in \mathbf{R}^2 \tag{4.95}$$

(here $\sigma = \pm$); in particular, it follows that the functions $X_\pm(\mathbf{p}',\eta)$ are continuous.

Exercise 4.8. (a) The two closed sets $\mathcal{M}_\pm$ in $\mathbf{R}^6 \equiv \mathbf{M} \times \mathbf{R}^2$ are defined as follows:

$$\mathcal{M}_+ = \{(p',\eta) \in \mathbf{R}^6 : p'^0 \ge X_+(\mathbf{p}',\eta)\}, \quad \mathcal{M}_- = \{(p',\eta) \in \mathbf{R}^6 : p'^0 \le X_-(\mathbf{p}',\eta)\}. \tag{4.96}$$

Verify that $\mathcal{M}_+$ (or $\mathcal{M}_-$) is the set of all points $(p',\eta) \in \mathbf{R}^6$ such that the upper (or lower) sheet of the hyperboloid $p' + \Gamma_{|\eta|}$ is contained in $\mathcal{R}_+$ (or $\mathcal{R}_-$).

(b) Prove that the set $\mathcal{B}(\mathcal{O})$ defined by (4.84) satisfies the relation

$$\mathcal{B}(\mathcal{O}) = \mathbf{R}^6 \backslash (\mathcal{M}_+ \cup \mathcal{M}_-). \tag{4.97}$$

[Hint: Use the characterizations of the sets $\mathcal{B}(\mathcal{O})$ and $\mathcal{M}_\pm$ given in the remark in §4.3.A and in part (a) of this exercise; it follows from them that $\mathcal{B}(\mathcal{O}) \subset \mathbf{R}^6\backslash(\mathcal{M}_+ \cup \mathcal{M}_-)$. Conversely, if $(p',\eta) \in \mathbf{R}^6\backslash(\mathcal{M}_+ \cup \mathcal{M}_-) = (\mathbf{R}^6\backslash\mathcal{M}_+) \cap (\mathbf{R}^6\backslash\mathcal{M}_-)$, then the upper sheet of the hyperboloid $p' + \Gamma_{|\eta|}$ contains a point $b \notin \mathcal{R}_+$, while the lower sheet contains a point $c \notin \mathcal{R}_-$. To prove that $(p',\eta) \in \mathcal{B}(\mathcal{O})$, it suffices to verify that $b \in \mathcal{O}$ and $c \in \mathcal{O}$. Suppose on the contrary that $c \notin \mathcal{O}$ for example; then we find that $c \in \mathcal{R}_+$; this together with the property $\mathcal{R}_+ + \overline{V}_+ \subset \mathcal{R}_+$ (see (4.92)) and the fact that $b - c \in \overline{V}_+$, leads to the conclusion that $b \in \mathcal{R}_+$, which is a contradiction.]

(c) Prove (4.93). [Hint: Use (4.96) and (4.97).]

Exercise 4.9. (a) Derive the estimates

$$X_+(\mathbf{p}',\eta) \ge -a^0 + |\mathbf{p}' - \mathbf{a}|, \quad X_-(\mathbf{p}',\eta) \le a^0 - |\mathbf{p}' - \mathbf{a}|. \tag{4.98}$$

[Hint: In the definition (4.94) use the estimates (4.88) and calculate the respective upper and lower bounds with respect to $\mathbf{p}$.]

(b) Prove that the function

$$X(\mathbf{p}', \eta) = \tfrac{1}{2}(X_+(\mathbf{p}', \eta) + X_-(\mathbf{p}', \eta)) \tag{4.99}$$

is bounded in modulus. [Hint: It follows from (4.94) that

$$\inf_{\mathbf{p}'} \chi(\mathbf{p}') \leq X(\mathbf{p}', \eta) \leq \sup_{\mathbf{p}'} \chi(\mathbf{p}'),$$

where $\chi(\mathbf{p})$ is the function introduced in Exercise 4.7(c).]

We say that the hyperboloid $p' + \Gamma_{\sqrt{\lambda}} \subset M$ is *admissible* for the pair of sets $\mathcal{R}_+$ and $\mathcal{R}_-$ defined above, if the upper sheet $p' + \Gamma^+_{\sqrt{\lambda}}$ lies entirely in $\mathcal{R}_+$ and the lower sheet $p' + \Gamma^-_{\sqrt{\lambda}}$ lies entirely in $\mathcal{R}_-$; the corresponding parameters p', λ of an admissible hyperboloid are called *admissible parameters* and we denote the set of them by $\mathrm{adm} \equiv \mathrm{adm}(\mathcal{R}_+, \mathcal{R}_-)$. We also introduce the notion of a *complexified admissible hyperboloid*: this is a set of points in complex Minkowski space $\mathbf{CM}$ of the form

$$p' + \mathbf{C}\Gamma_{\sqrt{\lambda}} = \{k \in \mathbf{CM} : \ (k - p')^2 = \lambda\}, \tag{4.100}$$

where p', λ are admissible parameters.

From Exercise 4.8(a) we have:

$$\mathrm{adm} = \{(p', \lambda) \in \mathbf{M} \times \overline{\mathbf{R}}_+ : \ X_+(\mathbf{p}', \eta)|_{|\eta|^2=\lambda} \leq p'^0 \leq X_-(p', \eta)|_{|\eta|^2=\lambda}\}, \tag{4.101}$$

while from (4.98) we have the inclusion:

$$\mathrm{adm} \subset Q[-a, a] \times \overline{\mathbf{R}}_+. \tag{4.102}$$

We now state a condition under which the support of the JLD spectral function can be chosen to be in the set $\mathrm{adm}(\mathcal{R}_+, \mathcal{R}_-)$ so that for this case, the problem we have been discussing has an affirmative solution.

Theorem 4.10. (a) *Let $\mathcal{R}_\pm$ and $\mathcal{R}$ be the closed subsets of Minkowski space defined by* (4.89), (4.90) (*where the $\chi_\pm(\mathbf{p})$ are functions satisfying* (4.87), (4.88)). *Then for any generalized function $g(p)$ of class $\sigma(\overline{V}; \mathcal{R})$ (that is, such that $g \in \mathcal{S}'(\mathbf{M})$, $\operatorname{supp} \tilde{g} \subset \overline{V}$, $\operatorname{supp} g \subset \mathcal{R}$) and any $\epsilon > 0$, there is a JLD representation* (4.77) *with spectral function $\Psi(p', \lambda) \in \mathcal{S}'(\mathbf{M} \times \mathbf{R}_+)$,having support in an ϵ-neighbourhood* (*in* $\mathbf{M} \times \overline{\mathbf{R}}_+$) *of the set** adm (4.101).

(b) *Suppose further that there exists a strictly spacelike hypersurface in $\mathbf{R}^6 \equiv \mathbf{M} \times \mathbf{R}^2$ that "mediates" between the hypersurfaces $p'^0 = X_-(\mathbf{p}', \eta)$ and $p'^0 = X_+(\mathbf{p}', \eta)$, that is, it lies entirely in the set***

$$\{(p', \eta) \in \mathbf{R}^6 : X_-(\mathbf{p}', \eta) \wedge X_+(\mathbf{p}', \eta) \leq p'^0 \leq X_-(\mathbf{p}', \eta) \vee X_+(\mathbf{p}', \eta)\}. \tag{4.104}$$

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* It is clear here that $\operatorname{supp} \Psi \subset Q[-b, b] + \overline{\mathbf{R}}_+$, where $b \in \mathbf{M}$ can be chosen (in its dependence on Ψ) to be arbitrarily close to a.

** Here we are using the "lattice" notation

$$\begin{aligned} \lambda \wedge \mu &= \min\{\lambda, \mu\} = \tfrac{1}{2}(\lambda + \mu - |\lambda - \mu|), \\ \lambda \vee \mu &= \max\{\lambda, \mu\} = \tfrac{1}{2}(\lambda + \mu + |\lambda - \mu|), \end{aligned} \tag{4.103}$$

where λ, μ are real numbers.

Then for any generalized function of the class $\sigma(\overline{V};\mathbf{R})$, *the JLD representation* (4.77) *holds, the spectral function* $\Psi(\mathbf{p}',\lambda)$ *having support in the set* adm.

■ We first consider the case when the additional hypothesis of part (b) of the theorem holds. We suppose, in fact, that the spacelike surface Σ in $\mathbf{M}\times\mathbf{R}^2$ (featuring there) is the one chosen in the course of the proof of Theorem 4.8 (that is, in formula (4.80)). We claim that the corresponding spectral function (4.82) is concentrated in $\mathrm{adm}(\mathcal{R}_+,\mathcal{R}_-)$; for this it suffices to verify that the generalized function H_Σ is concentrated in the set

$$\{(p',\eta)\in\mathbf{R}^6:\ X_+(\mathbf{p}',\eta)\le p'^0\le X_-(\mathbf{p}',\eta)\}. \tag{4.105}$$

In fact by construction, H_Σ vanishes in the set $\mathcal{B}(\mathcal{O})$ of (4.93) where $G(\mathbf{p},\eta)$ vanishes; furthermore, its support is concentrated on the hypersurface Σ and hence in the set (4.104). It is now clear that H_Σ has support in the set (4.105).

To prove the general case (that is, part (a) of the theorem), we have to introduce one or two alterations in the above argument. We consider the function $X(\mathbf{p}',\eta)$ (4.99), which is continuous, bounded and $O(2)$-invariant (with respect to η), and can therefore be written in the form $X(\mathbf{p}',\eta) = Y(\mathbf{p}',|\eta|^2)$, where $Y(\mathbf{p}',t)$ is a continuous bounded function of $\mathbf{p}'\in\mathbf{R}_3$ and $t\in\mathbf{R}$. We set $\omega(p',t)=\theta(p'^0-Y(\mathbf{p}',t))*\alpha(\mathbf{p}',t)$, where $\alpha(\mathbf{p}',t)$ is a function in $\mathcal{D}(\mathbf{R}^5)$ with support in a ball of radius ϵ and with integral $\int\alpha(p',t)d^4p'\,dt$ equal to one. For the solution of the wave equation, instead of the representation (4.79) (with H_Σ defined by (4.80)), we use the "smoothed" version of it (given in Exercise 4.4):

$$G(p,\eta)=D_{(6)}(p,\eta)*H(p,\eta),$$

where

$$H(p,\eta)=\square_{(6)}\{G(p,\eta)\omega(p,|\eta|^2)\}.$$

As a result we arrive at the JLD representation with spectral function defined by a formula of type (4.82) (with H_Σ replaced by H.) We suggest that the reader verifies (by means of arguments similar to those given above for part (b) of the theorem) that the generalized function $\Psi(p',\lambda)$ so constructed, has support in an ϵ-neighbourhood of the set adm. ■

As we have already remarked (in §4.1.B), the JLD representation explicitly builds in the property $\operatorname{supp}\tilde{g}(x)\subset\overline{V}$; and now it explicitly takes into account the condition $\operatorname{supp} g\subset\mathcal{R}$ of (4.5b) as well. We go into this point in more detail. It follows from the property of the support (with respect to p) of the kernel $K(p;p',\lambda)$ (4.16) and the spectral function $\Psi(p',\lambda)$ that $\operatorname{supp} g$ is concentrated in the closure of the union of the hyperboloids $p'+\Gamma_\lambda$, where the point (p',λ) runs through an ϵ-neighbourhood (in $\mathbf{M}\times\overline{\mathbf{R}}_+$) of the set $\mathrm{adm}(\mathcal{R}_+,\mathcal{R}_-)$. We conclude from the arbitrariness of $\epsilon>0$ that:*

$$\operatorname{supp} g\subset\check{\mathcal{R}}, \tag{4.106}$$

where

$$\check{\mathcal{R}}=\bigcup_{(p',\lambda)\in\mathrm{adm}}(p'+\Gamma_{\sqrt{\lambda}}); \tag{4.107}$$

(as is easy to see, $\check{\mathcal{R}}$ is a closed subset of $\mathbf{M}$). It follows from the definition of adm that

$$\check{\mathcal{R}}\subset\mathcal{R}, \tag{4.108}$$

so that the condition $\operatorname{supp} g\subset\mathcal{R}$ does in fact hold.

It may happen that $\check{\mathcal{R}}$ is a proper subset of $\mathcal{R}$:

$$\check{\mathcal{R}}\subsetneq\mathcal{R}$$

(see Exercise 4.11 below). We then have an interesting situation: the support of $g(p)$ automatically vanishes in a larger domain than was originally postulated in condition (4.5b) (namely, in $\mathbf{M}\backslash\check{\mathcal{R}}$).

* Here we could have referred to a more detailed argument (in the spirit of Exercise 4.15 below).

C. Examples

Let $g(p)$ be a generalized function in $\mathcal{S}'(\mathbf{M})$ with the properties

$$\operatorname{supp}\tilde{g}(x) \subset \overline{V}, \tag{4.109a}$$

$$\operatorname{supp} g(p) \subset (-a + \overline{V}_M^+) \cup (a' + \overline{V}_{M'}^-), \tag{4.109b}$$

where $a + a' \in V^+$, M and M' are two non-negative parameters (having the meaning of "threshold masses" in the context of quantum field theory). Clearly, the conditions (4.109) imply that $g \in \sigma(\overline{V}; \mathcal{R})$, where $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$, while $\mathcal{R}_+$ and $\mathcal{R}_-$ are defined by (4.90) with

$$\chi_+(\mathbf{p}) = -a^0 + \sqrt{(\mathbf{p}+\mathbf{a})^2 + M^2}, \quad \chi_-(\mathbf{p}) = a'^0 - \sqrt{(\mathbf{p}-\mathbf{a}')^2 + M'^2}. \tag{4.110}$$

The functions $\chi_\pm(\mathbf{p})$ satisfy (4.87), (4.88), therefore Theorem 4.10 is applicable to the generalized function $g(p)$.

In order to indicate the support of the JLD spectral function $\Psi(p', \lambda)$ it suffices to determine the set $\operatorname{adm}(\mathcal{R}_+, \mathcal{R}_-)$.

Exercise 4.10. (a) Prove that the upper sheet of the hyperboloid $p' + \Gamma_{\sqrt{\lambda}}$ is contained in $-a + \overline{V}_M^+$ if and only if $p' \in -a + \overline{V}^+$ and $\sqrt{\lambda} \geq M - \sqrt{(p'+a)^2}$.

(b) Prove that the lower sheet of the hyperboloid $p' + \Gamma_{\sqrt{\lambda}}$ is contained in $a' + \overline{V}_{M'}^-$ if and only if $p' \in a' + \overline{V}^-$ and $\sqrt{\lambda} \geq M' - \sqrt{(p'-a')^2}$.

It follows from Exercise 4.10 that

$$\operatorname{adm} = \{(p', \lambda) \in Q[-a, a'] \times \overline{\mathbf{R}}_+ : \lambda \geq \kappa^2(p')\}, \tag{4.111}$$

where

$$\kappa(p') = 0 \vee (M - \sqrt{(p'+a)^2}) \vee (M' - \sqrt{(p'-a')^2}). \tag{4.112}$$

Exercise 4.11. Let $a + a' \in V_+$ and $0 \leq M' < M - \sqrt{a+a')^2}$. Deduce from conditions (4.108) that

$$\operatorname{supp} g(p) \subset (-a + \overline{V}_M^+) \cup (a' + \overline{V}_{M''}), \tag{4.113}$$

where $M'' = M - \sqrt{(a+a')^2}$. [Hint: Verify that in this case

$$\operatorname{adm} = \{(p', \lambda) \in Q[-a, a'] \times \overline{\mathbf{R}}_+ : \sqrt{\lambda} \geq M - \sqrt{(p'+a)^2}\}$$

and that the set $\check{\mathcal{R}}$ (4.107) is the same as the right hand side of (4.113).]

We give special attention to the "symmetric" case of equal "threshold masses": $M = M'$; we suppose further that $a = a'$. We suppose for definiteness, that the vector $a \in V^+$ is directed along the time axis:

$$a = a' = (A, 0), \ A > 0.$$

In this case, the conditions of part (b) of Theorem 4.10 are satisfied; namely, we can choose for Σ the hypersurface $p'^0 = 0$. We then find, using (4.80), (4.82), that the spectral function $\Psi(p', \lambda)$ is concentrated at $p'^0 = 0$; furthermore it has the form

$$\Psi(p', \lambda) = \delta(p'^0)\Phi_0(\mathbf{p}', \lambda) + \delta'(p'^0)\Psi_1(\mathbf{p}', \lambda).$$

Accordingly, the "symmetric" version of the JLD representation (first obtained by Jost and Lehmann, 1957) assumes the form

$$g(p) = \int \epsilon(p^0)\delta(p^{02} - (\mathbf{p} - \mathbf{p}')^2 - \lambda)(\Phi_0(\mathbf{p}', \lambda) + p^0\Phi_1(\mathbf{p}', \lambda))d_3\mathbf{p}'d\lambda, \tag{4.114}$$

where $\Phi_0(\mathbf{p}', \lambda)$ and $\Phi_1(\mathbf{p}', \lambda)$ are generalized functions in $\mathcal{S}'(\mathbf{R}_3 \times \overline{\mathbf{R}}_+)$, concentrated on the set

$$\left\{(\mathbf{p}', \lambda) \in \mathbf{R}_3 \times \overline{\mathbf{R}}_+ : |\mathbf{p}'| \leq A, \sqrt{\lambda} \geq M - \sqrt{A^2 - |\mathbf{p}'|^2}\right\} \tag{4.115}$$

($\Phi_1(\mathbf{p}', \lambda)$ is related to $\Psi_1(\mathbf{p}', \lambda)$ via the formula $\Phi_1(\mathbf{p}', \lambda) = -2\frac{\partial}{\partial\lambda}\Psi_1(\mathbf{p}', \lambda)$).

In contrast to the JLD representation in (4.114), the generalized functions $\Phi_0(\mathbf{p}', \lambda)$ and $\Phi_1(\mathbf{p}', \lambda)$ are uniquely defined by $g(p)$. For further details on the Jost-Lehmann representation (4.114) for the "symmetric" case, see the survey by Vladimirov and Zav'yalov (1980).

D. Representations for Generalized Functions of Retarded and Advanced Types

We now turn to the problem of §4.1.A on expressing generalized functions $f_\pm(x)$ of retarded and advanced types in terms of the jumps (4.2) or, equivalently, in terms of the difference

$$f(x) = f_+(x) - f_-(x). \tag{4.116}$$

The problem essentially reduces to determining the products $\theta(\pm x^0)f(x)$, since the properties of the supports $\operatorname{supp} f_\pm \subset \overline{V}^\pm$ then enable us to write

$$f_\pm(x) = \theta(\pm x^0)f(x) + R(x), \tag{4.117}$$

where $R(x)$ is a generalized function with support at the origin. It is clear that the products $\theta(\pm x^0)f(x)$ are not uniquely determined in general, and here we use the device described in §A.3.

In terms of the Fourier transform, the product becomes the convolution:

$$\int \theta(x^0)f(x)e^{ipx}dx = \int \frac{i}{p^0 - q^0 + i0}g(q^0, \mathbf{p})d_1q^0. \tag{4.118}$$

Exercise 4.12. Prove the relation

$$\int \frac{i}{p^0 - q^0 + i0}\epsilon(q^0 - p'^0)\delta((q^0 - p'^0)^2 - (\mathbf{p} - \mathbf{p}')^2 - \lambda)d_1q^0 = \\ = \frac{i}{2\pi}\,\frac{1}{(p^0 - p'^0 + i0)^2 - (\mathbf{p} - \mathbf{p}')^2 - \lambda}. \tag{4.119}$$

Substituting (4.77) into (4.118) and using (4.119), we obtain the following formal relation:

$$\int \theta(x^0)f(x)e^{ipx}dx = \frac{i}{2\pi}\int \frac{1}{(p^0 - p'^0 + i0)^2 - (\mathbf{p} - \mathbf{p}')^2 - \lambda}\Psi(p', \lambda)d_4p'd\lambda. \tag{4.120}$$

The indeterminacy of the product $\theta(x^0)f(x)$ is now reflected in the fact that the expression (4.120) (invoking the convolution) can diverge with respect to λ. To give meaning to the right hand side of (4.120) we proceed as in §A.3: we suppose that the spectral function $\Psi(p',\lambda) \in \mathcal{S}'(Q[-b,b] \times \overline{\mathbf{R}}_+)$ can be extended to a generalized function in $\mathcal{S}'(Q[-b,b] \times [0,\infty])$. For this it suffices (see the next exercise) that the right hand side of (4.120) be taken in the sense of a weak integral representation. Furthermore, our definition (4.120) of the product $\theta(x^0)f(x)$ is equal to $f(x)$ for $x^0 > 0$ and is zero for $x^0 < 0$.

Exercise 4.13. (a) Prove that the expression

$$(\varkappa\tilde{\phi})(p',\lambda) = \frac{i}{2\pi}\int \frac{1}{(p^0 - p'^0 + i0)^2 - (\mathbf{p} - \mathbf{p}')^2 - \lambda}\tilde{\phi}(p)d_4p \tag{4.121}$$

defines a continuous linear operator $\varkappa : \mathcal{S}(\mathbf{M}) \to \mathcal{S}(Q[-b,b] \times [0,\infty])$. [Hint: It is not difficult to prove that $(\varkappa\phi)(p',\lambda)$ is a C^∞-function of p',λ if (4.121) is rewritten in the form

$$(\varkappa\tilde{\phi})(p',\lambda) = i\int \theta(t)e^{ip'^0t}\frac{\sin\sqrt{(\mathbf{p}-\mathbf{p}')^2+\lambda}}{\sqrt{(\mathbf{p}-\mathbf{p}')^2+\lambda}}\underset{\sim}{\phi}(t,\mathbf{p})dt\,d_3\mathbf{p}, \tag{4.122}$$

where $\underset{\sim}{\phi}(t,\mathbf{p}) = \int e^{-itp^0}\tilde{\phi}(p)d_1p^0$. To verify that the function $(\varkappa\phi)(p',\lambda)$ belongs to $\mathcal{S}(Q[-b,b] \times$ $\times\ [0,\infty])$, one needs to obtain from the representation (4.121) the asymptotic expansion in λ^{-1} as $\lambda \to \infty$ for $\varkappa\phi$ and its derivatives.]

(b) Prove that

$$(\varkappa\tilde{\phi})(p',\lambda) = \int \epsilon(p^0 - p'^0)\delta((p-p')^2 - \lambda)\tilde{\phi}(p)d_4p,$$

if the support of the function $\phi(x) = \int e^{-ipx}\tilde{\phi}(p)d_4p$ is concentrated at $x^0 > 0$ and that $(\varkappa\tilde{\phi})(p',\lambda) =$ $= 0$, if $\phi(x) = 0$ for $x^0 > 0$. [Hint: Use (4.122).]

Formula (4.120) together with (4.117) leads to the representations for the Fourier transform of generalized functions of retarded and (similarly) advanced types:

$$h_\pm(p) = \frac{i}{2\pi}\int \frac{1}{(p^0 - p'^0 \pm i0)^2 - (\mathbf{p}-\mathbf{p}')^2 - \lambda}\Psi(p',\lambda)d_4p'\,d\lambda + T(p), \tag{4.123}$$

where now $\Psi(p',\lambda) \in \mathcal{S}'(Q[-b,b] \times [0,\infty])$ and $T(p)$ is a polynomial in p.

Exercise 4.14. Let $T(p)$ be an arbitrary polynomial in $p \in \mathbf{M}$.

(a) Prove that $T(p)$ can be represented in the form

$$T(p) = \sum_{k=0}^{N} t_k(\partial_p)(p^2)^k, \tag{4.124}$$

where the $t_k(\partial_p)$ are differential polynomials with constant coefficients. [Hint: Apply induction on the degree of $T(p)$. Assuming that $T(p)$ has the form (4.124), prove that $T(p)p^\mu$ can also be written in a similar form; to this end, note that the difference between $T(p)p^\mu$ and $\sum_{k=0}^N t_k(\partial_p)[(p^2)^kp^\mu]$ is a polynomial of the form (4.124).]

(b) Prove that the polynomial $T(p)$ can be represented in the form

$$T(p) = \sum_{k=0}^{N}\int \Phi_k(p')(p-p')^{2k}d_4p', \tag{4.125}$$

where the $\Phi_k(p')$ are generalized functions in $\mathcal{S}'(\mathbf{M})$ with supports at the origin. [Hint: Use (4.124).]

(c) Prove that the polynomial $T(p)$ has the representation

$$T(p) = \frac{i}{2\pi}\int \frac{1}{(p^0 - p'^0 \pm i0)^2 - (\mathbf{p}-\mathbf{p}')^2 - \lambda}\Phi(p',\lambda)d_4p'\,d\lambda, \tag{4.126}$$

where $\Phi(p',\lambda)$ is a generalized function in $\mathcal{S}'(Q[-b,b]\times[0,\infty])$ with support on the set $Q[-b,b]\times\{+\infty\}$. [Hint: Use the representations (4.125) and (A.22).]

Exercise 4.14 enables us to assume that $T(p) = 0$, since the general case reduces to this special case by replacing the spectral function $\Psi(p', \lambda)$ by $\Psi(p', \lambda) + \Phi(p', \lambda)$. We note that the support of $\Phi(p', \lambda)$ is contained in $Q[-b, b] \times \{+\infty\}$, that is, in the same place where the non-uniqueness of the spectral function $\Psi(p', \lambda)$ occurs on extending it from $Q[-b, b] \times [0, \infty)$ to $Q[-b, b] \times [0, \infty]$.

We have obtained the following result.

Theorem 4.11. *Let $h_+(p)$ and $h_-(p)$ be Fourier transforms of generalized functions in* $\mathbf{M}$ *of retarded and advanced types respectively, and suppose that their difference $g(p) = h_+(p) - h_-(p)$ belongs to the class $\sigma(\overline{V}, \mathcal{R})$ of Theorem 4.10 (and consequently, the representation (4.77) holds for any $\epsilon > 0$, where $\Psi(p', \lambda)$ is a generalized function in $\mathcal{S}'(Q[-b, b] \times \overline{\mathbf{R}}_+)$ with support in an ϵ-neighbourhood of the set* $\mathrm{adm}(\mathcal{R}_+, \mathcal{R}_-)$). *Then the spectral function $\Psi(p', \lambda)$ has an extension to a generalized function in $\mathcal{S}'(Q[-b, b] \times [0, \infty])$ such that the following representations hold for $h_\pm(p)$:*

$$h_\pm(p) = i \int \frac{1}{(p^0 - p'^0 \pm i0)^2 - (\mathbf{p} - \mathbf{p}')^2 - \lambda} \Psi(p', \lambda) d_4p' d_1\lambda. \tag{4.127}$$

It follows from (4.127) that the $h_\pm(p)$ have the common analytic continuation

$$h(k) = i \int \frac{1}{(k - p')^2 - \lambda} \Psi(p', \lambda) d_1 p' d_1 \lambda \tag{4.128}$$

to the (open) set

$$D = \{k \in \mathbf{CM} : (k - p')^2 \neq \lambda \quad \text{for all } (p', \lambda) \in \mathrm{adm}\}, \tag{4.129}$$

that is, in the complement (in $\mathbf{CM}$) of the union of all the complexified admissible hyperboloids:

$$\mathbf{CM} \backslash D = \bigcup_{(p', \lambda) \in \mathrm{adm}} (p' + \mathbf{C}\Gamma_{\sqrt{\lambda}}). \tag{4.130}$$

Exercise 4.15. (a) Let r be a point of the set D of (4.129). Prove that $(k - p')^2 \neq \lambda$ for all k in some neighbourhood of r in $\mathbf{CM}$ and for all (p', λ) in some ϵ-neighbourhood of adm. [Hint: Suppose the contrary. Then there exist sequences $k_n \to r$ in $\mathbf{CM}$ and (p'_n, λ_n) in an ϵ_n-neighbourhood of adm such that $(k_n - p'_n)^2 = \lambda_n$; here $\epsilon_n > 0$, $\epsilon_n \to 0$. Since the sequence $\{p'_n\}$ is bounded, we can, by taking a suitable subsequence, assume that $\{p'_n\}$ converges to a point $p' \in \mathbf{M}$; then λ_n also converges to some number λ. Thus a contradiction is obtained, namely, $(r - p')^2 = \lambda$, where $(p', \lambda) \in$adm.]

(b) Prove that D is open and that the set of its real points contains $\mathcal{O} \equiv \mathbf{M} \backslash \mathcal{R}$. [Hint: Use part (a) of this exercise for the proof of the first assertion; then use (4.107), (4.108).]

Exercise 4.16. (a) Prove that (4.128) defines an analytic function on D. [Hint: The spectral function Ψ can be assumed to be a generalized function in $\mathcal{S}'(\Omega)$ for $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1 is the closure of the set of those points (p', λ) in an ϵ-neighbourhood of adm for which $0 \leq \lambda \leq \Lambda$, $\Omega_2 = Q[-b, b] \times [\Lambda, \infty]$; the numbers $\epsilon > 0, \Lambda > 0$ can be taken to be arbitrary and b is arbitrarily close to a. As in Exercise 4.15(a), for a fixed point r in D one can choose a number $\delta > 0$, a neighbourhood of r in D and parameters ϵ, Λ, b such that $|(k - p')^2 - \lambda| > \delta$ for all k in this neighbourhood of r and for all $(p', \lambda) \in \Omega$. It follows that $[(k - p')^2 - \lambda]^{-1}$ is a function in $\mathcal{S}'(\Omega)$ with respect to p', λ, which depends analytically on k as a parameter.]

(b) Prove the relations

$$h_\pm(p) = \lim_{q \to 0,\, q \in V^\pm} h(p + iq) \text{ in } \mathcal{S}'(\mathbf{M}). \tag{4.131}$$

[Hint: Consider the family of operators $\mathfrak{n}_q$ from $\mathcal{S}(\mathbf{M})$ to $\mathcal{S}(Q[-b,b] \times [0,\infty])$, depending on the parameter $q \in V^+$:

$$(\mathfrak{n}_q\tilde{\phi})(p',\lambda) = \frac{i}{2\pi}\int \frac{1}{(p+iq-p')^2-\lambda}\tilde{\phi}(p)d_4p. \tag{4.132}$$

Use the following representation of type (4.122)

$$(\mathfrak{n}_q\tilde{\phi})(p',\lambda) = i\int \theta(t)e^{ik^0t}\frac{\sin\sqrt{(\mathbf{k}-\mathbf{p}')^2+\lambda}}{\sqrt{(\mathbf{k}-\mathbf{p}')^2+\lambda}}\underset{\sim}{\phi}(t,\mathbf{p})dt\, d_3\mathbf{p} \tag{4.133}$$

to verify that $\mathfrak{n}_q\tilde{\phi} \to \mathfrak{n}\tilde{\phi}$ (see (4.121)) as $q \to 0$, $q \in V^+$ in the topology of $\mathcal{S}(Q[-b,b] \times [0,\infty])$; then apply the generalized function $\Psi(p',\lambda) \in \mathcal{S}'(Q[-b,b] \times [0,\infty])$ to this relation to obtain the first of the relations (4.131).]

On the basis of Exercise 4.15 and 4.16 we come to the following conclusion.

Theorem 4.12. *The generalized functions $h_\pm(p)$ of Theorem 4.11 are generalized boundary values (as $q \equiv \operatorname{Im} k \to 0$, $q \in V^\pm$) of their common analytic continuation $h(k)$ (4.128) on the open set D (4.129) in* CM *(containing, in particular, the tubes $T_1^\pm \equiv \mathbf{M} + iV^\pm$ and the subset of real points $\mathcal{O} \equiv \mathbf{M}\backslash\mathcal{R}$).*

We give an example. Suppose that the analytic functions $h_\pm(k)$ of $k \in T_1^\pm$ are the Laplace transforms of generalized functions in $\mathcal{S}'(\mathbf{M})$ of retarded and advanced types respectively, and that their generalized boundary values $h_\pm(p)$ (as $q \equiv \operatorname{Im} k \to$ $\to 0$, $q \in V^\pm$) have the following common values:

$$h_+(p) = h_-(p) \quad \text{for } p \notin (-a+\overline{V}_M^+) \cup (a'+\overline{V}_{M'}^-), \tag{4.134}$$

where $a + a' \in V^+$ and M and M' are two non-negative parameters. Then the set of admissible parameters is defined by (4.111), (4.112). Consequently, the functions $h_\pm(k)$ have a common analytic continuation $h(k)$ (with representation (4.128)) in the domain

$$D = \{k \in \mathbf{CM} : (k-p')^2 \neq \lambda \quad \text{for all } p' \in Q[-a,a'],\ \lambda \geq \kappa^2(p')\}. \tag{4.135}$$

Exercise 4.17. Let $h_\pm(k)$ be a pair of analytic functions of $k \in T_1^\pm$ that are Laplace transforms of generalized functions in $\mathcal{S}'(\mathbf{M})$ of retarded and advanced types respectively, and suppose that their generalized boundary values $h_\pm(p)$ (as $q \to 0$, $q \in V^\pm$) coincide for $p^2 < 0$. Prove that $h_\pm(p)$ have a common analytic continuation $h(k)$ in the domain $\{k \in \mathbf{CM} : k^2 \notin \overline{\mathbf{R}}_+\}$, where $h(k)$ has the following representations (with $N < \infty$)

$$\begin{aligned} h(k) &= \sum_{n=0}^{N} \sum_{\mu_1,\dots,\mu_n=0,\dots,3} \partial^{\mu_1}\dots\partial^{\mu_n} h_{\mu_1\dots\mu_n}(k^2) = \\ &= \sum_{n=0}^{N} \sum_{\mu_1,\dots,\mu_n=0,\dots,3} k^{\mu_1}\dots k^{\mu_n} t_{\mu_1\dots\mu_n}(k^2), \end{aligned} \tag{4.136}$$

where $h_{\mu_1\dots\mu_n}(z)$ and $t_{\mu_1\dots\mu_n}(z)$ are analytic functions in the cut complex plane $\mathbf{C}\backslash\overline{\mathbf{R}}_+$. [Hint: As in Exercise 4.5, the spectral function $\Psi(p',\lambda)$ in the representations (4.127), (4.128) can be chosen with support at $p' = 0$.]

It follows from (4.136) that the functions $h_\pm(k)$ transform according to a finite-dimensional representation of the Lorentz group (or are "finite-covariant" in the terminology of Bros et al, 1967).

CHAPTER 5

Analytic Functions of Several Complex Variables

5.1. Properties of Holomorphic Functions. Plurisubharmonic Functions

A. Space of Holomorphic Functions

In the previous chapters we have repeatedly made use of holomorphic functions of a complex vector. Now we discuss their properties in more detail.

We recall that the definition of a holomorphic function in a domain or an open set $D \subset \mathbf{C}^n$ (Appendix B.1) includes two requirements: the "catch-all" condition of continuous differentiability and satisfaction of the system of Cauchy-Riemann equations

$$\bar{\partial}_j f(z) \equiv \frac{\partial}{\partial \bar{z}_j} f(z) = 0, \quad j = 1, \ldots, n. \tag{5.1}$$

The "catch-all" condition can be weakened considerably. For example, there is the so-called (fundamental) *Hartogs theorem* ([V4],§4.2; [S8],§6): if the function $f(z)$ in the domain $D \subset \mathbf{C}^n$ is analytic in each component z_j for any (admissible) values of the remaining components, then it is jointly analytic (and, in particular, C^∞) in all the variables $z \equiv (z_1, \ldots, z_n)$. In such a "pure" form, the Hartogs theorem is of no great value for what we shall have to say, since we shall usually be dealing with functions (or distributions) $f(z)$ with more suitable "catch-all" properties. For example, it will be sufficient that $f(z)$ be a distribution in $\mathcal{D}'(D)$; it then follows automatically from the Cauchy-Riemann equations that $f(z)$ is a holomorphic function (see Proposition 5.13).

The space of all holomorphic functions in the domain $D \subset \mathbf{C}^n$ will be denoted by $\mathcal{H}(D)$. It is clearly a linear subspace of the space $\mathcal{C}(D)$ of all continuous functions in D. We endow $\mathcal{H}(D)$ with the LCS structure induced from $\mathcal{C}(D)$, that is, with the topology defined by the system of seminorms

$$\|f\|^K = \sup_{z \in K} |f(z)|, \tag{5.2}$$

where K runs through all the compacta in D (or a countable family of compacta whose interiors cover D). A more general class than the compact subsets of D is the class of subsets $K \subset D$ with the property: $\sup_{z \in K} |f(z)| < \infty$ for all $f \in \mathcal{H}(D)$; these subsets K are called $\mathcal{H}(D)$-*bounded*. In other words, an $\mathcal{H}(D)$-bounded subset is any subset of D on which any holomorphic function in D is bounded in modulus. It is clear that formula (5.2) associates a seminorm on $\mathcal{H}(D)$ with each $\mathcal{H}(D)$-bounded subset K. It turns out (see Proposition 5.4) that all such seminorms $\|f\|^K$ are continuous (therefore replacing all the compact subsets K in the definition of the topology of $\mathcal{H}(D)$ by all the $\mathcal{H}(D)$-bounded subsets K does not alter the topology).

Exercise 5.1. Prove that any $\mathcal{H}(D)$-bounded set K is bounded. [Hint: Use the boundedness over K of the coordinate functions $f_j(z) = z_j$, $j = 1, \ldots, n$.]

Underlying our further discussion is the integral representation for polycircular domains. By an (open) *polycircular domain* in $\mathbf{C}^n$ with centre at the point $w \in \mathbf{C}^n$ and *polyradius* ρ (which is an ordered set $\rho \equiv (\rho_1, \ldots, \rho_n)$ of positive numbers), we mean the set

$$P(w;\rho) = \{z \in \mathbf{C}^n : |z_j - w_j| < \rho_j \quad \text{for } j = 1, \ldots, n\}; \tag{5.3}$$

$\overline{P}(w;\rho)$ is the closure of $P(w;\rho)$. Clearly, $\overline{P}(w;\rho') \subset P(w;\rho)$ if and only if $\rho'_j < \rho_j$ for all $j = 1, \ldots, n$ (in this case, we write $\rho' < \rho$). The following proposition is obtained by an n-fold application of Cauchy's formula in the theory of functions of a complex variable.

Proposition 5.1. *Suppose that the (closed) polycircular domain $P(w;\rho)$ is contained in the domain $D \subset \mathbf{C}^n$. Then any function $f(z)$ that is holomorphic in D can be represented in $P(w;\rho)$ by the n-tuple Cauchy integral:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = \rho_1} \cdots \int_{\zeta_n - w_n| = \rho_n} \frac{f(\zeta) d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)}. \tag{5.4}$$

We use the following notation for an arbitrary value of the multi-index $\alpha \in \overline{\mathbf{Z}}^n_+$:

$$\alpha! = \alpha_1! \ldots \alpha_n!, \tag{5.5}$$

$$\partial^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n}, \tag{5.6}$$

$$z^\alpha = (z_1)^{\alpha_1} \ldots (z_n)^{\alpha_n}; \tag{5.7}$$

in particular, let $I = (1, \ldots, 1)$; then

$$z^I = z_1 \ldots z_n. \tag{5.8}$$

Lemma 5.2. *Any holomorphic function in the domain $D \subset \mathbf{C}^n$ is infinitely differentiable. For each compactum $K \subset D$, there exists a compactum $K' \subset D$ and a polyradius ρ (> 0) such that the following inequality holds for any function $f \in \mathcal{H}(D)$ and any multi-index $\alpha \in \overline{\mathbf{Z}}^n_+$:*

$$\frac{\rho^\alpha}{\alpha!} \|\partial^k f\|^K \leq \|f\|^{K'}. \tag{5.9}$$

■ The infinite differentiability of f is an immediate consequence of (5.4). For the proof of (5.9) we note that by virtue of the compactness of $K \subset D$, there exists a compactum $K' \subset D$ and a polyradius $\rho > 0$ such that $\overline{P}(w;\rho) \subset K'$ for all $w \in K$. Differentiating (5.4), we obtain

$$\frac{1}{\alpha!} \partial^\alpha f(w) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = \rho_1} \cdots \int_{|\zeta_n - w_n| = \rho_n} \frac{f(\zeta) d^n\zeta}{(\zeta - w)^{\alpha + I}}, \tag{5.10}$$

whence the estimate (5.9) follows. ■

Proposition 5.3 *The space $\mathcal{H}(D)$ is a closed subspace of $\mathcal{C}(D)$ and also of $\mathcal{E}(D)$; consequently it is a Fréchet space. The LCS structure on $\mathcal{H}(D)$ induced from $\mathcal{C}(D)$ is the same as that induced from $\mathcal{E}(D)$.*

■ By virtue of Lemma 5.2, we merely have to prove that $\mathcal{H}(D)$ is closed in $\mathcal{C}(D)$. Let $\{f_\nu\}$ be a sequence in $\mathcal{H}(D)$ that converges to a function $f \in \mathcal{C}(D)$ in the $\mathcal{C}(D)$ topology. For any ν there is a representation of type (5.4) with f replaced by f_ν. In this representation we can clearly pass to the limit as $\nu \to \infty$, as a result of which we find that f satisfies (5.4) in any polycircular domain $P(a;\rho)$ whose closure is contained in D. From this it clearly follows that f is a holomorphic function. ■

Proposition 5.4. *For any $\mathcal{H}(D)$ -bounded set $K \subset D$, the seminorm $\|f\|^K$ on $\mathcal{H}(D)$ is continuous; furthermore, there exists a compactum $K' \subset D$ such that*

$$\|f\|^K \leq \|f\|^{K'} \quad \textit{for all } f \in \mathcal{H}(D). \tag{5.11}$$

■ It is clear that for any $z \in K$, the seminorm $|f(z)|$ on $\mathcal{H}(D)$ is continuous; it then follows from the uniform boundedness principle (Theorem 1.7) that the seminorm $\|f\|^K$ is continuous. Consequently, there exists a compactum $K' \subset D$ and a number $c \geq 0$ such that $\|f\|^K \leq c\|f\|^{K'}$ for all $f \in \mathcal{H}(D)$. If $c \leq 1$, then (5.11) is proved. If $c > 1$, then on replacing f by f^N, where $N \in \mathbf{Z}_+$, we obtain $\|f\|^K \leq c^{1/N}\|f\|^{K'}$. Since N was arbitrary, (5.11) now follows. ■

B. HOLOMORPHY AND ANALYTICITY

We shall prove the equivalence of the notions of holomorphy and analyticity. It is well known that the function $(\zeta_j - z_j)^{-1}$ can be represented as the series

$$\frac{1}{\zeta_j - z_j} = \sum_{k=0}^{\infty} \frac{(z_j - w_j)^k}{(\zeta_j - w_j)^{k+1}} \quad \text{for } |\zeta_j - w_j| = \rho_j \text{ and } |z_j - w_j| < \rho_j,$$

which is absolutely and uniformly convergent in ζ_j and in z_j for $|z_j - w_j| \leq \rho'_j$, where $0 < \rho'_j < \rho_j$. Similarly, we have

$$\frac{1}{(\zeta - z)^I} = \sum_{\alpha \in \overline{\mathbf{Z}}^n_+} \frac{(z - w)^\alpha}{(\zeta - w)^{\alpha + I}}, \tag{5.12}$$

where the series converges absolutely and uniformly in ζ for $|\zeta_j - w_j| = \rho_j$ ($j = 1, \ldots, n$) and in $z \in \overline{P}(w, \rho')$ for $\rho' < \rho$. Upon substituting (5.12) into (5.4) and integrating term by term (which is possible in view of the above convergence properties of the series (5.12)), we obtain the following power series expansion in $z - w$ for the function $f \in \mathcal{H}(D)$:

$$f(z) = \sum_{\alpha \in \overline{\mathbf{Z}}^n_+} c_\alpha (z - w)^\alpha \quad \text{for } z \in P(w;\rho), \tag{5.13}$$

where

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{|\zeta_1 - w_1| = \rho_1} \cdots \int_{|\zeta_n - w_n| = \rho_n} \frac{f(\zeta) d^n\zeta}{(\zeta - w)^{\alpha + I}}, \tag{5.14}$$

the series (5.13) being absolutely and uniformly convergent for z in any polycircular domain $\overline{P}(w;\rho')$ with polyradius $\rho' < \rho$. (We recall that the polyradius ρ is merely assumed to be such that $\overline{P}(w;\rho) \subset D$.) A comparison of (5.14) with (5.10) yields the formula for the coefficients

$$c_\alpha = \frac{1}{\alpha!} \partial^\alpha f(w), \tag{5.15}$$

so that (5.13) is the Taylor expansion of $f(z)$ at the point w:

$$f(z) = \sum_{\alpha\in\overline{\mathbf{Z}}_+^n} \frac{1}{\alpha!}\partial^\alpha f(w)(z-w)^\alpha. \tag{5.16}$$

For the proof of the converse, we require the following lemma on power series.

Lemma 5.5. *Suppose that the series*

$$\sum_{\alpha\in\overline{\mathbf{Z}}_+^n} c_\alpha(z-w)^\alpha \tag{5.17}$$

is convergent (in some ordering) at a point $z = a$, where $\rho_j = |a_j - w_j| > 0$ for all $j = 1,\dots,n$. Then it is absolutely convergent (and therefore independently of the order of summation) for all z in the polycircular domain $P(w;\rho)$ to a holomorphic function $f(z)$ in $P(w;\rho)$, the convergence being uniform in any polycircular domain $\overline{P}(w;\rho')$, where $\rho' < \rho$.

■ It follows from the convergence of the series (5.17) at $z = a$ that the general term of the series is bounded: $c_\alpha\rho^\alpha \le M$; hence we have the following estimate for all $\rho' < \rho$ and $z \in \overline{P}(w;\rho')$:

$$\sum_\alpha |c_\alpha(z-w)^\alpha| \le M\sum_\alpha \frac{\rho'^\alpha}{\rho^\alpha} = M\prod_{j=1}^n\left(1-\frac{\rho_j'}{\rho_j}\right)^{-1} < +\infty.$$

Thus the series converges absolutely (and uniformly in $\overline{P}(w;\rho')$ for any $\rho' < \rho$) to a continuous function $f(z)$ in $P(w;\rho)$. The holomorphy of f is a consequence of Proposition 5.3 according to which, the limit in the $\mathcal{C}(D)$ topology of a sequence of holomorphic functions in D is holomorphic in D. ■

A combination of Lemma 5.5 and the preceding discussion completes the proof of the following theorem.

Theorem 5.6. *A function $f(z)$ in the domain $D \subset \mathbf{C}^n$ is holomorphic if and only if it is analytic (that is, it can be expanded in a convergent power series in $z - w$ in a neighbourhood of any point $w \in D$). Furthermore, the Taylor series (5.6) of a function $f \in \mathcal{H}(D)$ is (absolutely and uniformly) convergent in any closed polycircular domain $\overline{P}(w;\rho) \subset D$.*

Exercise 5.2. Prove that for any $\mathcal{H}(D)$-bounded set $K \subset D$ there exists a polyradius ρ (> 0) such that for any function $f \in \mathcal{H}(D)$ and any point $w \in K$, the Taylor series (5.16) is (absolutely and uniformly) convergent in the polycircular domain $\overline{P}(w;\rho)$. [Hint: It follows from Proposition 5.4 and Lemma 5.2 that there exists a compactum $K' \subset D$ and a polyradius ρ such that

$$\frac{\rho^\alpha}{\alpha!}|\partial^\alpha f(w)| \le \|f\|^{K'} \tag{5.18}$$

for all $w \in K$, $f \in \mathcal{H}(D)$, $\alpha \in \overline{\mathbf{Z}}_+^n$.]

C. Analytic continuation

We see that a number of important facts from the theory of analytic functions of a single variable (such as the Cauchy integral formula, infinite differentiability and the power series expansion property) go over to analytic functions of several complex variables. Continuing the analogy, we now give the uniqueness theorem and the maximum modulus principle.

Theorem 5.7 (Uniqueness theorem). *If the analytic function f in the domain $D \subset \subset \mathbf{C}^n$ vanishes together with all its derivatives $\partial^\alpha f$ (of all orders) at some point $a \in \in D$, then $f \equiv 0$ in D. In particular, if f vanishes in a complex or real neighbourhood of the point $a \in D$, then $f \equiv 0$ in D.*

■ The set A of all points of D at which f vanishes together with all its derivatives is clearly closed in D (since all the functions $\partial^\alpha f$ are continuous). But this set A is also open. For let $w \in A$ and let $\overline{P}(w;\rho)$ be a closed polycircular domain in D with centre at w. Then f can be expanded in a power series in $P(w;\rho)$, all the coefficients of which are zero. Consequently, the sets A and $D \backslash A$ are open; it now follows from the connectedness of D that $A = D$. ■

Theorem 5.8 (Maximum modulus principle). *The modulus of an analytic function $f(z) \not\equiv$ const in a domain $D \subset \mathbf{C}^n$ cannot assume a (local or global) maximum at any point of D. Furthermore, for any $f \in \mathcal{H}(D)$ and any compactum $K \subset D$, we have the equality:**

$$\|f\|^K = \|f\|^{\partial K}. \tag{5.19}$$

■ Let $f \in \mathcal{H}(D)$ and suppose that $|f|$ assumes a (local) maximum at some point $w \in D$, so that for some polycircular domain $\overline{P}(w;\rho) \subset D$, the inequality $|f(z)| \leq |f(w)|$ holds for all $z \in \overline{P}(w;\rho)$. Then we claim that $f(z) = f(w)$ in $\overline{P}(w;\rho)$, whence by the uniqueness theorem it follows that $f(z)$ =const in D. Thus without loss of generality we may suppose that $f(w) \geq 0$ (this can be achieved by multiplying f by a phase factor). As in the case of a single variable, it follows from the Cauchy integral formula that the following mean-value formula holds for a polycircular domain:

$$f(w) = \frac{1}{\operatorname{mes} P(w;\rho)} \int_{P(w;\rho)} f(z) |d^n z\, d^n z\, d^n \bar{z}|, \tag{5.20}$$

where

$$\operatorname{mes} P(w;\rho) = \int_{P(w;\rho)} |d^n z\, d^n \bar{z}|;$$

consequently

$$\int_{P(w;\rho)} \{f(w) - \operatorname{Re} f(z)\} |d^n z\, d^n \bar{z}| = 0.$$

Since the integrand is continuous and non-negative, we conclude that it is equal to zero: $f(w) - - \operatorname{Re} f(z) = 0$ in $P(w;\rho)$. This together with the inequality $|f(z)| \leq f(w)$ gives the equality $f(z) = = f(w)$ in $P(w;\rho)$ and hence in D. The second statement of the theorem follows from the first by noting that the modulus of a function $f \not\equiv$ const cannot attain its maximum on K at any interior point of K ■

Exercise 5.3. Prove that if an $\mathcal{H}(D)$-bounded subset $K \subset D$ contains the boundary of some compactum with non-empty interior, then $\|f\|^K$ is a norm. [Hint: Use (5.19) and the uniqueness theorem.]

The method of analytic continuation is based on the uniqueness theorem, as illustrated by the following proposition.

Proposition 5.9. *Let $\{D_\lambda\}_{\lambda \in \Lambda}$ be a family of domains in $\mathbf{C}^n$, and $\{f_\lambda\}_{\lambda \in \Lambda}$, a family of functions $f_\lambda \in \mathcal{H}(D_\lambda)$ such that for all λ, μ, the intersection $D_\lambda \cap D_\mu$ is connected (or, in particular, empty) and if it is non-empty, then the functions f_λ and f_μ coincide in a neighbourhood of at least one point of $D_\lambda \cap D_\mu$. Then there exists a (unique) function f that is defined and holomorphic in the open set $D = \bigcup_\lambda D_\lambda$ and is equal to f_λ in D_λ for all λ.*

Exercise 5.4. Prove Proposition 5.9. [Hint: It suffices to verify the compatibility condition $f_\lambda = f_\mu$ in $D_\lambda \cap D_\mu$ for any $\lambda, \mu \in \Lambda$.]

* Here and throughout, ∂A denotes the *boundary* of the subset A of $\mathbf{C}^n$ ($\partial A = \bar{A} \backslash \operatorname{int} A$).

The conditions in Proposition 5.9 can be replaced by somewhat more useful ones in the case when the index set Λ is a connected subset of Euclidean space* (say, in $\mathbf{R}^k$).

Proposition 5.10. *Suppose that the index set Λ is a connected space and that $\{D_\lambda\}_{\lambda\in\Lambda}$ is a family of domains in $\mathbf{C}^n$ that are the sections $\lambda = \text{const}$ of some domain $\Omega \subset \mathbf{C}^n \times \Lambda$, and that for any λ, μ, the intersection $D_\lambda \cap D_\mu$ is connected (or, in particular, empty); assume also that for any $\lambda \in \Lambda$, the set*

$$M_\lambda = \{\mu \in \Lambda :\ D_\lambda \cap D_\mu \textit{ is non-empty}\}$$

is connected. Let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of functions $f_\lambda \in \mathcal{H}(D_\lambda)$, such that for any $\lambda \in \Lambda$ there exists a point $w \in D_\lambda$ and a neighbourhood $U_\lambda \subset \Omega$ of the point (w, λ) such that

$$f_\lambda(z) = f_\mu(z) \quad \textit{for all} \quad (\mu, z) \in U_\lambda.$$

Then there exists a (unique) function f that is analytic in the domain $D = \bigcup_\lambda D_\lambda$ and is equal to f_λ in D_λ for any λ.

■ It suffices to prove that for any $\lambda \in \Lambda$ the subset $M'_\lambda = \{\mu \in M_\lambda : f_\lambda = f_\mu \text{ in } D_\lambda \cap D_\mu\}$ is equal to M_λ (since the conditions of Proposition 5.9 then hold). The set M'_λ contains a neighbourhood of λ in Λ since by hypothesis, $f_\lambda = f_\mu$ in a neighbourhood of some point in $D_\lambda \cap D_\mu$ for all points $\mu \in \Lambda$ sufficiently close to λ; it follows from the connectedness of $D_\lambda \cap D_\mu$ that $f_\lambda = f_\mu$ in $D_\lambda \cap D_\mu$. Along with λ, we then fix an arbitrary point μ in the closure of M'_λ in M_λ. Since $\mu \in M_\lambda$, the domains D_λ and D_μ have a point ζ in common. The point (ζ, μ) lies in Ω together with some neighbourhood, which we can take in the form $\Delta \times N_\mu$, where Δ is a neighbourhood of ζ in $\mathbf{C}^n$ and N_μ is a neighbourhood of μ in Λ. Since it was proved above that μ lies in M'_μ together with some neighbourhood in Λ, we can suppose that $N_\mu \subset M'_\mu$ (by shrinking N_μ if necessary). Then in particular, we have $f_\mu(z) = f_\nu(z)$ for all $\nu \in N_\mu$, $z \in \Delta$. Since by construction, $f_\lambda(z) = f_\mu(z)$ for $z \in \Delta$, the functions f_λ and f_ν are equal in Δ for any $\nu \in N_\mu$; hence they are equal in $D_\lambda \cap D_\nu$ (because $D_\lambda \cap D_\nu$ is connected). This means that N_μ is contained in M'_λ. Thus M'_λ is non-empty, open and closed in M_λ. But M_λ is connected, therefore $M'_\lambda = M_\lambda$. ■

In Propositions 5.9 and 5.10, the result of the analytic continuation turns out to be a single-valued analytic function. In the general case, we have to consider the possibility of the phenomenon of many-valuedness. In this connection we give the following definition. Let D_1 and D_2 be two domains in $\mathbf{C}^n$ such that $D_1 \cap D_2$ is non-empty and let $f_1 \in \mathcal{H}(D_1)$, $f_2 \in \mathcal{H}(D_2)$ be a pair of analytic functions such that $f_1 = f_2$ in some domain D_{12} contained in $D_1 \cap D_2$ and hence (by the uniqueness theorem) $f_1 = f_2$ in the connected component of $D_1 \cap D_2$ that contains D_{12}. Then we say that f_2 is a *direct analytic continuation* from D_1 to D_2 via the route (D_1, D_{12}, D_2). (This continuation is non-trivial if D_2 is not contained in D_1, which is equivalent to D_2 containing at least one boundary point of D_1.) Of course if $D_1 \cap D_2$ is disconnected, this does not yet mean that a single-valued analytic continuation of the function f_1 is defined in $D_1 \cup D_2$.** Continuing this process, we say that an analytic function f' in the domain D' is an *analytic continuation* of the analytic function f in the domain D via the *route*

$$(D = D_1, D_{12}, D_2, D_{23}, \ldots, D_{N-1}, D_{N-1,N}, D_N = D'), \tag{5.21}$$

* More generally, Λ can be taken to be any connected topological space.

** For this reason, the definition of analytic continuation (and holomorphic continuation) allows the possibility that $D_1 \not\subset D_2$

if $D = D_1, D_2, \ldots, D_N = D'$ is a finite sequence of domains in $\mathbf{C}^n$ in each of which the analytic function $f = f_1, f_2, \ldots, f_N = f'$ is defined, where $f_{\nu+1}$ is a direct analytic continuation of f_ν via the route $(D_\nu, D_{\nu,\nu+1}, D_{\nu+1})$, $\nu = 1, \ldots, N-1$. Such a continuation is *trivial* if $D' \subset D$ and $f' = f$ in D'.

Let $\{D_\lambda\}_{\lambda\in\Lambda}$ be a given family of domains in $\mathbf{C}^n$ and let $\{f_\lambda\}_{\lambda\in\Lambda}$ be a family of analytic functions $f_\lambda \in \mathcal{H}(D_\lambda)$, such that any two functions f_λ, f_μ can be analytically continued from one to the other; then we say that the family $\{f_\lambda\}_{\lambda\in\Lambda}$ represents (in general) a *many-valued analytic function* in the domain $D = \bigcup_\lambda D_\lambda$.

Remark. If the domain Ω is simply connected, then (according to the classical monodromy theorem [V4],§6.6) a many-valued analytic function reduces, in fact, to a single-valued one, that is, for any λ, the function f_λ is the restriction to D_λ of some function $f \in \mathcal{H}(D)$.

A domain D is called a *domain of holomorphy* if there exists a function $f \in \mathcal{H}(D)$ that has no non-trivial analytic continuations; in this case, we also call D a *natural domain of holomorphy of the function f*. It is well known that every domain in the complex plane is a domain of holomorphy.* However for $n > 1$, not every domain in $\mathbf{C}^n$ is a domain of holomorphy and the problem of continuing all holomorphic functions in a given domain beyond its boundary becomes an interesting one. The most important aspect of the theory of functions of several complex variables is to do with this phenomenon of forced analytic continuation.

We say that the domain $D \subset \mathbf{C}^n$ is *holomorphically extended* to the domain $D' \subset \mathbf{C}^n$ via the route (5.21) if every function $f \in \mathcal{H}(D)$ has an analytic continuation $f' \in \mathcal{H}(D')$ via this route. In the case when the route (D, D_{12}, D') is such that D_{12} is in the connected component of $D \cap D'$, we talk about a *direct holomorphic extension* from D to D'. It goes without saying that a *trivial "extension"* (when $D' \subset D$ and $f' = f$ in D' for all $f \in \mathcal{H}(D)$) is of no interest. It turns out (see §5.2.A) that a domain D has no non-trivial holomorphic extensions if and only if it is a domain of holomorphy.

We give an example of a domain which is not a domain of holomorphy.

Exercise 5.5. Let D be a domain in $\mathbf{C}^2$ that does not contain the origin and is a neighbourhood of the boundary of the polycircular domain P in $\mathbf{C}^2$ with centre at 0 and with polyradius (1,1). Prove that each function $f \in \mathcal{H}(D)$ admits an analytic continuation to P. [Hint: The Cauchy formula with respect to z_2 is valid for $f(z_1, z_2)$ in a neighbourhood of the points $|z_1| = 1$, $|z_2| \le 1$:

$$f(z_1, z_2) = \frac{1}{2\pi i}\int_{|\zeta_2|=1} \frac{f(z_1, \zeta_2)}{\zeta_2 - z_2}\, d\zeta_2,$$

which defines an analytic continuation to P.]

The method used in the preceding exercise enables one to obtain a multi-dimensional corollary of the well known theorem on the absence of isolated singularities for a bounded analytic function of a complex variable.

Exercise 5.6. Let $g(z) \not\equiv 0$ be a holomorphic function in the domain $D \subset \mathbf{C}^n$, and $f(z)$ a function that is analytic in the domain $D \backslash g^{-1}\{0\}$ and bounded on any set of the form $K \backslash g^{-1}\{0\}$, where K is a compactum in D. Prove that $f(z)$ is analytic in D. [Hint: It is enough to verify that $f(z)$ is analytic at any point $w \in D \cap g^{-1}\{0\}$; suppose for definiteness, that this point is $w = 0$. It can be assumed without loss of generality that the system of coordinates is chosen so that the analytic function $g(0, \ldots, 0, z_n)$ of z_n is not identically zero. Choose numbers $r_j > 0$ such that $g(z)$ is non-zero for $|z_1| < r_1, \ldots, |z_{n-1}| < r_{n-1}$, $|z_n| = r_n$. Use the theorem referred to in front of this exercise to

* This follows from Exercise 5.15 and Theorem 5.24.

obtain the representation

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta_n|=r_n} \frac{f(z_1, \ldots, z_{n-1}, \zeta_n)}{\zeta_n - z_n} \, d\zeta_n$$

in the polycircular domain $|z_1| < r_1, \ldots, |z_n| < r_n$.]

D. GENERALIZED PRINCIPLE OF ANALYTIC CONTINUATION; "EDGE OF THE WEDGE" THEOREM

There follows from the generalized uniqueness theorem B.10 (see Appendix B) a corresponding generalized principle of analytic continuation. Let Ω_1 and Ω_2 be two convex domains in $\mathbf{R}^n$ whose boundaries contain 0, and let D_1 and D_2 be two convex domains in $\mathbf{C}^n$ that are adjacent to the real domain $\mathcal{O} \subset \mathbf{R}^n$ from the sides $i\Omega_1$ and $i\Omega_2$ respectively (see §B.4). If f_1 and f_2 are two holomorphic functions in D_1 and D_2 respectively having the same generalized boundary values

$$\lim_{y \to 0,\, y \in \Omega_1} f_1(x + iy) = \lim_{y \to 0,\, y \in \Omega_2} f_2(x + iy) \text{ in } \mathcal{D}'(\mathcal{O}), \tag{5.22}$$

then we say that f_2 is a *generalized analytic continuation of the function f_1 along the generalized route* $(D_1 \cap \mathcal{T}^{\Omega_1}, \mathcal{O}, D_2 \cap \mathcal{T}^{\Omega_2})$. It turns out that a generalized analytic continuation reduces to an ordinary one, that is, there exists a domain $G \subset \mathbf{C}^n$ adjacent to $\mathcal{O}$ from the side $i\Omega$, where $\Omega = \Omega_1 + \Omega_2$, in which the functions f_1 and f_2 have a common (direct) analytic continuation. This statement is known as the "edge of the wedge" theorem (which expresses its obvious geometric meaning) and the domain G featuring in the theorem is called the *"edge of the wedge" domain* corresponding to the generalized route $(D_1 \cap \mathcal{T}^{\Omega_1}, \mathcal{O}, D_2 \cap \mathcal{T}^{\Omega_2})$. The simplest version of the "gluing" of analytic functions along a boundary is provided by the following lemma (which is nothing more than a restatement of Corollary B.3 taking Theorem B.11 into account).

Lemma 5.11. *Let Ω_1 and Ω_2 be two convex domains in $\mathbf{R}^n$ such that $0 \in \partial\Omega_1 \cap \cap \partial\Omega_2$, and let Ω be the convex hull of $\Omega_1 \cup \Omega_2$. If f_1 and f_2 are two holomorphic functions in the tubes $\mathcal{T}^{\Omega_1}$ and $\mathcal{T}^{\Omega_2}$ respectively, that are the Laplace transforms there of generalized functions in $\mathcal{S}'(\mathbf{R}_n)$, and if the generalized boundary values*

$$\lim_{y \to 0,\, y \in \Omega_1} f_1(x + iy) \quad \text{and} \quad \lim_{y \to 0,\, y \in \Omega_2} f_2(x + iy) \text{ in } \mathcal{S}'(\mathbf{R}^n),$$

are equal, then it follows that there exists a function f that is holomorphic in the tube $\mathcal{T}^{\Omega}$ and is equal to f_j in $\mathcal{T}^{\Omega_j}$ (for $j = 1, 2$).

The "edge of the wedge" theorem generalizes this lemma to the case when the domain of coincidence may not be the entire real space $\mathbf{R}^n$.

Theorem 5.12 ("edge of the wedge"). *Let Ω_1 and Ω_2 be two convex domains in $\mathbf{R}^n$ whose boundaries contain 0, and D_1 and D_2 two domains in $\mathbf{C}^n$ that are adjacent to the real domain $\mathcal{O} \subset \mathbf{R}^n$ from the sides $i\Omega_1$ and $i\Omega_2$ respectively. If the analytic f_2 in D_2 is a generalized analytic continuation of the analytic function f_1 in D_1 through $\mathcal{O}$, then f_1 and f_2 have the same (direct) analytic continuation f to some domain G that is adjacent to $\mathcal{O}$ from the side $i\Omega$, where $\Omega = \Omega_1 + \Omega_2$. In particular, if $D_\alpha \supset \mathcal{O} + i\Omega_\alpha$ $(\alpha = 1, 2)$, then G can be taken to be*

$$G = \{z = x + i(y^{(1)} + y^{(2)}) : \ x \in \mathcal{O}, \ y^{(\alpha)} \in \vartheta\Omega_\alpha,$$

$$|y^{(\alpha)}| < \vartheta\, d(x, \partial\mathcal{O}) \ \textit{for}\ \alpha = 1\ \textit{and}\ 2\}, \tag{5.23}$$

*where** ϑ *is a positive constant (less than unity), depending only on* n.

■ We may suppose without loss of generality that $D_\alpha \supset \mathcal{O}+i\Omega_\alpha$ ($\alpha = 1, 2$) (otherwise we replace the Ω_α by arbitrary truncated convex cones $K_\alpha^r \subset \Omega_\alpha$). We begin by carrying out a construction in a neighbourhood of an arbitrary point of $a \in \mathcal{O}$ which will enable us to obtain a local analytic continuation of the functions f_1 and f_2. The aim of this construction is to reduce the local situation to Lemma 5.11. The second stage of the proof consists in gluing the local continuations so obtained; here we merely have to verify that the local continuations together define a single-valued function f (with the required properties).

We fix an arbitrary point $a \in \mathcal{O}$ and an arbitrary set $e_1, \dots, e_n$ of linearly independent vectors lying in $\Omega_1 \cup \Omega_2$ and with norms $|e_j| < d(a, \partial\mathcal{O})$. What we have to prove is that there exists a number $\vartheta \in (0, 1)$ (not dependent on $a, e_1, \dots, e_n$) such that in the domain (parallelepiped)

$$G(a; e_1, \dots, e_n) = \Big\{z \in \mathbf{C}^n : z = a + \sum_{j=1}^n \zeta_j e_j, |\operatorname{Re}\zeta_j| < \vartheta,\ 0 < \operatorname{Im}\zeta_j < \vartheta \text{ for } j = 1, \dots, n\Big\}$$

a holomorphic function f is defined which coincides with f_α in $G(a; e_1, \dots, e_n) \cap D_\alpha$. Suppose for definiteness that $\{e_1, \dots, e_\nu\} \subset \Omega_1$ and $\{e_{\nu+1}, \dots, e_n\} \subset \Omega_2$ for some ν. We denote by $\zeta_j \equiv \zeta_j + i\eta_j$ the coordinates of the vector $z - a$ with respect to the basis $\{e_1, \dots, e_n\}$:

$$z = a + \sum_{j=1}^n \zeta_j e_j$$

(where z is an arbitrary point of $\mathbf{C}^n$) and we write $f_\alpha(\zeta_1, \dots, \zeta_n)$ instead of $f_\alpha(z)$. We now replace the sets $\mathcal{O}, \Omega_1$ and Ω_2 by the smaller but more convenient sets $Q \subset \mathcal{O}$, $\omega_1 \subset \Omega_1$, $\omega_2 \subset \Omega_2$, by setting (where $\rho = 1/n$)

$$Q = \{x \in \mathbf{R}^n : |\xi_j| < \rho \text{ for } j = 1, \dots, n\},$$

$$\omega_1 = \{x \in \mathbf{R}^n : 0 < \eta_j < \rho \text{ for } j = 1, \dots, \nu \text{ and } \eta_j = 0 \text{ for } j = \nu + 1, \dots, n\},$$

$$\omega_2 = \{x \in \mathbf{R}^n : \eta_j = 0 \text{ for } j = 1, \dots, \nu \text{ and } 0 < \eta_j < \rho \text{ for } j = \nu + 1, \dots, n\}.$$

Then the function f_α satisfies the following type of estimate on the set $Q + i\omega_\alpha$:

$$|f_\alpha(z)| \le A\Big(\max_{j=1,\dots,n} |\eta_j|\Big)^{-l} \quad \text{for } \zeta \in Q + i\omega_\alpha. \tag{5.24}$$

(This follows from Exercise B.9(b).)

We now introduce the conformal map $\phi(\zeta) = \frac{2}{\pi}\ln\frac{\rho+\zeta}{\rho-\zeta}$, which takes the disc of radius ρ and centre at 0 in $\mathbf{C}$ onto the strip $|\operatorname{Im} w| < 1$. We now change from $\zeta_1, \dots, \zeta_n$ to the new variables $w_1, \dots, w_n$ by setting

$$w_j = \phi(\zeta_j) = \frac{2}{\pi}\ln\frac{\rho + \zeta_j}{\rho - \zeta_j}, \text{ so that } \zeta_j = \phi^{-1}(w_j) = \rho\tanh\Big(\frac{\pi}{4}w_j\Big),$$

and we set

$$g_\alpha(w_1, \dots, w_n) = \Big\{\prod_{j=1}^n (\rho^2 - \zeta_j^2)^l\Big\} f_\alpha(\zeta_1, \dots, \zeta_n). \tag{5.25}$$

It is clear that $g(w_1, \dots, w_n)$, defined in the tube T^{S_1} with base

$$S_1 = \{v \in \mathbf{R}^n : 0 < v_j < 1 \text{ for } j = 1, \dots, \nu; \quad v_j = 0 \text{ for } j = \nu + 1, \dots, n\},$$

is continuously differentiable in all the variables and is holomorphic in $w_1, \dots, w_\nu$. Furthermore, it satisfies an estimate of type

$$|g_1(w_1, \dots, w_n)| \le A'(\max_{j=1,\dots,n} |\operatorname{Im} w_j|)^{-l} \text{ in } T^{\Omega_1}. \tag{5.26}$$

* Here $d(z, A)$ denotes the distance from $z \in \mathbf{C}^n$ to the set $A \subset \mathbf{C}^n$.

This estimate is obtained from (5.24), (5.25) and the inequality *

$$|\rho^2 - \zeta_j^2||\operatorname{Im}\zeta_j|^{-1} \le 4\rho|\operatorname{Im} w_j|^{-1} \quad \text{for } |\zeta_j| < \rho.$$

Similarly, the function g_2 defined in the tube $\mathcal{T}^{S_2}$ with base

$$S_2 = \{v \in \mathbf{R}^n : v_j = 0 \text{ for } j = 1, \dots, \nu; \quad 0 < v_j < 1 \text{ for } j = \nu+1, \dots, n\},$$

is infinitely differentiable in all the variables, holomorphic in $w_{\nu+1}, \dots, w_n$ and satisfies an estimate of type (5.26) in $\mathcal{T}^{S_2}$. Then by Propositions B.6 and B.11, each of the functions g_α ($\alpha = 1$ and 2) is the Laplace transform of a generalized function $T_\alpha \in \mathcal{S}'(\mathbf{R}_n)$ and has a generalized boundary value, say, $h_\alpha \in \mathcal{S}'(\mathbf{R}^n)$; furthermore, since we can pass to the generalized boundary value and take the conformal transformation in either order (see Proposition B.12) and by virtue of Exercise B.8, we have the following equality in $\mathcal{D}'(Q)$:

$$\lim_{\eta \to 0\, \eta \in \omega_\alpha} \Big\{\prod_{j=1}^{n} (\rho^2 - \zeta_j^2)^{-l}\Big\} g_\alpha(\phi(\zeta_1), \dots, \phi(\zeta_n)) = \Big\{\prod_{j=1}^{n} (\rho^2 - \xi_j^2)^{-l}\Big\} h_\alpha(\phi(\xi_1), \dots, \phi(\xi_n)). \tag{5.27}$$

According to (5.25), the left hand side of this equality is $\lim_{y \to 0,\, y \in \omega_\alpha} f_\alpha(x+iy)$, where for $\alpha = 1$ and 2 these distributions are the same in view of (5.22). It follows that for $\alpha = 1$ and 2, the right hand sides of (5.27) are also the same, that is, $h_1 = h_2$.

We have now arrived at the situation stated in Lemma 5.11 (but now the roles of f_α and $\mathcal{T}^{\Omega_\alpha}$ are assumed by g_α and $\mathcal{T}^{S_\alpha}$). This enables us to conclude that the functions g_1 and g_2 have a common holomorphic continuation g to the tube $\mathcal{T}^S$ with base $S = S_1 + S_2$, where

$$\lim_{\eta \to 0,\, \eta \in S} g(\xi + i\eta) = h(\xi) \quad \text{in} \quad \mathcal{S}'(\mathbf{R}^n) \tag{5.28a}$$

(and where $h = h_1 = h_2$). It should be noted that S contains the set $\{\eta \in \mathbf{R}^n : 0 < \eta_j < 1/2 \text{ for } j = 1, \dots, n\}$. Therefore, by choosing $\vartheta > 0$ sufficiently small** (namely, so that the image of the rectangle $\{\zeta \in \mathbf{C} : |\operatorname{Re}\zeta| < \vartheta,\ 0 < \operatorname{Im}\zeta < \vartheta\}$ under the conformal map $\zeta \to w = \frac{2}{\pi} \ln \frac{\rho+\zeta}{\rho-\zeta}$ be contained in the strip $\{w \in \mathbf{C} : |\operatorname{Im} w| < 1/2\}$), we conclude from (5.25) that the functions $f_1(z)$ and $f_2(z)$ have a common holomorphic continuation to the domain

$$G = \{z \in \mathbf{C}^n : |\xi_j| < \vartheta,\ 0 < \eta_j < \vartheta \quad \text{for } j = 1, \dots, n\};$$

here (again thanks to the general formula (B.59c)), (5.28a) implies that

$$\lim_{y \to 0,\, y \in \omega_\alpha} f(x+iy) = \lim_{y \to 0,\, y \in \omega_\alpha} f_\alpha(x+iy) \quad \text{for } |\xi_j| < \vartheta. \tag{5.28b}$$

Thus we have carried out the first stage of the proof, which establishes the existence of a common analytic continuation of f_1 and f_2 to any of the domains $G(a; e_1, \dots, e_n)$ of the form indicated at the beginning of the proof. It merely remains for us to check that these continuations together define a single-valued function in the union of such domains. Thus let $G_1 = U_1 + iV_1$ and $G_2 = U_2 + iV_2$ be an

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* In fact, if we take the branch of $\arg w$ to be defined in the usual way (with values in the interval $(-\pi/2, \pi/2)$ for $\operatorname{Re} w > 0$, we have

$$\begin{aligned}
\operatorname{Im} w_j| &\le \frac{2}{\pi}\{|\arg(\rho+\zeta_j)| + |\arg(\rho-\zeta_j)|\} \le \\
&\le \frac{2}{\pi}\Big\{\Big|\frac{\pi}{2}\sin\arg(\rho+\zeta_j)\Big| + \Big|\frac{\pi}{2}\sin\arg(\rho-\zeta_j)\Big|\Big\} \le \\
&\le |\rho+\zeta_j|^{-1}|\operatorname{Im}\zeta_j| + |\rho-\zeta_j|^{-1}|\operatorname{Im}\zeta_j| \le 4\rho|\rho^2 - \zeta_j^2|^{-1}|\operatorname{Im}\zeta_j|.
\end{aligned}$$

** For example, ϑ can be taken to be $(\sqrt{3}-1)/2n$.

arbitrary pair of such domains (where U_1, U_2, V_1 and V_2 are parallelepipeds in $\mathbf{R}^n$) and let f_1' and f_2' be holomorphic continuations of f_1, f_2 to the domains G_1 and G_2 respectively. We claim that $f_1' = f_2'$ in $G_1 \cap G_2$. For the difference $f_1' - f_2'$ is defined and holomorphic in $G_1 \cap G_2 = (U_1 \cap U_2) + i(V_1 \cap V_2)$, and according to (5.28b), converges to 0 in $\mathcal{D}'(U \cap U')$ as $y \to 0$, $y \in V \cap V'$. Then by the generalized uniqueness theorem B.10, we conclude that $f_1' - f_2' = 0$. ∎

The explicit form (5.23) of the "edge of the wedge" domain is not, in fact, of great significance (since this domain is not optimal, that is, is not maximal possible, apart from the case when the D_α are tubular cones and $\mathcal{O} = \mathbf{R}^n$). The main point is the very possibility of a common analytic continuation of the functions f_1 and f_2, which means that the generalized principle of analytic continuation reduces to the ordinary one. (It is interesting that as a result of this kind of continuation, the real domain of coincidence of the functions f_1 and f_2 can be extended by comparison with the original domain $\mathcal{O}$ exclusively "from geometric considerations"; we have already encountered this phenomenon in Exercise 4.11; see also [V4],§28.)

We now indicate a generalization in the statement of the "edge of the wedge" theorem. It is not necessary to assume that Ω_1 and Ω_2 are domains in $\mathbf{R}^n$; we can equally well suppose that Ω_1 and Ω_2 are two convex subsets of $\mathbf{R}^n$ whose boundaries contain 0 and are such that the convex hull of their union is a domain. The statement and proof of Theorem 5.12 then go over to the more general case.

This generalized version of the theorem is used in the following exercise.

Exercise 5.7. Let f be an analytic function in the domain $D \subset \mathbf{C}^n$ containing the tubes $\mathcal{T}^{\Omega_1}$ and $\mathcal{T}^{\Omega_2}$, where the bases Ω_1, Ω_2 of these tubes are convex subsets of $\mathbf{R}^n$ with a non-empty intersection* such that the convex hull of Ω of their union is a domain in $\mathbf{R}^n$.

(a) Let $a \in \Omega_1 \cap \Omega_2$, $\xi \in \Omega_1$, $\eta \in \Omega_2$. Prove that f has a direct analytic continuation to the tube $\mathcal{T}^\omega$, where ω is a convex neighbourhood of the triangle in $\mathbf{R}^n$ with vertices $a, a+\vartheta(\xi - a)$, $a+\vartheta(\eta - a)$, and ϑ is the same parameter as in (5.23). [Hint: After translating the tube through the vector $-ia$, the situation reduces to the "edge of the wedge" theorem in the generalized version.]

(b) Prove that f has a direct analytic continuation to the tube $\mathcal{T}^\Omega$. [Hint: Choose arbitrary points $a \in \Omega_1 \cap \Omega_2$, $\xi \in \Omega_1$, $\eta \in \Omega_2$ and let T be the set of all $t \in [0,1]$ such that f has a direct analytic continuation to $\mathcal{T}^\omega$ where ω is a convex neighbourhood of the closed triangle in $\mathbf{R}^n$ with vertices a, $a + t(\xi - a)$, $a + t(\eta - a)$. It is sufficient to prove that $T = [0,1]$. First verify that T is open in [0,1]. Use part (a) to prove that T is closed. Conclude from the fact that T is connected that $T = [0,1]$.]

Remark. One can draw the following conclusion from the proof of the "edge of the wedge" theorem (see formula (5.28)): the generalized boundary value $\lim\limits_{y\to 0,\, y\in\Omega} f(x+iy)$ exists for $x \in \mathcal{O}$, where

$$\lim_{y\to 0,\, y\in\Omega} f(x+iy) = \lim_{y\to 0,\, y\in\Omega_\alpha} f_\alpha(x+iy) \text{ in } \mathcal{D}'(\mathcal{O})\ (\alpha = 1,2). \tag{5.29}$$

E. HOLOMORPHIC DISTRIBUTIONS

As before, let D be a domain in $\mathbf{C}^n$. A distribution $f(z) \in \mathcal{D}'(D)$ is said to be *holomorphic* if it satisfies the system of Cauchy-Riemann equations (5.1). It is clear that every holomorphic function $f \in \mathcal{H}(D)$ corresponds to a holomorphic distribution according to the rule

$$(f,u) = \int f(z)u(z)|d^n z\, d^n \bar{z}|, \ u \in \mathcal{D}(D). \tag{5.30}$$

The converse also turns out to be true.

* Here it is not required that Ω_1 and Ω_2 be domains in $\mathbf{R}^n$.

Proposition 5.13. (a) *Every holomorphic distribution $f(z) \in \mathcal{D}'(D)$ is a function in $\mathcal{H}(D)$.*

(b) *If the sequence of holomorphic distributions $f_\nu(z)$ converges to the distribution $f(z)$ in (the weak topology of) $\mathcal{D}'(D)$, then $f \in \mathcal{H}(D)$ and $f_\nu \to f$ in the $\mathcal{H}(D)$ topology.*

■ (a) Let $\overline{P}(w;\rho)$ be an arbitrary closed polycircular domain in D and let $\chi(z)$ be a function in $\mathcal{D}(D)$ that is equal to one in some neighbourhood of $\overline{P}(w;\rho)$. We denote by $F(z)$ the generalized function in $\mathcal{S}'(\mathbf{C}^n)$ equal to $\chi(z)f(z)$ in D and vanishing outside the support of χ. Clearly the support of F is compact and is contained in some polycircular domain $\overline{P}(w;R)$ with $\rho < R$. Bearing in mind the formula

$$\frac{\partial}{\partial \bar{z}} z_j^{-1} = 2\pi\delta(z_j) \tag{5.31}$$

(see formulae (C.54)–(C.56)), we find that $F(z)$ has a representation in the form of a convolution:

$$F(z) = F(z) * \delta(z) = (2\pi)^{-n}\bar{\partial}^I F(z) * \frac{1}{z^I}. \tag{5.32}$$

We set $\psi(z) = (2\pi)^{-n}\bar{\partial}^I F(z)$. This is a generalized function in $\mathcal{S}'(\mathbf{C}^n)$ which (by virtue of the holomorphy of $f(z)$) has compact support in $Q = \overline{P}(w;R)\backslash P(w;\rho)$. Restricting the representation (5.32) to the polycircular domain $P(w;\rho)$, we find that $f(z)$ has the weak integral representation:

$$f(z) = \int \frac{1}{(z-\zeta)^I}\psi(\zeta)|d^n\zeta d^n\bar{\zeta}| = \left(\psi(\zeta), \frac{1}{(z-\zeta)^I}\right)_\zeta \quad \text{for } z \in P(w;\rho). \tag{5.33}$$

The kernel $1/(z-\zeta)^I$ of this representation is a test function with respect to ζ in the space $\mathcal{S}(Q)$ which is analytically dependent on the parameter $z \in P(w;\rho)$. This proves that $f(z)$ is holomorphic in $P(w;\rho)$ and hence (in view of the arbitrariness of $\overline{P}(w,\rho)$ in D) throughout D.

(b) The second part of Proposition 5.13 is also obtained by applying the representation (5.33) to each distribution $f_\nu(z)$:

$$f_\nu(z) = \left(\psi_\nu(\zeta), \frac{1}{(z-\zeta)^I}\right)_\zeta \quad \text{for } z \in P(w;\rho).$$

Since $\psi_\nu \to \psi$ in $\mathcal{S}'(Q)$ and $1/(z-\zeta)^I$ is a test function in $\mathcal{S}(Q)$ with respect to ζ which is analytically dependent on the parameter z, statement (b) follows from Corollary 1.10. ■

Exercise 5.8. Prove the following generalization of the classical Liouville's theorem: an arbitrary holomorphic generalized function $f(z) \in \mathcal{S}'(\mathbf{C}^n)$ is a polynomial in z. [Hint: It follows from the Cauchy-Riemann equations that the Fourier transform

$$\tilde{f}(k) = \int f(z)e^{i\,\mathrm{Re}\,kz}|d^n z\, d^n \bar{z}|$$

of $f(z)$ is concentrated at the origin.]

Let z and ξ be variables ranging respectively over the domains D in $\mathbf{C}^n$ and Ω in $\mathbf{R}^k$. Then the distribution $f(z,\xi) \in \mathcal{D}'(D\times\Omega)$ is said to be holomorphic in z (or *partially holomorphic*) if it satisfies the Cauchy-Riemann equations $\frac{\partial}{\partial \bar{z}_j} f(z,\xi) = 0$ ($j = 1,\ldots,n$). Then for any test function $v \in \mathcal{D}(\Omega)$, the distribution $\int f(z,\xi)v(\xi)d^k\xi$ with respect to z is holomorphic and hence (by Proposition 5.13) belongs to $\mathcal{H}(D)$. Thus $f(z,\xi)$ turns out to be a distribution with respect to ξ in $\mathcal{D}'(\Omega)$ which is holomorphically dependent on $z \in D$ as a parameter (see Proposition 2.11). By taking Proposition 2.10 into account, we obtain the following result (which we shall repeatedly use in the subsequent chapters).

Proposition 5.14. *If D is a domain in $\mathbf{C}^n$ and Ω is a domain in $\mathbf{R}^k$, then every partially holomorphic distribution (with respect to z) $f(z,\xi) \in \mathcal{D}'(D\times\Omega)$ is a distribution in $\mathcal{D}'(\Omega)$ with respect to ξ which is holomorphically dependent on $z \in \Omega$*

as a parameter. If for each $v \in \mathcal{D}(\Omega)$ the functions $\int f(z,\xi)v(\xi)d^n\xi$ are continued analytically to the domain D' ($D \subset D' \subset \mathbf{C}^n$), then they define a distribution with respect to ξ in $\mathcal{D}'(\Omega)$ which is holomorphically dependent on the parameter $z \in D'$.

F. Invariant and covariant analytic functions

We shall confine our attention to the concrete group $G = O_+(d)$ or $O(d)$ acting in $\mathbf{C}^{dk} \equiv \mathbf{C}^d \times \ldots \times \mathbf{C}^d$ according to formula (3.151). The number k is assumed to be less than d in the case $O_+(d)$ or not exceeding d in the case $O(d)$, so that there exists a set of algebraically independent standard polynomial invariants $I_1(z), \ldots, I_\nu(z)$ in $\mathbf{C}^n$ (here $n = dk$, $\nu = k(k+1)/2$). By identifying (as in §3.4.A) an arbitrary point $z \in \mathbf{C}^{dk}$ with the ordered set $z \equiv (z_1, \ldots, z_k)$ of vectors $z_1, \ldots, z_k \in \mathbf{C}^d$, we can take these invariants to be the scalar products (z_i, z_j), where $1 \leq i \leq j \leq k$.

We are interested in the possibility of representing an invariant analytic function $f(z)$ in the $O_{(+)}(d)$-invariant domain $D \subset \mathbf{C}^n$ as an analytic function of the invariants:

$$f(z) = \phi(I(z)). \tag{5.34}$$

We note that the polynomials $I_1(z), \ldots, I_\nu(z)$ are in fact invariants with respect to the larger group $O(d,C)$, namely, the complexification of $O(d)$. [More precisely, $O(d,C)$ (and similarly $O_+(d,C)$) is the group of complex orthogonal $d \times d$-matrices (respectively with determinant 1).] Consequently the representation (5.34) gives an analytic continuation of the function $f(z)$ to the $O(d,C)$-saturation of the domain D, that is, to the domain $O(d,C)D = \cup_{g \in O(d,C)} gD$. Therefore in the representation (5.34) it suffices to confine ourselves to $O_{(+)}(d,C)$-invariant domains (or functions).

For $d = 4$, the space $\mathbf{C}^d$ can be identified with the complex Minkowski space $\mathbf{CM}$, and $O_{(+)}(d,C)$ with the complex (proper) Lorentz group $L_{(+)}(C)$ (see §9.1.A).

Proposition 5.15. *Let D be an $O_{(+)}(d,C)$-invariant* domain in $\mathbf{C}^n = \mathbf{C}^{dk}$ (where $k < d$ for $O_+(d,C)$ and $k \leq d$ for $O(d,C)$), and suppose that the "technical" assumption holds: for any point $w \in I(D)$ there exists a compactum $Q \subset D$ such that $w \in \operatorname{int} I(Q)$, that is*

$$I(D) = \bigcup \operatorname{int} I(Q), \tag{5.35}$$

the union being taken over all compact subsets Q of D. (Here $I(z) \equiv (I_1(z), \ldots, I_\nu(z))$ is a map from $\mathbf{C}^{dk}$ to $\mathbf{C}^\nu$ for $\nu = k(k+1)/2$, consisting of the standard polynomial invariants $I_1(z), \ldots, I_\nu(z)$, which in our case are the scalar products (z_i, z_j), $1 \leq i \leq$ $\leq j \leq k$.) Then each $O_{(+)}(d)$-invariant analytic function $f(z)$ in D can be uniquely represented in the form (5.34), *where ϕ is an analytic function in the domain $I(D) \subset \mathbf{C}^\nu$.*

■ Our proof is somewhat sketchy: in certain places we refer the reader to the article by Hall and Wightman (1957) for the details. We begin by deriving the representation (5.34) for the domain $D_0 = \{z \in D : p(I(z)) \neq 0\}$, where $p(I(z))$ is the determinant of the $k \times k$-matrix (z_i, z_j) of the scalar products. (The polynomial $p(I(z))$ is not identically zero, since this would contradict the condition that the standard invariants $I_1, \ldots, I_\nu$ be algebraically independent in the present instance.) It is not difficult to see that if $z \in D_0$, then the vectors $z_1, \ldots, z_k$ are linearly independent. Hence the points $z \in D_0$ are non-singular for the map I, that is, when $z \in D_0$, the rank of the Jacobian matrix $DI(z)$

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* In this proposition it can also be assumed that D is merely $O_+(d)$-invariant if it is supposed in addition to (5.35) that the domain $I(D)$ is simply connected (the proof given below can even be simplified).

is equal to ν (see Lemma 7 in the Hall and Wightman article). Furthermore, at each such point z, the subspace of vectors in $\mathbf{C}^{dk}$ of the form $(\omega z_1, \ldots, \omega z_k)$, where ω runs through the skew-symmetric $d \times d$-matrices, has dimension $dk - \nu$ (see Lemma 6 of the Hall and Wightman article).

Let a be an arbitrary fixed point of D_0. Then in a sufficiently small neighbourhood $U_a \subset D_0$ of a we can choose the invariants $I_1, \ldots, I_\nu$ as the first ν local coordinates, while for the remaining local coordinates in U_a we can choose a set $\tilde{z} \equiv (z_j^\lambda)_{(j,\lambda)\in S}$ of $dk - \nu$ components. The condition of $O_+(d)$-invariance of the function $f(z)$ written in infinitesimal form, leads to the system of equations

$$\sum_{j=1}^{k} (\omega z_j)\partial_j f = 0,$$

where ω runs through the skew-symmetric $d \times d$-matrices. In this system we change to the new variables: $f(z) = \tilde{f}(I, \tilde{z})$. Since the invariants $I_1, \ldots, I_\nu$ satisfy these equations, we obtain a system of $dk - \nu$ linearly independent equations in the derivatives $(\partial\tilde{f}/\partial z_j^\lambda)_{(j,\lambda)\in S}$. Upon solving these we find that $\partial\tilde{f}/\partial\tilde{z} = 0$. This means that $\tilde{f}(I, \tilde{z})$ does not depend on $\tilde{z}$. Thus we have proved the representation in a sufficiently small neighbourhood of an arbitrary point $a \in D_0$:

$$f(z) = \phi_a(I(z));$$

here ϕ_a is an analytic function in some neighbourhood $V_a \subset I(D_0)$ of the point $I(a)$.

We now verify that the functions ϕ_a define a single-valued analytic function ϕ in $I(D_0)$. To this end we note that the condition of $O_{(+)}(d)$-invariance

$$f(gz) = f(z) \quad \text{for } g \in O_{(+)}(d),\ z \in D$$

can be analytically continued with respect to g; this means that the function $f(z)$ is in fact $O_{(+)}(d, C)$-invariant. (This technique is described in more detail in §9.1.B.) We now use the fact that (according to Hall and Wightman's Lemma 2) we can connect any pair of points $z, z' \in \mathbf{C}^{dk}$ with the same invariants $I(z) = I(z')$, by a transformation of the group $O_{(+)}(d, C)$, (that is, $z' = gz$, where $g \in O_{(+)}(d, C)$) provided that $p(I(z)) = p(I(z')) \neq 0$. Hence and from the $O_{(+)}(d, C)$-invariance of f, it follows that

$$f(z) = f(z'), \quad \text{if } z,\ z' \in D_0 \text{ and } I(z) = I(z'),$$

so that

$$\phi_a(w) = \phi_b(w), \quad \text{if } w \in V_a \cap V_b \text{ and } p(w) \neq 0.$$

Since $p(w)$ is a polynomial that does not vanish identically, we have (by the uniqueness theorem) $\phi_a = \phi_b$ in $V_a \cap V_b$. This means that there exists a single-valued analytic function $\phi(w)$ in the domain $I(D_0) \equiv I(D)\backslash p^{-1}\{0\}$ such that the representation (5.34) holds in D_0.

It remains to verify that $\phi(w)$ is analytic in $I(D)$. For this we note that it easily follows from the representation (5.34) in D_0 and the assumption (5.35), that the function $\phi(w)$ is bounded on any set of the form $K\backslash p^{-1}\{0\}$, where K is an arbitrary compactum in $I(D)$. According to Exercise 5.6, $\phi(w)$ can be continued analytically to $I(D)$; hence (by the uniqueness theorem) the representation (5.34) holds everywhere in D. ■

In the case of a G-covariant analytic function $f(z)$ (with values in a finite-dimensional complex vector space $\mathcal{X}$ transforming according to a representation T of the group $G = O_{(+)}(d)$), of interest is the possibility of the expansion

$$f(z) = \sum_{\rho=1}^{N} Q_\rho(z) h_\rho(z), \tag{5.36}$$

where $\{Q_\rho(z)\}_{\rho=1}^N$ is a polynomial basis of standard covariants in $\mathbf{C}^n$ (or $\mathbf{R}^n$) transforming according to the representation T, and the $h_\rho(z)$ are invariant analytic functions. In the derivation of the representation (5.36) we make the assumption that the domain $D \subset \mathbf{C}^n$ is *I-saturated*, that is, that

$$D = I^{-1}(I(D)).$$

Proposition 5.16. *Suppose that the $O_{(+)}(d,C)$-invariant domain $D \subset \mathbf{C}^{dk}$ (where $k < d$ for $O_+(d,C)$ and $k \leq d$ for $O(d,C)$) is I-saturated and that there exists a polynomial basis $\{Q_\rho\}_{\rho=1}^N$ of standard covariants in $\mathbf{C}^n$ (or in $\mathbf{R}^n$), that transform according to the representation T of the group $O_{(+)}(d)$ in $\mathcal{X}$. Then an arbitrary $\mathcal{X}$-valued covariant analytic function $f(z)$ in D can be (uniquely) decomposed into the sum* (5.36)*, where the $h_\rho(z)$ are invariant analytic functions in D (which in turn have a representation of the form* (5.34)*).*

For the proof, see the article by Hepp (1963c).*

We show in the following counterexample that the condition of I-saturatedness in Proposition 5.16 is essential (at least for $d = 2$). For an arbitrary point $z \equiv (z^1, z^2) \in \mathbf{C}^2$ we introduce the coordinates $z_\pm = z^1 \pm iz^2$. Then the function $f(z) = (z_+)^{-1}$ is $O_+(2)$-covariant in the domain $D = \{z \in \mathbf{C}^2 : z_+ \neq 0\}$ and transforms according to the (one-dimensional) representation with spin -1 of the group $O_+(2)$. However, in the covariant representation $f(z) = Q(z)\phi(z \cdot z)$ with the standard covariant $Q(z) = z_-$ and the standard invariant $I(z) = z \cdot z$, the function $\phi(w) = w^{-1}$ is not analytic in the domain $I(D) = \mathbf{C}$, since Proposition 5.16 is not applicable because the domain D is not I-saturated. (Nevertheless, according to Proposition 5.15, every $O_+(2)$-invariant analytic function $f(z)$ in this domain D has the form $f(z) = \phi(z \cdot z)$, where $\phi(w)$ is an entire analytic function.)

G. Plurisubharmonic Functions

For the characterization of domains of holomorphy we use not only analytic functions but also the so-called plurisubharmonic functions.

A function $p(z)$ in the domain (or in the open subset) D of the complex plane $\mathbf{C}$ is said to be *subharmonic* if it takes values in the extended real line $[-\infty, +\infty)$, is upper semicontinuous and satisfies the condition: for any subdomain G with compact closure $\overline{G} \subset D$ and any real harmonic function** $u(z)$ in G that is continuous in $\overline{G}$, the condition $p(z) \leq u(z)$ on ∂G implies that $p(z) \leq u(z)$ in G.

The upper semicontinuity condition means that

$$p(z) = \overline{\lim_{\zeta \to z, \zeta \in D}} p(\zeta) \tag{5.37}$$

(this notation presupposes that ζ can take the value z; otherwise the $=$ sign must be replaced by a $\geq$ sign). This condition is equivalent to the requirement that the inverse image $p^{-1}([-\infty, x))$ of any infinite interval of the form $[-\infty, x)$ (where $x \in \mathbf{R}$) under the map p is an open subset of D. Hence it follows that an upper semicontinuous function is measurable and that its restriction to any compactum is bounded above and attains its maximum. Therefore the integrals of the form $\int_0^{2\pi} p(z + re^{i\phi})d\phi$ and $\int_{|c|<r} p(z+c)|dc\, d\bar{c}|$, where $0 < r < d(z, \partial D)$, are defined provided that we allow the values of the integrals to have the value $-\infty$ as well as real values.

There is a useful criterion for subharmonicity.

Proposition 5.17. *In order that an upper semicontinuous function $p(z)$, defined in the domain (or in the open subset) $D \subset \mathbf{C}$ and taking values on the extended real*

*In view of the uniqueness of the expansion, the requirement in Proposition 2 of Hepp's article that D be a domain of holomorphy is not essential in the present instance.

** A *harmonic* function $u(z)$ in the domain $D \subset C$ is a function satisfying the Laplace equation $\Delta u \equiv 4\bar{\partial}\partial u = 0$. (As a "catch-all" condition, one can require that u be continuously differentiable; it is then infinitely smooth, as in the case of analytic functions.) The Laplace equation implies that ∂u is a holomorphic function. Hence it follows that a real harmonic function $u(z)$ in a simply connected domain $D \subset \mathbf{C}$ is the real part of some analytic function $h(z)$.

line $[-\infty, +\infty)$ *be subharmonic, it is necessary that*

$$p(z) \leq \frac{1}{2\pi} \int_0^{2\pi} p(z + re^{i\phi}) d\phi \tag{5.38}$$

for all $z \in D$ *and* $r > 0$ *such that* $r < d(z, \partial D)$*, and it is sufficient that* (5.38) *should hold for all* $z \in D$ *and for all* r *in some interval* $(0, r_0(z))$*, where* $r_0(z)$ *is an arbitrary positive number (depending on* z*).*

See [V4],§9.4 for the proof. There it is proved that in any closed disc in D with centre at the point w and radius r, a subharmonic function in D satisfies the following inequality, which is more general than (5.38):

$$p(z) \leq \int_0^{2\pi} \mathcal{P}(z - w, re^{i\phi}) p(w + re^{i\phi}) d\phi; \tag{5.39}$$

here $\mathcal{P}(z, re^{i\phi})$ is the Poisson kernel:

$$\mathcal{P}(\rho e^{i\theta}, re^{i\phi}) = \frac{1}{2\pi} \frac{r^2 - \rho^2}{r^2 - 2r\rho\cos(\phi - \theta) + \rho^2}.$$

Exercise 5.9. Let $f(z)$ be an analytic function in the domain $D \subset \mathbf{C}$. Prove that $f(z) = \ln |f(z)|$ is a subharmonic function. [Hint: At those points z where $f(z) = 0$, (5.38) is trivial; in a neighbourhood of any point where $f(z) \neq 0$, $p(z)$ is the real part of an analytic function; apply the mean value formula to it.]

Exercise 5.10. Prove that a subharmonic function $p \not\equiv$ const in the domain $D \subset \mathbf{C}$ cannot assume a (local or global) maximum at points of D. Moreover, for any compactum $K \subset D$ and any subharmonic function p in D, we have

$$\sup_{z \in K} p(z) = \sup_{z \in \partial K} p(z). \tag{5.40}$$

[Hint: The line of argument, based on (5.38), is the same as in Theorem 5.8.]

A function $p(z)$ defined in the domain (or in the open set) $D \subset \mathbf{C}^n$ is called *plurisubharmonic* if it takes values on the extended real line $[-\infty + \infty)$ is upper semicontinuous and its restriction to the intersection of D with any one-dimensional complex line in $\mathbf{C}^n$ is a subharmonic function. According to Proposition 5.17, this last condition is equivalent to the requirement that the inequality

$$p(z) \leq \frac{1}{2\pi} \int_0^{2\pi} p(z + re^{i\phi} b) d\phi \tag{5.41}$$

holds for any $z \in D$, any vector $b \in \mathbf{C}^n$ and for $0 < r < r_0(z, b)$, where $r_0(z, b)$ is a positive function of z, b. If in fact, this condition holds, then for $b \neq 0$, $r_0(z, b)$ can be taken to be the quantity $R_b(z, \partial D)$, which is the *distance along the ray* $\mathbf{C}b$ from the point $z \in D$ to the boundary ∂D of the domain; it is defined by the equality

$$R_b(z, \partial D) = |b| \cdot \sup\{r > 0 : \ z + cb \in D \quad \text{for } c \in \mathbf{C}, \ |c| < r\}. \tag{5.42}$$

If $p(z)$ is a plurisubharmonic function in the domain $D \subset \mathbf{C}^n$, then the function $-p(z)$ is called a *plurisuperharmonic* (and in the case $n = 1$, *superharmonic*) function.

Exercise 5.11. (a) Prove that $\ln |f(z)|$ is a plurisubharmonic function if $f(z)$ is an analytic function in the domain $D \subset \mathbf{C}^n$. [Hint: Use Exercise 5.9.]

(b) Let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a family of plurisubharmonic functions in the domain D such that the function $p(z) = \sup_\lambda p_\lambda(z)$ is upper semicontinuous. Prove that $p(z)$ is a plurisubharmonic function.

Integrating (5.41) with respect to dr^2 we obtain

$$p(z) \leq \int_{|c|<r} p(z+cb)\frac{|dc\,d\bar{c}|}{2\pi r^2} \tag{5.43}$$

for $0 < r < R_b(z, \partial D)$.

In fact the integral in this argument is meant as the iterated integral: $\int dr' \int d\phi \ldots$, where $c = r'e^{i\phi}$. However, by Fubini's theorem ([S4],Ch.IV,Theorem 77) applied to the non-negative measurable function $F(r', \phi) = A - p(z + r'e^{i\phi}b)$, where A is a sufficiently large number, this iterated integral is the same as the double integral (5.43).

Let $\overline{P}(w, \rho) \subset D$. Then a successive application of (5.43) leads to the following condition for a plurisubharmonic function:

$$p(w) \leq \frac{1}{\operatorname{mes} P(w;\rho)} \int_{P(w;\rho)} p(z)|d^n z\, d^n \bar{z}| \tag{5.44}$$

(here, as in (5.43), the iterated integral is the same as the multiple one).

Exercise 5.12. Prove that a plurisubharmonic function satisfies the conditions

$$p(z) = \lim_{\rho\to 0} \frac{1}{\operatorname{mes} P(z;\rho)} \int_{P(z;\rho)} p(\zeta)|d^n\zeta d^n\bar{\zeta}| = \varlimsup_{\zeta\to z,\, \zeta\in D\setminus(z)} p(\zeta), \tag{5.45a}$$

$$p(z) = \lim_{r\to+0} \int p(z+r\zeta)u(\zeta)|d^n\zeta d^n\bar{\zeta}| \tag{5.45b}$$

for all $z \in D$; here $u(z)$ is a non-negative function in $\mathcal{D}(\mathbf{C}^n)$ with integral 1. [Hint: For the proof of (5.45a), pass to the limit in (5.44) and use the condition (5.37) of upper semicontinuity. Equation (5.45b) is established by the same method.]

We note that along with the formulae (5.45), we have the following result.

Proposition 5.18. *Let $p(z)$ be a plurisubharmonic function in the domain $D \subset \mathbf{C}^n$ and let $z \in D$, $b \in \mathbf{C}^n\setminus\{0\}$. Then*

$$p(z) = \varlimsup_{r\to+0} p(z+rb). \tag{5.46}$$

The proof can be found in [V4],§9.16.

Proposition 5.19. *For any plurisubharmonic function $p(z)$ in the domain $D \subset \mathbf{C}^n$ we have the alternative: either $p \equiv -\infty$ or p is locally integrable in D.*

■ Suppose that p is not locally integrable: then we claim that $p \equiv -\infty$. For consider the set Ω of all points $w \in D$ such that $\int_{P(w;\rho)} p(z)|d^n z\, d^n \bar{z}| = -\infty$ for at least one polycircular domain $\overline{P}(w;\rho) \subset D$ with polyradius $\rho \equiv \rho_w$. Then in view of (5.45a), it suffices to prove that Ω coincides with D. Since Ω is non-empty and D is connected, we merely have to show that Ω is an open and closed subset of D. If $w \in \Omega$, then there exists a value of the polyradius ρ' and a neighbourhood U of w such that $\overline{P}(w;\rho_w) \subset \overline{P}(\zeta;\rho') \subset D$ for all $\zeta \in U$. Clearly $\int_{P(\zeta;\rho')} p(z)|d^n z\, d^n \bar{z}| = -\infty$ for $\zeta \in U$, which proves that Ω is open. We now prove that Ω is closed in D. Let w be a point of Ω that is in the closure of D. Then for any ρ such that $\overline{P}(w;\rho) \subset D$, the set $P(w;\rho)$ has a non-empty intersection with Ω. Since $p(z) = -\infty$ for $z \in \Omega$ and the set $p(w;\rho) \cap \Omega$ is open and non-empty (and hence has non-zero Lebesgue measure), the integral of $p(z)$ over $P(w;\rho)$ is equal to $-\infty$. This proves that Ω is closed. ■

The method of smoothing can be used to represent an arbitrary plurisubharmonic function as a limit of smooth plurisubharmonic functions.

Proposition 5.20. (a) *There exists for any plurisubharmonic function $p \not\equiv -\infty$ in the domain $D \subset \mathbf{C}^n$, an increasing sequence of domains $D_\nu \subset \mathbf{C}^n$ the union of which* *is D, and a monotone decreasing sequence of plurisubharmonic functions $p_\nu \in \mathcal{E}(D_\nu)$ converging to f.*

(b) *The limit of a monotone decreasing sequence p_ν of plurisubharmonic functions is a plurisubharmonic function.*

The proof can be found in [V4],§§9.6,10.9.

For a smooth real function f, the plurisubharmonic condition is in fact a sort of local requirement on the function at each point. In this connection we say that a matrix-valued function $u(z) \equiv (u_{jk}(z))_{j,k=1,\dots,n}$ in the domain D is of *positive type* if for each $z \in D$ the matrix $u(z)$ satisfies the positive-definiteness condition

$$\sum_{j,k=1,\dots,n} u_{jk}(z) a_j \bar{a}_k \geq 0 \quad \text{for all } a \in \mathbf{C}^n. \tag{5.47}$$

Similarly, we say that the matrix-valued distribution $\phi(z) \equiv (\phi_{jk}(z))_{j,k=1,\dots,n}$ with matrix entries in the space $\mathcal{D}'(D)$ is of *positive type* if

$$\sum_{j,k} \int \phi_{jk}(z) u_{jk}(z) |d^n z\, d^n \bar{z}| \geq 0 \tag{5.48}$$

for all matrix-valued functions $u(z)$ of positive type with matrix entries of class $\mathcal{D}(D)$.

Proposition 5.21. *A real twice continuously differentiable function $p(z)$ in the domain $D \subset \mathbf{C}^n$ is plurisubharmonic if and only if the matrix-valued function $(\partial_j \overline{\partial}_k p(z))$ is of positive type.*

■ The case of arbitrary dimension n easily follows from $n = 1$, therefore we suppose that D is a domain in the complex plane $\mathbf{C}$. We define the two functions

$$I(z;r) = \int_0^{2\pi} p(z + re^{i\phi}) d_1\phi, \quad J(z;r) = 4 \int_0^{2\pi} \partial\overline{\partial} p(z + re^{i\phi}) d_1\phi$$

for $z \in D$, $0 < r < d(z, \partial D)$. The function $I(w;r)$ is twice continuously differentiable in r and satisfies the "initial conditions"

$$I(z;0) = p(0), \quad \frac{\partial}{\partial r} I(z;0) = 0.$$

Bearing in mind that $4\partial\overline{\partial} = \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial \phi^2}$, we obtain the following relation between $I(w;r)$ and $J(w;r)$:

$$\frac{\partial^2}{\partial r^2} I(z;r) + \frac{1}{r}\frac{\partial}{\partial r} I(z;r) = J(z;r),$$

whence it follows that

$$\frac{\partial}{\partial r} I(z;r) = \int_0^r \frac{r'}{r} J(z;r') dr'. \tag{5.49}$$

Suppose that $\partial\overline{\partial} f(z) \geq 0$; then $J(z;r) \geq 0$ and (by (5.49)) $I(z;r)$ does not increase as r increases, which together with the " initial condition" implies that the plurisubharmonic condition (5.38) holds for all $z \in D$ and $r < d(z, \partial D)$. Conversely, suppose that $p(z)$ is subharmonic. We prove by contradiction that $\partial\overline{\partial} p(z)$ is positive. For if $\partial\overline{\partial} p(z) < 0$ at some point $z = w$, then $J(w;0) < 0$. It then follows from (5.49) that $\frac{\partial}{\partial r} I(w;r) < 0$ for $0 < r < r_0(w)$, where $r_0(w)$ is some positive number, so that $I(w;r) < I(w;0) = p(0)$ for such r, which is in contradiction with the subharmonicity condition (5.38). ■

Exercise 5.13. Prove that the function $p(z) = |z|^2$ (in $\mathbf{C}^n$) is plurisubharmonic. [Hint: Use Proposition 5.21.]

Proposition 5.21 suggests a method of characterizing plurisubharmonic functions in terms of distributions. We say that a real distribution $p(z) \in \mathcal{D}'(D)$ in the domain

$D \subset \mathbf{C}^n$ is *plurisubharmonic* if the matrix-valued distribution $(\partial_j \overline{\partial}_k p(z))$ is of positive type.

Proposition 5.22. *The formula of type* (5.30) (*or* (5.45b)) *sets up a one-to-one correspondence between the plurisubharmonic functions in the domain* $D \subset \mathbf{C}^n$ *that are not identically equal to* $-\infty$ *and the plurisubharmonic distributions in* D.

The proof is given in [V9],§10.10.

Proposition 5.22 enables us to identify the plurisubharmonic functions (in the domain D) that are not identically equal to $-\infty$ with the plurisubharmonic distributions.

Exercise 5.14. Let $\phi(z') \equiv (\phi_j(z'))_{j=1,\ldots,n}$ be a holomorphic map from the domain $G \subset \mathbf{C}^k$ to the domain $D \subset \mathbf{C}^n$ (that is, ϕ is a vector-valued holomorphic function in G with values in D) and let p be a plurisubharmonic function in D. Prove that $p(\phi(\zeta))$ is a plurisubharmonic function in G. [Hint: It suffices to consider the case $p \not\equiv -\infty$. If p is a smooth function, one can use Proposition 5.21. The general case reduces to the case of a smooth function p using Proposition 5.20.]

5.2. Domains of Holomorphy

A. Holomorphic convexity

A domain D in $\mathbf{C}^n$ is said to be *holomorphically convex* if every $\mathcal{H}(D)$-bounded closed subset of D is compact. As we shall see presently, this notion is equivalent to that of a domain of holomorphy.

The condition of holomorphic convexity clearly implies additionally that the closure in $\mathbf{C}^n$ of any $\mathcal{H}(D)$-bounded set $K \subset D$ does not intersect the boundary ∂D of D, which in turn is equivalent to the distance between K and ∂D, defined by

$$d(K, \partial D) = \inf\left\{ |z - w| \equiv \left(\sum_{j=1}^{n} |z_j - w_j|^2\right)^{1/2} \ : \ z \in K, \ w \in \partial D \right\} \tag{5.50}$$

being positive. The following conclusion is a direct corollary of this.

Lemma 5.23. *In order that a set* $D \subset \mathbf{C}^n$ *be holomorphically convex, it is necessary and sufficient that for any point* $w \in \partial D$ *and any set* $K \subset D$ *whose closure in* $\mathbf{C}^n$ *contains* w, *there exist a function* $f \in \mathcal{H}(D)$ *such that*

$$\overline{\lim_{z \to w,\, w \in K}} |f(z)| = \infty. \tag{5.51}$$

Exercise 5.15. Prove that every domain D in the complex plane $\mathbf{C}$ is holomorphically convex. [Hint: Condition (5.51) holds for $f(z) = (z - w)^{-1}$.]

Exercise 5.16. Prove that the direct product $G \times G' \subset \mathbf{C}^n$ of two holomorphically convex domains $G \subset \mathbf{C}^k$ and $G' \subset \mathbf{C}^{n-k}$ is holomorphically convex. [Hint: If K is an $\mathcal{H}(G \times G')$-bounded set, then functions in $\mathcal{H}(G \times G')$ that depend only on the projection of a point in $G \times G'$ onto G, are bounded on K. Deduce that the distance between the projections of K onto G and ∂G is positive. In precisely the same way, the distance between the projections of K onto G' and $\partial G'$ is positive.]

Exercise 5.17. Prove that any connected component G of the intersection of a holomorphically convex domain $D \subset \mathbf{C}^n$ with any complex plane in $\mathbf{C}^n$ is holomorphically convex. [Hint: The functions in $\mathcal{H}(D)$ are bounded on any $\mathcal{H}(G)$-bounded set $K \subset G$, therefore $d(K, \partial D) > 0$; on the other hand, $d(K, \partial G) \geq d(K, \partial D)$.]

According to the next exercise, holomorphic convexity is a generalization of the notion of convexity.

Exercise 5.18. Prove that every convex domain $D \subset \mathbf{C}^n$ is holomorphically convex. [Hint: If $w \in \partial D$ and $\mathrm{Re}(a, z - w) = 0$ is the equation of the support hyperplane through this point,* then the function $f(z) = (a, z - w)^{-1}$ is holomorphic in D; now apply Lemma 5.23.]

In the preceding exercises we have used the "sufficiency" in Lemma 5.23; part (b) of the following Exercise 5.19 relies on the "necessity".

Exercise 5.19. (a) Let $\mathcal{X}_1, \dots \mathcal{X}_k$ be a finite set of linear subspaces of the LCS $\mathcal{X}$, none of which coincides with $\mathcal{X}$ and let $\mathcal{Y}_\nu$ be the complement of $\mathcal{X}_\nu$ in $\mathcal{X}$. Prove that $\bigcap_{\nu=1}^k \mathcal{Y}_\nu$ is dense in $\mathcal{X}$. [Hint: For a finite-dimensional space, say $\mathbf{C}^k$, this assertion is elementary;** it can also easily be deduced from the Baire category theorem (Theorem 1.3); the conclusion remains valid, of course, if instead of linear subspaces we take planes in $\mathbf{C}^k$. For the proof of the general case, take a fixed $u_\nu \in \mathcal{Y}_\nu$ ($\nu = 1, \dots, k$); then any element $u \in \mathcal{X}_k$ can be approximated to any desired degree of accuracy by sums of the form $u(c) \equiv u + \sum_{\nu=1}^k c_\nu u_\nu \in \bigcap_{\nu=1}^k \mathcal{Y}_\nu$ for sufficiently small $c_\nu \in \mathbf{C}$, since the sets $\{c : u(c) \in \mathcal{X}_\nu\}$ are planes in $\mathbf{C}^k$ that are distinct from $\mathbf{C}^k$.]

(b) Let S be an arbitrary finite set of the boundary of the holomorphically convex domain D. Prove that the set of all functions $f \in \mathcal{H}(D)$ satisfying the condition

$$\overline{\lim_{z \to w,\, z \in D}} |f(z)| = \infty \quad \text{for all } w \in S,$$

is dense in $\mathcal{H}(D)$. [Hint: For $S = \{w^{(1)}, \dots, w^{(k)}\}$ define $\mathcal{X}_\nu$ to be the set of functions in $\mathcal{H}(D)$ for which

$$\overline{\lim_{z \to w^{(\nu)},\, z \in D}} |f(z)| < \infty;$$

by Lemma 5.23, it is not the whole of $\mathcal{H}(D)$. Now use part (a) of this exercise, noting that our set of functions in $\mathcal{H}(D)$ is $\bigcap_{\nu=1}^k \mathcal{Y}_\nu$, where $\mathcal{Y}_\nu = \mathcal{H}(D) \backslash \mathcal{X}_\nu$.]

The next theorem provides a number of equivalent descriptions of domains of holomorphy. In this connection we say that a point w of the boundary of the domain D is a *barrier point* for D if there exists a function $f \in \mathcal{H}(D)$ that does not admit a direct analytic continuation to the point w (that is, does not admit a direct analytic continuation to any neighbourhood of w in $\mathbf{C}^n$). Such a function is called a *barrier function* at the point w of the domain D.

Theorem 5.24. *The following conditions for a domain D in $\mathbf{C}^n$ are equivalent.*

(a.1) *D is a domain of holomorphy.*

(a.2) *Every point $w \in \partial D$ is a barrier point for D.*

(a.3) *D has no non-trivial holomorphic continuations to any other domain in $\mathbf{C}^n$.*

(a.4) *D is holomorphically convex.*

(a.5) *There exists a function $f \in \mathcal{H}(D)$ such that*

$$\overline{\lim_{z \to w,\, z \in D}} |f(z)| = \infty \tag{5.52}$$

for any point $w \in \partial D$.

■ The implications (a.1)⇒(a.2)⇒(a.3) are trivial.

(a.3)⇒(a.4). Let $K \subset D$ be an arbitrary $\mathcal{H}(D)$-bounded subset. According to Exercise 5.2 there exists a polyradius $\rho > 0$ such that D can be holomorphically continued to any polycircular domain of radius ρ with centre at an arbitrary point of K. It follows from condition (a.3) that any such polycircular domain is contained in D, so that K is at a positive distance from ∂D. It follows that (a.4) holds.

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* Given any point of the boundary of a convex domain in (real) Euclidean space, one can draw a so-called support hyperplane through it with the property that the given domain lies entirely on one side of it ([S2],p.64).

** In this case it holds, of course, for a countable family of subspaces.

(a.4)$\Rightarrow$(a.5). Let D be a domain of holomorphy. We fix a countable dense subset of points $\{w^{(k)}\}_{k=1,2,\dots}$ in ∂D and an increasing sequence of compacta $Q_k \subset D$ whose interiors cover D. We set $V_k^\epsilon = \{f \in \mathcal{H}(D) : \|f\|^{Q_k} < \epsilon\}$. We associate with each $k \in \overline{\mathbf{Z}}_+$ a number $j_k \in \mathbf{Z}_+$, a function $f_k \in \mathcal{H}(D)$, a number $\epsilon_k > 0$, and a finite set $S_k \subset D$ in the following recursive manner. We choose j_0, f_0, S_0 and ϵ_0 arbitrarily. Supposing that $j_{k-1}, f_{k-1}, S_{k-1}$ and ϵ_{k-1} have already been chosen, we choose for j_k a number greater than j_{k-1} such that $S_{k-1} \subset Q_{j_k}$. According to Exercise 5.19(b), the set of functions $f \in \mathcal{H}(D)$ such that (5.51) holds for $w = w^{(1)}, w^{(2)}, \dots, w^{(k)}$, is dense in $\mathcal{H}(D)$. We choose such a function for f_k, requiring further that $f_k \in f_{k-1} + V_{j_{k-1}}^{\epsilon_{k-1}}$. We choose for ϵ_k a number $\epsilon_k < \epsilon_{k-1}$ such that $f_k + V_{j_k}^{\epsilon_k} \in f_k + V_{j_{k-1}}^{\epsilon_{k-1}}$. We suppose that $\epsilon_k \to 0$ as $k \to \infty$. Finally, by virtue of the fact that f_k satisfies a condition of type (5.51) for $w = w^{(1)}, \dots, w^{(k)}$, we can choose a finite set of points $S_k \subset D$ such that

$$d(w^{(j)}, S_k) < 1/k \quad \text{for } j = 1, \dots, k \tag{5.53}$$

and

$$|f_k(z)| > k \quad \text{for } z \in S_k. \tag{5.54}$$

By construction, the sequence f_k is fundamental and therefore has a limit f in $\mathcal{H}(D)$. It follows from (5.54) that $\min_{z \in S_k} |f(z)| \to \infty$ as $k \to \infty$, which together with (5.53) proves (5.52).

(a.5)$\Rightarrow$(a.1). We claim that D is the unique domain of holomorphy for the function f in condition (a.5). To this end, it suffices to verify that every direct analytic continuation f' of f along the route (D, D_{12}, D') is trivial. We suppose that D_{12} is the connected component of $D \cap D'$ and that $\overline{D}_{12}$ is its closure in $\mathbf{C}^n$; then $D_{12} = \overline{D}_{12} \cap D' \cap D$. Since $f = f'$ in D_{12}, it follows that f can be continuously extended to the closure $\overline{D}_{12} \cap D'$ of D_{12} in D'. It follows from (a.5) that this closure contains no points of D, so that $\overline{D}_{12} \cap D' = \overline{D}_{12} \cap D' \cap D = D_{12}$. This proves that the open set D_{12} is closed in D' and (since D' is connected) coincides with D'. We have thus proved that $D' \subset D$ and $f' = f$ in D'. ∎

Exercise 5.20. Prove that any connected component D of the interior of the intersection of an arbitrary family $\{D_\lambda\}_{\lambda \in \Lambda}$ of domains of holomorphy is a domain of holomorphy. [Hint: Any direct holomorphic extension of D is a holomorphic extension for D_λ for any λ and is therefore trivial.]

Exercise 5.21. Prove that any connected component D' of the intersection $D \cap l$ of the domain of holomorphy $D \subset \mathbf{C}^n$ with the k-dimensional complex plane l in $\mathbf{C}^n$ is a domain of holomorphy (in l). [Hint: Any $\mathcal{H}(D')$-bounded subset of $K \subset D'$ that is closed in D' is $\mathcal{H}(D)$-bounded and closed in D and therefore compact.]

Exercise 5.22. Let ϕ be a holomorphic map of the domain of holomorphy $\Omega \subset \mathbf{C}^k$ into $\mathbf{C}^n$, and D a domain of holomorphy in $\mathbf{C}^n$. Prove that any connected component Ω' of the open set $\phi^{-1}(D) \subset \Omega$ is a domain of holomorphy. [Hint: It suffices to prove that any $\mathcal{H}(\Omega')$-bounded closed subset K of Ω' is compact. It follows from the $\mathcal{H}(\Omega)$-boundedness of K that $d(K, \partial\Omega) > 0$, therefore it remains to prove that K is closed in Ω. Prove this by contradiction: suppose that there exists a sequence $\zeta^{(\nu)} \in K$ converging to some point $\zeta^{(\infty)} \in \Omega \cap \partial\Omega'$; then $\zeta^{(\infty)} \notin \phi^{-1}(D)$. On the other hand, the sequence $\phi(\zeta^{(\nu)}) \in D$ converges to $\phi(\zeta^{(\infty)}) \in \mathbf{C}^n$. Now conclude from the boundedness on K of a function of the form $f(\phi(\zeta))$, where $f \in \mathcal{H}(D)$, that $\phi(K)$ is at a positive distance from ∂D and thus arrive at the contradiction: $\phi(\zeta^{(\infty)}) \in D$.]

B. PSEUDO-CONVEXITY

We now turn to the characterization of domains of holomorphy in terms of plurisubharmonic functions. Let $\mathcal{P}(D)$ be the space of all plurisubharmonic functions in the domain $D \subset \mathbf{C}^n$. The subset $K \subset D$ is said to be $\mathcal{P}(D)$-***bounded*** if any function in $\mathcal{P}(D)$ is bounded above on K.

Exercise 5.23. Prove that every $\mathcal{P}(D)$-bounded set is bounded. [Hint: Use Exercise 5.13.]

A domain $D \subset \mathbf{C}^n$ is said to be ***pseudo-convex*** if every $\mathcal{P}(D)$-bounded closed subset of D is compact. This means that the distance $d(K, \partial D)$ from any $\mathcal{P}(D)$-bounded set $K \subset D$ to the boundary of the domain is positive.

Exercise 5.24. Prove that every domain of holomorphy is pseudo-convex. [Hint: Use the holomorphic convexity of a domain of holomorphy and Exercise 5.11(a).]

Let G be an arbitrary bounded domain in the complex plane $\mathbf{C}$ (with closure $\overline{G}$), and $[0,1]$ the interval on the real line. We consider a continuous map

$$\phi : \overline{G} \times [0,1] \to \mathbf{C}^n \tag{5.55}$$

such that $\phi(\zeta, t)$ is holomorphic in $\zeta \in G$ for any $t \in [0,1]$. If the domain $D \subset \mathbf{C}^n$ has the property that for any such map ϕ, the conditions

$$\phi(G \times \{0\}) \subset D \quad \text{and} \quad \phi((\partial G) \times [0,1]) \subset D \tag{5.56}$$

always imply that

$$\phi(G \times [0,1]) \subset D, \tag{5.57}$$

then we say that the *principle of continuity* holds for the domain D.

Theorem 5.25. *The following conditions are equivalent for a domain $D \subset \mathbf{C}^n$.*

(b.1) *D is pseudo-convex.*

(b.2) *The principle of continuity holds for D.*

(b.3) *For any non-zero vector $b \in \mathbf{C}^n$, the function $-\ln R_b(z, \partial D)$ is plurisubharmonic in D (where $R_b(z, \partial D)$ is the distance along the ray $\mathbf{C}b$ from z to ∂D; see* (5.42)).

(b.4) *The function $-\ln d(z, \partial D)$ is plurisubharmonic in D.*

(b.5) *There exists a plurisubharmonic function $p(z)$ in D such that*

$$\lim_{z \to w,\, z \in D} p(z) = +\infty \tag{5.58}$$

for all $w \in \partial D$.

■ (b.1)⇒(b.2). Let D be a pseudo-convex domain. We consider the continuous map (5.55), which is holomorphic in ζ and satisfies conditions (5.56). We claim that (5.57) also holds. Thus let T denote the set of all $t \in [0,1]$ such that $\phi(\overline{G} \times [0,t]) \subset D$. It is clearly non-empty (since $0 \in T$) and open in $[0,1]$. We now show that it is closed. Clearly, to prove that $\phi(\overline{G} \times \overline{T}) \subset D$ it suffices to verify that the set $\phi(\overline{G} \times T)$ is $\mathcal{P}(D)$-bounded (since it will then be at a positive distance from ∂D). Let p be an arbitrary function in $\mathcal{P}(D)$, and $M = \sup\{p(z) : z \in \phi((\partial G) \times [0,1])\}$ ($M < \infty$ since the set $\phi((\partial G) \times [0,1])$ is compact). Then (according to Exercise 5.14) for any $t \in T$, the function $p(\phi(\lambda, t))$ is plurisubharmonic in $\lambda \in G$ and upper semicontinuous on $\overline{G}$. On the basis of Exercise 5.10, we conclude that $\sup_{\lambda \in \overline{G}} p(\phi(\lambda, t)) \leq M$ for all $t \in T$. This proves that $\phi(\overline{G} \times T)$ is $\mathcal{P}(D)$-bounded. Thus T is a non-empty open and closed subset of $[0,1]$. Since $[0,1]$ is connected, it is the whole of $[0,1]$.

(b.2)⇒(b.3). It is easy to see that the function $-\ln R_b(z, \partial D)$ is upper semicontinuous in D (for any domain D). Therefore the plurisubharmonicity of this function implies the following. For any points $z \in D$, $a \in \mathbf{C}^n$ such that $z + \zeta a \in D$ for all ζ in the closed disc $\overline{G} \subset \mathbf{C}$ with centre at the origin and radius 1, and for any function $h(\zeta)$ that is analytic in G and continuous in $\overline{G}$, the condition $R_b(z + \zeta a, \partial D) \geq |e^{h(\zeta)}|$ for all $\zeta \in \partial G$ implies that $R_b(z + \zeta a, \partial D) \geq |e^{h(\zeta)}|$ for all $\zeta \in \overline{G}$, that is, the condition $z + \zeta a + e^{h(\zeta)} b \in D$ for all $\zeta \in \partial G$ implies that

$$z + \zeta a + e^{h(\zeta)} b \in D \quad \text{for all } \zeta \in \overline{G}. \tag{5.59}$$

We set $\phi(\zeta, t) = z + \zeta a + t e^{h(\zeta)}$ for $\zeta \in \overline{G}$, $0 \leq t \leq 1$. By construction, ϕ satisfies (5.56) and since according to (b.2) the principle of continuity holds for D, ϕ satisfies (5.57) which, clearly, is equivalent to (5.59).

(b.3)⇒(b.4). The function $-\ln d(z, \partial D)$ is continuous in D (for any domain D.) Furthermore, it is clearly related to the function $-\ln R_b(z, \partial D)$ in the following way:

$$-\ln d(z, \partial D) = \sup_{b \in \mathbf{C}^n,\, |b|=1} (-\ln R_b(z, \partial D)).$$

Formula (b.4) is now an immediate corollary of (b.3) (and Exercise 5.11).

(b.4)$\Rightarrow$(b.5). It suffices to take $-\ln d(z,\partial D)$ for $p(z)$.

The implication (b.5)$\Rightarrow$(b.1) is trivial. ■

It turns out that the condition of pseudo-convexity is equivalent to that of holomorphic convexity (and hence, all the conditions (a.1)–(a.5) and (b.1)–(b.5) in Theorems 5.24 and 5.25 are equivalent to one another).

Theorem 5.26. *In order that the domain $D \subset \mathbf{C}^n$ be a domain of holomorphy, it is necessary and sufficient that it be pseudo-convex.*

This result is called *Oka's theorem*. With regard to the "necessity", see Exercise 5.23. The proof of the "sufficiency" is given in [S8],§45.

Exercise 5.25. Prove that the union $D = \bigcup_\nu D_\nu$ of an increasing sequence of domains of holomorphy $D_\nu \subset \mathbf{C}^n$ is a domain of holomorphy. [Hint: Apply the criterion (b.4) in Theorem 5.25. Use the fact that for $z \in D$, $d(z,\partial D)$ is the limit of the monotone increasing sequence $d(z,\partial D_\nu)$, and Proposition 5.20(b).]

C. Modified Principle of Continuity

The image of a bounded domain (say, a disc) $G \subset \mathbf{C}$ under a holomorphic map in $\mathbf{C}^n$ can be visually represented as a disc in $\mathbf{C}^n$. The principle of continuity then means that if the original disc is contained in a domain of holomorphy D and is then subjected to a continuous deformation (in the same class of "analytic discs") in such a way that its boundary remains all the time in D, then the final disc also lies entirely in D.

We give a further modification of the principle, when the condition that the final disc be contained in D is replaced by the weaker requirement that the final disc intersect D. On the other hand, we now require that the deformation be real analytic in the deformation parameter t. We note that the complex dimension of the domain G (the inverse image of the "disc") can in fact be arbitrary and not merely unity (just as the principle of continuity in §5.2.B holds for domains of holomorphy D, where G can be of any complex dimension).

Thus, let G be an arbitrary domain in $\mathbf{C}^k$. We consider maps

$$\phi:\ G \times [0,1] \to \mathbf{C}^n, \tag{5.60}$$

that are restrictions of holomorphic maps $\phi : \Omega \to \mathbf{C}^n$ defined in some domain $\Omega \subset \mathbf{C}^{k+1}$ containing $G \times [0,1]$. If for any such map ϕ, the conditions

$$\phi(G \times [0,1)) \subset D, \quad \phi(G \times \{1\}) \cap D \neq \emptyset \tag{5.61}$$

(where D is a domain in $\mathbf{C}^n$) always imply that

$$\phi(G \times \{1\}) \subset D, \tag{5.62}$$

then we say that the *modified principle of continuity* holds for the domain D.

Theorem 5.27. *The modified principle of continuity holds for any domain of holomorphy $D \subset \mathbf{C}^n$.*

■ Let ϕ be a mapping (5.60) of the class indicated above and satisfying (5.61). We denote by G' the set of all points $\zeta \in G$ such that $\phi(\zeta, 1) \in D$. This set is non-empty by hypothesis; it is clearly open. It suffices to prove that $G' = G$. We argue by contradiction. Thus supposing the opposite, the connectedness of G implies that there is a point $a \in G'$ such that $d(a,\partial G') < d(a,\partial G)$. We set

$r' = d(a, \partial G')$ and choose r with $r' < r < d(a, \partial G)$, such that the domain $\Omega \in \mathbf{C}^{k+1}$ (in which, by hypothesis, the map $\phi : \Omega \to \mathbf{C}^n$ is defined) contains the direct product $S(a, r) \times S(1, \epsilon)$, where $S(a, r)$ is an open ball in $\mathbf{C}^k$ with centre at a and radius r, and $S(1, \epsilon)$ is an open disc in $\mathbf{C}$ with centre at the point 1 and with a sufficiently small radius $\epsilon > 0$. We denote by Ω' the connected component of the intersection of $\phi^{-1}(D)$ with $S(a, r) \times S(1, \epsilon)$ that contains the point $(a, 1)$; this is a domain of holomorphy (by Exercise 5.22). By (5.61) (and by the construction of a, r, r'), the domain Ω' contains the sets $S(a, r') \times \{1\}$ and $S(a, r) \times \{t\}$ for all $t \in (1 - \epsilon, 1)$. Furthermore, the boundary of Ω' contains a point $(w, 1)$ such that $|w - a| = r'$. We set $b = \left(\frac{1}{r'}(w - a), 0\right) \in \mathbf{C}^{k+1}$ and we denote the distance along the ray $\mathbf{C}b$ from the point $(\zeta, t) \in \Omega'$ to the boundary $\partial\Omega'$ by $R_b((\zeta, t), \partial\Omega')$. Then by construction we have

$$R_b((a, 1), \partial\Omega') = r', \ R_b((a, t), \partial\Omega') > r > r' \quad \text{for } 1 - \epsilon < t < 1.$$

On the other hand, according to Theorem 5.25, $-\ln R_b((\zeta, t), \partial\Omega')$ is a plurisubharmonic function in Ω', therefore on the basis of Proposition 5.18 we can write

$$\overline{\lim_{t \to 1,\, 1-\epsilon<t<1}}(-\ln R_b((a, t), \partial\Omega')) = -\ln R_b((a, 1), \partial\Omega'),$$

from which there follows the contradiction: $r = r'$. ∎

We note that the converse of Theorem 5.27 also holds: if the modified principle of continuity holds for the domain $D \subset \mathbf{C}^n$ (at least when the domain G has complex dimension 1) then D is a domain of holomorphy. For the modified principle of continuity immediately implies the somewhat weaker version of the principle of continuity, given in §5.2.B, namely, the map ϕ (5.54) is also required to be real analytic with respect to the deformation parameter t. But since in the proof of the implication (b.2)⇒(b.3) such deformations are, in fact, used (they are even linear in t), it is clear that (b.3) can be derived from the modified principle of continuity.

As an example of the application of Theorem 5.27, we consider the situation in which the domain of holomorphy is (locally) saturated with respect to the holomorphic map.

Proposition 5.28. * *Let D be a domain of holomorphy in $\mathbf{C}^n$ and f a holomorphic function in some neighbourhood of the point $w \in \partial D$ such that $f(w) \in \mathbf{R}$ and $\partial_j f(w) \neq 0$ for at least one value $j = 1, \ldots, n$. Suppose further that*

$$f^{-1}(\mathbf{C} \backslash \mathbf{R}) \subset D. \tag{5.63}$$

Then there exists a neighbourhood Ω of w such that

$$D \cap \Omega = f^{-1}(\omega) \cap \Omega, \text{ where } \omega = f(D \cap \Omega). \tag{5.64}$$

∎ We suppose that $\partial_n f(w) \neq 0$. Since what we are trying to prove has a local character, we can carry out a biholomorphic change of coordinates in a neighbourhood of w (that is, a holomorphic map with a holomorphic inverse) $(z_1, \ldots, z_{n-1}, z_n) \to (z_1, \ldots, z_{n-1}, f)$. Thus we may suppose without loss of generality that $f(z) = z_n$. Condition (5.63) implies that there exists a polycircular domain $\Omega = P(0; \rho) \equiv S(0, \rho_1) \times \ldots \times S(0, \rho_n)$ in $\mathbf{C}^n$ with centre at 0 such that all points z of it with $\operatorname{Im} z_n \neq 0$ belong to D. We set $G = S(0, \rho_1) \times \ldots \times S(0, \rho_{n-1}) \subset \mathbf{C}^{n-1}$. Our proposition will be proved if we verify that for any point $a \in D \cap \Omega$ with $a_n \in \mathbf{R}$, all points of the form $(z_1, \ldots, z_{n-1}, a_n) \in \Omega$ (with fixed coordinate $z_n = a_n$) belong to D. To this end we set $G = S(0, \rho_1) \times \ldots \times S(0, \rho_{n-1}) \subset \mathbf{C}^n$, $\zeta = (z_1, \ldots, z_{n-1})$ and introduce the map $\phi(\zeta, t) = (z_1, \ldots, z_{n-1}, a_n + i\delta(1-t))$, where $\zeta \in G$, $0 \leq t \leq 1$. It is then clear that for sufficiently small $\delta > 0$ (5.61) holds. The result: $(z_1, \ldots, z_{n-1}, a_n) \equiv \phi(\zeta, 1) \in D$ now follows from Theorem 5.27. ∎

* This proposition is close to Lemma 3.5 of Epstein (1966).

D. SINGLE-SHEETED ENVELOPES OF HOLOMORPHY

For a domain $D \subset \mathbf{C}^n$, we consider the union

$$\tilde{D} = \cup D' \tag{5.65}$$

of all domains $D' \subset \mathbf{C}^n$ to which D has a holomorphic extension; we call $\tilde{D}$ the ***holomorphic container*** of D. An arbitrary function $f \in \mathcal{H}(D)$ admits an analytic continuation to each of the domains D', but it is possible that it does not define a single-valued function in $\tilde{D}$. To be able to treat such many-valued functions, it is necessary to consider, instead of $\tilde{D}$, a certain "many-sheeted domain" which covers $\tilde{D}$. If, on the other hand, it turns out that an arbitrary function $f \in \mathcal{H}(D)$ does have a single-valued analytic continuation to $\tilde{D}$, then in this case we call $\tilde{D}$ a ***single-sheeted envelope of holomorphy*** of D and denote it by $H(D)$.

It should be borne in mind that an arbitrary domain in $\mathbf{C}^n$ has an envelope of holomorphy which, in general, is many-sheeted, but not every domain has a single-sheeted envelope of holomorphy. Therefore in what follows, when we speak about the existence of $H(D)$, we have in mind a single-sheeted envelope of holomorphy.

It is clear that $H(D)$ is a maximal holomorphic extension of D. Since each holomorphic extension of $H(D)$ is also a holomorphic extension for D, it is trivial (in view of the fact that $H(D)$ is a maximal holomorphic extension of D). Consequently, $H(D)$ is a domain of holomorphy. On the other hand, if Ω is any other domain of holomorphy in $\mathbf{C}^n$ containing D, then it also contains $H(D)$ (since any holomorphic extension of D to a domain D' is also a holomorphic extension for Ω and is therefore trivial). Thus we have the following result.

Proposition 5.29. (a) *A single-sheeted envelope of holomorphy $H(D)$ is a domain of holomorphy containing D which is a holomorphic extension for D.*

(b) *A single-sheeted envelope of holomorphy $H(D)$ of the domain $D \subset \mathbf{C}^n$ (if it exists) coincides with the intersection of all domains of holomorphy in $\mathbf{C}^n$ containing D.*

There is a simple sufficient criterion for the existence of $H(D)$.

Proposition 5.30. *Let $D \subset \mathbf{C}^n$ be a domain that can be holomorphically extended to each of the domains D_λ of some family $\{D_\lambda\}_{\lambda \in \Lambda}$ whose union $\bigcup_\lambda D_\lambda$ is a simply connected domain of holomorphy containing D. Then a single-sheeted envelope of holomorphy exists and is equal to $\bigcup_\lambda D_\lambda$. In particular, if the holomorphic container $\tilde{D}$ of D is simply connected, then it is a single-sheeted envelope of holomorphy for D.*

This proposition is an immediate corollary of Proposition 5.29 and the remark in §5.1.C.

Exercise 5.26. Suppose that $H(D)$ exists.

(a) Prove that for any compactum $K \subset H(D)$ there is a compactum $K' \subset D$ such that $\|f\|^K \le \|f\|^{K'}$ for all $f \in \mathcal{H}(H(D))$. [Hint: Use the closed graph theorem to verify that the analytic continuation $\mathcal{H}(D) \to \mathcal{H}(H(D))$ is a continuous operator and hence that for any compactum $K \subset H(D)$, the expression $\|f\|^K$ defines a continuous seminorm on $\mathcal{H}(D)$. The rest of the argument is the same as in Proposition 5.4.]

(b) Let K_ν be an increasing sequence of compacta in D whose interiors cover D and let $\hat{K}_\nu = \{z \in H(D) : |f(z)| \le \|f\|^{K_\nu}$ for all $f \in \mathcal{H}(H(D)\}$. Prove that $\hat{K}_\nu$ is an increasing sequence of compacta in $H(D)$ whose interiors cover $H(D)$. [Hint: Use part (a) of this exercise.]

We now give a number of simple properties of envelopes of holomorphy.

Exercise 5.27. (a) Let $D_1 \subset D_2$ be two domains in $\mathbf{C}^n$. If $H(D_1)$ and $H(D_2)$ exist, then $H(D_1) \subset H(D_2)$.

(b) Prove that the envelope of holomorphy of a bounded domain is bounded. [Hint: Apply part (a) of this exercise to the case when D_2 is a ball.]

(c) Let $\{D_\lambda\}$ be a family of domains for which $H(D_\lambda)$ exist and let D be a connected component of the interior of their intersection. Prove that if $H(D)$ exists, then it is contained in $\bigcap_\lambda H(D_\lambda)$. [Hint: Use part (a) of this exercise.]

(d) Let Ω be a connected component of the intersection of the domain $D \subset \mathbf{C}^n$ with a k-dimensional complex plane $l \subset \mathbf{C}^n$ and suppose that $H(D)$ and $H(\Omega)$ exist. Prove that $H(\Omega) \subset$ $\subset H(D) \cap l$. [Hint: According to Exercise 5.21, $H(D) \bigcap l$ is a domain of holomorphy in l which contains Ω.]

Exercise 5.28. (a) Let $D = \bigcup D_\nu$ be the union of an increasing sequence of domains for which the $H(D_\nu)$ exist. Prove that $H(D)$ exists and is the same as $\bigcup_\nu H(D_\nu)$. [Hint: Use Exercise 5.25.]

(b) Let $D_1 \subset \mathbf{C}^k$ and $D_2 \subset \mathbf{C}^{n-k}$ be two domains having the single-sheeted envelopes of holomorphy $H(D_1)$ and $H(D_2)$. Prove that $H(D_1 \times D_2)$ exists and is the same as $H(D_1) \times H(D_2)$. [Hint: By Exercise 5.16 it suffices to prove that every function $f(\zeta, \zeta') \in \mathcal{H}(D_1 \times D_2)$ has a single-valued analytic continuation to $H(D_1) \times H(D_2)$. This continuation can be accomplished in two stages: first with respect to the first argument $\zeta \in D_1$ to the domain $H(D_1)$ for $\zeta' \in D_2$ and then with respect to the second argument $\zeta' \in D_2$ to $H(D_2)$ for fixed $\zeta \in H(D_1)$. The fact that for such a continuation, say during the first stage, one obtains an analytic function in $H(D_1) \times D_2$, can be derived from Propositions 5.13 and 5.14: for this it suffices to treat ζ' as the argument of the holomorphic distribution $f(\zeta, \zeta')$ and ζ as the parameter of the analytic continuation.]

The following proposition states that the process of holomorphic extension of a domain can only pass through a point of the boundary that is not a barrier point. (Of course, this fact can hardly be said to be unexpected.)

Proposition 5.31. *Suppose that $H(D)$ exists for the domain D. Then the set of barrier points for D is the same as the intersection of the boundaries of D and $H(D)$.*

■ If the point $w \in \partial D$ is an interior point of $H(D)$, then there is a neighbourhood of it in $H(D)$ into which all the functions in $\mathcal{H}(D)$ can be directly analytically continued. Therefore such a point is not a barrier point for D. If the point $w \in \partial D$ is a boundary point of $H(D)$, then (by the criterion (a.2) of Theorem 5.24) it is a barrier point for $H(D)$. It is now fairly easy to show by contradiction that w is also a barrier point for D. ■

It is clear that the operation of analytic continuation "commutes" with the algebraic operations on functions (as well as with differentiation with respect to an argument). From this we conclude the following.

Exercise 5.29. Prove that an analytic function in a domain D cannot take new values in $H(D)$.* [Hint: If $f(z) \neq c$ in D, then the function $(f(z) - c)^{-1}$ is holomorphic in D.]

The result of Exercise 5.27(d) admits a generalization to the case when the plane l intersects the boundary of the domain. We choose special coordinates for simplicity, by representing $\mathbf{C}^n$ as the product $\mathbf{C}^n = \mathbf{C}^k \times \mathbf{C}^{n-k}$ and supposing that the plane l has the form

$$l = \mathbf{C}^k \times \{\xi\}, \tag{5.66}$$

where ξ is a fixed point of $\mathbf{R}^{n-k}$. As was done in §B.4, we suppose that the domain $D \subset \mathbf{C}^n$ *is adjacent to the domain* $\Omega \subset l$ from the side iS (where S is a convex subset of $\mathbf{R}^k \times \{0\}$ whose boundary contains 0) in the sense that for any compactum $Q \subset \Omega$ and any truncated cone $K^r \subset S$ with compact closure in $S \cup \{0\}$, there exists $\rho \in (0, r)$ such that $D \supset Q + iK^\rho$.

Exercise 5.30. Suppose that the domain D in $\mathbf{C}^n$ is adjacent to the domain Ω on the k-dimensional complex plane (5.66) from the side iS. If $H(D)$ and $H(\Omega)$ exist, then $H(D)$ is also adjacent to $H(\Omega)$ from the side iS. [Hint: The union of all domains in l that are adjacent to $H(D)$ from the side iS is clearly an open set in l to which $H(D)$ is adjacent from the side iS. Let Ω' be the connected component of this union containing Ω. Then it suffices to prove that Ω' is a domain of holomorphy

* The corresponding assertion for vector-valued analytic functions would, in general, be false. (For $D \neq H(D)$, the vector-valued function $f(z) = z$ serves as a counterexample.)

(in l). One can use the criterion (b.2) of Theorem 5.25; thus the translations $\phi_y(\zeta,t) = \phi(\zeta,t) + iy$ by sufficiently small vectors $y \in K^r \subset S$ can be used as the arbitrary map $\phi : \overline{G} \times [0,1] \to l$ featuring in the principle of continuity.]

E. INVARIANT DOMAINS

Holomorphic maps of domains can also be continued analytically to envelopes of holomorphy. Thus let D' be a domain in $\mathbf{C}^k$ for which a single-sheeted envelope of holomorphy $H(D')$ exists, and let $\phi : D' \to \mathbf{C}^n$ be a holomorphic map. Then any component $\phi_j(\zeta)$ $(j = 1, \dots, n)$ can be continued (uniquely and) analytically to the domain $H(D')$ and so defines a holomorphic continuation $\phi : H(D') \to \mathbf{C}^n$. We suppose in addition that the image D' under the map ϕ is contained in a domain $D \subset \mathbf{C}^n$ for which $H(D)$ exists. It turns out that $\phi(H(D')) \subset H(D)$. To prove this, we introduce the set Ω of all points $\zeta \in H(D')$ such that $\phi(\zeta) \in H(D)$. This set is clearly non-empty (since it contains D') and open. To prove that Ω is closed in $H(D')$ we consider a set K of points of an arbitrary sequence in Ω that converges to a point $\zeta^{(\infty)} \in H(D')$. Then $\phi(K)$ is $\mathcal{H}(H(D))$-bounded (since for any function $f \in \mathcal{H}(H(D))$, $f(\phi(\zeta))$ belongs to $\mathcal{H}(H(D')$ which means that the set of numbers $f(\phi(K))$ is bounded). Therefore $\phi(K)$ is at a positive distance from $\partial H(D)$ and the limit point $\phi(\zeta^{(\infty)})$ belongs to $H(D)$. This means that Ω is closed in $H(D')$ and therefore is the same as $H(D')$. We have thus proved the following result.

Proposition 5.32. *Let ϕ be a holomorphic map from the domain $D' \subset \mathbf{C}^k$ to the domain $D \subset \mathbf{C}^n$ and suppose that $H(D')$ and $H(D)$ exist. Then ϕ has a single-valued continuation to a holomorphic map from $H(D')$ to $H(D)$.*

Corollary 5.33. Suppose that $H(D)$ exists. Then the action of any group G by analytic transformations onto D can be (uniquely) continued to an action of G by analytic transformations onto $H(D)$.

In other words, the envelope of holomorphy $H(D)$ inherits the symmetry properties of D.

We consider several illustrations of Proposition 5.32 and its corollary 5.33.

Tubular domains are domains in $\mathbf{C}^n$ that are invariant under the group $\mathbf{R}^n$ of real translations.

Proposition 5.34. *There exists a single-sheeted envelope of holomorphy $H(D)$ of any tubular domain $D \subset \mathbf{C}^n$ and it coincides with the convex hull of D.*

■ Let $\mathcal{M}(D)$ be the family of all domains $D' \in \mathbf{C}^n$ into which D can be holomorphically extended and let $\tilde{D} = \cup D'$. Without loss of generality we may suppose that D' is a convex domain (since any domain is the union of its convex subdomains). We can then suppose that D' is a tubular domain. In fact, if D' is a convex domain of $\mathcal{M}(D)$, then for any $a \in \mathbf{R}^n$, the domain $a + D'$ also belongs to $\mathcal{M}(D)$. Using Proposition 5.10, it is now fairly easy to deduce that the tubular domain $\bigcup_{a\in\mathbf{R}^n}(a + D')$ also belongs to $\mathcal{M}(D)$.

Now let D_1 and D_2 be two convex tubular domains in $\mathcal{M}(D)$ having a non-empty intersection. Suppose further that these domains (together with their intersection D_{12}) form the component (D_1, D_{12}, D_2) of some route of a holomorphic extension from D to D_2. It then follows from part (b) of Exercise 5.7 that D can be holomorphically extended to the tubular domain that is the convex hull of $D_1 \cup D_2$.

Finally, we consider an arbitrary route (5.21) of a holomorphic extension of D. According to what we have just said, all the domains featuring here, with the possible exception of the original domain $D = D_1$, may be assumed to be convex tubular ones. Then (by induction on N) we conclude that the convex hull of the union of these convex tubes also belongs to $\mathcal{M}(D)$. We now take some other route $(D = D_1, D'_{12}, D'_2, \dots)$. Since D is connected, we may suppose without loss of generality that D_{12} and D'_{12} have a non-empty intersection. The same argument shows that the convex hull of the union of the tubes $D_{12}, D_2, \dots, D_N, D'_{12}, D'_2, \dots, D'_N$ also belongs to $\mathcal{M}(D)$. This means that the domain $\tilde{D}$ is a convex tube and hence is a domain of holomorphy and that we obtain a single-valued analytic function in $\tilde{D}$ as a result of the analytic continuation of any function in $\mathcal{H}(D)$. This shows that $\tilde{D}$ is a single-sheeted envelope of holomorphy for D. ∎

We write an arbitrary point of $\mathbf{C}^n$ in the form (z,ζ) where $z \in \mathbf{C}^{n-1}$, $\zeta \in \mathbf{C}$ (in this way $\mathbf{C}^n$ is identified with the direct product $\mathbf{C}^{n-1} \times \mathbf{C}$). A domain $D \subset \mathbf{C}^n$ is called *semitubular* (with respect to ζ) if it is invariant with respect to the group $\mathbf{R}$ of translations $(z,\zeta) \to (z, \zeta + t)$, where t is an arbitrary real number. Clearly, D has the form

$$D = \{(z,\zeta) \in \mathbf{C}^n : z \in B,\ \alpha(z) < \operatorname{Im}\zeta < \beta(z)\}, \tag{5.67}$$

where B is some domain in $\mathbf{C}^{n-1}$ and $\alpha(z) < \beta(z)$ are two functions in B with values on $[-\infty, +\infty)$ and $(-\infty, +\infty]$ respectively, where $\alpha(z)$ is upper semicontinuous and $\beta(z)$ is lower semicontinuous.

Proposition 5.35. *Let D be a semitubular domain in $\mathbf{C}^n$ of the form* (5.67)*, where B is a domain of holomorphy in $\mathbf{C}^{n-1}$. Then $H(D)$ exists and is a semitubular domain of the form*

$$H(D) = \{(z,\zeta) \in \mathbf{C}^n : z \in B,\ a(z) < \operatorname{Im}\zeta < b(z)\}, \tag{5.68}$$

where $a(z)$ and $b(z)$ are respectively plurisubharmonic and plurisuperharmonic functions in B, with $a(z) \le \alpha(z)$, $\beta(z) \le b(z)$.

The proof can be found in [V4],§§21.4,21.5. A more complete characterization of the functions $a(z)$ and $b(z)$ in (5.68) is as follows: $a(z)$ is the largest of the plurisubharmonic functions in B that are bounded above by $\alpha(z)$; it is called the greatest plurisubharmonic minorant of $\alpha(z)$. Similarly $b(z)$ is the smallest plurisuperharmonic majorant of $\beta(z)$.

Let G be the group $U(1)^n$, the elements of which are the points $\omega \equiv (\omega_1, \dots, \omega_n)$ in $\mathbf{C}^n$ with $|\omega_j| = 1$ (for $j = 1, \dots, n$). It acts on $\mathbf{C}^n$ by componentwise multiplication: $(\omega z)_j = \omega_j z_j$. Every $U(1)^n$-invariant domain in $\mathbf{C}^n$ is called a *Reinhardt domain.** Clearly, such a domain is the union of (open) sets of the form $U_1 \times \dots \times U_n$, where each U_j is either an (open) disc or an (open) annulus in $\mathbf{C}$ with centre at the origin. In particular, a domain in $\mathbf{C}^n$ that is a union of (open) polycircular domains in $\mathbf{C}^n$ with centre at the origin is called a *complete Reinhardt domain.* A Reinhardt domain D is said to be *logarithmically convex* if for any finite set of points $z^{(1)}, \dots, z^{(N)} \in D$ and any finite set of non-negative numbers $t^{(1)}, \dots, t^{(N)}$ whose sum is 1, each point $z \in \mathbf{C}^n$ such that

$$|z_j| = \prod_{\nu=1}^{N} |z_j^{(\nu)}|^{t^{(\nu)}}, \tag{5.69}$$

also belongs to D. Clearly this condition implies that the image of D in $[-\infty, \infty) \times \dots \times [-\infty, \infty)$ under the map $z \to (\ln|z_1|, \dots, \ln|z_n|)$ is a convex set. The *logarithmically convex hull* of an arbitrary (complete) Reinhardt domain D (consisting of points $z \in \mathbf{C}^n$ having a representation (5.69)) is also a (complete) Reinhardt domain.

* More precisely, it is a Reinhardt domain relative to the origin. Translating this by an arbitrary vector $a \in \mathbf{C}^n$, we obtain a Reinhardt domain relative to the point a.

Proposition 5.36. *A single-sheeted envelope of holomorphy of a complete Reinhardt domain $D \subset \mathbf{C}^n$ exists and coincides with the logarithmically convex hull of the domain.**

■ We will show that D can be holomorphically extended to the logarithmically convex hull $L(D)$ of D. We have the representation (5.69) for an arbitrary $z \in L(D)$, where $z^{(1)}, \dots, z^{(N)} \in D$, and $t^{(1)}, \dots, t^{(N)}$ is a set of numbers ≥ 0 with sum equal to 1. The point $z^{(\nu)}$ is contained in some polycircular domain $P(0; \rho^{(\nu)})$ the closure of which lies in D. Consequently any function $f \in \mathcal{H}(D)$ can be expanded in a Taylor series $f(z) = \sum_\alpha c_\alpha z^\alpha$ in $P(0; \rho^{(\nu)})$, the coefficients of which have the following upper bound:

$$|c_\alpha| \leq M^{(\nu)}/\rho^{(\nu)\alpha} \quad \text{for all } a \in \overline{\mathbf{Z}}_+^n,$$

so that

$$|c_\alpha| \leq M\Big[\prod_{\nu=1}^{N} (\rho^{(\nu),\alpha})^{t^{(\nu)}}\Big]^{-1}, \quad \text{where } M = \sum_{\nu=1}^{N} (M^{(\nu)})^t.$$

Hence it follows that the Taylor series converges in a neighbourhood of the point $z \in L(D)$ (fixed above). Thus we have proved that every function $f \in \mathcal{H}(D)$ has an analytic continuation to any polycircular domain $P(0;\rho) \subset L(D)$. Furthermore, in the intersection of two such polycircular domains, the analytic continuations are represented by the same Taylor series and therefore are identical. This means that D extends holomorphically to $L(D)$.

Next, we verify that $L(D)$ is a domain of holomorphy. Let D' be an arbitrary domain in $\mathbf{C}^n$ lying outside the closure of $L(D)$ in $\mathbf{C}^n$. We fix an arbitrary $w \in D'$ with non-zero coordinates w_j $(j = 1, \dots, n)$. Let S be the image in $\mathbf{R}^n$ of the set of points $z \in D$ with non-zero coordinates z_j $(j = = 1, \dots, n)$ under the map $z \to (\ln|z_1|, \dots, \ln|z_n|)$. Since $L(D)$ is a logarithmically convex Reinhardt domain, S is a convex domain whose closure does not contain the point $\lambda = (\ln|w_1|, \dots, \ln|w_n|)$. Consequently there exists a hyperplane in $\mathbf{R}^n$ separating S and λ (see the hint to Exercise B.5(b)). In other words, there exist $\alpha \in \mathbf{R}^n$ and $b \in \mathbf{R}$ such that $\alpha\xi \leq b$ for $\xi \in S$, $\alpha\lambda > b$. Since any coordinate ξ_j can tend to $-\infty$ in S (for fixed values of the remaining coordinates), $\alpha_j \geq 0$ for all j. It is clear that for those j for which $|z_j|$ is not bounded in D, the equality $\alpha_j = 0$ must hold. Therefore, by slightly increasing the non-zero coordinates α_j and the number b if necessary, we can ensure that the following condition holds: all the α_j are non-negative rational numbers. Then by multiplying the α_j and b by the same integral factor, we can ensure that $a_j \in \overline{\mathbf{Z}}_+$ for all j. We have proved the existence of a multi-index $\alpha \in \overline{\mathbf{Z}}_+^n$ and a number $M = e^b$, for which $|z^\alpha| \leq M$ for $z \in L(D)$ and $|w^\alpha| > M$. Since z^α is an entire analytic function and can not take any new values under a holomorphic continuation (see Exercise 5.29), it follows that the point w cannot belong to any holomorphic extension of $L(D)$. Consequently $L(D)$ has no non-trivial holomorphic extensions. ■

A domain D in $\mathbf{C}^n$ is called *starlike* (or starlike relative to the origin), if $rD \subset D$ for all $r \in (0, 1]$. If D is a starlike domain, then the domain $a + D$ (for $a \in \mathbf{C}^n$) is said to be starlike relative to the point a. It is fairly easy to see that the holomorphic container $\tilde{D}$ of a starlike domain D is a starlike domain.

Proposition 5.37. *If the starlike domain $D \subset \mathbf{C}^n$ is such that its holomorphic container $\tilde{D}$ has a convex intersection with some neighbourhood of the origin, then a single-sheeted envelope of holomorphy of D exists: $H(D) = \tilde{D}$.*

■ In view of Proposition 5.30, it suffices to prove that the starlike domain $\tilde{D}$ is simply connected. Let $\overline{S}$ be the closed disc with centre at 0 and radius 1 in $\mathbf{C}$, ∂S its boundary, and $\phi : \partial S \to \tilde{D}$ an arbitrary continuous map. We claim that ϕ can be extended to a continuous map $\phi : \overline{S} \to \tilde{D}$. Thus we set $\phi(\zeta) = |\zeta|\phi(\zeta/|\zeta|)$ for $r \leq |\zeta| \leq 1$, where $r > 0$ will be chosen presently. Let D_0 be a neighbourhood of the origin having a convex intersection with $\tilde{D}$. Then we choose $r \in (0, 1]$ from the condition that $r\phi(\partial S) \subset \tilde{D} \cap D_0$. Since $\tilde{D} \cap D_0$ is convex and is therefore a simply connected domain, there exists a continuous map $\phi : r\overline{S} \to \tilde{D}$, that coincides on $r\partial S$ with the map $\phi : \overline{S}\backslash rS \to \tilde{D}$ constructed above. As a result we have constructed a map $\phi : \overline{S} \to \tilde{D}$, which is equal to the given map on ∂S. ■

* We have restricted ourselves to complete Reinhardt domains, although the assertion holds for arbitrary Reinhardt domains (see [V4]§19.3).

Corollary 5.38. A starlike domain $D \subset \mathbf{C}^n$ containing the origin has a single-sheeted envelope of holomorphy.

F. AN EXAMPLE OF A HOLOMORPHIC EXTENSION

We consider an example (from the article by Bros et al, 1964) in which domains feature that are related to the light cone (see §B.3). We denote by $\mathbf{CM}^n$ the direct product of n copies of complex four-dimensional Minkowski sapce; this is the space $\mathbf{C}^{4n}$ endowed with the bilinear form (B.44). We single out the two tubes $\mathcal{T}^{\Omega_1}$ and $\mathcal{T}^{\Omega_2}$ in $\mathbf{CM}^n$ with the bases

$$\Omega_1 = \{y \in \mathbf{M}^n : y_j \in V^+ \quad \text{for } j = 1, \ldots, n\} \tag{5.70}$$

$$\Omega_2 = \{y \in \mathbf{M}^n : y_1 \in V^-, \quad y_1 + y_j \in V^+ \quad \text{for } j = 2, \ldots, n\} \tag{5.71}$$

and the real open set (where $\mu > 0$)

$$\mathcal{O} = \{x \in \mathbf{M}^n : (x_1)^2 < \mu^2\}. \tag{5.72}$$

Then we have the following result.

Proposition 5.39. *(In the notation of* (5.70)–(5.72)*) let D be a starlike domain in $\mathbf{CM}^n$ containing the origin, the tubes $\mathcal{T}^{\Omega_1}$, $\mathcal{T}^{\Omega_2}$ and the "edge of the wedge" domain D_0 corresponding to the generalized route $(\mathcal{T}^{\Omega_1}, \mathcal{O}, \mathcal{T}^{\Omega_2})$. Then $H(D)$ exists and contains the domain*

$$G = \mathcal{T}^{\Omega} \backslash \gamma, \tag{5.73}$$

where Ω is the convex hull of the cones Ω_1 and Ω_2, and

$$\gamma = \{z \in \mathbf{CM}^n : (z_1)^2 \geq \mu^2\}. \tag{5.74}$$

The proof is divided into several lemmas. It serves as an illustration of the "method of sections" consisting in the following. The domain $D_1 = \mathcal{T}^{\Omega_1} \cup \mathcal{T}^{\Omega_2} \cup D_0$ is cut up by the complex planes $l \subset \mathbf{M}^n$ from a certain family Λ and in each of them, the holomorphic envelope $H(D_1 \cap l)$ is constructed and the results are formed into a union. (The fact that the set so obtained is contained in $H(D)$ follows from Exercise 5.27(d); the existence of $H(D)$ can be deduced from Corollary 5.38.)

In the following lemmas we shall also be dealing with the situation when we are given two convex cones, say, K_1 and K_2 in $\mathbf{R}^n$ and a domain $\mathcal{O}$ in $\mathbf{R}^n$. We denote the "edge of the wedge" domain corresponding to the generalized route $(\mathcal{T}^{K_1}, \mathcal{O}, \mathcal{T}^{K_2})$ (see §5.1.D) by $\mathcal{N}(\mathcal{O})$ (without further explanation, since it will be clear from the context which generalized route we are talking about).

Lemma 5.40. *Let Δ_n be a parallelepiped in $\mathbf{R}^n$ of the form*

$$\Delta_n = (a_1, b_1) \times \ldots \times (a_n, b_n), \tag{5.75}$$

*and $\phi(z) \equiv (\phi_j(z))$ the holomorphic map**

$$\phi_j(z) = \frac{1}{\pi} \ln \frac{z_j - a_j}{b_j - z_j}. \tag{5.76}$$

* The branch of the logarithm is defined in the plane cut along the negative real semi-axis by the condition $|\operatorname{Im} \ln z| < \pi$.

Then the envelope of holomorphy of the union of the domains $\mathbf{R}^n - i\mathbf{R}^n_+$, $\mathbf{R}^n + i\mathbf{R}^n_+$ *and* $\mathcal{N}(\Delta_n)$ *satisfies the relation*

$$\phi(H(\mathbf{R}^n - i\mathbf{R}^n_+) \cup (\mathbf{R}_n + i\mathbf{R}^n_+) \cup \mathcal{N}(\Delta_n)) = \mathbf{R}^n + iS_n, \tag{5.77}$$

where S_n *is the convex hull of the union of the sets* $S_n^{\pm}$:

$$S_n^+ = -S_n^- = \{y \in \mathbf{R}^n : 0 < y_j < 1,\ j = 1, \ldots, n\}.$$

This lemma is an immediate consequence of the "edge of the wedge" theorem which is applicable to the tubes $\mathbf{R}^n \pm S_n^+$ and to the domain of coincidence $\mathbf{R}^n$; in this case, the "edge of the wedge" domain is the convex hull of the tubes (by virtue of Proposition 5.34).

Lemma 5.40 easily extends to the case of an unbounded parallepiped Δ_n. For example, let

$$\Delta_n = \Delta_2 \times \mathbf{R}^{n-2}. \tag{5.78}$$

Then we have the relation

$$\begin{aligned} &H((\mathbf{R}^n - i\mathbf{R}^n_+) \cup (\mathbf{R}^n + i\mathbf{R}^n_+) \cup \mathcal{N}(\Delta_n)) = \\ &= H((\mathbf{R}^2 - i\mathbf{R}^2_+) \cup (\mathbf{R}^2 + i\mathbf{R}^2_+) \cup \mathcal{N}(\Delta_2 \times \mathbf{C}^{n-2})). \end{aligned} \tag{5.79}$$

In fact, if we replace the equality in (5.79) by the inclusion ($\subset$), the relation becomes trivial. To prove the reverse inclusion we note that the left hand side of (5.79) contains all sets of the form $H((\mathbf{R}^n - i\mathbf{R}^n_+) \cup (\mathbf{R}^n + i\mathbf{R}^n_+) \cup \mathcal{N}(\Delta'_n))$, where $\Delta'_n = \Delta_2 \times (-l, l) \times \ldots \times (-l, l)$. Using Lemma 5.40 it is not difficult to see that an arbitrary point of the right hand side of (5.79) is contained in such a set for sufficiently large l. This proves (5.79).

Lemma 5.41. *Let*

$$\delta = \{z \in \mathbf{C}^2 : z_1 z_2 \geq \mu^2\}, \tag{5.80}$$

where $\mu > 0$. *Then*

$$\mathbf{C}^2 \backslash \delta = \bigcup_{\Delta_2 \subset \mathbf{R}^2 \backslash \delta} H((\mathbf{R}^2 - i\mathbf{R}^2_+) \cup (\mathbf{R}^2 + i\mathbf{R}^2_+) \cup \mathcal{N}(\Delta_2)), \tag{5.81}$$

where Δ_2 *runs through all possible bounded parallelepipeds* (*of type* (5.75)) *in* $\mathbf{R}^2 \backslash \delta$.

■ For definiteness, we set $\mu = 1$. As in the discussion of (5.79) above, the inclusion one way round is trivial. Thus it suffices to prove that any point $z \in \mathbf{C}^2 \backslash \delta$ is contained in one of the sets on the right hand side of (5.81) for some $\Delta_2 = (a_1, b_1) \times (a_2, b_2) \subset \mathbf{R}^2 \backslash \delta$. It is obvious here that we may suppose that

$$a_1 = -a,\ a_2 = -1/a,\ b_1 = b,\ b_2 = 1/b,$$

where a, b are positive numbers. By virtue of the previous lemma, we are required to prove that

$$\frac{1}{\pi}\left(\arg \frac{z_1 - a_1}{b_1 - a_1},\ \arg \frac{z_2 - a_2}{b_2 - z_2}\right) \in S_2 \equiv \{y \in \mathbf{R}^2 : |y_1| < 1,\ |y_2| < 1,\ |y_1 - y_2| < 1\} \tag{5.82}$$

(here, $z_1 \neq a_1,\ z_1 \neq b_1,\ z_2 \neq a_2,\ z_2 \neq b_2$). If $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 > 0$, then $\arg \frac{z_1 - a_1}{b_1 - z_1}$ and $\arg \frac{z_2 - a_2}{b_2 - z_2}$ always have the same sign and (5.82) holds.

It remains to consider the case $\operatorname{Im} z_1 \cdot \operatorname{Im} z_2 < 0$. Condition (5.82) then has the following geometric interpretation: $\alpha + \beta > \pi$, where $\alpha \in [0, \pi]$ is the size of the angle of the triangle (a_1, z_1, b_1) with vertex at z_1 in the complex plane (if the triangle degenerates to a line segment, then $\alpha = 0$ or π according to whether z is an end point or an interior point of the segment); similarly β is the size of the angle of the triangle (a_2, z_2, b_2) with vertex at z_2. It is clear that this condition holds for some a, b if at least one of the numbers at z_1 or z_2 is zero, therefore we now suppose that $z_1 z_2 \neq 0$. If $|\arg z_1| > |\arg z_2|$, then as $a \to \infty$ and $b \to 0$ we have $\alpha \to |\arg z_1|$ and $\beta \to \pi - |\arg z_2|$, so that the condition $\alpha + \beta > \pi$ holds for sufficiently large a and b^{-1}. The case $|\arg z_1| < |\arg z_2|$ is symmetric. There remains the final case $\arg z_1 = -\arg z_2$. Together with the condition $z \in \mathbf{C}^2 \backslash \delta$ we have $0 < |z_1| \cdot |z_2| < 1$. Then for $a = b = |z_1|$, it is clear that $\alpha = \pi/2$ and $\beta > \pi/2$, so that $\alpha + \beta > \pi$ again holds. ∎

Lemma 5.42. *Let $\mathcal{T}^{\omega_1}$ and $\mathcal{T}^{\omega_2}$ be two convex tubular domains in $\mathbf{C}^n$ with bases ω_1 and ω_2 such that $\overline{\omega}_1 \supset \omega_1'$, $\overline{\omega}_2 \supset \omega_2'$, where*

$$\omega_1' \supset \{y \in \mathbf{R}^n : y_i \geq 0 \quad \text{for } i \in I,\ y_j = 0 \quad \text{for } j \in J\},$$
$$\omega_2' \supset \{y \in \mathbf{R}^n : y_i = 0 \quad \text{for } i \in I,\ y_j \leq 0 \quad \text{for } j \subset J\};$$

here I and J are two non-empty disjoint subsets of the set of indices $\{1, \ldots, n\}$ such that $I \cup J = \{1, \ldots, n\}$. Then for any bounded parallelepiped $\Delta_n \subset \mathbf{R}^n$ of type (5.75) *we have the relation*

$$H(\mathcal{T}^{\omega_1} \cup \mathcal{T}^{\omega_2} \cup \mathcal{N}(\Delta_n)) \supset ((\mathbf{R}^n - i\mathbf{R}^n_+) \cup (\mathbf{R}^n + i\mathbf{R}^n_+) \cup \mathcal{N}(\Delta_n)) \cap \mathcal{T}^{\omega}, \qquad (5.83)$$

where

$$\omega = \{y \in \mathbf{R}^n : y_i \geq 0 \quad \text{for } i \in I, \quad y_j \leq 0 \quad \text{for } j \in J\}.$$

As a corollary, when $1 \in I$ and $2 \in J$ we have

$$H(\mathcal{T}^{\omega_1} \cup \mathcal{T}^{\omega_2} \cup \mathcal{N}((\mathbf{R}^2 \backslash \delta) \times \mathbf{R}^{n-2})) \supset (\mathbf{C}^2 \backslash \delta) \cap \mathcal{T}^{\omega},$$

where δ is defined in (5.80).

∎ Instead of the condition $\overline{\omega}_\alpha \supset \omega_\alpha'$, we can suppose that the condition $\omega_\alpha \supset \omega_\alpha'$ holds (the general case reduces to this by making an arbitrarily small rotation of the axes and an arbitrarily small change in Δ_n). Let F be the domain of definition of the conformal map ϕ of Lemma 5.40. Then clearly,

$$\phi(\mathcal{T}^{\omega_\alpha} \cap F) \supset \mathcal{T}^{s_\alpha} \ (\alpha = 1, 2), \ \phi(\Delta_n) = \mathbf{R}^n,$$

where

$$s_1 = \{y \in \mathbf{R}^n : 0 < y_i < 1 \quad \text{for } i \in I, \quad y_j = 0 \quad \text{for } j \in J\},$$
$$s_2 = \{y \in \mathbf{R}^n : y_i = 0 \quad \text{for } i \in I, \quad -1 < y_j < 0 \quad \text{for } j \in J\}.$$

By the "edge of the wedge" theorem 5.12 and Proposition 5.34 we have*

$$H(\phi((\mathcal{T}^{\omega_1} \cup \mathcal{T}^{\omega_2} \cup \mathcal{N}(\Delta_n)) \cap F)) \supset \mathcal{T}^{s},$$

where s is the convex hull of $s_1 \cup s_2$. Hence $H(\mathcal{T}^{\omega_1} \cup \mathcal{T}^{\omega_2} \cup \mathcal{N}(\Delta_n)) \supset \phi^{-1}(\mathcal{T}^{s})$, which together with the relation $\phi^{-1}(\mathcal{T}^{s}) = H((\mathbf{R}^n - i\mathbf{R}^n_-) \cup (\mathbf{R}^n + i\mathbf{R}^n_+) \cup \mathcal{N}(\Delta_n)) \cap \mathcal{T}^{\omega}$ gives (5.83). To derive (5.84) we suppose that Δ_2 is an arbitrary bounded rectangle in $\mathbf{R}^2 \backslash \delta$. Then for all Δ_n of the form $\Delta_2 \times (-l, l) \times \ldots \times (-l, l)$ the left hand side of (5.84) contains the left hand side of (5.83), therefore $H((\mathbf{R}^2 - i\mathbf{R}^2_+) \cup (\mathbf{R}^2 + i\mathbf{R}^2_+) \cup \mathcal{N}(\Delta_2)) \times \mathbf{C}^{n-2}$ by virtue of the arbitrariness of l (see the corresponding argument in the derivation of (5.79)). An application of Lemma 5.41 now completes the proof of (5.84). ∎

The next lemma is an analogue of (5.73) for the two-dimensional time space (in the so-called "conical" variables). We write an arbitrary point $z \in \mathbf{C}^{2n}$ in the form $z = (z_1, z_1', z_2, z_2', \ldots, z_n, z_n')$.

Lemma 5.43. *Let $\mathcal{T}^{Q_1}$ and $\mathcal{T}^{Q_2}$ be two tubular domains in $\mathbf{C}^{2n}$ with bases*

$$Q_1 = \{y \in \mathbf{R}^{2n} : y_j > 0, \ y_j' > 0 \quad \text{for } j = 1, \ldots, n\},$$
$$Q_2 = \{y \in \mathbf{R}^{2n} : y_1 < 0, \ y_1' < 0, \ y_1 + y_j > 0, \ y_1' + y_j' > 0 \quad \text{for } j = 2, \ldots, n\}$$

* Compare the similar argument in Lemma 5.40.

and let $\lambda = \{z \in \mathbf{C}^{2n} : z_1 z_1' \le \mu^2\}$. *Then*

$$H(\mathcal{T}^{Q_1} \cup \mathcal{T}^{Q_2} \cup \mathcal{N}(\mathbf{R}^{2n}\backslash\lambda)) = (\mathcal{T}^{Q_1} + \mathcal{T}^{Q_2})\backslash\lambda. \tag{5.85}$$

■ Since the right hand side of (5.85) is a domain of holomorphy containing $\mathcal{T}^{Q_1} \cup \mathcal{T}^{Q_2} \cup (\mathbf{R}^{2n}\backslash\lambda)$, it suffices to prove the relation obtained from (5.85) in which the = sign is replaced by $\supset$. For $Q_1 + Q_2$ we have

$$Q_1 + Q_2 = \{y \in \mathbf{R}^{2n} : y_j > 0,\ y_1 + y_j > 0,\ y_j' > 0,\ y_1' + y_j' > 0 \quad \text{for } j = 2, \dots, n\}. \tag{5.86}$$

For the right hand side is a convex cone containing Q_1 and Q_2 and hence $Q_1 + Q_2$. Conversely, let y be an arbitrary point of the right hand side of (5.86): we set

$$\begin{aligned} a_1 &= y_1\theta(y_1) + \epsilon, & a_1' &= y_1'\theta(y_1') + \epsilon & a_j &= \epsilon y_j, & a_j' &= \epsilon y_j', \\ b_1 &= y_1\theta(-y_1) - \epsilon, & b_1' &= y_1'\theta(-y_1') - \epsilon, & b_j &= (1-\epsilon)y_j, & b_j' &= (1-\epsilon)y_j' \end{aligned}$$

(where $j = 2, \dots, n$). It is clear that for sufficiently small $\epsilon > 0$ these equalities define a pair of points $a \in \mathbf{R}^2_+$, $b \in -\mathbf{R}^2_+$, whose sum is equal to y. Thus (5.86) has been proved. It can be rewritten in the form

$$Q_1 + Q_2 = Q_1 \cup Q_2 \cup Q_3 \cup Q_4,$$

where

$$\begin{aligned} Q_3 &= \{y \in \mathbf{R}^{2n} : y_1 \ge 0,\ y_1' \le 0,\ y_j > 0,\ y_1' + y_j' > 0,\ j = 1, 2, \dots, n\}, \\ Q_4 &= \{y \in \mathbf{R}^{2n} : y_1 \le 0,\ y_1' \ge 0,\ y_1 + y_j > 0,\ y_j' > 0,\ j = 2, \dots, n\}. \end{aligned}$$

Consequently our formula (5.85) reduces to the assertion that the left hand side contains the sets $\mathcal{T}^{Q_3}\backslash\lambda$, $\mathcal{T}^{Q_4}\backslash\lambda$. In view of the symmetry of the situations, it is enough to consider just $\mathcal{T}^{Q_3}\backslash\lambda$. We introduce new coordinates $\zeta_1, \dots, \zeta_{2n}$ in $\mathbf{C}^{2n}$ by setting

$$\zeta_1 = z_1,\ \zeta_2 = z_1',\ \zeta_{2j-1} = z_j,\ \zeta_{2j} = -(z_1' + z_j') \quad \text{for } j = 2, \dots, n.$$

Let $\mathcal{T}^{\omega_1}$ and $\mathcal{T}^{\omega_2}$ be the new notation for the tubes $\mathcal{T}^{Q_1}$ and $\mathcal{T}^{Q_2}$ in the ζ coordinates; then they satisfy the conditions of Lemma 5.42 (with n replaced by $2n$) if for I and J we take the sets of even and odd indices in $\{1, \dots, 2n\}$ respectively. Formula (5.84) (in the ζ coordinates), rewritten in the previous z coordinates, means that the left hand side of (5.85) contains $\mathcal{T}^{Q_3}\backslash\lambda$. This completes the proof of (5.85). ■

As we have already said, we shall prove Proposition (5.39) by the "method of sections". We consider the family Λ of complex $(3n-1)$-dimensional planes l in $\mathbf{CM}^n$ of the form

$$l = a + \{z \in \mathbf{CM}^n : z_1 = z_1'e' + z_1''e'',\ z_j = z_j'e' + z_j''e'' + \zeta_j e_j \quad \text{for } j \ge 2\}.$$

Such a plane is characterized by a pair of 4-vectors $e', e'' \in \overline{V}^+$ such that $e'^2 = e''^2 = 0$, $e'e'' > 0$, the vectors $e_2, \dots, e_n$ in V^+ and the point $a \equiv (a_1, \dots, a_n) \in \mathbf{M}^n$, where $ae' = ae'' = 0$. The variables $z_1', z_1'', \dots, z_n', z_n'', \zeta_2, \dots \zeta_n$ form independent coordinates on l. We denote by $\operatorname{Re} l$ the real part of l corresponding to the real values of the coordinates z_j', z_j'', ζ_k ($j \ge 1$, $k \ge 2$). We single out the tubes $\mathcal{T}_1(l)$ and $\mathcal{T}_2(l)$ and the set $\gamma(l)$ in l by the conditions

$$\begin{aligned} &\mathcal{T}_1(l) : \ \operatorname{Im} z_j' > 0,\ \operatorname{Im} z_j'' > 0,\ \operatorname{Im}\zeta_k > 0\ (1 \le j \le n,\ 2 \le k \le n), \\ &\mathcal{T}_2(l) : \ \operatorname{Im} z_j' < 0,\ \operatorname{Im} z_1'' < 0,\ \operatorname{Im}(z_1' + z_j') > 0,\ \operatorname{Im}(z_1'' + z_j'') > 0,\ \operatorname{Im}\zeta_k > 0\ (2 \le j \le n,\ 2 \le k \le n), \\ &\gamma(l) : \ z_1'z_1'' \ge \mu^2 - a_1^2. \end{aligned}$$

It is then clear that the following relations hold: $\mathcal{T}_1(l) \subset \mathcal{T}^{\Omega_1}$, $\mathcal{T}_2(l) \subset \mathcal{T}^{\Omega_2}$ (and hence, that $\mathcal{T}(l) \subset \subset \mathcal{T}^{\Omega}$, where $\mathcal{T}(l)$ is the convex hull of $\mathcal{T}_1(l) \cup \mathcal{T}_2(l)$), $\operatorname{Re} l \subset \mathbf{M}^n$, $\gamma(l) = \gamma \cap l$. Furthermore, $D_0 \cap l$ contains a complex domain on l that is adjacent to the domain $\operatorname{Re} l\backslash\gamma(l)$ from the side of the tube $\mathcal{T}(l)$. Then (by the "edge of the wedge" theorem)

$$H(\mathcal{T}_1(l) \cup \mathcal{T}_2(l) \cup (\mathcal{N}(\operatorname{Re} l\backslash\gamma(l)))) = H(\mathcal{T}_1(l) \cup \mathcal{T}_2(l) \cup (D_0 \cap l) \subset \mathcal{T}^{\Omega}\backslash\gamma. \tag{5.87}$$

By representing (as in Lemma 5.41) $\operatorname{Re} l\backslash\gamma(l)$ as a union of bounded rectangles and applying the arguments given after Lemma 5.40, we see that the left hand side of (5.87) factorizes in the variables (z', z'') and (ζ). We then have the situation of Lemma 5.43 with respect to the variables (z', z''), which in all gives

$$H(\mathcal{T}_1(l) \cup \mathcal{T}_2(l) \cup (\mathcal{N}(\operatorname{Re} l\backslash\gamma(l)))) = \mathcal{T}(l)\backslash\gamma(l) = \mathcal{T}(l)\backslash\gamma. \tag{5.88}$$

It follows from (5.87), (5.88) and the explanation of the "method of sections" given above that for the proof of Proposition 5.39, it remains to verify the equality

$$\bigcup_l \mathcal{T}(l) = \mathcal{T}^{\Omega}. \tag{5.89}$$

For this it suffices to show that an arbitrary point $z \in \mathcal{T}^{\Omega}$ belongs to $\mathcal{T}(l)$ for some l. We may suppose without loss of generality that the real part of the vector $z \equiv x + iy$ is zero (the general case being obtained by translating by a real vector). By hypothesis we have $y = u + v$, where $u_j \in V^+$ for $1 \le j \le n$, $v_1 \in V^-$, $(v_1 + v_k) \in V^+$ for $2 \le k \le n$. We choose $e', e'' \in \overline{V}^+$ arbitrarily in the plane of the vectors u_1 and v_1 (but subject to the conditions $e'^2 = e''^2 = 0$, $e'e'' > 0$). We then set $e_j = u_j + (v_1 + v_j) + 2\epsilon v_1$, where $\epsilon > 0$ is small enough so that $e_j \in V^+$ $(2 \le j \le n)$. Finally, we choose $a = 0$. By definition of the plane l, the vector z is the sum of two vectors $ip \in \mathcal{T}_1(l)$ and $iq \in \mathcal{T}_2(l)$, if we set

$$p_1 = u_1,\ p_j = -\epsilon v_1 + \tfrac{1}{2}e_j,\ q_1 = v_1,\ q_j = -(1+\epsilon)v_1 + \tfrac{1}{2}e_j$$

(where $2 \le j \le n$). Thus we have proved (5.89) and now Proposition 5.39 is entirely proved.

The method of proof of Proposition 5.39 can be applied in other similar problems. For example, Bros et al (1961) have proved in similar fashion the following result on a domain of type (4.130) obtained from the "symmetric" JLD representation (see §4.3.C).

Proposition 5.44. *Let $\mathcal{N}(\mathcal{O})$ be the "edge of the wedge" domain corresponding to the generalized route $(\mathrm{M} - iV^+, \mathcal{O}, \mathrm{M} + iV^+)$, where $\mathcal{O}$ is the complement in* M *of the set $(-a + \overline{V}^+_M) \cup (a + \overline{V}^-_M)$ and $a = (A, 0)$ $(A > 0)$. Then the domain* (4.130) *(where* adm *is defined by* (4.111) *with $a = a', M = M'$) is an envelope of holomorphy of the domain $(\mathrm{M} - iV^+) \cup (\mathrm{M} + iV^-) \cup \mathcal{N}(\mathcal{O})$.*

For the proof, see Bros et al (1961).

Part II

Relativistic Quantum Systems

Synopsis

The starting object of the algebraic approach in quantum theory is the algebra of observables defined as a C^*-algebra $\mathfrak{A}$ with identity. By a state one means a positive functional ω on $\mathfrak{A}$ normalized by the condition $\omega(1) = 1$. The mean value of an observable A $(= A^*)$ in state ω is defined as the value of the functional $\omega(A)$ (A.I,§6.1.A). The extreme points of the (convex) set $S(\mathfrak{A})$ of all states are called pure states. The states that are expressible in the form $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, where $0 < \lambda < 1$, $\omega_1 \neq \omega_2$, describe statistical mixtures. The physical states form a convex subset $\mathfrak{S}$ of the set $S(\mathfrak{A})$ of all states. It is postulated that it coincides with the set S_π of states associated with the (physical) representation $\pi = \pi_{\text{phys}}$ of the algebra $\mathfrak{A}$ in (physical) Hilbert space $\mathcal{H} = \mathcal{H}_{\text{phys}}$ (A.II,§6.1.C). Superpositions of states of an electron and a neutron or a neutrino and a photon are not physically realizable pure states. The first (respectively second) of these superpositions is unrealizable by virtue of the superselection rule with respect to the electric charge (repectively, the "valency" (or parity of the number of fermions)). The electric charge Q and the valency are examples of operators that commute with all the observables of the algebra $\pi(\mathfrak{A})$. We suppose (Hypothesis (a)) that the vectors in physical space $\mathcal{H}$ defining the pure states of the algebra of observables form a total set (that is, their linear span is dense in $\mathcal{H}$). The discreteness of the superselection rules (Proposition 6.5) follows from this. If it is further supposed that the correspondence between the pure states of $\mathfrak{S}$ and the unit rays in $\mathfrak{M}$ is one-to-one (Hypothesis (b)), then the superselection rules are commutative (that is, the algebra $\pi(\mathfrak{A})$ has an abelian commutant in $\mathcal{B}(\mathcal{H})$, Proposition 6.6).

The standard (that is, discrete commutative) superselection rules are described by defining the abelian group $U(1) \times \ldots \times (1)$ of gauge transformations (of the first kind) which act in $\mathcal{B}(\mathcal{H})$ and leave the observables invariant (§6.2.C). An example is also given of the non-standard superselection rules corresponding to a non-abelian gauge group (§6.2.D).

An important example of an algebra of observables is the algebra of canonical commutation relations (CCR's). In its exponential form (due to Weyl) this algebra is generated by two (n-parameter) abelian groups $U(a) = e^{iap}$ $(ap \equiv \sum_{k=1}^n a_k p_k)$ and $V(b) = e^{iqb}$ satisfying the commutation relation

$$U(a)\,V(b) = e^{-iab}V(b)\,U(a)$$

The von Neumann uniqueness theorem (Theorem 6.14) holds, according to which, each irreducible Weyl system with n degrees of freedom is unitarily equivalent to the Schrödinger representation $\left(p_k = -i\frac{\partial}{\partial q_k}\right)$ in $\mathcal{L}^2(\mathbf{R}^n)$.

By contrast, in a system with an infinite number of degrees of freedom there exist many inequivalent CCR representations. It is proved in §6.4.B that each unitary representation of the Weyl algebra of CCR's with a cyclic vector is defined (to within unitary equivalence) by some characteristic functional. The choice of the CCR representation in quantum theory is physically closely related to the dynamical model. We return to this question in §9.4 in the interpretation of Haag's theorem.

The symmetries in the algebraic approach are defined by Jordan automorphisms, that is, linear bijections $\alpha : \mathfrak{A} \to \mathfrak{A}$, for which $\alpha(A^2) = (\alpha(A))^2$ for all $A \in \mathfrak{A}$. This property comes out of the more physical definition of symmetry as a pair of bijections $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\alpha' : S(\mathfrak{A}) \to S(\mathfrak{A})$, satisfying the compatibility condition $(\alpha'\omega)(\alpha A) = \omega(A)$ (Proposition 6.10). The proof of this assertion is based on Wigner's theorem (Theorem 6.8), that every bijection on the set of one-dimensional projectors that preserves the transition probabilities $\operatorname{tr}\Pi_1\Pi_2$ is generated either by a unitary or an anti-unitary transformation in $\mathcal{H}$.

Chapter 7 is devoted to an important example of geometric symmetry, namely the invariances of quantum theory with respect to the group of Poincaré transformations. The analysis of §6.3.C shows that (under the assumption that $\alpha'_{a,\Lambda}(\omega)$ is continuous in a, Λ in the norm topology of the set of physical states $\mathfrak{S}$), the Poincaré conjecture on the invariance of the theory leads to the existence of a unitary representation of the universal covering $\mathfrak{P}_0$ of the proper Poincaré group (also called the spinor or quantum-mechanical Poincaré group) (Axiom A.III,§.7.2.A).

The self-adjoint infinitesimal operators of the physical representation of the group $\mathfrak{P}_0$ are identified with the operators of the 4-momentum P^μ and the angular momentum $M^{\mu\nu}$. From these one can form two independent polynomials of the invariant (the Casimir operator): the square of the momentum $P^2 \equiv P_0^2 - \mathbf{P}^2$ (the physical meaning of which is the square of the mass of the system) and the square of the Pauli-Lubanski vector $W_\lambda = \frac{1}{2}\epsilon_{\lambda\mu\nu\rho}P^\mu M^{\nu\rho}$. The irreducible unitary representations of the Poincaré group are (partially) classified by the values of these invariants and the sign of the energy $\epsilon = \epsilon(P^0)\theta(P^2)$. The spectrality postulate states that the mass and the energy are non-negative, that is, the spectrum of the energy-momentum operator P belongs to the (closed) upper light cone $\overline{V}^+$. We append to this postulate the requirement of existence and uniqueness of the (normalized) translation-invariant state, namely, the vacuum. It is shown (Proposition 7.1) that this requirement is equivalent to the cluster property

$$\lim_{\lambda\to\infty}\langle\Phi, U(\lambda a, 1)\Psi\rangle = \langle\Phi, \Psi_0\rangle\langle\Psi_0, \Psi\rangle \quad (\text{for } a^2 < 0).$$

The state spaces of the elementary particles correspond to irreducible representations of the group $\mathfrak{P}_0$. For a particle of positive mass m, the second Casimir operator is defined by its spin s : $W^2 =$ $= -m^2 s(s+1)$; the components of the spin vector are defined by the formula $S^j = \frac{1}{m}\left(W^j - \frac{W^0 P^j}{m+P^0}\right)$. The one-particle states can be realized in the space $\mathfrak{H}^{[m,s]}$ of functions in which the commuting operators P^μ and S^3 are diagonal. The scalar product in $\mathfrak{H}^{[m,s]}$ is then defined by the formula

$$\langle\Phi, \Psi\rangle = \sum_{\sigma=-s}^{s}\int_{\Gamma_m^+}\overline{\Phi(p,\sigma)}\Psi(p,\sigma)\,(dp)_m,$$

where σ is the eigenvalue of the third projection of the spin S^3, Γ_m^+ is the hyperboloid $p^0 = \sqrt{m^2 + \mathbf{p}^2}$, and $(dp)_m = \frac{d_3 p}{2p^0} = (2\pi)^{-3}\frac{d^3 p}{2p^0}$ is the Lorentz-invariant measure on Γ_m^+. In §7.2.D we give a manifestly covariant realization of the physical irreducible representations of $\mathfrak{P}_0$ in terms of the homogeneous polynomials of the two-component spinors ω introduced in §3.1.D.

The Fock space of relativistic Bose (Fermi) particles of a given type is defined as the (anti-)symmetrized " exponent" of the space of one-particle states (§7.3.A). The theories that are invariant with respect to the general Poincaré group (including reflection) are considered in §7.3.D, where it is shown that the transformations containing time reversal are represented by anti-unitary operators in $\mathfrak{H}$. It is shown (§7.3.F) without any dynamical assumptions (or hypotheses of analyticity) that the spinor amplitude of the two-particle scattering of particles with positive masses admits a unique covariant decomposition (Proposition 7.7).

CHAPTER 6

Algebra of Observables and State Space

6.1. Algebraic Formulation of Quantum Theory

A. Algebra of Observables. States

Historically, operators first made their appearance in the construction of quantum mechanics (in Heisenberg's matrix formulation) as operators (or matrices) corresponding to physical observables; it was only later (in the work of Schrödinger and subsequently Born and von Neumann) that the modern notion of a physical state took shape.

In the standard exposition of the quantum mechanics of systems with a finite number of degrees of freedom, the observables are identified with Hermitian operators and the pure states with rays* in some Hilbert space. In this case we can say that the algebra of observables is concrete, that is, it is represented by operators in a Hilbert space. The fundamental observable quantities are the cartesian coordinates and momenta, regarded as Hermitian operators satisfying the canonical commutation relations. A fundamental fact of the quantum theory of systems with a finite number of degrees of freedom is that these representations essentially admit a unique physically interesting representation (see §6.4.A). Therefore no problems arise in the choice of the representation, and this allows us to construct the entire quantum-mechanical formalism on the basis of a fixed Hilbert space.

The situation is different with quantum systems with an infinite number of degrees of freedom. Whereas it is possible to construct the corresponding system of canonical commutation relations more or less uniquely as an abstract algebra, there exist a large number of unitarily inequivalent representations of it. It turns out that the choice of the representation is dictated, in particular, by the dynamics of the fields (so that, for example, the Fock representation is unsuitable for interacting fields). Thus the representation now becomes an unknown quantity subject to determination. In this connection it is advisable to begin the axiomatic construction with an algebraic structure corresponding to the set of observables and only then, introduce the representation of the algebra of observables in Hilbert space in its dependence on some or other physical situation.

Thus the starting object of the axiomatic discussion is a C^*-algebra $\mathfrak{A}$ with identity (see §1.5.) which we shall call the *algebra of observables* (or the algebra of

* By a *ray* in Hilbert space $\mathcal{H}$ we mean a set of vectors of the form $\lambda\Phi$, where Φ is a fixed non-zero vector and λ runs through all the non-zero complex numbers. If the extra condition $\|\Phi\| = |\lambda| = 1$ is laid down, then the resulting set is called a *unit ray* in $\mathcal{H}$. Vectors with norm 1 are called *unit* (or *normalized*) vectors.

bounded observables). The Hermitian elements of this algebra are called (bounded) ***observables***. A positive functional ω on $\mathfrak{A}$ normalized by the condition

$$\|\omega\| \equiv \omega(1) = 1, \tag{6.1}$$

is called a ***state*** of the algebra $\mathfrak{A}$. We denote the set of all states of $\mathfrak{A}$ by $S(\mathfrak{A})$. The quantity $\omega(A)$ for the case when $A = A^*$ is interpreted as the ***mean value*** of the observable A in the state ω. (It is real in view of the properties of positive functionals; see §1.5.C.)

It is not difficult to verify that $S(\mathfrak{A})$ is ***convex*** (in other words, if ω_1 and ω_2 are two states and $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$, then $\lambda_1\omega_1 + \lambda_2\omega_2 \in S(\mathfrak{A})$.) A state ω is said to be *mixed* (or a statistical mixture) if it is representable in the form $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, where $0 < \lambda < 1$ and ω_1, ω_2 are two distinct states of $\mathfrak{A}$. A state that is not mixed is called *pure*. (In the terminology of §1.5.C, the pure states are the extreme points of the set $S(\mathfrak{A})$.) We denote the set of all pure states of $\mathfrak{A}$ by $PS(\mathfrak{A})$.

In the abstract algebraic scheme we are generally dealing only with bounded observables so as to avoid the "technical" difficulties associated with the definition of algebraic operations on unbounded observables. However, in Hilbert space we shall freely go over to unbounded observables in view of the fact that every unbounded observable is completely characterized (in the spirit of the spectral theorem, §1.4.C) by the bounded functions of it. (The corresponding notion of unbounded observable is introduced below in §6.1.C.)

A priori we only attach meaning to the Hermitian elements of $\mathfrak{A}$, so that it would have been more logical to start with the *Jordan algebra* $\mathfrak{A}_h$ consisting of the Hermitian elements of $\mathfrak{A}$. In this algebra, only linear combinations with real coefficients and the square of each element are defined (and hence also the "symmetric product" $\frac{1}{2}(AB + BA) = \frac{1}{4}(A+B)^2 - \frac{1}{4}(A-B)^2$). In the algebraic definition of symmetry (§6.3.A) we give this remark due consideration by seeing to it that the symmetries reproduce just those operations that are inherent in $\mathfrak{A}_h$.

The set $S(\mathfrak{A})$ of all states of the algebra of observables $\mathfrak{A}$ is very extensive as a rule: it can be non-separable even if $\mathfrak{A}$ itself is separable; however, not all of $S(\mathfrak{A})$ is of physical interest. The class of states under consideration is restricted by physical considerations, depending on what processes are of interest and on what the "boundary conditions" (for example, at infinity) are. One of the problems of quantum field theory is the construction of the algebra of observables $\mathfrak{A}$ and the class of states describing the elementary excitations of the quantum fields, that is, the simplest objects forming the theoretical basis of the physics of elementary particles. Later in this chapter and in Chapters 7, 8, 10 we take up the characterization of the set of quantum-field states in greater detail (see, in particular, Postulates A.I–A.II in §6.1 and Postulate A.III in §7.2.). For the moment, we merely make the overall requirement that the set $\mathfrak{S}$ of so-called physical states be convex (that is, it contains along with any pair of states their statistical mixtures). We can also suppose that the set of states $\mathfrak{S}$ ***distinguishes the positive elements*** of $\mathfrak{A}$, that is, the condition $\omega(A) \geq 0$ for all $\omega \in \mathfrak{S}$ implies that A is a positive element of $\mathfrak{A}$ (that is, A is representable in the form $A = B^*B$).

According to a result of Kadison (1965) a convex set $\mathfrak{S} \subset S(\mathfrak{A})$ distinguishes the elements of $\mathfrak{A}$ if and only if it is dense in $S(\mathfrak{A})$ in the weak* topology.

Exercise 6.1. Prove that a set of states $\mathfrak{S}$ that distinguishes the positive elements of the algebra $\mathfrak{A}$ separates the elements of $\mathfrak{A}$, that is, the condition $\omega(A) = \omega(B)$ for all $\omega \in \mathfrak{S}$ implies that $A = B$. [Hint: If $\omega(A) = \omega(B)$ for all $\omega \in \mathfrak{S}$, then both elements $A - B$ and $B - A$ are positive.]

Thus as our starting point of the algebraic theory we make the following postulate.

A.I (Algebra of Observables). *A physical system* is characterized by a C^*-algebra $\mathfrak{A}$ with identity, the Hermitian elements of which are called (bounded) observables. The value of the functional $\omega(A)$, where $\omega \in S(\mathfrak{A})$, $A = A^*$, is the mean value of the observable A in the state ω.*

The set $\mathfrak{S}$ featuring above will be called the set of *physical states* (of the algebra of observables $\mathfrak{A}$) and the pair $(\mathfrak{A}, \mathfrak{S})$ is called the *physical system.*

In elementary quantum mechanics, the algebra of (bounded) observables is chosen to be the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators in a Hilbert space, while the physical states are in one-to-one correspondence with the density matrices ρ in $\mathcal{H}$, that is, the positive operators in $\mathcal{H}$ with trace equal to 1:

$$\omega_\rho(A) = \operatorname{tr}(\rho A), \quad A \in \mathcal{B}(\mathcal{H}). \tag{6.2}$$

Then the pure (physical) states are in one-to-one correspondence $\Pi \leftrightarrow \omega_\Pi$ with the one-dimensional (orthogonal) projectors Π in $\mathcal{H}$ or, equivalently, with the unit rays in $\mathcal{H}$. The fact that the set of all pure states of $\mathfrak{S}$ is parametrized by the collection of all unit rays in $\mathcal{H}$ is usually called the superposition principle. (We note that the set of all states of the algebra $\mathcal{B}(\mathcal{H})$ is considerably more extensive than the above set $\mathfrak{S}$; we discuss the reason for this restriction in §6.4.A.)

B. TRANSITION PROBABILITY

In elementary quantum mechanics, the transition probability is defined in the following way. If ω_1 and ω_2 are two pure states of the algebra $\mathcal{B}(\mathcal{H})$ corresponding to the one-dimensional projectors $\Pi_1 = |\Phi_1\rangle\langle\Phi_1|$ and $\Pi_2 = |\Phi_2\rangle\langle\Phi_2|$ (where Φ_1 and Φ_2 are two unit vectors in $\mathcal{H}$), then the probability of a transition from state ω_1 to state ω_2 (or between the states ω_1 and ω_2) is

$$|\langle\Phi_1, \Phi_2\rangle|^2 = \operatorname{tr}(\Pi_1\Pi_2), \tag{6.3}$$

which in turn is expressed in terms of $\|\omega_1 - \omega_2\|$ in accordance with the following exercise.

Exercise 6.2. Prove that

$$\|\omega_1 - \omega_2\|^2 = (\|\Pi_1 - \Pi_2\|_1)^2 = 4(1 - |\langle\Phi_1, \Phi_2\rangle|^2),$$

where $\|A\|_1$ is the trace norm (1.111). [Hint: The trace norm of an arbitrary Hermitian operator in $\mathcal{L}_1(\mathcal{H})$ is the sum of the absolute values of its eigenvalues. Show that the eigenvalues of the operator $\Pi_1 - \Pi_2 = |\Phi_1\rangle\langle\Phi_1| - |\Phi_2\rangle\langle\Phi_2|$ of rank ≤ 2 are equal to $\pm[1 - |\langle\Phi_1, \Phi_2\rangle^2]^{1/2}$.]

For an arbitrary C^*-algebra $\mathfrak{A}$, the *transition probability* (in zero time) between two pure states $\omega_1, \omega_2 \in PS(\mathfrak{A})$ is defined by the formula suggested by Exercise 6.2:

$$\omega_1 \cdot \omega_2 \equiv 1 - \tfrac{1}{4}\|\omega_1 - \omega_2\|^2. \tag{6.4}$$

It is clear that $\omega_1 \cdot \omega_2 = \omega_2 \cdot \omega_1$ and $\omega_1 \cdot \omega_2$ are always contained in $[0, 1]$. Here, $\omega_1 \cdot \omega_2 = 1$ if and only if $\omega_1 = \omega_2$. Using the quantum-mechanical terminology, we say that the two pure states ω_1 and ω_2 are *orthogonal* if the transition probability

* Here and in what follows, by a physical system we mean some mathematical-physical model, rather than a real system.

$\omega_1 \cdot \omega_2$ is zero. Similarly, two subsets S_1 and S_2 of $PS(\mathfrak{A})$ are said to be (mutually) orthogonal if $\omega_1 \cdot \omega_2 = 0$ for all $\omega_1 \in S_1$ and $\omega_2 \in S_2$.

We say that a non-empty subset S of pure states of the algebra of observables $\mathfrak{A}$ is *indecomposable* if it cannot be partitioned into two non-empty orthogonal subsets. Each maximal indecomposable set (that is, an indecompsable set that is not a proper subset of any other indecomposable set of pure states of $PS(\mathfrak{A})$) is called a *sector*. (A sector is the algebraic counterpart of a coherent subspace of physical Hilbert space: see below §6.2.B, in particular, Exercise 6.18.)

We will show that $PS(\mathfrak{A})$ divides into sectors (and at the same time verify the existence of sectors). To this end we introduce the following relation in $PS(\mathfrak{A})$: $\omega_1 \sim \omega_2$ if and only if there is an indecomposable subset of $PS(\mathfrak{A})$ containing ω_1 and ω_2.

Proposition 6.1. *The relation $\omega_1 \sim \omega_2$ is an equivalence relation on $PS(\mathfrak{A})$ and $PS(\mathfrak{A})$ can be uniquely partitioned into (mutually disjoint and mutually orthogonal) sectors, which are precisely the equivalence classes in $PS(\mathfrak{A})$.*

■ The reflexivity ($\omega \sim \omega$) and symmetry ($\omega_1 \sim \omega_2 \Rightarrow \omega_2 \sim \omega_1$) are obvious. We prove transitivity. Let $\omega_1 \sim \omega_2$ and $\omega_2 \sim \omega_3$. Then there exist indecomposable subsets S_1 and S_2 such that $\omega_1, \omega_2 \in S_1$ and ω_2, $\omega_3 \in S_2$. We claim that the union $S = S_1 \cup S_2$ is indecomposable. For if not, then S is the union of two non-empty orthogonal subsets S' and S''. We suppose for definiteness that $\omega_2 \in S'$. Then S_1 is the union of the orthogonal subsets $S_1 \cap S'$ and $S_1 \cap S''$, the first of these being non-empty (since it contains ω_2). Hence since S_1 is indecomposable, the second of these sets $S_1 \cap S''$ is empty. Similarly, we can show that $S_2 \cap S''$ is empty from which it follows that S'' itself (as the union of $S_1 \cap S''$ and $S_2 \cap S''$) is empty. The contradiction so obtained proves that S is indeed indecomposable. Thus we have shown that $\sim$ is an equivalence relation on $PS(\mathfrak{A})$. It is fairly easy to see that the maximal indecomposable sets (sectors) are equivalence classes in $PS(\mathfrak{A})$. Consequently $PS(\mathfrak{A})$ divides into sectors which, moreover, are mutually orthogonal (since if two distinct states ω_1, ω_2 in $PS(\mathfrak{A})$ are not orthogonal, then they form a two-point indecomposable subset and therefore are equivalent and hence are contained in the same equivalence class). The uniqueness of the partitioning of $PS(\mathfrak{A})$ into sectors now follows from the uniqueness of the partitioning of $PS(\mathfrak{A})$ into equivalence classes. ■

In the next section we give a characterization of sectors in terms of representations.

C. Relationship to representations

The notion of a state is closely related to the representations of the algebra. If π is some representation of the algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$, then for any non-zero vector $\Phi \in \mathcal{H}$, the expression

$$\omega_\Phi(A) = \frac{\langle \Phi, \pi(A)\Phi \rangle}{\langle \Phi, \Phi \rangle} \tag{6.5}$$

defines a state ω_Φ of $\mathfrak{A}$, called the *vector state* associated with the representation π (and corresponding to the vector Φ). If ρ is a density matrix (that is, a positive operator with trace 1) in $\mathcal{H}$, then it defines the state ω_ρ:

$$\omega_\rho(A) = \operatorname{tr}(\rho\pi(A)). \tag{6.6}$$

Similarly, ω_ρ is called the state associated with the representation π corresponding to the density matrix ρ. (Clearly, the states (6.6) are countable statistical mixtures of vector states of the form (6.5).) We denote the set of all states (or of all pure states) associated with the representation π by S_π (or PS_π). Two representations

π_1 and π_2 with the same set of associated states (that is, with $S_{\pi_1} = S_{\pi_2}$) are called *phenomenologically equivalent.* In particular, unitarily equivalent representations are so related.

Exercise 6.3. Show that every pure state associated with the representation π is a vector state of the form (6.5). [Hint: Diagonalize the density matrix of the state.]

Exercise 6.4. Prove that if π is an irreducible representation in Hilbert space, all the vector states associated with π are pure (in other words, the set of all vector states with π is precisely PS_π). Show further (using Exercise 1.66) that for any unit vectors $\Phi_1, \Phi_2 \in \mathcal{H}$, the transition probability between the states ω_{Φ_1} and ω_{Φ_2} is given by the usual quantum-mechanical formula

$$\omega_{\Phi_1} \cdot \omega_{\Phi_2} = |\langle \Phi_1, \Phi_2 \rangle|^2. \tag{6.7}$$

Exercise 6.5. Let π be a representation of the C^*-algebra $\mathfrak{A}$ in $\mathcal{H}$ and Φ a cyclic vector of π defining the state ω_Φ. Prove that π is irreducible if and only if ω_Φ is a pure state. [Hint: Identify $\pi, \mathcal{H}, \Phi$ with elements of the GNS construction and use Proposition 1.27.]

It turns out that the notion of a sector (§6.1.B) is closely related to irreducible representations.

Proposition 6.2. *The set $PS(\mathfrak{A})$ of all pure states of the algebra $\mathfrak{A}$ coincides with the set of all vector states associated with all the irreducible representations of $\mathfrak{A}$. Each sector in $PS(\mathfrak{A})$ is precisely the same as the set of all vector states associated with some irreducible representation of $\mathfrak{A}$; furthermore, two unitarily equivalent (or unitarily inequivalent) irreducible representations define the same sector (or mutually orthogonal sectors).*

The first assertion of Proposition 6.2 is an immediate corollary of Theorem 1.25 (that is, the GNS construction) and Exercise 6.5. The fact that the set PS_π of all vector states associated with an arbitrary irreducible representation π is indecomposable is a result of Exercise 6.5. To verify that PS_π is a maximal indecomposable set (that is, a sector), it suffices to prove that two states associated with unitarily inequivalent representations are orthogonal. We give this result in the form of the following two exercises. The remaining assertions of Proposition 6.2 will then easily follow.

Exercise 6.6. Let π_1 and π_2 be two unitarily inequivalent irreducible representations of the algebra $\mathfrak{A}$ in the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and let $\pi = \pi_1 \oplus \pi_2$ be their direct sum. Prove that the closure of $\pi(\mathfrak{A})$ in the weak operator topology contains the unitary operator $U = \Pi_1 - \Pi_2$ where Π_1 and Π_2 are the projectors from the direct sum $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ onto $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. [Hint: Use the bicommutant theorem and Exercises 1.54 and 1.55.]

Exercise 6.7. Let ω_1 and ω_2 be two states associated with the unitarily inequivalent irreducible representations π_1 and π_2 of the algebra $\mathfrak{A}$. Prove that $\|\omega_1 - \omega_2\| = 2$ and hence that the (pure) states ω_1 and ω_2 are orthogonal. [Hint: In the notation of the previous exercise we have $(\omega_1 - \omega_2)(A) =$ $= \mathrm{tr}((\Pi_1 - \Pi_2)\pi(A))$, where Π_1 and Π_2 are the corresponding one-dimensional projectors in $\mathcal{H}_1$ and $\mathcal{H}_2$. Then use (1.116) and the fact that $\mathrm{tr}((\Pi_1 - \Pi_2)U) = 2$ for the operator U of the preceding exercise.]

The next exercises characterize representations that are phenomenologically equivalent to an irreducible representation. Prior to this we recall that a representation π is called a *multiple* of the representation $\tilde{\pi}$ if π is the direct sum of a (finite non-zero or infinite) number of copies of $\tilde{\pi}$. A representation of a C^*-algebra $\mathfrak{A}$ is said to be *factorial of type* I if it is a multiple of an irreducible representation of $\mathfrak{A}$. Two factorial representations of type I of $\mathfrak{A}$ are said to be *disjoint* if they are multiples of some irreducible representations π_1 and π_2, where π_1 and π_2 are unitarily inequivalent.

Exercise 6.8. If the representation π of the algebra $\mathfrak{A}$ is a multiple of the representation $\tilde{\pi}$, then π and $\tilde{\pi}$ are phenomenologically equivalent. In particular, every factorial representation of type I is phenomenologically equivalent to an irreducible representation.

The converse also holds: every representation that is phenomenologically equivalent to an irreducible one is factorial of type I (see Exercise 6.12 below).

Exercise 6.9. If two factorial representations of type I of the algebra $\mathfrak{A}$ are disjoint, then the corresponding sets of states of $\mathfrak{A}$ associated with these representations have no elements in common. [Hint: If two vectors Φ_1 and Φ_2 correspond to the same pure state of $\mathfrak{A}$, then, for example, by Theorem 1.25, the irreducible representations of $\mathfrak{A}$ in the closures of $\mathfrak{A}\Phi_1$ and $\mathfrak{A}\Phi_2$ are unitarily equivalent.

The relation between states and representations of an algebra becomes even closer if we take into account that, according to the GNS construction (see Theorem 1.25), each state in its turn defines a representation of the algebra (the resulting representation being irreducible if and only if the state is pure).

From what has been said, it makes sense to regard the representations of an algebra as an effective means of organizing states. We therefore admit a further refinement of Postulate A.I.

A.II (States). *The set of physical states* $\mathfrak{S}$ *(of the algebra of observables* $\mathfrak{A}$*) coincides with the set of all states* S_π *associated with a representation* $\pi = \pi_{\text{phys}}$ *(called the physical representation) of* $\mathfrak{A}$ *in some Hilbert space* $\mathcal{H} = \mathcal{H}_{\text{phys}}$ *(called the physical space).*

If the representation π is faithful, then the states of $\mathfrak{S} = S_\pi$ distinguish the positive elements of the algebra. In accordance with Postulate A.I, the physical system can now be characterized by the pair $(\mathfrak{A}, \pi)$.

The introduction of the physical representation π at such an early stage of the exposition of the theory does not diminish the role of the abstract algebraic formulation. Running ahead, we can justify the selection of the purely algebraic level by an example from the Lagrangian theory. Corresponding to a given Lagrangian there can be various unitarily inequivalent representations of the quantum fields. Each such representation will, in general, possess less symmetry than that of the original Lagrangian. (In such a case we talk about spontaneous breakdown of symmetry.) In this context, the algebraic formulation corresponds to the Lagrangian level while the selection of the representation π_{phys} in the Hilbert space $\mathcal{H}_{\text{phys}}$ is equivalent to selecting one of the possible operator theories (pure phases in the terminology of statistical physics) that correspond to the given Lagrangian.

Once the physical representation π of the algebra of observables has been chosen and is faithful, we can (by abuse of notation) identify the original algebra of observables with the operator algebra $\pi(\mathfrak{A}) \subset \mathcal{B}(\mathcal{H})$:

$$\mathfrak{A} \Rightarrow \pi(\mathfrak{A}).$$

(In the physical literature, the passage from $\mathfrak{A}$ to the concrete algebra $\pi(\mathfrak{A})$ is sometimes called "dressing".) As for any operator algebra, $\mathfrak{A}$ can be endowed with the σ-weak (or weak) operator topology defined by the seminorms $|\omega(A)|$ with arbitrary (or arbitrary vector) states associated with the representation π. It is clear that the σ-weak operator topology does not change if π is replaced by a phenomenologically equivalent representation (since such a representation has the same collection of associated states). The closure $\bar{\mathfrak{A}}$ of the algebra $\mathfrak{A}$ in the σ-weak (or, what in the present case is the same, the weak) operator topology is called the *von Neumann algebra of observables.* According to Theorem 1.31 (von Neumann's bicommutant theorem), $\bar{\mathfrak{A}}$ is the same as the bicommutant $\mathfrak{A}^{cc}$ of $\mathfrak{A}$.

Of course, the algebra $\bar{\mathfrak{A}}$ is defined not merely by the algebra of observables $\mathfrak{A}$ but also by the choice of the physical representation π. Phenomenologically equivalent representations lead to (algebraically) isomorphic algebras $\bar{\mathfrak{A}}$, since $\bar{\mathfrak{A}}$ is the completion of $\mathfrak{A}$ in the LCS structure defined by the collection of physical states. All the same, it should be noted that (just as for arbitrary operators in $\mathcal{B}(\mathcal{H})$ in elementary quantum theory), the elements of the von Neumann algebra of observables are "observable" in a very non-constructive sense.

The condition that the von Neumann algebras $\overline{\pi_1(\mathfrak{A})}$ and $\overline{\pi_2(\mathfrak{A})}$ be naturally algebraically isomorphic for two different representations of the algebra $\mathfrak{A}$ is in fact equivalent to the notion of phenomenological equivalence of these representations. For the proof, it should be noted that S_π can be characterized in terms of $\overline{\pi(\mathfrak{A})}$ as the set of so-called normal positive functionals with norm 1 of the von Neumann algebra of observables ([D4],p.50).

As we have already noted, certain physically interesting quantities are represented by unbounded self-adjoint operators and therefore do not belong to the algebra $\mathfrak{A}$ or $\bar{\mathfrak{A}}$. In connection with this, the following definition is convenient. A self-adjoint operator A in $\mathcal{H}$ is said to be ***affiliated*** to the von Neumann algebra $\bar{\mathfrak{A}}$ if all the spectral projectors E_λ (in the spectral decomposition (1.64) of A) and hence all the bounded functions of A belong to $\bar{\mathfrak{A}}$. Self-adjoint operators that are affiliated to the von Neumann algebra $\bar{\mathfrak{A}}$ of observables can be interpreted as (possibly unbounded) observables associated with the given representation of the algebra $\mathfrak{A}$.

In field-theoretic constructions, one often uses global observables corresponding to generators of the symmetry group of the system (such as the electric charge operator, the operators of energy, momentum and total angular momentum, which are defined — at least formally — as the integral over the whole of three-space of the corresponding local densities) which possess spectral projectors belonging to the algebra $\bar{\mathfrak{A}}$ (but possibly not $\mathfrak{A}$).

Remark. If ω is a pure state on $\mathfrak{A}$, then we can associate with the elements B of $\mathfrak{A}$ for which $\omega(B^*B) > 0$, certain pure states of $\mathfrak{A}$. As was noted in §1.5.C, the set of exceptional elements B for which $\omega(B^*B) = 0$ forms a left ideal $\mathfrak{I}_\omega$ in $\mathfrak{A}$. Corresponding to each element $B \notin \mathfrak{I}_\omega$ is the state

$$\omega^B(A) = \frac{\omega(B^*AB)}{\omega(B^*B)}. \tag{6.8}$$

Exercise 6.10. Verify that for a pure state ω, the functional ω^B is positive and defines a pure state. [Hint: Use Theorem 1.25 and Exercise 6.5.]

It is fairly easy to verify that the transition probability between two such states ω^A and ω^B $(A, B \notin$ $\notin \mathfrak{I}_\omega)$ is given by the formula

$$\omega^A \cdot \omega^B = (1 - \tfrac{1}{4}\|\omega^A - \omega^B\|^2) = \frac{\omega(A^*B)\omega(B^*A)}{\omega(A^*A)\omega(B^*B)}. \tag{6.9}$$

Suppose that in the representation defined by the state ω the states ω^A and ω^B correspond to the unit vectors Ψ_A and Ψ_B; then the right hand side becomes the usual expression $|\langle\Psi_A, \Psi_B\rangle|^2$.

In accordance with this remark, Haag and Kastler (1964) attach a physical meaning to arbitrary (including non-Hermitian) elements of the algebra $\mathfrak{A}$ as *operations* on the state. (In a given state ω, only those operations that do not belong to the ideal $\mathfrak{I}_\omega$ are admissible.)

6.2. Superselection Rules

A. The role of pure vector states

Going over to the physical representation converts the abstract algebra of observables $\mathfrak{A}$ into a concrete operator algebra in the physical Hilbert space $\mathcal{H}$; in this case the states of $\mathfrak{S}$ are defined by density matrices in $\mathcal{H}$. We now state two extra conditions (called hypotheses below, so as to emphasize their optional character by comparison with the axioms), which strengthen the role of the vectors of the pure states in the construction of the physical Hilbert space.

In elementary quantum mechanics, a very close connection is realized between the physical Hilbert space and the pure state vectors in the sense that all the pure states are in one-to-one correspondence with all the unit rays of the Hilbert space. In the physics of elementary particles, the situation is more complex. Here we systematically use the description of pure states prepared under laboratory conditions in terms of the wave functions of incoming or outgoing particles (more precisely, by means of the vectors of Fock space, which will be discussed in Chapter 7). We therefore suppose that the entire physical Hilbert space is spanned by these vectors. However, in contrast to quantum mechanics, not every linear combination of such vectors forms a pure state (attention was first drawn to this fact of principle by Wick et al (1952)). Experiments show that certain quantum numbers of pure states, for example, electric charge and parity of the number of fermions, always have completely determinate values. In particular, a coherent superposition (that is, a pure state vector that is a sum of vectors) of a single-proton and single-neutron state has never been observed. It can also be concluded from theoretical considerations that certain vectors should correspond to mixed states. For example, the superposition $\Phi_1 + \Phi_2$ of the vector Φ_1 of a state with integral spin and the vector Φ_2 of a state with half-integral spin defines a mixed state,* since under a rotation of the system through an angle of 2π round some axis, the physical system is converted to itself, whereas the vector $\Phi_1 + \Phi_2$ is taken to the vector $\Phi_1 - \Phi_2$. Consequently the vectors $\Phi_1 + \Phi_2$ and $\Phi_1 - \Phi_2$ correspond to the same state of the system, whence it follows of necessity (see Exercise 6.11) that this state is mixed.

Exercise 6.11. Suppose that the physical Hilbert space decomposes to a direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2$ such that $\omega_{\Phi_1} \neq \omega_{\Phi_2}$, and $\omega_{\Phi_1+\Phi_2} = \omega_{\Phi_1-\Phi_2}$, for all non-zero vectors $\Phi_1 \in \mathcal{H}_1$ and $\Phi_2 \in \mathcal{H}_2$. Prove that the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant with respect to the algebra of observables $\mathfrak{A}$ and that superpositions of the form $\Phi_1 + \Phi_2$ define mixed states of $\mathfrak{A}$, where as before, Φ_1 and Φ_2 are arbitrary non-zero vectors in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. [Hint: $\omega_{\Phi_1+\lambda\Phi_2}(A) = \omega_{\Phi_1-\lambda\Phi_2}(A)$ for all complex λ.]

Guided by the foregoing discussion, we introduce the further requirement, ** which sets up a relation between the vectors of the Hilbert space and the pure states but which is less rigid than in quantum mechanics.

(a) Hypothesis of Discrete Superselection Rules.

All the vectors in the physical Hilbert space $\mathcal{H}$ that define the pure states of the algebra of observables $\mathfrak{A}$ form a total subset of $\mathcal{H}$.

As a motivation for the name of this hypothesis we point out (running ahead) that it is equivalent to the condition that the physical representation be decomposable into a discrete direct sum (that is, an ordinary direct sum instead of the possibly more general direct integral) of factorial representations of Type I.

The hypothesis admits a number of equivalent restatements. First we prove two simple lemmas which use the following definition. A non-empty set $\mathfrak{M}$ of non-zero

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* It is supposed that in $\mathcal{H}$ we are given a unitary representation of the group $O_+(3)$ of the three-dimensional (spatial) rotations. In this discussion, only the operator $\exp(i2\pi M^3)$ of a rotation through 2π is used, which takes the values $+1$ and -1 respectively on the subspaces of single-valued and two-valued irreducible representations of $O_+(3)$.

** In favour of this requirement (in the form of Condition (a.2) of Proposition 6.5) are theoretical considerations which are also based on the corpuscular interpretation of the states of relativistic quantum theory, given in the article by Doplicher et al. (1969).

vectors of the Hilbert space $\mathcal{H}$ is said to be a *linked system of vectors* if $\mathfrak{M}$ cannot be represented as the union of two (or more) non-empty mutually orthogonal subsets.

This definition is related to the notion of an indecomposable set of states (although generally speaking, for a C^*-algebra other than $\mathfrak{A}$): it is not difficult to see that $\mathfrak{M}$ is a linked system if and only if the set of all states of the algebra $\mathcal{B}(\mathcal{H})$ defined by vectors in $\mathfrak{M}$ is indecomposable.

Lemma 6.3. *Let $\mathfrak{M}$ be a total set of non-zero vectors of the Hilbert space $\mathcal{H}$. Then $\mathfrak{M}$ can be uniquely represented as the union of a family $\{\mathfrak{M}_\nu\}_{\nu\in N}$ of linked systems $\mathfrak{M}_\nu$ that are mutually orthogonal (and hence are pairwise disjoint). Accordingly, $\mathcal{H}$ can be uniquely decomposed into a direct sum of (non-zero) subspaces*

$$\mathcal{H} = \bigoplus_{\nu\in N} \mathcal{H}_\nu \tag{6.10}$$

such that $\mathfrak{M}$ is the union of the subsets $\mathfrak{M}_\nu = \mathfrak{M} \cap \mathcal{H}_\nu$, each of which is total in the corresponding subspace $\mathcal{H}_\nu$ and forms a linked system.

■ We say that the vectors $\Phi, \Psi \in \mathfrak{M}$ are connected by the relation $\Phi \sim \Psi$, if Φ and Ψ are vectors of some linked system of vectors in $\mathfrak{M}$. Arguing as in Proposition 6.1, it is now easy to see that this relation is an equivalence relation and that the equivalence classes in $\mathfrak{M}$ form a partitioning of $\mathfrak{M}$ into mutually orthogonal linked systems $\mathfrak{M}_\nu$, where $\{\nu\} = N$ is a family of indices. The uniqueness of such a decomposition is proved in entirely similar fashion. If we now take as $\mathcal{H}_\nu$ the closed linear span of $\mathfrak{M}_\nu$, we arrive at the required decomposition of $\mathcal{H}$ into a direct sum of mutually orthogonal subspaces $\mathcal{H}_\nu$. ■

Lemma 6.4. *A representation π of the C^*-algebra $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$ is factorial of Type I if and only if the set $\mathfrak{M}$ of all vectors in $\mathcal{H}$ that define pure states of $\mathfrak{A}$ form a linked system of vectors that is total in $\mathcal{H}$*

■ Let the representation π be factorial of Type I. Then $\mathcal{H}$ and π can be identified with $\bigoplus^K \tilde{\mathcal{H}}$ and $\bigoplus^K \tilde{\pi}$ respectively, these being respectively the direct sum of a (finite or infinite) family K of copies of $\tilde{\mathcal{H}}$ and an irreducible representation $\tilde{\pi}$ of $\mathfrak{A}$ in $\tilde{\mathcal{H}}$. Each vector $\Phi \in \mathcal{H}$ is now represented as the family $\{\Phi^{(\kappa)}\}_{\kappa\in K}$ of the projections $\Phi^{(\kappa)} \in \tilde{\mathcal{H}}$ (where $\|\Phi\|^2 = \sum_{\kappa\in K} \|\Phi^{(\kappa)}\|^2$). It is fairly easy to see that each non-zero vector Φ for which all the projections $\Phi^{(\kappa)}$ are (complex) collinear (here zero projections are allowed), defines a pure state of $\mathfrak{A}$. The set of all such vectors is total in $\mathcal{H}$ and, clearly, it cannot be partitioned into two non-empty mutually orthogonal subsets. The "only if" half of our lemma now follows.

Conversely, let $\mathfrak{M}$ be a linked system that is total in $\mathcal{H}$. The idea behind the subsequent argument can be made clear in the following way. We associate with each vector Φ in $\mathfrak{M}$ the subspace $\mathcal{H}^\Phi$ of $\mathcal{H}$ that is the closure of the set $\pi(\mathfrak{A})\Phi$ (of all vectors of the form $\pi(A)\Phi$ for $a \in \mathfrak{A}$). Then $\mathcal{H}^\Phi$ is clearly an invariant subspace with respect to π and the restriction of π to $\mathcal{H}^\Phi$ defines a representation π^Φ of $\mathfrak{A}$. The vector Φ is cyclic for π^Φ and, by hypothesis, corresponds to a pure state of $\mathfrak{A}$, therefore, (see Exercise 6.5) π^Φ is irreducible. Thus we can represent $\mathcal{H}$ as the closed linear span of the family $\{\mathcal{H}^\Phi\}$ of invariant spaces in which $\mathfrak{A}$ acts according to irreducible representations. This, however, does not mean that $\mathcal{H}$ is the direct (orthogonal) sum of this family, since these families are not mutually orthogonal (and may even have non-zero pairwise intersections). But this defect is easily removed by revising the above construction using a standard argument reminiscent, say, of the process of constructing an orthonormal basis in Hilbert space.

To this end we fix some vector Φ_1 in $\mathfrak{M}$. If $\mathcal{H} = \mathcal{H}^{\Phi_1}$, then the required decomposition is completed by this construction. Therefore we suppose that $\mathcal{H} \neq \mathcal{H}^{\Phi_1}$ and hence that $\mathcal{H}$ decomposes into a direct sum $\mathcal{H}^{\Phi_1} \oplus \mathcal{K}_1$ of subspaces that are invariant with respect to $\mathfrak{A}$. Now as a result of the hypothesis that $\mathfrak{M}$ is a linked system, however we decompose $\mathcal{H}$ into two non-zero orthogonal subspaces, there exists a vector in $\mathfrak{M}$ that has non-zero projections onto each of these subspaces. Hence we can fix a vector $\Phi \in \mathfrak{M}$ having non-zero projections, say, Φ_1' and Φ_2 onto $\mathcal{H}^{\Phi_1}$ and $\mathcal{K}_1$ respectively. The states defined by these projections coincide with ω_Φ (since they are subordinated to ω_Φ), so that Φ_1' and Φ_2 belong to $\mathfrak{M}$ and the representations of $\mathfrak{A}$ in $\mathcal{H}^{\Phi_1'} \subset \mathcal{H}^{\Phi_1}$ and $\mathcal{H}^{\Phi_2} \subset \mathcal{K}_1$ are unitarily equivalent (by

Theorem 1.25). In fact we have $\mathcal{H}^{\Phi_1'} = \mathcal{H}^{\Phi_1}$ (since π^{Φ_1} is irreducible). Thus we have now isolated two invariant mutually orthogonal subspaces $\mathcal{H}^{\Phi_1}$ and $\mathcal{H}^{\Phi_2}$ of $\mathcal{H}$ in which $\mathfrak{A}$ acts according to irreducible and unitarily equivalent representations. If the orthocomplement $\mathcal{K}_2$ of $\mathcal{H}^{\Phi_1} \oplus \mathcal{H}^{\Phi_2}$ is not $\{0\}$, then we can choose a vector $\Psi \in \mathfrak{M}$ with non-zero projections onto $\mathcal{K}_2$ and at least one of the subspaces $\mathcal{H}^{\Phi_1}$, $\mathcal{H}^{\Phi_2}$, and so on. This process of decomposing π into a direct sum of pairwise unitarily equivalent representations can be completed using a standard argument based on Zorn's lemma. (We recommend that the reader carry this through; cf. [N2], Proposition VII of §5.4 or §17.2.) This completes the proof of the lemma. ■

Proposition 6.5. *The hypothesis of discrete superselection rules is equivalent to either of the following two conditions.*

(a.1) *Every physical state in $\mathfrak{S}$ is a finite or countable convex linear combination of pure vector states associated with the physical representation.*

(a.2) *The physical Hilbert space $\mathcal{H}$ decomposes into a direct sum of a family of subspaces $\{\mathcal{H}_\nu\}_{\nu \in N}$ that are invariant with respect to the algebra $\mathfrak{A}$ and are such that the representations of $\mathfrak{A}$ in the subspaces $\mathcal{H}_\nu$ are factorial of Type I and are pairwise disjoint. (This decomposition is unique if we suppose that all the subspaces $\mathcal{H}_\nu$ are distinct from $\{0\}$.)*

■ Condition (a.1) is a direct restatement of the hypothesis of discrete superselection rules;† we leave the verification of this to the reader. We turn to the proof of the equivalence of (a.2) and Hypothesis (a). We denote by $\mathfrak{M}$ the set of all vectors in $\mathcal{H}$ defining the pure states of the algebra $\mathfrak{A}$. Suppose that condition (a.2) holds. Then each vector $\Phi \in \mathfrak{M}$ belongs to just one subspace $\mathcal{H}_\nu$, since if it had non-zero projections on two such subspaces, then these projections would define the same state as Φ (since ω_Φ is a pure state); but this is impossible (by Exercise 6.9). Thus $\mathfrak{M}$ is the union of a family of mutually orthogonal sets $\mathfrak{M}_\nu = \mathfrak{M} \cap \mathcal{H}_\nu$ each of which, as a set of all the pure states associated with a factorial representation of Type I, is (according to Lemma 6.4) total and indecomposable in the corresponding subspace $\mathcal{H}_\nu$. Consequently $\mathfrak{M}$ is total in $\mathcal{H}$. We have thus proved the implication (a.2)$\Rightarrow$(a). At the same time we have proved the uniqueness of the decomposition in condition (a.2), since we have seen that this decomposition is completely defined by a decomposition of $\mathfrak{M}$ into mutually orthogonal linked systems, which is unique (by Lemma 6.3).

We now prove the implication (a)$\Rightarrow$(a.2). According to Hypothesis (a), the set $\mathfrak{M}$ is total in $\mathcal{H}$, which enables us to use Lemma 6.3. It remains to verify that the decomposition (6.10) realized by this lemma has the required properties. In fact it suffices to prove merely that the subspaces $\mathcal{H}_\nu$ in (6.10) are invariant with respect to $\mathfrak{A}$, since (a.2) will then follow from Lemma 6.4 (applied to the representations of $\mathfrak{A}$ in the subspaces $\mathcal{H}_\nu$). Since $\mathfrak{M}_\nu$ is total in $\mathcal{H}_\nu$, we must show that for any $\Phi \in \mathfrak{M}_\nu$, the inclusion $\mathfrak{A}\Phi \subset \mathcal{H}_\nu$ holds. We have: for $A \in \mathfrak{A}$ with $A\Phi \neq 0$, the state $\omega_{A\Phi} \equiv \omega_\Phi^A$ is pure (see Exercise 6.10), therefore $\mathfrak{A}\Phi \backslash \{0\} \subset \mathfrak{M}$. On the other hand, $\mathfrak{A}\Phi$ is a linear subspace of $\mathcal{H}$, therefore the set $\mathfrak{A}\Phi\backslash\{0\}$ is clearly a linked system and hence lies entirely in one of the maximal linked systems of the family $\{\mathfrak{M}_\nu\}_{\nu \in N}$. This subset is in fact $\mathfrak{M}_\nu$ since $\mathfrak{A}\Phi\backslash\{0\}$ and $\mathfrak{M}_\nu$ have the common element Φ ($\equiv 1 \cdot \Phi$). Thus we have established the invariance of the subspaces $\mathcal{H}_\nu$ with respect to $\mathfrak{A}$, which completes the proof of Proposition 6.5. ■

Exercise 6.12. Prove that every representation that is phenomenologically equivalent to an irreducible one is factorial of Type I. [Hint: Use the equivalence of conditions (a.1) and (a.2) of Proposition 6.5.]

We now turn to the structure of the von Neumann algebra of observables $\bar{\mathfrak{A}}$. (As we have already remarked, we identify the algebra of observables $\mathfrak{A}$ with its image $\pi(\mathfrak{A}) \subset \mathcal{B}(\mathcal{H})$ under the physical representation π.) Condition (a.2) of Proposition

† Translator's note: Strictly speaking, condition (a.1) is not a direct restatement of Hypothesis (a) (of discrete superselection rules). The point is that in general context, totality of a subset $\{\Phi_n\} \subset \mathcal{H}$ does not guarantee that every vector in $\mathcal{H}$ can be expressed as a convergent series $\sum \lambda_n \Phi_n$. Call the latter property strong totality. Proposition 6.5 can be proved by establishing the implications (a)$\Rightarrow$(a.2)$\Rightarrow$(a.1) ((a.1)$\Rightarrow$(a) being trivial). The line of argument given in the authors' proof works because in Lemma 6.4, the "only if" half establishes strong totality, whereas the "if" half merely requires totality of the set of pure vector states.

6.5 enables us to represent each of the subspaces $\mathcal{H}_\nu$ as a direct sum $\bigoplus^{K_\nu} \tilde{\mathcal{H}}_\nu$ of some family of K_ν copies of the same space $\tilde{\mathcal{H}}_\nu$ and hence to represent $\mathcal{H}$ as the double direct sum

$$\mathcal{H} = \bigoplus_{\nu \in N} \Big(\bigoplus^{K_\nu} \tilde{\mathcal{H}}_\nu \Big). \tag{6.11}$$

Accordingly, the physical representation assumes the form of a double direct sum

$$A \equiv \pi(A) = \bigoplus_{\nu \in N} \Big(\bigoplus^{K_\nu} \tilde{\pi}_\nu(A) \Big), \tag{6.12}$$

where the $\tilde{\pi}_\nu$ are irreducible representations of $\mathfrak{A}$ in $\tilde{\mathcal{H}}_\nu$, $\tilde{\pi}_\nu$ and $\tilde{\pi}_{\nu'}$ being unitarily inequivalent for $\nu \neq \nu'$. This construction is a detailed deciphering of the meaning of condition (a.2). To find $\bar{\mathfrak{A}}$ we then have to use Theorem 1.31 (on the bicommutant): $\bar{\mathfrak{A}}$ is equal to the bicommutant of $\mathfrak{A}$ in $\mathcal{B}(\mathcal{H})$.

The form of the commutant $\mathfrak{A}^c$ can be found with the help of two propositions (which generalize Schur's lemma) which we give as exercises.

Exercise 6.13. Let π be the direct sum $\bigoplus^K \tilde{\pi}$ of a family of K copies of the irreducible representation $\tilde{\pi}$ of the algebra $\mathfrak{A}$; here $\tilde{\pi}$ acts in the space $\tilde{\mathcal{H}}$ and π acts in the space $\mathcal{H} = \bigoplus_{\kappa \in K} \tilde{\mathcal{H}}_\kappa$ where the $\tilde{\mathcal{H}}_\kappa$ are all identical to $\tilde{\mathcal{H}}$ and differ only by their indices $\kappa \in K$. We define an arbitrary operator X in $\mathcal{B}(\mathcal{H})$ by the matrix $\{X_{\kappa\lambda} \equiv E_\kappa^{-1} X E_\lambda\}_{\kappa,\lambda \in K}$ of operators in $\mathcal{B}(\tilde{\mathcal{H}})$, where each E_κ is an isomorphism from $\tilde{\mathcal{H}}$ onto $\tilde{\mathcal{H}}_\kappa$. Prove that $\pi(\mathfrak{A})^c$ consists precisely of all operators $X \in \mathcal{B}(\mathcal{H})$ for which the operator-valued matrix $\{X_{\kappa\lambda}\}$ is a scalar-valued matrix $\{x_{\kappa\lambda}\}$, where $x_{\kappa\lambda} \in \mathbf{C}$. [Hint: Use Exercise 1.54.]

Exercise 6.14. Let $\pi = \bigoplus_{\nu \in N} \pi_\nu$ be the direct sum of pairwise disjoint factorial representations π_ν of Type I of the algebra $\mathfrak{A}$, acting in the corresponding subspaces $\mathcal{H}_\nu$. Prove that $\pi(\mathfrak{A})^c = \bigoplus_\nu \pi_\nu(\mathfrak{A})^c$, that is, an operator X in $\mathcal{B}(\mathcal{H}) \equiv \mathcal{B}(\bigoplus_\nu \mathcal{H}_\nu)$ commutes with $\pi(\mathfrak{A})$ if and only if it leaves each subspace $\mathcal{H}_\nu$ invariant and commutes with $\pi_\nu(\mathfrak{A})$ in each such subspace $\mathcal{H}_\nu$. [Hint: Use Exercise 1.55.]

Using the above exercises it is fairly easy to find the general form of the operators of the bicommutant $\pi(\mathfrak{A})^{cc} \equiv \bar{\mathfrak{A}}$: they are the set of operators of the form

$$A = \bigoplus_\nu \Big(\bigoplus^{K_\nu} \tilde{A}_\nu \Big),$$

where $\{\tilde{A}_\nu\}_{\nu \in N}$ is any family of operators such that $\tilde{A}_\nu \in \mathcal{B}(\tilde{\mathcal{H}}_\nu)$ and $\sup_\nu \|\tilde{A}_\nu\| = \|A\| < \infty$. It follows, in particular, that the von Neumann algebra of observables $\bar{\mathfrak{A}}$ is isomorphic (as a C^*-algebra) to the direct sum $\bigoplus_{\nu \in N} \mathcal{B}(\tilde{\mathcal{H}}_\nu)$ (acting in $\bigoplus_{\nu \in N} \tilde{\mathcal{H}}_\nu$) of the family of algebras $\mathcal{B}(\tilde{\mathcal{H}}_\nu)$.

Exercise 6.15. Use the representation (6.12) to show that the centre $\mathfrak{Z}$ of the von Neumann algebra of observables $\bar{\mathfrak{A}}$ (defined as the set of operators in $\bar{\mathfrak{A}}$ that commute with all the operators in $\bar{\mathfrak{A}}$) consists of operators of the form

$$A = \bigoplus_\nu a_\nu \cdot 1_\nu, \quad a_\nu \in \mathbf{C},$$

where 1_ν is the identity operator in $\mathcal{B}(\mathcal{H}_\nu)$; show also that the condition $\sup_\nu |a_\nu| = \|A\| < \infty$ holds. Hence obtain the following characterization of the subspaces $\mathcal{H}_\nu$ in condition (a) of Proposition 6.5: the subspaces $\mathcal{H}_\nu$ are the maximal subspaces of $\mathcal{H}$ that are invariant with respect to the centre $\mathfrak{Z}$ and in which each operator in $\mathfrak{Z}$ is a multiple of the identity.

One result of this exercise is that the projectors E_ν onto the subspaces $\mathcal{H}_\nu$ belong to the von Neumann algebra of observables (and even belong to its centre).

B. Standard Superselection Rules

It is clear that a change of the multiplicity (from 1 to ∞) with which the irreducible representations enter into the physical representation leaves it in the same class of phenomenologically equivalent representations (see Exercise 6.8). Therefore if in the direct sum (6.12) we leave one representation from each class of unitarily equivalent irreducible representations, then as a result we obtain a different representation (in general) which, however, will be phenomenologically equivalent to the original one. The class of representations obtained in this manner can be characterized by the Hypothesis (a) on discrete superselection rules, supplemented by the following requirement.

(b) Hypothesis of Commutativity of (Discrete) Superselection Rules.

As before, we denote by $\mathfrak{M}$ *the total set of all vectors in the physical Hilbert space* $\mathcal{H}$ *defining the pure states of the algebra of observables* $\mathfrak{A}$*. Then the correspondence between the pure states of* $\mathfrak{S}$ *and the unit rays in* $\mathfrak{M}$ *is one-to-one.*

Theories in which Condition (a) together with Hypothesis (b) hold are called theories with *standard superselection rules.*

The next proposition justifies the use of the word "commutativity" in the title of Hypothesis (b).

Proposition 6.6. *Suppose that the hypothesis of discrete superselection rules holds and that* (6.10) *is the corresponding decomposition of* $\mathcal{H}$ *realized according to Proposition 6.5. Then the following conditions are equivalent.*

(b.1) *The pure states are in one-to-one correspondence with the rays in* $\mathcal{H}$ *defining the pure states.*

(b.2) *The representations of* $\mathfrak{A}$ *in the subspaces* $\mathcal{H}_\nu$ *are irreducible (and unitarily inequivalent to one another).*

(b.3) *The set of all vectors in* $\mathcal{H}$ *defining the pure states is the set of all non-zero vectors in* $\bigcup_{\nu\in N}\mathcal{H}_\nu$*.*

(b.4) *The (orthogonal) projector onto any vector in* $\mathcal{H}$ *defining a pure state belongs to the von Neumann algebra of observables* $\bar{\mathfrak{A}}$*.*

(b.5) *The algebra* $\mathfrak{A}$ *has an abelian commutant* $\mathfrak{A}^c$ *in* $\mathcal{B}(\mathcal{H})$*.*

We shall confine ourselves to proving the equivalence of (b.2) and (b.5).

Let $\pi(\mathfrak{A}) = \bigoplus \pi_\nu(\mathfrak{A})$, where the representations π_ν are irreducible and unitarily inequivalent and let B be a bounded operator in $\mathcal{H} = \bigoplus \mathcal{H}_\nu$ that commutes with all the operators in $\pi(\mathfrak{A})$. Then in view of Exercise 6.14 we have $B = \sum_\nu B_\nu \Pi_\nu$, where the Π_ν are the projection operators onto the invariant subspaces $\mathcal{H}_\nu$ and the $B_\nu \in \mathcal{B}(\mathcal{H}_\nu)$ commute with the operators of $\pi_\nu(\mathfrak{A})$. Since the π_ν are irreducible representations, B_ν is a multiple of the identity operator in $\mathcal{H}_\nu$. Thus the commutant $\pi(\mathfrak{A})^c$ consists of operators B of the form

$$B = \sum_\nu b_\nu \Pi_\nu, \quad \text{where } b_\nu \in \mathbf{C},\ \sup_\nu |b_\nu| = \|B\| < \infty.$$

This set of operators is clearly commutative, which proves the implication (b.2) $\Rightarrow$(b.5).

We prove the converse (b.5) $\Rightarrow$(b.2) by reductio ad absurdum. Thus we suppose that the commutant $\pi(\mathfrak{A})^c$ is abelian but that at least one of the subspaces $\mathcal{H}_\nu$ in the decomposition $\mathcal{H} = \oplus_\nu \mathcal{H}_\nu$ is irreducible. Then this $\mathcal{H}_\nu$ contains a pair of orthogonal subspaces, say $\hat{\mathcal{H}}_1$ and $\hat{\mathcal{H}}_2$ in which the subrepresentations $\hat{\pi}_1$ and $\hat{\pi}_2$ are irreducible and unitarily equivalent. We claim that the subalgebra of $\pi(\mathfrak{A})^c$ that stabilizes the subspace $\hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2$ of $\mathcal{H}$ contains non-commuting operators.

For let $V_{21} : \hat{\mathcal{H}}_1 \to \hat{\mathcal{H}}_2$ be an invertible map with inverse $V_{12} : \hat{\mathcal{H}}_2 \to \hat{\mathcal{H}}_1$ such that $V_{21}\hat{\pi}_1(A) = \hat{\pi}_2(A_2)V_{21}$, $\hat{\pi}_1(A)V_{12} = V_{12}\hat{\pi}_2(A)$ for all $A \in \mathfrak{A}$ (and $V_{21}V_{12} = 1 = V_{12}V_{21}$). By denoting the vectors of $\hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2$ as columns

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \quad \Psi_1 \in \hat{\mathcal{H}}_1, \quad \Psi_2 \in \hat{\mathcal{H}}_2,$$

we can define the operators in this space in the form of 2×2-matrices with operator-valued entries.

It is not difficult to verify that the operators $V = \begin{pmatrix} 0 & V_{12} \\ V_{21} & 0 \end{pmatrix}$ and $\Pi_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ commute with all the operators of the representation $\pi(A)|_{\hat{\mathcal{H}}_1 \oplus \hat{\mathcal{H}}_2} = \begin{pmatrix} \hat{\pi}_1(A) & 0 \\ 0 & \hat{\pi}_2(A) \end{pmatrix}$, but do not commute with each other: $V\Pi_1 = \begin{pmatrix} 0 & 0 \\ V_{21} & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & V_{12} \\ 0 & 0 \end{pmatrix} = \Pi_1 V$.

We thus arrive at a contradiction with the hypothesis that the superselection rules are commutative, which proves the implication (b.5) $\Rightarrow$(b.2).

Exercise 6.16. Prove the implications (b.1) $\Rightarrow$(b.2) $\Rightarrow$(b.3) $\Rightarrow$(b.4) $\Rightarrow$(b.1).

In particular we can state the hypothesis of standard superselection rules in the following way: the physical space $\mathcal{H}$ can be (uniquely) decomposed into a direct sum (6.10) of invariant subspaces $\mathcal{H}_\nu$ in which the representations of the algebra $\mathfrak{A}$ in these subspaces are irreducible and pairwise unitarily inequivalent. Because of this, the following analogue of one of the fundamental postulates of elementary quantum mechanics holds: the pure states are in one-to-one correspondence with the unit rays in $\bigcup_\nu \mathcal{H}_\nu$. The superposition rule now has a restricted form (namely, within the confines of the subspaces $\mathcal{H}_\nu$): a non-zero linear combination of vectors of pure states is the vector of a pure state provided that all the original vectors lie in the same subspace $\mathcal{H}_\nu$. Therefore (see Exercise 6.18 below) we call the subspaces $\mathcal{H}_\nu$ *coherent subspaces of* $\mathcal{H}$. The bounded operators of $\mathcal{B}(\mathcal{H})$ that are multiples of the identity in each of these subspaces (that is, in fact, operators in the centre $\mathfrak{Z}$ of the von Neumann algebra of observables $\bar{\mathfrak{A}}$) are called *superselection operators*, while the collection of all such operators (or simply any set of operators generating $\mathfrak{Z}$) are called *superselection rules*. (However, in the trivial case when the index ν in (6.10) runs through only one value, that is, when there is only one coherent subspace $\mathcal{H}_\nu = \mathcal{H}$, we then talk about the absence of (non-trivial) superselection rules.)

Exercise 6.17. Prove that in a theory with standard superselection rules, the von Neumann algebra of observables has the form

$$\bar{\mathfrak{A}} = \bigoplus_\nu \mathcal{B}(\mathcal{H}_\nu). \tag{6.13}$$

Exercise 6.18. Prove that (if the hypothesis of standard selection rules holds) the sectors (that is, the maximal indecomposable sets of states, see §6.1.B) are in one-to-one correspondence with the coherent subspaces.

The hypothesis of standard superselection rules singles out from among the physically identical representations (with discrete operator rules) the unique (to within unitary equivalence) most economical representation. If in the general decomposition (6.12) the multiplicities (that is the cardinalities of the sets K_ν) of the irreducible representations entering into the physical representation are interpreted as some kind of unobservable degrees of freedom, then we can say that conditions (a) and (b) reduce such freedom to a minimum. This circumstance is essential in the discussion of certain questions, in particular, symmetries. The physical symmetries are directly formulated in terms of observables and states and the transformation laws of the vectors of the physical space cannot be derived from them. In this connection, the value of the hypothesis of standard superselection rules consists in the fact (as will become clear from what follows) that it reduces the question of transformations of vectors under symmetries to the well known Wigner theorem. For an individual transformation(say, the so-called C-, P- and T-transformations), the indeterminacy of the state vectors reduces to a collection of phase factors (one for each coherent subspace) while for certain groups of symmetries (in particular, for the proper Poincaré group) the arbitrariness can be minimized or completely removed.

C. Connection with Gauge Groups

In field theory and elementary particle physics the standard superselection rules arise as a result of the following construction. We have a connected compact n-parameter abelian group $G = U(1)^n \equiv U(1) \times \ldots \times U(1)$, called in this context the *gauge group*. An arbitrary element of this group is a collection of n phase factors:

$$g = (s_1, \ldots, s_n) \equiv (e^{i\alpha_1}, \ldots, e^{i\alpha_n}),\ 0 \le \alpha_j < 2\pi. \tag{6.14}$$

A faithful unitary representation $\mathcal{U}$ of G is given in the physical Hilbert space $\mathcal{H}$. The corresponding gauge transformations in $\mathcal{H}$ have the form

$$\mathcal{U}(g) = s_1^{Q_1} \ldots s_n^{Q_n} \equiv e^{i(\alpha_1 Q_1 + \ldots + \alpha_n Q_n)}. \tag{6.15}$$

The generators $Q_1, \ldots, Q_n$ of the gauge transformations occurring here are mutually commuting self-adjoint operators with integer-valued spectrum; they are called the charges (corresponding to the given gauge group). Then $\mathcal{H}$ decomposes into the direct sum

$$\mathcal{H} = \bigoplus_{q_1, \ldots, q_n \in \mathbf{Z}} \mathcal{H}(q_1, \ldots, q_n) \tag{6.16}$$

of the corresponding spectral subspaces, consisting of all vectors Φ such that $(Q_j - -q_j)\Phi = 0$, $j = 1, \ldots, n$. We now assume that an arbitrary non-zero vector $\Phi \in \mathcal{H}$ defines a pure state of the algebra of observables $\mathfrak{A}$ if and only if it is an eigenvector for all the charges. Thus standard selection rules are defined in $\mathcal{H}$ for which (6.16) is the decomposition of $\mathcal{H}$ into a direct sum of coherent subspaces $\mathcal{H}(q_1, \ldots, q_n)$.

An equivalent method of prescribing these superselection rules can be brought about by choosing the von Neumann algebra of observables $\bar{\mathfrak{A}}$ as a subalgebra of $\mathcal{B}(\mathcal{H})$. For this we define the gauge transformations in $\mathcal{B}(\mathcal{H})$:

$$A \to \mathcal{U}(g)\, A\mathcal{U}(g)^{-1},\ A \in \mathcal{B}(\mathcal{H}). \tag{6.17}$$

Exercise 6.19. Prove that in the construction given above of the superselection rules, the bounded operator $A \in \mathcal{B}(\mathcal{H})$ belongs to the von Neumann algebra of observables if and only if it is invariant with respect to all the gauge transformations of G. Show also that

$$\omega_{\mathcal{U}(g)\Phi} = \omega_\Phi \quad \text{for all } \Phi \in \mathcal{H}\backslash\{0\},\ g \in G. \tag{6.18}$$

Thus we have the following characterization of the von Neumann algebra of observables: $\bar{\mathfrak{A}}$ consists precisely of all operators in $\mathcal{B}(\mathcal{H})$ that are left unchanged under the gauge transformations of G.

Exercise 6.20. In the above construction of the superselection rules, prove that a bounded operator in $\mathcal{H}$ is a superselection if and only if it is a (bounded) function of the charges and that the centre $\mathfrak{Z}$ of the von Neumann algebra is the bicommutant of the set $\mathcal{U}(G) \equiv \{\mathcal{U}(g) : g \in G\}$ of gauge transformations:

$$\mathfrak{Z} = \mathcal{U}(G)^{cc}. \tag{6.19}$$

The unitary operators in the centre $\mathfrak{Z}$ of the von Neumann algebra of observables are sometimes called generalized gauge transformations.

At the present stage of the exposition it is fairly difficult to see why from among all the unitary superselection operators, clear preference is given to the gauge transformations. (But then, of course, we could equally well have characterized the algebra $\bar{\mathfrak{A}}$ as the set of all operators $A \in \mathcal{B}(\mathcal{H})$ such that $UAU^{-1} = A$ for all unitary superselection operators U.) It is not merely that the gauge transformations provide a compact means of describing the superselection rules. Later on (starting at Chapter 8) we shall be considering local field structures, and the value of the gauge transformations is that (in contrast to general unitary superselection operators) they preserve the local structures.

This has a direct bearing on the fact that in the traditional canonical formalism of field theory, the charge operators (as opposed to arbitrary operators associated with $\mathfrak{Z}$) are three-dimensional integrals of local currents.

The modern situation in elementary particle physics is in fairly good accord with the hypothesis that the superselection rules can be described by the electric Q $(= Q_1)$, baryonic B $(= Q_2)$ and leptonic L $(= Q_3)$ charges,* so that the decomposition into coherent subspaces has the form

$$\mathcal{H} = \bigoplus_{q,b,l \in \mathbf{Z}} \mathcal{H}(q, b, l). \tag{6.20}$$

We note that the superselection rule according to valency (of the representations of the rotation group) mentioned at the beginning of §6.2.A is not independent because of the empirical relation

$$e^{i2\pi M_3} = (-1)^{B+L}; \tag{6.21}$$

the operator featuring here

$$F = B + L \tag{6.22}$$

can be called the fermionic charge.

In the interpretation of states in terms of Bose and Fermi particles (further details of which will be given in Chapter 7), F takes an odd value if and only if the state contains an odd number of Fermi particles.** Therefore (6.21) is an expression of the ***(normal) connection between spin and statistics***: states with an even (or odd) number of fermions transform according to single-valued (or two-valued) representations of the rotation group (also the Poincaré group). This rule is proved in the field theory (see Chapter 9).

D. Example of non-abelian gauge groups

We have already noted at the beginning of this section that the hypothesis of standard superselection rules is optional. Furthermore, in modern elementary particle theory, of considerable popularity is the hypothesis on the existence of a strict (gauge) $SU(3)$-symmetry, called the *colour $SU(3)$ group* (and denoted by $SU(3)_c$). The presence of such a symmetry calls for some explanation, because from the point of view of the quark model, all observable hadrons are bound states either of three quarks or of a quark and an antiquark. The inviolability of the colour symmetry is responsible in these representations for the fact that free quarks and (colour) gluons are not observed.

The operator of the quark field $\psi^{\alpha}_{a,i}(x)$ is dependent on the point in space-time and on three further indices: the spinor index α $(= 1, 2, 3, 4)$, which distinguishes two spin projections and differentiates the particles from the antiparticles, the index a giving the charged state of the quark (more precisely, its charge, hypercharge, "charm" etc.) and finally, the index i $(= 1, 2, 3)$ defining the "colour". We shall confine ourselves to the simplest model for illustrative purposes, in which only the degrees of freedom corresponding to colour and the "particle–antiparticle" states are preserved.†

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* Within the framework of models of the "grand unification" (GUT) which extrapolate the strong, weak and electromagnetic processes to energy ranges of order 10^{15} GeV, theoretical schemes are widely discussed for which it is possible that the baryonic and leptonic charges are not conserved (concerning this, see, for example Langacker, 1981; Slansky, 1981). The hypothesis of non-conservation of baryonic charge lies at the basis of the models of the formation of baryonic asymmetry of the universe after the big bang (Sakharov, 1967, 1979; Kuz'min, 1970; Weinberg, 1979).

** However, in contrast to F, the number of (say, incoming or outgoing) Fermi particles is not a superselection operator. Here we have in view states of observable particles. "States" corresponding to $B = 1/3$ are considered by us in the context of a non-abelian gauge group (see §6.2.D below).

† A more realistic examination of the model of coloured quarks and gluons and their field-theoretic foundations goes beyond the framework of this monograph (see, for example, the survey by Greenberg and Nelson, 1977, and the book [C5]).

We consider a system of six anticommuting (Fermi) operators q_i^ϵ, $i = 1,2,3$, $\epsilon = \pm$, and their Hermitian adjoints $q_i^{\epsilon *}$. We suppose that the operators q and q^* satisfy the canonical anticommutation relations

$$[q_i^\epsilon, q_j^{\eta *}]_+ \equiv q_i^\epsilon q_j^{\eta *} + q_j^{\eta *} q_i^\epsilon = \delta_{ij}\delta_{\epsilon\eta};\ i,j = 1,2,3,\ \epsilon,\eta = \pm \tag{6.23}$$

These operators generate a finite-dimensional algebra which we call the algebra of the field and denote by $\mathcal{F}$. (This terminology will be justified later, see Chapters 8,10: for the moment we note that $\mathcal{F}$ corresponds to the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators as set forth in the preceding subsection.)

We define the action in $\mathcal{F}$ of the non-abelian gauge group of automorphisms $U(3)$ in such a way that

$$\begin{aligned} &q_i^+ \to (u^{-1})_{ij} q_j^+, \quad q_i^- \to (\bar{u}^{-1})_{ij} q_j^-, \\ &(q_i^+)^* \to (\bar{u}^{-1})_{ij} (q_j^+)^*, \quad (q_i^-)^* \to (u^{-1})_{ij} q_j^{-*}; \quad u \equiv (u_{ij}) \in U(3). \end{aligned} \tag{6.24a}$$

The Hermitian generators of the Lie algebra of this group are simply expressed in terms of the operators q and q^*:

$$\begin{aligned} &Y_a = \tfrac{1}{2}(q^{+*}\lambda_a q^+ + q^-\lambda_a q^{-*}), \quad a = 1,\dots,8, \\ &B = \tfrac{1}{3}(N_+ - N_-), \quad N_\epsilon = q_i^{\epsilon *} q_i^\epsilon, \quad \epsilon = \pm, \end{aligned} \tag{6.24b}$$

where the λ_a are a complete system of orthonormal traceless Hermitian 3×3-matrices with the normalization condition tr $\lambda_a\lambda_b = \delta_{ab}$, $a,b = 1,\dots,8$ (called the Gell-Mann matrices). Here and in what follows, the repeated index i is to be summed from 1 to 3.

The algebra of observables $\mathfrak{A}$ is, by definition, the subalgebra of the gauge-invariant elements of the algebra $\mathcal{F}$. It is generated by (the identity and) its second-order elements

$$N_+,\ N_-,\ A = \tfrac{1}{3} q_i^+ q_i^-,\quad A^* = \tfrac{1}{3} q_i^{-*} q_i^{+*}$$

by the algebraic operations. The observables N_+ and N_- can be interpreted as the operators of the number of quarks and antiquarks respectively. The centre of $\mathfrak{A}$ is generated by the two Casimir operators of $SU(3)$ and the baryonic charge B (6.24b). The eigenvalues of the Casimir operators of $SU(3)$ are expressed in terms of the pair of non-negative integers (r_+, r_-) defining the irreducible representations of it.*

The space $\mathcal{H}$ in which the physical representation of the algebras $\mathfrak{A}$ and $\mathcal{F}$ is realized is defined as the Fock space for the operators q and q^* with the invariant vacuum vector $|0\rangle$ annihilated by the operators q : $q_j^\epsilon|0\rangle = 0$. In this definition, $\mathcal{F}$ coincides with the set of all linear operators in the finite-dimensional space $\mathcal{H}$ (and hence the representation of the algebra of the fields is irreducible). The representation of the algebra of observables $\mathfrak{A}$ is reducible and decomposes into factorial components indexed by the three numbers (r_+, r_-) and B, which give the irreducible representations of the gauge group:

$$\mathcal{H} = \Big(\bigoplus_{B=-1,0,1} \mathcal{H}_B^{(00)}\Big) \oplus \Big(\bigoplus_{B=1/3,-2/3} \mathcal{H}_B^{(10)}\Big) \oplus \Big(\bigoplus_{B=-1/3,2/3} \mathcal{H}_B^{(01)}\Big) \oplus \mathcal{H}_0^{(11)}. \tag{6.25a}$$

All the spaces $\mathcal{H}_B^{(r_+r_-)}$ contain the given irreducible representation of $\mathfrak{A}$ with multiplicity equal to the dimension of the representation (r_+, r_-) of the group $SU(3)$.

We note that the dimensions of the spaces $\mathcal{H}_B^{(r_+r_-)}$ themselves are higher than those of the corresponding representations of the gauge group. Thus the subspace $\mathcal{H}_0^{(00)}$ corresponding to the identity representation of G is four-dimensional. It is spanned by the vacuum vector $|0\rangle$ and the vectors of the ν-meson states ($\nu = 1,2,3$):

$$A^*|0\rangle,\ \tfrac{\sqrt{3}}{2}(A^*)^2|0\rangle,\ \tfrac{\sqrt{3}}{2}(A^*)^3|0\rangle.$$

* The three-dimensional representation $(1,0)$ of $SU(3)$ can be realized in the space with $B = 1/3$ (spanned by the vectors $q_i^{+*}|0\rangle$) or in the space with $B = -2/3$ (spanned by the vectors $\epsilon_{ijk} q_j^{-*} q_k^{-*}|0\rangle$). Similarly, the admissible values of B for the adjoint representation (0,1) are equal to $-1/3$ and $2/3$. The eight-dimensional representation (1,1) corresponds only to the baryonic number $B = 0$. The scalar representation (0,0) can be realized for the values $B = 0$ ($|0\rangle$), 1 ($q_1^{+*}q_2^{+*}q_3^{+*}|0\rangle$) and -1 ($q_1^{-*}q_2^{-*}q_3^{-*}|0\rangle$). The other representations of $SU(3)$ are not realized in this example.

(We leave the reader to verify, using the commutation relation $[A, A^*] = 1 - \frac{1}{3}N$, where $N = N_+ + N_-$, that the above basis vectors of $\mathcal{H}_0^{(00)}$ with quantum numbers of the vacuum are orthonormal.) The subspace $\mathcal{H}_{1/3}^{(10)}$ is nine-dimensional. It is spanned by the single-quark states, the states with one quark and one meson, and the states with one quark and two mesons. The states with different numbers of mesons differ physically (for example, in the average values of the observable A^*A), whereas states with a given number of mesons but with quarks of a different colour are indistinguishable. Thus the three-dimensional irreducible representation of the algebra of observables is triply contained in $\mathcal{H}_{1/3}^{(10)}$. The same representation of $\mathfrak{A}$ occurs eight times in the subspace $\mathcal{H}_0^{(11)}$ of state vectors with the quantum numbers of a single gluon.

In accordance with Proposition 6.6, the degeneracy that we have described of the representations of $\mathfrak{A}$ corresponds to the non-commutativity of the operators Y_a in the commutant of $\mathfrak{A}$. On the other hand, as we remarked in the general case in §6.2.B, there exists a phenomenologically equivalent representation of the algebra of observables in which the standard superselection rules are realized. In this case the space $\mathcal{H}_{ST}$ contains along with the six-dimensional subspace of "colourless hadrons"

$$\bigoplus_{B=0,\pm 1} \mathcal{H}_B^{(00)} \tag{6.25b}$$

(which remain unchanged), five more three-dimensional subspaces corresponding to the remaining summands in the decomposition (6.25a). In the general complexity, the (21-dimensional) space $\mathcal{H}_{ST}$ splits up into a direct sum of eight coherent subspaces in which inequivalent irreducible representations of the algebra of observables are realized.

We note that the hypothesis of the confinement of quarks corresponds in the present context to the restriction of the physical Hilbert space to the space (6.25b), which implies the restriction of the algebra of "physical" fields to the algebra of observables with the addition of the operators of creation and annihilation of baryons $(q_1^\epsilon q_2^\epsilon q_3^\epsilon)^{(*)}$, $\epsilon = +$ or $-$.

6.3. Symmetries in the Algebraic Approach

A. THE CONCEPT OF SYMMETRY

By a symmetry we mean a transformation of a physical system that does not change its structural properties. In the present context, a physical system is characterized by the algebra of observables $\mathfrak{A}$ and the set of states $S(\mathfrak{A})$. Accordingly we define a *symmetry* as a pair of bijections* $\alpha : \mathfrak{A} \to \mathfrak{A}$ and $\alpha' : S(\mathfrak{A}) \to S(\mathfrak{A})$ satisfying the compatibility condition:

$$(\alpha'\omega)(\alpha A) = \omega(A) \quad \text{for all } A \in \mathfrak{A},\ \omega \in S(\mathfrak{A}). \tag{6.26}$$

It would be reasonable, of course, to impose certain further conditions of continuity of the bijections α and α', and to require that α preserve the algebraic operations inherent in the set of observables. However, it is a remarkable fact that such properties of the maps α and α' can be extracted from (6.26). Furthermore, as we shall see, just one of the bijections α, α' suffices for the characterization of a symmetry (since the second bijection is then completely defined by the relations (6.26); see Exercise 6.21 below). Therefore, by abuse of notation, we can also call one of the bijections α, α' a symmetry.

It is clear that the set of all symmetries of a physical system form a group under multiplication defined by the composition of bijections. (The product of the two symmetries (α, α') and (β, β') is the symmetry$(\alpha\beta, \alpha'\beta')$, where $(\alpha\beta)(A) \equiv \alpha[\beta(A)]$ and $(a'\beta')(\omega) \equiv \alpha'[\beta'(\omega)]$.)

* A *bijection* $f : X \to Y$ is a one-to-one map from the set X onto the set Y.

If one adheres to the point of view that physically it is only necessary to define the law of transformation of the (Hermitian) observables, then property (6.26) can be used to *extend the definition* of the map α defined originally as a bijection $\mathfrak{A}_h \to \mathfrak{A}_h$ of the Hermitian elements of the algebra. (In fact, any element $A \in \mathfrak{A}$ can be uniquely represented in the form $A = A_1 + iA_2$, where $A_1, A_2 \in \mathfrak{A}_h$, and in order to satisfy (6.26) we must set $\alpha(A) = \alpha(A_1) + i\alpha(A_2)$.)

Formula (6.26) can be interpreted as though the symmetry transformation reduced merely to a change in the parametrization of the observables and states under which the directly measurable quantities (in the present instance, the mean values) do not change. This is the so-called "passive" point of view on symmetries. It is also possible to adopt the "active" point of view when it is supposed that under the action of the symmetry only the observables are changed by the map α, the states remaining unchanged (so-called Heisenberg picture) or when the states are changed by the map α', the observable remaining unchanged (Schrödinger picture). In essence all these ways of looking at a symmetry are equivalent (since the bijections α and α' are interconnected) and the choice of the point of view is merely dictated by considerations of convenience.

A map α' from the set of states $S(\mathfrak{A})$ into itself is said to be *affine* if it preserves convex linear combinations:

$$\alpha'[\lambda\omega_1 + (1-\lambda)\omega_2] = \lambda\alpha'(\omega_1) + (1-\lambda)\alpha'(\omega_2), \ 0 \le \lambda \le 1.$$

Exercise 6.21. Let (α, α') and (β, β') be two symmetries of the system $(\mathfrak{A}, S(\mathfrak{A}))$. Prove that $\alpha = \beta$ if and only if $\alpha' = \beta'$. [Hint: Use Exercise 6.1.]

Exercise 6.22. If (α, α') is a symmetry, then

$$\alpha(A^*) = \alpha(A)^*, \ A \in \mathfrak{A}. \tag{6.27}$$

[Hint: Set $\tilde{\alpha}(A) = \alpha(A^*)^*$; then $(\tilde{\alpha}, \alpha')$ is a symmetry; now use the previous exercise.]

Exercise 6.23. Let (α, α') be a symmetry. Prove that α' is an affine weak* continuous map and that α is a linear map. [Hint: Observe that $\alpha'[\lambda\omega_1 + (1-\lambda)\omega_2]\ (A) = [\lambda\omega_1 + (1-\lambda)\omega_2](\alpha^{-1}A)$ and recall that the weak* topology is the weakest topology for which the map $\omega \to \omega(A)$ is continuous for any $A \in \mathfrak{A}$. For the proof of the linearity of the map, use Exercise 6.1, from which it follows that the condition $(\alpha'\omega)(B) = 0$ for all $\omega \in S(\mathfrak{A})$ implies that $B = 0$.]

If the symmetry α' is originally defined only on the set of "physical" states $\mathfrak{S}$ (concerning which we suppose here that it distinguishes the positive elements of $\mathfrak{A}$ and hence is weak* dense in $S(\mathfrak{A})$), then the result of Exercise 6.23 enables us to extend α' by continuity to $S(\mathfrak{A})$ so that

$$(\alpha'\omega)(\alpha A) = \omega(A) \quad \text{for all } \omega \in S(\mathfrak{A}), \ A \in \mathfrak{A}. \tag{6.28}$$

This extension (which we again denote by α') is also a bijection from $S(\mathfrak{A})$ onto itself since it has an inverse, namely the extension of α'^{-1}.

Exercise 6.24. Let (α, α') be a symmetry. Prove that

$$\|\alpha(A)\| = \|A\| \quad \text{for all } A \in \mathfrak{A}_h. \tag{6.29}$$

[Hint: Use (6.28) and the formula for the norm of elements $A \in \mathfrak{A}_h$

$$\|A\| = \sup\{|\omega(A)| : \omega \in S(\mathfrak{A})\}.$$

This formula in turn is easily proved for operator algebras; for the states we can restrict ourselves to the vector states. For the case of abstract algebras, we can use the Gel'fand-Naimark theorem 1.28.]

It follows from the preceding exercises that α is a linear homeomorphism. Consequently, (6.26) (or (6.28)) can be interpreted in the following way: α' is the restriction to $\mathfrak{S}$ (or to $S(\mathfrak{A})$) of the adjoint of α^{-1}.

Proposition 6.7. *Let α' be the map of the set of physical states $\mathfrak{S}$ induced by a symmetry. Then α' is an affine weak* continuous homeomorphism such that*

$$\|\alpha'(\omega_1) - \alpha'(\omega_2)\| = \|\omega_1 - \omega_2\| \quad \textit{for all } \omega_1,\ \omega_2 \in \mathfrak{S} \tag{6.30}$$

(that is, α' is an isometry).

■ Thanks to Exercise 6.23 we merely have to prove the estimate (6.30). We apply Exercise 1.53 to the Hermitian functional $\alpha'(\omega_1) - \alpha'(\omega_2)$:

$$\|\alpha'(\omega_1) - \alpha'(\omega_2)\| = \sup_{A\in\mathfrak{A},\,\|A\|\le 1} |(\alpha'\omega_1)(A) - (\alpha'\omega_2)(A)| = \sup_{A\in\mathfrak{A}_h,\,\|A\|\le 1} |(\omega_1 - \omega_2)(\alpha^{-1}(A)|.$$

The estimate (6.30) now follows from (6.29). ■

Exercise 6.25. Let (α, α') be a symmetry. Prove that the bijection α' maps pure states to pure states with preservation of the transition probability, and that if S is a sector in $PS(\mathfrak{A})$, then $\alpha'(S)$ is also a sector. [Hint: It follows from the preservation of the transition probability under the map that $\alpha'(S)$ is an indecomposable set of pure states if and only S is; now use the definition of a sector as a maximal indecomposable subset of $PS(\mathfrak{A})$.]

In elementary quantum mechanics there is a remarkable theorem due to Wigner giving a characterization of the symmetries in terms of the map α'. In this case $\mathfrak{A} = \mathcal{B}(\mathcal{H})$, while the set $\mathfrak{S}$ of all pure physical states is (according to (6.2)) in one-to-one correspondence $\omega_\Pi \leftrightarrow \Pi$ with the set

$$\mathcal{P} = \{\Pi \in \mathcal{L}_1(\mathcal{H}) : \Pi^* = \Pi = \Pi^2\} \tag{6.31}$$

of all one-dimensional (orthogonal) projectors in $\mathcal{H}$; here the transition probability between two pure states has the form

$$\omega_{\Pi_1} \cdot \omega_{\Pi_2} = \mathrm{tr}(\Pi_1\Pi_2). \tag{6.32}$$

Consequently the map α' induced by the symmetries and restricted to $\mathfrak{S}$ can be regarded (after identifying $\mathfrak{S}$ with $\mathcal{P}$) as a bijection $\tau : \mathcal{P} \to \mathcal{P}$ that preserves the transition probabilities (6.32). Wigner's theorem states that this property of the map τ is a characteristic property of the symmetry; furthermore, each symmetry α is *unitarily* or *anti-unitarily induced* in the sense that there exists a unitary or anti-unitary operator U in $\mathcal{H}$ such that for all $A \in \mathcal{B}(\mathcal{H})$

$$\alpha(A) = UAU^{-1} \quad \text{for unitary } U, \tag{6.33a}$$

$$\alpha(A) = UA^*U^{-1} \quad \text{for anti-unitary } U. \tag{6.33b}$$

We now give (in a somewhat more general context) a statement of Wigner's theorem, the significance of which goes beyond the framework of elementary quantum mechanics.

Theorem 6.8 (Wigner). *Let $\mathcal{P}$ and $\mathcal{P}'$ be the sets of all one-dimensional projectors in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively. If a bijection $\tau : \mathcal{P} \to \mathcal{P}'$ $(\tau(\Pi) \equiv \Pi')$ is given, such that*

$$\mathrm{tr}(\Pi_1\Pi_2) = \mathrm{tr}(\Pi_1'\Pi_2') \textit{ for all } \Pi_1, \Pi_2 \in \mathcal{P}, \tag{6.34}$$

then there exists either a unitary or an anti-unitary operator $U : \mathcal{H} \to \mathcal{H}'$ such that*

$$\Pi' = U\Pi U^{-1} \textit{ for all } \Pi \in \mathcal{P}. \tag{6.35}$$

* Only in the trivial case when $\mathcal{H}$ is one-dimensional can each of these two alternatives be realized (for a fixed map τ).

The operator U is defined by this condition to within an arbitrary phase factor.

B. Proof and Discussion of Wigner's Theorem

We consider the algebra $\mathcal{F}$ of all operators of finite rank and the set $\mathcal{F}_h$ of all Hermitian elements of $\mathcal{F}$; $\mathcal{F}$ is a *-algebra (without an identity if $\dim \mathcal{H} = \infty$). The main step in the proof of Theorem 6.8 consists in the construction of the bijection $\tau : \mathcal{F} \to \mathcal{F}'$ extending the map $\mathcal{P} \to \mathcal{P}'$ and having the properties:

(α) the map τ is linear: $\tau(\lambda A + \mu B) = \lambda\tau(A) + \mu\tau(B)$,

(β) either

$$\tau(AB) = \tau(A)\tau(B) \quad \text{for all } A,\ B \in \mathcal{F}, \tag{6.36}$$

or

$$\tau(AB) = \tau(B)\tau(A) \quad \text{for all } A,\ B \in \mathcal{F}; \tag{6.37}$$

(γ) $\tau(A^*) = \tau(A)^*$ for all $A \in \mathcal{F}$.

Remark. Instead of condition (β) we can require that (6.36) always holds, but instead of (α) we allow the map τ to be either linear of antilinear. (In fact, in the case when (6.37) holds, $\tau(A)$ must be replaced by $\tau(A)^*$.)

Lemma 6.9. *Each operator $A \in \mathcal{F}$ is uniquely defined by the values of $\operatorname{tr}\Pi A$ for all $\Pi \in \mathcal{P}$, that is, $\operatorname{tr}\Pi A = 0$ for all $\Pi \in \mathcal{P}$ implies that $A = 0$.*

■ For each $\Phi, \Psi \in \mathcal{H}$, the matrix element $\langle\Phi, A\Psi\rangle$ is represented in the form $\operatorname{tr} AB$ where $B = |\Psi\rangle\langle\Phi| \in \mathcal{F}$. On the other hand each operator $B \in \mathcal{F}$ is a finite linear combination of one-dimensional projectors, since it can be represented in the form $B = B_1 + iB_2$, where $B_{1,2} \in \mathcal{F}_h$, while for the Hermitian elements of $\mathcal{F}$, this follows automatically from their spectral decomposition. (It is not difficult to construct the explicit decomposition of the operator $|\Psi\rangle\langle\Phi|$ into one-dimensional projectors.) ■

We now define the map $\tau : \mathcal{F} \to \mathcal{F}$ by the formula

$$\operatorname{tr}\tau(\Pi)\tau(A)\ (\equiv \operatorname{tr}\Pi'\tau(A)) = \operatorname{tr}\Pi A, \quad A \in \mathcal{F}. \tag{6.38}$$

By the above lemma, if the operator $\tau(A)$ satisfying (6.38) exists, then it is unique. The existence of such an operator follows from the statement used above, that $\mathcal{F}$ is the linear span of $\mathcal{P}$ and from the condition of the theorem.

Property (α) of the map τ is obvious from (6.38). We now verify property (γ). For all $\Pi \in \mathcal{P}$ and $A \in \mathcal{F}$ we have

$$\operatorname{tr}\Pi'\tau(A^*) = \operatorname{tr}\Pi A^* = \operatorname{tr} A^*\Pi = \overline{\operatorname{tr}\Pi A} = \overline{\operatorname{tr}\Pi'\tau(A)} = \operatorname{tr}\Pi'\tau(A)^*;$$

hence by the lemma, (γ) follows.

We note that equality (6.38) can be extended by linearity:

$$\operatorname{tr}\tau(A)\tau(B) = \operatorname{tr} AB \quad \text{for all } A,\ B \in \mathcal{F}. \tag{6.39}$$

By approximating the identity operator in $\mathcal{H}$ by finite-dimensional projectors, we also obtain

$$\operatorname{tr}\tau(A) = \operatorname{tr} A \quad \text{for all } A \in \mathcal{F}. \tag{6.40}$$

Before proving (β), we introduce the weaker property:

$$\tau(AB) + \tau(BA) = \tau(A)\tau(B) + \tau(B)\tau(A) \quad \text{for all } A,\ B \in \mathcal{F}_h. \tag{6.41}$$

In view of the identity $AB + BA = (A+B)^2 - A^2 - B^2$, property (6.41) follows from the more specific property $\tau(A^2) = [\tau(A)]^2$ for all $A \in \mathcal{F}_h$, which in its turn follows from the spectral decomposition of A into mutually orthogonal projectors ($A = \sum \lambda_j \Pi_j \Rightarrow A^2 = \sum \lambda_j^2 \Pi_j$).

Exercise 6.26. Prove that the alternative (β) is equivalent to the alternative: either

$$\operatorname{tr}\tau(A)\tau(B)\tau(C) = \operatorname{tr}(ABC) \quad \text{for all } A, B, C \in \mathcal{F}, \tag{6.42}$$

or

$$\operatorname{tr}\tau(A)\tau(B)\tau(C) = \operatorname{tr}(CBA) \quad \text{for all } A, B, C \in \mathcal{F}. \tag{6.43}$$

We claim that the alternative (6.42), (6.43) holds for operators of the form $|\Psi\rangle\langle\Psi|$. For the set $\overline{\mathbf{R}}_+\mathcal{P}$ of these operators (consisting of all non-negative operators in $\mathcal{H}$ of rank ≤ 1 or, equivalently, all operators of the form $\rho\Pi$, $\Pi \in \mathcal{P}$, $\rho \geq 0$) is taken to a set of the same type under the map τ, and its linear span is the whole of the algebra $\mathcal{F}$. Therefore it follows immediately from the validity of our alternative on $\overline{\mathbf{R}}_+\mathcal{P}$ that it holds on all of $\mathcal{F}$.

Exercise 6.27. Show that

$$|\operatorname{tr}(\tau(A)\tau(B)\tau(C))| = |\operatorname{tr}(ABC)|, \tag{6.44}$$

$$\operatorname{Re}\operatorname{tr}(\tau(A)\tau(B)\tau(C)) = \operatorname{Re}\operatorname{tr}(ABC) \tag{6.45}$$

for all $A, B, C \in \mathbf{R}_+\mathcal{P}$. [Hint: For the verification of (6.44) use the fact that the identity $|\operatorname{tr}(ABC)|^2 =$ $= \operatorname{tr}(AB)\operatorname{tr}(BC)\operatorname{tr}(CA)$ holds for all operators of the form $|\Psi\rangle\langle\Psi|$, and invoke (6.39); for the derivation of (6.45) note that $2\operatorname{Re}\operatorname{tr}(ABC) = \operatorname{tr}((AB + BA)C)$ for all $A, B, C \in \mathcal{F}_h$, and use Exercise 6.28.]

It follows from (6.44) and (6.45) that

$$\operatorname{Im}\operatorname{tr}(\tau(A)\tau(B)\tau(C)) = \pm\operatorname{Im}\operatorname{tr}(ABC) \quad \text{for all } A, B, C \in \mathbf{R}_+\mathcal{P}, \tag{6.46}$$

where *a priori*, the choice of the sign on the right hand side would appear to depend on the triple of operators A,B,C. We show in fact that the sign does not depend on the choice of the triple.

We prove this by contradiction. To this end we introduce the functions

$$f_\pm(A, B, C) = \operatorname{Im}[\operatorname{tr}(\tau(A)\tau(B)\tau(C)) \pm \operatorname{tr}(ABC)]$$

and suppose that there exist two sets (A_0, B_0, C_0) and (A_1, B_1, C_1) in $\mathbf{R}_+\mathcal{P}$ such that

$$f_+(A_0, B_0, C_0) \neq 0 \neq f_-(A_1, B_1, C_1). \tag{6.47}$$

Let $A_0 = |\Phi_0\rangle\langle\Phi_0|$, $A_1 = |\Phi_1\rangle\langle\Phi_1|$; we set $A_t = |\Phi_t\rangle\langle\Phi_t|$, where $\Phi_t = (1-t)\Phi_0 + t\Phi_1$. We define the operators B_t and C_t similarly. The functions $f_\pm(A_t, B_t, C_t)$ are polynomials in t none of which is identically zero (by (6.47)). Consequently there exists t for which they are simultaneously non-zero, which contradicts (6.46).

Thus the sign in (6.46) is the same for all $A, B, C \in \mathbf{R}_+\mathcal{P}$. If it is plus, then the equalities (6.45), (6.46) are together equivalent to (6.42); if it is minus, then they are equivalent to (6.43).

Thus we have completed the proof of properties (α), (β), (γ) and we now embark on the construction of the operator U. This operator turns out to be unitary if (6.42) holds and anti-unitary if (6.43) holds. We give this construction for the second case, leaving the reader to make the necessary modifications for the construction of the first case.

We fix a unit vector $\Phi \in \mathcal{H}$ and associate with each vector Ψ the pair of operators

$$S_\Psi = |\Psi\rangle\langle\Psi| \quad \text{and} \quad T_\Psi = |\Psi\rangle\langle\Phi|$$

of rank ≤ 1, which clearly enjoy the properties:

$$S_\Psi^* = S_\Psi, \; S_\Psi^2 = (\operatorname{tr} S_\Psi)S_\Psi; \tag{6.48}$$

$$T_\Psi^* T_\Psi = (\operatorname{tr} S_\Psi)S_\Phi, \; T_\psi T_\Psi^* = S_\Psi. \tag{6.49}$$

It follows from the properties of the map τ proved above, that the equalities (6.48) are preserved when S_Ψ is replaced by $\tau(S_\Psi)$, while (6.49) is converted (by virtue of (6.43)) to the relations

$$\tau(T_\Psi)\tau(T_\Psi)^* = \operatorname{tr}\tau(S_\Psi)\tau(S_\Psi), \quad \tau(T_\Psi)^*\tau(T_\Psi) = \tau(S_\Psi). \tag{6.50}$$

It follows from the new equality of type (6.48) that we can find a unit vector Φ' in $\mathcal{H}'$ such that

$$\tau(S_\Phi) = S_{\Phi'} = |\Phi'\rangle\langle\Phi'|. \tag{6.51}$$

Exercise 6.28. Prove that given Φ', there exists a uniquely defined vector $\Psi' \in \mathcal{H}'$ such that

$$\tau(T_\Psi) = |\Phi'\rangle\langle\Psi'|. \tag{6.52}$$

[Hint: Use the equalities (6.50) and Lemma 6.9.]

We now define the operator U by the equality

$$U\Psi = \Psi' \quad \text{for all } \Psi \in \mathcal{H}.$$

This operator U is, to begin with, antilinear; secondly it is an isometry (which follows from the second of the equalities (6.49) and from (6.40)); thirdly, U maps $\mathcal{H}$ onto the whole of $\mathcal{H}'$, since it follows from the equality $\tau(S_\Psi) = S_{\Psi'}$ and from the hypothesis of the theorem (that the map $\mathcal{P} \to \mathcal{P}'$ is a bijection) that $S_{\Psi'}$ runs through the whole of the set $\overline{\mathbf{R}}_+\mathcal{P}$ when Ψ runs through $\mathcal{H}$. Consequently U is anti-unitary. It then follows from the chain of trivial equalities

$$\tau(S_\Psi) = S_{\Psi'} = |\Psi'\rangle\langle\Psi'| = U|\Psi\rangle\langle\Psi|U^* = US_\Psi U^{-1}$$

that U also satisfies (6.35). Hence U is the required anti-unitary operator.

It should be noted that all the arbitrariness in the choice of U is in the arbitrariness in the choice of the vector Φ' satisfying condition (6.51).

Exercise 6.29. Prove that if U and U' are two vectors satisfying the condition of Theorem 6.8, then they differ only by a phase factor: $U' = e^{i\alpha}U$ ($\alpha \in \mathbf{R}$).

This completes the proof of Wigner's theorem.

Exercise 6.30. Suppose that the bijection $\tau : \mathcal{P} \to \mathcal{P}$ satisfies the condition of Wigner's theorem (with $\mathcal{H} = \mathcal{H}'$) and let $\sigma = \tau \circ \tau$. Prove that the symmetry σ is unitarily induced, that is, that there exists a unitary operator V in $\mathcal{H}$ such that $\sigma(\Pi) = V\Pi V^{-1}$ for all $\Pi \in \mathcal{P}$. [Hint: For V one can take U^2, where U is defined by Theorem 6.8.]

As a corollary of Wigner's theorem, there is the following algebraic characterization of a symmetry.

Proposition 6.10. *Let (α, α') be a symmetry; then the map $\alpha : \mathfrak{A} \to \mathfrak{A}$ is a Jordan automorphism, that is, α is a linear bijection of $\mathfrak{A}$ onto itself, such that*

$$\alpha(A^2) = (\alpha(A))^2 \quad \textit{for all} \quad A \in \mathfrak{A}. \tag{6.53}$$

■ Since the linearity of α was established earlier, it remains to verify (6.53). Let π be an arbitrary irreducible representation of $\mathfrak{A}$ in a Hilbert space $\mathcal{H}$. By Proposition 6.2, PS_π is a sector in $PS(\mathfrak{A})$ the states of which are in one-to-one correspondence with the one-dimensional projectors in $\mathcal{H}$. Exercise 6.25 shows that α' maps the sector PS_π onto (possibly some other) sector $PS_{\pi'}$ preserving the transition probabilities, where π' is an irreducible representation of $\mathfrak{A}$ into a Hilbert space $\mathcal{H}'$. Consequently a bijection $\Pi \to \Pi'$ of the set of all one-dimensional projectors is defined. Since (according to the remark in §6.1.C) the transition probability between two states in PS_π (and similarly $PS_{\pi'}$) has the usual quantum-mechanical form (6.7), the bijection τ satisfies the condition of Wigner's theorem 6.8. Let U be the corresponding unitary or anti-unitary operator satisfying (6.35). Then for any $\omega \equiv \omega_\Pi \in PS_\pi$ and $A \in \mathfrak{A}$ we have

$$(\alpha'\omega)(\alpha A) = \mathrm{tr}(\Pi'\pi'(\alpha A)) = \mathrm{tr}(U\Pi U^{-1}\pi'(\alpha A)) = \mathrm{tr}(\Pi U^{-1}(\pi'(\alpha A))^{(*)}U) \tag{6.54}$$

(where the $(*)$ sign denotes the Hermitian adjoint if the operator U is anti-unitary and leaves $\pi'(\alpha A)$ unchanged, if U is unitary). On the other hand, the left hand side of (6.54) is $\omega(A) = \mathrm{tr}(\Pi\pi(A))$, from which we conclude (in view of the arbitrariness of Π) that

$$\pi(A) = U^{-1}\pi'(\alpha A)^{(*)}U,$$

or, equivalently,

$$\pi'(\alpha A) = U\pi(A)^{(*)}U^{-1}. \tag{6.55}$$

It follows easily from the latter equality that $\pi'[\alpha(A^2)] = \pi'[(\alpha A)^2]$. Going over to the states, we obtain $\omega'[\alpha(A^2)] = \omega'[(\alpha A)^2]$ for all $\omega' \in PS_\pi$ and hence (in view of the arbitrariness of π as well as π') for all $\omega \in PS(\mathfrak{A})$. Since the set of all pure states separates the elements of $\mathfrak{A}$, we now arrive at formula (6.53). ∎

In particular, by applying Proposition 6.10 to the elements $A+B$, A and B, we find that

$$\alpha(AB+BA) = \alpha(A)\alpha(B) + \alpha(B)\alpha(A). \tag{6.56}$$

In fact this property (together with (6.27) and the linearity of α) is often taken as the definition of a homomorphism of a Jordan C^*-algebra.

We recall that *a priori*, a physical meaning is only assigned to the Hermitian elements of the algebra of observables and that the set $\mathfrak{A}_h$ of these Hermitian elements has the structure of a Jordan algebra. In the light of this, the result of Proposition 6.10 appears very natural, since it shows that the symmetry reproduces the algebraic operations inherent in $\mathfrak{A}_h$. The class of automorphisms of a Jordan algebra $\mathfrak{A}$ is generally much more extensive than the class of algebraic automorphisms (when in addition to (6.27) and the linearity of α it is required that $\alpha(AB) \equiv \alpha(A)\alpha(B)$ for $A, B \in \mathfrak{A}$). This extensiveness has an important physical meaning. It is clear from elementary quantum-mechanics, where there are anti-unitarily induced symmetries (for example time reversal) for which the bijection α is an algebraic anti-automorphism (that is α is linear, $\alpha(A^*) \equiv \alpha(A)^*$ and $\alpha(AB) \equiv \alpha(B)\alpha(A)$)).* The algebraic automorphisms and algebraic anti-automorphisms are fully adequate for applications of the classes of Jordan automorphisms.**

Exercise 6.31. Suppose that the symmetry (α, α') is the square of a symmetry (β, β') (that is, $\alpha = \beta \cdot \beta$). Prove that α is then an algebraic automorphism. [Hint: Using Exercise 6.30, show that the operator U used in the proof of Proposition 6.10 is in fact unitary; it then follows from (6.55) that $\pi'[\alpha(AB)] = \pi'(\alpha A)\pi'(\alpha B)$ for all $A, B \in \mathfrak{A}$ and for any irreducible representation π'.]

So far we have primarily dealt with the behaviour of observables in states under a symmetry transformation. The problem of the transformation properties of the vectors of the physical Hilbert space is to determine the law of the (bijective) transformation $\Phi \to \Phi'$ of the vectors so that it be consistent with the transformation of the vector states:

$$\omega_{\Phi'} = \alpha'(\omega_\Phi), \;\; \Phi \in \mathcal{H}. \tag{6.57}$$

We then say that such a bijection $\Phi \to \Phi'$ is induced by the symmetry (α, α') as a transformation of vectors in $\mathcal{H}$ (or even simply that it is a symmetry). However, the compatibility condition is not enough to fix uniquely the transformation properties of the vectors in $\mathcal{H}$. The fact is that the parametrization of the (vector) states of the vectors of the Hilbert space is non-unique and in the case of non-standard superselection rules, this arbitrariness can be highly significant (in §6.2.B we interpreted this

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*In this situation some authors use for $\alpha(A)$ what we have been calling $\alpha(A)^*$. The bijection α then turns out to be antilinear and $\alpha(AB) = \alpha(A)\alpha(B)$. The transformation properties of the (Hermitian) observables then remain the same as before, so that both points of view are physically equivalent.

** Strictly speaking, the confinement of the discussion of symmetries to automorphisms or anti-automorphisms is an additional hypothesis. *A priori* one can conceive of a symmetry that is represented by a unitary operator in one sector and by an anti-unitary operator in another. In such a case there would be nothing that we could add to Proposition 6.10 that would guarantee that α would be a Jordan automorphism. In what follows, however, we shall see that for the physically interesting symmetries, other postulates of the theory are called upon (such as the condition that the energy be positive) which exclude such "mixed representations". Therefore, to avoid further complications, we shall henceforth consider (where required) only symmetries induced by algebraic (anti-)automorphisms.

as unobservable degrees of freedom). Here we shall confine ourselves to the case of standard superselection rules by assuming that the symmetry α' leaves invariant the set of states $\mathfrak{S}$ associated with the physical representation of $\mathfrak{A}$ (see Postulate A.II).*

In particular, when the entire space $\mathcal{H}$ is coherent (which corresponds to the quantum-mechanical system of a fixed number and type of particles), Wigner's theorem enables us to define the law of transformation of vectors in the form

$$\Phi' = U\Phi, \tag{6.58}$$

where U is a unitary or anti-unitary operator defined to within a phase factor: the transformation

$$U \to U' = \chi U, \tag{6.59}$$

where $\chi \in \mathbf{C}$, $|\chi| = 1$, clearly does not change the (anti-)automorphism of $\mathfrak{A}$.

Proposition 6.11. *Suppose that the physical representation of the algebra of observables $\mathfrak{A}$ in the Hilbert space $\mathcal{H}$ has the standard superselection rules, and that the algebraic automorphism (or anti-automorphism) α of $\mathfrak{A}$ defines a symmetry (α, α') that leaves the set of physical states $\mathfrak{S}$ invariant. Then this symmetry is induced by a unitary (or anti-unitary) operator U in $\mathcal{H}$ and there exists a bijection $\nu \to \nu'$ of the index set N (enumerating the coherent subspaces of $\mathcal{H}$) onto itself such that U maps the coherent subspace $\mathcal{H}_\nu$ unitarily (or anti-unitarily) onto $\mathcal{H}_{\nu'}$. The operator U is defined by these conditions to within an arbitrary unitary superselection operator, that is, instead of U we can equally well take* (6.59), *where χ is an arbitrary unitary superselection operator.*

■ We shall confine ourselves to the case when α is an algebraic automorphism. Let S_ν be the sector of pure states defined by the vectors of the coherent subspace $\mathcal{H}_\nu$. According to Exercise 6.25, $\alpha'(S_\nu)$ is also a sector in $PS(\mathfrak{A})$. Furthermore, by definition, $\alpha'(S_\nu)$ is contained in $\mathfrak{S}$, the set of states defined by the density matrices in $\mathcal{H}$. It follows that $\alpha'(S_\nu) = S_{\nu'}$ for some index ν'. Arguing now as in the proof of Proposition 6.10 we arrive at the relation of type (6.55) for all $A \in \mathfrak{A}$:

$$\pi_{\nu'}(\alpha A) = U_\nu \pi_\nu(A)^{(*)} U_\nu^{-1}; \tag{6.60}$$

where $U_\nu : \mathcal{H}_\nu \to \mathcal{H}_{\nu'}$ is a unitary or anti-unitary operator. In fact, U_ν can be taken to be unitary. In the case $\dim \mathcal{H}_\nu = 1$ this is obvious (see the footnote to Theorem 6.8). In the case $\dim \mathcal{H}_\nu > 1$ we have: $\pi_\nu(\mathfrak{A})^{cc} = \mathcal{B}(\mathcal{H}_\nu)$ is a non-commutative algebra, therefore $\pi_\nu(\mathfrak{A})$ is also non-commutative (see Exercise 1.67); but then, if U_ν were anti-unitary, this would be inconsistent with (6.60) since by hypothesis α is an algebraic automorphism. For the operator U we are seeking we can now take $U = \sum_\nu U_\nu \Pi_\nu$, where Π is the projector from $\mathcal{H}$ onto $\mathcal{H}_\nu$. The arbitrariness in the construction of U follows from the structure of the commutant of $\mathfrak{A}$. (Namely, every bounded operator in $\mathcal{H}$ commuting with $\mathfrak{A}$ has the form $\sum_\nu a_\nu \Pi_\nu$, $a_\nu \in \mathbf{C}$.) ■

We see that the arbitrariness in the choice of the operator U in the case of standard superselection rules reduces to a family $\{\chi_\nu\}_{\nu \in N}$ of phase factors (one for each coherent subspace). We must emphasize, however, that Proposition 6.11 only provides a preliminary solution to the question of the transformation properties of the vectors of the states, since it does not take into account the additional structures of the theory (for example, the local structure of quantum theory) which are in a position to restrict substantially the arbitrariness in the choice of the operator U.

C. SYMMETRY GROUPS

A group G is called a *symmetry group* (or a *group of invariants*) of the algebra of observables $\mathfrak{A}$ if a homomorphism $g \to (\alpha_g, \alpha'_g)$ is defined from G to the group of all

* In the case when α' takes $\mathfrak{S}$ outside this class, an algebraic symmetry cannot be realized by an (anti-)unitary operator in the Hilbert space $\mathcal{H}_{\mathrm{phys}}$ (and we have spontaneous breakdown of symmetry on our hands).

symmetries of the system $(\mathfrak{A}, S(\mathfrak{A}))$. Of particular interest is the case when G is a Lie group. In this case we shall assume the following continuity condition: for any physical state $\omega \in \mathfrak{S}$ and any fixed $A \in \mathfrak{A}$, the function $g \to \omega(\alpha_g(A))$ is continuous in g.

In particular, if α'_g leaves $\mathfrak{S}$ invariant for all $g \in G$ we talk about the symmetry group of the system $(\mathfrak{A}, \mathfrak{S})$. A state $\omega \in \mathfrak{S}$ is called ***G-invariant*** if $\alpha'_g(\omega) = \omega$ (or, equivalently, if $\omega(\alpha_g(A)) = \omega(A))$ for all $g \in G$.

We note that it is no haphazard choice that we restrict the state to $\mathfrak{S}$ in imposing the continuity condition. This condition would be too strong for the set of all states $S(\mathfrak{A})$. This demonstrates the need to restrict the set of all possible states in describing a given physical system.

Our interest in considering concrete symmetry groups is maintained by the fact that such groups quite often reflect fundamental physical ideas. Thus, the non-relativistic ideas on space-time dictate invariance with respect to the Galilei group, whereas the homogeneity and isotropy of (Minkowski) relativistic space-time finds its expression in invariance with respect to the Poincaré group. Nevertheless, an important structural role is played in the theory by the so-called internal symmetries (not directly related to space-time).

Exercise 6.32. Let $\mathfrak{A}$ be the algebra of observables realized by the operators in $\mathcal{B}(\mathcal{H})$; as usual, $\mathfrak{S}$ is the set of states associated with the density matrices in $\mathcal{H}$. Prove that the continuity condition presupposed in the definition of the symmetry groups is equivalent to the requirement that for any fixed $A \in \mathfrak{A}$, the map $g \to \alpha_g(A)$ be continuous with respect to $g \in G$ in the weak (or, equivalently, in the strong) operator topology in $\mathcal{B}(\mathcal{H})$. [Hint: In the proof of the original continuity condition, use the fact that any state in $\mathfrak{S}$ is finite or a countable convex linear combination of state vectors and therefore can be approximated in norm to any desired degree of accuracy by a finite linear combination of vector states. For the proof of the continuity in the weak operator topology, use the fact that every functional $A \to (\Phi, A\Psi)$ on $\mathfrak{A}$ is a linear combination of vector states. Finally, for the proof of the continuity in the strong operator topology it suffices to pass to the limit $g \to e$ in the formula $\|\alpha_g(A)\Phi - \Phi\|^2 = (\Phi, \alpha_g(A^*A)\Phi) - (\alpha_g(A)\Phi, A\Phi) - (A\Phi, \alpha_g(A)\Phi) + (\Phi, A^*A\Phi)$.]

If the symmetry group G is connected, then without loss of generality we can always suppose it to be simply connected; otherwise G can be replaced by the universal covering group $\tilde{G}$ and we can set

$$\beta_{\tilde{g}} = \alpha_{\phi(\tilde{g})} \quad \text{for } \tilde{g} \in \tilde{G}, \tag{6.61}$$

where ϕ is the covering homomorphism. Such a replacement is very convenient in a number of questions (for example in the question of the integrability of representations of the Lie algebra of a group). We note that the symmetry group $\tilde{G}$ obtained in this way has the property:

$$\beta_{\tilde{g}} = \mathrm{id} \quad \text{if } \phi(\tilde{g}) = e, \tag{6.62}$$

where id is the identity transformation. As the next exercise shows, this property enables us to revert to the symmetry group G.

Exercise 6.33. Let ϕ be a homomorphism from the group $\tilde{G}$ onto the group G (here we are not assuming that $\tilde{G}$ is the universal covering group for G). Suppose that $\tilde{G}$ is a symmetry group and that (6.62) holds. Show that (6.61) well-defines G as a symmetry group.

Exercise 6.34. Let G be a connected Lie group of symmetries. Prove that for all $g \in G$, α_g is an algebraic automorphism of the algebra of observables. [Hint: Use Exercises 6.33 and D.7.]

In practice we confine ourselves to the most important class of symmetries when for any element $g \in G$, α_g is either an algebraic automorphism or an algebraic anti-automorphism; in what follows this condition will always be assumed to hold. (According to the preceding exercise, the case of an anti-automorphism can only occur for elements g that are not in the connected component of the identity of G.)

We say that a symmetry group G is unitarily/anti-unitarily realized if there exists a continuous representation $g \to U_g$ of G (in the weak, or equivalently, in the strong operator topology) by unitary or anti-unitary operators (according to whether α_g is an algebraic automorphism or an anti-automorphism) in the physical Hilbert space $\mathcal{H}$, such that for all $A \in \mathfrak{A}$, $g \in G$ we have

$$\alpha_g(A) = U_g A^{(*)} U_g^{-1}, \tag{6.63}$$

where $A^{(*)}$ stands for A if U_g is unitary and for A^* if U_g is anti-unitary. It is clear that the operators U_g are unitary for all g in the connected component of the identity of G (see Exercise 6.34.) We now turn our attention to the question whether the property of a symmetry group G being unitarily/anti-unitarily realized depends on the choice of the (physical) representation π of the algebra of observables. If a physical representation has this property, then for all elements g, at least in the connected component of the identity of G, the correspondence $A \to \alpha_g(A)$ defines a representation of $\mathfrak{A}$ in $\mathcal{H}$ that is phenomenologically equivalent to the original representation π. In this sense, we can talk about the invariance of the physical representation itself with respect to a given symmetry group. The value of a unitary realization of a (connected) symmetry group is that the generators of the representation as a rule allow a physical interpretation by attaching the meaning of self-adjoint operators to quantities that existed hitherto only in the heuristic sense (say, within the framework of the canonical formalism of quantum field theory). It therefore makes sense to require further that the generators of a unitary realization U of a connected symmetry group be self-adjoint operators in $\mathcal{H}$ that are affiliated to the von Neumann algebra of observables $\bar{\mathfrak{A}}$ or, what is the same, that the operators of the representation U of the connected symmetry group belong to $\bar{\mathfrak{A}}$. (This condition will be proved, under extra restrictions, in Propositions 6.12 and 6.13 below.)

The GNS construction provides the most effective criterion for judging whether a symmetry group is unitarily/anti-unitarily realized. True this criterion has its greatest significance in another context, namely, applied to field algebras rather than algebras of observables (see §10.3), since in that context the condition of cyclicity of an invariant state vector is more appropriate. For this reason we shall state this result in a somewhat more general form. As before, we call G a symmetry group of a C^*-algebra $\mathfrak{A}$ of operators in the Hilbert space $\mathcal{H}$ if we are given a homomorphism $g \to \alpha_g$ of G into the group of automorphisms or anti-automorphisms of $\mathfrak{A}$, where $\alpha_g(A)$ is continuous in g in the weak (or strong) operator topology for each fixed $A \in \mathfrak{A}$.

Proposition 6.12. *Let G be a symmetry group of the C^*-algebra $\mathfrak{A}$ of operators in Hilbert space $\mathcal{H}$, and Ω a cyclic vector of $\mathfrak{A}$ defining a G-invariant state. Then the symmetry group G is realized by a continuous (in the weak or strong operator topology) unitary/anti-unitary representation $g \to U_g$ of G in $\mathcal{H}$ which along with* (6.63) *has the property*

$$U_g\Omega = \Omega \quad \text{for all } g \in G. \tag{6.64}$$

The representation U with these properties is unique.

Exercise 6.35. Prove Proposition 6.12. [Hint: U_g is to be defined by the relation $U_g A\Omega = \alpha_g(A)\Omega$ for all $A \in \mathfrak{A}$; cf. Exercise 1.57.]

We consider the problem of the unitary realization of a connected symmetry group within the framework of the standard superselection rules hypothesis. According to Proposition 6.11, each individual element $g \in G$ can be associated with a unitary operator U_g satisfying (6.63). Therefore the problem can be replaced by two more concrete questions: firstly, does U_g depend continuously on g (say, in the strong operator topology); secondly, is the correspondence $g \to U_g$ a representation of G? In order to ensure that U_g be continuous in g, we have to strengthen the continuity requirement on the symmetry group by comparison with what we presupposed in the definition of a symmetry group. This will become clear from the following exercise.

Exercise 6.36. Prove that if the symmetry group G is unitarily/anti-unitarily realized, then $\alpha'_g(\omega)$ depends continuously on g (for any fixed $\omega \in \mathfrak{S}$) in the norm topology on $\mathfrak{S}$. [Hint: Any state of $\mathfrak{S}$ can be approximated to any desired degree of accuracy by a finite convex linear combination of state vectors, and for state vectors we have $\|\alpha'_g(\omega_\Phi) - \omega_\Phi\| \leq 2\|\Phi\| \cdot \|U_g\Phi - \Phi\|$.]

Exercise 6.37. Let G be a connected symmetry group in which $\alpha'_g(\omega)$ depends continuously on g in the norm topology on $\mathfrak{S}$ (for fixed $\omega \in \mathfrak{S}$). Prove that for any $g \in G$ the bijection α'_g maps any sector $S \subset \mathfrak{S}$ onto itself. [Hint: Deduce from the preceding exercise that for any fixed $\omega \in S$ there exists a neighbourhood $\mathcal{O}$ of the identity of G such that $\omega \cdot \alpha'_g(\omega) \neq 0$ for all $g \in \mathcal{O}$; hence ω and $\alpha'_g(\omega)$ are in the same sector; now use the fact that an arbitrary element of G is the product of a finite number of elements of $\mathcal{O}$.]

It follows from the last exercise that the continuity condition featuring there guarantees that each sector is left invariant under the symmetries of a connected group G. Consequently the question of the unitary realization of a connected symmetry group can be restated in terms of the coherent subspaces $\mathcal{H}_\nu$ taken individually. That is to say, corresponding to each map $\alpha'_g : \mathfrak{S} \to \mathfrak{S}$ is a transition-probability preserving bijection from the set of all unit rays in $\mathcal{H}_\nu$ (or what is the same, a bijection $\tau_\nu(g) : \mathcal{P}_n \to \mathcal{P}_\nu$ of the set of all one-dimensional projectors (in $\mathcal{H}_\nu$) onto itself that preserves the quantities $\mathrm{tr}(\Pi_1\Pi_2)$). Furthermore, we have the group property: $\tau_\nu(e)$ is the identity map and $\tau_\nu(g_1g_2) = \tau_\nu(g_1) \circ \tau_\nu(g_2)$. Also the continuity condition holds: $\tau(\nu(g)(\Pi)$ is continuous in g for any fixed Π, where $\mathcal{P}_\nu$ is endowed with the metric induced from $PS(\mathfrak{A})$, that is,

$$d(\Pi_1, \Pi_2) = \|\omega_{\Pi_1} - \omega_{\Pi_2}\| = 4(1 - \mathrm{tr}(\Pi_1\Pi_2)) \tag{6.65}$$

(see Exercise 6.4 and formula (6.4)). In this case we say that $\mathcal{H}_\nu$ is defined by a *unitary projective representation* τ_ν of the connected group G. Of interest to us is whether τ_ν has a unitary realization, that is, can τ_ν be represented in the form

$$\tau_\nu(g)(\Pi) = U_\nu(g)\Pi U_\nu(g)^{-1}, \tag{6.66}$$

where $U_\nu(g)$ is a continuous representation of G (in the weak operator topology)? This so-called lifting problem of projective representations has been studied in full generality by Bargmann (1954). In the general case, the answer to this question turns out to be negative. Nevertheless, for certain important classes of groups it can be solved affirmatively at the cost of replacing the given group by some covering group G_1 (we can "as a last resort" use the universal covering group $\tilde{G}$ for G_1). We shall confine our discussion to several typical examples (the proofs of which can be found in Bargmann's article).

Examples. 1) Let G be the additive group $\mathbf{R}$ of real numbers; in this case G_1 is also $\mathbf{R}$.

2) Let G be the connected compact abelian group tor $U(1)^n \equiv U(1) \times \ldots \times U(1)$; in this case we can take G for G_1.

3) Let G be a connected compact Lie group; in this case there exists a compact covering group G_1 for G that is isomorphic to $U(1)^n \times G'$, where G' is a connected compact semisimple Lie group.

4) Let G be a connected semisimple Lie group; then we must take the universal covering group $\tilde{G}$ for G_1. (Examples of such groups G are the rotation group $O_+(3)$ and the proper Lorentz group $L_+^\uparrow$; then $\tilde{G}$ is $SU(2)$ and $SL(2,C)$ respectively.)

5) Let G be the proper Poincaré group $\mathfrak{P}_+^\uparrow$ (see §7.1.A concerning this); then for G_1 we must take the universal covering group $\mathfrak{P}_0$ for $\mathfrak{P}_+^\uparrow$

Proposition 6.13. *Suppose that the connected Lie group G is (at least) one of the above types 1)–5) and let G_1 be the corresponding covering group for G. Suppose that the system $(\mathfrak{A}, \mathfrak{S})$ with standard superselection rules possesses a symmetry with respect to G (and, hence, with respect to G_1), where $\alpha'_g(\omega)$ is continuous in g in the norm topology (for any $\omega \in \mathfrak{S}$). Then this symmetry of G_1 is realized by unitary operators $U(g)$ $(g \in G_1)$ belonging to the von Neumann algebra of observables $\bar{\mathfrak{A}}$ and forming a representation of G_1. For elements g in the kernel of the covering homomorphism $G_1 \to G$, the operators $U(g)$ belong to the centre $\mathfrak{z}$ of $\bar{\mathfrak{A}}$.*

A physically important example of a Lie group for which Proposition 6.13 is not applicable is the (10-dimensional) Galilei group of symmetries of non-relativistic quantum mechanics. In this connection, there arises in non-relativistic quantum mechanics a special superselection rule with respect to the (total) mass. (A discussion of this question can also be found in Bargmann's article.)

6.4. Canonical Commutation Relations

A. THE ROLE OF THE SCHRÖDINGER REPRESENTATION

Among the systems playing a primary role in quantum theory are those provided by the canonical commutation relations (CCR). Here we take up the questions on the manner in which the algebra of observables of a canonical system is defined by the CCR's and why in the case of a finite number of degrees of freedom (after laying down the precise details concerning the kinematics), there is essentially a unique physically interesting representation of it. (By contrast, in quantum field theory, different inequivalent representations of the CCR's participate.)

As we have already remarked, in the usual formulation of quantum mechanics with n degrees of freedom, all the physical quantities are functions of the position coordinates $q \equiv (q_1, \ldots, q_n)$ and momenta $p \equiv (p_1, \ldots, p_n)$ which are Hermitian (more precisely, essentially self-adjoint) operators in Hilbert space $\mathcal{H}$ satisfying the CCR's*

$$[q_j, q_k] = 0 = [p_j, p_k], \; [q_j, p_k] = i\delta_{jk} \tag{6.67}$$

(here and in what follows $[A, B] \equiv AB - BA$ is the commutator of two operators). Each such set of operators p and q is also called a *representation of the CCR's*.

* In the standard system of units used in quantum mechanics one must put $i\hbar\delta_{jk}$ in (6.67) instead of $i\delta_{jk}$ where $\hbar$ is Planck's constant divided by 2π.

Exercise 6.38. Prove that the Hermitian operators p_j and q_j satisfying the CCR's cannot be simultaneously bounded. [Hint: Assuming that all these operators are bounded, deduce from (6.67) that for all $a \in \mathbf{R}^n$ the following relation holds:

$$[q_j, e^{ipa}] = -a_j e^{ipa}, \quad pa \equiv \sum_{j=1}^{n} p_j a_j; \tag{6.68}$$

now estimate the left and right hand sides in norm and let a_j tend to infinity.]

Since unbounded operators enter into (6.67), these relations are not well defined. To overcome this difficulty we must go over to bounded operators of p and q. We introduce the two n-parameter abelian groups of unitary operators

$$U(a) = e^{iap}, \; V(b) = e^{ibq}, \tag{6.69}$$

where $ap \equiv \sum_{j=1}^{n} a_j p_j$, $bq \equiv \sum_{j=1}^{n} b_j q_j$. The abelian property of the groups $U(a)$ (and of $V(b)$) is equivalent to the previous condition of commutativity of the operators $p_1, \dots, p_n$ (and of $q_1, \dots, q_n$):

$$U(a)U(a') = U(a + a'), \; V(b)V(b') = V(b + b'). \tag{6.70}$$

The remaining commutation relations between p and q can be expressed thus:

$$U(a)V(b) = e^{iab}V(b)U(a). \tag{6.71}$$

At the heuristic level, (6.71) follows from (6.67) by manipulating with series (cf. Exercise 6.38). But in fact, (6.70), (6.71) cannot be deduced from (6.67) without further "technical" assumptions (which are of no interest to us here).

We consider the traditional representation of the CCR's in the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^n)$ of complex square-integrable functions Ψ of n real variables $q \equiv (q_1, \dots, q_n)$. We define the operator q_j as the operator of multiplication by the variable q_j and set $p_j = -i\frac{\partial}{\partial q_j}$. For the domains of definition of the operators q_j and p_j we can take the Schwartz space $\mathcal{S}(\mathbf{R}^n) \subset \mathcal{L}^2(\mathbf{R}^n)$.

Exercise 6.39. Prove that the operators q_j and p_j in $\mathcal{L}^2(\mathbf{R}^n)$ are essentially self-adjoint on $\mathcal{S}(\mathbf{R}^n)$. [Hint: Verify that q_j and p_j are symmetric and then prove that $\mathcal{D}(\mathbf{R}^n)$ is a domain of analyticity for q_j, while the set of functions of $\mathcal{S}(\mathbf{R}^n)$, whose Fourier transforms belong to $\mathcal{D}(\mathbf{R}^n)$ forms a domain of analyticity for p_j.]

It is clear that these operators q and p form a representation of the CCR's, called the *Schrödinger representation.* They are interpreted as the coordinate and momentum variables of a pointwise quantum system in Euclidean n-space (or a system of N quantum particles in ordinary three-dimensional space, if $n = 3N$). In this case the operators $U(a)$ and $V(b)$ act in $\mathcal{L}^2(\mathbf{R}^n)$ as follows:

$$(U(a)\Phi)(q) = \Phi(q + a), \tag{6.72}$$
$$(V(b)\Phi)(q) = e^{iqb}\Phi(q). \tag{6.73}$$

It is easily checked that the relations (6.70), (6.71) hold.

In accordance with what we have said above, we adopt the following definition as the correct formulation of the CCR's dealing only with bounded operators. By a *representation of the CCR's in exponential form* for a system with n degrees of freedom, we mean a pair of abelian n-parameter groups $U(a)$, $V(b)$ of unitary operators in Hilbert space $\mathcal{H}$ satisfying (6.70), (6.71) for all $a, b \in \mathbf{R}^n$. (We recall that according to §1.4.C, the definition of such groups includes the requirement that they be continuous with respect to the parameters in the weak operator topology.) A representation of the CCR's in exponential form is also called a *Weyl system* (in $\mathcal{H}$). If we transfer from the operators $U(a)$, $V(b)$ to coordinates and momenta as afforded by Stone's theorem 1.17, we can write $U(a)$, $V(b)$ in the form (6.69).

As usual, if there is a second system $U'(a)$, $V'(b)$ of the CCR's in the Hilbert space $\mathcal{H}'$ which is related to the original system by the formulae $U'(a) = SU(a)S^{-1}$, $V'(b) = SV(b)V^{-1}$, where $S : \mathcal{H}' \to \mathcal{H}$ is a unitary operator, then these two representations of the CCR's are said to be unitarily equivalent. A Weyl system $U(a)$, $V(b)$ in $\mathcal{H}$ is said to be cyclic with cyclic vector $\Phi \in \mathcal{H}$ if the vectors of the form $U(a)V(b)\Phi$ for all $a, b \in \mathbf{R}^n$ form a total subset of $\mathcal{H}$. We give another standard definition of irreducibility of a Weyl system $U(a)$, $V(b)$ in $\mathcal{H}$; it reduces to the requirement that there should not be any non-trivial closed subspaces that are invariant with respect to all the operators $U(a)$, $V(b)$. (Conversely, if $\mathcal{H}$ has such an invariant subspace $\mathcal{H}'$, then its orthocomplement is also invariant and the original representation of the CCR's decomposes into a direct sum of two representations.)

Exercise 6.40. Prove that the condition of irreducibility of a Weyl system $U(a)$, $V(b)$ in $\mathcal{H}$ is equivalent to each of the following two conditions: (1) every non-zero vector in $\mathcal{H}$ is cyclic; (2) every bounded operator in $\mathcal{H}$ that commutes with all the operators $U(a)$, $V(b)$ is a multiple of the identity operator. [Hint: The arguments of Exercise 1.54 are applicable here.]

Exercise 6.41. Prove that the Schrödinger representation is irreducible. [Hint: Let Φ be an arbitrary fixed non-zero vector in $\mathcal{L}^2(\mathbf{R}^n)$, and $\Psi \in \mathcal{L}^2(\mathbf{R}^n)$ a function that is orthogonal to all functions of the form $U(a)V(b)\Phi$. It is required to prove that $\Psi = 0$. For this it suffices to verify that $\overline{\Psi(x+a)}\Phi(x) = 0$ in $\mathcal{L}^1(\mathbf{R}^n)$ for all $a \in \mathbf{R}^n$, whence it follows that $\overline{\Psi(x+y)}\Phi(x) = 0$ in $S'(\mathbf{R}^{2n})$ and hence that $\Psi = 0$.]

There is a remarkable theorem due to von Neumann which states that to within unitary equivalence and multiplicity, there exists a unique Weyl system of the CCR's for a system with n degrees of freedom.

Theorem 6.14 (von Neumann uniqueness theorem). *Each irreducible Weyl system with n degrees of freedom is unitarily equivalent to the Schrödinger representation in $\mathcal{L}^2(\mathbf{R}^n)$. Every reducible Weyl system (with n degrees of freedom) is a direct sum of irreducible representations and hence is a multiple of the Schrödinger representation.*

We give a proof of this theorem in §6.4.C. It is of great significance in quantum mechanics, since it allows us to confine attention to a single irreducible representation of the CCR's, say, the Schrödinger representation. It finds a second application in the construction of the abstract algebra of observables for a CCR system (with a finite or infinite number of degrees of freedom); thus it provides the possibility of including such systems into the general algebraic scheme. We shall return to this question in §6.4.C.

Representations of the CCR's in the Weyl form enable us in advance to avoid the possibility that one of the operators q or p is unbounded. In order to convince ourselves of this and to see what sort of a physical problem this situation corresponds to, we consider the motion of a particle around a circle. In this case the coordinate q varies over an interval of length 2π and a representation of Schrödinger type can be realized in the Hilbert space $\mathcal{H} = \mathcal{L}^2([-\pi, \pi])$. The momentum p is defined as the differentiation operator $p = -i\frac{d}{dq}$, the domain of which is the set D_p of absolutely continuous functions $\Psi(q)$ with square-integrable first derivative and satisfying the periodicity condition $\Psi(-\pi) = \Psi(\pi)$. It is easy to see that Ψ has a simple discrete spectrum, $p_n = n$ $(= 0, \pm 1, \pm 2, \ldots)$ (the corresponding normalized eigenfunctions are $\Psi_n = (2\pi)^{-1/2}e^{inq}$). The Weyl commutation relations

$$e^{iqb}e^{ipa}e^{-iqb} = e^{i(p-b)a}$$

can hold only for integral b, since otherwise the spectrum of the operator e^{ipb} would be altered (which is impossible for a unitary transformation). This contradicts the hypothesis that there exists a continuous group of unitary operators $V(b)$.

Apart from this, the motion of a particle round a circle has other peculiarities from the point of view of ordinary quantum mechanics. Thus we consider the Heisenberg inequality (or "uncertainty relation"):

$$(\Delta p)^2_\Phi (\Delta q)^2_\Phi \geq 1/4. \tag{6.74}$$

Here $(\Delta p)^2_\Phi = \langle \Phi, (p - \bar{p}_\Phi)^2 \Phi \rangle$, $(\Delta q)^2_\Phi = \langle \Phi, (q - \bar{q}_\Phi)^2 \Phi \rangle$ are the variances of the momentum and coordinate in the state defined by the unit vector Φ; $\bar{p}_\Phi = \langle \Phi, p\Phi \rangle$ and $\bar{q}_\Phi = \langle \Phi, q\Phi \rangle$ are the mean values of these quantities. If we write the variances in the form $(\Delta p)^2_\Phi = \|(p - \bar{p}_\Phi)\Phi\|^2$ and $(\Delta q)^2_\Phi = \|(q - \bar{q}_\Phi)\Phi\|^2$, we see that the quantities occurring in the uncertainty relation are well defined (or "finite") provided only that Φ belongs simultaneously to the domains of the closures of the operators p and q.

Exercise 6.42. Prove the uncertainty relation for the Schrödinger representation of the CCR's in $\mathcal{L}^2(\mathbf{R})$. [Hint: Prove this relation first for any unit vector $\Phi \in \mathcal{S}(\mathbf{R})$ by using the inequality

$$\|[\alpha(q - \bar{q}_\Phi) + i(p - \bar{p}_\Phi)]\Phi\|^2 \geq 0,$$

which holds for all real α. Then verify that for any unit vector Φ for which the uncertainty relation is meaningful, one can choose a sequence of vectors of the form $\Phi_n(x) = (\omega_n(x)\Phi(x)) * \chi_n(x)$, where ω_n and χ_n are suitable functions in $\mathcal{D}(\mathbf{R})$ such that $\|\Phi - \Phi_n\| \to 0$, $(\Delta p)^2_{\Phi_n} \to (\Delta p)^2_\Phi$, $(\Delta q)^2_{\Phi_n} \to (\Delta q)^2_\Phi$.]

One usually draws the conclusion from the result of this exercise that the Heisenberg inequality is universally applicable. Such a conclusion is in fact justified when we are dealing with the usual Schrödinger representation corresponding to motion along a line. But it is easy to see that for the case of the motion of a point round a circle, the uncertainty relation can break down, since the variance in the coordinate cannot exceed $(2\pi)^2$, whereas the variance in the momentum can be arbitrarily small. In particular,

$$(\Delta q)^2_\Phi (\Delta p)^2_\Phi = 0$$

for any eigenvector $\Phi = \Psi_n$ of p. (The product of operators pq is undefined at such a vector, therefore the standard proof of the uncertainty relation becomes inapplicable.)

B. Infinite Number of Degrees of Freedom

The Weyl system $U(a)$, $V(b)$ is constructed on a basis of n-dimensional Euclidean space (since $a, b \in \mathbf{R}^n$). With the aim of generalizing to the case of an arbitrary number of degrees of freedom, we start from a real vector space $\mathcal{E}$ in which a scalar product (f, g) is defined. We take this scalar product to be positive definite but we do not assume that $\mathcal{E}$ is complete with respect to the norm $\|f\| = (f, f)^{1/2}$. (In the context of field theory, the elements f of $\mathcal{E}$ play the role of test functions which smooth the field operators with respect to the spatial variables.) The dimension of $\mathcal{E}$ will be interpreted as the number of degrees of freedom of the canonical system. Here we are interested in the case of infinite-dimensional $\mathcal{E}$.

By a *CCR representation* (or *Weyl system*) *over* $\mathcal{E}$ we mean a pair of maps that associate with each vector $f \in \mathcal{E}$ unitary operators $U(f)$ and $V(f)$ in the Hilbert space $\mathcal{H}$ that satisfy the following relations (for all $f, g \in \mathcal{E}$)

$$U(f)U(g) = U(f+g), \quad V(f)V(g) = V(f+g), \tag{6.75a}$$

$$U(f)V(g) = e^{i(f,g)}V(g)U(f) \tag{6.75b}$$

and are continuous in the weak operator topology when f runs through an arbitrary finite-dimensional subspace of $\mathcal{E}$. In what follows, we shall only use this minimal requirement of continuity, which does not even require a topology on $\mathcal{E}$. (Usually $\mathcal{E}$ can be given a natural LCS topology, in which case the continuity is accordingly strengthened in some reasonable fashion.)

We consider as an example the CCR's in quantum field theory, restricting ourselves for simplicity to the case of a real scalar field. In this case the field operators $\phi(\mathbf{x})$ and the conjugate momenta $\pi(\mathbf{x})$ (where $\mathbf{x} \equiv (x^1, x^2, x^3) \in \mathbf{R}^3$) define a set of fundamental dynamical variables at a fixed moment of time. They are characterized by the CCR's

$$[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0 = [\pi(\mathbf{x}), \pi(\mathbf{y})], \tag{6.76a}$$

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \tag{6.76b}$$

and the Hermitian property

$$\phi^*(\mathbf{x}) = \phi(\mathbf{x}), \quad \pi^*(\mathbf{x}) = \pi(\mathbf{x}). \tag{6.77}$$

It is clear from the singular structure of the commutator that $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ are in fact operator-valued generalized functions of $\mathbf{x}$. We take as the space $\mathcal{E}$ of test functions, the space $\mathcal{S}_r(\mathbf{R}^3)$ of real functions of $\mathcal{S}(\mathbf{R}^3)$ with the usual scalar product

$$(f,g) = \int f(\mathbf{x})g(\mathbf{x})d^3x. \tag{6.78}$$

Let

$$\phi(f) = \int \phi(\mathbf{x})f(\mathbf{x})d^3x, \quad \pi(f) = \int \pi(\mathbf{x})f(\mathbf{x})d^3x. \tag{6.79}$$

Then the above conditions imply that $\phi(f)$ and $\pi(f)$ are essentially self-adjoint operators (on some domain) in Hilbert space $\mathcal{H}$ such that

$$[\phi(f), \phi(g)] = 0 = [\pi(f), \pi(g)], \tag{6.80a}$$

$$[\pi(f), \phi(g)] = -i(f,g). \tag{6.80b}$$

It goes without saying that the inadequacies, mentioned in the preceding subsection, of the formulation of the CCR's in terms of unbounded operators persist here as well. The correct formulation of the CCR's is achieved by the relations (6.75) for the operators $U(f)$, $V(f)$ related to $\phi(f)$ and $\pi(f)$ by the formulae

$$U(f) = e^{i\pi(f)}, \quad V(f) = e^{i\phi(f)}. \tag{6.81}$$

It is convenient to work with the complex version of the CCR's. To this end we introduce the complexification $\mathcal{E}_c \equiv \mathcal{E} + i\mathcal{E}$ of the space $\mathcal{E}$. We write the elements F of $\mathcal{E}_c$ in the form $F = f + ig$ $(f, g \in \mathcal{E})$ and define the scalar product in $\mathcal{E}_c$ by setting

$$\langle F_1, F_2 \rangle \equiv (f_1 + ig_1, f_2 + ig_2) = (f_1, f_2) + (g_1, g_2) + i(f_1, g_2) - i(f_2, g_1).$$

(In particular, the complexification of $\mathcal{S}_r(\mathbf{R}^3)$ is $\mathcal{S}(\mathbf{R}^3)$ with the scalar product inherited from $\mathcal{L}^2(\mathbf{R}^3)$.) We introduce the operator (for any $F \in \mathcal{E}_c$)

$$W(F) \equiv W(f + ig) = e^{i(f,g)}V(\sqrt{2}f)U(\sqrt{2}g). \tag{6.82}$$

It is clear that the operators $W(F)$ are unitary, satisfy the condition

$$W(F)^* = W(-F) \tag{6.83}$$

and are continuous in F in the weak operator topology when F runs through an arbitrary finite-dimensional subspace of $\mathcal{E}_c$. The commutation relations (6.75) are now written in the form of the single relation (satisfied for all F_1, F_2 in $\mathcal{E}_c$):

$$W(F_1)W(F_2) = \exp(-i\,\mathrm{Im}\langle F_1, F_2\rangle)W(F_1 + F_2), \tag{6.84}$$

We call the family of operators $W(F)$ with the properties indicated above, a system of CCR's in complex form (over $\mathcal{E}_c$ in the Hilbert space $\mathcal{H}$).

Exercise 6.43. (a) Verify the equivalence of (6.75) and (6.84).

(b) Prove that for the Weyl system with n degrees of freedom, the following relations hold

$$\begin{aligned} W(z) \equiv W(x+iy) &= e^{ixy}e^{i\sqrt{2}xq}e^{i\sqrt{2}yp} = e^{i\sqrt{2}(xq+yp)} = \\ &= e^{za^*-\bar{z}a} = e^{-\frac{1}{2}\bar{z}z}e^{za^*}e^{-\bar{z}a}, \end{aligned} \tag{6.85}$$

where

$$a_j = \frac{1}{\sqrt{2}}(p_j - iq_j), \quad a_j^* = \frac{1}{\sqrt{2}}(p_j + iq_j), \quad [a_j, a_k] = 0, \quad [a_j, a_k^*] = \delta_{jk}. \tag{6.86}$$

By fixing a unit vector Φ in the space of the representation, we can construct the functional over $\mathcal{E}_c$

$$E(F) = \langle \Phi, W(F)\Phi\rangle. \tag{6.87}$$

Exercise 6.44. Prove that $E(F)$ satisfies the properties

(1) *normalization* $E(0) = 1$,

(2) *positivity*

$$\sum_{j,k=1}^{n} \lambda_j\bar{\lambda}_k E(F_j - F_k)\exp(-i\,\mathrm{Im}\langle F_j, F_k\rangle) \geq 0 \tag{6.88}$$

for all sets of complex numbers $\lambda_1, \dots, \lambda_n$ and vectors $F_1, \dots, F_n$ in $\mathcal{E}_c$,

(3) *continuity* with respect to F, when F runs through an arbitrary finite-dimensional subspace of $\mathcal{E}_c$.

Each functional $E(F)$ with the properties (1)–(3) in Exercise 6.44 is called a *characteristic functional over* $\mathcal{E}_c$. The importance of this notion is that it provides the possibility of constructing cyclic representations of the CCR's (the other representations are then constructed from cyclic ones).

Exercise 6.45. Prove that every CCR representation is a direct sum of cyclic representations.

Proposition 6.15. *Every characteristic functional over $\mathcal{E}_c$ is defined by a unitary representation $W(F) \equiv W_E(F)$ of the CCR's over $\mathcal{E}_c$ with cyclic vector Φ such that* (6.87) *holds. The representation W_E defined by these requirements is unique to within unitary equivalence.*

■ We consider the set $\mathcal{F}(\mathcal{E}_c)$ of all complex functions $R(F)$ on $\mathcal{E}_c$ that are non-zero for only a finite number of $F \in \mathcal{E}_c$. Clearly $\mathcal{F}(\mathcal{E}_c)$ is a complex vector space. We define in $\mathcal{F}(\mathcal{E}_c)$ the product

$$R_1R_2(F) = \sum_{F' \in \mathcal{E}_c} R_1(F')R_2(F - F')\exp(-i\,\mathrm{Im}\langle F', F\rangle)$$

(this sum is finite in view of the conditions imposed on R_1 and R_2), and the involution

$$R^*(F) = \overline{R(-F)}.$$

It is not difficult to verify that $\mathcal{F}(\mathcal{E}_c)$ is an involutive algebra with identity and that the formula

$$\omega(R) = \sum_{F \in \mathcal{E}_c} R(F)E(F)$$

defines a positive functional on $\mathcal{F}(\mathcal{E}_c)$ (that is, $\omega(R^*R) \geq 0$). Then according to the GNS construction, there exists a (unique to within unitary equivalence) representation π of $\mathcal{F}(\mathcal{E}_c)$ in some Hilbert space $\mathcal{H}$ with cyclic vector $\Phi \in \mathcal{H}$, where $\omega(R) = \langle \Phi, \pi(R)\Phi \rangle$ for all $R \in \mathcal{E}_c$. We set $W(F) = \pi(R_F)$, where R_F is a functional on $\mathcal{E}_c$ equal to one at the point F and zero at the remaining points. It is easily checked that $W(F)$ is the required CCR representation on $\mathcal{E}_c$. ■

We use Proposition 6.15 for the construction of examples of representations of the CCR's.* We begin with the case of n degrees of freedom. For $z \in \mathbf{C}^n$ we set

$$E_0(z) = \exp(-\tfrac{1}{2}|z|^2). \tag{6.89}$$

It can be directly verified that $E_0(z)$ is a characteristic functional over $\mathbf{C}^n$ and hence defines a cyclic CCR representation $W_0(z)$ in some Hilbert space. Another more instructive proof of this result consists in the following explicit realization of E_0 in the form (6.87).

Exercise 6.46. Let $W(z)$ be the Schrödinger representation of the CCR's in $\mathcal{L}^2(\mathbf{R}^n)$ (see Exercise 6.43). Consider the normalized vector

$$\Phi(q) = \pi^{-n/4}\exp(-\tfrac{1}{2}|q|^2),$$

which, as is well known from quantum mechanics, defines the ground state of the Hamiltonian harmonic oscillator $H = \frac{1}{2}\sum_{j=1}^n (p_j^2 + q_j^2) = \sum_{j=1}^n (a_j^* a_j + \frac{1}{2})$. Prove that $E_0(z) = \langle \Phi, W(z)\Phi \rangle$. [Hint: Use (6.85), (6.86) and the fact that $a_j\Phi = 0$.]

Similarly, in the case of infinite-dimensional $\mathcal{E}_c$ we define the CCR representation $W_0(F)$ over $\mathcal{E}_c$ by means of the characteristic functional

$$E_0(F) = \exp(-\tfrac{1}{2}\|F\|^2). \tag{6.90}$$

It can be directly verified that this is indeed a characteristic functional over $\mathcal{E}_c$. Besides, this follows also from the finite-dimensional case considered above, according to which $E_0(F)$ satisfies the conditions of Exercise 6.44 when F runs through any finite-dimensional subspace of $\mathcal{E}_c$.

In particular, we can associate with any Hilbert space $\mathfrak{H}_1$ a cyclic representation $W_0(F)$ of the CCR's over $\mathfrak{H}_1$ in some Hilbert space $\mathfrak{H}$; this representation is defined by the characteristic functional (6.90). In §7.3 we shall construct this Hilbert space by the method of second quantization and we shall see that $W_0(F)$ is equivalent to the so-called Fock representation of the CCR's over $\mathfrak{H}_1$ and is irreducible.

Starting from the representation W_0 it is not difficult to construct other representations of the CCR's. Thus if we fix a positive number ρ, then we set

$$W^{(\rho)}(f + ig) = W_0(\rho f + i\tfrac{1}{\rho}g), \; f, g \in \mathcal{E}. \tag{6.91}$$

Exercise 6.47. Prove that $W^{(\rho)}(F)$ is a representation of the CCR's over $\mathcal{E}_c$.

The system $W^{(\rho)}$ is also irreducible. It can be shown that in the case of an infinite-dimensional space $\mathcal{E}$, the systems $W^{(\rho)}$ amd $W^{(\rho')}$ are unitarily inequivalent when $\rho \neq \rho'$ (see Exercise 7.24). Thus we have constructed a continuous family of irreducible pairwise unitarily inequivalent representations of the CCR's over $\mathcal{E}_c$ and so demonstrating the inapplicability of von Neumann's uniqueness theorem to systems with an infinite number of degrees of freedom. The above construction can

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* The proof of Lemma 6.16 in the next subsection is another application of Proposition 6.15.

be generalized to a certain extent. A real linear map $F \to F'$ from the space $\mathcal{E}_c$ onto itself is said to be a *symplectic transformation* if it preserves the (symplectic) form $\mathrm{Im}\langle F_1, F_2\rangle$, that is, if $\mathrm{Im}\langle F_1', F_2'\rangle = \mathrm{Im}\langle F_1, F_2\rangle$. By setting

$$W(F) = W_0(F'), \tag{6.92}$$

we again arrive at an irreducible representation of the CCR's over $\mathcal{E}_c$. In particular, the representation $W^{(\rho)}$ (6.91) over $\mathcal{E}_c$ corresponds to the symplectic transformation $f + ig \to \rho f + \frac{i}{\rho} g$ (for $f, g \in \mathcal{E}$) of $\mathcal{E}_c$.

C. Proof of von Neumann's uniqueness theorem

The main step of the proof is contained in the following lemma.

Lemma 6.16. *Let $W(z)$ be a system of CCR's over $\mathbf{C}^n$ in the Hilbert space $\mathcal{H}$. Then there exists in $\mathcal{H}$ a normalized vector Φ_0 for which*

$$\langle \Phi_0, W(z)\Phi_0\rangle = \exp\left(-\tfrac{1}{2}|z|^2\right) \equiv E_0(z),$$

so that the restriction of the given system to the closed linear span of the sets of vectors of the form $W(z)\Phi_0$ is equivalent to the Schrödinger representation.

■ We introduce the operator

$$\Pi = \int W(z)\rho(z)d\mu(z),$$

where

$$\rho(z) = \pi^{-n}\exp\left(-\tfrac{1}{2}|z|^2\right), \quad d\mu(x+iy) = d^n x d^n y.$$

We claim that the operator Π is non-zero. Thus consider the product $W(z)\Pi W(z')$, which (by means of the CCR's) is written in the form

$$W(z)\Pi W(z') = \exp\left(-\tfrac{1}{2}|z|^2 - \tfrac{1}{2}|z'|^2 - \overline{z'}z\right)\int W(\zeta)\exp(\overline{z'}\zeta + \overline{\zeta}z)\rho(\zeta)d\mu(\zeta); \tag{6.93}$$

here $\overline{z}z' \equiv \sum_{j=1}^n \overline{z}_j z_j'$. We choose a pair of vectors $\Phi, \Psi \in \mathcal{H}$ for which the matrix element $\langle \Phi, W(\zeta)\Psi\rangle$ is not identically zero. This matrix element is a continuous bounded function of ζ, therefore it can be approximated in the $\mathcal{L}^2(\mathbf{C}^n; \rho(\zeta)d\mu(\zeta))$ norm to any desired degree of accuracy by polynomials in $\overline{\zeta}$, ζ arising as a result of applying differential polynomials in the variables $z, \overline{z}$ to the exponentials $\exp(\overline{z}\zeta + \overline{\zeta}z)$ at the point $z = 0$. Consequently the right hand side of (6.93) cannot be identically zero, therefore $\Pi \neq 0$.

Exercise 6.48. Prove the relations

$$\Pi W(z)\Pi = \exp\left(-\tfrac{1}{2}|z|^2\right)\Pi. \tag{6.94}$$

[Hint: Use (6.93) with $z = 0$, $z' \in \mathbf{C}^n$ and with $z \in \mathbf{C}^n$, $z' = 0$.]

It is clear that $\Pi^* = \Pi$, therefore it follows from (6.94) with $z = 0$ that Π is an orthogonal projection operator and since it is non-zero, there exists a unit vector Φ_0 in $\mathcal{H}$ for which $\Pi\Phi_0 = \Phi_0$. We have

$$\begin{aligned}\langle \Phi_0, W(z)\Phi_0\rangle &= \langle \Pi\Phi_0, W(z)\Pi\Phi_0\rangle = \langle \Phi_0, \Pi W(z)\Pi\Phi_0\rangle = \\ &= \exp\left(-\tfrac{1}{2}|z|^2\right)\langle \Phi_0, \Pi\Phi_0\rangle = \exp\left(-\tfrac{1}{2}|z|^2\right).\end{aligned}$$

Thus we have constructed the required vector Φ_0. ■

Theorem 6.14 now follows easily from the lemma. Let $W(z)$ be a system of CCR's over $\mathbf{C}^n$ in $\mathcal{H}$. We choose the normalized vector Φ_0 according to Lemma 6.16 and we let $\mathcal{H}_0$ be the closed linear span of the set of vectors of the form $W(z)\Phi_0$. Then a representation of the CCR's is realized in $\mathcal{H}_0$ that is unitarily equivalent to the Schrödinger representation. If $\mathcal{H} = \mathcal{H}_0$ (and this is possible only when the given system $W(z)$ is irreducible), then our theorem is proved. If $\mathcal{H}_0 \neq \mathcal{H}$ then we have to introduce the orthocomplement $\mathcal{K}_1$ to $\mathcal{H}_0$ and again apply the lemma (this time to $\mathcal{K}_1$), and so on. Zorn's lemma enables us to carry out this process of decomposing $\mathcal{H}$ into a direct sum of subspaces that are invariant with respect to the operators $W(z)$, so that in each of these subspaces the system $W(z)$ is unitarily equivalent to the Schrödinger system. (We leave the reader to complete this argument.)

We saw in the preceding subsection that von Neumann's uniqueness theorem is not applicable to canonical systems with an infinite number of degrees of freedom. Even so, this theorem enables us to construct an algebra of observables corresponding to a CCR system both for a finite and for an infinite number of degrees of freedom.

To construct the algebra of observables we fix some representation $W(F)$ of the CCR's over $\mathcal{E}_c$ in the Hilbert space $\mathcal{H}$. (Such a representation exists by the results of the preceding subsection.) We define the *algebra of observables* $\mathfrak{A} \equiv \mathfrak{A}(\mathcal{E})$ *corresponding to the CCR's* over $\mathcal{E}$ (or over $\mathcal{E}_c$) as the minimal C^*-subalgebra in $\mathcal{B}(\mathcal{H})$ containing all the operators of $W(F)$. (We can call this algebra the minimal algebra of the CCR's over $\mathcal{E}_c$.) It is clear that $\mathfrak{A}$ is the closure (or at the abstract level, the completion) of the set $\mathfrak{B}$ of all linear combinations of operators of $W(F)$ in the norm topology, since this set is itself an involutive subalgebra of $\mathcal{B}(\mathcal{H})$. It turns out that $\mathfrak{A}$ does not depend on the choice of the original representation, in the sense that another representation of the CCR's gives an isomorphic algebra. In fact let $\tilde{W}(z)$ be another representation of the CCR's over $\mathcal{E}_c$ in some Hilbert space $\tilde{\mathcal{H}}$; then we can construct the corresponding algebra $\tilde{\mathfrak{B}}$ and its completion $\tilde{\mathfrak{A}}$. Clearly it suffices to construct a norm-preserving and involution-preserving isomorphism γ from $\mathfrak{B}$ onto $\tilde{\mathfrak{B}}$. In the case of a finite-dimensional space $\mathcal{E}$, it follows from von Neumann's uniqueness theorem that the required isomorphism can be defined by the formula

$$\gamma\Big(\sum_{j=1}^{n} \lambda_j W(F_j)\Big) = \sum_{j=1}^{n} \lambda_j \tilde{W}(F_j);$$

here $\lambda_1, \ldots, \lambda_n$ is an arbitrary set of complex numbers while $F_1, \ldots, F_n$ is an arbitrary set of vectors in $\mathcal{E}_c$. This same formula defines an isomorphism $\gamma : \mathfrak{B} \to \tilde{\mathfrak{B}}$ even in the case of an infinite-dimensional space $\mathcal{E}$, since the vectors $F_1, \ldots, F_n$ occurring here belong to the complexification of some finite-dimensional subspace $\mathfrak{M}$ (depending on $F_1, \ldots, F_n$) of $\mathcal{E}$, and for finite-dimensional subspaces the required properties of the map γ hold.

Thus we associate a canonical system over $\mathcal{E}$ with an abstract algebra of observables $\mathfrak{A}(\mathcal{E})$ such that, by construction, each element $F \in \mathcal{E}_c$ corresponds to an element $w(F) \in \mathfrak{A}$ for which

$$w^*(F) = w(-F), \tag{6.95}$$

$$w(F)w(F') = \exp(-i\,\mathrm{Im}\langle F, F'\rangle)w(F + F'). \tag{6.96}$$

It is interesting to trace the connection between the representations of the CCR's over $\mathcal{E}$ and those of the corresponding algebra of observables $\mathfrak{A}(\mathcal{E})$.

Exercise 6.49. Prove that corresponding to each representation $W(F)$ of the CCR's over $\mathcal{E}_c$ there is a unique representation of the algebra of observables $\mathfrak{A}(\mathcal{E})$ such that

$$W(F) = \pi(w(F)). \tag{6.97}$$

However, the relationship is more complicated in the reverse direction. If we are given a representation π of the algebra $\mathfrak{A}(\mathcal{E})$, then (6.97) does not necessarily define a representation of the CCR's over $\mathcal{E}$. In order that $W(F)$ be a representation of the CCR's we must further require continuity (entering into the definition of the CCR's): $\pi(w(F))$ must be continuous in F in the weak operator topology subject to the condition that F runs through an arbitrary finite-dimensional subspace of $\mathcal{E}_c$. Indeed such representations π have physical interest, since we can introduce "generalized coordinates" $q(f)$ and "generalized momenta" $p(g)$ into them:

$$\pi(w(f)) = e^{i\sqrt{2}q(f)}, \quad \pi(w(ig)) = e^{i\sqrt{2}p(g)}, \tag{6.98}$$

where $f, g \in \mathcal{E}$ (cf. (6.85)).

CHAPTER 7

Relativistic Invariance in Quantum Theory

7.1. The Poincaré Group

A. DEFINITION

By the *general Poincaré group* we mean the set $\mathcal{P}$ of all transformations $x \to x'$ of Minkowski space $\mathbf{M}$ that leaves the interval between any pair of points of $\mathbf{M}$ invariant, that is, transformations such that $(x' - y')^2 = (x - y)^2$ for all $x, y \in \mathbf{M}$. Each transformation of $\mathcal{P}$ automatically turns out to be an inhomogeneous linear (that is, affine) transformation; more precisely, it has the form

$$x' = \Lambda x + a, \tag{7.1}$$

where a is a fixed vector of $\mathbf{M}$ and Λ is a transformation of the general Lorentz group. Hence it is clear that the general Poincaré group can also be defined as the set of pairs (a, Λ), where $a \in \mathbf{M}$, $\Lambda \in L$, with the multiplication law

$$(a_1, \Lambda_1)(a_2, \Lambda_2) = (a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2). \tag{7.2}$$

The group $\mathcal{P}$ contains as subgroups the (abelian) group of translations in $\mathbf{M}$ (which we also denote by $\mathbf{M}$) and the general Lorentz group L.* Furthermore, as is obvious from (7.2) it is the semidirect product $\mathbf{M} \circ L$ of these subgroups. Like L, the general Poincaré group is disconnected and consists of four components $\mathcal{P}_+^\uparrow$, $\mathcal{P}_-^\uparrow$, $\mathcal{P}_+^\downarrow$, $\mathcal{P}_-^\downarrow$, corresponding to the four components of the general Lorentz group. The subgroups $\mathcal{P}_+^\uparrow$, $\mathcal{P}^\uparrow$ and $\mathcal{P}_+$ are called the *proper, orthochronous* and *special* Poincaré groups respectively.

The group $\mathcal{P}_+^\uparrow$ of proper Poincaré transformations is a connected but not simply connected group. By analogy with $L_+^\uparrow$ we introduce the universal covering group $\mathcal{P}_0$ for $\mathcal{P}_+^\uparrow$, called the *Poincaré spinor group*. It consists of all possible pairs $(a, \underset{\sim}{\Lambda})$, where a is an arbitrary 4-vector in $\mathbf{M}$ and $\underset{\sim}{\Lambda} \in SL(2, C)$, the law of group multiplication being analogous to (7.2):

$$(a_1, \underset{\sim}{\Lambda}_1)(a_2, \underset{\sim}{\Lambda}_2) = (a_1 + \Lambda(\underset{\sim}{\Lambda}_1) a_2, \underset{\sim}{\Lambda}_1 \underset{\sim}{\Lambda}_2). \tag{7.3}$$

The element $(a, \underset{\sim}{\Lambda}) \in \mathcal{P}_0$ is also written in the form $(\underset{\sim}{a}, \underset{\sim}{\Lambda})$ where $\underset{\sim}{a}$ is the corresponding Hermitian 2×2-matrix (see (3.33)).

* The term "inhomogeneous Lorentz transformations" is also used to denote the transformations of the Poincaré group.

Exercise 7.1. Verify that if the element $(a, \underset{\sim}{\Lambda})$ of $\mathfrak{p}_0$ is written as the 4×4-matrix

$$\begin{pmatrix} \underset{\sim}{\Lambda} & \underset{\sim}{a}\underset{\sim}{\Lambda}^{*-1} \\ 0 & \underset{\sim}{\Lambda}^{*-1} \end{pmatrix}, \tag{7.4}$$

the group multiplication (7.3) becomes ordinary matrix multiplication.

B. REFLECTIONS

The group $\mathfrak{p}_0$ is connected and simply connected; it is a double covering of $\mathfrak{p}_+^\uparrow$. It is possible to introduce the group $\tilde{\mathfrak{p}}$ with four connected components which doubly covers the Poincaré group $\mathfrak{p}$ and for which $\mathfrak{p}_0$ is a connected component. For this purpose we note that an arbitrary element (a, Λ) of the general Poincaré group $\mathfrak{p}$ can be identified with the automorphism $\sigma_{a,\Lambda}$ of the proper Poincaré group:

$$\sigma_{a,\Lambda}(a', \Lambda') \equiv (a, \Lambda)(a', \Lambda')(a, \Lambda)^{-1},$$

or, equivalently,

$$\sigma_{a,\Lambda}(a', \Lambda') = ((1 - \Lambda\Lambda'\Lambda^{-1})a + \Lambda a', \Lambda\Lambda'\Lambda^{-1}). \tag{7.5}$$

Instead of $\sigma_{I_s}, \ \sigma_{I_t}, \ \sigma_{I_{st}}$ we write $\sigma_s, \ \sigma_t, \ \sigma_{st}$.

Exercise 7.2. Prove that $\sigma(a, \Lambda)$ is the identity automorphism if and only if $a = 0, \ \Lambda = 1$.

If $\operatorname{Aut}\mathfrak{p}_+^\uparrow$ denotes the group of all automorphisms of $\mathfrak{p}_+^\uparrow$, then (as follows from Exercise 7.2) $\mathfrak{p}$ is isomorphic to a subgroup of $\operatorname{Aut}\mathfrak{p}_+^\uparrow$, so that we can identify $\mathfrak{p}$ with this subgroup.

Exercise 7.3. Let $\Im$ be the subgroup of $\mathfrak{p}$ consisting of the four elements $e, \ I_s, \ I_t$ and I_{st}, where e is the group identity, I_s is spatial reflection, I_t is time reversal and I_{st} is total reflection. Prove that $\mathfrak{p}$ is a semidirect product of the subgroups $\mathfrak{p}_+^\uparrow$ and $\Im$.

This relation between $\mathfrak{p}_+^\uparrow$ and $\mathfrak{p}$ can be used for the construction of $\tilde{\mathfrak{p}}$ as a semidirect product of $\mathfrak{p}_0$ and the group of reflections

$$\Im = \{e, \ I_s, \ I_t, \ I_{st}\}. \tag{7.6}$$

For this we define $\tilde{\sigma}_e$ as the identity automorphism of $\mathfrak{p}_0$, while the automorphisms $\tilde{\sigma}_{I_s} = \tilde{\sigma}_s, \ \tilde{\sigma}_t \equiv \tilde{\sigma}_{I_t}, \ \tilde{\sigma}_{st} \equiv \tilde{\sigma}_{I_{st}}$ are defined as follows:

$$\tilde{\sigma}_s(a, \underset{\sim}{\Lambda}) = (I_s a, \underset{\sim}{e}_0 \underset{\sim}{\Lambda}^{*-1} \tilde{e}_0), \tag{7.7}$$

$$\tilde{\sigma}_t(a, \underset{\sim}{\Lambda}) = (I_t a, \underset{\sim}{e}_0 \underset{\sim}{\Lambda}^{*-1} \tilde{e}_0), \tag{7.8}$$

$$\tilde{\sigma}_{st}(a, \underset{\sim}{\Lambda}) = (-a, \underset{\sim}{\Lambda}); \tag{7.9}$$

here e_0 is the unit time vector in $\mathbf{M}$.

Exercise 7.4. Prove that the map $J \to \tilde{\sigma}_J$ is a homomorphism from $\Im$ to $\operatorname{Aut}\mathfrak{p}_0$, where

$$\tilde{\sigma}_J(a, \underset{\sim}{\Lambda}) \to \sigma_J(a, \Lambda(\underset{\sim}{\Lambda})) \quad \text{for all } a \in M, \ \underset{\sim}{\Lambda} \in SL(2, C), \ J \in \Im, \tag{7.10}$$

where the arrow denotes the covering homomorphism $\mathfrak{p}_0 \to \mathfrak{p}_+^\uparrow$.

We now define $\tilde{\mathfrak{p}}$ as the semidirect product of $\mathfrak{p}_0$ and $\Im$. The elements of $\tilde{\mathfrak{p}}$ are written as triples $(a, \underset{\sim}{\Lambda}, J)$, where $a \in \mathbf{M}, \ \underset{\sim}{\Lambda} \in SL(2, C), \ J \in \Im$. The groups $\mathfrak{p}_0$ and

$\mathfrak{T}$ enter into $\tilde{\mathfrak{p}}$ as subgroups, therefore for these we can use the previous notations of type $(a, \underset{\sim}{\Lambda})$ and I_s, I_t, I_{st}.

Exercise 7.5. Prove that the map $(a, \underset{\sim}{\Lambda}, J) \to (a, \Lambda(\underset{\sim}{\Lambda})J)$ is a homomorphism from $\tilde{\mathfrak{p}}$ onto $\mathfrak{p}$ which doubly covers $\mathfrak{p}$.

C. THE LIE ALGEBRA OF THE POINCARÉ GROUP

The Poincaré group is a ten-dimensional (real) Lie group. We can take as parameters of a neighbourhood of the identity, the 4-vector a and the skew-symmetric real 4×4-matrix θ, by writing an element of the group in the form

$$(a, \Lambda) = (a, \exp(\tfrac{1}{2} l_{\lambda\mu} \theta^{\lambda\mu})) \tag{7.11}$$

(see (3.16)). The Lie algebra of the Poincaré group can be constructed starting from some faithful representation of the group. Since the Poincaré group is defined as a group of transformations of Minkowski space, it is natural to choose a representation of the Poincaré group in the space of functions on $\mathbf{M}$.

Thus let $T \equiv (T_g)$ be the representation of $\mathfrak{p}$, say, in the space $\mathcal{S}(\mathbf{M})$ of test functions on $\mathbf{M}$, for which the action of the element (a, Λ) on the function $f(x)$ has the form

$$(T_{a,\Lambda} f)(x) = f(\Lambda^{-1}(x - a)).$$

It is easily verified that, in fact, this formula defines a (continuous) representation of $\mathfrak{p}$ into $\mathcal{S}(\mathbf{M})$. We separate out the linear part of the expansion of the operators $T_{a,\Lambda} - 1$ in a Taylor series in powers of the parameters a^μ and $\theta^{\lambda\mu}$:

$$T_{a,\Lambda} - 1 = iP^\mu a_\mu + \frac{i}{2} M^{\lambda\mu} \theta_{\lambda\mu} + \text{ higher order terms,} \tag{7.12}$$

where $M^{\lambda\mu} = -M^{\mu\lambda}$. This formula defines P^μ and $M^{\lambda\mu}$ as first order linear differential operators in $\mathbf{M}$:

$$P^\mu = i \frac{\partial}{\partial x_\mu}, \tag{7.13}$$

$$M^{\lambda\mu} = i(x^\lambda \frac{\partial}{\partial x_\mu} - x^\mu \frac{\partial}{\partial x_\lambda}) = x^\lambda P^\mu - x^\mu P^\lambda. \tag{7.14}$$

They satisfy the following commutation relations:

$$[P^\lambda, P^\mu] = 0, \tag{7.15a}$$

$$[M^{\lambda\mu}, P^\nu] = i(g^{\mu\nu} P^\lambda - g^{\lambda\nu} P^\mu), \tag{7.15b}$$

$$[M^{\lambda\mu}, M^{\rho\sigma}] = -i(g^{\lambda\rho} M^{\mu\sigma} - g^{\mu\rho} M^{\lambda\sigma} + g^{\mu\sigma} M^{\lambda\rho} - g^{\lambda\sigma} M^{\mu\rho}). \tag{7.15c}$$

Exercise 7.6. Prove (7.15) by using the relation

$$[x^\lambda, P^\mu] = -ig^{\lambda\mu}.$$

The set of real linear combinations of the operators iP^λ and $iM^{\lambda\mu}$ is called the *Lie algebra of the Poincaré group*. The commutation relations in this algebra are defined by the formulae (7.15).

From the point of view of the abstract definition, only the formulae (7.15) are essential for the definition of the Lie algebra; the concrete realizations (7.13) and (7.14) have no great significance. We note that the use of the imaginary units in the definitions of P^λ and $M^{\lambda\mu}$ is dictated by considerations of later convenience, namely, the generators turn out to be Hermitian operators if the representation of $\mathfrak{P}_0$ is unitary.

It is clear that starting from an arbitrary representation of the proper Poincaré group, we could construct by means of (7.12) operators P^λ and $M^{\lambda\mu}$ satisfying the commutation relations (7.15). They are called *generators of the representation* (of the Poincaré group). Following the analogy with non-relativistic quantum mechanics and with the classical Noether's theorem in view (see, for example, [B10],Ch. 1) we can suppose, by definition, that P^μ is the (total) 4-momentum operator corresponding to the relativistic system,* and that the tensor $M^{\lambda\mu}$ is the operator of the four-dimensional "angular" momentum (or moment of momentum). The operators

$$M^j = \tfrac{1}{2}\epsilon^{jkl}M^{kl} \tag{7.16a}$$

form a (three-dimensional) "angular" momentum pseudo-vector, while the quantities

$$N^j = M^{0j} \tag{7.16b}$$

are the generators of the hyperbolic rotations (see Exercise 3.5). The corresponding commutation relations have the form

$$[M^j, M^k] = i\epsilon^{jkl}M^l,\ [M^j, N^k] = i\epsilon^{jkl}N^l,\ [N^j, N^k] = -i\epsilon^{jkl}M^l. \tag{7.17}$$

Two (and only two) independent Casimir operators can be formed from the generators of the Poincaré group (that is, two polynomials of the generators that commute with all the generators). One of them

$$P^2 \equiv P_\mu P^\mu \tag{7.18}$$

is called the *mass square* operator. To construct the second operator, we introduce the four-dimensional pseudo-vector called the *Pauli-Lubanski* operator

$$W_\lambda = \tfrac{1}{2}\epsilon_{\lambda\mu\nu\rho}P^\mu M^{\nu\rho}, \tag{7.19}$$

where $\epsilon_{\lambda\mu\nu\rho}$ is the completely antisymmetric tensor (normalized by the condition $\epsilon_{0123} = 1 = -\epsilon^{0123}$). The pseudo-vector W can be written in terms of the vectors $\mathbf{M}$ and $\mathbf{N}$ in the form

$$W^0 = \mathbf{PM} = P^jM^j,\ W^j = P^0M^j - \epsilon^{jkl}P^kN^l. \tag{7.20}$$

An important property of the operators W^λ is that they commute with the 4-momentum:

$$[W^\lambda, P^\mu] = 0. \tag{7.21}$$

The commutation relations of the operators W^λ among themselves and with the $M^{\mu\nu}$ are defined by the formulae

$$[W^\lambda, M^{\mu\nu}] = i(g^{\lambda\mu}W^\nu - g^{\lambda\nu}W^\mu), \tag{7.22}$$

$$[W^\lambda, W^\mu] = i\epsilon^{\lambda\mu\nu\rho}W_\nu P_\rho. \tag{7.23}$$

* P^0 is called the *energy* operator and $\mathbf{P}$ the *three-dimensional momentum* operator.

Direct verification shows that the square of the Pauli-Lubanski vector

$$W^2 \equiv W_\mu W^\mu \tag{7.24}$$

commutes with all the generators of the Poincaré group and hence is the second Casimir operator for $\mathfrak{p}_+^\uparrow$.

Exercise 7.7. Verify the relation

$$P_\mu W^\mu = 0. \tag{7.25}$$

7.2. Unitary Representations of the Proper Poincaré Group

A. POINCARÉ INVARIANCE CONDITION

In the previous chapter we introduced the postulates that were common to a non-relativistic and relativistic quantum theory. The class of relativistic theories is singled out by the following postulate.

The proper Poincaré group $\mathfrak{p}_+^\uparrow$ is the symmetry group of the physical system $(\mathfrak{A}, \mathfrak{S})$.

From *a priori* considerations it would be natural to expect that the symmetry group of a relativistic theory would also contain the operations of spatial reflection (I_s) and time reversal (I_t). This is what was thought until 1957 when the non-preservation of P-parity was discovered in the weak interactions of elementary particles. Nowadays these operations are regarded as approximate symmetries which "work" in those places where one can ignore the effects of the weak interactions (for example, in the strong interactions of elementary particles). Nevertheless, the discrete operation of total reflection (I_{st}) of space-time combined with the operation I_c of charge conjugation (replacement of a particle by the anti-particle) is currently regarded as an exact symmetry, and there are impressive theoretical grounds for this, namely, the celebrated TCP theorem (Chapter 9).

In the above form the Poincaré invariance condition gives no kind of indication of the specifics of the quantum systems. The following strengthening of the postulate reflects the requirement of Poincaré invariance of the physical representation of the quantum system.

A.III (Poincaré invariance). *In the physical Hilbert space $\mathcal{H}$, a unitary representation $(a, \underset{\sim}{\Lambda}) \to U(a, \underset{\sim}{\Lambda})$ of the Poincaré spinor group $\mathfrak{p}_0$ is defined (which is continuous in the weak operator topology), giving the law of transformation under translations and Lorentz transformations for the observables*

$$A \to \alpha_{a,\Lambda(\underset{\sim}{\Lambda})}(A) \equiv \alpha_{a,\underset{\sim}{\Lambda}}(A) = U(a, \underset{\sim}{\Lambda}) A U(a, \underset{\sim}{\Lambda})^{-1} \tag{7.26}$$

and the vector states

$$\Phi \to U(a, \underset{\sim}{\Lambda})\Phi \tag{7.27}$$

(here A is an arbitrary element of the algebra of observables $\mathfrak{A}$ or the von Neumann algebra of observables $\overline{\mathfrak{A}}$ and Φ is an arbitrary vector of $\mathcal{H}$). The generators of this representation, which are the operators of the total 4-momentum P^μ and the four-dimensional angular momentum $M^{\lambda\mu}$, are self-adjoint operators in $\mathcal{H}$ affiliated to the von Neumann algebra of observables.

In §6.3.C we gave sufficient conditions for the derivation of Postulate A.III from the requirement of relativistic invariance of the system $(\mathfrak{A}, \mathfrak{S})$ so that the required representation $U(a, \underset{\sim}{\Lambda})$ was uniquely defined. For this we have to suppose that the hypothesis of standard superselection rules holds and that the continuity condition for the action of the group $\mathfrak{p}_0$ on $\mathfrak{S}$ is strengthened by requiring that $\alpha'_{a,\underset{\sim}{\Lambda}}(\omega)$ be continuous with respect to $(a, \underset{\sim}{\Lambda})$ in the norm topology on $\mathfrak{S}$ for any fixed $\omega \in \mathfrak{S}$ (see

Proposition 6.13). In place of $\mathcal{P}_+^\uparrow$ we now take as the symmetry group $\mathcal{P}_0$ the universal covering group for $\mathcal{P}_+^\uparrow$. As we remarked at the beginning of §6.3.C, this modification has considerable technical advantages without altering the physical content.

B. Classification of Irreducible Representations of $\mathcal{P}_0$. Spectral Principle

The study of the possible unitary representations of $\mathcal{P}_0$ reduces to the classification of its irreducible representations, since any unitary representation of this group can be decomposed into a direct sum (or integral) of irreducible representations.

This result is non-trivial since the Poincaré group is non-compact (it is only locally compact). Concerning the theory of the decomposition of an arbitrary unitary representation of a locally compact group, see the survey by Naimark (1964).

By virtue of what we said in the preceding section, the operators P^2 and W^2 are multiples of the identity operator in the space in which any irreducible representation of the Poincaré group is realized, and their values in this space are used for the classification of the irreducible representations. For $P^2 \geq 0$ we can introduce another discrete invariant characteristic, namely, the sign of the energy

$$\epsilon \equiv \epsilon(P^0) \tag{7.28}$$

(if by definition we set $\epsilon = 0$ for $P^2 < 0$, then we can write $\epsilon = \theta(P^2)\epsilon(P^0)$ for all P; the functions of the self-adjoint operators are defined in the ordinary way via the spectral decomposition).

In their dependence on the values of the invariants P^2 and ϵ, the representations of $\mathcal{P}_0$ can be divided into the following classes.

(a) $P^2 = m^2 > 0$, $\epsilon = 1$ (that is, $P^0 > 0$). The corresponding representations describe the transformation properties of real particles with rest mass m.

($\bar{\text{a}}$) $P^2 = m^2 > 0$, $\epsilon = -1$ (that is, $P^0 < 0$). These representations are the complex conjugates of the representations of class (a).

(b) $P^2 = 0$, $P \neq 0$, $\epsilon = 1$ (that is, $P^0 > 0$). The corresponding representations relate to particles with zero rest mass (neutrino, photon).

($\bar{\text{b}}$) $P^2 = 0$, $P \neq 0$, $\epsilon = -1$ (that is, $P^0 < 0$). The representations of this class are the complex conjugates of the representations of class (b).

(c) $P = 0$ (that is, $P^0 = P^1 = P^2 = P^3 = 0$). All states with such a P are translation-invariant. All the unitary representations of this class, apart from the identity ($U(a, \underset{\sim}{\Lambda}) \equiv 1$) are infinite-dimensional. The identity transformation corresponds to states that are invariant with respect to all the Poincaré transformations. Such a state with ideal space-time symmetry is interpreted as the vacuum.

(d) $P^2 = -m^2 < 0$ (that is, P is spacelike). According to the principles of relativistic mechanics, particles with such a momentum have no real existence.*

The spectral axiom, which is the next axiom of our scheme, is a physical principle that excludes some of the possible transformation properties of vector states that have been listed.

A.IV (Spectral axiom). *The spectrum of the energy-momentum operator P belongs to the closed upper light cone $\overline{V}^+$.*

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* See, however, the articles by J. Feinberg (1967), Sudarshan (1968), and Ecker (1970) (and the literature cited therein) where an approach is developed that admits particles moving faster than light.

The spectral axiom is often stated as follows: there exists a complete system of states with non-negative energies in $\mathcal{H}$.

We say that $\mathcal{H}$ satisfies the *existence and uniqueness of the vacuum condition* if there exists a unique (to within a phase factor) normalized vector Ψ_0 in $\mathcal{H}$ that is invariant with respect to the translations $U(a, 1)$; this distinguished vector Ψ_0 is called the *vacuum vector* (or *vacuum*), denoted by $|0\rangle$. The vacuum is the "ground" state of the physical system (that is, the state with the lowest energy), the remaining being a kind of excitation of the vacuum. (The spectral axiom is necessary in order to ensure that the physical requirement of stability of the vacuum state with respect to infinitely small perturbations of the system be satisfied.)

A physical representation with a unique vacuum can also be characterized in the following way: the point $p = 0$ is a discrete non-degenerate eigenvalue of the operator P, that is, there exists a unique (to within a phase factor) normalized vector Ψ_0 in $\mathcal{H}$ for which $P\Psi_0 = 0$. Such a vector Ψ_0 is then automatically invariant with respect to all the proper transformations of the Poincaré group $\mathfrak{P}_0$. This follows from what was said earlier in connection with representations of the class (c), or from the following exercise.

Exercise 7.8. (a) Prove that each unitary representation of the Poincaré group $\mathfrak{P}_0$ by pairwise commuting operators $U(a, \underset{\sim}{\Lambda})$ is the identity: $U(a, \underset{\sim}{\Lambda}) = 1$ for all $(a, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0$. [Hint: $\mathfrak{P}_0$ is generated by the commutators $g_1 g_2 g_1^{-1} g_2^{-1}$ of its elements.]

(b) Every one-dimensional representation of $\mathfrak{P}_0$ is the identity.

Technically, the spectral principle means that a representation of $\mathfrak{P}_0$ in $\mathcal{H}$ decomposes into irreducible representations occurring only in the classes (a), (b) and (c). Among the representations of class (c), the only one considered is the trivial (identity) transformation corresponding to the vacuum. By means of Stone's theorem 1.17, the spectral condition can be stated in the following way: for all $\Phi, \Psi \in \mathcal{H}$ there is a representation

$$\langle \Phi, U(a, 1)\Psi \rangle = \int e^{ipa} du(p), \tag{7.29}$$

where u is a finite (complex) measure in Minkowski space with support in $\overline{V}^+$. In particular,

$$\operatorname{supp} \int \langle \Phi, U(a, 1)\Psi \rangle e^{-ipa} da \subset \overline{V}^+ \tag{7.30}$$

(in the sense of generalized functions in $\mathcal{S}'(\mathrm{M})$).

For field theory there is a convenient existence and uniqueness criterion for the vacuum based on the following notion. We say that the *cluster property* (also called the asymptotic factorization property) holds in the physical Hilbert space $\mathcal{H}$ if there exists a unit vector Ψ_0 in $\mathcal{H}$ such that

$$\langle \Phi, U(\lambda a, 1)\Psi \rangle \to \langle \Phi, \Psi_0 \rangle \langle \Psi_0, \Psi \rangle \;\text{ as } \lambda \to \infty; \tag{7.31}$$

here a is an arbitrary spacelike vector in M and Φ, Ψ are arbitrary vectors in $\mathcal{H}$.

Proposition 7.1. *The condition of existence and uniqueness of the vacuum in the physical space $\mathcal{H}$ is equivalent to the cluster condition.*

■ Suppose that the cluster condition holds. It is clear that Ψ_0 is uniquely defined by (7.1) to within a phase factor. We claim that it is translation-invariant. For if we replace Ψ in (5.3) by $U(-a,1)\Psi$, then we see that $\langle\Psi_0, U(-a,1)\Psi\rangle = \langle\Psi_0,\Psi\rangle$ for all Ψ, from which it follows that $U(a,1)\Psi_0 = \Psi_0$ for all spacelike vectors a in $\mathbf{M}$. Since such vectors a span the whole of $\mathbf{M}$, it follows that $U(a,1)\Psi_0 = \Psi_0$ for all $a \in \mathbf{M}$. Now let Ψ be a translation-invariant vector. Then it follows from (5.31) that $\langle\Psi,\Psi\rangle = |\langle\Psi_0,\Psi\rangle|^2$, so that $\Psi = \lambda\Psi_0$. Thus the cluster property implies the existence and uniqueness of the vacuum (where Ψ_0 is the vacuum vector).

Conversely, suppose that the existence and uniqueness condition holds for the vacuum. It is enough to prove the cluster property (7.31) only for the special vector $a = e_3 \equiv (0,0,0,1)$. The general case will then follow from this and from the formula

$$U(0,\underset{\sim}{\Lambda})U(a,1)U(0,\underset{\sim}{\Lambda})^{-1} = U(\Lambda(\underset{\sim}{\Lambda})a,1). \tag{7.32}$$

If one of the vectors Φ,Ψ in (7.31) is the vacuum, then the cluster property holds trivially. Therefore it suffices to prove (7.31) with $a = e_3$ for vectors Φ,Ψ in some dense subset $\mathfrak{M}$ of the subspace $\mathcal{H}_0^\perp$ of $\mathcal{H}$ orthogonal to Ψ_0. We now carry out the construction of the subset $\mathfrak{M}$. Let E_ϵ be the projector in $\mathcal{H}$ onto the subspace where $P^0 \geq \epsilon$ $(\epsilon > 0)$. It is clear that $E_\epsilon\Phi \to \Phi$ as $\epsilon \to 0$ for $\Phi \in \mathcal{H}_0^\perp$; therefore the set $\mathfrak{M}_1 = \bigcup_{\epsilon>0} E_\epsilon\mathcal{H}$ is dense in $\mathcal{H}_0^\perp$. We now choose for $\mathfrak{M}$ the set

$$\mathfrak{M} = D \cap \mathfrak{M}_1 = \bigcup_{\epsilon>0}(D \cap E_\epsilon\mathcal{H}),$$

where D is the domain of the operator $N^3 \equiv M^{03}$.

Exercise 7.9. (a) Prove that $\mathfrak{M}$ is dense in $\mathfrak{M}_1$ and hence in $\mathcal{H}_0^\perp$. [Hint: Verify that $D \cap E_\epsilon\mathcal{H}$ is dense in $E_\epsilon\mathcal{H}$; consider for this purpose the set of vectors of the form

$$\int f_n(t)U(0,\underset{\sim}{\Lambda}_t)\Phi,$$

where $\Phi \in E_\epsilon\mathcal{H}$, $\underset{\sim}{\Lambda}_t = \exp(t\sigma_3)$; $\{f_n\}$ is a δ-sequence of functions in $\mathcal{D}(\mathbf{R})$].

(b) Prove that the following equality holds on $\mathfrak{M}$:

$$[N^3, U(a,1)] = (P^0a^3 - P^3a^0)U(a,1). \tag{7.33}$$

[Hint: Replace $\underset{\sim}{\Lambda}$ by $\underset{\sim}{\Lambda}_t$ in (7.32) and differentiate the equality with respect to t at $t = 0$.]

Since the inverse operator $(P^0)^{-1}$ is defined on $\mathfrak{M}$, it follows from (7.33) with $a = \lambda e_3$ that

$$\langle N^3\Phi, U(\lambda e_3,0)(P^0)^{-1}\Psi\rangle - \langle\Phi, U(\lambda e_3,0)N^3(P^0)^{-1}\Psi\rangle = \lambda\langle\Phi, U(\lambda e_3,0)\Psi\rangle$$

for all $\Phi,\Psi \in \mathfrak{M}$, $\lambda \in \mathbf{R}$. Clearly the left hand side of this equality is bounded as $\lambda \to \infty$, whence it follows that $\langle\Phi, U(\lambda e_3,0)\Psi\rangle \to 0$ as $\lambda \to \infty$. This completes the proof. ■

Corollary 7.2. In the relativistic theory with a unique vacuum, the following property holds:

$$\langle\Psi_0, A_1\alpha_{\lambda a,1}(A_2)\Psi_0\rangle \to \langle\Psi_0, A_1\Psi_0\rangle\langle\Psi_0, A_2\Psi_0\rangle \text{ as } \lambda \to \infty; \tag{7.34}$$

here $a^2 < 0$, $A_1, A_2 \in \mathfrak{A}$.

This (weak) form of the spectral axiom together with the condition of existence and uniqueness of the vacuum is sufficient for the study of many of the structural properties of quantum field theory. It is, however, inadequate for the construction of an effective theory of scattering. It is clear from physical considerations that the notion of asymptotically free (incoming or outgoing) particles presupposes that the interaction between excitations of the vacuum that are localized in spatially separated regions must decrease sufficiently rapidly as the distance between these regions increases. For this it suffices to exclude the possibility of states with zero mass, for example, by requiring that the spectrum of the mass operator $\sqrt{P^2}$ has no points in common with a certain interval $(0,\mu)$. The corresponding parameter μ is called the *mass gap*. Thus in our discussion of some of these questions (principally relating to scattering theory), we shall need a certain strengthening of the spectral axiom.*

Strong form of the spectral axiom. *The spectrum of the energy-momentum operator is contained in the set*

$$\{0\} \bigcup \overline{V}_\mu^+ \tag{7.35}$$

(*see the notation of* (3.9)), *where the value of the parameter* μ (*the mass gap*) *is positive; the point* $p = 0$ *corresponds to the unique vacuum state.*

The strong spectral condition makes the representation (7.29) more specific: for all $\Phi, \Psi \in \mathcal{H}$ we have

$$\langle \Phi, U(a,1)\Psi \rangle = \langle \Phi|0\rangle\langle 0|\Psi\rangle + \int e^{ipa} dv(p), \tag{7.36}$$

where v is a finite complex measure in $\mathbf{M}$ with support in $\overline{V}_\mu^+$, so that

$$\operatorname{supp}\{\int \langle \Phi, U(a,1)\Psi\rangle e^{-ipa} da - \langle \Phi|0\rangle\langle 0|\Psi\rangle(2\pi)^4\delta(p)\} \subset \overline{V}_\mu^+. \tag{7.37}$$

Each discrete point m of the spectrum of the mass operator $\sqrt{P^2}$ indicates the presence of a physical particle of mass m. However, because of the superselection rule, the notion of a *single* relativistic particle is somewhat more complicated than a subspace of $\mathcal{H}$ with a fixed value of the mass operator. In theories with standard superselection rules, each subspace of $\mathcal{H}$ that lies entirely in some coherent subspace and that corresponds to a discrete point m of the spectrum of the mass operator $\sqrt{P^2}$ is interpreted as a space of relativistic particles with mass m and with superselection quantum numbers of the given coherent spectrum. The separation of particles into "particles" and "antiparticles" is related (along with the superselection rules) to the operation of charge conjugation or, say, the CP or CPT operation containing the charge conjugation. If such an operation takes a given coherent sector into another coherent sector, then the particles of one sector are appropriately called "particles" and those of the other "antiparticles". In the case where the coherent sector is invariant with respect to such an operation, then it makes sense to talk about the coincidence of the particle with the antiparticle. (The corresponding particles, for example, the photon, are called truly neutral, and in the case of spinor particles, Majorana particles.)

It must be borne in mind, however, that such a notion of a particle is a strong idealization, since the presence of a strictly fixed value of the mass of the particle presupposes its absolute stability. (In this connection, see §7.3.E.) The majority of real particles (which in some cases are interpreted as "elementary", while in other cases as "bound states" and " resonances") are unstable and as a consequence, the distribution of the quantity $\sqrt{P^2}$ corresponding to them has non-zero width (or dispersion). The idealization that we have referred to is admissible and is widely used in the construction of the scattering matrix; in the examination of decay processes, however, the width of the mass spectrum of the particle has to be taken into account. We see that the requirements in the scattering picture and in the decay picture come into conflict; the way out of this requires the development of a more realistic notion of a particle.

We now give some further facts touching upon the classification of the irreducible representations of the group $\mathfrak{P}_0$ of classes (a) and (b). The energy-momentum operator P in an irreducible representation of $\mathfrak{P}_0$ can be degenerate; the multiplicity of this degeneracy is interpreted physically as the number of independent polarizations (or the number of possible orthogonal spin states) of the particle associated with the given irreducible representation and with a fixed value of the momentum. All the particles known in physics have a finite number of (independent) polarizations.

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* In fact, scattering theory requires more detailed information on the spectrum of the energy-momentum masses. However, we shall defer this question until Chapter 12.

Therefore within the family of all irreducible representations of classes (a) and (b), the adopted convention is to call "physical" only those representations with a finite multiplicity of degeneracy of the energy-momentum operator. It turns out that all the representations of class (a) automatically fall into the class of "physical" representations. According to the analysis of Wigner (1939), to characterize an irreducible representation of class (a) to within unitary equivalence, one must indicate (along with the value of the mass $m > 0$ and the positive sign of the energy $\epsilon = 1$) the value of the square of the Pauli-Lubanski vector; furthermore, the admissible values for W^2 are covered by the formula

$$W^2 = -m^2 s(s+1). \tag{7.38}$$

Here s is an integer or half-integer, called the *spin* of the representation. The corresponding multiplicity of degeneracy of the spectrum of P is equal to $2s+1$. It follows from what has been said that an irreducible representation of class (a) is characterized to within unitary equivalence by the two parameters $[m, s]$, namely, the mass $m > 0$ and the spin $s = 0, 1/2, \ldots$

For irreducible representations of class (b), the spectrum of P is either non-degenerate or infinitely degenerate.* Consequently in "physical" irreducible representations of ρ_0 of class (b), the spectrum of P is non-degenerate. As a consequence of this, the Pauli-Lubanski vector is proportional to P:

$$W^\lambda = sP^\lambda. \tag{7.39}$$

The number s occurring here can take integer or half-integer values (positive, negative or zero); it is called the *helicity* of the representation. The absolute value $|s|$ of the helicity is sometimes called the spin of a particle with zero mass. The helicity characterizes the irreducible representations of class(b) to within unitary equivalence. We denote by $[0, s]$ the irreducible representation of ρ_0 with positive energy, zero mass and helicity s. It should be noted that according to (7.20) the helicity operator can be written in the form

$$s = \frac{1}{|\mathbf{P}|}\mathbf{PM}, \tag{7.40}$$

so that it has the meaning of the projection of the three-dimensional angular momentum in the direction of the three-dimensional momentum $\mathbf{P}$. Clearly, this formula can also be used to define helicity for representations of class (a), but it is only relativistically invariant for representations with zero mass.

We recommend that the reader consult the classical paper by Wigner (1939) in which there is a thorough account and proof of the above classification (see also the monographs [M2], [M8], where these questions are discussed in a more general setting). In §7.2.D below we give a construction of all the "physical" irreducible representations of ρ_0 in spaces of spinor wave functions and so give a visual demonstration of the properties of these representations mentioned above. In §7.2.C we give the more traditional (Wigner) construction of representations of class (a) which diagonalizes the operators P and S^3, where S^3 is the third projection of the non-relativistic spin operator. (Another popular realization of representations of class (a) in terms of the energy-momentum and helicity operators can be found in the article by Jacob and Wick, 1959.)

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* The corresponding representations of ρ_0 (called representations with positive energy, zero mass and infinite spin) are realized in the space of wave functions with an infinite number of components. (See Iverson and Mack, 1971, also Appendix I.)

C. DESCRIPTION OF REPRESENTATIONS CORRESPONDING TO PARTICLES WITH POSITIVE MASS

We give the Wigner construction for irreducible representations of $\mathfrak{P}_0$ of class (a). For this purpose we choose from among the Hermitian functions of generators of the group, a complete set of commuting operators.* This set always contains the Casimir operators $P^2 = m^2$ and W^2 of the Poincaré group, which commute with all the operators of the representation. In addition to these, it is convenient to choose the momentum operators P^μ ($\mu = 0, 1, 2, 3$) and the third projection of the spin

$$S^3 = \frac{1}{m}\Big(W^3 - \frac{W^\circ P^3}{m + P^\circ}\Big). \tag{7.41}$$

It is not difficult to see with the help of (7.21) that the operators P^μ and S^3 commute. They also commute if instead of S^3 we take any linear combination of the components of W. Therefore this special choice of (7.41) in the definition of the spin requires clarification.

At the points p of the spectrum of P, where $\mathbf{p} = 0$ (or, as they say, in the rest frame of the vector P), the spin vector is equal to

$$\mathbf{S}_R = \frac{1}{m}\mathbf{W}. \tag{7.42}$$

We postulate that in any frame, the vector $\mathbf{S}$ is a linear combination of components of $\mathbf{W}$ with coefficients depending only on the 4-momentum P. Here it is required that $\mathbf{S}$ be a three-dimensional vector, that is,

$$[M^j, S^k] = i\epsilon^{jkl}\mathbf{S}^l \tag{7.43}$$

and that the usual commutation relations hold for the spin operator:

$$[S^j, S^k] = i\epsilon^{jkl}\mathbf{S}^l. \tag{7.44}$$

We claim that the unique axial-vector-valued linear combination of the operators W^μ that satisfies (7.43) and (7.44) and is taken to the vector (7.42) in the rest frame has the form

$$S^j = \frac{1}{m}\Big(W^j - \frac{W^\circ P^j}{m + P^\circ}\Big) = \frac{1}{m}\Big[P^0 M^j - \frac{1}{2}\epsilon^{jkl}P^k N^l - P^j\frac{\mathbf{PM}}{m + P^\circ}\Big]. \tag{7.45}$$

For the condition that $\mathbf{S}$ be a pseudo-vector (this condition is implied by (7.43)) yields

$$S^j = \frac{a}{m}(W^j - bW^0 P^j), \tag{7.46}$$

where a and b are functions of the momentum P that are invariant with respect to the group of three-dimensional Euclidean rotations and consequently depend only on P^0 and m (since $\mathbf{P}^2 = (P^0)^2 - m^2$). Here we have taken into account the fact that $\mathbf{P}$ and W^0 change sign under spatial reflection, so that their product remains unchanged. On the other hand, a term of type $(\mathbf{WP})\mathbf{P}$ also has the same property but, in view of (7.25), it is equal to $W^0 P^0 \mathbf{P}$ and hence does not give anything new. To define the coefficients a and b we substitute the expression (7.46) into (7.44). A comparison of the left and right hand sides gives the two equations for a and b:

$$a[P^0 - b((P^0)^2 - m^2)] = m, \quad a(1 - bP^0) = mb.$$

This system has the two solutions: $a = 1$, $b = \frac{1}{m+P^0}$ and $a = -1$, $b = \frac{1}{P^0 - m}$. The second of these is ruled out by condition (7.42). Thus (7.45) is proved.

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* We say that a system of commuting Hermitian operators forms a complete set (in mathematical terminology, a maximal abelian set) if their simultaneous eigenvalues uniquely define (to within a factor) a common eigenvector (in other words, if their joint spectrum is simple). Our requirements are less strict, namely: any polynomial of the generators of the representation that commutes with all the operators of the complete set Π is a function of these operators. However, there may be other operators (for example, the electric charge or baryonic number operators) acting in the space of state vectors and commuting with the operators of the system Π, but which are not functions of these operators.

We note that the combination (7.45) is none other than the spatial components of the vector W after it has been transformed to the rest frame,

$$S^j = \frac{1}{m}(\Lambda_p^{-1}W)^j, \tag{7.47}$$

where Λ_p is the "pure" Lorentz transformation taking the time axis along the vector p:

$$\Lambda_p^{-1}p = (m, 0, 0, 0). \tag{7.48}$$

The explicit form of the matrix Λ_p^{-1} is given by

$$\begin{aligned}(\Lambda_p^{-1})^0_\mu &= (\Lambda_p^{-1})^\mu_0 = \frac{p_\mu}{m} = g_{\mu\nu}\frac{p^\nu}{m},\\ (\Lambda_p^{-1})^j_k &= \delta^j_k + \frac{p^j p^k}{m(m+p^0)}.\end{aligned} \tag{7.49}$$

By virtue of (7.44), the Hermitian operators S^j are generators of the Lie algebra of the group $O_+(3)$ of rotations of three-dimensional Euclidean space (or, what is the same, the Lie algebra of the group $SU(2)$). It is called the *little Poincaré group* corresponding to the given irreducible representation of $\mathfrak{P}_0$.* All the irreducible unitary representations of the algebra of operators S^j are finite-dimensional and can be classified by the value of the total spin s (which takes integral or half-integral non-negative values):

$$\mathbf{S}^2 = s(s+1). \tag{7.50}$$

Exercise 7.10. Prove that the invariant $-W^2$ is equal to the product of S^2 and m^2:

$$-W^2 = m^2\mathbf{S}^2 = m^2 s(s+1). \tag{7.51}$$

We realize the irreducible representation $[m, s]$ of $\mathfrak{P}_0$ of class (a) in the Hilbert space $\bigoplus^{2s+1}\mathcal{L}^2(\Gamma_m^+)$ of functions of the eigenvalues p^μ and σ of the operators P^μ and S^3 respctively. The variable σ takes the $2s+1$ values:

$$\sigma = -s, -s+1, \ldots, s-1, s; \tag{7.52}$$

the 4-vector p^μ ranges over the upper hyperboloid Γ_m^+ endowed with the invariant measure

$$(dp)_m \equiv \frac{d_3 p}{2p^0} \equiv \frac{d^3 p}{(2\pi)^3\cdot 2p^0}. \tag{7.53}$$

The scalar product in $\bigoplus^{2s+1}\mathcal{L}^2(\Gamma_m^+)$ is defined by the equality

$$(\Phi, \Psi) = \sum_{\sigma=-s}^{s}\int_{\Gamma_m^+}\overline{\Phi(p,\sigma)}\Psi(p,\sigma)(dp)_m. \tag{7.54}$$

As a dense subspace of $\bigoplus^{2s+1}\mathcal{L}^2(\Gamma_m^+)$, it is convenient to choose the space $\bigoplus^{2s+1}\mathcal{S}(\Gamma_m^+)$ consisting of all functions $\Phi(p,\sigma)$ that belong to $\mathcal{S}(\mathbf{R}^3)$ for each fixed σ, that is, the space of infinitely-differentiable rapidly decreasing functions of the 3-vector $\mathbf{p}$.

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* In fact the little group can be defined in all the four classes of representations listed above, and its type depends only on the class of the irreducible representation. The definition of the little group is as follows: let Γ be the spectrum of the operator P of an irreducible representation of the Poincaré group; then by the *little group* of the point $p \in \Gamma$ we mean the subgroup B_p of the transformations of the proper Lorentz group that leave p invariant. For any two points of Γ, the little groups are isomorphic, and this fact is essential for the characterization of the type of the little group. Thus for the class (a), as we shall see, it is the group of rotations of 3-dimensional Euclidean space; for class (b), it is the group of motions of the Euclidean plane (including the translations); for class (c), it is the entire proper Lorentz group, and for (d), it is the group of pseudo-Euclidean rotations of three-dimensional space (with signature $+--$).

The operators P^μ (like S^3) are defined on $\bigoplus \mathcal{S}(\Gamma_m^+)$ by multiplication and under this definition leave this space invariant:

$$P^\mu \Phi(p,\sigma) = p^\mu \Phi(p,\sigma), \quad S^3\Phi(p,\sigma) = \sigma\Phi(p,\sigma). \tag{7.55}$$

(By contrast, the action of the momentum operators is not defined for any vector of the space $\mathcal{H}$ itself, since it does not, in general, follow from the square integrability of $\Phi(p,\sigma)$ over Γ_m^+ that $p^\mu\Phi(p,\sigma)$ will be square integrable. The eigenfunctions of the momentum operator, just as in non-relativistic quantum mechanics, are generalized states belonging to the dual space $\bigoplus^{2s+1} \mathcal{S}'(\Gamma_m^+)$.)

The unitary operators $U(a,\underset{\sim}{\Lambda})$ of the representation of $\mathfrak{P}_0$, unlike the infinitesimal operators P^μ and $M^{\mu\nu}$, are defined on the whole of the Hilbert space $\bigoplus^{2s+1} \mathcal{L}^2(\Gamma_m^+)$. The representation of the subgroup of translations has an especially simple form:

$$(U(a,1)\Psi)(p,\sigma) = e^{ipa}\Psi(p,\sigma). \tag{7.56}$$

To construct the representation $U(0,\underset{\sim}{\Lambda})$ of $SL(2,C)$ we proceed as follows. We choose a momentum on the hyperboloid Γ_m^+ in the rest frame

$$p_R = (m,0) \tag{7.57}$$

(the choice of this vector is merely a matter of convenience). The representation of the little group $SU(2)_R$, which leaves this vector invariant, is defined at the point $\mathbf{p} = p_R$ by the formula

$$(U(V)\Psi)(p_R,\sigma) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(V)\Psi(p_R,\sigma'). \tag{7.58}$$

Here $V \in SU(2)$, that is, V is a two-row unitary matrix which can be written in the form

$$V \equiv \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \equiv \begin{pmatrix} V_{11} & V_{12} \\ -\overline{V}_{12} & \overline{V}_{11} \end{pmatrix}, \tag{7.59}$$

$$\det V \equiv |V_{11}|^2 + |V_{12}|^2 = 1, \tag{7.60}$$

and the $D^{(s)}_{\sigma\sigma'}(V)$ are the matrix elements of the irreducible representation of $SU(2)$ corresponding to the spin s:

$$D^{(s)}_{\sigma\sigma'}(V) = \sqrt{\frac{(s+\sigma)!(s-\sigma)!}{(s+\sigma')!(s-\sigma')!}}\, V_{11}^{\sigma+\sigma'} V_{12}^{\sigma-\sigma'} P_{s-\sigma}^{(\sigma-\sigma',\sigma+\sigma')}(|V_{11}|^2 - |V_{12}|^2), \tag{7.61}$$

where the $P_n^{(\alpha,\beta)}(z)$ are the Jacobi polynomials:

$$P_n^{(\alpha,\beta)}(z) = \frac{(-1)^n}{2^n n!}(1-z)^{-\alpha}(1+z)^{-\beta}\frac{d^n}{dz^n}[(1-z)^{\alpha+n}(1+z)^{\beta+n}] \tag{7.62}$$

(see, for example, [W5], also Joos,1962). We note that $D^{(1/2)}_{\sigma\sigma'}(V) = V_{3/2-\sigma,3/2-\sigma'}$. Any unimodular matrix $\underset{\sim}{\Lambda} \in SL(2,C)$ factorizes uniquely into a product of unitary and positive definite matrices:

$$\underset{\sim}{\Lambda} = V_{\underset{\sim}{\Lambda}} H_{\underset{\sim}{\Lambda}}, \; V_{\underset{\sim}{\Lambda}} = \underset{\sim}{\Lambda}\sqrt{\underset{\sim}{\Lambda}^{-1}\underset{\sim}{\Lambda}^{*-1}}, \; H_{\underset{\sim}{\Lambda}} = \sqrt{\underset{\sim}{\Lambda}^{*}\underset{\sim}{\Lambda}}. \tag{7.63}$$

(According to Exercise 3.5, $V_{\underset{\sim}{\Lambda}}$ corresponds to a rotation, while $H_{\underset{\sim}{\Lambda}}$ corresponds to a pure Lorentz transformation.) The representation of the transformation (7.63) at the point $p = p_R$ has the form

$$(U(0,\underset{\sim}{\Lambda})\Psi)(p_R,\sigma) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(V_{\underset{\sim}{\Lambda}})\Psi(\Lambda^{-1}(H_{\underset{\sim}{\Lambda}})p_R,\sigma'), \tag{7.64}$$

where $\Lambda(H_{\underset{\sim}{\Lambda}})$ is the Lorentz transformation corresponding to the matrix $H_{\underset{\sim}{\Lambda}}$ and $V_{\underset{\sim}{\Lambda}}$ is the unitary matrix defined in (7.63). We see that at the point p_R this Lorentz transformation acts only on the argument p of the function Ψ (note that $\Lambda(\underset{\sim}{\Lambda}^{-1})p_R = \Lambda(H_{\underset{\sim}{\Lambda}}^{-1}p_R)$. Only the three-dimensional rotation acts non-trivially on the spin index and this leaves the vector p_R invariant.

Using (7.64) and the fact that a product of matrices $\underset{\sim}{\Lambda}$ corresponds to a product of operators U, we can find the action of $U(0,\underset{\sim}{\Lambda})$ on Ψ at any point p. To do this we proceed as follows. Let Λ_p be the partial Lorentz transformation taking the vector p_R into a given vector p on the hyperboloid Γ_m^+:

$$(\Lambda_p x)^0 = \frac{p^0}{m}x^0 + \frac{1}{m}\mathbf{p}\mathbf{x}, \quad (\Lambda_p x)^j = x^j + \frac{p^j}{m}\Big(\frac{\mathbf{p}\mathbf{x}}{p^0+m} + x^0\Big). \tag{7.65}$$

Corresponding to this transformation is a unique positive-definite matrix H_p in $SL(2,C)$:

$$H_p = \frac{m+\underset{\sim}{p}}{\sqrt{2m(p^0+m)}} = \frac{1}{\sqrt{2m(p^0+m)}}\begin{pmatrix} p^0+m+p^3 & p^1-ip^2 \\ p^1+ip^2 & p^0+m-p^3 \end{pmatrix}, \tag{7.66}$$

such that

$$H_p \underset{\sim}{p}_R H_p = \underset{\sim}{p}. \tag{7.67}$$

To verify that the matrix (7.66) does indeed satisfy (7.67) it suffices to note that for $p^2 = m^2$

$$\underset{\sim}{p}^2 = 2p^0\underset{\sim}{p} - m^2$$

and therefore

$$(m+\underset{\sim}{p})^2 = 2(p^0+m)\underset{\sim}{p}.$$

Any matrix $\underset{\sim}{\Lambda} \in SL(2,C)$ can be represented in the form

$$\underset{\sim}{\Lambda} = H_p B, \tag{7.68}$$

where $H_p^2 = \underset{\sim}{p}$ and $\det B = 1$. By virtue of (7.63), (7.64) and (7.68) we have

$$\begin{aligned}
(U(0,\underset{\sim}{\Lambda})\Psi)(p,\sigma) &= (U(0,H_p)U(0,B)\Psi)(p,\sigma) = \\
&= (U(0,B)\Psi)(p_R,\sigma) = \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(V_B)\Psi(\Lambda^{-1}(B)p_R,\sigma') = \\
&= \sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}\Big(H_p^{-1}\underset{\sim}{\Lambda}\sqrt{\underset{\sim}{\Lambda}^{-1}H_p^2\underset{\sim}{\Lambda}^{*-1}}\Big)\Psi(\Lambda^{-1}(\underset{\sim}{\Lambda})p,\sigma')
\end{aligned} \tag{7.69}$$

(the matrix Λ_p^{-1} is given by (7.49)).

Exercise 7.11. (a) Prove that the matrix H_p (7.66) is the "positive" root of the matrix $\underset{\sim}{p}/m$:

$$H_p = \sqrt{\underset{\sim}{p}/m}, \quad p \in \Gamma_m^+. \tag{7.70}$$

(b) Consider the unimodular matrix $A = \underset{\sim}{a}$, where $a^{0^2} - \mathbf{a}^2 = 1$. Show that

$$A^{-1} \equiv \underset{\sim}{a}^{-1} = \tilde{a} \equiv a^0\sigma_0 - \mathbf{a}\boldsymbol{\sigma}. \tag{7.71}$$

(c) Verify directly that the matrix

$$V(\underset{\sim}{\Lambda},p) = H_p^{-1}\underset{\sim}{\Lambda}\sqrt{\underset{\sim}{\Lambda}^{-1}H_p^2\underset{\sim}{\Lambda}^{*-1}} = \sqrt{\underset{\sim}{p}^{-1}}\underset{\sim}{\Lambda}\sqrt{\underset{\sim}{\Lambda}^{-1}\underset{\sim}{p}\underset{\sim}{\Lambda}^{*-1}}, \tag{7.72}$$

entering into the argument of the representation D (7.69) is unitary.

The above results, which are due to Wigner (1939) can be summarized as follows.

Theorem 7.3. *Every irreducible unitary representation of* $\mathfrak{P}_0$ *of class* (a) *can be characterized (to within unitary equivalence) by a pair of numbers* $[m,s]$, *where* $m > 0$ *is the mass,* s *is the spin which takes integral or half-integral non-negative values. This representation can be realized in the* *space* $\bigoplus^{2s+1}\mathcal{L}^2(\Gamma_m^+)$ *with scalar product* (7.54), *and the action of the operators of the representation is given by the formula*

$$(U(a,\underset{\sim}{\Lambda})\Psi)(p,\sigma) = e^{ipa}\sum_{\sigma'=-s}^{s} D^{(s)}_{\sigma\sigma'}(V(\underset{\sim}{\Lambda},p))\Psi(\Lambda^{-1}(\underset{\sim}{\Lambda})p,\sigma'), \tag{7.73}$$

where $D^{(s)}$ *is the matrix of the irreducible representation* (s) *of* $SU(2)$ (7.61), *while the unitary unimodular matrix* $V(\underset{\sim}{\Lambda},p)$ *(called the Wigner rotation) is defined by* (7.72).

D. Manifestly Covariant Realization of "Physical" Irreducible Representations

It turns out that all the "physical" irreducible representations of $\mathfrak{P}_0$ can be constructed in the space of spinor wave functions (with a finite number of components). We start with the representation $\mathfrak{D}^{(j,0)}$ or $\mathfrak{D}^{(0,j)}$ of $SL(2,C)$ (where j is an arbitrary integral or half-integral non-negative number). The corresponding spinor wave function is the complex function $\Phi(p,\omega)$ (or $\Phi(p,\overline{\omega})$) of the 4-vector $p \in \mathbf{M}$ which is a homogeneous polynomial of degree $2j$ in the spinor variable $\omega \in \mathbf{C}^2$ (with respect to ω). Taking p to be the momentum variable, we define the law of transformation of the wave function with respect to the Poincaré group by the following manifestly covariant formula:

$$(\mathcal{U}(a,\underset{\sim}{\Lambda})\Phi)(p,\omega) = e^{ipa}\Phi(\Lambda^{-1}p,\underset{\sim}{\Lambda}^{-1}\omega) \tag{7.74}$$

or, respectively,

$$(\mathcal{U}(a,\underset{\sim}{\Lambda})\Phi)(p,\overline{\omega}) = e^{ipa}\Phi(\Lambda^{-1}p,\overline{\underset{\sim}{\Lambda}}^{-1}\overline{\omega}). \tag{7.75}$$

If a $\mathfrak{P}_0$-invariant positive-definite (or positive-semidefinite) Hermitian form is defined on such wave functions, then according to the standard construction (§1.1.D) the classes of equivalent wave functions (modulo functions of zero norm) form a dense subspace of some Hilbert space that is transformed according to a unitary representation of $\mathfrak{P}_0$. (We denote this representation again by $\mathcal{U}(a,\underset{\sim}{\Lambda})$.)

Thus, the problem consists in finding a scalar product on the wave functions that provides an irreducible representation of $\mathfrak{P}_0$. We add the further requirement that the class of wave functions so selected should contain the functions of the class $\mathcal{S}(\mathbf{M})$ and that the scalar product of such functions be continuous in the topology of $\mathcal{S}(\mathbf{M})$. Then by appealing to the kernel theorem and the translation-invariance condition, it is not difficult to see that the scalar product we are interested in must have the form

$$\langle\Phi,\Psi\rangle = \frac{1}{((2j)!)^4}\int\left(\frac{\partial}{\partial\omega}\epsilon\frac{\partial}{\partial w}\right)^{2j}\left(\frac{\partial}{\partial\overline{w}}\epsilon\frac{\partial}{\partial\overline{\omega}}\right)^{2j}\times$$
$$\times T(p,w,\overline{w})\delta(p-q)\overline{\Phi(p,\omega)}\Psi(q,\omega)d_4p\,d_4q, \tag{7.76}$$

where $T(p,w,\overline{w})$ is a Lorentz-invariant spinor generalized function transforming according to the representation $\mathfrak{D}^{(j,j)}$ of $SL(2,C)$. (Here for the sake of definiteness, we consider the case of the functions $\Phi(p,\omega)$.) The general form of such a generalized function is given by Proposition 3.6; in our case, $F(p)$ must be taken to be a positive Lorentz-invariant generalized function. In order that the resulting representation $\mathcal{U}(a,\underset{\sim}{\Lambda})$ have positive energy and fixed mass $m \geq 0$, we must set $F(p) = c\delta_m^+(p)$, where $c > 0$. Here we are using the notation

$$\delta_m^+(p) = 2\pi\theta(p^0)\delta(p^2-m^2); \tag{7.77}$$

similarly, we set

$$\delta_m^-(p) = 2\pi\theta(-p^0)\delta(p^2 - m^2). \tag{7.78}$$

Thus, apart from an inessential positive factor, our scalar product on the wave function $\Phi(p,\omega)$ has the form

$$\langle\Phi,\Psi\rangle = \int_{\Gamma_m^+} \frac{1}{((2j)!)^2}\left(\frac{\partial}{\partial\omega}\underset{\sim}{p}\frac{\partial}{\partial\overline{\omega}}\right)^{2j}\overline{\Phi(p,\omega)}\Psi(p,\omega)(dp)_m. \tag{7.79}$$

Similarly for the functions $\Phi(p,\overline{\omega})$ we have

$$\langle\Phi,\Psi\rangle = \int_{\Gamma_m^+} \frac{1}{((2j)!)^2}\left(\frac{\partial}{\partial\omega}\underset{\sim}{p}\frac{\partial}{\partial\overline{\omega}}\right)^{2j}\overline{\Phi(p,\overline{\omega})}\Psi(p,\overline{\omega})(dp)_m. \tag{7.80}$$

We analyse below the corresponding representations $\mathcal{U}(a,\underset{\sim}{\Lambda})$ and, in particular, we shall see that they are irreducible.

In a more general setting, we could have constructed unitary representations of $\mathfrak{P}_0$ in the class of wave functions $\Phi(p,\omega,\overline{\omega})$ that transform according to an arbitrary finite-dimensional irreducible representation $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$. The problem of finding the corresponding scalar product is almost identical to the problem on the general form and expansion in invariants of the Poincaré group, of the covariant two-point function of the quantum fields (see Appendices G,I). This question is closely related to the decomposition of a free Lorentz-covariant quantum field into fields with specified spin (or with specified helicity, in the case of a field of zero mass). We noted above that a representation with positive energy and mass m that is realized in the space of functions $\Phi(p,\omega)$ or $\Phi(p,\overline{\omega})$ is automatically irreducible. The corresponding assertion is false for representations in the class of wave functions $\Phi(p,\omega,\overline{\omega})$ that transform according to the representation $\mathfrak{D}^{j,k}$ for $j,k \neq 0$. These representations decompose according to irreducible representations realized on functions of type $\Phi(p,\omega)$ and $\Phi(p,\overline{\omega})$. For this reason, we restrict ourselves here to the technically simplest realization of "physical" irreducible representations on spinor wave functions of type $\mathfrak{D}^{(j,0)}$ and $\mathfrak{D}^{(0,j)}$ (see, however, Exercises 7.15 and 7.17 below).

Exercise 7.12. Prove that the Pauli-Lubanski vector in the space of wave functions $\Phi(p,\omega,\overline{\omega})$ is given by the following differential operator:

$$W^\lambda = -\frac{1}{4}\left\{\omega\epsilon(\underset{\sim}{e}^\lambda\tilde{p} - \underset{\sim}{p}\tilde{e}^\lambda)\epsilon^{-1}\frac{\partial}{\partial\omega} - \overline{\omega}(\tilde{e}^\lambda\underset{\sim}{p} - \tilde{p}\underset{\sim}{e}^\lambda)\frac{\partial}{\partial\overline{\omega}}\right\}. \tag{7.81}$$

[Hint: $M^{\lambda\mu} = L^{\lambda\mu} + X^{\lambda\mu}$, where $L^{\lambda\mu} = i\left(p^\lambda\frac{\partial}{\partial p_\mu} - p^\mu\frac{\partial}{\partial p_\lambda}\right)$ and $X^{\lambda\mu}$ is defined by (3.56) or (3.57).]

First we consider the case $m > 0$. We fix an integral or half-integral non-negative number $s = j$. Then associated with the scalar product (7.79) is the Hilbert space $\mathfrak{H}^{[m,s]}$ of all complex measurable functions* $\Phi(p,\omega)$ on $\Gamma_m^+ \times \mathbf{C}^2$ that are homogeneous polynomials of degree $2s$ in ω and satisfy the relation

$$\|\Phi\|^2 \equiv \int \frac{1}{((2j)!)^2}\left(\frac{\partial}{\partial\omega}\underset{\sim}{p}\frac{\partial}{\partial\overline{\omega}}\right)^{2j}|\Phi(p,\omega)|^2(dp)_m < \infty. \tag{7.82}$$

The action (7.74) of $\mathfrak{P}_0$ defines a unitary representation in $\mathfrak{H}^{[m,s]}$, denoted by $\mathcal{U}^{[m,s]}$.

Exercise 7.13. (a) Prove that every continuous $\mathfrak{P}_0$-invariant sesquilinear form $a(\Phi,\Psi)$ on $\mathfrak{H}^{[m,s]}$ has the form $a(\Phi,\Psi) = \alpha\langle\Phi,\Psi\rangle$, where $\alpha \in \mathbf{C}$. [Hint: Use the arguments in the derivation of the scalar product (7.79).]

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* Here and in what follows, by the measurability of a function $\Phi(p,\omega)$ we mean that $\Phi(p,\omega)$ is a polynomial in the spinor variable ω with coefficients that are complex measurable functions of $p \in \Gamma_m^+$.

(b) Prove that every bounded operator A in $\mathfrak{H}^{[m,s]}$ commuting with all the operators $\mathcal{U}^{[m,s]}(a,\underset{\sim}{A})$, is a multiple of the identity. [Hint: Consider the sesquilinear form $a(\Phi,\Psi) = \langle\Phi, A\Psi\rangle$.]

Exercise 7.14. Prove formula (7.38) for the square of the Pauli-Lubanski vector in the representation $\mathcal{U}^{[m,s]}$. [Hint: Use (7.81).]

It follows from these exercises that $\mathcal{U}^{[m,s]}$ is an irreducible representation of ρ_0 with positive energy, mass m and spin s.

In the construction of $\mathfrak{H}^{[m,s]}$ we could equally have started from the functions $\Psi(p,\overline{\omega})$ (instead of $\Phi(p,\omega)$) which are homogeneous polynomials of degree $2s = 2j$ in $\overline{\omega}$. The scalar product and the representation of ρ_0 are now defined by (7.80) and (7.75). We leave the reader to verify that the representation of ρ_0 obtained in this way is unitarily equivalent to the representation $\mathcal{U}^{[m,s]}$ described above, where the operator of the unitary equivalence can be defined as follows:

$$\mathfrak{H}^{[m,s]} \ni \Phi(p,\omega) \to \Psi(p,\overline{\omega}) = \frac{1}{(2s)!}\left(\overline{\omega}\frac{\tilde{p}}{m}\epsilon^{-1}\frac{\partial}{\partial\omega}\right)^{2s}\Phi(p,\omega). \tag{7.83}$$

We noted above that the use of the more general representations $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ provides other possibilities for the manifestly covariant realization of the representations $[m,s]$. In this connection, for integral spin the role of the spinor variable can be assigned to a 4-vector. (Cf. Exercise 3.9 and 3.10 in connection with the representations of the Lorentz group.)

Exercise 7.15. Let $m > 0$ and $l = 0,1,2,\ldots$. Consider the Hilbert space of complex functions $f(p,r)$ of the vectors $p \in \Gamma_m^+$ and $r \in \mathbf{M}$, that are homogeneous pseudo-harmonic polynomials of degree l with respect to r and satisfy the "transversality" condition (in the four-dimensional sense)

$$\left(p\frac{\partial}{\partial r}\right)f(p,r) = 0 \tag{7.84}$$

The scalar product is defined by the formula

$$\langle f,f'\rangle = \int \frac{2^l}{(l!)^2}\left(-\frac{\partial}{\partial r}\frac{\partial}{\partial r'}\right)^l \overline{f(p,r)}f'(p,r')(dp)_m, \tag{7.85}$$

and the action of ρ_0 by $f(p,r) \to e^{ipa}f(\Lambda^{-1}p,\Lambda^{-1}r)$. Prove that the representation of ρ_0 so obtained, is unitarily equivalent to $\mathcal{U}^{[m,l]}$. [Hint: Consider the map

$$f(p,r) \to \Phi(p,\omega) = \frac{1}{m^l}\frac{1}{(l!)^2}\left(\omega\epsilon\underset{\sim}{p}\frac{\tilde{\partial}}{\partial r}\omega\right)^l f(p,r).\ ]$$

We now turn to the case $m = 0$. First we suppose that s is an integral or half-integral negative number. We set $s = -j$ and consider the space of all complex measurable functions $\Phi(p,\omega)$ of $p \in \Gamma_0^+$ and $\omega \in \mathbf{C}^2$ that are homogeneous polynomials of degree $2j$ and satisfy (7.82). The equivalence classes of functions modulo functions of zero norm form a Hilbert space $\mathfrak{H}^{[0,s]}$. To avoid complicating the notation, we shall agree to identify each vector in $\mathfrak{H}^{[0,s]}$ with some representative function $\Phi(p,\omega)$ of it.* Formula (7.74) defines a unitary representation ρ_0, denoted by $\mathcal{U}^{[0,s]}$.

Similarly, if $m = 0$ and s is an integral or half-integral non-negative number, then in place of the functions $\Phi(p,\omega)$, we introduce the functions $\Phi(p,\overline{\omega})$ that are homogeneous polynomials of degree $2s$ and satisfy a condition of type (7.82). The

* This convention is not at all uncommon. For example, when one talks about a Hilbert space of functions (on a measure space) it is understood that this Hilbert space is formed not by the functions themselves, but by their equivalence classes (modulo functions that are zero almost everywhere).

corresponding Hilbert space with scalar product (7.80), where $j = s$, and the unitary representation (7.75) are again denoted by $\mathfrak{H}^{[0,s]}$ and $\mathcal{U}^{[0,s]}$.

We apply the result of Exercise 7.13 to the spaces $\mathfrak{H}^{[0,s]}$, which means that the representation $\mathcal{U}^{[0,s]}$ is irreducible. A direct calculation of the Pauli-Lubanski vector shows that $\mathcal{U}^{[0,s]}$ is a representation with positive energy, zero mass and polarization s. (In connection with this, see Exercise 7.16 below.)

We have seen that in the case $m > 0$, replacing the spinor variable ω by $\overline{\omega}$ leads to a unitarily equivalent representation of $\mathfrak{P}_0$. However, this is not so for the case $m = 0$, since the representations $\mathcal{U}^{[0,s]}$ and $\mathcal{U}^{[0,-s]}$ are unitarily inequivalent when $s > 0$. Nevertheless, there exists an explicit covariant construction of the representations $\mathcal{U}^{[0,s]}$ which uses the variable $\overline{\omega}$ for $s < 0$ and ω for $s > 0$. This realization of the representation is very instructive, since it visually demonstrates that a vector state in $\mathfrak{H}^{[0,s]}$ is effectively described by a single-component wave function and that the corresponding particle has just one state of polarization.*

We now consider the case $s < 0$ in detail. We introduce the complex measurable functions $X(p,\overline{\omega})$ on $\Gamma_0^+ \times \mathbf{C}^2$ that are homogeneous polynomials of degree $2|s|$ in $\overline{\omega}$ and satisfy the condition

$$\left(\omega\epsilon \underset{\sim}{p}\frac{\partial}{\partial\overline{\omega}}\right)X(p,\overline{\omega}) = 0. \tag{7.86}$$

This condition means that $X(p,\overline{\omega})$ has the form

$$X(p,\overline{\omega}) = (\overline{\omega}\epsilon\overline{\zeta}_p)^{2|s|}u(p), \tag{7.87}$$

where ζ_p is a vector in $\mathbf{C}^2$ such that $\overline{\omega}\tilde{p}\omega = (\tilde{\omega}\epsilon\overline{\zeta}_p)(\omega\epsilon\zeta_p)$ for all $\omega \in \mathbf{C}^2$ or, equivalently, $\overline{\zeta}_p\tilde{q}\zeta_p = 2pq$ for all $q \in \mathbf{M}$. Making use of this observation we can rewrite (7.86) in the form

$$\frac{1}{(2|s|)!}\left(\overline{\omega}\tilde{p}\underset{\sim}{q}\frac{\partial}{\partial\overline{\omega}}\right)^{2|s|}X(p,\overline{\omega}) = (2pq)^{2|s|}X(p,\overline{\omega}),\ q \in \mathbf{M}. \tag{7.88}$$

We define the norm in the space of such functions X by setting

$$\|X\|^2 = \int V_X(p)(dp)_0,$$

where the function V_X is defined by

$$\left(\frac{\partial}{\partial\omega}\underset{\sim}{q}\frac{\partial}{\partial\overline{\omega}}\right)^{2|s|}|X(p,\overline{\omega})|^2 = (2pq)^{2|s|}V_X(p),\ q \in \mathbf{M}. \tag{7.89}$$

We leave the reader to verify as an exercise that the representation of $\mathfrak{P}_0$ in the Hilbert space of functions X (defined by a formula of type (7.75) suitably modified) is equivalent to $\mathcal{U}^{[0,s]}$, the operator of the unitary equivalence being given by

$$\Phi(p,\omega) \to X(p,\overline{\omega}) = \frac{1}{(2|s|)!}\left(\overline{\omega}\tilde{p}\epsilon^{-1}\frac{\partial}{\partial\omega}\right)^{2|s|}\Phi(p,\omega). \tag{7.90}$$

Similarly, for the case $s > 0$ there is an alternative realization of the representation $\mathcal{U}^{[0,s]}$ in terms of functions $X(p,\omega)$ that are homogeneously and polynomially (of degree $2s$) dependent on ω (instead of $\overline{\omega}$).

* This circumstance can also be seen from the form of the scalar product (7.80), since the matrix $\underset{\sim}{p}$ featuring in it has rank 1 for $p \in \Gamma_0^+$.

Exercise 7.16. Derive formula (7.39) for the Pauli-Lubanski vector in the space $\mathfrak{H}^{[0,s]}$. [Hint: In the case $s < 0$, apply the differential operator (7.81) to $X(p,\overline{\omega})$ and use (7.86).]

The construction of the representation of $\mathfrak{P}_0$ described in Exercise 7.15 goes over with slight modifications to the case $m = 0$. Now for $l \neq 0$, the scalar product (7.85) becomes degenerate, therefore the Hilbert space is formed by equivalence classes of functions $f(p,r)$ modulo functions with zero norm. For $l \neq 0$ the resultant unitary representation of $\mathfrak{P}_0$ is reducible: it decomposes into a direct sum $\mathcal{U}^{[0,1]} \oplus \mathcal{U}^{[0,-1]}$. The case $l = 1$ is of particular practical interest, since the representation $\mathcal{U}^{[0,l]} \oplus \mathcal{U}^{[0,-l]}$ of $\mathfrak{P}_0$ describes a photon. We consider this particular case in greater detail in the following exercise.

Exercise 7.17. Let $m = 0$. Consider the space $\mathfrak{H}'$ of complex 4-vector-valued functions $\mathcal{A}^\mu(p)$ of the vector $p \in \Gamma_0^+$ satisfying the transversality condition (in the four-dimensional sense):

$$p_\mu \mathcal{A}^\mu(p) = 0. \tag{7.91}$$

The scalar product of two functions is defined as follows:

$$\langle \mathcal{A}, \mathcal{A}' \rangle = -\int \overline{\mathcal{A}_\mu(p)} \mathcal{A}'^\mu(p)(dp)_0. \tag{7.92}$$

Let $\mathfrak{H}''$ be the space of vectors in $\mathfrak{H}'$ with zero scalar square. Prove that

(a) the scalar product (7.92) is positive semidefinite,

(b) the vectors in $\mathfrak{H}''$ have the form

$$\mathcal{A}^\mu(p) = p^\mu b(p), \tag{7.93}$$

(c) the action

$$\mathcal{A}^\mu(p) \to e^{ipa} \Lambda^\mu_\nu \mathcal{A}^\nu(\Lambda^{-1}p) \tag{7.94}$$

defines a unitary representation of $\mathfrak{P}_0$ in the quotient space $\mathfrak{H} = \mathfrak{H}'/\mathfrak{H}''$, where this representation is unitarily equivalent to $\mathcal{U}^{[0,1]} \oplus \mathcal{U}^{[0,-1]}$. [Hint: Consider the two maps

$$\mathcal{A}(p) \to X_+(p,\omega) = \omega \epsilon \underset{\sim}{p} \tilde{\mathcal{A}}(p) \omega = 1/2 \omega \epsilon \underset{\sim}{e}^\lambda \tilde{e}^\mu \omega f^{(+)}_{\lambda\mu}(p), \tag{7.95}$$

$$\mathcal{A}(p) \to X_-(p,\overline{\omega}) = \overline{\omega} \tilde{p} \underset{\sim}{\mathcal{A}}(p) \epsilon \overline{\omega} = 1/2 \overline{\omega} \tilde{e}^\lambda \underset{\sim}{e}^\mu \epsilon \overline{\omega} f^{(-)}_{\lambda\mu}(p); \tag{7.96}$$

here

$$f_{\lambda\mu}(p) = p_\lambda A_\mu(p) - p_\mu A_\lambda(p), \tag{7.97}$$

$$f^{(\pm)}_{\lambda\mu}(p) = 1/2(f_{\lambda\mu}(p) \pm 1/2 i \epsilon_{\lambda\mu\rho\sigma} f^{\rho\sigma}(p)).\,] \tag{7.98}$$

7.3. Fock Space of Relativistic Particles

A. Second Quantization Space

In relativistic quantum theory, the result of scattering is described in terms of a system of an arbitrary finite number of non-interacting relativistic particles subject to quantum statistics (see §7.3.E below). Here we concern ourselves with the simplest state space of such a system.

In order to keep the notation short, we suppose that we are dealing only with one type of relativistic particle characterized by its mass $m \geq 0$ and spin $s = 0, 1/2, 1, \ldots$ (for $m > 0$) and helicity $s = 0, \pm 1/2, \pm 1, \ldots$ (for $m = 0$). The Hilbert space $\mathfrak{H}^{[m,s]}$ in §7.2.D will be called the space of one-particle states, which within this subsection will simply be denoted by $\mathfrak{H}_1$. The state of n (≥ 2) non-interacting identical particles will be described by the wave function $\Phi_n(p_1, \ldots, p_n; \overset{(-)}{\omega}_1, \ldots, \overset{(-)}{\omega}_n)$ of n independent momenta $p_1, \ldots, p_n \in \Gamma_m^+$ and the spinor vectors $\omega_1, \ldots, \omega_n \in \mathbf{C}^2$ or on the conjugate vectors $\overline{\omega}_1, \ldots, \overline{\omega}_n$; here the wave function is a homogeneous polynomial of degree $2|s|$ separately in each vector $\overset{(-)}{\omega_1}, \ldots \overset{(-)}{\omega_n}$.* We require further that Φ_n be a measurable function with finite norm:

$$\|\Phi_n\|^2 = \int \frac{1}{((2|s|)!)^n} \prod_{j=1}^{n} \left(\frac{\partial}{\partial \omega_j} \underset{\sim}{p_j} \frac{\partial}{\partial \overline{\omega}_j}\right)^{2|s|} \times$$
$$\times |\Phi_n(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n})|^2 \prod_{j=1}^{n} (dp_j)_m, \tag{7.99}$$

and satisfy the condition

$$\Phi_n^\pi = \Phi_n \quad \text{for all } \pi \tag*{7.100a)}$$

or

$$\Phi_n^\pi = \epsilon(\pi)\Phi_n \quad \text{for all } \pi; \tag{7.100b}$$

here π is an arbitrary permutation of the indices $1, 2, \ldots, n$ in the wave function $\Phi_n(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n})$, so that

$$\Phi_n^\pi(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n}) \equiv \Phi_n(p_{\pi(1)}, \ldots, p_{\pi(n)}; \overset{(-)}{\omega}_{\pi(1)}, \ldots, \overset{(-)}{\omega}_{\pi(n)}); \tag{7.101}$$

and $\epsilon(\pi)$ denotes the parity of the permutation π. Corresponding to the two possibilities in (7.100) we refer to the two "quantum statistics", namely, of the Bose particles (or bosons) and the Fermi particles (or fermions). Conditions (7.100) are sometimes called the principle of indistinguishability of particles subject to quantum statistics, since according to (7.100) any permutation of the variables in the wave function with respect to the individual particles does not alter the state.

Empirical evidence testifies to the fact that particles with integral (or half-integral) spin are subject to Bose (or Fermi) statistics. The theoretical foundation of this law of behaviour is the theorem on the connection between spin and statistics (see Chapter 9).

The space of all measurable functions $\Phi_n(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n})$ with finite norm (but possibly not satisfying any of the conditions (7.100)) forms a Hilbert space called the nth *tensor power* of the Hilbert space $\mathfrak{H}_1$, denoted by $\mathfrak{H}_1^{\otimes n}$. The scalar product of two vect ›rs Φ_n and Ψ_n is given by the formula

$$\langle \Phi_n, \Psi_n \rangle = \int \frac{1}{((2|s|)!)^n} \prod_{j=1}^{n} \left(\frac{\partial}{\partial \omega_j} \underset{\sim}{p} \frac{\partial}{\partial \overline{\omega}_j}\right)^{2|s|} \times$$
$$\times \overline{\Phi_n(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n})} \Psi_n(p_1, \ldots, p_n; \overset{(-)}{\omega_1}, \ldots, \overset{(-)}{\omega_n}) \prod_{j=1}^{n} (dp_j)_m. \tag{7.102}$$

The vectors satisfying (7.100a) (or (7.100b)) distinguish a subspace of $\mathfrak{H}_1^{\otimes n}$, called the nth *symmetric* (or *antisymmetric*) *tensor power* of $\mathfrak{H}_1$; we denote it by $\mathfrak{H}_1^{\vee n}$ (or $\mathfrak{H}_1^{\wedge n}$). The orthogonal projectors in $\mathfrak{H}_1^{\wedge n}$ onto the subspaces $\mathfrak{H}_1^{\vee n}$ and $\mathfrak{H}_1^{\wedge n}$ have the form

$$\operatorname{Sym} \Phi_n = \frac{1}{n!} \sum_{\pi} \Phi_n^\pi, \tag{7.103a}$$

$$\operatorname{Antisym} \Phi_n = \frac{1}{n!} \sum_{\pi} \epsilon(\pi) \Phi_n^\pi; \tag{7.103b}$$

* If we start with another realization of the one-particle space, then the role of these spinor vectors must be assigned to some other quantities or indices of spinor (or vector) type.

here the summation is taken over all permutations π of the indices $1, \ldots, n$. We shall agree that for $n = 1$, all the spaces $\mathfrak{H}_1^{\otimes n}$, $\mathfrak{H}_1^{\vee n}$, $\mathfrak{H}_1^{\wedge n}$ are the same as $\mathfrak{H}_1$ while for $n = 0$ they all coincide with the scalar field $\mathbf{C}$ endowed with the usual scalar products $\overline{\Phi}_0\Psi_0$.

In the complete (physical) Hilbert space, vectors of states with a different number of particles form mutually orthogonal subspaces. Hence for the representation of states with an unspecified number of particles we must introduce the direct sum of the n-particle subspaces.

The direct sum

$$\bigoplus_{n=0}^{\infty} \mathfrak{H}_1^{\otimes n} \tag{7.104}$$

is called the tensor algebra over the Hilbert space $\mathfrak{H}_1$. It is a Hilbert space whose elements (vectors) can be defined by arbitrary sequences $\Phi \equiv \{\Phi_n\}_{n=0}^{\infty}$ such that $\Phi_n \in \mathfrak{H}^{\otimes n}$ and

$$\|\Phi\|^2 \equiv \sum_n \|\Phi_n\|^2 < \infty, \tag{7.105}$$

the scalar product being given by

$$\langle \Phi, \Psi \rangle = \sum_{n=0}^{\infty} \langle \Phi_n, \Psi_n \rangle. \tag{7.106}$$

Clearly $\mathfrak{H}_1^{\otimes n}$ can be identified with the n-particle subspace in the direct sum (7.104) consisting of those vectors Φ for which $\Phi_k = 0$ for $k \neq n$. In particular, the one-dimensional 0-particle subspace $\mathfrak{H}_1^{\otimes 0}$ is called the vacuum; it is spanned by the vector $\Psi_0 \equiv |0\rangle$ with components

$$(\Psi_0)_0 = 1, \ (\Psi_0)_n = 0 \quad \text{for } n \neq 0, \tag{7.107}$$

called the *vacuum vector*. For any vector Φ in the direct sum we can regard Φ_n as the projection of Φ onto $\mathfrak{H}_1^{\otimes n}$. A vector Φ is said to be finite if it has only a finite number of non-zero projections Φ_n. The finite vectors clearly form a dense linear manifold in the direct sum (7.104). The name "tensor algebra" is explained by the fact that a (distributive, associative but not commutative) operation of tensor multiplication $\Phi \otimes \Psi$ is defined on the finite vectors:

$$(\Phi \otimes \Psi)_n(p_1, \ldots, p_n; \overset{(-)}{\omega}_1, \ldots, \overset{(-)}{\omega}_n) =$$

$$= \sum_{k=0}^{n} \Phi_k(p_1, \ldots, p_k; \overset{(-)}{\omega}_1, \ldots, \overset{(-)}{\omega}_k) \Psi_{n-k}(p_{k+1}, \ldots, p_n; \overset{(-)}{\omega}_{k+1}, \ldots, \overset{(-)}{\omega}_n). \tag{7.108}$$

The vectors Φ in the direct sum (7.104) for which each projection Φ_n satisfies (7.100a) and hence belongs to $\mathfrak{H}_1^{\vee n}$, form a Hilbert space

$$\mathcal{F}_{\vee}(\mathfrak{H}_1) = \bigoplus_{n=0}^{\infty} \mathfrak{H}_1^{\vee n}, \tag{7.109a}$$

called the *Fock space of Bose particles* over the one-particle space $\mathfrak{H}_1$. Similarly, if we replace (7.100a) by (7.100b) then we obtain the Hilbert space

$$\mathcal{F}_{\wedge}(\mathfrak{H}_1) = \bigoplus_{n=0}^{\infty} \mathfrak{H}_1^{\wedge n}, \tag{7.109b}$$

called the exterior algebra or *Fock space of Fermi particles* over $\mathfrak{H}_1$. The symmetric (distributive, associative and commutative) tensor product is defined on the finite vectors Φ, Ψ of $\mathcal{F}_\vee(\mathfrak{H}_1)$ by

$$\Phi \vee \Psi' = \operatorname{Sym} \Phi \otimes \Psi, \tag{7.110a}$$

while the (distributive and associative) exterior product is defined on the finite vectors Φ, Ψ in $\mathcal{F}_\wedge(\mathfrak{H}_1)$ by

$$\Phi \wedge \Psi = \operatorname{Antisym} \Phi \otimes \Psi. \tag{7.110b}$$

Exercise 7.18. Prove the identities

$$\Phi \vee \Psi = \Psi \vee \Phi, \tag{7.111a}$$

$$\Phi_k \wedge \Psi_l = (-1)^{kl} \Psi_l \wedge \Phi_k. \tag{7.111b}$$

Thus we have constructed the Fock spaces $\mathcal{F}_\vee(\mathfrak{H}_1)$ and $\mathcal{F}_\wedge(\mathfrak{H}_1)$ of Bose and Fermi particles over the one-particle subspace $\mathfrak{H}_1 = \mathfrak{H}^{[m,s]}$ which transform according to a unitary irreducible representation of the Poincaré group P_0. Although this is the case of particular interest to us, we draw attention to the fact that the specifics of the space $\mathfrak{H}_1$ are not, in fact, made use of anywhere (apart from in the definition of the tensor power $\mathfrak{H}_1^{\otimes n}$). Therefore the above construction of the Fock space can be carried out word for word on an arbitrary Hilbert space $\mathfrak{H}_1$; here we need to use the general definitions of tensor product and direct sum given in §1.1.E. The spaces $\mathfrak{H}_1^{\vee n}$ and $\mathfrak{H}_1^{\wedge n}$ which are the terms in the direct sums (7.109a) and (7.109b) can naturally be regarded as the eigenspaces of the (essentially self-adjoint) operator of the number of particles N defined (on the finite vectors) by the formula

$$(N\Phi)_n = n\Phi_n.$$

The construction of the Fock space $\mathcal{F}_\vee(\mathfrak{H}_1)$ or $\mathcal{F}_\wedge(\mathfrak{H}_1)$ in terms of a given Hilbert space $\mathfrak{H}_1$ is called (Bose or Fermi) *second quantization.* It has a functorial character in the sense that the operators between the one-particle spaces also are subject to second quantization. We restrict ourselves to the important case of the second quantization of unitary operators. Let U be a unitary operator in $\mathfrak{H}_1$; then for any $n = 1, 2, \ldots$, the tensor product $U^{\otimes n} \equiv U \otimes \ldots \otimes U$ of n copies of U is a unitary operator in $\mathfrak{H}_1^{\otimes n}$ (see Exercise 1.44). The operator $U^{\otimes n}$ leaves the subspaces $\mathfrak{H}_1^{\vee n}$ and $\mathfrak{H}_1^{\wedge n}$ invariant; the restriction of $U^{\otimes n}$ to each of these subspaces is denoted by $U^{\vee n}$ ($U^{\wedge n}$) and is called the n-th *(anti-)symmetric tensor power* of U. It is clearly a unitary operator. Furthermore, for $n = 0$ we may suppose that $U^{\otimes n} = U^{\vee n} = U^{\wedge n} = 1$ (the identity operator in $\mathbf{C}$). By the second quantization of the unitary operator U we mean the unitary operator $\mathcal{F}_\vee(U)$ in $\mathcal{F}_\vee(\mathfrak{H})$ (or $\mathcal{F}_\wedge(U)$ in $\mathcal{F}_\wedge(\mathfrak{H})$ such that

$$(\mathcal{F}_\vee(U)\Psi)_n = U^{\vee n}\Psi_n, \tag{7.112a}$$

$$(\mathcal{F}_\wedge(U)\Psi)_n = U^{\wedge n}\Psi_n. \tag{7.112b}$$

Exercise 7.19. (a) Prove that for a unitary representation U_g of the Lie group G in $\mathfrak{H}_1$, the operators $\mathcal{F}_\vee(U_g)$ and $\mathcal{F}_\wedge(U_g)$ form unitary representations of G in $\mathcal{F}_\vee(\mathfrak{H}_1)$ and $\mathcal{F}_\wedge(\mathfrak{H}_1)$.

(b) We define the one-parameter abelian gauge group U_α in $\mathfrak{H}_1$ by setting $(U_\alpha \Phi) = e^{i\alpha}\Phi$, $\Phi \in \mathfrak{H}_1$. Prove that $\mathcal{F}_{\vee,\wedge}(U_\alpha) = e^{i\alpha N}$, where N is the operator of the number of particles.

(c) Prove that the representation $\mathcal{U}^{[m,s]}(a, \underset{\sim}{\Lambda})$ of P_0 in $\mathfrak{H}^{[m,s]}$ induces a unitary representation $\mathcal{F}_{\vee,\wedge}(\mathcal{U}^{[m,s]}(a, \underset{\sim}{\Lambda}))$ in the Fock space $\mathcal{F}_{\vee,\wedge}(\mathfrak{H}^{[m,s]})$, which satisfies the spectral condition, each P_0-invariant vector in the Fock space being collinear with the vacuum vector; in the case $m > 0$, the strong spectral condition holds with mass gap m.

Part (c) of this exercise shows the consistency of the use of the term "vacuum vector" in two different contexts (in the spectral condition and in the second quantization formalism).

B. CONNECTION WITH (ANTI-)COMMUTATION RELATIONS

If the vector Φ in (7.110) is regarded as a fixed one-particle vector, then the map $\Psi \to \sqrt{N}\Phi \vee \Psi$ (or $\Psi \to \sqrt{N}\Phi \wedge \Psi$ is a linear operator on the finite vectors Ψ of the Fock space of Bose (or Fermi) particles.* It has the property that an n-particle vector is taken to an $(n+1)$-particle one, therefore, it is called the *creation operator* of a particle with wave function Φ, denoted by $a^*(\Phi)$. Thus in the Bose case, the creation operator is defined by the formula

$$a^*(\Phi)\Psi = \sqrt{N}\Phi \vee \Psi, \tag{7.113a}$$

or in more detail (in the realization of the previous subsection for $\mathfrak{H}^{[m,s]}$)

$$(a^*(\Phi)\Psi)_n(p_1, \dots, p_n; \overset{(-)}{\omega}_1, \dots \overset{(-)}{\omega}_n) =$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Phi(p_j; \overset{(-)}{\omega}_j)\Psi_{n-1}(p_1 \dots \widehat{p}_j \dots p_n; \overset{(-)}{\omega}_1 \dots \widehat{\overset{(-)}{\omega}}_j \dots \overset{(-)}{\omega}_n); \tag{7.114a}$$

here the sign $\widehat{\ }$ above an argument means that this argument is to be omitted; for $n = 0$, the left hand side of this equality is defined to be zero. For the Fermi case we have the analogous formula:

$$a^*(\Phi)\Psi = \sqrt{N}\Phi \wedge \Psi, \tag{7.113b}$$

$$(a^*(\Phi)\Psi)_n(p_1 \dots p_n; \overset{(-)}{\omega}_1 \dots \overset{(-)}{\omega}_n) =$$

$$= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{j-1} \Phi(p_j; \overline{\omega}_j)\Psi_{n-1}(p_1 \dots \widehat{p}_j \dots p_n; \overset{(-)}{\omega}_1 \dots \widehat{\overset{(-)}{\omega}}_j \dots \overset{(-)}{\omega}_n). \tag{7.114b}$$

It is easy to see that the adjoint operator $(a^*(\Phi))^* \equiv a(\Phi)$ is also defined on all finite vectors of the Fock space. It takes an n-particle vector to an $(n-1)$-particle one and is therefore called the *annihilation operator* of the particle with wave function Φ. Its action is defined by the formula

$$a(\Phi)(\Psi_1 \vee \dots \vee \Psi_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} \langle \Phi, \Psi_j \rangle \Psi_1 \vee \dots \widehat{\Psi}_j \dots \vee \Psi_{n+1} \tag{7.115a}$$

in the Bose case and by

$$a(\Phi)(\Psi_1 \wedge \dots \wedge \Psi_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n+1} (-1)^{j-1} \langle \Phi, \Psi_j \rangle \Psi_1 \wedge \dots \widehat{\Psi}_j \dots \wedge \Psi_{n+1} \tag{7.115b}$$

in the Fermi case; here $\Psi_1, \dots, \Psi_{n+1}$ are arbitrary one-particle vectors. In particular for $\mathfrak{H}_1 = \mathfrak{H}^{[m,s]}$ we have

$$(a(\Phi)\Psi)_n(p_1 \dots p_n; \overset{(-)}{\omega}_1 \dots \overset{(-)}{\omega}_n = \sqrt{n+1} \int \frac{1}{(2|s|)!} \Big(\frac{\partial}{\partial \omega} \tilde{p} \frac{\partial}{\partial \overline{\omega}} \Big)^{2|s|} \times$$

* The factor $\sqrt{N}$ (the positive square root of the operator of the number of particles) in the definition of the creation operator guarantees that the canonical commutation relations (7.119) (see below) between the Hermitian conjugate operators a and a^* will hold.

$$\times\overline{\Phi(p;\overset{(-)}{\omega})}\Psi_{n+1}(p,p_1,\ldots,p_n;\overset{(-)}{\omega},\overset{(-)}{\omega}_1,\ldots,\overset{(-)}{\omega}_n)(dp)_m \tag{7.116}$$

for both the Bose and Fermi cases. We must add to this, that the operator $a(\Phi)$ acts on the vacuum vector $|0\rangle$ to give zero. We draw attention to the fact that the operator $a(\Phi)$ depends antilinearly on Φ in contrast to $a^*(\Phi)$ which acts linearly on Φ.

Exercise 7.20. Prove the identities

$$a^*(\Phi_1)\ldots a^*(\Phi_n)|0\rangle = \sqrt{n!}\Phi_1 \vee \ldots \vee \Phi_n \tag{7.117a}$$

in the Bose case and

$$a^*(\Phi_1)\ldots a^*(\Phi_n)|0\rangle = \sqrt{n!}\Phi_1 \wedge \ldots \wedge \Phi_n \tag{7.117b}$$

in the Fermi case.

What is the role of the creation and annihilation operators ? Firstly, it follows from Exercise 7.20 that the polynomials of the creation operators act on the vacuum vector $|0\rangle$ to form a set of vectors that is dense in the Fock space. It then turns out that the creation and annihilation operators can be characterized by a very simple set of properties. Firstly, the creation and annihilation operators are adjoint to each other:*

$$a^*(\Phi) = a(\Phi)^*. \tag{7.118}$$

Secondly, they satisfy the following algebraic relations:

$$[a(\Phi), a(\Phi')]_{\mp} = 0 = [a^*(\Phi), a^*(\Phi')]_{\mp}, \tag{7.119a}$$

$$[a(\Phi), a^*(\Phi')]_{\mp} = \langle \Phi, \Phi'\rangle \cdot \mathbf{1}. \tag{7.119b}$$

Here and in what follows the expression $[A,B]_- \equiv [A,B] \equiv AB - BA$ denotes the commutator of A and B, while $[A,B]_+ \equiv AB + BA$ denotes the anticommutator. In (7.119) it is to be understood that the commutator is used for the Bose particles and the anticommutator for the Fermi particles. The third property expresses the action of the creation and annihilation operators on the vacuum:

$$a(\Phi)|0\rangle = 0, \tag{7.120}$$

$$a^*(\Phi)|0\rangle = \Phi. \tag{7.121}$$

Exercise 7.21. (a) Derive the (anti-)commutation relations (7.119) from the definition of the creation and annihilation operators.

(b) Prove that in the Fermi case, the creation and annihilation operators $a^*(\Phi)$ and $a(\Phi)$ are bounded and have norm $||\Phi||$.

In the Bose case, the relations (7.118) and (7.119) imply that the creation and annihilation operators satisfy the canonical commutation relations (CCR's) in the complex form. We mentioned in §6.4.B that it is more convenient to deal with the exponential form of the CCR's in which only bounded operators feature. Therefore in accordance with §6.4.B (in particular, Exercise 6.43) we define for any $\Phi \in \mathfrak{H}_1$ the operator:

$$W_0(\Phi) = e^{-1/2||\Phi||^2} e^{-a^*(\Phi)} e^{a(\Phi)}. \tag{7.122}$$

* This equality, as well as the later ones, is to be understood in the sense of equality of operators defined on all finite vectors.

The operator-valued exponents occurring here are well defined (as series of creation and annihilation operators) on the finite vectors and, more generally, on the set $\mathfrak{M}$ of vectors Ψ of the Fock space $\mathcal{F}_\vee(\mathfrak{H}_1)$ satisfying the estimate (in which the numbers $c(\Psi) \geq 0$, $d(\Psi) \geq 0$ depend on Ψ):

$$|\Psi_n| \leq \frac{c(\Psi)^n d(\Psi)}{\sqrt{n!}}, \ n = 0, 1, 2, \ldots \tag{7.123}$$

Exercise 7.22. (a) Prove that the operators $e^{-a^*(\Phi)}$ and $e^{a(\Phi)}$ take the set $\mathfrak{M}$ into itself. [Hint: Use the estimate

$$\|a^{(*)}(\Phi)^k \Psi_n\| \leq \sqrt{\frac{(n+k)!}{n!}} \|\Phi\|^k \cdot \|\Psi_n\|, \tag{7.124}$$

where $\Phi \in \mathfrak{H}_1$, $\Psi_n \in \mathfrak{H}_1^{\vee n}$, $a^{(*)}(\Phi)$ is either $a(\Phi)$ or $a^*(\Phi)$.]

(b) Prove the operator equality on $\mathfrak{M}$:

$$e^{a(\Phi)} e^{a^*(\Phi')} = e^{(\Phi,\Phi')} e^{a^*(\Phi')} e^{a(\Phi)}. \tag{7.125}$$

(c) Prove that the operators $W_0(\Phi)$ are unitary and define a representation of the CCR's in the sense of (6.84) over $\mathfrak{H}_1$ in the Hilbert space $\mathcal{F}_\vee(\mathfrak{H}_1)$ with cyclic vector $|0\rangle$.

This exercise shows that a system of CCR's $W_0(\Phi)$ is defined in $\mathcal{F}_\vee(\mathfrak{H}_1)$ with cyclic vector $|0\rangle$; it is called the Fock system or *Fock representation* of the CCR's over the Hilbert space $\mathfrak{H}_1$. According to Proposition 6.15, such a representation is uniquely defined (to within unitary equivalence) by the vacuum characteristic functional

$$E_0(\Phi) = \langle 0|W_0(\Phi)|0\rangle = \exp(-\tfrac{1}{2}\|\Phi\|^2) \tag{7.126}$$

(the first equality serves as the definition, while the second follows from (7.122)). We have already considered this characteristic functional in §6.4.B, therefore on the basis of Proposition 6.15 we can assert that the CCR representation corresponding to it is unitarily equivalent to the Fock representation. The following exercise demonstrates the irreducibility of the Fock representation.

Exercise 7.23. (a) Suppose that the boundary operator A in $\mathcal{F}_\vee(\mathfrak{H}_1)$ commutes with all the operators $W_0(\Phi)$. Prove that $A|0\rangle = a|0\rangle$, where $a = \langle 0|A|0\rangle$. [Hint: The left hand side of the relation

$$\langle 0|\exp(1/2\|\lambda\Phi\|^2)W_0(\lambda\Phi)A|0\rangle = \langle 0|A\exp(1/2\|\lambda\Phi\|^2)W_0(\lambda\Phi)|0\rangle$$

is complex-analytically dependent on $\lambda \in \mathrm{C}$, while the right hand side is complex-analytically dependent on $\bar{\lambda}$ and therefore does not, in general, depend on λ; use this fact to derive the equality $\langle 0|W_0(\Phi)A|0\rangle = a\langle 0|W_0(\Phi)|0\rangle$ and then use the cyclicity of the vacuum.]

(b) Prove that the system of CCR's $W_0(\Phi)$ is irreducible in $\mathcal{F}_\vee(\mathfrak{H}_1)$. [Hint: Use part (a) of this exercise.]

We can draw the following conclusion from what we have said so far.

Proposition 7.4. *The cyclic representation of the CCR's $W(\Phi)$ over the Hilbert space $\mathfrak{H}_1$ in some Hilbert space $\mathcal{H}$ with characteristic functional $E(\Phi)$ is unitarily equivalent to the Fock representation $W_0(\Phi)$ if and only if there exists in $\mathcal{H}$ a cyclic vector Ψ_0 such that*

$$\langle \Psi_0, W(\Phi)\Psi_0\rangle = \exp(-\tfrac{1}{2}\|\Phi\|^2) \quad \text{for all } \Phi \in \mathfrak{H}_1, \tag{7.127}$$

or, equivalently, if there exists in $\mathcal{F}_\vee(\mathfrak{H}_1)$ a vector Ψ such that

$$\langle \Psi, W_0(\Phi)\Psi\rangle = E(\Phi) \quad \text{for all } \Phi \in \mathfrak{H}_1. \tag{7.128}$$

Exercise 7.24. (a) Let $\mathfrak{H}_1$ be the complexification of the real Hilbert space $\mathcal{E}$ so that $\mathfrak{H}_1 = \mathcal{E} + i\mathcal{E}$ (see §6.4.B) and suppose that for $\rho > 0$, $W^{(\rho)}(f + ig)$ is a representation of the CCR's constructed from the Fock representation according to the formula

$$W^{(\rho)}(f + ig) = W_0\big(\rho f + i\tfrac{1}{\rho}g\big), \quad f, g \in \mathcal{E}.$$

Prove that for $\rho \neq 1$ the representations W_0 and $W^{(\rho)}$ are not unitarily equivalent. [Hint: Argue by the method of contradiction, supposing that there exists a vector Ψ such that

$$\langle\Psi|W_0(f + ig)|\Psi\rangle = \exp\Big(-\tfrac{1}{2}\big(\|\rho f\|^2 + \|\tfrac{1}{\rho}g\|^2\big)\Big), \quad f, g \in \mathcal{E}.$$

Show further that for any unit vector $f \in \mathcal{E}$, the projection of Ψ onto the subspace $M(f)$ in $\mathcal{F}_\vee(\mathfrak{H}_1)$ generated by the vectors $a^*(f)^n|0\rangle$ ($n = 1, 2, \ldots$), has non-zero length which is independent of f. Finally, choose an infinite orthonormal sequence $f_1, f_2, \ldots$ in $\mathcal{E}$ and use the orthogonality of the subspaces $M(f_1), M(f_2), \ldots$ in order to obtain the required contradiction.]

(b) With the same notation, prove that the representations $W^{(\rho_1)}$ and $W^{(\rho_2)}$ are not unitarily equivalent for $\rho_1 \neq \rho_2$. [Hint: It would follow from the unitary equivalence of $W^{(\rho_1)}$ and $W^{(\rho_2)}$ that $W^{(\rho)}$ and W_0 would be unitarily equivalent for $\rho = \rho_1/\rho_2$.]

We have been examining the characteristic properties of the creation and annihilation operators in the Bose case. In the Fermi case, the situation is similar and even simpler in view of the boundedness of the creation and annihilation operators. Here it is appropriate to give the general notion of a system (or representation) of *canonical anticommutation relations* (CAR's) over a Hilbert space $\mathfrak{H}_1$: by this we mean a set of bounded operators $a(\Phi)$ and $a^*(\Phi)$ depending on the vector $\Phi \in \mathfrak{H}_1$ acting in some Hilbert space $\mathcal{H}$ and satisfying the conditions (7.118) and (7.119) (using the anticommutator version in (7.119)). We must add to this some reasonable continuity condition (say, that $a(\Phi)$ and $a^*(\Phi)$ be weakly continuous with respect to Φ when Φ runs through an arbitrary finite-dimensional subspace of $\mathfrak{H}_1$). The creation and annihilation operators in the Fock space $\mathcal{F}_\wedge(\mathfrak{H}_1)$ form a system of CAR's called the *Fock representation* of the CAR's over $\mathfrak{H}_1$. In §6.4.C we introduced the notion of the algebra of CCR's starting from a single fixed (in fact Fock) representation of the CCR's. The algebra of CAR's over $\mathfrak{H}$ is introduced in precisely the same way: it is the C^*-algebra of operators in $\mathcal{F}_\wedge(\mathfrak{H}_1)$ generated by the operators $a(\Phi)$ and $a^*(\Phi)$; in other words, it is the minimal C^*-algebra containing the operators $a(\Phi)$ and $a^*(\Phi)$ in $\mathcal{F}_\wedge(\mathfrak{H}_1)$. In the case of infinite-dimensional $\mathfrak{H}_1$ there is a set of unitarily inequivalent representations of the CAR's as well as representations of the algebra of CAR's; the relation between the representations of the CAR's and the representations of the algebra of the CAR's is the same as for the case of the CCR's (see §6.4.C). The following result characterizes representations of the CAR's that are unitarily equivalent to the Fock representation.

Proposition 7.5. *A representation $a(\Phi)$, $a^*(\Phi)$ of the CAR's over $\mathfrak{H}_1$ is unitarily equivalent to the Fock representation if and only if there exists in the space of the representation a cyclic vector Ψ_0 (the vacuum vector) such that*

$$a(\Phi)\Psi_0 = 0 \quad \text{for all } \Phi \in \mathfrak{H}. \tag{7.129}$$

Exercise 7.25. Prove Proposition 7.5.

Examples of representations of the CAR's that are unitarily inequivalent to the Fock representation can be constructed according to the same scheme as for the case of the CCR's. Let $\mathfrak{H}_1$ be the complexification of the infinite-dimensional real Hilbert space $\mathcal{E}$ and let $a^{(*)}(\Phi)$ be the creation and annihilation operators in $\mathcal{F}_\wedge(\mathfrak{H}_1)$. It is not difficult to verify that for any $\rho > 0$ the operators

$$a^{(*)}_{(\rho)}(f + ig) = a^{(*)}(\rho f + i\tfrac{1}{\rho}g) \tag{7.130}$$

define a system of CAR's over $\mathfrak{H}_1$.

Exercise 7.26. Prove that the representations of the CAR's $a^{(*)}_{(\rho_1)}$ and $a^{(*)}_{(\rho_2)}$ over the infinite-dimensional space $\mathfrak{H}_1$ are unitarily inequivalent for $\rho_1 \neq \rho_2$. [Hint: Use the argument of Exercise 7.24.]

We can make another important conclusion from what has been said: the creation and annihilation operators in both the Bose and Fermi cases form an irreducible set of operators in the Fock space, therefore every (say, bounded) operator in Fock space is — in the generalized sense (see §6.1.C) — a function of them. Furthermore, there are formulae for reducing operators to normal form by representing operators in Fock space as a series in so-called normal products of creation and annihilation operators (the creation operators being placed to the left of the annihilation operators).

Exercise 7.27. Let $\{f_\alpha\}$ be an orthonormal basis in the one-particle Hilbert space $\mathfrak{H}_1$. Prove that the operator of the number of particles N in the Fock space $\mathcal{F}_{\vee,\wedge}(\mathfrak{H}_1)$ can be represented as a series that converges on the finite vectors:

$$N = \sum_\alpha a^*(f_\alpha)a(f_\alpha). \tag{7.131}$$

C. Covariant Creation and Annihilation Operators

In our previous discussion we have supposed for the sake of simplicity that there is only one type of particle. In a more realistic situation there is a finite or countable set of different species of particle; we distinguish them by the symbol κ, which in fact denotes a collection of quantum numbers of particles which must be prescribed along with m_κ, s_κ (the mass of the particle and spin for $m_\kappa > 0$ or helicity for $m_\kappa = 0$) for a complete characterization of the particle. The Hilbert space of one particle of type κ will be denoted by $\mathfrak{H}^{[m_\kappa, s_\kappa, \kappa]}$ or $\mathfrak{H}^{[\kappa]}$; it is transformed according to an irreducible representation of $\mathfrak{P}_0$ and therefore can be realized, for example, by spinor wave functions (§7.2.D). The superselection quantum numbers must be specially chosen (as was remarked in §6.2.C, in elementary particle physics the role of these numbers is played by the electric Q, baryonic B and leptonic L charge). We suppose that an operation of charge conjugation $\kappa \to \overline{\kappa}$ is defined on the set $\{\kappa\}$ of species of particle such that $\overline{\overline{\kappa}} = \kappa$, $m_{\overline{\kappa}} = m_\kappa$, $s_{\overline{\kappa}} = s_\kappa$ for $m_\kappa > 0$ and $s_{\overline{\kappa}} = -s_\kappa$ for $m_\kappa = 0$; here $\overline{\kappa}$ is called the antiparticle of the particle κ (in the scheme of superselection rules defined by the charges, the particle and antiparticle have charges that are equal in magnitude but of opposite sign). The case of $\overline{\kappa} = \kappa$ is not excluded. More generally, if κ and $\overline{\kappa}$ have the same superselection quantum numbers, then κ is called a purely neutral particle (or Majorana particle in the case of half-integral spin). An example of such a particle is the photon.*

We now turn to the construction of the state space of a system with an arbitrary number of particles of any type subject to quantum statistics. For this purpose we suppose that among the superselection quantum numbers there is a *statistics type*:

$$\epsilon \equiv \epsilon_\kappa = \begin{cases} + & \text{for bosons,} \\ - & \text{for fermions.} \end{cases} \tag{7.132}$$

* We recall that we classify particle type in terms of the invariants of the proper Poincaré group $\mathfrak{P}^\uparrow_+$. In using the classification one must distinguish the left and right photons by the sign of the helicity.

If there exists in the theory a fermionic charge F which is preserved, then $\epsilon(-1)^F = 1$ (as we recalled in §6.2.C, experiment does in fact point to this, with $F = B + L$).

The connection between spin and statistics is expressed by the formula

$$\epsilon_\kappa \mathcal{U}^{[\kappa]}(0, -1) = 1 \tag{7.133}$$

(cf. (6.21)).

We define the space $\mathfrak{H}_1^B$ of the boson one-particle state vectors and the space $\mathfrak{H}_1^F$ of fermion one-particle state vectors as the direct sum of the spaces $\mathfrak{H}^{[\kappa]}$ taken over all types κ of boson or fermion particles respectively:

$$\mathfrak{H}_1^B = \bigoplus_{\epsilon_\kappa=+} \mathfrak{H}^{[\kappa]}, \quad \mathfrak{H}_1^F = \bigoplus_{\epsilon_\kappa=-} \mathfrak{H}^{[\kappa]}. \tag{7.134}$$

A system with an unspecified number of particles of arbitrary type has as its state space the Fock space

$$\mathfrak{H} = \mathfrak{H}^B \otimes \mathfrak{H}^F, \tag{7.135a}$$

where $\mathfrak{H}^B = \mathcal{F}_\vee(\mathfrak{H}_1^B)$ and $\mathfrak{H}^F = \mathcal{F}_\wedge(\mathfrak{H}_1^F)$ are the direct sums of the (anti-)symmetric tensor powers of the corresponding one-particle spaces:

$$\mathfrak{H}^B = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n^B = \bigoplus_{n=0}^{\infty} (\mathfrak{H}_1^B)^{\vee n}, \tag{7.135b}$$

$$\mathfrak{H}^F = \bigoplus_{n=0}^{\infty} \mathfrak{H}_n^F = \bigoplus_{n=0}^{\infty} (\mathfrak{H}_1^B)^{\wedge n}. \tag{7.135c}$$

We note that $\mathfrak{H}_n^B$ and $\mathfrak{H}_n^F$ are canonically isomorphic to subspaces of $\mathfrak{H}$ and will therefore be identified with these subspaces. We denote by $|0\rangle$ the vacuum vector:

$$|0\rangle = 1 \otimes 1 \in \mathfrak{H}_0^B \otimes \mathfrak{H}_0^F \subset \mathfrak{H}.$$

It is clear that the representations $\mathcal{U}^{[\kappa]}(a, \underset{\sim}{\Lambda})$ of the Poincaré group $\mathfrak{P}_0$ in the one-particle subspaces $\mathfrak{H}^{[\kappa]}$ induce a unitary representation $\mathcal{U}(a, \underset{\sim}{\Lambda})$ of $\mathfrak{P}_0$ in $\mathfrak{H}$ which satisfies the spectral condition with a unique (to within a factor) $\mathfrak{P}_0$-invariant vector, namely the vacuum vector $|0\rangle$ (see Exercise 7.19(c)).

In accordance with the general scheme of second quantization we associate with each type of particle κ and with each one-particle vector $\Phi \in \mathfrak{H}^{[\kappa]}$ a pair of creation and annihilation operators $a_\kappa^*(\Phi)$, $a_\kappa(\Phi)$, defined at least on all the vectors with a finite number of particles. They are characterized by the following properties:

$$a_\kappa^*(\Phi) = (a_\kappa(\Phi))^*, \; a_\kappa(\Phi)|0\rangle = 0, \; a_\kappa^*(\Phi)|0\rangle = \Phi, \tag{7.136}$$

$$[a_\kappa(\Phi), a_{\kappa'}(\Phi')]_\mp = 0, \; [a_\kappa(\Phi), a_{\kappa'}^*(\Phi')]_\mp = \delta_{\kappa\kappa'}\langle\Phi, \Phi'\rangle. \tag{7.137}$$

We use the $+$ sign (that is, anticommutator) if both particles κ and κ' are fermions and the $-$ sign (that is, commutator) in the remaining cases. As usual, $\delta_{\kappa\kappa'}$ is the Kronecker delta.

It is convenient to introduce as well, the pair of operator-valued generalized functions $A^*(p, \omega, \kappa), \overline{A}(p, \bar{\omega}, \kappa)$ or $\overline{A}^*(p, \bar{\omega}, \kappa)$, $A(p, \omega, \kappa)$, which are homogeneous polynomials in ω or $\bar{\omega}$ of degree $2|s_\kappa|$. They are related to the creation and annihilation operators in the following way. For $m_\kappa > 0$ and for $m_\kappa = 0$, $s_\kappa \geq 0$, we associate with an arbitrary test function $f(p)$ in the Schwartz space and an arbitrary vector $\omega \in \mathbf{C}^2$ the one-particle vector $\Phi_{f,\omega} \in \mathfrak{H}^{[\kappa]}$:

$$\Phi_{f,\omega}(p,w) = \frac{1}{(2s_\kappa)!}(w\epsilon\omega)^{2s_\kappa} f(p) \quad \text{for } p \in \Gamma^+_{m_\kappa},\ w \in \mathbf{C}^2. \tag{7.138}$$

By definition, we set

$$\int A^*(p,\omega,\kappa) f(p) d_4p = a^*_\kappa(\Phi_{f,\omega}), \tag{7.139a}$$

$$\int \overline{A}(p,\bar{\omega},\kappa)\overline{f(p)} d_4p = \bar{a}_\kappa(\Phi_{f,\omega}). \tag{7.139b}$$

For $m_\kappa = 0,\ s_\kappa < 0$ we define $\overline{A}^*(p,\bar{\omega},\kappa)$ and $A(p,\omega,\kappa)$ by

$$\int \overline{A}^*(p,\bar{\omega},\kappa) f(p) d_4p = a^*_\kappa(\Psi_{f,\bar{\omega}}), \tag{7.140a}$$

$$\int A(p,\omega,\kappa)\overline{f(p)} d_4p = a_\kappa(\Psi_{f,\bar{\omega}}), \tag{7.140b}$$

where

$$\Psi_{f,\bar{\omega}}(p,\overline{w}) = \frac{1}{(2|s_\kappa|)!}(\overline{w}\epsilon\overline{\omega})^{2|s_\kappa|} f(p). \tag{7.141}$$

For $m_\kappa > 0$ we can equally well work with the operator-valued generalized functions $\overline{A}^*(p,\overline{\omega},\kappa)$, $A(p,\omega,\kappa)$ instead of $A^*(p,\omega,\kappa)$, $\overline{A}(p,\overline{\omega},\kappa)$ (cf. (7.83)):

$$\overline{A}^*(p,\overline{\omega},\kappa) = \frac{1}{(2s_\kappa)!}\left(\overline{\omega}\frac{\tilde{p}}{m_\kappa}\epsilon^{-1}\frac{\partial}{\partial\omega}\right)^{2s_\kappa} A^*(p,\omega,\kappa), \tag{7.142a}$$

$$A(p,\omega,\kappa) = \frac{1}{(2s_\kappa)!}\left(-\omega\epsilon\frac{\underset{\sim}{p}}{m_\kappa}\frac{\partial}{\partial\bar{\omega}}\right)^{2s_\kappa}\overline{A}(p,\overline{\omega},\kappa). \tag{7.142b}$$

It is clear that giving the operator-valued distributions $\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)$ is equivalent to giving the creation and annihilation operators (here $\overset{(-)}{A}{}^{(*)}$ is used instead of $A, \overline{A}, A^*$ or $\overline{A}^*$, and $\overset{(-)}{\omega}$ is used instead of ω or $\overline{\omega}$). We list their most important properties. These operator-valued generalized functions have support on $\Gamma^+_{m_\kappa}$ and satisfy the equation

$$(p^2 - m_\kappa^2)\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa) = 0. \tag{7.143}$$

For $m_k = 0$ the following additional condition holds:

$$\left(\overline{\omega}\tilde{p}\epsilon^{-1}\frac{\partial}{\partial\omega}\right)A^{(*)}(p,\omega,\kappa) = 0 \tag{7.144a}$$

$$\left(\omega\epsilon\underset{\sim}{p}\frac{\partial}{\partial\overline{\omega}}\right)\overline{A}^{(*)}(p,\overline{\omega},\kappa) = 0; \tag{7.144b}$$

as was pointed out in §7.2.D, this condition means that a particle with mass 0 (corresponding to the "physical" irreducible representation of ρ_0) has only one polarization. The characteristic properties (7.136) and (7.137) can be written in terms of $\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)$ in the obvious way. For example, we write down the non-trivial (anti-)commutators:

$$[\overline{A}(p,\overline{\omega},\kappa),\ A^*(p',\omega',\kappa')]_{\mp} = \delta_{\kappa\kappa'}(2\pi)^4\delta(p-p')\delta^+_{m_\kappa}(p)(\overline{\omega}\tilde{p}\omega')^{2|s_\kappa|} =$$

$$= [A(p',\omega',\kappa'),\ \overline{A}^*(p,\overline{\omega},\kappa)]_\mp \tag{7.145}$$

(see the notation of (7.77)). Finally, the operator-valued generalized functions $\overset{(-)}{A}{}^{(*)}$ are subject to the following law of transformation with respect to the symmetries of $\mathfrak{P}_0$:

$$\mathcal{U}(a,\underset{\sim}{\Lambda})\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)\mathcal{U}(a,\underset{\sim}{\Lambda})^{-1} = e^{i\Lambda pa}\overset{(-)}{A}{}^{(*)}(\Lambda p,\overset{(-)}{\underset{\sim}{\Lambda}}\overset{(-)}{\omega},\kappa), \tag{7.146a}$$

$$\mathcal{U}(a,\underset{\sim}{\Lambda})\overset{(-)}{A}(p,\overset{(-)}{\omega},\kappa)\mathcal{U}(a,\underset{\sim}{\Lambda})^{-1} = e^{-i\Lambda pa}\overset{(-)}{A}(\Lambda p,\overset{(-)}{\underset{\sim}{\Lambda}}\overset{(-)}{\omega},\kappa). \tag{7.146b}$$

The explicit Lorentz-covariance of this law makes the operator-valued generalized functions $\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)$ convenient to handle; for this reaon we call them simply the covariant creation and annihilation operators.

Exercise 7.28. Prove the formulae (7.146).

We note that instead of $\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)$ the analogous covariant creation and annihilation operators $\overset{(-)}{a}{}^{(*)}(p,\overset{(-)}{\omega},\kappa)$ are widely used on the mass shell. These are operator-valued generalized functions on the manifold $\Gamma^+_{m_\kappa}$, that is, generalized functions of the three-dimensional momentum $\mathbf{p}$, which runs over $\mathbf{R}^3$ for $m_\kappa > 0$ or over $\mathbf{R}^3\backslash\{0\}$ for $m_\kappa = 0$. The relation between $\overset{(-)}{A}{}^{(*)}$ and $\overset{(-)}{a}{}^{(*)}$ is given by the formula

$$\overset{(-)}{A}{}^{(*)}(p,\overset{(-)}{\omega},\kappa) = \delta^+_{m_\kappa}(p)\overset{(-)}{a}{}^{(*)}(p,\overset{(-)}{\omega},\kappa) = \frac{\pi}{p^0}\delta(p^0-\sqrt{\mathbf{p}^2+m_\kappa^2})\overset{(-)}{a}{}^{(*)}(p,\overset{(-)}{\omega},\kappa). \tag{7.147}$$

To avoid going into details concerning the definition of the product (7.147) for $m_\kappa = 0$, we suppose in this case that $\mathbf{p} \neq 0$ in (7.147). (The restriction $\mathbf{p} \neq 0$ in the construction of the Fock space of particles of zero mass is clearly inessential since the point $\mathbf{p} = 0$ has zero measure with respect to $(dp)_0$.) The non-trivial commutation relations (7.145) take the following form in terms of $\overset{(-)}{a}{}^{(*)}$:

$$\begin{aligned}[\overline{a}(p,\overline{\omega},\kappa), a^*(p',\omega',\kappa')]_\mp &= \delta_{\kappa\kappa'}\cdot 2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')(\overline{\omega}\tilde{p}\omega')^{2|s_\kappa|} = \\ &= [a(p',\omega',\kappa'),\overline{a}^*(p,\overline{\omega},\kappa)]_\mp;\end{aligned} \tag{7.148}$$

here $2p^0(2\pi)^3\delta(\mathbf{p}-\mathbf{p}')$ is the "invariant δ-function" on $\Gamma^+_{m_\kappa}$.

D. Symmetries of the General Poincaré Group

We consider the question of the behaviour of the covariant creation and annihilation operators under spatial reflections; but first we make some general comments.

The Poincaré group $\mathfrak{P}$ is a more extensive group of space-time symmetries than $\mathfrak{P}^\uparrow_+$; so also are its subgroups $\mathfrak{P}^\uparrow$ and $\mathfrak{P}_+$. These groups are generated by $\mathfrak{P}^\uparrow_+$ and the corresponding elements of the group of reflections $\mathfrak{I} = \{e, I_s, I_t, I_{st}\}$. If the automorphisms $\alpha_{a,\Lambda} \equiv \alpha_{a,\Lambda(\underset{\sim}{\Lambda})}$ with $(a,\underset{\sim}{\Lambda}) \in \mathfrak{P}_0$ are already fixed, then it only remains to define the automorphisms or anti-automorphisms $\alpha(I_s)$, $\alpha(I_t)$, $\alpha(I_{st})$ of the algebra of observables corresponding to the reflections. The group character of the symmetry imposes certain constraints on these automorphisms.

Exercise 7.29. (a) Suppose that the automorphisms $\alpha_{a,\Lambda}$ define a symmetry with respect to the group $\mathfrak{P}^\uparrow_+$. Prove that the automorphisms or anti-automorphisms $\alpha(I_s)$, $\alpha(I_t)$, $\alpha(I_{st})$ induce a symmetry with respect to $\mathfrak{P}$ if and only if the following relations hold

$$\alpha(J)\alpha_{a,\Lambda}\alpha(J)^{-1} = \alpha(\sigma_J(a,\Lambda)),\ J = I_s, I_t, I_{st}, \tag{7.149}$$

$$\alpha(I_s)^2 = \alpha(I_t)^2 = \alpha(I_{st})^2 = 1, \tag{7.150a}$$

$$\alpha(I_{st}) = \alpha(I_s)\alpha(I_t) = \alpha(I_t)\alpha(I_s), \tag{7.150b}$$

and $\sigma_J(a, \Lambda)$ is defined by (7.5).

(b) State and prove the corresponding assertions for the groups $\mathfrak{P}^\uparrow$ and $\mathfrak{P}_+$.

In this exercise all the group relations characterizing reflections appear. Nevertheless, one can use different sets of (anti-)automorphisms $\alpha(J)$ for one and the same physical system, and the physical interpretation of the reflections would then be different. The fact is that these relations only fix the transformations of the space-time degrees of freedom under reflections; therefore the physical interpretation of the reflections still depends on how the (anti-)automorphism acts on the internal degrees of freedom, in particular on the charge operators. It is also clear from this why different choices of the automorphisms $\alpha(J)$, $J \in \mathfrak{I}$ are possible: if the system has a group of internal symmetries (the automorphisms of which, by definition, commute with the operations of the space-time symmetries), then upon multiplying the (anti-)automorphisms $\alpha(J)$ by suitable automorphisms of the internal symmetries, it is generally possible to obtain new (anti-)automorphisms $\alpha'(J)$ with the previous group relations but with a new physical interpretation. It would be best, of course, if we could choose the various operations of reflection so that they could be interpreted as operations of pure spatial reflection P, pure time reversal T, pure total reflection PT or combinations of them of type CP, CPT with charge conjugation. However, for systems with internal symmetries, such definitions would inevitably have a conventional aspect.

We now suppose that the group of space-time symmetries with reflections are unitarily/anti-unitarily realized in physical Hilbert space with the standard superselection rules. In accordance with Postulate A.III, this means that there exist unitary or anti-unitary operators $U(I_s)$, $U(I_t)$, $U(I_{st})$ in the physical Hilbert space such that

$$\alpha(J)(A) = U(J)A^{(*)}U(J)^{-1}, \quad J = J_s, I_t, I_{st}, \tag{7.151}$$

for all elements A of the algebra of observables (here, as usual, $A^{(*)}$ denotes A if $U(J)$ is unitary, or A^* if $U(J)$ is anti-unitary). We cannot assert *a priori* that for operators $U(a, \underset{\sim}{\Lambda})$ and $U(J)$, the group relations analogous to the relations (7.149), (7.150) for the (anti-)automorphisms $\alpha_{a,\Lambda}$ and $\alpha(J)$ hold, since the equalities (7.151) define $U(J)$ to within an arbitrary unitary operator in the centre of the von Neumann algebra of observables (and this arbitrariness does not alter the physical interpretation of the reflections). Nevertheless, the relations of type (7.149) turn out to hold automatically; hence we can conclude from the spectral condition A.IV that $U(I_s)$ is unitary and that $U(I_t)$, $U(I_{st})$ are anti-unitary. Consequently corresponding to the spatial reflection I_s is the algebraic automorphism $\alpha(I_s)$, while the anti-automorphisms $\alpha(I_t)$, $\alpha(I_{st})$ correspond to the operations I_t, I_{st}, including time reversal.

Exercise 7.30. (a) Prove the relations

$$U(I_s)U(a, \underset{\sim}{\Lambda})U(I_s)^{-1} = U(\tilde{\sigma}_s(a, \underset{\sim}{\Lambda})) \equiv U(I_s a, e_0 \underset{\sim}{\Lambda}^{*-1}\tilde{e}_0), \tag{7.152a}$$

$$U(I_t)U(a, \underset{\sim}{\Lambda})U(I_t)^{-1} = U(\tilde{\sigma}_t(a, \underset{\sim}{\Lambda})) \equiv U(I_t a, e_0 \underset{\sim}{\Lambda}^{*-1}\tilde{e}_0) \tag{7.152b}$$

$$U(I_{st})U(a, \underset{\sim}{\Lambda})U(I_{st})^{-1} = U(\tilde{\sigma}_{st}(a, \underset{\sim}{\Lambda})) \equiv U(-a, \underset{\sim}{\Lambda}) \tag{7.152c}$$

for all $(a, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0$. [Hint: It follows from (7.26) and (7.151) that the left and right hand sides of any of these relations define the same automorphisms of the algebra of observables; consequently they can only differ by a unitary operator $V(a, \underset{\sim}{\Lambda}) \in \mathfrak{Z}$; verify that the commuting operators $V(a, \underset{\sim}{\Lambda})$ form a unitary representation of $\mathfrak{P}_0$ and use Exercise 7.8.]

(b) Prove that $U(I_s)$ is unitary and that $U(I_t)$ and $U(I_{st})$ are anti-unitary. [Hint: It follows from (7.152a) that

$$U(I_s)P^0U(I_s)^{-1} = \pm P^0;$$

here P^0 is the energy operator; on the right hand side of the equation the plus sign is used if $U(I_s)$ is unitary and the minus sign if $U(I_s)$ is anti-unitary; for $U(I_t)$ and $U(I_{st})$, the arrangement of the signs is the other way round; now use the fact that the energy operator is non-negative.]

We now turn to the question of the behaviour of the creation and annihilation operators under space-time transformations including spatial reflection and time reversal. For this we have to suppose that the unitary operator $\mathcal{U}(I_s)$ and anti-unitary operators $\mathcal{U}(I_t)$, $\mathcal{U}(I_{st})$ are defined in the Fock space $\mathfrak{H}$ and satisfy relations of type (7.152) (in which the role of $U(a,\underset{\sim}{\Lambda})$ is played by the second quantization operators $\mathcal{U}(a,\underset{\sim}{\Lambda})$ in $\mathfrak{H}$) and, furthermore,

$$\mathcal{U}(I_s)|0\rangle = \mathcal{U}(I_t)|0\rangle = \mathcal{U}(I_{st})|0\rangle = |0\rangle. \tag{7.153}$$

We are interested in the symmetries that leave the Fock structure of $\mathfrak{H}$ invariant. This means that

$$\mathcal{U}(J)a^*_\kappa(\Phi)\mathcal{U}(J)^{-1} = a^*_{\kappa_j}(\Phi_j), \quad J = I_s, I_t, I_{st} \tag{7.154}$$

for all types of particle κ and all one-particle vectors $\Phi \in \mathfrak{H}^{[\kappa]}$; here $\kappa_{I_t} = \kappa$, while $\kappa_{I_s} = \kappa_{I_{st}}$ if κ is a massive or scalar-valued massless particle. If not, then $\kappa_{I_s} = \kappa_{I_{st}}$ denotes a particle type with the same quantum numbers as κ apart from the helicity sign: $s_{\kappa_{I_s}} = -s_\kappa$. This definition agrees with the interpretation of the operators $\mathcal{U}(I_s)$, $\mathcal{U}(I_t)$, $U(I_{st})$ as "pure" reflections, since they leave the internal quantum numbers of the particles invariant. However, in theories with massless particles, such operators do not always exist; for example, simply as a result of the absence of the particle type denoted above by $\kappa_{I_s} = \kappa_{I_{st}}$ (see the survey [N3] devoted to the breakdown of P-parity in weak interactions). Therefore the operators $\mathcal{U}_{CP}$, $\mathcal{U}_T \equiv \mathcal{U}(I_t)$, ,$\mathcal{U}_{CPT}$ are considered to be more fundamental; these operators satisfy relations of type (7.152), (7.154) (where the role of I_s, I_t, I_{st} is played by CP, T, CPT), and also

$$\mathcal{U}_J a^*_\kappa(\Phi)\mathcal{U}_J^{-1} = a^*_{\kappa_J}(\Phi_J), \quad J = CP,\ T,\ CPT, \tag{7.155}$$

where $\kappa_{CP} = \kappa_{CPT} = \overline{\kappa}$, $\kappa_T = \kappa$. The operators $\mathcal{U}_{CP}$ and $\mathcal{U}_{CPT}$ take a particle into the antiparticle and are therefore interpreted as combinations of charge conjugation C with a spatial reflection P (or with a total reflection PT) (even if the operations C, P, PT do not exist separately). The general form of such operators $\mathcal{U}_J$ is defined by transformation formulae of the creation (and annihilation) operators which must be combined with a condition of invariance of the vacuum (of type (7.153));

$$\mathcal{U}_{CP}a^*(p,\omega,\kappa)\mathcal{U}_{CP}^{-1} = \eta_{CP}(\kappa)\frac{1}{(2|s_\kappa|)!}\left(\omega\epsilon\underset{\sim}{e}_0\frac{\partial}{\partial\overline{\omega}}\right)^{2|s_\kappa|}\overline{a}^*(I_sp,\overline{\omega},\overline{\kappa}), \tag{7.156a}$$

$$\mathcal{U}_T a^*(p,\omega,\kappa)\mathcal{U}_T^{-1} = \eta_T(\kappa)\frac{1}{(2|s_\kappa|)!}\left(\overline{\omega}\tilde{\underset{\sim}{e}}_0\epsilon^{-1}\frac{\partial}{\partial\omega}\right)^{2|s_\kappa|}a^*(I_sp,\omega,\kappa), \tag{7.156b}$$

$$\mathcal{U}_{CPT}a^*(p,\omega,\kappa)\mathcal{U}_{CPT}^{-1} = \eta_{CPT}(\kappa)\frac{1}{(2|s_\kappa|)!}\left(\overline{\omega}\frac{\partial}{\partial\overline{\zeta}}\right)^{2|s_\kappa|}\overline{a}^*(p,\overline{\zeta},\overline{\kappa}) \tag{7.156c}$$

and similarly for $\overline{a}^*(p,\overline{\omega},\kappa)$ instead of $a^*(p,\omega,\kappa)$; here $\eta_{CP}(\kappa)$, $\eta_T(\kappa)$, $\kappa_{CPT}(\kappa)$ are phase factors (that is, complex numbers with modulus 1).

Exercise 7.31. Starting from (7.155), deduce formulae (7.156).

Along with CP-invariance it is possible to have C-invariance and P-invariance; $\mathcal{U}_{CP}$ then factorizes into a product of unitary operators $\mathcal{U}_C$ and $\mathcal{U}_P \equiv \mathcal{U}(I_s)$ defined in the Fock space by the formulae

$$\mathcal{U}_C a^*(p,\omega,\kappa)\mathcal{U}_C^{-1} = \eta_C(\kappa)a^*(p,\omega,\overline{\kappa}_P), \tag{7.157a}$$

$$\mathcal{U}_P a^*(p,\omega,\kappa)\mathcal{U}_P^{-1} = \eta_P(\kappa)\frac{1}{(2|s_\kappa|)!}\left(\omega\epsilon\underset{\sim}{e}_0\frac{\partial}{\partial\overline{\omega}}\right)^{2|s_\kappa|}a^*(I_sp,\overline{\omega},\kappa_P) \tag{7.157b}$$

with suitable phase factors $\eta_C(\kappa)$, $\eta_P(\kappa)$; one must add to this the analogous formulae for the operators $\overline{a}^*(p,\overline{\omega},\kappa)$.

If the operators $\mathcal{U}_{CP}$, $\mathcal{U}_T$, $\mathcal{U}_{CPT}$ satisfied the same group relations (7.150) as the corresponding (anti-)automorphisms $\alpha(I_s)$, $\alpha(I_t)$, $\alpha(I_{st})$, then these operators, together with the operators $\mathcal{U}(a,\underset{\sim}{\Lambda})$ of the transformations of the proper Lorentz group would generate a unitary/anti-unitary representation of $\tilde{\mathfrak{p}}$ (the covering group of the general Poincaré group (see Exercise 7.5)), and these group operations would lead to definite restrictions on the phase factors $\eta_J(\kappa)$ in the formulae (7.156). In the general case, however, it can be asserted that the left and right hand sides of any of the relations of type (7.150) for the reflection operators will differ by some superselection factor (that is, a superselection unitary operator). If we make use of the freedom available in the definition of the (anti-)automorphisms $\alpha(I_s)$, $\alpha(I_t)$, $\alpha(I_{st})$ (which leads to different physical interpretations of the reflections) and take into account the fact that the reflection operators are defined by (7.154) only to within a unitary superselection operator (which has no physical meaning), then we can attempt to reduce the "multiplication table" of the reflection operators to one or other of the simple standard forms. An analysis of this kind is carried out in the article by Lee and Wick (1966) where conventions are also evolved for the standard sets of phase factors $\eta_J(\kappa)$. We refer the interested reader to this article for details. The following minimal requirements (which are usually confined to elementary particle theory) are common to the various standard forms of the reflection operators:

$$\mathcal{U}_{CPT} = \mathcal{U}_{CP}\mathcal{U}_T, \quad \mathcal{U}_{CPT}^2 = \mathcal{U}(0,-1), \quad \mathcal{U}_{CP}^4 = \mathcal{U}_T^4 = 1. \tag{7.158}$$

The various "multiplication tables" of the reflection operators can be expressed in the language of extensions of the group of reflections $\mathfrak{I}$ (or extensions of the general Poincaré group $\mathfrak{p}$; see Goldberg, 1969).

E. Relativistic Scattering Matrix

The quantum theory of the scattering of relativistic particles is defined by the following components: a physical Hilbert space $\mathcal{H}$ and a unitary representation U of $\mathfrak{p}_0$ in $\mathcal{H}$ (satisfying the spectral condition); the Fock space $\mathfrak{H}$ described in §7.3.C (which is called the Fock space of incoming or outgoing particles, depending on the context); and two Poincaré-invariant linear isometric embeddings Ω^{in} and Ω^{out} from $\mathfrak{H}$ into $\mathcal{H}$. We set

$$\Omega^{\text{in}}\mathfrak{H} = \mathcal{H}^{\text{in}} \subset \mathcal{H}, \quad \Omega^{\text{out}}\mathfrak{H} = \mathcal{H}^{\text{out}} \subset \mathcal{H} \tag{7.159}$$

and we call them spaces of asymptotic states of incoming and outgoing particles, respectively. The linear operator

$$S = (\Omega^{\text{out}})^*\Omega^{\text{in}} \tag{7.160}$$

in Fock space is called the *scattering matrix* (or *scattering operator*, or *S-matrix*). Its norm does not exceed unity. From physical considerations, we must require further that the condition

$$\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}, \tag{7.161}$$

holds, which is clearly equivalent to the requirement that the S-matrix be unitary. If this condition does not hold, the above description of the process of scattering becomes inadequate (because of its incompleteness).

A simple physical picture can be given for this formal scheme. At the initial instant a state is prepared which can be treated as a system of two moving particles separated by a macroscopic distance. Under a certain relation between the velocities and configurations of the particles we can extrapolate the evolution of this state "backwards" in time and suppose that at $t=-\infty$ we have a state vector describing a system of free relativistic particles. In the formal scheme, this corresponds to the isometric operator Ω^{in} mapping the Fock space $\mathfrak{H}$ of incoming particles into the physical Hilbert space $\mathcal{H}$. This operator parametrizes the vectors of the physical Hilbert space $\mathcal{H}$ by vectors of the Fock space. If this parametrization caters for all the vectors of $\mathcal{H}$ (that is, if $\mathcal{H}^{\text{in}} = \mathcal{H}$), then the states of the system under consideration admit a complete interpretation in terms of incoming particles. The corpuscular language becomes inappropriate for the detailed description of the evolution of the states in time for elementary relativistic processes, but — according to physical ideas — the asymptotic behaviour of the system after the act of scattering (mathematically, as $t \to +\infty$) again violates the features of the corpuscular picture; therefore in the formal scheme we introduce another parametrization of the physical space by the Fock space $\mathfrak{H}$ (the outgoing particles) by means of the isometric operator Ω^{out}. A complete corpuscular interpretation, both in terms of the incoming and outgoing particles, is achieved only if

$$\mathcal{H}^{\text{in}} = \mathcal{H} = \mathcal{H}^{\text{out}}; \tag{7.162}$$

in this case we say that the *asymptotic completeness condition* is satisfied.

Thus in the physical Hilbert space $\mathcal{H}$ we have two frames of reference $\Omega^{\text{in}} : \mathfrak{H} \to \mathcal{H}$ and $\Omega^{\text{out}} : \mathfrak{H} \to \mathcal{H}$, and hence there are (in general) two distinct vectors corresponding to one and the same state vector Φ of Fock space:

$$\Phi^{\text{in}} = \Omega^{\text{in}}\Phi, \quad \Phi^{\text{out}} = \Omega^{\text{out}}\Phi. \tag{7.163}$$

The matrix element $\langle \Psi, S\Phi \rangle$ is the amplitude $\langle \Psi^{\text{out}}, \Phi^{\text{in}} \rangle$ of the transition probability $|\langle \Psi^{\text{out}}, \Phi^{\text{in}} \rangle|^2$ between the two states (with vectors Φ^{in} and Ψ^{out}), the first of which shows up as the state of a system of incoming particles with vector $\Phi \in \mathfrak{H}$, and the second as the state of a system of outgoing particles with vector $\Psi \subset \mathfrak{H}$ (here it must be assumed that Φ^{in} and Ψ^{out} are pure state vectors).

The above scheme does not include processes of decaying particles (and the corresponding processes reversed in time) since it deals with an idealized situation when all the particles have a precisely fixed mass, and this automatically implies that they are stable. In fact, let Φ be an arbitrary one-particle state of Fock space, say, $\Phi \in \mathfrak{H}^{[\kappa]} \subset \mathfrak{H}$ and let Ψ be the projection of $S\Phi$ onto the $(n-1)$-particle subspace corresponding to the particles $\kappa_1, \ldots, \kappa_{n-1}$ (for $n > 2$); then Ψ is an eigenvector (with finite norm) for the operator of the square of the mass $P^2 \equiv P_\mu P^\mu$, therefore Ψ, as a function on $\Gamma^+_{m_1} \times \ldots \times \Gamma^+_{m_{n-1}}$ has support on a set of zero measure and hence $\Psi = 0$. It is clear from this that $\langle \Psi, S\Phi \rangle = 0$ no matter what the one-particle vector Φ and the $(n-1)$-particle vector Ψ in Fock space are (for $n > 2$). In other words, there are no decay processes. This result is no surprise from the point of view of the phenomenology of elementary particles where unstable particles are characterized by non-zero mass width. At the present moment there is no simple consistent scheme for processes of scattering and decay of (unstable) particles* with a non-discrete mass spectrum (and there is even no precise definition of this kind of particle).

We fix some reaction channel

$$\kappa_1 + \ldots + \kappa_\nu \to \kappa_{\nu+1} + \ldots + \kappa_n \tag{7.164}$$

in which n particles participate (with types $\kappa_1, \ldots, \kappa_\nu$ for the incoming particles and $\kappa_{\nu+1}, \ldots, \kappa_n$ for the outgoing particles). The part of the scattering operator

* In quantum field theory there are reduction formulae which express the scattering amplitude in terms of the field operators (more precisely, in terms of the Green's functions of the fields; see Chapter 13). In the phenomenological estimates of the probability of decay in terms of perturbation theory, suitable "reduction formulae" are used which "work" satisfactorily, even though they have (for the reason we have indicated) no proper theoretical justification.

corresponding to this channel can be characterized either by all the possible matrix elements $\langle \Psi, S\Phi \rangle$ between arbitrary vectors of the form

$$\Phi = a^*_{\kappa_\nu}(\Phi_\nu)\dots a^*_{\kappa_1}(\Phi_1)|0\rangle, \tag{7.165a}$$
$$\Psi = a^*_{\kappa_{\nu+1}}(\Phi_{\nu+1})\dots \Phi^*_{\kappa n}(\Phi_n)|0\rangle; \tag{7.165b}$$

or by the quantity

$$\begin{aligned}\langle 0|\overset{(-)}{\mathcal{A}}(p_n, \overset{(-)}{\omega}_n, \kappa_n)\dots\overset{(-)}{\mathcal{A}}(p_{\nu+1}, \overset{(-)}{\omega}_{\nu+1}, \kappa_{\nu+1})(S-1)\overset{(-)}{\mathcal{A}}{}^*(p_\nu, \overset{(-)}{\omega}_\nu, \kappa_\nu)\dots\\ \dots\overset{(-)}{\mathcal{A}}{}^*(p_1, \overset{(-)}{\omega}_1, \kappa_1)|0\rangle = (2\pi)^4 i\delta\Big(\sum_{j=1}^{\nu} p_j - \sum_{k=\nu+1}^{n} p_k\Big)\mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\dots\\ \dots; p_{\nu+1}, \overset{(-)}{\omega}_{\nu+1}, \overline{\kappa}_{\nu+1}|p_\nu, \overset{(-)}{\omega}_\nu, \kappa_\nu;\dots;p_1, \overset{(-)}{\omega}_1, \kappa_1).\end{aligned} \tag{7.166}$$

Both parts of this equality are generalized functions in $\mathcal{S}'(\mathrm{M}^n)$ in the variables $p_1, \dots, p_n$ and homogeneous polynomials in $\overset{(-)}{\omega}$ (of degree $2|s_{\kappa_j}|$ in $\overset{(-)}{\omega}_j$). The separation of the difference between the total 4-momenta of the incoming and outgoing particles as a multiple of the δ-function takes into account the law of conservation of 4-momentum which follows from the condition of translation invariance and the more general condition of Poincaré-invariance of the S-matrix:

$$\mathcal{U}(a, \underset{\sim}{\Lambda}) S \mathcal{U}(a, \underset{\sim}{\Lambda})^{-1} = S \tag{7.167}$$

(which in turn is a corollary of the Poincaré-invariance of the isometric operators Ω^{in} and Ω^{out}). Equality (7.166) serves as the definition of the generalized function $\mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\dots|\dots, p_1, \overset{(-)}{\omega}_1, \kappa_1)$, called the *spinor amplitude* of the n-particle process (7.164). By virtue of the law of conservation of 4-momentum, we can suppose that it is a generalized function of $p_1, \dots, p_n$ defined only on the $4(n-1)$-dimensional plane*

$$\Pi = \Big\{(p_1, \dots, p_n) \in \mathrm{M}^n : \sum_{j=1}^{\nu} p_j - \sum_{k=\nu+1}^{n} p_k = 0\Big\}. \tag{7.168}$$

The Lorentz-invariance of the S-matrix implies the Lorentz-covariance of the spinor amplitudes:

$$\begin{aligned}\mathcal{T}(\Lambda(\underset{\sim}{\Lambda})p_n, \overset{(-)}{\underset{\sim}{\Lambda}}\overset{(-)}{\omega}_n, \overline{\kappa}_n;\dots|\dots;\Lambda(\underset{\sim}{\Lambda})p_1, \overset{(-)}{\underset{\sim}{\Lambda}}\overset{(-)}{\omega}_1, \kappa_1) =\\ = \mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\dots|\dots;p_1, \overset{(-)}{\omega}_1, \kappa_1),\ \underset{\sim}{\Lambda} \in SL(2, C).\end{aligned} \tag{7.169}$$

The simplicity of the definition of spinor amplitudes combined with the explicit covariance under Lorentz transformations constitutes their main advantage (over other possible amplitudes). The possible invariance of the S-matrix with respect to the discrete operations CP, T, CPT:

$$\mathcal{U}_{CP} S \mathcal{U}_{CP}^{-1} = S, \quad \mathcal{U}_T S \mathcal{U}_T^{-1} = S^*, \quad \mathcal{U}_{CPT} S \mathcal{U}_{CPT}^{-1} = S^*, \tag{7.170}$$

---

* We use $\mathcal{S}'(\Pi)$ to preserve the symmetry between $p_1, \dots, p_n$. Every generalized function in $\mathcal{S}'(\Pi)$ can be regarded as a function in $\mathcal{S}'(\mathrm{M}^{n-1})$ whenever some $n-1$ of the momenta $p_1, \dots, p_n$ are chosen as the coordinates on Π.

can be restated in terms of the spinor amplitudes of the original channel (7.164) and the transformed channels

$$\overline{\kappa}_1 + \ldots + \overline{\kappa}_\nu \to \overline{\kappa}_{\nu+1} + \ldots + \overline{\kappa}_n, \tag{7.171a}$$

$$\kappa_n + \ldots + \kappa_{\nu+1} \to \kappa_\nu + \ldots + \kappa_1, \tag{7.171b}$$

$$\overline{\kappa}_n + \ldots + \overline{\kappa}_{\nu+1} \to \overline{\kappa}_\nu + \ldots + \overline{\kappa}_1. \tag{7.171c}$$

A similar remark applies to the case of C- and P-invariance.

Exercise 7.32. Write down the condition of CP-, T- and CPT-invariance in terms of the spinor amplitudes.

It follows from the properties of the support of the covariant creation and annihilation operators that $\mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots|\ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1)$ has support on the physical mass shell $\mathfrak{M}$ of the process (7.164):

$$\mathfrak{M} = \{(p_1, \ldots, p_n) \in \Pi:\ p_j^0 = \sqrt{m_j^2 + \mathbf{p}_j^2},\ j = 1, \ldots, n\} \tag{7.172}$$

(here and in what follows, we shall denote m_{κ_j} simply by m_j). Moreover it follows from (7.143) that

$$(p_j^2 - m_j^2)\mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots|\ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) = 0. \tag{7.173}$$

If some particles have zero masses, then S satisfies extra conditions of type (7.144) in the appropriate variables.

The information contained in the property of the support and in equation (7.173) is often expressed in the following explicit manner:

$$\mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots|\ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) =$$

$$= \Pi_{j=1}^n \delta_{m_j}^+(p_j) T(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots|\ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1), \tag{7.174}$$

where T is regarded as a generalized function on $\mathfrak{M}$. It should be noted that the definition of generalized functions on $\mathfrak{M}$ is not entirely trivial (since $\mathfrak{M}$ has singular points as a variety; more precisely, $\mathfrak{M}$ relates to the class of so-called semi-algebraic varieties, since it is defined by a finite system of algebraic equations and inequalities). However, these complications are in fact not all that important, since by virtue of the unitarity of the S-matrix, it is enough to define the spinor amplitude just on the set of regular points of $\mathfrak{M}$ (see Exercise 7.33 below with regard to this). It must also be noted that the most important physical characteristics of the scattering process can only be defined when T is an ordinary (measurable) function. For example, for the process $\kappa_1 + \kappa_2 \to \kappa_3 + \ldots + \kappa_n$ with spinless particles, the (effective) *differential cross section* of the process is defined by the formula

$$\frac{d\sigma_{\kappa_1\kappa_2 \to \kappa_3 \ldots \kappa_n}}{d\Xi_{\kappa_3 \ldots \kappa_n}} =$$

$$= \frac{1}{4p_1^0 p_2^0 v}(2\pi)^4 \delta(p_1 + p_2 - p_3 - \ldots - p_n)|T(p_n, \ldots, p_3|p_2, p_1)|^2 \tag{7.175a}$$

in any frame of reference associated with p_1, p_2, in which the vectors $\mathbf{p}_1$, $\mathbf{p}_2$ are collinear (for example, in the laboratory frame or the centre-of-mass frame); here

$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2}$, $v = \left|\frac{\mathbf{p}_1}{p_1^0} - \frac{\mathbf{p}_2}{p_2^0}\right|$ is the modulus of the difference of the velocities of the incoming particles and $d\Xi_{\kappa_3 \ldots \kappa_n}$ is the natural volume element of the momentum space of the outgoing particles:

$$d\Xi_{\kappa_3 \ldots \kappa_n} = \prod_{j=3}^{n} (dp_j)_{m_j}.$$

The *cross section* of this process is

$$\sigma_{\kappa_1 \kappa_2 \to \kappa_3 \ldots \kappa_n} = \frac{1}{N_1! \ldots N_r!} \int \frac{d\sigma_{\kappa_1 \kappa_2 \to \kappa_3 \ldots \kappa_n}}{d\Xi_{\kappa_3 \ldots \kappa_n}} d\Xi_{\kappa_3 \ldots \kappa_n}, \tag{7.175b}$$

where $N_1, \ldots, N_r$ are the numbers of particles of the various types $\kappa^{(1)}, \ldots, \kappa^{(r)}$ in the totality $\kappa_3, \ldots, \kappa_n$ of outgoing particles.

The factors $1/N_1! \ldots N_r!$ in (7.175b) appear as a result of the principle of indistinguishability of identical particles and the corresponding symmetry of the differential cross section (7.175a) with respect to the momenta of the identical particles. Points of the momentum space of the outgoing particles that differ only by a permutation of the momenta of identical particles correspond to the same physical event; therefore the cross section $\sigma_{\kappa_1 \kappa_2 \to \kappa_3 \ldots \kappa_n}$ is obtained either by integrating the differential cross section over the corresponding $1/N_1! \ldots N_r!$ fraction of the momentum space or, equivalently, by integrating over the whole of the momentum space but with the extra factor as in (7.175b).

In the next subsection we go into details on representations of type (7.174) for spinor amplitudes of two-particle processes. Here we make some observations with regard to the interpretation of the representation (7.174) in the spirit of generalized functions. The natural way to treat the quantity T is via the weak integral representation.

For this it is required that the result of smoothing the spinor amplitude with an arbitrary test function $f(p) \in \mathcal{S}(\Pi)$ with respect to the variables $p \equiv (p_1, \ldots, p_n)$ should depend only on the restriction of f to the mass shell $\mathfrak{M}$, that is,

$$\int \mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n, \ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) f(p_1, \ldots, p_n) d_4p_1 \ldots d_4p_n = 0, \tag{7.176a}$$

if

$$f(p) = 0 \quad \text{for } p \in \mathfrak{M}. \tag{7.176b}$$

It turns out that this property is equivalent to the property of the support of the spinor amplitude supplemented by the equations (7.173); however, the proof of this fact is considerably simplified if we use the provenance of the spinor amplitudes as generalized matrix elements of the unitary operator S.

Exercise 7.33. (a) Prove the following representation:

$$\begin{aligned}\int_\Pi \mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots | \ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) f(p) d_4p_1 \ldots d_4p_{n-1} = \\ = \lim_{r \to \infty} \lim_{s \to \infty} \int_\Pi \mathcal{T}(p_n, \overset{(-)}{\omega}_n, \overline{\kappa}_n;\ \ldots | \ldots;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) f_{r,s}(p) d_4p_1 \ldots d_4p_{n-1},\end{aligned} \tag{7.177}$$

where

$$f(p) \in \mathcal{S}(\Pi), \quad f_{r,s}(p) = w_r(p_1, \ldots, p_\nu) w'_s(p_{\nu+1}, \ldots, p_n) f(p),$$

and w_r and w'_s are sequences of C^∞-functions satisfying conditions of the following type (which we write out explicitly only for w_r):

1) $0 \leq w_r \leq 1$;

2) w_r has compact support (depending on r) on the set

$$\{(p_1, \ldots, p_\nu) \in \mathbf{M}^\nu : p_1^0 > 0, \ldots, p_\nu^0 > 0;\ p_1, \ldots, p_\nu \text{ are non-collinear}\},$$

3) the restriction of w_r to $\Gamma^+_{m_1} \times \ldots \times \Gamma^+_{m_\nu}$ converges to unity as $r \to \infty$ almost everywhere with respect to the measure $(dp_1)_{m_1} \ldots (dp_\nu)_{m_\nu}$. [Hint: First consider functions $f(p) \in \mathcal{S}(\mathrm{M}^n)$ of the form $f(p) = \overline{\Psi(p_1, \ldots, p_{\nu+1})}\Phi(p_\nu, \ldots, p_1)$, and then use the fact that the linear combinations of such functions are dense in $\mathcal{S}(\mathrm{M}^n)$.]

(b) Prove (7.176). [Hint: The map

$$\Pi \ni (p_1, \ldots, p_n) \to (p_1^2 - m_1^2, \ldots, p_n^2 - m_n^2)$$

is regular, that is, it has rank n on the support of the functions $f_{r,s}$ in part (a) of this exercise; deduce that if $f_{r,s}(p) = 0$ on $\mathfrak{M}$, then

$$f_{r,s}(p) = \sum_{j=1}^{n} (p_j^2 - m_j^2) h_{\nu,s,j}(p),$$

where the $h_{r,s,j}$ are C^∞-functions on $\mathfrak{M}$ with compact supports; finally, use equations (7.173).]

F. Kinematics of two-particle processes

We consider the two-particle process

$$\kappa_1 + \kappa_2 \to \kappa_3 + \kappa_4 \tag{7.178}$$

with momenta $-p_1$, $-p_2$ for the incoming particles and p_3, p_4 for the outgoing particles:

$$p_j \in \Gamma^-_{m_j} \text{ for } j = 1, 2; \quad p_j \in \Gamma^+_{m_j} \text{ for } j = 3, 4, \tag{7.179}$$

satisfying the conservation law

$$p_1 + p_2 + p_3 + p_4 = 0. \tag{7.180}$$

In this subsection Π denotes the plane (7.180) in M^4 and $\mathfrak{M}$ the mass shell of the process (7.178) (that is, the set of points of Π satisfying (7.179)).

We can form the following three Lorentz-invariants from the momenta of the particles:

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2. \tag{7.181}$$

The variable s is the square of the total energy in the centre-of-mass frame; it varies over the interval $s \geq s_{\text{phys}}$, where

$$s_{\text{phys}} = (m_1 + m_2)^2 \vee (m_3 + m_4)^2 \tag{7.182}$$

is the threshold of the reaction (7.178). The variables t and u are called the *invariant squares* of the momentum transfer (between the particles 1 and 3 or 1 and 4 respectively). The set of values of G_{phys} taken by the invariants s, t, u when the momenta run through $\mathfrak{M}$, is called the *physical region* of the process (see (7.191) below). The variables s, t, u are related by the formula

$$s + t + u = \sum_{j=1}^{4} m_j^2, \tag{7.183}$$

so that only two of them are independent.

Exercise 7.34. Let $(p_1, \ldots, p_4)$ be an arbitrary point of the plane Π; prove that the invariants s, t, u satisfy the condition

$$s + t + u = \sum_{j=1}^{4} p_j^2. \tag{7.184}$$

In the centre-of-mass frame ($\mathbf{p}_1 + \mathbf{p}_2 = 0$) the momenta of the particles are parametrized as follows:

$$\begin{aligned} -p_1 &= (E_1(s), K_{12}(s)\mathbf{n}_{12}), \quad -p_2 = (E_2(s), -K_{12}(s)\mathbf{n}_{12}), \\ p_3 &= (E_3(s), K_{34}(s)\mathbf{n}_{34}), \quad p_4 = (E_4(s), -K_{34}(s)\mathbf{n}_{34}); \end{aligned} \tag{7.185}$$

here the $E_j(s)$ are the energies of the particles:

$$\begin{aligned} E_1(s) &= \frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, \quad E_2(s) = \frac{s - m_1^2 + m_2^2}{2\sqrt{s}}, \\ E_3(s) &= \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}, \quad E_4(s) = \frac{s - m_3^2 + m_4^2}{2\sqrt{s}}; \end{aligned} \tag{7.186}$$

$K_{12}(s)$ and $K_{34}(s)$ are the lengths of the three-dimensional momenta of the particles before and after the reaction:

$$K_{12}^2(s) = \frac{1}{4s}(s - (m_1 + m_2)^2)(s - (m_1 - m_2)^2), \tag{7.187a}$$

$$K_{34}^2(s) = \frac{1}{4s}(s - (m_3 + m_4)^2)(s - (m_3 - m_4))^2; \tag{7.187b}$$

$\mathbf{n}_{12}$ and $\mathbf{n}_{34}$ are arbitrary vectors on the unit sphere S^2 in $\mathbf{R}^3$ (the directions of the three-dimensional momenta of the particles 1 and 3); their scalar product

$$\mathbf{n}_{12}\mathbf{n}_{34} = \cos\theta \tag{7.188}$$

is the cosine of the angle of scattering in the centre-of-mass frame. The parameter t can be regarded as a function of s and $\cos\theta$:

$$t = m_1^2 + m_3^2 - \frac{1}{2s}(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) + 2K_{12}(s)K_{34}(s)\cos\theta. \tag{7.189}$$

The range of values of the independent Lorentz-invariants of the process (7.178) is clearly defined by the inequalities

$$s \geq s_{\text{phys}}, \quad -1 \leq \cos\theta \leq 1. \tag{7.190}$$

Accordingly the physical region is given in terms of the variables s, t by *

$$G_{\text{phys}} = \{(s,t) \in \mathbf{R}^2 : s \geq s_{\text{phys}},\ t_{\min}(s) \leq t \leq t_{\max}(s)\}, \tag{7.191}$$

where

$$t_{\substack{\max\\ \min}}(s) = m_1^2 + m_3^2 - \frac{1}{2s}(s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2) \pm 2K_{12}(s)K_{34}(s). \tag{7.192}$$

Exercise 7.35. Prove that G_{phys} is a canonically closed regular subset of $\mathbf{R}^2$.

---

* For $\cos\theta = \pm 1$, the vectors p_1, p_2, p_3 are linearly dependent; consequently the boundary of the physical region lies on the curve $\det(p_i p_j)_{i,j=1,2,3} = 0$ in the (s,t)-plane. (Here the scalar products $p_i p_j$ are assumed to be expressed in terms of s, t).

By the law of conservation of momentum, the spinor amplitude

$$\mathcal{T}(p_4, \overset{(-)}{\omega}_4, \overline{\kappa}_4;\ p_3, \overset{(-)}{\omega}_3, \overline{\kappa}_3;\ p_2, \overset{(-)}{\omega}_2, \kappa_2;\ p_1, \overset{(-)}{\omega}_1, \kappa_1) \equiv$$
$$\equiv \mathcal{T}(p_4, \overset{(-)}{\omega}_4, \overline{\kappa}_4;\ p_3, \overset{(-)}{\omega}_3, \overline{\kappa}_3 | -p_2, \overset{(-)}{\omega}_2, \kappa_2;\ -p_1, \overset{(-)}{\omega}_1, \kappa_1)$$

of the two-particle process depends on three momenta (for example, p_1, p_2, p_3), that is, it is a generalized function of class $\mathcal{S}'(\Pi)$. To obtain the covariant representations it only remains to go over to the mass shell in the corresponding representations of §3.4.C.

Proposition 7.6. *The formula*

$$f(p_1, \dots, p_4) = \prod_{j=1}^{2} \delta^-_{m_j}(p_j) \prod_{k=3}^{4} \delta^+_{m_k}(p_k) F((p_1+p_2)^2, (p_1+p_3)^2) \equiv$$
$$\equiv \int \prod_{j=1}^{2} \delta^-_{m_j}(p_j) \prod_{k=3}^{4} \delta^+_{m_k}(p_k) \delta(s-(p_1+p_2)^2)\delta(t-(p_1+p_3)^2) F(s,t)\, ds\, dt \qquad (7.193)$$

realizes an isomorphism between the space of generalized functions $F(s,t) \in \mathcal{S}'(G_{\text{phys}})$ and the space of Lorentz-invariant generalized functions $f(p_1, \dots, p_4) \in \mathcal{S}'(\Pi)$ with support in $\mathfrak{M}$ and satisfying the equations

$$(p_j^2 - m_j^2) f(p_1, \dots, p_4) = 0, \quad j = 1, \dots, 4. \qquad (7.194)$$

Exercise 7.36. Prove Proposition 7.6. [Hint: According to Exercise 3.28, f is uniquely expressible in the form

$$f(p_1, \dots, p_4) = \phi(p_1^2, \dots, p_4^2, (p_1+p_2)^2, (p_1+p_3)^2), \qquad (7.195)$$

where $\phi(y_1, \dots, y_4, s, t) \in \mathcal{S}'(G)$ and G is a canonically closed regular subset of $\mathbf{R}^6$. By multiplying both sides of (7.195) by $p_j^2 - m_j^2$ and using (7.194), verify that $(y_j - m_j^2)\phi = 0$ for $j = 1, \dots, 4$.]

In particular, Proposition 7.6 yields the following important symmetry property.

Exercise 7.37. Let $m_1 = m_3$, $m_2 = m_4$. Prove that every Lorentz-invariant generalized function $f(p_1, \dots, p_4) \in \mathcal{S}'(\Pi)$ with support in $\mathfrak{M}$, and satisfying the equations (7.194), has the property

$$f(p_1, p_2, p_3, p_4) = f(-p_3, -p_4, -p_1, -p_2). \qquad (7.196)$$

[Hint: Use the representation (7.193).]

Exercise 7.38. Let $\mathcal{T}(p_4, \bar{\kappa}_2;\ p_3, \bar{\kappa}_1;\ p_2, \kappa_2;\ p_1, \kappa_1)$ be the amplitude of the elastic two-particle process $\kappa_1 + \kappa_2 \to \kappa_1 + \kappa_2$ with spinless particles:

$$(2\pi)^4 \delta(p_1 + \ldots + p_4) \mathcal{T}(p_4, \bar{\kappa}_2;\ p_3, \bar{\kappa}_1;\ p_2, \kappa_2;\ p_1, \kappa_1) =$$
$$= \langle 0 | A(p_4, \kappa_2) A(p_3, \kappa_1) T A^*(-p_2, \kappa_2) A^*(-p_1, \kappa_1) | 0 \rangle, \qquad (7.197)$$

where

$$iT \equiv S - 1.$$

Prove the relation

$$(2\pi)^4 \delta(p_1 + \ldots + p_4) \cdot 2i \operatorname{Im} \mathcal{T}(p_4, \bar{\kappa}_2;\ p_3, \bar{\kappa}_1;\ p_2, \kappa_2;\ p_1, \kappa_1) =$$
$$= \langle 0 | A(p_4, \kappa_2) A(p_3, \kappa_1) (T - T^*) A^*(-p_2, \kappa_2) A^*(-p_1, \kappa_1) | 0 \rangle. \qquad (7.198)$$

[Hint: Apply the result of the previous exercise to the second term on the right hand side of (7.198).]

In the covariant decompositions of the spinor amplitude we restrict ourselves to the case when the masses m_j are positive. A polynomial basis of standard covariants $Q_\rho(p_4, \overset{(-)}{\omega}_4; \ldots; p_1, \overset{(-)}{\omega}_1)$ is chosen according to Exercise 3.31 with $n = 3$ (the momentum p_4 is not a new variable by virtue of (7.180)); the role of $\mathcal{X}$ is played by the space of homogeneous polynomials $\psi(\overset{(-)}{\omega}_4, \ldots, \overset{(-)}{\omega}_1)$of degree $2s_j$ in $\overset{(-)}{\omega}_j$ (where s_j is the spin of the jth particle). The dimension of $\mathcal{X}$ is equal to

$$N = \prod_{j=1}^{4}(2s_j + 1),$$

so that (according to Exercise 3.31) the set $\{Q_\rho\}$ consists of N covariants forming a linear basis $\mathcal{X}$ at each point of rank 3. In accordance with (3.238) we introduce the $N \times N$-matrix $X \equiv (X_{\rho\sigma}(s,t))$, by setting

$$X_{\rho\sigma}(s,t) = Q_\rho\Big(p_4\epsilon^{-1}\frac{\partial}{\partial\overset{(-)}{\omega}_4}; \ \ldots; \ p_1, \epsilon^{-1}\frac{\partial}{\partial\overset{(-)}{\omega}_1}\Big)\times$$
$$\times Q_\sigma(p_4, \overset{(-)}{\omega}_4; \ \ldots; \ p_1, \overset{(-)}{\omega}_1)\big|_{p_j^2=m_j^2} \tag{7.199}$$

(its determinant is non-zero at points of rank 3).

Proposition 7.7. *The spinor amplitude of a two-particle process between particles with positive masses has the unique covariant decomposition**

$$\mathcal{T}(p_4, \overset{(-)}{\omega}_4, \bar{\kappa}_4; \ \ldots; \ p_1, \overset{(-)}{\omega}_1, \kappa_1) =$$
$$= \sum_{\rho=1}^{N} Q_\rho(p_4, \overset{(-)}{\omega}_4; \ \ldots; \ p_1, \overset{(-)}{\omega}_1)\prod_{j=1}^{2}\delta^-_{m_j}(p_j)\prod_{k=3}^{4}\delta^+_{m_k}(p_k)T_\rho(s,t), \tag{7.200}$$

where the set $T \equiv \{T_\rho(s,t)\}$ of invariant amplitudes is an element of the space $X^{-1}\bigoplus^N \mathcal{S}'(G_{\mathrm{phys}})$.

Exercise 7.39. Prove Proposition 7.7. [Hint: The line of argument is the same as in Exercise 7.36, except that Exercise 3.31 must be used instead of Exercise 3.28.]

Appendix E. Four-Component Spinors and the Dirac Equation

E.1. Clifford Algebra over Minkowski Space

An explicit covariant representation corresponding to spin 1/2 is realized by the Dirac spinors. Since this important example is constantly encountered in applications (including the later sections of this book), we set out the necessary information in this appendix.

Let $\mathcal{X} = \mathbf{C}^D$ be a D-dimensional complex vector space with a non-degenerate bilinear symmetric form (,) ("scalar product"). By a *Clifford algebra* Cliff$\mathcal{X}$ over $\mathcal{X}$ we mean the algebra of complex $2^{[D/2]} \times 2^{[D/2]}$-matrices (where $[D/2]$ is the integer part of $D/2$) together with an (injective) linear map

$$\gamma : \mathcal{X} \to \mathrm{Cliff}\,\mathcal{X}, \tag{E.1a}$$

satisfying the condition

$$[\gamma\xi, \gamma\xi']_+ \equiv (\gamma\xi)(\gamma\xi') + (\gamma\xi')(\gamma\xi) = 2(\xi,\xi') \tag{E.1b}$$

* The discrete symmetries (of type P, C) reduce the number of independent invariant amplitudes.

(we omit the identity matrix on the right hand side). If $\{e_a\}$ is a basis in $\mathcal{X}$ such that $(e_a, e_b) = g_{ab}$, then the corresponding units $\gamma_a = \gamma e_a$ of the Clifford algebra satisfy the anticommutation relations $[\gamma_a, \gamma_b]_+ = 2g_{ab}$.

An example of such an algebra is the Clifford algebra over three-dimensional Euclidean space. It is the algebra of 2×2 matrices, with the Pauli matrices σ_i serving as the units. The example that is next in complexity, which is of interest to us, is the Clifford algebra over four-dimensional (complex) Minkowski space **CM**, which is generated by the Dirac matrices γ_μ satisfying the conditions

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \quad (\mu, \nu = 0, 1, 2, 3). \tag{E.2}$$

In the study of a given Clifford algebra a fundamental role is played by the theorem on the uniqueness (to within equivalence) of an irreducible representation of it. We now state and prove this theorem for the case $D = 4$ of interest to us.

Theorem E.1. *The Clifford algebra of Minkowski space (generated by the matrices γ_μ satisfying* (E.2)) *has a unique (to within equivalence) irreducible complex representation; its dimension is equal to four.*

■ We give a proof based on certain properties of representations of finite groups.

If we look at (E.2) we can see that there are 16 independent products of the identities γ_μ. These are the products (of not more than four factors) in which no two of the γ_μ are the same:

$$\{\gamma_R\} = \{1, \gamma_\mu, \tfrac{1}{2}[\gamma_\mu, \gamma_\nu] \; (= \gamma_{\mu\nu}), \gamma_\mu\gamma_5, \gamma_5\}; \tag{E.3}$$

here

$$\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 \; (\gamma_5^2 = -1), \quad \gamma_\mu\gamma_5 = \frac{1}{3!}\epsilon_{\mu\kappa\lambda\nu}\gamma^\kappa\gamma^\lambda\gamma^\nu. \tag{E.4}$$

The monomials $\{\gamma_R\}$ and $\{-\gamma_R\}$ form a group G_{32} of 32 elements with centre $\mathbf{Z}_2 = \{1, -1\}$. The quotient group $G_{16} = G_{32}/\mathbf{Z}_2$ is abelian, since the products $\gamma_{R_1}\gamma_{R_2}$ and $\gamma_{R_2}\gamma_{R_1}$ in the original group merely differ in sign. We are interested in the faithful representations of G_{32} (in which the centre of the representation is non-trivial).

Exercise E.1. Let G be a finite group (with $N(G)$ elements), H a normal subgroup of G and $K = G/H$. Prove the formula

$$N(G) - N(K) = \sum_T d_T^2,$$

where T runs through the complete set of pairwise inequivalent (complex) irreducible representations of G (of dimension d_T) that are not lowered to representations of the group K (that is, the kernels of which do not contain H). [Hint: Use Burnside's theorem in the theory of representations of finite groups; [Z2],§34,Theorem 4.]

If we apply the result of this exercise to the group $G = G_{32}$ and the quotient group $K = G_{16}$ we see that if G_{32} has a faithful four-row (irreducible) representation, then it is the unique (to within equivalence) faithful irreducible representation of this group.

Exercise E.2. Verify that the matrices

$$\gamma_\mu = \begin{pmatrix} 0 & \underset{\sim}{e}_\mu \\ \tilde{e}_\mu & 0 \end{pmatrix} \tag{E.5}$$

satisfy the anticommutation relations (E.2).

We claim that all the matrices γ_R (E.3) are linearly independent (from which it will follow, in particular, that our representation of G_{32} is faithful). To this end we note to begin with that for each of the fifteen matrices γ_R, excluding the identity, there exists a matrix $\gamma_{R'}$ such that

$$\gamma_{R'}\gamma_R\gamma_{R'}^{-1} = -\gamma_R. \tag{E.6}$$

Consequently, in view of the relations

$$[\gamma_5, \gamma_\mu]_+ = 0 = [\gamma_5, \gamma_\mu\gamma_5]_+, \tag{E.7}$$

which follow from (E.2), we can choose $\gamma_{R'} = \gamma_5$ for the matrices γ_μ and $\gamma_\mu\gamma_5$; for the matrices $\gamma_{\mu\nu}$ and γ_5 we can set $\gamma_{R'} = \gamma_\nu$.

It follows from (E.2) (and from the fact that the representation (E.5) is a four-row one) that the matrices $\frac{1}{2}\gamma_\mu$ are pseudo-orthonormal with respect to the trace of the product:

$$\operatorname{tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu}. \tag{E.8}$$

Using (E.6), it can be shown that the analogous property is enjoyed by the matrices γ_R.

Exercise E.3. Prove that

$$\operatorname{tr}\gamma_R\gamma_S = 4\epsilon_R\delta_{RS}, \quad \text{where } \epsilon_R = \pm 1. \tag{E.9}$$

[Hint: Using (E.6) and the fact that one can cyclically permute the factors in the trace, verify to begin with that (E.9) holds for $\gamma_S = 1$.]

It is now fairly easy to verify that the quality $\sum_S c_S\gamma_S = 0$, where the c_S are complex numbers, can only hold if all the $c_S = 0$; in fact, according to (E.9), $c_R = \frac{\epsilon_R}{4}\operatorname{tr}\gamma_R\sum c_S\gamma_S$.

The irreducibility of the representation (E.5) follows from Schur's lemma and from the fact that any 4×4-matrix can be written in the form $\sum c_S\gamma_S$. ■

In particular, the so-called *Pauli lemma* follows from the above theorem.

Corollary E.2. If γ_μ and γ'_μ are two sets of four-row matrices satisfying (E.2), then there exists a non-singular matrix S such that $\gamma_\mu = S\gamma_\mu S^{-1}$.

Using the averaging device in the theory of representations of finite groups, it is possible to obtain an algorithm for constructing the matrix S realizing the similarity transformation.

Exercise E.4. Prove that for any choice of the four-row matrix F in $S = \sum_R \gamma'_R F\gamma_R$, we have

$$\gamma'_R S = S\gamma_R. \tag{E.10}$$

Having thus constructed the (complex) Clifford algebra corresponding to complex Minkowski space, we can also define the real Clifford algebra Cliff **M** the elements of which belong to the linear span (with real coefficients) of the matrices (E.3).

E.2. Spinor representation of the Lorentz group; various realizations of the γ-matrices

It follows from the anticommutation relation (E.2) that the matrices

$$s_{\mu\nu} = \frac{i}{2}\gamma_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \ \mu,\nu = 0,1,2,3, \tag{E.11}$$

satisfy the commutation relations for the "physical generators" of the Lie algebra $sl(2,C) \approx o(3,1)$:

$$[s_{\kappa\lambda}, s_{\mu\nu}] = i(g_{\kappa\nu}s_{\lambda\mu} + g_{\lambda\mu}s_{\kappa\nu} - g_{\kappa\mu}s_{\lambda\nu} - g_{\lambda\nu}s_{\kappa\mu}). \tag{E.12}$$

It is not difficult to verify that the representation $sl(2,C)$ with the generators (E.11) is reducible, since $[s_{\mu\nu}, \gamma_5] = 0$.

Exercise E.5. Prove that in the basis (E.5) the generators and matrix γ_5 have the reduced form:

$$s_{\mu\nu} = \begin{pmatrix} \sigma^{(1)}_{\mu\nu} & 0 \\ 0 & \sigma^{(2)}_{\mu\nu} \end{pmatrix}, \quad \sigma^{(1)}_{\mu\nu} = \frac{i}{4}(\underset{\sim}{e}_\nu\tilde{e}_\mu - \underset{\sim}{e}_\mu\tilde{e}_\nu), \quad \sigma^{(2)}_{\mu\nu} = \frac{i}{4}(\tilde{e}_\nu\underset{\sim}{e}_\mu - \tilde{e}_\mu\underset{\sim}{e}_\nu), \tag{E.13a}$$

$$i\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{E.13b}$$

Verify that

$$\begin{gathered} \underset{\sim}{\Lambda} = \exp(\tfrac{1}{2}i\sigma^{(1)}_{\mu\nu}\theta^{\mu\nu}) \textit{ and } \underset{\sim}{\Lambda}^{*-1} = \exp(\tfrac{1}{2}i\sigma^{(2)}_{\mu\nu}\theta^{\mu\nu}), \\ \theta^{\mu\nu} = -\,\theta^{\mu\nu}, \end{gathered} \tag{E.14}$$

define two inequivalent two-dimensional representations of $SL(2,C) : \underset{\sim}{\Lambda} = (\Lambda^a_b),\ \underset{\sim}{\Lambda}^{*-1} = ((\underset{\sim}{\Lambda}^{*-1})^{b'}_{a'})$.

The four-dimensional space of complex vectors $\psi = \{\psi^\alpha\}_{\alpha=1,\ldots,4}$, in which the irreducible representation of the Clifford algebra is realized is called the space of *Dirac spinors.* * According to Exercise E.5, the *Dirac representation* of $SL(2,C)$ decomposes into a direct sum $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of two irreducible representations.

The relation between the Dirac representation

$$V(\theta) = \exp(\tfrac{1}{2}i\theta^{\mu\nu}s_{\mu\nu}) \tag{E.15}$$

and the vector representation

$$\Lambda(\theta) = e^\theta \quad (\Lambda^\mu_\nu = \delta^\mu_\nu + \theta^\mu_\nu + \tfrac{1}{2}\theta^\mu_\lambda\theta^\lambda_\nu + \ldots) \tag{E.16}$$

of $SL(2,C)$ is given by the equalities

$$V(\theta)\gamma^\mu V^{-1}(\theta) = \Lambda(\theta)^\mu_\nu\gamma^\nu, \tag{E.17a}$$

$$V(\theta)\gamma_\nu V^{-1}(\theta) = \gamma_\mu\Lambda(\theta)^\mu_\nu. \tag{E.17b}$$

Exercise E.6. Verify (E.17) using the relation

$$[g_{\lambda\mu}, \gamma_\nu] = i(g_{\mu\nu}\gamma_\lambda - g_{\lambda\nu}\gamma_\mu). \tag{E.18}$$

It should be noted that the reducible representation (E.15) of $SL(2,C)$ becomes irreducible if we adjoin to the connected Lorentz group the spatial reflection

$$V(I_s) = \eta_s\gamma_0, \quad \text{where } |\eta_s| = 1 \ (V(I_s)\gamma_\mu V^{-1}(I_s) = g_{\mu\mu}\gamma_\mu). \tag{E.19}$$

The following exercises show how we can construct from the γ-matrices generators of the Lie algebra $su(2,2)$ of the conformal group and its subalgebra $sp(2,R) \approx o(3,2)$.

Exercise E.7. Show that the matrices

$$s_{\mu 6} = \frac{1}{2}\gamma_\mu, \quad s_{\mu 5} = \frac{i}{2}\gamma_\mu\gamma_5, \quad s_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \quad s_{56} = \frac{1}{2}\gamma_5 \tag{E.20}$$

satisfy the commutation relations for the "physical generators" of the Lie algebra $o(4,2) \approx su(2,2)$. Find a Hermitian matrix β (with signature $++--$), satisfying the relations

$$s^*_{ab}\beta = \beta s_{ab}, \quad a, b = 0, 1, 2, 3, 5, 6. \tag{E.21}$$

[Hint: In a basis for which $\gamma^*_\mu = g_{\mu\mu}\gamma_\mu$ (for example, in the basis (E.5)), one can set $\beta = \gamma_0$ (or $(\beta = -\gamma_0)$,]

Exercise E.8. Show that there exists an antisymmetric matrix C realizing a similarity between the matrices $-\gamma^T_\mu$ (where "T" denotes the transpose) and γ_μ:

$$C\gamma_\mu C^{-1} = -\gamma^T_\mu, \quad Cs_{\mu\nu}C^{-1} = -s^T_{\mu\nu}. \tag{E.22}$$

Verify that the generators $s_{\mu\nu}$ and $\frac{1}{2}\gamma_\mu$ form the Lie algebra $sp(2,R) \approx o(3,2)$. Prove that the operator realizes an equivalence between the dual representations:

$$CV(\theta)C^{-1} = V(\theta)^{T-1}. \tag{E.23}$$

[Hint: In the representation of type (E.5) in which $\gamma^T_\mu = (-1)^\mu\gamma_\mu$, $\mu = 0, 1, 2, 3$ and to which the matrices (E.24) given below belong , set $C = i\gamma_0\gamma_2$.]

The realization (E.5) of the γ-matrices is distinguished by the fact that in it, the representation of $SL(2,C)$ and of its Lie algebra have a reducible (block diagonal) form (E.13). In physics (at least)

*The Dirac spinors are also called bispinors. In E. Cartan's terminology, the four-component Dirac spinors are called spinors, while the two-component Lorentz-invariant constituents corresponding to the basis (E.13) are called half-spinors.

two further types of basis play a role: the *Pauli basis* in which the matrix γ_0 is diagonal and the *Majorana basis* in which γ_μ and $s_{\mu\nu}$ are purely imaginary (and the representation of the Dirac group $SL(2,C)$ is real.

Exercise E.9. Verify that the matrices

$$\gamma_0^P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k^P = \begin{pmatrix} 0 & -\tau_k \\ \tau_k & 0 \end{pmatrix} \quad \left(\gamma_5^P = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \tag{E.24}$$

satisfy the anticommutation relations (E.2). Prove that

$$\gamma_\mu^P = S_P \gamma_\mu S_P^{-1}, \quad \text{where } S_P = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = S_P^{-1}. \tag{E.25}$$

Exercise E.10. Verify that the matrices

$$\begin{aligned} \gamma_0^M &= i\gamma_1^P = i\begin{pmatrix} 0 & -\tau_1 \\ \tau_1 & 0 \end{pmatrix}, \quad \gamma_1^M = i\gamma_0^P = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \gamma_2^M &= \gamma_2^P = \begin{pmatrix} 0 & -\tau_2 \\ \tau_2 & 0 \end{pmatrix}, \quad \gamma_3^M = -\gamma_5^P = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \gamma_5^M &= \gamma_3^P = \begin{pmatrix} 0 & \tau_3 \\ -\tau_3 & 0 \end{pmatrix}, \quad C_M = -\gamma_0^M = (\beta^M)^T \end{aligned} \tag{E.26}$$

also satisfy the relations (E.2), (E.22). Show that

$$\gamma_\mu^M = S_{MP}\gamma_\mu^P S_{MP}^{-1}, \quad \text{where } S_{MP} = \frac{1+i\gamma_2^P}{2} + i\frac{i\gamma_3^P\gamma_5^P - \gamma_0^P\gamma_1^P}{2}, \tag{E.27}$$

$$S^{-1} = S^*(=\overline{S}) \quad (S^4 = 1).$$

The existence of a real representation (showing that it is possible to define real four-component spinors (*Majorana spinors*) in Lorentz-invariant fashion) is characteristic for the signature of the metric $g_{\mu\nu}$ of Minkowski space and the Lorentz group $L_+^\uparrow$. It is not difficult to verify that in the Euclidean case the spinor representation of the group $SU(2) \times SU(2)$ (which covers $O_+(4)$) is not equivalent to a real representation.

E.3. Dirac Equation; Representations of the Poincaré Group with Spin 1/2

The Dirac equation

$$(i\hat{\partial} - m)\psi(x) = 0, \quad \text{where } \hat{\partial} \equiv \gamma^\mu \frac{\partial}{\partial x^\mu}, \tag{E.28}$$

is invariant with respect to arbitrary transformations

$$\psi(x) \to V(\underline{\Lambda})\psi(\Lambda^{-1}(x-a)) \tag{E.29}$$

of the quantum-mechanical Poincaré group (including reflections); it is also invariant with respect the charge conjugation

$$\psi(x) \to \psi^c(x) = C^{-1}\tilde{\psi}(x). \tag{E.30}$$

Since $(m + i\hat{\partial})(m - i\hat{\partial}) = m^2 + \Box$, each solution of the Dirac equation is also a solution of the Klein-Gordon equation and can therefore be written in the form

$$\psi(x) = \int e^{-ipx} \cdot 2\pi\delta(m^2 - p^2)u(p)d_4p, \tag{E.31}$$

where the spinor $u(p)$ satisfies the linear system of equations

$$(\hat{p} - m)u(p) = 0 \quad \text{for } p^2 = m^2 (\hat{p} \equiv p_\mu\gamma^\mu) \tag{E.32}$$

or

$$\psi(x) = \int_{\Gamma_m^+} (e^{-ipx}u^{(+)}(p) + e^{ipx}u^{(-)}(p))(dp)_m, \tag{E.33}$$

where the spinors $u^{(\pm)}(p)$ are solutions with positive energy of the system of linear (algebraic) equations

$$(m - \hat{p})u^{(+)}(p) = 0 = (m + \hat{p})u^{(-)}(p). \tag{E.34}$$

Exercise E.11. Prove that

$$\det(\hat{p} \mp m) = (p^2 - m^2)^2 \quad (= 0);$$

and hence deduce that for

$$p^0 = \sqrt{m^2 + \mathbf{p}^2}$$

the rank of the matrix $\hat{p} \mp m$ is equal to two, so that each of the equations (E.32) has two linearly independent solutions.

It is possible to construct a basis of linearly independent solutions of (E.32) by supposing that $u^{(\pm)}(p)$ are the eigenvectors of the operator of the third projection of the spin $s^3(\mathbf{p})$, which can be defined by a Lorentz transformation from its value in the rest frame;

$$s^j(0)\ (\equiv s_j(\mathbf{p} = 0)) = \tfrac{1}{2}\epsilon^{jkl}s_{kl} = \tfrac{1}{2}i\gamma^5\gamma^0\gamma^j\ (\equiv \tfrac{1}{2}i\gamma_5\gamma_0\gamma_j). \tag{E.35}$$

The bispinor representation of the pure Lorentz transformation ("boost")

$$(\Lambda_p)^\mu_0 = \frac{p^\mu}{m}(= (\Lambda_p)^0_\mu),\ \Lambda^j_{pk} = \delta^j_k - \frac{p^j p_k}{m(m + p^0)}, \tag{E.36}$$

which takes the time axis to the direction of the vector p with $p^0 > 0$, has the form

$$V_p = \hat{b}_p\gamma^0, \tag{E.37a}$$

where

$$b^0_p = \frac{p^0 + m}{\sqrt{2m(p^0 + m)}}, \quad b^j_p = \frac{p^j}{\sqrt{2m(p^0 + m)}}. \tag{E.37b}$$

Exercise E.12. Verify the equality

$$\hat{p} = mV_p\gamma^0V_p^{-1} = mV_pV_p^*\gamma^0 \quad (V_p^* = V_p). \tag{E.38}$$

We now calculate the spin operator:

$$s(\mathbf{p}) = V_ps(0)V_p^{-1} = \hat{b}_ps(0)\hat{b}_p = \frac{1}{m}\left(\mathbf{W} - \frac{W^0\mathbf{p}}{p^0 + m}\right) \quad (p^0 = \sqrt{(m^2 + \mathbf{p}^2)}, \tag{E.39}$$

where W^μ is the (matrix-valued) Pauli-Lubanski vector:

$$W^\kappa = 1/2\epsilon^{\kappa\lambda\mu\nu}s_{\lambda\mu}p_\nu \tag{E.40}$$

(which commutes with p).

Exercise E.13. Prove that

$$s^2(\mathbf{p}) = 3/4 \tag{E.41}$$

and that $s_3(\mathbf{p})$ has eigenvalues $\pm 1/2$.

The solution of the equation

$$s_3(\mathbf{p})u^{(\pm)}_\zeta(p) = \zeta u^{(\pm)}_\zeta(p), \quad \zeta = \pm 1/2 \tag{E.42}$$

(where $u^{(\pm)}_\zeta$ satisfy the Dirac equation (E.32)), is defined to within normalization.

Exercise E.14. Verify the equivalence of the normalization conditions

$$\bar{u}_\zeta^{(\pm)}(p)u_\zeta^{(\pm)}(p) = 2\omega_p, \tag{E.43a}$$

$$\tilde{u}_\zeta^{(\pm)}(p)u_\zeta^{(\pm)}(p) = \pm 2m. \tag{E.43b}$$

We denote by $u_\zeta^{(e)}(p)$ ($e = \pm$, $\zeta = \pm 1/2$) the four linearly independent solutions of the system (E.32), (E.42) corresponding to the eigenvalues $p^0 = \sqrt{m^2 + \mathbf{p}^2}$ and $s_3(\mathbf{p}) = \zeta$.

Exercise E.15. Prove the formulae

$$\sum_{\zeta=\pm 1/2} u_\zeta^{(\pm)}(p) \otimes \tilde{u}_\zeta^{(\pm)}(p) = \hat{p} \pm m, \quad \text{where } p^0 = \sqrt{m^2 + \mathbf{p}^2}. \tag{E.44}$$

Exercise E.16. Prove that in the Pauli basis (E.24)

$$u_\zeta^{(+)}(p) = \frac{1}{\sqrt{m + p^0}} \begin{pmatrix} (m + p^0)e_\zeta \\ (\mathbf{p}\sigma)e_\zeta \end{pmatrix}, \quad \text{where } e_\zeta = \begin{pmatrix} \zeta + 1/2 \\ \zeta - 1/2 \end{pmatrix}, \tag{E.45a}$$

$$u_\zeta^{(-)}(p) = \frac{1}{\sqrt{m + p^0}} \begin{pmatrix} (\mathbf{p}\sigma)e_\zeta \\ (m + p^0)e_\zeta \end{pmatrix}. \tag{E.45b}$$

The formulae (E.45) demonstrate the convenience of the Pauli realization in the study of the non-relativistic limit of the spinors $u_\zeta^{(\pm)}(p)$: in this limit, $\mathbf{p}\sigma/(m + p^0) \to 0$ and the spinors (E.45) in effect become two-component ones.

The solutions of the Dirac equation can serve as a basis for some manifestly covariant realization of the unitary representation $[m, 1/2, +]$ of the quantum-mechanical Poincaré group which is different from the Wigner realization.

We consider the space $\mathfrak{H}_{m,1/2}$, consisting of the four-component complex (measurable) spinor functions $\Psi(p) \equiv (\Psi^\alpha(p))$ on the hyperboloid Γ_m^+ for which the integral

$$(\Psi, \Psi) = \frac{1}{m} \int_{\Gamma_m^+} \tilde{\Psi}(p)\hat{p}\Psi(p)(dp)_m \ (< \infty) \tag{E.46}$$

converges.

Exercise E.17. Prove that if the spinor-valued function $\Psi(p)$ does not vanish almost everywhere, then $(\Psi, \Psi) > 0$. [Hint: Use the fact that the matrix $\beta\hat{p}$ is positive for $p \in \Gamma_m^+$.]

The spinor representation of the Poincaré group $\tilde{\mathfrak{P}}$ (including the reflections) acts on the vectors in $\mathfrak{H}_{m,1/2}$ according to the formulae

$$[U(a, \underset{\sim}{\Lambda})\Psi](p) = e^{ipa}V(\underset{\sim}{\Lambda})\Psi(\Lambda^{-1}p), \tag{E.47}$$

$$U(I_s)\Psi(p) = \eta_s\gamma^0\Psi(I_s p) \quad (I_s(p^0, \mathbf{p}) = (p^0, -\mathbf{p})), \tag{E.48}$$

$$U(I_t)\Psi(p) = \eta_t\gamma_0\gamma_5 C^{-1}\bar{\Psi}(I_s p), \tag{E.49}$$

where the operator $U(I_t)$ is anti-unitary. It turns out that this representation is reducible.

Exercise E.18. Verify that the Dirac equation (E.32) is invariant with respect to the transformations (E.47)–(E.49). Deduce the transformation law of Ψ under reflections from this invariance condition (and from the requirement that the energy be positive).

The space $\mathfrak{H}_{m,1/2}$ splits up into a direct sum of two $\tilde{\mathfrak{P}}$-invariant subspaces:

$$\mathfrak{H}_{m,1/2} = \mathfrak{H}^{[m,1/2,+]} \oplus \mathfrak{H}^{[m,1/2,-]}, \quad \text{where } (m \mp \hat{p})\mathfrak{H}^{[m,1/2,\pm]} = 0, \tag{E.50}$$

and where

$$(\Phi, \Psi) = \int_{\Gamma_m^+} (\tilde{\Phi}^{(+)}(p)\Psi^{(+)}(p) - \tilde{\Phi}^{(-)}(p)\Psi^{(-)}(p))(dp)_m. \tag{E.51}$$

We now show that the operator U_c of charge conjugation can be defined as a unitary operator that intertwines the invariant subspaces $\mathfrak{H}^{[m,1/2,\pm]}$. Thus decomposing an arbitrary spinor $\Psi(p)$ on the hyperboloid Γ_m^+ in the basis $u_\zeta^{(e)}(p)$,

$$\Psi(p) = \sum_{\zeta,e} \Psi(p,\zeta,e)u_\zeta^{(e)}(p), \tag{E.52}$$

we set, by definition,

$$U_c\Psi(p) = \sum_{\zeta,e} \Psi(p,\zeta,e)u_\zeta^{(e)}(p)^c, \tag{E.53}$$

where u^c is defined by the equality $u^c(p) = C^{-1}\tilde{u}(p)$ (see (E.30)).

Exercise E.19. Prove that if $(m \mp \hat{p})u(p) = 0$, then $(m \pm \hat{p})u^c(p) = 0$. Verify in the Pauli basis in which

$$C^{-1} \equiv C_P^{-1} = -i\gamma_0^P\gamma_2^P = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (\epsilon\sigma^T \mathbf{p}\epsilon = \sigma\mathbf{p}), \tag{E.54}$$

that

$$U_C u_\zeta^{(\pm)}(p) = \mp 2\zeta u_{-\zeta}^{(\pm)}(p) \quad \left(-2\zeta = (-1)^{\zeta+\frac{1}{2}}\right). \tag{E.55}$$

The result of this exercise shows that the index e $(= \pm)$ plays the role of (the sign of) the charge; the operation of charge conjugation takes a particle into the antiparticle (and vice versa), simultaneously changing the sign of the spin projection.

Note that the decomposition (E.52) makes explicit the connection between the Wigner realization of a representation of $\tilde{\mathcal{P}}$ (in the space of functions $\Psi(p,\zeta,e)$ with fixed "charge" e) and the manifestly covariant realization of this subsection.

Part III

Local Quantum Fields and Wightman Functions

Synopsis

Both in classical and quantum physics, the notion of a field invokes the idea of short-range behaviour. To give due consideration to this idea, we add to the axioms of the algebraic approach the postulate A.V: corresponding to each bounded domain $\mathcal{O}$ of Minkowski space is a C^*-algebra $\mathfrak{A}(\mathcal{O})$ with identity such that the local commutativity condition holds (together with the requirements of isotony and Poincaré-covariance) (§8.1.A). The replacement of the points of space-time by domains is necessary if we want to deal with algebras of bounded operators. The notion of a local quantum field requires the use of a more singular object, namely, the operator-valued generalized function (§8.2). Physically, the Wightman axioms W.I–W.VII are realizations of the postulates of the algebraic approach supplemented by the requirements of uniqueness (W.III) and cyclicity (W.VII) of a translation-invariant vacuum in terms of these fields.

A system of (smoothed) fields $\phi^{(\kappa)}(f)$ is irreducible if the only bounded operators C that commute weakly with the $\phi^{(\kappa)}(f)$ are the multiples of the identity operator. It is proved (Proposition 8.1) that in the Wightman theory, the irreducibility of the fields is a consequence of the axioms. Less trivial is the result of Reeh-Schlieder which states that the vacuum is cyclic with respect to the algebra $\mathcal{P}(\mathcal{O})$ of polynomials of the fields smoothed with test functions and concentrated in an arbitrary non-empty open set $\mathcal{O}$ (§8.2.D, Proposition 8.2).

The main idea of the Wightman approach is that the entire content of the theory can be restated in the language of vacuum expectation values of products of fields, called Wightman functions (§8.3). In the simplest case of the theory of a neutral scalar field $\phi(x)$ we write

$$w^{[n]}(x_1,\ldots,x_n) = \langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle = W^{[n]}(\xi_1,\ldots,\xi_{n-1}),$$

$$\xi_k = x_k - x_{k+1}.$$

The requirement of uniqueness of the vacuum becomes the cluster condition W.VI:

$$w(x_1,\ldots,x_k,x_{k+1}+\lambda a,\ldots,x_n+\lambda a) \underset{\substack{\lambda\to\infty \\ a^2<0}}{\longrightarrow} w(x_1,\ldots,x_k)w(x_{k+1},\ldots,x_n).$$

The spectrality condition is written in the form

$$\operatorname{supp}\widetilde{W}(q_1,\ldots,q_{n-1}) \subset \overline{V}^+ \times \ldots \times \overline{V}^+,$$

where

$$\widetilde{W}(q_1,\ldots,q_{n-1}) = \int W(\xi_1,\ldots,\xi_{n-1})e^{i(q_1\xi_1+\cdots+q_{n-1}\xi_{n-1})}d^4\xi_1\ldots d^4\xi_{n-1}.$$

The Källén-Lehmann representation (§8.3.B) holds which enables us to express the two-point functions of an arbitrary field as a superposition of two-point functions of free fields with arbitrary (non-negative) masses.

The set of all Wightman functions enables us to define the linear functional

$$W(f) = w^{[0]}f^{[0]} + \int w^{[1]}(x)f^{[1]}(x)d^4x + \ldots + \int w^{[n]}(x_1,\ldots,x_n)f^{[n]}(x_1,\ldots,x_n)d^4x_1\ldots d^4x_n + \ldots$$

on the algebraic direct sum $\Omega = \bigoplus_{n=0}^{\infty} \mathcal{S}(\mathbf{M}^n)$ of test function spaces (consisting of finite sequences). The space Ω can be converted to an involutive algebra by defining (non-commutative) multiplication and conjugation on it by the formulae

$$(f \otimes g^{[n]}) = \sum_{k=0}^{n} f^{[k]}(x_1, \ldots, x_k) g^{[n-k]}(x_{k+1}, \ldots, x_n),$$
$$(f^{+})^{[n]} = \overline{f^{[n]}(x_n, \ldots, x_1)}.$$

Axioms W.I–W.VII are compactly written in the language of the Wightman functional $W(f)$. In particular, the positiveness of the scalar product in the Hilbert space of states (see W.I) reduces to the condition of multiplicative positiveness $W(f \otimes f^{+}) \geq 0$. The Wightman reconstruction theorem (Theorem 8.6,§8.3.C) enables us to recover the entire theory (that is, the Hilbert space $\mathcal{H}$, the representation of the group P_0 and the operators of the field ϕ) from the given functional W (satisfying appropriate conditions). The proof of this theorem reduces to a particular case of the GNS construction (§1.5.D). The free field ϕ is completely characterized by its two-point function (Proposition 8.8, §8.4.A). In §8.4.D we prove that a field whose commutator is a numerical function or depends only on the difference between the arguments is a superposition of free fields; a field with such properties is called a generalized free field.

In view of the spectrality condition, the Wightman functions $W^{[n+1]}(\xi_1, \ldots, \xi_n)$ have an analytic continuation to the past tube $T_n^- = (\mathbf{M} + iV^-)^n$ (Theorem 8.5,§8.3.A). On the other hand, every Lorentz-covariant function that is analytic in T_n^- has a unique analytic continuation to the extended tube

$$T_n = \bigcup_{\Lambda \in L_+(C)} \Lambda T_n^-$$

which is covariant with respect to the complex Lorentz group $L_+(C)$ (Theorem 9.1,§9.1.B). In contrast with T_n^-, the extended tube contains real points, called Jost points (a point $(\xi_1, \ldots, \xi_n) \in \mathbf{M}^n$ belongs to T_n if for any $\lambda_j \geq 0$, $\sum \lambda_j > 0$, the vector $\sum \lambda_j \xi_j$ is spacelike; Proposition 9.5,§9.1.C). The locality property enables one to further extend the domain of analyticity of the Wightman functions (Theorem 9.6,§9.1.D)

The reflection of all four axes in M is an element of $L_+(C)$. This enables us to prove (§9.2.A) using Theorem 9.1 (Bargmann-Hall-Wightman) the invariance of the Wightman theory with respect to anti-unitary TCP-transformations (Theorem 9.13). This result can, in fact, be sharpened by replacing the locality requirement by the less stringent weak locality condition (Theorem 9.14,§9.2.B). Using the TCP-operator we prove the transitivity of the (weak) locality properties for an irreducible system of fields. This leads to the notion of the Borchers class of mutually (weakly) local fields (§9.2.B) and, in particular, to the notion of a local composite field (§8.2.C). Later (§12.1.C) the coincidence of the S-matrices for mutually local fields will be shown.

The Bargmann-Hall-Wightman theorem also enables us to prove the normal connection between spin and statistics for the components of a field ψ that transforms according to an irreducible (finite-dimensional) representation of the Lorentz group and its Hermitian adjoint: if we assume that the components of the field either commute or anti-commute at spacelike separation of the arguments, then $\psi(x_1, \omega_1, \bar{\omega}_1)\psi(x_2, \omega_2, \bar{\omega}_2)^* = (-1)^{2(j+k)}\psi(x_2, \omega_2, \bar{\omega}_2)^*\psi(x_1, \omega_1, \bar{\omega}_1)$, where $(-1)^{2(j+k)}$ is the valency of the representation (Theorem 9.19,§9.3.A and Lemma 9.21,§9.3.B). (It is shown in Appendix I that for fields with an infinite number of components, the connection between spin and statistics does not follow from the axioms.) The commutation relations between different fields are not fixed by the remaining postulates. If, however, there are anomalous commutation relations in the system of fields, then the Wightman functions possess some additional symmetry. This enables us to define the (non-unitary) Klein transformations of the fields which preserve the Wightman functions and are such that the new fields have the regular connection between spin and statistics.

Also based on an analysis of the analytic properties of the Wightman functions is the proof of Haag's theorem on the non-existence of a representation of the interaction in the local relativistic theory (§9.4). This was historically the first of the indications on the need to use non-Fock representations of CCR's.

One of the most important applications of the analytic properties of the Wightman functions $w^{[n]}(x_1, \ldots, x_n)$, both in perturbative calculations and in constructive quantum field theory, is the possibility (based on the aforesaid properties) of passing to a Euclidean formulation of the theory. The

Schwinger functions $s^{[n]}(x_1,\dots,x_n)$, which are obtained from $w^{[n]}$ under purely imaginary changes of time $(x_k)_4 = -ix_k^0$, are analytic for (real) unequal arguments and satisfy the conditions e.1–e.7 (Osterwalder-Schrader axioms,§9.5.B). These conditions completely characterize the theory: if the functions $\{s^{[n]}\}$ satisfy e.1–e.7, then they are the Schwinger functions of some system of Wightman fields (Theorem 9.31,§9.5.C).

Our confidence in quantum field theory is based mainly on the successes of the perturbative calculations in quantum electrodynamics. Strictly speaking, however, quantum electrodynamics does not completely fit in to the Wightman approach (Chapters 8 and 9). In order to formulate it axiomatically in terms of local covariant gauges (Chapter 10) it is necessary to make two modifications of the apparatus: firstly, we must allow an indefinite metric in the space of "virtual state" vectors; secondly, we must use a modified Maxwell's equation (keeping the usual Maxwell's equation $\partial^\lambda \mathcal{F}_{\lambda\mu} - \mathcal{J}_\mu = 0$ only for the representation of the observable fields in the physical Hilbert space). In every theory with an indefinite metric, the collection of "pseudo-Wightman" axioms has to be supplemented by axioms enabling one to give a physical interpretation of the formalism (that is, to construct an algebra of observables and a physical Hilbert space;§§10.1.C,10.1.D). The specific character of theories with an indefinite metric emerges, in particular, in the question of spontaneous breaking of the internal symmetries generated by the local conserved currents: in the Wightman theory, spontaneous symmetry breaking is accompanied by a massless scalar boson (Goldstone's theorem,§10.3.B); in a theory with an indefinite metric, the "Goldstone boson" may turn out to be purely fictitious in that it is not found in the physical Hilbert space (§10.3.C).

One of the simplest field models with an indefinite metric is the massless scalar field in two-dimensional space-time (the one-particle subspace arising from the quantization of this field by the Fock method is not a Hilbert space but a so-called Pontryagin space). It is convenient to carry out the study of this model for the right and left components separately; these are the one-dimensional quantum fields on the isotropic lines of space-time (§§11.1.A,11.1.B). Using (local normal) exponentials of these fields, we obtain fields with any desired spin, in particular, the free massless Dirac field in two-dimensional space-time (§11.1.C), as well as the Thirring field (§11.2). The Schwinger model (that is, two-dimensional massless quantum electrodynamics, §11.3) is constructed by a similar method.

CHAPTER 8

The Wightman Formalism

8.1. Quantum Field Systems

A. Concept of Localization

In Chapter 6 we took as the definition of a physical system a pair $(\mathfrak{A}, \mathfrak{S})$, where $\mathfrak{A}$ is an abstract C^*-algebra (algebra of observables) and $\mathfrak{S}$ is the set of " physical" states, one of its properties being the capability of distinguishing the elements of $\mathfrak{A}$. The Hermitian elements of $\mathfrak{A}$ play the role of generalized variables which, in principle, can be measured experimentally (hence the terminology " algebra of observables"). We now single out the class of field systems. Intuitively we can imagine the observables of a field system to be certain functionals of a collection of "fundamental fields" which are functions on Minkowski space satisfying specified (field) equations. However, in the most interesting cases, the value of the quantum field at a point in space-time is devoid of meaning as will be shown presently, but the somewhat more general idea of localization comes to our aid for the correct definition of quantum field systems.

For this purpose we augment the previous axioms A.I–A.IV of the algebraic approach with the following axiom.

A.V. *Associated with each bounded open subset $\mathcal{O}$ of Minkowski space-time is a unital C^*-algebra $\mathfrak{A}(\mathcal{O})$ which is a subalgebra of the algebra of observables $\mathfrak{A}$ and is called the algebra of (local) observables associated with the set $\mathcal{O}$; here the algebra $\mathfrak{A}$ is the norm completion of the union*

$$\mathfrak{A}_{\text{loc}} = \bigcup_{\mathcal{O} \subset \mathbf{M}} \mathfrak{A}(\mathcal{O}), \tag{8.1}$$

called the algebra of local observables. Furthermore the following conditions hold:

(a) *isotony: if $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$;*

(b) *covariance with respect to the proper Poincaré group:*

$$\alpha_{a,\Lambda}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\Lambda\mathcal{O} + a); \tag{8.2}$$

(c) *local commutativity: the algebras $\mathfrak{A}(\mathcal{O}_1)$ and $\mathfrak{A}(\mathcal{O}_2)$ associated with regions $\mathcal{O}_1$ and $\mathcal{O}_2$ having spacelike separation, commute with each other.*

The content of the isotony and covariance conditions is clear; the physical meaning of the principle of local commutativity will be discussed in the next subsection.

A family $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O} \subset \mathbf{M}}$ of C^*-algebras satisfying the isotony condition is also known as a *net of C^*-algebras* over $\mathbf{M}$.

In the original statement of the axioms of algebraic quantum field theory (Haag and Schroer,1962) it was required that instead of abstract C^*-algebras, the algebras $\mathfrak{A}(\mathcal{O})$ had to be von Neumann algebras of operators in (physical) Hilbert space $\mathcal{H}$. The difference between these two variants is only of significance from the constructive point of view. It can happen that the algebra of observables is constructed before its physical representation π is chosen; in that case the abstract algebraic point of view is a reasonable one. On the other hand, when the physical representation is fixed, the algebras $\mathfrak{A}(\mathcal{O})$ can be considered to be "concrete" von Neumann algebras (replacing if necessary the abstract C^*-algebras $\mathfrak{A}(\mathcal{O})$ by von Neumann algebras $\overline{\pi(\mathfrak{A}(\mathcal{O}))}$).

As the norm completion of the algebra $\mathfrak{A}_{\text{loc}}$ of local observables, $\mathfrak{A}$ is also called the algebra of quasi-local observables. Not all observable quantities belong to these algebras. For example, the total energy (or momentum, or charge) of a quantum field system may not be a (quasi-) local observable. The standard method of constructing such "global" observables consists in the consideration of limits of local observables in the weak operator topology of the physical representation, that is, one goes over to the von Neumann algebra $\overline{\pi(\mathfrak{A})}$. If it is insisted (as Haag and Kastler, 1964, did) that the local algebras provide a full description of the field system which is amenable to physical experiment, then it must be accepted that all the observables are in some sense the outcome of local observables. In this connection it is most natural to assume that all the observables (including the "global" ones) are contained in the von Neumann algebra $\overline{\pi(\mathfrak{A})}$, or at any rate are affiliated to it.

B. Principle of local commutativity

One of the most essential requirements of the relativistic local theory is the postulate of local commutativity. It is a concrete expression of the general principle of causality according to which any two events in space-time that are separated by a spacelike interval cannot be in a cause-and-effect relationship.

In the approach that we are adopting, this principle is realized in the following way. In the spirit of Chapter 6, any unitary element U of the algebra of observables can be interpreted as the result of the action on the system which takes a state ρ into the state ρ_U, where $\rho_U(A) \equiv \rho(U^*AU)$, $A \in \mathfrak{A}$. The unitary elements U of the local algebra $\mathfrak{A}(\mathcal{O})$ can be interpreted as actions concentrated in the space-time region $\mathcal{O}$. If $\mathcal{O}$ is contained in the time slice $t_1 < t < t_2$, then according to the ordinary causality principle, a system subjected to such an action must be described by the state ρ for $t < t_1$ and by ρ_U for $t > t_2$; in other words, for local observables A associated with regions $\mathcal{Q}$ with $t < t_1$, the average value is given by $\rho(A)$, while for observables associated with regions $\mathcal{Q}$ with $t > t_2$, the average value is $\rho_U(A)$. In order that this prescription be consistent with the Einstein principle of causal independence, we must require that for regions $\mathcal{Q}$ that have a spacelike disposition with respect to $\mathcal{O}$, both expressions $\rho(A)$ and $\rho_U(A)$ should be the same (independently of whether $\mathcal{Q}$ is earlier or later than $\mathcal{O}$ in some coordinate frame, since in the given situation the notions "earlier" and "later" are not relativistic invariants).

For the sake of brevity we adopt the convention of writing the condition of spacelike separation of two regions $\mathcal{O}$ and $\mathcal{Q}$ in $\mathbf{M}$ in the form of the relation $\mathcal{O} \sim \mathcal{Q}$ (meaning: $(x-y)^2 < 0$ for all $x \in \mathcal{O}$, $y \in \mathcal{Q}$). As a result we arrive at the following version of Einstein's principle of causal independence:

$$\rho_U(A) = \rho(A) \quad \text{for all states } \rho,$$

where U is an arbitrary unitary operator in $\mathfrak{A}(\mathcal{O})$, A is an arbitrary element of $\mathfrak{A}(\mathcal{Q})$ and $\mathcal{O} \sim \mathcal{Q}$. From this it clearly follows that $U^*AU = A$, that is, $AU = UA$. Since the unitary elements U generate the entire algebra $\mathfrak{A}(\mathcal{O})$, we conclude that

$$AB = BA, \quad \text{if } A \in \mathfrak{A}(\mathcal{O}), \quad B \in \mathfrak{A}(\mathcal{Q}) \quad \text{and } \mathcal{O} \sim \mathcal{Q}. \tag{8.3}$$

This then is the formulation of condition A.V(c) of local commutativity.

It should be emphasized that although quantum field theory is based on the principle of local commutativity (and on its generalization, the principle of locality §8.2.A), and so reflects the modern idealized ideas on the structure (and in particular, the "microstructure") of space-time, it nevertheless allows us to realize the idea of diffuseness of the internal structure of an elementary particle. In fact, the average of such observables of the fields as the energy-momentum tensor, the electric current and, in particular, the electric charge density in a one-particle state are expressed, generally speaking, essentially non-locally in terms of the wave function of the particle. (The special concept of form factors of the particle are introduced for this recalculation.) In other words, the interaction of the local fields provides the elementary particles with a non-local structure which is both specifically relativistic and specifically quantum (that is, not reducible to "classical" analogues, such as the spatial extent of a body).

C. "FUNDAMENTAL" FIELDS AND "PHYSICAL" FIELDS

We have remarked that the observable quantities and states serve as a means of expressing the quantum phenomenology. However, for working out the details of the dynamics, we need to introduce dynamical variables or "fundamental" fields which may not enter into the set of observable quantities and serve as a kind of building material for the observables. This circumstance is not an exclusive property of quantum theory. For example, in classical electrodynamics, the four-vector potential $A_\mu(x)$ of an electromagnetic field is the "fundamental" field, but is not an observable quantity; the observable quantities are the gauge-invariant combinations of stress or current type, etc.*

It is clear from the example of electrodynamics, that unobservable fields can enter into the theory on a completely different footing from observable ones and there is no logical necessity to require that they be subjected to the scheme set out above for observables. In particular, we do not require that unobservable quantities (or fundamental fields) be operators acting in a physical Hilbert space. The mathematical nature of these fields can be very diverse: they can be operators in physical Hilbert space, they can also be operators in a non-physical pseudo-Hilbert space with indefinite metric (here there is scope for even further possibilities).

Fields that can be associated with operators in a physical Hilbert space are called "physical" fields, while the (C^*- or von Neumann) algebra of $\mathfrak{Y}$ bounded operators generated by the "physical" fields is called the algebra of physical quantities. The algebra of physical quantities occupies an imtermediate position in the hierarchy from "fundamental" fields to observable quantities: whereas the observables are reducible

* It is appropriate here to recall the thought experiment of Aharanov-Bohm (1959,1963), which turns attention to the specifics of the construction of observables for a non-simply connected space. It is asserted that if instead of ordinary space $\mathbf{R}^3$ one takes the outside of an infinite solenoid, then another independent observable emerges in quantum theory, namely, $\exp(ie \oint_C \mathbf{A}(x)d\mathbf{x})$, consisting of an electromagnetic field A_μ and which can not be expressed in terms of the stress tensor $F_{\lambda\mu}$. Here $\oint_C \mathbf{A}(x)dx$ is the circulation of the electromagnetic field around a contour C enclosing the solenoid and e is the quantum of electric charge. Thus, quantum physics allows the measurement of this circulation modulo g (where $g = 2\pi/e$ is the magnetic charge of the Dirac monopole), whereas in classical physics we have no means of measuring it (in the sense that one cannot "get into" the solenoid).

by superselection subspaces, the unobservable elements of the algebra of physical quantities are intertwining operators for these subspaces (for example, they serve as the operators of the creation of charged states from neutral ones, etc.).

For an algebra of physical quantities it is possible to state a set of axioms analogous to the axioms for the net of local algebras of observables (see Doplicher et al., 1969; Haag et al., 1970). The most important difference from the requirement of A.V is the addition of the axiom of cyclicity of the vacuum and a generalization of the axiom of local commutativity (we give these in the context of the Wightman formalism).

We now give an account of the Wightman formalism which is not as general as the algebraic one, but with its more advanced analytic apparatus, deals directly with the physical fields. (The vector potential of an electromagnetic field and, more generally, the gauge fields in an arbitrary covariant or local gauge are not "physical"; see Chapter 10 concerning these.)

8.2. Definition and Properties of a Local Quantum Field

A. WIGHTMAN'S AXIOMS

A fundamental role in the Wightman scheme is played by the notion of a relativistic quantum field, which has a well-known classical analogue. Since fields of various natures can feature in the theory (for example, scalar, spinor fields etc.), the quantum field ϕ can in fact be represented as a finite or infinite collection $\{\phi^{(\kappa)}\}$ of fields $\phi^{(\kappa)}$, where the index κ defines the type of the field. We suppose that each of the fields $\phi^{(\kappa)}$ is a tensor-valued or spin-tensor-valued quantity with a finite number of Lorentz components $\phi_l^{(\kappa)}$ ($l = 1, \dots, r_\kappa$) and with specified transformation properties under transformations from the proper Lorentz group $L_+^\uparrow$ or its covering group $SL(2, C)$. If $\phi_l^{(\kappa)}$ is not a Hermitian field, then we suppose that there exist indices $\bar{\kappa}$ and $\bar{l}$ in terms of which the Hermitian adjoint field is expressed:

$$\phi_l^{(\kappa)*} = \phi_{\bar{l}}^{(\bar{\kappa})}. \tag{8.4}$$

In its turn, $\phi_l^{(\kappa)} \equiv \phi_l^{(\kappa)}(x)$ is an operator-valued generalized function on space-time $\mathbf{M}$, so that the results of smoothing it

$$\phi_l^{(\kappa)}(f) \equiv \int \phi_l^{(\kappa)}(x) f(x) d^4x \tag{8.5}$$

with test functions $f(x)$ are linear operators in the physical Hilbert space $\mathcal{H}$. These operators are not assumed to be bounded; we suppose instead that all the operators $\phi_l^{(\kappa)}(f)$ have a common domain of definition D which is a dense linear subspace of $\mathcal{H}$. In order that the algebraic operations on the operators $\phi_l^{(\kappa)}$ should make sense, we must further suppose that any operator $\phi_l^{(\kappa)}(f)$ maps D into itself.

The precise definition of the relativistic quantum field is given by the following set of Wightman axioms. Since in the Wightman formalism we (explicitly or implicitly) use all the axioms of relativistic quantum theory listed earlier, we shall for convenience reproduce them here in connection with the present context.

W.I (Relativistic invariance of the state space). *There is a physical Hilbert space $\mathcal{H}$ in which a unitary representation $U(a, \underset{\sim}{\Lambda})$ of the Poincaré spinor group $\mathfrak{P}_0$ acts.*

W.II (Spectral property). *The spectrum of the energy-momentum operator P is concentrated in the closed upper light cone $\overline{V}^+$.*

W.III (Existence and uniqueness of the vacuum). *There exists in $\mathcal{H}$ a unique (to within a phase factor) unit vector Ψ_0 (also denoted by $|0\rangle$ and called the vacuum vector), which is invariant with respect to the space-time translations $U(a, 1)$.*

W.IV (Domain of definition of the fields). *The components $\phi_l^{(\kappa)}$ of the quantum fields $\phi^{(\kappa)}$ are operator-valued generalized functions $\phi_l^{(\kappa)}(x)$ over the Schwartz space $\mathcal{S}(\mathbf{M})$ with domain of definition D which is common to all the operators $\phi_l^{(\kappa)}(f)$ and is dense in $\mathcal{H}$; it is supposed that the vacuum vector Ψ_0 is contained in D and that D is taken into itself under the action of the operators $\phi_l^{(\kappa)}(f)$ and $U(a, \underset{\sim}{\Lambda})$.*

W.V (Poincaré-invariance of the fields).

$$U(a, \underset{\sim}{\Lambda})\phi_l^{(\kappa)}(x)U(a, \underset{\sim}{\Lambda})^{-1} = \sum_m V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda}^{-1})\phi_m^{(\kappa)}(\Lambda(\underset{\sim}{\Lambda})x + a); \tag{8.6}$$

here $V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda}))$ is a complex or real finite-dimensional matrix representation of $SL(2, C)$.

We suppose here that $V^{(\kappa)}(-1) = \pm 1$. If $V^{(\kappa)}(-1) = 1$, then the field $\phi^{(\kappa)}$ transforms according to a single-valued representation of $L_+^\uparrow$ and is called a field with integral spin (or a tensor field); if $V^{(\kappa)}(-1) = -1$, then the field transforms according to a two-valued representation of $L_+^\uparrow$ and is called a field with half-integral spin (or spinor field).

W.VI (Locality, or microcausality). *Any two field components $\phi_l^{(\kappa)}(x)$ and $\phi_m^{(\kappa')}(y)$ either commute or anticommute under a spacelike separation of the arguments x and y:*

$$[\phi_l^{(\kappa)}(x), \phi_m^{(\kappa')}(y)]_\mp = 0 \quad \text{for } (x-y)^2 < 0. \tag{8.7}$$

W.VII (Cyclicity of the vacuum). *The set D_0 of finite linear combinations of vectors of the form $\phi_{l_1}^{(\kappa_1)}(f_1) \ldots \phi_{l_n}^{(\kappa_n)}(f_n)\Psi_0$ $(n = 0, 1, \ldots)$ is dense in $\mathcal{H}$.*

B. Discussion of the Axioms

Even at a cursory glance at the Wightman scheme we can distinguish the axioms of principle which fix certain physical situations and the "technical" axioms, although it is not possible to give a sharp delineation between them. The essence of the formal Wightman scheme is, of course, the idea of a quantum field. The "technical" axioms serve as a concrete mathematical formulation of this idea; their justification is that they are balanced and they "work".

The first three axioms W.I–W.III have already been discussed in Chapter 7; we merely add that Axiom W.III plays an extremely vital role in the Wightman theory even though it has a "technical" nature to a certain extent and is not a necessary attribute of relativistic quantum theory. (Axiom W.III is equivalent to the cluster property (8.19) given below.)

Axiom W.IV introduces the notion of a quantum field. The definition of a field as an operator-valued generalized function on $\mathbf{M}$ is more in accordance with the real physical situation than the more customary notion of a field defined at each point of space-time (Bohr and Rosenfeld, 1950). This fact was already noted in §6.4.B (see also Exercise 8.7 below). The specific choice of the space of test functions is unquestionably a technical proposal. Originally Wightman (1956) proposed a more general version, using the Schwartz space $\mathcal{D}(\mathbf{M})$, but this was quickly replaced by a formulation in terms of $\mathcal{S}(\mathbf{M})$ (Schmidt and Baumann, 1956). The convenience of the space of tempered generalized functions lies in its naturalness and the simplicity of the apparatus of the Fourier transform which enables one simultaneously to consider local properties of such generalized functions both in the x- and the p-space. The effectiveness of tempered generalized functions is underlined by the experience of the Lagrange theory of perturbations of renormalizable quantum fields (see, for example, [B10] and [I4]), as well as the results of constructive quantum field theory (see [G8]). Apparently this apparatus is well suited to the treatment of field theories with renormalizable interactions. Various attempts have been made to generalize the Wightman formalism in application to non-renormalizable interactions by choosing suitable spaces of test and generalized functions compatible with the local properties in x-space * (Concerning this, see Meiman, 1964; Jaffe, 1967; Efimov, 1968, [E1]; Iofa and Fainberg, 1969a, b; Solov'ev M.A., 1971,1980a,b).

Wightman's theory works with unbounded operators, since the result of smoothing a field $\phi_l^{(\kappa)}(x)$ with a test function is generally an unbounded operator. This method has its advantages and disadvantages by comparison with the abstract algebraic formulation dealing with bounded operators. On the credit side is the simplicity of expressing the axioms of Poincaré covariance (W.V) and locality (W.VI) (which is fairly essential in the study of the analytic properties of scattering amplitudes; this also provides the possibility of a direct treatment of the renormalized field equations in concrete models). On the other hand, Axiom W.IV is full of technical assumptions which are necessary for handling the unbounded operators. In any case, they allow one to associate with each open subset $\mathcal{O} \subset \mathbf{M}$ an algebra of operators $\mathcal{P}(\mathcal{O})$, called the *polynomial algebra* associated with $\mathcal{O}$, and generated by the complex linear combinations of all possible operators A of the form

$$A = \int \phi_{l_1}^{(\kappa_1)}(x_n) \dots \phi_{l_n}^{(\kappa_n)}(x_n) f(x_1, \dots, x_n) dx_1 \dots dx_n, \tag{8.8}$$

where the supports of the test functions $f(x_1, \dots, x_n)$ are contained in the set $\mathcal{O} \times \times \dots \times \mathcal{O}$ (for $n = 0$, $A = f$ is a scalar). Since $\mathcal{P}(\mathcal{O})$ is closed with respect to taking the Hermitian adjoint ($A \to A^*|D$), it is actually a *-algebra, or an involutive algebra. We denote by $\alpha_{(a,\underset{\sim}{\Lambda})}$ the *-automorphism of the algebra $\mathcal{P}(\mathbf{M})$, acting according to the formula

$$\alpha_{(a,\underset{\sim}{\Lambda})}(A) = U(a, \underset{\sim}{\Lambda}) A U(a, \underset{\sim}{\Lambda})^{-1} \tag{8.9}$$

and which, according to Axiom W.V maps $\mathcal{P}(\mathcal{O})$ onto $\mathcal{P}(\Lambda(B)\mathcal{O} + a)$.

The assumption in Axiom W.IV is generally insufficient for fixing the field operators completely. As a further strengthening of Axiom W.IV we can require that all those linear combinations of smooth field operators that are Hermitian operators be essentially self-adjoint operators on the domain D. We then have the possibility of constructing a net of local field algebras $\mathfrak{z}(\mathcal{O})$ by defining $\mathfrak{z}(\mathcal{O})$ as the von Neumann algebra generated by all possible bounded functions of the operators indicated above, subject to the condition that the supports of the smoothing test functions be

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* In the remark in §9.1.E we shall indicate an approach to non-renormalizable theories in terms of analytic Wightman functions.

contained in the given open set $\mathcal{O}$. The field (quasi-local) algebra $\mathfrak{F}$ is defined as the norm closure of the net of algebras $\mathfrak{F}(\mathcal{O})$. In order that the net of algebras $\mathfrak{F}(\mathcal{O})$ so obtained should satisfy the condition of locality, that is, that the reformulation of Axiom W.VI be admitted, it is necessary to add further "technical" hypotheses (see Borchers and Zimmermann, 1964; Bisognano and Wichmann, 1975).

Hence it is possible to go over to the net of algebras of local observables $\mathfrak{A}(\mathcal{O})$. The basic method of constructing $\mathfrak{A}(\mathcal{O})$ in terms of $\mathfrak{F}(\mathcal{O})$ is as follows (Doplicher et al, 1969a). We are given a group of symmetries of the algebra $\mathfrak{F}$ which take the subalgebras $\mathfrak{F}(\mathcal{O})$ into themselves and which are called *gauge transformations*. Then $\mathfrak{A}(\mathcal{O})$ is defined as the subalgebra of gauge-invariant elements of $\mathfrak{F}(\mathcal{O})$. This rule is in accordance with the general principle according to which the gauge transformations of the fields do not affect, that is, do not alter, the observables. Accordingly, the algebra of quasi-local observables $\mathfrak{A}$ is defined as the norm completion of the union of the algebras $\mathfrak{A}(\mathcal{O})$.

Axiom W.V allows only fields with a finite number of Lorentz components (that is, finite-component fields, or fields that transform according to finite-dimensional representations of the Lorentz group). In principle, the Wightman formalism is capable of being adapted to the idea of infinite-component fields; however, in this case, a number of corollaries of the Wightman theory are lost (see Appendix I). Apart from this, since massless particles with infinite (Wigner) spin have not been experimentally observed (see §7.2.B), we shall exclude from this discussion fields with an infinite number of Lorentz components; we shall also confine ourselves in the main text to finite-component fields.

We now give another formulation of the covariance condition. Suppose for definiteness that the field $\phi^{(\kappa)}$ transforms according to an irreducible representation $V^{(\kappa)}$ of $SL(2,C)$ which is equivalent to the representation $\mathfrak{D}^{(j,k)}$ in the space $\mathfrak{p}^{(j,k)}$ (see §3.1.D). Then instead of the components $\phi_l^{(\kappa)}$ of the field, we can consider the operator-valued generalized function $\phi(x;\omega,\overline{\omega})$ of $x \in \mathrm{M}$, this being a homogeneous polynomial of degree $2j$ of the complex two-dimensional vector $\omega \equiv (\omega^1,\omega^2)$ and a homogeneous polynomial of degree $2k$ of the complex conjugate vector $\overline{\omega} \equiv (\overline{\omega}^1,\overline{\omega}^2)$:

$$\phi^{(\kappa)}(x,\omega,\overline{\omega}) = \sum_{\substack{\alpha_1\dots\alpha_{2j} \\ \beta'_1\dots\beta'_{2k}}} \phi^{(\kappa)}_{\alpha_1\dots\alpha_{2j};\beta'_1\dots\beta'_{2k}}(x)\omega^{\alpha_1}\dots\omega^{\alpha_{2j}}\overline{\omega}^{\beta'_1}\dots\overline{\omega}^{\beta'_{2k}}. \tag{8.10}$$

Here the law of transformation (8.6) assumes the form

$$U(a,\underset{\sim}{\Lambda})\phi^{(\kappa)}(x,\omega,\overline{\omega})U(a,\underset{\sim}{\Lambda})^{-1} = \phi^{(\kappa)}(\Lambda x + a, \underset{\sim}{\Lambda}\omega, \bar{\underset{\sim}{\Lambda}}\overline{\omega}). \tag{8.11}$$

The relationship between this description and the one given above is realized by the formula

$$\phi^{(\kappa)}(x,\omega,\overline{\omega}) = \sum_l \phi_l^{(\kappa)}(x)\psi_l(\omega,\overline{\omega}),$$

where ψ_l is a basis in the space $\mathfrak{p}^{(j,k)}$ such that

$$(\mathfrak{D}^{(j,k)}(\underset{\sim}{\Lambda})\psi_l)(\omega,\overline{\omega}) \equiv \psi_l(\underset{\sim}{\Lambda}^{-1}\omega, \bar{\underset{\sim}{\Lambda}}^{-1}\overline{\omega}) = \sum_m V_{m,l}^{(\kappa)}(\underset{\sim}{\Lambda})\psi_m(\omega,\overline{\omega}).$$

This language of the variables $\omega,\overline{\omega}$, is clearly also applicable to all the simply reducible representations $V^{(\kappa)}$ (and, as is clear from Chapter 3 as well as from Appendix G, its convenience shows up in the search for covariant representations).

We now turn to Axiom W.VI. Locality in quantum field theory is the property of quantum fields that ensures that the observables will satisfy local commutativity, which in turn expresses the Einstein principle of causal independence. In a theory in which all the fields are observable we have to require that any two fields with spacelike separation of the arguments are mutually commutative:

$$[\phi_l^{(\kappa)}(x), \phi_m^{(\kappa')}(y)] = 0 \quad \text{for } (x-y)^2 < 0. \tag{8.12}$$

Since the Wightman theory admits unobservable fields, a more general form of the locality condition (8.7) is postulated for them. From a consideration of Lorentz invariance, we suppose from now on that the sign (+ or -) in this condition depends only on the type of field (that is, on κ and κ') but does not depend on the Lorentz components l, m; then (8.7) can be rewritten in the form *

$$\phi_l^{(\kappa)}(x), \phi_m^{(\kappa')}(y) = \sigma^{(\kappa,\kappa')}\phi_m^{(\kappa')}(y)\phi_l^{(\kappa)}(x) \quad \text{for } (x-y)^2 < 0, \tag{8.13}$$

where all the elements of the matrix $\sigma \equiv \sigma^{(\kappa,\kappa')}$ consist only of the numbers $+1$ and -1. The field $\phi^{(\kappa)}$ is called a *boson* field if

$$[\phi_l^{(\kappa)}(x), (\phi_l^{(\kappa)}(y))^*] = 0 \quad \text{for } (x-y)^2 < 0, \tag{8.14}$$

and a *fermion* field if

$$[\phi_l^{(\kappa)}(x), (\phi_l^{(\kappa)}(y))^*]_+ = 0 \quad \text{for } (x-y)^2 < 0. \tag{8.15}$$

In the Wightman theory there is the Bose-Fermi alternative: each field is either a boson or a fermion field (where it cannot be either at the same time, except when it is identically zero). This postulate also has a historical origin: all known elementary particles are subject to either Bose or Fermi statistics, therefore for the quantum fields (regarded as potential operators of creation of particles), we must also introduce the two "statistics", namely Bose and Fermi. However, in addition to these two statistics intermediate ones are theoretically possible, the so-called *parastatistics* (see, for example, Green, 1953; Messiah and Greenberg, 1964; Greenberg and Messiah, 1965; Appendix H is devoted to the notion of a parafield).

Corresponding to the experimentally observed connection of spin with statistics (§7.3.A) in field theory is the notion of *normal connection of spin with statistics*; this means that fields with integral spin commute under spacelike separation with all the remaining fields, while all fields with half-integral spin anticommute under spacelike separation of the arguments. In terms of the matrix $\sigma^{(\kappa,\kappa')}$, this condition means:

$$\begin{aligned} &\sigma^{(\kappa,\kappa')} = 1, \text{ if at least one of the fields } \phi^{(\kappa)}, \phi^{(\kappa')} \text{ has integral spin,} \\ &\sigma^{(\kappa,\kappa')} = -1, \text{ if both fields } \phi^{(\kappa)}, \phi^{(\kappa')} \text{ have half-integral spin.} \end{aligned} \tag{8.16}$$

Exercise 8.1. (Doplicher et al., 1969) Verify that in the case of normal connection of spin with statistics, the condition of locality can be written in the form

$$A^t B = B A^t \quad \text{for } A \in \mathcal{P}(\mathcal{O}),\ B \in \mathcal{P}(\mathcal{Q}),\ \mathcal{O} \sim \mathcal{Q}; \tag{8.17}$$

* Condition (8.13) can be derived from (8.7) for specially defined types of fields; for this it suffices to suppose that the fields $\phi^{(\kappa)}$ transform according to real irreducible representations of $SL(2,C)$ (see [J3], §5.3).

here the *torsion* operation $A \to A^t$ is used in the algebra $\mathcal{P}(\mathbf{M})$, which is defined as follows:

$$A^t = A_+ + U(0,-1)A_-, \quad A_\pm \equiv \tfrac{1}{2}(A \pm \alpha_{(0,-1)}(A)). \tag{8.18}$$

In connection with the condition of locality given in Exercise 8.1, there arises the following strengthening of this concept. The set

$$\mathfrak{J}(\mathcal{O})^{tc} = \{B \in \mathcal{B}(\mathcal{H}) : A^t B = BA^t \quad \text{for all } A \in \mathfrak{J}(\mathcal{O})\}$$

is called the *commutant with torsion* of the algebra $\mathfrak{J}(\mathcal{O})$. We say that the condition of *duality with torsion* holds if

$$\mathfrak{J}(\mathcal{O}') = \mathfrak{J}(\mathcal{O})^{tc}$$

for any bounded open set $\mathcal{O} \subset \mathbf{M}$ with a piecewise-smooth boundary such that $\mathcal{O} = (\mathcal{O}')'$. Here the prime denotes the spacelike complement of the region:

$$\mathcal{O}' = \operatorname{int}\{x \in \mathbf{M} : x \sim \mathcal{O}\}.$$

(An example of such a set $\mathcal{O}$ is the diamond (4.76).) The condition of duality with torsion is more stringent than the condition of locality (which merely requires the inclusion $\mathfrak{J}(\mathcal{O}') \subset \mathfrak{J}(\mathcal{O})^{tc}$). In the case of boson fields, the terms "commutant with torsion" and "duality with torsion" are replaced by just "commutant" and "duality".

Although in Wightman's theory the normal connection between spin and statistics may not be observed, we do not in fact obtain any interesting generalization, * so that the Wightman axioms are sometimes supplemented by another axiom W.VIII, which simplifies the statement of certain results.

W.VIII (Spin and Statistics). *There is a normal connection between spin and statistics.*

However, when we use this axiom in the sequel, we shall explicitly mention it.

There is a remarkable theorem (going back to Pauli, 1940) on the relation between spin and statistics (§9.3). What it says is that the components of the tensor field commute, while the components of the spinor field anticommute under spacelike separation of the arguments. An anomalous association of spin with statistics can only occur for different fields; the local quantum fields can always be redefined (by means of a transformation due to Klein, 1938) in such a way that the normal connection between spin and statistics is restored.

Finally, Axiom W.VII can be regarded as a "technical" hypothesis. As we shall see in the next subsection, it means irreducibility of the given system of fields.

Using Axiom W.VII and Proposition 7.1 we can replace Axiom W.III by the following property.

Cluster property. There exists in D a unit vector $|0\rangle$ such that for any spacelike vector $a \in \mathbf{M}$ and any $A_1, A_2 \in \mathcal{P}(\mathbf{M})$ the following relation holds:

$$\langle 0|A_1 U(\lambda a, 1) A_2|0\rangle \to \langle 0|A_1|0\rangle\langle 0|A_2|0\rangle \quad \text{as } \lambda \to \infty. \tag{8.19}$$

C. Irreducibility of fields

We say that the field operators form an irreducible system if any bounded operator C in $\mathcal{H}$ that commutes in the weak sense with all the operators $\phi_l^{(a)}(f)$, that is,

$$\langle \phi_l^{(\kappa)}(f)^* \Phi, C\Psi\rangle = \langle \Phi, C\phi_l^{(\kappa)}(f)\Psi\rangle \tag{8.20}$$

* In the theory of infinite-component fields, this is no longer the case (see Appendix I).

for all κ, l and $\Phi, \Psi \in D$, $f \in \mathcal{S}(\mathbf{M})$, must be a multiple of the identity operator. It is clear that (8.20) is equivalent to the irreducibility of the polynomial algebra $\mathcal{P}(\mathbf{M})$, that is, the condition

$$\langle A^*\Phi, C\Psi\rangle = \langle \Phi, CA\Psi\rangle \tag{8.21}$$

for all $A \in \mathcal{P}(\mathbf{M})$, $\Phi, \Psi \in D$. We have the following result (Ruelle, 1962; Borchers, 1962).

Proposition 8.1. *In the Wightman theory (in which Axioms* W.I–W.VII hold) *the field operators form an irreducible system.*

■ Suppose that the bounded operator C satisfies (8.21). We need to prove that it is a multiple of the identity operator: $C = \lambda$. First we verify that the vacuum is an eigenvector of C:

$$C\Psi_0 = \lambda\Psi_0. \tag{8.22}$$

It follows from (8.21) that

$$\langle \alpha_{(-a,1)}(A)\Psi_0, C\Psi_0\rangle = \langle C^*\Psi_0, \alpha_{(-a,1)}(A)^*\Psi_0\rangle$$

for all $a \in \mathbf{M}$, $A \in \mathcal{P}(\mathbf{M})$. Using (8.9) we find that

$$\langle A\Psi_0, U(a,1)C\Psi_0\rangle = \langle C^*\Psi_0, U(-a,1)A^*\Psi_0\rangle, \tag{8.23}$$

and hence
$$\int \langle A\Psi_0, U(a,1)C\Psi_0\rangle e^{-ipa} d^4a = \int \langle C^*\Psi_0, U(-a,1)A^*\Psi_0\rangle e^{-ipa} d^4a.$$

By the spectrum condition, the left hand side of this equality has support with respect to the variable p in the cone $\overline{V}^+$ and the right hand side has support in the cone $\overline{V}^- \equiv -\overline{V}^+$; hence both sides are generalized functions in p with support at 0. Thus the expressions in (8.23) are polynomials in $a \in \mathbf{M}$, but since they are bounded as functions of a, they cannot depend on a:

$$\langle A\Psi_0, U(a,1)C\Psi_0\rangle = \langle A\Psi_0, C\Psi_0\rangle.$$

It follows from Axiom W.VII that the vectors of the form $A\Psi_0$ form a dense set in $\mathcal{H}$, therefore this last equality implies that $U(a,1)C\Psi_0 = C\Psi_0$, which together with the uniqueness of the vacuum implies (8.22).

For all $\Psi \in D$, $A \in \mathcal{P}(\mathbf{M})$ we have

$$\langle \Psi, CA\Psi_0\rangle = \langle A^*\Psi, C\Psi_0\rangle = \lambda\langle A^*\Psi, \Psi_0\rangle = \langle \Psi, \lambda A\Psi_0\rangle,$$

so that $(C - \lambda)A\Psi_0 = 0$. Again since the vectors of the form $A\Psi_0$ are dense in $\mathcal{H}$, we conclude that $C = \lambda$. ■

In the next subsection we prove that Axiom W.VII can be strengthened: the vacuum vector is cyclic for any local algebra $\mathcal{P}(\mathcal{O})$, where $\mathcal{O}$ is a non-empty open subset of $\mathbf{M}$. However, for bounded sets these algebras do not form irreducible systems of operators. This caused Haag to state the following extra condition which, after Haag and Schroer (1962), we call the condition of *primitive causality*: the algebra $\mathcal{P}(\mathcal{O}_{(t_1,t_2)})$ is irreducible for any domain of the form

$$\mathcal{O}_{(t_1,t_2)} = \{x \in \mathbf{M} : t_1 < x^0 < t_2\}.$$

(This is the slice between two planes with fixed values of time t_1, t_2 ($t_1 < t_2$) in space-time.)

The condition of primitive causality is a "technical" expression of the general *principle of causality* (or *determinism*) which is required in order that the field algebra in the entire Minkowski space-time be generated by fields in three-dimensional space in an arbitrarily small interval of time (t_1, t_2). (In contrast to the axiom of locality, the principle of causality retains its sense in the non-relativistic theory as well.) This requirement is automatically fulfilled when we are dealing with the canonical formalism of ordinary (non-degenerate) Lagrangian field systems. The fields then satisfy the equations of motion, and if we define them (together with their derivatives of sufficiently high order) on any spacelike surface, we can recover the fields in the entire space-time. On the other hand, the statement of the condition of primitive causality is considerably more flexible than the standard canonical Lagrangian (or Hamiltonian) formalism. In fact, in contrast to the canonical formalism, we do not even require the existence of the values of the field at a fixed moment of time; all that we require is that the values of $\phi(f)$ should be meaningful for $f(x) \in \mathcal{S}(\mathbf{M})$. The condition of primitive causality does not depend on axioms W.I–W.VII. This can be seen from the example of generalized free fields (§8.4.D) with slowly decreasing weight functions $\sigma(m^2)$ (Haag and Schroer, 1962). However this postulate does not exclude all the generalized free fields; it holds in the model with rapidly decreasing weight functions (Borchers et al., 1963; see also Garber, 1975).

D. Separating property of the vacuum vector

We show that the vacuum vector is cyclic not only for the whole algebra $\mathcal{P}(\mathbf{M})$, but also for any local polynomial algebra $\mathcal{P}(\mathcal{O})$.

Proposition 8.2 (Reeh and Schlieder, 1961). *For any non-empty open set $\mathcal{O} \subset \mathbf{M}$, the set of vectors of the form $A\Psi_0$, where $A \in \mathcal{P}(\mathcal{O})$, is dense in $\mathcal{H}$.*

■ Let $\Phi \in \mathcal{H}$ be orthogonal to all vectors of the form $A\Psi_0$, $A \in \mathcal{P}(\mathcal{O})$. It is required to prove that $\Phi = 0$. For this purpose we consider all possible generalized functions

$$\langle \Phi, \phi_{l_1}^{(\kappa_1)}(x_1) \dots \phi_{l_n}^{(\kappa_n)}(x_n)\Psi_0 \rangle. \tag{8.24}$$

Exercise 8.2. Prove that the Fourier transform

$$\langle \Phi, \tilde{\phi}_{l_1}^{(\kappa_1)}(p_1) \dots \tilde{\phi}_{l_n}^{(\kappa_n)}(p_n)\Psi_0 \rangle =$$
$$= \int \langle \Phi, \phi_{l_1}^{(\kappa_1)}(x_1) \dots \phi_{l_n}^{(\kappa_n)}(x_n)\Psi_0 \rangle \exp\Big(i \sum_{j=1}^{n} p_j x_j\Big) d^4x_1 \dots d^4x_n$$

is a generalized function (8.24) contained in the intersection of the sets $\{p \in \mathbf{M}^n : p_k + p_{k+1} + \dots + p_n \in \in \overline{V}^-\}$ for $k = 1, \dots, n$. [Hint: Use the identity

$$\langle \Phi, \phi(x_1) \dots \phi(x_{k-1})\phi(x_k + a) \dots \phi(x_n + a)\Psi_0 \rangle =$$
$$= \langle \Phi, \phi(x_1) \dots \phi(x_{k-1})U(a,1)\phi(x_k) \dots \phi(x_n)\Psi_0 \rangle$$

and the spectrum condition.]

It follows from this exercise and Theorem B.7 that (8.24) is the boundary value of a function $f(z_1, \dots, z_n)$ that is holomorphic in a tube of the form $M^n + iK$ (K being a convex cone with a non-empty interior). Furthermore, by hypothesis, the functions (8.24) vanish on a non-empty open set $x_1, \dots, x_n \in \mathcal{O}$. It follows from the generalized uniqueness theorem B.10 that (8.24) is identically zero. Thus $\langle \Phi, A\Psi_0 \rangle = 0$ for all $A \in \mathcal{P}(\mathbf{M})$, from which it follows that $\Phi = 0$. ■

The arguments used in Proposition 8.2 find further application in the following exercise.

Exercise 8.3. (a) Let $\mathcal{H}_{l_1 \dots l_n}^{(\kappa_1 \dots \kappa_n)}$ be the closure of the set of vectors $A\Psi_0$ where A has the form (8.8) with fixed $\kappa_1, \dots, \kappa_n, l_1, \dots, l_n$ and with the function f ranging over $\mathcal{S}(\mathbf{M}^n)$. Prove that $\mathcal{H}_{l_{\pi 1} \dots l_{\pi n}}^{(\kappa_{\pi 1} \dots \kappa_{\pi n})} = = \mathcal{H}_{l_1 \dots l_n}^{(\kappa_1 \dots \kappa_n)}$ for any permutation π of the indices $1, \dots, n$. [Hint: The assertion is equivalent to the following. If the vector $\Phi \in \mathcal{H}$ is such that the generalized function $\langle \Phi, \phi_{l_{\pi 1}}^{(\kappa_{\pi 1})}(x_{\pi 1}) \dots \phi_{l_{\pi n}}^{(\kappa_{\pi n})}(x_{\pi n})\Psi_0 \rangle$ is equal to zero in $\mathbf{M}^n$, then the generalized function (8.24) is also equal to zero in $\mathbf{M}^n$. To prove this, use the fact that on the set of points $(x_1, \dots, x_n)$ such that $x_j \sim x_k$ for $j \neq k$, the order of the operator-valued factors in (8.24) can be permuted (apart from a possible change in sign), so that (8.24) vanishes on the given set. Finally, use the fact that (8.24) is the boundary value of an analytic function in the tube.]

(b) Let $\mathcal{H}^{(\kappa_1 \dots \kappa_n)}$ be the closure of the set of vectors of the form

$$\sum_{l_1 \dots l_n} \int \phi_{l_1}^{(\kappa_1)}(x_1) \dots \phi_{l_n}^{(\kappa_n)}(x_n) f^{l_1 \dots l_n}(x_1 \dots x_n) dx_1 \dots dx_n \Psi_0,$$

where $\kappa_1, \ldots, \kappa_n$ are fixed and the $f^{l_1 \ldots l_n}$ range over $\mathcal{S}(\mathbf{M}^n)$. Prove that $\mathcal{H}^{(\kappa_{\pi 1} \ldots \kappa_{\pi n})} = \mathcal{H}^{(\kappa_1 \ldots \kappa_n)}$ for any permutation π. [Hint: $\mathcal{H}^{(\kappa_1 \ldots \kappa_n)}$ is the closed linear span of the union of all the subspaces $\mathcal{H}^{(\kappa_1 \ldots \kappa_n)}_{l_1 \ldots l_n}$ for all possible $l_1, \ldots, l_n$.]

A vector of the form $A\Psi_0$, where A is a unitary element of the local polynomial algebra $\mathcal{P}(\mathcal{O})$ can be called an excitation of the vacuum in the region $\mathcal{O}$ (or a state that is localized in $\mathcal{O}$). However, in order that this definition be of interest, we have to go over from $\mathcal{P}(\mathcal{O})$ to the local field algebra $\mathfrak{F}(\mathcal{O})$ in which there are a sufficiency of unitary elements. With regard to the concept of states that are localized in a region $\mathcal{O}$, see the articles by Knight (1961) and Licht (1963).

Exercise 8.4. Suppose that Axioms W.VIII hold and that the torsion $A \to A^t$ is defined by the formulae (8.18). Prove that for any non-empty open set $\mathcal{O} \subset \mathbf{M}$ the set of vectors of the form $A^t\Psi_0$, where $A \in \mathcal{P}(\mathcal{O})$, is dense in $\mathcal{H}$. [Hint: Verify that the set of vectors of the $A^t\Psi_0$, $A \in \mathcal{P}(\mathcal{O})$ is the same as the set of vectors of the form $B\Psi_0$, $B \in \mathcal{P}(\mathcal{O})$.]

We can now prove a very important property of the vacuum vector: it separates the elements of the local field algebra $\mathcal{P}(\mathcal{O})$ of a bounded open set $\mathcal{O}$. This means that if $A_1\Psi_0 = A_2\Psi_0$, where $A_1, A_2 \in \mathcal{P}(\mathcal{O})$, then $A_1 = A_2$; this is clearly equivalent to the property that the quality $A\Psi_0 = 0$ for $A \in \mathcal{P}(\mathcal{O})$ implies that $A = 0$. For the proof of the separation property we use Axiom W.VIII on the normal association of spin with statistics.

Proposition 8.3. *In the Wightman theory with a normal connection between spin and statistics, the vacuum vector separates the elements of the local polynomial algebras $\mathcal{P}(\mathcal{O})$ associated with bounded open sets $\mathcal{O}$.*

■ Let $A\Psi_0 = 0$, where $A \in \mathcal{P}(\mathcal{O})$. We choose a region $\mathcal{Q}$ such that $\mathcal{Q} \sim \mathcal{O}$; then

$$AB^t\Psi_0 = B^tA\Psi_0 = 0 \tag{8.25}$$

for all $B \in \mathcal{P}(\mathcal{Q})$. According to Exercise 8.4, the vectors of the form $B^t\Psi_0$ for $B \in \mathcal{P}(\mathcal{Q})$ form a dense subset of $\mathcal{H}$, while the operator A has a closure (since its adjoint has a dense domain). It therefore follows from (8.25) that $A = 0$ ■

Without assuming the normal connection between spin and statistics, it is not difficult to prove a somewhat weaker version. Henceforth by a *polylocal monomial* we mean an operator-valued generalized function of the form

$$X = \phi^{(\kappa_1)}_{l_1}(x_1) \ldots \phi^{(\kappa_n)}_{l_n}(x_n). \tag{8.26}$$

Proposition 8.4. *If the equality $X\Psi_0 = 0$ holds for some polylocal monomial X (8.26), then $X = 0$.*

■ Let $\mathcal{O}$ be an arbitrary bounded region. It clearly suffices to prove that $X = 0$ when $x_1, \ldots, x_n \in \mathcal{O}$. We choose a region $\mathcal{Q}$ such that $\mathcal{Q} \sim \mathcal{O}$. It then follows from the locality condition W.VI that for each $B \in \mathcal{P}(\mathcal{Q})$ there exists $B' \in \mathcal{P}(\mathcal{Q})$ such that $XB = B'X$; hence $XB\Psi_0 = B'X\Psi_0 = 0$. We now argue as in the proof of Proposition 8.3 to conclude that $X = 0$. ■

8.3. Wightman Functions

A. Characteristic Properties of Wightman Functions

Let $\{\phi^{(\kappa)}\}_{\kappa \in K}$ be a system of Wightman fields. The vacuum expectation value

$$w^{(\kappa_1 \ldots \kappa_n)}_{l_1 \ldots l_n}(x_1, \ldots, x_n) = \langle 0|\phi^{(\kappa_1)}_{l_1}(x_1) \ldots \phi^{(\kappa_n)}_{l_n}(x_n)|0\rangle \tag{8.27}$$

(denoted also by $\langle\phi_{l_1}^{(\kappa_1)}(x_1\dots\phi_{l_n}^{(x_n)}(x_n)\rangle_0)$ of the polylocal monomial $\phi_{l_1}^{(\kappa_1)}(x_1)\dots$ $\dots\phi_{l_n}^{(\kappa_n)}(x_n)$ is called the n-point *Wightman function** of the fields $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$. These are in fact not functions, but generalized functions in $x_1,\dots,x_n$ in the Schwartz space $\mathcal{S}'(\mathrm{M}^n)$, therefore they are sometimes called Wightman generalized functions (or distributions). For $n=0$ we define the 0-point function by setting $w^{[0]}=1$. Axioms W.I–W.VII impose certain conditions on the Wightman functions. The main result of Wightman (1956) is that if one takes a system $w\equiv\{w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}\}$ of generalized functions satisfying all these conditions, then there exists a system of Wightman fields $\phi\equiv\{\phi^{(\kappa)}\}$ whose vacuum expectation values are equal to w. In other words, the entire content of the quantum field theory can be restated in the language of Wightman functions: a knowledge of them enables us to reconstruct the Hilbert space of states, the unitary representation of the Poincaré spin group and the covariant operator fields so that all the Wightman axioms hold.

Here is a list of the characteristic properties of the Wightman functions.

w.1 (*Admissible nature of a singularity and growth*). The Wightman functions (8.27) are generalized functions of temperate growth in the variables $x_1,\dots,x_n$.

w.2 (*Property of the adjoint*).

$$\overline{w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1\dots x_n)}=w_{\bar{l}_n\dots\bar{l}_1}^{(\bar{\kappa}_n\dots\bar{\kappa}_1)}(x_n,\dots,x_1)\tag{8.28}$$

(see the notation (8.4)).

w.3 (*Positive definiteness*).

$$\sum_{m,n=0}^{\infty}\sum_{\substack{\kappa_1\dots\kappa_m\\ l_1\dots l_m}}\sum_{\substack{\kappa_1'\dots\kappa_n'\\ l_1'\dots l_n'}} w_{\bar{l}_m\dots\bar{l}_1 l_1'\dots l_n'}^{(\bar{\kappa}_m\dots\bar{\kappa}_1\kappa_1'\dots\kappa_n')}(x_m,\dots,x_1,x_1',\dots x_n')\times$$

$$\times\overline{f_{l_1\dots l_m}^{(\kappa_1\dots\kappa_m)}(x_1,\dots,x_m)}f_{l_1'\dots l_n'}^{(\kappa_1'\dots\kappa_n')}(x_1',\dots,x_n')d^4x_1\dots d^4x_m d^4x_1'\dots d^4x_n'\geq 0\tag{8.29}$$

for any finite system** $f\equiv\{f_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}\}$ of complex functions $f_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1,\dots,x_n)$ in the corresponding Schwartz spaces $\mathcal{S}(\mathrm{M}^n)$; for $n=0$, $f^{[0]}$ is an arbitrary complex number.

w.4 (*Poincaré-covariance*).

$$\sum_{m_1\dots m_n}V_{l_1m_1}^{(\kappa_1)}(\underset{\sim}{\Lambda}^{-1})\dots V_{l_nm_n}^{(\kappa_n)}(\underset{\sim}{\Lambda}^{-1})w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(\Lambda x_1+a,\dots,\Lambda x_n+a)=$$
$$=w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1,\dots,x_n)\quad\text{for all}\quad a\in\mathrm{M},\ \underset{\sim}{\Lambda}\in SL(2,C).\tag{8.30}$$

This property is equivalent to the following two properties.

w.4′ (*Translational invariance*). There exist generalized functions $W_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(\xi_1,\dots,\xi_{n-1})\in\mathcal{S}'(\mathrm{M}^{n-1})$, also called Wightman functions, such that

$$w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1,\dots,x_n)=W_{l_1\dots l_n}^{(x_1\dots\kappa_n)})(x_1-x_2,\dots,x_{n-1}-x_n).\tag{8.31}$$

* As a rule(when this does not lead to ambiguity) we shall omit the indices $l_1,\dots,l_n$ and $\kappa_1,\dots,\kappa_n$ of the Wightman functions and fields; thus instead of $w_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1\dots x_n)$, for example, we simply write $w(x_1,\dots x_n)$ and instead of $\phi_{l_j}^{(\kappa_j)}(x_j)$ we write $\phi_j(x_j)$.

** A system of functions is called finite if all except a finite number of the functions are identically zero.

w.4″ (*Lorentz-covariance*).

$$\sum_{m_1 \dots m_n} V^{(\kappa_1)}_{l_1 m_1}(\underset{\sim}{\Lambda}^{-1}) \dots V^{(\kappa_n)}_{l_n m_n}(\underset{\sim}{\Lambda}^{-1}) W^{(\kappa_1 \dots \kappa_n)}_{m_1 \dots m_n}(\Lambda\xi_1, \dots, \Lambda\xi_{n-1}) =$$
$$= W^{(\kappa_1 \dots \kappa_n)}_{l_1 \dots l_n}(\xi_1, \dots, \xi_{n-1}) \quad \text{for all } \underset{\sim}{\Lambda} \in SL(2, C). \tag{8.32}$$

w.5. (*Spectrum property*). The Fourier transform

$$\tilde{w}(p_1, \dots, p_n) = \int w(x_1, \dots, x_n) e^{i(p_1 x_1 + \dots + p_n x_n)} d^4x_1 \dots d^4x_n \tag{8.33}$$

is concentrated at $p_k + p_{k+1} + \dots + p_n \in \overline{V}^-$ $(k = 2, 3, \dots, n)$.

We also introduce the Fourier transform for W:

$$\widetilde{W}(q_1, \dots, q_{n-1}) = \int W(\xi_1, \dots, \xi_{n-1}) e^{i(q_1\xi_1 + \dots + q_{n-1}\xi_{n-1})} d^4\xi_1 \dots d^4\xi_{n-1}; \tag{8.34}$$

then

$$\tilde{w}(p_1, \dots, p_n) = (2\pi)^4 \delta(p_1 + \dots + p_n) \widetilde{W}(p_1, p_1 + p_2, \dots, p_1 + \dots + p_{n-1}), \tag{8.35}$$

and the spectrum property can be rewritten in the form

w.5′. $$\operatorname{supp} \widetilde{W}(q_1, \dots, q_{n-1}) \subset \overline{V}^+ \times \dots \times \overline{V}^+ \tag{8.36}$$

($n - 1$ direct factors).

w.6. (*Cluster property*). For any spacelike vector $a \in \mathbf{M}$, the following relation holds (in the sense of convergence of generalized functions with respect to $x_1, \dots, x_n$): as $\lambda \to \infty$

$$w^{(\kappa_1 \dots \kappa_n)}_{l_1 \dots l_n}(x_1, \dots, x_k, x_{k+1} + \lambda a, \dots, x_n + \lambda a) \to$$
$$\to w^{(\kappa_1 \dots \kappa_k)}_{l_1 \dots l_k}(x_1, \dots, x_k) w^{(\kappa_{k+1} \dots \kappa_n)}_{l_{k+1} \dots l_n}(x_{k+1}, \dots, x_n). \tag{8.37}$$

w.7 (*Locality*)

$$w^{(\kappa_1 \dots \kappa_n)}_{l_1 \dots l_n}(x_1, \dots, x_n) =$$
$$= \sigma^{(\kappa_k, \kappa_{k+1})} w^{(\kappa_1 \dots \kappa_{k-1} \kappa_{k+1} \kappa_k \dots \kappa_n)}_{l_1 \dots l_{k-1} l_{k+1} l_k \dots l_n}(x_1, \dots, x_{k-1}, x_{k+1}, x_k, \dots, x_n) \tag{8.38}$$

for $(x_k - x_{k+1})^2 < 0$.

Properties w.1–w.7 are a direct corollary of the Wightman axioms W.I–W.VII and we suggest that the reader verify this as an exercise. We merely note that the positive definiteness property (8.29) is a simple description of the relation $\langle 0|A^*A|0\rangle > 0$ for $A \in \mathcal{P}(\mathbf{M})$ in terms of the Wightman functions. (The property of the adjoint w.2 is a corollary of w.3; however, in view of its "linear" character, it is usually isolated as a separate preliminary condition for positive definiteness.)

We give another formulation of the Lorentz covariance condition of Wightman functions in the case when the fields are realized in the form (8.10). Now the Wightman functions

$$w^{(\kappa_1 \dots \kappa_n)}(x_1, \omega_1, \overline{\omega}_1; \dots; x_n, \omega_n, \overline{\omega}_n) \equiv$$
$$\equiv W^{(\kappa_1, \dots, \kappa_n)}(x_1 - x_2, \dots, x_{n-1} - x_n; \omega_1, \overline{\omega}_1; \dots \omega_n, \overline{\omega}_n) =$$
$$= \langle 0|\phi^{(\kappa_1)}(x_1, \omega_1, \overline{\omega}_1) \dots \phi^{(\kappa_n)}(x_n, \omega_n, \overline{\omega}_n)|0\rangle \tag{8.39}$$

are additionally homogeneous polynomials of degree $2j_i$ of the complex 2-dimensional vector $\omega_i \in \mathbf{C}^2$ and homogeneous polynomials of degree $2k_i$ in $\bar{\omega}_i$ $(i = 1, \ldots, n)$. The condition (8.33) of Lorentz-covariance assumes the form of $SL(2, C)$-invariance with respect to the totality of variables $x_1, \omega_1, \overline{\omega}_1, \ldots, x_n, \omega_n, \overline{\omega}_n$:

$$w(\Lambda x_1, \underset{\sim}{\Lambda}\omega_1, \overline{\underset{\sim}{\Lambda}}\overline{\omega}_1; \ldots; \Lambda x_n, \underset{\sim}{\Lambda}\omega_n, \overline{\underset{\sim}{\Lambda}}\overline{\omega}_n) =$$

$$= w(x_1, \omega_1, \overline{\omega}_1; \ldots; x_n, \omega_n, \overline{\omega}_n) \quad \text{for } \underset{\sim}{\Lambda} \in SL(2, C), \tag{8.40a}$$

or, equivalently,

$$W(\Lambda\xi_1, \ldots, \Lambda\xi_{n-1}; \underset{\sim}{\Lambda}\omega_1, \overline{\underset{\sim}{\Lambda}}\overline{\omega}_1, \ldots, \underset{\sim}{\Lambda}\omega_n, \overline{\underset{\sim}{\Lambda}}\overline{\omega}_n) =$$

$$= W(\xi_1, \ldots, \xi_{n-1}; \omega_1, \overline{\omega}_1, \ldots, \omega_n, \overline{\omega}_n) \quad \text{for } \underset{\sim}{\Lambda} \in SL(2, C). \tag{8.40b}$$

Exercise 8.5. Prove that if the number of fields with half-integral spin in the formula (8.27) for the vacuum expectation value is odd, then the Wightman function is identically zero. [Hint: Set $\underset{\sim}{\Lambda} = -1$ in the Lorentz-covariance condition (8.40).]

Applying Theorem B.7 (or Corollary B.9) to the spectrum condition, we obtain the following important result.

Theorem 8.5. *The Wightman function $W(\xi_1, \ldots, \xi_{n-1})$ is a boundary value in the class $S'(\mathbf{M}^{n-1})$ of a function $W(\zeta_1, \ldots, \zeta_{n-1})$ that is holomorphic in the so-called past tube*

$$T^-_{n-1} = (\mathbf{M} + iV^-)^{n-1} \tag{8.41}$$

(that is, for $\zeta_j \equiv \xi_j + i\eta_j$, $\xi_j \in \mathbf{M}$, $\eta_j \in V^-$) and satisfies the estimate

$$|W(\zeta_1, \ldots, \zeta_{n-1})| \leq A\Big(1 + \sum_{j=1}^{n} |\zeta_j|^2\Big)^m (\min_j \eta_j^2)^{-l} \tag{8.42}$$

for $\eta_j \equiv \operatorname{Im} \zeta_j \in V^-$; here A, m, l are non-negative numbers (different for different Wightman functions). Accordingly $w(x_1, \ldots, x_n)$ is the boundary value (in $S'(\mathbf{M}^n)$) of a function $w(z_1, \ldots, z_n)$ that is holomorphic in the lower tube $\mathcal{T}_n^-$:

$$\mathcal{T}_n^- = \{(z_1, \ldots, z_n) \in \mathbf{CM}^n : x_j \in \mathbf{M}, y_{k+1} - y_k \in V^-,\ j = 1, \ldots, n; k = 1, \ldots, n-1\} \tag{8.43}$$

(here and in what follows $x_j \equiv \operatorname{Re} z_j$ $y_j \equiv \operatorname{Im} z_j$).

B. Källén-Lehmann representation for a scalar field

The positiveness property mixes up all the n-point functions (with all values of n), therefore it is impossible to form a set of all characteristic properties of n-point functions with n fixed. Nevertheless, it is useful to have certain necessary conditions, say, for the two-point Wightman functions. By using in combination the conditions of Lorentz-covariance, positive definiteness and the spectrum property it is possible to obtain a simple integral representation for the two-point functions.

For simplicity we consider the case of a scalar Hermitian field (the general case being considered in Appendix G). For this case, the Fourier transform $\widetilde{W}(p)$ of the two-point function $W(\xi)$ is a Lorentz-invariant generalized function concentrated (in view of the spectrum condition) in the upper light cone $\overline{V}^+$. From the positive-definiteness condition w.2 we have

$$\int W(x-y)\overline{f(x)}f(y)d^4xd^4y \geq 0 \tag{8.44}$$

for all $f \in \mathcal{S}(\mathbf{M})$, that is,

$$\int \widetilde{W}(p)|\tilde{f}(p)|^2 d^4p \geq 0 \quad \text{for all } f \in \mathcal{S}(\mathbf{M}) \tag{8.45}$$

whence

$$\int \widetilde{W}(p)u(p)d_4p \geq 0 \tag{8.46}$$

for all non-negative $u \in \mathcal{S}(\mathbf{M})$.

Exercise 8.6. Derive (8.46) from (8.45). [Hint: It is sufficient to consider non-negative functions $u(p)$ with compact support; each such function can be represented as a limit

$$u(p) = \lim_{n\to\infty}(v_n(p))^2, \quad \text{where } v_n(p) = \left(\frac{1}{n}\chi^2(p) + u(p)\right)^{1/2},$$

and $\chi(p)$ is a non-negative function in $\mathcal{D}(\mathbf{M})$ equal to unity on supp u.]

Thus $\widetilde{W}(p)$ is a non-negative Lorentz-invariant generalized function with support in $\overline{V}^-$. The general form of such generalized functions (according to §3.2.C) is as follows:

$$\widetilde{W}(p) = a(2\pi)^4\delta(p) + \int_0^\infty 2\pi\theta(p^0)\delta(p^2-m^2)d\sigma(m^2), \tag{8.47}$$

where $a \geq 0$, and $\sigma(\lambda)$ is a monotone decreasing function of polynomial growth. Formula (8.47) is called the *Källén-Lehmann representation* for the two-point function.

In x-space, the Källén-Lehmann representation has the form

$$w(x,y) \equiv W(x-y) = a + \int_0^\infty \frac{1}{i}D_m^{(-)}(x-y)d\sigma(m^2). \tag{8.48}$$

Accordingly we have the following representation for the vacuum expectation value of the commutator of the fields:

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = \frac{1}{i}\int_0^\infty D_m(x-y)d\sigma(m^2). \tag{8.49}$$

Here we have used the generalized functions

$$D_m^{(\pm)} = \mp\int 2\pi i\theta(p^0)\delta(p^2-m^2)e^{\pm ipx}d_4p, \tag{8.50}$$

$$D_m(x) = \int 2\pi i\epsilon(p^0)\delta(p^2-m^2)e^{-ipx}d_4p \tag{8.51}$$

(see Appendix F for their explicit form).

Exercise 8.7. Obtain the following relation for the constant a in (8.48)

$$a = |\langle 0|\phi|0\rangle|^2 \equiv (w^{[1]})^2. \tag{8.52}$$

[Hint: Use the cluster property w.6.]

On the basis of the Källén-Lehmann representation we can exclude the case of a Wightman field defined at each point x as being of no physical interest (since then the fields are c-number constants not depending on x).

Exercise 8.8. Consider the hypothetical scalar Hermitian field $\phi(x)$ defined as an operator at each point $x \in \mathbf{M}$. Suppose that all the axioms W.I–W.VII hold, which must now be regarded as ordinary rather than generalized operator-valued functions; in particular, we now suppose that $\phi(x)D \subset D$ in Axiom W.IV, while in Axiom W.VII we suppose that the linear combinations of vectors of the form $\phi(x_1)\ldots\phi(x_n)|0\rangle$ (for $n = 0, 1, \ldots$) are dense in $\mathcal{H}$. Prove that $\phi(x) = c$, where c is a real numerical constant not depending on x. [Hint: By redefining the field $\phi(x) \to \phi(x) - \langle 0|\phi(x)|0\rangle$, we can suppose that $\langle 0|\phi(x)|0\rangle = 0$; now verify that $\phi(x)|0\rangle = 0$; consider for this purpose the two-point function

$$w(x,y) = \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\phi(0)U(y-x,1)\phi(0)|0\rangle$$

and verify that $\widetilde{W}(p)$ is a finite measure; on the other hand, (8.47) must hold for $\widetilde{W}(p)$ with $a = 0$; hence conclude that $d\sigma(\lambda) \equiv 0$.]

C. Reconstruction of the Theory from the Wightman Functional

We now state the results that we have obtained in somewhat more abstract terms. This will enable us to write the properties of the Wightman functions in a more compact form and to give a simple proof of Wightman's theorem. For the sake of simplicity we shall confine ourselves to the case of a single scalar neutral field $\phi(x)$ (the general case differs only in its more complicated notation).

Let Ω be the space of finite sequences $f \equiv \{f^{[n]}\}$ of functions $f^{[n]}(x_1,\ldots,x_n)$ in the Schwartz space $\mathcal{S}(\mathbf{M}^n)$ (recall that $f^{[0]} \in \mathbf{C}$). It is clear that Ω is a complex vector space (in which addition and multiplication by a scalar are defined componentwise); we call it the algebraic direct sum of the spaces $\mathcal{S}(\mathbf{M}^n)$:

$$\Omega = \bigoplus_{n=0}^{\infty} \mathcal{S}(\mathbf{M}^n) \tag{8.53}$$

(where $\mathcal{S}(\mathbf{M}^0) = \mathbf{C}$). We say that a sequence f_ν ($\nu = 1, 2, \ldots$) of elements of Ω converges to zero if firstly, there is a positive number N not depending on ν such that $f_\nu^{[n]} = 0$ for all ν and for all $n > N$, secondly, $f_\nu^{[n]}(x_1,\ldots,x_n) \to 0$ with respect to convergence in $\mathcal{S}(\mathbf{M}^n)$.

In place of convergence in Ω we can equally well use the natural topology of the direct sum ([S2],p.55). This makes Ω a LCS; it is not an F-space but a so-called inductive limit of F-spaces.

We define in Ω the (non-commutative) multiplication $\otimes$ by the formula

$$(f \otimes g)^{[n]} = \sum_{k=0}^{n} f^{[k]}(x_1,\ldots,x_k) g^{[n-k]}(x_{k+1},\ldots,x_n).$$

It is not difficult to check that this product is associative and induces a bilinear map from $\Omega \times \Omega$ onto Ω which is continuous with respect to the given convergence (or the natural topology) on Ω. Therefore Ω is also called the tensor algebra over $\mathcal{S}(\mathbf{M})$. The element $I = (1, 0, \ldots, 0, \ldots)$ serves as the identity in Ω. The antilinear operation of forming the adjoint $f \to f^+$ is defined in Ω by the formula

$$(f^+)_n(x_1,\ldots,x_n) = \overline{f}_n(x_n,\ldots,x_1).$$

It is clear that $(\lambda f+\mu g)^{+}=\overline{\lambda}f^{+}+\overline{\mu}g^{+}$, $(f\otimes g)^{+}=g^{+}\otimes f^{+}$, $(f^{+})^{+}=f$. Thus Ω is an involutive algebra (or *-algebra).

Summing up we can say that Ω is a topological involutive algebra with identity. Note that the Poincaré group acts on Ω by the automorphisms $f\to f_{\{a,\Lambda\}}$; here

$$f^{[n]}_{\{a,\Lambda\}}(x_1,\dots,x_n)=f^{[n]}(\Lambda^{-1}(x_1-a),\dots,\Lambda^{-1}(x_n-a)).$$

Let Ω' be the space of continuous linear functionals on Ω. Any functional F in Ω' can be written in the form

$$F(f)=F^{[0]}f^{[0]}+(F^{[1]},f^{[1]})+\dots+(F^{[n]},f^{[n]})+\dots, \tag{8.54}$$

where $F^{[n]}\in\mathcal{S}'(\mathbf{M}^n)$ (and $\mathcal{S}'(\mathbf{M}^0)=\mathbf{C}$). The functional F is said to be Poincaré-invariant if $F(f_{\{a,\Lambda\}})\equiv F(f)$; this is equivalent to the Poincaré invariance of all the generalized functions $F^{[n]}$ in the representation (8.54). We say that F is *normalized* if $F(I)=1$. We call F *Hermitian* if

$$F(f^{+})=\overline{F(f)}\quad\text{for all } f\in\Omega,$$

and *multiplicatively positive** if

$$F(f\otimes f^{+})\geq 0\quad\text{for all } f\in\Omega. \tag{8.55}$$

A multiplicatively positive functional is Hermitian and satisfies the Cauchy-Bunyakovsky-Schwarz inequality

$$|F(f\otimes g^{+})|^2\leq F(f\otimes f^{+})F(g\otimes g^{+}) \tag{8.56}$$

(in this connection see §1.5.C).

Let F be a multiplicatively positive functional. The set of elements

$$\Im_F=\{f\in\Omega: F(f\otimes f^{+})=0\} \tag{8.57}$$

is a right ideal (see Exercise 1.51). It is not difficult to see that $\Im_F$ is a closed subspace of Ω.

An important example of a multiplicatively positive functional on Ω is a functional W of the form

$$W(f)=w^{[0]}f^{[0]}+(w^{[1]},f^{[1]})+\dots+(w^{[n]},f^{[n]})+\dots, \tag{8.58}$$

where $w^{[n]}(x_1,\dots,x_n)$ is a sequence of Wightman functions of a Hermitian scalar field $\phi(x)$ satisfying Axioms W.I–W.VII. The ideal $\Im_W\equiv\Im$ is non-trivial, (that is, non-zero). In fact, it follows from the spectrum condition w.5 that $\Im$ contains the right ideal $\Im_{\text{sp}}$:

$$\Im_{\text{sp}}=\{f\in\Omega: f^{[0]}=0,\ f^{[n]}(p_1,\dots,p_n)=0\quad\text{for } p_1\in\overline{V}^-,\ p_1+p_2\in\overline{V}^-,\dots,p_1+$$

* In §1.5.C we called the analogous functionals on a C^*-algebra positive (since for that case there was an alternative equivalent characterization of them: they take non-negative values at the positive elements of the C^*-algebra).

$$+\dots+p_n \in \overline{V}^-, \ n = 1, 2, \dots\}. \tag{8.59}$$

Furthermore, it follows from the locality condition w.7 that $\Im$ contains the two-sided ideal $\Im_{\rm loc}$ consisting of all possible linear combinations of elements f of the form $f = (0, \dots, 0, f^{[n]}, 0, \dots)$, where

$$f^{[n]}(x_1, \dots, x_n) = h(x_1, \dots, x_j, x_{j+1}, \dots, x_n) - h(x_1, \dots, x_{j+1}, x_j, \dots, x_n),$$

and the functions $h(x_1, \dots, x_n)$ are concentrated at $(x_j - x_{j+1})^2 < 0$.

Exercise 8.9. Verify that $\Im_{\rm sp}$ is a right ideal and that $\Im_{\rm loc}$ is a two-sided ideal.

We now give the general definition of a Wightman functional. A normalized multiplicatively positive functional $F \in \Omega'$ is called a *Wightman functional* if it is Poincaré-invariant and its right ideal $\Im = \{f : W(f \otimes f^+) = 0\}$ contains the ideals $\Im_{\rm sp}$ and $\Im_{\rm loc}$.

Exercise 8.10. Verify that if W is a Wightman functional, then the generalized functions $w^{[n]}$ defined by (8.58) satisfy Conditions w.1–w.5 and w.7 (of §8.3.A). Conversely, if the sequence of generalized functions $\{w^{[n]}\}$ satisfies Conditions w.1–w.5 and w.7, then the corresponding functional W is a Wightman functional.

Wightman's result that properties w.1–w.7 completely characterize the quantum field can be restated in the following way.

Theorem 8.6 (Wightman's reconstruction theorem). *Let W be a Wightman functional on the algebra Ω satisfying the cluster property*

$$W(f \otimes g_{\{\lambda a, 1\}}) \to W(f)W(g) \ \textit{as} \ \lambda \to \infty; \tag{8.60}$$

here a is an arbitrary spacelike vector. Then there exist a separable Hilbert space $\mathcal{H}$, a unitary representation $U(a, \Lambda)$ of the proper Poincaré group and a (scalar Hermitian) quantum field $\phi(x)$ (which are uniquely defined to within unitary equivalence) such that all the Wightman axioms W.I–W.VII *hold and*

$$W(f) = \langle 0|A(f)|0\rangle,$$

where

$$A(f) \equiv f^{[0]} + \sum_{n=1}^{\infty} \int \phi(x_1) \dots \phi(x_n) f^{[n]}(x_1, \dots, x_n) dx_1 \dots dx_n.$$

The reconstruction theorem is, in fact, a special case of the GNS construction (§1.5.E), only instead of a C^*-algebra $\mathfrak{A}$, we are now dealing with a topological $*$-algebra Ω; and the proof is virtually a verbatim repetition of that of Theorem 1.25 and the addendum to it, Proposition 1.30.

Remark. If we exclude the cluster property from the reconstruction theorem, then the resultant quantum field satisfies all the Wightman axioms with the possible exception of the uniqueness of the vacuum, and the field will, in general, be reducible. The requirement of the uniqueness of the vacuum can also be stated in the form of the following indecomposability condition. We say that a Wightman functional W is *indecomposable* if the representation $W = \alpha W_1 + (1 - \alpha) W_2$, where $0 < \alpha < 1$ and W_1, W_2 are Wightman functionals, can only hold for $W_1 = W_2 = W$.

Proposition 8.7. *The indecomposability condition for a Wightman functional W is equivalent to the cluster property (as well as the condition of uniqueness of the vacuum ray in the Hilbert space $\mathcal{H}$ associated with W).*

For a proof and further discussion of decomposable functionals see Borchers (1962), Reeh and Schlieder (1962), Maurin (1963a,b).

8.4. Examples: Free and Generalized Free Fields

A. Free scalar neutral field

The examples of linear quantum fields that we shall be considering below show that the Wightman axioms are mathematically consistent and also enable us to judge whether or not they are independent. All these models are uniquely defined by one- and two-point Wightman functions; the higher Wightman functions are defined by an explicit recurrence method.

We say that a scalar neutral (that is, Hermitian) Wightman field $\phi(x)$ is a free field with mass $m \geq 0$ if (along with the Hermitian property $\phi^*(x) = \phi(x)$) it satisfies the *Klein-Gordon equation**

$$(\Box + m^2)\phi \equiv (\partial_\mu \partial^\mu + m^2)\phi(x) \equiv \Big(g^{\lambda\mu}\frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial x^\mu} + m^2\Big)\phi(x) = 0 \tag{8.61}$$

and the commutation relation

$$[\phi(x), \phi(y)] = \frac{1}{i}D_m(x-y). \tag{8.62}$$

The Pauli-Jordan commutation function $D_m(x)$ (8.51) occurring here is a fundamental solution of the Klein-Gordon equation, that is, it can be regarded as a generalized function of tempered growth in $\mathbf{x}$ which is C^∞-dependent on x^0 and satisfies the conditions

$$(\Box + m^2)D(x) = 0, \;\; D(0,\mathbf{x}) = 0, \;\; \frac{\partial}{\partial t}D(t,\mathbf{x})|_{t=0} = \delta(\mathbf{x}). \tag{8.63}$$

It follows from (8.61) (and Exercise 2.49) that the field $\phi(x)$ admits a restriction to the plane x^0 =const; hence for $\phi(t,\mathbf{x})$ and $\pi(t,\mathbf{x}) \equiv \frac{\partial}{\partial t}\phi(t,\mathbf{x})$ the commutation relations (8.62) at equal times go over to the canonical commutation relations (6.76):

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = 0 = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})], \tag{8.64a}$$

$$[\pi(t,\mathbf{x}), \phi(t,\mathbf{y})] = \frac{1}{i}\delta(\mathbf{x}-\mathbf{y}). \tag{8.64b}$$

Since the initial data $\phi(t,\mathbf{x})$, $\pi(t,\mathbf{x})$ (for fixed t) completely determine $\phi(x)$, we are dealing with a system of CCR's with an infinite number of degrees of freedom (§6.3.B).

A Wightman field $\phi(x)$ with the required properties indeed exists and can be defined in the Fock space $\mathfrak{H}$ of a scalar (bosonic) neutral particle of mass m with covariant creation and annihilation operators $A^*(p)$ and $A(p)$ satisfying the commutation relations of type (7.145):

$$[A(p), A(q)] = 0 = [A^*(p), A^*(q)], \tag{8.65a}$$

$$[A(p), A^*(q)] = (2\pi)^4\delta(p-q)\delta_m^+(p). \tag{8.65b}$$

By definition we set

$$\tilde{\phi}(p) = A(p) + A^*(-p) = 4\pi\omega(\mathbf{p})[\delta(p^0 - \omega(\mathbf{p}))a(p) + \delta(p^0 + \omega(\mathbf{p}))a^*(-p)]; \tag{8.66}$$

* The operator $K = \partial_\mu\partial^\mu + m^2$ is called the *Kleinian* (with mass m).

here we are also using the creation and annihilation operators $a^*(p)$, $a(p)$ on the mass shell, see (7.148); $\tilde{\phi}(p)$ is the Fourier transform of the field $\phi(x)$:

$$\tilde{\phi}(p) = \int \phi(x) e^{ipx} d^4x. \tag{8.67}$$

Thus the operators $\phi(f) \equiv \int \phi(x) f(x) d^4x$ are defined on the domain D consisting of finite sums of vectors of the Fock space $\mathfrak{H}$ of the form

$$\int A^*(p_1) \dots A^*(p_n) f(p_1, \dots, p_n) d_4 p_1 \dots d_4 p_n |0\rangle \tag{8.68}$$

(where $f \in \mathcal{S}(\mathbf{M}^n)$) and take D into itself. (For $n = 0$, the expression (8.68) is taken to $f \cdot |0\rangle$ so that the Fock vacuum $|0\rangle$ occurs in D.)

Exercise 8.11. (a) Verify that the quantum field $\phi(x)$ satisfies the Klein-Gordon equation and the commutation relations (8.62).

(b) Prove that the field $\phi(x)$ satisfies all the Wightman axioms with the Fock vacuum $|0\rangle$ playing the role of the Wightman vacuum.

(c) Prove that for real test functions f, the operators $\phi(f)$ are essentially self-adjoint. [Hint: Use Exercise 7.22 to verify that all vectors (8.68) with compact support are analytic vectors for $\phi(f)$.]

(d) We define the local field algebras $\mathfrak{F}(\mathcal{O})$ associated with the bounded open sets $\mathcal{O} \subset \mathbf{M}$ as the von Neumann algebra generated by the bounded functions of operators $\phi(f)$, where $f \in \mathcal{D}_r(\mathcal{O})$ (it suffices to confine oneself to imaginary exponents $\exp i\phi(f)$); let $\mathfrak{F}$ be the norm completion of the union of the algebras $\mathfrak{F}(\mathcal{O})$. Prove that the net of local algebras $\mathfrak{F}(\mathcal{O})$ satisfies the local commutativity condition and that $\mathfrak{F}$ is irreducible in $\mathfrak{H}$. [Hint: Use Exercise 7.23.]

It is not difficult to see that the operator of the number of particles (7.131) can be expressed in the following way using the covariant creation and annihilation operators:

$$N = \int_{\Gamma_m^+} a^*(p) a(p) (dp)_m; \tag{8.69}$$

the operator-valued generalized function of $\mathbf{p}$ occurring here

$$\nu(p) = a^*(p) a(p) \tag{8.70}$$

plays the role of the density of the number of particles with fixed momentum $p \in \Gamma_m^+$. We note in passing that a product of creation and annihilation operators of type $a^*(p)a(p)$ in which all the creation operators are on the left of the annihilation operators, is called *normal* (or reduced to normal form). In contrast to the normal product $a^*(p)a(p)$, the product $a(p)a^*(p)$ is not defined as an operator-valued generalized function; this follows from the relation

$$a(p)a^*(q) = a^*(q)a(p) - 2p^0(2\pi)^3\delta(\mathbf{p} - \mathbf{q});$$

here the first term on the right hand side is well defined as $\mathbf{q} \to \mathbf{p}$ whereas the second (c-number) term does not have a limit.

Once we have defined the operator of the density of the number of particles, it is not difficult to introduce a number of other dynamical variables as well. For example, the operator of the total 4-momentum of the field takes the form

$$P^\mu = \int p^\mu a^*(p) a(p) (dp)_m. \tag{8.71}$$

The expression (8.71) is obtained as a result of removing the c-numeric infinities from the formal expression

$$P^\mu = \int_{x^0=\text{const}} T^{\mu 0}(x) d^3x,$$

where $T^{\mu\nu}$ is the energy-momentum tensor of the field ϕ:

$$T^{\mu\nu}(x) = \tfrac{1}{2}\partial^\mu\phi\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \quad (\mathcal{L} \equiv \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2).$$

The correct expression (8.71) is obtained if the products of the operators occurring in $T^{\mu 0}$ are replaced by normal products. This is the simplest example of removal of divergences in quantum field theory (see [B10]).

Exercise 8.12. (a) Prove that the two-point Wightman function in the above model has the form

$$w^{[2]}(x,y) = \frac{1}{i} D_m^{(-)}(x-y) \tag{8.72}$$

(where $D_m^{(-)}(x-y)$ is the negative-frequency Pauli-Jordan function (8.50)).

(b) Prove that in the given model, the Wightman functions $w^{[2n+1]}$ (of odd order) are equal to zero, while the Wightman functions $w^{[2n]}$ (of even order) are defined by the recurrence relation

$$w^{[2n+2]}(x_1,\ldots,x_{2n+2}) = \sum_{j=1}^{2n+1} w^{[2]}(x_j, x_{2n+2}) w^{[2n]}(x_1, \widehat{x_j}, \ldots, x_{2n+1}), \tag{8.73}$$

where the $\frown$ sign over x_j means that this argument is to be omitted. [Hint: Introduce the positive- and negative-frequency parts $\phi^{(\pm)}(x)$ of the field $\phi(x)$ by setting

$$\tilde{\phi}^{(+)}(p) = \theta(p^0)\tilde{\phi}(p) = A^*(-p),\ \tilde{\phi}^{(-)}(p) = \theta(-p^0)\tilde{\phi}(p) = A(p); \tag{8.74}$$

then use the commutation relations $[\phi(x), \phi^{(+)}(y)] = \frac{1}{i}D^{(-)}(x-y)$.]

(c) Prove the formula

$$w^{[2n]}(x_1,\ldots,x_{2n}) = \sum_P w^{[2]}(x_{P1}, x_{P2}) \ldots w^{[2]}(x_{P(2n-1)}, x_{P(2n)}); \tag{8.75}$$

here the summation is taken over all partitions of the index set $\{1,\ldots,2n\}$ into pairs (that is, over all pairings); these can be identified with permutations P such that $P1 < P3 < \ldots < P(2n-1)$ and $P(2j-1) < P(2j)$, $j = 1,\ldots,n$.

Formula (8.72) is a characteristic property of free fields.

Proposition 8.8. *Suppose that in the Wightman theory of a single scalar Hermitian field $\phi(x)$ the two-point function is defined by (8.72). Then $\phi(x)$ is a free field of mass m.*

■ We shall only give the proof for the case $m > 0$.* To show that $\phi(x)$ satisfies the Klein-Gordon equation we introduce the Hermitian field $j(x) = (\partial_\mu\partial^\mu + m^2)\phi(x)$. By hypothesis, its two-point function is equal to zero, therefore $j(x)|0\rangle = 0$; hence (by the separating property of the vacuum), $j(x) = 0$. It remains to verify the commutation relation (8.62). Using the fact that the Fourier transform $\tilde{\phi}(p)$ of the field $\phi(x)$ is concentrated on the two-sheeted hyperboloid $p^2 = m^2$ (since $(p^2 - m^2)\tilde{\phi}(p) = 0$), we can partition the field $\phi(x)$, as for a free field, into positive- and negative-frequency parts:

$$\tilde{\phi}^{(\pm)}(p) = \theta(\pm p^0)\tilde{\phi}(p).$$

It follows from the spectrum postulate that the positive-frequency part of the field must annihilate the vacuum vector:

$$\tilde{\phi}^{(+)}(p)|0\rangle = 0. \tag{8.76}$$

* See Pohlmeyer (1969) for the case $m = 0$.

We now consider the expression

$$\tilde{\phi}^{(+)}(q)\tilde{\phi}^{(-)}(p)|0\rangle, \tag{8.77}$$

on which the energy-momentum operator acts according to the formula

$$P\tilde{\phi}^{(+)}(q)\tilde{\phi}^{(-)}(p)|0\rangle = -(p+q)\tilde{\phi}^{(+)}(q)\tilde{\phi}^{(-)}(p)|0\rangle.$$

Here $p+q$, being the sum of a positive and negative vector on a hyperboloid of mass m, is either a spacelike vector or zero. Furthermore, as p and q range over arbitrary compact subsets of the lower and upper sheets of the hyperboloid respectively, $p+q$ ranges over a compactum in $\mathbf{M}$ that intersects the cone $\overline{V}^{+}$ only at the origin. It now follows from this together with the postulates of the spectrum property and the uniqueness of the vacuum that (8.77) (when smoothed by test functions in p and q) is collinear with the vacuum vector. On the other hand, it follows from (8.76) and (8.72) that

$$\langle 0|\phi^{(+)}(x)\phi^{(-)}(y)|0\rangle = \langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{i}D_m^{(+)}(x-y). \tag{8.78}$$

Thus

$$[\phi^{(+)}(x),\phi^{(-)}(y)]|0\rangle = \phi^{(+)}(x)\phi^{(-)}(y)|0\rangle = \frac{1}{i}D_m^{(+)}(x-y)|0\rangle. \tag{8.79}$$

Hence and from the trivial equality

$$[\phi^{(-)}(x),\phi^{(-)}(y)]|0\rangle = 0$$

we obtain

$$[\phi(x),\phi(y)]|0\rangle = \frac{1}{i}D_m(x-y)|0\rangle + [\phi^{(-)}(x),\phi^{(-)}(y)]|0\rangle. \tag{8.80}$$

Exercise 8.13. Prove that (8.80) implies that

$$\langle\Psi,[\phi^{(-)}(x),\phi^{(-)}(y)]|0\rangle = 0 \tag{8.81}$$

for all $\Psi \in \mathcal{H}$. [Hint: It follows from the properties of the support of $\tilde{\phi}^{(-)}(p)$ that the left hand side of (8.81) has an analytical continuation to the tube $x \in T_1^-$, $y \in T_1^-$; on the other hand, according to (8.80) it vanishes for $(x-y)^2 < 0$.]

It follows from (8.80) and (8.81) that

$$\{[\phi(x),\phi(y)] - \frac{1}{i}D_m(x-y)\}|0\rangle = 0. \tag{8.82}$$

Hence by Proposition 8.3 it follows that the field $\phi(x)$ satisfies the commutation relation (8.62). ■

As we have already mentioned, expressions of the form

$$\phi^{(-)}(x_1)\dots\phi^{(-)}(x_k)\phi^{(+)}(x_{k+1})\dots\phi^{(+)}(x_n), \tag{8.83}$$

in which the creation operators are on the left of the annihilation operators, are called normal products. The normal products

$$:\phi(x_1)\dots\phi(x_n): = \sum_{k=0}^{n}\phi^{(-)}(x_1)\dots\phi^{(-)}(x_k)\phi^{(+)}(x_{k+1})\dots\phi^{(+)}(x_n) \tag{8.84}$$

are defined by linearity. They can also be defined recurrently by means of the following formula due to Wick (see [B10],§17 for details).

Exercise 8.14. Prove the formula

$$\phi(x_1)\dots\phi(x_n) = \sum_{k=0}^{n}\sum_{P}\langle 0|\phi(x_{P1})\dots\phi(x_{Pk})|0\rangle : \phi(x_{P(k+1)}\dots\phi(x_{Pn}); \tag{8.85}$$

here P runs over all partitions of the set $\{1, \ldots, n\}$ into two parts consisting of k and $n-k$ elements; they can be identified with all permutations P such that $P1 < P2 < \ldots < Pk$ and $P(k+1) < \ldots < Pn$.

A remarkable property of the normal products (8.84) is the fact that they can be regarded as operator-valued generalized functions of a single 4-vector variable, say, x_1, that are C^∞-dependent on the differences $x_1 - x_2,\ x_2 - x_3, \ldots,\ x_{n-1} - x_n$ as parameters.

Exercise 8.15. Prove the above property. [Hint: It suffices to consider the expression (8.83); furthermore, since the annihilation operators are paired off with creation operators of a vector Φ in the domain of definition D, it suffices to restrict attention to the case $k = n$ in (8.83); verify that in the variables $x_1,\ \xi_j = x_j - x_1\ (j = 2, \ldots, n)$, the operator $\phi(x_1) \ldots \phi(x_n)$ smoothed with a test function in x_1 has the form

$$\int A^*(p_1) \ldots A^*(p_n) f(p_1 + \ldots + p_n) \exp\Big\{ i \sum_{j=2}^{n} p_j \xi_j \Big\} d_4 p_1 \ldots d_4 p_n,$$

where $f \in \mathcal{S}(\mathrm{M})$; using the property of the support of $A^*(p)$, one can replace $f(p_1 + \ldots + p_n)$ here by a suitable function in $\mathcal{S}(\mathrm{M}^n)$.]

As a result we can now introduce the notion of a *Wick monomial* of a free field ϕ; this is an expression of the form

$$: D^{\alpha_1}\phi(x) \ldots D^{\alpha_n}\phi(x) : \; = \; : D^{\alpha_1}\phi(x_1) \ldots D^{\alpha_n}\phi(x_n) : \Big|_{x_1 = x, x_j - x_1 = 0, j = 2, \ldots, n} \tag{8.86}$$

(the differential operator D^α is defined by (1.39)). Then an arbitrary complex combination of Wick monomials is called a *Wick polynomial.*

Exercise 8.16. Prove that the system of all Wick monomials satisfies the Wightman axioms.

Exercise 8.17. Prove that there exists in $\mathfrak{H}$ a unique unitary operator $\mathcal{U}(I_s)$ (spatial reflection) and an anti-unitary operator $\mathcal{U}(I_t)$ (time reversal) such that

$$\mathcal{U}(I_s)|0\rangle = |0\rangle,\ \mathcal{U}(I_s)\phi(x)\mathcal{U}(I_s)^{-1} = \pm\phi(x^0, -\mathbf{x}), \tag{8.87a}$$
$$\mathcal{U}(I_t)|0\rangle = |0\rangle,\ \mathcal{U}(I_t)\phi(x)\mathcal{U}(I_t)^{-1} = \phi(-x^0, \mathbf{x}). \tag{8.87b}$$

In the case when the $+$ sign is chosen in (8.87a), the field ϕ is called a *"genuinely" scalar* field, while for the $-$ sign it is called a *pseudo-scalar* field.

It can be shown (Borchers et al., 1963) that in the case $m > 0$ there exists a unique Wightman quantum field satisfying (8.61) and (8.62). However for $m = 0$, apart from the solution (8.66), called the Fock representation of a massless field, there is a whole family of solutions ϕ' parametrized by the real number c. This situation relates to the phenomenon of spontaneous breaking of symmetry to which we return in Chapter 10. The field ϕ' is the result of translating the Fock solution by a fixed constant:

$$\phi'(x) = \phi(x) + c. \tag{8.88}$$

It is now clear that $w'^{[1]} = c,\ w'^{[2]}(x, y) = c^2 + w^{[2]}(x, y)$. From now on, unless otherwise stated, by a scalar free neutral field of zero mass we mean the standard Fock solution (8.66).

Exercise 8.18. Prove that the transformations (8.88) form a symmetry group of the field algebra which does not have a unitary realization in $\mathfrak{H}$. [Hint: Introduce the conserved current

$$\eta^\mu(x) = \partial^\mu\phi(x) \tag{8.89}$$

$(\partial_\mu \eta^\mu = 0)$ and set

$$Q_R(x) = \int_{x^0=0} \eta^0(x)\beta(R^{-1}\mathbf{x})d^3x, \tag{8.90}$$

where $\beta(\mathbf{x})$ is a function in $\mathcal{D}_r(R^3)$ that is equal to unity in a neighbourhood of the origin. Verify that

$$e^{iQ_R c} \int \phi(x)f(x)d^4x \cdot e^{-iQ_R C} = \int \phi'(x)f(x)d^4x \tag{8.91}$$

for $f \in \mathcal{D}(M)$ and sufficiently large R. For the proof of the second half of the exercise, suppose that the symmetry is realized by unitary operators $\mathcal{U}_c$. Deduce that $\mathcal{U}_c$ would then commute with the translation operators $\mathcal{U}(a, 1)$ and would therefore leave the vacuum invariant; now derive a contradiction by taking the vacuum expectation value of (8.91).]

A neutral scalar field $\phi(x)$ with positive mass can naturally be regarded as an observable field; there are no unobservable quantities in such a field model. In the case of zero mass, there is an alternative possibility. We can consider (8.88) to be gauge transformations; the gauge-invariant current (8.89) can then be regarded as an observable field. The multiplicity, noted above, of representations of the field ϕ do not have any observable consequences in this case (since η^μ does not depend on c).

Exercise 8.19. Show that the current $\eta^\mu(x)$ (8.89) acting in the Fock space $\mathfrak{H}$ of a scalar massless field ϕ, satisfies all the Wightman axioms. [Hint: For the proof of the cyclicity of the vacuum with respect to the current, it is sufficient to verify that vectors of the form $\exp(i\int\phi(x)f(x)d^4x)|0\rangle$ for arbitrary $f \in \mathcal{D}_r(\mathrm{M})$ are limits of vectors of the form $c \cdot \exp(i\int\phi(x)h(x)d^4x)|0\rangle$, where $h \in \mathcal{D}_r(\mathrm{M})$ and $\int h(x)d^4x = 0$; for this purpose choose $h(x)$ in the form $h(x) = f(x) - \chi(x+\lambda a)\int f(y)d^4y$, where $\chi \in \mathcal{D}_r(\mathrm{M})$, $\int \chi(x)d^4x = 1$, a is an arbitrary spacelike vector and $\lambda \in \mathbf{R}$; then let λ tend to infinity and use Corollary 7.2.]

It follows from this exercise that a representation of the current η^μ in the Fock representation of a massless field ϕ is irreducible.

B. FREE SCALAR CHARGED FIELD

As a further illustration of the possibility of a discrepancy between the notion of observable and unobservable quantities, we consider a free scalar complex (or charged) field $\phi(x)$. This is an operator-valued generalized function satisfying the Klein-Gordon equation (8.61) and the commutation relations

$$[\phi(x), \phi(y)] = 0 = [\phi^*(x), \phi^*(y)], \ [\phi^*(x), \phi(y)] = \frac{1}{i}D_m(x-y). \tag{8.92}$$

The field ϕ can be decomposed into real and imaginary parts:

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)), \quad \phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x)); \tag{8.93}$$

here $\phi_1(x)$ and $\phi_2(x)$ are a pair of Hermitian fields satisfying the Klein-Gordon equation and the commutation relation (8.62) and also

$$[\phi_1(x), \phi_2(y)] = 0. \tag{8.94}$$

Thus the complex scalar field reduces to a combination of two independent Hermitian free fields.

It is now clear how to construct the Wightman scheme for the field ϕ. For this, we have to choose as the physical space $\mathfrak{H}$ the tensor product $\mathfrak{H}^{(1)} \otimes \mathfrak{H}^{(2)}$ of two copies $\mathfrak{H}^{(1)}$ and $\mathfrak{H}^{(2)}$ of the Fock space of the preceding subsection with corresponding creation and annihilation operators $A_i^*(p)$, $A_i(p)$ $(i = 1, 2)$ and define the field ϕ by means of the equality

$$\tilde{\phi}(p) = A^*(-p, -1) + A(+p, +1), \tag{8.95}$$

where

$$A(p, \pm 1) = \frac{1}{\sqrt{2}}(A_1(p) \mp iA_2(p)), \; A^*(p, \pm 1) = A(p, \pm 1)^*. \tag{8.96}$$

Exercise 8.20. Let $\phi^{(*)}(x)$ denote either $\phi(x)$ or $\phi^*(x)$. Prove that the Wightman function $\langle 0|\phi^{(*)}(x_1)\ldots$ $\ldots\phi^{(*)}(x_n)|0\rangle$ is non-zero provided only that ϕ and ϕ^* occur the same number of times in the monomial $\phi^{(*)}(x_1)\ldots\phi^{(*)}(x_n)$ (and hence, n is even); write down the formula of type (8.75) for the Wightman function for this case (in terms of pairings).

Exercise 8.21. Prove that there exists in $\mathfrak{H}$ a (unique) unitary operator $\mathcal{U}_C$ such that

$$\mathcal{U}_C|0\rangle = |0\rangle, \quad \mathcal{U}_C\phi(x)\mathcal{U}_C^{-1} = \phi^*(x), \quad \mathcal{U}_C\phi^*(x)\mathcal{U}_C^{-1} = \phi(x).$$

The subspace of one-particle states in $\mathfrak{H}$ can be regarded as consisting of complex functions $\Psi(p, \sigma)$ defined on the hyperboloid Γ_m^+ and depending on the discrete index σ, which takes the two values $+1$ and -1; by construction, functions with different charges are orthogonal to one another. The entire Hilbert space $\mathfrak{H}$ is generated by the vector-valued generalized functions

$$A^*(p_1, \sigma_1)\ldots A^*(p_n, \sigma_n)|0\rangle. \tag{8.97}$$

We now define the charge operator Q as the difference between the number of particles with charge $+1$ and the number of particles with charge -1:

$$QA^*(p_1, \sigma_1)\ldots A^*(p_n, \sigma_n)|0\rangle = (\sigma_1 + \ldots + \sigma_n)A^*(p_1, \sigma_1)\ldots A^*(p_n, \sigma_n)|0\rangle. \tag{8.98}$$

It is easy to see that Q is actually a self-adjoint operator with integer-valued spectrum and that $\mathfrak{H}$ splits up into a direct sum

$$\mathfrak{H} = \bigoplus_{q=-\infty}^{+\infty} \mathfrak{H}^{[q]}$$

of subspaces $\mathfrak{H}^{[q]}$ with fixed charge q. By definition, ϕ^* increases and ϕ decreases the charge by one.

Exercise 8.22. Prove the formula

$$Q = \int \sum_{\sigma=\pm 1} \sigma a^*(p, \sigma)a(p, \sigma)(dp)_m = \int i\big(a_2^*(p)a_1(p) - a_1^*(p)a_2(p)\big)(dp)_m. \tag{8.99}$$

We define the gauge transformations (of the first kind) of the field ϕ as transformations of the form

$$\phi(x) \to \phi'(x) = e^{i\lambda}\phi(x), \quad \phi^*(x) \to \phi'^*(x) = e^{-i\lambda}\phi(x), \tag{8.100}$$

where λ is an arbitrary real parameter. It is clear that such transformations form a 1-parameter group (which is, in fact, the compact abelian group $U(1)$). The gauge transformations form a symmetry group of the field algebra $\mathfrak{F}$ and of the local field algebras $\mathfrak{F}(\mathcal{O})$ (that is, the von Neumann algebras generated by the operators

$\exp\left[i\int(\phi(x)f(x)+\phi^*(x)\overline{f(x)})d^4x\right]$, where $f\in\mathcal{D}(\mathcal{O})$). This follows from the next exercise.

Exercise 8.23. Let

$$V(\lambda)=e^{-i\lambda Q},\ \lambda\in\mathbf{R}. \tag{8.101}$$

Prove the relations

$$V(\lambda)\phi(x)V(\lambda)^{-1}=e^{i\lambda}\phi(x),\quad V(\lambda)\phi^*(x)V(\lambda)^{-1}=e^{-i\lambda}\phi^*(x). \tag{8.102}$$

In the canonical quantization of a free charged field ϕ, we start from the formal expression

$$Q=\int j^0(x)d^3x, \tag{8.103}$$

where

$$j_\mu(x)=i(\phi^*(x)\partial_\mu\phi(x)-\partial_\mu\phi^*(x)\phi(x)) \tag{8.104}$$

is the conserved Noether current; the expression (8.99) is obtained by removing the c-numbered infinities from (8.104), which is equivalent to replacing the ordinary products by normal ones (see [B10]).

The operators $V(\lambda)$ enable us to define the group of gauge transformations for the entire von Neumann field algebra $\overline{\mathfrak{F}}=\mathcal{B}(\mathfrak{H})$:

$$A\to A'=V(\lambda)AV(\lambda)^{-1}. \tag{8.105}$$

An element $A\in\mathcal{B}(\mathfrak{H})$ is called gauge-invariant if it commutes with all the operators $V(\lambda)$ of the gauge transformations.

We now define the algebra of local observables $\mathfrak{A}(\mathcal{O})$ associated with a bounded open set $\mathcal{O}$ as the set of gauge-invariant operators $\mathfrak{F}(\mathcal{O})$ (these are von Neumann algebras); accordingly the algebra of quasi-local observables $\mathfrak{A}$ is defined as the norm completion of the algebras $\mathfrak{A}(\mathcal{O})$. It is clear that $\mathfrak{A}$ is reduced by each of the subspaces $\mathfrak{H}^{[q]}$ with fixed charge, so that the representation $\pi^{[q]}$ of $\mathfrak{A}$ is defined in each of these subspaces. The representation $\pi^{[0]}$ of the algebra of observables is called the vacuum representation (since $\mathfrak{H}^{[0]}$ contains the vacuum vector); the other representations are vacuumless (the energy operator P^0 is strictly positive in them; furthermore its spectrum in $\mathfrak{H}^{[q]}$ is the interval $[m|q|,+\infty)$).

The operators of a charged field $\phi(x)$ smoothed with test functions serve as intertwining operators for the subspaces $\mathfrak{H}^{[q]}$. However, it is better to look for intertwining operators from among the unitary elements of $\mathfrak{F}(\mathcal{O})$. It turns out that this can, in fact, be done; for any open set $\mathcal{O}\subset\mathbf{M}$ there exists a unitary operator $\psi_{\mathcal{O}}\in\mathfrak{F}(\mathcal{O})$ such that

$$V(\lambda)\psi_{\mathcal{O}}V(\lambda)^{-1}=e^{i\lambda}\psi_{\mathcal{O}}. \tag{8.106}$$

This operator can be constructed as follows (Wilde, 1976). We fix a real function $f(x)$ in $\mathcal{S}_r(\mathbf{M})$ with support in $\mathcal{O}$ such that its Fourier transform $\tilde{f}(p)$ is not identically zero on Γ_m^+. Then

$$\phi(f)=\tfrac{1}{\sqrt{2}}(\phi_1(f)+i\phi_2(f))$$

is a linear combination of two essentially self-adjoint operators $\phi_1(f)$ and $\phi_2(f)$, each of which acts in the appropriate factor of the tensor product $\mathfrak{H}=\mathfrak{H}^{(1)}\otimes\mathfrak{H}^{(2)}$; therefore these operators commute with each other. Thus the closure A of $\phi(f)$ is a normal operator. Furthermore, the closures of $\phi_1(f)$ and $\phi_2(f)$ have no eigenvectors with zero eigenvalue. This is fairly easy to deduce from the fact that given f, we can choose $g\in\mathcal{S}_r(\mathbf{M})$ such that $[\phi_1(f),\phi_1(g)]=i$.

Exercise 8.24. Suppose that the self-adjoint operators p, q satisfy the CCR's (6.71), where $U(a)$, $V(b)$ are defined in (6.69) with $n = 1$. Prove that p has no eigenvectors with zero eigenvalue. [Hint: Use the von Neumann uniqueness theorem (6.14) and the Schrödinger representation for the CCR's with one degree of freedom.]

In this case the required operator $\psi_{\mathcal{O}}$ is obtained by setting

$$\psi_{\mathcal{O}} = (A^* A)^{-1/2} A,$$

where A is the closure of $\phi(f)$.

Exercise 8.25. Verify (8.106).

Using the intertwining operator $\psi_{\mathcal{O}}$, we construct the automorphisms $\rho_{\mathcal{O}}^{[q]}$ of $\mathfrak{A}$:

$$\rho_{\mathcal{O}}^{[q]}(A) = (\psi_{\mathcal{O}})^q A (\psi_{\mathcal{O}})^{-q}. \tag{8.107}$$

We say that the automorphism ρ of $\mathfrak{A}$ is localized in $\mathcal{O}$ if for any region $\mathcal{Q} \sim \mathcal{O}$ we have

$$\rho(A) = A \quad \text{for all } A \in \mathfrak{A}(\mathcal{Q}). \tag{8.108}$$

If the region $\mathcal{O}$ is not made concrete, then we say that the above automorphism ρ is a *localized automorphism.*

Exercise 8.26. (a) Prove that the operator $(\psi_{\mathcal{O}})^q$ maps $\mathfrak{H}^{[0]}$ unitarily onto $\mathfrak{H}^{[q]}$.

(b) Prove that the automorphism $\rho_{\mathcal{O}}^{[q]}$ is localized in $\mathcal{O}$.

(c) Consider the representation $\tilde{\pi}^{[q]}$ of $\mathfrak{A}$ obtained from the vacuum representation by means of the automorphism $\rho_{\mathcal{O}}^{[q]}$:

$$\tilde{\pi}^{[q]}(A) = \pi^{[0]}\left(\rho_{\mathcal{O}}^{[q]}(A)\right). \tag{8.109}$$

Prove that the representations $\tilde{\pi}^{[q]}$ and $\pi^{[q]}$ of $\mathfrak{A}$ are unitarily equivalent.

From this exercise there follows the important result: every representation $\pi^{[q]}$ of $\mathfrak{A}$ is obtained to within unitary equivalence from the vacuum representation $\pi^{[0]}$ by means of a localized automorphism.

It turns out that the representations $\pi^{[q]}$ of $\mathfrak{A}$ are faithful, irreducible and pairwise unitarily inequivalent. (This was proved in a more general situation by Doplicher et al., 1969.)

Note that, just as for neutral scalar fields, a charged field of zero mass also has a "shifted" Fock representation (of type (8.88)) of a field ϕ with non-zero vacuum expectation value $\langle 0|\phi|0\rangle$. Apart from this, there is now a new possibility in the definition of the gauge transformations

$$\phi(x) \to \phi'(x) = e^{i\lambda}\phi(x) + c \tag{8.110}$$

($\lambda \in \mathbf{R}$, $c \in \mathbf{C}$), so that the group of gauge transformations now becomes isomorphic to the group of motions of the plane. We recommend as an exercise that the reader determines the decomposition into sectors of the algebra of observables in the Fock representation of ϕ.

C. Free Dirac Field

The free spinor (charged) field with spin 1/2 and mass m is the operator-valued spinor generalized function $\psi(x) \equiv (\psi^\alpha(x))$ satisfying the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \tag{8.111}$$

and the following commutation (or anticommutation) relations at equal times:

$$\begin{aligned} [\psi^\alpha(t,\mathbf{x}),\ \psi^\beta(t,\mathbf{y})]_+ &= 0 = [\psi^{\alpha*}(t,\mathbf{x}), \psi^{\beta*}(t,\mathbf{y})]_+, \\ [\psi^\alpha(t,\mathbf{x}),\ \psi^{\beta*}(t,\mathbf{y})]_+ &= \delta^{\alpha\beta}\delta(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{8.112}$$

Using the equation of motion, it is not difficult to obtain the covariant commutation relations for the fields $\psi(x)$ and $\tilde{\psi}(x) \equiv \psi^*(x)\gamma^0$:

$$\begin{aligned} [\psi(x),\ \psi(y)]_+ &= 0 = [\tilde{\psi}_\alpha(x),\ \tilde{\psi}_\beta(y)]_+, \\ [\psi^\alpha(x),\ \tilde{\psi}_\beta(y)]_+ &= \frac{1}{i}(S_m)^\alpha_\beta(x-y), \end{aligned} \tag{8.113}$$

where

$$\begin{aligned} S_m(x) = (i\gamma^\mu\partial_\mu + m)D_m(x) &= i\int((\hat{p}+m)e^{-ipx} + (\hat{p}-m)e^{ipx})(dp)_m = \\ &= i\int 2\pi(\hat{p}+m)\epsilon(p^0)\delta(p^2-m^2)e^{ipx}d_4p. \end{aligned} \tag{8.114}$$

We will show that there exists a Wightman field $\psi(x)$ satisfying our requirements. For this purpose we introduce the Fock space $\mathfrak{H}$ of two relativistic particles with mass m and spin 1/2 of different charges $\sigma = \pm 1$. Let us agree to call the particle with charge $\sigma = +1$ (simply) the particle, and the particle with charge $\sigma = -1$, the antiparticle. We choose the operator of creation of the particle $A^*(p,\omega,+1)$ and the operator of annihilation of the antiparticle $A(p,\omega,-1)$ in the form of linear functionals of the two-dimensional complex vector ω. The non-trivial commutation relations (7.145) then take on the form

$$\begin{aligned} [A^*(p,\omega,+1),\ A(p',\bar{\omega},+1)]_+ &= [A^*(p,\omega,-1),\ A(p',\bar{\omega},-1)]_+ = \\ &= (2\pi)^4\omega\underline{p}\bar{\omega}\delta(p-p')\delta^+_m(p). \end{aligned} \tag{8.115}$$

We now define the field $\psi(x)$ in the bispinor representation

$$\psi(x,\omega,\bar{\omega}) \equiv \sum_{\alpha=1,2}\psi_\alpha(x)\omega^\alpha + \sum_{\alpha'=1,2}\psi_{\alpha'}(x)\bar{\omega}^{\alpha'} \tag{8.116}$$

by means of the Fourier transformation

$$\tilde{\psi}(p,\omega,\bar{\omega}) = A^*(-p,\bar{\omega},-1) + A(p,\omega,+1). \tag{8.117}$$

Exercise 8.27. Verify that the field $\psi(x)$ defined in this way, satisfies all the Wightman axioms as well as the Dirac equation and the commutation relations (8.113).

The vacuum expectation value of the polylocal products of the spinor fields ψ and $\tilde{\psi}$ are equal to zero if the number of spinor fields ψ differs from the number of Dirac conjugate fields $\tilde{\psi}$. The two-point Wightman functions of the free spinor field have the form

$$\begin{aligned} w_{\psi\psi}(x,y) &= \langle 0|\psi(x)\psi(y)|0\rangle = 0, \\ w_{\psi\tilde{\psi}}(x,y) &= \langle 0|\psi(x)\tilde{\psi}(y)|0\rangle = \frac{1}{i}S^{(-)}_m(x-y), \\ w_{\tilde{\psi}\psi}(x,y) &= \langle 0|\tilde{\psi}(x)\psi(y)|0\rangle = \frac{1}{i}S^{(+)}_m(y-x), \\ w_{\tilde{\psi}\tilde{\psi}}(x,y) &= \langle 0|\tilde{\psi}(x)\tilde{\psi}(y)|0\rangle = 0, \end{aligned} \tag{8.118}$$

where

$$S^{(\pm)}(x) = i\int(\hat{p}\mp m)e^{\pm ipx}(dp)_m =$$

$$= \mp i \int (\hat{p}+m)\theta(\mp p^0)\cdot 2\pi\delta(p^2-m^2)e^{-ipx}d_4p = (i\gamma^\mu\partial_\mu + m)D_m^{(\pm)}(x), \qquad (8.119)$$

where the $D_m^{(\pm)}(x)$ are the frequency functions of the scalar field. The higher Wightman functions are expressed as a sum of products of two-point functions by analogy with formula (8.75) for the scalar field.

Exercise 8.28. Prove the formula for the Wightman functions of a free Dirac field:*

$$w^{[2n]}(x_1,\ldots,x_{2n}) = \sum_P \epsilon(P)w^{[2]}(x_{P1},x_{P2})\ldots w^{[2]}(x_{P(2n-1)},x_{P(2n)}); \qquad (8.120)$$

here P has the same sense as in formula (8.75); $\epsilon(P)$ is the parity of the permutation P; $w^{[2]}(x_1,x_2)$ is now a 2×2-matrix with elements defined by formulae (8.118).

The gauge transformations (of the first kind) of the Dirac field are defined by the same formulae (8.100) as in the case of the charged scalar field.

Exercise 8.29. We define the charge operator Q by a formula of type (8.98). Verify that for $m>0$ it can be represented in the form

$$Q = \int \frac{1}{m^2}\left(\frac{\partial}{\partial\bar\omega}\tilde{p}\frac{\partial}{\partial\omega}\right)\{a^*(p,\omega,+1)a(p,\bar\omega,+1) - a^*(p,\bar\omega,-1)a(p,\omega,-1)\}(dp)_m. \qquad (8.121)$$

Prove that the unitary operators $V(\lambda)=e^{-i\lambda Q}$ serve as the gauge transformations of the field ψ, that is,

$$V(\lambda)\psi(\lambda)V(\lambda)^{-1} = e^{i\lambda}\psi(x). \qquad (8.122)$$

Note that under the canonical quantization ([B10]), the charge is defined by the formula

$$Q = \int_{x^0=0} j^0(x)d^3x; \qquad (8.123)$$

here

$$j^\mu(x) = :\tilde\psi(x)\gamma^\mu\psi(x): \qquad (8.124)$$

is the current of the Dirac field, chosen in the form of a normal product. (The ***normal product*** of free spinor fields is defined in such a way that all the creation operators in the product are positioned to the left of the annihilation operators and the necessary permutations are carried out as though all the anticommutators were equal to zero; see [B10].)

If the field algebra $\mathfrak{F}(\mathcal{O})$ of ψ, associated with the bounded region $\mathcal{O}\subset \mathrm{M}$, is the von Neumann algebra generated by the operators ** $\psi^\alpha(f)$, $\tilde\psi_\beta(f)$, where $f\in\mathcal{D}(\mathcal{O})$, then the corresponding algebra of local observables $\mathfrak{A}(\mathcal{O})$ is defined as the set of gauge-invariant operators of $\mathfrak{F}(\mathcal{O})$. It is clear that the algebra of observables $\mathfrak{A}$ is reduced by the subspaces $\mathfrak{H}^{[q]}$ with a fixed value q of the charge operator Q, so that the representation $\pi^{[q]}$ of $\mathfrak{A}$ acts in each subspace $\mathfrak{H}^{[q]}$. For these representations, the results given above in connection with the charged scalar field are valid.

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* The right hand side of (8.120) is called the Pfaffian of the skew-symmetric $2n\times 2n$-matrix a with elements $a_{ij}=w^{[2]}(x_i,x_j)$ for $i<j$; the remarkable property of the Pfaffian is that it is the square root of the determinant of the matrix a.

** These operators are bounded according to Exercise 7.21.

Exercise 8.30. (a) Show that there exist (uniquely defined) unitary operators $\mathcal{U}(I_s)$, $\mathcal{U}_C$ and an anti-unitary operator $\mathcal{U}(I_t)$ that leave the vacuum vector invariant and possess the properties:

$$\mathcal{U}(I_s)\psi(x)\mathcal{U}(I_s)^{-1} = \eta_s\gamma^0\psi(I_sx),\ |\eta_s| = 1, \tag{8.125a}$$
$$\mathcal{U}(I_s)\tilde{\psi}(x)\mathcal{U}(I_s)^{-1} = \overline{\eta}_s\tilde{\psi}(I_sx)\gamma^0, \tag{8.125b}$$
$$\mathcal{U}_C\psi(x)\mathcal{U}_C^{-1} = \eta_C\psi^C(x) \equiv \eta_C C\tilde{\psi}(x),\ |\eta_C| = 1, \tag{8.126a}$$
$$\mathcal{U}_C\tilde{\psi}(x)\mathcal{U}_C^{-1} = \overline{\eta}_C\tilde{\psi}^C(x) \equiv \overline{\eta}_C C^{-1}\psi(x) \equiv -\overline{\eta}_C C\psi(x), \tag{8.126b}$$
$$\mathcal{U}(I_t)\psi(x)\mathcal{U}(I_t)^{-1} = \eta_t\gamma^5C^{-1}\psi(I_tx), \tag{8.127a}$$
$$\mathcal{U}(I_t)\tilde{\psi}(x)\mathcal{U}(I_t)^{-1} = -\overline{\eta}_t\gamma^5C^{-1}\tilde{\psi}(I_tx) \equiv -\overline{\eta}_t\tilde{\psi}(I_tx)\gamma^5C. \tag{8.127b}$$

(b) Show that $\mathcal{U}(I_s)$ and $\mathcal{U}_C$ commute if and only if η_s is purely imaginary:

$$\eta_s^2 = -1. \tag{8.128}$$

(c) Show that $\mathcal{U}(I_t)$ and $\mathcal{U}_C$ commute if and only if $\eta_C\eta_t$ is real:

$$\eta_C^2\eta_t^2 = 1. \tag{8.129}$$

(d) Prove that under the conditions (8.128), (8.129), we have the following "multiplication table":

$$\mathcal{U}_C^2 = 1,\quad \mathcal{U}(I_s)^2 = \mathcal{U}(I_t)^2 = \mathcal{U}(0,-1), \tag{8.130a}$$
$$\mathcal{U}_C\mathcal{U}(I_s) = \mathcal{U}(I_s)\mathcal{U}_C,\quad \mathcal{U}_C\mathcal{U}(I_t) = \mathcal{U}(I_t)\mathcal{U}_C, \tag{8.130b}$$
$$\mathcal{U}(I_s)\mathcal{U}(I_t) = \mathcal{U}(0,-1)\mathcal{U}(I_t)\mathcal{U}(I_s). \tag{8.130c}$$

Note that no finer details concerning the factors η_s, η_C, η_t are necessary, since different choices of the phases satisfying (7.128) and (7.129) differ by a gauge transformation and are therefore physically equivalent. For example, the choice $\eta_s = i$ is physically equivalent to the choice $\eta_s = -i$. However, in the case of several spinor fields, the ratio of the coefficients η_s of different spinor fields may now be an observable quantity; it is called the *relative* (P-) *parity* of the spinors. Note that in the case of a system of spinor fields, the defining relations (8.125)–(8.127) do not exhaust all the possible representations of the discrete operations I_s, C, I_t (since in this case there is the possibility of introducing a more extensive group of gauge transformations). For example, in the case of two spinor fields, one can find an irreducible representation of these operations in a space in which two spinors are combined into a single eight-component spinor, which is reasonable in the presence of isotopic symmetry (for details, see Ogievetskii and Chew Huan Chow, 1959; Lee and Wick, 1966).

Along with the charged spinor field we could also have considered the neutral or Majorana field $\psi(x) = \psi^C(x)$. In this case, instead of two particles, we deal with a single particle with creation and annihilation operators $A^*(p,\bar{\omega})$ and $A(p,\omega)$ respectively; accordingly instead of (8.117) we now have

$$\tilde{\psi}(p,\omega,\bar{\omega}) = A^*(-p,\bar{\omega}) + A(p,\omega). \tag{8.131}$$

The gauge group of the Majorana field reduces to the group $\mathbf{Z}_2 = \{\pm 1\}$, in which the non-trivial element acts according to the rule

$$\psi(x) \to -\psi(x). \tag{8.132}$$

Exercise 8.31. Prove that the gauge transformation (8.132) is unitarily realized by the valency operator $\mathcal{U}(0,-1)$, that is,

$$\mathcal{U}(0,-1)\psi(x)\mathcal{U}(0,-1)^{-1} = -\psi(x). \tag{8.133}$$

D. Generalized free fields

A characteristic feature of a free (for definiteness, scalar) field is the fact that its commutator $[\phi(x),\phi(y)]$ is a c-number (that is, a multiple of the identity operator).

The question then arises whether it is possible to construct a physically non-trivial theory in which this property is preserved. The answer is in the negative: it turns out that such a field is, in point of fact, a superposition of free fields. This is the justification for calling a scalar field $\phi(x)$ a ***generalized free field*** if its commutator $[\phi(x), \phi(y)]$ is a c-number (Greenberg, 1961). In spite of the physical triviality of a generalized free field, its study has a certain theoretical interest (if only because it gives an indication of what to avoid in the search for a non-trivial theory).

It is not difficult to give the general construction of Wightman generalized free fields; we shall restrict ourselves here to the case of a scalar Hermitian field. By definition, the commutator $[\phi(x), \phi(y)]$ coincides with its vacuum expectation value, therefore the Källén-Lehmann representation for it (8.49) takes the form

$$[\phi(x), \phi(y)] = \frac{1}{i}\int_0^\infty D_m(x-y)d\sigma(m^2), \tag{8.134}$$

where $\sigma(\lambda)$ is a monotone increasing function of polynomial growth on $[0,\infty)$. The measure $d\sigma(\lambda)$ is uniquely defined by the commutator. Without loss of generality we may suppose that $w^{[1]} \equiv \langle 0|\phi(x)|0\rangle = 0$ (otherwise we replace ϕ by the generalized free field $\phi'(x) = \phi(x) - w^{[1]}$). Then for the two-point Wightman function $w^{[2]}$ we have the Källén-Lehmann representation

$$w^{[2]}(x,y) = \frac{1}{i}\int_0^\infty D_m^{(-)}(x-y)d\sigma(m^2). \tag{8.135}$$

There is a recurrence relation for the higher Wightman functions (see Exercise 8.12).

The space of the vector states of a generalized free field can be realized by analogy with the Fock space for neutral mesons with fixed mass. The space of one-particle states $\mathfrak{H}_1$ is now given by the completion of the pre-Hilbert space of complex continuous functions $\Phi_1(p)$ with compact support which are defined for $p^0 \geq 0$, $p^2 \in \operatorname{supp} d\sigma(\lambda)$ and have scalar product

$$\langle \Phi_1, \Psi_1\rangle = \int_0^\infty d\sigma(m^2)\int_{\Gamma_m^+} \overline{\Phi_1(p)}\Psi_1(p)(dp)_m \equiv \int \overline{\Phi_1(p)}\Psi_1(p)d\mu(p). \tag{8.136}$$

Now that we have the space $\mathfrak{H}_1$, we can define the space $\mathfrak{H}_n$ of n-particle states as the nth symmetric tensor power of the spaces $\mathfrak{H}_1$. The action of the generalized free field is then defined by the formula

$$\begin{aligned}(\phi(f)\Psi)_n(p_1,\dots,p_n) &= \sqrt{n+1}\int d\mu(p)\tilde{f}(-p)\Psi_{n+1}(p,p_1,\dots,p_n)+\\ &+\frac{1}{\sqrt{n}}\sum_{j=1}^n \tilde{f}(p_j)\Psi_{n-1}(p_1,\dots,\widehat{p}_j,\dots,p_n)\end{aligned} \tag{8.137}$$

(the $\widehat{\ }$ sign over the argument denotes the omission of that argument).

Exercise 8.32. (a) Prove that the field $\phi(x)$ defined by (8.137) satisfies all the Wightman axioms.

(b) Decompose the generalized free field into a direct integral of free fields with a specified mass $\sqrt{\lambda}$ (Licht, 1963; Wightman, 1964).

We give a criterion for a generalized free field.

Proposition 8.9 (Licht and Toll, 1961). *Suppose that a scalar neutral field $\phi(x)$ satisfies all the Wightman axioms. If the commutator $[\phi(x),\phi(y)]$ depends only on the difference $x-y$, that is,*

$$[\phi(x),\phi(y)] = B(x-y),$$

then the operator-valued generalized function $B(x-y)$ is a multiple of the identity operator and hence, $\phi(x)$ is a generalized free field.

■ We shall prove that

$$[\phi(z), B(x-y)] = 0. \tag{8.138}$$

We may suppose that x, y and z vary in arbitrarily fixed bounded domains. We can then choose a spacelike vector a such that the vectors $x+a-z$ and $y+a-z$ are spacelike. Using locality, we find that $[\phi(z), B(x-y)] = [\phi(z),[\phi(x+a),\phi(y+a)]] = [[\phi(z),\phi(x+a)],\phi(y+a)]+[\phi(x+a),[\phi(z),\phi(y+a)]] = 0$. It now follows from (8.138) and Proposition 8.1 that $B(x-y)$ is a multiple of the identity operator. (Note that the proof of Proposition 8.1 is valid not only for a bounded operator C commuting with the operators of $\mathcal{P}(\mathbf{M})$ but also for an arbitrary operator C in $\mathcal{P}(\mathbf{M})$ commuting with $\mathcal{P}(\mathbf{M})$.) ■

It is clear from (8.137) that the Fourier image of a generalized free field vanishes for spacelike momenta. It turns out that this property is characteristic for a generalized free field.

Proposition 8.10. *Let ϕ be a scalar field in the Wightman theory with the strong spectrum condition. If the Fourier image $\tilde{\phi}(p)$ of the field vanishes in some non-empty open set of spacelike momenta p, then ϕ is a generalized free field. A similar conclusion can be made if the measure $d\sigma(\lambda)$ in the Källén-Lehmann representation* (8.135) *has compact support.*

The proof of this assertion is based on the Jost-Lehmann-Dyson representation; we shall not give it here (see Greenberg, 1962; Robinson, 1962). There is also a criterion for a generalized free field in terms of the so-called truncated Wightman functions $w^{[n]T}$ which are defined in §12.2.A: it suffices that $w^{[n]T} = 0$ for all n greater than some number n_0 (see Greenberg and Licht, 1963; Robinson, 1966; a further strengthening of this result is given in the paper by Baumann (1975)). We note in conclusion that in four-dimensional Minkowski space-time, the Wightman fields of Lie type, the commutator of which is an inhomogeneous linear functional of the field operators:

$$[\phi(x),\phi(y)] = \frac{1}{i}\Delta(x-y) + \frac{1}{i}\int b(x,y,z)\phi(z)d^4z, \tag{8.139}$$

which at first glance appear to be more general, reduce to generalized free fields (see Robinson, 1964, Wightman, 1967a and Efremov, 1968/69).

Appendix F. Summary of Invariant Solutions and Green's Functions of the Klein-Gordon Equation

In what follows we shall repeatedly make use of the characteristic singular (in particular commutation) functions of the free fields that we have been dealing with in §8.4. Therefore for convenience we give a summary of these generalized functions, as well as the Green's functions related to them.

The Pauli-Jordan commutation function $D_m(x) \equiv D(x)$:

$$[\phi(x),\phi(y)] = \frac{1}{i}D(x-y); \tag{F.1}$$

$$(\partial_\mu\partial^\mu + m^2)D(x) \equiv (\Box + m^2)D(x) = 0, \tag{F.2a}$$

$$D(0,\mathbf{x}) = 0, \quad \frac{\partial}{\partial x^0}D(x)\big|_{x^0=0} = \delta(\mathbf{x}); \tag{F.2b}$$

$$D(x) = \frac{1}{i}\int 2\pi\epsilon(p^0)\delta(p^2-m^2)e^{ipx}d_4p = \tag{F.3a}$$

$$= \frac{1}{2\pi}\epsilon(x^0)\Big\{\delta(x^2) - \theta(x^2)\frac{m}{2\sqrt{x^2}}J_1(m\sqrt{x^2})\Big\}; \tag{F.3b}$$

here and in what follows, $J_1(z)$ and $K_1(z)$ are the functions of Bessel and MacDonald ([G11], 8.402, 8.432, 8.446):

$$J_1(z) \equiv -iI_1(iz) = \frac{z}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+1)!}\Big(\frac{z}{2}\Big)^{2k}, \tag{F.4}$$

$$K_1(z) = z\int_0^{\infty} e^{-z\cosh\tau}\sinh^2\tau d\tau \quad \text{for } |\arg z| < \pi/2, \tag{F.5a}$$

$$K_1(z) = \frac{1}{z} + \ln\frac{z}{2}I_1(z) - \frac{z}{4}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+2)}{k!(k+1)!}\Big(\frac{z}{2}\Big)^{2k} \tag{F.5b}$$

$(\psi(k) \equiv \Gamma'(k)/\Gamma(k))$.

Pauli-Jordan positive- and negative-frequency functions $D_m^{(\pm)}(x) \equiv D^{(\pm)}(x)$:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \frac{1}{i}D^{(-)}(x-y), \tag{F.6}$$

$$(\Box + m^2)D^{(\pm)}(x) = 0, \tag{F.7}$$

$$D^{(\pm)}(x) = \frac{\pm 1}{i}\int 2\pi\theta(\pm p^0)\delta(p^2-m^2)e^{ipx}d_4p = \tag{F.8a}$$

$$= \mp \lim_{y\to 0, y\in V^+}\frac{im^2}{4\pi^2}h(-m^2(x\pm iy)^2), \tag{F.8b}$$

where*

$$h(\zeta) = K_1(\sqrt{\zeta})/\sqrt{\zeta}, \tag{F.9}$$

$$D^{(-)}(x) = -D^{(+)}(-x) = \overline{D^{(+)}(x)}, \tag{F.10}$$

$$D^{(+)}(x) + D^{(-)}(x) = D(x). \tag{F.11}$$

The causal Green's function $D_m^c(x) \equiv D^c(x)$:

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{1}{i}D^c(x-y), \tag{F.12}$$

$$D^c(x) = \theta(x^0)D^{(-)}(x) - \theta(-x^0)D^{(+)}(x), \tag{F.13}$$

$$(\Box + m^2)D^c(x) = \delta(x), \tag{F.14}$$

$$D^c(x) = \int\frac{1}{m^2-p^2-i0}e^{ipx}d_4p. \tag{F.15}$$

The retarded and advanced Green's functions $D_m^{\overset{\text{ret}}{\text{adv}}}(x) = D^{\overset{\text{ret}}{\text{adv}}}(x)$:

$$(\Box + m^2)D^{\overset{\text{ret}}{\text{adv}}}(x) = \delta(x), \tag{F.16}$$

$$D^{\text{ret}}(x) = \theta(x^0)D(x) = D^{\text{adv}}(-x), \tag{F.17}$$

$$D^{\overset{\text{ret}}{\text{adv}}}(x) = \int\frac{1}{m^2-p^2\pm i0\cdot\epsilon(p^0)}e^{ipx}d_4p = \tag{F.18a}$$

$$= \frac{1}{2\pi}\theta(\pm x^0)\Big\{\delta(x^2) - \theta(x^2)\frac{m}{2\sqrt{x^2}}J_1(m\sqrt{x^2})\Big\}, \tag{F.18b}$$

$$D^{\overset{\text{ret}}{\text{adv}}}(x) = D^c(x) \pm D^{(\pm)}(x), \tag{F.19}$$

$$D^{\text{ret}}(x) - D^{\text{adv}}(x) = D(x). \tag{F.20}$$

* In (F.9), the branch of the function $\sqrt{\zeta}$ (in the complex plane cut along $\zeta \leq 0$) is chosen to be positive for $\zeta > 0$.

The characteristic singular functions of the spinor field $S_{\alpha\beta}(x)$ and $S^Z_{\alpha\beta}(x)$ for $Z = (\pm)$, c, ret, adv, are obtained from the corresponding singular functions of the scalar field according to the formula

$$S^Z_{\alpha\beta}(x) = (i\gamma^\mu\partial_\mu + m)_{\alpha\beta}D^Z(x); \tag{F.21}$$

they are the solutions or Green's functions (for $Z = c$, ret, adv) of the Dirac equation.

Appendix G. General Form of the Covariant Two-Point Function

G.1. Covariant decompositions compatible with locality

In this subsection we give the analogue of the Källén-Lehmann representation for fields that transform according to arbitrary finite-dimensional irreducible or simply reducible representations of the proper Lorentz group (or $SL(2,C)$, its simply-connected covering group). First we explain our notation.

The set of all complex 2×2-matrices has the structure of an associative algebra. However, we now regard it simply as a monoid (that is, a set with associative multiplication and an identity). For any non-negative integer or half-integer s ("spin"), we define the $2s+1$-dimensional matrix representation $A \to (D^{(s)}_{\sigma\sigma'}(A))$ of this monoid by setting

$$\begin{aligned} D^{(s)}_{\sigma\sigma'}(A) = \sqrt{\frac{(s+\sigma')!(s-\sigma')!}{(s+\sigma)!(s-\sigma)!}}(A_{11})^{\sigma'+\sigma}(A_{12})^{\sigma'-\sigma}(\det A)^{s-\sigma'}\times \\ \times P^{(\sigma'-\sigma,\sigma'+\sigma)}_{s-\sigma'}\left(\frac{A_{11}A_{22}+A_{12}A_{21}}{\det A}\right) \end{aligned} \tag{G.1}$$

for $\sigma, \sigma' = -s, -s+1, \ldots, s$ and A_{11}, A_{12}, $\det A \neq 0$* (for the remaining A, $D^{(s)}_{\sigma\sigma'}(A)$ is defined by continuity).

Let $\phi^{(\kappa_1)}(x,\omega,\bar\omega)$ and $\phi^{(\kappa_2)}(x,w,\bar w)$ be two quantized fields transforming according to the finite-dimensional simply reducible representations

$$\bigoplus_{(j_1,k_1)\in R_1} \mathcal{D}^{(j_1,k_1)} \quad\text{and}\quad \bigoplus_{(j_2,k_2)\in R_2} \mathcal{D}^{(j_2,k_2)} \tag{G.2}$$

of $SL(2,C)$, realized in the spaces of polynomials in $\omega,\bar\omega$ and $w,\bar w$ respectively (see §§3.1.E and 8.2.B); here, R_1 and R_2 are finite subsets of $\overline{\mathbb{Z}}_+/2 \times \overline{\mathbb{Z}}_+/2$. Then the two-point Wightman functions

$$\langle 0|\phi^{(\kappa_1)}(x,\omega,\bar\omega)\phi^{(\kappa_2)}(y,w,\bar w)|0\rangle = \int \widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,\bar\omega,w,\bar w)e^{ip(x-y)}d_4p \tag{G.3}$$

have the following expansion in the standard polynomial covariants.

Proposition G.1. *A two-point function of fields $\phi^{(\kappa_1)}$, $\phi^{(\kappa_2)}$ transforming according to finite-dimensional simply reducible representations of $SL(2,C)$, has the form*

$$\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,\bar\omega,w,\bar w) = \sum(\omega\epsilon w)^A(\overline{\omega\epsilon w})^B D^{(L)}_{jk}(P)f^{LAB}_{jk}(p), \tag{G.4}$$

where the summation is carried out over some finite set of numbers j,k,A,B,L satisfying the conditions

$$A,\ B,\ 2L,\ L-|j|,\ L-|k| \in \overline{\mathbb{Z}}_+; \tag{G.5}$$

P is the 2×2-matrix formed from the variables p,ω,w:

$$P = \begin{pmatrix} \bar\omega\tilde p\omega & \bar\omega\tilde p w \\ \bar w\tilde p\omega & \bar w\tilde p w \end{pmatrix}; \tag{G.6}$$

---

* The fact that formula (G.1) defines a representation of $GL(2,C)$ (the group of all complex non-singular 2×2-matrices) is easy to see from the fact that for unitary matrices A, the formula (G.1) becomes (7.61); it is obtained for matrices $A \in SL(2,C)$ by analytic continuation and for $A \in GL(2,C)$ from considerations of homogeneity.

the $f_{jk}^{LAB}(p)$ are Lorentz-invariant generalized functions in $S'(\mathrm{M})$ with supports in $\overline{V}^+$.

■ The covariants $(\omega\epsilon w)^A(\overline{\omega\epsilon w})^B D_{jk}^{(L)}(P)$ are clearly linearly independent, therefore it suffices, in view of Exercise 3.19, to prove that they are pseudo-harmonic in p:

$$\partial^\mu\partial_\mu D_{jk}^{(L)}(P) \equiv 4|\omega\epsilon w|^2\Big(\frac{\partial^2}{\partial P_{11}\partial P_{22}} - \frac{\partial^2}{\partial P_{12}\partial P_{21}}\Big)D_{jk}^{(L)}(P) = 0;$$

but this is a corollary of the differential equation for the Jacobi polynomials

$$\Big\{(1-z^2)\frac{d^2}{dz^2} + [(\beta-\alpha) - (\alpha+\beta+2)z]\frac{d}{dz} + n(n+\alpha+\beta+1)\Big\}P_n^{(\alpha,\beta)}(z) = 0$$

(see [S18],§4.2). ■

Generally speaking, the invariant generalized functions f_{jk}^{LAB} are defined from the expansion (G.4) to within arbitrary terms of the form $r_{jk}^{LAB}(\Box)\delta(p)$, where $r_{jk}^{LAB}(z)$ is a polynomial of degree $2L-1$ (and this arbitrariness is accounted for in the theories with indefinite metric). But in the Wightman formalism, where $\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,\bar{\omega},w,\bar{w})$ is a (complex) measure of power growth in the variable p with support in $\overline{V}^+$, and in which we can always suppose that the measure of the point $p=0$ is equal to 0,* the $f_{jk}^{LAB}(p)$ are uniquely defined by the expansion (G.4) if it is further stipulated that they are also (in general, complex) Lorentz-invariant measures of polynomial growth with the point $p=0$ having zero measure. We consider by way of explanation the representation

$$\tilde{F}(p;\omega,\bar{\omega}) = (\bar{\omega}\tilde{p}\omega)^n f(p), \tag{G.7}$$

where the left hand side is a Lorentz-covariant measure with respect to p with support in $\overline{V}^+$, the point $p=0$ having zero measure, and is a homogeneous polynomial in $\omega,\bar{\omega}$ of bidegree (n,n) (the case of arbitrary representations reduces to this case, see §3.3.B). Since for each point $p \neq 0$ there is a vector ω such that $\bar{\omega}\tilde{p}\omega \neq 0$, it follows from (G.7) that $f(p)$ is a Lorentz-invariant measure in $\mathrm{M}\backslash\{0\}$ (with support in $\overline{V}^+\backslash\{0\}$). If we extend the definition of the measure to M by making the measure at the point $p=0$ zero, then the representation (G.7) remains intact over all of M.

If the fields $\phi^{(\kappa_1)}$ and $\phi^{(\kappa_2)}$ transform according to the irreducible representations $\mathfrak{D}^{(j_1,k_1)}$ and $\mathfrak{D}^{(j_2,k_2)}$ of $SL(2,C)$, then the two-point function (G.3) is not identically zero only under the following conditions:

$$j_- - k_- \in \mathbf{Z}, \tag{G.8a}$$

$$|j_-| \vee |k_-| \leq j_+ \wedge k_+, \tag{G.8b}$$

where

$$j_\pm = j_1 \pm j_2, \quad k_\pm = k_1 \pm k_2. \tag{G.9}$$

In this case the covariant expansion takes the form

$$\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,\bar{\omega},w,\bar{w}) = \sum_L (\omega\epsilon w)^{j_+-L}(\overline{\omega\epsilon w})^{k_+-L}D_{j_-k_-}^{(L)}(P)f_L(p), \tag{G.10}$$

where

$$L = (|j_-|\vee|k_-|),\quad (|j_-|\vee|k_-|)+1,\dots,j_+\wedge k_+. \tag{G.11}$$

Formula (G.4) (or (G.10)) is called a covariant expansion compatible with locality, because here (as in Exercise 3.19) the projectors realizing this expansion are constructed only from the spinor variables ω, w and do not depend on p.

G.2. Decomposition with respect to spin

We suppose that in the Wightman theory, the strong spectrum condition holds with mass gap $\mu > 0$. Let E^J be the projector in the physical Hilbert space onto the subspace with spin J (where $2J \in \overline{\mathbf{Z}}_+$).

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* The only exception that can occur is for the case of scalar fields (see §8.3.B), but this can be removed by redefining the fields by subtracting the one-point Wightman functions from them.

Consider the two-point function *

$$\langle 0|(\phi^{(\kappa_1)}(x,\omega,\bar{\omega}))^*\phi^{(\kappa_2)}(y,w,\bar{w})|0\rangle = \int \widetilde{W}^{(\bar{\kappa}_1\kappa_2)}(p;\omega,\bar{\omega},w,\bar{w})e^{ip(x-y)}d_4p, \tag{G.12}$$

where $\phi^{(\kappa_1)}(x,\omega,\bar{\omega})$ and $\phi^{(\kappa_2)}(x,w,\bar{w})$ are fields transforming according to the irreducible representations $\mathfrak{D}^{(j_1,k_1)}$ and $\mathfrak{D}^{(j_2,k_2)}$ of $SL(2,C)$. By inserting between the field operators in (G.12) the expansion $1 = \sum_J E^J$ of the identity operator into a sum of projectors E^J, we obtain the expansion of the two-point function in terms of the spin:

$$\widetilde{W}^{(\bar{\kappa}_1\kappa_2)} = \sum_J \widetilde{W}_J^{(\bar{\kappa}_1\kappa_2)}. \tag{G.13}$$

It turns out that the expansion in terms of spin has the following covariant structure:

$$\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,\bar{\omega},w,\bar{w}) = \sum (\bar{\omega}\tilde{p}\omega)^{s_1-J}(\bar{w}\tilde{p}w)^{s_2-J}D^{(J)}_{\sigma_1\sigma_2}(Q)h_J(p), \tag{G.14}$$

where Q is the 2×2-matrix formed from the variables p,ω,w:

$$Q = \begin{pmatrix} \bar{\omega}\tilde{p}w & \sqrt{p^2}w\epsilon\omega \\ \sqrt{p^2}\overline{\omega\epsilon w} & \bar{w}\tilde{p}\omega \end{pmatrix}; \tag{G.15a}$$

the $h_J(p)$ are Lorentz-invariant measures with supports in $\overline{V}_\mu^+$;

$$\sigma_i = j_i - k_i, \quad s_i = j_i + k_i \quad (i = 1,2), \tag{G.15b}$$

and J runs through the values

$$J = |\sigma_1| \vee |\sigma_2|, (|\sigma_1| \vee |\sigma_2|) + 1, \ldots, s_1 \wedge s_2. \tag{G.16}$$

Here for the case $\widetilde{W}^{(\bar{\kappa}_1\kappa_2)} \not\equiv 0$ we must suppose that the following conditions hold:

$$\sigma_1 - \sigma_2 \in \mathbf{Z}, \quad |\sigma_1| \vee |\sigma_2| \le s_1 \wedge s_2. \tag{G.17}$$

For the proof of the expansion (G.14) we go over in the generalized function $\widetilde{W}_J^{(\bar{\kappa}_1\kappa_2)}(p;\omega,\bar{\omega},w,\bar{w})$ to the rest frame of the vector p (that is, we set $\mathbf{p} = 0$, which is possible by the condition $p \in \overline{V}_\mu^+$; see §3.4.C). We smooth the expression so obtained with an arbitrary test function in the variable $p^0 \in [\mu,\infty)$ and we fix ω. We obtain a function of w which for $w \neq 0$ (taking into account the partial homogeneity condition in $w,\bar{w}$) can be written in the form

$$(\bar{w}w)^{s_2}F_\omega(Y), \tag{G.18}$$

where Y is the following matrix in $SU(2)$:

$$Y = \frac{1}{\bar{w}w}\begin{pmatrix} w^1 & w^2 \\ -\bar{w}^2 & \bar{w}^1 \end{pmatrix},$$

and $F_\omega(Y)$ is a function on the group $SU(2)$ depending on the parameter ω. We use the fact ([Z2],§25) that every complex continuous function on $SU(2)$ can be expanded in a series of elements proportional to the matrix elements $D^{(J)}_{\sigma\sigma'}(Y)$:

$$F_\omega(Y) = \sum_{J',\sigma,\sigma'} a^{J'}_{\sigma'\sigma}D^{(J')}_{\sigma\sigma'}(Y). \tag{G.19}$$

* For convenience we have replaced the field $\phi^{(\kappa_1)}$ (in contrast to (G.3)) by the Hermitian adjoint field.

On the other hand, the projectors E^J in $\mathcal{H}$ were defined by the expansion $1 = \sum_J E^J$ and the condition

$$-\frac{1}{p_\mu p^\mu} W_\mu W^\mu E^J = J(J+1)E^J,$$

where W^μ is the Pauli-Lubanski vector. This imposes on $\widetilde{W}^{(\bar{\kappa}_1\kappa_2)}$ in the expansion (G.13) a certain condition, which in the rest frame of p implies that $F_\omega(Y)$ as a function of w transforms according to a representation of $SU(2)$ with spin J and this, as is easy to see, is equivalent to the property that only the terms $J' = J$ are present in the expansion (G.19). We have considered the vector ω above to be a fixed parameter. It is now not difficult to take into account the dependence $F_\omega(Y)$ on ω by using the (partial) homogeneity conditions in $\omega, \bar{\omega}$ and the $SU(2)$-invariance of $\widetilde{W}_J^{(\bar{\kappa}_1\kappa_2)}$ in ω, w in the system $\mathbf{p} = 0$:

$$F_\omega(Y) = (\bar{\omega}\omega)^{s_1} F_{\omega_0}(YX^*), \tag{G.20}$$

where

$$X = \frac{1}{\bar{\omega}\omega}\begin{pmatrix} \omega^1 & \omega^2 \\ -\bar{\omega}^2 & \bar{\omega}^1 \end{pmatrix}.$$

Substituting (G.19) (with $\omega = \omega_0$) into (G.20) and taking fully into account the conditions of polynomiality and homogeneity in $\omega, \bar{\omega}, w, \bar{w}$, we obtain

$$(\bar{w}w)^{s_2} F_\omega(Y) = a(\bar{\omega}\omega)^{s_1}(\bar{w}w)^{s_2} D^{(J)}_{\sigma_1\sigma_2}(YX^*);$$

here (in the case $a \not\equiv 0$) conditions (G.17) must be satisfied. Thus we have proved the expansion (G.14) in the rest frame of the vector p, whence it follows (by Corollary 3.16) that it holds for all p. Note that the representation (G.14) is obtained by means of a re-expansion of a covariant representation of type (G.10), therefore the generalized functions $h_J(p)$ in (G.14) are measures of power growth just as the $f_L(p)$ are in (G.10).

We see that corresponding to each admissible value of the spin J there is one covariant structure in the expansion (G.14). This is a reflection of the fact that the subspace $\mathcal{H}^{(\kappa\alpha)}$ ($\alpha = 1, 2$) of $\mathcal{H}$, defined as the closure of the linear span of the vectors of the form $\int \phi^{(\kappa\alpha)}(x, \omega, \bar{\omega})u(x)d^4x$ (for $u \in \mathcal{S}(\mathrm{M})$) carries a simply reducible representation of the group $\mathfrak{P}_0$, that is,

$$\mathcal{H}^{(\kappa\alpha)} = \bigoplus_J \int_\mu^\infty \mathfrak{H}^{[mJ]} d\rho_J(m). \tag{G.21}$$

In the case when the fields $\phi^{(\kappa\alpha)}$ transform according to reducible representations of $SL(2, C)$, this is no longer in general true; in the expansion of type (G.14) there can then exist for each J several covariant structures. For fields that transform according to simply reducible representations of $SL(2, C)$, we obtain the following result.

Proposition G.2. *Suppose that in the Wightman theory with the strong spectrum condition, the fields* $\phi^{(\kappa_1)}(x, \omega, \bar{\omega})$ *and* $\phi^{(\kappa_2)}(x, w, \bar{w})$ *transform according to the simply reducible representations* (G.2) *of the Lorentz group. Then the expansion of the two-point function* (G.12) *with respect to the spin* J *has the following form:*

$$\widetilde{W}^{(\bar{\kappa}_1\kappa_2)}(p; \omega, \bar{\omega}, w, \bar{w}) = \sum (\bar{\omega}\tilde{p}\omega)^{n_1}(\bar{w}\tilde{p}w)^{n_2} D^{(J)}_{\sigma\sigma'}(Q) h^{Jn_1n_2}_{\sigma\sigma'}(p), \tag{G.22}$$

where the summation is taken over a finite set of numbers $n_1, n_2, \sigma, \sigma', J$, *satisfying the conditions*

$$n_1,\ n_2,\ 2J,\ J - |\sigma|,\ J - |\sigma'| \in \overline{\mathbf{Z}}_+; \tag{G.23}$$

$h^{Jn_1n_2}_{\sigma\sigma'}(p)$ *are (possibly complex) Lorentz-invariant measures of power growth with supports in* $\overline{V}^+_\mu$ *(where* μ *is the mass gap).*

CHAPTER 9

Analytic Properties of Wightman Functions in Coordinate Space

9.1. Bargmann-Hall-Wightman Theorem and its Corollaries

A. Complex Lorentz transformations

We saw (Theorem 8.5) that by virtue of the spectrum condition, the Wightman functions $w(x_1,\dots,x_n) = W(\xi_1,\dots,\xi_{n-1})$ are boundary values of functions $W(\zeta_1,\dots$ $\dots,\zeta_{n-1})$ of $\zeta_1,\dots,\zeta_{n-1}$ that are holomorphic in the past tubes T^-_{n-1} (8.41). It turns out that the combination of the spectrum property and Lorentz-invariance provides extra information about analyticity, in particular, it allows one to continue the functions analytically to a wider domain (called the extended tube). This theorem, due to Bargmann, Hall and Wightman, lies at the basis of the TCP theorem and the theorem on the connection between spin and statistics, which we shall become acquainted with in this chapter.

Exercise 9.1. Prove that the analytically continued Wightman functions $W^{(\kappa_1\dots\kappa_n)}(\zeta_1,\dots,\zeta_{n-1})$ satisfy the Lorentz-covariance condition

$$\sum_{m_1,\dots,m_n} V^{(\kappa_1)}_{l_1 m_1}(\underset{\sim}{\Lambda}^{-1})\dots V^{(\kappa_n)}_{l_n m_n}(\underset{\sim}{\Lambda}^{-1}) W^{(\kappa_1\dots\kappa_n)}_{m_1\dots m_n}(\Lambda\zeta_1,\dots,\Lambda\zeta_{n-1}) =$$
$$= W^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(\zeta_1,\dots,\zeta_{n-1}) \quad \text{for } \zeta \in T^{-1}_{n-1},\ \underset{\sim}{\Lambda} \in SL(2,C). \tag{9.1}$$

[Hint: Apply the uniqueness theorem B.10.]

The idea of the Bargmann-Hall-Wightman construction consists in extending the Lorentz-covariance condition to complex Lorentz transformations.

The group $L(C)$ of complex Lorentz transformations is defined as the collection of homogeneous linear transformations of four-dimensional complex Minkowski space $\mathbf{CM} \equiv \mathbf{C}^4$ that preserve the bilinear form

$$zw \equiv g_{\lambda\mu} z^\lambda w^\mu \equiv z^0 w^0 - \mathbf{zw} \tag{9.2}$$

(here the indices λ, μ are summed from zero to three). In other words, $L(C)$ is the group of complex 4×4-matrices Λ satisfying the orthogonality condition (3.13):

$$\Lambda^T g \Lambda = g, \tag{9.3}$$

where g is the metric tensor in Minkowski space.

Note that the complex Lorentz group is isomorphic to the complex orthogonal group $O(4,C)$. In fact, if we introduce the new coordinates $z'^0 = z^0$, $z'^j = iz^j$ (since the z^λ are complex numbers, the new coordinates have the same status as the old ones), then the minus sign in (9.2) is replaced by plus and the metric tensor $g = (g_{\lambda\mu})$ in the Minkowski space is accordingly replaced by the unit tensor $\delta_{\lambda\mu}$

in $\mathbf{C}^4$. We ordinarily use the Lorentz basis in which equations (9.2) and (9.3) hold, since in this basis the passage to the real Minkowski space and correspondingly the real Lorentz group appears more natural.

If we take the determinant of both sides of (9.3), then we obtain for $\Lambda \in L(C)$ (just as for the real Lorentz transformations)

$$(\det \Lambda)^2 = 1. \tag{9.4}$$

Thus the complex Lorentz group is partitioned into two subsets $L_+(C)$ and $L_-(C)$ defined respectively by the equations $\det \Lambda = 1$ and $\det \Lambda = -1$. In contrast to the real case, $L_+(C)$ is a connected group (it is called the *proper complex Lorentz group*).

Exercise 9.2. Prove that $L_+(C)$ is connected. [Hint: The line of argument is the same as that in Exercise 3.2.]

It is a remarkable fact that $L_+(C)$ contains the total space-time reflection $PT = = -1$ $(z \to -z)$. The fact that this transformation can be joined by a continuous curve to the identity of the group is established by a simple construction. In fact, the matrices

$$\Lambda(\alpha) = \begin{pmatrix} \cos\alpha & 0 & 0 & i\sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ i\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \tag{9.5}$$

are in $L_+(C)$ and depend continuously on α and $\Lambda(0) = 1$, $\Lambda(\pi) = -1$.

In §3.1.C we constructed the covering homomorphism $SL(2,C) \to L_+^{\uparrow}$. A similar construction can be carried out for the complex Lorentz group as well. In place of $\underset{\sim}{\Lambda}$ and $\overline{\underset{\sim}{\Lambda}}$ in (3.39) we introduce the two independent variables Λ_l, $\Lambda_r \in SL(2,C)$ and we set

$$\underset{\sim}{\Lambda(\Lambda_l,\Lambda_r)z} = \Lambda_l \underset{\sim}{z} \Lambda_r^T, \quad \widetilde{\Lambda(\Lambda_l,\Lambda_r)z} = \Lambda_r^{T^{-1}} \tilde{\underset{\sim}{z}} \Lambda_l^{-1}, \tag{9.6}$$

where as before,

$$\underset{\sim}{z} = \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^1 + iz^2 & z^0 - z^3 \end{pmatrix}, \quad \tilde{\underset{\sim}{z}} = \begin{pmatrix} z^0 - z^3 & -z^1 + iz^2 \\ z^1 + iz^2 & z^0 + z^3 \end{pmatrix}. \tag{9.7}$$

Clearly, $\det \underset{\sim}{z} = z^2$ $(\equiv zz)$, so that $\Lambda(\Lambda_l,\Lambda_r) \in L(C)$. Then since $SL(2,C)$ is a connected set, $\det \Lambda(\Lambda_l, \mathrm{L}_r) = 1$, and therefore $\Lambda(\Lambda_l,\Lambda_r) \in L_+(C)$. As a result we obtain the homomorphism

$$SL(2,C) \times SL(2,C) \ni (\Lambda_l,\Lambda_r) \to \Lambda(\Lambda_l,\Lambda_r) \in L_+(C). \tag{9.8}$$

The same arguments as for the case $L_+^{\uparrow}$ show that the homomorphism (9.8) maps $SL(2,C) \times SL(2,C)$ onto the whole group and its kernel consists of the two elements ± 1. Thus we have constructed a double covering homomorphism of the direct product $SL(2,C) \times SL(2,C)$ onto $L_+(C)$; since it is connected and simply connected, $SL(2,C) \times SL(2,C)$ serves as the universal covering group of $L_+(C)$. By the *real section* of $SL(2,C) \times SL(2,C)$ we mean the set of elements of the form $(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})$; these elements form a subgroup isomorphic to $SL(2,C)$, while the transformations $\Lambda(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}}) \equiv \Lambda(\underset{\sim}{\Lambda})$ correspond to real transformations of $L_+(C)$, that is, transformations of $L_+^{\uparrow}$.

As for the case of the proper Lorentz group $L_+^\uparrow$, the above construction enables us to replace $L_+(C)$ by $SL(2,C) \times SL(2,C)$, which simplifies the treatment of the two-valued representations of the Lorentz group.

The group $SL(2,C) \times SL(2,C)$ (and also the group $L_+(C)$ which is locally isomorphic to it) is a complex Lie group of dimension 6. This means that locally (say, in a neighbourhood of the identity) we can choose six of the matrix elements of the matrices $\Lambda_l, \Lambda_r \in SL(2,C)$ as independent complex variables (called local coordinates) in terms of which the remaining two matrix elements are expressed analytically. For example, when $(\Lambda_l)_{22} \neq 0$, $(\Lambda_r)_{22} \neq 0$, we can choose $(\Lambda_l)_{12}$, $(\Lambda_l)_{21}$, $(\Lambda_l)_{22}$, $(\Lambda_r)_{12}$, $(\Lambda_r)_{21}$, $(\Lambda_r)_{22}$ as local coordinates. A complex function $f(\Lambda_l, \Lambda_r)$ on $SL(2,C) \times \times SL(2,C)$ (or on some open subset of it) is said to be holomorphic (or complex analytic) if it is a holomorphic function of the local coordinates. Below we shall also encounter functions $f(\zeta_1, \ldots, \zeta_n; \Lambda_l, \Lambda_r)$ which depend on the complex 4-vectors $\zeta_1, \ldots, \zeta_n$ and on the matrices $\Lambda_l, \Lambda_r \in SL(2,C)$; in this case there is the obvious notion of holomorphy in all the variables $\zeta_1, \ldots, \zeta_n, \Lambda_l, \Lambda_r$.

We shall need the following simple uniqueness theorem.

Exercise 9.3. Let $f(\zeta_1, \ldots, \zeta_n; \Lambda_l, \Lambda_r)$ be a holomorphic function in some domain $\mathcal{O} \subset \mathbf{CM}^n \times \times SL(2,C) \times SL(2,C)$, having a non-empty intersection with the set of points of the form

$$\{(\zeta_1, \ldots, \zeta_n; \Lambda_l, \Lambda_r) : (\zeta_1, \ldots, \zeta_n) \in \mathbf{CM}^n,\ \Lambda_r = \overline{\Lambda}_l\}, \tag{9.9}$$

and suppose that f vanishes on this intersection. Prove that $f \equiv 0$ in $\mathcal{O}$.

In §3.1.D we considered the finite-dimensional irreducible representations $\mathfrak{D}^{(j,k)}(\underset{\sim}{\Lambda})$ of $SL(2,C)$ which now play the role of a real section of $SL(2,C) \times SL(2,C)$. It is easy to see that this representation can be holomorphically continued to a representation of the entire group $SL(2,C) \times SL(2,C)$ (such an extension is unique according to Exercise 9.3 with $n = 0$). To do this, we set

$$\mathfrak{D}^{(j,k)}(\Lambda_l, \Lambda_r)\psi)^{\alpha_1 \ldots \alpha_{2j}; \beta'_1 \ldots \beta'_{2k}} =$$

$$= \sum_{\gamma_1 \ldots \gamma_{2j} \delta'_1 \ldots \delta'_{2k}} \left(\prod_s^{2j} (\Lambda_l)^{\alpha_s}_{\gamma_s}\right) \left(\prod_{t=1}^{2k} (\Lambda_r)^{\beta'_t}_{\delta'_t}\right) \psi^{\gamma_1 \ldots \gamma_{2j}; \delta'_1 \ldots \delta'_{2k}}, \tag{9.10}$$

if $\mathfrak{D}^{(j,k)}$ is realized in the space of spin-tensors ψ (cf. (3.51)), or

$$(\mathfrak{D}^{(j,k)}(\Lambda_l, \Lambda_r)\psi)(\omega, w) = \psi(\Lambda_l^{-1}\omega,\ \Lambda_r^{-1}w), \tag{9.11}$$

if $\mathfrak{D}^{(j,k)}$ is realized in the space of all complex polynomials $\psi(\omega, w)$ in $\omega, w \in \mathbf{C}^2$, that are homogeneous of degree $2j$ in ω and homogeneous of degree $2k$ in w (cf. (3.52)). If $\mathfrak{D}^{(j,k)}(-A, -B) = \mathfrak{D}^{(j,k)}(A, B)$, that is, if $j + k$ is an integer, then we can define a single-valued representation of $L_+(C)$, by setting

$$\mathfrak{D}^{(j,k)}(\Lambda(\Lambda_l, \Lambda_r)) \equiv \mathfrak{D}^{(j,k)}(\Lambda_l, \Lambda_r).$$

If, on the other hand $j + k$ is half-integral , then this formula defines a two-valued representation of $L_+(C)$.

Since any complex representation $V(\underset{\sim}{\Lambda}) \equiv V(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})$ of $SL(2,C)$ is a direct sum of irreducible representations, it can also be uniquely continued to a holomorphic representation $V(\Lambda_l, \Lambda_r)$ of $SL(2,C) \times SL(2,C)$.

B. Lorentz-Covariant Analytic Functions in the Past Tube

Suppose that we are given a (complex) representation $V(\Lambda) \equiv V(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})$ of dimension r of the group $SL(2,C)$ and that r functions $f_1, \ldots, f_r$ of the complex 4-vectors $\zeta_1, \ldots, \zeta_n$ are given such that these functions are analytic in the past tube T_n^- and satisfy the condition

$$f_i(\Lambda(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})\zeta_1 \ldots, \Lambda(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})\zeta_n) = \sum_j V_{ij}(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}}) f_j(\zeta_1, \ldots, \zeta_n)$$

$$\text{for} \quad \zeta \equiv (\zeta_1, \ldots, \zeta_n) \in T_n^-, \ \underset{\sim}{\Lambda} \in SL(2,C) \tag{9.12a}$$

or briefly,

$$f(\Lambda(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})\zeta) = V(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}}) f(\zeta) \quad \text{for } \zeta \in T_n^-, \ \underset{\sim}{\Lambda} \in SL(2,C). \tag{9.12b}$$

Such a family of functions $f_1, \ldots, f_r$ will be called a covariant analytic function in T_n^- with respect to $L_+^\uparrow$ (or $SL(2,C)$).

Theorem 9.1 (Bargmann-Hall-Wightman). *Every $L_+^\uparrow$-covariant analytic function $f(\zeta)$ in T_n^- has a unique analytic continuation into the so-called extended tube**

$$T_n = \bigcup_{\Lambda \in L_+(C)} \Lambda T_n^- \tag{9.13}$$

and is covariant in it with respect to $L_+(C)$:

$$f(\Lambda(\Lambda_l, \Lambda_r)\zeta) = V(\Lambda_l, \Lambda_r) f(\zeta) \quad \textit{for } \zeta \in T_n, \Lambda_l, \Lambda_r \in SL(2,C). \tag{9.14}$$

Scheme of the proof. We set

$$f^{(\Lambda_l, \Lambda_r)}(\zeta) = V(\Lambda_l, \Lambda_r)^{-1} f(\Lambda(\Lambda_l, \Lambda_r)\zeta),$$

$\Lambda_l, \Lambda_r \in SL(2,C)$. This is a holomorphic function of ζ in the domain $\Lambda(\Lambda_l, \Lambda_r)^{-1} T_n^-$. If we prove that the family of holomorphic functions $f(\Lambda_l, \Lambda_r)$ satisfies the compatibility conditions

$$f^{(\Lambda_l', \Lambda_r')}(\zeta) = f^{(\Lambda_l'', \Lambda_r'')}(\zeta) \quad \text{for } \zeta \in \Lambda(\Lambda_l', \Lambda_r')^{-1} T_n^- \cap \Lambda(\Lambda_l'', \Lambda_r'')^{-1} T_n^- \tag{9.15}$$

for all (Λ_l', Λ_r') and $(\Lambda_l'', \Lambda_r'')$ in $SL(2,C) \times SL(2,C)$, then the formula

$$f(\zeta) = f^{(\Lambda_l, \Lambda_r)}(\zeta) \quad \text{for } \zeta \in \Lambda(\Lambda_l, \Lambda_r)^{-1} T_n^-$$

is well defined and admits an analytic continuation of $f(\zeta)$ to the extended tube T_n. It is not difficult to see that $f(\zeta)$ then satisfies the covariance condition (9.14). Thus this function is the required analytic continuation of the original function f.

It remains to prove that the compatibility conditions (9.15) hold.

Exercise 9.4. Prove that the compatibility conditions (9.15) are equivalent to the following reduced compatibility conditions:

$$f(\zeta) = f^{(\Lambda_l, \Lambda_r)}(\zeta) \quad \text{for } \zeta \in T_n^- \cap \Lambda(\Lambda_l, \Lambda_r)^{-1} T_n^- \tag{9.16}$$

and for all $(\Lambda_l, \Lambda_r) \in SL(2,C) \times SL(2,C)$. [Hint: Apply (9.16) for $\Lambda_l = \Lambda_l' \Lambda_l''^{-1}$, $\Lambda_r = \Lambda_r' \Lambda_r''^{-1}$ and replace ζ by $\Lambda(\Lambda_l'', \Lambda_r'')\zeta$].

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* We draw attention to the fact that the extended tube is not a tube, that is, a set of the form $\mathbf{M}^n + i\mathcal{O}$ (the same comment also applies to the notion of symmetrized tube in §9.1.D).

Thus to prove the theorem, it suffices to verify the reduced compatibility conditions. We regard $f^{(\Lambda_l,\Lambda_r)}(\zeta)$ as a holomorphic function $f(\zeta;\Lambda_l,\Lambda_r)$ of the vector ζ and the variables Λ_l,Λ_r, ranging over the open set

$$\mathcal{O} = \{(\zeta,\Lambda_l,\Lambda_r) \in T_n^- \times SL(2,C) \times SL(2,C) : \zeta \in \Lambda(\Lambda_l,\Lambda_r)^{-1}T_n^-\}. \tag{9.17}$$

For $\Lambda_r = \overline{\Lambda}_l = \overline{\Lambda}$, $f(\zeta,\Lambda,\overline{\Lambda})$ is the same as the original function $f(\zeta)$; therefore if the open set $\mathcal{O}$ is connected, then by Exercise 9.3, $f(\zeta,\Lambda_l,\Lambda_r) \equiv f(\zeta)$ in $\mathcal{O}$, which is equivalent to the reduced compatibility conditions.

In the next two lemmas we prove that the open set $\mathcal{O}$ is connected; this will complete the proof of Theorem 9.1.

Lemma 9.2. *Let $\mathcal{A}$ be a set of elements $(\Lambda_l,\Lambda_r) \in SL(2,C) \times SL(2,C)$ such that the set $T_n^- \cap \Lambda(\Lambda_l,\Lambda_r)^{-1}T_n^-$ is non-empty; then $\mathcal{A}$ is a domain (that is, a non-empty connected open set).*

■ It is clear that $\mathcal{A}$ is open. We denote by $\mathcal{A}_0$ the connected component of $\mathcal{A}$ containing the identity of $SL(2,C) \times SL(2,C)$; in other words $\mathcal{A}_0$ is the set of elements in $\mathcal{A}$ that can be joined to the identity by a continuous curve in $\mathcal{A}$. Our problem is to prove that $\mathcal{A}_0 = \mathcal{A}$.

Exercise 9.5. (a) We introduce in $SL(2,C) \times SL(2,C)$ the equivalence relation: $(\Lambda_l',\Lambda_r') \sim (\Lambda_l'',\Lambda_r'')$ if there exist $X,Y \in SL(2,C)$ such that $\Lambda_l'' = X\Lambda_l'Y$, $\Lambda_r'' = \overline{X}\Lambda_r'\overline{Y}$, that is, $(\Lambda_l'',\Lambda_r'') = (X,\overline{X})(\Lambda_l',\Lambda_r') \times (Y,\overline{Y})$. Prove that if $(\Lambda_l',\Lambda_r') \in \mathcal{A}$ (or $(\Lambda_l',\Lambda_r') \in \mathcal{A}_0$) and $(\Lambda_l'',\Lambda_r'') \sim (\Lambda_l',\Lambda_r')$, then $(\Lambda_l'',\Lambda_r'') \in \mathcal{A}$ (or $(\Lambda_l'',\Lambda_r'') \in \mathcal{A}_0$).

(b) Prove that each element $(\Lambda_l,\Lambda_r) \in SL(2,C) \times SL(2,C)$ is equivalent to some element of the form $(C,1)$ where the matrix C has one of the following three forms:

$$C = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{9.18a}$$

$$C = -\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{9.18b}$$

$$C = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda \in \mathbf{C}\backslash\{0\}. \tag{9.18c}$$

[Hint: Choose the matrices X, C so that C has the indicated "canonical form" and $A_l = XCX^{-1}$; then verify that $(\Lambda_l,\Lambda_r) = (X,\overline{X})(C,1)(Y,\overline{Y})$ for $Y = X^{-1}$.]

(c) Prove that any element $(\Lambda_l,\Lambda_r) \in \mathcal{A}$ is equivalent either to an element of the form $(C,1)$ where C has the form (9.18a), or to an element of the form $(K,\overline{K}^{-1})$, where

$$K = \begin{pmatrix} \exp(\frac{1}{2}(\alpha+i\beta)) & 0 \\ 0 & \exp(-\frac{1}{2}(\alpha+i\beta)) \end{pmatrix}; \tag{9.19}$$

here α and β are real and $0 \le \beta < \pi$. [Hint: Consider separately the three possibilities in part (b) of this exercise. If C has the form (9.18a), then one can find a vector $\zeta \in T_n^-$, for example, $\zeta_j^\mu = -i\delta_{\mu,0}$ $(j = 1,\ldots,n)$ such that $\Lambda(C,1)^{-1}\zeta \in T_n^-$; hence $(C,1) \in \mathcal{A}$. If C has the form (9.18b), then $(C,1) \notin \mathcal{A}$; for the proof, take an arbitrary vector $\zeta \in T_n^-$, set $\zeta' = \Lambda(C,1)^{-1}\zeta$ and verify that $y_j'^0 - y_j'^3 = -(y_j^0 - y_j^3) > 0$. Finally, if C has the form (9.18c), and the matrix K is chosen so that $\exp(\frac{1}{2}(\alpha+i\beta)) = \sqrt{\lambda}$ and $0 \le \beta \le \pi$, then $(C,1) \sim (K,\overline{K}^{-1})$; it therefore suffices to show that the element $(K,\overline{K}^{-1})$ belongs to $\mathcal{A}$ if and only if $0 \le \beta < \pi$. For this, we have to calculate the matrix $\Lambda(K,\overline{K}^{-1})$:

$$\Lambda(K,\overline{K}^{-1}) = \begin{pmatrix} \cos\beta & 0 & 0 & i\sin\beta \\ 0 & \cosh\alpha & -i\sinh\alpha & 0 \\ 0 & i\sinh\alpha & \cosh\alpha & 0 \\ i\sin\beta & 0 & 0 & \cos\beta \end{pmatrix}. \tag{9.20}$$

Here if $\cos\beta = -1$, then for an arbitrary vector $\zeta \in T_n^-$, the vector $\zeta' = \Lambda(K,\overline{K}^{-1})^{-1}\zeta$ has the property $y_j'^0 = -y_j^0 > 0$ $(j = 1,\ldots,n)$ and therefore $\zeta' \notin T_n^-$ and $(K,\overline{K}^{-1}) \notin \mathcal{A}$. If, on the other hand, $\cos\beta \ne -1$, then we have to prove that $(K,\overline{K}^{-1}) \in \mathcal{A}$; to this end, we take a vector $\zeta \in T_n^-$ of the form $\zeta_j = -(ia,0,0,b)$, where $a > 0$, $b\sin\beta < a\cos\beta$ and verify that the imaginary part of the vector $\zeta' = \Lambda(K, \overline{K}^{-1})^{-1}\zeta$ is given by the formula $y'_j = -(a\cos\beta - b\sin\beta, 0, 0, 0)$ $(j = 1, \ldots, n)$; hence it follows that $(K, \overline{K}^{-1}) \in \mathcal{A}$.]

Exercise 9.5 reduces the proof of the lemma to a consideration of the following two situations.

Firstly, we are required to prove that the element C of (9.18a) is such that $(C, 1)$ belongs to $\mathcal{A}_0$, that is, it can be joined to the identity by a continuous curve in $\mathcal{A}$. For this purpose, we fix a vector $\zeta \in T_n^-$ with components $\zeta_j^\mu = -i\delta_{\mu,0}$ $(j = 1, \ldots, n)$ and set

$$C(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad 0 \le t \le 1.$$

Then $\Lambda(C(t), 1)\zeta_j = (-i, \frac{1}{2}it, -\frac{1}{2}t, 0)$, so that $\Lambda(C(t), 1)^{-1}\zeta \in T_n^-$. Hence, $(C(t), 1)$ is a continuous curve in $\mathcal{A}$ joining $(C, 1)$ to the identity.

Secondly, we must prove that the element K of the form (9.19) is such that $(K, \overline{K}^{-1})$ belongs to $\mathcal{A}_0$. We set

$$K(t) = \begin{pmatrix} \exp\left[\frac{t}{2}(\alpha + i\beta)\right] & 0 \\ 0 & \exp\left[-\frac{t}{2}(\alpha + i\beta)\right] \end{pmatrix}, \quad 0 \le t \le 1.$$

According to Exercise 9.5(c) the elements $(K(t), \overline{K(t)}^{-1})$ lie in $\mathcal{A}$ and hence they define a continuous curve in $\mathcal{A}$ joining $(K, \overline{K}^{-1})$ to the identity. ∎

Lemma 9.3. *The set $\mathcal{O}$ (9.17) is connected.*

∎ Since the set of all points of $\mathcal{O}$ at which $\Lambda_l = \Lambda_r = 1$ is clearly connected, it suffices to prove that an arbitrary point $(\zeta; \Lambda_l, \Lambda_r)$ of $\mathcal{O}$ can be joined by a continuous curve in $\mathcal{O}$ to some point of the form $(\zeta', 1, 1)$ where $\zeta' \in T_n^-$; in other words, it is possible to construct a continuous curve $(\zeta(t); \Lambda_l(t), \Lambda_r(t)) \in \mathcal{O}$ $(0 \le t \le 1)$ such that $\zeta(0) = \zeta$, $\Lambda_l(0) = \Lambda_l$, $\Lambda_r(0) = \Lambda_r$ and $\Lambda_l(1) = \Lambda_r(1) = 1$, $\zeta(1) \in T_n^-$. We use Lemma 9.2 which enables us to construct a continuous curve $(\Lambda_l(t), \Lambda_r(t))$ in $SL(2, C) \times SL(2, C)$ with the property that for any $t \in [0, 1]$, there exists a vector $z(t) \in$ $\in T_n^- \cap \Lambda(\Lambda_l(t), \Lambda_r(t))^{-1} T_n^-$. It follows from the continuity of $\Lambda_l(t)$ and $\Lambda_r(t)$ in t, that for each t there exists a relatively open interval * I_t in $[0, 1]$ containing t and such that $\Lambda(\Lambda_l(s), \Lambda_r(s))z(t) \in T_n^-$ for $s \in I_t$. By the classical Heine-Borel theorem, we can select from the covering $\{I_t\}$ of $[0, 1]$, a finite subcovering. Thus there exists a finite set of intervals $[a_0, b_0), (a_1, b_1), \ldots, (a_N, b_N]$, contained in $[0, 1]$ and covering it, and there exists a finite set of points $z^{(0)}, \ldots, z^{(N)} \in T_n^-$ such that $\Lambda(\Lambda_l(t), \Lambda_r(t))z^{(k)} \in T_n^-$ for $t \in [a_k, b_k]$. By decreasing these intervals if necessary and also by decreasing their number N and by suitably enumerating them, we may suppose further that

$$a_k < b_{k-1} < a_{k+1} < b_k \quad \text{for } k = 1, \ldots, N-1.$$

We now set

$$\begin{aligned} \zeta(t) &= \tfrac{b_{k-1} - t}{b_{k-1} - a_k} z^{(k-1)} + \tfrac{t - a_k}{b_{k-1} - a_k} z^{(k)} && \text{for } a_k \le t \le b_{k-1}, \\ \zeta(t) &= z^{(k)} && \text{for } b_{k-1} \le t \le a_{k+1}, \\ \zeta(t) &= \tfrac{b_k - t}{b_k - a_{k+1}} z^{(k)} + \tfrac{t - a_{k+1}}{b_k - a_{k+1}} z^{(k+1)} && \text{for } a_{k+1} \le t \le b_k. \end{aligned}$$

It is not difficult to see that $\zeta(t)$ is a continuous curve in T_n^- such that $\Lambda(\Lambda_l(t), \Lambda_r(t))\, \zeta(t) \in T_n^-$ for all $0 \le t \le 1$. Hence it follows that $(\zeta(t); \Lambda_l(t), \Lambda_r(t))$ can be chosen as the required curve in $\mathcal{O}$. ∎

By applying Theorem 9.1 to the Wightman functions

$$w_{l_1 \ldots l_n}^{(\kappa_1 \ldots \kappa_n)}(z_1, \ldots, z_n) \equiv W_{l_1 \ldots l_n}^{(\kappa_1 \ldots \kappa_n)}(z_1 - z_2, \ldots, z_{n-1} - z_n), \tag{9.21}$$

we obtain the corollary.

Corollary 9.4. The Wightman functions (9.21) are analytic with respect to the variables $z_1, \ldots, z_n$ in the extended tube

$$\mathcal{T}_n = \bigcup_{\Lambda \in L_+(C)} \Lambda T_n^- \tag{9.22}$$

* This means the intersection of an open interval in $\mathbf{R}$ with $[0, 1]$.

and satisfy the covariance condition with respect to $SL(2,C)\times SL(2,C)$ (or $L_+(C)$):

$$\sum_{m_1\dots m_n} V^{(\kappa_1)}_{l_1m_1}(\Lambda_l^{-1},\Lambda_r^{-1})\dots V^{(\kappa_n)}_{l_nm_n}(\Lambda_l^{-1},\Lambda_r^{-1})w^{(\kappa_1\dots\kappa_n)}_{m_1\dots m_n}(\Lambda(\Lambda_l,\Lambda_r)z_1,\dots$$
$$\dots,\Lambda(\Lambda_l,\Lambda_r)z_n) = w^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(z_1,\dots,z_n). \tag{9.23}$$

Exercise 9.6. Suppose that the fields $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$ transform according to the irreducible representations $\mathfrak{D}^{(j_1,k_1)},\dots,\mathfrak{D}^{(j_n,k_n)}$ respectively, of $SL(2,C)$ and let

$$J = j_1+\dots+j_n. \tag{9.24}$$

Prove that

$$w^{(\kappa_1\dots\kappa_n)}(-z_1,\dots,-z_n) = (-1)^{2J}w^{(\kappa_1\dots\kappa_n)}(z_1,\dots,z_n) \tag{9.25}$$

for $z\in\mathcal{T}_n$. [Hint: Set $\Lambda_l=-1$, $\Lambda_r=1$ in the covariance condition (9.23).]

Theorem 9.1 is applied to two-point functions in the next two exercises.

Exercise 9.7. (a) Prove that the image of the tube T_1^- under the map $\zeta\to t=\zeta^2$ is the cut plane $\mathbf{C}\backslash\overline{\mathbf{R}}_+\equiv\mathbf{C}\backslash[0,\infty)$.

(b) Prove the following formula for the extended tube T_1:

$$T_1 = \{\zeta\in\mathbf{CM}:\zeta^2\in\mathbf{C}\backslash\overline{\mathbf{R}}_+\} \tag{9.26}$$

and verify that the subsets $\zeta^2=\text{const}$ are the orbits of the group $L_+(C)$ in T_1. [Hint: If ζ and ζ' are a pair of points in $\mathbf{CM}$ and $\zeta^2=\zeta'^2\neq0$, then $\zeta'=\Lambda(1,B)\zeta$ for $B=(\underset{\sim}{\zeta})^{-1}\underset{\sim}{\zeta}'$.]

(c) Prove that every analytic $L_+^\uparrow$-invariant function $f(\zeta)$ in T_1^- can be represented in the form

$$f(\zeta)=h(\zeta^2),$$

where $h(t)$ is an analytic function of t in the cut plane $\mathbf{C}\backslash\overline{\mathbf{R}}_+$. [Hint: Use Theorem 9.1 and the previous parts of the exercise.]

Exercise 9.8. (a) Let $f(\zeta)$ be a Lorentz-covariant analytic function of $\zeta\in T_1^-$ transforming according to the representation $\mathfrak{D}^{(j,0)}\otimes\mathfrak{D}^{(k,0)}$ of $SL(2,C)$; it can be regarded as a polynomial function $f(\zeta,\omega,w)$ of the additional spinor variables $\omega,w\in\mathbf{C}^2$, it being homogeneous of degree $2j$ in ω and homogeneous of degree $2k$ in w; the covariance condition can be written in the form

$$f(\Lambda(\Lambda_l,\Lambda_r)\zeta;\Lambda_l\omega,\Lambda_lw) = f(\zeta,\omega,w);\quad \Lambda_l,\Lambda_r\in SL(2,C). \tag{9.27}$$

(Here we have used the analytic continuation of the covariance condition to the complex Lorentz transformations.) Prove that $f(z,\omega,w)$ has the form

$$f(z,\omega,w)=\delta_{jk}(\omega\epsilon w)^{2j}h(\zeta^2),$$

where $h(t)$ is an analytic function of $t\in\mathbf{C}\backslash\overline{\mathbf{R}}_+$. [Hint: First use the covariance condition (9.27) with $\Lambda_l=1$ and prove that $f(\zeta,\omega,w)$ depends on ζ only as a function of ζ^2; then use the covariance condition with $\Lambda_r=1$.]

(b) Let $f(\zeta,\omega,\bar\omega)$ be a Lorentz-covariant analytic function of $\zeta\in T_1^-$ transforming according to the representation $\mathfrak{D}^{(j,k)}$; it can be represented as a homogeneous polynomial of degree $2j$ in $\omega\in\mathbf{C}^2$ and a homogeneous polynomial of degree $2k$ in $\bar\omega$. Prove that $f(\zeta,\omega,\bar\omega)$ has the form

$$f(\zeta,\omega,\bar\omega)=\delta_{jk}(\bar\omega\tilde{\zeta}\omega)^{2j}h(\zeta^2)= \tag{9.28a}$$
$$=\delta_{jk}(\bar\omega\tilde{\partial}_\zeta\omega)^{2j}g(\zeta^2), \tag{9.28b}$$

where $h(t)$ and $g(t)$ are analytic functions of $t\in\mathbf{C}\backslash\overline{\mathbf{R}}_+$. [Hint: For the proof of (9.28a) introduce the function

$$F(\zeta,\omega,w)=\left(w\tilde{\zeta}\epsilon^{-1}\frac{\partial}{\partial\bar\omega}\right)^{2k}f(\zeta,\omega,\bar\omega)$$

and use part (a) of this exercise; for the proof of (9.28b) use the identity

$$(\bar{\omega}\tilde{\partial}_\zeta\omega)^{2j}g(\zeta^2) \equiv (2\bar{\omega}\tilde{\zeta}\omega)^{2j}g^{(n)}(\zeta^2).]$$

C. Real points of the extended tube

It is clear from the definition of the tubes $T_n^\pm$ that they do not contain any real points. It is therefore of special interest that the extended tube $\mathcal{T}_n$ does contain real points for all n. It follows in particular that the Wightman functions $w^{[n]}$, which were originally defined as generalized functions, are not only limiting values of holomorphic functions (in the tube $\mathcal{T}_n^-$), but are themselves analytic functions in some real domain of the space-time variables.

This result is obvious for the case of two-point functions from formula (9.26) for $\mathcal{T}_1$, which shows that the real points of $\mathcal{T}_1$ are precisely all the spacelike points of $\mathbf{M}$.

The next proposition provides a description of the real points of the extended tube $\mathcal{T}_n$.

Proposition 9.5 (Jost). *In order that the real point $(\xi_1,\ldots,\xi_n) \in \mathbf{M}^n$ belong to the extended tube $\mathcal{T}_n$, it is necessary and sufficient that for any choice of non-negative numbers λ_j not all zero, the vector*

$$\rho = \sum_{j=1}^{n} \lambda_j \xi_j \tag{9.29}$$

be spacelike.

■ *Necessity.* By hypothesis, $(\xi_1,\ldots\xi_n) = (\Lambda\zeta_1,\ldots,\Lambda\zeta_n)$, where $\Lambda \in L_+(C)$, $(\zeta_1,\ldots,\zeta_n) \in T_n^-$. Then $\rho = \Lambda z$, where $z = \sum_{j=1}^n \lambda_j\zeta_j \in T_1^-$. Thus ρ is a real point of the extended tube $\mathcal{T}_1$ and, according to the above remark, ρ is a spacelike vector.

Sufficiency. Let K be a set of vectors of the form $\rho = \sum_{j=1}^n \lambda_j x_j$, where $\lambda_j \geq 0$, $\sum_{j=1}^n \lambda_j > 0$, and the vector x_j $(j = 1,\ldots,n)$ ranges over a sufficiently small neighbourhood $\mathcal{O}_j$ of ξ_j.

Exercise 9.9. Prove that the cone K defined above is for sufficiently small neighbourhoods $\mathcal{O}_j$ of ξ_j, an open convex cone in $\mathbf{M}$ consisting of spacelike vectors. [Hint: The vector ρ in (9.29) depends continuously on λ_j, x_j, therefore there exists for any point λ of the set $I = \{\lambda \equiv (\lambda_1,\ldots,\lambda_n) : \lambda_j \geq \geq 0,\ \sum_{j=1}^n \lambda_j = 1\}$, a neighbourhood $\mathcal{Q}^{(\lambda)}$ of λ in I and neighbourhoods $\mathcal{O}_j^{(\lambda)}$ of the points ξ_j in $\mathbf{M}$ such that the vector $\sum_{j=1}^n \mu_j x_j$ is spacelike for $\mu \in \mathcal{Q}^{(\lambda)}$, $x_j \in \mathcal{O}_j^{(\lambda)}$. By the Heine-Borel theorem, a finite subcovering of the compactum I can be chosen from the system of neighbourhoods $\mathcal{Q}^{(\lambda)}$. Then for the required $\mathcal{O}_j$, one can choose $\bigcap_\lambda \mathcal{O}_j^{(\lambda)}$, where the λ enumerate the neighbourhoods $\mathcal{Q}^{(\lambda)}$ of the above finite subcovering.]

Thus we may suppose that K is an acute convex cone in $\mathbf{M}$ consisting of spacelike vectors and hence is disjoint from V^+. There is a simple geometric separation theorem ([S2],p.64) which applied to our case, asserts that the cones K and V^+ can be strictly separated by a hyperplane, that is, there exists a vector $a \in \mathbf{M}$ such that

$$ax > 0 \text{ for } x \in V^+, \quad a\rho < 0 \text{ for } \rho \in K.$$

The cones $-K$ and V^+ can be similarly separated; consequently there exists $b \in \mathbf{M}$ such that

$$bx > 0 \text{ for } x \in V^+, \quad b\rho > 0 \text{ for } \rho \in K.$$

It is clear that the vectors a and b are non-collinear and positive timelike or isotropic, therefore we can choose a Lorentz coordinate frame such that

$$a = \alpha(1,0,0,u), \quad b = \beta(1,0,0,-v),$$

where α, β, u, v are positive numbers with $u \leq 1$, $v \leq 1$. It now follows from the conditions $a\rho < 0$ and $b\rho > 0$ for $\rho \in K$ that $|\rho^0| < \rho^3$ for $\rho \in K$. In particular,

$$|\xi_j^0| < \xi_j^3 \quad \text{for } j = 1, \ldots, n.$$

We choose the complex Lorentz rotation $\Lambda \in L_+(C)$ in the $(0,3)$-plane:

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}.$$

Then $\Lambda\xi_j = (-i\xi_j^3, \xi_j^1, \xi_j^2, -i\xi_j^0) \in T_1^-$, whence it follows that $(\Lambda\xi_1, \ldots, \Lambda\xi_n) \in T_n^-$, and hence, $(\xi_1, \ldots, \xi_n) \in \mathcal{T}_n$. ■

We now apply this result to the Wightman functions. The real points of the extended tube $\mathcal{T}_n$ are called *Jost points*; we denote the set of them by $\mathcal{J}_n$:

$$\mathcal{J}_n = \mathcal{T}_n \cap \mathbf{M}^n. \tag{9.30}$$

The Jost points form a non-empty subset of $\mathbf{M}^n$ which is contained in the domain of holomorphy of the Wightman function $w^{(\kappa_1 \ldots \kappa_n)}(z_1, \ldots, z_n)$.

Exercise 9.10. Prove that the point $(x_1, \ldots, x_n) \in \mathbf{M}^n$ is a Jost point if and only if

$$\Big(\sum_{j=1}^{n} \lambda_j x_j\Big)^2 < 0$$

for all real $\lambda_1, \ldots, \lambda_n$ satisfying the conditions

$$\lambda_1 + \ldots + \lambda_n = 0, \; \lambda_1 + \ldots + \lambda_k \geq 0 \quad \text{for } k = 1, \ldots, n-1$$

and

at least one of the $\lambda_1 + \ldots + \lambda_k$ is non-zero.

We give another related concept: a point $(x_1, \ldots, x_n) \in \mathbf{M}^n$ is said to be a *totally spacelike point* if $x_j - x_k$ is a spacelike vector for all $j \neq k$.

Exercise 9.11. Prove that the Jost points are totally spacelike. [Hint: Use the preceding exercise.]

The result of Exercise 9.11 enables us to obtain relations between the various Wightman functions.

Exercise 9.12. (a) Prove that under the normal connection between spin and statistics, the following relation between the analytic Wightman functions in $\mathcal{T}_n$ holds:

$$w^{(\kappa_n \ldots \kappa_1)}(z_n, \ldots, z_1) = (-1)^{F/2} w^{(\kappa_1 \ldots \kappa_n)}(z_1, \ldots, z_n), \tag{9.31}$$

where F is the number of fields with half-integer spin featuring on each side of this formula. [Hint: The equality holds at the Jost points where the locality of the fields can be used; also bear in mind that $w = 0$ for odd F and $(-1)^{F(F-1)/2} = (-1)^{F/2}$ for even F.]

(b) Prove that under the normal connection between spin and statistics, the following relation holds:

$$w^{(\kappa_n \ldots \kappa_1)}(-x_n, \ldots, -x_1) = (-1)^{2J + F/2} w^{(\kappa_1 \ldots \kappa_n)}(x_1, \ldots, x_n); \tag{9.32}$$

it is assumed here that the fields $\phi^{(\kappa_1)}, \ldots, \phi^{(\kappa_n)}$ transform according to the irreducible representations $\mathfrak{D}^{(j_1, k_1)}, \ldots, \mathfrak{D}^{(j_n, k_n)}$ of $SL(2, C)$; J and F have the same meaning as in (9.24), (9.31). [Hint: It follows from Exercise 9.6 and part (a) of this exercise that

$$w^{(\kappa_n \ldots \kappa_1)}(-z_n, \ldots, -z_1) = (-1)^{2J + F/2} w^{(\kappa_1 \ldots \kappa_n)}(z_1, \ldots, z_n) \tag{9.33}$$

for $(z_1,\dots,z_n) \in \mathcal{T}_n$; confine attention here to points $(z_1,\dots,z_n) \in \mathcal{T}_n^-$ and pass to the boundary values $y_j \to 0$ in the cone $(y_j - y_{j+1}) \in V^-$; here $((-y_{j+1} - (-y_j))$ also lies in V^-.]

Exercise 9.13. Assuming the normal connection between spin and statistics, prove the following relation between the two-point Wightman functions of the field $\phi^{(\kappa)}(x)$ and its Hermitian adjoint field $\phi^{(\bar\kappa)}(x) \equiv (\phi^{(\kappa)}(x))^*$:

$$\langle 0|\phi^{(\kappa)}(x)\phi^{(\bar\kappa)}(y)|0\rangle = \langle 0|\phi^{(\bar\kappa)}(x)\phi^{(\kappa)}(y)|0\rangle. \tag{9.34}$$

[Hint: Use Exercise 9.12(b) and the fact that a two-point function depends on x, y only in terms of their difference $x - y$.]

D. Analyticity of Wightman functions in a symmetrized tube

Theorem 9.1 enables us to continue the Wightman functions $w^{[n]}(z_1,\dots,z_n)$ analytically from the tube $\mathcal{T}_n^-$ to the extended tube $\mathcal{T}_n$. We now show that the locality of the fields enables us to continue analytically the Wightman functions to the so-called *symmetrized tube*

$$\mathcal{T}_n^S = \bigcup_{\pi\in S_n} \pi\mathcal{T}_n; \tag{9.35}$$

here S_n is the group of permutations of the indices $1,\dots,n$ which acts on $(z_1,\dots,z_n) \in$ $\in \mathbf{C}M^n$ as follows:

$$\pi(z_1,\dots,z_n) = (z_{\pi^{-1}(1)},\dots,z_{\pi^{-1}(n)}). \tag{9.36}$$

We suppose for simplicity of statement that throughout this subsection there is the normal connection between spin and statistics. For this case we introduce the notion of fermionic parity of the permutation π:

$$\epsilon_F(\pi) = \prod_{i<j}{}' \operatorname{sgn}(\pi j - \pi i), \tag{9.37}$$

where the product $\prod'$ is only taken over pairs of indices i,j $(i<j)$ for which $\phi^{(\kappa_i)}$ and $\phi^{(\kappa_j)}$ are fermionic fields (thus, the fixed monomial $\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_n)}(x_n)$ of the fields features in the definition of $\epsilon_F(\pi)$).

Note that the numbers $\epsilon_F(\pi)$ form a representation of S_n (that is, the relation $\epsilon_F(\pi_1)\epsilon_F(\pi_2) =$ $= \epsilon_F(\pi_1\pi_2)$ for all $\pi_1,\pi_2 \in S_n$) only in the case when all the fields of the above monomial are either bosonic or fermionic. The fact is that $\epsilon_F(\pi)$ depends not only on π but also on the partitioning of the indices $\{1,\dots,n\}$ into two parts, bosonic and fermionic. In the general case there is a modified formula for the product $\epsilon_F(\pi_1)\epsilon_F(\pi_2)$.

Exercise 9.14. For $\tau,\pi \in S_n$ we set

$$\epsilon_F^{(\tau)}(\pi) = \prod_{i<j}{}'' \operatorname{sgn}(\pi j - \pi i),$$

where the product $\prod''$ is taken over pairs of indices i,j $(i<j)$ for which $\phi^{(\kappa_{\tau(i)})}$ and $\phi^{(\kappa_{\tau(j)})}$ are fermionic fields (here we are supposing the monomial $\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_n)}(x_n)$ to be fixed). Prove the relation

$$\epsilon_F(\pi_1)\epsilon_F(\pi_2) = \epsilon_F^{(\pi_2^{-1})}(\pi_1\pi_2^{-1}) \quad \text{for all } \pi_1,\pi_2 \in S_n. \tag{9.38}$$

Exercise 9.15. Prove that under the normal connection between spin and statistics the following relation holds:

$$w(x_{\pi 1},\dots,x_{\pi n}) = \epsilon_F(\pi)w(x_1,\dots,x_n)$$

at all Jost points $(x_1,\dots,x_n) \in \mathcal{T}_n$.

Theorem 9.6. *In the Wightman theory with normal connection between spin and statistics, the Wightman functions $w(z_1, \ldots, z_n)$ admit a single-valued analytic continuation to the symmetrized tube $\mathcal{T}_n^S$ and satisfy there the commutation condition**

$$w(z_{\pi 1}, \ldots, z_{\pi n}) = \epsilon_F(\pi) w(z_1, \ldots, z_n), \ \pi \in S_n. \tag{9.39}$$

Scheme of the proof. The line of argument is the same as in the proof of Theorem 9.1. Suppose that the field $\phi^{(\kappa_j)}$ transforms according to the representation $V_j(\underset{\sim}{\Lambda}) \equiv V_j(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})$ of $SL(2, C)$; we set

$$V^{(1,\ldots,n)}(\Lambda_1, \Lambda_r) = V_1(\Lambda_l, \Lambda_r) \otimes \ldots \otimes V_n(\Lambda_l, \Lambda_r).$$

For $\pi \in S_n$, $\Lambda_l, \Lambda_r \in SL(2, C)$ we introduce the holomorphic function of $z \in \pi\Lambda(\Lambda_l, \Lambda_r)^{-1}\mathcal{T}_n^-$:

$$f^{(\Lambda_l,\Lambda_r;\pi)}(z) = \epsilon_F(\pi) V^{(\pi 1 \ldots \pi n)}(\Lambda_l, \Lambda_r) w^{(\kappa_{\pi 1} \ldots \kappa_{\pi n})}(\pi^{-1}\Lambda(\Lambda_l, \Lambda_r) z).$$

The theorem will be proved if we can verify that the following compatibility conditions hold (for all $\pi_1, \pi_2 \in S_n$, $\Lambda_l', \Lambda_r', \Lambda_l'', \Lambda_r'' \in SL(2, C)$):

$$f^{(\Lambda_l',\Lambda_r';\pi_1)}(z) = f^{(\Lambda_l'',\Lambda_r'';\pi_2)} \quad \text{for } z \in \pi_1\Lambda(\Lambda_l', \Lambda_r')^{-1}\mathcal{T}_n^- \bigcap \pi_2\Lambda(\Lambda_l'', \Lambda_r'')^{-1}\mathcal{T}_n^-.$$

Exercise 9.16. Prove that the compatibility conditions are equivalent to the reduced compatibility conditions (for all $\pi \in S_n$, $\Lambda_l, \Lambda_r \in SL(2, C)$):

$$f^{(\Lambda_l,\Lambda_r;\pi)}(z) = w^{(\kappa_1 \ldots \kappa_n)}(z) \quad \text{for } z \in \mathcal{T}_n^- \bigcap \pi\Lambda(\Lambda_l, \Lambda_r)^{-1}\mathcal{T}_n^-. \tag{9.40}$$

[Hint: See Exercise 9.4; also use (9.38).]

For the proof of the reduced compatibility condition (9.40) we regard $f^{(\Lambda_1,\Lambda_r;\pi)}(z)$ as a holomorphic function $f^{(\pi)}(z; \Lambda_l, \Lambda_r)$ of the vector $z \in \mathcal{T}_n^-$ and of the matrices $\Lambda_j, \Lambda_r \in SL(2, C)$, defined on the open set

$$\mathcal{O}^{(\pi)} = \{(z; \Lambda_l, \Lambda_r) \in \mathcal{T}_n^- \times SL(2, C) \times SL(2, C) : z \in \pi\Lambda(\Lambda_l, \Lambda_r)^{-1}\mathcal{T}_n^-\}. \tag{9.41}$$

In the next three lemmas we prove the connectedness (and non-emptiness) of the $\mathcal{O}^{(\pi)}$ and the fact that for any $\pi \in S_n$, the equality

$$f^{(\pi)}(z; \Lambda_l, \Lambda_r) = w(z) \tag{9.42}$$

holds in a neighbourhood of some point $(z; \Lambda_l, \Lambda_r) \in \mathcal{O}^{(\pi)}$. In view of the uniqueness theorem for holomorphic functions, this will complete the proof of the reduced compatibility conditions and, hence, the proof of the theorem.

Lemma 9.7. *Let $\mathcal{A}^{(\pi)}$ be the set of elements $(\Lambda_l, \Lambda_r) \in SL(2, C) \times SL(2, C)$ such that the intersection $\mathcal{T}_n^- \cap \pi\Lambda(\Lambda_l, \Lambda_r)^{-1}\mathcal{T}_n^-$ is non-empty; then $\mathcal{A}^{(\pi)}$ is a non-empty connected open subset (that is, a domain).*

■ Lemma 9.7 was proved earlier (see Lemma 9.2) for the case when π is equal to the identity e of S_n. We denote by π_0 the following permutation: $\pi_0(j) = n - j + 1$. Then $\mathcal{A}^{(\pi_0)}$ contains the element $(-1, 1)$ and $\mathcal{A}^{(\pi_0)} = (-1, 1) \cdot \mathcal{A}^{(e)}$; hence this case reduces to the preceding one. It remains to consider the case when $\pi \neq e$, $\pi \neq \pi_0$.

Exercise 9.17. (a) Prove that if $(\Lambda_l', \Lambda_r') \sim (\Lambda_l'', \Lambda_r'')$ (in the sense of Exercise 9.5) and $(\Lambda_l', \Lambda_r') \in \mathcal{A}^{(\pi)}$, then $(\Lambda_l'', \Lambda_r'') \in \mathcal{A}^{(\pi)}$.

(b) Let $(\Lambda_l, \Lambda_r) \sim (C, 1)$, where C has the form (9.18a). Prove that $(\Lambda_l, \Lambda_r) \in \mathcal{A}^{(\pi)}$ only for $\pi = e$. [Hint: Let $z \in \mathcal{T}_n^-$; then the quantity $y_j^0 - y_j^3$ increases as j increases; under the transformation

* In the case of bosonic fields, the commutation property reflects the symmetry of the Wightman functions with respect to permutations of the arguments and indices of the fields. We draw attention to the fact that in (9.39) it is to be understood that the permutations are with respect to both the arguments and the indices of the Wightman functions.

$z \to z' = \Lambda(C,1)^{-1}z$ we have $z_j'^0 - z_j'^3 = z_j^0 - z_j^3$; hence deduce that $y'^0_{\pi^{-1}(j)} - y'^3_{\pi^{-1}(j)}$ increases with increasing j only for $\pi = e$.]

(c) Let $(\Lambda_l, \Lambda_r) \sim (C,1)$, where C has the form (9.18b). Prove that $(\Lambda_l, \Lambda_r) \in \mathcal{A}^{(\pi)}$ only for $\pi = \pi_0$. [Hint: Use the fact that $\mathcal{A}^{(\pi_0)} = (-1,1)\mathcal{A}^{(e)}$.]

(d) Let $\pi \neq e$, $\pi \neq \pi_0$. Prove that each element $(\Lambda_l, \Lambda_r) \in \mathcal{A}^{(\pi)}$ is equivalent to an element of the form $(K, \overline{K}^{-1})$, where K is given by (9.19), α is real and $0 < \beta < \pi$. [Hint: Use Exercise 9.5. The cases $(\Lambda_l, \Lambda_r) \sim (C,1)$, where C is defined either by (9.18a) or (9.18b), is excluded. There remains the case when $\Lambda_l, \Lambda_r) \sim (K, \overline{K}^{-1})$, where K is given by (9.19) with $\alpha \in \mathbf{R}$, $0 \leq \beta \leq \pi$, and $\Lambda(K, \overline{K}^{-1})$ is given by (9.20). Arguing as in parts (b) and (c) of this exercise, we can exclude the cases $\beta = 0$ and $\beta = \pi$, since for $\beta = 0, \Lambda(K, \overline{K}^{-1})$ leaves the zero components of the vectors z_j invariant, while for $\beta = \pi$, $\Lambda(K, \overline{K}^{-1})$ changes the sign of the zero components of the vectors z_j. Finally, for $0 < \beta < \pi$ there exists a point $z \in \mathcal{T}_n^- \cap \pi\Lambda(K, \overline{K}^{-1})^{-1}\mathcal{T}_n^-$; it suffices to take $z_j = (ia_j, 0, 0, b_j)$, where a_j and b_j are real, a_j increases as j increases and $a_j \cos\beta + b_j \sin\beta$ increases as πj increases.]

It follows from Exercise 9.17(d) that $\mathcal{A}^{(\pi)}$ is a non-empty open set and that any element $(\Lambda_l, \Lambda_r) \in$ $\in \mathcal{A}^{(\pi)}$ is equivalent to an element of the form $(K, \overline{K}^{-1})$ with $\alpha \in \mathbf{R}$, $0 < \beta < \pi$. Thus if $\mathcal{K}$ is the set of all such elements of the form $(K, \overline{K}^{-1})$ (with $\alpha \in \mathbf{R}$, $0 < \beta < \pi$), then any element $(\Lambda_l, \Lambda_r) \in \mathcal{A}^{(\pi)}$ can be joined by a continuous curve in $\mathcal{A}^{(\pi)}$ with an element of $\mathcal{K}$ and since, clearly, $\mathcal{K}$ is itself a connected subset of $\mathcal{A}^{(\pi)}$, we have proved that $\mathcal{A}^{(\pi)}$ is connected. ■

Lemma 9.8. *The set $\mathcal{O}^{(\pi)}$ is a domain.*

■ As in the preceding lemma, the cases $\pi = e$ or $\pi = \pi_0$ are trivial (they reduce to Lemma 9.3). Therefore we may suppose that $\pi \neq e$, $\pi \neq \pi_0$. We fix an element K of the form (9.19), where $\alpha \in \mathbf{R}$, $0 < \beta < \pi$. According to Exercise 9.17, $(K, \overline{K}^{-1}) \in \mathcal{A}^{(\pi)}$. Clearly the set of all points of $\mathcal{O}^{(\pi)}$ at which $\Lambda_l = K$, $\Lambda_r = \overline{K}^{-1}$, is a (non-empty) intersection of two convex cones and is therefore connected. It therefore suffices to prove that any point of $\mathcal{O}^{(\pi)}$ can be joined by a continuous curve in $\mathcal{O}^{(\pi)}$ to a point $(z; K, \overline{K}^{-1})$, where $z \in \mathcal{T}_n^- \cap \pi\Lambda(K, \overline{K}^{-1})^{-1}\mathcal{T}_n^-$. The rest of the argument is almost a verbatim repetition of the proof of Lemma 9.3, which we omit. ■

Lemma 9.9. *For any $\pi \in S_n$, equation* (9.42) *holds on some non-empty open subset of $\mathcal{O}^{(\pi)}$.*

■ We fix a real point r of $\pi\mathcal{T}_n$, setting

$$r_j = (0,0,0,b_j),$$

where the b_j are real numbers that increase as πj increases. The point r, along with a neighbourhood of it, is in the domain of analyticity of the function $\epsilon_F(\pi)w^{(\kappa_{\pi 1} \cdots \kappa_{\pi n})}(\pi^{-1}z)$. Moreover, the point r is totally spacelike; it therefore follows from the locality condition that the (generalized) Wightman function $w^{(\kappa_1 \cdots \kappa_n)}(x)$ is equal to $\epsilon_F(\pi)w^{(\kappa_{\pi 1} \cdots \kappa_{\pi n})}(\pi^{-1}x)$ in some real neighbourhood of r and hence can also be continued analytically to a complex neighbourhood of r in which it is equal to $\epsilon_F(\pi)w^{(\kappa_{\pi 1} \cdots \kappa_{\pi n})}(\pi^{-1}z)$. Using the covariance of the Wightman functions with respect to complex Lorentz transformations, we can write this result in the form

$$\epsilon_F(\pi)V^{(\pi 1 \dots \pi n)}(\Lambda_l, \Lambda_r)w^{(\kappa_{\pi 1} \cdots \kappa_{\pi n})}(\pi^{-1}\Lambda(\Lambda_l, \Lambda_r)z) = w^{(\kappa_1 \cdots \kappa_n)}(z) \tag{9.43}$$

for $z \in \mathcal{R}$, $(\Lambda_l, \Lambda_r) \in \mathcal{U}$, where $\mathcal{R}$ is a complex neighbourhood of r and $\mathcal{U}$ is a complex neighbourhood of the identity element in $SL(2,C) \times SL(2,C)$. Since (9.43) is the same as (9.42), our lemma will be proved if we can verify that the set $\mathcal{R} \times \mathcal{U}$ has non-empty intersection with $\mathcal{O}^{(\pi)}$. For this it suffices to show that there exist elements $(z; \Lambda_l, \Lambda_r)$ in $\mathcal{O}^{(\pi)}$ that are arbitrarily close to $(r; 1, 1)$. To this end we set $z_j \equiv z_j(\epsilon) = (ia_j, 0, 0, b_j)$, $\Lambda_l \equiv \Lambda_r(\epsilon) = K$, $\Lambda_r \equiv \Lambda_r(\epsilon) = \overline{K}^{-1}$, where $a_j = j\epsilon^2$ and K is the matrix of the form (9.19) with $\alpha = 0$, $\beta = \epsilon$. It is not difficult to see that for sufficiently small positive ϵ, $(z; \Lambda_l, \Lambda_r) \in \mathcal{O}^{(\pi)}$ (see the hint in Exercise 9.17(d)). Furthermore, $(z; \Lambda_l, \Lambda_r) \to (r; 1, 1)$ as $\epsilon \to +0$. Hence it follows that the intersection of $\mathcal{R} \times \mathcal{U}$ and $\mathcal{O}^{(\pi)}$ is non-empty. ■

Theorem 9.6 has an important corollary (Schwinger, 1958). With each point $x \equiv (x^1, \dots, x^4) \equiv (\mathbf{x}, x^4) \in \mathbf{E} = \mathbf{R}^4$, we associate the point $x' \in \mathbf{CM}$ according to the rule

$$x' = (ix^4, \mathbf{x}) \tag{9.44}$$

and we call *Euclidean points*, points in $\mathbf{CM}^n$ of the form

$$(z_1, \ldots, z_n) = (x'_1, \ldots, x'_n), \tag{9.45}$$

where $x_1, \ldots, x_n \in \mathbf{E}$. If $x_j \neq x_k$ for all $j \neq k$, then such a point is called a *non-exceptional Euclidean point.*

Proposition 9.10. *The symmetrized tube $\mathcal{T}_n^S$ (in which the Wightman functions $w^{(\kappa_1 \ldots \kappa_n)}(z_1, \ldots, z_n)$ are analytic) contains all the non-exceptional Euclidean points.*

■ We begin by considering the simplest situation when all the quantities x_j^4 $(j = 1, \ldots, n)$ are distinct. Let π be a permutation in S_n such that x_j^4 increases as πj increases. Then, clearly, $(x'_1, \ldots, x'_n) \in \pi \mathcal{T}_n^- \subset \mathcal{T}_n^S$. We now turn to the case when at least two of the x_j^4 are the same. We may suppose without loss of generality that

$$x_1^4 \leq x_2^4 \leq \ldots, \leq x_n^4 \tag{9.46}$$

(the general case is obtained by permuting the indices j of the vectors x_j). Consider the finite collection of vectors in $\mathbf{R}^3$

$$\{x_j - x_k : \; j, k = 1, \ldots, n; x_j^4 = x_k^4\}.$$

By hypothesis, all these vectors are non-zero, therefore we can find a three-dimensional unit vector s that is not orthogonal to any of them. It is clearly sufficient to consider the case when s is a vector along the third axis; we then have: $x_j^3 - x_k^3 \neq 0$ if $x_j^4 = x_k^4$. Again without loss of generality, we may suppose that

$$x_j^3 < x_k^3, \quad \text{if } j < k \text{ and } x_j^4 = x_k^4 \tag{9.47}$$

(the general case is obtained by a suitable permutation of the numbers j in each of the groups of vectors x_j with the same coordinate x_j^4). Consider the complex transformation $\Lambda \equiv \Lambda_\beta$ of the form (9.20) for $\alpha = 0$, $\beta > 0$. We then conclude without difficulty from (9.46), (9.47) that for sufficiently small β, the point $\Lambda_\beta(x'_1, \ldots, x'_n)$ belongs to $\mathcal{T}_n^-$, so that $(x'_1, \ldots, x'_n) \in \mathcal{T}_n \subset \mathcal{T}_n^S$. ■

We can apply the geometric arguments used in the proof of this proposition to prove the following stronger result needed in §9.5.

Exercise 9.18. Prove that for any point $(x_1, \ldots, x_n) \in \mathbf{E}^n$, where $x_j \neq x_k$ when $j \neq k$, there exists a unit vector $s \in \mathbf{E}$ such that

$$|(s, x_i - x_j)| \geq c|x_i - x_j| \quad \text{for all } i \neq j,$$

and that there exists a permutation π of the indices $1, \ldots, n$ such that

$$\min_{i=1,\ldots,n-1} (s, x_{\pi(i+1)} - x_{\pi i}) \geq c \min_{i \neq j} |x_i - x_j|; \tag{9.48}$$

here c is a positive constant depending only on n. [Hint: Define the function $f(\{e_{ij}\}_{i<j})$ of the $n(n-1)/2$ unit vectors e_{ij} $(i < j)$ in $\mathbf{E}$ by the formula

$$f(\{e_{ij}\}_{i<j}) = \sup_{s \in \mathbf{E}, |s|=1} \min_{i<j} |(s, e_{ij})|;$$

verify that this function is continuous and strictly positive and, since its domain of definition is a compactum, that $f(\{e_{ij}\}_{i<j}) \geq c > 0$ for some constant c.]

E. Global Nature of Locality

The commutation property (9.39) is a corollary of the locality of the fields. This property is, in fact, equivalent to the axiom of locality, so that Theorem 9.6 has a converse. We restrict ourselves for the sake of simplicity to the case of a single scalar field $\phi(x)$ (when the commutation property (9.39) becomes a condition of symmetry with respect to permutations of the arguments).

Proposition 9.11. *Consider the theory of a scalar field $\phi(x)$ satisfying all the Wightman axioms with the possible exception of the axiom of locality* **W.VI**, *and suppose that all the Wightman functions $w^{[n]}(z_1,\dots,z_n)$ are analytically continued to the tube $\mathcal{T}_n^S$ and are symmetric in $z_1,\dots,z_n$. Then the axiom of locality holds (the axiom of local commutativity in the present instance).*

■ We shall prove that for real values of the argument, the Wightman functions satisfy the condition

$$w(x_1,\dots,x_k,x_{k+1},\dots,x_n)=w(x_1,\dots,x_{k+1},x_k,\dots,x_n) \tag{9.49}$$

for $(x_k-x_{k+1})^2<0$ $(k=1,2,\dots,n-1)$. By hypothesis, we have

$$w(z_1,\dots,z_k,z_{k+1},\dots,z_n)=w(z_1,\dots,z_{k+1},z_k,\dots,z_n) \tag{9.50}$$

in $\mathcal{T}_n^S$. In particular, this equality holds if $z_j-z_{j+1}\in T_1^-$ for $j\neq k$, and

$$z_k-z_{k+1}=x_k-x_{k+1} \tag{9.51}$$

is a real spacelike vector. In fact, such points $(z_1,\dots,z_n)$ lie in the extended tube $\mathcal{T}_n$ and it suffices to verify this for the case when

$$x_k-x_{k+1}=(0,0,0,a),$$

where $a<0$. For this purpose we choose the complex Lorentz transformation $\Lambda\equiv\Lambda_\beta$ in the form (9.20) for $\alpha=0$, $\beta>0$; it is not difficult to verify that $\Lambda_\beta z\in\mathcal{T}_n^-$ for sufficiently small β, from which it follows that $z\in\mathcal{T}_n$. (Similarly it can be proved that when $\beta<0$, $z\in\pi\mathcal{T}_n$, where $\pi:(1,\dots,k,k+1,\dots)\to(1,\dots,k+1,k,\dots,n)$). Assuming (9.51) to hold, we pass to the limit in (9.50) (in the sense of generalized functions with respect to $x_1,\dots,x_n$ in the domain $(x_k-x_{k+1})^2<0$) as $\operatorname{Im}(z_j-z_{j+1})\to 0$ in the cone $\operatorname{Im}(z_j-z_{j+1})\in V^-$, where $j\neq k$. The right and left hand sides of (9.50) will then have as their respective limits, the right and left hand sides of the equation (9.49) which we are trying to prove.

This fact requires explanation. Consider, say,

$$w(x_1,\dots,x_k,x_{k+1},\dots,x_n)\equiv W(\xi_1,\dots,\xi_k,\dots,\xi_{n-1})$$

as a generalized function of the variables $\xi_k=x_k-x_{k+1}$ $(k=1,\dots,n-1)$. We know that $W(\xi_1,\dots,\xi_{n-1})$ is the Fourier transform of a generalized function with support in the cone $(\overline{V}^-)^{\times(n-1)}$ (see (8.34), (8.36)), therefore it is the boundary value (in the sense of generalized functions) of the analytic function $W(\zeta_1,\dots,\zeta_{n-1})$ as $\operatorname{Im}\zeta_j\to 0$, $\operatorname{Im}\zeta_j\in V^-$ $(j=1,\dots,n-1)$ (see Theorem 8.5, which is itself a particular case of Theorem B.7). However, as is easy to see from the proof of Theorem B.7, the order in which the limit $\operatorname{Im}\zeta_j\to 0$ is carried out in the cone $\operatorname{Im}\zeta_j\in V^-$ (or even in the cone $\operatorname{Im}\zeta_j\in\overline{V}^-$), $j=1,\dots,n-1$, is quite immaterial. Therefore in the present situation we can first pass to the limit $\operatorname{Im}\zeta_k\to 0$ (in the cone $\operatorname{Im}\zeta_k\in V^-$, where the $\operatorname{Im}\zeta_j\in V^-$ are kept constant for $j\neq k$). Since we are interested in the region $\xi_k^2<0$, we have (as was mentioned above) so far stayed within the domain of analyticity of the function $W(\zeta_1,\dots,\zeta n-1)$. By passing to the limit $\operatorname{Im}\zeta_j\to 0$ in the cone $\operatorname{Im}\zeta_j\in V^-$ $(j\neq k)$, we then obtain the required relation

$$W(\xi_1,\dots,\xi_k,\dots,\xi_{n-1})=\lim_{\substack{\operatorname{Im}\zeta_j\to 0,\ \operatorname{Im}\zeta_j\in V^-\\ j\neq k}} W(\zeta_1,\dots,\zeta_{k-1},\xi_k,\zeta_{k+1},\dots,\zeta_{n-1}). \tag{9.52}$$

The right hand sides of (9.49), (9.50) are dealt with in similar fashion:

$$\begin{aligned} &W(\xi_1,\dots,\xi_{k-1}+\xi_k,-\xi_k,\xi_k+\xi_{k+1},\dots,\xi_{n-1})=\\ &=\lim_{\substack{\operatorname{Im}\zeta_j\to 0,\ \operatorname{Im}\zeta_j\in V^-\\ j\neq k}} W(\zeta_1,\dots,\zeta_{k-1}+\xi_k,-\xi_k,\xi_k+\zeta_{k+1},\dots,\zeta_{n-1}). \end{aligned} \tag{9.53}$$

This completes the proof of condition (9.49). ■

The arguments used in Proposition 9.11 are also applicable for the proof of the global nature of locality (which will also be stated below as applied to the case of a single scalar field). Prior to this we give a note of explanation. We have already mentioned that the postulate of local commutativity is one of the most restrictive principles of quantum field theory. Misgivings could arise concerning the experimental justification of this postulate: we have no special reason for supposing that the measurement of a component of a Hermitian field at some point has no influence on the value of the components of this field at another point separated from the first point by an arbitrarily small spacelike interval. It turns out, however, that the property of local commutativity can be *proved* if we make the assumption, which at a first glance seems to be weaker, that the fields commute only at sufficiently large spacelike separations. It follows from this that if the remaining requirements of relativistic quantum theory hold in the non-local theory, then, roughly speaking, the commutator of the fields must be non-zero everywhere. It is therefore no surprise that the attempts to introduce "non-locality in the small" at the same time require a rejection of some other requirements of the Wightman formalism, for example, "renormalizability" (that is, the conditions that the $\tilde{w}^{[n]}(p_1,\ldots,p_n)$ be tempered generalized functions).

Proposition 9.12. *Suppose that in the theory of a scalar field $\phi(x)$, all the Wightman axioms hold with the possible exception of locality, instead of which the (at first glance) weaker condition is postulated, namely that $[\phi(x),\phi(y)] = 0$, when x and y vary in some (arbitrarily small) domains with spacelike separation. Then the field $\phi(x)$ also satisfies the locality condition (more precisely, local commutativity).*

■ In view of the translation-covariance of the fields, we may suppose that the equality $[\phi(x), \phi(y)] = 0$ holds when $x - y$ varies in some domain $\mathcal{O} \subset \mathrm{M}$ of spacelike vectors. Thus we start with the conditions

$$w(x_1,\ldots,x_k,x_{k+1},\ldots,x_n) = w(x_1,\ldots,x_{k+1},x_k,\ldots,x_n) \tag{9.54}$$

for $x_k - x_{k+1} \in \mathcal{O}$ $(k = 1,\ldots,n-1)$. We are required to prove that (9.54) holds for all spacelike $x_k - x_{k+1}$. Consider the two Wightman functions

$$w(z_1,\ldots,z_k,z_{k+1},\ldots,z_n),\ w(z_1,\ldots,z_{k+1},z_k,\ldots,z_n). \tag{9.55}$$

We saw in the proof of Proposition 9.11 that the points $(z_1,\ldots,z_n)$ belong to $\mathcal{T}_n \cap \pi\mathcal{T}_n$ if $\mathrm{Im}(z_j - z_{j+1}) \in$ $\in V^-$ for $j \neq k$ and $z_k - z_{k+1} = \xi_k$ is a real spacelike vector; consequently such points z are in the domain of analyticity of both functions (9.55). We rewrite (9.54) in terms of the distances $\xi_j = x_j - x_{j+1}$.

$$W(\xi_1,\ldots,\xi_{k-1},\xi_k,\xi_{k+1},\ldots,\xi_{n-1}) = W(\xi_1,\ldots,\xi_{k-1}+\xi_k,-\xi_k,\xi_k+\xi_{k+1},\ldots,\xi_{n-1}) \tag{9.56}$$

for $\xi_k \in \mathcal{O}$, it being assumed that this equation is smoothed with an arbitrary test function (in the variable ξ_k) from $\mathcal{D}(\mathcal{O})$. Then both sides of (9.56) are boundary values with respect to the remaining variables ξ_j $(j \neq k)$, of functions that are analytic in the tube $\mathrm{Im}\,\zeta_j \in V^-$ $(j \neq k)$ (see (9.52) and (9.53)). It follows from the uniqueness theorem B.10 that the functions themselves on the right hand sides of (9.52), (9.53) must be equal (at least for $\xi_k \in \mathcal{O}$). Thus we have proved that the functions (9.55) are equal if $\mathrm{Im}(z_j - z_{j+1}) \in V^-$ for $j \neq k$ and $z_k - z_{k+1} \equiv \xi_k \in \mathcal{O}$. But since both functions (9.55) are analytic in $\mathcal{T}_n \cap \pi\mathcal{T}_n$, it follows that they are equal in the connected component of $\mathcal{T}_n \cap \pi\mathcal{T}_n$ containing a point z with $\mathrm{Im}(z_j - z_{j+1}) \in V^-$ for $j \neq k$ and $z_k - z_{k+1} \in \mathcal{O}$. In particular, the functions (9.55) are equal for all z such that $\mathrm{Im}(z_j - z_{j+1}) \in V^-$ for $j \neq k$ and $z_k - z_{k+1}$ is a real spacelike vector. It then remains to use the arguments employed in the proof of Proposition 9.11 and conclude that (9.54) holds for $(x_k - x_{k+1})^2 < 0$. ■

We can suppose without loss of generality that in the condition of Proposition 9.12,

$$[\phi(x),\phi(y)] = 0 \quad \text{for } -l_1^2 < (x-y)^2 < l^2 \tag{9.57}$$

(where $l_1 < l$ are positive numbers), since (9.57) follows immediately from the condition of Proposition 9.12 and the Poincaré-invariance of the fields. Another result was proved in Vladimirov's book [V4],§29.6; see also his earlier paper (1960) and Petrina (1961). It follows from translation-invariance, the spectrum condition and (9.57) that ordinary commutativity holds: $[\phi(x), \phi(y)] = 0$ for $(x-y)^2 < 0$. This result is the strongest possible since it does not presuppose covariance with respect to the Lorentz group* $L_+^\uparrow$, but follows from the theorem on C-convex hulls in the theory of functions of several complex variables, proved by Vladimirov in the above-mentioned works.

Remark. Proposition 9.11 suggests the following approach to (possibly) "non-renormalizable" quantum field theories. Consider a theory (for definiteness, of a single scalar Hermitian field ϕ) in which the requirement that the field $\phi(x)$ be an operator-valued generalized function over the space $\mathcal{S}(\mathrm{M})$ is replaced by the weaker requirement: the field in the momentum space $\tilde{\phi}(p)$ is an operator-valued distribution over $\mathcal{D}(\mathrm{M})$. All the Wightman axioms apart from the locality axiom go over to such a theory without any difficulty. Thus in terms of the Wightman functions, condition w.1 (§8.3.A) is now replaced by the condition $\tilde{w}^{[n]}(p_1, \ldots, p_n) \in \mathcal{D}'(\mathrm{M}^n)$; the other characteristic properties w.2–w.6 of the Wightman functions are similarly rewritten in terms of $\tilde{w}^{[n]}(p_1, \ldots, p_n)$. The corresponding reconstruction theorem gives the field ϕ as an operator-valued distribution $\tilde{\phi}(p)$ over $\mathcal{D}(\mathrm{M})$ in the momentum space. However, there is as yet no analogue of the locality axiom in such a theory (since the x-space has not yet been introduced). In order to ensure that we can go over to the coordinate space and have the possibility of postulating, more or less, the usual locality (without any "non-locality in the small") we introduce the following notion. We say that a quantum field theory model has pre-exponential growth (in p-space) if the distributions $\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1}) \in \mathcal{D}'(\mathrm{M}^{n-1})$ have growth that is slower than any $\exp(\epsilon|q|)$ (where $|q|$ is the Euclidean norm of the vector $q \equiv (q_1, \ldots, q_{n-1}) \in \mathrm{M}^{n-1}$); more precisely, this means that for any $\epsilon > 0$

$$[\cosh(\epsilon|q|)]^{-1}\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1})$$

is a tempered generalized function (that is, belongs to $\mathcal{S}'(\mathrm{M}^{n-1})$). Taking into account the spectrum property, we obtain

$$[\cosh(\epsilon|q|)]^{-1}\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1}) \in \mathcal{S}'(\mathrm{M}^{n-1}|(\overline{V}^+)^{n-1}) \quad \text{for any } \epsilon > 0.$$

It is not difficult to verify that the C^∞-functions $[\cosh(\epsilon|q|)]\exp(\sum_{j=1}^{n-1} p_j\eta_j)$ (for any $\eta \in (V^-)^{n-1}$ and some $\epsilon = \epsilon(\eta) > 0$) and $[\cosh(\epsilon|q|)]^{-1}\exp(-\sum_{j=1}^{n-1} p_j\eta_j)$ (for any $\epsilon > 0$ and some $\eta = \eta(\epsilon) \in (V^-)^{n-1}$) are multiplicators in the space $\mathcal{S}'(\mathrm{M}^{n-1}|(\overline{V}^+)^{n-1})$. Therefore the condition of pre-exponential growth means that

$$\exp\Big(\sum_{j=1}^{n-1} p_j\eta_j\Big)\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1}) \in \mathcal{S}'(\mathrm{M}^{n-1}|(\overline{V}^+)^{n-1})$$

for all $\eta \equiv (\eta_1, \ldots, \eta_{n-1}) \in (V^-)^{n-1}$. According to §B.1, this is equivalent to the property that the distributions $\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1})$ and $\tilde{w}^{[n]}(p_1, \ldots, p_n)$ have Laplace transforms $W^{[n]}(\zeta_1, \ldots, \zeta_{n-1})$ and $w^{[n]}(z_1, \ldots, z_n)$, which are analytic in the tubes $\mathcal{T}_{n-1}^-$ (8.41) and $\mathcal{T}_n^-$ (8.43) respectively.

We can now state the locality condition in a theory of pre-exponential growth: it is required that (for any $n = 1, 2, \ldots$) the Wightman functions $w^{[n]}(z_1, \ldots, z_n)$ have an analytic continuation to the symmetrized tubes $\mathcal{T}_n^S$ and be symmetric in $z_1, \ldots, z_n$.

There is an alternative equivalent statement of locality in such theories: one postulates the existence of generalized boundary values

$$w^{[n]}(x_1, \ldots, x_n) = \lim_{y \equiv \mathrm{Im}\, z \to 0,\ y \in \mathrm{Im}\, \mathcal{T}_n^-} w^{[n]}(z_1, \ldots, z_n)$$

in $\mathcal{D}'(\mathcal{O}_n)$, where $\mathcal{O}_n$ is the set of all totally spacelike points of M^n, these generalized boundary values being symmetric with respect to permutations of the argument.

The equivalence of the two definitions of locality follows from the following considerations. According to a result of Ruelle (1959) (see [J3], Theorem on p.88), totally spacelike points of M^n are contained in the envelope of holomorphy of the symmetrized tube $\mathcal{T}_n^S$, therefore it follows from the

* The only vestiges of Lorentz-covariance are in the spectrum condition.

condition of locality in the first sense that locality holds in the second sense. Conversely, suppose that locality holds in the second sense. According to the modified Streeter theorem (1962a) (see [J3],Theorem on p.97), it follows (even without supposing the fields to be Lorentz-covariant) that the Wightman functions $w^{[n]}(z_1,\ldots,z_n)$ can be continued analytically to the extended tubes $\mathcal{T}_n$. By virtue of the symmetry of the boundary values in the class $\mathcal{D}'(\mathcal{O}_n)$, the families of functions $\{w^{[n]}(z_{\pi 1},\ldots,z_{\pi n})\}$ (where π runs through the permutations of the indices $1,\ldots,n$) in general form many-valued analytic functions in the symmetrized tubes $\mathcal{T}_n^S$, and since the $\mathcal{T}_n^S$ are simply connected domains, (Tomozawa, 1963b), the locality condition holds in the first sense.

As we saw in Proposition 9.11, the locality condition stated above (in the first sense) goes over to the ordinary locality condition in models of Wightman type (which are "renormalizable" by definition, that is, the $\tilde{w}^{[n]}(p_1,\ldots,p_n)$ are temperate generalized functions in them). Therefore all the Wightman fields are special cases of local models of pre-exponential growth (in p-space).

9.2. *TCP*-Theorem

A. *TCP*-INVARIANCE

In §8.4 we considered the discrete symmetries P, C, T for free fields. In this case the TCP-operator Θ can be defined as the anti-unitary operator equal to the product of the operations T, C and P in the space of state vectors:

$$\Theta = U(I_t)U_cU(I_s). \tag{9.58}$$

Starting from a specified law of transformation of the fields under T-, C- and P-transformations, it is not difficult to find conditions on the Wightman functions making the theory T-, C-, P- or TCP-invariant. A remarkable result of the Wightman theory is that although each of the symmetries T, C or P imposes certain new restrictions on the Wightman functions, the symmetries with respect to the TCP transformations are a corollary of the basic requirements of local quantum field theory and impose no such new restrictions. Thus we can conceive of a theory in which there is TCP-symmetry but which is not invariant* with respect to T, C and P separately. (Furthermore, as experimental data show, this situation holds in reality, if we take into account all the types of elementary particle interaction; in this connection see, for example, Lee, 1966,1967, Arbuzov and Filippov, 1966, Okun', 1966; references to the earlier original papers can be found in the monograph [K1].) In this case the definition of the operator Θ as the product of the operators $U(I_t)$, U_c, $U(I_s)$ should be regarded merely as a guiding consideration in the construction of Θ.

Another interesting feature of the TCP-Theorem is that it prescribes a certain universal law of transformation of fields under the TCP-transformation.** As is well known, *a priori* some freedom is allowed in this kind of transformation (which reduces, say, to a variation in the phase factors in the law of transformation of fields that transform according to irreducible representations of the Lorentz group). In principle, the Wightman theory also allows deviation from the above-mentioned

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* As in the case of internal symmetries (see §11.3), the term "non-invariance" can mean that either the specified symmetry of the field algebra does not exist, or such a symmetry is not realizable (anti-)unitarily.

** Here the normal connection between spin and statistics is assumed. (Otherwise in the law of the TCP-transformation of fields that transform according to irreducible representations of $SL(2,C)$, the additional factors ± 1 would appear, as follows from the Klein transformation formulae in §9.3.)

universal law of transformation of the fields; however, such freedom is available only in the presence of a suitable internal symmetry of the field algebra.

We illustrate this by the example of the free Dirac field. In this case, we have for the operator Θ (according to Exercise 8.30)

$$\begin{aligned}\Theta\psi(x)\Theta^{-1} &= \eta\gamma^5\psi^*(-x) = \eta\gamma^5\gamma^0\tilde{\psi}(-x),\\ \Theta\tilde{\psi}(x)\Theta^{-1} &= \bar{\eta}\gamma^5\gamma^0\psi(-x),\end{aligned} \tag{9.59}$$

where $\eta = \bar{\eta}_t\bar{\eta}_c\bar{\eta}_s$ ($\eta^4 = 1$). The group of gauge transformations enables us to choose the phase η in (9.59) equal to unity (if $\eta \neq 1$, then we can redefine Θ by introducing a new operator Θ' equal to the product of Θ and the operator of the corresponding gauge transformation). We consider the two-component formalism

$$\psi = \begin{pmatrix} \xi^\alpha \\ \chi_{\alpha'} \end{pmatrix}$$

in the representation in which the matrix γ^5 is diagonal (see Appendix E); the formulae (9.59) (for $\eta = 1$) then reduce to the form

$$\begin{aligned}\Theta\begin{pmatrix} \xi^\alpha(x) \\ \chi_{\alpha'}(x)\end{pmatrix}\Theta^{-1} &= \begin{pmatrix} -i\xi^\alpha(-x)^* \\ i\chi_{\alpha'}(-x)^*\end{pmatrix},\\ \Theta\begin{pmatrix} \chi^*_\alpha(x) \\ \xi^{*\alpha'}(x)\end{pmatrix}\Theta^{-1} &= \begin{pmatrix} -i\chi_{\alpha'}(-x) \\ i\xi^\alpha(-x)\end{pmatrix}.\end{aligned} \tag{9.60}$$

This is the law of transformation of spinors with spin 1/2 we are choosing in the TCP-Theorem.

In order to clarify the law of transformation of arbitrary spin-tensor fields under the TCP-operation, we consider the spin-tensor fields consisting of free spinor fields, for example the field

$$\phi^{\alpha_1\ldots\alpha_m}_{\beta_1'\ldots\beta_n'}(x) = :\ \xi^{(1)\alpha_1}(x)\ldots\xi^{(m)\alpha_m}(x)\chi^{(1)}_{\beta_1'}(x)\ldots(\chi)^{(n)}_{\beta_n'}(x) :\,;$$

here we have introduced a finite collection of free Dirac fields $\psi^{(1)}$, $\psi^{(2)}, \ldots$ in order to obtain fields ϕ with an arbitrary number of primed and unprimed indices (where distinct Dirac fields anticommute). It is clear that under the action of the proper Lorentz group (or $SL(2,C)$) the field $\phi(x)$ transforms according to the tensor product of m copies of the representation $\mathfrak{D}^{(1/2,0)}$ and n copies of $\mathfrak{D}^{(0,1/2)}$. Starting from the law of transformation (9.60) for the Dirac spinors and taking into account the anticommutativity of the spinor fields under the law of the normal product, it is not difficult to verify that

$$\Theta\phi^{\alpha_1\ldots\alpha_m}_{\beta_1'\ldots\beta_n'}(x)\Theta^{-1} = \epsilon(-i)^m i^n \phi^{\alpha_1\ldots\alpha_m}_{\beta_1'\ldots\beta_n'}(-x)^*,$$

where ϵ is the parity of the permutation $(1,2,\ldots,m+n) \to (m+n,\ldots,2,1)$:

$$\epsilon = \begin{cases} (-1)^{(m+n)/2}, & \text{if } m+n \text{ is even},\\ (-1)^{(m+n-1)/2}, & \text{if } m+n \text{ is odd}.\end{cases}$$

Thus,

$$\Theta\phi^{\alpha_1\ldots\alpha_m}_{\beta_1'\ldots\beta_n'}(x)\Theta^{-1} = (-1)^m i^F \phi^{\alpha_1\ldots\alpha_m}_{\beta_1'\ldots\beta_n'}(-x)^*, \tag{9.61}$$

where F is the fermion number of ϕ (that is, $F = 0$ or 1 according as $m + n$ is even or odd).

Formula (9.61) is again the universal law of transformation of the fields in the TCP-Theorem (we see that it is chosen as if all the fields consisted of Dirac spinors).

We now state the TCP-Theorem.

Theorem 9.13 (TCP-Theorem). *In the Wightman theory with normal connection between spin and statistics, there exists a (unique) anti-unitary operator* Θ, *called the* TCP*-operator, that leaves the vacuum vector* Ψ_0 *invariant and acts according to the formula*

$$\Theta\phi^{(\kappa)}(x)\Theta^{-1} = (-1)^{2j} i^{F^{(\kappa)}} \phi^{(\kappa)}(-x)^* \tag{9.62a}$$

*or, equivalently,**

$$\Theta\phi^{(\kappa)}(x)\Theta^{-1} = i^{F^{(\kappa)}} (V^{(\kappa)}(-1,1)\phi^{(\kappa)}(-x))^* \tag{9.62b}$$

on a field that transforms according to the irreducible representation $\mathfrak{D}^{(j,k)}$ *of* $SL(2,C)$; *here* $F^{(\kappa)}$ *is the spinor number of the field* $\phi^{(\kappa)}$:

$$F^{(\kappa)} = \begin{cases} 0, & \textit{if } j+k \textit{ is an integer,} \\ 1, & \textit{if } j+k \textit{ is a half-integer.} \end{cases} \tag{9.63}$$

■ In order that an operator Θ with the required properties exist, it is necessary and sufficient that the Wightman functions satisfy the relation

$$w^{(\kappa_n \ldots \kappa_1)}(-x_n, \ldots, -x_1) = (-1)^{2J}(-i)^F w^{(\kappa_1 \ldots \kappa_n)}(x_1, \ldots, x_n); \tag{9.64}$$

here $J = j_1 + \ldots + j_n$; $F = F^{(\kappa_1)} + \ldots + F^{(\kappa_n)}$. The sketch proof of this criterion given below is a modification of the standard argument used in the appendix to the GNS Theorem (see Proposition 1.30), applied to the case of a symmetry that is realized anti-unitarily.

The necessity of (9.64) is established by applying the vacuum expectation value to the operator $\Theta\phi^{(\kappa_1)}(x_1)\ldots\phi^{(\kappa_n)}(x_n)\Theta^{-1}$ and using conditions (9.62) (as well as the invariance of the vacuum with respect to the anti-unitary operator Θ). For the proof of the sufficiency, we associate with an arbitrary monomial A of the form (8.8) the monomial A' obtained from A by means of the substitution $\phi_l^{(\kappa)}(x) \to (-1)^{2j} i^{F^{(\kappa)}} \phi_{\bar{l}}^{(\bar{\kappa})}(-x)$, $f(x) \to \overline{f(x)}$. It then follows from (9.64) that

$$\langle A_2\Psi_0, A_1\Psi_0\rangle = \langle A_1'\Psi_0, A_2'\Psi_0\rangle,$$

from which it clearly follows that there exists an anti-unitary operator Θ such that $\Theta A\Psi_0 = A'\Psi_0$.

To complete the proof of the theorem, it remains to verify (9.64); in fact we noted earlier (see Exercise 9.12(b)) that these relations are a corollary of the Wightman axioms. ■

Exercise 9.19. Prove that the operator Θ is related to the representation $U(a,\underset{\sim}{\Lambda})$ of the Poincaré spinor group $\mathfrak{P}_0$ by the formulae: $\Theta U(a,\underset{\sim}{\Lambda})\Theta^{-1} = U(-a,\underset{\sim}{\Lambda})$, $\Theta^2 = U(0,-1)$.

Exercise 9.20. Let A be an element of $\mathcal{P}(\mathcal{O})$ of the form (8.8) and let f run through the whole of $\mathcal{S}(\mathbf{M}^n)$. Prove that the closure of the set of vectors of the form $A^*\Psi_0$ is obtained from the closure of the set of vectors of the form $A\Psi_0$ by applying the TCP-operator Θ. [Hint: Use the TCP-Theorem and Exercise 8.3.]

* The formula written down in this way is applicable to Bose and Fermi fields that transform according to arbitrary (possibly reducible) finite-dimensional representations $V^{(\kappa)}$ of $SL(2,C)$.

B. WEAK LOCALITY

The proof of the TCP-Theorem was based on the relations (9.64), which in turn were deduced from the relations

$$\langle 0|\phi^{(\kappa_n)}(x_n)\dots\phi^{(\kappa_1)}(x_1)|0\rangle = i^F\langle 0|\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_n)}(x_n)|0\rangle, \tag{9.65}$$

where $(x_1,\dots,x_n)$ is an arbitrary Jost point and F is the number of fields with half-integral spin in the set $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$ (see Exercise 9.12). The relation (9.65) is weaker than the requirement of locality of the fields: they are called the *weak locality* condition of the fields. Thus we have the following modification of the TCP-Theorem due to Jost (1957).

Theorem 9.14 (Modified TCP-theorem). *In a quantum field theory in which all the Wightman axioms hold with the possible exception of locality, the weak locality condition is equivalent to the existence of a TCP-operator Θ satisfying the relations* (9.62) *and leaving the vacuum vector invariant.*

It is not difficult to see from the next example that the condition of weak locality is indeed weaker than locality. Let $\phi(x)$ be a free scalar Hermitian field of mass m

$$\phi(x) = \int_{\Gamma_m^+}(a^*(p)e^{ipx} + a(p)e^{-ipx})(dp)_m,$$

where this time, the operator-valued generalized functions $a(p)$ and $a^*(p)$ satisfy the following anticommutation relations (instead of the commutation relations (8.65)):

$$[a(p),a(q)]_+ = 0, \quad [a(p),a^*(q)]_+ = 2\omega(\mathbf{p})(2\pi)^3\delta(\mathbf{p}-\mathbf{q}).$$

Then the field $\phi(x)$ is not local; both the commutator and the anticommutator

$$[\phi(x),\phi(y)]_+ = D^{(1)}(x-y) \equiv \int 2\pi\delta(p^2-m^2)e^{ip(x-y)}d_4p$$

are non-zero for spacelike intervals $x-y$. Nevertheless, there clearly exists an anti-unitary operator Θ with the properties

$$\Theta\Psi_0 = \Psi_0, \quad \Theta\phi(x)\Theta^{-1} = \phi(-x), \quad \Theta^2 = 1.$$

Consequently, according to Theorem 9.14, the Wightman functions satisfy the weak local commutativity condition (that is, weak locality is applicable to fields with integral spin).

Exercise 9.21. Verify directly that the condition of weak local commutativity holds for the above model. [Hint: The Wightman functions have an expansion in pairings of type (8.120) for the Dirac fields; now verify that

$$W^{[2]}(x-y) - W^{[2]}(y-x) = 0 \quad \text{for } (x-y)^2 < 0;$$

then use the fact that every odd Lorentz-invariant function in M vanishes at a spacelike argument.]

C. BORCHERS CLASSES; THE NOTION OF A LOCAL COMPOSITE FIELD

The modified TCP-Theorem found an interesting application in the study of classes of fields that are local among themselves (Borchers, 1960), which provided a fresh approach to the problem of defining a local composite field. Roughly speaking, the irreducibility of the fields (proved in §8.2.C) implies that every linear operator in $\mathcal{H}$ is a functional of the fields, so that the general notion of a functional of fields becomes trivial; for this reason, the notion of a local functional is of greater interest. The fact is that quantum fields do not exist at a point taken in isolation, and the naïve definition, say, of the square $\phi^2(x)$ of the scalar field $\phi(x)$ is devoid of meaning. The general definition of a local composite field is as follows. Let $\phi \equiv \{\phi^{(\kappa)}\}_{\kappa \in K}$ be a system of Wightman quantum fields, for the sake of definiteness, with the normal connection between spin and statistics. Let $\psi \equiv \{\psi^{(\tau)}\}_{\tau \in T}$ be a second system of tensor or spin-tensor operator-valued generalized functions acting in the same space. Then we say that the fields $\psi^{(\tau)}$ are *local composite fields* with respect to the basis fields ϕ if the extended collection of fields obtained by adjoining the fields $\psi^{(\tau)}$ to the original system of fields ϕ, again satisfies the Wightman axioms W.I–W.VIII (with the previous domain of definition D).

In order to justify this definition, it is important to satisfy ourselves that if ψ' is a field that is also a local composite field of the fields ϕ, then ψ and ψ' are mutually local and hence, the collection of fields obtained by combining ψ and ψ' with the original system of fields ϕ also satisfies the Wightman axioms. This property (see Proposition 9.17 below) is called *transitivity of the local condition* (Borchers, 1962). Any collection of local composite fields of the system of fields ϕ that contains this original system ϕ is called a *Borchers extension* of the system of fields ϕ. The totality of all local composite fields for a given system of fields ϕ is called the *Borchers class* of the fields ϕ. Thus the Borchers class is the maximal extension of the system of fields ϕ that satisfies all the Wightman axioms (and such a maximal extension is unique).

Examples of a local composite field are furnished by the Wick polynomials of a free field (§8.4.A). Epstein (1963) proved the stronger result: the Wick polynomials exhaust the Borchers class of a free field. (In this connection, see also the result of Baumann, 1982.)

We now turn to the derivation of Borchers' result on the transitivity of the property of locality and of weak locality.

Proposition 9.15. *Let $\phi \equiv \{\phi^{(\kappa)}\}$ be a system of quantum fields satisfying all the Wightman axioms with the possible exception of locality, instead of which weak locality is postulated. Let $\psi(x)$ be another quantum field such that the union of the collection of fields $\{\phi^{(\kappa)}\}$ and $\{\psi\}$ satisfy all the Wightman axioms with the possible exception of locality. Suppose that the relations of the form*

$$\langle 0|\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_k)}(x_k)\psi(x)\phi^{(\kappa_{k+1})}(x_{k+1})\dots\phi^{(\kappa_n)}(x_n)|0\rangle =$$

$$= i^F\langle 0|\phi^{(\kappa_n)}(x_n)\dots\phi^{(\kappa_{k+1})}(x_{k+1})\psi(x)\phi^{(\kappa_k)}(x_k)\dots\phi^{(\kappa_1)}(x_1)|0\rangle \tag{9.66}$$

*hold at all the Jost points $(x_1,\dots,x,\dots,x_n) \in \mathcal{T}_{n+1}$ (for all $n = 1,2,\dots$ and for any types $\kappa_1,\dots,\kappa_n$ of fields); here F is the number of spinor fields in the monomial $\phi^{(\kappa_1)}\dots\phi^{(\kappa_n)}\psi$. Then the fields ϕ and ψ have a common TCP-operator Θ; therefore $\psi(x)$ is weakly local and the fields $\phi^{(\kappa)}(x)$ and $\psi(x)$ are mutually weakly local.**

■ It follows from (9.66) that the equality

$$\langle 0|\phi^{(\kappa_n)}(x_n)\dots\phi^{(\kappa_{k+1})}(x_{k+1})\psi(x)\phi^{(\kappa_k)}(x_k)\dots\phi^{(\kappa_1)}(x_1)|0\rangle =$$

* The fields ϕ and ψ are said to be *mutually weakly local* if the vacuum expectation values of products of any number of fields of ϕ and ψ satisfy the weak locality condition.

$$= (-1)^{2J}(-i)^F\langle 0|\phi^{(\kappa_1)}(-x_1)\dots\psi(-x)\dots\phi^{(\kappa_n)}(-x_n)|0\rangle \tag{9.67}$$

holds in $\mathbf{M}^{n+1}$ (see the argument in Exercise 9.12). Let Θ be a TCP-operator for the field ϕ and let

$$\Phi = \int \phi^{(\kappa_k)}(x_k)\dots\phi^{(\kappa_1)}(x_1)f(x_1,\dots,x_k)d^4x_1\dots d^4x_k|0\rangle,$$

$$\Psi = \int \phi^{(\kappa_{k+1})}(-x_{k+1})\dots\phi^{(\kappa_n)}(-x_n)g(x_{k+1},\dots,x_n)d^4x_{k+1}\dots d^4x_n|0\rangle;$$

such vectors Φ (or Ψ) form a total set in the Hilbert space $\mathcal{H}$. Equality (9.67) is now written in the form

$$\langle\Theta^{-1}\Psi,\psi(x)\Phi\rangle = (-1)^{2j}(-i)^{F^{(\kappa)}}\langle\Theta\Phi,\psi(-x)\Psi\rangle,$$

or, taking into account the fact that Θ is anti-unitary:

$$\overline{\langle\Psi,\Theta\psi(x)\Phi\rangle} = (-1)^{2j}(-i)^{F^{(\kappa)}}\overline{\langle\Psi,\psi(-x)^*\Theta\Phi\rangle}.$$

Hence, Θ is also a TCP-operator for the field ψ:

$$\Theta\psi(x)\Theta^{-1} = (-1)^{2j}i^{F^{(\kappa)}}\psi(-x)^*.$$

It now follows from the modified TCP-Theorem that ψ is weakly local and that ϕ and ψ are mutually weakly local. ∎

Corollary 9.16. The property of mutual weak locality is transitive in the following sense. Let $\phi \equiv \{\phi^{(\kappa)}\}$ be a system of weakly local fields (satisfying all the Wightman axioms except possibly for locality; in particular, it is assumed that the vacuum is cyclic for ϕ); suppose further that the quantum fields $\psi^{(1)}$ and $\psi^{(2)}$ are defined in $\mathcal{H}$ and that they satisfy the conditions of Proposition 9.14 and are hence mutually weakly local with the field ϕ. Then $\psi^{(1)}$ and $\psi^{(2)}$ are mutually weakly local.

In fact by Proposition 9.15, the fields ϕ, $\psi^{(1)}$ and $\psi^{(2)}$ have a common TCP-operator and hence (according to the modified TCP-Theorem) they are all mutually weakly local.

Thus for a given system ϕ of weakly local fields (satisfying all the Wightman axioms, except possibly for locality) we can form the class of all the mutually weakly local fields containing ϕ.* Entering into this class are all the weakly local fields ψ with a common domain of definition and a common TCP-operator.

We will show that the transitivity property also holds for mutually local fields.

Proposition 9.17. *Let $\phi \equiv \{\phi^{(\kappa)}\}$ be a Wightman system of local fields with normal connection between spin and statistics and let $\psi^{(1)}$ and $\psi^{(2)}$ be a pair of quantum fields such that the combined totality of the fields ϕ, $\psi^{(1)}$, $\psi^{(2)}$ satisfies all the Wightman axioms with the possible exception of locality, in place of which it is postulated that each of the fields $\psi^{(1)}$, $\psi^{(2)}$ are mutually local with ϕ:*

$$[\phi^{(\kappa)}(x),\psi^{(1)}(y)]_\pm = 0 = [\phi^{(\kappa)}(x),\psi^{(2)}(y)]_\pm \quad for\ (x-y)^2 < 0$$

(the $\pm$ sign is chosen here in accordance with the rule (8.13), (8.16) *for the normal connection between spin and statistics). Then the fields $\psi^{(1)}$ and $\psi^{(2)}$ are mutually local:*

$$[\psi^{(1)}(x),\psi^{(2)}(y)]_\pm = 0 \quad for\ (x-y)^2 < 0, \tag{9.68}$$

* Note that classes of mutually local fields are equivalence classes in the mathematical sense only if one considers fields (or systems of fields) satisfying the cyclic vacuum condition. (A similar remark needs to be made in connection with the class of all mutually local fields containing ϕ, that is, the Borchers class of the fields ϕ.)

so that the combined system of fields ϕ, $\psi^{(1)}$, $\psi^{(2)}$ satisfies all the Wightman axioms (with the normal connection between spin and statistics).

■ The fields ϕ, $\psi^{(1)}$, $\psi^{(2)}$ satisfy all the conditions in the corollary of Proposition 9.15, therefore they are mutually weakly local. It follows from this and from the locality of the fields ϕ and the mutual locality of each of the fields $\psi^{(1)}$, $\psi^{(2)}$ with ϕ, that the following relations hold at the Jost points $(x_1, \dots, x_k, y, z, x_{k+1}, \dots, x_n) \in \mathcal{J}_{n+2}$:

$$\begin{aligned}\langle 0|\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_k)}(x_k)\psi^{(1)}(y)\psi^{(2)}(z)\phi^{(\kappa_{k+1})}(x_{k+1})\dots\phi^{(\kappa_n)}(x_n)|0\rangle = \\ = (-1)^{F/2}\langle 0|\phi^{(\kappa_n)}(y_n)\dots\phi^{(\kappa_{k+1})}(x_{k+1})\psi^{(2)}(z)\psi^{(1)}(y)\phi^{(\kappa_k)}(x_k)\dots\phi^{(\kappa_1)}(x_1)|0\rangle = \\ = \pm\langle 0|\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_k)}(x_k)\psi^{(2)}(z)\psi^{(1)}(y)\phi^{(\kappa_{k+1})}(x_{k+1})\dots\phi^{(\kappa_n)}(x_n)|0\rangle;\end{aligned} \tag{9.69}$$

here F is the number of fields with half-integral spin in the monomials featuring in these relations; the $\pm$ sign denotes $-$ if $\psi^{(1)}$ and $\psi^{(2)}$ have half-integral spin and $+$ otherwise.

Exercise 9.22. Deduce from (9.69) the equality

$$\langle 0|\phi^{(\kappa_1)}(x_1)\dots\phi^{(\kappa_k)}(x_k)[\psi^{(1)}(y), \psi^{(2)}(z)]_{\mp}\phi^{(\kappa_{k+1})}(x_{k+1})\dots\phi^{(\kappa_n)}(x_n)|0\rangle = 0 \tag{9.70}$$

for any $x_1, \dots, x_n \in \mathbf{M}$ and $(y-z)^2 < 0$. [Hint: Smooth the left hand side of (9.70) with respect to y, z with an arbitrary function $f(y, z)$ in $\mathcal{D}(\mathbf{M}^2)$ with support in a sufficiently small neighbourhood of the point $(a, b) \in \mathbf{M}^2$, where a, b is an arbitrary pair of points with spacelike separation in $\mathbf{M}$; the generalized function so obtained (with respect to the remaining variables $x_1, \dots, x_n$) is then the boundary value of a function which is analytic in the tube

$$\{z \in \mathbf{CM}^n : \operatorname{Im} z_1 \in V^-,\ \operatorname{Im} z_n \in V^+,\ \operatorname{Im}(z_j - z_{j+1}) \in V^- \\ \text{for } j = 1, \dots, k-1 \text{ and } j = k+1, \dots, n-1\};$$

then use the uniqueness theorem B.10.]

Formula (9.68) now follows from (9.70) and the cyclicity of the vacuum with respect to the field ϕ. ■

Corollary 9.18. If the system of fields ϕ satisfies the axioms W.I–W.VIII and ψ is mutually local with ϕ (with the normal connection between spin and statistics), then ψ is itself local. (It suffices to set $\psi^{(1)} = \psi^{(2)} = \psi$ in Proposition 9.15.)

In §12.1.C we shall give another remarkable result of Borchers which states that the S-matrices for mutually local fields are equal. From this it follows in particular that in the theory of a field with non-trivial S-matrix (that is, for $S \neq 1$) the fields ϕ and their asymptotic fields $\phi^{\text{in}}_{\text{out}}$ may not be mutually local (so that the field ϕ may not be in the Borchers class of the free field ϕ^{in}).

Remark. At the beginning of this subsection we required the intuitive notion of local functional of fields. In order that a local composite field can be regarded as a local functional of the basis fields, there must be the assurance that the local composite field in an arbitrarily small neighbourhood $\mathcal{O} \subset \mathbf{M}$ can in some sense be generated by the algebra of the original fields in the same domain. The most natural route for achieving a situation in which the notions of a locally composite field and a local functional of fields can be considered to be on a completely equal footing is to suppose that corresponding to the given system of fields is a net of local field (von Neumann) algebras $\mathcal{F}(\mathcal{O})$ satisfying the condition of duality with torsion (§8.2.B). (In this connection there arises the problem, whether in the case when the original net of local field algebras does not satisfy the condition of duality with torsion, one can use the process of Borchers extension for "saturating" the local field algebras so as to obtain a new net of local field algebras satisfying the condition of duality with torsion.)

9.3. Connection between Spin and Statistics

A. Statement of the Results

As we remarked in §8.2.B, the Wightman theory allows the Bose-Fermi alternative: any two fields $\phi^{(\kappa)}$ and $\phi^{(\kappa')}$ either commute or anticommute at spacelike separation of the arguments:

$$\phi^{(\kappa)}(x)\phi^{(\kappa')}(y) = (-1)^{F(\kappa)F(\kappa') + \omega(\kappa,\kappa')}\phi^{(\kappa')}(y)\phi^{(\kappa)}(x) \quad \text{for } (x-y)^2 < 0; \tag{9.71}$$

here $\omega \equiv \{\omega(\kappa, \kappa')\}$ is a symmetric matrix consisting of zeros and ones; $F^{(\kappa)}$ is the spinor number of the field $\phi^{(\kappa)}$ (see (9.63)). Previously we have not infrequently made a simplifying assumption (Axiom W.VIII): the connection between spin and statistics is normal, that is, the matrix ω in (9.71) is equal to zero. It turns out that the following remarkable result (going back to Pauli, 1940) holds.

Theorem 9.19 (on the connection between spin and statistics). *In the Wightman theory where all the fields either only commute or only anticommute at spacelike separation of the arguments, there is the normal connection between spin and statistics (that is, the matrix ω in (9.71) is equal to zero).*

More generally, the Wightman theory allows anomalous commutation relations between different fields and Theorem 9.19 can be generalized in this case by Lemmas 9.21 and 9.23 below (from which it follows as a particular case). Analysis shows that a theory with anomalous commutation relations possesses an additional discrete symmetry group as a result of which one can redefine the fields so that they satisfy the normal commutation relations. But it is necessary here to make a "technical" assumption: the index κ distinguishing the fields ranges over a finite number of values, say, $1, 2, \dots, N$. (As usual, the index $\bar{\kappa}$ corresponds to the Hermitian adjoint field: $\phi^{(\bar{\kappa})} = \phi^{(\kappa)*}$.)

It is clear that by incorporating the derivatives of the fields with respect to the coordinates and defining the various composite fields (of type corresponding to Wick polynomials of free fields and their analogues, the composite fields of a given Borchers class in the general theory, see §9.2.C), the number of local fields in the Wightman theory can always be taken to infinity. What we have in mind implies the existence of a finite generating set of local fields. The hypothesis of the finiteness of the set of "basis" fields always holds in all the existing physical models of quantum fields.

We now state the main result on the reduction of commutation relations to normal form.

Theorem 9.20. *Given a system of Wightman fields $\phi^{(\kappa)}$ ($\kappa = 1, \dots, N$), which are not identically zero, one can associate with it a system of commuting unitary operators $V^{(\kappa)}$ ($\kappa = 1, \dots, N$) that leave the vacuum vector invariant and have the following commutation properties with the fields:*

$$V^{(\kappa)}\phi^{(\kappa')}(x) = (-1)^{\alpha(\kappa,\kappa')}\phi^{(\kappa')}(x)V^{(\kappa)}; \tag{9.72}$$

here $\alpha \equiv \{\alpha(\kappa, \kappa')\}$ is a matrix of zeros and ones such that

$$\alpha(\kappa, \kappa') + \alpha(\kappa', \kappa) = \omega(\kappa, \kappa') \pmod 2. \tag{9.73}$$

We define new fields by means of the so-called Klein transformation:

$$\phi'^{(\kappa)}(x) = i^{\alpha(\kappa,\kappa)}V^{(\kappa)}\phi^{(\kappa)}(x). \tag{9.74}$$

Then the fields $\phi'(x)$ satisfy all the Wightman axioms (including the Hermitian adjoint condition $\phi'^{(\kappa)*} = \phi'^{(\bar{\kappa})}$); moreover, they have the normal connection between spin and statistics.

We shall prove Theorem 9.20 in §9.3.D after we have made a preliminary study of the properties of the matrix ω.

The next exercise contains the simplest example of a Klein transformation.

Exercise 9.23. Consider a system of fields consisting of a scalar field ϕ and a Dirac spinor ψ, where these fields anticommute at spacelike intervals:

$$[\phi(x), \psi(y)]_+ = 0 \quad \text{for } (x-y)^2 < 0$$

(while the commutator $[\phi(x), \phi(y)]$ and the anticommutator $[\psi(x), \psi(y)]_+$ vanish for $(x-y)^2 < 0$ as in the case of the normal connection between spin and statistics). Prove that the Klein transformation can be defined by the formulae $\psi'(x) = \psi(x)$, $\phi'(x) = U(0,-1)\phi(x)$, where $U(0,-1)$ is the valency operator.

B. Necessary conditions for anomalous commutation relations

It is convenient to express the commutation properties of fields in terms of polylocal monomials X formed from the fields and having the form

$$X = \phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n), \tag{9.75}$$

it being supposed throughout that the point $(x_1,\dots,x_n)$ in (9.75) only ranges over totally spacelike values (that is, $(x_j - x_k)^2 < 0$ for $j \neq k$). If X and Y are two such monomials, then

$$XY = (-1)^k YX \tag{9.76}$$

(under spacelike separation of the arguments); here k is 0 or 1 in accordance with the formula

$$k = \sum_{\kappa,\kappa'} \mathfrak{m}^{(\kappa)}(X)\mathfrak{m}^{(\kappa')}(Y)(F^{(\kappa)}F^{(\kappa')} + \omega(\kappa,\kappa')) \pmod 2,$$

and $\mathfrak{m}^{(\kappa)}(x)$ is the number of occurrences modulo 2 of the field $\phi^{(\kappa)}$ in the monomial X. Let $\mathfrak{B}$ be the space of all vectors $\mathfrak{m} \equiv (\mathfrak{m}^{(1)},\dots,\mathfrak{m}^{(N)})$, whose components are either zero or one. Then associated with each monomial X is the vector $\mathfrak{m}(X)$ in $\mathfrak{B}$ according to the rule indicated above:

$$\mathfrak{m}(X) \equiv (\mathfrak{m}^{(1)}(X),\dots,\mathfrak{m}^{(N)}(X)). \tag{9.77}$$

In particular, the vector $\mathfrak{e}^{(\kappa)} \equiv (0,\dots,1,\dots,0)$ (with the 1 in the κth position) is associated with the field $\phi^{(\kappa)}$. The formula for k is now written in the form

$$k = F(\mathfrak{m}(X))F(\mathfrak{m}(Y)) + \omega(\mathfrak{m}(X),\mathfrak{m}(Y)) \pmod 2, \tag{9.78}$$

$$F(\mathfrak{m}) \equiv \sum_{\kappa} F^{(\kappa)}\mathfrak{m}^{(\kappa)} \pmod 2 \tag{9.79}$$

$$\omega(\mathfrak{m},\mathfrak{m}') = \sum_{\kappa,\kappa'} \omega(\kappa,\kappa')\mathfrak{m}^{(\kappa)}\mathfrak{m}'^{(\kappa')} \pmod 2 \tag{9.80}$$

In what follows we shall omit the mod 2 which corresponds to treating $\mathfrak{B}$ as a vector space over the field $\mathbf{Z}_2$ of the numbers 0 and 1 (addition in $\mathbf{Z}_2$ is carried out modulo 2, while multiplication is carried out using the ordinary multiplication table). The usual concepts of linear algebra make sense for $\mathfrak{B}$ (only it must be remembered that 1 is the only non-zero scalar). In particular, a family of vectors $f_1,\dots,f_k$ in $\mathfrak{B}$ is linearly independent if the sum of any non-empty family of the $f_1,\dots,f_k$ is non-zero.

Henceforth we shall always assume that none of the fields is identically zero. The fundamental properties of the matrix $\omega(\kappa,\kappa')$ can then be expressed by Lemmas 9.21 and 9.23 which follow.

Lemma 9.21. *Every field $\phi^{(\kappa)}$ has the normal commutation properties with its Hermitian adjoint field $\phi^{(\kappa)*} \equiv \phi^{(\bar\kappa)}$:*

$$\phi^{(\kappa)}(x)\phi^{(\bar\kappa)}(y) = (-1)^{F^{(\kappa)}F^{(\bar\kappa)}}\phi^{(\bar\kappa)}(y)\phi^{(\kappa)}(x) \quad \textit{for } (x-y)^2 < 0,$$

that is, for all $\kappa = 1, \ldots, N$

$$\omega(\kappa, \bar{\kappa}) = 0. \tag{9.81}$$

■ In Exercise 9.13 we saw that if the field $\phi^{(\kappa)}$ has the normal commutation property with its adjoint $\phi^{(\bar{\kappa})}$ (that is, if $\omega(\kappa, \bar{\kappa}) = 0$), then (9.34) holds (in the sense of the equality of generalized functions in $\mathcal{S}'(\mathbf{M}^2)$). Conversely, if we suppose that $\omega(\kappa, \bar{\kappa}) = 1$, then proceeding as in Exercise 9.13, we obtain instead of (9.34):

$$\langle 0|\phi_l^{(\kappa)}(x)(\phi_{\bar{l}}^{(\bar{\kappa})}(y)|0\rangle + \langle 0|\phi_{\bar{l}}^{(\bar{\kappa})}(y)\phi_l^{(\kappa)}(x)|0\rangle = 0$$

(equality in $\mathcal{S}'(\mathbf{M}^2)$). We smooth this relation with an arbitrary test function $f(x)$ in x and the complex conjugate function in y; we obtain

$$\langle 0|(\phi_l^{(\kappa)}, f)(\phi_{\bar{l}}^{(\bar{\kappa})}, \bar{f})|0\rangle + \langle 0|(\phi_{\bar{l}}^{(\bar{\kappa})}, f)(\phi_l^{(\kappa)}, f)|0\rangle = 0,$$

that is,

$$\|(\phi_l^{(\kappa)}, f)\Psi_0\|^2 + \|(\phi_l^{(\kappa)}, f)^*\Psi_0\|^2 = 0,$$

from which it follows that $\phi_l^{(\kappa)}(x)\Psi_0 = 0$. But according to Proposition 8.4 this implies that $\phi^{(\kappa)}(x) \equiv$ $\equiv 0$, which contradicts our agreement to exclude fields that are identically zero. ■

Lemma 9.22. *If two polynomials X and Y of the fields anticommute at spacelike separation (of the arguments of X from the arguments of Y), then either $\langle 0|X|0\rangle \equiv 0$, or $\langle 0|Y|0\rangle \equiv 0$.*

■ Let a be a vector in $\mathbf{M}$ converging to infinity along some spacelike ray. We suppose that the arguments of the monomials X and Y vary in arbitrary bounded regions; then for sufficiently large a, the monomials X and $\alpha_{(a,1)}(Y)$ anticommute, so that

$$\langle 0|X\alpha_{(a,1)}(Y)|0\rangle + \langle 0|\alpha_{(a,1)}(Y)X|0\rangle = 0.$$

According to the cluster property, this equality implies in the limit as $a \to \infty$:

$$\langle 0|X|0\rangle\langle 0|Y|0\rangle + \langle 0|Y|0\rangle\langle 0|X|0\rangle = 0,$$

that is,

$$\langle 0|X|0\rangle\langle 0|Y|0\rangle \equiv 0. \quad ■$$

Lemma 9.23. *Any pair of fields $\phi^{(\kappa)}$, $\phi^{(\kappa')}$ has the same type of commutation relations as the pair of fields $\phi^{(\kappa)*}$, $\phi^{(\kappa')}$, that is*

$$\omega(\kappa, \kappa') = \omega(\bar{\kappa}, \kappa') = \omega(\kappa, \bar{\kappa}'). \tag{9.82}$$

■ Suppose, on the contrary, that $\omega(\kappa, \kappa') + \omega(\bar{\kappa}, \kappa') = 1$. Then the polynomials

$$X = \phi^{(\kappa)}(x_1)^*\phi^{(\kappa)}(x_2) \text{ and } Y = \phi^{(\kappa')}(y_1)^*\phi^{(\kappa')}(y_2)$$

anticommute for $(x_j - y_k)^2 < 0$ $(j, k = 1, 2)$. Then by Lemma 9.22, either $\langle 0|X|0\rangle \equiv 0$, or $\langle 0|Y|0\rangle \equiv 0$. But it follows from the positive definiteness of the generalized functions $\langle 0|X|0\rangle$ and $\langle 0|Y|0\rangle$ (as in the proof of Lemma 9.21) that either $\phi^{(\kappa)}$ or $\phi^{(\kappa')}$ is equal to zero, which contradicts our agreement. ■

Corollary 9.24. The matrix ω has the properties

$$\omega(\kappa, \kappa) = \omega(\kappa, \bar{\kappa}) = \omega(\bar{\kappa}, \kappa) = 0, \tag{9.83}$$

and the form ω is skew-symmetric in the sense that

$$\omega(\mathfrak{n}, \mathfrak{n}) = 0 \quad \text{for all } \mathfrak{n} \in \mathfrak{B}. \tag{9.84}$$

In fact (9.83) is a direct corollary of Lemmas 9.21 and 9.23. Formula (9.84) now follows from the fact that in the expression of type (9.80) for $\omega(\mathfrak{m}, \mathfrak{m})$ the only terms present are the non-diagonal terms (with $\kappa < \kappa'$ and with $\kappa > \kappa'$) which cancel one another in view of the symmetry of the matrix ω (recall that addition is performed modulo 2).

C. Reduction of ω to canonical form

Thus we are dealing with a skew-symmetric form ω on $\mathfrak{B}$; there arises the natural problem of reducing it to canonical form.

We call a subspace (that is, a linear subspace) *isotropic* if $\omega(\mathfrak{m}, \mathfrak{m}') = 0$ for all $\mathfrak{m}$, $\mathfrak{m}'$ in this subspace.

Lemma 9.25. *The set of vectors $\mathfrak{I} = \{\mathfrak{m}(X)\}$ of vectors in $\mathfrak{B}$ corresponding to polynomials X whose vacuum expectation values are not identically zero ($\langle 0|X|0\rangle \not\equiv 0$) forms an isotropic subspace of $\mathfrak{B}$.*

■ Let X and Y be two arbitrary monomials with non-zero vacuum expectation values. Since $\mathfrak{m}(X) + \mathfrak{m}(Y) = \mathfrak{m}(XY)$, it suffices for the proof of the linearity of $\mathfrak{I}$, to verify that the vacuum expectation of the monomial XY is not identically zero. Suppose on the contrary that $\langle 0|XY|0\rangle \equiv 0$; then $\langle 0|X\alpha_{(a,1)}(Y)|0\rangle \equiv 0$, where a is a vector on some spacelike ray in $\mathbf{M}$. As $a \to \infty$, we obtain $\langle 0|X|0\rangle\langle 0|Y|0\rangle \equiv 0$. This contradiction proves that $\langle 0|XY|0\rangle \not\equiv 0$.

We now prove that $\omega(\mathfrak{m}(X), \mathfrak{m}(Y)) = 0$. Since $\langle 0|X|0\rangle \not\equiv 0$, the number of spinor fields in X is even (see Exercise 8.5), that is, $F(\mathfrak{m}(X)) = 0$; for the same reason $F(\mathfrak{m}(Y)) = 0$. Thus if we suppose that, on the contrary, $\omega(\mathfrak{m}(X), \mathfrak{m}(Y)) = 1$, this would mean that X and Y would anticommute at spacelike separation of the arguments of X and the arguments of Y. But then (according to Lemma 9.22) one of the vacuum expectation values $\langle 0|X|0\rangle$, $\langle 0|Y|0\rangle$ would be identically zero, contrary to hypothesis.

We thus conclude that $\omega(\mathfrak{m}(X), \mathfrak{m}(Y)) = 0$. ■

As usual, an isotropic subspace of $\mathfrak{B}$ is said to be *maximal isotropic* if it is not a proper subspace of another isotropic subspace of $\mathfrak{B}$. It is not difficult to see that every isotropic subspace of $\mathfrak{B}$ is contained in some maximal isotropic subspace of $\mathfrak{B}$. In fact, given an isotropic subspace that is not maximal isotropic, then it is contained in an isotropic subspace of greater dimension. If the new subspace is not maximal isotropic, then the extension process can be continued. Since $\mathfrak{B}$ is finite-dimensional, we obtain a maximal isotropic subspace containing the original subspace after a finite number of steps.

In particular, the isotropic subspace $\mathfrak{I}$ in Lemma 9.24 is contained in a maximal isotropic subspace $\mathfrak{M}$ of $\mathfrak{B}$. Proceeding from this subspace $\mathfrak{M}$, we can reduce ω to canonical form.

Lemma 9.26. *Let $\mathfrak{M}$ be a maximal isotropic subspace of $\mathfrak{B}$. Then there exists in $\mathfrak{B}$ a basis of vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}_1, \ldots \mathfrak{b}_r, \mathfrak{c}_1, \ldots, \mathfrak{c}_s$ (where $2r$ is the rank of ω and $2r + s = 2^N$) such that the vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{c}_1, \ldots, \mathfrak{c}_s$ form a basis in $\mathfrak{M}$ and*

$$\omega(\mathfrak{a}_i, \mathfrak{a}_j) = \omega(\mathfrak{b}_i, \mathfrak{b}_j) = 0, \quad \omega(\mathfrak{a}_i, \mathfrak{b}_j) = \delta_{ij}, \quad i, j = 1, \ldots, r; \tag{9.85}$$

$$\omega(\mathfrak{c}_l, \mathfrak{m}) = 0 \quad \textit{for all} \quad l = 1, \ldots, s; \ \mathfrak{m} \in \mathfrak{B}.$$

Thus in this basis the matrix ω has the block form

$$\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & & 0 \\ & \ddots & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & \\ 0 & & & 0 \end{pmatrix}$$

■ The lemma is proved by the standard arguments of linear algebra; we reproduce them here for the sake of completeness. If $\mathfrak{M} = \mathfrak{B}$ (that is, $r = 0$), then the lemma is trivially true. We now consider the case $\mathfrak{M} \neq \mathfrak{B}$. Then there exists in $\mathfrak{B}$ a vector $\mathfrak{b}_1 \notin \mathfrak{M}$. Moreover there exists a vector $\mathfrak{a}_1$ in $\mathfrak{M}$ such that $\omega(\mathfrak{a}_1, \mathfrak{b}_1) = 1$ (since otherwise, the linear span of the union of $\mathfrak{M}$ and $\mathfrak{b}_1$ would be an isotropic subspace, contradicting the maximality of $\mathfrak{M}$). We now proceed by induction. Suppose that vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_\nu \in \mathfrak{M}$ and $\mathfrak{b}_1, \ldots, \mathfrak{b}_\nu \in \mathfrak{B}\backslash\mathfrak{M}$ have been constructed such that

$$\omega(\mathfrak{a}_i, \mathfrak{a}_j) = \omega(\mathfrak{b}_i, \mathfrak{b}_j) = 0, \quad \omega(\mathfrak{a}_i, \mathfrak{b}_j) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, \nu.$$

If the linear span of the union of $\mathfrak{M}$ and the vectors $\mathfrak{b}_1, \ldots, \mathfrak{b}_\nu$ is the whole of $\mathfrak{B}$, then we have completed the construction of the vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}_1, \ldots, \mathfrak{b}_r$. We therefore suppose that the above linear span, denoted by $\mathfrak{N}$, is not the whole of $\mathfrak{B}$. Then there exists a vector $\mathfrak{b}_{\nu+1}$ in $\mathfrak{B}\backslash\mathfrak{N}$ such that

$$\omega(\mathfrak{a}_i, \mathfrak{b}_{\nu+1}) = \omega(\mathfrak{b}_i, \mathfrak{b}_{\nu+1}) = 0 \quad \text{for } i = 1, \ldots, \nu.$$

It suffices for this to take an arbitrary vector $\mathfrak{y} \in \mathfrak{B}\backslash\mathfrak{N}$ and set

$$\mathfrak{b}_{\nu+1} = \mathfrak{y} - \sum_{i=1}^{\nu} (\omega(\mathfrak{a}_i, \mathfrak{y})\mathfrak{b}_i + \omega(\mathfrak{b}_i, \mathfrak{y})\mathfrak{a}_i).$$

Then there exists a vector $\mathfrak{x}$ in $\mathfrak{M}$ such that $\omega(\mathfrak{x}, \mathfrak{b}_{\nu+1}) = 1$ (otherwise the linear span of the union of $\mathfrak{M}$ and $\mathfrak{b}_{\nu+1}$ would be an isotropic subspace, contradicting the maximality of $\mathfrak{M}$). We set

$$\mathfrak{a}_{\nu+1} = \mathfrak{x} - \sum_{i=1}^{\nu} \omega(\mathfrak{b}_i, \mathfrak{x})\mathfrak{a}_i.$$

As a result, the system $\mathfrak{a}_1, \ldots, \mathfrak{a}_\nu, \mathfrak{b}_1, \ldots, \mathfrak{b}_\nu$ obtained from the previous stage has been extended by the two vectors $\mathfrak{a}_{\nu+1}, \mathfrak{b}_{\nu+1}$. This process can be continued as long as the set $\mathfrak{M}$ and the vectors $\mathfrak{b}_1, \ldots, \mathfrak{b}_r$ together do not span the whole of $\mathfrak{B}$. It remains to choose the vectors $\mathfrak{c}_1, \ldots, \mathfrak{c}_s$. For this purpose we extend the system of vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \in \mathfrak{M}$ by the vectors $\mathfrak{x}_1, \ldots, \mathfrak{x}_s \in \mathfrak{M}$ to form a basis in $\mathfrak{M}$ and set

$$\mathfrak{c}_l = \mathfrak{x}_l - \sum_{i=1}^{r} \omega(\mathfrak{x}_l, \mathfrak{b}_i)\mathfrak{a}_r, \quad l = 1, \ldots, s.$$

It is now clear that the system of vectors $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}_1, \ldots, \mathfrak{b}_r, \mathfrak{c}_1, \ldots, \mathfrak{c}_s$ has the required properties. ■

Exercise 9.24. In the notation of Lemma 9.26, prove the identity

$$\omega(\mathfrak{m}, \mathfrak{m}') = \sum_{j=1}^{r} (\omega(\mathfrak{a}_j, \mathfrak{m})\omega(\mathfrak{b}_j, \mathfrak{m}') + \omega(\mathfrak{b}_j, \mathfrak{m})\omega(\mathfrak{a}_j, \mathfrak{m}'))$$

for all $\mathfrak{m}, \mathfrak{m}' \in \mathfrak{B}$. [Hint: An arbitrary vector $\mathfrak{m} \in \mathfrak{B}$ can be expanded in the basis $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}_1, \ldots, \mathfrak{b}_r, \mathfrak{c}_1, \ldots, \mathfrak{c}_s$; here the coefficient of $\mathfrak{a}_j$ is $\omega(\mathfrak{m}, \mathfrak{b}_j)$ and the coefficient of $\mathfrak{b}_j$ is $\omega(\mathfrak{m}, \mathfrak{a}_j)$.]

D. Construction of the Klein transformation

We now return to the proof of Theorem 9.20.

■ Let $\mathfrak{I}$ be the subspace defined in Lemma 9.25, and $\mathfrak{M}$ a maximal isotropic subspace of $\mathfrak{B}$ containing $\mathfrak{I}$. According to Lemma 9.26, it is possible to choose two systems of vectors $\{\mathfrak{a}_1, \dots, \mathfrak{a}_r\}$ and $\{\mathfrak{b}_1, \dots, \mathfrak{b}_r\}$ in $\mathfrak{B}$ (where $2r$ is the rank of ω) satisfying (9.85) with $\omega(\mathfrak{a}_j, \mathfrak{m}(X)) = 0$ for all monomials X with zero vacuum expectation values.

We claim that there exist r unitary operators $V_1, \dots, V_r$ in $\mathcal{H}$ defining unitary symmetries of the field algebra via the formula

$$\phi^{(\kappa)}(x) \to (-1)^{\omega(\mathfrak{e}^{(\kappa)}, \alpha_j)} \phi^{(\kappa)}(x) = V_j \phi^{(\kappa)}(x) V_j^{-1} \tag{9.86}$$

$(j = 1, \dots, r)$. To this end, we associate with an arbitrary monomial X of the fields, the monomial

$$\beta_j(X) = (-1)^{\omega(\mathfrak{m}(X), \mathfrak{a}_j)} X \tag{9.87}$$

(where $j = 1, \dots, r$ is fixed). From general considerations concerning internal symmetries,* in order that the correspondence $X \to \beta_j(X)$ define a symmetry of the field algebra that is unitarily realized by an operator V_j leaving the vacuum vector invariant, it is necessary and sufficient that

$$\beta_j(X^*) = \beta_j(X)^* \tag{9.88}$$

and

$$\langle 0|\beta_j(X)|0\rangle = \langle 0|X|0\rangle \tag{9.89}$$

for all monomials X. The property (9.88) of the adjoint follows from the fact that $\omega(\mathfrak{m}(X^*), \mathfrak{a}_j) \equiv$ $\equiv \omega(\mathfrak{m}(X), \mathfrak{a}_j)$, which in turn follows from (9.82) (since $\mathfrak{m}^{(\kappa)}(X^*) = \mathfrak{m}^{(\bar{\kappa})}(X)$). Property (9.89) means that $\omega(\mathfrak{m}(X), \mathfrak{a}_j) = 0$, if $\langle 0|X|0\rangle \not\equiv 0$, which holds in view of our choice of the vectors $\mathfrak{a}_j$.

By construction, the operators V_j leave the vacuum vector invariant. It is not difficult to verify that they commute with one another and that their squares are equal to unity. We set

$$V^{(\kappa)} = \sum_{j=1}^{r} V_j^{\omega(\mathfrak{e}^{\kappa}, \mathfrak{b}_j)} \tag{9.90}$$

$(\kappa = 1, \dots, N)$. The commutation relations (9.72) now follow from (9.87), where

$$\alpha(\kappa, \kappa') = \sum_{j=1}^{r} \omega(\mathfrak{e}^{(\kappa)}, \mathfrak{b}_j)\omega(\mathfrak{e}^{(\kappa')}, \mathfrak{a}_j). \tag{9.91}$$

Formula (9.73) follows from Exercise 9.24.

We now take into consideration the new fields $\phi'^{(\kappa)}$ (9.74). We leave the verification of the property $\phi'^{(\kappa)} = \phi^{(\bar{\kappa})'}$ as an exercise for the reader (use the commutation property (9.72) of the fields $\phi^{(\kappa)}$ with the operators $V^{(\kappa)}$, the relation $V^{(\kappa)*} = V^{(\kappa)}$, and the fact that according to Lemma 9.23, $\omega(\mathfrak{e}^{(\bar{\kappa})}, \mathfrak{m}) \equiv \omega(\mathfrak{e}^{(\kappa)}, \mathfrak{m})$ for all $\mathfrak{m} \in \mathfrak{B}$). The new fields $\phi'^{(\kappa)}$ have the normal connection between spin and statistics. In fact, using the commutation properties (9.72) of the fields $V^{(\kappa)}$ with the operators $\phi^{(\kappa')}$, we obtain for $(x - y)^2 < 0$

$$\phi'^{(\kappa_1)}(x)\phi'^{(\kappa_2)}(y) = (-1)^{F^{(\kappa_1)}F^{(\kappa_2)} + \omega(\kappa_1, \kappa_2) + \alpha(\kappa_1, \kappa_2) + \alpha(\kappa_2, \kappa_1)} \phi'^{(\kappa_2)}(y)\phi'^{(\kappa_1)}(x) =$$
$$= (-1)^{F^{(\kappa_1)}F^{(\kappa_2)}} \phi'^{(\kappa_2)}(y)\phi'^{(\kappa_1)}(x).$$

It is not difficult to see that the fields $\phi'^{(\kappa)}$ satisfy all the remaining Wightman axioms as well. ■

We illustrate the Klein transformation with some more complicated examples.

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* See §10.3.A for details on internal symmetries.

Exercise 9.25. Suppose that the system of two Hermitian scalar fields $\phi^{(1)}(x)$ and $\phi^{(2)}(x)$ satisfies the normal commutation relations

$$[\phi^{(\kappa)}(x), \phi^{(\kappa)}(y)] = 0 \quad \text{for } (x-y)^2 < 0, \ \kappa = 1, 2,$$

and the anomalous commutation relations between distinct fields

$$[\phi^{(1)}(x), \phi^{(2)}(y)]_+ = 0 \quad \text{for } (x-y)^2 < 0.$$

Suppose that the two-point function $\langle 0|\phi^{(1)}(x)\phi^{(2)}(y)|0\rangle$ is not identically zero. Prove that the Klein transformation can be defined by $\phi'^{(1)} = \phi^{(1)}, \phi'^{(2)} = iV\phi^{(2)}$, where $V = 1$ (or -1) at vectors obtained by the action of an even (or odd) number of smoothed fields $\phi^{(1)}$ and $\phi^{(2)}$ on the vacuum vector. [Hint: It follows from Lemma 9.22 that the vacuum expectation values of monomials containing an odd number of fields $\phi^{(1)}$ and $\phi^{(2)}$ vanish; therefore there exists a unitary operator V with the properties indicated above.]

Exercise 9.26. Let $\phi(x)$ be a Hermitian scalar field, and $\psi^{(1)}$ and $\psi^{(2)}$ two spinor fields. Suppose that the normal commutation relations

$$[\psi^{(1)}(x), \phi(y)] = [\psi^{(1)}(x), \psi^{(2)}(y)]_+ = 0 \quad \text{for } (x-y)^2 < 0$$

hold between distinct fields and at the same time we have the anomalous commutation relations

$$[\psi^{(2)}(x), \phi(y)]_+ = 0 \quad \text{for } (x-y)^2 < 0.$$

Suppose that for some odd n the nth order vacuum expectation $\langle 0|\phi(x_1)\ldots\phi(x_n)|0\rangle$ of ϕ is not identically zero. Prove in this case that the Klein transformation can be defined as a transformation leaving the fields $\psi^{(1)}$ and $\psi^{(2)}$ invariant and replacing ϕ by $\phi' = V\phi$, where $V = 1$ (or -1) at vectors obtained by the action of an even (or odd) number of smoothed fields $\psi^{(2)}$ and an arbitrary number of fields $\phi, \phi^{(1)}$. [Hint: According to Lemma 9.22 applied to the monomial $X = \phi(x_1)\ldots\phi(x_n)$ with n odd and with non-zero vacuum expectation value, and to an arbitrary monomial Y containing an odd number of fields $\psi^{(2)}$, the vacuum expectation value $\langle 0|Y|0\rangle$ is identically zero, whence the existence of the operator V with the required properties follows.]

9.4. Equal-Time Commutation Relations. Haag's Theorem

A. Three-dimensional version of Haag's theorem

As was mentioned in the introduction, the axiomatic approach to quantum field theory enables us to look at the difficulties of the traditional canonical (or Hamiltonian) formulation of relativistic quantum theory from a new point of view. It has turned out that the so-called interaction representation, often used in the usual formulation of the quantum field theory of perturbations, does not strictly speaking exist (Haag's theorem). It follows that in the quantum theory of interacting relativistic fields with "equal-time" commutation relations, we are always dealing with representations of commutation relations that are not unitarily equivalent to representations for free fields. Historically this conclusion had great significance as one of the first results prescribing the use of CCR's representations in quantum field theory that are not equivalent to Fock representations.

The most important assumption in the Hamiltonian (or Lagrangian) formulation, which is appended to the Wightman axioms, consists in ascribing a definite meaning to fields at a fixed moment of time.* Whereas we postulated in §8.2.A that fields

* In the Lagrangian theory of perturbations, fields featuring in the Wightman axioms are often called *Heisenberg* (or fields "in the the Heisenberg picture"), in contrast to auxiliary free fields (or fields "in the interaction picture").

$\phi(f)$ smoothed with respect to all four coordinates had the meaning of (possibly unbounded) operators in the vector state space $\mathcal{H}$, in the Hamiltonian approach we must further require that only fields smoothed with respect to the three spatial coordinates

$$\phi(h;t) = \int_{\mathbf{R}^3} \phi(t,\mathbf{x})h(\mathbf{x})d^3x \ (h \in \mathcal{S}(\mathbf{R}^3)),$$

have the meaning of operators defined together with their adjoints on an invariant domain D in $\mathcal{H}$. Thus $\phi(x)$ is now regarded as an operator-valued generalized function of $\mathbf{x} \in \mathbf{R}^3$ depending on x^0 as a parameter.

The connection between the two treatments of the field $\phi(x)$, as an operator-valued generalized function of x and of $\mathbf{x}$, is realized by the formula

$$\int \phi(x)h(\mathbf{x})u(x^0)d^4x = \int \phi(h;t)u(t)dt \tag{9.92}$$

for all $h \in \mathcal{S}(\mathbf{R}^3)$, $u \in \mathcal{S}(\mathbf{R})$. Here it is enough to suppose, for example, that for all $h \in \mathcal{S}(\mathbf{R}^3)$, $\Phi \in D$, the vector-valued function $\phi(h;t)\Phi$ is a continuous function of t whose norm is a bounded polynomial of $|t|$.

We saw (§8.2.C) that the Wightman fields form an irreducible system of operators in Hilbert space. We shall therefore suppose that the fields $\phi^{(\kappa)}(x)$ can be chosen so that at a fixed moment of time t the operators $\phi_l^{(\kappa)}(h;t)$ also form an irreducible system of operators in $\mathcal{H}$. In the Hamiltonian formalism, it suffices for this purpose that the system should contain along with each field $\phi(x)$ its conjugate "generalized momentum" $\pi(x)$. (In the case of a free scalar neutral field, $\pi(x)$ is none other than the time derivative $\dot{\phi}(x) \equiv \partial_0\phi(x)$ of $\phi(x)$, that is, the zeroth component of the vector field $\partial_\mu\phi(x)$.) This irreducibility assumption is sufficient for our purposes (the standard canonical commutation relations between fields at equal times will not be used).

The "interaction picture" in the Hamiltonian scheme of quantum field theory (see, for example, [W5],[W2]) is based on the supposition that at a fixed instant t, the fields ϕ and π are related to the free fields $\phi_{(0)}$ and $\pi_{(0)}$ by a unitary transformation. Haag's theorem shows that if we add to this usual scheme the requirement of relativistic invariance (and, in more detail, the Wightman axioms), then the theory becomes trivial: the fields turn out to be free.

Haag's theorem falls naturally into two parts. In the first part (suitable for the Hamiltonian formalism), invariance is assumed only with respect to the three-dimensional Euclidean motions. This part relates both to the relativistic and the non-relativistic theory. The second part (given in the next subsection) is based on relativistic invariance and contains the main result; here the use of Corollary 9.4 of Theorem 9.1 for the Wightman functions is essential.

Theorem 9.27. *Let $\phi^{(\kappa)}(x)$ and $\phi_{(0)}^{(\kappa)}(x)$ be two systems of fields in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{(0)}$ respectively, where each system is irreducible at the fixed instant $x^0 = t$ (here it is understood that the fields exist at a fixed time). Suppose that unitary representations $U(\mathbf{a},R)$ and $U_{(0)}(\mathbf{a},R)$ of the group $E_+(3)$ of proper Euclidean motions of three-dimensional space are realized in $\mathcal{H}$ and $\mathcal{H}_{(0)}$, under which the fields $\phi^{(\kappa)}$ and*

$\phi_{(0)}^{(\kappa)}$ *transform covariantly:*

$$\begin{aligned} U(\mathbf{a},R)\phi_l^{(\kappa)}(x)U(\mathbf{a},R)^{-1} &= \sum_m T_{lm}^{(\kappa)}(R^{-1})\phi_m^{(\kappa)}(t,R\mathbf{x}+\mathbf{a}), \\ U_{(0)}(\mathbf{a},R)\phi_{(0)l}^{(\kappa)}(x)U(\mathbf{a},R)^{-1} &= \sum_m T_{lm}^{(\kappa)}(R^{-1})\phi_{(0)m}^{(\kappa)}(t,R\mathbf{x}+\mathbf{a}), \end{aligned} \tag{9.93}$$

where $T^{(\kappa)}(R)$ is the matrix representation of the three-dimensional rotation group. If at some instant $x^0 = t$ the fields $\phi^{(\kappa)}(x)$ and $\phi_{(0)}^{(\kappa)}(x)$ are related by the unitary transformation*

$$\phi^{(\kappa)}(t,\mathbf{x}) = V\phi_{(0)}^{(\kappa)}(t,\mathbf{x})V^{-1}, \tag{9.94}$$

then the representations U and $U_{(0)}$ of $E_+(3)$ are unitarily equivalent:

$$U(\mathbf{a},R) = VU_{(0)}(\mathbf{a},R)V^{-1}. \tag{9.95}$$

If it is supposed further that there exists in $\mathcal{H}_{(0)}$ a unique (to within a phase) normalized vector $\Psi_{(0)0}$ that is invariant under the representation $U_{(0)}$ of $E_+(3)$, then there also exists in $\mathcal{H}$ a unique (to within a phase) normalized vector

$$\Psi_0 = V\Psi_{(0)0} \tag{9.96}$$

that is invariant under the representation U of $E_+(3)$.

■ It follows from (9.93) and (9.94) that the operator $V^{-1}U(\mathbf{a},R)^{-1}VU_{(0)}(\mathbf{a},R)$ commutes with all the operators $\phi_{(0)}^{(\kappa)}(t,\mathbf{x})$; since the fields $\phi_{(0)}^{(\kappa)}(x)$ are irreducible at the instant $x^0 = t$, the above operator is a multiple of the identity, so that

$$U(\mathbf{a},R) = \omega(\mathbf{a},R)VU_{(0)}(\mathbf{a},R)V^{-1},$$

where $\omega(\mathbf{a},R)$ is a complex number. It is not difficult to see that $\omega(\mathbf{a},R)$ defines a one-dimensional (irreducible) representation of $E_+(3)$; hence** $\omega(\mathbf{a},R) \equiv 1$. Thus (9.95) is proved. The final assertion of the theorem follows trivially from the unitary equivalence of U and $U_{(0)}$. ■

It can be shown that the requirement of uniqueness of a state that is invariant with respect to three-dimensional Euclidean motions is too stringent since all states with zero three-dimensional momentum and zero spin (for example, the state of a spinless particle at rest) are invariant with respect to $E_+(3)$. However, such "states" are non-normalizable (they are, in fact, generalized states) and are therefore excluded from our discussion.

Exercise 9.27. Prove that in the Wightman theory, the vacuum vector $|0\rangle$ is the unique (to within a factor) vector in $\mathcal{H}$ that is invariant with respect to the operators $U((0,\mathbf{a}),1)$ of the representation of the group of three-dimensional translations of $\mathbf{R}^3$. [Hint: It is required to prove that $U((0,\mathbf{a}),1)\Phi = \Phi$ and $\langle\Phi|0\rangle = 0$ imply that $\Phi = 0$. Since $\Phi = \lim_{\epsilon\to 0} E_\epsilon\Phi$, it suffices to suppose that $\Phi = E_\epsilon\Phi$ for some $\epsilon > 0$; here E_ϵ is the projector onto the subspace where $P^0 \geq \epsilon$. To see that $\Phi = 0$, introduce the vector $\Phi_\Lambda = U(0,\Lambda)\Phi$, where $\Lambda \neq 1$ is a pure Lorentz rotation; then the support of the Fourier transform of

* We also admit two-valued representations of $O_+(3)$, that is, representations of $SU(2)$. It would therefore be more accurate to say that U and $U_{(0)}$ are unitary representations of the universal covering group of $E_+(3)$.

** It is fairly easy to see that the only one-dimensional representation of $E_+(3)$ is the trivial representation (see Exercise 7.8 where the corresponding result for the Poincaré group is stated).

the expression $\langle \Phi, U(a,1)\Phi_\Lambda \rangle$ as a function of $a \in \mathbf{M}$ is the empty set, so that $\langle \Phi, \Phi_\Lambda \rangle = 0$; on the other hand, $\langle \Phi, \Phi_\Lambda \rangle \to \|\Phi\|^2$ as $\Lambda \to 1$.]

B. HAAG'S THEOREM IN THE RELATIVISTIC THEORY

The second (most important) part of Haag's theorem deals with relativistic field theory. We shall state it in the somewhat more generalized formulation due to Hall and Wightman. For the sake of simplicity we confine ourselves to the case of a scalar neutral field.

Theorem 9.28 (Generalized Haag Theorem). *Let ϕ and $\phi_{(0)}$ be two scalar neutral Wightman fields acting in the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_{(0)}$ respectively. Suppose that at each instant $x^0 = t$, the fields $\phi(t,\mathbf{x})$, $\dot{\phi}(t,\mathbf{x})$ and the fields $\phi_{(0)}(t,\mathbf{x})$, $\dot{\phi}_{(0)}(t,\mathbf{x})$ (the existence of which is postulated) form two irreducible systems of fields in $\mathcal{H}$ and $\mathcal{H}_{(0)}$ respectively. Suppose finally that at the instant $x^0 = t$, these two systems of fields are related by the (time-dependent) unitary transformation $V \equiv V(t)$:*

$$\phi(t,\mathbf{x}) = V(t)\phi_{(0)}(t,\mathbf{x})V(t)^{-1}, \quad \dot{\phi}(t,\mathbf{x}) = V(t)\dot{\phi}_{(0)}(t,\mathbf{x})V(t)^{-1}. \tag{9.97}$$

*Then the first four Wightman functions of the respective fields $\phi(x)$ and $\phi_{(0)}(x)$ are the same in both models. If, in addition, $\phi_{(0)}(x)$ is a free field of mass $m \geq 0$, then $\phi(x)$ is also a free field of mass m and both models coincide (to within a unitary transformation V).**

■ It follows from (9.94) and (9.96) that the Wightman functions in both theories are the same at the instant $x^0 = t$:

$$w^{[n]}(t,\mathbf{x}_1;\ldots;t,\mathbf{x}_n) = w^{[n]}_{(0)}(t,\mathbf{x}_1;\ldots;t,\mathbf{x}_n),$$

that is,

$$W^{[n]}(\xi_1,\ldots,\xi_{n-1}) = W^{[n]}_{(0)}(\xi_1,\ldots,\xi_{n-1}) \quad \text{for } \xi_1^0 = \ldots = \xi_{n-1}^0 = 0. \tag{9.98}$$

Let S_{n-1} be the set of real points $(\xi_1,\ldots,\xi_{n-1})$ of the extended tube T_{n-1} that can be taken by a proper Lorentz transformation into the plane $\xi_1^0 = \xi_2^0 = \ldots = \xi_{n-1}^0 = 0$. Then it follows from formula (9.98) of Theorem 9.1 that $W^{[n]} = W^{[n]}_{(0)}$ on S_{n-1}. We claim that for $n \leq 4$, S_{n-1} contains a non-empty open subset of $\mathbf{M}^{n-1}$, from which it will follow that $W^{[n]} = W^{[n]}_{(0)}$ in the extended tube T_{n-1} and hence in $\mathbf{M}^{n-1}$ as well (since they are then generalized boundary values of the same analytic functions). For $n = 2$, the real points ξ of T_1 are precisely the spacelike points of $\mathbf{M}$; clearly they can be taken into the plane $\xi^0 = 0$ by a Lorentz transformation, therefore S_1 is a domain in $\mathbf{M}$. We now turn to the cases $n = 3$ and $n = 4$. Consider the open subset of points $(\xi_1,\ldots,\xi_{n-1}) \in \mathbf{M}^{n-1}$ such that the matrix $\{-\xi_i\xi_j\}$ is positive definite; this means that all the real linear combinations $\sum_{j=1}^{n-1} \lambda_j\xi_j$ of the vectors $\xi_1,\ldots,\xi_{n-1}$ form an $(n-1)$-dimensional spacelike plane. Hence it is clear that such points $\xi_1,\ldots,\xi_{n-1}$ are real points of the extended tube. On the other hand, the orthocomplement of the above $(n-1)$-dimensional plane contains a timelike vector η. In the Lorentz frame in which $\eta = 0$, we have $\xi_1^0 = \ldots = \xi_{n-1}^0 = 0$, so that $(\xi_1,\ldots,\xi_{n-1}) \in S_{n-1}$.

We have thus proved that the first four Wightman functions are the same in both theories. If now $\phi_{(0)}$ is a free field of mass m, then by Proposition 8.8 it follows from the equality $W^{[2]} = W^{[2]}_{(0)}$ that ϕ is also a free field. ■

Exercise 9.28. Prove that for $n \geq 5$, the set of points $x \equiv (x_1,\ldots,x_n)$, that can be taken by a Lorentz transformation onto the plane of equal times $\xi_1^0 = \ldots = \xi_{n-1}^0 = 0$, no longer forms a complete neighbourhood in the set of Jost points $\mathcal{J}_n$. In particular, verify that for $n = 5$, the number of independent scalar products formed from the vectors $\xi_1,\ldots,\xi_4$ is equal to ten, while the number of independent scalar products of the vectors in the plane of equal times is equal to nine.

* This final statement forms the content of Haag's original theorem (1955).

C. COMMENTS ON HAAG'S THEOREM

Haag's Theorem implies that the representation of the interaction with which one usually begins to set forth perturbation theory, does not in fact exist. In other words, there is no well defined operator V in the space of vector space $\mathcal{H}$ associating the free field with the interacting field according to (9.94). We briefly analyse the canonical Lagrangian approach in quantum field theory for this situation.

The canonical quantization scheme starts from the classical Lagrangian $\mathcal{L}$ which is a function of the fields $\phi_\alpha(x)$ and their first derivatives:

$$\mathcal{L} = \mathcal{L}\Big(\phi_\alpha(x), \frac{\partial \phi_\alpha(x)}{\partial x^\nu}\Big).$$

The "conjugate momentum" to the field ϕ_α at the instant t is defined by the formula

$$\pi_\alpha(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \frac{\partial \phi_\alpha(t,\mathbf{x})}{\partial t}}.$$

For example, for the free scalar field with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\big(g^{\mu\nu}\tfrac{\partial \phi}{\partial x^\mu}\tfrac{\partial \phi}{\partial x^\nu} - m^2\phi^2\big)$$

we have

$$\pi(t, \mathbf{x}) = \frac{\partial \phi(t, \mathbf{x})}{\partial t};$$

for the free spinor field with Lagrangian

$$\mathcal{L} = i\tilde{\psi}(x)\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - m\tilde{\psi}(x)\psi(x),$$

the "momentum" conjugate to ψ is

$$\pi_\psi(x) = i\tilde{\psi}(x)\gamma^0 = i\psi^*(x).$$

It is postulated that the conjugate fields ϕ_α and π_α (more precisely, the smoothed fields $\phi_\alpha(f; t)$) are the basis elements of an algebra defined (in the case of Bose fields) by the canonical commutation relations *

$$\begin{aligned} [\phi_\alpha(t, \mathbf{x}), \phi_\beta(t, \mathbf{y})] &= [\pi_\alpha(t, \mathbf{x}), \pi_\beta(t, \mathbf{y})] = 0, \\ [\phi_\alpha(t, \mathbf{x}), \pi_\beta(t, \mathbf{y})] &= i\delta_{\alpha\beta}\delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{9.99}$$

Thus the quantum field theory is formulated as a quantum mechanical system with an infinite number of degrees of freedom (the $\mathbf{x}$ and $\mathbf{y}$ emerging here as the "indices" of the generalized coordinate and generalized momentum). It is also convenient to deal with a countable (instead of a continuum) basis. For this purpose it suffices to introduce a complete orthonormal system of functions $h_\nu(\mathbf{x})$ in three-dimensional space (for example, the Hermite functions $N_\nu e^{-x^2/2}H_{\nu_1\nu_2\nu_3}(\mathbf{x})$) and to define the "coordinates" and "momenta" of the field by the formulae

$$\begin{aligned} Q_n(t) &= \int \phi_\alpha(t, \mathbf{x})\bar{h}_\nu(\mathbf{x})d^3x, \\ P_n(t) &= \int \pi_\alpha(t, \mathbf{x})h_\nu(\mathbf{x})d^3x, \end{aligned} \tag{9.100}$$

* For Fermi fields, the commutators are replaced by anticommutators.

where $\mathbf{n}$ is the composite discrete index: $\mathbf{n} \equiv (\alpha, \nu)$.

Exercise 9.29. Using the orthonormality of the system of functions h_ν, prove that the commutation relations (9.99) in terms of $Q_\mathbf{n}$ and $P_\mathbf{n}$ assume the form

$$\begin{aligned} &[Q_\mathbf{n}(t), Q_{\mathbf{n}'}(t)] = [P_\mathbf{n}(t), P_{\mathbf{n}'}(t)] = 0, \\ &[Q_\mathbf{n}(t), P_{\mathbf{n}'}(t)] = i\delta_{nn'}. \end{aligned} \tag{9.101}$$

We then realize the abstract algebra of elements $Q_\mathbf{n}$ and $P_\mathbf{n}$ satisfying (9.101) as an algebra of unbounded operators in a Hilbert space $\mathcal{H}$ (of vector states). In the case of a system with a finite number of degrees of freedom, any two irreducible representations of the commutation relations (9.101) realized by self-adjoint operators in $\mathcal{H}$ are unitarily equivalent (von Neumann's Theorem 6.14). In particular, there always exists a unitary operator $V(t_2, t_1)$ relating the operators $P_\mathbf{n}$ and $Q_\mathbf{n}$ at different moments of time:

$$\begin{aligned} Q_\mathbf{n}(t_2) &= V(t_2, t_1) Q_\mathbf{n}(t_1) V^{-1}(t_2, t_1), \\ P_\mathbf{n}(t_2) &= V(t_2, t_1) P_\mathbf{n}(t_1) V^{-1}(t_2, t_1). \end{aligned} \tag{9.102}$$

This is no longer the case for systems with an infinite number of degrees of freedom. Moreover, linear canonical transformations (that is, transformations of the variables $P_\mathbf{n}$ and $Q_\mathbf{n}$ that preserve the form of the commutation relations (9.101)) do not in general correspond to a transformation of a unitary equivalence (see Exercise 7.24).

From among the infinite set of inequivalent representations of the canonical commutation relations in the theory of free fields, the Fock representation (§8.4.A) is selected by means of the additional requirement that this space has a unique normalized invariant state Ψ_0 that is annihilated under the action of positive-frequency operators:

$$\tilde{\phi}^{(+)}(p)\Psi_0 = 0, \tag{9.103}$$

in other words, the requirement of the existence of the vacuum. The Fock representation is uniquely defined (to within unitary equivalence). It might be supposed (and is often implicitly assumed) that in the theory of interacting fields, one can choose the Fock representation with a physical vacuum which is of necessity preserved in time. But Haag's theorem shows, in fact, that this is not the case and that at any finite moment of time we have to use non-Fock representations of the commutation relations (in which, roughly speaking, each state contains an "actual" infinity of "bare" particles).

The discrepancy between the interaction representation in the standard theory (with local Hamiltonian) and the general requirements of quantum theory also consists in the following. The time-shift generator P^0 must always make the vacuum vector vanish. However, if we represent P^0 as an integral of the local Hamiltonian function corresponding to a non-trivial interaction, this will not be the case; in this connection, in the standard approach one talks about the mathematical (or "bare") and physical vacua. But according to Haag's theorem, these two vacua are not related by a unitary equivalence. (All this is illustrated by simple non-relativistic models, see, for example, [V5].) We note, however, that Haag's theorem does not exclude the existence of a representation of the interaction, if one violates the translational invariance of the model (as is not infrequently done, see, for example, [B10]) by the introduction of a spatial cut-off of the interaction. If a representation of the interaction exists for a Hamilton cut off in this way, then we are dealing with the so-called *local Fock representation* of the canonical commutation relations; (an equality of type (9.94) then holds for any bounded domain $\mathcal{O}$ of values $\mathbf{x}$ with t fixed; however, the operator V realizing the unitary equivalence varies as $\mathcal{O}$ varies, since it is defined by a spatially cut-off interaction). Such a "local" treatment of the representation of the interaction which bypasses Haag's theorem, has been proposed by Guenin (1966a) and Segal (1967).

It must be said that apart from the need to take into account the inequivalent representations of the canonical commutation relations, one also has to bear in mind the possibility that the commutation relations themselves have a more singular character. Even for the free complex vector field $V^\mu(x)$, the commutation relations have the form (see [B10],§11.3)

$$[V^\mu(x), V^{*\nu}(y)] = \frac{1}{i}\Big(g^{\mu\nu} + \frac{1}{m^2}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\Big)D_m(x-y). \qquad (9.104)$$

Bearing in mind that the "momentum" conjugate to $V^\mu(x)$ is $\partial V^{*\mu}/\partial t$, instead of (9.99) we have

$$\Big[V^\mu(x), \frac{\partial}{\partial t}V^{*\mu}(y)\Big]\Big|_{x^0=y^0} = \frac{1}{im^2}\Delta\delta(\mathbf{x}-\mathbf{y}), \qquad (9.105)$$

where Δ is the Laplace operator. In principle, the commutator of interacting fields can be so singular that it is necessary to smooth it with respect to all four coordinates and it cannot be regarded at a fixed instant of time as a generalized operator-valued function of the spatial coordinates; in this case, simultaneous CCR's have no meaning.

Remark. As we shall see in §14.2, the difficulties connected with Haag's theorem do not arise if instead of the interaction representation one uses the asymptotic representation, that is, if in the construction of the interacting fields from the free fields, they are equated not at a finite instant t, but at $t = -\infty$ (or $+\infty$). Formula (9.97) (which, as we have said, is ill defined in its application to interacting fields) is then replaced by a well defined formula (see (14.67) later). This, of course, does not exclude the appearance of "divergences" in the use of an asymptotic representation in perturbation theory. Nevertheless, it can happen that all the divergences reduce merely to "ultraviolet" ones (connected with the non-existence at a fixed moment of time of a field corresponding to the Lagrangian of the interaction; in other words, this relates to a somewhat different circle of phenomena than Haag's theorem which explains the "infrared" divergences).

9.5. Euclidean Green's Functions

A. Group of Rotations of Four-Dimensional Euclidean Space

We saw (§9.1.D) that the Wightman functions $w(z_1, \ldots, z_n)$ are analytic at the non-exceptional Euclidean points $(x'_1, \ldots, x'_n)$ defined by (9.44), (9.45) (where $x_j \neq x_k$ if $j \neq k$). Each vector x'_j has a real three-dimensional projection $\mathbf{x}_j$ and a purely imaginary time coordinate $x'^0 = ix^4$. The scalar product of a pair of such vectors x' and y' differs only in sign from the Euclidean scalar product of their Euclidean inverse images* $x, y \in \mathbf{E}$ $(\equiv \mathbf{R}^4)$:

$$-x'y' = (x,y) \equiv x^\mu y_\mu \equiv \delta_{\lambda\mu}x^\lambda y^\mu. \qquad (9.106)$$

Thus Proposition 9.10 opens up the interesting possibility of associating with relativistic quantum theory a mathematical image based exclusively on the four-dimensional Euclidean space $\mathbf{E}$. In the Euclidean formulation, the role of the Lorentz (or Poincaré) group must be played by the group $O(4,R) \equiv O(4)$ of orthogonal transformations of $\mathbf{E}$ (or the group $E(4)$ of motions of $\mathbf{E}$).

The group $O(4)$ is clearly a subgroup ("real section") of the group $O(4,C)$, which is isomorphic (as noted in §9.1.A) to the group $L(C)$ of complex Lorentz transformations. Therefore facts relating to the group $O(4)$ are in total correspondence with those that we gave for the group $L(C)$ in §9.1.A.

* We identify the components of vectors of Euclidean space with upper and lower indices.

Exercise 9.30. Prove that $O(4)$ consists of two connected components and that the connected component of the identity (called the rotation group of $\mathbf{E}$) is the subgroup $O_+(4)$ of all elements of $O(4)$ with determinant $+1$.

It is clear that $O_+(4)$ contains the transformation -1 (total reflection of $\mathbf{E}$) which generates the centre of the group. The group $O_+(4)$ is not simply connected. It is locally isomorphic to the simply connected group $SU(2) \times SU(2)$, which is thus a universal covering of $O_+(4)$. For the construction of the covering map from $SU(2) \times SU(2)$ onto $O_+(4)$, we associate with each vector $x \in \mathbf{E}$ the complex 2×2-matrix (or quaternion in the matrix realization) $\hat{x}$ according to the formula

$$\hat{x} = -i\underset{\sim}{x}' = \begin{pmatrix} x^4 - ix^3 & -x^2 - ix^1 \\ x^2 - ix^1 & x^4 + ix^3 \end{pmatrix}. \tag{9.107}$$

We denote by $\hat{x}^*$ the conjugate quaternion:

$$\hat{x}^*(\equiv \widehat{Px} \equiv \epsilon \hat{x}^T \epsilon^{-1}) = \begin{pmatrix} x^4 + ix^3 & x^2 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 \end{pmatrix};$$

here P is the three-dimensional reflection:

$$Px \equiv (-\mathbf{x}, x_4). \tag{9.108}$$

The quantity $|\hat{x}| = (\det \hat{x})^{1/2}$ is called the norm of $\hat{x}$.

Exercise 9.31. Prove the following identities:

$$|\hat{x}|^2 = |x|^2 \; (\equiv \mathbf{x}^2 + x_4^2);$$

$$\hat{x}\hat{x}^* = |\hat{x}|^2 \cdot \mathbf{1}, \quad \operatorname{tr}(\hat{x}\hat{y}^*) = 2(x, y);$$

$$\hat{e}_\lambda \hat{e}_\mu^* + \hat{e}_\mu \hat{e}_\lambda^* = 2\delta_{\lambda\mu} \cdot \mathbf{1} = \hat{e}_\lambda^* \hat{e}_\mu + \hat{e}_\mu^* \hat{e}_\lambda;$$

$$\hat{e}_\lambda \hat{e}_\mu^* - \hat{e}_\mu \hat{e}_\lambda^* = -\tfrac{1}{2}\epsilon_{\lambda\mu\rho\sigma}(\hat{e}_\rho \hat{e}_\sigma^* - \hat{e}_\sigma \hat{e}_\rho^*);$$

$$\hat{e}_\lambda^* \hat{e}_\mu - \hat{e}_\mu^* \hat{e}_\lambda = \tfrac{1}{2}\epsilon_{\lambda\mu\rho\sigma}(\hat{e}_\rho^* \hat{e}_\sigma - \hat{e}_\sigma^* \hat{e}_\rho);$$

here $\{e_\lambda\}_{\lambda=1,\dots,4}$ is the standard basis in $\mathbf{E}$ ($e_\lambda^\mu \equiv \delta_{\lambda\mu}$); $\epsilon_{\lambda\mu\rho\sigma}$ is the absolute antisymmetric tensor with $\epsilon_{1234} = 1$.

The covering map

$$SU(2) \times SU(2) \ni (U_l, U_r) \to R(U_l, U_r) \in O_+(4) \tag{9.109}$$

can now be defined by the formula

$$(R(U_l, U_r)x)\hat{} = U_l \hat{x} U_r^T. \tag{9.110}$$

Exercise 9.32. (a) Prove that $R(U_l, U_r) \in O_+(4)$. [Hint: Verify that the transformation $R(U_l, U_r)$ does not change the norm of the vectors; now use the connectedness of the group $SU(2) \times SU(2)$.]

(b) Prove that the map (9.109) is a local homeomorphism. [Hint: Verify that the rank of this map at the point $U_l = U_r = 1$ is equal to six.]

(c) Prove that the kernel of the homomorphism (9.109) consists of the two elements: $U_l = U_r = 1$ and $U_l = U_r = -1$.

It follows from Exercise 9.32 that $SU(2) \times SU(2)$ is a universal covering for $O_+(4)$ and is a double covering of it.

Exercise 9.33. Prove that under the covering homomorphism, the subgroup of $SU(2) \times SU(2)$ of elements of the form $(U_l, \overline{U}_l)$ doubly covers the group of rotations $O_+(3)$ of three-dimensional space. ($O_+(3)$ is regarded as the subgroup of $O_+(4)$ of elements that leave the vector $e_4 \equiv (0,0,0,1)$ invariant.)

The above construction enables us to replace $O_+(4)$ by $SU(2) \times SU(2)$ in what follows. The irreducible (complex) representations of $SU(2)\times SU(2)$ are parametrized by the pair (j,k) of non-negative integers or half-integers ("left" and "right" spins). This representation acts in the space of spinors $\psi^{\alpha_1 \dots \alpha_{2j};\beta'_1 \dots \beta_{2k}}$, which are symmetric in $\alpha_1, \dots, a_{2j}$ and $\beta'_1, \dots, \beta'_{2k}$, according to (9.10) (with Λ_l, Λ_r being replaced by U_l, U_r):

$$(\mathfrak{D}^{(j,k)}(U_l, U_r)\psi)^{\alpha_1 \dots \alpha_{2j};\beta'_1 \dots \beta'_{2k}} =$$

$$= \sum_{\substack{\gamma_1 \dots, \gamma_{2j} \\ \delta'_1 \dots \delta'_{2k}}} \prod_{s=1}^{2j} (U_l)^{\alpha_s}_{\gamma_s} \prod_{t=1}^{2k} (U_r)^{\beta'_t}_{\delta'_t} \psi^{\gamma_1 \dots \gamma_{2j};\delta'_1 \dots \delta'_{2k}}. \tag{9.111}$$

Similarly we associate with an arbitrary finite-dimensional (complex) representation $V(\underset{\sim}{\Lambda}) \equiv V(\underset{\sim}{\Lambda}, \overline{\underset{\sim}{\Lambda}})$ of $SL(2,C)$ the representation $V(U_l, U_r)$, which is the restriction of the representation $V(\Lambda_l, \Lambda_r)$ of $SL(2,C) \times SL(2,C)$ (see §9.1.A) to the subgroup $SU(2) \times SU(2)$. (This construction bears the name of Weyl's "unitary trick", see [Z2],§42.)

B. PROPERTIES OF THE SCHWINGER FUNCTIONS

We introduce the inverse image in $\mathbf{E}^n$ of the set of non-exceptional Euclidean points of $\mathbf{CM}^n$:

$$\mathbf{E}^n_{\neq} = \{(x_1, \dots, x_n) \in \mathbf{E}^n : x_j \neq x_k \quad \text{for } j \neq k)\}. \tag{9.112}$$

Then the restriction of the Wightman function $w(z_1, \dots, z_n)$ to the set of non-exceptional Euclidean points of $\mathbf{CM}^n$ defines a function on $\mathbf{E}^n_{\neq}$, called the *Schwinger function*:*

$$s^{(\kappa_1 \dots \kappa_n)}_{l_1 \dots l_n}(x_1, \dots, x_n) = w^{(\kappa_1 \dots \kappa_n)}_{l_1 \dots l_n}(x'_1, \dots, x'_n); \tag{9.113}$$

here, as in (9.44), $x'_j \equiv (ix^4_j, \mathbf{x}_j)$.

As is well known from complex analysis, a holomorphic function in a complex domain can be completely recovered from a knowledge of its values in a real neighbourhood of some point. Consequently, the Schwinger functions enable us in principle to recover the Wightman functions and hence the entire Wightman field theoretic model. There arises the natural problem of reformulating the characteristic properties of the Wightman functions directly in terms of the Schwinger functions. We state here the results of Osterwalder and Schrader along these lines.

In addition to the subset $\mathbf{E}^n_{\neq}$ of $\mathbf{E}^n$ we shall require

$$\mathbf{E}^n_{<} = \{x \in \mathbf{E}^n : 0 < x^4_1 < x^4_2 < \dots < x^4_n\}, \tag{9.114}$$

$$\mathbf{E}^{n-1}_{-} = \{\xi \in \mathbf{E}^{n-1} : \xi^4_j < 0, \ j = 1, \dots, n-1\}, \tag{9.115}$$

---

* Here we are using the same basis in the space of spin-tensors for two different situations; in this way, certain of the important properties of the Schwinger functions, positive definiteness, for example, (see (9.123) below), appear more natural. But for practical purposes it is often convenient to choose a different basis in the Euclidean space from the one in the pseudo-Euclidean space (so as to simplify the description of discrete transformations of C, P, CP type in terms of the Schwinger functions).

as well as

$$\overline{\mathbf{M}}_{+}^{n-1} = \{q \in \mathbf{M}^{n-1} : q_j^0 \geq 0,\ j = 1, \ldots, n-1\}. \tag{9.116}$$

Then every generalized function $T(q_1, \ldots, q_{n-1})$ in $\mathcal{S}'(\overline{\mathbf{M}}_{+}^{n-1}) \approx \mathcal{S}'(\mathbf{M}^{n-1}|\overline{\mathbf{M}}_{+}^{n-1})$ can be put in correspondence with a generalized function $F(\xi_1, \ldots, \xi_{n-1})$ in $\mathcal{S}'(\mathbf{E}_{-}^{n-1})$ by means of the Fourier-Laplace-type transformation:

$$F(\xi_1, \ldots \xi_{n-1}) = \int \exp\Big[\sum_{j=1}^{n-1}(\xi_j^4 q_j^0 + i\xi_j \mathbf{q}_j)\Big] T(q_1, \ldots, q_{n-1}) d_4 q_1 \ldots d_4 q_{n-1}. \tag{9.117}$$

In fact in view of the property of the support of T, the Fourier transform of T can be analytically continued with respect to the zeroth components of the vectors to the domain $(\mathbf{R} - i\mathbf{R}_+)^n$ and the expression (9.117) is the restriction of this continuation to the set of vectors with zeroth components $i\xi_j^4$, $\xi_j^4 \in \mathbf{R}_-$. The functional treatment of the transform (9.117) is given by the formula

$$\int F(\xi_1, \ldots, \xi_{n-1}) u(\xi_1, \ldots, \xi_{n-1}) d^4\xi_1 \ldots d^4\xi_{n-1} =$$
$$= \int T(q_1, \ldots, q_{n-1}) \hat{u}(q_1, \ldots, q_{n-1}) d_4 q_1 \ldots d_4 q_{n-1}, \tag{9.118}$$

where $u(\xi_1, \ldots, \xi_{n-1})$ is an arbitrary function of $\mathcal{S}(\mathbf{E}_{-}^{n-1})$ and $\hat{u}(q_1, \ldots, q_{n-1})$ is the function in $\mathcal{S}(\overline{\mathbf{M}}_{+}^{n-1})$ defined by the formula

$$\hat{u}(q_1, \ldots, q_{n-1}) = \int \exp\Big[\sum_{j=1}^{n-1}(\xi_j^4 q_j^0 + i\xi_j \mathbf{q}_j)\Big] u(\xi_1, \ldots, \xi_{n-1}) d^4\xi_1 \ldots d^4\xi_{n-1}. \tag{9.119}$$

The transformations of type (9.117), (9.119) have already been considered by us (in §B.6 of Appendix B); the only difference is that in §B.6 we only dealt with the time components of the momenta, leaving out the spatial components as being inessential. Therefore Exercise B.14 and Proposition B.15 easily extend to our case and we come to the following conclusion.

Proposition 9.29 *The map $u \to \hat{u}$ (9.119) is a continuous linear operator from $\mathcal{S}(\mathbf{E}_{-}^{n-1})$ to $\mathcal{S}(\overline{\mathbf{M}}_{+}^{n-1})$; its null space is the origin and its range is everywhere dense in $\mathcal{S}(\overline{\mathbf{M}}_{+}^{n-1})$. The adjoint map $T \to F$ (9.118) sets up a one-to-one correspondence between the generalized functions $F \in \mathcal{S}'(\mathbf{E}_{-}^{n-1})$ and the generalized functions $F \in \mathcal{S}'(\mathbf{E}_{-}^{n-1})$ that are continuous linear functionals on $\mathcal{S}(\mathbf{E}_{-}^{n-1})$ in the weakened topology.*

Here the weakened topology on $\mathcal{S}(\mathbf{E}_{-}^{n-1})$ is defined by the family of norms

$$\|u\|_{l,m}^{\wedge} = \|\hat{u}\|_{l,m}^{\overline{\mathbf{M}}_{+}^{n-1}} \quad (l, m = 0, 1, \ldots). \tag{9.120}$$

We now list the most important properties of the Schwinger functions.

e.1 (*Admissible character of singularity and growth*). The Schwinger function $s_{l_1 \ldots l_n}^{(\kappa_1 \ldots \kappa_n)}(x_1, \ldots, x_n)$ is defined on $\mathbf{E}_{\neq}^n$ as a generalized function in $\mathcal{S}'(\mathbf{E}_{\neq}^n)$. (For $n = 0$, we define the 0-point Schwinger function $s^{[0]} = 1$.)

e.2 (*Property of the Hermitian adjoint*).

$$s_{\bar{l}_1 \ldots \bar{l}_n}^{(\bar{\kappa}_1 \ldots \bar{\kappa}_n)}(x_1, \ldots, x_n) = \overline{s_{l_n \ldots l_1}^{(\kappa_n \ldots \kappa_1)}(Tx_n, \ldots, Tx_1)}; \tag{9.121}$$

here

$$T(\mathbf{x},x^4)\equiv(\mathbf{x},-x^4). \tag{9.122}$$

e.3 (*Positive definiteness*).

$$\sum_{m,n=0}^{\infty}\sum_{\substack{\kappa_1\dots\kappa_m\\ l_1\dots l_m}}\sum_{\substack{\kappa'_1\dots\kappa'_n\\ l'_1\dots l'_n}}\int s^{(\bar\kappa_m\dots\bar\kappa_1,\kappa'_1\dots\kappa'_n)}_{\bar l_m\dots\bar l_1,l'_1\dots l'_n}(Tx_m,\dots,Tx_1,y_1,\dots,y_n)\times$$

$$\times\overline{f^{(\kappa_1\dots\kappa_m)}_{l_1\dots l_m}(x_1,\dots,x_m)}f^{(\kappa'_1\dots\kappa'_n)}_{l'_1\dots l'_n}(y_1,\dots,y_n)d^4x_1\dots d^4y_n\geq 0 \tag{9.123}$$

for any finite system $f\equiv\{f^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}\}$ of functions $f^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}$ in the respective space $\mathcal{S}(\mathbf{E}^n_<)$; for $n=0$, $f^{[0]}$ is a complex number.

e.4 (*Euclidean covariance*).

$$\sum_{m_1\dots m_n}V^{(\kappa_1)}_{l_1m_1}(U_l^{-1},U_r^{-1})\dots V^{(\kappa_n)}_{l_nm_n}(U_l^{-1},U_r^{-1})s^{(\kappa_1\dots\kappa_n)}_{m_1\dots m_n}(Rx_1+a,\dots,Rx_n+a)=$$

$$=s^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(x_1,\dots,x_n)\quad\text{for all } a\in\mathbf{E},\ U_l,U_r\in SU(2). \tag{9.124}$$

e.5 (*Spectrum property*). The Schwinger function is representable in the domain $\mathbf{E}^n_<$ in the form

$$s^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(x_1,\dots,x_n)=S^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(x_1-x_2,\dots,x_{n-1}-x_n), \tag{9.125}$$

where $S^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(\xi_1,\dots,\xi_{n-1})$ is a generalized function in $\mathcal{S}'(\mathbf{E}^{n-1}_-)$ which is a continuous linear functional on $\mathcal{S}(\mathbf{E}^{n-1}_-)$ in the weakened topology.

e.6 (*Cluster property*). The limit formula

$$\lim_{\lambda\to\infty}\int(s^{[n]}(x_1,\dots,x_k,x_{k+1}+\lambda a,\dots,x_n+\lambda a)-s^{[k]}(x_1,\dots,x_k)\times$$

$$\times s^{[n-k]}(x_{k+1},\dots,x_n))f(Tx_k,\dots,Tx_1)g(x_{k+1}\dots x_n)dx_1\dots dx_n=0 \tag{9.126}$$

holds for any vector $a=(\mathbf{a},0)\neq 0$ and any $f(x_1,\dots,x_k)\in\mathcal{S}(\mathbf{E}^k_<)$, $g(x_{k+1},\dots,x_n)\in$ $\in\mathcal{S}(\mathbf{E}^{n-k}_<)$.

In formula (9.126) we have omitted the indices of the Schwinger function for brevity.

e.7 (*Permutation property*). For any permutation π of the indices $(1,\dots,n)$ the following relation holds:

$$s^{(\kappa_{\pi 1}\dots\kappa_{\pi n})}_{l_{\pi 1}\dots l_{\pi n}}(x_{\pi 1},\dots,x_{\pi n})=\epsilon_F(\pi)s^{(\kappa_1\dots\kappa_n)}_{l_1\dots l_n}(x_1,\dots,x_n), \tag{9.127}$$

($\epsilon_F(\pi)$ is the parity of the permutation of the Fermi fields and is defined by (9.37)).

We see that properties e.1–e.7 completely duplicate properties w.1–w.7 of the Wightman functions. According to Osterwalder and Schrader's result, they are equivalent to the Wightman axioms, therefore they are called the *Osterwalder-Schrader axioms*.

We begin by proving the implication (w)$\Rightarrow$(e).

Theorem 9.30. *The Schwinger functions in the Wightman theory (with the normal connection between spin and statistics) satisfy the conditions* e.1–e.7.

■ To shorten the notation we restrict the proof to the case of a single Hermitian scalar field $\phi(x)$. The Schwinger function $s^{[n]}(x_1, \ldots, x_n)$ is, by definition, a real analytic function in $\mathbf{E}^n_{\neq}$ and properties e.4 (Euclidean invariance) and e.7 (symmetry under permutations) follow directly from the properties of invariance (Corollary 9.4) and symmetry (Theorem 9.6) of an analytic Wightman function in the symmetrized tube $\mathcal{T}_n^S$.

In order that the Schwinger function $s^{[n]}(x_1, \ldots, x_n)$ should define a functional in $\mathcal{S}'(\mathbf{E}^n_{\neq})$, it is clearly sufficient that it satisfies the estimate

$$|s^{[n]}(x_1, \ldots, x_n)| \leq a(1 + |x|)^m \left(\min_{j \neq k} |x_j - x_k|\right)^{-l}, \quad x \in \mathbf{E}_{\neq}, \tag{9.128}$$

where a, m, l are non-negative numbers. For the proof of (9.128) we use the estimate (8.42) for an analytic Wightman function in the upper tube; for $\mathbf{E}^n_{<}$ we then obtain

$$|s^n(x_1, \ldots, x_n)| \leq A(1 + |x|)^m \left(\min_{j \neq k} |x_j^4 - x_k^4|\right)^{-l}, \quad x \in \mathbf{E}_{<}. \tag{9.129}$$

Then, according to Exercise 9.18, there exists for any point $x \in \mathbf{E}^n_{\neq}$ a permutation π, a four-dimensional rotation R and a translation a such that $(Rx_{\pi 1} + a, \ldots, Rx_{\pi n} + a) \in \mathbf{E}^n_{<}$ and

$$\min_{j \neq k} |(Rx_j)^4 - (Rx_k)^4| \geq c \min_{j \neq k} |x_j - x_k|,$$

where c is a positive constant (depending only on n). The symmetry under permutations and Euclidean invariance * of the Schwinger functions now enable us to rewrite (9.129) in the form (9.128).

We have shown that the Schwinger function $s^{[n]}(x_1, \ldots, x_n)$ can be regarded as a generalized function in $\mathcal{S}'(\mathbf{E}^n_{\neq})$ satisfying the conditions e.4 (Euclidean invariance) and e.7 (symmetry under permutations). The spectrum property e.5 holds by virtue of Proposition 9.29 (where the roles of $T(\xi_1, \ldots$ $\ldots, \xi_{n-1})$ and $F(q_1, \ldots, q_{n-1})$ are played respectively by $S^{[n]}(\xi_1, \ldots, \xi_{n-1})$ and $\widetilde{W}^{[n]}(q_1, \ldots, q_{n-1})$).

Properties e.2 and e.3 follow directly from the next exercise.

Exercise 9.34.(a) Use the spectrum condition to prove that the vector-valued generalized functions

$$\Psi^{[n]}(x_1, \ldots, x_n) = \phi(x_1) \ldots \phi(x_n)|0\rangle \quad (x_j \in \mathbf{M}) \tag{9.130}$$

admit an analytic continuation with respect to the variables $z_j = x_j + iy_j$ to the tube

$$y_1 \in V^+, \quad y_{j+1} - y_j \in V^+ \quad (j = 1, \ldots, n-1). \tag{9.131}$$

(b) Let

$$\chi^{[n]}(x_1, \ldots, x_n) = \Psi^{[n]}(x_1', \ldots, x_n'), \quad (x_1, \ldots, x_n) \in \mathbf{E}^n_{<}$$

(the corresponding points $(x_1', \ldots, x_n')$ are contained in the tube (9.131)). Prove the relation

$$\langle \chi^{[m]}(x_1, \ldots, x_m), \chi^{[n]}(y_1, \ldots, y_n) \rangle = s^{[m+n]}(Tx_m, \ldots, Tx_1, y_1, \ldots, y_n). \tag{9.132}$$

Finally, the cluster property e.6 is also an immediate corollary of (9.132) and the cluster property in the Wightman theory (see Proposition 7.1). ■

The Schwinger functions $s^{[n]}(x_1, \ldots, x_n)$ were defined above on the set $\mathbf{E}^n_{\neq}$. However, in certain cases they can be naturally extended to the whole of $\mathbf{E}^n$ as generalized functions in $\mathcal{S}'(\mathbf{E}^n)$; these extensions are also denoted by $s^{[n]}(x_1, \ldots, x_n)$. We give an example of the free scalar field.

Exercise 9.35. (a) Prove that the two-point Schwinger function of the scalar neutral field $\phi(x)$ is the restriction to $\mathbf{E}^2_{\neq}$ of the following generalized function $s^{[2]}(x, y) \equiv S^{[2]}(x - y) \in \mathcal{S}'(\mathbf{E}^2)$:

$$S^{[2]}(x - y) = \int e^{i(p, x-y)} \frac{1}{|p|^2 + m^2} d_4 p; \tag{9.133}$$

---

* In the case of spin-tensor fields, we must also take into account the uniform boundedness of the matrices $V(U_e, U_r)$ in (9.124).

here (p,x) is the Euclidean scalar product. [Hint: It suffices to restrict attention to the domain $x_1^4 - x_2^4 < 0$, carry out the integration with respect to p^4 in (9.133) and compare the result with the Wightman function $W^{[2]}(x-y)$, given by formulae (8.72) and (8.50).]

(b) Prove that the k-point Schwinger function of the scalar neutral field is equal to zero when k is odd, while for even k $(=2n)$, it is the restriction to $\mathbf{E}^k_{\neq}$ of the following generalized function in $\mathcal{S}'(\mathbf{E}^k)$:

$$s^{[2n]}(x_1,\ldots,x_{2n}) = \sum_P s^{[2]}(x_{P_1},x_{P_2})\ldots s^{[2]}(x_{P(2n-1)},x_{P_n}); \tag{9.134}$$

here P ranges over the permutations of the indices $1,\ldots,2n$ such that $P1 < P3 < \ldots < P(2n-1)$ and $P(2j-1) < P(2j)$ $(j=1,\ldots,n)$. [Hint: Cf. Exercise 8.12.]

The discrete C,P,CP (or T) symmetries, if they occur in the theory, can clearly be expressed in terms of the Schwinger functions as well. We consider, for example, the case of a charged (genuinely) scalar field.

Exercise 9.36. In the theory of the charged scalar field we introduce the following abbreviations for the Schwinger functions:

$$s(x_1,\ldots,x_m|y_1,\ldots,y_n) = s^{\left(\overbrace{\phi\ldots\phi}^{m}\overbrace{\phi^*\ldots\phi^*}^{n}\right)}(x_1,\ldots,x_m,y_1,\ldots,y_n).$$

Prove that the conditions of C-, P- and/or CP-invariance in terms of the Schwinger functions have the following form respectively:

$$s(x_1,\ldots,x_m|y_1,\ldots,y_n) = s(y_1,\ldots,y_n|x_1,\ldots,x_m), \tag{9.135a}$$
$$s(x_1,\ldots,x_m|y_1,\ldots,y_n) = \overline{s(y_1,\ldots,y_n|x_1,\ldots,x_m)}, \tag{9.135b}$$
$$s(x_1,\ldots,x_m|y_1,\ldots,y_n) = \overline{s(x_1,\ldots,x_m|y_1,\ldots,y_n)}. \tag{9.135c}$$

C. Reconstruction Theorem in Terms of Schwinger Functions

Properties e.1–e.7 completely characterize the Schwinger functions so that the converse of Theorem 9.30 holds as well, namely, the implication (e)⇒(w).

Theorem 9.31 (Osterwalder-Schrader Reconstruction Theorem). *Every set of (generalized) functions* $\{s^{(\kappa_1,\ldots,\kappa_n)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n)\}$*, satisfying the axioms* e.1–e.7 *is the set of Schwinger functions of some system* $\{\phi^{(\kappa)}\}$ *of Wightman fields (with normal connection between spin and statistics).*

■ As in the proof of Theorem 9.30, we restrict ourselves to the case of a single scalar Hermitian field. According to Proposition 9.29, property e.5 enables us to represent (uniquely) the Schwinger function in the domain $\mathbf{E}^{n-1}_-$ in the form

$$S^{[n]}(\xi_1,\ldots,\xi_{n-1}) = \int \exp\left[\sum_{j=1}^{n-1}(q_j^0\xi_j^4 + i\mathbf{q}_j\boldsymbol{\xi}_j)\right]\widetilde{W}^{[n]}(q_1,\ldots,q_{n-1})d_4q_1\ldots d_4q_{n-1}, \tag{9.136}$$

where $\widetilde{W}^{[n]} \in \mathcal{S}'(\overline{\mathbf{M}^{n-1}_+})$; $\widetilde{W}^{[n]}$ is the candidate for the Fourier transform of the Wightman function; we regard it as a generalized function in $\mathcal{S}'(\mathbf{M}^{n-1})$ with support on the set $q_j^0 \geq 0$ $(j=1,\ldots,n-1)$. We will show that the set of generalized functions in $\mathcal{S}'(\mathbf{M}^n)$

$$w^{[n]}(x_1,\ldots,x_n) = \int \exp\left[-i\sum_{j=1}^{n-1} q_j(x_j - x_{j+1})\right]\widetilde{W}^{[n]}(q_1,\ldots,q_{n-1})d_4q_1\ldots d_4q_n \tag{9.137}$$

enjoys the characteristic properties of the Wightman functions (and we shall therefore refer to them as Wightman functions).

The property of the Hermitian adjoint for the Wightman functions is obtained by substituting the representation (9.136) into (9.121).

Next we turn to the positive definiteness condition e.3 of the Schwinger functions

$$\sum_{m,n=0}^{\infty} \int s^{[m+n]}(Tx_m, \dots, Tx_1, y_1, \dots, y_n)\overline{f^{[m]}(x_1, \dots, x_m)} f^{[n]}(y_1, \dots, y_n) d^4x_1, \dots, d^4y_n \geq 0$$

for any finite system $\{f^{[n]}\}$ of functions $f^{[n]} \in \mathcal{S}(\mathbf{E}^n_<)$. Clearly the connection (9.136), (9.137) between the generalized functions $s^{[n]}$ and $w^{[n]}$ can be rewritten in the form

$$s^{[n]}(x_1, \dots, x_n) = \int \exp\left[\sum_{j=1}^{n}(p_j^0 x_j^4 + i\mathbf{p}_j \mathbf{x}_j)\right] \tilde{w}^{[n]}(p_1, \dots, p_n) d_4p_1 \dots d_4p_n \tag{9.138}$$

for $0 < x_1^4 < x_2^4 \dots < x_n^4$. Substituting this representation into (9.123) yields

$$\sum_{m,n=0}^{\infty} \int \tilde{w}^{[m+n]}(-p_m, \dots, -p_1, q_1, \dots, q_n)\overline{F^{[m]}(p_1, \dots, p_m)} F^{[n]}(q_1, \dots, q_n) \times$$
$$\times d_4p_1 \dots d_4q_n \geq 0; \tag{9.139}$$

here $\tilde{w}^{[m+n]}(-p_m, \dots, -p_1, q_1, \dots, q_n)$ is regarded as a generalized function in $\mathcal{S}'(\mathbf{M}^{m+n}|X_m \times X_n) \approx$ $\approx \mathcal{S}'(X_m \times X_n)$, while

$$F^{[n]}(p_1, \dots, p_n) = \int \exp\left[\sum_{j=1}^{n}(p_j^0 x_j^4 + i\mathbf{p}_j \mathbf{x}_j)\right] f^{[n]}(x_1, \dots, x_n) dx_1 \dots dx_n \tag{9.140}$$

are functions in $\mathcal{S}(X_n)$; here the set X_n is given by

$$X_n = \left\{(p_1, \dots, p_n) \in \mathbf{M}^n : \sum_{j=k} p_j^0 \leq 0,\ k = 1, 2, \dots, n\right\}. \tag{9.141}$$

Exercise 9.37. Prove that the map $f^{[n]} \to F^{[n]}$ given by (9.140) defines a continuous linear map from $\mathcal{S}(\mathbf{E}^n_<)$ onto an everywhere dense subset of $\mathcal{S}(X_n)$ [Hint: The argument here is the same as in Exercise B.14 or Proposition 9.29.]

In view of the density of the functions (9.140) in $\mathcal{S}(X_n)$, we may consider $F \equiv \{F^{[n]}\}$ in (9.139) to be an arbitrary finite sequence of functions $F^{[n]} \in \mathcal{S}(X_n)$, while in view of the properties of the support of $\tilde{w}^{[m+n]}$, we can regard the $F^{[n]}$ as arbitrary functions in $\mathcal{S}(\mathbf{M}^n)$ (such that the sequence $F^{[n]}$ remains finite as before). We have thus proved the positive definiteness condition for the Wightman functions $w^{[n]}$.

By construction, the Wightman functions $w^{[n]}$ are invariant with respect to translations and three-dimensional rotations. For the proof of Poincaré-invariance, it suffices to prove invariance with respect to pure Lorentz rotations; this invariance can be expressed in the infinitesimal form:

$$\sum_{j=1}^{n-1}\left(q_j^0 \frac{\partial}{\partial q_j^\alpha} + q_j^\alpha \frac{\partial}{\partial q_j^0}\right)\widetilde{W}^{[n]}(q_1, \dots, q_{n-1}) = 0, \quad \alpha = 1, 2, 3. \tag{9.142}$$

We use the Euclidean invariance of the Schwinger functions, from which it follows that

$$\sum_{j=1}^{n-1}\left(\xi_j^4 \frac{\partial}{\partial \xi_j^\alpha} - \xi_j^\alpha \frac{\partial}{\partial \xi_j^4}\right) S^{[n]}(\xi_1, \dots, \xi_{n-1}) = 0, \quad \alpha = 1, 2, 3,$$

in the domain $\mathbf{E}_+^{n-1}$. Hence by substituting the representation (9.138) and using the uniqueness of this kind of representation, we arrive at (9.142).

Thus the generalized functions $\widetilde{W}^{[n]}(q_1, \dots, q_{n-1})$ are Lorentz-invariant and since by construction they have support in $\overline{\mathbf{M}}_+^{n-1}$ their supports are in fact concentrated in the largest closed Lorentz-invariant subset contained in $\overline{\mathbf{M}}_+^{n-1}$, that is, in $\overline{V}_+^{n-1}$. Thus the spectrum property of the Wightman functions is proved.

For the proof of locality w.7, we use the fact that (in view of the spectrum condition) the Wightman functions $w^{[n]}(x_1,\ldots,x_n)$ can be continued analytically to the lower tube $\mathcal{T}_n^-$ and hence (in view of Lorentz-invariance), to the extended tube $\mathcal{T}_n$. Formula (9.138) expresses the relation between the Schwinger function and the analytic Wightman function

$$s^{[n]}(x_1,\ldots,x_n) = w^{[n]}(x'_1,\ldots,x'_n)$$

for $(x_1,\ldots,x_n) \in \mathbf{E}^n_<$ (the notation being the same as in (9.113)). Consequently, the Schwinger function is in fact a real analytic function in the domain $\mathbf{E}^n_<$ and since, furthermore, it is Euclidean-invariant and symmetric with respect to $x_1,\ldots,x_n$, it is real analytic in the entire domain of definition $\mathbf{E}^n_{\neq}$.

Let π be a permutation of the indices $1,\ldots,n$. We fix a point $a \equiv (a_1,\ldots,a_n) \in \mathbf{E}^n_{\neq}$ such that the numbers a^4_j and $a^3_{\pi j}$ are positive and increase as j increases. Clearly, $a \in \mathbf{E}^n_<$, so that $a' \in \mathcal{T}_n^-$. The point a can be taken into $\pi^{-1}\mathbf{E}^n_<$ by a four-dimensional rotation, therefore $a' \in \mathcal{T}_n^- \cap \pi^{-1}\mathcal{T}_n^-$. The symmetry relation $s^{[n]}(x_1,\ldots,x_n) = s^{[n]}(x_{\pi 1},\ldots,x_{\pi n})$ can be written in some neighbourhood $\mathcal{O}$ of a in the form of the equality

$$w^{[n]}(x'_1,\ldots,x'_n) = w^{[n]}(x'_{\pi 1},\ldots,x'_{\pi n}), \quad x \in \mathcal{O},$$

whence it follows that the functions $w^{[n]}(z_1,\ldots,z_n)$ and $w^{[n]}(z_{\pi 1},\ldots,z_{\pi n})$ are equal in a complex neighbourhood of the point $a' \in \mathcal{T}_n^- \cap \pi^{-1}\mathcal{T}_n^-$ and, hence, throughout $\mathcal{T}_n^- \cap \pi^{-1}\mathcal{T}_n^-$. We conclude that the Wightman function $w^{[n]}(z_1,\ldots,z_n)$ is analytic in the extended tube $\mathcal{T}_n^S$ and symmetric in $z_1,\ldots,z_n$, therefore (by Proposition 9.11) the generalized Wightman function $w^{[n]}(x_1,\ldots,x_n)$ satisfies the locality condition w.7.

We have now checked all the properties of the Wightman function apart from the cluster property w.6. We conclude from the Wightman reconstruction theorem 8.6 (and the remark following it) that the generalized functions $w^{[n]}(x_1,\ldots,x_n)$ are in fact the Wightman functions of some Hermitian scalar field $\phi(x)$ satisfying all the Wightman axioms with the possible exception of the uniqueness of the vacuum. It remains to check this property. To this end, we note that the vectors $\chi^{[n]}(x_1,\ldots,x_n)$ defined by (9.130), (9.131) for $(x_1,\ldots,x_n) \in \mathbf{E}^n_<$, form a total subset of the Hilbert state space $\mathcal{H}$. (In fact, it follows by the principle of analytic continuation, that if the vector $\Phi \in \mathcal{H}$ is orthogonal to all such vectors $\chi^{[n]}(x_1,\ldots,x_n)$, then $\langle\Phi,\phi(x_1)\ldots\phi(x_n)\Psi_0\rangle \equiv 0$, which together with the cyclicity of the vacuum vector yields $\Phi = 0$.) The cluster property e.6 for the Schwinger functions, which holds by hypothesis, can be written in the form

$$\langle\chi^{[m]}(x_1,\ldots,x_m), U(\lambda a,1)\chi^{[n]}(y_1,\ldots,y_n)\rangle \to$$
$$\to \langle\chi^{[m]}(x_1,\ldots,x_m),\Psi_0\rangle\langle\Psi_0,\chi^{[n]}(y_1,\ldots,y_n)\rangle \quad \text{as } \lambda\to\infty;$$

here $a = (0,\mathbf{a}) \neq 0$. We conclude from the totality of the set of vectors $\chi^{[n]}(x_1,\ldots,x_n)$ that

$$\langle\Phi, U(\lambda a,1)\Psi\rangle \to \langle\Phi,\Psi_0\rangle\langle\Psi_0,\Psi\rangle \quad \text{as } \lambda\to\infty$$

for all $\Phi,\Psi \in \mathcal{H}$. Finally, the Lorentz-covariance enables us here to consider a to be an arbitrary spacelike vector in $\mathbf{M}$.

Thus we have proved the cluster property (8.19) as well, which completes the verification of the Wightman axioms. ∎

Theorems 9.30 and 9.31 show that in quantum field theory, the approach is possible in which one deals with four-dimensinal Euclidean space rather than four-dimensional Minkowski space. However, in some of the axioms e.1–e.7, the traces of the Wightman formalism and Minkowski space-time are too much in evidence, and this seems to be a shortcoming from the point of view of an independent Euclidean approach. In particular, this relates to the spectrum axiom e.5 which (according to Proposition 9.29) in fact postulates the existence of the Fourier transform of Wightman functions with the required properties of the supports. Attempts have been made to give up axiom e.5 entirely; a system of axioms is then obtained which is weaker than the Wightman axiom scheme. Thus Glaser (1974) showed that the existence of a quantum field theory and Wightman functions follows from the Osterwalder-Schrader axioms, but without axiom e.5; however, in general, they will be more singular than the Wightman theory allows (the language for their description can no longer be the theory of generalized functions of temperate growth, but the theory of hyperfunctions*). In order to preserve the admissible character of the singularities, as prescribed by the Wightman theory, Osterwalder and Schrader (1975) put forward a new axiom scheme which was sufficient to carry out the reconstruction theorem in its full extent: in place of axiom e.5, they postulated a certain *a priori* restriction on the growth of the Schwinger functions as the number of points n increases. Other sets of axioms of the Euclidean field theory were also suggested, from which the Wightman formalism follows (Nelson, 1973b; Hegerfeldt, 1974; Fröhlich, 1974; Nelson's axiom scheme, proposed on the basis of work of Symanzik (1969a), is in fact chronologically the first).

Appendix H. Parastatistics

H.1. FREE PARAFIELDS AND PARACOMMUTATION RELATIONS

The hypothesis of total symmetry or total antisymmetry of the state vector of a system of identical particles is a stronger assumption than the hypothesis of physical identity of particles. Other types of symmetry are theoretically possible and correspondingly, other *parastatistics.* In the quantum theory of free fields, third order commutation relations are possible with respect to the components of the field, defining *parafields* instead of the usual second order commutation relations for bosonic and fermionic fields. As was shown by Greenberg and Messiah (1964,1965), none of the known observable particles is a paraparticle of order $p > 1$. In connection with the attempts to use parastatistics for the construction of a consistent spectroscopy of hadrons within the framework of the quark model (see Greenberg, 1964) it should be noted that according to the analysis of a number of articles (see Govorkov,1982; Bogolubov et al.,1983), Greenberg's conjecture on the parafermi-statistics of quarks does not enable one to introduce the unitary gauge $SU(3)$-symmetry which lies at the basis of quantum chromodynamics and is thus a physically inequivalent alternative conjecture concerning coloured fermi-quarks.

It is convenient to construct the free parafields in a space of state vectors with a discrete orthonormal basis. Thus let $|\Phi_\nu\rangle$ be basis vectors of one-particle states labelled by the index ν ranging over a countable set of values. We introduce the operator a_ν^* of the creation of the state vector $|\Phi_\nu\rangle$ from the vacuum $|0\rangle$ and the adjoint annihilation operator a_ν. We also introduce the operator of the number of particles in state ν by the formula

$$n_\nu = a_\nu^* a_\nu \equiv \tfrac{1}{2}[a_\nu^*, a_\nu]_{-\sigma} + \sigma \cdot \tfrac{1}{2}, \tag{H.1}$$

where $\sigma = -$ for bose-particles and $\sigma = +$ for fermi-particles and $[A, B]_\sigma \equiv AB + \sigma BA$. The way n_ν commutes with the creation and annihilation operators does not depend on the type of the statistics, that is, on whether the operators a_ν commute or anticommute among themselves; in both cases they have the form

$$[n_\mu, a_\nu^*] = \delta_{\mu\nu} a_\nu^*, \; [a_\nu, n_\mu] = \delta_{\mu\nu} a_\nu. \tag{H.2}$$

It is natural to require that these relations also be preserved in the theory of free fields corresponding to generalized statistics (*parabose* or *parafermi*). But in the case of abnormal statistics, the first of equations (H.1) no longer holds and we must take for n_ν an (anti-)symmetrized expression of the same type as the right hand side of (H.1). Following Green (1953), we postulate the following more stringent system of commutation relations, from which (H.2) will follow:

$$[[a_\lambda^*, a_\mu]_{-\sigma}, a_\nu] = -2\delta_{\lambda\nu} a_\mu, \tag{H.3a}$$

$$[[a_\lambda, a_\mu]_{-\sigma}, a_\nu] = 0, \tag{H.3b}$$

where $\sigma = \pm$.

Exercise H.1. Prove that (H.3) implies the relation

$$[[a_\lambda, a_\mu]_{-\sigma}, a_\nu^*] = 2(\delta_{\mu\nu} a_\lambda - \sigma \delta_{\lambda\nu} a_\mu). \tag{H.4}$$

[Hint: Use the identity

$$[[A, B]_\sigma, C] + [[C, A]_\sigma, B] + [[B, C]_\sigma, A] \equiv 0, \tag{H.5}$$

* The resultant scheme can be applied to non-renormalizable quantum models (in this connection see also the remark in §9.1.E, where a direct approach in Minkowski space-time is indicated).

which generalizes the Jacobi identity.]

There follow from (H.3)–(H.5) a whole series of commutation relations of the same type (as well as the relations obtained from their Hermitian adjoints).

It is clear that if, for a representation of the a_λ's and a_λ^*'s with the cyclic vacuum vector $0|\rangle$, we define the number of particles in state ν by the equality

$$n_\nu = \tfrac{1}{2}[a_\nu^*, a_\nu]_{-\sigma} + C \tag{H.6}$$

(the constant C being defined by the condition $n_\nu|0\rangle = 0$), then the commutation relations (H.2) are obtained as a special case from (H.3).

There exists for each sign of σ a countable set of inequivalent fields satisfying (H.3) and indexed by a positive number p (*the order of the parastatistics*). For a given p, the operators a_ν are defined by the so-called *Green's rule*:

$$a_\nu = \sum_{\alpha=1}^{p} b_\nu^{(\alpha)}, \tag{H.7}$$

where for given α, the fields $b_\nu^{(\alpha)}$ satisfy the commutation relations

$$[b_\mu^{(\alpha)}, b_\nu^{(\alpha)*}]_\sigma = \delta_{\mu\nu}, \quad [b_\mu^{(\alpha)}, b_\nu^{(\alpha)}]_\sigma = 0, \tag{H.8}$$

while for different α, the commutation relations are of opposite type:

$$[b_\mu^{(\alpha)}, b_\nu^{(\beta)*}]_{-\sigma} = [b_\mu^{(\alpha)}, b_\nu^{(\beta)}]_{-\sigma} = 0 \quad (a \neq \beta). \tag{H.9}$$

In order to obtain a single valued (to within unitary equivalence) representation of the operators a_ν in Fock space, we require that all the $b_\nu^{(\alpha)}$ annihilate the vacuum vector:

$$b_\nu^{(\alpha)}|0\rangle = 0 \quad \text{for all } \nu \text{ and } \alpha. \tag{H.10}$$

The Hilbert space $\mathcal{B}_\sigma^{(p)}$ in which the operators $b_\nu^{(*)(\alpha)}$ act, is defined in the usual manner as the closure of the set of vectors of the form $P(b^*)|0\rangle$, where P is an arbitrary polynomial of the creation operators $b_\nu^{(\alpha)*}$. In view of (H.7), a representation of the algebra of the original operators a_ν and a_ν^* in the space $\mathcal{B}_\sigma^{(p)}$ can be realized in which the commutation relations (H.3) hold together with the conditions

$$a_\nu|0\rangle = 0 \tag{H.11}$$
$$a_\mu a_\nu^*|0\rangle = p\delta_{\mu\nu}|0\rangle. \tag{H.12}$$

Exercise H.2. Prove that in a representation with the cyclic vacuum vector $|0\rangle$ in which (H.11) and (H.12) hold, the operator of the number of particles (H.6) assumes the form

$$n_\nu = \tfrac{1}{2}([a_\nu^*, a_\nu]_{-\sigma} + \sigma p) \qquad \text{H.13)}$$

(that is, the constant C defined by the condition $n_\nu|0\rangle = 0$ is equal to $\frac{1}{2}\sigma p$).

It is not difficult to see that the representation of the algebra of the operators $a_\nu^{(*)}$ in $\mathcal{B}_\sigma^{(p)}$ is reducible.

This situation holds even for the finite-dimensional case (that is, when the index ν ranges over a finite number of values) and can be illustrated by the simple example of the *Duffin-Kemmer algebra*. In this example, $p = 2$ and the set of possible values of ν is also equal to two. We give the matrix realization of the Duffin-Kemmer algebra. We set

$$b_1^{(\alpha)} = \tfrac{1}{2}(i\gamma_1^{(\alpha)} - \gamma_2^{(\alpha)}), \quad b_2^{(\alpha)} = \tfrac{1}{2}(\gamma_0^{(\alpha)} - \gamma_3^{(\alpha)}), \quad \alpha = 1, 2. \tag{H.14}$$

By definition

$$[\gamma_\mu^{(1)}, \gamma_\nu^{(2)}] = 0, \quad \mu, \nu = 0, \ldots, 3. \tag{H.15}$$

Exercise H.3. Show that the matrices

$$a_j = b_j^{(1)} + b_j^{(2)} \tag{H.16}$$

and their Hermitian adjoints a_j^* satisfy the commutation relations for parafermi-fields (that is, the relations (H.3) with $\sigma = +$).

In order to reduce the triple commutations to the standard Duffin-Kemmer relations (see, for example, [U1],§5.2) we define the matrices β_μ via the formulae

$$\beta_0 = \tfrac{1}{2}(a_2 + a_2^*), \quad \beta_1 = \tfrac{1}{2i}(a_1 + a_1^*), \quad \beta_2 = \tfrac{1}{2}(a_1^* - a_1), \quad \beta_3 = \tfrac{1}{2}(a_2^* - a_2). \tag{H.17}$$

Exercise H.4. Show that *

$$\beta_\mu \beta_\nu \beta_\lambda + \beta_\lambda \beta_\nu \beta_\mu = g_{\lambda\nu}\beta_\mu + g_{\mu\nu}\beta_\lambda. \tag{H.18}$$

It is fairly easy to verify directly ([U1],§5.2) that the representation (H.16), (H.19) of the commutation relations (H.3) or (H.18) is reducible. This representation is 16-dimensional. In fact, in order that (H.15) be satisfied, it is necessary to define the $\gamma_\mu^{(\alpha)}$ as the Kronecker products

$$\gamma_\mu^{(1)} = \gamma_\mu \otimes 1, \quad \gamma_\mu^{(2)} = 1 \otimes \gamma_\mu. \tag{H.20}$$

It decomposes into three irreducible representations of the algebra generated by the operators a_ν, a_ν^*: one is ten-dimensional, one five-dimensional and one one-dimensional (trivial). Note that, apart from the trivial representation, only the ten-dimensional representation has a unique vacuum satisfying the condition

$$a_1|0\rangle = a_2|0\rangle = 0.$$

This fact has a general character: the space $\mathcal{B}_\sigma^{(p)}$ contains exactly one non-trivial subspace $\mathcal{H}_\sigma^{(p)}$ that is invariant with respect to the algebra of operators a_ν, a_ν^* and in which there is a unique vacuum vector satisfying (H.11).

Exercise H.5. Prove that an (unnormalized) basis in the space $\mathcal{H}_+^{(2)}$ of the ten-dimensional representation of the Duffin-Kemmer algebra can be defined (without introducing the operators $b_j^{(\alpha)}$) by the formulae

$$|0\rangle,\ a_j^*|0\rangle,\ a_j^*a_k^*|0\rangle,\ a_1^{*2}a_2^*|0\rangle = -a_2^*a_1^{*2}|0\rangle,$$

$$a_1^*a_2^{*2}|0\rangle = -a_2^{*2}a_1|0\rangle,\ a_1^{*2}a_2^{*2}|0\rangle,\ j,k = 1,2.$$

[Hint: Use the fact that in the case of parastatistics of order $p = 2$, the commutation relations (H.3) can be replaced by the simpler relations:

$$\begin{aligned} a_\lambda a_\mu^* a_\nu + \sigma a_\nu a_\mu^* a_\lambda &= 2(\delta_{\lambda\mu}a_\nu + \sigma\delta_{\mu\nu}a_\lambda), \\ a_\lambda^* a_\mu a_\nu + \sigma a_\nu a_\mu a_\lambda^* &= 2\sigma\delta_{\lambda\mu}a_\nu, \\ a_\lambda a_\mu a_\nu + \sigma a_\nu a_\mu a_\lambda &= 0.] \end{aligned} \tag{H.21}$$

Greenberg and Messiah (1965) have shown that for an infinite-dimensional one-particle space, all irreducible representations of the commutation relations (H.3) in Hilbert space with a unique cyclic vacuum for which (H.11) holds, also satisfy (H.12) for some positive p and are defined to within unitary equivalence by the equalities (H.11), (H.12). Each such representation is contained in a reducible representation in the space $\mathcal{B}_\sigma^{(p)}$ given by Green's formula (H.7). In particular, for $p = 1$ one obtains the usual Bose and Fermi statistics.

* If we take into account the fact that

$$\beta_\mu = \tfrac{1}{2}(\gamma_\mu^{(1)} + \gamma_\mu^{(2)}), \tag{H.19}$$

then we can obtain (H.18) immediately from (D.2) and (H.15).

H.2. Comment on the TCP-theorem and the connection between spin and parastatistics for local parafields

To formulate the locality property of parafields we use the following generalization of Green's rule (H.7). We define (after Dell-Antonio et al., 1964) the local parafield $A(x)$ as the sum of fields

$$A(x) = \sum_{\alpha=1}^{p} B^{(\alpha)}(x), \tag{H.22}$$

where the $B^{(\alpha)}$ satisfy the locality conditions of anomalous type:

$$[B^{(\alpha)}(x), B^{(\alpha)(*)}(y)]_\sigma = 0 \quad \text{for } (x-y)^2 < 0, \tag{H.23a}$$

$$(1-\delta_{\alpha\beta})[B^{(\alpha)}(x), B^{(\beta)(*)}(y)]_{-\sigma} = 0. \tag{H.23b}$$

Exercise H.6. Prove that (H.22)–(H.23) imply the local paracommutation relations for the parafield

$$[[A(x), A(y)]_{-\sigma}, A(z)] = 0, \tag{H.24a}$$

if at the same time

$$(x-z)^2 < 0 \text{ and } (y-z)^2 < 0. \tag{H.24b}$$

In connection with the definition (H.22) we note that by requiring (H.22), (H.23), we have imposed certain restrictions on the auxiliary fields $B^{(\alpha)}$ in the extended Hilbert space $\mathcal{B}_\sigma^{(p)}$, but not in the physical space of state vectors $\mathcal{H}_\sigma^{(p)}$ in which the parafield A acts. The restriction (H.23b) is particularly stringent; it is in fact equivalent to the supposition that there is no interaction between the different fields $B^{(\alpha)}$. Furthermore, we recall that it is only for free fields that it has been proved that Green's rule exhausts all the representations of the paracommutation relations.* Furthermore, the introduction of the system of fields $B^{(\alpha)}$ in the very definition of the parafield A reduces the parafields to a system of ordinary (Bose- or Fermi-) fields with anomalous connection between spin and statistics (for $p > 1$) of the same type as those considered in §9.3 (hence, such a system of fields possesses additional symmetry). Therefore in making the "technical" assumptions, the consideration of questions of TCP-invariance and the connection between spin and statistics becomes completely trivial: it simply reduces to the corresponding questions of the Wightman theory; namely, in order to establish the connection between spin and parastatistics, it suffices to note that if the field A is to have specified transformation properties with respect to the Poincaré group, one must require that each field $B^{(\alpha)}$ has the same transformation properties. It now follows from (H.22) and Lemma 9.21 that A has integral (or half-integral) spin if it is subject to Bose (or Fermi) parastatistics. As noted in §9.2.A, TCP-invariance also holds in this case (because the locality condition of the system of fields $B^{(\alpha)}$ can be reduced to normal form by means of the Klein transformation). Furthermore, the law of the TCP-transformation of the fields can be taken in the same form (see §9.2.A) as in the case of normal connection between spin and statistics.

To prove this, it suffices to verify that the Wightman functions of the fields $B^{(\alpha)}(x)$ satisfy the weak locality condition (§9.2.B). This condition clearly holds for Wightman functions of the fields $B^{(\alpha)}(x)$ with the same value of α (since according to (H.23a), these fields satisfy the ordinary locality condition with normal connection between spin and statistics). The general case easily follows from (H.23b) and the factorization property:

$$\langle X^{(1)} \dots X^{(p)} \rangle_0 = \langle X^{(1)} \rangle_0 \dots \langle X^{(p)} \rangle_0, \tag{H.25}$$

where the $X^{(\alpha)}$ is an arbitrary polynomial formed from fields $B^{(\alpha)}(x)$ with a fixed value of α.

Exercise H.7. Prove the factorization property (H.25). [Hint: In order to prove, for example, the relation

$$\langle X^{(1)} \dots X^{(p-1)} X^{(p)} \rangle_0 = \langle X^{(1)} \dots X^{(p-1)} \rangle_0 \langle X^{(p)} \rangle_0,$$

*In the case of free parafields, the equivalence of locality and paralocality has also been proved (see Araki et al., 1966).

one can use (H.23b), from which it follows that

$$\langle X^{(1)} \dots X^{(p-1)} U(a,1) X^{(p)} \rangle_0 = \pm \langle X^{(p)} U(-a,1) X^{(1)} \dots X^{(p-1)} \rangle_0;$$

now deduce from the spectrum condition that both sides of the latter equation are independent of $a \in \mathbf{M}$ and hence,

$$\langle X^{(1)} \dots X^{(p-1)} U(a,1) X^{(p)} \rangle_0 = \langle X^{(1)} \dots X^{(p-1)} \rangle_0 \langle X^{(p)} \rangle_0.]$$

Appendix I. Infinite-Component Fields

I.1. Elementary representations of $SL(2,C)$

The concept of an infinite-component field (ICF for short) is the result of abandoning the "technical" requirement that the representations of the Lorentz group according to which the fields transform (say, in the Wightman formalism) be finite-dimensional. This idea turned up at the earliest stages of quantum field theory: in 1932, Majorana gave an example of an infinite-dimensional wave equation $(i\Gamma^\mu \partial_\mu - M)\psi(x) = 0$ without negative-energy solutions of non-negative square mass, that is, without "antiparticles". The subsequent development of the ICF's was stimulated by one of the ideas of the theory of dynamical symmetries, namely, the reduction of a composite system with infinite mass spectrum* (examples being the hydrogen atom and an infinite series of hadrons of Regge trajectory type in the quark scheme) to a single ICF, the latter being a more "elementary" object (see, for example, Barut and Kleinert, 1967; Fronsdal, 1967; Ruegg et al., 1967; Delbourgo et al., 1967; Budini, 1968). On the other hand, the rejection of the requirement of a finite number of Lorentz components is of interest from the standpoint of the algebraic approach (which has assumed greater and greater universal significance for the axiomatic construction of quantum field theory), since the local algebras constructed on the basis of a local quantum ICF can have specific properties (some of which we shall encounter in due course). Running ahead (see §I.3), it should be noted, however, that the description of composite systems by means of ICF's has met with difficulties which, it would seem, require a weakening of the postulate of (strict) locality.**

Here by ICF's we mean fields that transform according to infinite-dimensional irreducible representations of $SL(2,C)$ (the covering group for $L_+^\uparrow$) or (the more general case) according to so-called elementary representations of $SL(2,C)$.† These representations are parametrized by the ordered pair $\chi \equiv (\lambda, \mu)$ of complex numbers λ, μ such that $\lambda - \mu \in \mathbf{Z}$. In Appendix C.5 we called such a pair χ an index and denoted the set of them by $\mathcal{X}$. For different purposes, the space of the representation of the index χ can be chosen in different (topologically inequivalent) ways. Of special interest is the representation T_χ of $SL(2,C)$ in the space $\mathfrak{D}_\chi \equiv \mathfrak{D}_\chi(\mathring{\mathbf{C}}^2)$ of all complex C^∞-functions $u(\omega)$ in $\mathring{\mathbf{C}}^2 \equiv \mathbf{C}^2 \backslash \{0\}$ that are homogeneous of index χ or, equivalently, homogeneous of bidegree $(\lambda - 1, \mu - 1)$. (This means that they satisfy (C.48) with $n = 2$.) The topology in $\mathfrak{D}_\chi$ is induced by the natural topology of $\mathcal{E}(\mathring{\mathbf{C}}^2)$. The element $\underset{\sim}{\Lambda} \in SL(2,C)$ acts on the function $u \in \mathfrak{D}_\chi$ according to the formula

$$(T_\chi(\underset{\sim}{\Lambda})u)(\omega) = u(\underset{\sim}{\Lambda}^{-1}\omega). \tag{I.1}$$

The valency operator is given by

$$T_\chi(\underset{\sim}{\Lambda})|_{\underset{\sim}{\Lambda}=-1} = (-1)^{\lambda-\mu}, \tag{I.2}$$

so that T_χ defines a single-valued or two-valued representation of the Lorentz group $L_+^\uparrow$ depending on whether $\lambda - \mu$ is even or odd. (In the context of §I.2, we then say that the corresponding ICF carries

* Or energy in the centre of mass frame.

** A more radical approach is to do away with the local fields, replacing them by extended objects of string type (as suggested by dual resonance models).

† The elementary representations of $SL(2,C)$ are obtained by analytic continuation with respect to the (continuous) parameter of the principal series of unitary irreducible representations ([G3], [G2]).

integral or half-integral spin.) The representation T_χ is irreducible for all $\chi \in \mathcal{X}$ except for the cases $\chi \in \Xi_+^{[2]}$ or $\chi \in \Xi_-^{[2]}$, that is, the cases

$$\text{a)}\quad \lambda,\mu = 1,2,3,\ldots \quad \text{or} \quad \text{b)}\quad \lambda,\mu = -1,-2,-3,\ldots; \tag{I.3}$$

here, two such non-exceptional representations T_χ and $T_{\chi'}$ are equivalent if and only if $\chi' = \pm\chi$. In the exceptional case $\chi \in \Xi_+^{[2]}$ (that is, (I.3a)), $\mathfrak{D}_\chi$ contains the finite-dimensional subspace $\mathfrak{p}^{(j,k)}$ of polynomials in $\omega, \bar{\omega}$ that are homogeneous of degree $2j = \lambda - 1$ in ω and homogeneous of degree $2k = \mu - 1$ in $\bar{\omega}$; the finite-dimensional irreducible representation $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ is realized in this space. In the exceptional case $\chi \in \Xi_-^{[2]}$ (that is, (I.3b)), T_χ is also reducible: $\mathfrak{D}_\chi$ contains a closed subspace of finite-codimension, so that there is realized in the quotient space a finite-dimensional representation which is equivalent to $\mathfrak{D}^{(j,k)}$ for $2j = -\lambda - 1,\ 2k = -\mu - 1$.

In a number of situations, one can (and sometimes has to) choose as the "carrier" of the representation of index χ, the completion of $\mathfrak{D}_\chi$ in some weaker (for example, the Hilbert) topology; in this case, the role of the space $\mathfrak{D}_\chi$ is that it is a kind of "kernel" of the (completed) representation. We shall make use of this remark below in a situation when the (weak) topology in $\mathfrak{D}'(\mathring{C}^2)$ is chosen as the weaker topology; it is not difficult to see that corresponding to the completion of $\mathfrak{D}_\chi$ is a space $\mathfrak{d}_\chi \equiv \mathfrak{d}_\chi(\mathring{C}^2)$ of generalized functions $u(\omega)$ of $\omega \in \mathring{C}^2$ that are homogeneous of index χ. According to Proposition C.8(b), $\mathfrak{d}_\chi$ is canonically isomorphic to the space $\mathfrak{D}'_{-\chi}$ of all continuous linear functionals on $\mathfrak{D}_\chi$:

$$\mathfrak{D}_\chi \subset \mathfrak{d}_\chi \approx \mathfrak{D}'_{-\chi}; \tag{I.4}$$

we shall also use this property in the definition of an IFC that transforms according to a representation of index χ.

I.2. Concept of a quantum IFC

If we assume that we can associate with one IFC an infinite set of particles with arbitrarily large spin, then for the IFC it becomes even more desirable than for the case of ordinary fields with a finite number of components, that non-renormalizable models be taken into consideration (that is, it is possible to have greater than polynomial growth in the p-space). We restrict ourselves to theories of pre-exponential growth, when it is still possible more or less to preserve the usual locality condition (as in the Remark in §9.1.D).

A quantum field that transforms according to the finite-dimensional representation $\mathfrak{D}^{(j,k)}$ of $SL(2,C)$ is an operator-valued distribution $\tilde{\phi}^{(\kappa)}(p,\omega,\bar{\omega})$ over the space $\mathcal{D}(\mathrm{M})$ in the momentum variable p, this distribution being a homogeneous polynomial of degree $2j$ in $\omega \in \mathrm{C}^2$ and of degree $2k$ in $\bar{\omega}$. Accordingly, we define the quantum IFC transforming according to an elementary representation of $SL(2,C)$ of index χ, as an operator-valued distribution $\tilde{\phi}^{(\kappa)}(p,\omega)$ over the space $\mathcal{D}(\mathrm{M} \times \mathring{C}^2)$ in the variables $p \in \mathrm{M},\ \omega \in \mathring{C}^2$, satisfying the condition of homogeneity of index χ with respect to ω:

$$\tilde{\phi}^{(\kappa)}(p, a\omega) = \phi_\chi^{[2]}(a)\tilde{\phi}^{(\kappa)}(p,\omega),\ a \in \mathring{C} \equiv \mathrm{C}\backslash\{0\}. \tag{I.5}$$

We suppose that in the Hilbert space $\mathcal{H}$ there is a common dense domain D on which the operators

$$\int \tilde{\phi}^{(\kappa)}(p,\omega)F(p,\omega)d_4p|d^2\omega d^2\bar{\omega}| \quad \text{for all } F \in \mathcal{D}(\mathrm{M} \times \mathring{C}^2)$$

are defined and that this domain is invariant with respect to these operators. It is clear that the ordinary case of a field with a finite number of components which transforms according to the representation $\mathfrak{D}^{(j,k)}$ enters into this scheme as the particular case with $\chi \equiv (\lambda,\mu) = (2j+1, 2k+1) \in \Xi_+^{[2]}$.

There is an alternative equivalent definition. Suppose that we are given the continuous bilinear functional

$$(f,u) \to \tilde{\phi}^{(\kappa)}(f;u) \equiv \int \tilde{\phi}^{(\kappa)}(p;u)f(p)d_4p \tag{I.6}$$

on $f \in \mathcal{D}(\mathrm{M}),\ u \in \mathfrak{D}_{-\chi}$, whose values are (in general, unbounded) operators which are defined on D along with their adjoints and which leave D invariant. In accordance with this, the ICF $\tilde{\phi}^{(\kappa)}(p,\omega)$ is defined as the kernel of the operator-valued functional $\tilde{\phi}^{(\kappa)}(f;u)$ in the sense that the correspondence between the two definitions is realized in the spirit of the isomorphism (I.4) (or Proposition C.8(b)):

$$\int \tilde{\phi}^{(\kappa)}(p,\omega)f(p)g(\omega)d_4p|d^2\omega d^2\bar{\omega}| = \tilde{\phi}^{(\kappa)}(f;I_{-\chi}g)$$

for all $f \in \mathcal{D}(\mathrm{M})$, $g \in \mathcal{D}(\mathring{\mathrm{C}}^2)$.

As always, $\phi^{(\bar{\kappa})}$ denotes the Hermitian conjugate field:

$$\tilde{\phi}^{(\bar{\kappa})}(p,\omega) \equiv (\tilde{\phi}^{(\kappa)}(-p,\omega))^*; \tag{I.8}$$

it has index of homogeneity $\chi^+ \equiv (\bar{\mu},\bar{\lambda})$.

The Wightman functions of the ICF in p-space

$$\begin{aligned}\tilde{w}^{(\kappa_1\ldots\kappa_n)}(p_1,\omega_1;\ldots;p_n,\omega_n) \equiv \\ \equiv (2\pi)^4\delta(p_1+\ldots+p_n)\widetilde{W}^{(\kappa_1\ldots\kappa_n)}(p_1,p_1+p_2,\ldots,p_1+\ldots+p_{n-1};\omega_1,\ldots,\omega_n) = \\ = \langle 0|\tilde{\phi}^{(\kappa_1)}(p_1,\omega_1)\ldots\tilde{\phi}^{(\kappa_n)}(p_n,\omega_n)|0\rangle\end{aligned} \tag{I.9}$$

are distributions in $\mathcal{D}'(\mathrm{M}^n \times (\mathring{\mathrm{C}}^2)^n)$ with respect to $p_1,\ldots,p_n,\omega_1,\ldots,\omega_n$, satisfying the conditions of separate homogeneity of index χ_j with respect to ω_j:

$$\begin{aligned}\tilde{w}^{(\kappa_1\ldots\kappa_n)}(p_1,a_1\omega_1;\ldots;p_n,a_n\omega_n) = \prod_{j=1}^{n}\phi^{[2]}_{\chi_j}(a_j)\times \\ \times\tilde{w}^{(\kappa_1\ldots\kappa_n)}(p_1,\ldots,p_n;\omega_1,\ldots,\omega_n) \quad \text{for } a_1,\ldots,a_n \in \mathring{\mathrm{C}}.\end{aligned} \tag{I.10}$$

It is not difficult to reformulate in terms of the Wightman functions (I.9) all the characteristic properties of §8.3.A (apart from locality), in other words, the properties of the adjoint, the spectrum, positive definiteness, Poincaré-covariance, and the cluster property. In particular, translational invariance is already taken into account in the first of the relations (I.9), while Lorentz-covariance means that

$$\begin{aligned}\widetilde{W}^{(\kappa_1\ldots\kappa_n)}(\Lambda(\underset{\sim}{A})q_1,\ldots,\Lambda(\underset{\sim}{A})q_{n-1};\underset{\sim}{A}\omega_1,\ldots,\underset{\sim}{A}\omega_n) = \\ = \widetilde{W}^{(\kappa_1\ldots\kappa_n)}(q_1,\ldots,q_{n-1};\omega_1,\ldots,\omega_n), \quad \underset{\sim}{A} \in SL(2,C).\end{aligned} \tag{I.11}$$

The spectrum property implies the following property of the support:

$$\operatorname{supp}\widetilde{W}^{(\kappa_1\ldots\kappa_n)}(q_1,\ldots,q_{n-1};\omega_1,\ldots,\omega_n) \subset (\overline{V}^+)^{n-1} \times (\mathring{\mathrm{C}}^2)^n. \tag{I.12}$$

Finally, for the formulation of locality (in the spirit of the Remark in §9.1.D) we confine ourselves to pre-exponential growth, that is, we suppose that the Laplace transform

$$\begin{aligned}w^{(\kappa_1\ldots\kappa_n)}(z_1,\omega_1;\ldots;z_n,\omega_n) \equiv W^{(\kappa_1\ldots\kappa_n)}(z_1-z_2,\ldots,z_{n-1}-z_n;\omega_1,\ldots,\omega_n) = \\ = \int \tilde{w}^{(\kappa_1\ldots\kappa_n)}(p_1,\omega_1;\ldots;p_n,\omega_n)e^{-i(p_1z_1+\ldots+p_nz_n)}d_4p_1\ldots d_4p_n\end{aligned} \tag{I.13}$$

is defined (in the sense of §B.1) for all $(z_1,\ldots,z_n) \in \mathcal{T}_n^-$ (here we take it that all the expressions in (I.13) are smoothed with respect to $\omega_1,\ldots,\omega_n$ with an arbitrary test function $g(\omega_1,\ldots,\omega_n) \in \mathcal{D}((\mathring{\mathrm{C}}^2)^n)$). The locality axiom now implies that the Wightman functions (I.13) can be analytically continued with respect to $(z_1,\ldots,z_n)$ to the symmetrized tubes $\mathcal{T}_n^S$ and have commutation relations of type (9.39):

$$w^{(\kappa_{\pi1}\ldots\kappa_{\pi n})}(z_{\pi1};\omega_{\pi1};\ldots;z_{\pi n},\omega_{\pi n}) = \epsilon_F(\pi)w^{(\kappa_1\ldots\kappa_n)}(z_1,\omega_1;\ldots;z_n,\omega_n), \quad \pi \in S_n. \tag{I.14}$$

Here we are supposing that all the fields are divided into "bosonic" ($F^{(\kappa)} = 0$) and "fermionic" ($F^{(\kappa)} = 1$) ones (but no connection between spin and statistics is assumed; as we shall see in §I.4, there may not be any).

On the basis of Propositions 5.13(a) and 5.14, we can equally well regard the Wightman function in the coordinate space as the partially holomorphic distribution:

$$w^{(\kappa_1\ldots\kappa_n)}(z_1,\omega_1;\ldots;z_n,\omega_n) \in \mathcal{D}'(T_n^S \times (\mathring{\mathbf{C}}^2)^n), \tag{I.15a}$$

$$\frac{\partial}{\partial \bar{z}_j^\mu} w^{(\kappa_1\ldots\kappa_n)}(z_1,\omega_1;\ldots;z_n,\omega_n) = 0, \quad j = 1,\ldots,n;\ \mu = 0,\ldots,3. \tag{I.15b}$$

I.3. Covariant structure of the two-point function. Infinite degeneracy of mass with respect to spin

The Wightman two-point functions are the simplest objects reflecting the basic principles of the theory (and in the case of free fields, completely determine the model). For this reason it is natural to begin the study of the properties of ICF's with the two-point functions.

To begin with we give the covariant representation in the momentum space without taking locality into account (but assuming the strong spectrum condition with mass gap $\mu > 0$); after this we go over to the coordinate space.*

Lemma I.1. *Every distribution $K(p;\omega,w) \in \mathcal{D}'(\overline{V}_\mu^+ \times \mathring{\mathbf{C}}^2 \times \mathring{\mathbf{C}}^2)$ satisfying the $SL(2,C)$-invariance condition*

$$K(\Lambda(\underset{\sim}{A})p;\ \underset{\sim}{A}\omega,\ \underset{\sim}{A}w) = K(p;\omega,w) \quad \textit{for } \underset{\sim}{A} \in SL(2,C), \tag{I.16}$$

is representable in the form

$$K(p;\omega,w) = \mathcal{K}(p^2, \bar{\omega}\tilde{p}\omega, \bar{w}\tilde{p}w, V), \tag{I.17}$$

where V is the matrix in $SU(2)$ constructed from the variables p,ω,w:

$$V = (\bar{\omega}\tilde{p}\omega \cdot \bar{w}\tilde{p}w)^{-1/2} \begin{pmatrix} \bar{\omega}\tilde{p}w & \sqrt{p^2}\, w\epsilon\omega \\ \sqrt{p^2}\,\overline{\omega\epsilon w} & \bar{w}\tilde{p}\omega \end{pmatrix}, \tag{I.18}$$

and

$$\mathcal{K} \in \mathcal{D}'([\mu^2,\infty) \times \mathbf{R}_+ \times \mathbf{R}_+ \times SU(2)). \tag{I.19}$$

By supplementing the $SL(2,C)$-invariance condition in Lemma I.1 with the condition of homogeneity with respect to ω, w, we obtain the covariant structure of the two-point function. Here we set

$$\sigma_i = \tfrac{1}{2}(\lambda_i - \mu_i) \quad (i = 1,2). \tag{I.20}$$

It is not difficult to see that for half-integral $\sigma_1 - \sigma_2$ the two-point function $\widetilde{W}^{(\bar{\kappa}_1\kappa_2)}$ is identically zero; therefore in what follows, we assume that

$$\sigma_1 - \sigma_2 \in \mathbf{Z}. \tag{I.21}$$

Proposition I.2. *In the momentum space, the two-point function of ICF's $\phi^{(\bar{\kappa}_1)}$ and $\phi^{(\kappa_2)}$ transforming according to the elementary representation of $SL(2,C)$ of indices χ_1^+ and χ_2 has, for $\sigma_2 \geq |\sigma_1|$,** the form*

$$\widetilde{W}^{(\bar{\kappa}_1\kappa_2)}(p;\omega,w) = (\bar{\omega}\tilde{p}w)^{(\bar{\lambda}_1+\bar{\mu}_1)/2-\sigma_2-1}(\bar{w}\tilde{p}w)^{\mu_2-1}(\bar{\omega}\tilde{p}w)^{\sigma_2+\sigma_1}(w\epsilon\omega)^{\sigma_2-\sigma_1} H(p^2,\nu), \tag{I.22}$$

where

$$\nu = \frac{|\bar{\omega}\tilde{p}w|^2 - p^2|\omega\epsilon w|^2}{\bar{\omega}\tilde{p}\omega \cdot \bar{w}\tilde{p}w}, \tag{I.23}$$

$$H(s,\nu) \in \mathcal{D}'_{1/X}([\mu^2,\infty) \times [-1,1]) \equiv (1/X)\cdot\mathcal{D}'([\mu^2,\infty)\times[-1,1]), \tag{I.24}$$

$$X \equiv X(\nu) = \Big(\frac{1-\nu}{2}\Big)^{|\sigma_1-\sigma_2|}\Big(\frac{1+\nu}{2}\Big)^{|\sigma_1+\sigma_2|}. \tag{I.25}$$

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* The results of this appendix are stated without proof (for which, see the bibliography).

** The other cases are similar.

The space $\mathcal{D}'_{1/X} \equiv (1/X) \cdot \mathcal{D}'$ featuring in (I.24) is defined in the spirit of formula (3.172) as the space of continuous linear functionals over $X \cdot \mathcal{D}([\mu^2, \infty) \times [-1,1])$, that is, the equality $H = (1/X)G$ means:

$$(H, Xf) = (G, f) \quad \text{for all } f \in \mathcal{D}([\mu^2, \infty) \times [-1,1]).$$

For the expansion of the two-point function in terms of the spin, we use the result of representing distributions on the interval $[-1,1]$ as a series in the Jacobi polynomials $P_n^{(\alpha,\beta)}(\nu)$.

Theorem I.3. *Let $\alpha, \beta > -1$ and $X(\nu) = \left(\frac{1-\nu}{2}\right)^\alpha \left(\frac{1+\nu}{2}\right)^\beta$ for $\nu \in [-1,1]$. An arbitrary distribution $h(\nu) \in \mathcal{D}'_{1/X}([-1,1])$ can be expanded in a series*

$$h(\nu) = \sum_{n=0}^{\infty} h_n P_n^{(\alpha,\beta)}(\nu), \tag{I.26}$$

converging in the weak topology of $\mathcal{D}'_{1/X}([-1,1])$. Here the Jacobi polynomials are interpreted as functionals on $X \cdot \mathcal{D}([-1,1])$ defined by the integral

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(\nu) g(\nu) d\nu, \quad g(\nu) \in X \cdot \mathcal{D}([-1,1]);$$

$\{h_n\}$ is a sequence of complex numbers of polynomial growth with respect to n:

$$|h_n| \le a(1+n)^r. \tag{I.27}$$

Conversely, every sequence $\{h_n\}$ satisfying (I.27) *defines a distribution in $\mathcal{D}'_{1/X}([-1,1])$ via the formula* (I.26).

Using Theorem I.3, we can expand the distribution $H(s,\nu)$ of Proposition I.2 in a series (first for $\sigma_2 \ge |\sigma_1|$ and then for the other cases in the same way):

$$H(s,\nu) = \sum_{J \ge |\sigma_1| \vee |\sigma_2|} P^{(J)}_{\sigma_1\sigma_2}(\nu) i^{\sigma_1 - \sigma_2} H_J(s), \tag{I.28}$$

where the $P^{(J)}_{\sigma\sigma'}(\nu)$ are the functions associated with the Jacobi polynomials ([V3],§3.3.9):

$$P^{(J)}_{\sigma\sigma'}(\nu) \equiv P^{(J)}_{-\sigma,-\sigma'} \equiv P^{(J)}_{\sigma'\sigma} = i^{\sigma'-\sigma} \sqrt{\frac{(J+\sigma')!(J-\sigma')!}{(J+\sigma)!(J-\sigma)!}} \times$$
$$\times \left(\frac{1-\nu}{2}\right)^{\frac{\sigma'-\sigma}{2}} \left(\frac{1+\nu}{2}\right)^{\frac{\sigma'+\sigma}{2}} P^{(\sigma'-\sigma, \sigma'+\sigma)}_{J-\sigma'}(\nu); \tag{I.29}$$

$$H_J(s) \in \mathcal{D}'([\mu^2, \infty)). \tag{I.30}$$

As a result we arrive at the following expansion of a two-point function in terms of the spin (the notation being the same as in (G.14)):

$$\widetilde{W}^{(\bar\kappa_1 \kappa_2)}(p;\omega,w) = \sum_{J \ge |\sigma_1| \vee |\sigma_2|} (\bar\omega \tilde p \omega)^{\frac{\bar\lambda_1 + \bar\mu_1}{2} - J - 1} (\bar w \tilde p w)^{\frac{\lambda_2 + \mu_2}{2} - J - 1} D^{(J)}_{\sigma_1\sigma_2}(Q) h_J(p), \tag{I.31}$$

where the $h_J(p)$ are Lorentz-invariant measures in $\mathcal{D}'(\overline{V}^+_\mu)$ (more precisely, they are complex Lorentz invariant measures on $\overline{V}^+_\mu$).

To reflect the locality property, we use the fact that in coordinate space the two-point function $W^{(\kappa_1\kappa_2)}(\zeta;\omega,w)$ is a distribution in $\mathcal{D}'(T_1 \times \mathring{\mathbb{C}}^2 \times \mathring{\mathbb{C}}^2)$ with the following properties:

(a) holomorphy with respect to ζ:

$$\frac{\partial}{\partial \bar\zeta^\mu} W^{(\kappa_1\kappa_2)}(\zeta;\omega,w) = 0, \quad \mu = 0, \ldots, 3;$$

(b) $SL(2,C)$-invariance:

$$W^{(\kappa_1\kappa_2)}(\Lambda(\underset{\sim}{A})\zeta;\underset{\sim}{A}\omega,\underset{\sim}{A}w) = W^{(\kappa_1\kappa_2)}(\zeta;\omega,w) \quad \text{for } \underset{\sim}{A}\in SL(2,C);$$

(c) separate homogeneity in ω, w:

$$W^{(\kappa_1\kappa_2)}(\zeta;a\omega,bw) = \phi^{[2]}_{\chi_1}(a)\phi^{[2]}_{\chi_2}(b)W^{(\kappa_1\kappa_2)}(\zeta;\omega,w) \quad \text{for } a,b\in \mathring{C};$$

(d) spectrum property: for $\zeta\in T_1^-$, $W^{(\kappa_1\kappa_2)}(\zeta;\omega,w)$ is the Laplace transform with respect to p of the distribution $\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,w)\in \mathcal{D}'(\overline{V}^+\times\mathring{C}^2\times\mathring{C}^2)$;

(e) $W^{(\kappa_1\kappa_2)}(\zeta;\omega,w)\mp W^{(\kappa_2\kappa_1)}(-\zeta,w,\omega)=0$.

This list of properties can be effectively taken into account by a manifestly covariant representation generalizing the representation of Källén-Lehmann type for finite-component fields. In the next proposition, $\Gamma^{\mu_1\cdots\mu_n}(\omega,w)$ denotes the kernel of an $SL(2,C)$-covariant tensor-valued continuous bilinear functional $\Gamma^{\mu_1\cdots\mu_n}(u,v)$ on $\mathfrak{D}_{-\chi_1}\times\mathfrak{D}_{-\chi_2}$ (see §C.5) which transforms with respect to the indices $\mu_1\ldots\mu_n$ according to the irreducible representation $\mathfrak{D}^{(n/2,n/2)}$; this means that $\Gamma^{\mu_1\cdots\mu_n}(\omega,w)$ is a symmetric tensor with respect to $\mu_1\ldots\mu_n$ with zero trace:

$$g_{\mu_1\mu_2}\Gamma^{\mu_1\mu_2\cdots\mu_n}(\omega,w)=0.$$

Proposition I.4. *A two-point function of an ICF in coordinate space $W^{(\kappa_1\kappa_2)}(\zeta;\omega,w)$ (satisfying conditions (a)–(e)) has the form*

$$\widetilde{W}^{(\kappa_1\kappa_2)}(\zeta;\omega,w)=\sum_n i^n\Gamma^{\mu_1\cdots\mu_n}(\omega,w)\partial_{\mu_1}\ldots\partial_{\mu_n}F_n(\zeta); \tag{I.32a}$$

correspondingly,

$$\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,w)=\sum_n \Gamma^{\mu_1\cdots\mu_n}(\omega,w)p_{\mu_1}\ldots p_{\mu_n}\widetilde{F}_n(p). \tag{I.32b}$$

Here the $\widetilde{F}_n(p)$ are scalar Lorentz-invariant distributions in $\mathcal{D}'(\mathrm{M})$ with supports in $\overline{V}^+$ and admitting Laplace transforms $F_n(\zeta)$ in the tube T_1^-; here only a finite number of the $\widetilde{F}_n$ are non-zero.

The above representations can be applied to the problem of "splitting" the mass in terms of the spin. We recalled above that the notion of an ICF can be used as a method for describing composite systems. Of special interest are models in which the mass of one-particle states depends on the spin in such a way that at each (discrete) value of the mass, the number of possible values of the spin is finite. There arises the question whether the corresponding ICF is local. It turns out that this is impossible: Grodsky and Streater (1968) proved the "no-go" theorem according to which locality and the spectrum property in an ICF theory implies that the mass is infinitely degenerate with respect to the spin. In terms of the representation (I.31) this means that if we exclude the finite-component case, then in each interval $M^2 < s < M'^2$, the series (I.31) cannot have just a finite number of non-zero terms. Only the case of renormalizable theories was considered in the Grodsky-Streater theorem and the result was obtained by a direct application of the Bogolubov-Vladimirov theorem (see Exercise 4.17). It might be thought that the degeneracies in the mass can be avoided by going over to non-renormalizable theories. But it turns out that the Grodsky-Streater result still holds if we go to the class of non-renormalizable theories of pre-exponential growth in p-space (which again have a natural formulation of locality).

Theorem I.5 (Generalization of the Grodsky-Streater theorem). *Suppose that (in the model of an ICF of pre-exponential growth in p-space) $\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,w)\in\mathcal{D}'(\overline{V}^+\times\mathring{C}^2\times\mathring{C}^2)$ is a two-point function (in momentum space) of the local fields $\phi^{(\kappa_1)}$ and $\phi^{(\kappa_2)}$ that transform according to elementary representations of $SL(2,C)$ of indices χ_1 and χ_2 respectively. If the conditions $\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,w)\not\equiv 0$ hold in the domain*

$$p^0>0,\ M^2<p^2<M'^2, \tag{I.33}$$

(where $0<M<M'$) and the expansion (I.31) of the two-point function in terms of the spin gives only a finite number of non-zero terms, then $\chi_1,\chi_2\in\Xi_+^{[2]}$ (and hence the spaces of the representations of indices χ_1 and χ_2 contain the invariant subspaces of homogeneous polynomials in $\omega,\bar\omega$ and

$w, \bar{w})$. *In the domain* (I.33), *only the "finite-component part" is present in* $\widetilde{W}^{(\kappa_1\kappa_2)}(p;\omega,w)$; *that is,* $W^{(\kappa_1\kappa_2)}(p;\omega,w)$ *is a polynomial in* $\omega,\bar{\omega},w,\bar{w}$.

The fact that composite systems without infinite degeneracy of the mass spectrum in terms of the spin do not admit an "elementary" description by a local field is an indication of the presence of an internal structure of these systems which is not realizable by local fields.

I.4. Absence of $\mathfrak{P}_+$-covariance and connection between spin and statistics in ICF models

We confine ourselves here to renormalizable theories, that is, in accordance with §I.2, an ICF $\phi^{(\kappa)}$ that transforms according to a representation of index $\chi \equiv \chi^{(\kappa)}$ of $SL(2,C)$ is regarded as an operator-valued generalized function $\phi^{(\kappa)}(x;u^{(\kappa)})$ over $\mathcal{S}(\mathrm{M})$ with respect to the variable x, which is linearly and continuously dependent on $u^{(\kappa)} \in \mathfrak{D}_{-\chi}$.

The ICF theory contrasts with the Wightman formalism of finite-component fields in the question of TCP-symmetry. In the Wightman theory (with axiom W.VIII (§8.2)), there is the anti-unitary operator Θ of total space-time reflection (TCP-operator) which gives a transformation of the fields according to the universal law $\phi(x) \to \phi'(x) = (i^{-F}V(-1,1) \times \phi(-x))^*$ (see (9.62b)). But infinite (in the present instance, elementary) representations of $SL(2,C)$ (or $L_+^\uparrow$) do not have an analytic continuation to $SL(2,C) \times SL(2,C)$ (or $L_+(C)$); in view of this, the previous formulation and, of course, the proof of the TCP theorem cannot now be carried out. Incidentally, a formal extrapolation $T_\chi(-1,1) = (-1)^{\lambda-1}$ of the finite-component formula $\mathfrak{D}^{(j,k)}(-1,1) = (-1)^{2j}$ to the infinite-dimensional case clearly leads to non-single-valuedness (for non-integral λ,μ).

In this situation it makes sense to replace the previous universal law (9.62b) of transformation of fields under TCP-symmetry, by allowing a more general law of transformation of fields with respect to total space-time reflection under which the matrices $i^{-F}V(-1,1)$ are replaced by suitable $SL(2,C)$-invariant operators which act, as before, on the spin variables (or spin indices) of the fields. For such a "generalized TCP-invariance" (if it exists) we shall use the general "neutral" term $\mathfrak{P}_+$-covariance. Thus by $\mathfrak{P}_+$-covariance of a system of ICF's, we mean firstly, the existence of an anti-automorphism of the field algebra $\mathcal{P}(\mathrm{M})$ having the meaning of total space-time reflection and acting on the basis fields $\phi^{(\kappa)}(x;u^{(\kappa)})$ according to a "pointwise" transformation of the form

$$\phi^{(\kappa)}(x;u^{(\kappa)}) \to \sum_{\kappa'} \phi^{(\kappa')}(-x;A^{(\kappa'\kappa)}u^{(\kappa)}), \tag{I.34}$$

where the $A^{(\kappa'\kappa)} : \mathfrak{D}_{-\chi} \to \mathfrak{D}_{-\chi'}$ are intertwining (that is $SL(2,C)$-invariant) operators* for the representation $T_{-\chi}$ and $T_{-\chi'}$; secondly, the realizability

$$U(\tilde{I}_{st})\phi^{(\kappa)}(x;u^{(\kappa)})U(\tilde{I}_{st})^{-1} = \sum_{\kappa'} \phi^{(\kappa')}(-x;A^{(\kappa'\kappa)}u^{(\kappa)})^* \tag{I.35}$$

of this anti-isomorphism via an anti-unitary operator $U(\tilde{I}_{st})$ with the correct group property with respect to the given representation of the Poincaré group **

$$U(\tilde{I}_{st})U(a,\underset{\sim}{\Lambda})U(\tilde{I}_{st})^{-1} = U(-a,\underset{\sim}{\Lambda}). \tag{I.36a}$$

Without imposing any essential restriction, we can suppose that the vacuum is invariant:

$$U(\tilde{I}_{st})|0\rangle = |0\rangle. \tag{I.36b}$$

*It is assumed that $A^{(\kappa'\kappa)} = 0$ if one of the fields $\phi^{(\kappa)}$, $\phi^{(\kappa')}$ is fermionic while the other is bosonic.

** In connection with this, see (7.152c). We recall that the special Poincaré group $\mathfrak{P}_+$ is the semi-direct product of the subgroup $\mathfrak{P}_+^\uparrow$ and the subgroup $\{e, I_{st}\}$ generated by the total reflection I_{st}. Equation (I.36a) in its turn ensures the existence of a unitary/anti-unitary representation U in $\mathcal{H}$ of a covering group for $\mathfrak{P}_+$, the latter being the semi-direct product of the spinor proper Poincaré group $\mathfrak{P}_0$ and some (possibly free) group generated by the element $\tilde{I}_{st}$. Since $I_{st}^2 = 1$, the unitary operator $U(\tilde{I}_{st})^2$ (if it is different from the identity) must define a superselection rule (that is, it must be a superselection operator).

It turns out that even in this most general form (I.35) (with "pointwise" transformation law of the fields), $\mathfrak{P}_+$-covariance of the system of ICF's may be missing. The aim of the present subsection is to prove this for the simplest models of ICF's. Another specific phenomenon, which turns up in our example, consists in the following: the system of ICF's may not be $\mathfrak{P}_+$-covariant, but $\mathfrak{P}_+$-covariance can emerge after a Borchers extension of the original system of fields (here, the Borchers extension is non-trivial, since the appended fields are expressed non-locally in terms of the original basis fields in x-space).

We now turn to the examples of ICF's. In our models we shall be dealing with a finite system $\{\phi^{(\kappa)}(x;u)\}_{\kappa=1,\dots,K}$ $(2 \leq K < \infty)$ of free Hermitian fields transforming according to a self-adjoint elementary representation * $\chi = (\frac{1}{2}, \frac{1}{2})$ of $SL(2,C)$. The Hermitian property of the fields means that:

$$\phi^{(\kappa)}(x;u)^* = \phi^{(\kappa)}(x;\bar{u}), \ \kappa = 1,\dots,K. \tag{I.37}$$

The fields $\phi^{(\kappa)}$ are free in the sense that they satisfy the Klein-Gordon equation with mass $m > 0$ and the (anti-)commutator $[\phi^{(\kappa)}(x;u),\ \phi^{(\kappa')}(y;v)]_\mp$ is a c-number, therefore such fields are uniquely determined by the two-point functions. Before defining them, we give a number of properties of the Majorana representation in $\mathfrak{D}_{-\chi}$. It is well known ([G2],§III.6) that $\mathfrak{D}_{-\chi}$ is a pre-Hilbert space with respect to the $SL(2,C)$-invariant scalar product $(u|v)$ (where $u, v \in \mathfrak{D}_{-\chi}$). We denote by H the Hilbert space that is the completion of $\mathfrak{D}_{-\chi}$. The closures $V(\underset{\sim}{A})$ of the operators $T_{-\chi}(\underset{\sim}{A})$ form a unitary representation in H. There then exists in H ([G3],§2.9.7) a covariant 4-vector of Hermitian operators Γ^μ $(\mu = 0,\dots,3)$ defined on $\mathfrak{D}_{-\chi}$ together with all the polynomials of Γ^μ, which commute with conjugation $u \to \bar{u}$ (covariance of Γ^μ means $V(\underset{\sim}{A})\Gamma^\mu V(\underset{\sim}{A}^{-1}) = \Lambda(\underset{\sim}{A})^\mu_\nu T^\nu$). Furthermore, Γ^0 is strictly positive, so that (multiplying by a constant if necessary) we can suppose that

$$\Gamma^0 \geq 1. \tag{I.38a}$$

It follows from the covariance of Γ^μ that

$$p_\mu \Gamma^\mu \geq m \quad \text{for } p^2 = m^2,\ p^0 > 0. \tag{I.38b}$$

We now define the free Hermitian ICF's $\phi^{(\kappa)}(x;u)$ $(\kappa = 1,\dots,K)$ by the two-point functions

$$\begin{aligned}\langle 0|\phi^{(\kappa)}(x;u)\phi^{(\kappa')}(y;v)|0\rangle &= (\bar{u}|M^{(\kappa\kappa')}\Big(i\Gamma^\mu \frac{\partial}{\partial x^\mu}\Big)\Big|v)\frac{1}{i}D_m^{(-)}(x-y) \equiv \\ &\equiv \int (\bar{u}|M^{(\kappa\kappa')}(p_\mu\Gamma^\mu)|v) 2\pi\theta(p^0)\delta(p^2-m^2)e^{-ip(x-y)}d_4p;\end{aligned} \tag{I.39}$$

here

$$M(t) = \sum_{j=0}^{n} M_j t^j \tag{I.40}$$

is a polynomial in the real variable t taking values in the set of Hermitian $K \times K$-matrices; we assume that $\det M_n \neq 0$. For the positive definiteness of the two-point function it suffices that

$$M(t) \geq 0 \quad \text{for all } t \geq m. \tag{I.41}$$

Locality imposes the condition

$$M(-t) \mp \overline{M(t)} = 0; \tag{I.42}$$

here the $\mp$ sign means $-$ for Bose-statistics and $+$ for Fermi-statistics, according as n is even or odd respectively.

It is not difficult to see by means of the Wightman reconstruction theorem that every $K \times K$-matrix $M(t)$ (I.40), being polynomially dependent on t and satisfying (I.41), (I.42), uniquely defines a system of free Hermitian bosonic (for n even) or fermionic (for n odd) ICF's that transform according to an elementary representation of index χ of $SL(2,C)$ and have the two-point functions (I.39).

* It is called the *Majorana representation* (and the same name is borne by a finite-dimensional representation of γ-matrices in the space of Majorana (that is, "real" Dirac) spinors in the Majorana basis; see §E.2).

The fields so constructed have half-integral spin and can be bosonic (for n even) or fermionic (for n odd). This shows that there is no connection between spin and statistics for ICF's.

Condition (I.35) of covariance with respect to total space-time reflection (that is, in fact, $\mathfrak{P}_+$-covariance) in our case means that

$$U(\tilde{I}_{st})\phi^{(\kappa)}(x;u)U(\tilde{I}_{st})^{-1} = \sum_{\kappa'} B^{\kappa'\kappa}\phi^{(\kappa')}(-x;\bar{u}), \tag{I.43}$$

where $B \equiv (B^{\kappa'\kappa})$ is a real* $K \times K$-matrix and $U(\tilde{I}_{st})$ is an anti-unitary operator. The anti-unitarity of $U(\tilde{I}_{st})$ (together with the invariance of the vacuum) imposes the following restriction on the two-point function:

$$\overline{\langle 0|\phi^{(\kappa)}(x;u)\phi^{(\kappa')}(y;v)|0\rangle} = \sum_{\kappa_1,\kappa_1'} B^{\kappa_1\kappa}B^{\kappa_1'\kappa'}\langle 0|\phi^{(\kappa_1)}(-x,\bar{u})\phi^{(\kappa_1')}(-y;\bar{v})|0\rangle, \tag{I.44}$$

which is written in terms of the matrix $M(t)$ in the form

$$\overline{M(t)} = B^T M(t)B. \tag{I.45}$$

We choose $M(t)$ so that the real linear span of the matrices M_j ($j = 0,\ldots,n$) coincides with the set of all Hermitian $K \times K$-matrices. Condition (I.45) then means that $a^T = B^T aB$ for all Hermitian $K \times K$-matrices a. Now it is easy to see that such matrices B do not exist (since otherwise we would have $ab = ba$ for all $K \times K$-matrices a, b). Hence the corresponding ICF model is not $\mathfrak{P}_+$-covariant.

Finally, we encounter the possibility of reinstating $\mathfrak{P}_+$-covariance at the cost of replacing the original system of fields by the Borchers extension.

Exercise I.1. Suppose that there exists a complex $K \times K$-matrix $X(t)$ that is polynomially dependent on the real parameter t and satisfies the relations

$$X(-t) \mp \overline{X(t)} = 0, \tag{I.46}$$

$$X^*(t)M(t)^{-1}X(t) = \overline{M(t)}; \tag{I.47}$$

let

$$Y(t) = M(t)^{-1}X(t). \tag{I.48}$$

(a) Prove that the fields

$$\psi^{(\kappa)}(x;u) = \sum_{\kappa'} \phi^{(\kappa')}\Big(x;Y^{(\kappa'\kappa)}\Big(i\Gamma^\mu \frac{\overleftarrow{\partial}}{\partial x^\mu}\Big)u\Big) \equiv \tag{I.49a}$$

$$\equiv \sum_{\kappa'} \int \tilde{\phi}^{(\kappa')}(p;Y^{(\kappa'\kappa)}(p_\mu\Gamma^\mu)u)e^{-ipx}d_4p \tag{I.49b}$$

are Hermitian (in the sense of (I.37)) and belong to the Borchers class of fields $\phi^{(\kappa)}$, where

$$\langle 0|\psi^{(\kappa)}(x;u)\psi^{(\kappa')}(y;v)|0\rangle = \Big(\bar{u}\Big|M^{(\kappa'\kappa)}\Big(i\Gamma^\mu\frac{\partial}{\partial x^\mu}\Big)\Big|v\Big)\frac{1}{i}D_m^{(-)}(x-y). \tag{I.50}$$

(b) Prove that there exists an anti-unitary operator $U(\bar{I}_{st})$ satisfying (I.36) and such that

$$U(\tilde{I}_{st})\phi^{(\kappa)}(x;u)U(\tilde{I}_{st})^{-1} = \psi^{(\kappa)}(-x;\bar{u}). \tag{I.51}$$

We give a concrete example. Let $n = 2l + n_0$, where n_0 is zero in the Bose case or unity in the Fermi case, and $l \in \mathbf{Z}_+$. Let

$$M(t) = t^{n_0}N^*(t)N(t), \tag{I.52a}$$

where $N(t) \equiv \sum_{j=1}^{l} N_j t^j$ is an arbitrary complex $K \times K$-matrix depending on t as a polynomial of degree l which is continuous for $t \geq m$ and is such that

$$N(-t) = \overline{N(t)}.$$

* The fact that B is real follows from the fact that Hermitian fields are taken to Hermitian fields under the transformation (I.43).

It is obvious that for sufficiently large l we can choose $N(t)$ such that the real span of the matrices M_j ($j = 0, \dots, n$) is the same as the set of all Hermitian $K \times K$-matrices. According to what has been said above, the corresponding system of ICF's $\phi^{(\kappa)}$ (with the two-point functions (I.39)) is not $\mathcal{P}_+$-covariant. Nevertheless, if we set

$$X(t) = t^{n_0} N^*(t) N(t), \tag{I.52b}$$

$$Y(t) = M(t)^{-1} X(t) \equiv N(t)^{-1} \overline{N(t)}, \tag{I.52c}$$

then (I.46), (I.47) will hold. According to Exercise I.1, this implies the existence of an anti-unitary operator $U(\tilde{I}_{st})$ satisfying (I.36) such that

$$U(\tilde{I}_{st}) \phi^{(\kappa)}(x; u) U(\tilde{I}_{st})^{-1} = \psi^{(\kappa)}(-x; \bar{u}) \tag{I.53a}$$

and, as is easily seen,

$$U(\tilde{I}_{st}) \psi^{(\kappa)}(x; u) U(\tilde{I}_{st})^{-1} = \phi^{(\kappa)}(-x; \bar{u}) \tag{I.53b}$$

where the fields $\psi^{(\kappa)}(x; u)$ are defined by (I.49) and belong to the Borchers extension of the system $\phi^{(\kappa)}$. If it is further supposed, for example, that $N_l = 1$ and $\overline{N(t)} \not\equiv N(t)$, then $Y(t) \not\equiv$const is a matrix-valued rational (non-polynomial) function of the parameter t, so that the fields $\psi^{(\kappa)}$ are expressed in terms of $\phi^{(\kappa)}$ in x-space in an essentially non-local way.

CHAPTER 10

Fields in an Indefinite Metric

10.1. Pseudo-Wightman Formalism

A. PSEUDO-HILBERT SPACE

We need to have recourse to a picture of fields in a space with an indefinite metric when attempting to formulate the idea of a virtual (or "potential") state (not unlike the way one uses representations to organize the physical states of quantum systems). The notion of a virtual state can be illustrated by the following "classical" example. As is well known, the state of a free classical electromagnetic field in space-time M is defined by the stress tensor field $F_{\lambda\mu}(x)$ satisfying Maxwell's equations. This state can also be defined by a vector potential $A_\mu(x)$; however, such a characterization has a certain redundancy since two configurations, say, $A_\mu(x)$ and $A'_\mu(x)$ of the vector potential define the same physical state (that is, they are equivalent) if their stress tensor fields are the same:

$$F_{\lambda\mu}(\equiv \partial_\lambda A_\mu - \partial_\mu A_\lambda) = F'_{\lambda\mu}(\equiv \partial_\lambda A'_\mu - \partial_\mu A'_\lambda).$$

In this situation we can say that the configuration $A_\mu(x)$ of the vector potential is a virtual state of the (classical) electromagnetic field and that the physical state is the equivalence class of virtual states. Here we could allow only those configurations $A_\mu(x)$ satisfying some gauge condition, rather than arbitrary configurations; we would then obtain different representations of the physical states of equivalence classes of virtual states (depending on the gauge). As we shall see presently, the situation is similar in the quantum case; here too, the space of virtual state vectors (for the same "physics") can generally be chosen in various ways (corresponding to different gauges).

The representation of (fundamental) quantum fields in a space with indefinite metric is developed in complete analogy with the Wightman formalism. The main difference is that the Hilbert space of physical state vectors is now replaced by a pseudo-Hilbert space of virtual state vectors (defined below), therefore the formalism has to include supplementary axioms in order to give it a physical interpretation.

Let $\mathfrak{H}$ be a complex F-space with a Hilbert topology. This means that the LCS structure on $\mathfrak{H}$ is defined by a norm $\|\Phi\| = \sqrt{(\Phi,\Phi)}$, where (Φ,Ψ) is the scalar product in $\mathfrak{H}$ with respect to which $\mathfrak{H}$ is a Hilbert space. We emphasize here that in the definition of the Hilbert topology, the specific choice of the scalar product is to a certain degree inessential: if $(\Phi,\Psi)_1$ is another scalar product in $\mathfrak{H}$, then it gives the same topology if there exist positive numbers C, C' such that

$$C\cdot(\Phi,\Phi) \leq (\Phi,\Phi)_1 \leq C'\cdot(\Phi,\Phi)$$

(in other words, if the norms $\|\Phi\|$ and $\|\Phi\|_1$ are subordinated to each other). Various other concepts are formulated in such a way that they are invariant with respect to the above change of the Hilbert scalar product in $\mathfrak{H}$. (An example of such a concept is that of a bounded linear operator A in $\mathfrak{H}$; on the other hand, the concept of an operator being Hermitian is not invariant.)

By a ***pseudo-Hilbert space*** we mean an F-space $\mathfrak{H}$ with a Hilbert topology that is also endowed with an indefinite scalar product $\langle\Phi,\Psi\rangle$ (so that $\mathfrak{H}$ is a space with an indefinite metric, see §1.1.D); furthermore, the indefinite scalar product and the Hilbert topology are compatible in the following sense: every continuous linear functional F on $\mathfrak{H}$ is uniquely representable in the form

$$F(\Psi)=\langle\Phi,\Psi\rangle,\quad \Psi\in\mathfrak{H}, \tag{10.1}$$

where Φ is a vector in $\mathfrak{H}$ and conversely, for any $\Phi\in\mathfrak{H}$, (10.1) defines a continuous linear functional $F(\Psi)$ on $\mathfrak{H}$. In this definition, the indefinite scalar product is the essential element of the structure, whereas the Hilbert topology plays an auxiliary role, therefore the Hilbert scalar product (Φ,Ψ) defining it can vary within admissible limits. The condition of compatibility of the scalar product $\langle\Phi,\Psi\rangle$ and the Hilbert topology can be restated as follows: there exists a bounded linear operator η with bounded inverse η^{-1} in $\mathfrak{H}$ that is Hermitian with respect to the Hilbert scalar product (Φ,Ψ) and is such that

$$\langle\Phi,\Psi\rangle=(\Phi,\eta\Psi)\quad\text{for }\Phi,\Psi\in\mathfrak{H}. \tag{10.2}$$

Here is the proof. Suppose that the compatibility condition holds. It follows from Riesz's theorem (see Example 2 in §1.1.F) that the general form of a continuous linear functional F on $\mathfrak{H}$ is given by the formula $F(\Psi)=(\Phi_1,\Psi)$; on the other hand, we have (10.1). Therefore the formula $\Phi_1=\eta\Phi$ defines a linear automorphism $\eta:\mathfrak{H}\to\mathfrak{H}$. The operator η is Hermitian:

$$(\eta\Phi,\Psi)=\langle\Phi,\Psi\rangle=\overline{\langle\Psi,\Phi\rangle}=\overline{(\eta\Psi,\Phi)}=(\Phi,\eta\Psi),$$

consequently it has a closed graph and (by Theorem 1.11) is bounded. The inverse operator η^{-1} is also Hermitian and bounded for the same reason. Thus the representation (10.2) is proved. The converse, namely, that the compatibility condition holds in the presence of the representation (10.2), is trivial.

Exercise 10.1. Prove that the Hilbert scalar product (Φ,Ψ) can be redefined so that the operator η in (10.2) satisfies the additional condition

$$\eta^2=1. \tag{10.3.}$$

[Hint: If (10.3) does not hold for the representation (10.2), then one can introduce the new Hilbert scalar product $(\Phi,\Psi)_1=(\Phi,\sqrt{\eta^2}\Psi)$ and a new operator η_1 such that $\langle\Phi,\Psi\rangle=(\Phi,\eta_1\Psi)_1$; hence deduce that $\eta_1=\eta(\eta^2)^{-1/2}$ and $\eta_1^2=1$.]

The Hermitian operator η satisfying (10.3) is the difference $\eta=\eta_+-\eta_-$ of orthogonal projectors (with respect to the Hilbert scalar product); the pair (q,p), where $q=\dim\eta_+\mathfrak{H}$, $p=\dim\eta_-\mathfrak{H}$ is called the ***signature*** of the pseudo-Hilbert space $\mathfrak{H}$. Separable spaces with signature (∞,p) where p is a natural number, are called ***Pontryagin spaces*** (see Pontryagin, 1944).

Exercise 10.2. Suppose that the separable infinite-dimensional pseudo-Hilbert space $\mathfrak{H}_1$ has a closed subspace $\mathfrak{H}_1'$ of finite codimension p, (that is, $\dim\mathfrak{H}_1/\mathfrak{H}_1'=p<\infty$) such that the restriction of the scalar product $\langle\cdot,\cdot\rangle$ to $\mathfrak{H}_1'$ is non-negative definite and the dimension of the subspace $\mathfrak{H}_1''=\{\Phi\in\mathfrak{H}_1':\langle\Phi,\Phi\rangle=0\}$ of $\mathfrak{H}_1'$ is equal to p. Prove that $\mathfrak{H}_1$ is a Pontryagin space with signature (∞,p). [Hint: $\mathfrak{H}_1$ can be decomposed into a direct sum of two orthogonal (with respect to $\langle\cdot,\cdot\rangle$) closed subspaces, one of which $\mathcal{K}$ is an infinite-dimensional Hilbert space (with scalar product $\langle\cdot,\cdot\rangle$), the other being spanned by vectors $a_1,\dots,a_p,b_1,\dots,b_p$ with the properties $\langle a_i,a_j\rangle=\langle b_i,b_j\rangle=0$, $\langle a_i,b_j\rangle=\delta_{ij}$. For the proof, choose an arbitrary basis $a_1, \ldots, a_p$ in $\mathfrak{H}_1''$ and vectors $b_1', \ldots, b_p' \in \mathfrak{H}_1$ such that $\langle a_i, b_j' \rangle = \delta_{ij}$ and set $b_i = b_i' - \sum_j \langle a_j, b_i' \rangle a_j$. Finally, define $\mathcal{K} = \{\Phi \in \mathfrak{H}_1' : \langle \Phi, b_i \rangle = 0,\ i = 1, \ldots, p\}$.]

Proposition 10.1. *The Hilbert topology of a pseudo-Hilbert space is uniquely defined by the condition of compatibility of the indefinite scalar product and the Hilbert topology.*

■ Let $\mathfrak{H}$ be a pseudo-Hilbert space with indefinite scalar product $\langle \Phi, \Psi \rangle$, whose Hilbert topology can be defined by a scalar product (Φ, Ψ); then (10.2) holds. Suppose now that another Hilbert scalar product $(\Phi, \Psi)_1$ has been defined and is compatible with the indefinite scalar product $\langle \Phi, \Psi \rangle$. Then we also have the representation

$$\langle \Phi, \Psi \rangle = (\Phi, \eta_1 \Psi)_1, \tag{10.4}$$

where η_1 is a Hermitian operator in $(\mathfrak{H}, (\cdot, \cdot)_1)$ with a bounded inverse. We claim that the norms $\|\Phi\|$ and $\|\Phi\|_1$ on $\mathfrak{H}$ are equivalent. Thus it follows from (10.2) and (10.4) that $(\Phi, \Psi)_1 = (\Phi, \eta\eta_1^{-1}\Psi)$. From this it is clear that the operator $\eta\eta_1^{-1}$ is Hermitian with respect to the scalar product $(\cdot, \cdot)$ and since it is defined on the whole of $\mathfrak{H}$, it is bounded (again by Theorem 1.11). Hence there exists a number $c' > 0$ such that $(\Phi, \Phi)_1 \le c'(\Phi, \Phi)$. Similarly, one proves that $(\Phi, \Phi) \le c''(\Phi, \Phi)_1$ for some number $c'' > 0$. ■

Certain concepts relating to Hilbert space have the obvious generalization to a pseudo-Hilbert space $\mathfrak{H}$. For a linear operator A with dense domain D_A the (*pseudo-Hermitian conjugate*) or *adjoint* operator A^* is given by the defining relation

$$\langle D_{A^*}\Phi, \Psi \rangle = \langle \Phi, D_A \Psi \rangle \quad \text{for } \Phi \in D_{A^*},\ \Psi \in D_A. \tag{10.5}$$

A continuous linear operator A in $\mathfrak{H}$ is called *pseudo-Hermitian* if $A^* = A$, that is, if

$$\langle A\Phi, \Psi \rangle = \langle \Phi, A\Psi \rangle \quad \text{for } \Phi, \Psi \in \mathfrak{H}. \tag{10.6}$$

A linear automorphism $\mathcal{U}$ of $\mathfrak{H}$ is said to be *pseudo-unitary* if it is defined on a dense linear subspace $\mathfrak{D} \subset \mathfrak{H}$ and maps it (in one-to-one fashion) onto a dense subspace of $\mathfrak{H}$ and is such that

$$\langle \mathcal{U}\Phi, \mathcal{U}\Psi \rangle = \langle \Phi, \Psi \rangle \quad \text{for } \Phi, \Psi \in \mathfrak{D}. \tag{10.7}$$

If a pseudo-unitary operator $\mathcal{U}$ is a linear automorphism of $\mathfrak{H}$ (so that $\mathfrak{D} = \mathcal{U}\mathfrak{D} = \mathfrak{H}$), then it is easy to see, using the closed graph theorem 1.11, that $\mathcal{U}$ and $\mathcal{U}^{-1}$ are continuous (and in this case, the Hilbert norms $\|\Phi\| \equiv (\Phi, \Phi)^{1/2}$ and $\|\Phi\|_1 \equiv \|\mathcal{U}\Phi\|$ are equivalent).

Exercise 10.3. Prove that a pseudo-unitary operator $\mathcal{U}$ is unitary with respect to the auxiliary scalar product $(\cdot, \cdot)$ if and only if it commutes with η.

A pseudo-unitary representation $\mathcal{U}(g)$ of a Lie group G is defined in natural fashion (here we suppose that $\mathcal{U}(g_1)\mathfrak{D}$ is in the domain of definition of $\mathcal{U}(g_2)$ for any $g_1, g_2 \in G$; furthermore, as in the case of unitary representations, we postulate that all the matrix elements $\langle \Phi, \mathcal{U}(g)\Psi \rangle$ are continuous functions of g for any $\Phi, \Psi \in \mathfrak{D}$).

When we talk about the algebra $\mathcal{B}(\mathfrak{H})$ of all continuous linear operators in a pseudo-Hilbert space $\mathfrak{H}$ as a Banach *-algebra, we always take it that the operators in $\mathcal{B}(\mathfrak{H})$ are endowed with the norm defined by the Hilbert norm in $\mathfrak{H}$ (§1.4.A) and that the operator η in the representation (10.2) satisfies (10.3). (The operator η is then Hermitian and unitary with respect to the Hilbert scalar product, therefore the condition $\|A^*\| = \|A\|$ entering into the definition of a Banach *-algebra holds.)

Exercise 10.4. Let $\mathfrak{A}$ be a Banach *-subalgebra of the algebra $\mathcal{B}(\mathfrak{H})$ of all continuous linear operators in the pseudo-Hilbert space $\mathfrak{H}$. Let π be a *-representation of $\mathfrak{A}$ by operators in a Hilbert space $\mathcal{H}$ (where all the operators $\pi(A)$ have a common dense domain of definition $\mathfrak{D}$ in $\mathcal{H}$ such that $\pi(A)\mathfrak{D} \subset \mathfrak{D}$). Prove that all the operators $\pi(A)$ $(A \in \mathfrak{A})$ are bounded and that the representation π is norm continuous: $\|\pi(A)\| \le \|A\|$ for all $A \in \mathfrak{A}$. [Hint: Verify that for $A \in \mathfrak{A}$, $\|A\| < 1$, there exists $B = B^* \in \mathfrak{A}$ such that $A^*A + B^2 = 1$; for this it suffices to define B as a series $f(A^*A)$, where $f(z)$ is the Taylor series of the function $\sqrt{1-z}$. Deduce that $\|\pi(A)\| \le 1$ for all $A \in \mathfrak{A}$, $\|A\| < 1$.]

We consider as an example, the space $\mathfrak{H}_1$ of virtual one-photon states in the Gupta-Bleuler gauge. It consists of all complex 4-vector-valued measurable functions $\mathcal{A}^\mu(p)$ on the cone $\Gamma_0^+ \equiv \{p \in \mathbf{M} : p^0 = |\mathbf{p}|\}$ for which the square of the Hilbert norm

$$\|\mathcal{A}\|^2 \equiv (\mathcal{A}, \mathcal{A}) = \int_{\Gamma_0^+} |\mathcal{A}(p)|^2 (dp)_0 \tag{10.8}$$

is finite, where

$$|\mathcal{A}|^2 = |\mathcal{A}^0|^2 + |\mathbf{A}|^2, \quad (dp)_0 = d^3p/[(2\pi)^3 2p_0].$$

We define a representation in $\mathfrak{H}_1$ of the Poincaré group $\mathfrak{P}^\uparrow$ by the formula

$$(\mathcal{U}_1(a,\Lambda)\mathcal{A})^\mu(p) = e^{ipa}\Lambda^\mu_\nu \mathcal{A}^\nu(\Lambda^{-1}p). \tag{10.9}$$

It is easily verified that $\mathfrak{H}_1$ is a pseudo-Hilbert space with Hermitian form

$$\langle \mathcal{A}, \mathcal{A}' \rangle = -\int_{\Gamma_0^+} \overline{\mathcal{A}}_\mu(p)\mathcal{A}'^\mu(p)(dp)_0 \tag{10.10}$$

and that the operators $\mathcal{U}_1(a,\Lambda)$ form a pseudo-unitary representation of the Poincaré group.

We see that the "virtual" photon in the Gupta-Bleuler gauge has four polarizations in contrast to the "physical" photon which has two. According to Exercise 7.17, the Hilbert space of state vectors of the physical photon is $\mathcal{H}_1 = \mathfrak{H}_1'/\mathfrak{H}_1''$, where $\mathfrak{H}_1'$ is the subspace of vectors $\mathcal{A} \in \mathfrak{H}_1$ with transverse wave function $(p_\mu \mathcal{A}^\mu(p) = 0)$, and $\mathfrak{H}_1''$ is the subspace of vectors in $\mathfrak{H}_1'$ with zero scalar square. This example demonstrates a possible relation between the spaces of virtual and physical state vectors.

In the above example, the Hilbert scalar product (Φ, Ψ) is invariant with respect to the seven-dimensional "Aristotle group" including the three-dimensional rotations and the four-dimensional translations of space-time. In the case of more general covariant gauges of a free electromagnetic field, the Hilbert norm may turn out not to be translation-invariant.

We can apply the operations of direct sum and tensor product to a given collection of pseudo-Hilbert spaces to construct new ones. Thus if $\mathfrak{H}_j$ $(j = 1, \dots, n)$ is a finite collection of pseudo-Hilbert spaces in which the Hilbert and pseudo-Hilbert scalar products are connected by relations of type (10.2): $\langle \Phi_j, \Psi_j \rangle = (\Phi_j, \eta_j \Psi_j)$, then we must take the Hilbert tensor product as the tensor product $\mathfrak{H} = \mathfrak{H}_1 \otimes \dots \otimes \mathfrak{H}_n$ and define in it the indefinite norm of type (10.2) with $\eta = \eta_1 \otimes \dots \otimes \eta_n$. It is easy to see using Exercise 1.44, that $\mathfrak{H}$ is a pseudo-Hilbert space.

In precisely the same way, we can start with a given "one-particle" pseudo-Hilbert space $\mathfrak{H}_1$ and use the method of second quantization to construct the pseudo-Hilbert Fock spaces $\mathcal{F}_\vee(\mathfrak{H}_1)$ and $\mathcal{F}_\wedge(\mathfrak{H}_1)$ of type (7.109) of virtual Bose and Fermi particles.

B. Axioms of Pseudo-Wightman Type

It is natural to suppose that the quantum fields in the formalism with indefinite metric are operator-valued generalized functions whose values are operators in a pseudo-Hilbert space $\mathfrak{H}$; accordingly, the field algebra $\mathfrak{F}$ is a *-algebra of operators in $\mathfrak{H}$. However, this assumption is too strict in the general case. Instead we require that the field *-algebra $\mathfrak{F}$ be realized by sesquilinear forms on a dense subspace $\mathfrak{D}$ of $\mathfrak{H}$. This means that associated with each element $A \in \mathfrak{F}$ is a unique sesquilinear form $\langle\Phi|A|\Psi\rangle$ where $\Phi, \Psi \in \mathfrak{D}$; here $\langle\Phi|A^*|\Psi\rangle \equiv \overline{\langle\Psi|A|\Phi\rangle}$. Since there is no natural definition of a product of sesquilinear forms, we shall suppose that we are given the following constructive correspondence between the product in the algebra $\mathfrak{A}$ and the product of operators in $\mathfrak{H}$: each element $A \in \mathfrak{A}$ is the limit (in the sense of convergence of sesquilinear forms over $\mathfrak{D}$) of a sequence of operators A_n in $\mathfrak{H}$ such that the product AB is the limit of the sequence of products of the operators A_nB_n. The situation when the field algebra is represented by operators remains an important particular case.

We now give the list of axioms in the pseudo-Wightman approach.

PW.I (Relativistic invariance). *In the pseudo-Hilbert space $\mathfrak{H}$ of virtual vector states there is defined a pseudo-unitary representation $\mathcal{U}(a, \underset{\sim}{\Lambda})$ of the Poincaré spinor group $\mathfrak{P}_0$. It is supposed that $\|\mathcal{U}(a,1)\|$ is a continuous function of $a \in \mathrm{M}$ of polynomial growth.*

PW.II (Spectrum property). *For any vectors $\Phi, \Psi \in \mathfrak{H}$, the generalized function*

$$\int \langle\Phi, \mathcal{U}(a,1)\Psi\rangle e^{-ipa} d^4a \tag{10.11}$$

has support in the upper light cone $\overline{V}^+$.

PW.III (Existence and essential uniqueness of the vacuum).

(a) *There is a distinguished $\mathfrak{P}_0$-invariant vector $|0\rangle$ in $\mathfrak{H}$, called the vacuum vector, which is normalized by the condition $\langle 0|0\rangle = 1$.*

(b) *If $\Phi, \Psi \in \mathfrak{H}$ are such that the generalized function* (10.11) *is a (complex) measure with respect to p, then the contribution of this measure at the point $p = 0$ is equal to $(2\pi)^4\delta(p)\langle\Phi|0\rangle\langle 0|\Psi\rangle$.*

PW.IV (Field algebra). *A (field) *-algebra $\mathfrak{F}$ is defined which is realized by sesquilinear forms on a Poincaré-invariant dense linear subspace $\mathfrak{D}$ of $\mathfrak{H}$ containing the vacuum vector. The components $\phi_l^{(\kappa)}$ of the fundamental quantum fields $\phi^{(\kappa)}$ are $\mathfrak{F}$-valued operator-valued generalized functions $\phi_l^{(\kappa)}(x)$ on the Schwartz space $\mathcal{S}(\mathrm{M})$.*

Here we also suppose that each field $\phi^{(\kappa)}$ occurs in the given collection of fields along with its (pseudo-Hermitian) adjoint $\phi_l^{(\kappa)}(x)^* = \phi_{\bar{l}}^{(\bar{\kappa})}(x)$. We denote by $\mathcal{P}(\mathrm{M})$ the polynomial field algebra consisting of all possible finite sums of elements of the form (8.8). If the elements of (8.8) are restricted to functions $f \in \mathcal{D}(\mathcal{O})$, where $\mathcal{O}$ is an open subset of M, then we obtain the subalgebra $\mathcal{P}(\mathcal{O}) \subset \mathcal{P}(\mathrm{M})$. An element A belonging to $\mathcal{P}(\mathcal{O})$ for some bounded open set $\mathcal{O}$ is called a local quantity of $\mathcal{P}(\mathrm{M})$. Accordingly, an arbitrary element of $\mathcal{P}(\mathrm{M})$ can be interpreted as a quasi-local quantity. As we shall see later (§10.2.B), such quantities cannot, in general, form a sufficient set of operators of creation of charged physical states (from the vacuum). For this reason, in addition to $\mathcal{P}(\mathrm{M})$ we introduce the larger field *-algebra $\mathfrak{F} \supset \mathcal{P}(\mathrm{M})$ (with identity) which is generated by the algebra $\mathfrak{P}(\mathrm{M})$ (in the sense that each element of $\mathfrak{F}$ is the limit of a sequence $A_n \in \mathcal{P}(\mathrm{M})$, the convergence here being with respect to the sesquilinear forms on $\mathfrak{D}$) and which can contain "global" quantities.

As before, the vacuum expectation (8.27) is called the Wightman function of the fields $\phi^{(\kappa_1)}, \ldots, \phi^{(\kappa_n)}$ (which is a generalized function in $\mathcal{S}'(\mathrm{M}^n)$).

PW.V (Poincaré-covariance of the fields).

$$\mathcal{U}(a,\underset{\sim}{\Lambda})\phi_l^{(\kappa)}(x)\mathcal{U}(a,\underset{\sim}{\Lambda})^{-1} = \sum_m V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda}^{-1})\phi_m^{(\kappa)}(\Lambda x + a), \tag{10.12}$$

where the $V^{(\kappa)}(\underset{\sim}{\Lambda})$ *are finite-dimensional matrix representations of* $SL(2,C)$ *(here* (10.12) *is to be understood in the sense of sesquilinear forms on* $\mathfrak{D}$ *which are generalized functions of* x*).*

Axiom PW.V enables us to define the action of the Poincaré group $\mathfrak{P}_0$ on the field algebra $\mathfrak{F}$ as *-automorphisms $\alpha(a,\underset{\sim}{\Lambda})$ (so that a formula of type (8.9) holds in the sense of sesquilinear forms). This action must, of course, satisfy a reasonable continuity condition. We must at least suppose that if $A_n \to A$ and $B_n \to B$ (in the sense of sesquilinear forms on $\mathfrak{D}$), then $\langle \Phi, A_n\alpha_{(a,1)}(B_n)\Psi\rangle \to \langle \Phi, A\alpha_{(a,1)}(B)\Psi\rangle$ in the sense of generalized functions with respect to $a \in \mathbf{M}$ (here $\Phi, \Psi \in \mathfrak{D}$). It then follows from the spectrum condition PW.II that

$$\operatorname{supp}\int \langle \Phi|A\alpha_{(a,1)}(B)|0\rangle e^{-ipa}d_4p \subset \overline{V}^+ \quad \text{for } A,B \in \mathfrak{F}, \Phi \in \mathfrak{D}. \tag{10.13}$$

(Here we do not suppose that A and B are necessarily operators, since otherwise this condition would not contain anything new by comparison with PW.II.) In exactly the same way, in order to have the possibility of applying the condition of uniqueness of the vacuum in the context of sesquilinear forms, we suppose that the following supplement to Axiom PW.III(b) holds.

PW.III. (b′) *If the vector* $\Phi \in \mathfrak{D}$ *and the elements* $A, B \in \mathfrak{F}$ *are such that the generalized function* $\int\langle\Phi|A\alpha_{(a,1)}(B)|0\rangle e^{-ipa}d^4a$ *is a (complex) measure with respect to p, then the contribution of this measure at the point* $p = 0$ *is equal to* $(2\pi)^4\delta(p)\langle\Phi|A|0\rangle\langle 0|B|0\rangle$.

PW.VI (Locality).

$$\phi_l^{(\kappa)}(x)\phi_m^{(\kappa')}(y) = \sigma^{(\kappa,\kappa')}\phi_m^{(\kappa')}(y)\phi_l^{(\kappa)}(x) \quad \textit{for } (x-y)^2 < 0, \tag{10.14}$$

where the matrix $\sigma \equiv (\sigma^{(\kappa,\kappa')})$ *consists entirely of the numbers* ± 1.

PW.VII (Cyclicity of the vacuum). *If for some strongly convergent sequence* $\Phi_n \in \mathfrak{D}$ *the sequence* $\langle\Phi_n|A|0\rangle$ *converges to* 0 *for all* $A \in \mathcal{P}(\mathbf{M})$, *then* $\lim \Phi_n = 0$.*

In the case when the auxiliary Hilbert scalar product $(\cdot,\cdot)$ is translation-invariant, that is, all the operators $\mathcal{U}(a,1)$ are unitary (and hence, according to Exercise 10.3, commute with the operator η in the representation (10.2)), Stone's theorem is applicable, which gives the representation (1.68) for the operators $\mathcal{U}(a,1)$; accordingly, the 4-momentum operator P_μ (defined by the equality $P^\mu = -i\frac{\partial}{\partial a_\mu}\mathcal{U}(a,1)|_{a=0}$) has a spectral decomposition of type (1.64). Thus (by Stone's theorem) in order that the auxiliary Hilbert scalar product $(\cdot,\cdot)$ on $\mathfrak{H}$ be translation-invariant, it is necessary that the expression (10.11) be a (finite complex) measure on $\mathbf{M}$ for any $\Phi, \Psi \in \mathfrak{H}$. This condition does not hold in general and the earlier spectral decompositions (1.68) and (1.64) for $U(a,1)$ and P^μ become inapplicable; accordingly the decomposition (§7.2.B) of state vectors into vectors (not necessarily eigenvectors) with specified invariants of the Poincaré group also loses its meaning.

In the pseudo-Wightman approach, the vacuum vector $|0\rangle$ turns out to be a cyclic vector for the field algebras $\mathcal{P}(\mathcal{O})$ associated with arbitrary non-empty open subsets $\mathcal{O} \subset \mathbf{M}$ (the proof of Theorem 8.2 being the same as before). At the same time, the algebra $\mathcal{P}(\mathbf{M})$ is irreducible (as a corollary of

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* In the case when the algebra $\mathcal{P}(\mathbf{M})$ is realized by operators in $\mathfrak{H}$, this condition becomes the usual condition of cyclicity of the vacuum (see Axiom W.VII in §8.2.A).

the cyclicity condition of the vacuum, see Theorem 8.1) provided we insist that there exists in $\mathfrak{H}$ a unique (to within a factor) translation-invariant vector, namely, the vacuum vector $|0\rangle$ (Morchio and Strocchi, 1980, Theorem 11).

Models with a non-one-dimensional subspace of translation-invariant vectors sometimes admit a description by an alternative method, when, instead of the condition that $\mathfrak{H}$ contain a vacuum virtual state, it is required that $\mathfrak{H}$ contain a vacuum virtual generalized state (which is unique to within a factor). Here we shall not formulate the concepts relating to this in the general case, but shall incorporate them in Chapter 11 for specific models.

In the locality axiom PW.VI, no connection between spin and statistics is assumed (cf. Condition W.VIII). It can now break down, as the following example shows.

Let $c(x)$ be a free charged scalar field of mass m which we assume to be fermionic rather than bosonic. This means that it satisfies the following canonical anti-commutation relations:

$$[c(x), c(y)]_+ = 0 = [c^*(x), c^*(y)]_+, \tag{10.15a}$$

$$[c(x), c^*(y)]_+ = \frac{1}{i} D_m(x-y). \tag{10.15b}$$

It is not difficult to construct the corresponding representation by the method of second quantization; we choose for $\mathfrak{H}_1$ the same one-particle subspace as for the ordinary quantum free charged scalar (bosonic) field (§8.4.B). Let $\mathfrak{H} = \mathcal{F}_\wedge(\mathfrak{H}_1)$ be the corresponding Fock space of Fermi particles with scalar product $(\cdot,\cdot)$; as in (8.95), (8.96) we set

$$\tilde{c}(p) = A^+(-p,-1) + A(p,1),$$
$$A(p,\pm 1) = \frac{1}{\sqrt{2}}(A_1(p) \mp iA_2(p)),$$

where $A^+_{1,2}(p)$ and $A_{1,2}(p)$ are the creation and annihilation operators for Fermi particles:

$$[A_a(p), A^+_b(p')]_+ = \delta_{ab} \cdot (2\pi)^4 \delta(p-p') \delta^+_m(p).$$

We define in $\mathfrak{H}$ the operator N of the number of particles and the charge operator Q in the obvious way (cf. (8.98)). We define a pseudo-Hilbert structure in $\mathfrak{H}$ by formula (10.2) with $\eta = (-1)^{(N-Q)/2}$.

Exercise 10.5. Verify that the field $c(x)$ constructed in this way satisfies axioms PW.I–VII. Here the two-point Wightman functions have the form

$$\langle 0|c(x)c(y)|0\rangle = 0 = \langle 0|c^*(x)c^*(y)|0\rangle, \tag{10.15c}$$

$$\langle 0|c(x)c^*(y)|0\rangle = \frac{1}{i} D_m^{(-)}(x-y) = -\langle 0|c^*(x)c(y)|0\rangle. \tag{10.15d}$$

Exercise 10.6. Prove that (for any choice of the phase factor ϑ) the above scalar fermionic field $c(x)$ has no pseudo-anti-unitary operator Θ (that is, a TCP-operator) such that $\Theta c(x)\Theta^* = \vartheta c^*(-x)$, $\Theta c^*(x)\Theta^* = \bar{\vartheta} c(-x)$. [Hint: If such an operator Θ existed, it would have the property $\Theta \mathcal{U}(a,\Lambda)\Theta^* = \mathcal{U}(-a,\Lambda)$, which is proved using the irreducibility of the fields c, c^* in exactly the same way as (7.152c) was proved in Exercise 7.30; it would then follow from the uniqueness of the translation-invariant vector in $\mathfrak{H}$ that $\Theta|0\rangle = \omega|0\rangle$ (where ω is a phase factor), so that the two-point functions would then satisfy the property $\overline{\langle 0|c(x)c^*(y)|0\rangle} = \langle 0|c^*(-x)c(-y)|0\rangle$, which contradicts (10.15d).]

The earlier proof of the TCP-theorem 9.13 is inapplicable in this example because the connection between spin and statistics does not hold. (As we noted in §9.2.A, the TCP-theorem holds in the Wightman formalism even without the connection between spin and statistics because the Klein transformation is possible there.)

C. Vacuum Sector and Charged States

In order that the formalism with an indefinite metric have a physical interpretation, we have to bring in additional requirements. To this end we suppose that a *-algebra $\mathfrak{A}$, called the algebra of observables, can be constructed from the (fundamental) fields in the space with indefinite metric $\mathfrak{H}$. The basic principle for choosing $\mathfrak{A}$ is gauge invariance. This means that some group $\mathcal{G}$ (called in this context the large gauge group) acts on the field algebra $\mathfrak{F}$ by *-automorphisms $A \to \gamma_g(A)$. It is natural here to impose the following (sequential) continuity condition: if $A_n \to A$ (in the sense of sesquilinear forms on $\mathfrak{D}$) in $\mathfrak{F}$, then $\gamma_g(A_n) \to \gamma_g(A)$ (in the same sense) for any $g \in \mathcal{G}$. The algebra of observables $\mathfrak{A}$ is defined as the subalgebra of all $\mathcal{G}$-invariant elements of $\mathfrak{F}$. To ensure that the algebra of observables is taken into itself under the action of automorphisms $\alpha_{(\alpha,\underset{\sim}{\Lambda})}$ of the Poincaré group $\mathfrak{P}_0$, we suppose that $\mathfrak{P}_0$ acts by automorphisms on $\mathcal{G}$ ($g \to g' \equiv g^{(\alpha,\underset{\sim}{\Lambda})}$), so that the $\mathfrak{P}_0$-covariance condition holds:

$$\alpha_{(\alpha,\underset{\sim}{\Lambda})}\gamma_g\alpha^{-1}_{(\alpha,\underset{\sim}{\Lambda})} = \gamma_{g'} \quad \text{for all } g \in \mathcal{G}, (\alpha, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0. \tag{10.16}$$

Furthermore, according to the physical sense, the observables must transform according to a single-valued representation of the Lorentz group; we therefore require that one of the automorphisms γ_g ($g \in G$) is a transformation $\alpha_{(0,-1)}$ such that

$$\alpha_{(0,-1)}(A) = A \quad \text{for } A \in \mathfrak{A}. \tag{10.17}$$

The following axiom gives a possible physical interpretation of the formalism.

PW.VIII (Properties of the algebra of observables). *A large gauge group $\mathcal{G}$ acting by *-automorphisms on the field algebra $\mathfrak{F}$ is defined, such that*

(a) *the Poincaré-covariance condition* (10.16) *holds;*

(b) *the vacuum functional $\langle 0|A|0\rangle$ on the algebra of observables $\mathfrak{A}$ (that is, on the subalgebra of all $\mathcal{G}$-invariant elements of $\mathfrak{F}$) is a positive linear functional:*

$$\langle 0|A^*A|0\rangle \geq 0 \quad \textit{for } A \in \mathfrak{A}. \tag{10.18}$$

The vacum expectation value (as a positive functional) defines via the GNS construction a cyclic representation π_{vac} of $\mathfrak{A}$ in a Hilbert space $\mathcal{H}_{\text{vac}}$ with cyclic (vacuum) vector Ψ_{vac}; we call this the vacuum representation of $\mathfrak{A}$.

Proposition 10.2 *The vacuum representation of the algebra of observables is irreducible; the Poincaré group $\mathfrak{P}_0$ is unitarily realized in $\mathcal{H}_{\text{vac}}$. Furthermore, the spectrum condition and uniqueness (to within a factor) of the translation-invariant vector hold in $\mathcal{H}_{\text{vac}}$.*

■ The fact that the symmetry of $\mathfrak{A}$ with respect to the Poincaré group $\mathfrak{P}_0$ is unitarily realized in $\mathcal{H}_{\text{vac}}$ follows from Proposition 1.30. In view of the cyclicity of the vacuum, it suffices to verify the spectrum condition $\operatorname{supp} \int \langle \Phi_1, U(a,1)\Phi_2\rangle e^{-ipa} d^4a \subset \overline{V}^+$ for vectors of the form $\Phi_j = \pi_{\text{vac}}(A_j)\Psi_{\text{vac}}$, where $A_j \in \mathfrak{A}$; then $\langle \Phi_1, U(a,1)\Phi_2\rangle = \langle 0|A_1^*\alpha_{(a,1)}(A_2)|0\rangle$. The spectrum condition now follows from (10.13).

To prove that Ψ_{vac} is the unique (to within a factor) translation-invariant vector in $\mathcal{H}_{\text{vac}}$, we denote by Q the projector onto the subspace of translation-invariant vectors of $\mathcal{H}_{\text{vac}}$. It suffices to verify that $\langle \Phi_1, Q\Phi_2\rangle = \langle \Phi, \Psi_{\text{vac}}\rangle\langle \Psi_{\text{vac}}, \Phi_2\rangle$ for all $\Phi_1, \Phi_2 \in \mathcal{H}_{\text{vac}}$. As before, we can confine ourselves to vectors of the form $\Phi_j = \pi_{\text{vac}}(A_j)\Psi_{\text{vac}}$. The required equality now follows from the fact that the contribution of the measure $\int \langle 0|A_1^*\alpha_{(a,1)}(A_2)|0\rangle e^{-ipa} d^4a$ at the point $p = 0$ is equal to $(2\pi)^4\delta(p)\langle \Phi_1, Q\Phi_2\rangle$, and from condition PW.III(b′).

Finally, the irreducibility of π_{vac} is proved in exactly the same way as Proposition 8.1 (since the conditions of cyclicity relative to $\pi_{\text{vac}}(\mathfrak{A})$ and the uniqueness of the vacuum hold). ■

By generalizing the above construction we can in principle obtain the states of the non-vacuum sector. Suppose that there is a distinguished normal subgroup $\mathcal{G}_0$ of $\mathcal{G}$ that is taken into itself under the action of the Poincaré group $\mathfrak{P}_0$; in this context we call $\mathcal{G}_0$ the little gauge group. Here for simplicity we shall restrict ourselves to the case when the quotient group

$$\Gamma = \mathcal{G}/\mathcal{G}_0 \tag{10.19}$$

is a compact Lie group; it is called the effective gauge group.* By analogy with the definition of the algebra of observables, we call the set $\mathfrak{B}$ of $\mathcal{G}_0$-invariant elements of the field algebra $\mathfrak{F}$ the algebra of physical quantities (subject to the conditions PW.IX given below). Since for $g \in \mathcal{G}$, $A \in \mathfrak{B}$, $\gamma_g(A)$ depends only on the coset $h \in \Gamma$ of g, we shall write, by abuse of notation, $\gamma_g(A) = \gamma_h(A)$;** thus Γ acts by *-automorphisms of $\mathfrak{B}$. The next natural assumption is that for any $A \in \mathfrak{B}$, $\Phi, \Psi \in \mathfrak{D}$, the function $\langle\Phi|\gamma_h(A)\Psi\rangle$ is continuous in $h \in \Gamma$ and that we can associate with any element $A \in \mathfrak{B}$ its averaging $\int \gamma_h(A)dh$ over Γ, which is an element of $\mathfrak{A}$ such that

$$\langle\Phi| \int_\Gamma \gamma_h(A)dh|\Psi\rangle = \int_\Gamma \langle\Phi|\gamma_h(A)|\Psi\rangle dh \tag{10.20}$$

(here dh is the invariant measure on Γ normalized to unity).

Thus we add the following axiom so as to make it possible to construct the charged sectors of the algebra of observables.

PW.IX (Properties of the algebra of physical quantities). *A little gauge group $\mathcal{G}_0$ is given which is a Poincaré-invariant normal subgroup of $\mathcal{G}$ with compact quotient group Γ and is such that*

(a) *the automorphisms γ_h $(h \in \Gamma)$ of the *-algebra $\mathfrak{B}$ of physical quantities (defined as the subalgebra of all $\mathcal{G}_0$-invariant elements of $\mathfrak{F}$) commute with the automorphisms $\alpha_{(a,1)}$ of $\mathfrak{P}_0$;*

(b) *formula* (10.18) *defines the averaging operator (over the group Γ) from $\mathfrak{B}$ to $\mathfrak{A}$;*

(c) *the expression*

$$s(A) = \int_\Gamma \langle 0|\gamma_h(A)|0\rangle dh, \quad A \in \mathfrak{B}, \tag{10.21}$$

is a positive linear functional on $\mathfrak{B}$, that is,

$$s(A^*A) \geq 0 \quad \textit{for } A \in \mathfrak{B}. \tag{10.22}$$

Again the GNS construction applied to the positive linear functional $s(A)$ on $\mathfrak{B}$ enables us to construct a cyclic representation π of $\mathfrak{B}$ with cyclic vacuum vector Ψ_0 in some (physical) Hilbert space $\mathfrak{H}$ such that

$$s(A) \equiv \langle\Psi_0, \pi(A)\Psi_0\rangle, \quad A \in \mathfrak{B}. \tag{10.23}$$

Clearly the functional $s(A)$ is $\mathfrak{P}_0$- and Γ-invariant, therefore (as in Proposition 1.30) unitary representations $U(a, \underset{\sim}{\Lambda})$ of $\mathfrak{P}_0$ and $V(h)$ of Γ can be defined in which the

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* We use the term "effective gauge group" so as to distinguish Γ from the "gauge group" G featuring in models with gauge invariance of the second kind (and in which Γ and G may not be the same). By confining our discussion to compact groups Γ (rather than the wider class, for example, of locally compact groups) we are able to avoid additional complications (we shall encounter an example of a non-compact groupΓ in §11.1.B).

** Of greatest interest is the case when $\mathcal{G}$ is the (semi)direct product of the subgroups $\mathcal{G}_0$ and Γ; the element $\gamma_h(A)$ is then clearly defined for all $A \in \mathfrak{F}$, $h \in \Gamma$.

operators $U(a, \underset{\sim}{\Lambda})$ commute with the $V(h)$; they leave the vacuum sector Ψ_0 invariant. But this time, the subspace of all translation-invariant vectors in $\mathcal{H}$ can, in general, have dimension greater than unity (we then speak of a degenerate vacuum) and the representation π may be reducible (this occurs under spontaneous breaking of gauge invariance, see §10.3.C).

According to the theory of unitary representations of compact groups ([V3],§1.4.2), the physical Hilbert space $\mathcal{H}$ can be decomposed uniquely into a direct sum of orthogonal Γ-invariant subspaces $\mathcal{H}^{(\tau)} \neq \{0\}$

$$\mathcal{H} = \bigoplus_{\tau \in \mathcal{T}} \mathcal{H}^{(\tau)} \tag{10.24}$$

with the following property: the index τ ranges over a family $\mathcal{T}$ of pairwise inequivalent (finite-dimensional) unitary irreducible representations of Γ and the subrepresentation of Γ in $\mathcal{H}^{(\tau)}$ is unitarily equivalent to a representation of the form $\tau(h) \otimes 1$ (here $\mathcal{H}^{(\tau)}$ is isomorphic to the tensor product of the space of the representation τ with some Hilbert space).

Exercise 10.7. Suppose (for simplicity) that the algebra of observables $\mathfrak{A}$ consists of bounded operators in $\mathfrak{H}$ and is a Banach *-algebra. Prove that the subspaces $\mathcal{H}^{(\tau)}$ are invariant with respect to the operators $U(a, \underset{\sim}{\Lambda})$ and the operators $\pi(A)$, $A \in \mathfrak{A}$. [Hint: Apply Schur's lemma.]

According to the above exercise, in the case when the algebra of observables is a Banach *-algebra consisting of bounded operators in $\mathfrak{H}$, formula (10.21) gives the decomposition

$$\pi = \bigoplus_{\tau \in \mathcal{T}} \pi^{(\tau)} \tag{10.25}$$

of the representation π of $\mathfrak{A}$ into subrepresentations $\pi^{(\tau)}$ in spaces which transform according to a representation that is a multiple of τ of Γ. The set $\mathcal{T}$ featuring in (10.21) is called *the physical spectrum* of Γ. The vacuum representation π_{vac} of $\mathfrak{A}$ is clearly contained in the representation $\pi^{(0)}$ corresponding to the trivial representation $\tau_0(h) \equiv 1$ of Γ. The states corresponding to vectors in the subspaces $\mathcal{H}^{(\tau)}$ with $\tau \neq \tau_0$ are called the *charged states* of the algebra of observables.

In the more general case when the algebra $\pi(\mathfrak{A})$ admits unbounded operators, we have to add the following "technical" assumption to ensure that it will be reduced by the subspaces $\mathcal{H}^{(\tau)}$: for any element $A \in \mathfrak{B}$ and any matrix element $\tau_{mn}(h)$ (of an irreducible representation of Γ) the integral $\int \tau_{mn}(h)\gamma_h(A)dh \equiv A^{(\tau)}_{mn} \in \mathfrak{B}$ is defined. The decomposition (10.25) then holds as before, where $\mathcal{H}^{(\tau)}$ coincides with the closed linear span of the set of vectors of the form $\pi(A^{(\tau)}_{mn})\Psi_0$ for fixed τ. In particular,

$$\mathcal{H}^{(0)} \text{ is the closure of } \pi(\mathfrak{A})\Psi_0. \tag{10.26}$$

The following proposition holds for the representation π.

Proposition 10.3. *Suppose that the following natural "technical" assumptions hold: for any elements A, B in the algebra $\mathfrak{B}$ of physical quantities, the expression $\langle 0|\gamma_h(A\alpha_{(a,1)}(B))|0\rangle$, as a generalized function in $a \in \mathrm{M}$, depends continuously on $h \in \Gamma$ (as a parameter). Then:*

(a) *the symmetry of $\mathfrak{B}$ with respect to the Poincaré group $\mathfrak{P}_0$ has a unitary realization in $\mathcal{H}$ such that the spectrum condition holds;*

(b) *if Ψ_0 is the unique (to within a factor) translation-invariant vector in $\mathcal{H}$, then the representation π of $\mathfrak{B}$ is irreducible.*

Exercise 10.8. Prove Proposition 10.3. [Hint: Proceed as in the proof of Proposition 10.2.]

Remark. In conclusion it is worth noting that the role of the calculus of operators in the pseudo-Hilbert space can in principle be regarded as exhausted with the construction of the field *-algebra $\mathfrak{F}$ and the vacuum functional on it, since once this has been done, the basic elements of the pseudo-Wightman formalism can be restated in purely algebraic terms, that is, in terms of the algebra $\mathfrak{F}$ and the vacuum functional on it, in particular, without further explicit reference to the pseudo-Hilbert space. The only "vestige" of the indefinite metric is that the vacuum functional on $\mathfrak{F}$ is not positive. We shall not dwell on this purely algebraic formulation here, called the *weak form of the pseudo-Wightman approach* (which can be deduced from the axioms PW.I–IX without difficulty).

D. PHYSICAL SUBSPACE OF PSEUDO-HILBERT SPACE

We now look in more detail at the construction of the physical Hilbert space $\mathcal{H}$ in a particular but important case. First, we make the additional assumption that the effective gauge group Γ is a subgroup of $\mathcal{G}$ (so that $\mathcal{G}$ is the semidirect product of the subgroups $\mathcal{G}_0$ and Γ); secondly, the automorphisms γ_h ($h \in \Gamma$) of the field algebra $\mathfrak{F}$ are implemented by a pseudo-unitary representation $h \to \mathcal{V}(h)$ of Γ in $\mathfrak{H}$ for which the vacuum vector $|0\rangle$ is invariant with respect to the operators $\mathcal{V}(h)$. Finally, we suppose that the elements of $\mathfrak{F}$ are realized not simply by sesquilinear forms on $\mathfrak{D}$, but by operators on $\mathfrak{D}$ that leave $\mathfrak{D}$ invariant.

Under the above hypotheses, we denote by $\mathfrak{H}'$ the set of vectors Φ of the pseudo-Hilbert space $\mathfrak{H}$ of the form $\Phi_A = A|0\rangle$, where $A \in \mathfrak{B}$, that is,

$$\mathfrak{H}' = \mathfrak{B}|0\rangle; \tag{10.27}$$

we call $\mathfrak{H}'$ the *physical subspace* of the space $\mathfrak{H}$ of virtual states. The scalar product of Φ_A and Φ_B is clearly given by $\langle \Phi_A, \Phi_B\rangle = \langle 0|A^*B|0\rangle$. On the other hand, we have

$$s(A) = \int_\Gamma \langle 0|\mathcal{V}(h)A\mathcal{V}(h)^{-1}|0\rangle dh = \langle 0|A|0\rangle \tag{10.28}$$

for all $A \in \mathfrak{B}$, so that

$$\langle \Phi_A, \Phi_B\rangle = s(A^*B) = \langle \pi(A)\Psi_0, \pi(B)\Psi_0\rangle \tag{10.29}$$

for all $A, B \in \mathfrak{B}$, where π is the physical representation of $\mathfrak{B}$. Hence it follows that the restriction to $\mathfrak{H}'$ of the scalar product is a non-negative definite form; consequently the quotient space $D = \mathfrak{H}'/\mathfrak{H}''$ is a pseudo-Hilbert space; here

$$\mathfrak{H}'' = \{\Phi \in \mathfrak{H}' : \langle \Phi, \Phi\rangle = 0\}. \tag{10.30}$$

Furthermore, (10.29) allows us to identify the completion of D with the physical Hilbert space. In fact, it easily follows from (10.29) that the map associating the equivalence class $[\Phi_A]$ in $D = \mathfrak{H}'/\mathfrak{H}''$ of $\Phi_A \in \mathfrak{H}'$ with the vector $\pi(A)\Psi_0$ is well defined and preserves the scalar product. Thus D and $\pi(\mathfrak{B})\Psi_0$ are isomorphic; hence the completion of D and $\mathcal{H}$ are also isomorphic as Hilbert spaces and can be identified (which we shall do).

As is easy to see, once we have made this identification, the operators $\pi(A)$ (where $A \in \mathfrak{B}$) are given by the formulae

$$\pi(A)[\Phi] = [A\Phi] \quad \text{for } \Phi \in \mathfrak{H}'; \tag{10.31}$$

similarly for the representation of ρ_0 and Γ we have

$$U(a, \underset{\sim}{\Lambda})[\Phi] = [\mathcal{U}(a, \underset{\sim}{\Lambda})\Phi], \quad V(h)[\Phi] = [\mathcal{V}(h)\Phi] \quad \text{for } \Phi \in \mathfrak{H}'. \tag{10.32}$$

Exercise 10.9. Suppose that the assumptions made at the beginning of this subsection hold.

(a) Prove that Ψ_0 is the unique (to within a factor) translation-invariant vector in $\mathcal{H}$ and that the representation π of $\mathfrak{B}$ in $\mathcal{H}$ is irreducible. [Hint: Here all the arguments in the proof of Proposition 10.2 are applicable.]

(b) Prove that the vacuum representation π_{vac} of the algebra of observables $\mathfrak{A}$ is the same as the subrepresentation $\pi^{(0)}$ of $\mathfrak{A}$ (corresponding to the trivial representation $\tau_0(h) \equiv 1$ in the decomposition (10.25)). [Hint: Use (10.26) and the fact that $\mathcal{H}_{\text{vac}}$ is the closure of $\pi_{\text{vac}}(A)\Psi_{\text{vac}}$; then verify that there exists an isomorphism of $\mathcal{H}_{\text{vac}}$ and $\mathcal{H}_0$ uniquely defined by the condition: the vector $\pi_{\text{vac}}(A)\Psi_{\text{vac}}$ is taken to $\pi(A)\Psi_0$ for any $A \in \mathfrak{A}$.]

Exercise 10.10. Suppose that in addition to the assumptions at the beginning of this subsection, the following conditions hold: the large gauge group $\mathcal{G}$ is abelian and the automorphisms γ_g ($g \in \mathcal{G}$) of the field algebra are realized by pseudo-unitary operators $\mathcal{V}(g)$ in $\mathfrak{H}$ that form a representation of $\mathcal{G}$ and belong to the algebra of observables $\mathfrak{A}$, where

$$\langle 0|\mathcal{V}(g)|0\rangle = 1 \quad \text{for all } g \in \mathcal{G}. \tag{10.33}$$

(a) Prove the relations*

$$\pi(\mathcal{V}(g))\Psi_0 = \Psi_0 \quad \text{for } g \in \mathcal{G}, \tag{10.34}$$

$$\pi(\mathcal{V}(g)) = 1 \quad \text{for } g \in \mathcal{G}_0. \tag{10.35}$$

[Hint: Condition (10.33) means that $\langle \Psi_0, \pi(\mathcal{V}(g))\Psi_0\rangle = 1$; whence (10.34) follows. To prove (10.35), let the operator $\pi(\mathcal{V}(g))$ (for $g \in \mathcal{G}_0$) act on the vector $\pi(A)\Psi_0$ where $A \in \mathcal{B}$; then use the fact that $\mathcal{V}(g)$ and A commute, together with the relation (10.34).]

(b) Prove that

$$\langle \Phi, \mathcal{V}(g)\Psi\rangle = \langle \Phi, \Psi\rangle \quad \text{for all } g \in \mathcal{G}_0,\ \Phi, \Psi \in \mathfrak{H}'. \tag{10.36}$$

[Hint: Rewrite (10.36) in terms of the physical Hilbert space and use (10.35).]

10.2. Abelian Models with Gauge Invariance of the 2nd Kind

A. The Field of the Dipole Ghost and the Gradient Model

The models discussed in this subsection are not devoid of interest, despite their primitiveness, since the characteristic features of theories with gauge invariance of the 2nd kind are clearly manifested in them.

By a *dipole ghost field* we mean a scalar neutral field $\chi(x) \equiv \chi^*(x)$ satisfying the 4th order equation

$$\Box^2\chi(x) = 0 \tag{10.37}$$

and the canonical commutation relations (CCR's)**

$$[\chi(x), \chi(y)] = \int 2\pi\epsilon(p^0)\delta'(p^2)e^{-ip(x-y)}d_4p = -\frac{i}{8\pi}\epsilon(x^0)\theta(x^2). \tag{10.38}$$

The dimension of the field χ with respect to the mass is equal to zero, that is, one less than the "canonical" dimension of a free scalar field (equal to $d/2 - 1 = 1$, where $d = 4$ is the dimension of

* The relations (10.35) mean that for $g \in \mathcal{G}_0$, the quantities $\mathcal{V}(g)$ are "observable" but only in a very trivial sense: they are all identically equal to unity in the physical representation of the algebra of observables.

**Another variant of the dipole ghost model is obtained if the commutator $[\chi(x), \chi(y)]$ is taken with the opposite sign in (10.38); the analogous constructions can be carried out for this.

Minkowski space), therefore the field is also called a free massless *subcanonical* (or "dimensionless") *scalar field*. It is obtained as the result of the quantization of a system of two scalar neutral fields $\chi(x)$ and $\Lambda(x)$ with quadratic Lagrangian

$$\mathcal{L}(x) = \partial_\mu \chi \partial^\mu \Lambda + \tfrac{1}{2}\Lambda^2.$$

The Euler-Lagrange equations

$$\Box \chi = \Lambda, \tag{10.39a}$$
$$\Box \Lambda = 0 \tag{10.39b}$$

enable us to eliminate Λ to obtain (10.37). The simultaneous CCR's, the only non-trivial ones of which are

$$[\partial_0 \Lambda(x), \chi(y)]|_{x^0=y^0} = \frac{1}{i}\delta(x-y) = [\partial_0 \chi(x), \Lambda(y)]|_{x^0=y^0},$$

together with the equations of motion (10.39) lead to (10.38).

We shall construct a representation of $\chi(x)$ by the method of second quantization and by postulating the following form of the two-point function:

$$w(x-y) \equiv \langle 0|\chi(x)\chi(y)|0\rangle, \tag{10.40}$$

where

$$\begin{aligned} w(x) = \int 2\pi\theta(p^0)\delta'(p^2)e^{-ipx}d_4p &= -\frac{1}{(4\pi)^2}\ln(-\kappa^2 x^2 + i\theta(x^0)) = \\ &= -\frac{1}{(4\pi)^2}\{\ln|\kappa^2 x^2| + i\pi\epsilon(x^0)\theta(x^2)\}. \end{aligned} \tag{10.41}$$

Here we run into the indeterminacy of the product $\theta(p^0)\delta'(p^2)$ of generalized functions ("infrared divergence"): this generalized function is uniquely defined only for test functions $u(p)$ vanishing at the point $p=0$.* The result of extending the definition of this functional to a generalized function in $\mathcal{S}'(\mathrm{M})$ is given by (10.41) which contains an arbitrary parameter κ with the dimension of mass (parameter of infrared regularization). It is clear that $w(x)$ is an associated homogeneous generalized function (of zero degree of the first kind):

$$w(\rho x) = w(x) - \frac{1}{8\pi^2}\ln\rho \quad \text{for } \rho > 0. \tag{10.42}$$

Exercise 10.11. Prove that there exists a real function $h \in \mathcal{S}(\mathrm{M})$ such that $h(0) = 1$ and

$$\langle 0|\chi(h)\chi(h)|0\rangle \quad \left(\equiv \int w(x-y)h(x)h(y)d^4x d^4y\right) = 0. \tag{10.43}$$

[Hint: Let $f(x)$ be an arbitrary real function in $\mathcal{S}(\mathrm{M})$ such that $\tilde{f}(0) = 0$ and let $f_\rho(x) = \rho^{-4}f(\rho^{-1}x)$ for $\rho > 0$; it follows from (10.42) that we can choose as h the function f_ρ for a suitable choice of ρ.]

Exercise 10.12. Let n be a fixed vector in V^+ such that $n^2 = 1$ and let $u(p)$ be an arbitrary function in $\mathcal{S}(\mathrm{M})$ such that $u(0) = 0$. Prove the relation

$$2\pi \int \theta(p^0)\delta'(p^2)u(p)d_4p = \int_{\Gamma_0^+} \frac{1}{2np}\left(-n\partial + \frac{1}{np}\right)u(p)(dp)_0. \tag{10.44}$$

[Hint: For $n = (1,0)$ the relation is proved by rewriting the left hand side in the form

$$-\frac{\partial}{\partial m^2}\int 2\pi\theta(p^0)\delta(p^2 - m^2)u(p)d_4p = -\frac{\partial}{\partial m^2}\int \frac{d_3p}{2\omega}u(\omega, p);$$

the general case follows from the Lorentz-invariance of $\tilde{w}(p)$.]

*By contrast, the generalized function $\epsilon(p^0)\delta'(p^2)$ in (10.38) is well defined as an odd homogeneous generalized function in $\mathcal{S}'(\mathrm{M})$ (see Exercise C.10 or §3.2.C).

The Hermitian form $\langle 0|\chi(\bar{f})\chi(g)|0|\rangle$ $(f,g \in \mathcal{S}(\mathrm{M}))$ on $\mathcal{S}(\mathrm{M})$ is clearly not non-negative definite. (This follows from the fact that $\tilde{w}(p)$, which is the Fourier transform of a two-point function, is not a (positive) measure; it also follows from (10.42); cf. Exercise C.6.) Therefore the representation of $\chi(x)$ by the method of second quantization must be constructed in a space with indefinite metric. To this end we fix a real function $h \in \mathcal{S}(\mathrm{M})$ with $\tilde{h}(0) = 1$, for which $\langle 0|\chi(h)\chi(h)|0\rangle = 0$ (see Exercise 10.11). We also fix a vector $n \in V^+$ such that $n^2 = 1$. We associate with an arbitrary function $f(x) \in \mathcal{S}(\mathrm{M})$ the pair of functions $F^{(1)}(p)$, $F^{(2)}(p)$ defined on Γ_0^+ as follows:

$$F^{(1)}(p) = \frac{1}{2np}(\tilde{f}(p) - \tilde{f}(0)\tilde{h}(p))\Big|_{p^0=|\mathbf{p}|}, \tag{10.45a}$$

$$F^{(2)}(p) = \Big(-n\partial + \frac{1}{2np}\Big)(\tilde{f}(p) - \tilde{f}(0)\tilde{h}(p))\Big|_{p^0=|\mathbf{p}|} \tag{10.45b}$$

and the pair of complex functions

$$F^{(3)} = \langle 0|\chi(h)\chi(f)|0\rangle, \quad F^{(4)} = \tilde{f}(0). \tag{10.45c}$$

Then using Exercise 10.12 we obtain

$$\langle 0|\chi(\bar{f})\chi(g)|0\rangle = \int_{\Gamma_0^+}\{\overline{F^{(1)}(p)}G^{(2)}(p) + \overline{F^{(2)}(p)}G^{(1)}(p)\}(dp)_0 + \overline{F^{(3)}}G^{(4)} + \overline{F^{(4)}}G^{(3)}. \tag{10.46}$$

We now choose as the one-particle subspace the Hilbert space

$$\mathfrak{H}_1 = \mathcal{L}^2(\Gamma_0^+,(dp)_0) \oplus \mathcal{L}^2(\Gamma_0^+,(dp)_0) \oplus \mathbf{C}^2 \tag{10.47}$$

with scalar product

$$(F,G) = \int_{\Gamma_0^+}\{\overline{F^{(1)}}(p)G^{(1)}(p) + \overline{F^{(2)}}(p)G^{(2)}(p)\}(dp)_0 + \overline{F^{(3)}}G^{(3)} + \overline{F^{(4)}}G^{(4)}$$

and define the indefinite form $\langle F,G\rangle$ on it using the right hand side of (10.46); as a result, $\mathfrak{H}_1$ becomes a pseudo-Hilbert space.

Exercise 10.13. Prove that formulae (10.45) define a linear map from $\mathcal{S}(\mathrm{M})$ onto an everywhere dense subset of $\mathfrak{H}_1$.

It is possible to define a pseudo-unitary representation $\mathcal{U}_1(a,\Lambda)$ of $\mathfrak{P}_0$ in $\mathfrak{H}_1$ such that the function $f_{\{a,\Lambda\}}(x) \equiv f(\Lambda^{-1}(x-a))$ is taken under the map (10.45) to an element $\mathcal{U}_1(a,\Lambda)F$ in $\mathfrak{H}_1$. (We suggest as an exercise that the reader finds the explicit form of the operators $\mathcal{U}_1(a,\Lambda)$.) Suppose now that $\mathfrak{H} = \mathcal{F}_V(\mathfrak{H}_1)$ is a pseudo-Hilbert Fock space of Bose particles with creation and annihilation operators $a^*(F)$, $a(F')$ satisfying (7.119). We define these by setting

$$\int A^*(p)\tilde{f}(p)d_4p = a^*(F), \int A(p)\overline{\tilde{f}(p)}d_4p = a(F), \tag{10.48}$$

$$\tilde{\chi}(p) = A(p) + A^*(-p). \tag{10.49}$$

Exercise 10.14. Prove that the above-defined field satisfies the relations (10.37), (10.38), (10.40).

Exercise 10.15. Let B be a vector in $\mathfrak{H}_1$ defined by the conditions

$$B^{(1)}(p) = B^{(2)}(p) = 0, \quad B^{(3)} = \tfrac{1}{2}i, \quad B^{(4)} = 0. \tag{10.50}$$

(a) Prove that the one-dimensional subspace $\mathcal{E}_1$ of $\mathfrak{H}_1$ spanned by B is Poincaré-invariant. [Hint:

$$\langle 0|\chi(\tilde{f})a^*(B)|0\rangle = \tfrac{1}{2}i\overline{\tilde{f}(0)} \tag{10.51}$$

for all $f \in \mathcal{S}(\mathrm{M})$.]

(b) Prove that the subspace $\mathcal{E} = \mathcal{F}_V(\mathcal{E}_1)$ of $\mathfrak{H} = \mathcal{F}_V(\mathfrak{H}_1)$ is Poincaré-invariant, where

$$\langle \Phi, \Psi \rangle = \langle \Phi | 0 \rangle \langle 0 | \Psi \rangle \quad \text{for } \Phi, \Psi \in \mathcal{E}. \tag{10.52}$$

(c) Let

$$Q = a(B) + a^*(B). \tag{10.53}$$

Prove that for $f \in \mathcal{S}_r(\mathbf{M})$ and $c \in R$

$$\exp(iQc)\exp\Bigl(i\int \chi(x)f(x)d^4x\Bigr)\exp(-iQc) = \exp\Bigl(i\int \chi'(x)f(x)d^4x\Bigr), \tag{10.54}$$

where

$$\chi'(x) = \chi(x) + c. \tag{10.55}$$

[Hint: Use the CCR's in exponential form (that is, in the Weyl form)

$$\exp(iQc)\exp\Bigl(i\int \chi(x)f(x)d^4x\Bigr) =$$
$$= \exp\Bigl(-\Bigl[Q, \int \chi(x)f(x)d^4x\Bigr]c\Bigr)\exp\Bigl(i\int \chi(x)f(x)d^4x\Bigr)\exp(iQc);$$

here

$$\Bigl[\int \chi(x)f(x)d^4x, Q\Bigr] = i\tilde{f}(0) \tag{10.56}$$

as a corollary of (10.51), 10.53).]

An interesting feature of the field $\chi(x)$ is the possibility of constructing a local normal exponential $:\exp(ig\chi(x)):$, as well as the polylocal normal exponentials

$$:\exp\bigl(ig\sum_{i=1}^{N} n_j\chi(x_j)\bigr): \tag{10.57}$$

as tempered generalized functions in x and in $x_1, \ldots, x_N$ respectively (here and in what follows, the n_j are arbitrary integers and g is a fixed positive constant). Upon smoothing these with test functions, we obtain sesquilinear forms on the space $\mathfrak{D}_0$ of all possible linear combinations of (so-called coherent) vectors in $\mathfrak{H}$ of the form

$$\Phi = \exp\Bigl(i\int \chi(x)f(x)dx\Bigr)|0\rangle, \quad \text{where } f \in \mathcal{S}(\mathbf{M}). \tag{10.58}$$

To define the expressions of type (10.57) we proceed as follows. Let ϵ be a vector in V^+; we introduce the regularized field

$$\chi^{(\epsilon)}(x) = \int (A(p)e^{-ip(x-i\epsilon/2)} + A^*(p)e^{ip(x+i\epsilon/2)})d^4x. \tag{10.59}$$

It follows from the property of the support ($\operatorname{supp} A^{(*)}(p) \subset \overline{V}^+$) that $\chi^{(\epsilon)}(x)$ is a smooth operator-valued function of x. We now define the local normal exponent of the regularized field:

$$:\exp(\pm ig\chi^{(\epsilon)}(x)): = \frac{\exp(\pm ig\chi^{(\epsilon)}(x))}{\langle 0|\exp(\pm ig\chi^{(\epsilon)}(x))|0\rangle}; \tag{10.60a}$$

more generally, for $n_j \in \mathbf{Z}$ we set

$$:\exp\Bigl(ig\sum_{j=1}^{N} n_j\chi^{(\epsilon)}(x_j)\Bigr): = \frac{\exp\Bigl(ig\sum_{j=1}^{N} n_j\chi^{(\epsilon)}(x_j)\Bigr)}{\langle 0|\exp\Bigl(ig\sum_{j=1}^{N} n_j\chi^{(\epsilon)}(x_j)\Bigr)|0\rangle}. \tag{10.60b}$$

Exercise 10.16. Prove the equality

$$\langle 0|\exp(\pm ig\chi^{(\epsilon)}(x))|0\rangle = (\kappa^2\epsilon^2)^{-\frac{g^2}{32\pi^2}}. \tag{10.61}$$

[Hint: Use the relation

$$\langle 0|\exp\Big(i\int \chi(x)f(x)dx\Big)\Big|0\rangle = \exp\Big(-\tfrac{1}{2}\int\langle 0|\chi(x)\chi(y)|0\rangle f(x)f(y)dx\,dy\Big), \tag{10.62}$$

which follows from the Wick formulae of type (8.75), (8.85).]

We now set

$$\langle\Phi\Big|\int u(x_1,\dots,x_N) : \exp\Big(ig\sum_{j=1}^{N} n_j\chi(x_j)\Big) : dx_1\dots dx_N\Big|\Psi\rangle =$$

$$= \lim_{\epsilon\to 0}\langle\Phi\Big|\int u(x_1,\dots,x_N) : \exp\Big(ig\sum_{j=1}^{N} n_j\chi^{(\epsilon)}(x_j)\Big) : dx_1\dots dx_N\Big|\Psi\rangle \tag{10.63}$$

for $\Phi,\Psi\in\mathfrak{D}_0$, $u(x_1,\dots,x_N)\in\mathcal{S}(\mathbf{M}^N)$.

Exercise 10.17. (a) Verify that $\mathfrak{D}_0$ is a dense linear subspace of $\mathfrak{H}$. [Hint: Replace $f(x)$ in (10.58) by $\sum_{j=1}^{N}\lambda_j f_j(x)$ and differentiate $\Phi\equiv\Phi(\lambda_1,\dots,\lambda_N)$ with respect to $\lambda_1,\dots,\lambda_N$ at $\lambda_1=\dots=\lambda_N=0$ to obtain the vectors in the form $\chi(f_1)\dots\chi(f_N)|0\rangle$.]

(b) Starting from the formula of type (10.62), prove the existence of the limit (10.63).

Formula (10.63) defines $\int : \exp\Big(ig\sum_{j=1}^{N} n_j\chi(x_j)\Big) : u(x_1,\dots,x_N)dx_1\dots dx_N$ as a sesquilinear form over $\mathfrak{D}_0$; the linear combinations of such forms clearly form a *-algebra $\mathfrak{J}$ of quantum fields $:\exp(img\chi(x)):$ with law of multiplication defined from the definition (10.63):

$$:\exp\Big(ig\sum_{j=1}^{N} n_j\chi(x_j)\Big) :: \exp\Big(ig\sum_{k=1}^{N'} n'_k\chi(y_k)\Big) : =$$
$$= \prod_{j=1}^{N}\prod_{k=1}^{N'} d(x_j-y_k)^{-n_j n'_k} : \exp\Big(ig\sum_{j=1}^{N} n_j\chi(x_j) + \sum_{k=1}^{N'} n'_k\chi(y_k)\Big) :, \tag{10.64}$$

where

$$d(x) = (-\kappa^2x^2 + i0\cdot x^0)^{-g^2/16\pi^2} \tag{10.65}$$

(this and the analogous equalities given below are to be understood in the sense of generalized functions with respect to $x_1,\dots,x'_{N'}$).

We now turn to the gauge transformations of the field χ. By the group of global gauge transformations (or gauge transformations of the first kind) of χ, we mean the group R of *-automorphisms of the field algebra χ, realized in $\mathfrak{H}$ by the pseudo-unitary operators e^{iQc} and acting according to formulae (10.54), (10.55). Note that the Wightman functions are not invariant under these transformations (with $c\neq 0$); for example

$$\langle 0|\chi'(x)\chi'(y)|0\rangle = \langle 0|\chi(x)\chi(y)|0\rangle + c^2. \tag{10.66}$$

This situation is possible because the Fock vacuum vector $|0\rangle$ is not invariant with respect to the operators e^{iQc}: it is taken into the Poincaré-invariant vector $e^{iQc}|0\rangle$ which lies in the subspace $\mathcal{E}$ of Exercise 10.15 (in view of (10.52), the difference $e^{iQc}|0\rangle - |0\rangle$ is a vector with zero scalar square).

By the local gauge transformations (or special gauge transformations of the second kind) of χ, we mean the transformations $\chi(x)\to\chi(x)+\alpha(x)$, where $\alpha(x)$ is a smooth real solution of the equation

$$\Box\alpha(x) = 0. \tag{10.67}$$

Here $\alpha(x)$ is a member of the Schwartz space $\mathcal{S}(\mathbf{R}^3)$ with respect to the variable $\mathbf{x}$ for fixed x^0; these transformations form an abelian group $\mathcal{G}_0$ of symmetries which are pseudo-unitarily realized in $\mathfrak{H}$.

Exercise 10.18. For $\alpha \in \mathcal{G}_0$ let

$$\lambda(\alpha) = \int_{x^0=\mathrm{const}} \{\alpha(x)\partial_0\Lambda(x) - \Lambda(x)\partial_0\alpha(x)\}d^3x; \tag{10.68}$$

prove the relation

$$e^{i\lambda(\alpha)}\chi(x)e^{-i\lambda(\alpha)} = \chi(x) + \alpha(x). \tag{10.69}$$

The groups $\mathcal{G}_0$ and $\mathcal{G}_0 \times \mathbf{R}$ play the roles of small and large gauge groups respectively (in the sense of §10.1.C);* this means that the algebra of observables $\mathfrak{A}$ and the algebra of physical quantities $\mathfrak{B}$ are defined as the subalgebras of $\mathcal{G}_0$- and $\mathcal{G}$-invariant elements respectively of the field algebra. The physical content of the model coming out of this turns out to be trivial (the algebra of observables $\mathfrak{A}$ is the same as the algebra of physical quantities $\mathfrak{B}$ and is formed from the field Λ (10.39) with commutation relations $[\Lambda(x), \Lambda(y)] \equiv 0$; as a result, the construction of §10.1.C provides us with only one state of the commutative algebra of observables, namely, the vacuum state).

More interesting are the models of the interaction of a field χ with other fields. The simplest of these is the gradient model in which the Dirac field $\psi(x)$ features alongside the scalar field $\chi(x)$. The "classical" version of this model is characterized by the Lagrangian

$$\mathcal{L}(x) = \tilde{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \partial_\mu\chi\partial^\mu\Lambda + \tfrac{1}{2}\Lambda^2 - g\tilde{\psi}\gamma^\mu\psi\partial_\mu\chi.$$

The solution of the classical Euler-Lagrange equation

$$(i\gamma^\mu\partial_\mu - m)\psi = g\gamma^\mu\partial_\mu\chi\cdot\psi \tag{10.70}$$

and (10.39) has the form $\psi(x) = e^{-ig\chi(x)}\psi^{(0)}(x)$, where $\chi(x)$ is a solution of (10.37) and $\psi^{(0)}(x)$ is a solution of the free Dirac equation.

In the quantum case we define $\psi(x)$ by the formula

$$\psi(x) =: e^{-ig\chi(x)} : \psi^{(0)}(x), \tag{10.71}$$

where $\chi(x)$ is the free dipole ghost field (realized in the pseudo-Hilbert Fock space $\mathfrak{H}$) and $\psi^{(0)}(x)$ is a free Dirac field (acting in the Fock space $\mathcal{H}$). As a result, the *-algebra of the fields ψ, $\tilde{\psi}$, χ is realized by sesquilinear forms on some subspace $\mathfrak{D} \otimes D_0 \subset \mathfrak{H} \otimes \mathcal{H}$.

Exercise 10.19. Verify that the Wightman functions of the fields ψ, $\tilde{\psi}$ (with the same number of occurrences of ψ and $\tilde{\psi}$) are related to the Wightman functions of the free Dirac field (see (8.118)–(8.120)) by formulae of type **

$$\begin{aligned}\langle 0|\psi(x_1)\dots\psi(x_n)\tilde{\psi}(y_1)\dots\tilde{\psi}(y_n)|0\rangle &= \langle\psi^{(0)}(x_1)\dots\psi^{(0)}(x_n)\tilde{\psi}^{(0)}(y_1)\dots\tilde{\psi}^{(0)}(y_n)\rangle_0\times\\ &\times\prod_{1\le j<k\le n} d(x_j-x_k)^{-1}d(y_j-y_k)^{-1}\prod_{1\le p\le n}\prod_{1\le q\le n} d(x_p-y_q).\end{aligned} \tag{10.72}$$

[Hint: Use the law of multiplication (10.64) and the fact that the vacuum expectation value of the normal exponential of χ is equal to unity.]

The field ψ satisfies the renormalized quantum field equation (analogous to the classical field equation (10.70)):

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = g\gamma^\mu N(\partial_\mu\chi(x)\cdot\psi(x)), \tag{10.73}$$

where N is the sign of the normal product defined as the limit of the expression

$$N(\partial_\mu\chi(x)\psi(y)) = \partial_\mu(\chi(x) + igw(x-y))\psi(y) \tag{10.74}$$

as $y \to x$.

Exercise 10.20. Prove (10.73).

* The fact that the quotient group $(\mathcal{G}_0 \times \mathbf{R})/\mathcal{G}_0 \approx \mathbf{R}$ is non-compact in the present instance is of no consequence: we shall see presently that $\mathfrak{A} = \mathfrak{B}$, so that $\mathbf{R}$ acts trivially on $\mathfrak{B}$.

** Here the product of the generalized functions is well defined (see the example in §2.6.C).

The group $\mathcal{G}_0$ of local gauge transformations defined above is pseudo-unitarily realized in $\mathfrak{H} \otimes \mathcal{H}$ and acts on the field according to the formula

$$\chi(x) \to \chi(x) + \alpha(x), \quad \psi(x) \to e^{-ig\alpha(x)}\psi(x) \tag{10.75}$$

Here the fields $\psi^{(0)}$ and $\Lambda = \Box\chi$ remain $\mathcal{G}_0$-invariant; it is not difficult to see that they generate the algebra $\mathfrak{B}$ of physical (that is, $\mathcal{G}_0$-invariant) quantities. For the large gauge group we choose $\mathcal{G} = \mathcal{G}_0 \otimes \Gamma$; the group $\Gamma = \mathbf{R} \times U(1)$ with typical element $(c, e^{ig\beta})$ is pseudo-unitarily realized in $\mathfrak{H} \otimes \mathcal{H}$ and acts on the fields according to the transformations

$$\chi'(x) = e^{iQc}e^{iQ_F\beta}\chi(x)e^{-iQc}e^{-iQ_F\beta} = \chi(x) + c, \tag{10.76a}$$
$$\psi'(x) = e^{iQc}e^{iQ_F\beta}\psi(x)e^{-iQc}e^{-iQ_F\beta} = e^{-igc}e^{-ig\beta}\psi(x), \tag{10.76b}$$

where Q_F is the fermionic charge in g units (that is, Q_F/g is the operator in $\mathcal{H}$ defined by (8.121)); furthermore,

$$\Lambda'(x) = \Lambda(x), \quad \psi^{(0)'}(x) = e^{-ig\beta}\psi^{(0)}(x). \tag{10.76c}$$

Hence it follows that the algebra $\mathfrak{A}$ of observables (that is, the $\mathcal{G}$-invariant quantities) consists of elements of $\mathfrak{B}$ that are invariant with respect to the gauge group (10.76c), that is, in fact, with respect to $U(1)$ (so that the non-compactness of the quotient group $\mathcal{G}/\mathcal{G}_0 \approx \Gamma$ is not in evidence here). In the physical representation, the algebra goes over to the algebra of the spinor field $\psi^{(0)}$ (and $\tilde{\psi}^{(0)}$) in the Fock space $\mathcal{H}$ while the algebra of observables is taken to the subalgebra $U(1)$ of invariant operators.

We arrive at the conclusion that the gradient model has the same physical content as the model of the free Dirac field (see §8.4.C).

B. Local Formulation of Quantum Electrodynamics

In quantum electrodynamics, gauges of a different type are used, and here we restrict ourselves to local (Lorentz-) covariant gauges. Of these, the most commonly used are the so-called ξ-gauges, * as well as the Lorentz or Landau gauge (which is a formal limit of ξ-gauges as $\xi \to 0$).

We recall the formulation of classical electrodynamics in the ξ-gauge. The total Lagrangian

$$\mathcal{L} = \tilde{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{4}F_{\lambda\mu}F^{\lambda\mu} - A_\mu j^\mu - \tfrac{1}{2}\xi\Lambda^2 \tag{10.77}$$

consists of the gauge-invariant Lagrangian $\mathcal{L}_{\text{inv}}$ (first three terms) and a gauge fixing term or g.f.t. (last term); here $\psi, \tilde{\psi}, A_\mu$ are the fundamental fields, while the field-stress tensor, the current and the auxiliary field Λ are given by

$$F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda, \tag{10.78}$$
$$j^\mu(x) = e\tilde{\psi}(x)\gamma^\mu\psi(x), \tag{10.79}$$
$$\xi\Lambda = \partial_\mu A^\mu, \tag{10.80}$$

where e is the electromagnetic interaction constant and ξ is a fixed gauge parameter.

The Euler-Lagrange equations have the form

$$(\gamma^\mu(i\partial_\mu - eA_\mu) - m)\psi = 0, \tag{10.81a}$$

$$\partial^\lambda F_{\lambda\mu} = j_\mu - \partial_\mu\Lambda. \tag{10.81b}$$

From (10.81a) we have "conservation" of current: $\partial_\mu j^\mu = 0$, so that from (10.81b) and the antisymmetry of $F_{\lambda\mu}$ we have $\Box\Lambda(x) = 0$.

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* In particular, the value $\xi = 1$ corresponds to the Gupta-Bleuler (otherwise known as Feynman) gauge.

We shall not write down the equations of the canonical formalism of classical electrodynamics in full, restricting ourselves to the simultaneous Poisson brackets involving the current or the field Λ:

$$\{j^0(x), \psi(y)\}||_{x^0=y^0} = ie\delta(\mathbf{x}-\mathbf{y})\psi(y), \{j^0(x), \tilde{\psi}(y)\}|_{x^0=y^0} = -ie\delta(\mathbf{x}-\mathbf{y})\tilde{\psi}(y), \tag{10.82a}$$

$$\{j^\lambda(x), A^\mu(y)\}|_{x^0=y^0} = 0 = \{j^\lambda(x), \Lambda(y)\}|_{x^0=y^0}, \tag{10.82b}$$

$$\{\Lambda(x), \psi(y)\}|_{x^0=y^0} = 0 = \{\Lambda(x), \tilde{\psi}(y)\}|_{x^0=y^0}, \tag{10.83a}$$

$$\{\Lambda(x), A_\mu(y)\}|_{x^0=y^0} = \delta_{\mu 0}\cdot\delta(\mathbf{x}-\mathbf{y}), \ \{\Lambda(x), \Lambda(y)\}|_{x^0=y^0} = 0. \tag{10.83b}$$

Exercise 10.21. (a) Let

$$Q = \int j^0(x)d^3x \tag{10.84}$$

be the electric charge. Prove the equalities

$$\{Q, \psi(x)\} = ie\psi(x), \ \{Q, \tilde{\psi}(x)\} = -ie\tilde{\psi}(x), \ \{Q, A_\mu(x)\} = 0. \tag{10.85}$$

(b) Derive the simultaneous Poisson brackets

$$\{\partial_0\Lambda(x), \psi(y)\}|_{x^0=y^0} = -ie\delta(\mathbf{x}-\mathbf{y})\psi(y), \tag{10.86a}$$

$$\{\partial_0\Lambda(x), A_0(y)\}|_{x^0=y^0} = 0, \ \{\partial_0\Lambda(x), A_j(y)\}|_{x^0=y^0} = \partial_j\delta(\mathbf{x}-\mathbf{y}). \tag{10.86b}$$

[Hint: Use the field equation (10.81b), the relations (10.82) and the Poisson bracket

$$\{F_{0j}(x), A_\mu(y)\}|_{x^0=y^0} = -\delta_{j\mu}\cdot\delta(\mathbf{x}-\mathbf{y}).]$$

(c) Derive the Poisson brackets at all times:

$$\{\Lambda(x), \psi(y)\} = -ieD_0(x-y)\psi(y), \ \{\Lambda(x), \tilde{\psi}(y)\} = ieD_0(x-y)\tilde{\psi}(y), \tag{10.87a}$$

$$\{\Lambda(x), A_\mu(y)\} = \partial_\mu D_0(x-y), \ \{\Lambda(x), \Lambda(y)\} = 0. \tag{10.87b}$$

[Hint: Use the simultaneous Poisson brackets (10.83), (10.86) and the equation $\Box\Lambda = 0$.]

The electrodynamical Lagrangian $\mathcal{L}_{\text{inv}}$ is invariant with respect to the gauge transformation of the second kind:

$$A'_\mu(x) = A_\mu(x) + \partial_\mu\alpha(x), \ \psi'(x) = e^{-ie\alpha(x)}\psi(x), \tag{10.88a}$$

where $\alpha(x)$ is an arbitrary real function in $\mathbf{M}$. At the same time, equations (10.81) are invariant with respect to the special gauge transformations of the second kind under which $\alpha(x)$ is subject to the equation $\Box\alpha = 0$; here

$$\Lambda'(x) = \Lambda(x). \tag{10.88b}$$

If we suppose that $\alpha(x)$ rapidly decreases at spacelike infinity, and if we call the corresponding gauge transformations local, then it is not too difficult to check by means of (10.87) that the expression (10.68) defines a generator of the local gauge transformations:

$$\{\lambda(\alpha), \psi(x)\} = ie\alpha(x)\psi(x), \quad \{\lambda(\alpha), \tilde{\psi}(x)\} = -ie\alpha(x)\tilde{\psi}(x), \tag{10.89a}$$

$$\{\lambda(\alpha), A_\mu(x)\} = -\partial_\mu\alpha(x), \quad \{\lambda(\alpha), \Lambda(x)\} = 0. \tag{10.89b}$$

On the other hand, the relations (10.85) imply that the electric charge is a generator of the global gauge transformations (corresponding to the constant functions $\alpha(x)$ =const). In the electrodynamics we can only give a physical meaning to those quantities f that are invariant with respect to the local gauge transformations ($\{\lambda(\alpha), f\} \equiv 0$; quantities satisfying this condition together with global gauge invariance ($\{Q, f\} = 0$) are said to be observable.

As examples of locally gauge-invariant quantities, we can take objects of the type

$$Y(x_1, \dots, x_N; y_1, \dots, y_{N'}; \eta) = \psi(x_1)\dots\psi(x_N)\tilde{\psi}(y_1)\dots\tilde{\psi}(y_{N'})\times$$
$$\times\exp\Big(-i\int A_\mu(x)\eta^\mu(x;\ x_1, \dots, x_N;\ y_1, \dots, y_{N'})dx\Big); \tag{10.90a}$$

here $\eta^\mu(x)$ is an arbitrary external current satisfying the equation

$$\partial_\mu \eta^\mu(x; x_1, \ldots, x_N; y_1, \ldots, y_{N'}) = e \sum_{j=1}^{N} \delta(x - x_j) - e \sum_{k=1}^{N'} \delta(x - y_k). \tag{10.90b}$$

It is clear that $\eta^\mu(x)$ has singularities at the points $x_1, \ldots, y_{N'}$. For $N \neq N'$, it cannot decrease arbitrarily rapidly as $|x| \to \infty$ (otherwise if the current $\eta^\mu(x)$ belonged, say, to the class $\mathcal{O}_C(\mathrm{M})$ with respect to x, then the integral of the left hand side of (10.90b) with respect to x would vanish). Consequently, the expression (10.90a) is not invariant under global gauge transformations (that is, for $N \neq N'$) and cannot be a functional of the fields in a bounded domain. By contrast, if $N = N'$, then the expression (10.90a) is invariant both under local and global gauge transformations, and if $x_1, \ldots, y_N$ vary over a bounded domain $\mathcal{O}$, then Y can also be a functional of the fields in $\mathcal{O}$. Another well known example of an observable quantity is the stress tensor field $F_{\lambda\mu}(x)$ or a functional of it of the form $\exp(i \int F_{\lambda\mu}(x) h^{\lambda\mu}(x) dx)$, where $h^{\lambda\mu} \in \mathcal{S}(\mathrm{M})$; (this functional is a special case of (10.90a) for $N = N' = 0$).

In quantum electrodynamics in a ξ-gauge, we assume that the fields ψ, $\tilde{\psi}$, A_μ satisfy the axiom of the pseudo-Wightman approach and the renormalized field equations that are the quantum analogues of the classical equations (10.70)–(10.81). The question of the existence of a solution of these equations has so far remained open (only within the framework of perturbation theory using the renormalization technique has the existence of a solution of the field equations been proved;* concerning this, see, for example, Brandt, 1970a). Here we shall go into details on the specific features of quantum electrodynamics (assuming it has solutions) under the conditions of gauge invariance of the first and second kinds.

The local gauge transformations in a ξ-gauge are generated by a local pseudo-Hermitian scalar field $\Lambda(x)$ that is related to $A^\mu(x)$ by the formula

$$\partial_\mu A^\mu(x) = \xi \Lambda(x), \tag{10.91}$$

under the assumption that it satisfies the d'Alembert equation

$$\Box \Lambda(x) = 0 \tag{10.92}$$

and has the trivial CCR's:

$$[\Lambda(x), \Lambda(y)] = 0, \tag{10.93}$$

and the trivial Wightman functions:

$$\langle 0 | \Lambda(x_1) \ldots \Lambda(x_n) | 0 \rangle = 0 \quad (n \geq 1). \tag{10.94}$$

Let $\mathcal{G}_0$ be the abelian group of local gauge transformations consisting of real smooth solutions $\alpha(x)$ of the wave equation $\Box \alpha(x) = 0$ such that $\alpha(x) \in \mathcal{S}(\mathbf{R}^3)$ as a function of $\mathbf{x}$. Corresponding to it are *-automorphisms of the field algebra realized in $\mathfrak{H}$ by the pseudo-unitary operators $\exp(i\lambda(\alpha))$ (imaginary exponentials of the operators $\lambda(\alpha)$ (10.68)):

$$e^{i\lambda(\alpha)} \psi(x) e^{-i\lambda(\alpha)} = e^{ie\alpha(x)} \psi(x), \quad e^{i\lambda(\alpha)} \tilde{\psi}(x) e^{-i\lambda(\alpha)} = e^{-ie\alpha(x)} \tilde{\psi}(x), \tag{10.95a}$$

$$e^{i\lambda(\alpha)} A_\mu(x) e^{-i\lambda(\alpha)} = A_\mu(x) - \partial_\mu \alpha(x), \quad e^{i\lambda(\alpha)} \Lambda(x) e^{-i\lambda(\alpha)} = \Lambda(x); \tag{10.95b}$$

* This circumstance does not detract from the physical value of quantum electrodynamics, since it turns out, as a result of the smallness of the constant of the electromagnetic interaction ($e^2/4\pi \approx$ $\approx 1/137$), that the low order terms of perturbation theory describe the electromagnetic processes very well throughout the range of their applicability.

here (in accordance with (10.94)) we require that

$$\langle 0|e^{i\lambda(\alpha)}|0\rangle = 0 \quad \text{for all } \alpha \in \mathcal{G}_0. \tag{10.96}$$

The global (or 1st kind) gauge transformations relate to the electric current $j^\mu(x)$ of a fermionic field. We suppose that $j^\mu(x)$ is a local (composite) field, that is invariant with respect to the local gauge transformations and satisfies the modified Maxwell equation

$$\partial^\lambda F_{\lambda\mu}(x) = j_\mu(x) + j_\mu^{(\text{fict})}(x), \tag{10.97}$$

where

$$j_\mu^{(\text{fict})}(x) = -\partial_\mu \Lambda \tag{10.98}$$

is a fictitious current. In accordance with (10.91), (10.97) the currents $j_\mu(x)$ and $j_\mu^{(\text{fict})}(x)$ are preserved:

$$\partial^\mu j_\mu(x) = 0 = \partial^\mu j_\mu^{(\text{fict})}(x). \tag{10.99}$$

We take as the quantum analogue of the Poisson brackets (10.85)*

$$-D(X) \equiv -i\int_{y^0=\text{const}} [j^0(y), X]d^3y = -iq_X X, \tag{10.100a}$$

where X denotes any of the local fields $\psi(x)$, $\tilde{\psi}(x)$, $A_\mu(x)$, $\Lambda(x)$, $j_\mu(x)$, and q_X is the electric charge carried by the given field:

$$q_\psi = -e,\ q_{\tilde{\psi}} = e,\ q_A = q_\Lambda = q_j = 0. \tag{10.100b}$$

Equality (10.100a) defines an infinitesimal gauge transformation of the fields (of the first kind). In electrodynamics it is usually postulated that it is realized by a pseudo-Hermitian operator Q. This means that there exists a one-parameter group of pseudo-unitary operators $\exp(iQc)$ in $\mathfrak{H}$ (with generator Q) giving the global gauge transformations of the fields:

$$e^{iQc}Xe^{-iQc} = e^{iq_X c}X \tag{10.101}$$

(X has the same meaning as in (10.100a)). In every case $U(1)$ acts on the field algebra according to the formula

$$\gamma_h(X) = e^{iq_X c}X \quad \text{for } h = e^{iec},\ c \in \mathbf{R}. \tag{10.102}$$

Equation (10.100a) sheds further light on the fact that the Maxwell equation $\partial^\lambda F_{\lambda\mu} - j_\mu = 0$ cannot hold in quantum electrodynamics in any local covariant gauge. More precisely, it follows from these equations that there are no local charged fields, since otherwise the left hand side of (10.100a) could be written as an integral $\int_{y^0=\text{const}} \partial^j [F_{0j}(y), X]d^3y$ of the divergence of a vector-valued (generalized) function with bounded support (here we have used the property of local commutativity of the fields $F_{\lambda\mu}$ and X) and would therefore vanish, and together with (10.100a) this would give $q_X = 0$ for all local fields X.

Exercise 10.22. Let $\mathcal{O}$ be an arbitrary bounded domain in $\mathbf{M}$; prove that there exists for any global gauge transformation a local gauge transformation coinciding with γ_h on the field subalgebra associated with $\mathcal{O}$. [Hint: Let $h = e^{iec}$, $c \in \mathbf{R}$; then there exists a function $\alpha \in \mathcal{G}_0$ equal to c in $\mathcal{O}$; now compare the right hand sides (10.95) and (10.102) with $x \in \mathcal{O}$.]

* Here we shall not go into the problem of the definition of the integral in this relation (which is discussed in §10.3.A below).

The property expressed by Exercise 10.22 is characteristic for any local covariant gauge of quantum electrodynamics.

According to the general scheme, the algebra $\mathfrak{B}$ of physical quantities (or the algebra $\mathfrak{A}$ of observables) is defined as the set of elements of the field algebra that are invariant with respect to the local (or local and global) gauge transformations. This definition presupposes that axioms PW.VIII and PW.IX, which guarantee the physical interpretation of the theory are valid. Here the roles of Γ and $\mathcal{G}$ are respectively played by $U(1)$ and $\mathcal{G}_0 \times U(1)$, while the axioms PW.VIII and PW.IX reduce, in fact, to the positiveness condition (10.22) of the functional $s(A)$ (10.21) on $\mathfrak{B}$ (which, in particular, subsumes the positiveness condition (10.18) of the vacuum functional on the algebra of observables). In order to reflect the real physical situation we have to assume that the physical space $\mathcal{H}$ contains subspaces $\mathcal{H}^{(n)}$ with any value $q = ne$ $(n \in \mathbf{Z})$ of the electric charge.

The fields $F_{\lambda\mu}(x), j_\mu(x), j_\mu^{\text{fict}}(x), \Lambda(x)$ featuring in equations (10.97), (10.98) are observable, * therefore on going over to the physical representation π, we obtain the corresponding operator-valued generalized functions $\mathcal{F}_{\lambda\mu}(x)$, $\mathcal{J}_\mu(x)$, $\mathcal{J}_\mu^{(\text{fict})}(x)$, $\hat{\Lambda}(x)$ in the physical Hilbert space $\mathcal{H}$. It turns out that $\hat{\Lambda}(x) = 0$ and hence,

$$\mathcal{J}_\mu^{(\text{fict})}(x) = 0, \tag{10.103}$$

so that equation (10.97) goes over to the ordinary Maxwell equation.

Proposition 10.4. *In the physical Hilbert space of the quantum electrodynamics, the Maxwell equations hold:*

$$\partial^\lambda \mathcal{F}_{\lambda\mu} - \mathcal{J}_\mu = 0. \tag{10.104}$$

■ In view of the connection between $\Lambda(x)$ and the operators $e^{i\lambda(\alpha)}$, the assertion that $\hat{\Lambda}(x) = 0$ is equivalent to the equality $\pi(e^{i\lambda(\alpha)}) = 1$ for all $\alpha \in \mathcal{G}_0$. For the proof of the latter equality, we note that according to (10.96),

$$\langle \Psi_0, \pi^{(0)}(e^{i\lambda(\alpha)} - 1)^* \pi^{(0)}(e^{i\lambda(\alpha)} - 1)\Psi_0\rangle = \langle 0|(e^{-i\lambda(\alpha)} - 1)(e^{i\lambda(\alpha)} - 1)|0\rangle = 0;$$

whence and from the positive definiteness of the scalar product in $\mathcal{H}^{(0)}$, it follows that $\pi^{(0)}(e^{i\lambda(\alpha)})\Psi_0 =$ $= \Psi_0$ for all $\alpha \in \mathcal{G}_0$. We now use the fact that the operators $e^{i\lambda(\alpha)}$ commute with the physical quantities. Then for any $X_1, X_2 \in \mathfrak{B}$

$$\langle \Psi_0, \pi(X_1)\pi(e^{i\lambda(\alpha)})\pi(X_2)\Psi_0\rangle = s(X_1 e^{i\lambda(\alpha)} X_2) = s(X_1 X_2 e^{i\lambda(\alpha)}) =$$
$$= \langle 0|X e^{i\lambda(\alpha)}|0\rangle = \langle \Psi_0, \pi^{(0)}(X)\pi^{(0)}(e^{i\lambda(\alpha)})\Psi_0\rangle = \langle \Psi_0, \pi^{(0)}(X)\Psi_0\rangle$$

(where X is the averaging of $X_1 X_2$ over the group $\Gamma = U(1)$). Thus the above matrix elements of the operator $\pi(e^{i\lambda(\alpha)})$ do not depend on α, which together with the cyclicity of the representation π gives: $\pi(e^{i\lambda(\alpha)}) = 1$ for all $\alpha \in \mathcal{G}_0$. ■

It must not be assumed, however, that the stress tensor field $\mathcal{F}_{\lambda\mu}$ generates the entire algebra of observables $\pi(\mathfrak{A})$ in the physical Hilbert space. (In classical electrodynamics, in addition to the functionals of $F_{\lambda\mu}$, all possible gauge-invariant expressions involving fermionic fields of type $\tilde{\psi}_\alpha(\partial_{\mu_1} + ieA_{\mu_1})\ldots(\partial_{\mu_n} + ieA_{\mu_n})\psi_\beta$ or (10.90a)

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* An alternative point of view is possible according to which, the field $\Lambda(x)$ (or even $\exp(2\pi i\Lambda(x)/e)$) is physical but unobservable. However, the fictitious current $j_\mu^{(\text{fict})}(x)$ corresponding to this field is "observable" (and is taken to $\mathcal{J}_\mu^{(\text{fict})}(x) \equiv 0$ in the physical representation). In accordance with this, the effective gauge group Γ must be extended (so that its action on $\Lambda(x)$ will be non-trivial). We shall illustrate this second possibility in §11.3.C by the example of the Lorentz gauge.

with $N = N'$, are also observables; the analogous quantities, which are accordingly renormalizable, must be added to the algebra of observables of quantum electrodynamics.) Nevertheless, it must be assumed that the local observables associated with all possible bounded domains $\mathcal{O}$ of space-time generate the entire algebra of observables (in the sense that the von Neumann algebras $(\cup\pi(\mathfrak{A}(\mathcal{O}))^{cc}$ and $\pi(\mathfrak{A})^{cc}$ are the same). With regard to the algebra $\pi(\mathfrak{B})$ of physical quantities on the other hand, a similar assumption (on the coincidence of $(\cup\pi(\mathfrak{B}(\mathcal{O})))^{cc}$ and $\pi(\mathfrak{B})^{cc}$) is inapplicable to a theory in which there are charged physical states. In fact, every physical quantity associated with a bounded domain is locally gauge invariant and is therefore (by Exercise 10.22) globally gauge-invariant as well, that is, an observable quantity; therefore it can only produce neutral states from the vacuum. This argument applies to any local covariant gauge. As a result we arrive at the following result which reveals the important role of the essentially non-local physical quantities in quantum electrodynamics.

Proposition 10.5. *In a local covariant gauge of quantum electrodynamics, every physical quantity associated with a bounded domain of space-time is an observable and hence carries no electric charge.*

In the above example from classical electrodynamics, we have already drawn attention to such a phenomenon: the expression (10.90a) can be a functional of the field A_μ in a bounded domain (with $x_1, \dots, y_{N'}$, varying in this bounded domain) only if $N = N'$, that is, only if this expression is an observable quantity. Note that the (accordingly renormalized) expressions of type (10.90a) are also convenient in quantum electrodynamics. Thus for the construction of a covariant operator of creation from the vacuum (generalized) state with momentum $-p$ and charge $-e$ (in particular, the electron), the following expression may prove useful:

$$\tilde{\psi}_{\text{phys}}(p) = \text{ren} \int dx e^{ipx} \psi(x) \exp\Big(-ie \int e^{-ikx} \tilde{A}_\mu(k) F^\mu(k,p) d_4 k\Big) \tag{10.105a}$$

under the condition

$$kF(k,p) = 1 \tag{10.105b}$$

(where ren is the symbol of renormalization, possibly introducing a regularization parameter, which corresponds to the case $N = 1$, $N' = 0$; here the external source $\eta^\mu(x, x_1)$ characterizing the photon cloud is chosen to be dependent on the momentum of the external momentum p (as well as being dependent on $x - x_1$).

We suggest as an exercise that the reader looks at the case of a free electromagnetic field with arbitrary ξ-gauge * and, in particular, constructs (by the method of second quantization as was done in §10.2.A) an operator realization of it in pseudo-Hilbert space. Here the two-point function is defined by the equality

$$\langle 0|A_\lambda(x)A_\mu(y)|0\rangle =$$
$$= 2\pi \int \theta(p^0)(-g_{\lambda\mu}\delta(p^2) + (\xi - 1)\delta'(p^2)p_\lambda p_\mu)e^{-ip(x-y)}d_4p =$$
$$= \frac{1}{(2\pi)^2} g_{\lambda\mu}((x-y)^2 - i0(x^0 - y^0))^{-1} + \frac{\xi - 1}{(4\pi)^2}\partial_\lambda\partial_\mu \ln\{\kappa^2[-(x-y)^2 + i0(x^0 - y^0)]\}. \tag{10.106}$$

Exercise 10.23. Derive (10.106) from the covariance condition, the field equations

$$\partial^\lambda F_{\lambda\mu} + \partial_\mu\Lambda = 0, \quad \xi\Lambda = \partial_\mu A^\mu, \quad \Box\Lambda = 0$$

* We are assuming that the reader is familiar with the quantization of a free electromagnetic field in the Gupta-Bleuler gauge ($\xi = 1$) for example, as in [B10],§12. With regard to other gauges, see, for example, Strocchi and Wightman (1974), Rideau (1975).

and the vacuum expectation values $\langle 0|\Lambda(x)\Lambda(y)|0\rangle = 0$ (see (10.94)), $\langle 0|\Lambda(x)A_\mu(y)|0\rangle = i\partial_\mu D_0(x-y)$; the second of these vacuum expectation values follows from the commutation relations $[\Lambda(x), A_\mu(y)] = i\partial_\mu D_0(x-y)$ (cf. (10.95b)).

Remark. The above analysis is in fact applicable to all abelian gauge theories (with gauge group $U(1)$ or $\mathbf{R}$) in which the gauge (that is electromagnetic) field A_μ interacts with the matter fields in "minimal" fashion.* In the local covariant gauge, the modified Maxwell equations and Propositions 10.4 and 10.5 hold as previously. (In what follows in §10.3.C, the "minimality" of the interaction is meant in the sense that the usual Maxwell equation (10.104) holds in the physical Hilbert space and the modified Maxwell equation (10.97) in the space of virtual states, for definiteness, in the ξ-gauge.)

The Higgs model (1964a) falls into this class; this was put forward in connection with the phenomenon of spontaneous symmetry breaking (§10.3.C); its Lagrangian has the form

$$\mathcal{L} = -\tfrac{1}{4}F_{\lambda\mu}F^{\lambda\mu} + |(\partial_\mu - ieA_\mu)\phi|^2 - \lambda^2(|\phi|^2 - f^2) + \text{g.f.t.},$$

where e, λ, f are constants and ϕ is a scalar charged field. (The existence of this model for four-dimensional space-time at a constructive level is also highly problematical; important work has been carried out for two-dimensional space-time in the direction of a constructive definition of the model; see Bridges et al., 1979;1981.)

10.3. Internal Symmetries

A. Symmetries and Currents in the Wightman Formalism

We consider a Lie group G of internal symmetries of Wightman fields. This means that associated with each element $g \in G$ is a *-automorphism σ_g of the polynomial algebra $\mathcal{P}(\mathbf{M})$ having the form

$$\sigma_g(\phi_l^{(\kappa)}(x)) = \sum_{\kappa'} \tau^{\kappa\kappa'}(g^{-1})\phi_l^{(\kappa')}(x). \tag{10.107}$$

Here $\{\phi^{(\kappa)}\}$ is a basic set of Wightman fields in the Hilbert space $\mathcal{H}$ and $\tau(g) \equiv (\tau^{\kappa\kappa'}(g))$ is a matrix representation of G where the types κ of the fields play the role of matrix indices, so that the *-automorphisms σ_g commute with the transformations of the Poincaré group and leave the field subalgebras $\mathcal{P}(\mathcal{O})$ ($\mathcal{O} \subset \mathbf{M}$) invariant (hence the name "internal symmetries"). We assume that $\tau^{\bar\kappa\bar\kappa'}(g) = \overline{\tau^{\kappa\kappa'}(g)}$ (where $\bar\kappa$ is the index of the Hermitian adjoint field: $(\phi^{(\kappa)})^* = \phi^{(\bar\kappa)}$). In the case when the set of fields is infinite, we suppose further that the fields are partitioned into finite subfamilies such that $\tau^{\kappa\kappa'}(g) = 0$ for indices κ, κ' from different subfamilies (so that the summation problem in (10.107) does not arise).

We shall mainly be interested in the case when G is connected. Let $I_1, \dots, I_N$ be a basis in the (real) Lie algebra $\mathcal{L}$ of G (so that $[I_k, I_l] = c_{kl}^m I_m$), and $T_1, \dots, T_N$ the corresponding generators of the representation τ

$$T_k = \frac{d}{dt}\tau(\exp tI_k)\Big|_{t=0}. \tag{10.108}$$

We associate with each element I_k the operator D_k on the field algebra:

$$D_k(X) = \frac{d}{dt}\sigma_{\exp(tI_k)}(X)\Big|_{t=0}, \tag{10.109}$$

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* This means that the Lagrangian of such a theory is constructed from the Lagrangians $\mathcal{L}(\phi, \partial\phi)$ of the "matter" fields ϕ by replacing the ordinary derivatives $\partial_\mu\phi$ by the covariant ones $\nabla_\mu\phi = (\partial_\mu - iqA_\mu)\phi$ and adding the Lagrangian of the electromagnetic field $-\frac{1}{4}F_{\lambda\mu}F^{\lambda\mu}$+g.f.t. (see the lecture by Coleman, 1975a for details).

so that

$$D_k(\phi^{(\kappa)}(x)) = -\sum_{\kappa'} T_k^{\kappa\kappa'}\phi^{(\kappa')}(x). \tag{10.110}$$

Since the σ_g are *-automorphisms of $\mathcal{P}(\mathbf{M})$, the operators D_k satisfy the conditions

$$D_k(X^*) = D_k(X)^*, \quad D_k(XY) = XD_k(Y) + D_k(X)Y; \tag{10.111}$$

Operators with the above properties are called (*-)*derivations* of the field algebra. At the same time, the D_k are generators of the representation of the Lie algebra $\mathcal{L}$, therefore the following commutation relations hold:

$$[D_k, D_l] = c_{kl}^m D_m. \tag{10.112}$$

Conversely, suppose that (10.109) defines *-derivations of the field algebra satisfying the relations (10.112) (where the T_k are defined by (10.108)). It is easy to see that G then acts by *-automorphisms of the field algebra of the form (10.107). In other words, the internal symmetry with respect to a connected group is completely determined by the set of derivations $D_1, \ldots, D_N$.

Quantum field theory borrows from the Lagrangian formalism the method of constructing symmetries from the conserved currents ([B10],§2). Here by a *conserved current* we mean a covariant vector field $j^\mu(x)$ that is a local functional of the Wightman field with zero divergence: $\partial_\mu j^\mu(x) = 0$. Moreover, we shall restrict ourselves to the case of Hermitian currents: $j^{\mu*}(x) = j^\mu(x)$ (in accordance with the fact that we chose a real basis $I_1, \ldots, I_N$ in the Lie algebra $\mathcal{L}$). Each such current defines a *-derivation D on the field algebra according to the formula

$$D(X) = i\int_{x^0=\text{const}} [j^0(x), X]d^3x. \tag{10.113}$$

The integral entering into (10.113) is well defined, as the following argument shows. Let X be a monomial of the form (8.8); then in view of the locality condition, the corresponding commutator

$$[j^\mu(x), \phi(x_1)\ldots\phi(x_n)]$$

vanishes if $(x - x_j)^2 < 0$ for all $j = 1, \ldots, n$. It follows from Exercise 2.32 that this operator-valued generalized function is a convolute with respect to the variable $x \in \mathbf{R}^3$; therefore the integral

$$\int [j^0(x), \phi(x_1)\ldots\phi(x_n)]d^3x \tag{10.114}$$

is defined as an operator-valued generalized function with respect to $x^0, x_1, \ldots, x_n$. It follows from the conservation of the current and differentiation under the integral sign (see Exercise 2.28) that:

$$\partial_0 \int [j^0(x), \phi(x_1)\ldots\phi(x_n)]d^3x = \int [\partial\mathbf{j}(x), \phi(x_1)\ldots\phi(x_n)]d^3x = 0,$$

so that (10.114) does not depend on x^0 which means that the variable x^0 can be regarded as a parameter (see Exercise 2.51(b)) which does not affect the actual value of the integral (10.114).

Exercise 10.24. (a) Prove that for any conserved Hermitian vector current $j^\mu(x)$, the formula (10.113) defines a *-derivation of the field algebra.

(b) Let $j_1^\mu(x), \ldots, j_N^\mu(x)$ be a set of conserved Hermitian vector currents, and $D_1, \ldots, D_N$ the derivations of the field algebra corresponding to them, where

$$D_k(j_l^\mu(x)) = c_{kl}^m j_m^\mu(x) \tag{10.115}$$

for all $k, l = 1, \dots, N$; $\mu = 0, \dots, 3$ (here the c_{kl}^m are the structure constants of the Lie algebra $\mathcal{L}$). Prove that $D_1, \dots, D_N$ satisfy the commutation relations (10.112).

It is clear from the above exercise how to define an internal symmetry in terms of currents. Thus we have to be given a set $j_k^\mu(x)$ $(k = 1, \dots, N)$ of conserved Hermitian vector currents with the corresponding derivations D_k on the field algebra satisfying (10.110), (10.115). The commutation relation (10.112) will then hold, and formula (10.107) will define an action of G by *-automorphisms of the field algebra. In this case, we say that the symmetry of the field algebra with respect to the (connected) group G is generated by the conserved currents j_k^μ $(k = 1, \dots, N)$. Unless otherwise stated, we shall always understand a symmetry to be defined in this way.

With regard to an internal symmetry, there arises the question whether it is unitarily realizable, that is, whether there exist unitary operators $V(g)$ in the physical Hilbert space $\mathcal{H}$ that are continuously dependent on $g \in G$ and form a representation of G such that

$$\sigma_g(X) = V(g)XV(g)^{-1} \quad \text{for all } g \in G. \tag{10.116}$$

If not, then we say that the symmetry is spontaneously broken (formulae (8.88) give an example of a spontaneously broken symmetry in the model of a free scalar massless field.* There is a simple criterion in terms of the Wightman functions enabling one to judge whether a symmetry is unitarily realizable. It reduces to the condition of G-covariance of the Wightman functions:

$$\sum_{\kappa_1' \dots \kappa_n'} \tau^{\kappa_1 \kappa_1'}(g) \dots \tau^{\kappa_n \kappa_n'}(g) w^{(\kappa_1' \dots \kappa_n')}(x_1, \dots, x_n) = w^{(\kappa_1 \dots \kappa_n)}(x_1, \dots, x_n); \tag{10.117}$$

when G is connected, this condition can clearly be rewritten in infinitesimal form:

$$\sum_{j=1}^n T_k^{\kappa_j \kappa_j'} w^{(\kappa_1 \dots \kappa_j' \dots \kappa_n)}(x_1, \dots, x_n) = 0 \quad \text{for } k = 1, \dots, N. \tag{10.118}$$

Exercise 10.25. Verify that in terms of the conserved currents, (10.118) means that

$$\int_{x^0=\text{const}} (\Psi_0, [j_k^0(x), X]\Psi_0) d^3x = 0 \quad \text{for } k = 1, \dots, N \tag{10.119}$$

for all $X \in \mathcal{P}(\mathrm{M})$. [Hint: The left hand side of (10.118) is $\langle \Psi_0, D_k(\phi^{(\kappa_1)}(x_1) \dots \phi^{(\kappa_n)}(x_n))\Psi_0 \rangle$.]

Proposition 10.6. *The G-covariance condition* (10.117) (*or* (10.118) *for a connected group G*) *is necessary and sufficient for formula* (10.107) *to define a unitarily realizable internal symmetry of the Wightman fields with respect to G.*

■ For the proof of the necessity it suffices to verify that the vacuum is an eigenvector of the operators $V(g)$. It follows from (10.107), (10.116) and (8.6) that the operators $V(g)^{-1}U(a,1)V(g)U(a,1)^{-1}$ commute with all the operators of the field algebra $\mathcal{P}(\mathrm{M})$ and (according to Proposition 8.1) are therefore multiples of the identity operator. It follows that the vectors $V(g)\Psi_0$ are eigenvectors of the translation operators $U(a,1)$, and since the vacuum vector is the unique (to within a factor) such vector, we have $V(g)\Psi_0 = \lambda(g)\Psi_0$, where $\lambda(g)$ is a complex function on G of unit modulus. It is clear that the factors $\lambda(g)$ form a one-dimensional unitary representation of G. If $\lambda(g) \not\equiv 1$, then we redefine the representation $V(g)$ by replacing $V(g)$ by $\lambda(g^{-1})V(g)$. The equality (10.116) is not affected by

* Formula (8.88) can be regarded (in the spirit of (10.107)) as a linear transformation of the fields if we include the trivial field $\chi(x) \equiv 1$ along with $\phi(x)$.

this redefinition, so that the vacuum vector becomes invariant for all operators $V(g)$. By taking the vacuum expectation value in (10.116), we now obtain (10.117).

The proof of sufficiency (like the proof of Poincaré invariance in the Wightman reconstruction theorem) repeats the arguments of Proposition 1.30 which is an addendum to the GNS construction. We merely note that in the spirit of this construction, the operators $V(g)$ are defined by the formula

$$V(g)X\Psi_0 = \sigma_g(X)\Psi_0, \tag{10.120}$$

where $X \in \mathcal{P}(\mathbf{M})$, $g \in G$. ■

As is clear from the proof of Proposition 10.5, we may suppose without loss of generality that the vacuum is invariant with respect to the operators $V(g)$ and that these operators act according to formula (10.120). In the case of a connected group, the representation $V(g)$ is uniquely defined by the charge operators Q_k ($k = 1, \dots, N$), defined by the relation

$$V(\exp(tI_k)) = \exp(itQ_k); \tag{10.121}$$

it follows from (10.109), (10.120) that

$$Q_k X\Psi_0 = -iD_k(X)\Psi_0. \tag{10.122}$$

The operators Q_k are clearly Hermitian; in fact they are essentially self-adjoint since vectors of the form $X\Psi_0$ ($X \in \mathcal{P}(\mathbf{M})$) are analytic vectors on them. This follows from the fact that the representation $\tau(g)$ of G and the representation $\{T_k\}_{k=1,\dots,N}$ of the Lie algebra split up into finite-dimensional subrepresentations; therefore we see via the relations (10.107), (10.109) that the series

$$V\left(\exp\left(i\textstyle\sum_k \lambda^k I_k\right)\right)X\Psi_0 = \sigma_g(X)\Psi_0 = \textstyle\sum_{n=0}^{\infty}\frac{1}{n!}\left(i\sum_k \lambda^k Q_k\right)^n X\Psi_0$$

are convergent for all $\lambda \in \mathbf{R}^N$.

B. GOLDSTONE'S THEOREM

The question of unitary realizability of internal symmetries of Wightman fields turns out to be closely related to the spectrum condition.

Lemma 10.7. *Let $j^\mu(x)$ be a conserved vector current in a system of Wightman fields. Then the vacuum expectation value of the derivation D of the field algebra $\mathcal{P}(\mathbf{M})$ corresponding to it satisfies the formula*

$$\langle\Psi_0, D(X)\Psi_0\rangle = i\int_{x^0=\text{const}} \langle\Psi_0, (j^0(x)F(M^2)X - XF(M^2)j^0(x))\Psi_0\rangle d^3x; \tag{10.123}$$

here $F(\lambda)$ is an arbitrary function in $\mathcal{O}_M(\mathbf{R})$ equal to one at the origin and M is the mass operator in the physical Hilbert space (here the integrand in (10.123) is a convolute with respect to $\mathbf{x}$).

■ By definition (10.113), $\langle\Psi_0, D(X)\Psi_0\rangle = i\int f(x)d^3x$, where $f(x) = \langle\Psi_0, [j^0(x), X]\Psi_0\rangle$. As we have already remarked in connection with this definition, $f(x)$ is a convolute with respect to $\mathbf{x}$, therefore the Fourier transform $\tilde f(p)$ is C^∞-dependent on $\mathbf{p}$ as a parameter (see Exercise 2.48(b)). In view of the fact that a (partial) integral corresponds to a restriction of the Fourier transform (see Exercise 2.28), we have

$$\int f(x)d^3x = \int e^{-ip^0x^0}\tilde f(p)\big|_{\mathbf{p}=0} d_1p^0;$$

since the left hand side is independent of x^0 and is equal to $-i\langle 0|D(X)|0\rangle$, it follows that

$$\tilde f(p)\big|_{\mathbf{p}=0} = -2\pi i\langle\Psi_0, D(X)\Psi_0\rangle\delta(p^0).$$

Let $g(x)$ denote the integrand in (10.123); then $\tilde{g}(p) = F(p^2)\tilde{f}(p)$. This generalized function also depends in C^∞-fashion on $\mathbf{p}$ as a parameter (see Exercise 2.48(a)), therefore $g(x)$ is a convolute with respect to $\mathbf{x}$. We have

$$\int g(x)d^3x = \int e^{-ip^0x^0}\tilde{g}(p)\big|_{\mathbf{p}=0}d_1p^0 = \int e^{-ip^0x^0}F((p^0)^2)\tilde{f}(p)\big|_{\mathbf{p}=0}d_1p^0 =$$
$$= \int e^{-ip^0x^0}F((p^0)^2)(-2\pi i)\langle\Psi_0, D(X)\Psi_0\rangle\delta(p^0)d_1p^0 = -i\langle\Psi_0, D(X)\Psi_0\rangle.$$

Thus (10.123) is proved. ■

We now give a sharpening of Lemma 10.7 for the case when $X \in \mathcal{P}(\mathcal{O})$, where $\mathcal{O}$ is a bounded domain. For definiteness, we consider the diamond $\mathcal{O} = \{x \in \mathbf{M} : |x^0| + |\mathbf{x}| < r\}$; then for $X \in \mathcal{P}(\mathcal{O})$ the integrand $g(x)$ in (10.123) has support in $\overline{\mathcal{O}} + \overline{V} = \{x \in \mathbf{M} : |\mathbf{x}| \le |x^0| + r\}$. In fact, by construction we have $\tilde{g}(p) = F(p^2)\tilde{f}(p)$, where the generalized function $f(x)$ has the properties $\operatorname{supp} f(x) \subset \overline{\mathcal{O}} + \overline{V}$, $\operatorname{supp}\tilde{f}(p) \subset \overline{V}$. According to Exercise 4.6 (where the positions of the variables x and p have to be interchanged), $g(x)$ also enjoys these properties.

We use this observation in the following exercise.

Exercise 10.26. Let $u(x) = a(x^0)b(\mathbf{x})$, where $a(x^0)$ is a function in $\mathcal{D}_r(\mathbf{R})$ with support in the interval $|x^0| < \epsilon$ and with integral equal to 1, and $b(\mathbf{x})$ is a function in $\mathcal{D}_r(\mathbf{R}^3)$ that is equal to one for $|\mathbf{x}| \le r + \epsilon$. Let

$$J = \int j^0(x)u(x)d^4x. \tag{10.124}$$

Prove that for all $X \in \mathcal{P}(\mathcal{O})$, where $\mathcal{O} = \{x \in \mathbf{M} : |x^0| + |\mathbf{x}| < r\}$, formula (10.123) can be rewritten in the form

$$\langle\Psi_0, D(X)\Psi_0\rangle = i\langle\Psi_0, (JF(M^2)X - XF(M^2)J)\Psi_0\rangle. \tag{10.125}$$

Theorem 10.8. *In the Wightman theory with the strong spectrum condition (that is, with mass gap), the internal symmetry generated by the conserved currents is unitarily implementable. The charge operators Q_k are representable as integrals of the "densities" $j^0_k(x)$ in the following sense:**

$$\langle\Psi, Q_k\Phi\rangle = \int_{x^0=\text{const}} \langle\Psi, j^0_k(x)\Phi\rangle d^3x, \tag{10.126}$$

where Φ, Ψ are arbitrary vectors in the dense linear subspace $\mathcal{P}(\mathbf{M})\Psi_0$ of $\mathcal{H}$.

■ We introduce the function $F(\lambda) \in \mathcal{O}_M(\mathbf{R})$ equal to one at $\lambda = 0$ and vanishing for $\lambda \ge \mu^2$, where μ is the mass-gap parameter. Then $F(M^2)$ is the projection operator onto the vacuum, and the right hand side of (10.123) (in which we must make the substitutions $D = D_k$, $j^0 = j^0_k$), along with $\langle\Psi_0, D_k(X)\Psi_0\rangle$, vanishes identically. According to Proposition 10.6 (and Exercise 10.25) this means that the symmetry is unitarily realizable.

We begin by proving (10.126) for the case $\Phi = \Psi_0$. It is enough to consider the case when $\Psi = X\Psi_0$, where X is a monomial of the form (8.8). Moreover, in view of the spectrum condition, we can assume that the support of the Fourier transform $\tilde{f}(p_1, \dots, p_n)$ of the function f in (8.8) vanishes for $p^0_1 + \dots + p^0_n < -\mu$ (otherwise, we can redefine f by multiplying $\tilde{f}$ by a suitable factor without changing $\Psi = X\Psi_0$). Then (thanks to the strong spectrum condition) $X^*\Psi_0$ is a vector which is collinear with Ψ_0 : $X^*\Psi_0 = \langle\Psi_0, X^*\Psi_0\rangle \cdot \Psi_0$. As a result we obtain

$$\langle\Psi, j^0_k(x)\Psi_0\rangle = -\langle\Psi_0, [j^0_k(x), X^*]\Psi_0\rangle$$

(here we have taken into account the fact that the vacuum expectation value of the current $j^\mu_k(x)$ is equal to zero by Lorentz-covariance). This shows that $\langle\Psi, j^0_k(x)\Psi_0\rangle$ is a convolute with respect to $\mathbf{x}$ whose integral (with respect to $\mathbf{x}$) is equal to zero:

$$\int\langle\Psi, j^0_k(x)\Psi_0\rangle d^3x = -\int\langle\Psi_0, [j^0_k(x), X^*]\Psi_0\rangle d^3x = i\langle\Psi_0, D_k(X^*)\Psi_0\rangle = 0.$$

* As follows from the proof, the integrand here is a convolute with respect to $\mathbf{x}$.

We now prove (10.126) in the general case. Let $\Psi = X\Psi_0, \Phi = Y\Psi_0$, where $X, Y \in \mathcal{P}(\mathbf{M})$. Then

$$\langle\Psi, j_k^0(x)\Phi\rangle = \langle\Psi, [j_k^0(x), Y]\Psi_0\rangle + \langle Y^*\Psi, j_k^0(x)\Psi_0\rangle.$$

Now we have just proved that the second term is a convolute with respect to $\mathbf{x}$ whose integral (with respect to $\mathbf{x}$) is equal to zero. Therefore

$$\int\langle\Psi, j_k^0(x)\Phi\rangle d^3x = \int\langle\Psi, [j_k^0(x), Y]\Psi_0\rangle d^3x = -i\langle\Psi, D_k(Y)\Psi_0\rangle,$$

which together with (10.122) yields the formula (10.126). ■

Thus in a Wightman theory with spontaneously broken internal symmetries, the strong spectrum condition cannot hold. The following theorem shows the presence in such a theory of a particle of zero mass and zero spin (or zero helicity); this particle is called the *scalar Goldstone boson.*

Theorem 10.9 (Goldstone). *If the internal symmetry of Wightman fields with respect to a connected Lie group generated by the conserved currents $j_k^\mu(x)$ ($k = 1, \ldots, N$) is spontaneously broken, then there exists in the physical Hilbert space a vector Φ of zero mass and zero spin, where $\langle\Phi, j_k^\mu(x)\Psi_0\rangle \not\equiv 0$ for at least one such vector Φ and at least one value of $k = 1, \ldots, N$.*

■ According to Proposition 10.6, the spontaneous breaking of symmetry means that (10.119) fails for at least one value of $k = 1, \ldots, N$ and at least one operator $X \in \mathcal{P}(\mathbf{M})$. We fix this value of k and write j^0 in D instead of j_k^0 in D_k.

We can assume that the operator X has the form (8.8). Since (by the partial integral property, see Exercise 2.30) the left hand side of (10.119) is a continuous functional of the function $f \in \mathcal{S}(\mathbf{M}^n)$ featuring in (8.8) and this function is not identically zero, it is not identically zero at the functions $f \in \mathcal{D}(\mathbf{M}^n)$. In other words, we can assume that the above operator X has been chosen so that $\langle\Psi_0, D(X)\Psi_0\rangle \neq 0$ belongs to the algebra $\mathcal{P}(\mathcal{O})$, where $\mathcal{O}$ is a bounded subset of $\mathbf{M}$ (for example, a diamond). It then follows from Exercise 10.26 that there exists a real function $u(x) \in \mathcal{D}(\mathbf{M})$ such that

$$\langle\Psi_0, (JF(M^2)X - XF(M^2)J)\Psi_0\rangle \neq 0, \tag{10.127}$$

where the left hand side of this inequality does not depend on the choice of the function $F(\lambda) \in \mathcal{O}_M(\mathbf{R})$ equal to one at $\lambda = 0$; here J is the operator defined by (10.124). Consider a decreasing sequence of non-negative functions $F_n(\lambda) \in \mathcal{D}(\mathbf{R})$ equal to one at the point $\lambda = 0$ and with supports contracting to the point $\lambda = 0$. Then $F_n(M^2)$ converges (in the weak operator topology) to the sum of the projection operator $|0\rangle\langle 0|$ onto the vacuum and the projection operator E_0 onto the subspace of vectors in $\mathcal{H}$ of zero mass. Replacing F by F_n in (10.127) and passing to the limit as $n \to \infty$, we obtain

$$\langle\Psi_0, (JE_0X - XE_0J)\Psi_0\rangle \neq 0.$$

It follows that the vector $E_0J\Psi_0$ is non-zero.

We have verified that the closure in $\mathcal{H}$ of the set of vectors of the form $E_0\left(\int j^\mu(x)u_\mu(x)d^4x\right)\Psi_0$, where $u^\mu(x) \in \mathcal{S}(\mathbf{M})$ is not $\{0\}$. It remains to show that it transforms according to a representation of $\mathfrak{P}_0$ with zero mass and zero spin. In fact, the generalized function $\langle\Psi_0, j^\lambda(x)E_0j^\mu(y)\Psi_0\rangle \equiv V^{\lambda\mu}(x-y)$ is Lorentz-covariant, of negative frequency and satisfies the wave equation $\Box V^{\lambda\mu}(x-y) = 0$ and the equation $\partial_\lambda V^{\lambda\mu}(x-y) = 0$. Hence it follows that it has the form

$$\langle\Psi_0, j^\lambda(x)E_0j^\mu(y)\Psi_0\rangle = ic\partial^\lambda\partial^\mu D_0^{(-)}(x-y) = c\int 2\pi p^\lambda p^\mu\theta(-p^0)\delta(p^2)e^{-ip(x-y)}d_4p,$$

where $c > 0$. This formula shows that there is a Poincaré-invariant isomorphism between the space of vectors of the form $E_0\left(\int j^\mu(x)u_\mu(x)d^4x\right)\Psi_0$ and a dense subspace of the Hilbert space $\mathfrak{H}^{[0,0]}$ of massless spinless particles. ■

In the example of Exercise 8.18, the states of the Goldstone particle were formed by the action on the vacuum of a free scalar massless field.

The phenomenon of spontaneous symmetry breaking (that is, in fact, the absence of a corresponding symmetry of the vacuum) has its attractions as a method of describing approximate symmetries in elementary quantum processes. However, the massless scalar particle predicted by the Goldstone theorem creates a difficulty for the physical interpretation of such a mechanism, since Goldstone particles have not been observed experimentally (by contrast, the photon, which is a massless boson with spin 1 is connected with the breaking of the gauge invariance of electrodynamic processes). We shall see below how this difficulty can be removed in theories with an indefinite metric.

C. Spontaneous symmetry breaking in abelian gauge theories

The specifics of gauge theories become apparent, in particular, in the question of a symmetry with respect to the (global) gauge transformations of the algebra $\mathfrak{B}$ of physical quantities (but with regard to other symmetries, they can behave like ordinary theories and can also have a Goldstone boson). As in §10.2.B, we restrict ourselves here to abelian gauge theories (with gauge group $U(1)$ and "minimal" interaction of the gauge field with the matter fields). The little gauge group $\mathcal{G}_0$ is the same as in §10.2. The role of the large gauge group is played by $\mathcal{G} = \mathcal{G}_0 \times \Gamma$, where the (compact) effective gauge group Γ is generated by the subgroup $U(1)$ and possibly some other subgroup H and where the subgroup $U(1)$ is related to the gauge invariance of the second kind of the given model and acts on the field by automorphisms γ_h of the form (10.102). By construction (§10.1.C), the automorphisms γ_h (for $h \in \Gamma$) leave the state s of the algebra $\mathfrak{B}$ invariant and are realized by unitary operators $V(h)$ in the physical Hilbert space $\mathcal{H}$. We say that the gauge symmetry of $U(1)$ is spontaneously broken if the group of operators $V(h)$ ($h \in U(1)$) is not contained in the von Neumann algebra $\pi(\mathfrak{B})^{cc}$ of physical quantities (and hence is not in the von Neumann algebra $\pi(\mathfrak{A})^{cc}$ either). The total charge operator in such a theory, as an unobservable quantity, has a purely fictitious meaning and cannot be represented in any reasonable sense as an integral $\int \mathcal{J}^0(x)d^3x$ of the zeroth component of the current (since the current in an abelian gauge theory is an observable field).

In the case of an irreducible representation of $\mathfrak{B}$ (as in the preceding subsection), by a spontaneously unbroken symmetry we mean a symmetry of $\mathfrak{B}$ with respect to a group of *-automorphisms which are realized by a unitary representation of the group. In the case when representation π of $\mathfrak{B}$ may be reducible, then (to avoid considering physically uninteresting realizations of the symmetry) we impose an additional restriction in the definition of a spontaneous unbroken symmetry: the operators of the group representation must belong to the von Neumann algebra of physical quantities.

The picture of spontaneous symmetry breaking in abelian gauge theories at the level of the non-physical pseudo-Hilbert space $\mathfrak{H}$ shows a remarkable similarity to what occurs in the Wightman theory (§10.3.B); in particular, the matrix elements of the current $\langle\Phi|\tilde{j}^\mu(p)|0\rangle$ have singularities at $p^2 = 0$, which can be interpreted as the presence of a Goldstone boson in $\mathfrak{H}$. This property is, however, a purely formal one, and the Goldstone boson disappears when we go over to the physical representation.

This is one of the indications of the Higgs mechanism, namely, the effect of the acquired mass of the gauge vector field under spontaneous breaking of the gauge group (or by the absorption of the Goldstone boson, which is how this mechanism is characterized by Coleman, 1975a).

The next propositions 10.10 and 10.11 show how spontaneous breaking of gauge symmetry appears in the pseudo-Wightman formalism at the level of the "fundamental" fields in a local Lorentz covariant gauge. Since in the first instance we are interested in spontaneous breaking of the symmetry group $U(1)$, we shall suppose that with regard to the other possible subgroup $H \subset \Gamma$, the symmetry with respect to H is not spontaneously broken and that the automorphisms γ_h $(h \in H)$ of the field algebra $\mathfrak{F}$ are realized by pseudo-unitary operators in $\mathfrak{H}$ that leave the vacuum vector $|0\rangle$ invariant.

Proposition 10.10. *Suppose that in an abelian gauge theory (with "minimal" interaction) the (compact) effective gauge group Γ is generated by the subgroups $U(1)$ and H and that the symmetry with respect to H enjoys the property given in the above paragraph, while the symmetry of the algebra $\mathfrak{B}$ (of physical quantities) with respect to the gauge subgroup $U(1)$ is spontaneously broken. Then:*

(a) *the group of automorphisms γ_h $(h \in U(1))$ of the field algebra $\mathfrak{F}$ (in the local Lorentz-covariant gauge) is not realized by a group of pseudo-unitary operators in the pseudo-Hilbert space $\mathfrak{H}$ that leave the vacuum vector $|0\rangle$ invariant;*

(b) *there exists an element X of the polynomial field algebra $\mathcal{P}(\mathrm{M})$ such that*

$$\langle 0|D(X)|0\rangle \equiv \langle 0|i \int_{x^0=\mathrm{const}} [j^0(x), X]|0\rangle d^3x \neq 0. \tag{10.128}$$

■ For the proof of the first assertion we assume that the group of automorphisms γ_h $(h \in U(1))$ of the field algebra $\mathfrak{F}$ is realized by a group of pseudo-unitary operators $\mathcal{U}(h)$ in $\mathfrak{H}$ that leave the vacuum vector invariant. Then it is easy to see that $\langle 0|\gamma_h(X)|0\rangle = \langle 0|X|0\rangle$ for $X \in \mathfrak{F}$, $h \in U(1)$. Furthermore, this same equality holds for $h \in H$, by hypothesis; hence it holds for all $h \in \Gamma$. As a result we see that the state (10.21) on $\mathfrak{B}$ is $s(X) = \langle 0|X|0\rangle$. Hence, as in §10.1.D (see Exercise 10.9(a)), it follows that the vacuum in the physical Hilbert space $\mathcal{H}$ is non-degenerate, that the representation π of $\mathfrak{B}$ in $\mathcal{H}$ is irreducible and that the gauge group $U(1)$ is realized by unitary operators $V(h)$ in $\mathcal{H}$ (see (10.32)). Since π is irreducible, this implies that the gauge symmetry of $U(1)$ is not spontaneously broken. This contradicts the condition of the proposition, so that we have proved (a).

The second assertion is proved in similar fashion. In fact, if we suppose that $\langle 0|D(X)|0\rangle = 0$ for all $X \in \mathcal{P}(\mathrm{M})$, then $\langle 0|\gamma_h(X)|0\rangle = \langle 0|X|0\rangle$ for all $X \in \mathcal{P}(\mathrm{M})$, $h \in U(1)$; in view of the condition of continuity of the action of $\mathcal{G}$ on $\mathfrak{F}$ (given in §10.1.C in the definition of the large gauge group), the above equality holds for all $X \in \mathfrak{F}$, $h \in U(1)$. Arguing as in the proof of the first assertion, we then find that the gauge symmetry of $U(1)$ is not spontaneously broken; this contradiction proves (b). ■

Under spontaneous symmetry breaking with respect to the gauge group, there is a result analogous to Goldstone's theorem for the pseudo-Hilbert space $\mathfrak{H}$. The essential difference, arising from the indefiniteness of the scalar product in $\mathfrak{H}$, is that because of the possibility that the translation operators $\mathcal{U}(a, 1)$ may be non-unitary in $\mathfrak{H}$, the assertion that the subspace of vectors in $\mathfrak{H}$ of zero mass is non-trivial may be devoid of meaning; also the presence of a Goldstone particle in $\mathfrak{H}$ has to be judged in terms of the singularities of expressions of type (10.11) near the light cone $\Gamma_0 = \{p \in \mathrm{M} : p^2 = 0\}$. In this connection we say that a generalized function $f(p) \in \mathcal{S}'(\mathrm{M})$ is *not less singular than* $\delta(p^2)$ near the light cone Γ_0 if it is not a (Borel) measure in any neighbourhood of the light cone of the form $|p^2| < \epsilon$ (where $\epsilon > 0$), or if it is a measure $f(p)d_4p = d\mu$ in some such neighbourhood, but with $\int_{\Gamma_0 \setminus \{0\}} |d\mu(p)| \neq 0$. Similarly, if $(p^2)^k f(p)$ (for some $k = 1, 2, \ldots$) is not less singular than $\delta(p^2)$ near Γ_0, then we say that $f(p)$ is *not less singular* than $\delta^{(k)}(p^2)$ near Γ_0.

This definition merely serves as a qualitative estimate of a singularity of $f(p)$ near Γ_0 and does not touch upon the question of what "a contribution proportional to $\delta^{(k)}(p^2)$" is equal to in a generalized function $f(p)$; such a question is not really well posed unless we restrict ourselves to certain special classes of generalized functions $f(p)$.

Proposition 10.11. *Suppose that the condition of Proposition* 10.10 *holds in an abelian gauge model with spontaneous symmetry breaking with respect to the gauge* *group $U(1)$ and that the "fundamental" fields in the Lorentz-covariant gauge act in the pseudo-Hilbert space $\mathfrak{H}$ of virtual states. Then*

(a) *for some element $X \in \mathcal{P}(\mathbf{M})$, the generalized function $\langle 0|X\tilde{j}^0(p)|0\rangle$ is not less singular than $\delta(p^2)$ near the light cone Γ_0;*

(b) *in the ξ-gauge model (for $\xi \neq 0$), the generalized function $\langle 0|X\tilde{A}^0(p)|0\rangle$ is, for some $X \in \mathcal{P}(\mathbf{M})$ not less singular than $\delta'(p^2)$ near Γ_0 and the auxiliary scalar Hilbert product $\{\cdot,\cdot\}$ cannot be translation-invariant in $\mathfrak{H}$.*

■ Assertion (a) is proved in almost the same way as Theorem 10.9 (with the difference that we now consider the non-physical pseudo-Hilbert space $\mathfrak{H}$ rather than the physical Hilbert space $\mathcal{H}$); we therefore merely give an outline. It follows from (10.127) that $\langle 0|D(X)|0\rangle \neq 0$ for some element $X \in \mathcal{P}(\mathcal{O})$, where $\mathcal{O}$ is a bounded subset of $\mathbf{M}$ (for example, a diamond). Therefore there exists a real function $u(x) \in \mathcal{D}(\mathbf{M})$ such that

$$\langle 0|(JF(M^2)X - XF(M^2)J)|0\rangle \neq 0, \tag{10.129}$$

where the operator J is defined by (10.124) and $F(\lambda)$ is an arbitrary function in $\mathcal{O}_M(\mathbf{R})$ that is equal to one at the point $\lambda = 0$ (the left hand side of (10.129) does not depend on the choice of F). Inequality (10.129) can be rewritten in the form

$$\int F(p^2)\overline{\tilde{u}(p)}[\langle 0|X\tilde{j}^0(p)|0\rangle - \overline{\langle 0|X^*\tilde{j}^0(-p)|0\rangle}]d_4p \neq 0. \tag{10.130}$$

If we suppose that for some $Y \in \mathcal{P}(\mathbf{M})$, the generalized function $\langle 0|Y\tilde{j}^0(p)|0\rangle$ is not a measure in any neighbourhood of the form $|p^2| < \epsilon$ ($\epsilon > 0$), then statement (a) is trivial. It therefore remains to consider the case when each of the expressions in the square brackets in (10.130) is a measure in some region of the form $|p^2| < \epsilon$. It follows from axiom PW.III(b′) that the contribution at the point $p = 0$ is zero. We choose a sequence F_n as in the proof of Theorem 10.9 and substitute it into (10.130). On passing to the limit as $n \to \infty$, we obtain

$$\int_{\Gamma_0\setminus\{0\}} \overline{\tilde{u}(p)}[\langle 0|X\tilde{j}^0(p)|0\rangle - \overline{\langle 0|X^*\tilde{j}^0(-p)|0\rangle}]d_4p \neq 0,$$

which completes the proof of statement (a).

For the proof of (b) we must use the modified Maxwell equation (10.97) in the ξ-gauge, where $\Lambda = \xi^{-1}\partial_\mu A^\mu$. By substituting $j^0(x) = -\partial^j F^{0j} + \partial^0\Lambda = (-1+\xi^{-1})\partial^j F^{0j} - \xi^{-1}\Box A^0$ in (10.128) and taking into account the fact that $[F^{0j}(x), X]$ is a convolute with respect to $\mathbf{x}$, we obtain instead of (10.128):

$$\langle 0|\int_{x^0=\text{const}} [\Box A^0(x), X]|0\rangle d^3x \neq 0$$

for some element $X \in \mathcal{P}(\mathbf{M})$. The subsequent line of argument is an exact repetition of the proof of (a), except that $\Box A^0(x)$ now features in place of $j^0(x)$. As a result we find that for some element $X \in \mathcal{P}(\mathbf{M})$, the generalized function $\langle 0|X\tilde{A}^0(p)|0\rangle$ is not less than $\delta'(p^2)$ near the light cone. As we noted in §10.1.B, the auxiliary Hilbert scalar product in $\mathfrak{H}$ is not translation-invariant in this situation. ■

A singularity ("not less than $\delta(p^2)$") near the light cone in the matrix element $\langle\Phi|\tilde{j}^\mu(p)|0\rangle$ can be interpreted as the presence of a fictitious "Goldstone boson"; its fictitiousness can be inferred from the fact that it is not observable (that is, it is not to be found in the physical Hilbert space).

Proposition 10.12. *Let $\mathcal{J}^\mu(x)$ be a conserved current (associated with the gauge group $U(1)$) in the physical representation in an abelian gauge theory (with "minimal" interaction). Then there is no vector Φ in the physical Hilbert space $\mathcal{H}$ with zero mass for which the matrix element $\langle\Phi, \mathcal{J}^\mu(x)\Psi_0\rangle$ is non-zero.*

■ The proof is based on the Maxwell equation $\partial_\lambda \mathfrak{F}^{\lambda\mu} = \mathcal{J}^\mu$ and group considerations. Assuming the contrary, we prove that the closure in $\mathcal{H}$ of the set of vectors of the form $E_0\left(\int \mathcal{J}^\mu(x)u_\mu(x)d^4x\right)\Psi_0$ is not merely the origin; here E_0 is the projector in $\mathcal{H}$ onto the subspace of vectors of zero mass and $u^\mu \in \mathcal{S}(\mathbf{M})$. As we observed in the proof of Theorem 10.9, it follows that this subspace of $\mathcal{H}$ transforms according to an irreducible representation of $\mathfrak{P}_0$ with zero mass and zero spin; consequently we can define in it a basis of vectors $|k\rangle$ ($k \in \Gamma_0^+$) normalized by the condition $\langle k|k'\rangle = (2\pi)^3 \cdot 2k^0\delta(\mathbf{k}-\mathbf{k}')$. We then have

$$\langle \Psi_0, \mathcal{J}^\lambda(x)E_0\mathfrak{F}^{\mu\nu}(y)\Psi_0\rangle = \int \langle \Psi_0, \mathcal{J}^\lambda(x)|k\rangle\langle k|\mathfrak{F}^{\mu\nu}(y)\Psi_0\rangle(dk)_0. \tag{10.131}$$

It follows from $\mathfrak{P}_0$-covariance that $\langle k|\mathfrak{F}^{\mu\nu}(y)\Psi_0\rangle = e^{iky}f^{\mu\nu}(k)$, where $f^{\mu\nu}(k)$ is a Lorentz-covariant tensor function of $k \in \Gamma_0^+$ that is antisymmetric in the indices μ, ν. Now it is obvious that the only such function is $f^{\mu\nu}(k) = 0$, therefore the generalized function (10.131) is equal to zero and so also is the generalized function $\langle \Psi_0, \mathcal{J}^\lambda(x)E_0\mathcal{J}^\mu(y)\Psi_0\rangle$. This contradicts our assumption; thus we have completed the proof "by contradiction". ■

We explain why the previous arguments of Goldstone's theorem are inapplicable here. In a theory with gauge invariance of the second kind, a physical quantity X carrying a charge (that is, such that $D(X) \neq 0$) cannot be local (see Proposition 10.5), therefore the physical quantity X featuring in the condition $\langle \Psi_0, \pi(D(X))\Psi_0\rangle \neq 0$ must be non-local. Thus one of the essential starting points of the proof of Goldstone's theorem no longer holds.

CHAPTER 11

Examples: Explicitly Soluble Two-Dimensional Models

11.1. Free Scalar Massless Field in Two-Dimensional Space-Time*

A. ONE DIMENSIONAL NON-CANONICAL SCALAR FIELD

The two-dimensional free scalar massless field $\phi(x)$ is of interest for several reasons. Because of the infrared singularities of the Wightman functions, it goes beyond the scope of the Wightman formalism (and requires a modification of the Wightman formalism by allowing an indefinite metric in the non-physical representation, and by replacing the ordinary vacuum vector by a generalized one in the physical representation). The model enables us to illustrate the importance of non-local quantities, since there are non-local quantities among the physically interesting ones; here by non-local quantities we mean those not generated by the algebra of local quantities (for example, the operators of creation of states with non-zero "topological" charge). Finally, this model lies at the basis of a number of other explicitly soluble models.

We consider a scalar massless neutral field $\phi(x) \equiv \phi^*(x)$. By definition, it satisfies the d'Alembert wave equation

$$\partial_\mu \partial^\mu \phi(x) \equiv ((\partial_0)^2 - (\partial_1)^2)\phi(x) = 0 \tag{11.1}$$

and the CCR's

$$[\phi(x), \phi(y)] = \frac{1}{i} D_0(x-y), \tag{11.2}$$

$$D_0(x-y) = 2\pi i \textstyle\int \epsilon(p^0)\delta(p^2)e^{-ip(x-y)}d_2p = \frac{1}{2}\epsilon(x^0-y^0)\theta((x-y)^2). \tag{11.3}$$

In the usual local-field treatment of a field $\phi(x)$ as an operator-valued generalized function, we have to smooth the product $\phi(x_1)\dots\phi(x_n)$ with test functions in $\mathcal{S}(\mathrm{M}^n)$, as a result of which one obtains the quasi-local quantities. We call this approach local quantization.

A second method of quantization (inequivalent to the local one), which we call canonical, consists in the following. Let Ξ be the space of all real "classical" (that is, c-number) solutions $\xi(x)$ of the d'Alembert wave equation such that the derivatives

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* In this chapter only, $\mathbf{M}$ denotes two-dimensional Minkowski space-time; an arbitrary point $x \in \mathbf{M}$ is a pair of real numbers $x \equiv (x^0, x^1)$, and the pseudo-Euclidean scalar product in $\mathbf{M}$ has the form $xy = x^0y^0 - x^1y^1$.

$\partial_0\xi(x)$, $\partial_1\xi(x)$ belong to the Schwartz space $\mathcal{S}(\mathbf{R})$ with respect to x^1 and are C^∞-dependent on x^0 as a parameter. We define on Ξ the non-degenerate skew symmetric (or symplectic) form

$$\sigma(\xi,\eta) = \int_{x^0=\text{const}} (\xi(x)\partial_0\eta(x) - (\partial_0\xi(x))\eta(x))dx^1, \tag{11.4}$$

so that Ξ can be regarded as the phase space of some "classical" linear bosonic system (subjected to quantization). We associate with each element $\xi \in \Xi$ a linear functional σ_ξ on Ξ : $\sigma_\xi(\eta) = \sigma(\eta,\xi)$; the Poisson brackets for σ_ξ are defined by $\{\sigma_\xi,\sigma_\eta\} = \sigma(\xi,\eta)$. For the canonical quantization of ϕ, we postulate that each element $\xi \in \Xi$ is associated with a Hermitian operator ϕ_ξ which is linearly dependent on ξ, and is such that the following canonical CCR's hold:

$$[\phi_\xi,\phi_\eta] = i\sigma(\xi,\eta). \tag{11.5}$$

The quantities of the local approach are contained in the algebra of the canonical quantization; for this it suffices that each test function $f \in \mathcal{S}_r(\mathbf{M})$ be associated with the operator

$$\int \phi(x)f(x)d^2x = \phi_\xi, \quad \xi(x) = \int D_0(x-y)f(y)dy \tag{11.6}$$

(which thus defines $\phi(x)$ as a local field). However, the algebra of the canonically quantized field is considerably more extensive: it contains quantities (for example, $e^{i\phi_\xi}$ for certain ξ carrying a "topological" charge) that cannot be obtained as limits of local quantities (in the weak operator topology of the physical representation). Therefore the local quantization leads to a much richer structure than the local-field method.

There is yet another method ("quantization on the light cone") which we now discuss in detail. It turns out that it is completely equivalent to canonical quantization (in the sense that there exists a natural isomorphism between the von Neumann algebras of the physical quantities in the physical representation). At the basis of the method is the simple observation: every classical solution $\xi(x)$ of the wave equation (of the above class Ξ) can be represented in the form

$$\xi(x) = \frac{1}{\sqrt{2}}(\xi^R(x^R) + \xi^L(x^L)); \tag{11.7}$$

here x^R and x^L are the variables associated with the light cone:

$$x^R = x^0 - x^1, \quad x^L = x^0 + x^1; \tag{11.8}$$

$\xi^R(x^R)$ and $\xi^L(x^L)$ are a pair of real C^∞-functions whose derivatives belong to $\mathcal{S}_r(\mathbf{R})$. In similar fashion we construct the quantum field $\phi(x)$ from the right and left components ϕ^R and ϕ^L, which play the role of the "fundamental" fields in this model. Here we assume that $\phi^R(x^R)$ and $\phi^L(x^L)$ are operator-valued generalized functions on $\mathcal{S}(\mathbf{R})$ satisfying the CCR's:

$$\begin{gathered}[\phi^R(x^R),\phi^R(y^R)] = \frac{1}{i}D^R(x^R-y^R), \quad [\phi^R(x^R),\phi^L(y^L)] = 0, \\ [\phi^L(x^L),\phi^L(y^L)] = \frac{1}{i}D^L(x^L-y^L),\end{gathered} \tag{11.9}$$

where

$$D^R(x^R - y^R) = i \int P\tfrac{1}{\lambda} \exp[-i\lambda(x^R - y^R)] d_1\lambda = \tfrac{1}{2}\epsilon(x^R - y^R) \tag{11.10}$$

(similarly, $D^L(x^L - y^L)$ is obtained by replacing R by L). The advantage of the third method of quantization is that here we are essentially dealing only with the two independent one-dimensional scalar fields $\phi^R(x^R)$ and $\phi^L(x^L)$ (each of which can be called a one-dimensional non-canonical scalar field). Therefore in the construction of the representation, we can concentrate on just one of the components, say, ϕ^R.

For the construction of the physical representation of the field $\phi^R(s)$ ($s \in \mathbf{R}$), we use the indefinite metric formalism (although here, this formalism is not an essential stage in the argument; an alternative possibility leading to the same physical representation is to start from the abstract algebra of the CCR's for the field $\phi^R(s)$).

We postulate the following form of the two-point function:

$$\langle 0|\phi^R(s)\phi^R(t)|0\rangle = W^R(s-t), \tag{11.11}$$

where

$$W^R(s-t) = \int \theta(\lambda)\lambda^{-1} e^{-i\lambda(s-t)} d_1\lambda = -\frac{1}{2\pi}\ln[0 + i\kappa(s-t)]. \tag{11.12}$$

The product $\theta(\lambda)\lambda^{-1}$ is ill defined (it is the so-called "infrared divergence"); the result of extending the definition, given by this formula, contains the mass parameter $\kappa > 0$. Since the generalized function $\theta(\lambda)\lambda^{-1}$ is not positive (see Exercise C.6), the metric in the space of virtual states cannot be positive definite. Let $f(s)$ be an arbitrary function in $\mathcal{S}(\mathbf{R})$. We associate with it the triple

$$F^{(0)}(\lambda) = \tilde{f}(\lambda) - \tilde{f}(0)\tilde{h}(\lambda),\ F^{(1)} = \langle 0|\phi^R(h)\phi^R(f)|0\rangle,\ F^{(2)} = \tilde{f}(0), \tag{11.13}$$

where $h(s)$ is a fixed function in $\mathcal{S}_r(\mathbf{R})$ such that $\tilde{h}(0) = 1$ and $\langle 0|\phi^R(h)\phi^R(h)|0\rangle = 0$. It is not difficult to see that the operator $f \to F$ maps $\mathcal{S}(\mathbf{R})$ onto a dense linear subspace of the one-particle Hilbert space

$$\mathfrak{H}_1^R = \mathcal{L}^2(\mathbf{R}_+, \lambda^{-1}d_1\lambda) \oplus \mathbf{C}^2$$

with positive scalar product

$$(F,G) = \int_{\mathbf{R}_+} \overline{F^{(0)}(\lambda)} G^{(0)}(\lambda)\lambda^{-1} d_1\lambda + \overline{F^{(1)}}G^{(1)} + \overline{F^{(2)}}G^{(2)} \tag{11.14}$$

and with indefinite form

$$\langle F,G\rangle = \int_{\mathbf{R}_+} \overline{F^{(0)}(\lambda)} G^{(0)}(\lambda)\lambda^{-1} d_1\lambda + \overline{F^{(1)}}G^{(2)} + \overline{F^{(2)}}G^{(1)}. \tag{11.15}$$

The space $\mathfrak{H}_1^R$ is a Pontryagin space with signature $(\infty, 1)$ with respect to the scalar product (11.15).

Exercise 11.1. (a) Verify the relation

$$\langle 0|\phi^R(\bar{f})\phi^R(g)|0\rangle = \langle F,G\rangle \quad \text{for } f,g \in \mathcal{S}(\mathbf{R}).$$

(b) Prove that there is a pseudo-unitary representation of the group of translations of $\mathbf{R}$ in $\mathfrak{H}_1^R$ such that the function $f_{\{a^R\}}(x^R) \equiv f(x^R - a^R)$ is taken under the map $f \to F$ to the element $\mathcal{U}_1^R(a^R)F$ in $\mathfrak{H}_1^R$.

(c) Verify that the vectors F for which $F^{(0)} = 0$, $F^{(2)} = 0$, are translation-invariant.

Let $\mathfrak{H}^R = \mathcal{F}_V(\mathfrak{H}_1^R)$ be the pseudo-Hilbert space of Bose particles with creation and annihilation operators $a^{R*}(F)$, $a^R(F)$. We now define the non-physical representation of the one-dimensional non-canonical scalar field $\phi^R(s)$ by the formula

$$\tilde{\phi}^R(\lambda) = A^R(\lambda) + A^{R*}(-\lambda), \tag{11.16}$$

where

$$\int A^R(\lambda)\tilde{f}(\lambda)d_1\lambda = a^{R*}(F), \quad \int A^R(\lambda)\overline{\tilde{f}(\lambda)}d_1\lambda = a^R(F). \tag{11.17}$$

It is not difficult to see that for any real number c^R the transformation

$$\gamma_{c^R}(\phi^R(s)) = \phi^R(s) + c^R \tag{11.18}$$

is a canonical transformation. Thus there is defined the gauge group $\Gamma^R = \mathbf{R}$ of transformations of the right field (in the terminology of §10.1.C, Γ^R plays the role of the "large" and "effective" gauge groups, while the "little" gauge group is trivial in the present instance. According to the next exercise, it is realized by pseudo-unitary operators in $\mathfrak{H}^R$.

Exercise 11.2. Suppose that the element $B \in \mathfrak{H}_1^R$ is defined by the conditions

$$B^{(0)}(\lambda) \equiv 0, \quad B^{(1)} = i/2, \quad B^{(2)} = 0 \tag{11.19}$$

and let

$$k^R = a^R(B) + a^{R*}(B). \tag{11.20}$$

Prove the relations:

$$[\phi^R(s), k^R] = i, \tag{11.21}$$

$$\exp(ik^Rc^R)\exp\Big(i\int \phi^R(s)f(s)ds\Big)\exp(-ik^Rc^R) = \exp\Big(i\int (\phi^R(s) + c^R)f(s)ds\Big) \tag{11.22}$$

for all $f \in \mathcal{S}_r(R)$, $c^R \in R$.

It is clear that the field

$$v^R(s) = \frac{\partial}{\partial s}\phi^R(s) \tag{11.23}$$

is gauge-invariant; we call it the right component of the current. The algebra $\mathfrak{A}^R$ generated by it is called the ("right") algebra of observables. Clearly the two-point function

$$\langle 0|v^R(s)v^R(t)|0\rangle = \int \theta(\lambda)\lambda e^{-i\lambda(s-t)}d_1\lambda = -\frac{1}{2\pi}(s - t - i0)^{-2} \tag{11.24}$$

is positive definite. Therefore the vacuum representation $\pi^{(0)}$ of $\mathfrak{A}^R$ is the Fock representation in the corresponding Fock Hilbert space $H^{R(0)}$ (the one-particle subspace of which is the completion of the set of vectors of the form $\int v^R(s)f(s)ds|0\rangle$, $f \in \mathcal{S}(\mathbf{R})$).

A similar construction can be carried out for the left component $\phi^L(x^L)$. The total non-physical space $\mathfrak{H}$ in which the scalar massless field $\phi(x)$ acts is the tensor product $\mathfrak{H} = \mathfrak{H}^R \otimes \mathfrak{H}^L$. Accordingly, the entire algebra of observables is the tensor product $\mathfrak{A} = \mathfrak{A}^R \otimes \mathfrak{A}^L$ of $\mathfrak{A}^R$ and $\mathfrak{A}^L$.

B. PHYSICAL REPRESENTATION

Since our gauge group $\Gamma_R = \mathbf{R}$ (for the field ϕ^R) is non-compact, the scheme of §10.1.C requires some modification guaranteeing the convergence of the integrals with respect to the gauge group.

Let $\mathfrak{M}^R$ be the set of all possible finite sums of operators in $\mathfrak{H}^R$ of the form

$$A = \int_{\mathcal{N}} \exp\Bigl(i \int \phi^R(s) f(s) ds\Bigr) F(f) df; \tag{11.25}$$

here $\mathcal{N}$ is an arbitrary finite-dimensional affine subspace of $\mathcal{S}_r(\mathbf{R})$ such that the functional $f \to \tilde{f}(0) \equiv \int f(s)ds$ maps $\mathcal{N}$ onto $\mathbf{R}$; $F(f)$ is an arbitrary function of class $\mathcal{S}(\mathcal{N})$, df is a (translation-invariant) Lebesgue measure on $\mathcal{N}$. It is easy to see that $\mathfrak{M}^R$ is a *-algebra (without identity) which is taken into itself under (left or right) multiplication by an operator of the form $\exp\Bigl(i \int \phi^R(s) f(s) ds\Bigr)$ ($f \in \mathcal{S}_r(\mathbf{R})$). As in §10.1.C, we define the (now generalized) vacuum functional s^R on $\mathfrak{M}^R$ by means of the integral over the gauge group:

$$s^R(A) = \int \langle 0|\gamma_{C^R}(A)|0\rangle dc^R, \tag{11.26a}$$

which at elements of the form (11.25) is equal to

$$s^R(A) = \textstyle\int_{\mathcal{N}} 2\pi\delta(\tilde{f}(0)) \exp(-\frac{1}{2} W^R(s-t) f(s) f(t) ds\, dt) F(f) df. \tag{11.26b}$$

It turns out that the positive-definiteness condition also holds:

$$s^R(A^*A) \geq \quad \text{for all } A \in \mathfrak{M}^R. \tag{11.27}$$

For the proof, we introduce the infrared regularized field $\phi^R_m(s)$ with two-point function $\langle 0_m|\phi^R_m(s) \times \times \phi^R_m(t)|0_m\rangle = W_m(s-t) = \int \theta(\lambda)(\lambda + m)^{-1} e^{-i\lambda(s-t)} d_1\lambda$, where $m > 0$ is the regularization parameter. Since $W_m(s-t)$ is positive definite, the field $\phi^R_m(s)$ is constructed by second quantization in the Hilbert space $\mathfrak{H}^R_m$ with vacuum vector $|0_m\rangle$. We suppose that the *-algebra $\mathfrak{M}^R_m$ is defined in the same way as $\mathfrak{M}^R$, that is, by replacing ϕ^R by ϕ^R_m in (11.25). It then follows from the positiveness of the metric in $\mathfrak{H}_m$ that

$$a_m \langle 0_m|A^*_m A_m|0_m\rangle \geq 0 \quad \text{for all } A_m \in \mathfrak{M}^R_m; \tag{11.28}$$

here the factor $a_m = (\ln m_0/m)^{1/2}$ (with arbitrary fixed $m_0 > 0$) is chosen so that

$$s^R(A) = \lim_{m \to 0} a_m \langle 0_m|A_m|0_m\rangle$$

at elements A of the form (11.25) (here A_m is obtained by replacing $\phi^R(s)$ by $\phi^R_m(s)$ in (11.25)). By passing to the limit as $m \to 0$ in (11.28), we now obtain (11.27).

The functional s^R is no longer normalized (and cannot be normalized) to unity, therefore we call it the *generalized vacuum state* of the algebra $\mathfrak{F}^R$ of physical quantities of the right field. The GNS construction (§1.5.D) easily extends to a generalized state. The physical representation π of ϕ^R is constructed in a complex Hilbert space

$\mathcal{H}^R$ in which the linear subspace formed by vectors $X(A)$ ($A \in \mathfrak{M}^R$), linearly dependent on A and having scalar product

$$\langle X(A), X(B)\rangle = s^R(A^*B), \quad A, B \in \mathfrak{M}^R \tag{11.29}$$

is a dense linear subspace. The right component $\Phi^R(s) \equiv \pi(\phi^R(s))$ in the physical representation is defined by the formula

$$\exp\Big(i\int \Phi^R(s)f(s)ds\Big)X(A) = X\Big(\exp\Big(i\int \phi^R(s)f(s)ds\Big)A\Big), \; A \in \mathfrak{M}^R \tag{11.30a}$$

for all $f \in \mathcal{S}_r(\mathbf{R})$ or, equivalently,

$$\langle X(A), \exp\Big(\int \Phi^R(s)f(s)ds\Big)X(B)\rangle =$$

$$= s^R(A^* \exp\Big(i\int \phi^R(s)f(s)ds\Big)B\Big), \quad A, B \in \mathfrak{M}^R. \tag{11.30b}$$

We shall use the abbreviated notation:

$$E_f^R = \exp\Big(i\int \Phi^R(s)f(s)ds\Big) \equiv \pi\Big(\exp\Big(i\int \Phi^R(s)f(s)ds\Big)\Big) \quad \text{for } f \in \mathcal{S}_r(\mathbf{R}). \tag{11.31}$$

The right current in the physical representation is denoted by $V^R(s)$:

$$V^R(s) = \frac{\partial}{\partial s}\Phi^R(s). \tag{11.32}$$

As usual, we call the algebra $\pi(\mathfrak{F}^R)^{cc}$ (or $\pi(\mathfrak{A}^R)^{cc}$) the von Neumann algebra of physical (or observable) quantities.

We define in $\mathcal{H}^R$ a unitary representation $U^R(a^R, r^R)$ of the group $\mathbf{R} \circ \mathbf{R}_+$ of inhomogeneous linear transformations of the variable x^R:

$$x^R \to r^R x^R + a^R \quad (r^R > 0,\; a^R \in \mathbf{R}). \tag{11.33}$$

This is done as follows: we define the *-automorphisms $\alpha_{(a^R, r^R)}$ of $\mathfrak{M}^R$ at the elements (11.25) by the formula

$$\alpha_{(a^R,r^R)}(A) = \int_{\mathcal{N}} \exp\Big(i\int \phi^R(r^R s + a^R)f(s)ds\Big)F(f)df.$$

Then we set

$$U^R(a^R, r^R)X(A) = X\big(\alpha_{(a^R,r^R)}(A)\big), \; A \in \mathfrak{M}^R. \tag{11.34}$$

Exercise 11.3. Prove that (11.34) defines a unitary representation of $\mathbf{R} \circ \mathbf{R}_+$ in $\mathcal{H}^R$ such that

$$U^R(a^R, r^R)\Phi^R(x^R)U^R(a^R, r^R)^{-1} = \Phi^R(r^R x^R + a^R). \tag{11.35}$$

Similarly, the gauge transformations (11.18) are unitarily realized in $\mathcal{H}^R$ by the operators $\exp(iK^R c^R)$ acting according to the formula

$$e^{iK^R c^R}X(A) = X(\gamma_{c^R}(A)), \quad A \in \mathfrak{M}^R. \tag{11.36}$$

Exercise 11.4. Prove that (11.36) defines a unitary representation of the gauge group $\Gamma^R = \mathbf{R}$ in $\mathcal{H}^R$ such that

$$e^{iK^R c^R}\Phi^R(x^R)e^{-iK^R c^R} = \Phi^R(x^R) + c^R. \tag{11.37}$$

In §10.1.C, the representation of the algebra of observables in $\mathcal{H}$ was decomposed into a direct sum of subrepresentations $\pi^{(\tau)}$ in space $\mathcal{H}^{(\tau)}$ with "charge" τ. In the present case, since the gauge group is non-compact, the representation of the algebra of observables in $\mathcal{H}$ is decomposed, not into a direct sum, but into a direct integral of representations $\pi^{(\zeta^R)}$ corresponding to the value ζ^R of the (right) charge Q^R:

$$\pi^R = \int_R^{\oplus} \pi^{(\zeta^R)} d_1\zeta^R. \tag{11.38}$$

The representation $\pi^{(0)}$ is simply the Fock representation of the (right) current (§11.1.A), while the representations $\pi^{(\zeta^R)}$ for $\zeta^R \neq 0$ are the so-called ***displaced Fock representations.*** They are defined by the following construction.

Let $\mathfrak{n}^{(\zeta^R)}$ be the set of all possible finite complex linear combinations of operators $\exp\left(i\int \phi^R(s)f(s)ds\right)$ where f ranges over the affine subspace $\mathcal{X}^{(\zeta^R)}$ of $\mathcal{S}_r(\mathbf{R})$:

$$\mathcal{X}^{(\zeta^R)} = \left\{f \in \mathcal{S}_r(\mathbf{R}) : \tilde{f}(0) \equiv \int f(s)ds = \zeta^R\right\}.$$

Then we claim that the expression $\langle 0|A^*B|0\rangle$ (where $A, B \in \mathfrak{n}^{(\zeta^R)}$) defines a positive definite Hermitian form on $\mathfrak{n}^{(\zeta^R)}$. In fact, every function $f \in \mathcal{X}^{(0)}$ can be uniquely written in the form $f(s) = -\partial_s g(s)$ for $g \in \mathcal{S}_r(\mathbf{R})$, so that

$$\int \phi^R(s)f(s)ds = \int v^R(s)g(s)ds,$$

and for $\zeta^R = 0$, our claim is a corollary of the positive definiteness of the scalar product in the Fock representation for the (right) current. If $\zeta^R \neq 0$, then

$$\mathcal{X}^{(\zeta^R)} = \zeta^R h + \mathcal{X}^{(0)}, \tag{11.39}$$

where h is a fixed function in $\mathcal{S}_r(\mathbf{R})$ such that $\tilde{h}(0) = 1$. Hence it easily follows that

$$\langle 0|A^*B|0\rangle = \langle 0|\tilde{A}^*\tilde{B}|0\rangle \quad \text{for } A, B \in \mathfrak{n}^{(\zeta^R)}, \tag{11.40}$$

where we have associated the operator $A = \sum_f c_f \exp\left(i\int \phi^R(s)f(s)ds\right)$ in $\mathfrak{n}^{(\zeta^R)}$ with the operator

$$\tilde{A} = \sum_f c_f \exp\left(\frac{i}{2}\zeta^R \int D^R(s-t)h(s)f(t)ds\,dt\right) \exp\left(i\int \phi^R(s)(f(s) - \zeta^R h(s))ds\right) \tag{11.41}$$

in $\mathfrak{n}^{(0)}$. The positive definiteness of the Hermitian form $\langle 0|A^*B|0\rangle$ on $\mathfrak{n}^{(\zeta^R)}$ is now obvious from (11.40).

We define the complex Hilbert space $\mathcal{H}^{(\zeta^R)}$ as the completion of the set of vectors $Y^{(\zeta^R)}(A)$, that are linearly dependent on $A \in \mathfrak{n}^{(\zeta^R)}$ and form a pre-Hilbert space with scalar product

$$\langle Y^{(\zeta^R)}(A), Y^{(\zeta^R)}(B)\rangle = \langle 0|A^*B|0\rangle \quad A, B \in \mathfrak{n}^{(\zeta^R)}. \tag{11.42}$$

The displaced Fock representation $\pi^{(\zeta^R)}$ for the (right) current is now defined in $\mathcal{H}^{(\zeta^R)}$ by the formula

$$\pi(\zeta^R)\Big(\exp\Big(i\int v^R(s)g(s)ds\Big)\Big)Y^{(\zeta^R)}(A) =$$

$$= Y^{(\zeta^R)}\Big(\exp\Big(i\int v^R(s)g(s)ds\Big)A\Big), \quad A \in \mathfrak{N}^{(\zeta^R)}, \tag{11.43a}$$

or by the matrix elements

$$\langle Y^{(\zeta^R)}(A), \pi^{(\zeta^R)}\Big(\exp\Big(i\int v^R(s)g(s)ds\Big)\Big)Y^{(\zeta^R)}(B)\rangle =$$

$$= \langle 0|A^* \exp\Big(i\int v^R(s)g(s)ds\Big)B|0\rangle \quad \text{for } A, B \in \mathfrak{N}^{(\zeta^R)},\ g \in \mathcal{S}_r(\mathbf{R}). \tag{11.43b}$$

Exercise 11.5. (a) Prove that the formula

$$U^{(\zeta^R)}(a^R, r^R)Y^{(\zeta^R)}(A) = Y^{(\zeta^R)}(\delta_{(a^R, r^R)}A), \quad A \in \mathfrak{N}^{(\zeta^R)}, \tag{11.44}$$

defines a unitary representation of $\mathbf{R} \circ \mathbf{R}_+$ in $\mathcal{H}^{(\zeta^R)}$, where

$$U^{(\zeta^R)}(a^R, r^R)\pi^{(\zeta^R)}(v^R(x^R))U^{(\zeta^R)}(a^R, r^R)^{-1} =$$
$$= r^R\pi^{(\zeta^R)}(v^R(r^Rx^R + a^R)), \quad r^R > 0,\ a^R \in \mathbf{R}. \tag{11.45}$$

(b) Prove that the generator of the translations $P^{L(\zeta^R)}$ defined by the equality

$$U^{(\zeta^R)}(a^R, 1) = \exp(\tfrac{1}{2}iP^{L(\zeta^R)}a^R) \tag{11.46}$$

is strictly positive for $\zeta^R \neq 0$.* [Hint: Let $A = \exp(i\int \phi^R(s)f(s)ds)$ for $f \in \mathcal{X}^{(\zeta^R)}$, $\zeta^R \neq 0$; verify that $\langle Y^{(\zeta^R)}(A), U^{(\zeta^R)}(a^R, 1)Y^{(\zeta^R)}(A)\rangle$ is an analytic function in a^R for $\operatorname{Im} a^R > 0$ which is continuous for $\operatorname{Im} a^R \geq 0$ and converges to zero for $a^R = ir$, $r \to +\infty$.]

We now state a number of properties of the physical representation of the right component of the field $\phi(x)$.

Proposition 11.1. (a) *The physical representation of the right component $\phi^R(x^R)$ of a free massless scalar field $\phi(x)$ in the space $\mathcal{H}^R$ is irreducible, and its restriction to the algebra of observables $\mathfrak{A}^R$ decomposes into a direct integral* (11.38) *of pairwise inequivalent irreducible representations $\pi^{(\zeta^R)}$ of the displaced Fock representations.*

(b) *The von Neumann algebra of observables $\pi(\mathfrak{A}^R)^{cc}$ is the same as the set of all gauge-invariant operators in $\mathcal{B}(\mathcal{H}^R)$ (that is, with the set of all bounded operators in $\mathcal{H}^R$ that commute with the operators $\exp(iK^Rc^R)$). In particular, the charge operator K^R is adjoined to $\pi(\mathfrak{A}^R)^{cc}$.*

(c) *The group of symmetries $\mathbf{R} \circ \mathbf{R}_+$ of the field $\phi^R(x^R)$ is realized by the unitary operators $U^R(a^R, r^R)$ in $\mathcal{H}^R$ satisfying* (11.34), (11.35) *and belonging to the von*

* Recall that a self-adjoint operator A in a Hilbert space $\mathcal{H}$ is strictly positive if and only if $\langle \Psi, A\Psi\rangle \geq 0$ for all $\Psi \in D_A$, the equality sign holding only when $\Psi = 0$.

Neumann algebra of observables $\pi(\mathfrak{A}^R)^{cc}$, where the generator of the translations P^L is strictly positive.

■ It follows from the construction of the space $\mathcal{H}^{(\zeta^R)}$ that it can be identified with the Fock space $\mathcal{H}^{R(0)}$; for this it suffices to associate the vector $Y^{(\zeta^R)}(A)$, where $A \in \mathfrak{N}^{(\zeta^R)}$, with the vector $Y^{(0)}(\tilde{A})$ (where $\tilde{A} \in \mathfrak{N}^{(0)}$ is defined in accordance with (11.41)). Under such an identification the direct integral $\tilde{\mathcal{H}} = \int^{\oplus} \mathcal{H}(\zeta)d_1\zeta$ is the completion of the pre-Hilbert space of (norm) continuous vector-valued functions $\Psi(\zeta)$ on $\mathbf{R}$ with compact supports and with values in $\mathcal{H}^{R(0)}$. The scalar product in $\tilde{\mathcal{H}}$ is defined by the formula $\langle\Psi_1,\Psi_2\rangle = \int\langle\Psi_1(\zeta),\Psi_2(\zeta)\rangle d_1\zeta$. We define the representation $\tilde{\pi}$ of the right field $\phi^R(s)$ in $\tilde{\mathcal{H}}$ by setting

$$\Big(\tilde{\pi}\Big(\exp\Big(i\int\phi^R(s)f(s)ds\Big)\Big)\Psi\Big)(\zeta) =$$

$$= \exp\Big(i\zeta\int D^R(s-t)f(s)h(t)ds\,dt\Big)\pi^{(0)}\Big(\exp\Big(i\int\phi^R(s)f(s)ds\Big)\Big)\Psi(\zeta) \quad \text{for } \widetilde{f(0)} = 0, \tag{11.47a}$$

$$\Big(\tilde{\pi}\Big(\exp\Big(i\zeta'\int\phi^R(s)h(s)ds\Big)\Big)\Psi\Big)(\zeta) = \Psi(\zeta-\zeta') \quad \text{for } \zeta' \in \mathbf{R}. \tag{11.47b}$$

Since $\mathcal{S}_r(\mathbf{R})$ is the direct sum of the subspaces $\mathcal{X}^{(0)}$ and $\mathbf{R}\cdot h$, these equalities combined with the CCR's define $\tilde{\pi}\big(\exp(i\int\phi^R(s)f(s)ds)\big)$ for all $f \in \mathcal{S}_r(\mathbf{R})$

Exercise 11.6. (a) Prove that there exists an isomorphism $W : \mathcal{H}^R \to \tilde{\mathcal{H}}$ of Hilbert spaces that associates the vector $X(A) \in \mathcal{H}^R$ (for $A \in \mathfrak{M}^R$) with the vector-valued function

$$\Psi(\zeta) = \pi^{(0)}\Big(\int\gamma_{cR}\Big(\exp\Big(-i\zeta\int\phi^R(s)v(s)ds\Big)A\Big)\Big)\Psi^{R(0)},$$

where $\Psi^{R(0)}$ is the vacuum vector in $\mathcal{H}^{R(0)}$.

(b) Prove that

$$\tilde{\pi}\Big(\exp\Big(i\int\phi^R(s)f(s)ds\Big)\Big) = W\pi\big(\exp(i\int\phi^R(s)f(s)ds)\big)W^{-1}$$

for all $f \in \mathcal{S}_r(\mathbf{R})$.

Exercise 11.6 shows that the representation π of ϕ^R in $\mathcal{H}$ is unitarily equivalent to the representation $\tilde{\pi}$ in $\tilde{\mathcal{H}}$ constructed above; therefore we can replace π by $\tilde{\pi}$ in Proposition 11.1.

Formula (11.47a) shows that each operator $\tilde{\pi}\big(\exp(i\int\phi^R(s)f(s)ds)\big)$ where $f \in \mathcal{X}^{(0)}$ reduces to a direct integral of the family of operators $\pi^{(\zeta)}\big(\exp(i\int\phi^R(s)f(s)ds)\big)$, $\zeta \in \mathbf{R}$. It is clear that in the new realization, the gauge transformation operators act as multiplication operators $\Psi(\zeta) \to \exp(i\zeta c^R)\Psi(\zeta)$, so that the von Neumann algebra generated by the operators $\exp(iK^Rc^R)$ is isomorphic to the algebra $\mathcal{L}^\infty(\mathbf{R},d\zeta)$ of all complex measurable essentially bounded functions on $\mathbf{R}$ (see [N2],§§6.13 and 26.5). We will show that the gauge transformation operators are contained in the weak closure of the system of operators $\tilde{\pi}\big(\exp(i\int\phi^R(s)f(s)ds)\big)$, $f \in \mathcal{X}^{(0)}$.

Exercise 11.7. (a) Construct a sequence of functions $\omega_n(t)$ in $\mathcal{S}_r(\mathbf{R})$ such that $\omega_n(t) \to 1$ in $\mathcal{S}'(\mathbf{R})$ and

$$\int\langle 0|v^R(s)v^R(t)|0\rangle\omega_n(s)\omega_n(t)ds\,dt \to 0 \quad \text{for } n \to \infty. \tag{11.48}$$

[Hint: By setting $\tilde{f}_n(\lambda) = \frac{\pi}{n}\theta(1-|\lambda|)|\lambda|^{-1+1/n}$, verify that $\tilde{f}_n(\lambda) \to 2\pi\delta(\lambda)$ and $\int_0^\infty \lambda|\tilde{f}_n(\lambda)|^2d\lambda \to 0$ as $n \to \infty$. For $\tilde{\omega}_n(\lambda)$ one can choose $\chi_n(|\lambda|)\tilde{g}_n(\lambda)$, where $\chi_n(\lambda)$ is a smooth function with values in $[0,1]$ that is equal to one for $2\epsilon_n < \lambda < 1-2\epsilon_n$ and has support in $\epsilon_n \le \lambda \le 1-\epsilon_n$, where ϵ_n is a sufficiently small positive number.]

(b) Prove that for any sequence $\omega_n \in \mathcal{S}_r(\mathbf{R})$ satisfying (11.48), $\pi^{(0)}\big(\exp(i\int v^R(s)\omega_n(s)ds)\big) \to 1$ as $n \to \infty$ in the weak operator topology. [Hint: By virtue of the unitarity of the above operators, it suffices to prove this limit for the matrix elements between smooth finite vectors of the Fock space; here one can use an estimate of type (7.124).]

It follows from (11.47a) that for any such sequence ω_n in Exercise 11.7, the following limit formula holds (in the weak operator topology):

$$\pi\big(\exp(ic^R \int v^R(s)\omega_n(s)ds)\big) \to \exp(iK^R c^R) \quad \text{as } n \to \infty. \tag{11.49}$$

This proves that the charge operators are affiliated to the von Neumann algebra. It follows from (11.49) that $\pi^{(\zeta^R)}\big(\exp(ic^R \int v^R(s)\omega_n(s)ds)\big) \to \exp(i\zeta^R c^R)$ as $n \to \infty$, from which it clearly follows that these representations $\pi^{(\zeta^R)}$ for the (right) current are pairwise unitarily inequivalent.

We now show that the algebra $\pi(\mathfrak{F}^R)^{cc}$ coincides with the algebra $\mathcal{B}(\mathcal{H}^R)$. To this end we identify the direct integral $\tilde{\mathcal{H}} = \int^{\oplus} \mathcal{H}^{(\zeta)} d_1\zeta$ with the tensor product $\mathcal{H}^{R(0)} \otimes \mathcal{L}^2(\mathbf{R}, d_1\zeta)$. Since the von Neumann algebra of observables $\pi(\mathfrak{A}^R)^{cc}$ is generated by elements of the form $\pi^{(0)}\big(\exp(i \int v^R(s)g(s)ds)\big) \otimes$ $\otimes u$, where $u \in \mathcal{L}^\infty(\mathbf{R}, d_1\zeta)$, and since for any $g \in \mathcal{S}_r(\mathbf{R})$ the family $\pi^{(0)}\big(\exp(i \int v^R(s)g(s)ds)\big)$ is an irreducible collection of operators in $\mathcal{H}^{R(0)}$, it follows that $\pi(\mathfrak{A}^R)^{cc} = \mathcal{B}(\mathcal{H}^{R(0)}) \otimes \mathcal{L}^\infty(\mathbf{R}, d_1\zeta)$. Accordingly, its commutant is $\pi(\mathfrak{A}^R)^c = 1 \otimes \mathcal{L}^\infty(\mathbf{R}, d_1\zeta)$ (see [N2],§26.5). Finally, it follows from (11.47) that the von Neumann algebra $\pi(\mathfrak{F}^R)^{cc}$ of $\phi^R(s)$ is generated by the algebra $\pi(\mathfrak{A}^R)^{cc}$ and the shift operators $1 \otimes T_\zeta$, where $(T_{\zeta'}u)(\zeta) = u(\zeta - \zeta')$; therefore its commutant consists of the elements of the algebra $\pi(\mathfrak{A}^R)^c = 1 \otimes \mathcal{L}^\infty(\mathbf{R}, d_1\zeta)$, that are invariant with respect to shifts, that is, it consists of the scalars: $\pi(\mathfrak{F}^R)^c = \mathbf{C}$. Hence $\pi(\mathfrak{F}^R)^{cc} = \mathcal{B}(\mathcal{H}^R)$, so that the algebra of the field $\Phi^R(s)$ acts irreducibly in $\mathcal{H}^R$.

Thus statements (a), (b) of Proposition 11.1 are proved. Statement (c) was in fact proved in Exercises 11.3 and 11.5 (except for the fact that the operators $U^R(a^R, r^R)$ belong to the von Neumann algebra of observables $\pi(\mathfrak{A}^R)^{cc}$, which follows easily from statement (b)). ∎

A similar assertion holds for the left component $\Phi^L(x^L)$. Both fields $\Phi^R(x^R) \otimes 1$ and $1 \otimes \Phi^L(x^L)$ are defined in the physical Hilbert space $\mathcal{H} = \mathcal{H}^R \otimes \mathcal{H}^L$, but we shall keep to the previous notation $\Phi^R(x^R)$ and $\Phi^L(x^L)$. The decomposition

$$\Phi(x) = \frac{1}{\sqrt{2}}(\Phi^R(x^R) + \Phi^L(x^L)) \tag{11.50}$$

enables us to define a local field $\Phi(x)$ in $\mathcal{H}$ (that is, an operator-valued generalized function of class $\mathcal{S}'(\mathbf{M})$). The canonically quantized field also acts in $\mathcal{H}$ as follows. An arbitrary classical solution of the wave equation $\xi \in \Xi$ can be uniquely expressed in the form

$$\xi(x) = \frac{1}{\sqrt{2}}(c + \int D^R(x^R - y^R)f^R(y^R)dy^R + \int D^L(x^L - y^L)f^L(y^L)dy^L), \tag{11.51}$$

where $c \in R$, $f^R, f^L \in \mathcal{S}(\mathbf{R})$; the corresponding operators $\Phi_\xi \equiv \pi(\phi_\xi)$ of the canonical quantization are defined by the formula

$$\Phi_\xi = -c\frac{Q^R + Q^L}{2} + \int \Phi^R(x^R)f^R(x^R)dx^R + \int \Phi^L(x^L)f^L(x^L)dx^L \tag{11.52}$$

(here, as before, the connection between the local and canonical quantized fields is realized by a formula of type (11.6)).

We define the representation of $\mathcal{P}_+^\uparrow \circ \mathbf{R}_+$ in $\mathcal{H}$ by transformations of space-time $\mathbf{M}$ of the form

$$x \to r\Lambda x + a, \tag{11.53}$$

where $r > 0$ is a scale parameter of the transformation, $a \in \mathbf{M}$ is a translation vector and Λ is the proper Lorentz transformation:

$$\Lambda = \begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}, \tag{11.54a}$$

$$(\Lambda x)^R = e^{-\chi}x^R, \quad (\Lambda x)^L = e^{\chi}x^L. \tag{11.54b}$$

For this we must set

$$U(a, r\Lambda) = U^R(a^R, r^R) \otimes U^L(a^L, r^L), \quad \text{where } r^R = re^{-\chi},\ r^L = re^{\chi}. \tag{11.55}$$

It follows from Exercises 11.3, 11.5 that the field $\Phi(x)$ is covariant in the physical representation:

$$U(a, r\Lambda)\Phi(x)U(a, r\Lambda)^{-1} = \Phi(r\Lambda x + a), \tag{11.56}$$

and that the spectrum condition holds: the spectrum of the energy-momentum operator $P \equiv (P^0, P^1)$ lies in the closed upper light cone. (However, there is no vacuum state of the field $\Phi(x)$ in the physical representation; it is replaced by the generalized vacuum state $s = s^R \otimes s^L$.)

Along with the right and left charges K^R, K^L we introduce the operators

$$K = \frac{1}{\sqrt{2}}(K^R + K^L), \quad K' = \frac{1}{\sqrt{2}}(-K^R + K^L). \tag{11.57}$$

As the next exercise shows, they are expressed in terms of the zeroth components of the conserved currents

$$V^\mu(x) = \partial^\mu\Phi(x), \quad V'^\mu(x) = \epsilon^{\mu\nu}\partial_\nu\Phi(x), \tag{11.58}$$

(where $\epsilon^{\mu\nu}$ is the "absolute" antisymmetric tensor ($\epsilon^{01} = 1$)) associated with the right and left currents (see (11.23)) $V^R(x^R) \equiv \pi(v^R(x^R))$, $V^L(x^L) \equiv \pi(v^L(x^L))$ in the physical representation by the relations

$$V^R(x^R) = \frac{1}{\sqrt{2}}(V^0(x) + V^1(x)), \quad V^L(x^L) = \frac{1}{\sqrt{2}}(V^0(x) - V^1(x)). \tag{11.59}$$

Exercise 11.8. Let $\omega_n(t)$ be the sequence defined in Exercise 11.7(a). Prove the relations

$$\begin{aligned} \exp\Big(ic\int_{x^0=\text{const}} V^0(x)\omega_n(x^1)dx^1\Big) &\to \exp(icK), \\ \exp\Big(ic'\int_{x^0=\text{const}} V'^0(x)\omega_n(x^1)dx^1\Big) &\to \exp(ic'K') \end{aligned} \tag{11.60}$$

as $n \to \infty$ in the weak operator topology. [Hint: Use the limits of type (11.49).]

Formulae (11.60) clarify the meaning of the standard definition of charge as a spatial integral of the zeroth component of the current:

$$K = \int_{x^0=\text{const}} V^0(x)dx^1, \quad K' = \int_{x^0=\text{const}} V'^0(x)dx^1 \equiv \int_{x^0=\text{const}} \partial_1\Phi(x)dx^1. \tag{11.61}$$

The second of these formulae shows that the charge K' has a "topological" interpretation: K is, in essence, the jump between the right and left spatial infinities; therefore K' is called the "topological" charge. It is not difficult to see that in the local quantization, $\Phi(x)$ is invariant with respect to the gauge transformations $\exp(ic'K')$, therefore the local field algebra $\Phi(x)$ does not contain operators of creation of "topological" charge and is reducible with respect to this charge in the physical space $\mathcal{H}$ (by contrast, the canonically quantized field is clearly irreducible in $\mathcal{H}$).

Remark. We suppose for definiteness that the field $\Phi(x)$ behaves like a "genuine" scalar under a spatial reflection: $\Phi(x) \to \Phi(I_s x)$, where $I_s x \equiv (x^0, -x^1)$; here the right and left components are transformed according to the rule

$$\Phi^R(s) \to \Phi^L(s), \quad \Phi^L(s) \to \Phi^R(s).$$

In this interpretation of the reflection, the currents V^μ and V'^μ are a "genuine" vector and pseudo-vector field respectively, while the field

$$\Phi'(x) = \frac{1}{\sqrt{2}}(-\Phi^R(x^R) + \Phi^L(x^L)) \tag{11.62}$$

is a pseudo-scalar.

C. Free "quark" fields; bosonization of fermions

Let E^R_f be an exponential of the form (11.31) of the smoothed field $\Phi^R(x^R)$. We call the expression

$$NE^R_f = \exp\left(\tfrac{1}{2}\int W^R(s-t)f(s)f(t)ds\,dt\right)E^R_f, \tag{11.63}$$

where $f \in \mathcal{S}_r(\mathbf{R})$, a non-local *normal exponential* of the field $\Phi^R(x^R)$. We define the N-*product* of the non-local exponentials $NE^R_{f_1}, \dots, NE^R_{f_n}$ by the formula

$$N((NE^R_{f_1})\dots(NE^R_{f_n})) \equiv N\Big(\prod_{j=1}^n NE^R_{f_j}\Big) = NE^R_{f_1+\dots+f_n}. \tag{11.64}$$

It is clear that this N-product is commutative.

Exercise 11.9. (a) Verify that the operators (11.63) satisfy the conditions

$$NE^R_0 = 1, \quad NE^{R*}_f = NE^R_{-f}, \quad NE^{R*}_f \cdot NE^R_g = \Omega(f,g)NE^R_{-f+g}, \tag{11.65}$$

where

$$\Omega(f,g) = \exp\Big(\int W^R(s-t)f(s)g(t)ds\,dt\Big). \tag{11.66}$$

(b) Verify the relations

$$(NE^R_{f_1})\dots(NE^R_{f_n}) = \prod_{j<k}\Omega(f_j,f_k)^{-1}N\Big(\prod_{j=1}^n(NE^R_{f_j})\Big). \tag{11.67}$$

The *local normal exponentials* $N\exp(i\alpha\Phi^R(s))$, as well as the polylocal normal exponentials

$$N\exp\Big(i\sum_{j=1}^n \alpha_j\Phi^R(s_j)\Big) \quad (\alpha_1,\dots,\alpha_n \in \mathbf{R},\ s_1,\dots,s_n \in \mathbf{R}) \tag{11.68}$$

of the field $\Phi^R(s)$ are defined as limits of non-local normal exponentials in the sense of operator-valued generalized functions in the variables $s, s_1, \dots, s_n \in \mathbf{R}$. Formally the expression (11.68) corresponds to extending the operator-valued functional NE^R_f to generalized functions $f = \mu$ of the form

$$\mu(t) = \sum_{j=1}^n \alpha_j\delta(t-s_j). \tag{11.69}$$

For such μ we take as the definition:

$$NR_{\mu}^{R} \equiv N\exp\Big(i\sum_{j=1}^{n}\alpha_j\Phi^R(s_j)\Big) = \lim_{\epsilon\to+0} NE^R_{\mu*\chi_\epsilon}, \tag{11.70}$$

where $\tilde{\chi}_\epsilon(\lambda) = e^{-|\lambda|\epsilon}$.

Note that although the functions $\tilde{f}(\lambda) = \tilde{\mu}(\lambda)\tilde{\chi}_\epsilon(\lambda)$ are not of class $\mathcal{S}(\mathbf{R})$, they are continuous and their restrictions to $\pm\overline{\mathbf{R}}_+$ belong to $\mathcal{S}(\pm\overline{\mathbf{R}}_+)$. It is not difficult to see that the functional NE_f^R can be extended by continuity (in the weak operator topology) to such f.

Exercise 11.10. Prove that the limit (11.70) exists as an operator-valued generalized function in the variables $s_1,\dots,s_n$, and that the result of smoothing NE_μ^R with a test function is an operator defined on the dense subset of $\mathcal{H}^R$ consisting of vectors of the form $X(A)$, $A \in \mathfrak{M}^R$. [Hint: It suffices to verify the existence of the limit of the matrix elements $\langle X(A), NE^R_{\mu*\chi_\epsilon} NE^R_{\nu*\chi_{\epsilon'}} X(B)\rangle$ as $\epsilon,\epsilon'\to+0$, where $A, B \in \mathfrak{M}^R$. The problem reduces, by virtue of the relations (11.64), to a consideration of the behaviour of the quantities $\Omega(f,\mu*\chi_\epsilon)$ as $\epsilon\to+0$ and $\Omega(\mu*\chi_\epsilon,\nu*\chi_{\epsilon'})$ as $\epsilon,\epsilon'\to+0$. It is clear that $\Omega(f,\mu*\chi_\epsilon)$ has the limit $\Omega(f,\mu)$, which is a smooth function of the parameters $s_1,\dots,s_n$. The quantities of the form

$$\Omega(\mu*\chi_\epsilon,\nu*\chi_{\epsilon'}) = \prod_{j=1}^{n}\prod_{k=1}^{n'}(\epsilon+\epsilon'+i\kappa(s_j-t_k))^{-\alpha_j\beta_k/2\pi}$$

have a limit as $\epsilon,\epsilon'\to+0$ in the sense of generalized functions in $s_1,\dots,t_{n'}$:

$$\Omega(\mu,\nu) = \prod_{j=1}^{n}\prod_{k=1}^{n'}(0+i\kappa(s_j-t_k))^{-\alpha_j\beta_k/2\pi}; \tag{11.71}$$

the product of such quantities also has a well defined limit.]

Exercise 11.11. (a) Prove the relations

$$NE_\mu^{R*} = NE^R_{-\mu}, \quad (NE^R_{\mu_1})\dots(NE^R_{\mu_k}) = \prod_{i<j}\Omega(\mu_i,\mu_j)^{-1}NE^R_{\mu_1+\dots+\mu_k} \tag{11.72}$$

(in the sense of operator-valued generalized functions in $s_1,\dots,s_n,s_1^{(1)},\dots,s_{n_k}^{(k)}$.)

(b) Prove that (11.70) is an operator-valued function in s_1 that is C^∞-dependent on $s_1-s_2,\dots$ $\dots,s_{n-1}-s_n$ as parameters.

Since the charge K^R is in a certain sense a linear functional of the field Φ^R (see (11.60), (11.61)), it is fairly easy to define the normal exponentials of the field $\Phi^R(x^R)$ and the charge K^R; for this we set

$$N\exp\Big(i\int\Phi^R(x^R)f(x^R)dx^R + ic^RK^R\Big) = \lim_{n\to\infty} NE^R_{f+c^R\omega_n}, \tag{11.73}$$

where ω_n is the same sequence as in Exercise 11.7(a).

Exercise 11.12. Prove the relations

$$N\exp(ic^RK^R) = \exp(ic^RK^R), \tag{11.74}$$

$$N((NE^R_\mu)(N\exp(ic^RK^R))) \equiv N\exp\Big(i\Big(\textstyle\sum_{j=1}^n\alpha_j\Phi^R(s_j) + c^RK^R\Big)\Big) =$$

$$= \exp\Big(\frac{i}{2}c^R\int\mu(s)ds\Big)(NE^R_\mu)\exp(ic^RK^R) = \exp\Big(-\frac{i}{2}c^R\int\mu(s)ds\Big)\exp(ic^RK^R)NE^R_\mu. \tag{11.75}$$

It is clear that local normal exponentials of $\Phi^R(x^R)$ and $\Phi^L(x^L)$ can be constructed in the space $\mathcal{H} = \mathcal{H}^R \otimes \mathcal{H}^L$. In particular, we introduce the two-component field

$$q(x) \equiv \begin{pmatrix} q^R(x) \\ q^L(x) \end{pmatrix} = \begin{pmatrix} N\exp[i\alpha(\Phi^R(x^R) + \frac{1}{2}K^L)] \\ N\exp[i\alpha(\Phi^L(x^L) - \frac{1}{2}K^R)] \end{pmatrix}, \tag{11.76}$$

where α is a real parameter. It is not difficult to see that the component $q^{R,L}(x)$ carries a right (or left) charge equal to α:

$$\exp[i(c^RK^R + c^LK^L)]q(x)\exp[-i(c^RK^R + c^LK^L)] = \begin{pmatrix} \exp(i\alpha c^R)q^R(x) \\ \exp(i\alpha c^L)q^L(x) \end{pmatrix}. \tag{11.77}$$

Let $\alpha \neq 0$ and $H_{(\alpha)} = \left\{(c^R, c^L) \in \mathbf{R}^2 : \frac{2\pi}{\alpha}c^R,\ \frac{2\pi}{\alpha}c^L \in \mathbf{Z}\right\}$. It is clear that the operators $\exp[i(c^RK^R + c^LK^L)]$ with parameters c^R, c^L in $H_{(\alpha)}$ act trivially on the quark field, therefore the gauge group $\Gamma_{(\alpha)}$ of the field $q(x)$ is actually compact:

$$\Gamma_{(\alpha)} = \mathbf{R}^2/H_{(\alpha)} \approx U(1) \times U(1). \tag{11.78}$$

It is not difficult to check that the field $q(x)$ has the following transformation properties with respect to the group $\mathfrak{P}_+^\uparrow \circ \mathbf{R}_+$:

$$U(a, r\Lambda)q(x)U(a, r\Lambda)^{-1} = r^d \exp(-l\chi\gamma^3)q(r\Lambda x + a), \tag{11.79}$$

where

$$d = l = \alpha^2/4\pi, \tag{11.80}$$

$$\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11.81}$$

In particular, it follows that $q(x)$ has Lorentz spin $\pm l$. Furthermore, it clearly satisfies the Dirac equation

$$\gamma^\mu \partial_\mu q(x) = 0, \tag{11.82}$$

where the γ^μ are the Dirac matrices in two-dimensional Minkowski space-time, taken in the representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{11.83}$$

We call $q(x)$ the *free quark field.**

We note a number of interesting features of this field. Since the components of $q(x)$ are local normal exponentials of the fields $\Phi^R(x^R)$, $\Phi^L(x^L)$, there is a natural definition of the N-products $N(q^{(*)}(x_1)\dots q^{(*)}(x_n))$ for the free quark field; they satisfy the following multiplication formula:

$$N(q(x_1; m_1^R, m_1^L)\dots q(x_p; m_p^R, m_p^L)) \cdot N(q(y_1; n_1^R, n_1^L)\dots q(y_q; n_q^R, n_q^L)) =$$

$$= \prod_{j=1}^{p}\prod_{k=1}^{q}(e^{-i\pi l(m_j^R n_k^L - m_j^L n_k^R)} \cdot (0 + i\kappa(x_j^R - y_k^R))^{2lm_j^R n_k^R} \cdot (0 + i\kappa(x_j^L - y_k^L))^{2lm_j^L n_k^L})\times$$

* As we shall see in the next section, it is a constituent part of the Thirring field.

$$\times N(q(x_1; m_1^R,, n_1^L) \dots q(y_q; n_q^R, n_q^L)); \tag{11.84}$$

here we have introduced the notation $q(x; m^R, m^L)$ for the components of $q(x)$ and its conjugate field, which is defined by the rule

$$\begin{aligned} q(x;1,0) &= q^R(x); \quad q(x;-1,0) = q^R(x)^*; \\ q(x;0,1) &= q^L(x); \quad q(x;0,-1) = q^L(x)^*. \end{aligned} \tag{11.85}$$

In accordance with the commutativity of the N-product of normal exponentials, the N-product is not altered under a permutation of the quark fields.

Exercise 11.13. Prove the relation

$$q(x; m^R, m^L) q(y; n^R, n^L) = e^{-i \cdot 2\pi l \sigma} q(y; n^R, n^L) q(x; m^R, m^L) \quad \text{for } (x-y)^2 < 0, \tag{11.86}$$

where

$$\sigma = \epsilon(x^1 - y^1)(m^R n^R - m^L n^L) + (m^R n^L - m^L n^R) = \pm 1. \tag{11.87}$$

It follows from (11.86) that for (half-)integral spin l, the components of the quark field and the conjugate field (anti-)commute under a spacelike separation of the arguments. For other values of the spin, (11.86) is a generalization of the boson-fermion alternative in the locality condition; (in this situation we talk about "generalized statistics" of the quark fields).

According to Exercise 11.11(b), the N-product $N(q^{(*)}(x_1) \dots q^{(*)}(x_n))$ of the quark fields is an operator-valued function of x_1 which is C^∞-dependent on the differences $x_1 - x_2, \dots, x_{n-1} - x_n \in \mathbf{M}$. Therefore we can equate the arguments in the N-product when obtaining composite fields.

Exercise 11.14. Prove the relations

$$N(q^R(x)^* q^R(x)) = N(q^L(x)^* q^L(x)) = 1, \tag{11.88}$$

$$N(q^R(x)^* \partial_\mu q^R(x)) = i\alpha g_{\mu\mu} V^R(x^R) = i\frac{\alpha}{\sqrt{2}}(V_\mu(x) - V'_\mu(x)), \tag{11.89a}$$

$$N(q^L(x)^* \partial_\mu q^L(x)) = i\alpha V^L(x^L) = i\frac{\alpha}{\sqrt{2}}(V_\mu(x) + V'_\mu(x)). \tag{11.89b}$$

In particular, the N-products of the quark field $q \equiv q_{(\alpha)}$ whose components have the same labels as the quark field $q_{(\beta)}$ (obtained by replacing α by β in (11.75)) form the quark field $q_{(\alpha+\beta)}$:

$$N(q^R_{(\alpha)}(x) q^R_{(\beta)}(x)) = q^R_{(\alpha+\beta)}(x), \quad N(q^L_{(\alpha)}(x) q^L_{(\beta)}(x)) = q^L_{(\alpha+\beta)}(x); \tag{11.90a}$$

furthermore,

$$q^*_{(\alpha)}(x) = q_{(-\alpha)}(x). \tag{11.90b}$$

The above procedure for obtaining a quark (in particular, fermionic) field is called ***bosonization*** of the quark field. The resultant representation of the quark field in the physical space $\mathcal{H}$ of the scalar massless field turns out to be reducible. This follows from the fact that the field $q(x)$ commutes with the gauge transformation operators for $(c^R, c^L) \in \mathcal{H}_{(\alpha)}$ (see (11.76)). It can be shown by the method of Proposition 11.1 that the representation of q in $\mathcal{H}$ decomposes into a direct integral of pairwise inequivalent representations (parametrized by a point ϑ of a two-dimensional torus, therefore the corresponding irreducible representations are called

ϑ-representations). Only one of them has a vacuum vector (in all the other representations, the energy operator is strictly positive). For the construction of this vacuum representation, we introduce the auxiliary *-algebra $\mathfrak{F}_{(\alpha)}$ of bounded operators in $\mathcal{H}$ that are linear combinations of the operators $NE_f^R NE_g^R \exp i(c^R K^R + c^L K^L)$, where $f, g \in \mathcal{S}_r(\mathbf{R})$, $\alpha^{-1}\int f(s)ds \in \mathbf{Z}$, $\alpha^{-1}\int g(t)dt \in \mathbf{Z}$.

Exercise 11.15. (a) Prove that the von Neumann algebra $\mathfrak{F}_{(\alpha)}^{cc}$ generated by $\mathfrak{F}_{(\alpha)}$ consists of all operators in $\mathcal{H}$ that commute with the operators $\exp i(c^R K^R + c^L K^L)$ for $(c^R, c^L) \in H_{(\alpha)}$. [Hint: The algebra $\mathfrak{F}_{(\alpha)}^{cc}$ clearly contains the von Neumann algebra generated by the current $V^\mu(x)$, that is, the von Neumann algebra of observables; it therefore follows from Proposition 11.1 that $\mathfrak{F}_{(\alpha)}^c$ is contained in the algebra of functions $f(K^R, K^L)$ of K^R and K^L. It remains to verify that such an operator $f(K^R, K^L)$ commutes with $\mathfrak{F}_{(\alpha)}$ if and only if $f(\zeta^R, \zeta^L)$ is periodic with period α in ζ^R and in ζ^L.]

(b) Prove that the von Neumann algebra generated by the quark field $q(x)$ is the same as $\mathfrak{F}_{(\alpha)}^{cc}$. [Hint: For the determination of the commutant of the quark field, use the same arguments as in part (a) of this exercise.]

We define the vacuum state $s_{(\alpha)}$ on the algebra $\mathfrak{F}_{(\alpha)}$ by setting

$$s_{(\alpha)}(A) = \frac{\alpha^2}{(2\pi)^2}\left\langle \Psi_0^R \otimes \Psi_0^L, \pi^{(0)}\Big(\int_{\Gamma_{(\alpha)}} \gamma_c(A) dc^R dc^L\Big)\Psi_0^R \otimes \Psi_0^L\right\rangle,\ A \in \mathfrak{F}_{(\alpha)}, \tag{11.91}$$

where $\pi^{(0)}$ is the vacuum representation of the algebra of observables in the space $\mathcal{H}^{(0)R}\otimes\mathcal{H}^{(0)L}$ (see Proposition 11.1) with vacuum vector $\Psi_0^R\otimes\Psi_0^L$. Here if we integrate over the gauge group in (11.90), we clearly obtain an operator in the algebra of observables. The functional $s_{(\alpha)}$ is positive (since integrating a non-negative operator over $\Gamma_{(\alpha)}$ yields a non-negative operator).

The state $s_{(\alpha)}$ defines the vacuum representation $\pi_{(\alpha)}$ of the algebra $\mathfrak{F}_{(\alpha)}$ in some Hilbert space $\mathcal{H}_{(\alpha)}$ with cyclic vector $\Psi_{0(\alpha)}$. As in the case of the representations $\mathcal{H}^{(\zeta^R)}$ for the current $v^R(x^R)$ (§11.1.B), it is not difficult to verify that a unitary representation of the Poincaré group (also $\mathfrak{P}_+^\uparrow \circ \mathbf{R}_+$) acts in $\mathcal{H}_{(\alpha)}$ and that the spectrum condition holds; furthermore, the condition of the uniqueness of the vacuum holds. As usual (see Proposition 8.1), it follows that the representation $\pi_{(\alpha)}$ is irreducible.

We now define the vacuum representation $\pi_{(\alpha)}(q(x))$ of the quark field in terms of the limit of operators in $\mathfrak{F}_{(\alpha)}$, in the same way that $q(x)$ was defined in $\mathcal{H}$; for example:

$$\pi_{(\alpha)}(q^R(x)) = \lim_{\epsilon\to 0}\pi_{(\alpha)}\big(N\exp\big(i\alpha\int \Phi^R(s) * \chi_\epsilon(s)ds + \tfrac{1}{2}K^R\big)\big). \tag{11.92}$$

Exercise 11.16. Prove the relations

$$\langle \Psi_{0(\alpha)}, \pi_{(\alpha)}\Big(N\prod_{j=1}^n q(x_j; m_j^R, m_j^L)\Big)\Psi_{0(\alpha)}\rangle = \delta_{\Sigma m_j^R,0}\delta_{\Sigma m_j^L,0}, \tag{11.93a}$$

$$\begin{aligned}\langle \Psi_{0(\alpha)}, \pi_{(\alpha)}(q(x_1; m_1^R, m_1^L))\dots\pi_{(\alpha)}(q(x_n; m_n^R, m_n^L))\Psi_{0(\alpha)}\rangle = \\ = \delta_{\Sigma m_j^R,0}\delta_{\Sigma m_j^L,0}\prod_{j<k}\{e^{-i\pi l}(m_j^R m_k^L - m_j^L m_k^R)\times \\ \times(0 + i\kappa(x_j^R - x_k^R))^{2lm_j^R m_k^R}\cdot(0 + i\kappa(x_j^L - x_k^L))^{2lm_j^L m_k^L}.\end{aligned} \tag{11.93b}$$

[Hint: (11.93a) follows from (11.91); (11.93b) follows from (11.93a) and (11.84).]

The case $l \equiv \alpha^2/4\pi = \frac{1}{2}$ reduces to a free massless Dirac field $\psi(x)$ in two-dimensional space-time. Taking α to be equal to $\sqrt{2\pi}$, we set

$$\psi(x) = \sqrt{\frac{\kappa}{2\pi}}q(x) = \sqrt{\frac{\kappa}{2\pi}}\begin{pmatrix} N\exp(i\sqrt{2\pi}(\Phi^R(x^R) + \frac{1}{2}K^L)) \\ N\exp(i\sqrt{2\pi}(\Phi^L(x^L) - \frac{1}{2}K^R)) \end{pmatrix} \tag{11.94}$$

Exercise 11.17. Prove that the field $\psi(x)$ satisfies the following relations

$$\begin{gathered}[\psi^R(x)^*, \psi^R(y)]_+ = \delta(x^R - y^R), \quad [\psi^L(x)^*, \psi^L(y)]_+ = \delta(x^L - y^L), \\ [\psi^R(x)^*, \psi^L(y)]_+ = 0, \quad [\psi(x), \psi(y)]_+ = 0.\end{gathered} \tag{11.95}$$

Like the more general quark field $q(x)$, the Dirac field $\psi(x)$ is reducible in the physical Hilbert space $\mathcal{H}$ of the scalar massless field. Its vacuum representation $\pi_{(\alpha)}$ is defined by a formula of type (11.93) with $\alpha = \sqrt{2\pi}$, $l = \frac{1}{2}$. On the other hand, the free Dirac field has an ordinary Fock representation (constructed in the same way as in four-dimensional space-time; see §8.4.C). A direct comparison shows that all the Wightman functions of the field $\psi(x)$ are the same for these two representations, therefore the representation $\pi_{(\alpha)}$ (where $\alpha^2 = 2\pi$) is the Fock representation of the free massless Dirac field, but obtained by the method of bosonization.*

The other ϑ-representations contained in the decomposition of the field ψ in $\mathcal{H}$ into a direct integral of irreducible representations, are not equivalent to the Fock representation (the energy operator in them is strictly positive, so that they are vacuumless). The corresponding states can be interpreted as "quanta" of the fields ψ, ψ^* in the background of the soliton-like excitation (of arbitrarily weak energy) of the vacuum (the "condensate") whose right and left charges K^R, K^L are non-integer multiples of $\sqrt{2\pi}$. The ϑ-representation is parametrized by the pair $(\exp(i\vartheta^R), \exp(i\vartheta^L))$ of complex numbers of unit modulus. It can be obtained from the Fock representation via the linear canonical transformation

$$\psi(x) \to \psi'(x) = \begin{pmatrix} \exp(i\sqrt{2\pi}\alpha^R(x^R)) \cdot \psi^R(x^R) \\ \exp(i\sqrt{2\pi}\alpha^L(x^L)) \cdot \psi^L(x^L) \end{pmatrix} \tag{11.96}$$

where $\alpha^{R,L}(s)$ are arbitrary real C^∞-functions whose derivatives belong to $\mathcal{S}_r(\mathbf{R})$ and

$$\exp[i\sqrt{2\pi}(\alpha^{R,L}(+\infty) - \alpha^{R,L}(-\infty))] = \exp(i\vartheta^{R,L}). \tag{11.97}$$

In particular, a representation obtained from the Fock representation by a transformation of the form (11.96) is unitarily equivalent to the Fock representation (in this case, the canonical transformation is called proper) if and only if

$$\exp(i\sqrt{2\pi}\zeta^{R,L}) = 1, \quad \text{where } \zeta^{R,L} = \alpha^{R,L}(+\infty) - \alpha^{R,L}(-\infty). \tag{11.98}$$

This condition has a simple topological sense. It means that each map $s \to \exp(i\sqrt{2\pi} \cdot \alpha^{R,L}(s))$ can be uniquely extended (by continuity) to a smooth map of the manifold $\mathbf{R}_\infty \equiv \mathbf{R} \cup \{\infty\}$ (which is diffeomorphic to a circle) to the unit circle $U(1)$. The integers $\zeta^{R,L}/\sqrt{2\pi}$ are the number of times the circle $U(1)$ is covered (taking orientation into account) by $\mathbf{R}_\infty$ under these maps. These numbers distinguish the connected components of the group of proper canonical transformations of the form (11.96) (they define the charges carried by the unitary operator realizing the proper canonical transformation (11.96) in the Fock space).

Exercise 11.18. (a) Verify that the two-point Wightman functions of the free massless Dirac field $\psi(x)$ (taken in the vacuum representation) have the form

$$\langle\psi^{R*}(x)\Psi^R(y)\rangle_0 = \frac{1}{2\pi}[0 + i(x^R - y^R)]^{-1}, \tag{11.99a}$$

$$\langle\psi^{L*}(x)\Psi^L(y)\rangle_0 = \frac{1}{2\pi}[0 + i(x^L - y^L)]^{-1}, \tag{11.99b}$$

$$\langle\psi^{R*}(x)\psi^L(y)\rangle_0 = \langle\Psi^{L*}(x)\psi^R(y)\rangle_0 = \langle\psi^{R,L}(x)\psi^{R,L}(y)\rangle_0 = 0. \tag{11.99c}$$

* Note that a normal product of free Dirac fields (see §8.4.C) is not the same as an N-product (the essential difference being that the fields under the N-product commute).

(b) The current of the free Dirac field is defined by the formula (cf. (8.124))

$$j^\mu(x) = :\tilde{\psi}(x)\gamma^\mu\psi(x): \equiv \lim_{y\to x}\{\tilde{\psi}(x)\gamma^\mu\psi(y) - \langle\tilde{\psi}(x)\gamma^\mu\psi(y)\rangle_0\},$$

where $\tilde{\psi}(x) \equiv \psi^*(x)\gamma^0$. Verify the relation

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}V^\mu(x). \tag{11.100}$$

(c) Deduce the following expression for the current of a free massless Dirac field in terms of the N-products of the fields ψ^* and ψ at the same instant:

$$j^\mu(x) = N(\psi^*(x)(i\kappa^{-1}\gamma^1\partial_1)\gamma^\mu\psi(x)). \tag{11.101}$$

[Hint: Use (11.89) with $\alpha = \sqrt{2\pi}$ and (11.100).]

In the next part of the exercise we consider an example of a real local exponential of the fields $\Phi^{R,L}$.

(d) Prove that the field

$$\chi(x) = \sqrt{\frac{\kappa}{2\pi}}\begin{pmatrix} N\exp[\sqrt{2\pi}(\Phi^R(x^R) + \frac{1}{2}K^L)] \\ N\exp[\sqrt{2\pi}(\Phi^L(x^L) - \frac{1}{2}K^R)] \end{pmatrix} \tag{11.102}$$

transforms under the action of $\mathfrak{P}_+^\uparrow \circ \mathbf{R}_+$ according to (11.79) with $d = l = -1/2$ and satisfies the anticommutation relations

$$\begin{gathered}[\chi^R(x^R), \chi^R(y^R)]_+ = \delta(x^R - y^R)N(\chi^R(x^R)^2), \quad [\chi^R(x^R), \chi^L(y^L)]_+ = 0, \\ [\chi^L(x^L), \chi^L(y^L)]_+ = \delta(x^L - y^L)N(\chi^L(x^L)^2).\end{gathered} \tag{11.103}$$

D. FREE SCALAR MASSLESS "GHOST" FIELD

We consider a scalar field $b(x)$ subject to the wave equation (11.1), but instead of the CCR's (11.2), the following CCR's are satisfied:

$$[b(x), b(y)] = iD_0(x - y); \tag{11.104}$$

we call this a "ghost" field. It is not of interest in its own right, but can serve as a constituent part of more interesting models (such as the Schwinger model). We suppose that the field $b(x)$, like the field $\phi(x)$, has the form

$$b(x) = \frac{1}{\sqrt{2}}(b^R(x^R) + b^L(x^L)), \tag{11.105}$$

where the right and left components $b^{R,L}$ satisfy the CCR's

$$\begin{gathered}[b^R(x^R), b^R(y^R)] = iD^R(x^R - y^R), \quad [b^R(x^R), b^L(y^L)] = 0, \\ [b^L(x^L), b^L(y^L)] = iD^L(x^L - y^L).\end{gathered} \tag{11.106}$$

It is not difficult to construct the Fock representation for the field $b^R(x^R)$ (and similarly for $b^L(x^L)$), by postulating the form of the two-point function

$$\langle 0|b^R(s)b^R(t)|0\rangle = \omega^R(s - t) \equiv -W^R(s - t). \tag{11.107}$$

For this, we consider the auxiliary field $\phi^R(s)$ in §11.1.A, acting in the pseudo-Hilbert space $\mathfrak{H}^R$. The scalar product (11.15) is now the auxiliary product and is denoted by $\langle F, G\rangle_1$. We introduce the new scalar product in $\mathfrak{H}^R$:

$$\langle F, G\rangle = \langle F, (-1)^N G\rangle_1, \tag{11.108}$$

where N is the operator of the number of particles, and we define the field $b^R(s)$ by the formula $b^R(s) = i\phi^R(s)$. It is clear that this field is pseudo-Hermitian (with respect to the new scalar product) and satisfies the first of the commutation relations (11.106). The fields $b^L(s)$ in $\mathfrak{H}^L$ and $b(x)$ in $\mathfrak{H} = \mathfrak{H}^R \otimes \mathfrak{H}^L$ are defined in exactly the same way.

In the pseudo-Hilbert space $\mathfrak{H}^R$ (or $\mathfrak{H}$) the subspace of translation-invariant vectors is infinite-dimensional (since the gauge transformation operators of the fields $b^{R,L}(s)$ as well as those of the field $\phi^R(s)$ in Exercise 11.2 do not leave the vacuum vector invariant). We saw in §11.1.B that the field $\phi^R(s)$ has the alternative realization $\pi(\phi^R(s)) \equiv \Phi^R(s)$ in $\mathcal{H}^R$ with a one-dimensional vacuum generalized subspace. We construct the analogue of this realization for the "ghost" field $b^R(s)$; we denote $b^R(s)$ in the new realization by $B^R(s)$. The Hilbert scalar product (11.29) in $\mathcal{H}^R$ is now the auxiliary one, denoted by $(\cdot,\cdot)$.

Exercise 11.19. Let $\mathfrak{M}^R$ be the set of all possible finite sums of operators in $\mathfrak{H}^R$ of the form (11.25); we define on $\mathfrak{M}^R$ the linear map $A \to \hat{A} \equiv (-1)^N A(-1)^N$, which associates with an operator A of the form (11.25) the operator

$$\hat{A} = \int_{\mathcal{N}} \exp\Bigl(-i\int \phi^R(s)f(s)ds\Bigr)F(f)df. \tag{11.109}$$

Prove that the formula

$$\eta X(A) = X(\hat{A}) \quad \text{for } A \in \mathfrak{M}^R \tag{11.110a}$$

or, equivalently,

$$(X(A), \eta X(B)) = s^R(A^*\hat{B}) \quad \text{for } A, B \in \mathfrak{M}^R \tag{11.110b}$$

defines a Hermitian operator η on $\mathcal{H}^R$ such that $\eta^2 = 1$.

We define a new scalar product in $\mathcal{H}^R$ by setting $\langle\Psi_1, \Psi_2\rangle = (\Psi_1, \eta\Psi_2)$; we denote this pseudo-Hilbert space by $\mathfrak{H}^R_B$. It is clear that we have the following representation π of the field $b^R(s)$ in $\mathfrak{H}^R_B$:

$$B^R(s) \equiv \pi(b^R(s)) = i\pi(\phi^R(s)) \equiv i\Phi^R(s).$$

The pseudo-Hilbert space $\mathfrak{H}^L_B$ and the field $B^L(s) = i\Phi^L(s)$ are defined in precisely the same way. The scalar field

$$B(x) = \frac{1}{\sqrt{2}}(B^R(x^R) + B^L(x^L)) \tag{11.111}$$

and the pseudo-scalar field

$$B'(x) = \frac{1}{\sqrt{2}}(-B^R(x^R) + B^L(x^L)) \tag{11.112}$$

act in the pseudo-Hilbert space $\mathfrak{H}_B = \mathfrak{H}^R_B \otimes \mathfrak{H}^L_B$.

The charges of the fields $B^{R,L}(s)$ are defined by the equality $K^{R,L}_B = iK^{R,L}$.

Exercise 11.20. Prove that the operators $K_B^{R,L}$ are generators of the gauge transformations of the fields $B^{R,L}(s)$:

$$\exp[i(K_B^R c^R + K_B^L c^L)]B^{R,L}(s)\exp[-i(K_B^R c^R + K_B^L c^L)] = B^{R,L}(s) - c^{R,L}. \tag{11.113}$$

[Hint: cf. (11.37).]

The normal exponentials of the fields $B^{R,L}(s)$ and the charges $K_B^{R,L}$ (and their N-products) are defined in the same way as for the case of the fields $\Phi^{R,L}(s)$; for example:

$$N\exp(i\textstyle\int B^R(s)f(s)ds) = \\ = \exp(\frac{1}{2}\displaystyle\int \omega^R(s-t)f(s)f(t)ds\,dt)\exp(i\int B^R(s)f(s)ds) \tag{11.114}$$

(cf. (11.63)).

Exercise 11.21. Prove that the field

$$\chi(x) = \sqrt{\frac{\kappa}{2\pi}}\begin{pmatrix} N\exp[i\sqrt{2\pi}(B^R(x^R) + \frac{1}{2}K_B^L)] \\ N\exp[i\sqrt{2\pi}(B^L(x^L) - \frac{1}{2}K_B^R)] \end{pmatrix} \tag{11.115}$$

has the transformation property (11.79) with $d = l = -1/2$ and satisfies the anticommutation relations

$$\begin{gathered}[\chi^R(x^R), \chi^R(y^R)]_+ = \delta(x^R - y^R)N(\chi^R(x^R)^2), \quad [\chi^R(x^R)^*, \chi^R(y^R)]_+ = 0, \\ [\chi^R(x^R)^*, \chi^R(y^R)^*]_+ = \delta(x^R - y^R)N(\chi^R(x^R)^{*2})\end{gathered} \tag{11.116}$$

(and similarly the relations involving $\chi^{L(*)}$).

11.2. The Thirring Model

A. Solution of the Field Equation

The "classical" version of the Thirring model is characterized by the Lagrangian

$$\mathcal{L} = i\tilde{\psi}\gamma^\mu\partial_\mu\psi - \frac{g}{2}j_\mu j^\mu; \tag{11.117}$$

here g is a real parameter ("coupling constant"), $\psi(x) \equiv (\psi_1(x), \psi_2(x))$ is a field in two-dimensional space-time $\mathbf{M}$ transforming according to a representation of the Lorentz group with spin $\pm 1/2$, $\tilde{\psi}(x) \equiv \psi^*(x)\gamma^0$ is the Dirac conjugate field and j^μ is the current:

$$j^\mu = \tilde{\psi}\gamma^\mu\psi. \tag{11.118}$$

From this we obtain the field equation

$$i\gamma^\mu\partial_\mu\psi - g\gamma^\mu j_\mu\psi = 0. \tag{11.119}$$

It follows from invariance of the Lagrangian with respect to the compact two-parameter group $\Gamma = U(1) \times U(1)$ of gauge transformations

$$\psi(x) \to e^{i(-c+c'\gamma^3)}\psi(x) \text{ (where } e^{ic}, e^{ic'} \in U(1)) \tag{11.120}$$

that the ("genuine" vector and pseudo-vector) currents j^μ and j'^μ are conserved, where

$$j'^\mu = \tilde{\psi}\gamma^3\gamma^\mu\psi = \epsilon^{\mu\nu}j_\nu. \tag{11.121}$$

Furthermore, we have the following simultaneous Poisson brackets between j^μ and ψ:

$$\{j^\mu(x), \psi(y)\}|_{x^0=y^0} = i\delta(x^1 - y^1)\gamma^0\gamma^\mu\psi(x). \tag{11.122}$$

The conservation of the currents j^μ, j'^μ means that j^μ is the divergence of some field:

$$j^\mu = -\frac{1}{\sqrt{\pi}}\partial^\mu\phi,$$

where

$$\Box\phi = 0.$$

As a result it is not difficult to write the general solution of the "classical" equation (11.119):

$$\psi(x) = a\exp\big(i\frac{g}{\sqrt{\pi}}\phi(x)\big)q(x), \tag{11.123}$$

where a is a fixed multiple (defining the dimension of ψ) and $q(x)$ is a solution of the free Dirac equation (the dimension of $q(x)$ being zero in units of mass).

Exercise 11.22. (a) Derive the Poisson bracket for all values of x, y:

$$\{j^\mu(x), \psi(y)\} = i(g^{\mu\nu} - \epsilon^{\mu\nu}\gamma^3)\partial_\nu D_0(x-y)\psi(y). \tag{11.124}$$

(b) Let

$$Q = \int_{x^0=\text{const}} j^0(x)dx^1, \quad Q' = \int_{x^0=\text{const}} j'^0(x)dx^1;$$

prove that

$$\{Q, \psi(x)\} = i\psi(x), \quad \{Q', \psi(x)\} = -i\gamma^3\psi(x). \tag{11.125}$$

It turns out the above discussion can in large measure be taken over to the Thirring quantum model by postulating the existence of an N-product of the fields ψ, $\tilde{\psi}$ with a property similar to those of the N-products of free quark fields (§11.1.C). We therefore assume* that all the fields occurring in the N-product commute and that their arguments can be equated. We then postulate for the description of the field equation that there exist conserved vector currents $j^\mu(x)$ and $j'^\mu(x) = \epsilon^{\mu\nu}j_\nu(x)$ which are bilinear functionals of the fields ψ, $\tilde{\psi}$. However, the explicit expression (11.118) is clearly no longer suitable because of the inevitable "divergences". Instead we now characterize the current $j^\mu(x)$ by commutation relations with the field ψ analogous to (11.124) (but we still cannot write out these relations because the constants in them are subject to "quantum corrections"). Finally, we postulate that the Thirring field satisfies the field equation

$$i\gamma^\mu\partial_\mu\psi(x) - g\gamma^\mu N(j_\mu(x)\psi(x)) = 0. \tag{11.126}$$

Since the fields occurring in the N-product commute, it is natural to seek a solution of the field equation in a form analogous to the classical solution (11.123):

$$\psi(x) = N\big(a\exp\big(i\frac{g}{\sqrt{\pi}}\Phi(x)\big)q(x)\big); \tag{11.127}$$

* The compatibility of all these assumptions will be clear from the sequel.

here a is a normalizing factor, * $\Phi(x)$ is a free massless scalar field and $q(x)$ is a free quark field. In the method of bosonization (§11.1.C) the components of the quark field $q(x) \equiv q_{(\delta)}(x)$ are normal local exponentials of the fields $\Phi^R(x^R)$, $\Phi^L(x^L)$ and the charges K^R, K^L. The field $\Phi(x)$ is chosen in the form (11.116) from covariance considerations (as well as from the condition that it behaves like a "genuine" scalar under reflection). Although the factor $1/\sqrt{\pi}$ in the exponent in (11.127) is to a certain extent inessential and only shows up in the definition of the "coupling constant" g, we include it in (11.127) since here the current

$$j_\mu(x) = -\frac{1}{\sqrt{\pi}}\partial_\mu\Phi(x) \equiv -\frac{1}{\sqrt{\pi}}V_\mu(x) \tag{11.128}$$

is related to the field Φ by the same formulae as the current of a free Dirac field (see (11.100)). As a result we obtain the following expression for the Thirring field, under a suitable choice of the factor a in (11.127):

$$\psi(x) \equiv \begin{pmatrix}\psi_1(x)\\ \psi_2(x)\end{pmatrix} = \sqrt{\frac{\kappa}{2\pi}}\begin{pmatrix} N\ \exp[i(\alpha\Phi^R(x^R) + \beta\Phi^L(x^L) + \frac{1}{2}\beta K^R - \frac{1}{2}\alpha K^L)] \\ N\ \exp[i(\beta\Phi^R(x^R) + \alpha\Phi^L(x^L) + \frac{1}{2}\alpha K^R - \frac{1}{2}\beta K^L)]\end{pmatrix}; \tag{11.129}$$

here $\beta = g/\sqrt{2\pi}$ and α is a parameter to be determined presently. Thus the Thirring field (to within a factor $\sqrt{\kappa/2\pi}$) consists of N-products of differently labelled components of the free quark fields $q_{(\alpha)}(x)$ and $q_{(\beta)}(x)$:

$$\psi_1(x) = \sqrt{\frac{\kappa}{2\pi}}N(q^R_{(\alpha)}(x)q^L_{(\beta)}(x)), \quad \psi_2(x) = \sqrt{\frac{\kappa}{2\pi}}N(q^L_{(\alpha)}(x)q^R_{(\beta)}(x)). \tag{11.130}$$

Since the Thirring field is expressed in terms of the free fields $\Phi^R(x^R)$, $\Phi^L(x^L)$ and the charges K^R, K^L, it is clear that the N-products with the required properties are defined for it (the N-products can be expressed in terms of ordinary products of formulae of type (11.84)).

Exercise 11.23. Prove that the field $\psi(x)$ transforms with respect to the group $\mathcal{P}^\uparrow_+ \circ \mathbf{R}_+$ according to formula (11.79), where

$$d = (\alpha^2+\beta^2)/4\pi, \quad l = (\alpha^2-\beta^2)/4\pi. \tag{11.131}$$

It follows from Exercise 11.23 that the components $\psi_1(x)$, $\psi_2(x)$ of the field (11.129) have Lorentz spin $+1/2$ and $-1/2$ respectively if the parameters α and β are subject to the condition

$$\alpha^2 - \beta^2 = 2\pi, \tag{11.132}$$

from which we obtain the dependence of α on the "coupling constant" g:**

$$\alpha = [(g^2+4\pi^2)/2\pi]^{1/2}, \quad \beta = g/\sqrt{2\pi}. \tag{11.133}$$

* More precisely, there is one normalizing factor a for each component of the field ψ, since below we shall include in it components that are independent of x and are functions of the charges K^R, K^L.

** The positive value of α is chosen from the condition that for $g = 0$ the commutator (11.144) corresponds to the classical Poisson brackets (11.124). The case of negative α does not give any essentially new possibilities (since a solution with parameters $-\alpha$, $-\beta$ is obtained from a solution with parameters α, β by transformations of the internal symmetries, namely, the gauge transformation $\psi \longrightarrow -\gamma^3\psi$ and the charge conjugation $\psi \longrightarrow -\gamma^3\psi^*$).

Exercise 11.24. Prove that the components of the fields $\psi(x)$, $\tilde{\psi}(x)$ anticommute at spacelike separation of the arguments:

$$[\psi(x), \psi(y)]_+ = 0 = [\psi(x), \tilde{\psi}(y)]_+ \quad \text{for } (x-y)^2 < 0. \tag{11.134}$$

It is easy to see that under gauge transformations of a free scalar massless field, ψ behaves in the following way:

$$\exp[i(c^R K^R + c^L K^L)]\psi(x)\exp[-i(c^R K^R + c^L K^L)] =$$
$$= \exp[i(-c + c'\gamma^3)]\psi(x), \tag{11.135a}$$

where

$$c = -\tfrac{1}{2}(\alpha+\beta)(c^R + c^L), \quad c' = \tfrac{1}{2}(\alpha-\beta)(c^R - c^L). \tag{11.135b}$$

Consequently the gauge group Γ of the Thirring field can be identified with the quotient group $\mathbf{R}^2/H$:

$$\Gamma \approx \mathbf{R}^2/H, \tag{11.136a}$$

where

$$H = \{(c^R, c^L) : \frac{1}{4\pi}(\alpha + \epsilon\beta)(c^R + \epsilon c^L) \in \mathbf{Z},\ \epsilon = \pm 1\}. \tag{11.136b}$$

Remark. We could equally well have stipulated that the field $\psi(x)$ transform according to a representation of the Lorentz group with spin $\pm l$, where l is an arbitrary non-zero real number; then instead of (11.132), the connection between the parameters α and β $(= g/\sqrt{2\pi})$ would be expressed by the relation $\alpha^2 - \beta^2 = 4\pi l$. The resulting field can be interpreted as an interacting "quark" field (in contrast to the free "quark" field in §11.1.C) for which the locality condition corresponds to "generalized statistics".

It is clear that the transformation

$$\psi(x) \to \psi'(x) = \gamma^0\psi(x) \tag{11.137}$$

converts the Thirring field into a spinor field $\psi'(x)$ corresponding to $l = -1/2$ (that is, with components transforming according to the Lorentz group with spins $-1/2, 1/2$). The field $\psi'(x)$ is obtained by substituting

$$\alpha = \beta', \quad \beta = \alpha' \tag{11.138}$$

in (11.129), so that

$$\alpha'^2 - \beta'^2 = -2\pi. \tag{11.139}$$

Consequently $\psi'(x)$ satisfies the equation

$$i\gamma^\mu\partial_\mu\psi'(x) - g'\gamma^\mu N(j_\mu(x)\psi'(x)) = 0, \tag{11.140}$$

where $g' = \sqrt{2\pi}\beta'$, that is,

$$g' = \sqrt{g^2 + 4\pi^2}. \tag{11.141}$$

We conclude that the Thirring field satisfies not only equation (11.126), but also the equation

$$i\gamma'^\mu\partial_\mu\psi(x) - g'\gamma'^\mu N(j_\mu(x)\psi(x)) = 0, \tag{11.142a}$$

where

$$\gamma'^0 = \gamma^0, \quad \gamma'^1 = -\gamma^1. \tag{11.142b}$$

In particular, the free Dirac field (11.94) corresponding to $g = 0$ also satisfies the equation of an interacting field with $g' = 2\pi$:

$$i\gamma'^\mu\partial_\mu\psi(x) - 2\pi\gamma'^\mu N(j_\mu(x)\psi(x)) = 0. \tag{11.143}$$

It must be borne in mind however, that the covariant properties of this equation are different from those of the Dirac or Thirring equation (11.126) (the operators $i\gamma'^{\mu}\partial_{\mu} - g'\gamma'^{\mu}N(j_{\mu}(x)\ldots)$ on the left hand sides of (11.142a) or (11.143) take spinors with Lorentz spins $+1/2$, $-1/2$ into spinors with Lorentz spins $-1/2$, $+1/2$).

B. CURRENTS AND CHARGES; VACUUM REPRESENTATION

The main characteristic property of the current $j^{\mu}(x)$ in the field equation (11.126) is expressed by the commutation relations with the Thirring field, which are obtained without difficulty from (11.128), (11.129):

$$[j^{\mu}(x), \psi(y)] = (-zg^{\mu\nu} + z^{-1}\epsilon^{\mu\nu}\gamma^{3})\partial_{\nu}D_{0}(x-y)\psi(y), \tag{11.144}$$

where

$$z = (\alpha+\beta)/\sqrt{2\pi}. \tag{11.145}$$

It also follows trivially from (11.128) that

$$[j_{\lambda}(x), j_{\mu}(y)] = \frac{i}{\pi}\partial_{\lambda}\partial_{\mu}D_{0}(x-y). \tag{11.146}$$

We now turn to the expression of the current in terms of the Thirring field.

Exercise 11.25. Obtain the relations

$$N(\psi_1^*(x)\partial_{\mu}\psi_1(x)) = \frac{\kappa}{2i}(zj_{\mu}(x) - z^{-1}j'_{\mu}(x)), \tag{11.147a}$$

$$N(\psi_2^*(x)\partial_{\mu}\psi_2(x)) = \frac{\kappa}{2i}(zj_{\mu}(x) + z^{-1}j'_{\mu}(x)), \tag{11.147b}$$

where

$$j'^{\mu}(x) = \epsilon^{\mu\nu}j_{\nu}(x). \tag{11.148}$$

We obtain from (11.147) the "renormalized" expression for the current j^{μ} in terms of the fields ψ, ψ^* at the same instant of time:

$$j^{\mu}(x) = z^{g_{\mu\mu}}N(\psi^*(x)(i\kappa^{-1}\gamma^{1}\partial_{1})\gamma^{\mu}\psi(x)). \tag{11.149}$$

We introduce the charges

$$Q = z^{-1}\int_{x^0=\text{const}} j^{0}(x)dx^{1}, \quad Q' = z\int_{x^0=\text{const}} j'^{0}(x)dx^{1} \tag{11.150}$$

(where the integrals are to be understood in the same sense as in (11.61)). Using (11.128) we obtain

$$Q = -(1/z\sqrt{\pi})K, \quad Q' = -(z/\sqrt{\pi})K'. \tag{11.151}$$

Exercise 11.26. Derive the commutation relations

$$[Q, \psi(x)] = -\psi(x), \quad [Q', \psi(x)] = \gamma^{3}\psi(x). \tag{11.152}$$

Like the quark model (§11.1.C), the Thirring field obtained by the method of bosonization is reducible in the space of the physical representation of a free scalar massless field; it decomposes into a direct integral of pairwise inequivalent irreducible representations ("ϑ-representations"). The Thirring field $\pi_{\text{vac}}(\psi(x))$ in the vacuum representation π_{vac} is constructed by the same method as the vacuum representation $\pi_{(\alpha)}$ of the free quark field $q_{(\alpha)}(x)$, that is, by inducing the vacuum state of the Thirring field from the vacuum state of the algebra of observables (generated by the current) by averaging over the gauge group $\Gamma = \mathbf{R}^2/H$ (cf. 11.91)). As a result we obtain the following characterization of the vacuum state of the Thirring field:

$$\Big\langle N\Big(\prod_{j=1}^{k}\psi(x_j;m_j,n_j)\Big)\Big\rangle_0 = \Big(\frac{\kappa}{2\pi}\Big)^{k/2}\delta_{\Sigma m_j,0}\delta_{\Sigma n_j,0}, \tag{11.153}$$

where $\psi(x;m,n)$ denotes the components of the fields ψ, ψ^*:

$$\begin{aligned} &\psi(x;1,0)=\psi_1(x), \quad \psi(x;-1,0)=\psi_1^*(x),\\ &\psi(x;0,1)=\psi_2(x), \quad \psi(x;0,-1)=\psi_2^*(x). \end{aligned} \tag{11.154}$$

Exercise 11.27. Derive the formula for the Wightman functions of the Thirring field in the vacuum representation:

$$\begin{aligned} \langle\psi(x_1;m_1,n_1)\dots\psi(x_k;m_k,n_k)\rangle_0 &= \Big(\tfrac{\kappa}{2\pi}\Big)^{k/2}\delta_{\sum m_j,0}\delta_{\sum n_j,0}\times\\ &\times\prod_{j<j'} e^{i\pi(\zeta_j^R\zeta_{j'}^L-\zeta_j^L\zeta_{j'}^R)/2}\cdot(0+i\kappa(x_j^R-x_{j'}^R))^{\zeta_j^R\zeta_{j'}^R}\cdot(0+i\kappa(x_j^L-x_{j'}^L))^{\zeta_j^L\zeta_{j'}^L}, \end{aligned} \tag{11.155}$$

where

$$\zeta_j^R=\frac{1}{\sqrt{2\pi}}(\alpha m_j+\beta n_j), \quad \zeta_j^L=\frac{1}{\sqrt{2\pi}}(\beta m_j+\alpha n_j). \tag{11.156}$$

Exercise 11.28. Verify that the Thirring field in the vacuum representation satisfies all the Wightman axioms (with the dimensionality correction, of course).*

11.3. The Schwinger Model

A. Solution in the Lorentz Gauge

An example of an explicitly soluble model with gradient invariance is two-dimensional massless quantum electrodynamics, called the Schwinger model. In the "classical" version of (two-dimensional) electrodynamics, the specific character of masslessness** shows up in the additional "chiral" symmetry. Alongside the ordinary gauge invariance of the first kind with respect to the transformations $\psi \to e^{ic'}\psi$, guaranteeing the conservation of the vector current $j^\mu(x)=\tilde{\psi}(x)\gamma^\mu\psi(x)$, here there is a symmetry with respect to the "chiral" transformations $\psi \to e^{ic'\gamma^3}\psi$, from which it follows that the pseudo-vector current

$$j'^\mu = \tilde{\psi}\gamma^3\gamma^\mu\psi = \epsilon^{\mu\nu}j_\nu \tag{11.157}$$

is conserved. This circumstance has a decisive role for the explicit solubility of the field equations.

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* There is no vacuum state in the other ϑ-representations of the Thirring field.

** Masslessness of a model means that the Dirac equation (11.160a) for the field $\psi(x)$ is written in the field $A_\mu(x)$ with zero mass; as we shall see, in this model the photon acquires a mass in "dynamical" fashion.

To some extent, the Schwinger model is similar to the Thirring model. Here too, it is possible to define the N-product of fields non-trivially, within which the fields commute (and the arguments can be equated). At the same time, the gradiant invariance brings about an important difference; namely, the condition of gauge invariance of the second kind for the current $j^\mu(x)$ forces us to abandon the previous relation $j'^\mu = \epsilon^{\mu\nu} j_\nu$ between the conserved currents which we had in the "classical" version and the Thirring model. The fact is that the "naïve" expressions for the currents $j^\mu = \tilde{\psi}\gamma^\mu\psi$ and $j'^\mu = \tilde{\psi}\gamma^3\gamma^\mu\psi$, from which the relation $j'^\mu = \epsilon^{\mu\nu} j_\nu$ follows, are clearly ill defined and must be replaced by "renormalized" currents. In the Thirring model and, in particular, in the model of a free massless Dirac field, the "renormalized" current $j^\mu(x)$ is expressed in terms of the N-product of $\psi^*(x)$ and $\partial_\mu\psi(x)$ (see (11.147), (11.149), (11.101)). Such expressions clearly do not possess gauge invariance of the second kind and must be "corrected " by replacing the ordinary derivatives by the covariant derivatives $\partial_\mu + ieA_\mu$. This "correction" of the definition of the vector current taking gradient invariance into account is the source of the anomaly of the pseudo-vector current in the Schwinger model.

We now turn to the construction of the Schwinger model from the general principles of quantum electrodynamics (§10.2.B) supplemented by the hypothesis of the existence of N-products.* We consider the model in the Lorentz gauge ($\xi = 0$):

$$\partial_\mu A^\mu = 0, \tag{11.158}$$

when we can set

$$A^\mu(x) = \frac{\sqrt{\pi}}{e}\epsilon^{\mu\nu}\partial_\nu\rho(x), \tag{11.159}$$

where $\rho(x)$ is a pseudo-scalar neutral field (the dimensional factor ** $\sqrt{\pi}/e$ is taken for the sake of convenience). We write the field equations in the form

$$i\gamma^\mu\partial_\mu\psi(x) - e\gamma^\mu N(A_\mu(x)\psi(x)) = 0, \tag{11.160a}$$

$$\partial^\lambda F_{\lambda\mu}(x) = e(j_\mu(x) + j_\mu^{(\text{fict})}(x)), \tag{11.160b}$$

where

$$j_\mu^{(\text{fict})}(x) = -\partial_\mu\Lambda(x), \tag{11.160c}$$

$$\Box\Lambda(x) = 0. \tag{11.160d}$$

Since the expression

$$\lambda(\alpha) = \int_{x^0=\text{const}} (\alpha(x)\partial_0\Lambda(x) - \Lambda(x)\partial_0\alpha(x))dx^1$$

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* We note, however, (running ahead) that the "general principles" in essence fix the model to within two arbitrary parameters α and Z (together with the electromagnetic interaction constant e); to remove the arbitrariness of these parameters, we invoke extra considerations (namely, "asymptotic freedom" and the simultaneous commutation relations (11.179) for $A_\mu(x)$).

** In two-dimensional electrodynamics, the electromagnetic interaction constant e has dimensions of mass.

provides the generators of the local gauge transformations (where $\alpha(x)$ is an arbitrary solution of the d'Alembert equation of class $\mathcal{S}_r(\mathbf{R})$ with respect to x^1), the commutators $\Lambda(x)$ with the other fields have the form

$$[\Lambda(x), \psi(y)] = D_0(x-y)\psi(y), \tag{11.161a}$$
$$[\Lambda(x), A_\mu(y)] = ie^{-1}\partial_\mu D_0(x-y), \tag{11.161b}$$
$$[\Lambda(x), \Lambda(y)] = 0 \tag{11.161c}$$

(cf. 10.95)). As we did in the model of the free massless scalar field (§11.1.A), we suppose that the field $\Lambda(x)$ is formed from right and left components $\Lambda^R(x^R)$, $\Lambda^L(x^L)$, and we obtain

$$\Lambda(x) = \frac{1}{\sqrt{2}}(\Lambda^R(x^R) + \Lambda^L(x^L)), \quad \Lambda'(x) = \frac{1}{\sqrt{2}}(-\Lambda^R(x^R) + \Lambda^L(x^L)). \tag{11.162}$$

In order that (11.161c) be satisfied, we postulate that

$$[\Lambda^R(x^R), \Lambda^R(y^R)] = [\Lambda^R(x^R), \Lambda^L(y^L)] = [\Lambda^L(x^L), \Lambda^L(y^L)] = 0. \tag{11.163}$$

By substituting (11.159) into (11.160a) and taking into account the commutativity of the fields inside the N-product, we can look for a solution of (11.160a) in the form

$$\psi(x) = \sqrt{\kappa/2\pi}N(a\exp(-i\sqrt{\pi}\gamma^3\rho(x)) \cdot q(x)), \tag{11.164}$$

where the $a \equiv a_{1,2}$ are constant factors (depending possibly on the charges) such that $a^*a = 1$; $q(x) \equiv q_{(\alpha)}(x)$ is a free quark field chosen in the "bosonized" form (11.76) and we stay in the simplest version corresponding to the parameter $\alpha = \sqrt{2\pi}$ when $\sqrt{\kappa/2\pi}q(x)$ is the free Dirac field (11.94).

We can take as α any non-zero real number, obtaining as a result other (pairwise inequivalent for $\alpha' \neq \alpha$, $-2\pi/\alpha$) solutions of the equations. It turns out that if the second (essential) free parameter Z which, as we shall see, occurs in the model, is fixed at unity, then the value $\alpha = \sqrt{2\pi}$ (or the equivalent value $\alpha = -\sqrt{2\pi}$) which we shall confine ourselves to, is distinguished by the property of "asymptotic freedom'. This means that the degrees of the principal singularities of the Wightman functions (for example, $\langle \dots \psi(x)\psi^{(*)}(x+y)\dots\rangle_0$) at the points where the pair of arguments coincide (that is, as $y \to 0$) are the same as in the free theory. We note that, as is clear from (11.155), the Thirring model for $g \neq 0$ is an example of a model "without asymptotic freedom".

We take for the "renormalized" current in the Schwinger model the expression of type (11.101) with ∂_1 replaced by the covariant derivative $\partial_1 + ieA_1$:

$$j^\mu(x) = N(\psi^*(x)i\kappa^{-1}\gamma^1(\partial_1 + ieA_1)\gamma^\mu\psi(x)). \tag{11.165}$$

Substituting (11.164) into (11.165) gives

$$j^\mu(x) = -\frac{e}{\pi}A^\mu(x) - \frac{1}{\sqrt{\pi}}\partial^\mu\Phi. \tag{11.166}$$

The modified Maxwell equation (11.160b) now takes on the form

$$(\Box + m^2)A^\mu(x) = -\frac{e}{\sqrt{\pi}}\partial^\mu\Phi(x) - e\partial^\mu\Lambda(x), \tag{11.167}$$

where

$$m = e/\sqrt{\pi}, \tag{11.168}$$

so that

$$\epsilon^{\mu\nu}\partial_\nu\left\{\frac{\sqrt{\pi}}{e}(\Box + m^2)\rho(x) + \frac{e}{\sqrt{\pi}}\Phi'(x) + e\Lambda'(x)\right\} = 0. \tag{11.169}$$

Hence we find that*

$$\frac{\sqrt{\pi}}{e}(\Box + m^2)\rho(x) + \frac{e}{\sqrt{\pi}}\Phi'(x) + e\Lambda'(x) = 0. \tag{11.170}$$

Consequently the field $\rho(x)$ is represented in the form

$$\rho(x) = \tau(x) - B'(x), \tag{11.171}$$

where $\tau(x)$ is a neutral pseudo-scalar field of mass m, $B'(x)$ is a neutral pseudo-scalar field of zero mass which is expressed in terms of $\Phi'(x)$ and $\Lambda'(x)$ by the relation

$$\Lambda'(x) = \frac{1}{\sqrt{\pi}}(-\Phi'(x) + B'(x)). \tag{11.172a}$$

Consequently we set for the right and left components:

$$\Lambda^R(x^R) = \frac{1}{\sqrt{\pi}}(-\Phi^R(x^R) + B^L(x^L)), \quad \Lambda^L(x^L) = \frac{1}{\sqrt{\pi}}(-\Phi^L(x^L) + B^L(x^L)). \tag{11.172b}$$

Similarly, the charges of the fields $\Lambda^{R,L}$ are defined by the equalities

$$K^R_\Lambda = \frac{1}{\sqrt{\pi}}(-K^R_\Phi + K^R_B), \quad K^L_\Lambda = \frac{1}{\sqrt{\pi}}(-K^L_\Phi + K^L_B) \tag{11.173a}$$

(where $K^{R,L} \equiv K^{R,L}_\Phi$ and $K^{R,L}_B$ are the charges of the fields $\Phi^{R,L}$ and $B^{R,L}$; see Exercises 11.4 and 11.20). As in (11.57), we introduce

$$K_X = \frac{1}{\sqrt{2}}(K^R_X + K^L_X), \quad K'_X = \frac{1}{\sqrt{2}}(-K^R_X + K^L_X) \quad \text{for } X = \Phi, B, \Lambda. \tag{11.173b}$$

From (11.161b) we obtain the commutator of $\Lambda(x)$ and the field B defined as $B(x) \equiv \frac{1}{\sqrt{2}}(B^R(x^R) + B^L(x^L))$:

$$[\Lambda(x), B(y)] = \frac{i}{\sqrt{\pi}}D_0(x-y) \tag{11.174}$$

(in fact this commutator is defined from (11.161b) to within an arbitrary inessential purely imaginary constant). Since we have extended the algebra of the massless fields by introducing right and left components, we postulate the following commutation relations between $\Lambda^{R,L}$ and $B^{R,L}$ in order that the commutation relation (11.174) be satisfied:

$$[\Lambda^R(x^R), B^R(y^R)] = \frac{i}{\sqrt{\pi}}D^R(x^R - y^R), \; [\Lambda^L(x^L), B^L(y^L)] = \frac{i}{\sqrt{\pi}}D^L(x^L - y^L), \tag{11.175a}$$

* We have omitted the "arbitrary constant" which arises in going over from (11.169) to (11.170) and leads to a redefinition of the constant factors a in (11.164).

$$[\Lambda^R(x^R), B^L(y^L)] = 0 = [\Lambda^L(x^L), B^R(y^R)]. \tag{11.175b}$$

We find the commutation relations for $B^{R,L}$ from (11.172) and from the commutation relations (11.9), (11.163), (11.175a), (11.175b):

$$[B^R(x^R), B^R(y^R)] = iD^R(x^R - y^R), \quad [B^R(x^R), B^L(y^L)] = 0, \tag{11.175c}$$

$$[B^L(x^L), B^L(y^L)] = iD^L(x^L - y^L), \tag{11.175d}$$

and the mutual commutativity of the fields $B^{R,L}$ and $\Phi^{R,L}$:

$$[B^{R,L}(x^{R,L}), \Phi^{R,L}(x^{R,L})] = 0. \tag{11.175e}$$

As regards the commutation relations with the field $\tau(x)$ (which is free with mass m) we assume that they are

$$[\tau(x), \tau(y)] = \frac{1}{i} D_m(x-y) \equiv 2\pi \int \epsilon(p^0)\delta(p^2 - m^2)e^{-ip(x-y)} d_2p, \tag{11.176}$$

$$[\tau(x), \Phi^{R,L}(y^{R,L})] = 0 = [\tau(x), B^{R,L}(y^{R,L})]. \tag{11.177}$$

Of course, covariance considerations allow an arbitrary "renormalization" constant Z in the relation

$$[\tau(x), \tau(y)] = \frac{Z}{i} D_m(x-y). \tag{11.178}$$

We have set $Z = 1$ so that the simultaneous commutation relations

$$[A_\lambda(x), A_\mu(y)]|_{x^0=y^0} = 0 \tag{11.179}$$

be trivial (as in the canonical quantization).

Exercise 11.29. Verify the commutation relations for the field $A_\mu(x)$:

$$[A_\lambda(x), A_\mu(y)] = im^{-2}(\partial_\lambda\partial_\mu - g_{\lambda\mu}\Box)(D_m(x-y) - D_0(x-y)). \tag{11.180}$$

Upon substituting (11.171) into (11.159), (11.164) (under a suitable choice of the factors $a_{1,2}$ in (11.164)) we obtain the final expression for the fields in the Schwinger model:

$$A^\mu(x) = -\frac{1}{m}(\partial^\mu B(x) - \epsilon^{\mu\nu}\partial_\nu \tau(x)) \tag{11.181a}$$

$$\psi(x) = \sqrt{\frac{\kappa}{2\pi}} N\left\{ N \exp[i\sqrt{\pi}(B(x) - \gamma^3\tau(x))] \begin{pmatrix} \exp(i\vartheta^R(x^R)) \\ \exp(i\vartheta^L(x^L)) \end{pmatrix} \right\}, \tag{11.181b}$$

$$\Lambda(x) = \frac{1}{\sqrt{\pi}}(-\Phi(x) + B(x)), \tag{11.181c}$$

where

$$\vartheta^R(x^R) = -\sqrt{2}\pi\Lambda^R(x^R) + \frac{\pi}{\sqrt{2}}K^L_\Lambda, \quad \vartheta^L(x^L) = -\sqrt{2}\pi\Lambda^L(x^L) - \frac{\pi}{\sqrt{2}}K^R_\Lambda. \tag{11.182a}$$

It is clear that the fields $\vartheta^{R,L}(x^{R,L})$ satisfy the trivial commutation relations:

$$[\vartheta^R(x^R), \vartheta^R(y^R)] = [\vartheta^R(x^R), \vartheta^L(y^L)] = [\vartheta^L(x^L), \vartheta^L(y^L)] = 0. \tag{11.182b}$$

The (pseudo-Hilbert) space in which the fields featuring in (11.181) act can be taken to be $\mathcal{H}_\Phi \otimes \mathfrak{H}_B \otimes \mathcal{H}_\tau$, where $\mathcal{H}_\Phi \equiv \mathcal{H}_\Phi^R \otimes \mathcal{H}_\Phi^L$ is the Hilbert space in §11.1.B; $\mathfrak{H}_B \equiv \mathfrak{H}_B^R \otimes \mathfrak{H}_B^L$ is the pseudo-Hilbert space constructed in §11.1.D (the auxiliary fields $\Phi^{R,L}$ acting there in $\mathfrak{H}_B$ are not to be identified with the fields $\Phi^{R,L}$ defined in $\mathcal{H}_\Phi$); $\mathcal{H}_\tau$ is the space of the Fock representation of $\tau(x)$ (see §8.4.A) defined by the two-point function

$$\langle 0|\tau(x)\tau(y)|0\rangle = \frac{1}{i} D_m^{(-)}(x-y) \equiv 2\pi \int \theta(p^0)\delta(p^2 - m^2)e^{-ip(x-y)} d_2p. \qquad (11.183)$$

The non-local normal exponentials for the field $\tau(x)$ are given by the formula

$$N \exp\Big(i \int \tau(x)F(x)d^2x\Big) =$$
$$= \exp\Big(\frac{1}{2i} \int D_m^{(-)}(x-y)F(x)(y)d^2x\, d^2y\Big) \exp\Big(i \int \tau(x)F(x)d^2x\Big), \quad F \in \mathcal{S}_r(\mathrm{M});$$

the local normal exponentials are defined as the corresponding limits of the non-local normal exponentials (see §11.1.C).

Clearly the conserved current (11.166) can be written in the form

$$j^\mu(x) = -\frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu \tau(x) - j^{\mu(\text{fict})}(x) \qquad (11.184)$$

(where $j_\mu^{(\text{fict})}(x)$ is defined according to (11.160c)). It generates the global gauge transformations of the local fields:

$$e \int [j^0(x), X]dx^1 = q_X X, \qquad (11.185)$$

where X is any of the fields ψ, $\tilde{\psi}$, A_μ, Λ, j_μ, and q_X is defined as in (10.100b).

Exercise 11.30. (a) Verify the commutation relations

$$[j^\mu(x), \psi(y)] = (-\partial^\mu D_0(x-y) + \epsilon^{\mu\nu}\gamma^3\partial_\nu D_m(x-y))\psi(y), \qquad (11.186a)$$
$$[j_\lambda(x), A_\mu(y)] = \frac{i}{\pi}(\partial_\lambda\partial_\mu - g_{\lambda\mu}\Box)(D_m(x-y) - D_0(x-y)), \qquad (11.186b)$$
$$[j_\lambda(x), \Lambda(x)] = 0, \qquad (11.186c)$$
$$[j_\lambda(x), j_\mu(y)] = \frac{i}{\pi}(\partial_\lambda\partial_\mu - g_{\lambda\mu}\Box)D_m(x-y). \qquad (11.186d)$$

(b) Verify that the relations (11.185) hold and that

$$e^{iQ_c}Xe^{-iQ_c} = e^{iq_X}X, \qquad (11.187)$$

where Q is the electric charge:

$$Q = eK_\Lambda \equiv \int_{x^0=\text{const}} \partial_0\Lambda(x)dx^1 = -Q^{(\text{fict})} \qquad (11.188)$$

(the integral is meant in the same sense as in (11.61)).

On the other hand, the pseudo-vector current $\epsilon^{\mu\nu}j_\nu(x)$ has anomalous divergence (that is, it is not conserved):

$$\partial_\mu\epsilon^{\mu\nu}j_\nu(x) = \frac{e}{\pi}F(x), \qquad (11.189)$$

where $F(x) \equiv F^{01}(x)$ is the stress field:

$$F^{\lambda\mu}(x) = \epsilon^{\lambda\mu} F(x), \quad F(x) = m\tau(x). \tag{11.190}$$

Therefore the definition of the pseudo-vector current $j'^{\mu}(x)$ generated by the chiral symmetry must be modified:

$$j'^{\mu}(x) = \epsilon^{\mu\nu}(j_\nu(x) + \frac{e}{\pi}A_\nu(x)) = -\frac{1}{\sqrt{\pi}}\partial^\mu\Phi' \equiv -\frac{1}{\sqrt{\pi}}V'^{\mu}(x). \tag{11.191}$$

In contrast to the vector current $j^\mu(x)$, the pseudo-vector current $j'^{\mu}(x)$ is clearly not invariant with respect to the local gauge transformations (and is not an observable field).

Exercise 11.31. Verify the relations

$$\int [j'^0(x), \psi(y)]dx^1 = \gamma^3\psi(y),$$

$$\int [j'^0(x), A_\mu(y)]dx^1 = 0 = \int [j'^0(x), \Lambda(y)]dx^1.$$

B. Vacuum Functional

The defining representation of the *-algebra of the local fields $\psi(x)$, $A_\mu(x)$ in the space $\mathcal{H}_\Phi \otimes \mathfrak{H}_B \otimes \mathcal{H}_\tau$ does not satisfy all the axioms of the pseudo-Wightman approach (§10.1); in particular, the axiom of the existence of vacuum does not hold. (This situation is similar to that of the case of a free quark field or the Thirring field, which are reducible in the space $\mathcal{H}_\Phi$ of the representation of the free scalar massless field.) We now construct the vacuum "virtual" state of the field algebra by the technique used earlier of inducing the vacuum functional from the vacuum of the subalgebra.

For this we consider the space $\mathcal{H}_V \otimes \mathfrak{H}_\sigma \otimes \mathcal{H}_\tau$ of the (irreducible) Fock representation of the currents

$$V^\mu(x) = \partial^\mu\Phi(x), \quad \sigma^\mu(x) = \partial^\mu B(x)$$

and the field $\tau(x)$. It is constructed by means of the corresponding two-point functions

$$\langle 0|V^\lambda(x)V^\mu(y)|0\rangle = i\partial^\lambda\partial^\mu D_0(x-y) = -\langle 0|\sigma^\lambda(x)\sigma^\mu(y)|0\rangle$$

and (11.183). We introduce the auxiliary *-subalgebra $\mathfrak{C}$ of the field algebra generated by the fields $V^\mu(x)$, $\sigma^\mu(x)$, $\tau(x)$ and operators of the form

$$N\exp\Big\{i\sqrt{\pi}\int(B'(x) + \sqrt{2}\Phi^R(x^R) + \sqrt{\frac{\pi}{2}}K^L_\Lambda)F_1(x)d^2x\Big\}, \tag{11.192a}$$

$$N\exp\Big\{i\sqrt{\pi}\int(-B'(x) + \sqrt{2}\Phi^L(x^L) - \sqrt{\frac{\pi}{2}}K^R_\Lambda)F_2(x)d^2x\Big\}, \tag{11.192b}$$

where the $F_j(x)$ are arbitrary functions in $\mathcal{S}_r(\mathrm{M})$ such that $n_j \equiv \int F_j(x)d^2x \in \mathbf{Z}$. It follows from (11.144) that the components of $\psi(x)$ can be represented to within the factor $\sqrt{\kappa/2\pi}$ as limits of such operators. The group $U(1)\times U(1)$ acts on the auxiliary algebra $\mathfrak{C}$ by *-automorphisms which leave the fields $V^\mu(x)$, $\sigma^\mu(x)$, $\tau(x)$ invariant and which multiply the operators (11.192) by $e^{in_1c_1}$ and $e^{in_2c_2}$ respectively

(c_1, $c_2 \in \mathbf{R}$). The result of averaging the elements of $\mathfrak{C}$ over the group $U(1)\times U(1)$ belongs to the *-algebra generated by $V^\mu(x)$, $\sigma^\mu(x)$, $\tau(x)$ and the formula of type (11.91) defines the vacuum functional on $\mathfrak{C}$:

$$\langle A\rangle_0 = \frac{1}{(2\pi)^2}\langle 0|\int_{U(1)\times U(1)} \gamma_c(A)dc_1\, dc_2|0\rangle. \tag{11.193}$$

By representing the field $\sqrt{2\pi/\kappa}\psi(x)$ in the form of limits of the operators (11.192) (as we did for the free quark field; see (11.92)), we thereby obtain the Wightman functions $\langle\cdots\rangle_0$ of the local fields $A_\mu(x)$, $\psi(x)$, $\tilde{\psi}(x)$.

It is easily verified that the algebra of the fields $A_\mu(x)$, $\psi(x)$, $\tilde{\psi}(x)$ together with the vacuum functional on it satisfies the axioms of the pseudo-Wightman approach at least in the weak form (see the Remark in §10.1.C).

Exercise 11.32. Verify that the two-point Wightman functions of the spinor field in the Schwinger model (in the Lorentz gauge) are given by

$$\langle\psi_i^{(*)}(x)\psi_j(y)\rangle_0 = \exp\{i\pi(D_m^{(-)}(x-y) - D_0^{(-)}(x-y))\}\langle\psi_i^{(\mathrm{fr})(*)}(x)\psi_j^{(\mathrm{fr})}(y)\rangle_0, \tag{11.194}$$

where the $\langle\psi_i^{(\mathrm{fr})(*)}(x)\psi_j^{(\mathrm{fr})}(y)\rangle_0$ are the two-point Wightman functions of the free massless Dirac field (see (11.99)).

C. Physical fields; observables

Acting on the *-algebra generated by the fields $A_\mu(x)$, $\psi(x)$, $\Lambda^{R,L}(x^{R,L})$ is the group $\mathcal{G}_0$ of local gauge transformations:

$$\begin{gathered}\psi(x)\to e^{ie\alpha(x)}\psi(x),\quad A_\mu(x)\to A_\mu(x)-\partial_\mu\alpha(x),\\ \Lambda^{R,L}(x^{R,L})\to\Lambda^{R,L}(x^{R,L}),\end{gathered} \tag{11.195}$$

where $\alpha(x)$ is an arbitrary real solution of the d'Alembert equation of class $\mathcal{S}_r(\mathbf{R})$ with respect to x^1. The fields $\tau(x)$, $\Lambda^{R,L}(x^{R,L})$ and $\vartheta^{R,L}(x^{R,L})$ are obviously $\mathcal{G}_0$-invariant. For the algebra of physical quantities we choose the *-algebra $\mathfrak{B}$ generated by the fields $\tau(x)$ and $\exp(i\vartheta^{R,L}(x^{R,L}))$.

This is a natural choice for the following reasons. Consider the fields

$$A^{(\mathrm{phys})\mu}(x) = -\frac{1}{m}\epsilon^{\mu\nu}\partial_\nu\tau(x), \tag{11.196a}$$

$$\psi^{(\mathrm{phys})}(x) = \sqrt{\frac{\kappa}{2\pi}}Ne^{-i\gamma^5\tau(x)}\begin{pmatrix}\exp(i\vartheta^R(x^R))\\ \exp(i\vartheta^L(x^L))\end{pmatrix}. \tag{11.196b}$$

It is natural to assume that the fields (11.196) contain all the physical information of the model. On the other hand, they are clearly $\mathcal{G}_0$-invariant, therefore we have taken as the algebra of physical quantities, the algebra generated by them (which is precisely $\mathfrak{B}$).

The transformation (11.196) alters the properties of the fields in an essential manner. Whereas the fields $A^\mu(x)$ and $\psi(x)$ were vector and spinor fields respectively and satisfied the locality condition (with normal connection between spin and statistics), the fields $A^{(\mathrm{phys})\mu}(x)$ and $\psi^{(\mathrm{phys})}(x)$ are vector and scalar fields respectively (disregarding reflection) and enjoy the property of local commutativity. (The fact is that the transformation (11.196) is essentially non-local, since the field $B(x)$ does not belong to the local field algebra of the fields A_μ and ψ; therefore the system of physical fields goes beyond the limits of the scheme of §10.2.B which is based on the ordinary canonical formalism.) Another distinguishing property of the physical fields (11.196) is their commutativity with the electric charge (11.188). Thus all the physical quantities are electrically neutral and there are no electrically charged physical states in the model.*

Nevertheless, the algebra of physical fields has the obvious symmetry with respect to the group $U(1)$:

$$A_\mu^{(\mathrm{phys})}(x) \to A_\mu^{(\mathrm{phys})}(x), \quad \psi^{(\mathrm{phys})}(x) \to e^{ic}\psi^{(\mathrm{phys})}(x), \tag{11.197a}$$

which is a corollary of the extra symmetry of the system of fields $A_\mu(x)$, $\psi(x)$, $\Lambda^{R,L}(x^{R,L})$ in the Lorentz gauge:

$$\begin{gathered} A_\mu(x) \to A_\mu(x), \quad \psi(x) \to e^{-ic}\psi(x), \\ \Lambda^{R,L}(x^{R,L}) \to \Lambda^{R,L}(x^{R,L}) + \frac{1}{\pi\sqrt{2}}c. \end{gathered} \tag{11.197b}$$

(these are not the usual electrodynamical gauge transformations under which the fields $\Lambda^{R,L}(x^{R,L})$ are left invariant; see (10.100)). It should be borne in mind that neither the current $\frac{1}{\sqrt{\pi}}V^\mu(x)$, generating these transformations, nor the generator $\frac{1}{\sqrt{\pi}}K_\Phi$ bears any relation to the electric current and charge (and they are not physical quantities in general).

Just as for the chiral transformations, the physical fields behave (by virtue of (11.31)) according to the rule

$$A_\mu^{(\mathrm{phys})}(x) \to A_\mu^{(\mathrm{phys})}(x), \quad \psi^{(\mathrm{phys})}(x) \to e^{ic'\gamma^3}\psi^{(\mathrm{phys})}(x), \tag{11.198}$$

but neither the current $j'^\mu(x) = -\frac{1}{\sqrt{\pi}}V'^\mu(x)$, nor the charge $Q' = -\frac{1}{\sqrt{\pi}}K'_\Phi$ are physical quantities. According to §10.3 this means that the chiral symmetry of the physical fields (as well as the symmetry (11.197a)) is spontaneously broken. (There is no Goldstone boson in the physical Hilbert space since there is no physical conserved current generating these transformations.)

There are at least two possibilities for defining the observables. In the first version (as in the general scheme of §10.2.B), by the observables we mean all the electrically neutral physical quantities; hence in the given situation all the physical quantities are regarded as observables. The second version, which we shall look into, takes into account the specific character of the massless electrodynamics in the Lorentz gauge, in other words, the extra symmetries (11.197a), (11.198).

We postulate that the "chiral" transformations (11.198) along with the representations (11.197a) generate the effective gauge group Γ (in the sense of §10.1.C). Thus by definition, the algebra of observables $\mathfrak{A}$ consists of those physical quantities that are invariant with respect to the transformations (11.197a) and (11.198). They are clearly generated by the field $\tau(x)$ (proportional to the field strength $F(x)$) and the fictitious current $j_\mu^{(\mathrm{fict})}(x)$. The vacuum representation of the algebra of observables is clearly defined by the Fock representation $\pi_{\mathrm{vac}}(\tau(x)) \equiv \tau(x)$ of the field $\tau(x)$ in the Hilbert space $\mathcal{H} = \mathcal{H}_\tau$ (where $\pi_{\mathrm{vac}}(j_\mu^{(\mathrm{fict})}(x)) \equiv 0$). It is also easy to see that the

* This phenomenon is called *confinement* of the electric charge.

physical representation of the algebra $\mathfrak{B}$ of physical quantities (constructed according to the scheme of §10.1.C) decomposes into a direct integral

$$\pi = \int^{\oplus}_{\vartheta^R, \vartheta^L \in [0,2\pi)} \pi^{(\vartheta^R, \vartheta^L)} d_1\vartheta^R \, d_1\vartheta^L \tag{11.199}$$

of irreducible representations (ϑ-representations) realized in $\mathcal{H}$ and in which the physical fields have the form (11.195), where now

$$\vartheta^R(x^R) \equiv \vartheta^R, \quad \vartheta^L(x^L) \equiv \vartheta^L \tag{11.200}$$

are arbitrary real constants (say, in the interval $[0, 2\pi)$). The presence of the ϑ-representations does not, however, lead to new states of the algebra of observables (by comparison with the vacuum sector). This is the result of the spontaneous breaking of the symmetries (11.197a) and (11.198) and, in accordance with this, the fictitiousness (unobservability) of the charges ϑ^R, ϑ^L.

Since the field $\exp(2\pi i\Lambda(x))$ is constant in the physical representation π:

$$\pi(\exp(2\pi i \Lambda(x)) \equiv \exp(-i(\vartheta^R + \vartheta^L)),$$

the fictitious current $\mathcal{J}_\mu^{(\text{fict})}(x) \equiv -\partial_\mu \hat{\Lambda}$ is equal to zero and the Maxwell equations assume the usual form:

$$\partial^\mu \mathcal{F}_{\lambda\mu} = e\mathcal{J}_\mu(x),$$

where

$$\mathcal{J}^\mu(x) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \tau(x).$$

Note that the latter relation (together with the definition of the current (11.165)) enables us to interpret the field $\tau(x)$ as a local field of a scalar boson of mass $m = e/\sqrt{\pi}$ which is a fermion-antifermion bound state. The Schwinger model reduces to this free field at the level of the observables.

Part IV

Collision Theory. Axiomatic Theory of the S-Matrix

Synopsis

In non-relativistic quantum mechanics, standard scattering theory uses some assumption concerning the short-range action of the forces in question (and the Coulomb potential requires special consideration in the modified theory). In precisely the same way, relativistic quantum-field theory of scattering begins with the hypothesis concerning the mass spectrum corresponding to the short-range action. In this part we replace W.II in the Wightman axioms (§8.2.A) by the stronger spectrum condition W.II′ which assumes the existence of a mass gap between the vacuum and one-particle state (with positive minimal mass). We also supplement these axioms with the postulate W.III′ on the structure of the one-particle states (§12.1.A) which contains a hypothesis on the existence of an operator A in the polynomial field algebra that creates the (given) one-particle state from the vacuum. In the space of one-particle states (of a particle of specified type), the representation of the quantum-mechanical Poincaré group $\mathfrak{P}_0$ is assumed to be irreducible. Thus the particles are characterized by mass and spin (as well as the values of the charges). The TCP-operator converts particles into antiparticles (with opposite values of the charges). If the operator A creates a particle of mass m, then the vector $A(x)\Psi_0$ (for the non-local "field" $A(x) = U(x,1)AU(x,1)^{-1}$) satisfies the Klein-Gordon equation $(m^2 + \Box)A(x)\Psi_0 = 0$ (§12.1.B). Hence it follows that if

$$A^t = \int_{x^0=t} A(x)\overleftrightarrow{\partial_0} D_m(x)d^3x,$$

then $A^t\Psi_0 = A\Psi_0$. Theorem 12.1 (Haag-Ruelle) establishes the existence of n-particle asymptotic states

$$\Phi^{\text{in}}_{\text{out}}(A_1,\ldots,A_n) = \lim_{t\to\mp\infty} A_1^t \ldots A_n^t \Psi_0$$

and two isometric Poincaré-invariant operators $\Omega^{\text{in}}_{\text{out}}$ mapping the vectors of some auxiliary Fock space $\mathfrak{H}$ to vectors of the form $\Phi^{\text{in}}_{\text{out}}$ of the physical space $\mathcal{H}$. The scattering matrix is given by the formula $S = (\Omega^{\text{out}})^*\Omega^{\text{in}}$. It leaves the one-particle states invariant (§12.1.C). If Θ_0 is the TCP-operator in the Fock space $\mathfrak{H}$ of free (asymptotic) particles, and Θ is the TCP-operator in the physical space $\mathcal{H}$ of the interacting fields, then under the asymptotic completeness assumption (§7.3.E and 13.1.C) and identifying $\mathcal{H}^{\text{in}}$ with $\mathfrak{H}$ (that is, when $\Omega^{\text{in}} = 1$), the equality $S = \Theta_0^{-1}\Theta$ holds.

The truncated vacuum expectation values $\langle A_1 \ldots A_n\rangle^T$ are defined by a recurrence formula (symmetric difference of products of lowest vacuum expectation values in $\langle A_1 \ldots A_k\rangle_0$, §12.2.A). In theories with a mass gap, $\langle A_1(x_1)\ldots A_n(x_n)\rangle^T$ is a function in $\mathcal{S}(\mathbf{R}^{3(n-1)})$ with respect to the spatial variables $\mathbf{x}_1 - \mathbf{x}_2, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_n$, which is smoothly dependent on the time difference (Proposition 12.4,§12.2.B). This (strengthened) cluster property together with a certain property of the spread of wave packets satisfying the Klein-Gordon equation (§12.2.C) enables us to prove Theorem 12.1 (§12.2.D) stated above.

In the Lehmann-Symanzik-Zimmermann (LSZ) approach, the Wightman axioms (of a theory with a mass gap) are supplemented by three further hypotheses (§13.1.C). It is required that the operator A, creating a one-particle state with mass m from the vacuum, should be the same as the operator $\phi(f)$ if the test function $\tilde{f}$ in momentum space is concentrated in a sufficiently small neighbourhood of the mass hyperboloid $p^2 = m^2$ (LSZ.I). It is postulated that the theory be asymptotically complete (LSZ.II) and that there exist T-products of local fields satisfying the usual conditions (13.2) (LSZ.III). Instead of the Wightman functions, the basic objects in the LSZ theory are the τ-functions, which are vacuum expectation values of T-products.

Let $f(x)$ be a smooth solution of the Klein-Gordon equation that is rapidly decreasing in the spatial directions. Then there exist free fields $\phi^{\substack{\text{in}\\\text{out}}}$ of mass m and a dense set of vectors Φ, Ψ in $\mathcal{H}$ such that

$$\lim_{t\to\mp\infty}\Big(\Phi, \int_{x^0=t} f(x)\overleftrightarrow{\partial}_0\phi(x)d^3x\Psi\Big) = \Big(\Phi, \int_{x^0=t} f(x)\overleftrightarrow{\partial}_0\phi^{\substack{\text{in}\\\text{out}}}(x)d^3x\Psi\Big).$$

Furthermore, if $(\Box + m^2)\phi(x) = j(x)$, then the Yang-Feldman equation holds:

$$\phi(x) = \phi^{\text{in}}(x) + \int D_m^{\text{ret}}(x-y)j(y)d^4y = \phi^{\text{out}}(x) + \int D_m^{\text{adv}}(x-y)j(y)d^4y.$$

In §13.2 we give Hepp's proof of the validity of these statements. There is a reduction formula in LSZ according to which, the elements of the scattering matrix can be expressed in terms of amputated τ-functions (Theorem 13.9,§13.2.D).

An alternative approach to LSZ is given by the method of the extended scattering matrix (Ch.14). This theory is based on the following postulates (which, for the sake of simplicity, we summarize for the case of a theory of spinless particles of mass m of one type). S.I. Let $\mathfrak{H}$ be the Fock space of the free field $\phi^{\text{in}}(x)$ of mass m. A unitary Poincaré-invariant S-operator is defined in $\mathfrak{H}$ which leaves the vacuum and the one-particle states invariant. S.II. There exists the formal series $S(\chi) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \chi(x_n)\dots\chi(x_1)H_n(x_1,\dots,x_n)dx_1\dots dx_n$ in the classical field $\chi(x)$, where $H_0 \equiv S(\chi)|_{\chi=0} = S$. The operator-valued generalized functions H_n are defined on a domain $D \subset \mathfrak{H}$ which enters into the domain of definition of the closures of the operators $\phi^{\text{in}}(f)$; these operators take values in the domain $\widetilde{D} = SD$, where the closures of $\phi^{\text{out}}(f)$ are defined for $f \in \mathcal{S}(\mathrm{M})$. The extended S-matrix is Poincaré-invariant (S.III), unitary (S.IV) and causal (S.V); the causality condition has the form $\mathfrak{S}(\chi_1+\chi_2) = \mathfrak{S}(\chi_1)\mathfrak{S}(\chi_2)$ if $\operatorname{supp}\chi_1 \gtrsim \operatorname{supp}\chi_2$, where $\mathfrak{S}(\chi) = S^*S(\chi)$ ($\mathfrak{S}(0) = 1$). The analogue of Axiom LSZ.I is the strengthened condition (S.I′) of stability of the one-particle states:

$$\int \frac{\delta S}{\delta\phi^{\text{in}}(x)}u(x)d^4x|0\rangle = 0$$

for all $\tilde{u}$ in p-space with support in a sufficiently small neighbourhood of the hyperboloid $p^2 = m^2$.

An important role in the extended S-matrix theory is played by the radiation operators and in the first place by the current operator

$$J(x;\chi) = -iS^*(\chi)\frac{\delta S(\chi)}{\delta\chi(x)} = -i\mathfrak{S}(\chi)^{-1}\frac{\delta\mathfrak{S}(\chi)}{\delta\chi(x)}.$$

We recall from among its properties, the causality condition (§14.1.D):

$$\frac{\delta J(x;\chi)}{\delta\chi(y)} = 0 \quad \text{for } x \lesssim y.$$

The interacting (Heisenberg) field $\phi(x)$ is defined in the S-matrix formalism by the Yang-Feldman equation:

$$\phi(x) = \phi^{\text{in}}(x) + \int D_m^{\text{ret}}(x-y)J(y)dy,$$

where $J(y) = J(y;\chi=0)$; it is equivalent to the representation

$$\phi(x) = S^*T(S\phi^{\text{in}}(x)).$$

Defining the T-product of the fields ϕ by the equality

$$T(\phi(x_1)\dots\phi(x_n)) = S^*T(S\phi^{\text{in}}(x_1)\dots\phi^{\text{in}}(x_n)),$$

we derive the LSZ axioms from the postulates S.I–S.V and S.I′ (Theorem 14.1, §14.2.B). The generating functional for the T-products can be written in the form

$$\mathfrak{T}(\eta) \equiv T\exp\Big(i\int \eta(x)\phi(x)dx\Big) = S^*T\Big\{S\exp\Big(i\int \eta(x)\phi^{\text{in}}(x)dx\Big)\Big\}.$$

CHAPTER 12

Haag-Ruelle Scattering Theory

12.1. Scheme of the Quantum Field Theory of Scattering

A. THE ONE-PARTICLE PROBLEM IN QUANTUM FIELD THEORY

In order that quantum field theory may serve as a basis for elementary particle theory, it must contain the necessary components of relativistic collision theory. According to §7.3.E, this requires the Fock space $\mathfrak{H}$ of (asymptotically) free particles and two Poincaré-invariant isometric operators Ω^{in} and Ω^{out} from $\mathfrak{H}$ to the physical Hilbert space $\mathcal{H}$. (We shall use the common notation Ω^{ex}, where the symbol ex takes the two values: in and out). *

In §7.2.B we discussed the notion of a particle in relativistic quantum theory. The question arises how to introduce the notion of a system of particles within the scheme of the Wightman theory of interacting fields. The intuitive answer is that as $x^0 = t \to \pm\infty$, the Wightman field $\phi(x)$ must act as a free field and hence, one must form operators of creation and annihilation of particles from it. (This kind of requirement was originally put forward by Lehmann, Symanzik and Zimmermann in 1955 and bears the name of LSZ asymptotic conditions.) However, only after the work of Haag (1958; 1959a,b) did it become possible to give a precise formulation of this intuitive idea and to settle questions of compatibility of the asymptotic conditions with the remaining Wightman postulates and to what extent they are independent requirements. Definitive answers to this circle of questions can be found in the work of Ruelle (1962) and for this reason, the construction given below bears the title of Haag-Ruelle scattering theory.

As we have already mentioned (§7.2.B) relativistic scattering theory requires more detailed information on the energy-momentum spectrum. To this end, we replace the spectrum axiom W.II (in the Wightman axioms W.I–W.VIII) by the stronger variant W.II′, while Axiom W.III (on the existence and uniqueness of the vacuum) is supplemented by the requirement W.III′ on the structure of one-particle states.

W.II′ (**Strong Spectrum Condition**) *The spectrum of the energy-momentum operator lies in the set* $\{0\} \cup \overline{V_\mu^+}$, *where* $\mu > 0$.

The minimum such μ is called the *mass gap*.

W.III′ (**Structure of One-Particle States**). *The physical Hilbert space* $\mathcal{H}$ *contains a finite or countable family of mutually orthogonal subspaces* $\mathfrak{H}^{[\kappa]}$ *(so-called one-particle subspaces of particles of respective type* κ*), which transform according to irreducible representations of the proper Poincaré group* $\mathfrak{P}_0$ *and are taken into* $\mathfrak{H}^{[\bar{\kappa}]}$

* Ex is short for exterior.

under CPT transformations. For each particle type κ occurring in the model, there exists an operator $A \in \mathcal{P}(\mathrm{M})$ (in the polynomial field algebra associated with the whole of space-time M) such that $A\Psi_0$ is the zero vector in $\mathfrak{H}^{[\kappa]}$ and $A^*\Psi_0 \in \mathfrak{H}^{[\bar{\kappa}]}$.

We denote by K the collection $\{\kappa\}$ of all particle types occurring in the concrete model. Each particle type κ has a fixed positive mass m_κ and spin $s_\kappa = 0, \frac{1}{2}, 1, \ldots$. As in §7.3.C, we suppose in Axiom W.III′ that the operation of charge conjugation $\kappa \to \bar{\kappa}$ defined on the set K is such that $\bar{\bar{\kappa}} \equiv \kappa$ and $m_{\bar{\kappa}} = m_\kappa$, $s_{\bar{\kappa}} = s_\kappa$ (the case $\bar{\kappa} = \kappa$ is not excluded). We can construct the Hilbert space $\mathfrak{H}$ of asymptotically free particles from the given one-particle subspaces $\mathfrak{H}^{[\kappa]}$ by the method of second quantization (§7.3.C); here, of course, we are supposing the normal connection between spin and statistics. The vacuum vectors in $\mathfrak{H}$ and $\mathcal{H}$ will be denoted by $|0\rangle$ and Ψ_0 respectively (although, of course, these vacuum vectors can be identified as can the corresponding one-particle vectors in $\mathfrak{H}$ and $\mathcal{H}$). According to the rules of §7.3.C , we can associate with each $\Phi \in \mathfrak{H}$ operators of creation $a^*_\kappa(\Phi)$ and annihilation $a_\kappa(\Phi)$ acting in the Fock space $\mathfrak{H}$.

The operator$A$ featuring in Axiom W.III′ is called the solution of the quantum-field problem of one-particle states. The fact that the one-particle problem has already been solved then provides the starting point for the quantum field theory of scattering. In the next subsection we show that on the basis of this, we can construct the isometric operators $\Omega^{\mathrm{ex}} : \mathfrak{H} \to \mathcal{H}$ that create the scattering picture.

Exercise 12.1. Deduce from Axiom W.III′ that for any particle type $\kappa \in K$, there exists a linear subspace $\mathcal{A}^{[\kappa]} \subset \mathcal{P}(\mathrm{M})$ that is taken to $\mathcal{A}^{[\bar{\kappa}]}$ under Hermitian conjugation, is invariant with respect to the automorphisms of the proper Poincaré group and is such that $\mathcal{A}^{[\kappa]}\Psi_0$ is an everywhere dense subset of $\mathfrak{H}^{[\kappa]}$. [Hint: Let $\mathcal{B}^{[\kappa]}$ be the linear span of the operators $\alpha_{(a,\underline{\Lambda})}(A)$, where A is the operator in Condition W.III′ and $(a, \underline{\Lambda}) \in \mathfrak{P}_0$; one can now set $\mathcal{A}^{[\kappa]} = \mathcal{B}^{[\kappa]} + \mathcal{B}^{[\bar{\kappa}]*}$.]

We discuss two typical situations in which the one-particle problem has a simple solution. *

First we suppose that the point $m > 0$ is an isolated point of the spectrum of the mass operator $M = (P_\mu P^\mu)^{1/2}$. Consequently, there exists $\epsilon > 0$ such that the point m is the only point in the interval $(m - \epsilon,\ m + \epsilon)$ in the spectrum M. We take a fixed function $\chi(p) \in \mathcal{S}(\mathrm{M})$ such that $\chi(p) = 1$ for $p^2 = m^2$ and $\chi(p) = 0$ for $|(p^2)^{1/2} - m| \geq \epsilon$. It then follows from the hypothesis of the cyclicity of the vacuum that the vectors of the form $\chi(P)A\Psi_0$, $A \in \mathcal{P}(\mathrm{M})$, form a dense subset of the subspace of one-particle vectors in $\mathcal{H}$ with mass m. But the vectors $\chi(P)A\Psi_0$ can also be written in the form $B\Psi_0$ for suitable $B \in \mathcal{P}(\mathrm{M})$ (if A is a sum of expressions of type (8.8), then B is the corresponding sum obtained by making the substitution $\tilde{f}(p_1, \ldots, p_n) \to \chi(p_1 + \ldots + p_n)\tilde{f}(p_1, \ldots, p_n)$). Thus in the present instance, the one-particle problem is solved by applying the axiom of the cyclicity of the vacuum.

Suppose now that the mass m of the subspace of one-particle states lies in the continuous spectrum of the operator M, but that there exists a superselection rule separating these states (for example, the single-nucleon state is in the continuous spectrum corresponding to the two- and, more generally, many-meson states, but it is distinguished by the fact that its baryonic number B is equal to unity). The exact meaning of this hypothesis consists in the following. We are given a compact gauge

* It is true that here we have only considered the construction of states with a fixed mass, leaving aside the question of the other quantum numbers of the particles, which is not all that important for our purposes (and which, as a rule, is solved by group theoretic methods and does not give rise to any practical difficulties).

group G of internal symmetries of the fields which is unitarily realized in $\mathcal{H}$ and leaves the vacuum invariant. We suppose that the representations of this group in the one-particle subspace with mass m and in the subspace of the continuous spectrum of the mass in the interval $(m-\epsilon,\ m+\epsilon)$ (where $\epsilon > 0$) are disjoint (that is, they do not contain any equivalent subrepresentations). Consider vectors of the form $A\Psi_0$, where the A are operators in $\mathcal{P}(\mathbf{M})$ that transform according to irreducible representations of G contained in the representation of the one-particle subspaces with mass m. It is not difficult to see that the corresponding vectors $\chi(P)A\Psi_0$ form a total subset of the one-particle subspace in question (here the $\chi(P)$ are the same functions as in the preceding example). Clearly, these vectors can also be written in the form $B\Psi_0$, where $B \in \mathcal{P}(\mathbf{M})$, and this gives the solution of the one-particle problem.

In the scheme under consideration, no distinction is made between elementary (or fundamental) and composite particles. Such a distinction can be interpreted in terms of fields by dividing the fields into fundamental and composite ones. Particles obtained from the vacuum by the action of fundamental fields in the first degree are called elementary (or fundamental and carrying a single "quantum" of the fundamental field); the other particles are called composite particles (or bound states of some number of "quanta" of the fundamental fields). In practice, the internal symmetries and "quantum numbers" (charges) of the fields play an important role in such a classification.

B. Construction of in- and out-states

By virtue of hypothesis W.III′, each particle type $\kappa \in K$ can be associated with a linear subspace $A^{[\kappa]} \subset \mathcal{P}(\mathbf{M})$ such that $\mathcal{A}^{[\kappa]} = \mathcal{A}^{[\bar{\kappa}]}$, and $\mathcal{A}^{[\kappa]}\Psi_0$ is a dense linear subspace of $\mathfrak{H}^{[\kappa]}$ (see Exercise 12.1). For $A \in \mathcal{A}^{[\kappa]}$ we denote by $\hat{A}$ the following operator in the Fock space $\mathfrak{H}$:

$$\hat{A} = a^*_\kappa(A\Psi_0) + a_{\bar{\kappa}}(A^*\Psi_0) \tag{12.1}$$

(clearly, $\hat{A}$ depends linearly on A). Then all vectors of the form

$$\Phi(A_1,\dots,A_n) = \hat{A}_1\dots\hat{A}_n|0\rangle,\ A_j \in \mathcal{A}^{[\kappa_j]}, \tag{12.2}$$

evidently form a total subset of the Fock space (for $n=0$, the expression (12.2) is equal to $|0\rangle$ by definition).

Exercise 12.2. Let $A_j \in \mathcal{A}^{[\kappa_j]}$ and $A'_j = U(a,\underset{\sim}{\Lambda})A_jU(a,\underset{\sim}{\Lambda})^{-1}$, where $(a,\underset{\sim}{\Lambda}) \in \mathfrak{P}_0$. Prove that

$$\mathcal{U}(a,\underset{\sim}{\Lambda})\Phi(A_1,\dots,A_n) = \Phi(A'_1,\dots,A'_n),$$

where $\mathcal{U}(a,\underset{\sim}{\Lambda})$ is a representation of the Poincaré spin group $\mathfrak{P}_0$ in $\mathfrak{H}$.

We associate with an arbitrary operator $A \in \mathcal{A}^{[\kappa]}$ the non-local (and relativistically non-covariant) "field"

$$A(x) = U(x,1)AU(x,1)^{-1},\ x \in \mathbf{M}. \tag{12.3}$$

We intend to construct the asymptotic states using such "fields" $A(x)$ as $x^0 \to \pm\infty$; therefore we go into further details on their properties.

First of all, $A(x)$ is C^∞-dependent on x as a parameter. (In fact, if A is a sum of expressions of type (8.8), then $A(x)$ is the corresponding sum obtained by replacing the test functions $f(x_1,\dots,x_n)$ by $f(x_1-x,\dots,x_n-x)$.)

The action of $A(x)$ on the vacuum gives a vector function satisfying the Klein-Gordon equation

$$(\partial_\mu\partial^\mu + m^2_\kappa)A(x)\Psi_0 = 0. \tag{12.4}$$

In fact, the left hand side of this equation is equal to $U(x,1)(-P_\mu P^\mu + m_\kappa^2)A\Psi_0$; this is equal to zero by the hypothesis that $A\Psi_0 \in \mathfrak{H}^{[\kappa]}$.

Exercise 12.3. Suppose that the smooth function (or vector-valued function) $\Phi(x)$ satisfies the Klein-Gordon equation with mass m. Prove that for all t,

$$\Phi(0) = \int_{x^0=t} \Phi(x)\overleftrightarrow{\partial_0} D_m(x)d^3x \equiv \int_{x^0=t} \{\Phi(x)\partial_0 D_m(x) - \partial_0\Phi(x)D_m(x)\}d^3x; \tag{12.5}$$

here $D_m(x)$ is the Pauli-Jordan commutation function (the three-dimensional integration — in the sense of generalized functions — is taken over a bounded domain by virtue of the properties of the support of $D_m(x)$).

We now associate with the operator $A \in \mathcal{A}^{[\kappa]}$ the family of operators $A^t (t \in \mathbf{R})$ according to the rule

$$A^t = \int_{x^0=t} A(x)\overleftrightarrow{\partial_0} D_m(x)d^3x. \tag{12.6}$$

It follows from Exercise 12.3 that

$$A^t\Psi_0 = A\Psi_0 \quad \text{for all } t \in \mathbf{R}. \tag{12.7}$$

Thus the operators A^t form the same one-particle states from the vacuum as the operator A. However, as is clear from the definition, as $t \to \pm\infty$ they depend effectively on the local fields $\phi(x)$ as the modulus of the time x^0 increases to infinity; it is therefore natural to assume that in the limit $t \to \pm\infty$, the operators A^t take on the role of operators of creation of asymptotic states. In fact, we have the following fundamental result.

Theorem 12.1 (Haag-Ruelle). (a) *For any operators $A_j \in \mathcal{A}^{[\kappa_j]}$ ($j = 1, \ldots, n$, where $n = 0, 1, \ldots$), the following limits exist (in the norm topology):*

$$\Phi^{\substack{\text{in}\\ \text{out}}}(A_1, \ldots, A_n) = \lim_{t \to \mp\infty} A_1^t \ldots A_n^t \Psi_0. \tag{12.8}$$

(b) *There exist two linear isometric Poincaré-invariant maps $\Omega^{\text{ex}} : \mathfrak{H} \to \mathcal{H}$ (ex=in, out), satisfying the properties (which uniquely define Ω^{ex}):*

$$\Omega^{\text{ex}}(\Phi(A_1, \ldots, A_n)) = \Phi^{\text{ex}}(A_1, \ldots, A_n) \tag{12.9}$$

for all $A_j \in \mathcal{A}^{[\kappa_j]}$ ($n = 0, 1, \ldots$; $j = 1, 2, \ldots, n$).

The proof of this theorem is based on the properties of truncated vacuum expectation values and is set out in §12.2.

C. S-MATRIX AND TCP-OPERATORS IN THE ASYMPTOTICALLY COMPLETE THEORY

Theorem 12.1 clearly settles the various questions relating to the construction of the scattering matrix which (according §7.3.E) is given by

$$S = (\Omega^{\text{out}})^*\Omega^{\text{in}}. \tag{12.10}$$

It follows from (12.7) that Ω^{ex} behaves like the identity operator on the one-particle states, therefore

$$S\Phi = \Phi \tag{12.11}$$

for all Φ in the one-particle subspace $\mathfrak{H}_1 \subset \mathfrak{H}$. This property expresses the ***stability of the one-particle states.***

We bring to attention a certain point connected with formula (12.9): the asymptotic state $\Phi^{ex}(A_1, \ldots, A_n)$ depends on the operator $A_j \in \mathcal{A}^{[\kappa_j]}$ in terms of the operator $\hat{A}_j$ only (see (12.1)), that is, in terms of the one-particle states $A_j\Psi_0$ and $A_j^*\Psi_0$ formed by the action of A_j and A_j^* on the vacuum. This means that if $A_j' \in \mathcal{A}^{[\kappa_j]}$ and

$$\hat{A}_j' = \hat{A}_j \ (j = 1, \ldots, n),$$

then

$$\Phi^{ex}(A_1', \ldots, A_n') = \Phi^{ex}(A_1, \ldots, A_n).$$

This fact lies at the basis of the proof of the Poincaré invariance of the maps Ω^{ex}. We mention another simple application of this remark to the Borchers classes of local fields.

Corollary 12.2. Suppose that two systems of Wightman fields $\{\phi^{(\kappa)}\}$ and $\{\phi'^{(\nu)}\}$ act in the same Hilbert space $\mathcal{H}$ (with a fixed representation of the proper Poincaré group) and that they are mutually local. Suppose further that each system satisfies Axioms W.II′ and W.III′. Then the corresponding maps Ω^{ex} and the scattering matrices defined on each of these two systems of fields, coincide.

In fact, the union of the system of fields $\{\phi^{(\kappa)}\} \cup \{\phi'^{(\nu)}\}$ satisfies all the axioms, so that this corollary follows directly from Theorem 12.1 (or the remark given above).

By construction, the S-matrix is invariant with respect to the proper Poincaré group $\mathfrak{P}_0$. Another important property of it, namely, TCP-invariance, is a corollary of the TCP-theorem.

Exercise 12.4. Let $A_j \in \mathcal{A}^{[\kappa_j]}$ and $B_j = \Theta A_j \Theta^{-1}$, where Θ is the TCP-operator.

(a) Prove that $B_j \in \mathcal{A}^{[\bar{\kappa}_j]}$ and that the formula

$$\Theta_0\Phi(A_1, \ldots, A_n) = \Phi(B_1, \ldots, B_n) \tag{12.12}$$

defines an anti-unitary operator Θ_0 in $\mathfrak{H}$ such that

$$\Theta_0|0\rangle = |0\rangle, \ \Theta_0\hat{A}\Theta_0^{-1} = \hat{B} \quad \text{for } A \in \mathcal{A}^{[\kappa]}. \tag{12.13}$$

(b) Prove the relations

$$\Theta_0\mathcal{U}(a, \underset{\sim}{\Lambda})\Theta_0^{-1} = \mathcal{U}(-a, \underset{\sim}{\Lambda}) \quad \text{for } (a, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0, \tag{12.14}$$

$$\Theta_0^2 = \mathcal{U}(0, -1) \tag{12.15}$$

(the properties (12.13)–(12.15) imply that Θ_0 has the meaning of a TCP-operator of free particles; see §7.3.D).

(c) Prove the relations

$$\Phi^{\text{in}}_{\text{out}}(B_1, \ldots, B_n) = \Theta\Phi^{\text{out}}_{\text{in}}(A_1, \ldots, A_n), \tag{12.16}$$

which set up the following connection between the TCP-operators Θ and Θ_0:

$$\Omega^{\text{in}}_{\text{out}}\Theta_0 = \Theta\Omega^{\text{out}}_{\text{in}}. \tag{12.17}$$

Hence deduce the TCP-invariance of the S-matrix:

$$\Theta_0 S\Theta_0^{-1} = S^*. \tag{12.18}$$

[Hint: To obtain (12.18), multiply the equality $\Omega^{\text{in}}\Theta_0 = \Theta\Omega^{\text{out}}$ and the equality obtained from $\Omega^{\text{out}}\Theta_0 = \Theta\Omega^{\text{in}}$ by taking the Hermitian adjoint.]

In order to safeguard the complete corpuscular interpretation of the theory, we must add to our postulates the requirement of asymptotic completeness (§7.3.E):

$$\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}} = \mathcal{H}. \tag{12.19}$$

Thanks to the TCP-theorem, this requirement is equivalent to the following (which at first glance appears to be weaker):

$$\mathcal{H}^{\text{in}} = \mathcal{H} \tag{12.20}$$

(see Exercise 12.5 below). Condition (12.20) makes it possible to naturally identify the Fock space $\mathfrak{H}$ with the physical space $\mathcal{H}$ via the isomorphism $\Omega^{\text{in}} : \mathfrak{H} \to \mathcal{H}$, which is ordinarily done. This means that we parametrize the physical Hilbert space by vectors of the Fock space of incoming particles. Thereupon, the operator Ω^{in} becomes the identity operator, while the S-matrix becomes $(\Omega^{\text{out}})^*$:

$$\Omega^{\text{in}} = 1, \quad (\Omega^{\text{out}})^* = S. \tag{12.21}$$

In the asymptotically complete theory there is an interesting expression for the S-matrix in terms of the TCP-operators. If we make the above identification of $\mathfrak{H}$ and $\mathcal{H}$, then the expression for the S-matrix has the form

$$S = \Theta_0^{-1}\Theta. \tag{12.22}$$

Exercise 12.5. (a) Prove the equivalence of (12.19) and (12.20). [Hint: Use formulae (12.16).]

(b) Deduce (12.22). [Hint: Use (12.17), (12.21).]

The asymptotic completeness condition is independent of the remaining Wightman postulates, as the following example, in which this axiom does not hold, shows.*

Exercise 12.6. Let $\phi(x)$ be a Hermitian scalar generalized free field with two-point function

$$w^{[2]}(x,y) = \frac{1}{i}D_m^{(-)}(x-y) + \frac{1}{i}\int_M^\infty D_\mu^{(-)}(x-y)\rho(\mu)d\mu;$$

here $0 < m < M$, $\rho(\mu)$ is a continuous non-negative function not identically zero. Prove that the asymptotic completeness condition does not hold in this model:

$$\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}} \neq \mathcal{H}. \tag{12.23}$$

Verify also that $\Omega^{\text{in}} = \Omega^{\text{out}}$ and $S = 1$. [Hint: To verify (12.23), show that the one-particle states with mass greater than m are orthogonal to the Fock subspace $\mathfrak{H} \subset \mathcal{H}$ describing states of a free field of mass m.]

12.2. Existence of Asymptotic States

A. Truncated Vacuum Expectation Values

The presence of a mass gap in the energy-momentum spectrum enables us to prove a strengthened cluster property for the Wightman functions; this plays a decisive role

* In this example $\mathcal{H}^{\text{in}} = \mathcal{H}^{\text{out}}$ just the same. Note that formula (12.22) for the S-matrix (from which its unitarity follows immediately) is proved only under the assumption (12.19).

in the proof of Theorem 12.1. In order to formulate this property we need the notion of truncated vacuum expectation values.

The vacuum expectation value $\langle \Psi_0, A_1 \ldots A_n \Psi_0 \rangle$ of the product of elements $A_1, \ldots$ $\ldots, A_n$ of the polynomial field algebra $\mathcal{P}(\mathbf{M})$ will also be denoted by $\langle A_1 \ldots A_n \rangle_0$. We suppose for definiteness that each operator A_j contains either an even number of Fermi-fields (it is then regarded as bosonic) or an odd number (regarded as fermionic):

$$U(0,-1)A_j U(0,-1)^{-1} = \pm A_j.$$

We define the *truncated expectation values* $\langle A_1 \ldots A_n \rangle^T$ by the recurrence relations:

$$\langle A_1 \ldots A_n \rangle_0 = \sum \epsilon_F(\pi) \langle A_{l_1^1} \ldots A_{l_1^{k_1}} \rangle^T \ldots \langle A_{l_\nu^1} \ldots A_{l_\nu^{k_\nu}} \rangle^T; \tag{12.24}$$

here the summation is over all partitions of the indices $1, \ldots, n$ into groups (clusters), where the indices increase from left to right in each group; $\epsilon_F(\pi)$ is the fermionic valency of the permutation $\pi : (1, \ldots, n) \to (l_1^1, \ldots, l_\nu^{k_\nu})$ (see (9.37)). A typical partition is given by

$$\{1, \ldots, n\} = \{l_1^1, \ldots, l_1^{k_1}\} \cup \ldots \cup \{l_\nu^1, \ldots, l_\nu^{k_\nu}\}; \tag{12.25}$$

here ν $(= 1, \ldots, n)$ is the number of clusters; the superfix s in the notation l_α^s is the ordinal number of this number in the αth cluster; the numbers $k_1, \ldots, k_\nu$ are called the lengths of the clusters. The above recurrence relations enable us to express the truncated vacuum expectation values in terms of the ordinary ones; for example, for Bose operators we have

$$\langle A_1 \rangle^T = \langle A_1 \rangle_0, \ \langle A_1 A_2 \rangle^T = \langle A_1 A_2 \rangle_0 - \langle A_1 \rangle_0 \langle A_2 \rangle_0,$$
$$\langle A_1 A_2 A_3 \rangle^T = \langle A_1 A_2 A_3 \rangle_0 - \langle A_1 A_2 \rangle_0 \langle A_3 \rangle_0 - \langle A_1 A_3 \rangle_0 \langle A_2 \rangle_0 -$$
$$- \langle A_2 A_3 \rangle_0 \langle A_1 \rangle_0 + 2 \langle A_1 \rangle_0 \langle A_2 \rangle_0 \langle A_3 \rangle_0,$$

and so on. The *truncated Wightman functions* are defined in similar fashion:

$$w_{l_1 \ldots l_n}^{(\kappa_1 \ldots \kappa_n)T}(x_1, \ldots, x_n) \equiv \langle \phi_{l_1}^{(\kappa_1)}(x_1) \ldots \phi_{l_n}^{(\kappa_n)}(x_n) \rangle^T \tag{12.26}$$

(these are generalized functions in the variables $x_1, \ldots, x_n$).

Exercise 12.7. Let $A_j \in \mathcal{A}^{[\kappa_j]}$ and let $\hat{A}_j$ be defined by (12.1). Prove the relation

$$\langle \hat{A}_1 \ldots \hat{A}_n \rangle_0 = \sum \epsilon_F(\pi) \langle A_{l_1^1} A_{l_1^2} \rangle^T \ldots \langle A_{l_\nu^1} A_{l_\nu^2} \rangle^T; \tag{12.27}$$

here the summation is over all partitions of the indices $1, \ldots, n$ into clusters of length 2, so that $n = 2\nu$ (for odd n, the right hand side of (12.27) is zero by definition); the rest of the notation is the same as in (12.24).

We are interested in the properties of the truncated expectation value $\langle A_1(x_1) \ldots$ $\ldots A_n(x_n) \rangle^T$ of the non-local fields $A_j(x)$ defined by (12.3) for $A_j \in \mathcal{P}(M)$. It is clearly a $\mathcal{C}^\infty$-function in the variables $x_1, \ldots, x_n$. It is also translation-invariant and therefore depends only on the distances $x_1 - x_2, \ldots, x_{n-1} - x_n$. It follows from the definition of the vacuum expectation values $\langle A_1(x_1) \ldots A_n(x_n) \rangle_0$ that they are of polynomial growth in $x_1, \ldots, x_n$, therefore the truncated vacuum expectation values are (at worst) of polynomial growth and define tempered generalized functions; the Fourier transform is defined for them:

$$\langle A_1(x_1)\dots A_n(x_n)\rangle^T = \int \exp\Big(-i\sum_{j=1}^{n} p_j x_j\Big)\langle \tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle^T d_4p_1\dots d_4p_n.$$

We now give some less trivial results on truncated vacuum expectation values.

Lemma 12.3. *The truncated vacuum expectation values have the following properties:*

(a) *locality — if $A_k\in\mathcal{P}(\mathcal{O})$, $A_{k+1}\in\mathcal{P}(\mathcal{Q})$ and $\mathcal{O}\sim\mathcal{Q}$, then*

$$\langle A_1\dots A_kA_{k+1}\dots A_n\rangle^T = \pm\langle A_1\dots A_{k+1}A_k\dots A_n\rangle^T; \tag{12.28}$$

the sign on the right hand side is $-$ if A_k and A_{k+1} are fermionic operators and $+$ otherwise;

(b) *spectrum property — the support of $\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle^T$ is contained in the subset of $\mathbf{M}^n$ defined by the conditions*

$$p_1+\dots+p_n=0,\ p_1+\dots+p_k\in\overline{V}^+_\mu \quad \textit{for } k=1,\dots,n-1 \tag{12.29}$$

(μ is the mass gap).

■ (a) The assertion is trivial for $n=2$. We now proceed by induction. Suppose that the result holds for any set of m operators, where $m\le n-1$. We will show that (12.28) then holds. Consider a term in (12.24) with ν clusters, $\nu\ge 2$. Two cases are possible. If k and $k+1$ fall in the same cluster, then by the induction hypothesis, the analogous term in the expansion of type (12.24) for $\langle A_1\dots A_{k+1}A_k\dots A_n\rangle_0$ differs only by $\pm$ signs. If, on the other hand, k and $k+1$ are in different clusters, then, clearly, the same term (perhaps with a sign change) occurs in the expansion of type (12.24) for $\langle A_1\dots A_{k+1}A_k\dots A_n\rangle_0$. Thus we have

$$\langle A_1\dots A_kA_{k+1}\dots A_n\rangle_0 = \langle A_1\dots A_kA_{k+1}\dots A_n\rangle^T+\dots,$$
$$\langle A_1\dots A_{k+1}A_k\dots A_n\rangle_0 = \langle A_1\dots A_{k+1}A_k\dots A_n\rangle^T\pm\dots,$$

where the dots denote the same expression. Formula (12.28) now follows trivially from the locality axiom.

(b) The fact that $\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle^T$ is confined to the plane $p_1+\dots+p_n=0$ follows from translation invariance. We now proceed by induction. Since the assertion holds trivially for $n=1$, we are required to prove that for $k=1,\dots,n-1$,

$$\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle^T=0 \quad \text{for } p_1+\dots+p_k\notin\overline{V}^+_\mu, \tag{12.30}$$

if this property holds for truncated vacuum expectation values of $<n$ non-local fields. Since the only state in the region of momenta $p\notin\overline{V}^+_\mu$ is the vacuum, we have the following representation in the region (12.30):

$$\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle_0 = \langle\tilde{A}_1(p_1)\dots\tilde{A}_k(p_k)\rangle_0\langle\tilde{A}_{k+1}(p_{k+1})\dots\tilde{A}_n(p_n)\rangle_0. \tag{12.31}$$

We replace each factor on the right hand side by its representation in terms of the truncated vacuum expectation values; as a result, $\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle_0$ can be written in the form of a sum of type (12.24):

$$\langle\tilde{A}_1(p_1)\dots\tilde{A}_n(p_n)\rangle_0 = {\sum}'\epsilon_F(\pi)\prod_{\alpha=1}^{\nu}\langle\tilde{A}_{l^1_\alpha}(p_{l^1_\alpha})\dots\tilde{A}^{k_\alpha}_{l_\alpha}(p^{k_\alpha}_{l_\alpha})\rangle^T, \tag{12.32}$$

only now the sum is taken over all partitions of the indices $1,\dots,n$ into clusters (with the number of clusters $\nu\ge2$), where each cluster $\{l^1_{\alpha_1},\dots,l^{k_\alpha}_\alpha\}$ lies either entirely in $\{1,\dots,k\}$, or in $\{k+1,\dots,n\}$.

We claim that (12.32) still holds if we allow the sum to be extended to all partitions of the indices $1,\ldots,n$ with number of clusters $\nu \geq 2$. To this end, we verify that any extra term vanishes in the region (12.30). Suppose that this term corresponds to the partition (12.25); we then have

$$p_k + \ldots + p_n = \sum_{\alpha} (p_{l_\alpha}^{s_\alpha} + p_{l_\alpha}^{s_\alpha+1} + \ldots + p_{l_\alpha}^{k_\alpha}); \tag{12.33}$$

here the summation is taken only over those clusters of the partition (12.25) that have non-empty intersection $\{l_\alpha^{s_\alpha}, l_\alpha^{s_\alpha+1}, \ldots, l_\alpha^{k_\alpha}\}$ with $\{k+1,\ldots,n\}$. Clearly $s_\alpha > 1$ for at least one such intersection (otherwise this would contradict the hypothesis that the partition (12.25) is absent in (12.32)). As follows from (12.33), the region where (12.30) is satisfied is contained in the union of the regions $\bigcup_\alpha \mathcal{O}_\alpha$, where the $\mathcal{O}_\alpha$ are defined by the condition

$$\begin{aligned} p_\alpha^1 + \ldots + p_\alpha^{k_\alpha} &\notin \overline{V}^+ \quad \text{for } s_\alpha = 1, \\ p_\alpha^{s_\alpha} + \ldots + p_\alpha^{k_\alpha} &\notin \overline{V}_\mu^+ \quad \text{for } s_\alpha > 1. \end{aligned}$$

By the induction hypothesis $\langle \tilde{A}_{l_\alpha^{s_\alpha}}(p_{l_\alpha^{s_\alpha}}) \ldots \tilde{A}_{l_\alpha^{k_\alpha}}(p_{l_\alpha^{k_\alpha}}) \rangle^T = 0$ in $\mathcal{O}_\alpha$; hence the term corresponding to our partition (12.25) vanishes in the region for which (12.30) holds.

We have shown that in the region (12.30) we have the expansion (12.32), where the summation is over all partitions into $\nu \geq 2$ clusters. On the other hand, according to (12.24) the difference between the left and right hand sides of (12.32) is identically equal to $\langle \tilde{A}_1(p_1) \ldots \tilde{A}_n(p_n) \rangle^T$. Consequently, $\langle \tilde{A}_1(p_1) \ldots \tilde{A}_n(p_n) \rangle^T = 0$ in the region (12.30), which completes the induction argument. ∎

We consider the case where each operator A_j is a smoothed monomial in the fields

$$A_j = \int \phi(\xi_{j,1}) \ldots \phi(\xi_{j,r_j}) f_j(\xi_{j,1}, \ldots, \xi_{j,r_j}) d\xi_{j,1} \ldots d\xi_{j,r_j}. \tag{12.34}$$

We can write the truncated vacuum expectation value in the form

$$\langle A_1 \ldots A_n \rangle^T = \int v(\xi_{1,1}, \ldots, \xi_{1,r_1} | \ldots | \xi_{n,1}, \ldots, \xi_{n,r_n}) \times \\ \times \prod_{j=1}^{n} f_j(\xi_{j,1}, \ldots, \xi_{j,r_j}) d\xi_{j,1} \ldots d\xi_{j,r_j}, \tag{12.35}$$

where v is a generalized function in $\mathcal{S}'(\mathrm{M}^{r_1+\ldots+r_n})$, called the *n-cluster truncated Wightman function*. In particular, if all the r_j are equal to unity, then the n-cluster truncated Wightman function $v(x_1|,\ldots|x_n)$ is the same as the truncated Wightman function $w^{[n]T}(x_1,\ldots,x_n)$ introduced earlier. The above properties of the truncated vacuum expectation values can be expressed in the language of the generalized functions v, as is clear from the following exercise.

Exercise 12.8. Prove the following properties for the generalized functions v (12.35):

(a) locality: $v(\ldots|\xi_{k,1},\ldots,\xi_{k,r_k}|\xi_{k+1,1},\ldots,\xi_{k+1,r_{k+1}}|\ldots) = \pm v(\ldots|\xi_{k+1,1},\ldots,\xi_{k+1,r_{k+1}}|\xi_{k,1},\ldots$ $\ldots,\xi_{k,r_k}|\ldots)$, if $(\xi_{k,\alpha} - \xi_{k+1,\beta})^2 < 0$ for $\alpha = 1,\ldots,r_k$; $\beta = 1,\ldots,r_{k+1}$;

(b) spectrum property: the support of $\tilde{v}(p_{1,1}\ldots|\ldots p_{n,r_n})$ is contained in the intersection of the sets

$$\sum_{j=1}^{n}\sum_{\alpha=1}^{r_j} p_{j,\alpha} = 0, \quad \sum_{j=1}^{k}\sum_{\alpha=1}^{r_j} p_{j,\alpha} \in \overline{V}_\mu^+, \quad \text{where } k = 1,\ldots,n-1.$$

The cluster property is formulated in terms of truncated vacuum expectation values in the following way.

Exercise 12.9. Prove that $\langle A_1(x_1)A_2(x_2)\rangle^T \to 0$ for $|\mathbf{x}_1 - \mathbf{x}_2| \to \infty$ (x_1^0, x_2^0 are fixed).

B. STRENGTHENED CLUSTER PROPERTY

The presence of a mass gap enables us essentially to strengthen the cluster property.

Proposition 12.4. *For any $A_1, \ldots, A_n \in \mathcal{P}(\mathbf{M})$, the truncated vacuum expectation value $\langle A_1(x_1) \ldots A_n(x_n)\rangle^T$ is an element of $\mathcal{S}(\mathbf{R}^{3(n-1)})$ with respect to the variables $\mathbf{x}_1 - \mathbf{x}_2, \ldots, \mathbf{x}_{n-1} - \mathbf{x}_n$, which is C^∞-dependent on $x_1^0 - x_2^0, \ldots, x_{n-1}^0 - x_n^0$. Furthermore, for any differential polynomial $p(\partial/\partial x)$, any α in the interval $0 < \alpha < 1$ and any natural number N there exists a number c such that*

$$|d(x)^N p(\partial/\partial x)\langle A_1(x_1) \ldots A_n(x_n)\rangle^T \leq c \tag{12.36}$$

for all x in the cone Q:

$$Q = \{x \in \mathbf{M}^n : \ |x_i^0 - x_j^0| \leq \alpha |\mathbf{x}_i - \mathbf{x}_j|, \ i, j = 1, \ldots, n\}; \tag{12.37}$$

here

$$d(x) \equiv \sup_{i<j} |x_i - x_j| \tag{12.38}$$

is the diameter (in the Euclidean metric) of the configuration $x \equiv (x_1, \ldots, x_n)$ of points in $\mathbf{M}$.

We break up the proof of this proposition into separate lemmas.

The first lemma is purely geometrical. We denote by I a non-empty proper subset of the indices $\{1, \ldots, n\}$; J is the (non-empty) complement of I in $\{1, \ldots, n\}$; $\delta_I(x)$ is the distance (in the Euclidean metric) between the sets $\{x_i\}_{i \in I}$ and $\{x_j\}_{j \in J}$:

$$\delta_I(x) = \inf_{i \in I, j \in J} |x_i - x_j|. \tag{12.39}$$

Lemma 12.5. *The sets*

$$Q_I = \left\{x \in Q : \ \delta_I(x) \geq \frac{1}{n} d(x)\right\} \tag{12.40}$$

cover the cone Q. There exists a number $\sigma > 0$ such that the points $a + x_i$ and $b + x_j$ are separated by a spacelike interval for any $x \in Q_I$, $i \in I$, $j \in J$, $a, b \in \mathbf{M}$ with $|a| < \sigma d(x), |b| < \sigma d(x)$

■ Suppose that the diameter $d(x)$ is equal to the distance between x_i and x_j. We project all the x_k onto the vector $x_i - x_j$. There are at most n points on the interval and the entire interval splits up into at most $n-1$ intervals. Hence there is an interval of length $\geq d(x)/n$. The plane perpendicular to $x_i - x_j$ and passing through this interval partitions the points $x_1, \ldots, x_n$ into two halves which define the required partitioning $I \cup J$ of the set of indices $\{1, \ldots, n\}$. We have thus proved that the Q_I cover Q.

For $x \in Q_I$ with $d(x) \neq 0$, $i \in I$, $j \in J$, $a, b \in \mathbf{M}$ we have

$$\begin{aligned}(a + x_i - b - x_j)^2 &= (x_i - x_j)^2 + 2(x_i - x_j)(a - b) + (a - b)^2 \leq \\ &\leq (\alpha^2 - 1)|x_i - x_j|^2 + 2|x_i - x_j|\,|a - b| + |a - b|^2 \leq \\ &\leq -\frac{1 - \alpha^2}{1 + \alpha^2}\frac{1}{n^2} d^2(x) + 2d(x)(|a| + |b|) + (|a| + |b|)^2.\end{aligned}$$

It is clear that the right hand side of this relation is negative if $|a|/d(x)$ and $|b|/d(x)$ are sufficiently small. ■

In what follows we suppose that the operators A_j are smoothed by monomials of the fields and hence, we can use the language of n-cluster truncated Wightman functions v (12.35).

Lemma 12.6. *Let the indices in each of the groups $I = \{l_1, \dots, l_k\}$ and $J = \{l_{k+1}, \dots, l_n\}$ of the partition of the set $\{1, \dots, n\}$ be arranged in increasing order and let π and π' be the permutations taking $(1, \dots, n)$ to $(l_1, \dots, l_k, l_{k+1}, \dots, l_n\}$ and $(l_{k+1}, \dots, l_n, l_1, \dots, l_k)$ respectively. Then for any $f \in S(\mathrm{M}^{r_1+\dots+r_n})$ and $N > 0$, there exists $c_N(f)$ such that*

$$\Big|\int (v(\xi) - \epsilon_F(\pi) v_\pi(\xi)) f_x(\xi) d\xi\Big| \le c_N(f) d(x)^{-N}, \tag{12.41a}$$

$$\Big|\int (v(\xi) - \epsilon_F(\pi') v_{\pi'}(\xi)) f_x(\xi) d\xi\Big| \le c_N(f) d(x)^{-N} \tag{12.41b}$$

for all $x \in Q_I$; here by definition

$$v_\pi(\xi) = v(\xi_{\pi 1,1}, \dots, \xi_{\pi 1, r_{\pi 1}} | \dots | \xi_{\pi n,1}, \dots, \xi_{\pi n, r_{\pi n}}),$$
$$f_x(\xi) = f(\xi_{1,1} - x_1, \dots, \xi_{1,r_1} - x_1 | \dots | \xi_{n,1} - x_n, \dots, \xi_{n,r_n} - x_n).$$

■ Let $|\xi|^2 = \sum_{k=1}^n \sum_{\alpha=1}^{r_n} |\xi_{k,\alpha}|^2$ and let $\xi + \bar{x}$ be the point $\xi' \in \mathrm{M}^{r_1+\dots+r_n}$ with coordinates $\xi'_{i,\alpha} = \xi_{i,\alpha} + x_i$. By Lemma 12.5, $(\xi'_{i,\alpha} - \xi'_{j,\beta})^2 < 0$ for $|\xi| < \sigma d(x)$ and $d(x) \neq 0$, where $i \in I$, $j \in J$, $\alpha = 1, \dots, r_i$, $\beta = 1, \dots, r_j$. It follows from the locality property of the generalized function v (Exercise 12.8) that the support of $v(\xi + \bar{x}) - \epsilon_F(\pi) v_\pi(\xi + \bar{x})$ lies in the set $|\xi| \ge \sigma d(x)$; hence the left hand side of (12.41a) is equal to

$$\int (v(\xi + \bar{x}) - \epsilon_F(\pi) v_\pi(\xi + x)) s(|\xi|/d(x)) f(\xi) d\xi; \tag{12.42}$$

here $s(t)$ is a smooth function of the real variable t equal to zero for $|t| < 1/(2\sigma)$ and one for $|t| \ge 1/\sigma$. The generalized function $v(\xi) - \epsilon_F(\pi) v_\pi(\xi)$ can be expressed as a sum of derivatives of continuous functions of polynomial growth; as a result, the absolute value of (12.42) is majorized by the expression

$$\mathrm{const} \cdot \sum_{|\alpha|+|\beta| \le M} \int_{|\xi| \ge \sigma d(x)/2} (1 + |\xi| + d(x))^m |D^\alpha f(\xi)| \, |D^\beta u(|\xi|/d(x)|) d\xi.$$

Since $D^\alpha f(\xi)$ decreases more rapidly than any negative power of $|\xi|$, this expression is in turn majorized by a quantity of the form $c_N(f) d(x)^{-N}$ for any N. This proves the estimate (12.41a). The second estimate (12.41b) is entirely analogous. ■

Lemma 12.7. *Under the same conditions as in the previous lemma, the following estimate holds:*

$$\Big|\int v_\pi(\xi) f_x(\xi) d\xi\Big| \le c'_N(f) d(x)^{-N}, \ x \in Q_I. \tag{12.43}$$

■ Let $\chi(p) \in \mathcal{O}_M(\mathrm{M})$, where $\chi(p) = 0$ in a neighbourhood of $\overline{V}_\mu^-$ and $\chi(p) = 1$ in a neighbourhood of $\overline{V}_\mu^+$. We define $f'(\xi)$ in terms of the Fourier transform:

$$\tilde{f}'(p_{1,1}, \dots, p_{n,r_n}) = \chi\Big(-\sum_{j=1}^k \sum_{\alpha=1}^{r_{\pi j}} p_{\pi j, \alpha}\Big) \tilde{f}(p_{1,1}, \dots, p_{n,r_n}).$$

It then follows from the property of the support of v (Exercise 12.8) that

$$\int v_\pi(\xi) f'_x(\xi) d\xi = \int v_\pi(\xi) f_x(\xi) d\xi,$$
$$\int v_{\pi'}(\xi) f'_x(\xi) d\xi = 0.$$

Replacing f by f' in Lemma 12.6 and using (12.41) we obtain the estimate (12.43) for $c'_N(f) = 2c_N(f')$. ■

Lemma 12.8. *For any monomials A_j (12.34), the truncated expectation value $\langle A_1(x_1) \dots A_n(x_n) \rangle^T$, as well as its derivatives of any order with respect to x, decreases more rapidly than any negative power of $d(x)$ as $d(x) \to \infty$ in the cone Q.**

■ It follows from (12.35) that the truncated vacuum expectation value $\langle A_1(x_1) \dots A_n(x_n) \rangle^T$ along with any derivative of it with respect to x can be written in the form $\int v(\xi) f_x(\xi) d\xi$ for a suitable test function $f(\xi)$. According to Lemmas 12.6 and 12.7, the expressions $\int (v(\xi) - \epsilon_F(\pi) v_\pi(\xi)) f_x(\xi) d\xi$ and $\int v_\pi(\xi) f_x(\xi) d\xi$ and hence, $\int v(\xi) f_x(\xi) d\xi$ tend to zero more rapidly than any negative power of $d(x)$ for $x \in Q_I$, $d(x) \to \infty$. Since the sets Q_I cover Q, this proves the lemma. ■

Proposition 12.4 now follows immediately from Lemma 12.8.

Corollary 12.9. For any $f_j \in \mathcal{S}(\mathbf{M})$ the generalized function

$$\int \tilde{w}^T(p_1, \dots, p_n) \tilde{f}_j(-p_1) \dots \tilde{f}_n(-p_n) d_4 p_1 \prod_{j=2}^{n} d_1 p_j^0 \tag{12.44}$$

is in fact a function in $\mathcal{S}(\mathbf{R}^{3(n-1)})$ with respect to the variables $\mathbf{p}_2, \dots, \mathbf{p}_n$.

According to Exercise 12.4, a truncated vacuum expectation value of the form

$$\int w^T(x_1 - y_1, \dots, x_n - y_n) f_1(y_1) \dots f_n(y_n) d^4 y_1 \dots d^4 y_n \tag{12.45}$$

depends only on the distances $x_1 - x_2, \dots, x_{n-1} - x_n$ and is a function in $\mathcal{S}(\mathbf{R}^{3(n-1)})$ with respect to $\mathbf{x}_1 - \mathbf{x}_2, \dots, \mathbf{x}_{n-1} - \mathbf{x}_n$. Setting $x_1^0 = \dots = x_n^0 = 0$ and writing (12.45) in terms of the Fourier transform, we obtain the result stated in the corollary.

C. SPREAD OF RELATIVISTIC WAVE PACKETS

We require a further auxiliary result touching on the behaviour of the negative-frequency solutions $F(x)$ of the Klein-Gordon equation with positive mass m at large values of time:

$$F(x) = \int e^{-i(\omega x^0 - \mathbf{p}\mathbf{x})} h(\mathbf{p}) d_3 p, \quad \omega \equiv \sqrt{m^2 + \mathbf{p}^2}; \tag{12.46}$$

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* If we assume that there is a minimum positive mass m, we can show in fact that the truncated expectation values decrease exponentially at spacelike separation of the arguments; here the exponent depends on m; (we must suppose, however, that the test functions $f(x_1, \dots, x_n)$ in the definition (8.8) of the elements A of the polynomial algebra are of class $\mathcal{D}(\mathbf{M}^n)$). This is in accordance with the fact that in a theory in which only particles with positive masses are considered, all the forces are short-range (of Yukawa potential type $e^{-\mu r}/r$). For a theory in which there are particles with zero mass, the convergence to zero of the truncated vacuum expectation values can be inversely proportional to the square of the distance, as for Coulomb forces (see Araki et al., 1962).

here $h(\mathbf{p})$ is a "sufficiently good" function. The analogous result for positive-frequency solutions clearly holds.

Lemma 12.10. *Let k_0 be a fixed natural number. There exist natural numbers L, M and a constant c such that the following estimate holds for all solutions of the Klein-Gordon equation of the form (12.46) for $h \in \mathcal{S}(\mathbf{R}^3)$:*

$$\sup_{\mathbf{x}}(|\mathbf{x}|^k |F(t,\mathbf{x})|) \leq c\,\|h\|_{L,M}(1+|t|)^{-3/2+k}, \quad k = 0, 1, \ldots, k_0; \tag{12.47}$$

$$\int |F(t,\mathbf{x})| d^3x \leq c\,\|h\|_{L,M}(1+|t|)^{3/2}. \tag{12.48}$$

Generally speaking, these estimates hold if it is supposed that $h \in \mathcal{S}_{L,M}(\mathbf{R}^3)$ (see §1.1.C).

■ We introduce the continuous bounded solution of the Klein-Gordon equation:

$$T(x) = \int e^{i(\omega x^0 - \mathbf{p}\mathbf{x})} \frac{1}{\omega(1+\omega)^3} d_3p.$$

Then $F(x)$ is represented as a three-dimensional convolution

$$F(t,\mathbf{x}) = T(t,\mathbf{x}) * v(\mathbf{x}), \tag{12.49}$$

where

$$v(x) = \int e^{i\mathbf{p}\mathbf{x}} \omega(1+\omega)^3 h(\mathbf{p}) d_3p.$$

Thus the problem reduces to obtaining estimates for $T(x)$.

We write down the representation

$$T(x) = \int \Big(\int_0^\infty e^{-(1+\omega)\alpha} \alpha^2 dx\Big) e^{-(\omega x^0 - \mathbf{p}\mathbf{x})} \frac{d_3p}{2\omega};$$

here we can clearly change the order of integration:

$$\begin{aligned} T(x) = \int_0^\infty \alpha^2 e^{-\alpha} \Big\{ \int e^{-i[\omega(x^0 - i\alpha) - \mathbf{p}\mathbf{x}]} \frac{d_3p}{2\omega} \Big\} &= \text{const} \cdot \int_0^\infty D_m^{(-)}(x - i\alpha e_0) \alpha^2 e^{-\alpha} d\alpha = \\ &= \text{const} \cdot (|\mathbf{x}|^2 - (x^0 - i\alpha)^2)^{-1/2} K_1(m\sqrt{|\mathbf{x}|^2 - (x^0 - i\alpha)^2}) \alpha^2 e^{-\alpha} d\alpha, \end{aligned} \tag{12.50}$$

where e_0 is the unit vector along the time axis; in the latter equality we can use an explicit formula for $D_m^{(-)}(x)$ (see Appendix F). Using the well known asymptotic estimates for the MacDonald function of the first kind $K_1(z)$ for $\operatorname{Re} z > 0$:

$$zK_1(z) \to 1 \quad \text{for } z \to 0, \quad \sqrt{z}e^z K_1(z) \to \sqrt{\pi/2} \quad \text{for } z \to \infty$$

(see, for example [B3], Vol.2,§7.4.1), we obtain the estimate

$$|K_1(z)| \leq \text{const} \cdot (|z|^{-1/2} + |z|^{-1}) e^{-\operatorname{Re} z} \quad \text{for } \operatorname{Re} z > 0. \tag{12.51}$$

Since for $s > 0$ and $\operatorname{Re} z > 0$

$$|z|^{-s} e^{-\operatorname{Re} z} \leq |\operatorname{Im}(z^2)|^{-s} (2\operatorname{Re} z)^s e^{-\operatorname{Re} z} \leq \text{const} \cdot |\operatorname{Im}(z^2)|^{-s}$$

(where the constant depends on s), it follows from (12.51) that

$$|z^{-1} K_1(z)| \leq \text{const} \cdot \{|\operatorname{Im}(z^2)|^{-3/2} + |\operatorname{Im}(z^2)|^{-2}\} \quad \text{for } \operatorname{Re} z > 0. \tag{12.52}$$

This enables us to estimate the integral (12.50):

$$|T(t,\mathbf{x})| \leq \text{const} \cdot \int_0^\infty (|\alpha t|^{-3/2} + |\alpha t|^{-2})\alpha^2 e^{-\alpha} d\alpha \leq \text{const} \cdot (|t|^{-3/2} + |t|^{-2}).$$

Since $T(t,\mathbf{x})$ is uniformly bounded, it follows that

$$|T(t,\mathbf{x})| \leq \text{const} \cdot (1+|t|)^{-3/2}. \tag{12.53}$$

A stronger bound holds in the region $|\mathbf{x}| > 2|t|$. We set $z = m\sqrt{|\mathbf{x}|^2 - (t-i\alpha)^2}$. Then $|z| \geq$ $\geq \text{const} \cdot \alpha$ and $\operatorname{Re} z > \sqrt{\operatorname{Re}(z^2)} > \frac{m}{2}|\mathbf{x}|$ when $|\mathbf{x}| > 2|t|$, so that (12.51) yields:

$$\left|\frac{1}{z}K_1(z)\right| \leq \text{const} \cdot (\alpha^{-3/2} + \alpha^{-2})e^{-m|\mathbf{x}|/2} \quad \text{for } |\mathbf{x}| > 2|t|,$$

whence

$$|T(t,\mathbf{x})| \leq \text{const} \cdot \int_0^\infty (\alpha^{-3/2} + \alpha^{-2})e^{-m|\mathbf{x}|/2}\alpha^2 e^{-\alpha} d\alpha \leq \text{const} \cdot e^{-m|\mathbf{x}|/2} \quad \text{for } |\mathbf{x}| > 2|t|. \tag{12.54}$$

Hence and from (12.53) it clearly follows that

$$|\mathbf{x}|^k |T(t,\mathbf{x})| \leq \text{const} \cdot (1+|t|)^{-3/2+k}, \ 0 \leq k \leq k_0. \tag{12.55}$$

Similarly, by combining (12.54) and (12.55) we obtain

$$\begin{aligned}\int |T(t,\mathbf{x})| d^3x &\leq \text{const} \cdot \int_{|\mathbf{x}| \geq 2|t|} e^{-m|\mathbf{x}|/2} d^3x + \text{const} \cdot \int_{|\mathbf{x}| \leq 2|t|} (1+|t|)^{-3/2} d^3x \leq \\ &\leq \quad \text{const} \cdot (1+|t|)^{3/2}.\end{aligned} \tag{12.56}$$

It is not difficult to estimate the convolution (12.49). By applying the inequalities

$$|\mathbf{x}|^k \leq 2^k(|\mathbf{x}-\mathbf{y}|^k + |\mathbf{y}|^k) \leq 2^k(1+|\mathbf{x}-\mathbf{y}|^k)(1+|\mathbf{y}|^k),$$

we obtain with the help of (12.55)

$$\sup_{\mathbf{x}}(|\mathbf{x}|^k |F(t,\mathbf{x})|) \leq \text{const} \cdot (\int (1+|\mathbf{y}|^k|v(\mathbf{y})|d^3y)\,(1+|t|)^{-3/2+k}$$

for $0 \leq k \leq k_0$; similarly it follows from (12.56) that

$$\int |F(t,\mathbf{x})| d^3x \leq \text{const}(\int |v(\mathbf{y})| d^3y)\,(1+|t|)^{3/2}. \quad \blacksquare$$

We give the following supplement to Lemma 12.10 as an exercise.

Exercise 12.10. Let $h(\mathbf{p})$ be a function in $D(\mathbf{R}^3)$, $K \subset \mathbf{R}^3$ a neighbourhood of the set $\{\omega^{-1}(\mathbf{p})\mathbf{p} : \mathbf{p} \in \operatorname{supp} h\}$ and let $F(x)$ be defined by (12.46). Prove that for any $l, m \in \overline{\mathbf{Z}}_+$ there exist numbers $c_{l,m} \geq 0$ such that

$$\|F\|_{l,m}^Q \leq c_{l,m}, \tag{12.57}$$

where

$$Q = \{x \in \mathbf{M} : \ \mathbf{x} \notin x^0 K\}. \tag{12.58}$$

[Hint: Let $u \in Q$, $|u| = 1$; then

$$F(\lambda u) = \int e^{i\lambda s}\phi_u(s) ds,$$

where $\phi_u(s)$ is a function of s of class $\mathcal{S}(\mathbf{R})$ that is continuously dependent on u with respect to the topology of $\mathcal{S}(\mathbf{R})$; deduce that $|\lambda^m f(\lambda u)| \leq c_m$ uniformly in u; the estimate for the derivatives of F is similar.]

We use Lemma 12.10 in the following situation.

Lemma 12.11. *Let $A_1, \dots, A_n$ be a fixed collection of operators in $\mathcal{P}(\mathbf{M})$ and let $m_1, \dots, m_n$ be a set of positive masses (where $n \geq 2$). Then for any polynomial $p(x)$ of $x \in \mathbf{M}$, any differential polynomials $q(\partial/\partial x)$, $q'(\partial/\partial x)$ and any compactum $B \subset L_+^\uparrow$ there exists a number c such that the following estimate holds for all $t \in \mathbf{R}$, $\Lambda \in B$:*

$$\left|\int_{x_1^0=\dots=x_n^0=t} p(x)\Big(q\Big(\frac{\partial}{\partial x}\Big)\prod_{j=1}^n D_{m_j}(x_j)\Big)q'\Big(\frac{\partial}{\partial x}\Big)\langle A_1(\Lambda x_1)\dots A_n(\Lambda x_n)\rangle^T d\mathbf{x}_1\dots d\mathbf{x}_n\right| \leq$$
$$\leq c(1+|t|)^{-3(n-2)/2+k}; \qquad (12.59)$$

here k is the degree of the polynomial $p(x)$.

■ Note that the integral in (12.59) (understood in the sense of generalized functions) exists by virtue of the compactness of the support of $D_m(t, \mathbf{x})$ in $\mathbf{x}$. Clearly it is enough to consider the case when $q(\partial/\partial x)$ has the form of a product $\prod_{j=1}^n q_j(\partial/\partial x_j)$ of differential polynomials in each variable x_j. We fix a natural number N and introduce the functions

$$F_j(t, \mathbf{x}_j) = (1-\Delta_j)^{-N} q_j\Big(\frac{\partial}{\partial x_j}\Big) D_{m_j}(x_j),$$

where Δ_j is the Laplace operator in the variable $\mathbf{x}_j$. The function $F_j(x)$ is clearly a solution of the Klein-Gordon equation (for $m = m_j$); for sufficiently large N its negative-frequency part has the form (12.46) with $h(\mathbf{p}) \in \mathcal{S}_{L,M}(\mathbf{R}^3)$. A similar representation holds for the positive-frequency part of $F_j(x)$. Therefore each of the $F_j(x)$ satisfies estimates (12.47), (12.48). The left hand side of (12.59) can be rewritten in the form

$$\left|\int\Big(\prod_{j=1}^n F_j(x_j)\Big)u(x_1, \dots, x_n; \Lambda)d\mathbf{x}_1\dots d\mathbf{x}_n\Big|_{x_1^0=\dots=x_n^0=t}\right|, \qquad (12.60)$$

where

$$u(x_1, \dots x_n; \Lambda) = \prod_{j=1}^n (1-\Delta_j)^N p(x) q'\Big(\frac{\partial}{\partial x}\Big)\langle A_1(\Lambda x_1)\dots A_n(\Lambda x_n)\rangle^T.$$

Taking into account the translation invariance of the truncated expectation value, we can write down the representation for the restriction of $u(x_1, \dots, x_n; \Lambda)$ to the plane $x_1^0 = \dots = x_n^0 = t$:

$$u(x_1, \dots, x_n; \Lambda)\big|_{x_1^0=\dots=x_n^0=t} = \sum_{|\alpha|\leq k} t^{\alpha_0}\prod_{j=1}^3 (x_1^j)^{\alpha_j} v_\alpha(\mathbf{x}_1-\mathbf{x}_2, \dots, \mathbf{x}_{n-1}-\mathbf{x}_n; \Lambda), \qquad (12.61)$$

where the v_α are functions of the distances $\mathbf{x}_1-\mathbf{x}_2, \dots, \mathbf{x}_{n-1}-\mathbf{x}_n$ and Λ only. According to Proposition 12.4, $v_\alpha \in \mathcal{S}(\mathbf{R}^{3(n-1)})$ as a function of the variables $\mathbf{x}_1-\mathbf{x}_2, \dots, \mathbf{x}_{n-1}-\mathbf{x}_n$. In every case, Proposition 12.4 gives the following bound for v_α which is valid for all $\Lambda \in B$:

$$\int |v_\alpha(\xi_1, \dots, \xi_{n-1}, \Lambda)| d\xi_1 \dots \xi_{n-1} \leq c_1 \quad \text{for } \Lambda \in B. \qquad (12.62)$$

Substituting (12.61) into (12.60) enables us to majorize (12.60) by the expression

$$\int \sum_{|\alpha|\leq k} |t|^{\alpha_0}|x_1|^{\alpha_1+\alpha_2+\alpha_3}|F_1(t, \mathbf{x}_1)|\prod_{j=2}^{n-2}|F_j(t, \mathbf{x}_j)|\,|F_n(t, \mathbf{x}_n)|\times$$
$$\times|v_\alpha(\mathbf{x}_1-\mathbf{x}_2, \dots, \mathbf{x}_{n-1}-\mathbf{x}_n; \Lambda)|d\mathbf{x}_1\dots d\mathbf{x}_n. \qquad (12.63)$$

We now use the estimate (12.47) for the function $F_j(t, \mathbf{x}_j)$ with $j = 1, \dots, n-1$ and the estimate (12.48) for $F_n(t, \mathbf{x}_n)$. As a result, for $\Lambda \in B$, (12.59) is majorized by the expression

$$A(1+|t|)^{-3(n-1)/2+k}\int |F_n(t, \mathbf{x}_n)|\,|v_\alpha(\xi_1, \dots, \xi_{n-1}; \Lambda)|d\xi_1\dots d\xi_{n-1}d\mathbf{x}_n \leq A'(1+|t|)^{-3(n-2)/2+k},$$

which proves the estimate (12.59). ■

The results of the next exercise are an immediate corollary of Lemma 12.8.

Exercise 12.11. For $A_j \in \mathcal{P}(\mathrm{M})$, $t \in \mathbf{R}$, $\Lambda \in L_+^\uparrow$, set

$$A_j^{t,\Lambda} = \int_{x^0=t} A_j(\Lambda x)\overleftrightarrow{\partial}_0 D_{m_j}(x)d^3x. \tag{12.64}$$

(a) Prove that for any $A_1, \dots, A_n \in \mathcal{P}(\mathrm{M})$ $(n = 2, 3, \dots)$ and any compactum $B \subset L_+^\uparrow$ we have the estimate

$$|\langle A_1^{t,\Lambda} \dots A_n^{t,\Lambda}\rangle^T| \leq \mathrm{const}(1+|t|)^{-3(n-2)/2} \quad \text{for } \Lambda \in B; \tag{12.65}$$

the same estimates hold if some of the $A_j^{t,\Lambda}$ are replaced by $\frac{\partial}{\partial t}A_j^{t,\Lambda}$.

(b) Let

$$\Lambda = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}, \quad \eta \in \mathbf{R}. \tag{12.66}$$

Prove that if k $(1 \leq k \leq n)$ of the operators $A_j^{t,\Lambda}$ in the truncated vacuum expectation value $\langle A_1^{t,\Lambda} \dots A_n^{t,\Lambda}\rangle^T$ are replaced by the operators $\frac{\partial}{\partial \eta}A_j^{t,\Lambda}$, then the resultant expression is majorized in modulus by the expression $\mathrm{const}\cdot(1+|t|)^{-3(n-2)/2+k}$ uniformly in η on any fixed compactum in $\mathbf{R}$.

D. Proof of the Main Result

We now give the proof of Theorem 12.1.

■ (a) We set $\Phi^t = A_1^t \dots A_n^t \Psi_0$. For the limit (12.8) to exist, it suffices that the estimate

$$\left\|\frac{d}{dt}\Phi^t\right\|^2 \leq \mathrm{const}\cdot(1+|t|)^{-3} \tag{12.67}$$

holds, since then $\int \|\frac{d}{dt}\Phi^t\|dt < \infty$ and we can set, for example, $\Phi^{\mathrm{out}} = \Phi^0 + \int_0^\infty \left(\frac{d}{dt}\Phi^t\right)dt$. We have

$$\left\|\frac{d}{dt}\Phi^t\right\|^2 = \left\langle \frac{d}{dt}(A_n^{*t} \dots A_1^{*t})\frac{d}{dt}(A_1^t \dots A_n^t)\right\rangle_0.$$

We expand the right hand side in a sum of products of truncated vacuum expectation values formed from the operators A_j^t, A_j^{*t}. Clearly all the (truncated) vacuum expectation values of the form $\langle A_j^{(*)t}\rangle^T$ are equal to zero by virtue of the conditions $A_j\Psi_0 \in \mathfrak{H}^{[\kappa]}$, $A_j^{(*)}\Psi_0 \in \mathfrak{H}^{[\bar\kappa]}$, therefore it suffices to consider clusters of length ≥ 2. We begin by considering an arbitrary term in the expansion consisting only of truncated vacuum expectation values of length 2. Clearly, at least one of the factors will be of the form $\langle \Psi_0, \dots, \frac{d}{dt}A_j^{(*)t}\Psi_0\rangle$ and will be equal to zero by virtue of (12.7), therefore such terms give zero contribution. It remains to consider the terms containing at least two clusters of length 3 or at least one cluster of length ≥ 4. According to the estimate (12.65) (for $\Lambda = 1$), such terms are bounded in modulus by the expression $(1+|t|)^{-3}$. This proves (12.67).

(b) The existence of the operators Ω^{ex} with the required properties (apart from Lorentz-invariance) is an immediate corollary of the formula

$$\langle \Phi^{\mathrm{ex}}(A_1, \dots, A_k),\ \Phi^{\mathrm{ex}}(A'_1, \dots, A'_n)\rangle = \langle \Phi(A_1, \dots, A_k),\ \Phi(A'_1, \dots, A'_n)\rangle \tag{12.68}$$

for any $A_i \in \mathcal{A}^{[\kappa_i]}$, $A'_j \in \mathcal{A}^{[\kappa_j]}$; it is therefore enough to prove this formula. The left hand side is the limit as $t \to \mp\infty$ of the expression

$$\langle A_k^{*t} \dots A_1^{*t} A_1'^t \dots A_n'^t\rangle_0. \tag{12.69}$$

To find the limit we expand (12.69) in truncated vacuum expectation values. According to the argument used in the derivation of (12.67), as $t \to \infty$ all the terms vanish except those corresponding to clusters of length 2. Using the result of Exercise 12.7 we see that the expression (12.69) converges to the right hand side of (12.68) as $t \to \mp\infty$, thus proving that (12.68) holds.

It remains to prove the Poincaré-invariance of the expressions Ω^{ex}. First we show that in spite of the fact that the operators A^t (12.6) explicitly use the frame of reference in their definition, the asymptotic states are in fact independent of the choice of the frame. In a basis of a second Lorentz frame differing from the standard one by the transformation $\Lambda \in L_+^\uparrow$, we would have to consider the operators

$$A^{t,\Lambda} = \int \delta(xn - t)\{D_m(x) n^\mu \overleftrightarrow{\partial}_\mu A(x)\} d^4x$$

instead of the A^t, where $n = \Lambda e_0$; clearly, $A^{t,\Lambda}$ can also be defined by (12.64).

Exercise 12.12. Prove that $A^{t,\Lambda}\Psi_0$ does not depend on Λ for $A \in \mathcal{A}^{[\kappa]}$.

The argument used in part (a) suffices to prove the existence of the limits

$$\lim_{t \to \mp\infty} A_1^{t,\Lambda} \dots A_n^{t,\Lambda} \Psi_0. \tag{12.70}$$

The claim that the asymptotic states are independent of the frame of reference means that the state (12.70) is independent of Λ. For the case when Λ belongs to the group of rotations of three-dimensional space, this claim is trivial, therefore it suffices to consider just the case when Λ is a pure Lorentz transformation, for example, in the $(0,3)$-plane, that is, of the form (12.66). We introduce the vector

$$\Phi^{t,\Lambda} = A_1^{t,\Lambda} \dots A_n^{t,\Lambda} \Psi_0$$

and show that for any compactum $T \subset \mathbf{R}$, there exists a number c such that

$$\|\frac{\partial}{\partial \eta} \Phi^{t,\Lambda}\|^2 \le c(1 + |t|)^{-1} \quad \text{for } t \in \mathbf{R},\ \eta \in T. \tag{12.71}$$

It will then follow that

$$\|\Phi^{t,\Lambda} - \Phi^t\| \le \int_0^\eta \|\frac{\partial}{\partial \eta'} \Phi^{t,\Lambda'}\| d\eta' \le c'(1 + |t|)^{-1/2}|\eta| \to 0 \quad \text{for } t \to \infty,$$

which will prove that the expressions (12.70) and (12.8) are equal.

We now turn to the derivation of the estimate (12.71). Once again we expand the scalar square

$$\left\|\frac{\partial}{\partial \eta} \Phi^{t,\Lambda}\right\|^2 = \left\langle \frac{\partial}{\partial \eta}(A_n^{*,t,\Lambda} \dots A_1^{*,t,\Lambda}) \frac{\partial}{\partial \eta}(A_1^{t,\Lambda} \dots A_n^{t,\Lambda}) \right\rangle_0$$

in a series of truncated vacuum expectation values. Truncated expectation values not containing the derivatives $\partial/\partial\eta$ are estimated in accordance with (12.64). Next we consider a term in the expansion, one of the clusters of which contains two operators of the form $\frac{\partial}{\partial \eta} A_j^{(*)t,\Lambda}$. Clearly, it makes sense to restrict our attention to a cluster of length ≥ 4 (otherwise, by Exercise 12.12 we would have zero). Such a vacuum expectation value is majorized by the expression const $\cdot(1 + |t|)^{-1}$ (by Exercise 12.11(b)). There remains one further possibility, when in some term of the above decomposition there are two clusters containing at least one operator of the form $\frac{\partial}{\partial \eta} A_j^{(*)t,\Lambda}$; again we need only consider clusters of length ≥ 3. According to Exercise 12.11(b), the corresponding truncated vacuum expectation values are bounded in modulus by the expression const$\cdot(1 + |t|)^{-1/2}$, so that the corresponding term in the expansion is bounded in modulus by the expression const$\cdot(1 + |t|)^{-1}$. As a result we arrive at the estimate (12.71).

There is now no difficulty in proving the Poincaré-invariance of the operators Ω^{ex}. Since the invariance of the Ω^{ex} under translations and three-dimensional rotations is trivial, it is enough to consider Lorentz transformations of the form (12.66). It follows from definitions (12.8), (12.64) that

$$U(0,\Lambda)\Phi^{ex}(A_1, \dots, A_n) = \lim_{t \to \mp\infty} A_1'^{t,\Lambda} \dots A_n'^{t,\Lambda} \Psi_0,$$

where $A_j' = U(0, \underset{\sim}{\Lambda})\ A_j U(0, \underset{\sim}{\Lambda})^{-1}$. Since we have proved that the asymptotic states are independent of the frame of reference, we can omit the symbol Λ on the right hand side of the above formula; as a result,

$$U(0, \underset{\sim}{\Lambda})\Phi^{ex}(A_1, \dots, A_n) = \Phi^{ex}(A_1', \dots, A_n').$$

Using Exercise 12.2 we can write this relation in the operator form

$$U(0, \underset{\sim}{\Lambda})\Omega^{ex} = \Omega^{ex}\mathcal{U}(0, \underset{\sim}{\Lambda}),$$

which proves the Lorentz-invariance of the operators Ω^{ex}. ∎

CHAPTER 13

Lehmann-Symanzik-Zimmermann Formalism

13.1. Basic Concepts

A. T-PRODUCTS OF FIELDS

The quantum theory of scattering is considerably simplified if we suppose, as was done by Lehmann, Symanzik and Zimmermann, that there is a correspondence between particles and fields. A scheme of this sort, which is a special case of the Wightman approach, is called the LSZ (Lehmann-Symanzik-Zimmermann) formalism. An essential element of it is the notion of chronological products and Green's functions, which we now set forth.

Formally a *T-product* (or chronological, or time-ordered product) $T(\phi_{l_1}^{(\kappa_1)}(x_1)\ldots\ldots\phi_{l_n}^{(\kappa_n)}(x_n))$ and *anti-T-* (or $\overline{T}$-) *product* $\overline{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\ldots\phi_{l_n}^{(\kappa_n)}(x_n))$ of Wightman fields is defined by the formula

$$T(\phi_{l_1}^{(\kappa_1)}(x_1)\ldots\phi_{l_n}^{(\kappa_n)}(x_n)) = \\ = \sum_{\pi}\epsilon_F(\pi)\theta(x_{\pi 1}^0 - x_{\pi 2}^0)\ldots\theta(x_{\pi(n-1)}^0 - x_{\pi n}^0)\phi_{l_{\pi 1}}^{(\kappa_{\pi 1})}(x_{\pi 1})\ldots\phi_{l_{\pi n}}^{(\kappa_{\pi n})}(x_{\pi n}), \tag{13.1a}$$

$$\overline{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\ldots\phi_{l_n}^{(\kappa_n)}(x_n)) = \\ = \sum_{\pi}\epsilon_F(\pi)\theta(x_{\pi 2}^0 - x_{\pi 1}^0)\ldots\theta(x_{\pi n}^0 - x_{\pi(n-1)}^0)\phi_{l_{\pi 1}}^{(\kappa_{\pi 1})}(x_{\pi 1})\ldots\phi_{l_{\pi n}}^{(\kappa_{\pi n})}(x_{\pi n}), \tag{13.1b}$$

where the sum is taken over all permutations π of the indices $1,\ldots,n$. However, these formulae are not suitable as a precise definition. The point is that they contain products of operator-valued generalized functions with discontinuous functions and such objects in the Wightman theory are generally not defined (for the definition of the products one must in addition assume the existence of fields at fixed instants of time). Another undesirable feature of the expressions (13.1) is that even in the case when they are well defined, they can have unacceptable properties (such as dependence on the Lorentz coordinate frame). A well known example of this kind is the T-product of a free vector field: it is not difficult to see (see, for example, [B10],§15) that the T-product $T(A_\lambda(x)A_\mu(y))$ calculated according to (13.1a) is not covariant, although this non-covariance is easily removed by adding to the expression so obtained a term concentrated on the plane $x^0 = y^0$.

It is therefore more convenient to define the T-product by the collection of properties that it should possess. The characteristic properties of T-products (as operator-valued tempered generalized functions) are as follows.*

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* Here and in what follows we shall frequently use the convention $\phi_{l_j}^{(\kappa_j)}(x_j) \equiv \phi_j$.

(a) *Permutation property:*

$$T(\phi_{\pi 1}\dots\phi_{\pi n}) = \epsilon_F(\pi)T(\phi_1\dots\phi_n) \tag{13.2a}$$

for any permutation π.

(b) *Poincaré-covariance:*

$$\begin{aligned} U(a,\underset{\sim}{\Lambda})T(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n))U(a,\underset{\sim}{\Lambda})^{-1} = \\ = \sum_{m_1,\dots,m_n} V_{l_1m_1}^{(\kappa_1)}(\underset{\sim}{\Lambda}^{-1})\dots V_{l_nm_n}^{(\kappa_n)}(\underset{\sim}{\Lambda}^{-1})T(\phi_{m_1}^{(\kappa_1)}(\Lambda x_1+a)\dots\phi_{m_n}^{(\kappa_n)}(\Lambda x_n+a)) \end{aligned} \tag{13.2b}$$

for all $(a,\underset{\sim}{\Lambda}) \in \rho_0$.

(c) *Causality:*

$$T(\phi_1\dots\phi_n) = T(\phi_1\dots\phi_k)T(\phi_{k+1}\dots\phi_n), \tag{13.2c}$$

if $\{x_1,\dots,x_k\} \gtrsim \{x_{k+1},\dots,x_n\}$.*

(d) *Property of the Hermitian adjoint:*

$$T(\phi_1\dots\phi_n)^* = \overline{T}(\phi_n^*\dots\phi_1^*). \tag{13.2d}$$

(e) *Unitarity:* for $n = 1,2,\dots$

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!}\sum_{\pi}\epsilon_F(\pi)\overline{T}(\phi_{\pi 1}\dots\phi_{\pi k})T(\phi_{\pi(k+1)}\dots\phi_{\pi n}) = 0 \tag{13.2e}$$

(in the last formula we use the convention: for $n = 0$, $\overset{(-)}{T}(\phi_1\dots\phi_n)$ is taken to be unity).

As a motivation for these properties it can be said that they are all obtained by formal manipulations on expressions of type (13.1) (although these manipulations cannot be regarded as a proof for the reasons mentioned earlier).

The next exercise establishes the relation between the above definition of a T-product and the heuristic formula (13.1).

Exercise 13.1. Prove that

$$T(\phi_1\dots\phi_n) = \phi_1\dots\phi_n \quad \text{for } x_1^0 > x_2^0 > \dots > x_n^0, \tag{13.3a}$$

$$\overline{T}(\phi_1\dots\phi_n) = \phi_1\dots\phi_n \quad \text{for } x_1^0 < x_2^0 < \dots < x_n^0. \tag{13.3b}$$

[Hint: Employ (13.2c) and use induction on n.]

The vacuum expectation values

$$\tau_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1,\dots,x_n) = \langle 0|T(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n)|0\rangle, \tag{13.4a}$$

$$\overline{\tau}_{l_1\dots l_n}^{(\kappa_1\dots\kappa_n)}(x_1,\dots,x_n) = \langle 0|\overline{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n)|0\rangle \tag{13.4b}$$

are called *causal chronologically ordered* (or *antichronologically ordered*) n-point *Green's functions* of the fields $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$.

---

* For a pair of sets $\mathcal{O}$ and $\mathcal{Q}$ in M we introduce the notation $\mathcal{O} \lesssim \mathcal{Q}$ to mean that $x - y \notin \overline{V}^+$ for all $x \in \mathcal{O}$, $y \in \mathcal{Q}$ (the same relation is also written in the form $\mathcal{Q} \gtrsim \mathcal{O}$).

The existence of T-products of fields has not been proved from the Wightman axioms in the general case; therefore in the LSZ formalism the existence of T-products is accepted as an additional postulate. The significance will be clear from the reduction formulae (§13.2). Another strong argument in favour of this postulate is furnished by the Lagrangian quantum field theory in which T-products play a fundamental role; at any rate, there they are constructed in terms of perturbation theory (see [B10]).

If all the fields under consideration are bosonic, then clearly the permutation property expresses the symmetry of the T-products with respect to the indices:

$$T(\phi_{\pi 1}\dots\phi_{\pi n}) = T(\phi_1\dots\phi_n). \tag{13.5}$$

In this case the operator-valued generalized functions $T(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_1)}(x_n))$ are uniquely defined as the result of smoothing them with test functions of the form $\eta^{(\kappa_1)l_1}(x_1)\dots\eta^{(\kappa_n)l_n}(x_n)$, where $\eta^{(\kappa)l}$ is a finite family of functions in $S(\mathbf{M})$, called the classical source. The adjoint source is defined by $\eta^{+(\kappa)l}(x) = \overline{\eta^{(\bar\kappa)\bar l}(x)}$. (In formula (13.29) below we give a more general definition of the adjoint source suitable for Fermi fields as well.) For brevity we shall write $\int \eta(x)\phi(x)dx$ instead of $\sum_{\kappa,l}\int\eta^{(\kappa)l}(x)\phi_l^{(\kappa)}(x)dx$; we shall also use the notation

$$\overset{(-)}{T}\Big(\Big(\int\eta(x)\phi(x)dx\Big)^n\Big) =$$
$$= \sum_{\substack{\kappa_1\dots\kappa_n\\ l_1\dots l_n}}\int\eta^{(\kappa_n)l_n}(x_n)\dots\eta^{(\kappa_1)l_1}(x_1)\overset{(-)}{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n))dx_1\dots dx_n. \tag{13.6}$$

It is convenient to characterize the entire collection of T-products by the generating functional

$$\mathfrak{T}(\eta) = T\exp\Big(i\int\eta(x)\phi(x)dx\Big) \equiv \sum_{n=0}^{\infty}\frac{i^n}{n!}T\Big(\Big(\int\eta(x)\phi(x)dx\Big)^n\Big), \tag{13.7}$$

which has to be regarded as a formal series in the classical source $\eta \equiv (\eta^{(\kappa)})^l$ (thus one can only ascribe the meaning of operator to each individual term of the series (13.7)). In similar fashion we define the generating functional

$$\bar{\mathfrak{T}}(\eta) = \overline{T}\exp\Big(i\int\eta(x)\phi(x)dx\Big). \tag{13.8}$$

The $\overset{(-)}{T}$-product is now expressed in terms of the variational derivatives of $\overset{(-)}{\mathfrak{T}}(\eta)$ with respect to the source:

$$\overset{(-)}{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\dots\phi_{l_n}^{(\kappa_n)}(x_n)) = (-i)^n\frac{\delta^n\overset{(-)}{\mathfrak{T}}(\eta)}{\delta\eta^{(\kappa_1)l_1}(x_1)\dots\delta\eta^{(\kappa_n)l_n}(x_n)}\Big|_{\eta=0}. \tag{13.9}$$

The characteristic properties of the T-products in terms of the generating functionals can be written down as follows.

(a) "*Initial conditions*":

$$\mathfrak{T}(\eta)|_{\eta=0} = 1, \quad \frac{\delta}{\delta\eta^{(\kappa)l}(x)}\mathfrak{T}(\eta)|_{\eta=0} = i\phi_l^{(\kappa)}(x). \tag{13.10a}$$

(b) *Poincaré-covariance:*

$$U(a,\underset{\sim}{\Lambda})\mathfrak{T}(\eta)U(a,\underset{\sim}{\Lambda})^{-1} = \mathfrak{T}(\eta_{(a,\Lambda)}) \quad \text{for } (a,\underset{\sim}{\Lambda}) \in \mathcal{P}_0, \tag{13.10b}$$

where

$$\eta^{(\kappa)l}_{(a,\underset{\sim}{\Lambda})}(x) = \sum_m V^{(\kappa)}_{m_l}(\underset{\sim}{\Lambda}^{-1})\eta^{(\kappa)m}(\Lambda^{-1}(x-a)).$$

(c) *Causality:*

$$\mathfrak{T}(\xi+\eta) = \mathfrak{T}(\xi)\mathfrak{T}(\eta), \quad \text{if } \operatorname{supp}\xi \gtrsim \operatorname{supp}\eta. \tag{13.10c}$$

(d) *Property of the adjoint:*

$$\mathfrak{T}(\eta)^* = \bar{\mathfrak{T}}(-\eta^+). \tag{13.10d}$$

(e) *Unitarity:*

$$\mathfrak{T}(\eta)^*\mathfrak{T}(\eta^+) = 1. \tag{13.10e}$$

Exercise 13.2. Prove the equivalence of (13.2) and (13.10).

Exercise 13.3. Prove the relations

$$\bar{\mathfrak{T}}(-\eta)\mathfrak{T}(\eta) = 1 = \mathfrak{T}(\eta)\bar{\mathfrak{T}}(-\eta), \tag{13.11}$$

which we write as the single equation

$$\mathfrak{T}(\eta)^{-1} = \bar{\mathfrak{T}}(-\eta), \tag{13.12}$$

and which, of course, are understood in the sense of formal series in η. [Hint: By the condition $\mathfrak{T}(0) = 1$ there exists a formal series $\mathfrak{T}(\eta)^{-1}$ such that $\mathfrak{T}(\eta)^{-1}\mathfrak{T}(\eta) = 1 = \mathfrak{T}(\eta)\mathfrak{T}(\eta)^{-1}$.]

We illustrate the notion of T-product with the example of a free scalar field.

Exercise 13.4. Let $\phi(x)$ be a free Hermitian scalar field of mass m.

(a) We define the two-point causal Green's function of the field $\phi(x)$ by setting

$$\begin{aligned}\tau^{[2]}(x,y) &= \frac{1}{i}D^c_m(x-y) = \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle = \\ &= \frac{1}{i}\{\theta(x^0-y^0)D^{(-)}_m(x-y) + \theta(y^0-x^0)D^{(-)}_m(y-x)\}.\end{aligned}$$

Prove the formula

$$\tau^{[2]}(x,y) = \frac{1}{i}D^c_m(x-y) = i\int \frac{e^{-ip(x-y)}}{p^2-m^2+i0}d_4p; \tag{13.13}$$

hence deduce that the causal Green's function satisfies the equation

$$(\Box + m^2)\tau^{[2]}(x,y) = \frac{1}{i}\delta(x-y). \tag{13.14}$$

[Hint: Carry out the integration in (13.13) with respect to p^0 and compare with the definition.]

(b) We define the formal series with respect to the source η

$$:\!\exp\Big(i\int \phi(x)\eta(x)dx\Big)\!:= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int :\!\phi(x_1)\dots\phi(x_n)\!:\, \eta(x_1)\dots\eta(x_n)dx_1\dots dx_n.$$

Prove the equality of the formal series

$$\begin{aligned}&\exp\Big(i\int \phi(x)\eta(x)dx\Big) = \\ &\quad = \exp\Big(-\frac{1}{2i}\int D^{(-)}_m(x-y)\eta(x)\eta(y)dx\,dy\Big) :\!\exp\Big(\int i\phi(x)\eta(x)dx\Big)\!:.\end{aligned} \tag{13.15}$$

[Hint: Use (8.85).]

(c) We define the generating functional of the $\overset{(-)}{T}$-products of the field ϕ by the formula

$$\mathfrak{T}(\eta) = \exp\Big(-\frac{1}{2i}\int D_m^c(x-y)\eta(x)\eta(y)dx\,dy\Big) :\exp\Big(i\int \phi(x)\eta(x)dx\Big):, \tag{13.16a}$$

$$\bar{\mathfrak{T}}(\eta) = \exp\Big(\frac{1}{2i}\int \overline{D_m^c(x-y)}\eta(x)\eta(y)dx\,dy\Big) :\exp\Big(i\int \phi(x)\eta(x)dx\Big):. \tag{13.16b}$$

Prove that the characteristic properties (13.10) hold for these. [Hint: For the proof of unitarity, use the relation

$$\mathfrak{T}(\eta) = \exp\Big(-\frac{1}{2i}D_m^{\text{adv}}(x-y)\eta(x)\eta(y)dx\,dy\Big)\exp\Big(i\int \phi(x)\eta(x)dx\Big), \tag{13.17}$$

where

$$D_m^{\text{adv}}(x-y) = -\theta(y^0-x^0)D_m(x-y) = D_m^c(x-y) - D_m^{(-)}(x-y) = -\int \frac{e^{-ip(x-y)}}{p^2-m^2-i\epsilon p^0}\,d_4p\Big|_{\epsilon=+0} \tag{13.18}$$

is the so-called advanced Green's function of the free scalar field.]

(d) Prove the formulae for the causal Green's functions:

$$\tau^{[n]}(x_1,\ldots,x_n) = 0 \quad \text{for } n \text{ odd},$$

$$\tau^{[2n]}(x_1,\ldots,x_{2n}) = \sum_P \tau^{[2]}(x_{P1},x_{P2})\ldots\tau^{[2]}(x_{P(2n-1)},x_{P(2n)}) \tag{13.19b}$$

(P has the same meaning as in (8.75)),

$$\bar{\tau}^{[n]}(x_1,\ldots,x_n) = \overline{\tau^{[n]}(x_1,\ldots,x_n)}. \tag{13.19c}$$

(e) Prove the Wick formula for the $\overset{(-)}{T}$-products:

$$\overset{(-)}{T}(\phi(x_1)\ldots\phi(x_n)) = \sum_{k=0}^{n}\sum_P \overset{(-)}{\tau}{}^{[k]}(x_{P1},\ldots,x_{Pk}) :\phi(x_{P(k+1)})\ldots\phi(x_{Pn}); \tag{13.20}$$

(P has the same meaning as in (8.85)).

Proposition 13.1. *The causality condition* (13.10c) *is equivalent to the condition (in the sense of formal series in the classical sources ξ and η)*

$$\frac{\delta}{\delta\eta(x)}\Big\{\mathfrak{T}(\eta)^{-1}\mathfrak{T}(\xi+\eta)\Big\} = 0 \quad \textit{for } \operatorname{supp}\xi \lesssim \mathcal{N}_x, \tag{13.21}$$

where $\mathcal{N}_x$ is an (arbitrarily small) neighbourhood of x.

Equality (13.21) is a series of relations between operator-valued generalized functions of temperate growth; since such generalized functions are uniquely defined by their values on test functions with compact supports, it suffices for the proof of (13.21) to suppose that ξ and η are functions with compact support (so that the condition $\operatorname{supp}\xi \lesssim \mathcal{N}_x$ can be replaced by the condition $\operatorname{supp}\xi \lesssim x$).

It is clear that (13.21) is equivalent to the series of relations (for $n = 1,2,\ldots$)

$$\frac{\delta^n}{\delta\eta(x_1)\ldots\delta\eta(x_n)}\Big\{\mathfrak{T}(\eta)^{-1}\mathfrak{T}(\xi+\eta)\Big\}\Big|_{\eta=0} = 0 \tag{13.22}$$

in the open set

$$\{(x_1,\ldots,x_n)\in \mathrm{M}^n :\ x_j \gtrsim \operatorname{supp}\xi \quad \text{for some } j\}. \tag{13.23}$$

It is not difficult to see that for any point of this set there exists a non-empty subset J of the set of indices $N \equiv \{1,\ldots,n\}$ such that

$$\{x_\alpha\}_{\alpha\in J} \gtrsim \operatorname{supp}\xi \cup \{x_\beta\}_{\beta\in N\setminus J}. \tag{13.24}$$

Thus the (open) sets (13.24) cover the set (13.23), so that it suffices to prove (13.22) on each of the sets (13.24). In other words, we must prove that for any $k = 1, 2, \dots$ and $m = 0, 1, 2, \dots$

$$\frac{\delta^{k+m}}{\delta\eta(x_1)\dots\delta\eta(x_k)\delta\eta(y_1)\dots\delta\eta(y_m)}\{\mathfrak{T}(\eta)^{-1}\mathfrak{T}(\xi+\eta)\}\Big|_{\eta=0} = 0 \tag{13.25}$$

in the domain $\{x_1, \dots, x_k\} \gtrsim \operatorname{supp}\xi \cup \{y_1, \dots, y_m\}$, that is,

$$\frac{\delta^k}{\delta\eta(x_1)\dots\delta\eta(x_k)}\,\frac{\delta^m}{\delta\eta'(y_1)\dots\delta\eta'(y_m)}\{\mathfrak{T}(\eta+\eta')^{-1}\mathfrak{T}(\xi+\eta+\eta')\}\Big|_{\eta=\eta'=0} = 0, \tag{13.26}$$

if $\operatorname{supp}\eta \gtrsim \operatorname{supp}\xi \cup \operatorname{supp}\eta'$. But for such ξ, η, η' (according to the causality condition (13.10c)) we have

$$\mathfrak{T}(\eta+\eta') = \mathfrak{T}(\eta)\mathfrak{T}(\eta'), \quad \mathfrak{T}(\xi+\eta+\eta') = \mathfrak{T}(\eta)\mathfrak{T}(\xi+\eta')$$

and hence,

$$\mathfrak{T}(\eta+\eta')^{-1}\mathfrak{T}(\xi+\eta+\eta') = \mathfrak{T}(\eta')^{-1}\mathfrak{T}(\xi+\eta').$$

Hence (13.26) follows. Conversely (13.10c) is easily derived from (13.21). ∎

One can introduce the generating functional of T-products by (13.6), (13.7) even in the general case, including fermionic fields. However, since the T-product is antisymmetric with respect to fermionic fields, the classical sources of fermionic fields can no longer be regarded as ordinary numerical functions. We shall consider them to be merely symbols $\eta^{(\kappa)l}(x)$ depending on x and commuting with bosonic fields and bosonic sources and anticommuting with fermionic fields and fermionic sources.* This time the individual terms of the series (13.7) (as well as the entire series itself) no longer have the meaning of operators (apart from the case when one substitutes $\eta = 0$). As a rule we shall use the left variational derivative $\delta/\delta\eta(x)$ of the formal series with respect to the fermionic sources, which is defined by the formulae

$$\frac{\delta}{\delta\eta^{(\kappa)l}(x)}\eta^{(\kappa')m}(y) = \delta_{\kappa\kappa'}\delta_{lm}\delta(x-y), \tag{13.27a}$$

$$\frac{\delta}{\delta\eta(x)}\{A(\eta)B(\eta)\} = \left(\frac{\delta A(\eta)}{\delta\eta}\right)B(\eta) + (-1)^{N_A}A(\eta)\frac{\delta B(\eta)}{\delta\eta(x)}, \tag{13.27b}$$

where N_A ($= 0$ or 1) is the degree modulo 2 in which the fermionic fields and fermionic sources enter into the expression $A(\eta)$. For the higher variational derivatives, the order of the derivatives becomes important; we use the rule

$$\frac{\delta^n A(\eta)}{\delta\eta(x_1)\dots\delta\eta(x_n)} \equiv \frac{\delta}{\delta\eta(x_1)}\cdots\frac{\delta A(\eta)}{\delta\eta(x_n)}.$$

The characteristic properties (13.10) of the generating functional of the T-products extends to the general case. When taking the Hermitian adjoint in (13.10d) we must use rules of type

$$(A\eta^{(\kappa_1)l_1}(x_1)\dots\eta^{(\kappa_n)l_n}(x_n))^* = \overline{\eta^{(\kappa_n)l_n}(x_n)}\dots\overline{\eta^{(\kappa_1)l_1}(x_1)}A^* \tag{13.28}$$

* Fermionic sources can be treated as elements of a Grassmann algebra; such Grassmann variables have found wide application in field theory with supersymmetry (see, for example, Wess, 1976, Salam and Strathdee, 1977; [S17]).

(where A is an operator or operator-valued generalized function). The conjugate source η^+ is defined by the formula

$$\eta^{+(\kappa)l}(x) = (-1)^F \overline{\eta^{(\bar{\kappa})\bar{l}}(x)}, \tag{13.29}$$

where $F = 0$ or 1 respectively at the bosonic or fermionic components of the source. We also have to clarify the meaning of the expression $\operatorname{supp}\xi \underset{\sim}{>} \operatorname{supp}\eta$ in the causality condition (13.10c). This now means that the relations between the operator-valued generalized functions

$$\frac{\delta^m}{\delta\xi(x_1)\ldots\delta\xi(x_m)}\frac{\delta^n}{\delta\eta(y_1)\ldots\delta\eta(y_n)}(\mathfrak{T}(\xi+\eta) - \mathfrak{T}(\xi)\mathfrak{T}(\eta))\Big|_{\xi=\eta=0} = 0$$

(where $m,\ n = 0,1,2,\ldots$) corresponding to the equality (13.10c) of the formal series, is valid only in the domain $\{x_1,\ldots,x_m\} \underset{\sim}{>} \{y_1,\ldots,y_n\}$.

With a similar proviso concerning the condition $\operatorname{supp}\xi \underset{\sim}{<} \mathcal{N}_x$ (or $\operatorname{supp}\xi \underset{\sim}{<} x$), Proposition 13.1 goes over to the general case including fermionic fields.

B. RETARDED PRODUCTS

Retarded products are objects closely connected with T-products and playing a primary role in the study of the analytic properties of the scattering amplitude. Formally, the *retarded* (or *R-*)*product* of the fields $\phi^{(\kappa)}(x), \phi^{(\kappa_1)}(x_1), \ldots, \phi^{(\kappa_n)}(x_n)$ is defined by the formula *

$$\begin{aligned} R(\phi^{(\kappa_n)}(x_n)\ldots\phi^{(\kappa_1)}(x_1);\ \phi^{(\kappa)}(x)) = \\ = (-i)^n \sum_\pi \epsilon_F(\pi)\theta(x^0 - x^0_{\pi 1})\theta(x^0_{\pi 1} - x^0_{\pi 2})\ldots\theta(x^0_{\pi(n-1)} - x^0_{\pi n})\times \\ \times[\phi^{(\kappa_{\pi n})}(x_{\pi n}),\ldots[\phi^{(\kappa_{\pi 2})}(x_{\pi 2}),\ [\phi^{(\kappa_{\pi 1})}(x_{\pi 1}),\ \phi^{(\kappa)}(x)]_\mp]_\mp\ldots]_\mp; \end{aligned} \tag{13.30}$$

here as usual, the expression $[A,B]_\mp$ is a commutator if at least one of the operators A, B is of bosonic type and an anticommutator if both operators A, B are fermionic; the sum is taken over all permutations of the indices $1,\ldots,n$. The expression (13.30) suffers from the same drawback as (13.1), namely, it is ill-defined.

By supposing that the $\overset{(-)}{T}$-products of the fields are already defined in accordance with conditions (13.2), we can give a well-defined expression for the retarded product:

$$R(\phi_n\ldots\phi_1;\phi) = i^n \sum_{k=0}^n \frac{(-1)^k}{k!(n-k)!}\sum_\pi \epsilon_F(\pi)\overline{T}(\phi_n\ldots\phi_{k+1})T(\phi_k\ldots\phi_1\phi) = \tag{13.31a}$$

$$= \pm i^n \sum_{k=0}^n \frac{(-1)^{n-k}}{k!(n-k)!}\sum_\pi \epsilon_F(\pi)\overline{T}(\phi\phi_n\ldots\phi_{k+1})T(\phi_k\ldots\phi_1) \tag{13.31b}$$

(here $\pm$ is the $-$ sign if both operators ϕ and $\phi_1\ldots\phi_n$ are fermionic and $+$ otherwise). Formulae (13.31) clearly enable us to recover recursively the $\overset{(-)}{T}$-products from the retarded products.

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* In connection with the fact that we use left variational derivatives with respect to the Fermi sources, we shall use the ordering $(x_n,\ldots,x_1;x)$ of the arguments for the general case; this differs from the generally accepted $(x;x_1,\ldots,x_n)$ for the bosonic case.

By definition, the retarded products (13.31) are symmetric with respect to the bosonic fields in the collection $\phi_{l_1}^{(\kappa_1)}(x_1), \dots, \phi_{l_n}^{(\kappa_n)}(x_n)$ and antisymmetric with respect to the fermionic fields in the same collection. It is therefore convenient to characterize them by the generating functional

$$\mathfrak{R}_l^{(\kappa)}(x;\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{\kappa_1 \dots \kappa_n \\ l_1 \dots l_n}} \int \eta^{(\kappa_n) l_n}(x_n) \dots \eta^{(\kappa_1) l_1}(x_1) \times$$
$$\times R(\phi_{l_1}^{(\kappa_1)}(x_1) \dots \phi_{l_n}^{(\kappa_n)}(x_n);\ \phi_l^{(\kappa)}(x)) dx_1 \dots dx_n. \tag{13.32}$$

This is a formal series in the source $\eta(x)$ and the coefficient functions of this series are operator-valued generalized functions in the variables $x, x_1, \dots, x_n$:

$$R(\phi_{l_n}^{(\kappa_n)}(x_n) \dots \phi_{l_1}^{(\kappa_1)}(x_1);\ \phi_l^{(\kappa)}(x)) = \frac{\delta^n}{\delta\eta^{(\kappa_n) l_n}(x_n) \dots \delta\eta^{(\kappa_1) l_1}(x_1)} \mathfrak{R}_l^{(\kappa)}(x;\eta)\Big|_{\eta=0} \tag{13.33}$$

It is not difficult to see that (13.31) can now be rewritten in the form

$$\mathfrak{R}_l^{(\kappa)}(x;\eta) = -i\mathfrak{T}(\eta)^{-1} \frac{\delta\mathfrak{T}(\eta)}{\delta\eta^{(\kappa) l}(x)}. \tag{13.34}$$

The characteristic properties of the retarded products (in terms of the functional $\mathfrak{R}$) are as follows:

(a) "*Initial condition*":

$$\mathfrak{R}_l^{(\kappa)}(x;\eta)|_{\eta=0} = \phi_l^{(\kappa)}(x). \tag{13.35a}$$

(b) *Property of the Hermitian adjoint:*

$$\mathfrak{R}_l^{(\kappa)}(x;\eta)^* = \mathfrak{R}_{\bar{l}}^{(\bar{\kappa})}(x;\eta^+). \tag{13.35b}$$

(c) *Poincaré-covariance*:

$$U(a,\underset{\sim}{\Lambda})\mathfrak{R}_l^{(\kappa)}(x;\eta)U(a,\underset{\sim}{\Lambda})^{-1} = \sum_m V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda}^{-1})\mathfrak{R}_m^{(\kappa)}(\Lambda x + a;\ \eta_{(a,\underset{\sim}{\Lambda})}) \tag{13.35c}$$

for $(a, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0$.

(d) *Resolvability condition:*

$$\frac{\delta}{\delta\eta^{(\kappa')m}(y)}\mathfrak{R}_l^{(\kappa)}(x;\eta) \mp \frac{\delta}{\delta\eta^{(\kappa)l}(x)}\mathfrak{R}_m^{(\kappa')}(y;\eta) = -i[\mathfrak{R}_m^{(\kappa')}(y;\eta),\ \mathfrak{R}_l^{(\kappa)}(x;\eta)]_{\mp}. \tag{13.35d}$$

(e) *Causality:*

$$\frac{\delta}{\delta\eta(y)}\mathfrak{R}(x;\eta) = 0 \quad \text{for } x \underset{\sim}{<} y. \tag{13.35e}$$

Exercise 13.5. Derive the properties (13.35) and (13.10). [Hint: For the proof of the causality property use the equality

$$\mathfrak{R}(x;\eta) = -i\frac{\delta}{\delta\xi(x)}\{\mathfrak{T}(\eta)^{-1}\mathfrak{T}(\xi+\eta)\}\big|_{\xi=0} \tag{13.36}$$

and Proposition 13.1.]

The converse also holds: if a formal series $\Re(x;\eta)$ is given with the properties (13.35), then it can be shown that it uniquely defines the series $\mathfrak{T}(\eta)$ related to $\Re(x;\eta)$ by the formula (13.34) and satisfying the conditions (13.10).

Clearly the causality condition (13.35e) is a simple property of the support of the R-products:

$$\operatorname{supp} R(\phi_n(x)\dots\phi_1(x_1);\phi(x)) \subset \{(x,x_1,\dots,x_n)\in \mathbf{M}^{n+1} : x_j - x \in \overline{V}^- \text{ for } j=1,\dots,n\} \tag{13.37}$$

The following exercise (together with (13.37)) establishes a connection between the definition (13.31) of R-products and the heuristic formula (13.30).

Exercise 13.6. Prove that

$$R(\phi_n(x_n)\dots\phi_1(x_1);\phi(x)) = (-i)^n[\phi_n(x_n)\dots[\phi_2(x_2),[\phi_1(x_1),\phi(x)]_\mp]_\mp\dots]_\mp \tag{13.38}$$

for $x_n^0 < x_{n-1}^0 < \dots < x_1^0 < x^0$. [Hint: It follows from (13.35d) and (13.35e) that

$$\frac{\delta}{\delta\eta(y)}\Re(x;\eta) = -i[\Re(y;\eta),\Re(x;\eta]_\mp \quad \text{for } x \gtrsim y; \tag{13.39}$$

now use (13.33).]

The vacuum expectation value

$$r_{l_n\dots l_1;l}^{(\kappa_n\dots\kappa_1;\kappa)}(x_n,\dots,x_1;x) = \langle 0|R(\phi_{l_n}^{(\kappa_n)}(x_n)\dots\phi_{l_1}^{(\kappa_1)}(x_1);\phi_l^{(\kappa)}(x))|0\rangle \tag{13.40}$$

is called the ($(n+1)$-point) *retarded Green's function* (or r-function) of the fields $\phi^{(\kappa)}$, $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$. Clearly it has the same property of the support as the retarded product.

Exercise 13.7. Let $\phi(x)$ be a free scalar Hermitian field of mass m. Prove the formulae

$$R(\phi(y);\phi(x)) = r^{[2]}(y;x) = D_m^{\text{ret}}(x-y), \tag{13.41a}$$

$$R(\phi_n(x_n)\dots\phi(x_1);\phi(x)) = 0 \quad \text{for } n\ge 2; \tag{13.41b}$$

here

$$D_m^{\text{ret}}(x-y) = \theta(x^0-y^0)D_m(x-y) = D_m^c(x-y) - D_m^{(-)}(y-x) = -\int \frac{e^{-ip(x-y)}}{p^2-m^2+i\epsilon p^0}d_4p\Big|_{\epsilon=+0} \tag{13.42}$$

is the so-called retarded Green's function of the free scalar field. [Hint: Using (13.17) show that

$$\mathfrak{T}(\xi+\eta) = \mathfrak{T}(\eta)\Big(1 + i\int\phi(x)\xi(x)dx + i\int D_m^{\text{ret}}(x-y)\xi(x)\eta(y)dx\,dy) + \dots\Big),$$

where "..." are terms of order ≥ 2 in ξ; now use (13.36).]

The *advanced* (or A-) *products of fields* are defined similarly:

$$A(\phi_{l_n}^{(\kappa_n)}(x)\dots\phi_{l_1}^{(\kappa_1)}(x_1);\phi_l^{(\kappa)}(x)) = -i\frac{\delta^n}{\delta\eta^{(\kappa_n)l_n}(x_n)\dots\delta\eta^{(\kappa_1)l_1}(x_1)}\Big(\frac{\delta\mathfrak{T}(\eta)}{\delta\eta(x)}\mathfrak{T}(\eta)^{-1}\Big)\Big|_{\eta=0}. \tag{13.43}$$

They enjoy the same symmetry properties with respect to the indices $1,\dots,n$ as the retarded products and the property of their support is expressed thus:

$$\operatorname{supp} A(\phi_n(x_n)\dots\phi_1(x_1);\phi(x)) \subset \{(x,x_1,\dots,x_n)\in\mathbf{M}^{n+1} : x_j - x\in\overline{V}^+ \text{ for } j=1,\dots,n\}. \tag{13.44}$$

The vacuum expectation value

$$a^{(\kappa_n\dots\kappa_1;\kappa)}_{l_n\dots l_1;l}(x_n\dots x_1;x) = \langle 0|A(\phi^{(\kappa_n)}_{l_n}(x_n)\dots\phi^{(\kappa_1)}_{l_1}(x_1);\phi^{(\kappa)}_l(x))|0\rangle \tag{13.45}$$

is called the ($(n+1)$-point) *advanced Green's function* (or a-function) of the fields $\phi^{(\kappa)}$, $\phi^{(\kappa_1)},\dots,\phi^{(\kappa_n)}$.

The analogue of (13.30) is as follows:

$$A(\phi^{(\kappa_n)}_n\dots\phi^{(\kappa_1)}_1;\phi^{(\kappa)}) = i^n\sum_\pi \epsilon_F(\pi)\theta(x^0_{\pi 1}-x^0)\theta(x^0_{\pi 2}-x^0_{\pi 1})\dots\theta(x^0_{\pi n}-x^0_{\pi(n-1)})[\phi_{\pi n}(x_{\pi n}),\dots$$
$$\dots,[\phi_{\pi 2}(x_{\pi 2}),\ [\phi_{\pi 1}(x_{\pi 1}),\ \phi(x)]_\mp]_\mp\dots]_\mp. \tag{13.46}$$

Exercise 13.8. Let $\phi(x)$ be a free scalar Hermitian field of mass m. Prove the formula

$$A(\phi(y);\phi(x)) = a^{[2]}(y;x) = D^{\mathrm{adv}}_m(x-y). \tag{13.47}$$

In what follows we shall use the following abbreviated notation for the Fourier transform of the T-product $T(\phi_1(x_1)\dots\phi_n(x_n))$:

$$T(\tilde\phi_1(p_1)\dots\tilde\phi_n(p_n)) = \int e^{i(p_1x_1+\dots+p_nx_n)}T(\phi_1(x_1)\dots\phi_n(x_n))dx_1\dots dx_n;$$

the Fourier transforms of the retarded and advanced products $R(\tilde\phi_n(p_n)\dots\tilde\phi_1(p_1);\tilde\phi(p))$ and $A(\tilde\phi_n(p_n)\dots\tilde\phi_1(p_1);\tilde\phi(p))$ are defined similarly.

There are important relations connecting the Fourier transforms of τ-functions with the Fourier transforms of the r- and a-functions in certain domains of p-space. Here we use the notation

$$p_J = \sum_{j\in J} p_j, \tag{13.48}$$

where J is an arbitrary subset of $\{1,\dots,n\}$.

Proposition 13.2. *The following relations hold:*

$$\tilde\tau^{(\kappa_n\dots\kappa_1\kappa)}(p_n,\dots,p_1,p) = (-i)^n\tilde r^{(\kappa_n\dots\kappa_1;\kappa)}(p_n,\dots,p_1;p) \tag{13.49a}$$
$$\text{for } p_J\notin\overline{V}^+ \quad \text{for all } J\subset\{1,\dots,n\},$$
$$\tilde\tau^{(\kappa_n\dots\kappa_1\kappa)}(p_n,\dots,p_1,p) = (-i)^n\tilde a^{(\kappa_n\dots\kappa_1;\kappa)}(p_n,\dots,p_1;p) \tag{13.49b}$$
$$\text{for } p_J\notin\overline{V}^- \quad \text{for all } J\subset\{1,\dots,n\}.$$

■ For the proof of (13.49a) it suffices to take the vacuum expectation value of the relation (13.31a); by virtue of the spectrum condition, all the terms in the sum, apart from $k=n$, then vanish. Formula (13.49b) is proved in similar fashion. ■

Relations of type (13.49) are the starting point for the study of the analytic properties of the Green's function in p-space, since it follows from the properties of the support (13.37) and (13.44) that the generalized functions $\tilde r$ and $\tilde a$ are boundary values of functions that are analytic in tubes (in fact, the same analytic function); we return to this question in Ch.16.

C. LSZ AXIOMS

The original statements of LSZ (in the modern account) consist of two halves: the Wightman axioms W.I–W.VIII and the supplementary requirements.* These sup-

* The combined system of the Wightman and Lehmann-Symanzik-Zimmermann axioms are also called the LSZW formalism. Note that the requirement LSZ.I differs from W.III′ only in that the operators A of creation of one-particle states are linear with respect to the fields $\phi(x)$. The requirement LSZ.II was discussed in §12.1.C and can also, in principle, be stated in the Wightman approach. However, it is only made essential use of in the LSZ scheme.

plementary requirements (LSZ.I–LSZ.III), which determine the specific character of the LSZ formalism, are as follows.

LSZ.I (Connection between particles and fields). *In addition to the strong spectrum condition, it is supposed that the masses of the one-particle states in the physical Hilbert space $\mathcal{H}$ form a finite (or locally finite) sequence $0 < m_1 < m_2 < \ldots$. Corresponding to each field $\phi^{(\kappa)}$ with zero matrix elements between the vacuum vector and the one-particle vectors, is a parameter m_κ such that vectors of the form*

$$\sum_l \int \phi_l^{(\kappa)}(x) f^l(x) dx \Psi_0 \tag{13.50}$$

*are vectors (which do not vanish identically) with fixed mass** m_κ, if $f^l \in \mathcal{S}(\mathbf{M})$ *and the support of $f^l(p)$ is contained in the region*

$$G_\kappa = \{p \in \mathbf{M} : \; p^2 < m_\kappa'^2\} \tag{13.51}$$

(here m'_κ is a parameter exceeding m_κ). The closed linear span of the vectors of the form (13.50) account for the entire subspace $\mathfrak{H}_1$ of one-particle states of $\mathcal{H}$.

The interpretation of particles as elementary particles or as bound states (as in the general case considered in §12.1.A) is in accordance with the division of the fields $\phi^{(\kappa)}$ giving rise to these particles into fundamental and composite ones.

Exercise 13.9. Prove that the condition LSZ.I implies the existence of free fields $\phi_{(0)l}^{(\kappa)}$ of mass m_κ (acting in the Fock space $\mathfrak{H}$ on the one-particle subspace $\mathfrak{H}_1$) such that

$$\langle \Psi_0, \tilde{\phi}_l^{(\kappa)}(p) \tilde{\phi}_m^{(\kappa')}(p') \Psi_0 \rangle = \langle 0 | \tilde{\phi}_{(0)l}^{(\kappa)}(p) \tilde{\phi}_{(0)m}^{(\kappa')}(p') | 0 \rangle \tag{13.52}$$

in the region $p \in G_\kappa$. In particular, for a scalar field $\phi(x)$, the Källén-Lehmann representation has the form

$$\langle \Psi_0, \phi(x)\phi(y)\Psi_0 \rangle = Z \frac{1}{i} D_m^{(-)}(x-y) + \frac{1}{i} \int_{m'}^{\infty} D_\lambda^{(-)}(x-y) d\rho(\lambda) \; (Z > 0, \; m' > m). \tag{13.53}$$

Axiom LSZ.I enables us to apply the Haag-Ruelle construction for the construction of the asymptotic states; here the role of the operators of creation of one-particle states from the vacuum is played by the operators $\sum_l \int \phi_l^{(\kappa)}(x) f^l(x) dx$, where $f^l \in \mathcal{S}(\mathbf{M})$, $\operatorname{supp} \tilde{f}^l \subset G_\kappa$.

It follows from the next LSZ axiom that the S-matrix is unitary.

LSZ.II (Asymptotic Completeness Condition).

$$\mathcal{H}^{\text{in}} = \mathcal{H} = \mathcal{H}^{\text{out}}. \tag{13.54}$$

It also follows from this axiom that in the Hilbert space of physical states $\mathcal{H}$ there are two (distinct when $S \neq 1$) irreducible systems of free fields acting, namely in-fields and out-fields:

$$\phi^{\text{ex}(\kappa)}(x) = \Omega^{\text{ex}} \phi_{(0)}^{(\kappa)}(x) \Omega^{\text{ex}*}, \quad \text{ex} = \text{in}, \text{out} \tag{13.55}$$

* Here it is immaterial for us whether these vectors are reducible or irreducible subspaces with respect to the group $\mathfrak{P}_0$.

(see Exercise 13.9). The S-matrix connects the asymptotic fields with one another. By agreeing to set $\Omega^{\text{in}} = 1$, $\Omega^{\text{out}} = S^*$, we obtain

$$\phi^{\text{out}(\kappa)}(x) = S^* \phi^{\text{in}(\kappa)}(x) S. \tag{13.56}$$

This relation together with the condition $S|0\rangle = |0\rangle$ completely defines the S-matrix in terms of the asymptotic fields. In particular, the elementary process

$$\kappa_1 + \ldots + \kappa_k \to \bar{\kappa}_{k+1} + \ldots + \bar{\kappa}_n \tag{13.57}$$

(where the "quantum numbers" κ of the particles are parametrized by the "types" of the fields $\phi^{(\kappa)}$) corresponds to the matrix element of the S-matrix between the generalized state vectors and the specified momenta of the particles:

$$\begin{aligned}&\langle \tilde{\phi}^{\text{in}(\bar{\kappa}_{k+1})}(-p_{k+1}) \ldots \tilde{\phi}^{\text{in}(\bar{\kappa}_n)}(-p_n)\Psi_0,\\ &\qquad\qquad S\tilde{\phi}^{\text{in}(\kappa_k)}(-p_k) \ldots \tilde{\phi}^{\text{in}(\kappa_1)}(-p_1)\Psi_0\rangle = \\ &= \langle \tilde{\phi}^{\text{out}(\bar{\kappa}_{k+1})}(-p_{k+1}) \ldots \tilde{\phi}^{\text{out}(\bar{\kappa}_n)}(-p_n)\Psi_0,\ \tilde{\phi}^{\text{in}(\kappa_k)}(-p_k) \ldots \tilde{\phi}^{\text{in}(\kappa_1)}(-p_1)\Psi_0\rangle.\end{aligned} \tag{13.58a}$$

We agree to write the right hand side of this relation in the alternative form

$$\langle 0|\tilde{\phi}^{\text{out}(\kappa_n)}(p_n) \ldots \tilde{\phi}^{\text{out}(\kappa_{k+1})}(p_{k+1})\tilde{\phi}^{\text{in}(\kappa_k)}(-p_k) \ldots \tilde{\phi}^{\text{in}(\kappa_1)}(-p_1)|0\rangle \tag{13.58b}$$

(in this notation it is understood that the out-field acts to the left of the " bra-vector", as in formula (13.58a)).

In fact the expression (13.58b) also has a direct meaning. Let $\mathfrak{D}$ be the set of vectors of the Hilbert space $\mathcal{H}$ entering into the domain of definition of any positive integral power $(p^0)^n$ of the energy operator.

Exercise 13.10. (a) Prove that $\mathfrak{D}$ is a set of vectors $\Psi \in \mathcal{H}$ such that the vector-valued function $U(a)\Psi$ is C^∞-dependent on $a \in M$,

(b) Prove that for any function $f \in \mathcal{S}(\mathrm{M})$ the operator $\phi^{\text{ex}(\kappa)}(f)$ is defined (along with its adjoint) on the domain $\mathfrak{D}$ and takes it into itself. [Hint: It follows from the estimate (7.124) for the (bosonic) creation and annihilation operators that the $\phi^{\text{ex}(\kappa)}(f)^{(*)}$ are defined (by closure) on the domain of definition of the operator $\sqrt{N_\kappa}$, where N_κ is the number of particles of type κ; since $N_\kappa \leq m_\kappa^{-1}P^0$, it follows that $\phi^{\text{ex}(\kappa)}(f)^{(*)}$ is defined on $\mathfrak{D}$. It follows from the relation

$$U(a)\phi^{\text{ex}(\kappa)}(f)^{(*)}\Psi = \phi^{\text{ex}(\kappa)}(f_a)^{(*)}U(a)\Psi,$$

where $f_a(x) \equiv f(x-a)$, that $\phi^{\text{ex}(\kappa)}(f)^{(*)}$ takes $\mathfrak{D}$ into itself.]

It follows from Exercise 13.10 that the expressions $\tilde{\phi}^{\text{ex}(\kappa)}(p)$ are operator-valued generalized functions on the domain of definition $\mathfrak{D}$, where for any $f(p_1, \ldots, p_n) \in \mathcal{S}(\mathrm{M}^n)$, the operator

$$\int \tilde{\phi}^{\text{ex}_1(\kappa_1)}(p_1) \ldots \tilde{\phi}^{\text{ex}_n(\kappa_n)}(p_n) f(p_1, \ldots, p_n) d_4p_1 \ldots d_4p_n$$

takes $\mathfrak{D}$ into itself.

In the next exercise we consider the connection between the amplitudes of the process (13.57) and the TCP-transformed process

$$\kappa_n + \ldots + \kappa_{k+1} \to \bar{\kappa}_k + \ldots + \bar{\kappa}_1. \tag{13.59}$$

Exercise 13.11. (a) Prove the transformation law for asymptotic fields under a TCP-transformation:

$$\Theta \phi^{\substack{\text{in}\\ \text{out}}(\kappa)}(\kappa)\Theta^{-1} = (-1)^{2j} i^{F(\kappa)} \phi^{\substack{\text{out}\\ \text{in}}(\kappa)}(-\kappa)^* \tag{13.60}$$

(cf. (9.62)).

(b) Prove the following relations (expressing TCP-invariance of the S-matrix):

$$\begin{aligned}\langle 0|\tilde{\phi}^{\text{out}(\kappa_n)}(p_n)\ldots\tilde{\phi}^{\text{out}(\kappa_{k+1})}(p_{k+1})\tilde{\phi}^{\text{in}(\kappa_k)}(-p_k)\ldots\tilde{\phi}^{\text{in}(\kappa_1)}(-p_1)|0\rangle = \\ = (-1)^{F/2+2J}\langle 0|\tilde{\phi}^{\text{out}(\kappa_1)}(p_1)\ldots\tilde{\phi}^{\text{out}(\kappa_k)}(p_k)\tilde{\phi}^{\text{in}(\kappa_{k+1})}(-p_{k+1})\ldots\tilde{\phi}^{\text{in}(\kappa_n)}(-p_n)|0\rangle \end{aligned} \tag{13.61}$$

(the notation is the same as in (9.64)). [Hint: Use the equalities (13.60).]

Finally, the third axiom enables us to employ the apparatus of Green's functions.

LSZ.III (Existence of T-products). *The $\overset{(-)}{T}$-products $\overset{(-)}{T}(\phi_{l_1}^{(\kappa_1)}(x_1)\ldots\phi_{l_n}^{(\kappa_n)}(x_n))$ are operator-valued generalized functions of temperate growth satisfying all the conditions* (13.2). *The results of smoothing them with test functions are operators which are defined together with their adjoints on a domain D (of the same type as in the Wightman axiom W.IV) and leave it invariant.*

In the next section we concern ourselves with corollaries of the above axioms.

13.2. Asymptotic Conditions and Reduction Formulae

A. LSZ ASYMPTOTIC CONDITIONS

Lehmann, Symanzik and Zimmermann (1955) stated the asymptotic conditions in terms of weak convergence for the operators *

$$\int_{x^0=t} f(x)\overleftrightarrow{\partial}_0\phi^{(\kappa)}(x)d^3x, \tag{13.62}$$

where $f(x)$ is a solution of the Klein-Gordon equation in the class $\mathcal{S}(\mathbf{R}^3)$, that is, it has the form

$$f(x) = \int(e^{ipx}h_+(\mathbf{p}) + e^{-ipx}h_-(\mathbf{p}))\frac{d_3p}{2\omega}, \tag{13.63}$$

$$\omega \equiv \omega_\kappa = \sqrt{\mathbf{p}^2 + m_\kappa^2}; \tag{13.64}$$

here the functions $h_\pm(\mathbf{p})$ are of class $\mathcal{S}(\mathbf{R}^3)$. It was also postulated that

$$\lim_{t\to\mp\infty}\langle\Phi, \int_{x^0=t} f(x)\overleftrightarrow{\partial}_0\phi^{(\kappa)}(x)d^3x\Psi\rangle = \langle\Phi, \int_{x^0=t} f(x)\overleftrightarrow{\partial}_0\phi^{\text{ex}(\kappa)}(x)d^3x\Psi\rangle \tag{13.65}$$

on some dense set of vectors Φ, Ψ in $\mathfrak{H}$. The asymptotic fields $\phi^{\text{ex}(\kappa)}$ occurring here satisfy the Klein-Gordon equation

$$K_x^{(\kappa)}\phi^{\text{ex}(\kappa)}(x) \equiv (\partial_\mu\partial^\mu + m_\kappa^2)\phi^{\text{ex}(\kappa)} = 0.$$

We introduce

$$K_x^{(\kappa)}\phi^{(\kappa)}(x) = j^{(\kappa)}(x); \tag{13.66}$$

* In fact the expression (13.62) generally requires smoothing with respect to the variable t to give it the meaning of an operator.

this is the so-called *current-like field.* Using the asymptotic condition we obtain the following relation between the quantum field $\phi(x)$ and the asymptotic fields ϕ^{ex}:

$$\phi^{(\kappa)}(x) = \phi^{\text{in}(\kappa)}(x) + \int D^{\text{ret}}_{m_\kappa}(x-y)j^{(\kappa)}(x)d^4y = \tag{13.67a}$$

$$= \phi^{\text{out}(\kappa)}(x) + \int D^{\text{adv}}_{m_\kappa}(x-y)j^{(\kappa)}(x)d^4y \tag{13.67b}$$

(where $D^{\substack{\text{ret}\\ \text{adv}}}_m(x)$ is the retarded and advanced Green's function of the free scalar field). On subtracting these two equations we obtain the relation between the asymptotic fields:

$$\phi^{\text{out}(\kappa)}(x) = \phi^{\text{in}(\kappa)} + \int D_{m_\kappa}(x-y)j^{(\kappa)}(y)d^4y \tag{13.68}$$

(where $D_m(x)$ is the Pauli-Jordan commutation function). The relations (13.67), (13.68) are called the *Yang-Feldman equations.*

Following Hepp (1965a), we show how to derive the asymptotic condition and the Yang-Feldman formulae from the LSZ axioms. To this end (using formulae of type (12.6)), we introduce the operator-valued generalized functions

$$\phi^{t(\kappa)}(x) = \int_{y^0=t} D_m(y)\overleftrightarrow{\partial}_0\phi^{(\kappa)}(x+y)d^3y, \tag{13.69}$$

that is

$$\tilde{\phi}^{t(\kappa)}(p) = \left\{\frac{\omega_\kappa + p^0}{2\omega_\kappa}e^{-i(p^0-\omega_\kappa)t} + \frac{\omega_\kappa - p^0}{2\omega_\kappa}e^{-i(p^0+\omega_\kappa)t}\right\}\tilde{\phi}^{(\kappa)}(p), \tag{13.70}$$

and, as usual, we set

$$\phi^{t(\kappa)}(f) = \int \phi^{t(\kappa)}(x)f(x)d^4x \quad \text{for } f \in \mathcal{S}(\mathbf{M}). \tag{13.71}$$

Then according to Theorem 12.1 (Haag-Ruelle), the limit

$$\lim_{t\to\mp\infty} \Phi^t = \Phi^{\text{ex}} \tag{13.72}$$

exists (in the strong topology in $\mathcal{H}$), where

$$\Phi^t = \phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(f_n)|0\rangle, \tag{13.73}$$

$$\Phi^{\text{ex}} = \phi^{\text{ex}(\kappa_1)}(f_1)\dots\phi^{\text{ex}(\kappa_n)}(f_n)|0\rangle \tag{13.74}$$

subject to the condition that $\tilde{f}_j \in \mathcal{S}(G_{\kappa_j})$.

Exercise 13.12. Prove that for $f_j \in \mathcal{S}(\mathbf{M})$ the expression $\langle 0|\phi^{t(\kappa_1)}(f_1)\phi^{t(\kappa_2)}(f_2)|0\rangle$ has a limit as $t \to \mp\infty$; if $m_1 \neq m_2$, then this limit is equal to zero. [Hint: This expression consists of four terms of the form $\int e^{i(\pm\omega_1(\mathbf{p})\pm\omega_2(\mathbf{p}))t}\tilde{W}(p)F(p)d_4p$, which, upon integrating with respect to p^0 can be written in the form

$$\int e^{i(\pm\omega_1(\mathbf{p})\pm\omega_2(\mathbf{p}))t}h(\mathbf{p})d_3p, \quad \text{where } h(\mathbf{p}) \in \mathcal{S}(\mathbf{R}^3)$$

in view of Corollary 12.9; now use the well-known Riemann-Lebesgue theorem on the Fourier transform of integrable functions; see [T2],Vol.3,Ch.XIX,§717.]

Lemma 13.3. *Let $f_j \in S(G_{\kappa_j})$, $f \in S(\mathrm{M})$. Then*

$$\lim_{t\to\mp\infty} \phi^{t(\kappa)}(t)\phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(f_n)|0\rangle =$$
$$= \phi^{\mathrm{ex}(\kappa)}(f)\phi^{\mathrm{ex}(\kappa_1)}(f_1)\dots\phi^{\mathrm{ex}(\kappa_n)}(f_n)|0\rangle \qquad (13.75)$$

in the weak topology in $\mathcal{H}$.

■ Denote the expression under the limit sign by Ψ^t and the right hand side of (13.75) by Ψ^{ex}. First we show that the vector-valued function Ψ^t is bounded in norm. For this, we expand $\langle\Psi^t, \Psi^t\rangle$ in truncated vacuum expectation values. As in the proof of the Haag-Ruelle theorem (§12.2.D), the n-point truncated vacuum expectation values with $n \geq 3$ tend to zero as $t \to \infty$. The two-point vacuum expectation values also have a limit as $t \to \infty$ (see Exercise 13.12); therefore $\langle\Psi^t, \Psi^t\rangle$ has a limit and hence is bounded. It now suffices to prove that

$$\lim_{t\to\mp\infty} \langle\Phi, \Psi^t\rangle = \langle\Phi, \Psi^{\mathrm{ex}}\rangle \qquad (13.76)$$

for vectors of the form $\Phi = \Phi^{\mathrm{ex}}$ (13.74), since such vectors form a total subset of $\mathcal{H}$. We use the fact that $\Phi^{\mathrm{ex}} = \lim_{t\to\mp\infty} \Phi^t$ in norm; equality (13.76) then follows from the limit

$$\lim_{t\to\mp\infty} \langle\Phi^t, \Psi^t\rangle = \langle\Phi^{\mathrm{ex}}, \Psi^{\mathrm{ex}}\rangle,$$

which is proved by the same argument as in the proof of the existence of the limit $\lim_{t\to\mp\infty} \langle\Psi^t, \Psi^t\rangle$ (that is, we expand in truncated vacuum expectation values and apply Exercise 13.2 to the two-point vacuum expectation values). ■

We now investigate the speed of convergence to the limit in (13.72) under the hypothesis that the supports of the $\tilde{f}_j$ are pairwise disjoint in the velocity space, that is,

$$\frac{1}{\omega_{\kappa_i}}\mathbf{p}_i \neq \frac{1}{\omega_{\kappa_j}}\mathbf{p}_j \quad \text{for } i \neq j, \text{ if } p_l \in \operatorname{supp} \tilde{f}_l,\ l = 1, \dots, n. \qquad (13.77)$$

In this case we say that the functions $\tilde{f}_j$ are non-intersecting in the velocity space. It turns out that for such f_j, the expression under the limit sign in (13.72) differs from the right hand side by the quantity $o(|t|^{-N})$, where N is an arbitrary natural number. More precisely, we have the following lemma.

Lemma 13.4. *For non-intersecting functions $\tilde{f}_1, \dots, \tilde{f}_n$ in the velocity space, where $\tilde{f}_j \in \mathcal{D}(G_{\kappa_j})$, the following inequalities hold for all $N > 0$:* *

$$\left\|\frac{d}{dt}(\phi_{l_1}^{t(\kappa_1)}(f_1)\dots\phi_{l_n}^{t(\kappa_n)}(f_n))\Psi_0\right\| \leq c_N(1 + |t|)^{-N} \qquad (13.78)$$

(*where $c_N < \infty$*).

■ We expand the square of the left hand side of (13.78) in a sum of products of truncated vacuum expectation values. As in the proof of the Haag-Ruelle theorem, all one-point functions and all products containing only two-point functions are equal to zero. Therefore we only have to consider l-point truncated functions with $l \geq 3$. We can write a typical term in the form

$$\int \tilde{w}^T(p_1, \dots, p_l)\overline{\tilde{f}_1(p_1)}\dots\overline{\tilde{f}_k(p_k)}\tilde{f}_{k+1}(-p_{k+1})\dots\tilde{f}_n(-p_l)\times$$
$$\times\prod_{j=1}^{l}\Big(\frac{\omega_j \mp p_j^0}{2\omega_j}e^{-i(p_j^0\pm\omega_j)t}\Big)d_4p_1\dots d_4p_l, \qquad (13.79)$$

* We recall that for $N = 3/2$, the inequality (13.78) was proved in §12.2.D without the non-intersection assumption. It follows from Lemma 13.4 that when the velocities are non-intersecting, the Haag-Ruelle limit (13.72) is approached more rapidly than any power of $|t|^{-1}$.

where $\omega_j \equiv \omega_{\kappa_j}$. By virtue of the condition $l \geq 3$, the general term reduces to one of the following three possibilities: a) $k \geq 3$, b) $l-k \geq 3$, c) $k \geq 1$, $l-k \geq 1$.

We shall go into details for the first possibility $k \geq 3$. Then for some pair of values of $j = 1, 2, \ldots, k$, the signs of $\pm\omega_j$ in (13.79) are the same. Suppose for definiteness that there is a + sign in front of ω_1 and ω_2 (the other cases are completely similar). Taking translation invariance into account, we can rewrite (13.79) in the form

$$\int F(p_2, \ldots, p_l) e^{-i\Omega(p_2,\ldots,p_l)t} \prod_{j=2}^{l} d_3 p_j, \tag{13.80}$$

where

$$F(p_2, \ldots, p_l) = \int \tilde{w}^T(p_1, \ldots, p_l)\overline{\tilde{f}_1(p_1)} \ldots \overline{\tilde{f}_k(p_k)}\tilde{f}_{k+1}(-p_{k+1}) \ldots f_l(-p_l) \prod_{j=1}^{l} \left(\frac{\omega_j \mp p_j^0}{2\omega_j}\right) d_4 p_1 \prod_{j=2}^{l} d_1 p_j^0$$

is a function in $\mathcal{S}(\mathbf{R}^{3(l-1)})$ (by virtue of Corollary 12.9); furthermore, this is a function in $\mathcal{D}(\mathbf{R}^{3(l-1)})$, since $f_j \in \mathfrak{D}(G_{\kappa_j})$; moreover we set

$$\Omega(\mathbf{p}_2, \ldots, \mathbf{p}_l) = \omega_1\Big(\sum_{j=2}^{l} \mathbf{p}_j\Big) + \omega_2(\mathbf{p}_2) + \sum_{k=3}^{l} \pm\omega_k(\mathbf{p}_k).$$

This is a smooth function in $\mathbf{p}_2, \ldots, \mathbf{p}_l$. Since $\tilde{f}_1$ and $\tilde{f}_2$ are non-intersecting in the velocity space, we have

$$\frac{\partial\Omega}{\partial\mathbf{p}_2} = \frac{1}{\omega_1}\sum_{j=2}^{l} \mathbf{p}_j + \frac{1}{\omega_2}\mathbf{p}_2 = \frac{1}{\omega_2}\mathbf{p}_2 - \frac{1}{\omega_1}\mathbf{p}_1 \neq 0$$

at points in the support of F. It follows that there is a partition of unity $\alpha_i(\mathbf{p}_2, \ldots, \mathbf{p}_l)$, namely a triple of C^∞-functions α_i with sum $\sum_{i=1}^{3} \alpha_i \equiv 1$ such that $\partial\Omega/\partial p_2^i \neq 0$ for $(\mathbf{p}_2, \ldots, \mathbf{p}_l) \in \operatorname{supp} F \cdot \alpha_i$. The change of variables $p_2^i \to \Omega$ is then regular in a neighbourhood of $\operatorname{supp} F \cdot \alpha_i$, consequently the function

$$\int F(\mathbf{p}_2, \ldots, \mathbf{p}_l)\alpha_i(\mathbf{p}_2, \ldots, \mathbf{p}_l) e^{i\Omega t} \prod_{j=2}^{l} d_3 p_j$$

with respect to the variable t belongs to $\mathcal{S}(\mathbf{R})$ for $i = 1, 2, 3$. Hence the expression (13.79) decreases more rapidly than any negative power of $|t|$ as $t \to \infty$, which completes the proof of the estimate (13.78).

The case (b) is completely similar to (a). Finally, we consider the case (c). It follows from Axiom LSZ.I that we have $\frac{\omega_1 - p_1^0}{2\omega_1}\tilde{w}^T(p_1, \ldots, p_n) = 0$ in the domain $p_1 \in G_{\kappa_1}$, so that we can restrict ourselves to the case of a $-$ sign in front of ω_1 in the exponent in (13.79). For a similar reason we can suppose that $\pm\omega_l$ in (13.79) denotes $+\omega_l$. As in case (a), we write (13.79) in the form (13.80), where this time

$$\Omega(\mathbf{p}_2, \ldots, \mathbf{p}_l) = -\omega_1\Big(\sum_{j=2}^{l} \mathbf{p}_j\Big) + \omega_l(\mathbf{p}_l) + \sum_{k=2}^{l-1} \pm\omega_k(\mathbf{p}_k).$$

Since $\tilde{f}_1(p_1)$ and $\tilde{f}_l(p_l)$ are non-intersecting in the velocity space, we have

$$\frac{\partial\Omega}{\partial\mathbf{p}_l} = -\frac{1}{\omega_1}\Big(\sum_{j=2}^{l} \mathbf{p}_j\Big) + \frac{1}{\omega_l}\mathbf{p}_l = \frac{1}{\omega_1}\mathbf{p}_1 + \frac{1}{\omega_l}\mathbf{p}_l \neq 0$$

at points of the support of F. The rest of the argument is the same as for case (a). ∎

Vectors that can be written in the form (13.74) with functions $\tilde{f}_j \in \mathfrak{D}(G^+_{\kappa_j})$ that are non-intersecting in the velocity space are vectors of the Fock space $\mathcal{H}^{\rm in}$ with pairwise non-collinear momenta of the particles; here

$$G^\pm_n = \{p \in \mathbf{M} : \pm p^0 > 0,\ 0 < p^2 < m'^2_\kappa\}. \tag{13.81}$$

We denote by $D^{\rm ex}_0$ (ex = in, out) the set of linear combinations of all such vectors. It clearly follows from the definition of the Fock space of relativistic particles (§7.3.A) that $D^{\rm ex}$ is an everywhere-dense Poincaré-invariant subset of $\mathcal{H}^{\rm ex} = \mathcal{H}$.

It turns out that the operators A in the polynomial field algebra $\mathcal{P}(\mathbf{M})$ can be defined (by closure) on the vectors in $D^{\rm ex}_0$. We have

$$\Phi^{\rm out} = \Phi^t - \int_t^\infty \frac{d}{dt}\Phi^s ds, \tag{13.82}$$

therefore we set

$$A\Phi^{\rm out} = A\Phi^t - \int_t^\infty A\frac{d}{ds}\Phi^s ds \tag{13.83}$$

($A\Phi^{\rm in}$ is defined similarly). Here we use the estimate

$$\left\|A\frac{d}{ds}\Phi^s\right\|^2 \leq \left\|\frac{d}{ds}\Phi^s\right\| \cdot \left\|A^*A\frac{d}{ds}\Phi^s\right\|. \tag{13.84}$$

By Lemma 13.4, the first factor on the right hand side of this inequality decreases more rapidly than any negative power of $|s|$ as $s \to \infty$; with regard to the second factor, it is not difficult to see (using the definition of Φ^s) that it is of polynomial growth in s. As a result, $\|A\frac{d}{ds}\Phi^s\|$ decreases more rapidly than any negative power of $|s|$, therefore (13.83) is well-defined. In particular, if it is a smoothed monomial of type (8.8), then by improving the estimate for $\|A^*A\frac{d}{ds}\Phi^s\|$:

$$\left\|A^*A\frac{d}{ds}\Phi^s\right\| \leq c\|f\|^2_{l,m}(1+|s|)^L$$

(where c, l, m, L are certain numbers), it is not difficult to see that $A\Phi^{\rm ex} \to 0$ as $f \to 0$ in $\mathcal{S}(\mathbf{M})$, that is, expressions of type $\phi^{(\kappa_1)}_{l_1}(x_1)\dots\phi^{(\kappa_n)}_{l_n}(x)$ can be regarded as operator-valued generalized functions which are also defined on $D^{\rm ex}_0$.

On this basis we can now derive the LSZ asymptotic condition.

Theorem 13.5. (a) *Let $\Phi^{\rm ex} \in D^{\rm ex}_0$ and $f_j \in \mathcal{S}(G_{\kappa_j})$ for $j = 1,\dots,n$. Then the following limiting relation holds (in the sense of convergence in norm in $\mathcal{H}$):*

$$\lim_{t\to t^{\rm ex}} \phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(f_n)\Phi^{\rm ex} = \phi^{{\rm ex}(\kappa_1)}(f_1)\dots\phi^{{\rm ex}(\kappa_n)}(f_n)\Phi^{\rm ex} \tag{13.85}$$

($t^{\rm ex} = -\infty$ for ex = in and $t^{\rm ex} = +\infty$ for ex = out).

(b) *Let $\Phi^{\rm ex} \in D^{\rm ex}_0$, $f \in \mathcal{S}(\mathbf{M})$, then*

$$\lim_{t\to t^{\rm ex}} \phi^{t(\kappa)}(f)\Phi^{\rm ex} = \phi^{{\rm ex}(\kappa)}(f)\Phi^{\rm ex} \tag{13.86}$$

in the weak topology in $\mathcal{H}$.

■ (a) We set ex=out for definiteness. According to (13.83) we have:

$$\|\phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(f_n)\Phi^{\text{out}} - \phi^{t(\kappa_1)}(f_1)\dots\phi^{(\kappa_n)}(f_n)\Phi^t\| \leq$$
$$\leq \int_t^\infty \left\|\phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(t_n)\frac{d}{ds}\Phi^s\right\| ds. \tag{13.87}$$

An estimate of type (13.84) shows that the integrand in (13.87) can be majorized by the function $c_N(1+|t|)^L(1+|s|)^{-N}$, where L is fixed and N is arbitrary. It follows that the left hand side of (13.87) tends to zero as $t \to +\infty$. The limiting relation (13.85) now follows from the fact that (by the Haag-Ruelle theorem)

$$\phi^{t(\kappa_1)}(f_1)\dots\phi^{t(\kappa_n)}(f_n)\Phi^t \to \phi^{\text{out}(\kappa_1)}(f_1)\dots\phi^{\text{out}(\kappa_n)}(f_n)\Phi^{\text{out}} \quad \text{as } t \to +\infty$$

in norm.

(b) An estimate of type (13.87) gives

$$\|\phi^{t(\kappa)}(f)\Phi^{\text{ex}} - \phi^{t(\kappa)}(f)\Phi^t\| \to 0 \quad \text{as } t \to t^{\text{ex}}.$$

According to Lemma 13.3, $\phi^{t(\kappa)}(f)\Phi^t \to \phi^{\text{ex}(\kappa)}(f)\Phi^{\text{ex}}$ as $t \to t^{\text{ex}}$ in the weak topology, whence (13.86) follows. ■

Remark. The actual form of the factor $(\omega \pm p^0)/2\omega$ in (13.70) is not in fact particularly important; the only thing that matters is that $(\omega + p^0)/2\omega$ should be equal to unity on Γ_m^+ and zero on Γ_m^- and that $(\omega - p^0)/2\omega$ should be equal to zero on Γ_m^+ and unity on Γ_m^-. In certain situations it is more convenient to use factors with specific properties of the support. We introduce the arbitrary real function $\Lambda^{(\kappa)}(p)$ of class $\mathcal{O}_M(\mathbf{M})$ with the properties:

$$\operatorname{supp}\Lambda^{(\kappa)} \subset G_\kappa^+,\ \Lambda^{(\kappa)}(p) = 1 \quad \text{for } p^0 = \omega_\kappa(\mathbf{p}),$$

and define the operators (for $f \in \mathcal{S}(\mathbf{M})$)

$$\phi^{(\kappa)}(f;t) = \int \phi^{(\kappa)}(x)f(x;t)dx, \tag{13.88}$$

where

$$f(x;t) = \int e^{-ipx}\left(\Lambda^{(\kappa)}(p)e^{i(p^0-\omega_\kappa(\mathbf{p}))t} + \Lambda^{(\kappa)}(-p)e^{i(p^0+\omega_\kappa(\mathbf{p}))t}\right)\tilde{f}(p)d_4p. \tag{13.89}$$

By (an almost verbatim) repetition of the proof of the Haag-Ruelle theorem (§12.2.D), it is easy to show that we have the following limit (in norm) for $f_j \in \mathcal{S}(\mathbf{M})$

$$\lim_{t\to\mp\infty} \phi^{(\kappa_1)}(f_1;t)\dots\phi^{(\kappa_n)}(f_n;t)|0\rangle = \phi^{\text{ex}(\kappa_1)}(f_1)\dots\phi^{\text{ex}(\kappa_n)}(f_n)|0\rangle. \tag{13.90}$$

For test functions $\tilde{f}_j \in D(G_{\kappa_j}^+)$, that are non-intersecting in the velocity space, the vector (13.90) belongs to D_0^{ex}. Supposing that $f_j \in \mathcal{S}(\mathbf{M})$ and $\Phi^{\text{ex}} \in D_0^{\text{ex}}$, we obtain along with (13.85) the following limiting relation in the norm topology:

$$\lim_{t\to t^{\text{ex}}} \phi^{(\kappa_1)}(f_1;t)\dots\phi^{(\kappa_n)}(f_n;t)\Phi^{\text{ex}} = \phi^{\text{ex}(\kappa_1)}(f_1)\dots\phi^{\text{ex}(\kappa_n)}(f_n)\Phi^{\text{ex}}. \tag{13.91}$$

B. Yang-Feldman equations

We now show that the formally written Yang-Feldman equations (13.63) can also be interpreted as equalities on D_0^{ex}; here the four-dimensional convolutions are to be understood in the sense of limits (of vector-valued generalized functions):

$$\int D_m^{\substack{\text{ret}\\ \text{adv}}}(x-y)j(y)d^4y\Phi = -\lim_{\epsilon\to+0}\int \frac{e^{-ipx}}{(p^0 \pm i\epsilon)^2 - \mathbf{p}^2 - m^2}\tilde{j}(p)\Phi d_4p.$$

Theorem 13.6 *For $\Phi^{\text{ex}} \in D_0^{\text{ex}}$ and $m \equiv m_\kappa$ the Yang-Feldman equations hold:*

$$\phi^{(\kappa)}(x)\Phi^{\substack{\text{in}\\ \text{out}}} = \phi^{\substack{\text{in}\\ \text{out}}(\kappa)}\Phi^{\substack{\text{in}\\ \text{out}}} + \int D_m^{\substack{\text{ret}\\ \text{adv}}}(x-y)j^{(\kappa)}(y)\Phi^{\substack{\text{in}\\ \text{out}}} d^4y. \tag{13.92}$$

■ We shall confine attention to the case ex=out. Any function $f \in \mathcal{S}(\mathbf{M})$ can be split up into a sum $f = f_1 + f_2$ such that $\operatorname{supp}\tilde{f}_1 \in G_\kappa$, and $\operatorname{supp}\tilde{f}_2$ is contained in the set $|p^2 - m^2| > \alpha$ ($\alpha > 0$). We set

$$F(t) = \frac{d}{dt}\phi^{t(\kappa)}(f_1)\Phi^{\text{out}}.$$

Using estimates of type (13.87) it is easy to see that

$$\|F(t) - \frac{d}{dt}\phi^{t(\kappa)}(f_1)\Phi^t| \to 0 \quad \text{as } t \to \infty$$

more rapidly than any negative power of t. On the other hand, it follows from the proof of the Haag-Ruelle theorem that

$$\left\|\left(\frac{d}{dt}\phi^{t(\kappa)}(f_1)\right)\Phi^t\right\| \le \text{const}\cdot(1+|t|)^{-3/2},$$

therefore

$$\|F(t)\| \le \text{const}\cdot(1+|t|)^{-3/2}. \tag{13.93}$$

Hence we obtain

$$\phi^{(\kappa)}(f_1)\Phi^{\text{out}} - \phi^{\text{out}(\kappa)}(f_1)\Phi^{\text{out}} = -\lim_{t\to+\infty}\int_0^t F(s)ds = -\lim_{\epsilon\to+0}\lim_{t\to+\infty}\int_0^\infty e^{-\epsilon t}F(t)dt \tag{13.94}$$

(the first equality is based on Theorem 13.5 and the second, on the estimate (13.93)). We use the definition of $F(t)$:

$$\lim_{t\to+\infty}\int_1^t e^{-\epsilon s}F(s)ds = -i\lim_{t\to+\infty}\int_0^t\Big(\int\frac{1}{2\omega}(e^{-i(p^0-i\epsilon-\omega)s} - e^{-i(p^0-i\epsilon+\omega)s})\tilde{j}^{(\kappa)}(p)\tilde{f}_1(-p)\Phi^{\text{out}}d_4p\Big)dt.$$

It is not difficult to see that

$$-i\int_0^t\frac{1}{2\omega}(e^{-i(p^0-i\epsilon-\omega)s} - e^{-i(p^0-i\epsilon+\omega)s})\tilde{f}_1(-p)ds \to -\frac{1}{(p^0-i\epsilon)^2-\mathbf{p}^2-m^2}\tilde{f}_1(-p)$$

as $t\to+\infty$ in the $\mathcal{S}(\mathbf{M})$ topology; therefore

$$\lim_{t\to+\infty}\int_0^t e^{-\epsilon t}F(t)dt = -\int\frac{1}{(p^0-i\epsilon)^2-\mathbf{p}^2-m^2}\tilde{f}_1(-p)\tilde{j}^{(\kappa)}(p)d_4p.$$

Substituting this into (13.94) we see that (13.92) holds on smoothing with the test function $f_1(x)$.

Since

$$\frac{1}{(p^0-i\epsilon)^2-\mathbf{p}^2-m^2}(p^2-m^2)\tilde{f}_2(-p) \to \tilde{f}_2(-p) \quad \text{as } \epsilon\to+0$$

in the $\mathcal{S}(\mathbf{M})$ topology, we have the identity

$$\phi^{(\kappa)}(f_2) = \int D^{\text{ret}}(x-y)j^{(\kappa)}(y)f_2(x)dx\,dy.$$

Since, by construction, the support of $\tilde{f}_2$ does not intersect the mass shell, $\phi^{\text{out}(\kappa)}(f_2) = 0$, so that (13.94) also holds on smoothing with $f_2(x)$. ■

C. Partial Reduction Formulae

Lehmann, Symanzik and Zimmermann found the following interesting formula, suggested by perturbation theory:

$$\langle 0|\tilde{\phi}_1^{\rm out}(p_1)\ldots\tilde{\phi}_k^{\rm out}(p_k)\tilde{\phi}_{k+1}^{\rm in}(-p_{k+1})\ldots\tilde{\phi}_n^{\rm in}(-p_n)|0\rangle =$$
$$= i^n\prod_{i=1}^n \delta^+_{m_i}(p_i)\int K_{x_1}\ldots K_{x_n}\langle 0|T(\phi_1(x_1)\ldots\phi_n(x_n))|0\rangle\times$$
$$\times\prod_{i=1}^k e^{ip_jx_j}\prod_{j=k+1}^n e^{-ip_jx_j}\,dx_1\ldots dx_n \qquad (13.95)$$

(here the momenta $p_1\in G^+_{\kappa_1},\ldots,\ p_n\in G^+_{\kappa_n}$ are pairwise non-collinear). According to this formula, the calculation of the matrix element of the scattering operator between two generalized vector states with specified particle momenta can be split up into two stages: the calculation of the Fourier transform of the amputated* causal Green's function of the interacting fields and the restriction of the expression so obtained to the mass shell. Following Lehmann, Symanzik and Zimmermann, we call the extensive class of relations in which this type of replacement "particle $\to$ fields" occurs, reduction formulae. In this section we give reduction formulae, of importance for the subsequent development, in which the transition from particles to fields only takes place for part of the momenta.

The restriction of the generalized function $\widetilde{F}(p)$ from $\mathcal{S}'(\mathbf{M})$ to the mass shell Γ^+_m is well defined, for example, in the case when $\widetilde{F}(p)$ can be regarded as a generalized function of $\mathbf{p}$ in a neighbourhood of Γ^+_m that is continuously dependent on the parameter $\lambda = p^0-\omega(\mathbf{p})$ (the case of several momenta is similar). But generally speaking, the restriction of the generalized function $\widetilde{F}(p)$ to the mass shell Γ^+_m and the operation of multiplication of $\widetilde{F}(p)$ by $\delta^+_m(p)$ (which is equivalent to it) are not defined. We shall define this multiplication by

$$\int \delta^+_m(p)\widetilde{F}(p)\tilde{f}(-p)d_4p = \lim_{t\to+\infty}\int\frac{\sin(p^0-\omega)t}{(p^0-\omega)\omega}\widetilde{F}(p)\tilde{f}(-p)d_4p \qquad (13.96)$$

in the case when the limit on the right hand side exists for all test functions $\tilde{f}\in$ $\in\mathcal{S}(G^-_\kappa)$ and depends only on the restriction of $\tilde{f}$ to Γ^-_m. We then use for the right hand side the coordinate representation

$$\int\delta^+_m(p)\tilde{F}(p)\tilde{f}(-p)d_4p \equiv \int F(x)u(x)d^4x, \qquad (13.97)$$

where

$$u(x) = \int\delta^+_m(-p)\tilde{f}(p)e^{-ipx}d_4p. \qquad (13.98)$$

(It is clear that the right hand side of (13.97) is not defined by itself, since generally speaking, $u(x)$ is not a test function; formula (13.97) fulfills the role of a definition.) The products of $\widetilde{F}(p)$ and $\delta^-_m(p)$ and of $\widetilde{F}(p)$ and $2\pi\epsilon(p^0)\delta(p^2-m^2)\equiv\delta^+_m(p)-\delta^-_m(p)$

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* By the amputated T-product we mean the expression $K_{x_1}\ldots K_{x_n}T(\phi(x_1)\ldots\phi(x_n))$. (The amputated R-products and amputated Green's functions are defined similarly.)

are defined similarly. Multiplication of generalized functions of several momenta by $\prod_{i=1}^{n} \delta_{m_i}^{+}(p_i)$ (as in (13.95)) is carried out with respect to each momentum in succession; unless otherwise specified, it will be understood that such a product does not depend on the order of multiplication.

Exercise 13.13. Prove that the definition given above of the product $2\pi\epsilon(p^0)\delta(p^2 - m^2)$ and $(p^2 - m^2)\widetilde{F}(p)$ can be written in the form

$$-\int 2\pi i\epsilon(p^0)\delta(p^2 - m^2)\{(p^2 - m^2)\tilde{F}(p)\}\tilde{f}(-p)d_4p = \Big(\lim_{t\to+\infty} - \lim_{t\to-\infty}\Big)\int F(x)f(x;t)dx, \quad (13.99)$$

where $\tilde{f}(p) \in \mathcal{S}(G_\kappa^{\pm})$.

The notion of an essential support of a test function also proves to be very useful for the derivation of reduction formulae. Let $f_\tau(x)$ be a function in $\mathcal{S}(\mathbf{R}^n)$ in the variable x which is continuously dependent on the real parameter τ tending to ∞ (the case of several parameters is similar). We say that an *essential support* of the test function f_τ as $\tau \to \infty$ is *concentrated* in the open set $\mathcal{O} \subset \mathbf{R}^n$ if

$$\tau^N \int F(x)f_\tau(x)dx \to \infty \quad \text{as } \tau \to \infty$$

for any N and any generalized function $F \in \mathcal{S}'(\mathbf{R}^n)$ that vanishes on $\mathcal{O}$. In calculating the limit $\lim_{\tau\to\infty} \int F_1(x)f_\tau(x)dx$, this notion allows us to replace (to within an error of order $o(\tau^{-N})$) the generalized function $F_1(x)$ by another generalized function $F_2(x)$ coinciding with $F_1(x)$ in $\mathcal{O}$.

Exercise 13.14. Prove that if $Q \equiv \mathbf{R}^n \backslash \mathcal{O}$ is a canonically closed regular set (see Appendix A.2) and

$$\tau^N \|f_\tau\|_{l,m}^{Q} \to 0 \quad \text{as } \tau \to \infty$$

for all natural numbers l, m, then an essential support of the functions f_τ as $\tau \to \infty$ is concentrated in $\mathcal{O}$.

Lemma 13.7. *Let $f, g \in \mathcal{S}(\mathbf{M})$. Then an essential support of the function $f(x;t)g(y)$ as $t \to +\infty$ (or as $t \to -\infty$) is concentrated in the set $K^+ = \{(x,y) \in \mathbf{M}^2 : x^0 > y^0\}$ (or in the set $K^- = \{(x,y) \in \mathbf{M}^2 : x^0 < y^0\}$).*

■ For any polynomials $P_1(x)$, $Q_1(\partial/\partial x)$ we have *

$$P_1(x)Q_1\Big(\frac{\partial}{\partial x}\Big)f(x;t) = \int e^{-ipx}P_1\Big(-i\frac{\partial}{\partial p}\Big)\Big\{e^{i(p^0-\omega)t}Q_1(-ip)\tilde{f}(p)\Big\}d_4p =$$

$$= \sum_{l=0}^{L} t^l \int e^{-ip^0(x^0-t)}u_l(p)e^{i(\mathbf{px}-\omega t)}d_4p,$$

where L is the degree of the polynomial P_1; the u_l are functions in $\mathcal{S}(\mathbf{M})$. Integrating with respect to p^0, we obtain

$$P_1(x)Q\Big(\frac{\partial}{\partial x}\Big)f(x;t) = \sum_{t=0}^{L} t^l \int v_l(\mathbf{p}, x^0 - t)e^{i(\mathbf{px}-\omega t)}d_3p,$$

where $v_l \in \mathcal{S}(\mathbf{R}^4)$. Using an estimate of type (12.47) for solutions of the Klein-Gordon equation, we obtain the following estimate for any $N > 0$:

$$\Big|P_1(x)Q_1\Big(\frac{\partial}{\partial x}\Big)f(x;t)\Big| \le c_N(1+t)^L(1+|x^0 - t|)^{-N}, \ t > 0. \quad (13.100)$$

* For definiteness, we shall consider only the case $\tilde{f} \subset G_\kappa^{+}$; the general case is similar.

As a result, we obtain the following estimates for any polynomials P_1, P_2, Q_1, Q_2, where N is any number greater than L:

$$\sup_{(x,y)\in K^-}\left|P_1(x)P_2(y)Q_1\Big(\frac{\partial}{\partial x}\Big)Q_2\Big(\frac{\partial}{\partial y}\Big)f(x;t)g(y)\right| \leq$$
$$\leq c'_N \sup_{(x,y)\in K^-}\{(1+t)^L(1+|x^0-t|))^{-N}(1+|y^0|)^{-N}\} \leq$$
$$\leq c'_N \sup_{(x,y)\in K^-}\{(1+t)^L(1+|x^0-y^0-t|)^{-N}\} \leq c''_N(1+t)^{L-N} \quad \text{for } t>0.$$

Thus $\|f(x;t)g(y)\|_{l,m}^{\overline{K^-}} \to 0$ more rapidly than any negative power of t as $t \to +\infty$, whence the required result follows. ■

In the next theorem we give the reduction formulae used in the study of the analytic properties of amplitudes of two-particle processes defined by the formula

$$\begin{aligned}
&\langle 0|\tilde{\phi}^{\mathrm{out}(\kappa_4)}(p_4)\tilde{\phi}^{\mathrm{out}(\kappa_3)}(p_3)\tilde{\phi}^{\mathrm{in}(\kappa_2)}(-p_2)\tilde{\phi}^{\mathrm{in}(\kappa_1)}(-p_1)|0\rangle - \\
&\quad -\langle 0|\tilde{\phi}^{\mathrm{in}(\kappa_4)}(p_4)\tilde{\phi}^{\mathrm{in}(\kappa_3)}(p_3)\tilde{\phi}^{\mathrm{in}(\kappa_2)}(-p_2)\tilde{\phi}^{\mathrm{in}(\kappa_1)}(-p_1)|0\rangle = \\
&\quad = i(2\pi)^4\delta(p_1+p_2-p_3-p_4)\mathcal{T}^{(\kappa_4\kappa_3\kappa_2\kappa_1)}(p_4,p_3,-p_2,-p_1),
\end{aligned} \tag{13.101}$$

where $p_j \in G^+_{m_j}$.

Theorem 13.8. *The following relations hold for $p_j \in G^+_{\kappa_j}$:* *

$$\begin{aligned}
&(2\pi)^4\delta(p_1+p_2-p_3-p_4)\mathcal{T}^{(\kappa_4\kappa_3\kappa_2\kappa_1)}(p_4,p_3,-p_2,-p_1) = \\
&\quad = \delta^+_{m_2}(p_2)\delta^+_{m_3}(p_3)\{(p_2^2-m_2^2)(p_3^2-m_3^2)\langle 0|\tilde{\phi}^{(\kappa_4)}(p_4)\times \\
&\qquad \times A(\tilde{\phi}^{(\kappa_3)}(p_3);\ \tilde{\phi}^{(\kappa_2)}(-p_2))\tilde{\phi}^{(\kappa_1)}(-p_1)|0\rangle\},
\end{aligned} \tag{13.102}$$

$$\begin{aligned}
&\tilde{\phi}^{\mathrm{out}(\kappa_2)}(-p_2)\tilde{\phi}^{\mathrm{out}(\kappa_1)}(-p_1)|0\rangle - \tilde{\phi}^{\mathrm{in}(\kappa_2)}(-p_2)\tilde{\phi}^{\mathrm{in}(\kappa_1)}(-p_1)|0\rangle = \\
&\quad = -i\delta^+_{m_1}(p_1)\delta^+_{m_2}(p_2)\{(p_1^2-m_1^2)(p_2^2-m_2^2)A(\tilde{\phi}^{(\kappa_2)}(-p_2);\tilde{\phi}^{(\kappa_2)}(-p_1))|0\rangle\},
\end{aligned} \tag{13.103}$$

$$\begin{aligned}
&\tilde{j}^{(\kappa_2)}(-p_2)\tilde{\phi}^{\mathrm{in}(\kappa_1)}(-p_1)|0\rangle = \delta^+_{m_1}\{(p_1^2-m_1^2)(p_2^2-m_2^2)A(\tilde{\phi}^{(\kappa_2)}(-p_2); \\
&\qquad \tilde{\phi}^{(\kappa_1)}(-p_1))|0\rangle\}.
\end{aligned} \tag{13.104}$$

■ Let $\tilde{f}_j \in \mathcal{S}(G^-_{\kappa_j})$ for $j=3,4$ and $\tilde{f}_j \in (G^+_{\kappa_j})$ for $j=1,2$; then by definition, we have

$$\begin{aligned}
&i\int(2\pi)^4\delta(p_1+p_2-p_3-p_4)\mathcal{T}(p_4,p_3,-p_2,-p_1)\tilde{f}_4(-p_4)\tilde{f}_3(-p_3)\times \\
&\qquad \times\tilde{f}_2(p_2)\tilde{f}_1(p_1)d_4p_1\dots d_4p_4 = \\
&\quad = \Big(\lim_{s\to+\infty}-\lim_{s\to-\infty}\Big)\lim_{t\to-\infty}\int\langle 0|\phi(f_4)\phi(x)\phi(y)\phi(f_1)|0\rangle f_3(x;s)f_2(y;t)dx\,dy.
\end{aligned} \tag{13.105}$$

We now use Lemma 13.7 and the fact that

$$\phi(x)\phi(y) = -iA(\phi(x);\ \phi(y)) \pm \phi(y)\phi(x) \quad \text{for } x^0>y^0.$$

As a result we can write

$$\begin{aligned}
&\Big(\lim_{s\to+\infty}-\lim_{s\to-\infty}\Big)\lim_{t\to-\infty}(-i)\int\langle 0|\phi(f_4)A(\phi(x);\ \phi(y))\phi(f_1)|0\rangle f_3(x;s)f_2(y;t)dx\,dy\pm \\
&\qquad \pm\langle 0|\phi^{\mathrm{in}}(f_4)\phi^{\mathrm{in}}(f_2)(\phi^{\mathrm{out}}(f_3)-\phi^{\mathrm{in}}(f_3))\phi^{\mathrm{in}}(f_1)|0\rangle.
\end{aligned} \tag{13.106}$$

* In (13.104) we suppose that $p_2 \in G_{\kappa_2}$.

Taking into account the support of $\tilde{f}_3$, we have

$$\langle 0|\phi^{\text{in}}(f_4)\phi^{\text{in}}(f_2) = \langle 0|\phi^{\text{in}}(f_4)\phi^{\text{in}}(f_2)|0\rangle\langle 0|,$$

so that the second term in (13.106) is equal to zero; according to Lemma 13.7, we can replace $\lim\limits_{t\to-\infty}$ by $\lim\limits_{t\to-\infty} - \lim\limits_{t\to+\infty}$. Exercise 13.13 now allows us to write (13.105) in the form

$$i\int (2\pi)^4\delta(p_1+p_2-p_3-p_4)\mathcal{T}(p_4,p_3,-p_2,-p_1)\tilde{f}_4(-p_4)\tilde{f}_3(-p_3)\tilde{f}_2(p_2)\times$$
$$\times\tilde{f}_1(p_1)d_4p_1\ldots d_4p_4 = i\int \delta^+_{m_2}(p_2)\delta^+_{m_3}(p_3)\{(p_2^2-m_2^2)(p_3^2-m_3^2)\times$$
$$\times\langle 0|\tilde{\phi}(p_4)A\Big(\tilde{\phi}(p_3);\tilde{\phi}(-p_2)\Big)\tilde{\phi}(-p_1)|0\rangle\}\tilde{f}_4(-p_4)\tilde{f}_3(-p_3)\tilde{f}_2(p_2)\tilde{f}_1(p_1)d_4p_1\ldots d_4p_4.$$

We now prove (13.103). For $\tilde{f}_j \in \mathcal{S}(G^+_{\kappa_j})$ we have

$$\int\{\tilde{\phi}^{\text{out}}(-p_2)\tilde{\phi}^{\text{out}}(-p_1) - \tilde{\phi}^{\text{in}}(-p_2)\tilde{\phi}^{\text{in}}(-p_1)\}|0\rangle\tilde{f}_2(p_2)\tilde{f}_1(p_1)d_4p_1\,d_4p_2 =$$
$$= \Big(\lim_{s\to+\infty} - \lim_{s\to-\infty}\Big)\lim_{t\to-\infty}\int \phi(x)\phi(y)|0\rangle f_2(x;s)f_1(y;t)dx\,dy. \tag{13.107}$$

Using Lemma 13.7, we can rewrite the right hand side in the form

$$\Big(\lim_{s\to+\infty} - \lim_{s\to-\infty}\Big)\lim_{t\to-\infty}(-i)\int A(\phi(x);\ \phi(y))|0\rangle f_2(x;s)f_1(y;t)dx\,dy\pm$$
$$\pm\Big(\lim_{s\to+\infty} - \lim_{s\to-\infty}\Big)\lim_{t\to-\infty}\int \phi(y)\phi(x)|0\rangle f_2(x;s)f_1(y;t)dx\,dy.$$

Here the second term is clearly equal to zero (since $\int\phi(x)f_2(x,s)dx|0\rangle$ does not depend on s), while in the first term we can (by Lemma 13.7) replace $\lim\limits_{t\to-\infty}$ by $\lim\limits_{t\to-\infty} - \lim\limits_{t\to+\infty}$. As a result (taking Exercise 13.13 into account) (13.107) can be written in the form

$$-i\int \delta^+_{m_1}(p_1)\delta^+_{m_2}(p_2)\{(p_1^2-m_1^2)(p_2^2-m_2^2)A(\tilde{\phi}(-p_2);\ \tilde{\phi}(-p_1))|0\rangle\}\tilde{f}_2(p_1)\tilde{f}_1(p_2)d_4p_1\,d_4p_2,$$

which proves (13.103). Finally, for the proof of (13.104) we smooth the right hand side with the functions $\tilde{f}_1 \in \mathcal{S}(G^+_{\kappa_1})$, $\tilde{f}_2 \in (G_{\kappa_2})$:

$$\int \tilde{\jmath}(-p_2)\tilde{\phi}^{\text{in}}(-p_1)|0\rangle\tilde{f}_2(p_2)\tilde{f}_1(p_1)dp_1\,dp_2 = \lim_{t\to-\infty}\int K_x\phi(x)\phi(y)|0\rangle f_2(x)f_1(y;t)dx\,dy.$$

We again use Lemma 13.7 to convert this expression to the form

$$\lim_{t\to-\infty}(-i)\int K_xA(\phi(x);\ \phi(y))f_2(x)f_1(y;t)dx\,dy \pm \lim_{t\to-\infty}\phi(f_1;t)j(f_2)|0\rangle.$$

Here the second term is equal to zero by virtue of the fact that

$$\tilde{\jmath}^{(\kappa)}(-p)|0\rangle = 0 \quad \text{for } p^2 < m'^2_\kappa \tag{13.108}$$

(according to Axiom LSZ.I). In the first term $\lim\limits_{t\to-\infty}$ can be replaced by $\lim\limits_{t\to-\infty} - \lim\limits_{t\to+\infty}$. As a result we arrive at formula (13.104). ∎

The same arguments can be used to prove the following result.

Exercise 13.15. Prove that if $\Phi^{\text{ex}}, \Psi^{\text{ex}} \in D_0^{\text{ex}}$, then the following reduction formulae hold (in the sense of generalized functions with respect to $p, p_1, \dots, p_n$):

$$\begin{aligned}
&\langle \Phi^{\text{out}}, \{\tilde\phi_n^{\text{out}}(p_n) T(\tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1)) \mp T(\tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1)) \tilde\phi_n^{\text{in}}(p_n)\} \Psi^{\text{in}} \rangle = \\
&\quad = -2\pi i \epsilon(p_n^0) \delta(p_n^2 - m_n^2) \{(p_n^2 - m_n^2) \langle \Phi^{\text{out}}, T(\tilde\phi_n(p_n) \tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1)) \Psi^{\text{in}} \rangle\}, \qquad (13.109) \\
&\langle \Phi^{\text{in}}, [\tilde\phi_n^{\text{in}}(p_n),\ R(\tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1);\ \tilde\phi(p))]_\mp \Psi^{\text{in}} \rangle = \\
&\quad = -2\pi \epsilon(p_n^0) \delta(p_n^2 - m_n^2) \{(p_n^2 - m_n^2) \langle \Phi^{\text{in}}, R(\tilde\phi_n(p_n) \tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1); \tilde\phi(p)) \Psi^{\text{in}} \rangle\}, \qquad (13.110) \\
&\langle \Phi^{\text{out}}, [\tilde\phi_n^{\text{out}}(p_n),\ A(\tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1); \tilde\phi(p))]_\mp \Psi^{\text{out}} \rangle = -2\pi \epsilon(p_n^0) \delta(p_n^2 - m_n^2) \times \\
&\quad \times \{(p_n^2 - m_n^2) \langle \Phi^{\text{out}},\ A(\tilde\phi_n(p_n) \tilde\phi_{n-1}(p_{n-1}) \dots \tilde\phi_1(p_1); \tilde\phi(p)) \Psi^{\text{out}} \rangle\}. \qquad (13.111)
\end{aligned}$$

(Here, for example in (13.109), it is to be understood that $\phi^{\text{out}}(f)$ acts to the left on the "bra-vector" Φ^{out}.)

D. Reduction formulae for the scattering matrix

We now turn to the reduction formula (13.95). It is obtained by a repeated application of (13.109). The pairwise non-collinearity of the momenta essentially eases the passage to the mass shell. To obtain results relating to the behaviour of the Fourier transform of the amputated causal Green's function, we need the following lemma.

Lemma 13.9. *Let $t_1, \dots, t_n$ be real variables ranging through the sector $t_1 = t \equiv \equiv \max\limits_{j=1,\dots,n} |t_j|$ and suppose that the test functions $\tilde f_1 \in \mathfrak{D}(G_{\kappa_j}^+)$ have non-intersecting supports in the velocity space. An essential support of the function* $\prod_{j=1}^n \overset{(-)}{f_j}(x_j, t_j)$ as $t \to \infty$ is then concentrated in the set*

$$\mathcal{O} = \{(x_1 \dots x_n) \in \mathbf{M}^n :\ x_1 - x_j \in \mathbf{M} \backslash U \quad for\ j = 2, \dots, n\},$$

where $U = \{x \in \mathbf{M} :\ x^0 \geq \rho|\mathbf{x}|\}$ (ρ is an arbitrary fixed number in the interval $0 < \rho < 1$).

■ It suffices to show that

$$t^N \sup_{x: x_1 - x_k \in U} \left| \prod_{j=1}^n P_j(x_j) Q_j\Big(\frac{\partial}{\partial x_j}\Big) f_j(x_j; t_j) \right| \to 0 \quad \text{as } t \to \infty,$$

where P_j and Q_j are arbitrary polynomials and $k = 2, \dots, n$. For definiteness, we consider the case $n = 2$. Since an estimate of type (13.100) holds for $f_j(x_j; t_j)$, it clearly suffices to show that

$$t^L \sup_{x_1 - x_2 \in U} |F(x_1, x_2; t_1, t_2)| \qquad (13.112)$$

tends to zero as $t \to \infty$ for all natural numbers L, where

$$F(x_1, x_2; t_1, t_2) = \prod_{j=1}^2 P_j(x_j) Q_j\Big(\frac{\partial}{\partial x_j}\Big) f_j(x_j, t_j).$$

First we consider the case $t - t_2 > \eta t$ ($t \equiv \max\{|t_1|, |t_2|\} = t_1$), where η is a fixed positive number. In this case an estimate of type (13.100) enables us to majorize (13.112) by an expression of the form

$$c_N (1 + t)^M (1 + |x_1^0 - t|)^{-N} (1 + |x_2^0 - t_2|)^{-N} \leq c_N' (1 + |t|)^{-N+M} \qquad (13.113)$$

for some M and any N; here we have used the fact that $x_2^0 - x_1^0 > 0$ and $t_2 - t_1 > \eta t$.

---

* Here $\overset{(-)}{f_j}$ is either f_j or $\bar f_j$.

Next we suppose that $t - t_2 < \eta t$, where η is a positive number which will be fixed presently. Let K_j be a neighbourhood of the set $\{\omega_j^{-1}\mathbf{p}_j : p_j \in \operatorname{supp} \tilde{f}_j\}$, $j = 1, 2$. According to the conditions of the lemma, K_1 and K_2 can be chosen to be disjoint bounded sets; moreover we can suppose that there exists a positive number η in the interval $0 < \eta < 1$ such that the distance between the sets K_1 and αK_2 exceeds some positive number r for all $\eta < \alpha < 1$. The result of Exercise 12.10 enables us to sharpen the estimate (13.100):

$$\left|P_j(x)Q_j\left(\frac{\partial}{\partial x}\right)f_j(x)\right| \leq c_N(1 + |\mathbf{x}_j| + |t_j|)^{-N} \quad \text{for } \mathbf{x}_j \notin t_jK_j.$$

Therefore if we suppose further that either $\mathbf{x}_1 \notin t_1K_1$ or $\mathbf{x}_2 \notin t_2K_2$ in (13.112), then the required estimate becomes trivial. It remains to consider the case $\mathbf{x}_1 \in t_1K_1$, $\mathbf{x}_2 \in t_2K_2$. Then $|\mathbf{x}_2 - \mathbf{x}_1| \geq rt$ and (since $x_1 - x_2 \in U$) $x_2^0 - x_1^0 \geq \rho rt$, where ρ and r are fixed numbers. Again we have an estimate of type (13.113) for F, which completes the proof. ■

The main result of this subsection is stated as follows.

Theorem 13.10. *The amputated causal Green's function in the momentum space* $\prod_{j=1}^n(-p_j^2 + m_j^2)\tilde{\tau}(p_1, \dots, p_k, -p_{k+1}, \dots, -p_n)$ *becomes a* C^∞*-function in the variables* $p_j^0 - \omega_1, \dots, p_n^0 - \omega_n$ *in a neighbourhood of the origin after (making the change of variables* $p_j^0 \to p_j^0 - \omega_j$ *and) smoothing with an arbitrary test function* $\psi(\mathbf{p}_1, \dots, \mathbf{p}_n)$ *in* $\mathcal{D}(\mathcal{N})$*, where*

$$\mathcal{N} = \{(\mathbf{p}_1, \dots, \mathbf{p}_n) \in R^{3n} : \ \omega_j^{-1}\mathbf{p}_j \neq \omega_k^{-1}\mathbf{p}_k \quad for\ j \neq k\}. \tag{13.114}$$

For pairwise non-collinear momenta $p_1, \dots, p_n \in V^+$ *the scattering amplitude is related to the causal Green's function by the formula*

$$\begin{aligned}\langle 0|\tilde{\phi}^{\text{out}(\kappa_1)}(p_1)\dots\tilde{\phi}^{\text{out}(\kappa_k)}(p_k)\tilde{\phi}^{\text{in}(\kappa_{k+1})}(-p_{k+1})\dots\tilde{\phi}^{\text{in}(\kappa_n)}(-p_n)|0\rangle &= \\ = (-i)^n\Big(\prod_{j=1}^n\delta_{m_j}^+(p_j)\Big)\Big\{\Big(\prod_{j=1}^n(p_j^2 - m_j^2)\Big)\tilde{\tau}^{(\kappa_1\dots\kappa_n)}(p_1, \dots, p_k,& \\ -p_{k+1}, \dots, -p_n)\Big\}.&\end{aligned} \tag{13.115}$$

■ As we have already noted, the reduction formula (13.115) is obtained by repeated application of (13.109). It remains to prove that the generalized functions

$$\prod_{j=k+1}^{l}(p_j^2 - m_j^2)\langle 0|\tilde{\phi}^{\text{out}}(p_1)\dots\tilde{\phi}^{\text{out}}(p_k)T(\tilde{\phi}(\pm p_{k+1})\dots\tilde{\phi}(\pm p_l))\tilde{\phi}^{\text{in}}(-p_{l+1})\dots\tilde{\phi}^{\text{in}}(-p_n)|0\rangle \tag{13.116}$$

are of class C^∞ with respect to $p_{k+1}^0 - \omega_{k+1}, \dots, p_l^0 - \omega_l$ in a neighbourhood of the origin after integrating with a test function $\psi(\mathbf{p}_1, \dots, \mathbf{p}_n)$ in $\mathcal{D}(\mathcal{N})$ and integrating with respect to $p_1^0, \dots, p_k^0, \dots, p_{l+1}^0, \dots, p_n^0$. It suffices to restrict attention to the case when $\psi(\mathbf{p}_1, \dots, \mathbf{p}_n)$ is of the form of a product $\prod_{j=1}^n \psi_j(\mathbf{p}_j)$. (In fact, the only question of substance is the property of the support and the general case is not that much more complicated: if the support of ψ lies in a direct product of sets that are disjoint in the velocity space, then the proof is entirely similar; the general case reduces to this by means of a suitable partition of unity in $\mathcal{N}$.)

The statement that we are trying to prove means that the expression

$$\begin{aligned}I(t_{k+1}, \dots, t_l) = \int\langle\phi^{\text{out}}(f_k)\dots\phi^{\text{out}}(f_1)\Psi_0, T(\phi(x_{k+1})\dots\phi(x_l))\phi^{\text{in}}(f_{l+1})\dots\phi^{\text{in}}(f_n)\Psi_0\rangle\times& \\ \times\prod_{j=k+1}^{l}\frac{\partial}{\partial t_j}\overset{(-)}{f_j}(x_j; t_j)dx_j&\end{aligned} \tag{13.117}$$

is a test function of class $\mathcal{S}(\mathbf{R})$ with respect to the variables $t_{k+1}, \dots, t_l$ if the supports of the functions $\tilde{f}_j \in \mathcal{D}(G^+_{\kappa_j})$ are disjoint in the velocity space. Since all the derivatives with respect to t_j of the infinitely differentiable function (13.117) again have the form (13.117), it is enough to show that for any natural number N

$$t^N I(t_{k+1}, \dots, t_l) \to 0 \quad \text{as } t \equiv \max_j |t_j| \to \infty.$$

For definiteness, we consider the sector $t_{k+1} = t$ (the other sectors are entirely similar). According to Lemma 13.9, an essential support of the function $\prod_{j=k+1}^{l} \overset{(-)}{f_j}(x_j; t_j)$ as $t \to \infty$ is concentrated in the set

$$\mathcal{O} = \{(x_{k+1}, \dots, x_l) \in \mathbf{M}^{l-k} : x_{k+1} - x_j \notin U,\ j = k+2, \dots, l\}.$$

Then, to within error terms $o(t^{-N})$, where N is arbitrary, we can replace $T(\phi(x_{k+1}) \dots \phi(x_l))$ by $\phi(x_{k+1}) T(\phi(x_{k+2}) \dots \phi(x_l))$ in (13.117), since both expressions coincide in $\mathcal{O}$.

It remains to prove that the expression

$$I'(t_{k+1}, \dots, t_l) = \int \langle \phi^{\text{out}}(f_k) \dots \phi^{\text{out}}(f_1)\Psi_0,\ \phi(x_{k+1}) \times \\ \times T(\phi(x_{k+2}) \dots \phi(x_l)) \phi^{\text{in}}(f_{l+1}) \dots \phi^{\text{in}}(f_n)\Psi_0 \rangle \prod_{j=k+1}^{l} \frac{\partial}{\partial t_j} \overset{(-)}{f_j}(x_j; t_j) dx_j \tag{13.118}$$

decreases more rapidly than any negative power of t as $t \to \infty$. The Schwarz inequality enables us to majorize the modulus of (13.118) by the expression

$$\left\| \frac{\partial}{\partial t} \left(\int \phi(x_{k+1}) \overset{(-)}{f}(x_{k+1}; t) dx_{k+1} \right)^* \phi^{\text{out}}(f_k) \dots \phi^{\text{out}}(f_1)\Psi_0 \right\| \times \\ \times \left\| \int T(\phi(x_{k+2}) \dots \phi(x_n)) \prod_{j=k+2}^{l} \overset{(-)}{f_j}(x_j; t_j) dx_j \phi^{\text{in}}(f_{l+1}) \dots \phi^{\text{in}}(f_n)\Psi_0 \right\|. \tag{13.119}$$

As follows from the proof of (13.85) (or its analogue, (13.90)), the first factor in (13.119) tends to zero more rapidly than any negative power of t; the second factor is of polynomial growth in t. As a result, (13.118) decreases more rapidly than any negative power of t. ■

We draw attention to the fact that the scattering amplitudes of different reactions (so-called channels associated with the given Green's function) are expressed in terms of one and the same Green's function . For example, the amplitudes of the six elementary processes

$$\kappa_j + \kappa_k \to \bar{\kappa}_l + \bar{\kappa}_m, \tag{13.120}$$

where (j, k, l, m) is a permutation of the numbers 1,2,3,4, are expressed in terms of the four-point causal Green's function $\tau^{(\kappa_1 \dots \kappa_4)}(p_1, \dots, p_4)$. This fact lies at the basis of the crossing relation (that is, the relation via analytic continuation between the amplitudes of, say, the channels $\kappa_1 + \kappa_2 \to \bar{\kappa}_3 + \bar{\kappa}_4$ and $\kappa_1 + \kappa_3 \to \bar{\kappa}_2 + \bar{\kappa}_4$, which are possibly distinct and are not related to each other by the usual symmetry of CPT type and the like). As we saw in §7.3.F, the kinematics of the processes (13.120) are conveniently described by the invariant variables s, t, u (7.181), where the momenta range throughout the mass shell of the processes (13.120):

$$p_1 + p_2 + p_3 + p_4 = 0, \tag{13.121}$$

$$p_j \in \Gamma^{\pm}_{m_j}, \quad j = 1, \dots, 4. \tag{13.122}$$

In the centre-of-mass frame, the role of the square of the total energy of the processes

$$\kappa_1 + \kappa_2 \to \bar{\kappa}_3 + \bar{\kappa}_4,\ \kappa_1 + \kappa_3 \to \bar{\kappa}_2 + \bar{\kappa}_4,\ \kappa_1 + \kappa_4 \to \bar{\kappa}_2 + \bar{\kappa}_3 \tag{13.123}$$

is played by the variables s, t, u respectively, therefore these processes (or TCP-transformed processes) are called s-, t- and u-*channels* respectively.

The formula due to Zimmermann (1959) on the one-particle singularities of the causal Green's function follows from Theorem 13.10.

Exercise 13.16. Prove that for pairwise non-collinear momenta $p_1, \dots, p_n$, the following formula holds in a neighbourhood of the mass shell $p_j^2 = m_j^2$:

$$\tilde{\tau}(p_1, \dots, p_n) = \lim_{\epsilon \to +0} \prod_{j=1}^{n} (p_j^2 - m_j^2 + i\epsilon)^{-1} \Big\{ \prod_{j=1}^{n} (p_j^2 - m_j^2) \tilde{\tau}(p_1, \dots, p_n) \Big\} \tag{13.124}$$

(where the expression in the curly brackets is C^∞-dependent on $p_1^0 \pm \omega_1, \dots, p_n^0 \pm \omega_n$ in a neighbourhood of the origin). [Hint: Suppose that the supports of $\tilde{f}_j \in \mathcal{D}(G^+_{\kappa_j})$ are pairwise disjoint in the velocity space; then for $1 \le k < n$ the function

$$F(t_1, \dots, t_n) = \int \tau(x_1, \dots, x_n) \prod_{j=1}^{k} f_j(x_j; t_j) \prod_{j=k+1}^{n} \overline{f_j(x_j; -t_j)} dx_1 \dots dx_n$$

is of class C^∞ and $\frac{\partial^n}{\partial t_1 \dots \partial t_n} F(t_1, \dots, t_n) \in \mathcal{S}(\mathbf{R}^n)$; furthermore,

$$\lim_{t_l \to +\infty} \frac{\partial^{l-1}}{\partial t_1 \dots \partial t_{l-1}} F(t_1, \dots, t_n) = 0 \quad (\text{for } 1 \le l \le n).$$

Hence obtain the formula

$$F(0) = \lim_{\epsilon \to +0} (-1)^n \int_0^\infty dt_1 \dots \int_0^\infty dt_n e^{-\epsilon(t_1 + \dots + t_n)} \frac{\partial^n}{\partial t_1 \dots \partial t_n} F(t_1, \dots, t_n)$$

and reduce it to the form

$$F(0) = \lim_{\epsilon \to +0} \int \tilde{\tau}(p_1, \dots, p_k, p_{k+1}, \dots, p_n) \prod_{j=1}^{k} \frac{p_j^2 - m_j^2}{p_j^2 - m_j^2 + i\epsilon} \tilde{f}_j(-p_j) \prod_{j=k+1}^{n} \frac{p_j^2 - m_j^2}{p_j^2 - m_j^2 + i\epsilon} \times$$
$$\times \overline{\tilde{f}_j(p_j)} d_4p_1 \dots d_4p_n.]$$

Of physical interest are reactions in which two particles are converted into several particles. In this case, the scattering amplitude can be obtained by passing to the mass surface in the Fourier transforms of the advanced Green's functions. It is possible to do this in view of the following theorem, which is proved in the same way as Theorem 13.10 (only now we use (13.111) as well as (13.103)).

Theorem 13.11. *The amputated advanced Green's function in momentum space*

$$\prod_{j=1}^{n} (-p_j^2 + m_j^2) \tilde{a}^{(\kappa_n \dots ; \kappa_1)}(p_n, \dots, p_3, -p_2; -p_1)$$

is of class C^∞ with respect to the $p_j^0 - \omega_j$ after integrating with respect to $\mathbf{p}_1, \dots, \mathbf{p}_n$ with a test function in $\mathcal{D}(\mathcal{N})$ ($\mathcal{N}$ is defined by (13.114)). The following reduction formula holds for the process $\kappa_1 + \kappa_2 \to \bar{\kappa}_3 + \dots + \bar{\kappa}_{n-2}$ in a domain in which the particle momenta are pairwise non-collinear:

$$\langle 0 | \tilde{\phi}^{\text{out}(\kappa_n)}(p_n) \dots \tilde{\phi}^{\text{out}(\kappa_3)}(p_3) \tilde{\phi}^{\text{in}(\kappa_2)}(-p_2) \tilde{\phi}^{\text{in}(\kappa_1)}(-p_1) | 0 \rangle =$$
$$= -i \prod_{j=1}^{n} \delta^+_{m_j}(p_j) \Big\{ \prod_{j=1}^{n} (p_j^2 - m_j^2) \tilde{a}^{(\kappa_n \dots ; \kappa_1)}(p_n, \dots, p_3, -p_2; -p_1) \Big\}. \tag{13.125}$$

A successive application of (13.110), (13.111) enables us to obtain reduction formulae for the matrix elements of the field and also for the retarded and advanced products. In particular, the matrix elements of the retarded products between the in-states are expressed in terms of the retarded Green's functions (r-functions). This enabled Glaser, Lehmann and Zimmermann (1957; see also Steinmann, 1968) to reformulate the LSZ theory in terms of r-functions alone.

CHAPTER 14

The S-Matrix Method

14.1. S-Matrix Formulation of the Basic Requirements of the Local Theory

A. The concept of extending the S-matrix beyond the mass shell

The S-matrix method, put forward by Bogolubov, Medvedev and Polivanov is a special version of the LSZ theory. It is a development of an original idea to Heisenberg (1943) according to which, the content of relativistic quantum theory can be expounded in the language of the S-matrix. The S-matrix method arose in the process of generalizing Lagrangian quantum field theory (although there is no mention of the Lagrangian in it). It starts from the assumption that there is a more primitive object than the Heisenberg field and its T-products (which are the basic concepts of the LSZ formalism). The role of this fundamental object is fulfilled by the extension of the S-matrix beyond the mass shell on the basis of which the quantum fields and their T-products have already been constructed. In the S-matrix method, the derivation of the reduction formulae is considerably simplified. (The apparatus of formal variational derivatives makes the derivation of the various reduction formulae an automatic operation.) Another essential feature of the S-matrix method is that it is well suited to the treatment of the dynamical equations of quantum field theory (see, for example, [Z1]).

We turn to the exposition of the S-matrix method. First it is postulated that the scattering operator S is a unitary Poincaré-invariant operator in the Fock space $\mathfrak{H}$ of a system of free relativistic particles (of type given in §7.3.C); for definiteness, $\mathfrak{H}$ is identified with the Hilbert space of incoming particles. An irreducible system of (Wightman) free (or in-) fields $\phi^{\mathrm{in}(\kappa)}(x)$ acts in $\mathfrak{H}$ with canonical commutation relations (under the normal connection between spin and statistics)

$$[\phi_l^{\mathrm{in}(\kappa)}(x), \phi_{l'}^{\mathrm{in}(\kappa')}(y)]_{\mp} = \frac{1}{i} D_{ll'}^{(\kappa\kappa')}(x-y) \tag{14.1}$$

and with two-point functions of type (13.52)

$$\langle 0|\phi_l^{\mathrm{in}(\kappa)}(x)\phi_{l'}^{\mathrm{in}(\kappa')}(y)|0\rangle = \frac{1}{i} D_{ll'}^{(-)(\kappa\kappa')}(x-y). \tag{14.2}$$

As always, the out-fields are related to the in-fields by the formula

$$\phi^{\mathrm{out}(\kappa)}(x) = S^{*}\phi^{\mathrm{in}(\kappa)}(x)S. \tag{14.3}$$

$D_{ll'}^{(\kappa\kappa')}(x-y)$ is a Lorentz-covariant solution of the Klein-Gordon equation with mass $m \equiv m_\kappa > 0$; it is represented in the form

$$D_{ll'}^{(\kappa\kappa')}(x-y) = Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) D_m(x-y), \tag{14.4}$$

where $Q_{ll'}^{(\kappa\kappa')}(-i\frac{\partial}{\partial x})$ is a Lorentz-covariant polynomial in $\partial/\partial x$ with the properties *

$$Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) = (-1)^{F^{(\kappa)}} Q_{l'l}^{(\kappa'\kappa)}\Big(-i\frac{\partial}{\partial x}\Big), \tag{14.5a}$$

(where $F^{(\kappa)} = 0$ or 1 depending on whether the field $\phi^{\mathrm{in}(\kappa)}$ is bosonic or fermionic),

$$\overline{Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big)} = Q_{\bar{l}'\bar{l}}^{(\bar{\kappa}'\bar{\kappa})}\Big(i\frac{\partial}{\partial x}\Big). \tag{14.5b}$$

$D_{ll'}^{(-)(\kappa\kappa')}$ is the negative-frequency part of $D_{ll'}^{(\kappa\kappa')}$:

$$D_{ll'}^{(-)(\kappa\kappa')}(x-y) = Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) D_m^{(-)}(x-y). \tag{14.6}$$

Similarly we define:

$$D_{ll'}^{c(\kappa\kappa')}(x-y) = Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) D_m^{c}(x-y), \tag{14.7}$$

$$D_{ll'}^{\bar{c}(\kappa\kappa')}(x-y) = Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) \overline{D_m^{c}(x-y)}, \tag{14.8}$$

$$D_{ll'}^{\substack{\mathrm{ret}(\kappa\kappa') \\ \mathrm{adv}}}(x-y) = Q_{ll'}^{(\kappa\kappa')}\Big(-i\frac{\partial}{\partial x}\Big) D_m^{\substack{\mathrm{ret} \\ \mathrm{adv}}}(x-y). \tag{14.9}$$

We say that an extension of the S-matrix beyond the mass shell is defined if formal (left) variational derivatives of the S-matrix with respect to the in-fields

$$\frac{\delta^n S}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\dots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)} \equiv H^{(\kappa_1\dots\kappa_n)l_1\dots l_n}(x_1,\dots,x_n) \tag{14.10}$$

are defined and satisfy the conditions listed below. It is assumed that (14.10) is an operator-valued generalized function with respect to $x_1,\dots,x_n$ (with domain of definition in the Fock space $\mathfrak{H}$). The expression (14.10) is symmetric in its variables with respect to bosonic fields and antisymmetric with respect to fermionic fields; it is called a bosonic (or fermionic) operator if the sequence $\phi^{\mathrm{in}(\kappa_1)},\dots,\ \phi^{\mathrm{in}(\kappa_n)}$ contains an even (or odd) number of fermionic fields.

Next, it is supposed that the Fourier transform of (14.10) admits a restriction to the mass shell with respect to the momentum p_j conjugate to any of the variables x_j (for example, by means of the procedure of restricting generalized functions to the mass shell described in §13.2.C). The definition of such a restriction is expressed in coordinate space by convolution with the commutator function $D_{m_j}(x_j)$. Thus we write the assumption of the existence of the restriction to the mass shell, say, with respect to p_1 in the form of the existence of the convolution

$$\int D_{m_1}(x_1 - x_1')\frac{\delta^n S}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1')\delta\phi_{l_2}^{\mathrm{in}(\kappa_2)}(x_2)\dots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)} dx_1'. \tag{14.11}$$

* Formulae (14.5) guarantee the requisite property of (anti)symmetry of the (anti)commutator (14.1) and the property of the adjoint for the two-point function (14.2). (Concerning the explicit form of the covariants $Q^{(\kappa\kappa')}(p)$, see Appendix G, for example, the covariant expansion (G.4).)

For the variational derivative (14.10) we postulate a rule associating its restriction with respect to the momentum, say, p_1 to the mass shell with the lower order variational derivatives; namely, we have the following commutation formula:

$$\left[\phi_l^{\mathrm{in}(\kappa)}(x_1), \frac{\delta^{n-1}S}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_2)\dots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)}\right]_\mp =$$

$$= \sum_{\kappa',l'}\int[\phi_l^{\mathrm{in}(\kappa)}(x_1), \phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1')]_\mp \frac{\delta^n S}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1')\delta\phi_{l_2}^{\mathrm{in}(\kappa_2)}(x_2)\dots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)}dx_1', \quad (14.12)$$

where, as usual, the $\mp$ signs depend on the "statististics" of the operators. This relation lies at the basis of the reduction formulae of the S-matrix approach (see, for example, Exercises 14.4 and 14.10 below).

An equivalent method of dealing with the extension of the S-matrix beyond the mass shell is to introduce the generating functional for the operator-valued generalized functions (14.10), called the *S-matrix extended beyond the mass shell*:

$$S(\chi) = \sum_{n=0}^{\infty}\frac{1}{n!}\int \chi_{l_n}^{(\kappa_n)}(x_n)\dots\chi_{l_1}^{(\kappa_1)}(x_1)H^{(\kappa_1\dots\kappa_n)l_1\dots l_n}(x_1,\dots,x_n)dx_1\dots dx_n. \quad (14.13)$$

This is a formal series in the classical field $\chi(x)$ with operator-valued coefficient functions. As for the case of the classical source $\eta(x)$ (§13.1.A), the bosonic components $\chi_l^{(\kappa)}(x)$ of the classical field (which can be regarded as ordinary functions, say, of class $\mathcal{F}^{(\kappa)}$ defined in the next subsection) commute with all the components of the classical field and with all the operators, whereas the fermionic components of the classical field (which are merely symbols) anticommute with one another and with the fermionic operators. * Using the same rules of variational differentiation as in §14.1.A, we can write the formal variational derivatives of the S-matrix in terms of the ordinary variational derivatives of the functional $S(\chi)$:

$$\frac{\delta^n S}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\dots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)} \equiv \left.\frac{\delta^n S(\chi)}{\delta\chi_{l_1}^{(\kappa_1)}(x_1)\dots\delta\chi_{l_n}^{(\kappa_n)}(x_n)}\right|_{\chi=0}. \quad (14.14)$$

Here it is understood that the "initial condition" holds:

$$S(\chi)|_{\chi=0} = S. \quad (14.15)$$

It is also convenient to use the formal series

$$\mathfrak{S}(\chi) = S^*S(\chi); \quad (14.16)$$

its "initial condition" takes the universal form

$$\mathfrak{S}(\chi)|_{\chi=0} = 1. \quad (14.17)$$

The extension of the adjoint S^* of the S-matrix beyond the mass shell is defined by the formal series

$$S^*(\chi) = S(\chi^+)^* \equiv$$

$$\equiv \sum_{n=0}^{\infty}\frac{1}{n!}\int H^{(\kappa_1\dots\kappa_n)l_1\dots l_n}(x_1,\dots,x_n)^*\overline{\chi_{l_1}^{+(\kappa_1)}(x_1)}\dots\overline{\chi_{l_n}^{+(\kappa_n)}(x_n)}dx_1\dots dx_n, \quad (14.18)$$

* In this connection, see the footnote on p 508.

where the conjugate of the classical field χ^+ is defined by the equality

$$\chi_l^{+(\kappa)}(x) = \overline{\chi_{\bar{l}}^{(\bar{\kappa})}(x)} \tag{14.19}$$

(cf. the corresponding definition (13.29) of the adjoint for the classical source $\eta(x)$). For the formal variational derivatives of S^* with respect to the in-fields we have

$$\frac{\delta^n S^*}{\delta\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\ldots\delta\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)} \equiv \frac{\delta^n S^*(\chi)}{\delta\chi_{l_1}^{(\kappa_1)}(x_1)\ldots\delta\chi_{l_n}^{(\kappa_n)}(x_n)}\bigg|_{\chi=0}. \tag{14.20}$$

In what follows we have to distinguish between the relations between the operators S, S^* and their formal variational derivatives "outside the mass shell" and "on the mass shell". The relations "outside the mass shell" are in fact identities between the formal series with respect to the classical field χ; it is only these relations that admit further differentiation with respect to the asymptotic fields. The relations "on the mass shell" are ordinary operator equalities (or relations that hold upon substituting $\chi = 0$ in the formal series with respect to χ). Thus, adhering to the established tradition, here we are using the terminology "outside the mass shell" and "inside the mass shell" in a figurative sense * (as opposed to the literal usage, for example, in §13.2.C). It goes without saying that not every relation "on the mass shell" is valid "outside the mass shell". (An example of a relation that holds only "on the mass shell" is (14.35) given below.)

Bearing in mind the rules (14.14), (14.20) for reducing formal variational derivatives with respect to asymptotic fields to ordinary variational derivatives with respect to classical fields, we can define more complicated derivatives according to usual rules. For example, the formula

$$\frac{\delta}{\delta\phi^{\mathrm{in}}(x)}\left(S^*\frac{\delta S}{\delta\phi^{\mathrm{in}}(y)}\right) = S^*\frac{\delta^2 S}{\delta\phi^{\mathrm{in}}(x)\delta\phi^{\mathrm{in}}(y)} + \frac{\delta S^*}{\delta\phi^{\mathrm{in}}(x)}\frac{\delta S}{\delta\phi^{\mathrm{in}}(y)}$$

is an alternative expression of the identity

$$\frac{\delta}{\delta\chi(x)}\left(S^*(\chi)\frac{\delta S(\chi)}{\delta\chi(y)}\right) = S^*(\chi)\frac{\delta^2 S(\chi)}{\delta\chi(x)\delta\chi(y)} + \frac{\delta S^*(\chi)}{\delta\chi(x)}\frac{\delta S(\chi)}{\delta\chi(y)}.$$

We give a further clarification of the definition of the extension of the S-matrix beyond the mass shell. It is well known from the theory of second quantization (see [B4] and Berezin, 1967) that every bounded linear operator in the Fock space $\mathfrak{H}$ can be expanded in a series of normal products of creation and annihilation operators. In particular, S can be represented in the form

$$S = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int\sum_{\substack{\kappa_1\ldots\kappa_n\\ l_1\ldots l_n}}\tilde{\sigma}^{(\kappa_1\ldots\kappa_n)l_1\ldots l_n}(-p_1,\ldots,-p_n)\ :\tilde{\phi}_{l_1}^{\mathrm{in}(\kappa_1)}(p_1)\ldots\tilde{\phi}_{l_n}^{\mathrm{in}(\kappa_n)}(p_n):\, d_{4n}p. \tag{14.21}$$

(In any case the series $S = \sum_n S_n$ converges in the sense of convergence of the matrix elements $\sum_n\langle\Phi,\ S_n\Psi\rangle$ between "finite particle" elements Φ, Ψ of $\mathfrak{H}$, since only a finite number of terms of the series $\sum_n\langle\Phi, S_n\Psi\rangle$ are non-zero.) Since the Fourier transform of the free field $\tilde{\phi}^{(\kappa)}(p)$ contains the δ-function $\delta(p^2 - m^2)$, the coefficient functions $\tilde{\sigma}(p_1,\ldots,p_n)$ are essential for this expansion only on the mass shell $p_j^2 = m_j^2$. However, we are supposing that these coefficient functions can be extended

* A further motivation for this use of words is provided by the commutation formula (14.12) (since, in the case of a single scalar field, it shows that differentiation with respect to the asymptotic field followed by going over to the momentum on the mass shell, reduces to taking the commutator, that is, to an expression with "ordinary" operators).

by some means or other beyond the mass shell and are temperate generalized functions in $\mathbf{M}^n$; the expression (14.21) can then be rewritten in x-space:

$$S=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int\sum_{\substack{\kappa_1\ldots\kappa_n\\ l_1\ldots l_n}}\sigma^{(\kappa_1\ldots\kappa_n)l_1\ldots l_n}(x_1,\ldots,x_n):\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\ldots\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n):dx_1\ldots dx_n.$$

We can now introduce the S-matrix extended beyond the mass shell:

$$S(\chi)=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int\sum_{\substack{\kappa_1\ldots\kappa_n\\ l_1\ldots l_n}}\sigma^{(\kappa_1\ldots\kappa_n)l_1\ldots l_n}(x_1,\ldots,x_n):(\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)+\chi_{l_1}^{(\kappa_1)}(x_1))\ldots$$

$$\ldots(\phi_{l_n}^{\mathrm{in}(\kappa_n)}(\chi_n)+\chi_{l_n}^{(\kappa_n)}(x_n)):dx_1\ldots dx_n,\qquad(14.22)$$

where χ is a classical field. Rule (14.14) now enables us to calculate the formal variational derivatives of the S-matrix with respect to the asymptotic field. However, in view of the complication of dealing with the domains of definition of the (series of) operators in this language, we look upon the above construction as merely a heuristic consideration.

B. CHOICE OF THE CLASS OF TEST FUNCTIONS

The hypothesis of the existence of restrictions of the Fourier transforms of the formal variational derivatives (14.10) to the mass shell (which has such an important position in the S-matrix approach) can be regarded as automatic if the test function space is suitably chosen. We look at the formal variational derivative $\delta S/\delta\phi^{(\kappa)}(x)$ more closely. Here the result of smoothing $\int\frac{\delta S}{\delta\phi^{(\kappa)}(x)}u(x)d^4x$ with solutions $u(x)$ of the Klein-Gordon equation of the form

$$u(x)=\int D_{m_\kappa}(x-y)f(y)d^4y,\quad\text{where } f\in\mathcal{S}(\mathbf{M})\qquad(14.23)$$

must be defined. (In fact in the p-representation, the definition of this smoothing is equivalent to the definition of the product with the generalized function $\delta(p^2-m_\kappa^2)$ or to the restriction to the mass shell.) It is therefore natural to allow the space of test functions on which the operator-valued generalized function $\delta S/\delta\phi^{(\kappa)}(x)$ is defined, to be wider than $\mathcal{S}(\mathbf{M})$ and, in particular, to contain solutions of the Klein-Gordon equation of the form (14.23). It is convenient to choose as this space the space $\mathcal{F}^{(\kappa)}$, defined as follows: the functions $u(x)\equiv u(x^0,\mathbf{x})$ in $\mathcal{F}^{(\kappa)}$ are complex functions of class $\mathcal{S}(\mathbf{R}^3)$ of the vector $\mathbf{x}$ that depend in C^∞-fashion on x^0 and are such that

$$K_x^{(\kappa)}u(x)\equiv(\partial_\mu\partial^\mu+m_\kappa^2)u(x)\in\mathcal{S}(\mathbf{M}).\qquad(14.24)$$

It is clear that this definition of the space $\mathcal{F}^{(\kappa)}$ satisfies the conditions given above.

Exercise 14.1. (a) Prove that every function $u(x)$ in $\mathcal{F}^{(\kappa)}$ is uniquely representable in either of the following forms:

$$u(x)=u^{\mathrm{in}}(x)+\int D_m^{\mathrm{ret}}(x-y)f(y)dy=u^{\mathrm{out}}(x)+\int D_m^{\mathrm{adv}}(x-y)f(y)dy,\qquad(14.25)$$

where $m\equiv m_\kappa$, $f=K_x^{(\kappa)}u\in\mathcal{S}(\mathbf{M})$and $u^{\mathrm{ex}}(x)$ is a fundamental solution of the Klein-Gordon equation of class $\mathcal{S}(\mathbf{R}^3)$, * and where

$$u^{\mathrm{out}}(x)=u^{\mathrm{in}}(x)+\int D_m(x-y)f(y).\qquad(14.26)$$

* This means that $u(x)$, $\partial_0u(x)\in\mathcal{S}(\mathbf{R}^3)$ with respect to $\mathbf{x}$ for any fixed x^0.

[Hint: Consider the difference between $u(x)$ and functions defined by the convolutions in (14.25).]

(b) Prove that every function in $\mathcal{F}^{(\kappa)}$ is (uniquely) representable in the form

$$u(x) = \int D_m^{\text{ret}}(x-y)f^{(r)}(y)dy + \int D^{\text{adv}}(x-y)f^{(a)}(y)dy, \tag{14.27}$$

where $f^{r,a} \in \mathcal{S}(\mathbf{M})$. [Hint: Represent $u^{\text{in}}(x)$ in the form $\int D(x-y)h(y)dy$ with $h \in \mathcal{S}(\mathbf{M})$.]

It is clear from the representations (14.25) that $\mathcal{F}^{(\kappa)}$ is isomorphic to the direct sum $\mathcal{S}(\mathbf{M}) \oplus \mathcal{S}(\mathbf{R}^3) \oplus \mathcal{S}(\mathbf{R}^3)$; the topology (Fréchet space) on $\mathcal{F}^{(\kappa)}$ is defined via this isomorphism. The topology on $\mathcal{F}^{(\kappa)}$ can be defined equivalently as the inductive topology with respect to the map $\mathcal{S}(\mathbf{M}) \oplus \mathcal{S}(\mathbf{M}) \to \mathcal{F}^{(\kappa)}$ (which by (14.27) associates a pair of functions $f^{(r)}$, $f^{(a)}$ with the function u).

We now replace the requirement that there exist a restriction of the operator-valued generalized function $\int e^{ipx}\frac{\delta S}{\delta\phi^{(\kappa)}(x)}d^4x$ to the mass shell $p^2 = m_\kappa^2$ by the condition that $\frac{\delta S}{\delta\phi^{(\kappa)}(x)}$ be a generalized function defined on the test function space $\mathcal{F}^{(\kappa)}$.

Similarly for the formal variational derivatives of higher order we have to introduce the space $\mathcal{F}^{(\kappa_1\dots\kappa_n)}$ of complex functions $u(x_1,\dots,x_n)$, belonging to $\mathcal{F}^{(\kappa_j)}$ with respect to each of the variables x_j; more precisely, such functions admit the representation

$$u(x_1,\dots,x_n) = \sum_{\substack{\text{ret}\\\text{adv}}}\cdots\sum_{\substack{\text{ret}\\\text{adv}}}\int D_{m_1}^{\substack{\text{ret}\\\text{adv}}}(x_1-y_1)\dots D_{m_n}^{\substack{\text{ret}\\\text{adv}}}(x_n-y_n)\times$$
$$\times f^{\left(\substack{r\dots r\\a\dots a}\right)}(y_1,\dots,y_n)dy_1\dots dy_n \tag{14.28}$$

with arbitrary $f^{\left(\substack{r\dots r\\a\dots a}\right)} \in \mathcal{S}(\mathbf{M}^n)$ (the topology on $\mathcal{F}^{(\kappa_1\dots\kappa_n)}$ is defined as the inductive topology). Thus we postulate that the formal variational derivatives (14.10) are defined on the test function space $\mathcal{F}^{(\kappa_1\dots\kappa_n)}$.

C. Axioms of the S-matrix approach

Not every extension of the S-matrix beyond the mass shell is of interest from the point of view of local quantum theory. We list below the requirements that the extension of interest to us must satisfy.

S.I (Postulate of relativistic quantum collision theory). *In the Fock space $\mathfrak{H}$ of the system of free (incoming) relativistic particles, a (finite or countable) irreducible collection of free Wightman fields* (in-*fields*) $\phi^{\text{in}(\kappa)}(x)$ *acts with canonical commutation relations* (14.1) *and two-point functions* (14.2). *The S-matrix defined in $\mathfrak{H}$ is a unitary Poincaré-invariant operator that leaves the vacuum vector and the one-particle state vectors invariant.**

S.II (Extension of the S-matrix beyond the mass shell). *There exists a formal series $S(\chi)$* (14.13) *in the classical field $\chi(x)$ with operator-valued coefficient functions for which the "initial condition"* (14.15) *holds. The coefficient functions $H^{(\kappa_1\dots\kappa_n)l_1\dots l_n}(x_1,\dots,x_n)$ are operator-valued generalized functions on the corresponding test function spaces $\mathcal{F}^{(\kappa_1\dots\kappa_n)}$. On smoothing with test functions, they become*

* This last supposition is called the condition of stability of the one-particle states (see §12.1.C for the motivation of this).

operators defined on some Poincaré-invariant dense linear subspace D of $\mathfrak{H}$ taking values in $\tilde{D} \equiv SD$; the adjoint operators act from $\tilde{D}$ to D. It is also assumed that the spaces D and $\tilde{D}$ are in the domain of definition of the closures of the operators $\phi_l^{\text{in}(\kappa)}(f)$, $\phi_l^{\text{out}(\kappa)}(f)$, which are originally defined (for $f \in \mathcal{S}(\mathbf{M})$) respectively on the vectors (with compact support) of the in- *or* out-*states.*

S.III (Poincaré-invariance of the extended S-matrix).

$$U(a, \underset{\sim}{\Lambda})S(\chi)U(a, \underset{\sim}{\Lambda})^{-1} = S(\chi_{(a,\underset{\sim}{\Lambda})}) \quad \textit{for all } (a, \underset{\sim}{\Lambda}) \in \mathfrak{P}_0, \tag{14.29}$$

where

$$(\chi_{(a,\underset{\sim}{\Lambda})})_l^{(\kappa)}(x) = \sum_m V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda})\chi_m^{(\kappa)}(\Lambda^{-1}(x-a)).$$

S.IV (Extended unitarity).

$$S^*(\chi)S(\chi) = 1. \tag{14.30}$$

S.V (Causality, or microcausality).

$$\mathfrak{S}(\chi_1 + \chi_2) = \mathfrak{S}(\chi_1)\mathfrak{S}(\chi_2), \quad \textit{if}^* \ \operatorname{supp}\chi_1 \underset{\sim}{>} \operatorname{supp}\chi_2 \tag{14.31}$$

($\mathfrak{S}(\chi)$ is related to $S(\chi)$ by formula (14.16).)

Like the Wightman axiom W.IV, Axiom S.II is full of "technical" assumptions motivated by the same considerations as in the Wightman formalism. The causality condition S.V generalizes the Lagrangian formalism of quantum field theory, where the S-matrix is formally defined by the T-exponent in the interaction representation, which reflects our intuitive ideas on causality. These questions are discussed in detail in the book [B10]; we do not dwell on them here.

Exercise 14.2. (a) Prove the relation

$$S(\chi)S^*(\chi) = 1, \tag{14.32}$$

thanks to which, the unitarity condition can also be written in the form $S^*(\chi) = S(\chi)^{-1}$. [Hint: See Exercise 13.3.]

(b) Prove the relation

$$\frac{\delta}{\delta\chi(x)}(\mathfrak{S}(\chi)^{-1}\mathfrak{S}(\chi + \chi_1)) = 0 \quad \text{for } \operatorname{supp}\chi_1 \underset{\sim}{<} x. \tag{14.33}$$

[Hint: See the analogous Proposition 13.1.]

Exercise 14.3. Consider an S-matrix theory with a single real scalar field. Prove that

$$\int \frac{\delta S}{\delta\phi^{\text{in}}(x)}|0\rangle u(x)d^4x = 0$$

for any solution $u(x)$ of the Klein-Gordon equation (of mass m) of class $\mathcal{S}(\mathbf{R}^3)$. [Hint: Use the relation $[\phi^{\text{in}}(f),\ S]|0\rangle = 0$, which follows from the condition of stability of the vacuum and one-particle states and the commutativity formula (14.12).]

The result of Exercise 14.3 can be expressed thus: $\int e^{ipx}\frac{\delta S}{\delta\phi^{\text{in}}(x)}d^4x$ does not provide any contribution on the mass shell $p^2 = m^2$ when acting on the vacuum.

So as to have an analogue of Axiom LSZ.I (or its corollary (13.108)), we add one further axiom (which strengthens the result of Exercise 14.3).

* The meaning of the condition $\operatorname{supp}\chi_1 \underset{\sim}{>} \operatorname{supp}\chi_2$ is explained in §13.1.A (but there, classical sources feature rather than classical fields).

S.I′ (Strengthened condition of stability of one-particle states). *There exist parameters $m'_\kappa > m_\kappa$ such that*

$$\int \frac{\delta S}{\delta\phi^{\text{in}(\kappa)}(x)} u(x) d^4x|0\rangle = 0 \tag{14.34}$$

for all functions $u \in \mathcal{F}^{(\kappa)}$ whose Fourier transforms have support in $p^2 < m'^2_\kappa$.

D. RADIATION OPERATORS; CURRENT

By virtue of Postulate S.II, the expression

$$\frac{\delta^k S^*}{\delta\phi^{\text{in}}(x_1)\ldots\delta\phi^{\text{in}}(x_k)}\,\frac{\delta^{n-k} S}{\delta\phi^{\text{in}}(x_{k+1})\ldots\phi^{\text{in}}(x_n)}$$

becomes, upon smoothing with a test function, an operator which (together with its adjoint) is defined on the domain D and takes it into itself. Therefore, such operators generate an algebra (with involution) which we call the algebra of *radiation operators.** In essence, the axioms of the S-matrix method (apart from the usual statements of relativistic scattering theory) represent an infinite system of relations between the radiation operators ("on the mass shell"). For example, the causality condition is the following system of relations ("on the mass shell"):

$$S^* \frac{\delta^n S}{\delta\phi^{\text{in}}(x_1)\ldots\delta\phi^{\text{in}}(x_n)} = S^* \frac{\delta^k S}{\delta\phi^{\text{in}}(x_1)\ldots\delta\phi^{\text{in}}(x_k)}\; S^* \frac{\delta^{n-k} S}{\delta\phi^{\text{in}}(x_{k+1})\ldots\delta\phi^{\text{in}}(x_n)}, \tag{14.35}$$

if $\{x_1,\ldots,x_k\} \gtrsim \{x_{k+1},\ldots,x_n\}$.

In the LSZ formalism, (13.115) holds; this gives the field-theoretic expression for the S-matrix elements. A similar formula holds in the S-matrix method; but now the role of the amputated T-products is taken on by the radiation operators.

Exercise 14.4. Prove that if the momenta of the incoming particles are pairwise-non-collinear with the momenta of the outgoing particles, then the following reduction formula holds:

$$\begin{aligned}
&\langle 0|\tilde\phi^{\text{in}(\kappa_1)}_{l_1}(p_1)\ldots\tilde\phi^{\epsilon(\kappa_k)}_{l_k}(p_k) S \tilde\phi^{\text{in}(\kappa_{k+1})}_{l_{k+1}}(-p_{k+1})\ldots\tilde\phi^{\text{in}(\kappa_n)}_{l_n}(-p_n)|0\rangle = \\
&\qquad = (-i)^k i^{n-k} \sum_{\substack{\kappa'_1\ldots\kappa'_n \\ l'_1\ldots l'_n}} \widetilde D^{(\kappa_1\kappa'_1)}_{l_1 l'_1}(p_1)\ldots\widetilde D^{(\kappa_n\kappa'_n)}_{l_n l'_n}(-p_n)\times \\
&\qquad\times \int \prod_{j=1}^{k} e^{i\sum p_j x_j} \prod_{l=k+1}^{n} e^{-ip_l x_l} \langle 0|S^* \frac{\delta^n S}{\delta\phi^{\text{in}(\kappa'_1)}_{l'_1}(x_1)\ldots\delta\phi^{\text{in}(\kappa'_n)}_{l'_n}(x_n)}|0\rangle dx_1\ldots dx_n;
\end{aligned} \tag{14.36}$$

here $p_1,\ldots,p_n \in V^+$. [Hint: Use the commutation formula (14.12) and the stability of the vacuum $\langle 0|S^* = \langle 0|$.]

Among the radiation operators, a special role is played by the *current* defined by the formula

$$J^{(\kappa)l}(x) = -iS^* \frac{\delta S}{\delta\phi^{\text{in}(\kappa)}_l(x)}. \tag{14.37}$$

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* This terminology indicates the provenance of the scattering operator.

The origin of this name can be found in the Lagrangian form of quantum electrodynamics. In this case the interaction Lagrangian is written in the form

$$\mathcal{L}(x) = -j^\mu(x)A_\mu(x), \quad \text{where } j^\mu(x) = e\bar{\psi}(x)\gamma^\mu\psi(x) = -\frac{\partial\mathcal{L}}{\partial A_\mu(x)}$$

is the electromagnetic current. As usual, by setting $S = T\{\exp(i\int \mathcal{L}(x)d^4x)\}$, it is not difficult to verify that in first order perturbation theory (with respect to the coupling constant e), the radiation operator

$$-iS^*\frac{\delta S}{\delta A_\mu(x)}$$

is, apart from the sign, the same as the electromagnetic current.

As in the case of the S-matrix, a consideration of the current "outside the mass shell" is, of course, equivalent to the introduction of a formal series in the classical field χ:

$$J^{(\kappa)l}(x;\chi) = -iS^*(\chi)\frac{\delta S(\chi)}{\delta\chi_l^{(\kappa)}(x)} \equiv -i\mathfrak{S}(\chi)^{-1}\frac{\delta\mathfrak{S}(\chi)}{\delta\chi_l^{(\kappa)}(x)}. \tag{14.38}$$

Exercise 14.5. Prove the formula

$$\phi_l^{\text{out}(\kappa)}(x) \equiv S^*\phi_l^{\text{in}(\kappa)}(x)S = \phi_l^{\text{in}(\kappa)}(x) + \sum_{\kappa',l'}\int D_{ll'}^{(\kappa\kappa')}(x-y)J^{(\kappa')l'}(y)d^4y. \tag{14.39}$$

[Hint: Apply the commutation formula (14.12) to $[\phi^{\text{in}}(x), S]$.]

The current has the following characteristic properties.

(a) *Property of the Hermitian adjoint*:

$$J^{(\kappa)l}(x;\chi)^* = J^{(\bar\kappa)\bar l}(x;\chi^+). \tag{14.40a}$$

(b) *Poincaré-covariance*:

$$U(a,\underset{\sim}{\Lambda})J^{(\kappa)l}(x;\chi)U(a,\underset{\sim}{\Lambda})^{-1} = \sum_m V_{lm}^{(\kappa)}(\underset{\sim}{\Lambda}^{-1})J^{(\kappa)m}(\Lambda x + a;\chi_{(a,\underset{\sim}{\Lambda})}) \tag{14.40b}$$

for $(a,\underset{\sim}{\Lambda}) \in \mathfrak{P}_0$.

(c) *Resolvability condition*:

$$\frac{\delta J^{(\kappa)l}(x;\chi)}{\delta\chi_{l'}^{(\kappa')}(y)} \mp \frac{\delta J^{(\kappa')l'}(y;\chi)}{\delta\chi_l^{(\kappa)}(x)} = -i[J^{(\kappa')l'}(y;\chi),\ J^{(\kappa,l)}(x;\chi)]_\mp. \tag{14.40c}$$

(d) *Causality*:

$$\frac{\delta J^{(\kappa)l}(x;\chi)}{\delta\chi_{l'}^{(\kappa')}(y)} = 0 \quad \text{for } x \lesssim y. \tag{14.40d}$$

Properties (14.40c), (14.40d) are usually simply written in the form

$$\frac{\delta J^{(\kappa)}(x)}{\delta\phi^{\text{in}(\kappa')}(y)} \mp \frac{\delta J^{(\kappa')}(y)}{\delta\phi^{\text{in}(\kappa)}(x)} = -i[J^{(\kappa')}(y),\ J^{(\kappa)}(x)]_\mp, \tag{14.41}$$

$$\frac{\delta J^{(\kappa)}(x)}{\delta\phi^{\text{in}(\kappa')}(y)} = 0 \quad \text{for } x \lesssim y, \tag{14.42}$$

bearing in mind that these equalities hold "outside the mass shell" (that is, they allow further formal differentiation with respect to $\phi^{\rm in}$).

Exercise 14.6. Derive the properties (14.40) from the axioms of the S-matrix approach. [Hint: Use Exercise 14.2(b) for the proof of causality (14.40d).]

Note that the current completely characterizes the theory. In fact it is not difficult to see that $\mathfrak{S}(\chi)$ is uniquely defined in terms of $J(x;\chi)$; in the next subsection we show how the quantum fields and their T-products are constructed from the functional $\mathfrak{S}(\chi)$.

Exercise 14.7. Prove that the current satisfies the locality condition

$$[J^{(\kappa)}(x),\ J^{(\kappa')}(y)]_{\mp} = 0 \quad \text{for } (x-y)^2 < 0, \tag{14.43}$$

as well as the relation

$$\frac{\delta J^{(\kappa)}(x)}{\delta \phi^{\mathrm{in}(\kappa')}(y)} = \begin{cases} 0 & \text{for } x \lesssim y, \\ -i[J^{(\kappa')}(y),\ J^{(\kappa)}(x)]_{\mp} & \text{for } x \gtrsim y. \end{cases} \tag{14.44}$$

[Hint: Use the resolvability and causality conditions.]

The variational product

$$\frac{\delta^n J(x)}{\delta\phi^{\rm in}(x_1)\dots\delta\phi^{\rm in}(x_n)} \tag{14.45}$$

is called the *retarded radiation operator* since it follows from the causality property that its support is concentrated on $x - x_j \in \overline{V}^+$ $(j = 1, \dots, n)$. To define the advanced radiation operators, we introduce the operation of formal differentiation with respect to the out-fields:

$$\frac{\delta Q}{\delta \phi_l^{\mathrm{out}(\kappa)}(x)} = S^* \frac{\delta(SQS^*)}{\delta \phi_l^{\mathrm{in}(\kappa)}(x)} S; \tag{14.46}$$

here Q is any algebraic expression consisting of the operators S, S^* and their formal variational derivatives with respect to the in-fields.

In this definition, a commutation formula of type (14.12) holds for the out-fields.

Exercise 14.8. Prove the formula of commutation with the out-fields:

$$[\phi_l^{\mathrm{out}(\kappa)}(x), Q]_{\mp} = \sum_{\kappa', l'} \int [\phi_l^{\mathrm{out}(\kappa)}(x), \phi_{l'}^{\mathrm{out}(\kappa')}(y)]_{\mp} \frac{\delta Q}{\delta \phi_{l'}^{\mathrm{out}(\kappa')}(y)} dy. \tag{14.47}$$

Exercise 14.9. (a) Prove that the definition of the current can be written in terms of the out-fields in the form

$$J^{(\kappa)l}(x) = -i \frac{\delta S}{\delta \phi_l^{\mathrm{out}(\kappa)}(x)} S^*. \tag{14.48}$$

(b) Prove the relation

$$\frac{\delta J^{(\kappa)l}(x)}{\delta \phi_{l'}^{\mathrm{out}(\kappa')}(y)} = \pm \frac{\delta J^{(\kappa')l'}(y)}{\delta \phi_l^{\mathrm{in}(\kappa)}(x)}. \tag{14.49}$$

Using Exercise 14.9(b), we can rewrite the causality condition (14.42) in the following form:

$$\frac{\delta J_l^{(\kappa)}(x)}{\delta \phi_{l'}^{\mathrm{out}(\kappa')}(y)} = 0 \quad \text{for } x \gtrsim y. \tag{14.50}$$

The *advanced radiation operator* can now be defined as

$$\frac{\delta^n J(x)}{\delta\phi^{\rm out}(x_1)\dots\delta\phi^{\rm out}(x_n)}. \tag{14.51}$$

It follows from (14.50) that its support is concentrated in $x - x_j \in \overline{V}^-$ $(j = 1, \ldots, n)$.

Exercise 14.10. Prove the following reduction formula for the amplitude of the process $2 \to n-2$ in the region where the momenta of the ingoing particles are pairwise-non-collinear with the momenta of the outgoing particles:

$$\langle 0|\tilde{\phi}_{l_n}^{\mathrm{in}(\kappa_n)}(p_n)\ldots\tilde{\phi}_{l_3}^{\mathrm{in}(\kappa_3)}(p_3)S\tilde{\phi}_{l_2}^{\mathrm{in}(\kappa_2)}(-p_2)\tilde{\phi}_{l_1}^{\mathrm{in}(\kappa_1)}(-p_1)|0\rangle =$$

$$-(-i)^n \sum_{\substack{\kappa_1'\ldots\kappa_n' \\ l_1'\ldots l_n'}} \tilde{D}_{l_n l_n'}^{(\kappa_n \kappa_n')}(p_n)\ldots\tilde{D}_{l_1 l_1'}^{(\kappa_1\kappa_1')}(-p_1)\int e^{i(-p_1x_1-p_2x_2+p_3x_3+\ldots+p_nx_n)}\times$$

$$\times\langle 0|\frac{\delta^{n-1}J^{(\kappa_1)l_1}(x_1)}{\delta\phi_{l_n'}^{\mathrm{out}(\kappa_n')}(x_n)\ldots\delta\phi_{l_3'}^{\mathrm{out}(\kappa_3')}(x_3)\delta\phi_{l_2'}^{\mathrm{out}(\kappa_2')}(x_2)}|0\rangle dx_1\ldots dx_n;$$

here $p_1, \ldots, p_n \in V^+$. [Hint: Write the left hand side in the form

$$\langle 0|\tilde{\phi}_{l_n}^{\mathrm{out}(\kappa_n)}(p_n)\ldots\tilde{\phi}_{l_3}^{\mathrm{out}(\kappa_3)}(p_3)S\tilde{\phi}_{l_2}^{\mathrm{out}(\kappa_2)}(-p_2)\tilde{\phi}_{l_1}^{\mathrm{out}(\kappa_1)}(-p_1)|0\rangle$$

and then use the commutation formula (14.47) and the stability of the one-particle states.]

Using mixed formal variational derivatives of the current with respect to the in- and out-fields, we can define the so-called *generalized retarded radiation operators*

$$\frac{\delta^n J(x)}{\delta\phi^{\mathrm{ex}_1}(x_1)\ldots\delta\phi^{\mathrm{ex}_n}(x_n)} \tag{14.52}$$

(where ex_j =in or out). The retarded (or advanced) radiation operators are clearly obtained by substituting ex_j =in (or ex_j =out) in (14.52). In the LSZ formalism we can also introduce the analogous concept of generalized retarded products of the fields (we return to this in §16.1, where we establish the corresponding property of the support of the generalized retarded products; the generalized retarded radiation operators have the same support property *).

It turns out that the conditions of resolvability (14.41) and causality (14.42) for the current can be regarded as the dynamical equations of quantum field theory (Medvedev and Polivanov, 1961,1964; Medvedev, 1961,1964,1965). An essential point of this programme is the extension of the relation (14.44) to a domain including coincident arguments (that is, for all $x, y \in \mathrm{M}$):

$$\frac{\delta J(x)}{\delta\phi^{\mathrm{in}}(y)} = -i\theta(x^0 - y^0)[J(y),\ J(x)]_\mp + \Lambda(x,y); \tag{14.53}$$

here $\theta(x^0 - y^0)[J(y),\ J(x)]_\mp$ is in a sense a regularized product of the (anti-)commutator $[J(y),\ J(x)]_\mp$ with the discontinuous θ-function, while $\Lambda(x,y)$ is a quasi-local operator with support in the region $x = y$. In particular, one can manage to reproduce the results of the usual Lagrangian perturbation theory by this means. It is interesting that the role of the Lagrangian and the counter-terms at each order of the perturbation theory in the above scheme is played by the quasi-local operator $\Lambda(x,y)$ (which is also defined at each order of the perturbation theory); thus the entire dynamics of the model is contained in the quasi-local operator.

14.2. Fields in the Asymptotic Representation

A. Construction of Quantum Fields and Their T-Products

In order to go over to the LSZ formalism, we define the quantum field $\phi^{(\kappa)}(x)$ by means of the equation of Yang-Feldman type

$$\phi_l^{(\kappa)}(x) = \phi_l^{\mathrm{in}(\kappa)}(x) + \sum_{\kappa',l'}\int D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-y)J^{(\kappa')l'}(y)dy. \tag{14.54}$$

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* In accordance with the more precise definition given in §16.1.A, the expressions (14.52) could be called generalized retarded radiation operators of Steinmann type (so as to underline the possibility of defining a wider class of similar objects of Ruelle type).

Formula (14.54) defines an operator-valued temperate generalized function. In fact, the result of smoothing $\phi_l^{(\kappa)}(x)$ with the test functions $f^{(\kappa)l} \in \mathcal{S}(\mathrm{M})$ can be written in the form

$$\sum_{\kappa,l}\int \phi_l^{(\kappa)}(x)f^{(\kappa)l}(x)dx = \sum_{\kappa,l}\int \phi_l^{\mathrm{in}(\kappa)}(x)f^{(\kappa)l}(x)dx+ \\ +\sum_{\kappa',l'}\int D_{l'}^{\mathrm{ret}(\kappa')}(f;y)J^{(\kappa')l'}(y)dy, \tag{14.55}$$

where

$$D_{l'}^{\mathrm{ret}(\kappa')}(f;y) \equiv \sum_{\kappa,l}\int f^{(\kappa)l}(x)D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-y)dx \tag{14.56}$$

is an element of $\mathcal{F}^{(\kappa)}$ so that (14.55) makes sense.

Exercise 14.11. Prove the formula

$$\phi_l^{(\kappa)}(x) = \phi_l^{\mathrm{out}(\kappa)}(x) + \sum_{\kappa',l'} D_{ll'}^{\mathrm{adv}(\kappa\kappa')}(x-y)J^{(\kappa')l'}(y)dy. \tag{14.57}$$

[Hint: Use the relation $D^{\mathrm{ret}(\kappa\kappa')}(x) - D^{\mathrm{adv}(\kappa\kappa')}(x) = D^{(\kappa\kappa')}(x)$ and (14.47).]

We will show that the fields $\phi^{(\kappa)}$ satisfy all the LSZW axioms and for this purpose we shall need to define the T-products of the $\phi^{(\kappa)}$. Prior to this, we recall certain facts touching upon T-products of the free fields ϕ^{in}.

The generating functional of the $\overset{(-)}{T}$-products of free fields has the form

$$\mathfrak{T}_0(\eta) \equiv T\exp\Big(i\int \eta(\kappa)\phi^{\mathrm{in}}(x)dx\Big) = \\ = \exp\Big(-\frac{1}{2i}\sum_{\substack{\kappa\kappa'\\ ll'}}\int D_{ll'}^{c(\kappa\kappa')}(x-y)\eta^{(\kappa)l}(x)\eta^{(\kappa')l'}(y)dx\,dy\Big)\times \\ \times :\exp\Big(i\int \eta(x)\phi^{\mathrm{in}}(x)dx\Big):, \tag{14.58a}$$

$$\bar{\mathfrak{T}}_0(\eta) = \exp\Big(\frac{1}{2i}\sum_{\substack{\kappa\kappa'\\ ll'}} D_{ll'}^{\bar{c}(\kappa\kappa')}(x-y)\eta^{(\kappa)l}(x)\eta^{(\kappa')l'}(y)dx\,dy\Big)\times \\ \times :\exp\Big(i\int \eta(x)\phi^{\mathrm{in}}(x)dx\Big):. \tag{14.58b}$$

Exercise 14.12. Prove that the functionals (14.58) satisfy the characteristic properties of the $\overset{(-)}{T}$-products (13.10). [Hint: Use the line of argument in Exercise 13.4.]

Let $\mathcal{O}$ be an algebraic expression involving the fields $\phi^{\mathrm{in}(\kappa)}$, for example,

$$\mathcal{O} = \int F(x_1,\dots,x_n)\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\dots\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)dx_1\dots dx_n, \tag{14.59a}$$

where F is a coefficient function. Then the *Wick T-ordering* of the functional $\mathcal{O}$ is defined by the formula

$$T\mathcal{O} = \int F(x_1,\dots,x_n)T(\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)\dots\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n))dx_1\dots dx_n. \tag{14.59b}$$

Exercise 14.13. (a) Prove the formula

$$T(\phi_l^{\mathrm{in}(\kappa)}(x)\mathcal{O}) = \phi_l^{\mathrm{in}(\kappa)}(x)(T\mathcal{O}) + \sum_{\kappa' l'} \int \frac{1}{i} D_{ll'}^{\mathrm{adv}(\kappa\kappa')}(x-y) T \frac{\delta \mathcal{O}}{\delta \phi_{l'}^{\mathrm{in}(\kappa')}(y)} dy =$$
$$= \pm (T\mathcal{O})\phi_l^{\mathrm{in}(\kappa)}(x) + \sum_{\kappa' l'} \frac{1}{i} \int D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-y) T \frac{\delta \mathcal{O}}{\delta \phi_{l'}^{\mathrm{in}(\kappa')}(y)} dy, \qquad (14.60)$$

where the sign (+ or −) is chosen in accordance with the "statistics" of the operators $\phi^{\mathrm{in}(\kappa)}$ and $\mathcal{O}$.

(b) Prove the formula (in the sense of formal series in the source η):

$$T\Big(\exp\Big(i\int \eta(x)\phi^{\mathrm{in}}(x)dx\Big)\mathcal{O}\Big) = T\Big(\exp\Big(i\int \eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big)\times$$
$$\times T\Big(\exp\Big(\int D^{\mathrm{adv}}(\eta; y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)\mathcal{O}\Big) =$$
$$= T\Big(\Big(\exp\int D^{\mathrm{ret}}(\eta; y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)\mathcal{O}\Big) T\Big(\exp\Big(i\int \eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big). \qquad (14.61)$$

[Hint: It suffices to choose $\mathcal{O}$ in the form of the formal series $\exp(i\int \xi(x)\phi^{\mathrm{in}}(x)dx)$ in the source $\xi(x)$.]

The variational derivatives in (14.60), (14.61) are to be interpreted literally (that is, in accordance with the rules of type (13.27)). Since the formal variational derivatives of the S-matrix with respect to the in-fields are defined, we can apply analogous formulae to define expressions of type $T(\phi^{\mathrm{in}}(x_1)\ldots\phi^{\mathrm{in}}(x_n)A)$, where A denotes some formal variational derivative of the S-matrix with respect to the in-fields:

$$A = \frac{\delta^m S}{\delta\phi^{\mathrm{in}}(y_1)\ldots\delta\phi^{\mathrm{in}}(y_n)}. \qquad (14.62)$$

Namely, such expressions are defined recursively:

$$TA = A, \qquad (14.63a)$$

$$T(\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)A) =$$
$$= \phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n) T(\phi_{l_{n-1}}^{\mathrm{in}(\kappa_{n-1})}(x_{n-1})\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)A) +$$
$$+ \sum_{\kappa_n', l_n'} \frac{1}{i} \int D_{l_n l_n'}^{\mathrm{adv}(\kappa_n\kappa_n')}(x_n - y) T \frac{\delta}{\delta\phi_{l_n'}^{\mathrm{in}(\kappa_n')}(y)} (\phi_{l_{n-1}}^{\mathrm{in}(\kappa_{n-1})}(x_{n-1})\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)A)dy.$$
$$(14.63b)$$

In precisely the same way, if B denotes a formal variational derivative of S^* with respect to the in-fields:

$$B = \frac{\delta^m S^*}{\delta\phi^{\mathrm{in}}(y_1)\ldots\delta\phi^{\mathrm{in}}(y_m)}, \qquad (14.64)$$

then the anti-chronologically ordered products $\overline{T}(\phi^{\mathrm{in}}(x_1)\ldots\phi^{\mathrm{in}}(x_n)B)$ are defined by the following recursive procedure:

$$\overline{T}B = B, \qquad (14.65a)$$

$$\overline{T}(\phi_{l_n}^{\mathrm{in}(\kappa_n)}(x_n)\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)B) = \phi_{l_n}^{\mathrm{in}(\kappa_n)}\overline{T}(\phi_{l_{n-1}}^{\mathrm{in}(\kappa_{n-1})}(x_{n-1})\ldots$$
$$\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)B) - \sum_{\kappa_n', l_n'} \frac{1}{i} \int D_{l_n l_n'}^{\mathrm{ret}(\kappa_n\kappa_n')}(x_n - y) \overline{T} \frac{\delta}{\delta\phi_{l_n'}^{\mathrm{in}(\kappa_n')}(y)} \times$$
$$\times(\phi_{l_{n-1}}^{\mathrm{in}(\kappa_{n-1})}(x_{n-1})\ldots\phi_{l_1}^{\mathrm{in}(\kappa_1)}(x_1)B)dy. \qquad (14.65b)$$

The operators under the $\overset{(-)}{T}$-product sign commute or anticommute in accordance with their "statistics".

Exercise 14.14. Prove the formulae

$$T\Big(\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)S\Big) =$$
$$= \Big(T\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big)\exp\Big(\int D^{\mathrm{adv}}(\eta;y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)S =$$
$$= \Big(\exp\Big(\int D^{\mathrm{ret}}(\eta;y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)S\Big)T\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big); \qquad (14.66\mathrm{a})$$
$$\overline{T}\Big(S^*\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big) =$$
$$= \Big(\overline{T}\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big)\exp\Big(-\int D^{\mathrm{ret}}(\eta;y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)S^* =$$
$$= \Big(\exp\Big(-\int D^{\mathrm{adv}}(\eta;y)\frac{\delta}{\delta\phi^{\mathrm{in}}(y)}dy\Big)S^*\Big)\overline{T}\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big). \qquad (14.66\mathrm{b})$$

Like equation (14.54), formulae (14.66) are well defined, as can be seen by taking the variational derivatives of these equalities with respect to η for $\eta = 0$ and smoothing them with test functions in $\mathcal{S}(\mathbf{M}^n)$.

Using the above definition we can rewrite the definition (14.54) of the quantum field in the form

$$\phi(x) = S^*T(S\phi^{\mathrm{in}}(x)). \qquad (14.67)$$

Similar expressions are written for the $\overset{(-)}{T}$-products of the quantum fields $\phi(x)$:

$$T(\phi(x_1)\dots\phi(x_n)) = S^*T(S\phi^{\mathrm{in}}(x_1)\dots\phi^{\mathrm{in}}(x_n)), \qquad (14.68\mathrm{a})$$
$$\overline{T}(\phi(x_1)\dots\phi(x_n)) = \overline{T}(\phi^{\mathrm{in}}(x_1)\dots\phi^{\mathrm{in}}(x_n)S^*)S. \qquad (14.68\mathrm{b})$$

We write this definition of the $\overset{(-)}{T}$-products in the form

$$\mathfrak{T}(\eta) \equiv T\exp\Big(i\int\eta(x)\phi(x)dx\Big) = S^*T\Big(S\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)\Big), \qquad (14.69\mathrm{a})$$
$$\overline{\mathfrak{T}}(\eta) \equiv \overline{T}\exp\Big(i\int\eta(x)\phi(x)dx\Big) = \overline{T}\Big(\exp\Big(i\int\eta(x)\phi^{\mathrm{in}}(x)dx\Big)S^*\Big)S. \qquad (14.69\mathrm{b})$$

The expressions (14.67) and (14.68) are called the quantum fields and their $\overset{(-)}{T}$-products in the *asymptotic* (more precisely, the *in-*) *representation.*

Exercise 14.15. Prove the formulae

$$\mathfrak{T}(\eta) = S^*\mathfrak{T}_0(\eta)S(\chi) = S^*S(\chi')\mathfrak{T}_0(\eta), \qquad (14.70\mathrm{a})$$
$$\overline{\mathfrak{T}}(\eta) = \overline{\mathfrak{T}}_0(\eta)S^*(-\chi')S = S^*(-\chi)\overline{\mathfrak{T}}_0(\eta)S, \qquad (14.70\mathrm{b})$$

where

$$\chi_{l'}^{(\kappa')}(y) = D_{l'}^{\mathrm{adv}(\kappa')}(\eta;y) \equiv \sum_{\kappa,l}\int\eta^{(\kappa)l}(x)D_{ll'}^{\mathrm{adv}(\kappa\kappa')}(x-y)dx, \qquad (14.71)$$
$$\chi_{l'}^{\prime(\kappa')}(y) = D_{l'}^{\mathrm{ret}(\kappa')}(\eta;y) \equiv \sum_{\kappa,l}\int\eta^{(\kappa)l}(x)D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-y)dx. \qquad (14.72)$$

[Hint: Use the relations (14.66).]

Exercise 14.16. In the S-matrix theory of the scalar Hermitian field $\phi(x)$ prove the formula

$$K_x^{(\kappa)}K_y^{(\kappa)}T(\phi(x)\phi(y)) = \frac{1}{i}K_x^{(\kappa)}\delta(x-y) - S^*\frac{\delta^2 S}{\delta\phi^{\text{in}}(x)\delta\phi^{\text{in}}(y)}. \tag{14.73}$$

[Hint: Use the relation

$$S^*T(S\phi^{\text{in}}(x)\phi^{\text{in}}(y)) = \phi^{\text{in}}(x)\phi(y) + \phi(x)\phi^{\text{in}}(y) - \phi^{\text{in}}(x)\phi^{\text{in}}(y) + \\ + \frac{1}{i}D^{\text{ret}}(x-y) - \int D^{\text{ret}}(x-x')D^{\text{ret}}(y-y')S^*\frac{\delta^2 S}{\delta\phi^{\text{in}}(x')\delta\phi^{\text{in}}(y')}dx'\,dy'.]$$

B. Fulfillment of the LSZ axioms

We verify that the $\overset{(-)}{T}$-products of the fields $\phi^{(\kappa)}$ defined above satisfy the characteristic properties (13.10). The "initial condition" (13.10a) and the Poincaré-covariance are obvious from the definition (14.69); the property of the adjoint (13.10d) also follows immediately from (14.70). We now prove causality (13.10c). Let $\operatorname{supp}\xi \gtrsim \operatorname{supp}\eta$. We set $\chi_1(x) = D^{\text{adv}}(\xi; x)$, $\chi_2(x) = D^{\text{ret}}(\eta; x)$. Then we have

$$\mathfrak{T}(\xi+\eta) = S^*T\Big(S\exp\Big(i\int(\xi(x)+\eta(x))\phi^{\text{in}}(x)dx\Big)\Big) = \\ = S^*\mathfrak{T}_0(\xi)T\Big(\exp\Big(\int D^{\text{adv}}(\xi;y)\frac{\delta}{\delta\phi^{\text{in}}(y)}dy\Big)\exp\Big(i\int\eta(x)\phi^{\text{in}}(x)dx\Big)S\Big) = \\ = S^*\mathfrak{T}_0(\xi)T\Big(\exp\Big(i\int\eta(x)(\phi^{\text{in}}(x)+\chi_1(x))dx\Big)S(\chi_1)\Big).$$

Since

$$\int\eta(x)\chi_1(x)dx \equiv \sum_{\kappa\kappa'}\int D_{ll'}^{\text{adv}(\kappa\kappa')}(x-y)\eta^{(\kappa')l'}(y)\xi^{(\kappa)l}(x)dx\,dy = 0$$

(because of the condition $\operatorname{supp}\xi \gtrsim \operatorname{supp}\eta$), this last equality can be written in the form

$$\mathfrak{T}(\xi+\eta) = S^*\mathfrak{T}_0(\xi)T\Big(\exp\Big(i\int\eta(x)\phi^{\text{in}}(x)dx\Big)S(\chi_1)\Big) = \\ = S^*\mathfrak{T}_0(\xi)\Big(\exp\Big(\int D^{\text{ret}}(\eta;y)\frac{\delta}{\delta\phi^{\text{in}}(y)}dy\Big)S(\chi_1)\Big)\mathfrak{T}_0(\eta) = \\ = S^*\mathfrak{T}_0(\xi)S(\chi_1+\chi_2)\mathfrak{T}_0(\eta).$$

Since $\operatorname{supp}\chi_1 \gtrsim \operatorname{supp}\chi_2$, we have $S(\chi_1+\chi_2) = S(\chi_1)S^*S(\chi_2)$ (by virtue of the causality condition (14.31)). As a result we obtain

$$\mathfrak{T}(\xi+\eta) = S^*\mathfrak{T}_0(\xi)S(\chi_1)S^*S(\chi_2)\mathfrak{T}_0(\eta) = \mathfrak{T}(\xi)\mathfrak{T}(\eta),$$

which proves (13.10c).

It remains to prove unitarity (13.10e) or the equivalent equality (13.11). Defining $\chi(x)$ by formula (14.71), we have

$$\bar{\mathfrak{T}}(-\eta)\mathfrak{T}(\eta) = (S^*(\chi)\bar{\mathfrak{T}}_0(-\eta)S)\;(S^*\mathfrak{T}_0(\eta)S(\chi)) = 1;$$

here we have used the fact that $\mathfrak{T}_0(\eta)$ satisfies (13.11) (according to Exercise 14.12), in addition to the extended unitarity condition (14.30).

Thus the properties (13.10) all hold.

Exercise 14.17. Prove that the quantum fields $\phi^{(\kappa)}(x)$ constructed in the above manner satisfy the locality condition. [Hint: If $\operatorname{supp}\xi \sim \operatorname{supp}\eta$, then $\mathfrak{T}(\xi+\eta) = \mathfrak{T}(\xi)\mathfrak{T}(\eta) = \mathfrak{T}(\eta)\mathfrak{T}(\xi)$; now take the terms bilinear in ξ and η in the latter equality.]

Thus we have constructed a system of quantum fields $\phi^{(\kappa)}(x)$ and verified that all the LSZW axioms hold for them with the exception of the cyclicity of the vacuum and the asymptotic completeness condition. For the proof of these conditions, it clearly suffices to verify that the original free fields $\overset{\mathrm{in}(\kappa)}{\phi_{\mathrm{out}}}(x)$ are precisely the asymptotic fields for the $\phi^{(\kappa)}(x)$ (which justifies the use of the "in out" notation). According to the Yang-Feldman equations (13.92) for the derivation of which, the cyclicity and asymptotic completeness axioms were inessential, the asymptotic in-field is given at vectors $\Phi^{\mathrm{in}} \in D_0^{\mathrm{in}}$ by the expression

$$\left\{\phi^{(\kappa)}(x) - \int D_m^{\mathrm{ret}}(x-y)K_y^{(\kappa)}\phi^{(\kappa)}(y)dy\right\}\Phi^{\mathrm{in}} \tag{14.74}$$

(where $m \equiv m_\kappa$). Here we replace $\phi^{(\kappa)}$ by the expression (14.53) and use the identity

$$\int D_m^{\mathrm{ret}}(x-y)(K_y^{(\kappa)}D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(y-z))dy = D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-z).$$

As a result, for (14.74) we obtain

$$\left\{\phi_l^{(\kappa)}(x) - \sum_{\kappa'l'}\int D_{ll'}^{\mathrm{ret}(\kappa\kappa')}(x-y)J^{(\kappa')l'}(y)dy\right\}\Phi^{\mathrm{in}} = \phi_l^{\mathrm{in}(\kappa)}(x)\Phi^{\mathrm{in}}.$$

Thus the asymptotic in-field coincides with $\phi^{\mathrm{in}(\kappa)}$ on D_0^{in} (and, by taking the closure, on the entire domain of definition D). The case of the out-field is similar.

The above result can be summarized as follows.

Theorem 14.1. *In the S-matrix theory (in which Axioms* S.I–S.V *and* S.I′ *hold), the formulae* (14.67), (14.68) *define quantum fields* $\phi^{(\kappa)}$ *and their* $\overset{(-)}{T}$*-products, satisfying all the* LSZW *axioms.*

We have described the passage from the S-matrix approach to the LSZ formalism. In principle, the reverse is possible, that is, going from the LSZ formalism to the S-matrix method; (in the case of scalar fields, formulae of type (14.73) which define the formal variational derivatives of the S-matrix, provide the starting point); here, however, there remain difficulties in bringing the "technical" conditions into correspondence.

Part V

Causality and the Spectral Property: The Origins of the Analytic Properties of the Scattering Amplitude

Synopsis

It is possible to form 32 different generalized retarded functions (GRF's) from the vacuum expectation values of products of four fields. To give a description of them, it is convenient to introduce two types of variational derivatives, defined as follows.

Let $\mathfrak{T}(\eta)$ be the generating functional (13.7) for the T-products and $F(\eta)$ any arbitrary formal series in the classical source $\eta(x)$ with operator-valued "coefficient functions". Then we set

$$\frac{\delta F}{\delta\eta(x,-)} = \frac{\delta F}{\delta\eta(x)}, \quad \frac{\delta F}{\delta\eta(x,+)} = \mathfrak{T}^{-1}(\eta)\frac{\delta}{\delta\eta(x)}[\mathfrak{T}(\eta)F(\eta)\mathfrak{T}(\eta)^{-1}]\mathfrak{T}(\eta),$$
$$\mathfrak{R}(x;\eta) = -i\mathfrak{T}(\eta)^{-1}\frac{\delta\mathfrak{T}(\eta)}{\delta\eta(x,-)} = -i\frac{\delta\mathfrak{T}(\eta)}{\delta\eta(x,+)}\mathfrak{T}(\eta)^{-1}$$

and for any permutation j,k,l,n of the indices 1,2,3,4 we define the generalized functions

$$r_n(x) = \langle 0\left|\frac{\delta}{\delta\eta(x_j,-)}\frac{\delta}{\delta\eta(x_k,-)}\frac{\delta}{\delta\eta(x_l,-)}\mathfrak{R}(x_n;\eta)\right|_{\eta=0}\left|0\right\rangle,$$
$$a_n(x) = \langle 0\left|\frac{\delta}{\delta\eta(x_j,+)}\frac{\delta}{\delta\eta(x_k,+)}\frac{\delta}{\delta\eta(x_l,+)}\mathfrak{R}(x_n;\eta)\right|_{\eta=0}\left|0\right\rangle,$$
$$r_{jn}(x) = \langle 0\left|\frac{\delta}{\delta\eta(x_j,+)}\frac{\delta}{\delta\eta(x_k,-)}\frac{\delta}{\delta\eta(x_l,-)}\mathfrak{R}(x_n;\eta)\right|_{\eta=0}\left|0\right\rangle,$$
$$a_{jn}(x) = \langle 0\left|\frac{\delta}{\delta\eta(x_j,-)}\frac{\delta}{\delta\eta(x_k,+)}\frac{\delta}{\delta\eta(x_l,+)}\mathfrak{R}(x_n;\eta)\right|_{\eta=0}\left|0\right\rangle.$$

(For notational simplicity, the fields featuring here are assumed to be of boson type.) The main result of Chapter 16 is a proof of the assertion that under "linear postulates", the Fourier transforms of the GRF's are boundary values of a single analytic function whose domain of holomorphy contains complex neighbourhoods of the physical domains of all the two-particle processes that can be constructed from the four given particles. In the proof an essential role is played by the following stability condition. If m_j is the mass of particle j, $m_{jk} = m_{ln}$ is the mass of a one-particle state in the channel $jk \to ln$ (if there is such a channel), $M_{jk} = M_{ln}(> m_{jk})$ is the threshold in this channel and M_j is the threshold of the "channel" $j \to kln$, then the following inequalities are assumed to hold:

$$m_j < m_k + m_l + m_n, \quad M_{jk} \le m_j + m_k, \quad M_j \le m_{jk} + m_k$$

(in the last inequality, m_{jk} must be replaced by M_{jk} in the case when there is no one-particle pole in the channel $jk \to ln$). En route we establish the Steinmann identity

$$r_{jn}(x) - r_{lk}(x) = a_{kl}(x) - a_{nj}(x).$$

In Ch. 15 we derive the dispersion relations with respect to the energy for the scattering amplitude at a fixed value of the momentum transfer t (in some interval); here the amputated Green's function is involved on the mass shell under certain extra restrictions imposed on the masses and the thresholds (which hold for $\pi\pi$- and πN-scattering but not for NN-scattering). In the case of the invariant amplitude T of pion-pion, say $\pi^0\pi^0$-forward scattering (with the energy E of the incoming particle in the laboratory frame), the dispersion relation has the form

$$T(s,t)\Big|_{\substack{s=2m(E+m)\\ t=0}} \equiv f(E) = f(0) + \frac{4m}{\pi}\int_m^\infty \frac{E^2\sqrt{E'^2-m^2}\sigma_{\text{tot}}(E')dE'}{E'(E'^2-E^2-i0)}$$

or

$$f(E) = f(m) + \frac{4m}{\pi}\int_m^\infty \frac{(E^2-m^2)E'\sigma_{\text{tot}}(E')dE'}{\sqrt{E'^2-m^2}(E'^2-E^2-i0)},$$

where $\sigma_{\text{tot}}(E)(= \operatorname{Im} T/(2m\sqrt{E^2-m^2}))$ is the total cross section of the $\pi^0\pi^0$-interaction, which (possibly after averaging over a finite energy interval $[E-\delta/2, E+\delta/2]$) is bounded above by the quantity const$\cdot(\ln E)^2$ (called the Froissart boundary, §17.1.A).

In the proof of the dispersion relation we use the analytic continuation with respect to the variable $\zeta = p_2^2 - m_2^2 = p_4^2 - m_4^2$ from negative values to the point $\zeta = 0$. The same idea is used in the derivation of results in the more complicated situation considered in Ch. 16. Namely, the situation here requires us to set out the material of Part V in inductive fashion — from the particular to the general — rather than in deductive fashion: the scheme of the proof in Ch. 16 (especially in §16.3) in large measure duplicates the detailed arguments of Ch. 15 for the scattering amplitudes.

By exploiting the unitarity of the S-matrix we can continue the dispersion relation analytically with respect to t (in a disc $|t| < \mu^2$ not depending on s), while for physical s we can continue its absorptive part analytically with respect to t; these analytic continuations are into the small and large Martin ellipses respectively (Appendix J). A joint application of analyticity and unitarity lies at the basis of the derivation of estimates of the behaviour of the cross sections of two-particle processes as $s \to \infty$ (§17.1.A), whereas for assertions of Pomeranchuk theorem type (§17.1.B) the crossing property is essential. For inclusive processes (for example, $\kappa_1 + \kappa_2 \to \kappa_3 + \ldots$, where κ_3 is a detectable particle and the dots denote any allowable complex of particles) it is again possible to establish analytic properties of the differential cross sections and asymptotic estimates as $s \to \infty$ (§17.2).

CHAPTER 15

Analyticity with respect to Momentum Transfer and Dispersion Relations

15.1. The Lehmann Small Ellipse

A. INTRODUCTORY REMARKS

The amplitudes of processes in relativistic quantum theory possess certain analytic properties with respect to the momentum variables. These properties establish relations between the experimentally observed quantities and are therefore of great importance for the theory of interactions of elementary particles as compared with the significance of analyticity in quantum mechanics. Thus pole-type singularities indicate the presence of bound states in a given channel (more generally, localization of singularities of amplitudes is defined by physical parameters such as the masses of the particles and resonances, the energies of the bound states, reaction threshholds and so on). The dispersion relations derived from analyticity express the amplitude of an elastic process in terms of its imaginary part (which in forward scattering is expressed, according to the optical theorem, in terms of the total cross section). One of the specific peculiarities of the relativistic S-matrix (in contrast to the ordinary quantum mechanical one) is the crossing property which compares the value of the amplitude of the process (in the physical region) with the amplitude of some other (crossing-) process at non physical points; an understanding of relations of this kind can only be achieved by going over to the complex domain and by analytic continuation of the physical amplitude.

In this connection, analytic properties that follow from the principles of quantum field theory acquire a special significance. On the basis of these, a number of exact results have been obtained concerning the asymptotic behaviour of the amplitudes at high energies. On the other hand, analyticity can serve as the starting point for the construction of models and approximate schemes (such as the model of Regge poles and the dual resonance models) for collision processes (see, for example, [D1, N4]).

The derivation of the analytic properties of the scattering amplitude is based on reduction formulae according to which, the scattering amplitude is (to within a constant factor) the amputated (for example, causal) Green's function in the momentum representation restricted to the mass shell. The formulae of type (13.49) identifying such functions (in domains of the momentum space) with retarded or advanced (and, as we shall see in Ch.16, generalized retarded) Green's functions enables us to go over to the complex domain in the momentum space. This leads us to a certain (so-called primitive) domain of analyticity of the Green's function. Using the technique of analytic extension of domains, we then obtain by this means, a number of important qualitative results on the analytic properties of the amplitudes of two-particle processes (see Ch.16).

In this chapter we deal with the simpler problem of the analytic properties of two-particle processes with respect to one of the invariant variables $s = (p_1 + p_2)^2$ or $t = (p_1 + p_3)^2$, the other variable being fixed (in this context, of course, the generalized functions are to be smoothed with a test function in this "fixed" variable).

To determine the analyticity properties of the amplitude with respect to t (for fixed s), it suffices to use a partial reduction formula (of type (13.95)) with respect to the momenta $-p_1, -p_2$ of the incoming particles and in this way go over to the complex domain with respect to p_1, p_2; here the momenta p_3, p_4 of the outgoing particles as well as the variable s take only physical values. * An examination of this (§15.1.C) gives for the domain of analyticity (with respect to t) of the amplitude of the process

$$\kappa_1 + \kappa_2 \to \bar{\kappa}_3 + \bar{\kappa}_4 \tag{15.1}$$

the so-called *Lehmann small ellipse* $E(s) \equiv E_{12,34}(s)$: **

$$E(s) \equiv E_{12,34}(s) = \{t \in \mathbf{C} : |t - t_{\min}(s)| + |t - t_{\max}(s)| < 4K_{12}(s)K_{34}(s)x(s)\}, \tag{15.2}$$

encircling the interval $[t_{\min}(s),\ t_{\max}(s)]$ of physical values for t; here

$$x(s) = x_{12}(s) \vee x_{34}(s), \tag{15.3}$$

$$x_{12}(s) = \left\{1 + \frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{K_{12}^2(s)[s - (M_1 - M_2)^2]}\right\}^{1/2},$$

$$x_{34}(s) = \left\{1 + \frac{(M_3^2 - m_3^2)(M_4^2 - m_4^2)}{K_{34}^2(s)[s - (M_3 - M_4)^2]}\right\}^{1/2} \tag{15.4}$$

($K_{12}(s)$ and $K_{34}(s)$ are defined in (7.187)).

The Lehmann small ellipse is defined by the masses of the particles $m_{\kappa_j} \equiv m_j$ participating in the reaction and by the thresholds $M_{\kappa_j} \equiv M_j$ ($> m_j$) associated with the corresponding fields. Here M_κ is defined as the greatest lower bound of the continuous mass spectrum in the subspace $\mathcal{H}^{(\kappa)}$, which is the closure of the set of vectors of the form

$$\sum_l \phi_l^{(\kappa)}(f^l)\Psi_0, \quad \text{where } f^l \in \mathcal{S}(\mathbf{M})$$

($M_\kappa > m_\kappa$ according to Axiom LSZ.I †). We confine our attention to the typical case when the mass spectrum in $\mathcal{H}^{(\kappa_j)}$ contains no discrete points in the interval $(0, M_j)$ other than m_j. (The generalization to the case of several discrete points presents no difficulties; in fact, by acting on the field $\phi^{(\kappa)}(x)$ by Kleinians with the relevant masses, we can generally exclude the superfluous one-particle contributions.)

In precisely the same way we define the threshold of the continuous spectrum of the mass $M_{\kappa_1 \ldots \kappa_n}$ and the discrete mass $m_{\kappa_1 \ldots \kappa_n} \in (0, M_{\kappa_1 \ldots \kappa_n})$ (or the discrete

* Although p_1 and p_2 are now complex, they still lie on the plane in $\mathbf{C}M^4$ given by the equation $p_1 + p_2 + p_3 + p_4 = 0$.

** Here and in what follows, by ellipse we mean the interior of the ellipse.

† We suppose in this chapter and the following one that the LSZW axioms or their variant, the axioms of the S-matrix method, hold.

masses if there are several) in the space $\mathcal{H}^{(\kappa_1 \ldots \kappa_n)}$ defined in Exercise 8.3. (According to this exercise, $M_{\kappa_1 \ldots \kappa_n}$ and $m_{\kappa_1 \ldots \kappa_n}$ does not depend on the ordering of the indices $\kappa_1, \ldots, \kappa_n$.) The parameters $m_{ij} \equiv m_{\kappa_i \kappa_j} = m_{\bar{\kappa}_i \bar{\kappa}_j}$ and $M_{ij} \equiv M_{\kappa_i \kappa_j} = M_{\bar{\kappa}_i \bar{\kappa}_j}$ (where $1 \leq i < j \leq 4$) are essential for the characterization of the reaction (15.1). We shall suppose that *

$$m_{ij} = m_{kl}, \; M_{ij} = M_{kl} \; (> m_{kl}), \tag{15.5}$$

where i, j, k, l is an arbitrary permutation of the indices (1,2,3,4).

In the case of the elastic process $\kappa_1 + \kappa_2 \to \kappa_1 + \kappa_2$ it is clear that the conditions

$$m_1 = m_3, \; m_2 = m_4, \; M_1 = M_3, \; M_2 = M_4 \tag{15.6}$$

hold. In general, if the conditions (15.6) hold, then we call the process (15.1) *quasi-elastic*. Of such a type is the important class of processes in which the incoming and outgoing particles are related by a symmetry transformation (for example, an isotopic one).

Finally, we recall that in the scattering theory that we shall be considering, all the particles are assumed to be stable and decay processes are excluded (§7.3.E). In particular, we exclude the processes $\kappa_i \to \bar{\kappa}_j + \bar{\kappa}_k + \bar{\kappa}_l$ (where i, j, k, l is an arbitrary permutation of the indices 1,2,3,4). Thus we suppose that the conditions

$$m_i < m_j + m_k + m_l \tag{15.7}$$

hold, which in energy terms disallows such decays. Following Epstein (1966), for the purposes of this and the following chapters, we supplement this list (under the general name of *stability conditions*) with the following hypotheses:

$$M_{jk} \leq m_j + m_k, \tag{15.8}$$

$$M_j \leq m_{jk} + m_k, \tag{15.9}$$

which have the obvious meaning. In the absence of a discrete mass m_{jk}, (15.9) has to be modified as follows:

$$M_j \leq M_{jk} + m_k.$$

Exercise 15.1. (a) Deduce from (15.8), (15.9) that

$$|m_j - m_k| < m_{jk} < m_j + m_k, \quad |m_j - m_k| < M_{jk} \leq m_j + m_k, \tag{15.10}$$

$$|M_j - M_k| < m_{jk} < M_j + M_k, \quad |M_j - M_k| < M_{jk} < M_j + M_k. \tag{15.11}$$

[Hint: Use the fact that $m_j < M_j$, $m_{jk} < M_{jk}$.]

(b) Deduce (15.7) from (15.5), (15.8), (15.9).

Exercise 15.2. Consider the process

$$\kappa_j + \kappa_k \to \bar{\kappa}_l + \bar{\kappa}_m \tag{15.12}$$

with particle momenta

$$-p_j \in \Gamma^+_{m_j}, \; -p_k \in \Gamma^+_{m_k}, \; p_l \in \Gamma^+_{m_l}, \; p_m \in \Gamma^+_{m_m}. \tag{15.13}$$

* If not, we can arrange for these conditions to be satisfied by a change of parameters, setting $M'_{ij} = M'_{kl} = M_{ij} \vee M_{kl}$ instead of M_{ij} and M_{kl}. (If m_{ij} and m_{kl} are not the same, then these discrete masses can be completely ignored).

Prove that

$$
\begin{aligned}
(p_j + p_k)^2 \geq (m_j + m_k)^2 \geq M_{jk}^2,\\
(p_j + p_l)^2 \leq (m_j - m_l)^2 < M_{jl}^2, \quad (p_j + p_m)^2 \leq (m_j - m_m)^2 < M_{jm}^2.
\end{aligned} \tag{15.14}
$$

B. JLD REPRESENTATION FOR RETARDED AND ADVANCED (ANTI)COMMUTATORS

In this chapter we repeatedly make use of the JLD representation for retarded and advanced (anti)commutators and here we restate the results of §4.3 as applied to the quantum field context.

First, we can write down the JLD representation for the ordinary (anti)commutator of the fields $\phi_{l-1}(x_{l-1})$ and $\phi_l(x_l)$:

$$
\begin{aligned}
&\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-2}(p_{l-2})[\tilde{\phi}_{l-1}(p_{l-1}),\ \tilde{\phi}_l(p_l)]_{\mp}\tilde{\phi}_{l+1}(p_{l+1}) \dots \tilde{\phi}_n(p_n)\rangle_0 = \\
&\qquad = (2\pi)^4\delta(p_1 + \dots + p_n) \int \epsilon(p_l^0 - p_l'^0)\delta((p_l - p_l')^2 - \lambda)\times \\
&\qquad\qquad \times \Psi(p_1, \dots, p_{l-2}, p_l', p_{l+1}, \dots, p_n, \lambda)d_4p_l'd\lambda
\end{aligned} \tag{15.15}
$$

(here we can replace the fields ϕ_j for $j = 1, \dots, l-2, l+1, \dots, n$ by the asymptotic fields ϕ_j^{ex}).

For the proof we note that, thanks to translational symmetry, the generalized function

$$
\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-2}(p_{l-2})[\phi_l(x_{l-1}),\ \phi_l(x_l)]_{\mp}\tilde{\phi}_{l+1}(p_{l+1}) \dots \tilde{\phi}_n(p_n)\rangle_0
$$

is a C^∞-function of x_{l-1}. Consequently it can be restricted to the plane $x_{l-1} = 0$ and the left hand side of (15.15) is expressed in terms of this restriction:

$$
\begin{aligned}
&\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-2}(p_{l-2})[\tilde{\phi}_{l-1}(p_{l-1}),\ \tilde{\phi}_l(p_l)]_{\mp}\tilde{\phi}_{l+1}(p_{l+1}) \dots \tilde{\phi}_n(p_n)\rangle_0 = \\
&\qquad = (2\pi)^4\delta(p_1 + \dots + p_n)g(p_1, \dots, p_{l-2}, p_l, \dots, p_n),
\end{aligned} \tag{15.16}
$$

where

$$
\begin{aligned}
&g(p_1, \dots, p_l, \dots, p_n) = \\
&\qquad = \int e^{ip_lx_l}\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-2}(p_{l-2})[\phi_{l-1}(0),\ \phi_l(x_l)] \dots \tilde{\phi}_n(p_n)\rangle_0 dx_l
\end{aligned} \tag{15.17}
$$

is a generalized function in $\mathcal{S}'(\mathbf{M}^{n-1})$. It follows from the locality condition that the generalized function in the Fourier integral in (15.17) has support in the light cone $\overline{V}$, so that $g(p_1, \dots, p_l, \dots, p_n)$ is of class $\sigma(\overline{V})$ with respect to the momentum p_l. It then follows from the (strong) spectrum condition that the support of the generalized function

$$
\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-1}(p_{l-1})\tilde{\phi}_l(p_l) \dots \phi_n(p_n)\rangle_0 \tag{15.18}
$$

lies in the subset of $\mathbf{M}^n$ defined by

$$
p_l + p_{l+1} + \dots + p_n \in \{0\} \cup \overline{V}_{M'}^-, \tag{15.19}
$$

while the support of

$$
\langle \tilde{\phi}_1(p_1) \dots \tilde{\phi}_{l-2}(p_{l-2})\tilde{\phi}_l(p_l)\tilde{\phi}_{l-1}(p_{l-1}) \dots \tilde{\phi}_n(p_n)\rangle_0 \tag{15.20}
$$

is contained in the set of points such that

$$p_{l-1} + p_{l+1} + \ldots + p_n \in \{0\} \cup \overline{V}^-_{M}, \tag{15.21}$$

where M' and M are the greatest lower bounds of the mass spectra in the orthocomplement of the vacuum in the subspaces $\mathcal{H}^{(\bar\kappa_1 \ldots \bar\kappa_{l-2}\bar\kappa_{l-1})} \cap \mathcal{H}^{(\kappa_l \ldots \kappa_n)}$ and $\mathcal{H}^{(\bar\kappa_1 \ldots \bar\kappa_{l-2}\bar\kappa_l)} \cap \mathcal{H}^{(\kappa_{l-1}\kappa_{l+1} \ldots \kappa_n)}$ respectively. Thus the support of $g(p_1, \ldots, p_l, \ldots, p_n)$ lies in the union of the sets (15.19) and (15.21). However, (according to the next exercise), the point 0 must be excluded here and we come to the conclusion that the support of $g(p_1, \ldots, p_l, \ldots, p_n)$ is contained in the set of points such that

$$p_l \in [-(p_1 + \ldots + p_{l-2}) + \overline{V}^+_M] \cup [-(p_{l+1} + \ldots + p_n) + \overline{V}^-_{M'}]. \tag{15.22}$$

Exercise 15.3. Prove that $g(p_1, \ldots, p_n) = 0$, if p_l lies outside the set (15.22). [Hint: Expand (15.18) and (15.20) in truncated expectation values; then the terms in which the fields ϕ_{l-1} and ϕ_l fall into distinct clusters cancel each other in (15.16). The spectrum property of Exercise 12.8 can be applied to the remaining terms.]

It is clear that the usual spectrum condition holds alongside (15.22); the support of $g(p_1, \ldots, p_l, \ldots, p_n)$ lies in the set of points such that

$$\begin{aligned} &p_1 + \ldots + p_j \in \{0\} \cup \overline{V}^+_{\mu_j} \quad \text{for } j, \ldots, l-2, \\ &p_j + \ldots + p_n \in \{0\} \cup \overline{V}^-_{\mu_j} \quad \text{for } j = l+1, \ldots, n \end{aligned} \tag{15.23}$$

(the $\mu_j > 0$ being mass parameters). We suppose that at least one of the momenta $p_1 + \ldots + p_{l-2}$, $p_{l+1} + \ldots + p_n$ is non-zero (since otherwise (15.16) reduces to the vacuum expectation value of the (anti)commutator $[\tilde\phi_{l-1}(p_{l-1}),\ \tilde\phi_l(p_l)]$, which is of no interest to us). Then in addition to (15.23) we have

$$p_1 + \ldots + p_{l-2} - p_{l+1} - \ldots - p_n \in V^+. \tag{15.24}$$

A comparison of the conditions of the support (15.22) and (4.109b) shows that here we are dealing with the situation in §4.3.C. (An inessential difference is that now $g(p_1, \ldots, p_l, \ldots, p_n)$ is a generalized function in the other momenta in addition to p_l; however, the derivation of the JLD representation remains valid equally in this case.)

As a result we arrive at the JLD representation (15.15) for the (anti)commutator, where the spectral function $\Psi(p_1, \ldots, p_{l-2}, p_l', p_{l+1}, \ldots, p_n; \lambda)$ is a generalized function in $\mathcal{S}'(\mathbf{M}^{n-1} \times \overline{\mathbf{R}}_+)$ with the usual property at the support (15.23) with respect to $p_1, \ldots, p_{l-2}, p_{l+1} \ldots, p_n$; furthermore, for any $\epsilon > 0$ Ψ can be chosen with its support with respect to the variables $p_l' \in \mathbf{M}$, $\lambda \geq 0$ in an ϵ-neighbourhood (in $\mathbf{M} \times \overline{\mathbf{R}}_+$) of the set

$$\mathrm{adm} \equiv \mathrm{adm}(p_1 + \ldots + p_{l-2}, p_{l+1} + \ldots + p_n),$$

where

$$\begin{aligned} \mathrm{adm} = \{(p_l', \lambda) \in \mathbf{M} \times \overline{\mathbf{R}}_+ : p_l' \in Q[-(p_1 + \ldots + p_{l-2}), -(p_{l+1} + \ldots + p_n)], \\ \lambda \geq \kappa^2(p_1 + \ldots + p_{l-2} + p_l', p_l' + p_{l+1} + \ldots + p_n)\}, \end{aligned} \tag{15.25}$$

$$\begin{aligned} \kappa(p_1 + \ldots + p_{l-2} + p_l', p_l' + p_{l+1} + \ldots + p_n) = 0 \vee \left(M - \sqrt{(p_1 + \ldots + p_{l-2} + p_l')^2}\right) \vee \\ \vee \left(M' - \sqrt{(p_l' + p_{l+1} + \ldots + p_n)^2}\right). \end{aligned} \tag{15.26}$$

Then, proceeding as in §4.3.D, we obtain the representations for the retarded and advanced (anti)commutators of ϕ_{l-1} and ϕ_l

$$\langle\tilde{\phi}_1(p_1)\ldots\tilde{\phi}_{l-2}(p_{l-2})R(\tilde{\phi}_{l-1}(p_{l-1});\tilde{\phi}_l(p_l))\tilde{\phi}_{l+1}(p_{l+1})\ldots\tilde{\phi}_n(p_n)\rangle_0 =$$
$$= (2\pi)^4\delta(p_1+\ldots+p_n)h_r(p_1,\ldots,p_{l-2},p_l,\ldots,p_n), \tag{15.27a}$$
$$\langle\tilde{\phi}_1(p_1)\ldots\tilde{\phi}_{l-2}(p_{l-2})A(\tilde{\phi}_{l-1}(p_{l-1});\tilde{\phi}_l(p_l))\tilde{\phi}_{l+1}(p_{l+1})\ldots\tilde{\phi}_n(p_n)\rangle_0 =$$
$$= (2\pi)^4\delta(p_1+\ldots+p_n)h_a(p_1,\ldots,p_{l-2},p_l,\ldots,p_n), \tag{15.27b}$$

where

$$h_{r,a}(p_1,\ldots,p_l,\ldots,p_n) = \int \frac{\Psi(p_1,\ldots,p_{l-2},p'_l,p_{l+1},\ldots,p_n,\lambda)}{(p_l^0-p_l'^0\pm i0)^2-(\mathbf{p}_l-\mathbf{p}'_l)^2-\lambda}d_4p'_l d_1\lambda; \tag{15.28}$$

here Ψ is a suitable extension of the spectral function from the representation (15.15) to a generalized function in the space $\mathcal{S}'(A\times[0,\infty])$ and the set A consists of points $(p_1,\ldots,p_{l-2},p'_l,p_{l+1},\ldots,p_n)$, satisfying the conditions (15.22) and

$$p'_l \in Q[-(p_1+\ldots+p_{l-2}),\ -(p_{l+1}+\ldots+p_n)]. \tag{15.29}$$

As we know from §4.3.D, the $h_{r,a}(p_1,\ldots,p_l,\ldots,p_n)$ are boundary values (as $\operatorname{Im}p_l\to 0$, $\operatorname{Im}p_l\in V^{\pm}$) of the same analytic function $h(p_1,\ldots,p_l,\ldots,p_n)$ with respect to p_l (and generalized functions with respect to the other momenta). There is a representation for it analogous to (15.28) and the domain of analyticity is defined by the condition

$$(p_l-p'_l)^2-\lambda\neq 0 \quad \text{for } (p'_l,\lambda)\in \text{adm}. \tag{15.30}$$

Remark. It can be assumed that the JLD spectral function has the same Lorentz-covariance properties as $h_{r,a}(p_1,\ldots,p_l,\ldots,p_n)$. To see this, it suffices to go over to a frame of reference in which the vector (15.24) is directed along the time axis:

$$p_1+\ldots+p_{l-2}-p_{l+1}-\ldots-p_n = (\sqrt{\sigma},0). \tag{15.31}$$

Since the construction of the spectral function (in §§4.3.A, 4.3.B) preserves the covariance properties with respect to the rotation group, Ψ in this chosen system has the same covariance properties with respect to rotations as $h_{r,a}$. Reverting to the original frame of reference, we obtain (according to Corollary 3.16) the requisite properties of Lorentz-covariance of Ψ.

We have said that for any $\epsilon>0$, the support of Ψ can be chosen with respect to the variables p'_l, λ in an ϵ-neighbourhood of the set adm. In fact, ϵ can here be taken to be dependent on the momenta $p_1+\ldots+p_{l-2}$ and $p_{l+1}+\ldots+p_n$ (although not in an arbitrary manner). For definiteness we consider the same frame of reference (15.31). We fix arbitrary numbers $\epsilon>0$, $N\geq 0$ and set

$$\Delta(\sigma)=\epsilon\sigma^{-N}. \tag{15.32}$$

Then it follows from the construction of the spectral function (§4.3.B) applied to the present case that the support of Ψ with respect to p'_l, λ can be chosen to be in a $\Delta(\sigma)$-neighbourhood of the set adm.

C. Analyticity with respect to t

We now turn to the proof of the analyticity of the amplitude of the two-particle process (15.1) in the Lehmann small ellipse; for the sake of simplicity, the particle spins are taken to be zero.

It follows from the definition (13.101) and the reduction formula (13.103) that the amplitude has the representation:

$$(2\pi)^4\delta(p_1+p_2+p_3+p_4)\mathcal{T}(p_4,p_3,p_2,p_1) = -\delta^-_{m_1}(p_1)\delta^-_{m_2}(p_2)\times$$
$$\times\{(p_1^2-m_1^2)(p_2^2-m_2^2)\langle\tilde{\phi}_4^{\text{out}}(p_4)\tilde{\phi}_3^{\text{out}}(p_3)A(\tilde{\phi}_2(p_2);\tilde{\phi}_1(p_1))\rangle_0\}. \tag{15.33}$$

Entering here is the amputated advanced commutator $K_{x_1}K_{x_2}A(\phi_2(x_2);\phi_1(x_1))$, for which we have the representation of type (15.27b), (15.28):

$$(p_1^2-m_1^2)(p_2^2-m_2^2)\langle\tilde\phi_4^{\rm out}(p_4)\tilde\phi_3^{\rm out}(p_3)A(\tilde\phi_2(p_2);\tilde\phi_1(p_1))\rangle_0 = \\ = (2\pi)^4\delta(p_1+\ldots+p_4)h(p_4,p_3,p_1), \qquad (15.34)$$

where

$$h(p_4,p_3,p_1) = \int \frac{\Psi(p_4,p_3,p_1',\lambda)}{(p_1^0-p_1'^0-i0)^2-(\mathbf{p}_1-\mathbf{p}_1')^2-\lambda}d_4p_1'd_1\lambda. \qquad (15.35)$$

In contrast to the considerations of §15.1.B, in deriving the representation (15.35) we must start from the JLD representation for the commutator of the current-like fields $[j_2(x_2),\ j_1(x_1)] \equiv K_{x_1}K_{x_2}[\phi_2(x_2),\ \phi_1(x_1)]$ (not the commutator of the ordinary fields). Therefore the role of the mass parameters M' and M in §15.1.B is now played by M_1 and M_2 (the greatest lower bounds of the mass spectra of the states $j_1(f_1)|0\rangle$ and $j_2(f_2|0\rangle$ respectively).

On the basis of the preceding subsection, (15.35) reduces to the analyticity of $h(p_4,p_3,p_1)$ with respect to p_1 in the domain characterized by the condition

$$(p_1-p_1')^2-\lambda\neq 0 \text{ for all } p_1'\in Q(-p_3-p_4,0),\quad \lambda\geq\kappa^2(p_3+p_4+p_1',p_1'), \qquad (15.36)$$

where

$$\kappa(p_3+p_4+p_1',p_1') = 0\vee\left(M_2-\sqrt{(p_3+p_4+p_1')^2}\right)\vee\left(M_1-\sqrt{p_1'^2}\right). \qquad (15.37)$$

To find the intersection of the domain of analyticity of $h(p_4,p_3,p_1)$ with the complex mass shell ($p_1^2=m_1^2$, $p_2^2=m_2^2$) we go over to the centre-of-mass frame of the vectors p_3, p_4 (where $\mathbf{p}_3+\mathbf{p}_4=0$). The corresponding momenta $p_1,\ldots,p_4$ on the partially complexified mass shell of the process (15.1) clearly has a parametrization of type (7.185), only $\mathbf{n}_{12}$ is now a vector on the complex quadric $\mathbf{C}S^2$:

$$\mathbf{C}S^2 = \{\mathbf{n}\in\mathbf{C}^3 : \mathbf{n}^2\equiv(n^1)^2+(n^2)^2+(n^3)^2=1\}. \qquad (15.38)$$

Condition (15.36) now means that

$$\mathbf{n}_{12}\frac{\mathbf{p}_1'}{|\mathbf{p}_1'|}\neq\frac{1}{2K_{12}(s)|\mathbf{p}_1'|}[K_{12}^2(s)+|\mathbf{p}_1'^2|-(p_1'^0+E_1(s))^2+\lambda] \qquad (15.39a)$$

for all p_1, λ such that

$$|p_1'^0+\sqrt{s}/2|+|\mathbf{p}_1|\leq\sqrt{s}/2,\ \lambda\geq\kappa^2(p_3+p_4+p_1',p_1'). \qquad (15.39b)$$

This condition selects a domain of analyticity with respect to $\mathbf{n}_{12}$ on $\mathbf{C}S^2$.

Exercise 15.4. Prove that the conditions (15.39) are equivalent to the following:

$$|\operatorname{Re}\mathbf{n}_{12}| < x_{12}(s), \qquad (15.40)$$

where

$$x_{12}(s) = \inf_v\frac{1}{2K_{12}(s)|\mathbf{v}|}\left[K_{12}^2(s)+|\mathbf{v}|^2-\left(v^0+\frac{m_1^2-m_2^2}{2\sqrt{s}}\right)^2+\kappa^2(v,s)\right], \qquad (15.41)$$

where the greatest lower bound is taken over the set of points * $v \in \mathbf{M}$ such that $|v^0| + |v| \le \sqrt{s}/2$, and

$$\kappa(v,s) = 0 \vee \left(M_1 - \sqrt{\left(v^0 - \frac{\sqrt{s}}{2}\right)^2 - |\mathbf{v}|^2}\right) \vee M_2 - \sqrt{\left(v^0 + \frac{\sqrt{s}}{2}\right)^2 - |\mathbf{v}|^2}\Bigg). \tag{15.42}$$

[Hint: The right hand side of (15.41) takes (all) values in the integral $[x_{12}(s), \infty)$, as p_1', λ vary over the set (15.39b), therefore for $|\operatorname{Re}\mathbf{n}_{12}| < x_{12}(s)$, inequality (15.39a) is observed. If $|\operatorname{Re}\mathbf{n}_{12}| \ge x_{12}(s)$, then the right hand side of (15.39a) takes the value $|\operatorname{Re}\mathbf{n}_{12}|$ for some $p_1'^0$, $|\mathbf{p}_1'|$, λ. It remains to choose $-\mathbf{p}_1'$ along $\operatorname{Re}\mathbf{n}_{12}$, so that the inequality in (15.39a) becomes an equality.]

Exercise 15.5 (Lehmann, 1958). ** Prove that the quantity defined in (15.41) has the value given in (15.4).

By virtue of Lorentz-invariance, the amplitude $T(s,t)$ of the process is a distribution with respect to s for $s > s_{\text{phys}}$ and an analytic function with respect to t in the domain that t ranges over (see (7.189)) when $\mathbf{n}_{34}$ ranges over S^2 and $\mathbf{n}_{12}$ over the part of the quadric $\mathbf{C}S^2$ characterized by (15.40).

We explain this. In the frame of reference chosen above, refined by the requirement that the vector $\mathbf{n}_{34}$ be directed along the $\mathbf{e}_3$ axis, $h(p_4, p_3, p_1)$ can be written in the form

$$h(p_4,p_3,p_1) = \delta^+_{m_3}(p_3)\delta^+_{m_4}(p_4)H(\mathbf{n}_{12},s),$$

where $H(\mathbf{n}_{12},s)$ is a distribution with respect to s for $s \ge s_{\text{phys}}$ and an analytic function with respect to $\mathbf{n}_{12}$. Thus instead of dealing with a function on a quadric, we can extend the definition of $H(\mathbf{N},s)$ with respect to the three-dimensional vector $\mathbf{N}$ in the domain $\operatorname{Re}\mathbf{NN} > 0$ by setting $H(\mathbf{N},s) = H((\mathbf{NN})^{-1/2}\mathbf{N},s)$. The Lorentz-invariance of $h(p_4,p_3,p_1)$ now reduces to $O_+(2)$-invariance of $H(\mathbf{N},s)$ with respect to the coordinates N^1, N^2. According to Proposition 5.15 (and the remark following it), the dependence of $H(\mathbf{N},s)$ on N^1, N^2 reduces to analytic dependence on the scalar square $(N^1)^2 + (N^2)^2 \equiv \mathbf{NN} - (N^3)^2$. As a result, $H(\mathbf{n}_{12},s)$ depends only on s (as a distribution) and the third projection n_{12}^3 (as an analytic function):

$$H(\mathbf{n}_{12},s) = \widetilde{H}(n_{12}^3,s);$$

thus we arrive at the assertion given above concerning the amplitude $T(s,t)$ which, in the coordinates chosen, means that $H(\mathbf{n}_{12},s)$ depends only on s (as a distribution) and on the third projection n_{12}^3 of the vector $\mathbf{n}_{12}$ (as an analytic function): $H(\mathbf{n}_{12},s) = \widetilde{H}(n_{12}^3,s)$.

Exercise 15.6. (a) For $\alpha > 0$ set

$$\mathfrak{N}(\alpha) = \{\mathbf{n} \in \mathbf{C}S^2 : |\operatorname{Re}\mathbf{n}| < \cosh\alpha\}. \tag{15.43}$$

Prove that

$$\mathfrak{N}(\alpha) = \bigcup_{0 \le \beta < \alpha} O_+(3,R)R_3(i\beta)e_1, \tag{15.44}$$

where $R_3(i\beta)$ is the complex rotation around the $\mathbf{e}_3$ axis through an angle $i\beta$:

$$R_3(i\beta) = \begin{pmatrix} \cosh\beta & i\sinh\beta & 0 \\ -i\sinh\beta & \cosh\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{15.45}$$

[Hint: Every vector in $\mathfrak{N}(\alpha)$ has the form $\mathbf{n} = \cosh\beta\,\mathbf{a} + i\sinh\beta\,\mathbf{b}$ with $\beta \in [0,\alpha)$; here $\mathbf{a}$ and $\mathbf{b}$ are a pair of orthonormal vectors in $\mathbf{R}^3$ which can be written in the form $\mathbf{a} = R\mathbf{e}_1$, $\mathbf{b} = R\mathbf{e}_2$, where $R \in O_+(3,R)$.]

(b) Prove that

$$\mathfrak{N}(\alpha) = \bigcup_{0 \le \beta < \alpha} O_+(3,R)R_3(i\beta)O_+(3,R)\mathbf{e}_1. \tag{15.46}$$

* Here we have made the substitution $v^0 = p_1'^0 + \sqrt{s}/2$, $\mathbf{v} = \mathbf{p}_1'$.

** In connection with this exercise, see the paper by Vladimirov and Logunov (1959),§2.

[Hint: For any $R, R' \in O_+(3, R)$, $|\mathrm{Im}\, RR_3(i\beta)R'e_1| = |\mathrm{Im}\, R_3(i\beta)R'e_1| \leq \cosh\beta|R'e_1| = \cosh\beta$, therefore the right hand side of (15.46) is contained in $\mathfrak{N}(\alpha)$; the reverse inclusion is clear from (15.44).]

(c) Prove that the set of complex numbers

$$\{z \equiv x + iy : z = \mathbf{mn}, \mathbf{m} \in S^2,\ \mathbf{n} \in \mathfrak{N}(\alpha)\}, \tag{15.47}$$

is the ellipse

$$(x/\cosh\alpha)^2 + (y/\sinh\alpha)^2 < 1$$

with foci at the points ± 1 and with major semi-axis $\cosh\alpha$. [Hint: According to part (a) of this exercise, (15.47) is the set of points of the form $z = \mathbf{m}R_3(i\beta)e_1$, where $\mathbf{m} \in S^2$, $0 \leq \beta < \alpha$, that is, it is the set of points $z = \cosh\beta x_1 + i\sinh\beta x_2$, where $x_1^2 + x_2^2 < 1$, $0 \leq \beta < \alpha$.]

Thus the set of values of $\cos\theta = \mathbf{n}_{12}\mathbf{n}_{34}$, as $\mathbf{n}_{34}$ ranges over S^2 (or takes just a fixed value) and $\mathbf{n}_{12}$ ranges over the part (15.40) of the quadric, is the ellipse

$$|\cos\theta - 1| + |\cos\theta + 1| < 2x_{12}(s). \tag{15.48}$$

If we now go over to the variable t, we see from (7.189) that the invariant amplitude $T(s,t)$ is analytic in the ellipse

$$E_{12}(s) = \{t \in \mathbf{C} : |t - t_{\min}(s)| + |t - t_{\max}(s)| < 4K_{12}(s)K_{34}(s)x_{12}(s)\}. \tag{15.49}$$

It now remains to observe that we could have started from the reduction formula with respect to the momenta p_3, p_4 of the final particles leaving the momenta p_1, p_2 physical. As a result, the domain of analyticity with respect to $\cos\theta$ would be the same ellipse (15.48) but with major semiaxis $x_{34}(s)$ instead of $x_{12}(s)$. It is the greater of the quantities $x_{12}(s)$, $x_{34}(s)$ that corresponds to the ellipse (15.2). As a result we arrive at the following result due to Lehmann.

Theorem 15.1. *The invariant amplitude $T(s,t)$ of the two-particle process* (15.1) *is a distribution with respect to s for $s > s_{\mathrm{phys}}$ and an analytic function with respect to t in the Lehmann small ellipse* (15.2).

15.2. Dispersion Relations

A. The main steps for the derivation of the dispersion relations

The starting point for establishing the analytic properties of the two-particle amplitude with respect to the variable s (for fixed t) is the partial reduction formula (of type (12.98)) enabling one to go over to a complex domain in the variables p_2, p_4; here the momenta p_1, p_3 (together with t) remain physical. However, the domain of analyticity in p_2, p_4 obtained in this way does not intersect the complexified mass shell $p_2^2 = m_2^2$, $p_4^2 = m_4^2$.

The method put forward by Bogolubov is as follows. It is to be noted that for certain negative values of p_2^2 and p_4^2 (that is, for "non-physical masses") and for certain values of t there is a sufficiently extensive domain of analyticity with respect to s (namely, the plane with two real cuts) enabling one to write down the dispersion relation with respect to s for these non-physical masses (see 15.2.B, 15.2.C). Thus the amplitude is expressed in terms of its jumps at the cuts (differing by the coefficient of the absorptive parts of the amplitude of the s- and u-channels respectively). The subsequent programme reduces to establishing the analytic properties of the absorptive parts of the amplitude with respect to the variables p_2^2, p_4^2 and extending the dispersion relation to physical values of the masses. For this purpose the methods of §15.1 are suitable, which show up the analyticity of the absorptive part immediately in the three variables p_2^2, p_4^2, t. The resulting domain of analyticity (§15.2.D) turns out to be larger than that of the amplitude (due to the fact that the absorptive part, in contrast to the amplitude, satisfies the iterated JLD representation); in particular, the domain of analyticity of the absorptive part with respect to t at physical values of p_2^2, p_4^2 is the so-called Lehmann large ellipse $\mathcal{E}(s)$. The final stage (§15.2.E), which is the analytic continuation with respect to p_2^2, p_4^2 in the dispersion relation, cannot be carried out for all processes, nor for all values of t. Under a certain restriction on the masses and the threshholds of the s- and u-channels, there is an interval $(\tau_{\min}, \tau_{\max})$ of physical values of t for which the dispersion relation with respect to s holds for physical values of the masses.

In view of the notorious awkwardness of the derivation of the dispersion relations, we shall restrict ourselves to the case of scalar particles (although the methods and results carry over to the cases of particles of arbitrary spin).

B. PASSAGE TO THE COMPLEX DOMAIN WITH RESPECT TO THE MOMENTA p_2, p_4

We use the reduction formula (13.102) for $p_j \in G^-_{\kappa_j}$ $(j = 1,2)$, $p_j \in G^+_{\kappa_j}$ $(j = 3,4)$:

$$(2\pi)^4\delta(p_1+p_2+p_3+p_4)\mathcal{T}(p_4,p_3,p_2,p_1) = \delta^-_{m_2}(p_2)\delta^+_{m_4}(p_4)\times$$
$$\times\{(p_2^2-m_2^2)(p_4^2-m_4^2)\langle\tilde{\phi}_3(p_3)A(\tilde{\phi}_4(p_4);\ \tilde{\phi}_2(p_2))\tilde{\phi}_1(p_1)\rangle_0\}. \tag{15.50}$$

The JLD representation applied to the advanced commutator expresses the amplitude as the boundary value of an analytic function with respect to p_2:

$$\mathcal{T}(p_4,p_3,p_2,p_1) = \delta^-_{m_2}(p_2)\delta^+_{m_4}(p_4)\lim_{\gamma\in V^-,\gamma\to 0} h(p_3,p_2+i\gamma,p_1), \tag{15.51}$$

where

$$h(p_3,p_2,p_1) = \int\frac{\Psi(p_3,p_2',p_1,\lambda)}{(p_2-p_2')^2-\lambda}d_4p_2'd_1\lambda. \tag{15.52}$$

A similar representation clearly holds for the matrix elements of the commutator of the current-like fields j_4 and j_2:

$$\langle\tilde{\phi}_3(p_3)[\tilde{j}_4(p_4),\tilde{j}_2(p_2)]\tilde{\phi}_1(p_1)\rangle_0 = (2\pi)^4\delta(p_1+\ldots+p_4)g(p_3,p_2,p_1), \tag{15.53}$$

where

$$g(p_3,p_2,p_1) = \int\epsilon(p_2'-p_2'^0)\delta((p_2-p_2')^2-\lambda)\Psi(p_3,p_2',p_1,\lambda)d_4p_2'd\lambda. \tag{15.54}$$

The points p_2 in the past and future tubes $T_1^\pm \equiv \mathbf{M}+iV^\pm$ automatically lie in the domain of analyticity of $h(p_3,p_2,p_1)$ (with respect to p_2) where we have the invariant representation

$$h(p_3,p_2,p_1) = \delta^+_{m_3}(p_3)\delta^-_{m_1}(p_1)T(s,t,\zeta_2,\zeta_4); \tag{15.55}$$

here

$$p_1\in\Gamma^-_{m_1},\ p_3\in\Gamma^+_{m_3},\quad p_1 \text{ and } p_3 \text{ non-collinear}, \tag{15.56}$$

$$\zeta_2 = p_2^2, \quad \zeta_4 = p_4^2. \tag{15.57}$$

The invariant amplitude $T(s,t,\zeta_2,\zeta_4)$ is a distribution with respect to t and an analytic function with respect to s,ζ_2,ζ_4 in a domain in $\mathbf{C}^3$ containing at least the set of values of (s,ζ_2,ζ_4) taken by the quantities $(p_1+p_2)^2$, p_2^2, p_4^2 when $p_1 \in \Gamma_{m_1}^-$, $p_3 \in \Gamma_{m_3}^+$, $p_2 \in T_1^+ \cup T_1^-$ for a fixed value of $t=(p_1+p_3)^2 < (m_1-m_3)^2$.

For the derivation of (15.55) we can go over to a frame of reference related to p_2 and p_3, for example, by supposing that $-p_1$ and $\mathbf{p}_3$ are directed along the vectors $\mathbf{e}_0$ and $\mathbf{e}_3$ respectively. The Lorentz-invariance condition then reduces to $O_+(2)$-invariance of $h(p_3,p_2,p_1)$ with respect to the components p_2^1 and p_2^2 of the vector p_2. The subsequent argument is the same as in §15.1.C. It suffices to restrict attention to points for which $p_2^1 \neq 0$; if in the infinitesimal condition of $O_+(2)$-invariance we go over from p_2^1, p_2^2 to the new variables $(p_2^1)^2+(p_2^2)^2$ and p_2^2, we see that $h(p_3,p_2,p_1)$ only depends on p_2^1,p_2^2 via the combination $(p_2^1)^2+(p_2^2)^2$. It now remains to express $p_3^0, p_2^0, p_2^3, (p_2^1)^2+(p_2^2)^2$ in terms of the invariants s,t,ζ_2,ζ_4.

In accordance with §15.1.A, M_{12} will denote the infimum of the continuous mass spectrum in the subspace of vectors of the form $\int j_2(x_2)\phi_1(x_1)u(x_1,x_2)dx_1dx_2|0\rangle$. We further suppose that the interval $(0,M_{12})$ contains at most one discrete point m_{12} of the mass operator in this subspace (in other words, there is at most one bound state * in the s-channel). We can then write down the following representation for the first term on the left hand side of (15.53):

$$\begin{aligned}\langle\tilde\phi_3(p_3)\tilde\jmath_4(p_4)\tilde\jmath_2(p_2)\tilde\phi_1(p_1)\rangle_0 &= (2\pi)^4\delta(p_1+\ldots+p_4)\times\\ &\times\delta_{m_3}^+(p_3)\delta_{m_1}^-(p_1)\{\delta(s-m_{12}^2)B_s(p_2^2,p_4^2)+\frac{1}{\pi}A_s(s,t,p_2^2,p_4^2)\}.\end{aligned} \tag{15.58}$$

The quantities $B_s(p_2^2,p_4^2)$ and $A_s(s,t,p_2^2,p_4^2)$ are given the suffix s (which should not be confused with the variable s) to indicate that they are related to the s-channel. In the invariant amplitude, the contribution of the (possible) bound state in the s-channel is shown explicitly; therefore the support of A_s is concentrated in the region $s \geq M_{12}^2$. The second term on the left hand side of (15.53) has a similar representation:

$$\begin{aligned}\langle\tilde\phi_3(p_3)\tilde\jmath_2(p_2)\tilde\jmath_4(p_4)\tilde\phi_1(p_1)\rangle_0 &= (2\pi)^4\delta(p_1+\ldots+p_4)\times\\ \times\delta_{m_3}^+(p_3)\delta_{m_1}^-(p_1)\{\delta(u-m_{14}^2)B_u(p_2^2,p_4^2)&+\frac{1}{\pi}A_u(u,t,p_2^2,p_4^2)\},\end{aligned} \tag{15.59}$$

with the support of A_u concentrated in the region $u \geq M_{14}^2$. We recall that (according to (7.184)) the variables s,t,u are here related by the formula

$$s+t+u = m_1^2+\zeta_2+m_3^2+\zeta_4. \tag{15.60}$$

In connection with the decompositions (15.58), (15.59) it should be further noted that $B_s(\zeta_2,\zeta_4)$ and $B_u(\zeta_2,\zeta_4)$ do not depend on t.

We give an explanation of this for B_s. Suppose that the bound state in the s-channel corresponds to a massless particle with in-operators of creation and annihilation $a^*(p)$, $a(p)$. Then its contribution in (15.58) is equal to

$$\int\langle\tilde\phi_3(p_3)\tilde\jmath_4(p_4)a^*(p)\rangle_0\langle a(p)\tilde\jmath_2(p_2)\tilde\phi_1(p_1)\rangle_0(dp)_{m_{12}}. \tag{15.61}$$

The form-factors occurring here have invariant representations of the form

$$\langle a(p)\tilde\jmath_2(p_2)\tilde\phi_1(p_1)\rangle_0 = (2\pi)^4\delta(p_1+p_2+p)\delta_{m_1}^-(p_1)g_1(p_2^2).$$

---

* The case of several bound states is entirely similar.

Substituting them into (15.61) and performing the integration with respect to p, we obtain the following expression for (15.61):

$$(2\pi)^4\delta(p_1+p_2+p_3+p_4)\delta^+_{m_3}(p_3)\delta^-_{m_1}(p_1)\delta^-_{m_{12}}(p_1+p_2)g_1(p_2^2)g_2(p_4^2).$$

We now turn to the representation (15.52). For the reasons given in §15.1.B, $h(p_3,p_2,p_1)$ is analytic with respect to p_2 at points satisfying the condition

$$(p_2-p_2')-\lambda\neq 0 \quad \text{for all } p_2'\in Q[-p_3,-p_1], \quad \lambda\geq\kappa^2(p_1+p_2',p_2'+p_3), \qquad (15.62)$$

where

$$\kappa(p_1+p_2',p_2'+p_3)=0\vee\left(m_{12}-\sqrt{(p_1+p_2')^2}\right)\vee\left(m_{14}-\sqrt{(p_2'+p_3)^2}\right). \qquad (15.63)$$

It turns out that this domain of analyticity does not contain points with physical values of t, p_2^2, p_4^2.

In fact, m_{12} (also M_{12}) does not exceed $(m_1+m_2)\wedge(m_3+m_4)$; hence m_{12} does not exceed the half-sum of the masses of all the particles in the reaction. We fix a point $p_2'=-\frac{1}{2}(p_1+p_3)$. Then $m_{12}-\sqrt{(p_1+p_2')^2}\leq m_{12}-\frac{1}{2}(m_1+m_3)\leq\frac{1}{2}(m_2+m_4)$. Similarly, $m_{12}-\sqrt{(p_2'+p_3)^2}$ and hence, κ in (15.63) does not exceed $\frac{1}{2}(m_2+m_4)$. We now consider the expression $(p_2-p_2')^2$ featuring in (15.62) for this same value of p_2'. For fixed values of t, p_2^2, p_4^2, this is a positive number not less than $\frac{1}{4}(m_2+m_4)^2$; consequently (15.62) does not hold at such points. Clearly this conclusion is not related to the presence or absence of bound states in the s- or u-channel since replacing m_{12} by M_{12} and m_{14} by M_{14} has no bearing on the above argument.

In this paragraph we confine attention to non-physical (more precisely, negative) values of the parameters ζ_2,ζ_4.

Lemma 15.2. *The invariant amplitude $T(s,t,\zeta_2,\zeta_4)$ (15.55) is analytic in s,ζ_2,ζ_4 at points satisfying the conditions*

$$|\sqrt{-\zeta_2}-\sqrt{-\zeta_4}|<\sqrt{-t}<\sqrt{-\zeta_2}+\sqrt{-\zeta_4}, \qquad (15.64)$$

$$\operatorname{Im}s\neq 0 \qquad (15.65)$$

(here $\zeta_2,\zeta_4,t<0$).

■ Suppose that the conditions (15.64), (15.65) hold. It suffices to find a quadruple of four-vectors $(p_1,\dots,p_4)$ (with sum 0) such that (15.56), (15.57) hold and $p_2\in T_1^+\cup T_1^-$ (here, of course, $t=(p_1+p_3)^2=(p_2+p_4)^2$, $s=(p_1+p_2)^2$). For definiteness we suppose that $\operatorname{Im}s=b>0$. Clearly, there exists an interval $0<\delta<\delta_0(t,\zeta_2,\zeta_4)$ of values of δ for which (15.64) holds together with the inequalities

$$|\sqrt{-\zeta_2-\delta^2}-\sqrt{-\zeta_4-\delta^2}|<\sqrt{-t}<\sqrt{-\zeta_2-\delta^2}+\sqrt{-\zeta_4-\delta^2}.$$

We interpret this as a triangle inequality. It then follows that there exist two vectors $\mathbf{p}_2,\mathbf{p}_4\in\mathbf{R}^3$ such that

$$|\mathbf{p}_2+\mathbf{p}_4|^2=-t, \quad |\mathbf{p}_2|^2=-\zeta_2-\delta^2, \quad |\mathbf{p}_4|^2=-\zeta_4-\delta^2.$$

We now set

$$p_2=(-i\delta,\mathbf{p}_2), \quad p_4=(i\delta,\mathbf{p}_4).$$

It is clear that $p_2\in T_1^+$, $(p_2+p_4)=t$, $p_2^2=\zeta_2$, $p_4^2=\zeta_4$. Note that δ is not fixed here; below we use the fact that it can be arbitrarily small; the vectors $\mathbf{p}_2\equiv\mathbf{p}_2(\delta)$ and $\mathbf{p}_4\equiv\mathbf{p}_4(\delta)$ can be chosen to be functions of δ having a limit as $\delta\to 0$. We then set

$$p_1=(-b/2\delta,\mathbf{p}_1), \quad p_3=(b/2\delta,-\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_4),$$

where $\mathbf{p}_1$ is a vector in $\mathbf{R}^3$ such that

$$|\mathbf{p}_1|^2=(b/2\delta)^2-m_1^2, \quad |\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_4|^2=(b/2\delta)^2-m_3, \quad |\mathbf{p}_1+\mathbf{p}_2|^2=(b/2\delta)^2-\delta^2-\operatorname{Re}s. \quad (15.66)$$

If such a vector $\mathbf{p}_1$ exists, then we have thereby constructed the required quadruple of momenta $p_1, \dots, p_4$. The conditions (15.66) determine the distance from $\mathbf{p}_1$ to three ("almost fixed" in the limit $\delta \to 0$) points in $\mathbf{R}^3$ and since these distances increase without bound while the difference between them tends to zero as $\delta \to 0$, it is clear that such a point $\mathbf{p}_1$ exists for sufficiently small δ/b. ∎

Remark. We can draw the following further conclusion from the proof of Lemma 15.2. For a sufficiently small neighbourhood of an arbitrary real point $(s_{(0)}, t_{(0)}, \zeta_{2(0)}, \zeta_{4(0)})$ in $\mathbf{C} \times \mathbf{R}^3$ satisfying (15.64), there exists a continuous map $(s, t, \zeta_2, \zeta_4) \to (p_1, p_2, p_3, p_4)$, where $(p_1, \dots, p_4)$ is a quadruple of 4-momenta with sum 0, satisfying the conditions (15.56), (15.57), $t = (p_1 + p_3)^2$, $s = (p_1 + p_2)^2$, where $\operatorname{Im} p_2 = 0$ $(\in V^+, \in V^-)$, if the corresponding condition $\operatorname{Im} s = 0$ $(< 0, > 0)$ holds.

C. DISPERSION RELATION FOR NON-PHYSICAL "MASSES"

We have seen that for t, ζ_2, ζ_4 in the domain (15.64) the amplitude $T(s, t, \zeta_2, \zeta_4)$ is analytic with respect to s (at least) in the plane with a cut along the real axis. This is a necessary property for the description of the dispersion relation which expresses T in terms of its jumps at the real axis. But it does not follow from the above discussion that the amplitude is of polynomial growth in $|s|$ and in $|\operatorname{Im} s|^{-1}$, which is also necessary for the definition of the dispersion integral. Apart from this, we still have to establish a connection between the jump of the amplitude and the commutator (15.53). To obtain the requisite information we use the JLD representation (15.52).

In what follows we suppose that the momentum p_2 is contained in the tube T_1^+ or T_1^- (each of which is in the domain of analyticity of the amplitude) and that the variables t, ζ_2, ζ_4 are negative and range over the domain (15.64); here the variable s ranges over the half-plane $\operatorname{Im} s < 0$ or $\operatorname{Im} s > 0$ (depending on whether p_2 lies in T_1^+ or in T_1^-). In relation (15.52) we now go over to the laboratory frame of particle 1 (where $\mathbf{p}_1 = 0$) and direct the 3-axis along $\mathbf{p}_3$. We have the following expression for the momenta in this frame in terms of the invariants s, t, ζ_2, ζ_4:

$$
\begin{gathered}
p_1 = (-m_1, 0, 0, 0),\ p_3 = \Big(\frac{-t + m_1^2 + m_3^2}{2m_1}, 0, 0, ((p_3^0)^2 - m_3^2)^{1/2}\Big), \\
p_2^0 = \frac{-s + m_1^2 + \zeta_2}{2m_1}, p_2^3 = \frac{1}{2p_3^3}(2p_2^0 p_3^0 + s + t - m_1^2 - \zeta_4),\ (\mathbf{p}_2)^2 = (p_2^0)^2 - \zeta_2, \quad (15.67) \\
p_4^0 = -(p_1^0 + p_2^0 + p_3^0),\ p_4^3 = -(p_1^3 + p_2^3 + p_3^3),\ (\mathbf{p}_4)^2 = (p_4^0)^2 - \zeta_4,
\end{gathered}
$$

Since $t < 0$, the coordinates p_j^0, p_j^3 of all the momenta can in this frame be expressed as regular functions of s, t, ζ_2, ζ_4. The dependence on the momentum p_2 in (15.52) is contained in the expression $((p_2 - p_2')^2 - \lambda)^{-1}$ in which the quantities $(p_2)^2$, p_2^0, p_2^3 depend regularly on the invariants, whereas the two-dimensional scalar product $\tilde{p}_2 \tilde{p}_2'$ is not expressed in terms of them (here $\tilde{p}$ denotes the projection of the 4-vector p onto the (1,2)-plane). In order to exclude this scalar product, we use the invariance of the spectral function $\Psi(p_3, p_2', p_1, \lambda)$. We make the substitution $p_2' \to R^{-1}(\phi) p_2'$ in (15.52) and integrate with respect to ϕ from 0 to 2π, where $R(\phi)$ is the rotation through an angle ϕ in the (1,2)-plane. By virtue of the Lorentz-invariance of the spectral function we have

$$
h(p_3, p_2, p_1) = \int Y(s, t, \zeta_2, \zeta_4; p_2', \lambda) \Psi(p_3, p_2', p_1, \lambda) d_4 p_2' d_1 \lambda, \tag{15.68}
$$

where

$$
Y(s, t, \zeta_2, \zeta_4; p_2', \lambda) = \int_0^{2\pi} [(R(\phi) p_2 - p_2')^2 - \lambda]^{-1} d_1 \phi. \tag{15.69}
$$

An explicit form for Y is found by means of the integral *

$$\int_0^{2\pi} (a + b\cos\phi + c\sin\phi)^{-1} d_1\phi = (a^2 - b^2 - c^2)^{-1/2}, \tag{15.70}$$

so that

$$Y(s,t,\zeta_2,\zeta_4;p_2',\lambda) = G(s,t,\zeta_2,\zeta_4;p_2',\lambda)^{-1/2}, \tag{15.71}$$

where

$$G(s,t,\zeta_2,\zeta_4;p_2',\lambda) = \\ = (\zeta_2 + (p_2')^2 - 2p_2^0 p_2'^0 + 2p_2^3 p_2'^3 - \lambda)^2 - 4[(p_2^1)^2 + (p_2^2)^2][(p_2'^1)^2 + (p_2'^2)^2] \tag{15.72}$$

(the details relating to the choice of the sign of the square root in (15.71) are discussed below after (15.77)).

We now turn to the properties of the function Y. It is clear from the representation (15.69) that it is analytic in the tubes $T_1^{\pm}$, consequently it is analytic with respect to s for $\operatorname{Im} s \neq 0$ (as is the amplitude in Lemma 15.2). Since $G(s,t,\zeta_2,\zeta_4;p_2',\lambda)$ is a quadratic in s with real coefficients (depending in C^∞ fashion on the other variables), it follows from the analyticity of $G^{-1/2}$ with respect to s for $\operatorname{Im} s \neq 0$ that both roots of this polynomial are real and that Y has a uniform bound with respect to s of type $|Y| \leq A|\operatorname{Im} s|^{-1}$. It is of class C^∞ with respect to the other variables. Furthermore it is of class $\mathcal{S}(Q[-p_3,-p_1] \times [0,\infty])$ with respect to (p_2',λ) and, as is not difficult to see, each of its seminorms in this class of functions is bounded in modulus by an expression of the form

$$A'(1+|s|)^m |\operatorname{Im} s|^{-l} \tag{15.73}$$

uniformly in s (the same is true if Y is replaced by any of its derivatives with respect to the parameters t, ζ_2, ζ_4 in the domain (15.64)). It follows from (15.68) that if $T(s,t,\zeta_2,\zeta_4)$ is smoothed with an arbitrary test function of class $\mathcal{D}$ with respect to the variables t,ζ_2,ζ_4, then it becomes an analytic function in the domain (15.64) with respect to s for $\operatorname{Im} s \neq 0$ with an estimate of type (15.73). According to §B.5, it has boundary values as $\operatorname{Im} s \to \pm 0$ in the class $\mathcal{S}'(\mathbf{R}_\infty)$ and a dispersion relation holds which expresses it in terms of jumps of class $\mathcal{S}'(\mathbf{R}_\infty)$.

Thus we have obtained the dispersion relation for $T(s,t,\zeta_2,\zeta_4)$:

$$T(s,t,\zeta_2,\zeta_4) = \frac{1}{\pi}\int \frac{A(s',t,\zeta_2,\zeta_4)}{s'-s} ds', \tag{15.74}$$

where the quantity $A(s,t,\zeta_2,\zeta_4)$, called the *absorptive part of the amplitude*, is a generalized function with respect to s of class $\mathcal{S}'(\mathbf{R}_\infty)$ (see §A.3) and a distribution (of class $\mathcal{D}'$) with respect to t,ζ_2,ζ_4 in the domain (15.64). It is proportional to the jump of the amplitude $T(s,t,\zeta_2,\zeta_4)$:

$$A(s,t,\zeta_2,\zeta_4) = \frac{1}{2i}(T_+(s,t,\zeta_2,\zeta_4) - T_-(s,t,\zeta_2,\zeta_4)), \tag{15.75}$$

(where the $T_\pm$, the boundary values of T as $\operatorname{Im} s \to \pm 0$, are defined in accordance with §B.5).

* In general, this integral is either zero or $\pm(a^2-b^2-c^2)^{-1/2}$, but considerations of analyticity with respect to s (and the fact that $Y \not\equiv 0$) exclude the first possibility.

It remains to establish the connection between the absorptive part of the amplitude and the matrix element of the commutator:

$$\delta^+_{m_3}(p_3)\delta^-_{m_1}(p_1)A(s,t,\zeta_2,\zeta_4) = \tfrac{1}{2}g(p_3,p_2,p_1).$$

For this we note that it follows from the above properties of the function Y that it has boundary values $Y_\pm(s,t,\zeta_2,\zeta_4;p_2',\lambda)$ of class $\mathcal{S}'(\mathbf{R})$ (in fact, of class $\mathcal{S}'(\mathbf{R}_\infty)$) as $\operatorname{Im} s \to \pm 0$:

$$Y_\pm(s,t,\zeta_2,\zeta_4;p_2',\lambda) = G(s\pm i0,t,\zeta_2,\zeta_1;p_2',\lambda)^{-1/2}. \tag{15.76}$$

For these we also have the expression

$$\lim_{\substack{y\to 0\\ y\in V^\mp}} \int_0^{2\pi} [(R(\phi)(p_2+iy)-p_2')^2-\lambda]^{-1}d_1\phi. \tag{15.77}$$

In fact it follows from (15.70) that the integral (15.77) is an expression of the form $Q^{-1/2}$, where Q is a quadratic in s, the coefficients of which tend to the coefficients of the polynomial $G(s,t,\zeta_2,\zeta_4;p_2',\lambda)$ as $y\to 0$. It remains for us to make sure that we are dealing with one and the same branch of the root $G^{-1/2}$ in (15.71) and (15.77); for this it is enough to check that (15.71) and (15.77) have the same phases at the points of regularity with respect to s, and this again follows from (15.76) and the fact that as $y\to 0$, $y\in V^\mp$, we have $\operatorname{Im} s\to \pm 0$ (see the remark in §15.2.B; the conclusion made there does not depend on the choice of the Lorentz frame and, in particular, is applicable to the laboratory frame we are using here). Using the expression (15.77) for $Y_\pm$, we can now with the help of (15.68) write down the expression for the jump in $h(p_3,p_2,p_1)$ at $\operatorname{Im} s = 0$:

$$2\pi\int_0^{2\pi}\Bigl(\int i\epsilon(p_2^0-p_2'^0)\delta((R(\phi)p_2-p_2')^2-\lambda)\Psi(p_2,p_2',p_1,\lambda)d_4p_2'd_1\lambda\Bigr)d_1\phi,$$

whence (15.75) follows.

Substituting in (15.75) the expressions for g following from (15.58), (15.59), we obtain the relation

$$\begin{aligned}A(s,,t,\zeta_2,\zeta_4) = \pi\delta(s-m_{12}^2)B_s(\zeta_2,\zeta_4) - \pi\delta(u-m_{14}^2)B_u(\zeta_2,\zeta_4)+\\ +A_s(s,t,\zeta_2,\zeta_4) - A_u(u,t,\zeta_2,\zeta_4),\end{aligned} \tag{15.78}$$

where this equality holds only when $s\neq\infty$ (since in a neighbourhood of $s=\infty$, the right hand side, in contrast to the left hand side, is not defined). The first two terms in (15.78) arising from the presence of bound states in the s- and u-channels give a pole-type contribution to the scattering amplitude at the points $s=m_{12}^2$, $u=m_{14}^2$ respectively.

Here then is our preliminary result which we use in the proof of the dispersion relations.

Lemma 15.3. *The amplitude of a two-particle process is expressed by formulae* (15.51), (15.55). *Moreover the invariant amplitude $T(s,t,\zeta_2,\zeta_4)$ is analytic for $t<0$ with respect to s, ζ_2, ζ_4 in a domain which contains all points satisfying* (15.64), (15.65). *The dispersion relation* (15.74) *holds at these points, the absorptive part $A(s,t,\zeta_2,\zeta_4)$ of the amplitude being a generalized function of class $\mathcal{S}'(\mathbf{R}_\infty)$ with respect to s and a distribution with respect to t, ζ_2, ζ_4 in the domain* (15.64); *for $s\neq\infty$ the relation* (15.78) *holds.*

D. Analytic properties of the absorptive part of the amplitude

We use the reduction formula (13.104) twice (with respect to p_1 and with respect to p_3) for the expression (15.58):

$$\langle\tilde{\phi}_3(p_3)\tilde{j}_4(p_4)\tilde{j}_2(p_2)\tilde{\phi}_1(p_1)\rangle_0 = \delta^+_{m_3}(p_3)\delta^-_{m_1}(p_1)\Big\{\prod_{j=1}^{4}(p_j^2 - m_j^2)\times$$
$$\times\langle A(\tilde{\phi}_3(p_3);\tilde{\phi}_4(p_4))A(\tilde{\phi}_2(p_2);\tilde{\phi}_1(p_1))\rangle_0\Big\}. \tag{15.79}$$

A double application of the JLD representation to the advanced commutators then leads to the formula

$$\langle\tilde{\phi}_3(p_3)\tilde{j}_4(p_4)\tilde{j}_2(p_2)\tilde{\phi}_1(p_1)\rangle_0 =$$
$$= (2\pi)^4\delta(p_1 + \ldots + p_4)\delta^+_{m_3}(p_3)\delta^-_{m_1}(p_1)\frac{1}{\pi}a_s(p_4,\ldots,p_1), \tag{15.80}$$

where

$$a_s(p_4,\ldots,p_1) = \int\frac{\Psi(p_4',p_2',p_1+p_2,\lambda,\mu)d_4p_2'd_1\lambda d_4p_4'd_1\mu}{[(p_4^0-p_4'^0-i0)^2-(\mathbf{p}_4-\mathbf{p}_4')^2-\mu][(p_2^0-p_2'^0-i0)^2-(\mathbf{p}_2-\mathbf{p}_2')^2-\lambda]}. \tag{15.81}$$

The support of $a_s(p_4,\ldots,p_1)$ (and of Ψ) with respect to the variable p_1+p_2 is concentrated on the set $\Gamma^-_{m_{12}}\cup\overline{V}^-_{m_{12}}$. According to §15.1.B, it follows from (15.81) that $a_s(p_4,\ldots,p_1)$ is analytic with respect to p_2,p_4 in the domain defined by the conditions

$$(p_2-p_2')^2-\lambda\neq 0 \quad \text{for all } p_2'\in Q[-p_3-p_4,0],\ \lambda\geq\kappa_{12}^2(p_3+p_4+p_2',p_2'), \tag{15.82a}$$
$$(p_4-p_4')^2-\mu\neq 0 \quad \text{for all } p_4'\in Q[0,-p_1-p_2],\ \mu\geq\kappa_{34}^2(p_4',p_1+p_2+p_4'), \tag{15.82b}$$

where

$$\kappa_{12}(p_3+p_4+p_2',p_2') = 0\vee(M_1-\sqrt{(p_3+p_4+p_2')^2})\vee(M_2-\sqrt{p_2'^2}), \tag{15.83a}$$
$$\kappa_{34}(p_4',p_1+p_2+p_4') = 0\vee(M_3-\sqrt{(p_1+p_2+p_4')^2})\vee(M_4-\sqrt{p_4'^2}). \tag{15.83b}$$

We turn to a characterization of the domain of analyticity in the invariant representation:

$$a_s(p_4,\ldots,p_1) = \pi\delta(s-m_{12}^2)B_s(\zeta_2,\zeta_4) + A_s(s,t,\zeta_2,\zeta_4). \tag{15.84}$$

It suffices for our purposes to regard just one of the variables ζ_2,ζ_4 as independent; therefore we set

$$\zeta_j = m_j^2+\zeta,\quad j=2,4; \tag{15.85}$$

here we are mainly interested in the real values of the parameter $\zeta\leq 0$ (the value 0 now corresponds to the mass shell). In the centre-of-mass frame ($\mathbf{p}_1+\mathbf{p}_2=0$), the momenta admit a parametrization of type (7.185). Now the zeroth components $E_j(s,\zeta)$ and the squares of the three-dimensional momenta $K_{12}^2(s,\zeta)$, $K_{34}^2(s,\zeta)$ still depend on the parameter ζ:

$$E_1(s,\zeta) = \frac{s+m_1^2-\zeta_2}{2\sqrt{s}},\quad E_2(s,\zeta) = \frac{s-m_1^2+\zeta_2}{2\sqrt{s}},$$
$$E_3(s,\zeta) = \frac{s-m_3^2+\zeta_4}{2\sqrt{s}},\quad E_4(s,\zeta) = \frac{s-m_3^2+\zeta_4}{2\sqrt{s}}, \tag{15.86}$$

$$
\begin{aligned}
K_{12}^2(s,\zeta) &= \frac{1}{4s}(s^2 + m_1^4 + \zeta_2^2 - 2sm_1^2 - 2s\zeta_2 - 2m_1^2\zeta_2), \\
K_{34}^2(s,\zeta) &= \frac{1}{4s}(s^2 + m_3^2 + \zeta_4^2 - 2sm_3^2 - 2s\zeta_4 - 2m_3^2\zeta_4).
\end{aligned} \tag{15.87}
$$

(Note that K_{12}^2 and K_{34}^2 can also take negative values.) Since we now admit complex momenta, the three-dimensional vectors $\mathbf{n}_{12}$ and $\mathbf{n}_{34}$ lie on the complex quadric $\mathbf{C}S^2$.

Using the arguments of §15.1.C we find that $a_s(p_4,\dots,p_1)$ is analytic with respect to $\mathbf{n}_{12}$ in the domain

$$|\operatorname{Re}\mathbf{n}_{12}| < x_{12}(s,\zeta) \quad \text{for } K_{12}^2(s,\zeta) > 0, \tag{15.88a}$$

$$|\operatorname{Im}\mathbf{n}_{12}| < x_{12}(s,\zeta) \quad \text{for } K_{12}^2(s,\zeta) < 0, \tag{15.88b}$$

where

$$x_{12}(s,\zeta) = \inf_v \frac{1}{2|K_{12}|\,|\mathbf{v}|}\Big[K_{12}^2 + |\mathbf{v}|^2 - \Big(v^0 + \frac{m_1^2 - \zeta_2}{2\sqrt{s}}\Big)^2 + \kappa^2(v,s)\Big]; \tag{15.89}$$

the lower bound is taken over the set of points $v \in \mathbf{M}$ such that $|v^0| + |\mathbf{v}| \le \sqrt{s}/2$; $\kappa(v,s)$ is defined by (15.42). For x_{12} we have the formula of type (15.4):

$$x_{12}^2(s,\zeta) = \operatorname{sgn} K_{12}^2 + \frac{(M_1^2 - m_1^2)(M_2^2 - \zeta_2)}{|K_{12}|^2(s - (M_1 - M_2)^2)}. \tag{15.90}$$

In precisely the same way, $a_s(p_4,\dots,p_1)$ is analytic with respect to $\mathbf{n}_{34}$ in the domain

$$|\operatorname{Re}\mathbf{n}_{34}| < x_{34}(s,\zeta) \quad \text{for } K_{34}^2(s,\zeta) > 0, \tag{15.91a}$$

$$|\operatorname{Im}\mathbf{n}_{34}| < x_{34}(s,\zeta) \quad \text{for } K_{34}^2(s,\zeta) < 0, \tag{15.91b}$$

where

$$x_{34}^2(s,\zeta) = \operatorname{sgn} K_{34}^2 + \frac{(M_3^2 - m_3^2)(M_4^2 - \zeta_4)}{|K_{34}|^2(s - (M_3 - M_4)^2)}. \tag{15.92}$$

Exercise 15.7. Taking definition (15.43) into account, prove that the set of complex numbers

$$\{z : z = \mathbf{mn},\ \mathbf{m} \in \mathfrak{N}(\alpha),\ \mathbf{n} \in \mathfrak{N}(\alpha')\} \tag{15.93}$$

is an ellipse with foci at the points ± 1 and major semi-axis $\cosh(\alpha + \alpha')$. [Hint: Using (15.46), one can write the set (15.93) in the form

$$\cup_{0 \le \beta < \alpha}\{z : z = \mathbf{m}R_3(i\beta)\mathbf{n},\ \mathbf{m} \in S^2,\ \mathbf{n} \in \mathfrak{N}(\alpha')\}; \tag{15.94}$$

here the vector $R_3(i\beta)\mathbf{n}$ belongs to $\mathfrak{N}(\alpha + \alpha')$, since

$$|\operatorname{Im} R_3(i\beta)\mathbf{n}| \le \cosh\beta \cdot |\operatorname{Im}\mathbf{n}| + \sinh\beta \cdot |\operatorname{Re}\mathbf{n}| < \sinh(\alpha + \alpha').$$

By Exercise 15.6(c), the set (15.93) is contained in the ellipse with foci ± 1 and major semi-axis $\cosh(\alpha + \alpha')$. Conversely, every point of this ellipse is representable in the form $z = \mathbf{m}R_3(i\gamma)e_1$, where $0 < \gamma < \alpha + \alpha'$; it can clearly be written in the form $z = \mathbf{m}R_3(i\beta)R_3(i\beta')e_1$, where $0 \le \beta < \alpha$, $0 \le \beta' < \alpha'$, so that z is a member of (15.94).]

It follows from Exercise 15.7 that for $s \ge m_{12}^2$ $A_s(s,t,\zeta_2,\zeta_4)$ is analytic with respect to t and ζ in a domain containing the points $\zeta \le 0$, $t \in \mathcal{E}_{12,34}(s,\zeta)$, where $\mathcal{E}_{12,34}(s,\zeta)$ is the *Lehmann large ellipse* with foci at the points

$$t_\pm(s,\zeta) = m_1^2 + m_3^2 - 2E_1(s,\zeta)E_3(s,\zeta) \pm 2K_{12}(s,\zeta)K_{34}(s,\zeta) \tag{15.95}$$

and major semi-axis

$$2|K_{12}|\cdot|K_{34}|\cosh(\alpha_{12}+\alpha_{34}), \tag{15.96}$$

where

$$\sinh\alpha_{12}=\begin{cases}\sqrt{x_{12}^2-1} & \text{for } K_{12}^2>0,\\ x_{12} & \text{for } K_{12}^2<0\end{cases}\quad \sinh\alpha_{34}=\begin{cases}\sqrt{x_{34}^2-1} & \text{for } K_{34}^2>0,\\ x_{34} & \text{for } K_{34}^2<0.\end{cases} \tag{15.97}$$

The Lehmann large ellipse is non-empty if both numbers α_{12},α_{34} defined by (15.97) are positive, which is equivalent to the conditions

$$K_{12}^2\theta(-K_{12}^2)+\frac{(M_1^2-m_1^2)(M_2^2-\zeta_2)}{s-(M_1-M_2)^2}>0,\quad K_{34}^2\theta(-K_{34}^2)+\frac{(M_3^2-m_3^2)(M_4^2-\zeta_4)}{s-(M_3-M_4)^2}>0. \tag{15.98}$$

We shall suppose that they are satisfied for all $\zeta\le 0,\ s\in\{m_{12}^2\}\cup[M_{12}^2,\infty)$.

Exercise 15.8. Let

$$m_j\le m_k\quad \text{for } j=2,4 \text{ and } k=1,3. \tag{15.99}$$

Prove that for conditions (15.98) to hold for $\zeta\le 0$, it is necessary and sufficient that they hold for $\zeta=0$:

$$K_{12}^2(s)\theta(-K_{12}^2(s))+\frac{(M_1^2-m_1^2)(M_2^2-m_2^2)}{s-(M_1-M_2)^2}>0,\quad K_{34}^2(s)\theta(-K_{34}^2(s))+\frac{(M_3^2-m_3^2)(M_4^2-m_4^2)}{s-(M_3-M_4)^2}>0 \tag{15.100}$$

for $s=m_{12}^2$ and $s\ge M_{12}^2$. [Hint: It follows from (15.99) that K_{12}^2 and K_{34}^2 decrease as ζ increases.]

The real points of the Lehmann large ellipse $\mathcal{E}_{12,34}(s,\zeta)$ form an interval $(\chi_1(s,\zeta),\ \chi_2(s,\zeta))$ with end-points

$$\chi_{1,2}(s,\zeta)=m_1^2+m_3^2-2E_1(s,\zeta)E_3(s,\zeta)\mp 2\lambda(s,\zeta), \tag{15.101}$$

where

$$\lambda(s,\zeta)=|K_{12}|\,|K_{34}|\cdot\begin{cases}\cosh(\alpha_{12}+\alpha_{34}) & \text{for } K_{12}^2K_{34}^2>0,\\ \sinh(\alpha_{12}+\alpha_{34}) & \text{for } K_{12}^2K_{34}^2<0,\end{cases}$$

or (taking (15.97) into account)

$$\lambda(s,\zeta)=|K_{12}|\,|K_{34}|x_{12}x_{34}+\sqrt{|K_{12}x_{12}|^2-K_{12}^2}\sqrt{|K_{34}x_{34}|^2-K_{34}^2}. \tag{15.102}$$

By substituting (15.90), (15.92) into this, we obtain

$$\begin{aligned}\lambda(s,\zeta)=&\Big[\frac{(M_1^2-m_1^2)(M_2^2-\zeta_2)}{s-(M_1-M_2)^2}\,\frac{(M_3^2-m_3^2)(M_4^2-\zeta_4)}{s-(M_3-M_4)^2}\Big]^{1/2}+\\ &+\Big[K_{12}^2+\frac{(M_1^2-m_1^2)(M_2^2-\zeta_2)}{s-(M_1-M_2)^2}\Big]^{1/2}\Big[K_{34}^2+\frac{(M_3^2-m_3^2)(M_4^2-\zeta_4)}{s-(M_3-M_4)^2}\Big]^{1/2}.\end{aligned} \tag{15.103}$$

Thus, the following holds.

Theorem 15.4. *Suppose that for (fixed) $s\ge M_{12}^2$, the conditions* (15.100) *hold. Then the absorptive part $A_s(s,t)$ of the amplitude of the process* (15.1) *is analytic with respect to t in the Lehmann large ellipse $\mathcal{E}(s)\equiv\mathcal{E}_{12,34}(s)$ with foci*

$$t_\pm(s)=m_1^2+m_3^2-2E_1(s)E_3(s)\pm 2K_{12}(s)K_{34}(s) \tag{15.104}$$

and with real interval $(\chi_1(s), \chi_2(s))$ *(the end-points of which are defined by* (15.102), (15.103) *with* $\zeta_2 = m_2^2,\ \zeta_4 = m_4^2$*).*

In particular, for $s > s_{\text{phys}}$, the Lehmann large ellipse can be written in the form

$$\mathcal{E}_{12,34}(s) = \{t \in \mathbf{C} : |t - t_{\min}(s)| + |t - t_{\max}(s)| < 4K_{12}(s)K_{34}(s)X(s)\}, \qquad (15.105)$$

where

$$X(s) \equiv X_{12,34}(s) = \left[\frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{K_{12}^2(s)(s - (M_1 - M_2)^2)} \frac{(M_3^2 - m_3^2)(M_4^2 - m_4^2)}{K_{34}^2(s)(s - (M_3 - M_4)^2)}\right]^{1/2} +$$
$$+\left[1 + \frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{K_{12}^2(s)(s - (M_1 - M_2)^2)}\right]^{1/2}\left[1 + \frac{(M_3^2 - m_3^2)(M_4^2 - m_4^2)}{K_{34}^2(s)(s - (M_3 - M_4)^2)}\right]^{1/2}. \qquad (15.106)$$

With regard to the values of s in a neighbourhood of m_{12}^2 we know (from (15.84)) that $a_s(p_4, \ldots, p_1)$ depends only on ζ and not on t, so that (under conditions (15.98)) the domain of analyticity of the absorptive part for s contains the points $\zeta \leq 0$, $t \in \mathbf{C}$. With regard to the values $s \geq M_{12}^2$, the domain of analyticity of $A_s(s, t, \zeta_2, \zeta_4)$ contains at least the real points $\zeta \leq 0$, $t \in (\tau_1, \tau_2)$, where

$$\tau_1 = \sup_{\substack{\zeta \leq 0 \\ s \geq M_{12}}} \chi_1(s, \zeta), \quad \tau_2 = \inf_{\substack{\zeta \leq 0 \\ s \geq M_{12}^2}} \chi_2(s, \zeta); \qquad (15.107)$$

clearly this interval (τ_1, τ_2) is non-empty only when

$$\tau_1 < \tau_2. \qquad (15.108)$$

Exercise 15.9. (a) Prove that

$$\tau_2 \leq 0. \qquad (15.109)$$

[Hint: $\lim_{s \to \infty} \chi_2(s, \zeta) = 0$.]

(b) Prove that in the case of a quasi-elastic process, we have the equality

$$\tau_2 = 0. \qquad (15.110)$$

[Hint: In the case under discussion, $\chi_2(s, \zeta) = 2(M_1^2 - m_1^2)(M_2^2 - m_2^2 - \zeta)[s - (M_1 - M_2)^2]^{-1}$.]

(c) Suppose that conditions (15.99) hold. Prove that

$$\tau_1 = \sup_{s \geq M_{12}^2} \chi_1(s, m_2^2). \qquad (15.111)$$

[Hint: It follows from (15.99) that $E_1(s, \zeta)$ and $E_3(s, \zeta)$ decrease as ζ increases; so also do $K_{12}^2(s, \zeta)$, $K_{34}^2(s, \zeta)$ and (by (15.103)) $\lambda(s, \zeta)$; consequently the least upper bound with respect to ζ in (15.107) is attained at $\zeta = 0$.]

Lemma 15.5. *Under conditions* (15.98), (15.108), *the absorptive part* (15.84) *of the amplitude of the two-particle process (extended beyond the mass shell), is analytic with respect to* (t, ζ) *in a domain containing the points* $t \in (\tau_1, \tau_2)$, $\zeta \leq 0$. *Furthermore, for any* $\Lambda < 0$, $t \in (\tau_1, \tau_2)$, *the domain of analyticity of* $A_s(s, t, \zeta_2, \zeta_4)$ *with respect to* ζ *contains the set*

$$\Lambda < \operatorname{Re}\zeta < \epsilon, \quad |\operatorname{Im}\zeta| < \epsilon, \qquad (15.112)$$

where $\epsilon \equiv \epsilon(\Lambda, t) > 0$ *depends (continuously) on* Λ, t *and is independent of* s.

■ The first statement of the lemma was proved above. It follows from this that a set of the form (15.112) enters into the domain of analyticity of $A_s(s,t,\zeta_2,\zeta_4)$ with respect to ζ for any $\Lambda < 0$, $t \in (\tau_1, \tau_2)$, $s \geq M_{12}^2$, but with $\epsilon \equiv \epsilon(\Lambda, t, s) > 0$ (continuously) dependent on Λ, t, s. Thus it remains to prove that ϵ can be chosen independent of s. Clearly, on any finite interval Δ of variation of s, the dependence of ϵ on s can be eliminated (by replacing ϵ by $\inf_{s\in\Delta}\epsilon$), therefore in what follows we shall choose a certain fixed s_0 and suppose that $s \geq s_0$.

We have so far confined attention to real values of ζ for the characterization of the domain of analyticity with respect to t, ζ. We fix $t \in (\tau_1, \tau_2)$, $\Lambda < 0$ and suppose now that ζ is complex in the rectangle (15.112) with some parameter ϵ in the fixed interval $0 < \epsilon < \epsilon_0$. With regard to the domain of analyticity of $a_s(p_4, \dots, p_1)$ with respect to $\mathbf{n}_{12}$, in place of (15.88), (15.89) we can now say that (for sufficiently large s_0 and $s \geq s_0$) it contains at least the points of CS^2 such that

$$|\operatorname{Re}\mathbf{n}_{12}| < g_{12}(s,\zeta), \tag{15.113}$$

where

$$g_{12}(s,\zeta) = \inf_v \frac{1}{2|\mathbf{v}|}\operatorname{Re}\frac{1}{K_{12}(s,\zeta)}\left[K_{12}^2(s,\zeta) + |\mathbf{v}|^2 - \left(v^0 + \frac{m_1^2 - \zeta_2}{2\sqrt{s}}\right)^2 + \kappa^2(v,s)\right] \tag{15.114}$$

(the notation is the same as (15.89). Taking into account the explicit form of $K_{12}^2(s,\zeta)$, it is not difficult to estimate the right hand side of (15.114) from below for $s > s_0$:

$$g_{12}(s,\zeta) \geq \inf_v \frac{1}{2|\mathbf{v}|K_{12}(s,\operatorname{Re}\zeta)}\left[K_{12}^2(s,\operatorname{Re}\zeta) + |\mathbf{v}|^2 - \left(v^0 + \frac{m_1^2 - \zeta_2}{2\sqrt{s}}\right)^2 + \kappa^2(v,s) - C_1\epsilon s^{-1}\right], \tag{15.115}$$

(where the constant C_1 depends on t, Λ, ϵ_0, s_0). It is easy to see that the greatest lower bound in (15.115) with respect to those v for which $|\mathbf{v}| > a$ (where $a > 0$ is a fixed constant) has the minorant

$$x_{12}(s,\operatorname{Re}\zeta) - C_2\epsilon s^{-2}. \tag{15.116}$$

In order to obtain a similar estimate for $g(s,\zeta)$, it remains to consider the set of those v in (15.115) for which $|\mathbf{v}| < a$. This splits into three parts (for sufficiently large s_0):

$$-\sqrt{s}/2 + |\mathbf{v}| \leq v^0 \leq -\sqrt{s}/2 + \sqrt{M_2^2 + |\mathbf{v}|^2}, \quad \text{where } \kappa = M_2 - \sqrt{(\sqrt{s}/2 + v^0)^2 - |\mathbf{v}|^2}, \tag{15.117a}$$

$$-\sqrt{s}/2 + \sqrt{M_2^2 + |\mathbf{v}|^2} < v^0 < \sqrt{s}/2 - \sqrt{M_1^2 + |\mathbf{v}|^2}, \quad \text{where } \kappa(v,s) = 0, \tag{15.117b}$$

$$\sqrt{s}/2 - \sqrt{M_1^2 + |\mathbf{v}|^2} \leq v^0 \leq \sqrt{s}/2 - |\mathbf{v}|, \quad \text{where } \kappa(v,s) = M_1 - \sqrt{(\sqrt{s}/2 - v^0)^2 - |\mathbf{v}|^2}. \tag{15.117c}$$

In the interval (15.116a), the variable $w \equiv \sqrt{s}/2 + v^0$ is uniformly bounded in s and the expression under the inf sign in (15.115) reduces on this interval to the form

$$\frac{1}{2|\mathbf{v}|K_{12}(s,\operatorname{Re}\zeta)}\left[-\left(1 - \frac{w}{\sqrt{s}}\right)\operatorname{Re}\zeta_2 + |\mathbf{v}| + w^2 + w\frac{s - m_1^2}{\sqrt{s}} + \kappa^2(v,s) - C_1\epsilon s^{-1}\right];$$

here the term $-C_1\epsilon s^{-1}$ can be combined with $-(1 - \frac{w}{\sqrt{s}})\operatorname{Re}\zeta_2$ and we then find that the greatest lower bound in (15.115) over the set (15.117a) has the minorant

$$K_{12}(s,\operatorname{Re}\zeta)^{-1}K_{12}(s,\operatorname{Re}\zeta + C_3\epsilon s^{-1})x_{12}(s,\operatorname{Re}\zeta + C_3\epsilon s^{-1}), \tag{15.118}$$

where C_3 is a constant. In the second interval (15.117b), the expression under the inf sign in (15.115) is a quadratic polynomial in v^0 attaining its minimum at one of the end points of the interval (that is, either on the set (15.117a) or on (15.117c)), therefore we do not have to consider this separately. Finally, the greatest lower bound over the set (15.117c) has a lower estimate of type (15.118), only now, instead of the rider concerning $\operatorname{Re}\zeta$, we make a rider concerning m_1^2:

$$K_{12}(s,\operatorname{Re}\zeta)^{-1}\left\{K_{12}(s,\operatorname{Re}\zeta)x_{12}(s,\operatorname{Re}\zeta)\big|_{m_1^2 \to m_1^2 + C_4\epsilon s^{-1}}\right\}. \tag{15.119}$$

Taken together, the estimates (15.116), (15.118), (15.119) give the estimate for $g(s,\zeta)$ in (15.114):

$$g_{12}(s,\zeta) \geq x_{12}(s,\operatorname{Re}\zeta) - C\epsilon s^{-2}. \tag{15.120}$$

Similarly, in the same region of variation of the variables $s,\ t,\ \zeta$, the function $a_s(p_4,\dots,p_1)$ is analytic with respect to $\mathbf{n}_{34}$ on the set

$$|\operatorname{Re}\mathbf{n}_{34}| < g_{34}(s,\zeta), \tag{15.121}$$

where

$$g_{34}(s,\zeta) \geq x_{34}(s,\operatorname{Re}\zeta) - C'\epsilon s^{-2}. \tag{15.122}$$

It follows from (15.120), (15.122), that the domain of analyticity of $A_s(s,t,\zeta_2,\zeta_4)$ with respect to t for $s \geq s_0$ and ζ in the rectangle (15.112) contains a set differing from the Lehmann large ellipse $\mathcal{E}_{12,34}(s,\operatorname{Re}\zeta)$ by an arbitrarily small amount (for sufficiently small ϵ) uniformly in $s > s_0$; in particular, when ϵ is sufficiently small, it contains the point t (together with some interval in (τ_1,τ_2)), which is a point common to all the ellipses $\mathcal{E}_{12,34}(s,\operatorname{Re}\zeta)$ for $s \geq M_{12}^2$, $\operatorname{Re}\zeta \leq \epsilon$. This completes the proof of the lemma. ■

Remark. According to the construction of $g_{12}(s,\zeta)$, $g_{34}(s,\zeta)$, the inequalities (15.113), (15.121) guarantee the fulfillment of the conditions (15.82). One could impose the stronger conditions on $\mathbf{n}_{12}$, $\mathbf{n}_{34}$:

$$|(p_2 - p_2')^2 - \lambda| > \delta\epsilon s^{-1}, \quad |(p_4 - p_4')^2 - \mu| > \delta\epsilon s^{-1} \tag{15.123}$$

for all p_4', p_2', λ, μ, s in the support of the spectral function in the representation (15.81) (see §15.1.B concerning the support of the spectral function Ψ). In order that these hold, it suffices as before that $\mathbf{n}_{12}$ and $\mathbf{n}_{34}$ satisfy the inequalities (15.113), (15.121), but with modified g_{12}, g_{34}. For example, $g_{12}(s,\zeta)$ now has a lower estimate of type (15.115) but with a somewhat larger constant C_1. It is clear that the effect of this alteration of g_{12}, g_{34} is not important, since (by suitably redefining the constants C and C') they will satisfy estimates of type (15.120), (15.122) as before.

Similar results hold for the absorptive part

$$a_u(p_4,\dots,p_1) = \pi\delta(u - m_{14}^2)B_u(\zeta_2,\zeta_4) + A_u(u,t,\zeta_2,\zeta_4), \tag{15.124}$$

related to the u-channel and defined by formula (15.59); they are obtained by altering the notation.

The variable u (related to $s,\ t,\ \zeta$ by (15.60)) now takes the values $u = m_{14}^2$ and $u \geq M_{14}^2$. In order that the Lehmann large ellipse $\mathcal{E}_{14,23}(u,\zeta)$ related to the u-channel be non-empty, we shall suppose that the conditions analogous to (15.98) hold:

$$K_{14}^2\theta(-K_{14}^2) + \frac{(M_1^2 - m_1^2)(M_4^2 - \zeta_4)}{u - (M_1 - M_4)^2} > 0, \quad K_{23}^2\theta(-K_{34}^2) + \frac{(M_2^2 - \zeta_2)(M_3^2 - m_3^2)}{u - (M_2 - M_3)^2} > 0 \tag{15.125}$$

for $\zeta \leq 0,\ u \in \{m_{14}^2\} \cup [M_{14}^2, \infty)$; here $K_{14}^2(u,\zeta)$ and $K_{23}^2(u,\zeta)$ are defined by

$$\begin{aligned} K_{14}^2(u,\zeta) &= \frac{1}{4u}(u^2 + m_1^4 + \zeta_4^2 - 2um_1^2 - 2u\zeta_4 - 2m_1^2\zeta_4), \\ K_{23}^2(u,\zeta) &= \frac{1}{4u}(u^2 + \zeta_2^2 + m_3^2 - 2u\zeta_2 - 2um_3^2 - 2\zeta_2 m_3^2). \end{aligned} \tag{15.126}$$

For $\zeta \leq 0,\ u \geq M_{14}^2$ the absorptive part of the u-channel $A_u(u,t,\zeta_2,\zeta_4)$ is analytic with respect to t in the Lehmann large ellipse $\mathcal{E}_{14,23}(u,\zeta)$ with foci at the points

$$t'_{\pm}(u,\zeta) = m_1^2 + m_3^2 - 2E_1'(u,\zeta)E_3'(u,\zeta) \pm 2K_{14}(u,\zeta)K_{23}(u,\zeta) \tag{15.127}$$

and with real interval $(\chi_1'(u,\zeta),\ \chi_2'(u,\zeta))$ with end-points

$$\chi_{12}'(u,\zeta) = m_1^2 + m_3^2 - 2E_1'(u,\zeta)E_3'(u,\zeta) \mp 2\lambda'(u,\zeta). \tag{15.128}$$

Here

$$E_1'(u,\zeta) = \frac{u + m_1^2 - \zeta_4}{2\sqrt{u}}, \quad E_3'(u,\zeta) = \frac{u + m_3^2 - \zeta_2}{2\sqrt{u}}, \tag{15.129}$$

$$\lambda'(u,\zeta) = \Big[\frac{(M_1^2 - m_1^2)(M_4^2 - \zeta_4)}{u - (M_1 - M_4)^2}\, \frac{(M_2^2 - \zeta_2)(M_3^2 - m_3^2)}{u - (M_2 - M_3)^2}\Big]^{1/2} +$$
$$+\Big[K_{14}^2 + \frac{(M_1^2 - m_1^2)(M_4^2 - \zeta_4)}{u - (M_1 - M_4)^2}\Big]^{1/2} \Big[K_{23}^2 + \frac{(M_2^2 - \zeta_2)(M_3^2 - m_3^2)}{u - (M_2 - M_3)^2}\Big]^{1/2}. \tag{15.130}$$

In particular, for $\zeta = 0,\ u > u_{\text{phys}} \equiv (m_1 + m_4)^2 \vee (m_2 + m_3)^2$ the Lehmann large ellipse $\mathcal{E}_{14,23}(u)$ has the form

$$\mathcal{E}_{14,23}(u) = \{t \in \mathbf{C} : |t - t'_{\min}(u)| + |t - t'_{\max}(u)| < 4K_{14}(u)K_{23}(u)X_{14,23}(u)\}, \tag{15.131}$$

where

$$t'_{\substack{\max\\ \min}}(u) = m_1^2 + m_3^2 - 2E_1'(u)E_3'(u) \pm 2K_{14}'(u)K_{23}'(u), \tag{15.132}$$

$$X_{14,23}(u) = \Big[\frac{(M_1^2 - m_1^2)(M_4^2 - m_4^2)}{K_{14}^2(u)(u - (M_1 - M_4)^2)}\, \frac{(M_2^2 - m_2^2)(M_3^2 - m_3^2)}{K_{23}^2(u)(u - (M_2 - M_3)^2)}\Big]^{1/2} +$$
$$+\Big[1 + \frac{(M_1^2 - m_1^2)(M_4^2 - m_4^2)}{K_{14}^2(u)(u - (M_1 - M_4)^2)}\Big]^{1/2} \Big[1 + \frac{(M_2^2 - m_2^2)(M_3^2 - m_3^2)}{K_{14}^2(u)(u - (M_2 - M_3)^2)}\Big]^{1/2}. \tag{15.133}$$

The interval (τ_1', τ_2') of the points common to the ellipses $\mathcal{E}_{14,23}(u)$ for $\zeta \leq m_2^2$, $u \geq M_{14}^2$ are defined by the formulae

$$\tau_1' = \sup_{\substack{\zeta \leq 0\\ u \geq M_{14}^2}} \chi_1'(u,\zeta), \quad \tau_2' = \inf_{\substack{\zeta \leq 0\\ u \geq M_{14}^2}} \chi_2'(u,\zeta). \tag{15.134}$$

We set

$$\tau_{\min} = \tau_1 \vee \tau_1', \quad \tau_{\max} = \tau_2 \wedge \tau_2'. \tag{15.135}$$

Then the above considerations show that if the conditions (15.98), (15.125) hold and in addition

$$\tau_{\min} < \tau_{\max}, \tag{15.136}$$

then all the quantities $B_s(\zeta_2, \zeta_4),\ B_u(\zeta_2, \zeta_4),\ A_s(s, t, \zeta_2, \zeta_4),\ A_u(u, t, \zeta_2, \zeta_4)$ occurring on the right hand side of the dispersion relation (15.74) are analytic with respect to t, ζ in a domain containing the points $t \in (\tau_{\min},\ \tau_{\max}),\ \zeta \leq 0$; furthermore, for any $\Lambda < 0,\ t \in (\tau_{\min},\ \tau_{\max})$ the domain of analyticity with respect to ζ contains a set of the form (15.112).

Exercise 15.10. Suppose that for a quasi-elastic process the conditions $m_2 \leq m_1$ and

$$\frac{(m_1^2 - m_2^2)^2}{s^2} + \frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{(s - (M_1 - M_2)^2)^2} \leq 2 \quad \text{for } s = M_{12}^2 \wedge M_{14}^2 \tag{15.137}$$

hold. Prove that

$$\tau_{\min} = -4\Big[K_{12}^2(s) + \frac{(M_1^2 - m_1^2)(M_2^2 - m_2^2)}{(s - (M_1 - M_2)^2)}\Big]\Big|_{s = M_{12}^2 \wedge M_{14}^2}, \quad \tau_{\max} = 0. \tag{15.138}$$

[Hint: It follows from (15.137) that the least upper bound in (15.111) is attained at $s = M_{12}^2$; a similar result holds for τ_1'.]

E. DISPERSION RELATION ON THE MASS SHELL

We turn to the analytic continuation with respect to the masses in the dispersion relation (15.74); here (as in the preceding subsection) we use (15.85) to introduce the single mass parameter ζ in place of ζ_2, ζ_4.

First of all, we must attach a meaning to the right hand side of (15.74) when ζ varies throughout the rectangle (15.112).

Lemma 15.6. *Under the conditions of Lemma 15.5, the absorptive part $A_s(s,t,\zeta,\zeta_4)$ of the amplitude of a two-particle process (extended beyond the mass shell) is a generalized function of class $S'([M_{12}^2,\infty))$ with respect to s, which is C^∞-dependent on the parameter $t \in (\tau_1,\tau_2)$ and analytically dependent on ζ in the rectangle* (15.112), *where $\epsilon > 0$ is a positive parameter (continuously dependent on Λ, t). The same is true of a quasi-analytic process if t ranges over the half-closed interval $(\tau_1, 0]$.*

■ A similar statement was proved in Lemma 15.5, but only in the class $\mathcal{D}'(\mathbf{R})$ with respect to the variable s, so that it remains to prove that $A_s(s,t,\zeta_2,\zeta_4)$ is of temperate growth with respect to s. Since it is only large s that matter here, we fix s_0 sufficiently large (as in the proof of Lemma 15.5) and assume that $s \geq s_0$.

The representation (15.81) can be written in the centre-of-mass frame (and in a parametrization of type (7.185)) in the form

$$A_s(s,t,\zeta_2,\zeta_4) = \int W(\mathbf{n}_{12},\mathbf{n}_{34},\zeta;s,p_4',p_2',\lambda,\mu)\psi(p_4',p_2',s,\lambda,\mu)d_4p_2'd_1\lambda d_4p_4'd_1\mu, \tag{15.139}$$

where

$$W(\mathbf{n}_{12},\mathbf{n}_{34},\zeta;p_4',p_2',s,\lambda,\mu) = \int_{O_+(3)} \frac{\Omega(p_4',p_2',s,\lambda,\mu)}{[(Rp_2-p_2')^2-\lambda][(Rp_4-p_4')^2-\mu]}dR. \tag{15.140}$$

Here we have availed ourselves of the possibility of averaging over the group $O_+(3)$, which is permissible in view of the $O_+(3)$-invariance of the spectral function $\psi(p_4',p_2',s,\lambda,\mu)$ with respect to p_2', p_4'. (As before, we suppose that ζ_2, ζ_4 are expressed by (15.85) in terms of the single mass parameter ζ.) The function $\Omega(p_4',p_2',s,\lambda,\mu)$ takes into account the property of the support of the spectral function: this is a multiplicator in the space

$$\mathcal{S}(\mathbf{M}\times\mathbf{M}\times[s_0,\infty)\times[0,\infty]\times[0,\infty]) \tag{15.141}$$

with the properties $\Omega\cdot\psi = \psi$ and $\Omega = 0$ in a $\Delta(s)$-neighbourhood of the support of ψ with respect to the variables p_4', p_2', λ, μ (see the remark in §15.1.B).

As was pointed out in the remark of the previous subsection, for $s \geq s_0$ and $\mathbf{n}_{12}$, $\mathbf{n}_{34}$ satisfying (15.113), (15.121), the denominator in (15.140) has the estimate (15.123). On the other hand, (by virtue of $O_+(3)$-invariance), the function (15.140) depends on $\mathbf{n}_{12}$, $\mathbf{n}_{34}$ only via the scalar product * $\mathbf{n}_{12}\mathbf{n}_{34}$ and hence,

$$W(\mathbf{n}_{12},\mathbf{n}_{34},\zeta;p_4',p_2',s,\lambda,\mu) = w(t,\zeta;p_4',p_2',s,\lambda,\mu).$$

As is clear from the proof of Lemma 15.5, when $\epsilon > 0$ is sufficiently small, t can take values in a (sufficiently small) neighbourhood of any point in (τ_1,τ_2). **

Let $f(s)$ be an arbitrary test function in $\mathcal{S}([s_0,\infty))$. It then follows trivially from the above estimate for the denominator in (15.140) that the product $f(s)w(t,\zeta;p_4',p_2',s,\lambda,\mu)$ is a test function in the space (15.141) with respect to the variables p_4', p_2', s, λ, μ which is dependent on t, ζ as parameters (the dependence on ζ being analytic and that on t being of class C^∞). The statement of the lemma now follows from the equality

$$\int A_s(s,t,\zeta_2,\zeta_4)f(s)d_1s = \int f(s)w(t,\zeta;p_4',p_2',s,\lambda,\mu)\Psi(p_4',p_2',s,\lambda,\mu)d_4p_4'd_4p_2'd_1sd_1\lambda d_1\mu. \quad ■$$

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* The reasoning here is the same as in §15.1.C.

** In the case of a quasi-elastic process, t can take values in $(\tau_1, 0]$, since 0 is then a point common to all the large Lehmann ellipses $\mathcal{E}_{12,34}(s)$ for $s > s_{\text{phys}}$.

For the sake of simplicity, we supposed above that ζ_2 and ζ_4 were related by (15.85). It is clear that we could have considered them to be independent in the domain

$$\Lambda < \operatorname{Re}\zeta_2 < m_2^2, \quad |\operatorname{Im}\zeta_2| < \epsilon, \quad |\zeta_4 - \zeta_2 - m_4^2 + m_2^2| < \epsilon, \tag{15.142}$$

where $\epsilon \equiv \epsilon(\Lambda, t)$ depends (continuously) on $\Lambda < 0$ and $t \in (\tau_{\min}, \tau_{\max})$ (and in the case of a quasi-elastic process, for $t \in (\tau_{\min}, 0]$); the derivation of Lemma 15.6 remains intact under these circumstances. The analogue of Lemma 15.6 holds for $A_u(u, t, \zeta_2, \zeta_4)$ as well. Therefore the right hand side of (15.78) is a generalized function in $\mathcal{S}'(\mathbf{R})$ with respect to s which is C^∞-dependent on $t \in (\tau_{\min},\ \tau_{\max})$ (or on $t \in (\tau_{\min}, 0]$ for a quasi-elastic process) and is analytic with respect to ζ_2, ζ_4 in the domain (15.142). We denote by $A^{(1)}(s, t, \zeta_2, \zeta_4)$ its extension to a generalized function in $\mathcal{S}'(\mathbf{R}_\infty)$ with respect to s; this extension can be brought about by a constructive device (see Exercise A.5) as a result of which $A^{(1)}(s, t, \zeta_2, \zeta_4)$ inherits the C^∞-dependence on t and the analytic dependence on ζ_2, ζ_4. On the other hand, in §15.2.C we constructed the absorptive part of the amplitude $A(s, t, \zeta_2, \zeta_4)$ which is a generalized function of class $\mathcal{S}'(\mathbf{R}_\infty)$ with respect to s and a generalized function with respect to t, ζ_2, ζ_4 in the domain (15.64).

It follows from (15.78) and the construction of $A^{(1)}$ that if the conditions $t \in$ $\in (\tau_{\min}, \tau_{\max})$, (15.64), (15.142) hold simultaneously, then the difference $A - A^{(1)}$ is concentrated at $s = \infty$; consequently (on the basis of Exercise A.7) if we replace A by $A^{(1)}$ in the dispersion relation (15.74), then the right hand side is altered by a polynomial in s. Let the degree of this polynomial be $n - 1$. Then

$$\frac{\partial^n}{\partial s^n} T(s, t, \zeta_2, \zeta_4) = \frac{1}{\pi} \frac{\partial^n}{\partial s^n} \int \frac{A^{(1)}(s, t, \zeta_2, \zeta_4)}{s' - s} ds'. \tag{15.143a}$$

This formula brings about an analytic continuation of $\frac{\partial^n}{\partial s^n} T(s, t, \zeta_2, \zeta_4)$ with respect to s with $\operatorname{Im} s > 0$ (or with $\operatorname{Im} s < 0$) and with respect to ζ_2, ζ_4 in the domain (15.142). It follows from the principle of uniqueness of the analytic continuation (and simple connectedness of our domain for s, ζ_2, ζ_4) that the functions $\frac{\partial^n}{\partial s^n} T(s, t, \zeta_2, \zeta_4)$ defined by the equalities (15.55) (for $\operatorname{Im} p_2 \in V^-$) and (15.143a) (for $\operatorname{Im} s > 0$), coincide on the intersection of their domains of definition. In particular, as $\zeta_2 \to m_2^2$, $\zeta_4 \to m_4^2$ we obtain the following relation for the scattering amplitude on the mass shell:

$$\frac{\partial^n}{\partial s^n} \left[T(s, t) - \frac{1}{\pi} \int \frac{A^{(1)}(s, t, m_2^2, m_4^2)}{s' - s} ds' \right] = 0 \quad \text{for } \operatorname{Im} s \neq 0. \tag{15.143b}$$

From this it follows that the expression in the square brackets is a polynomial in s with coefficients that are C^∞-dependent on t.* On the basis of Exercise A.7, this polynomial contribution can be included in the dispersion integral and we obtain the dispersion relation

$$T(s, t) = \frac{1}{\pi} \int \frac{A(s', t)}{s' - s} ds' \quad \text{for } \operatorname{Im} s \neq 0. \tag{15.144}$$

* This follows from the behaviour of $A^{(1)}(s, t, m_2^2, m_4^2)$ with respect to t and from the property of $T(s, t)$ as a function of t for $s \equiv s_1 + i0$, $s_1 > s_{\text{phys}}$ (§15.1.C).

Here $A(s,t)$ differs from $A^{(1)}(s,t,m_2^2,m_4^2)$ by a generalized function concentrated at the point $s=\infty$.

According to §B.5, $A(s,t)$ can be chosen in (15.144) to have zero integral (with respect to s; then (on the basis of Exercise B.13) $2iA(s,t)$ is the same as the jump of class $\mathcal{S}'(\mathbf{R}_\infty)$ (with respect to s) of the amplitude $T(s,t)$.

We can isolate the s- and u-channel contributions in the absorptive part (as well as the contributions of the one-particle intermediate states; cf. (15.78)):

$$A(s,t)=\pi B_s\delta(s-m_{12}^2)-\pi B_u\delta(u-m_{14}^2)+A_s(s,t)-A_u(u,t), \tag{15.145}$$

where $A_s(s,t)$ and $A_u(u,t)$ are concentrated in the regions $s\geq M_{12}^2$ and $u\geq M_{14}^2$ respectively; we suppose here that they are suitably extended to the point $s=\infty$ (or $u=\infty$), so that they belong to the classes $\mathcal{S}'([M_{12}^2,\infty])$ (with respect to s) and $\mathcal{S}'([M_{14}^2,\infty])$ (with respect to u). Corresponding to the decomposition (15.145) is the decomposition of the dispersion integral

$$T(s,t)=\frac{B_s}{m_{12}^2-s}+\frac{B_u}{m_{14}^2-u}+\frac{1}{\pi}\int\frac{A_s(s',t)ds'}{s'-s}+\frac{1}{\pi}\int\frac{A_u(u',t)du'}{u'-u}\quad\text{for } \operatorname{Im} s\neq 0. \tag{15.146}$$

It is clear that corresponding to the one-particle intermediate states are pole-type contributions to the amplitude.

As a result we have a proof of the following.

Theorem 15.7. *Under conditions* (15.98), (15.125), (15.136) *and for* $t\in(\tau_{\min},\tau_{\max})$ (*and in the case of a quasi-linear process** for $t\in(\tau_{\min},0]$) *the amplitude* $T(s,t)$ *of the two-particle process* (15.1) *is a generalized boundary value* (*as* $\operatorname{Im} s\to+0$) *of the analytic function* $T(s,t)$ *with respect to* s *for* $\operatorname{Im} s\neq 0$, *satisfying the dispersion relation* (15.146). *The* s- *and* u-*channel absorptive parts* $A_s(s,t)$, $A_u(u,t)$ *occurring here are generalized functions of class* $\mathcal{S}'([M_{12}^2,\infty])$ *with respect to* s *and* $\mathcal{S}'([M_{14}^2,\infty])$ *with respect to* u *respectively, which are* C^∞-*dependent on* t *as a parameter.*

In the case of scattering of particles with spin, there is an analogous statement for the invariant amplitudes $T_p(s,t)$ (of the expansion of the amplitude in a polynomial basis of standard covariants). In this case $B_{p,s}(t)$ and $B_{p,u}(t)$ are polynomially dependent on t.

We consider as an illustration the three processes involving π-mesons and nucleons (that is, protons and neutrons):**

$$\pi+\pi\to\pi+\pi, \tag{15.147a}$$

$$\pi+N\to\pi+N, \tag{15.147b}$$

$$N+N\to N+N. \tag{15.147c}$$

* Although in writing down the representation (15.55) earlier we supposed p_1 and p_3 to be non-collinear, which in the case of a quasi-elastic process corresponds to the restriction $t<0$, in the equality (15.145) we can pass to the limit $t\to-0$ in the physical domain (that is, for $s\equiv s_1+i0$, $s_1>s_{\text{phys}}$), since (according to what was proved in §15.1.C and above) both sides of (15.146) are C^∞-functions of $t\leq 0$ at the point $t=0$. With regard to points $t=0$ with non-physical values of s, the equality (15.146) can naturally be adopted as an extension of the definition of $T(s,t)$ there.

** There is an extensive literature on dispersion relations for the first two processes (see, for example [S10], where, in particular, isotopic- and Lorentz-covariant decompositions for the amplitudes can be found).

Here under the hypothesis of exact isotopic invariance, the masses m_N of the nucleons are considered to be equal (and the masses m_π of the π-mesons are equal).

In the first process $m_j = m_\pi$, $M_j = 3m_\pi$ for all $j = 1, \ldots, 4$; $M_{12} = M_{14} = 2m_\pi$. (There are no one-particle contributions and hence, no parameters m_{12}, m_{14}.) The dispersion relation holds for $t \in (\tau_{\min}, 0]$, where $\tau_{\min} = -28m_\pi^2$.

In the second process $m_1 = m_3 = m_N$, $m_2 = m_4 = m_\pi$, $M_1 = M_3 = m_\pi + m_N$, $M_2 = M_4 = 3m_\pi$, $M_{12} = M_{14} = m_\pi + m_N$, $m_{12} = m_{14} = m_N$. In this case the dispersion relation holds for $t \in (\tau_{\min}, 0]$, where (according to (15.138)) $\tau_{\min} = -\frac{32}{3}\frac{2m_N+m_\pi}{2m_N-m_\pi}m_\pi^2$.

In the third process $m_j = m_N$, $M_j = m_\pi + m_N$ for $j = 1, \ldots, 4$; $M_{12} = 2m_N$, $M_{14} = 2m_\pi$. In this case $\tau_{\min} = -4m_\pi(2m_N + m_\pi)^2 + 4m_N^2$ would be negative only for $m_\pi/m_N > \sqrt{2} - 1$. In fact this inequality does not hold. This is why the derivation of the dispersion relation for NN-scattering has not been carried out within the framework of the axiomatic approach.*

We refer the reader to the survey by Sommer (1970), where a table is given of two-particle processes of strong interactions for which the dispersion relations have been proved.

Exercise 15.11. Prove the equality

$$\operatorname{Im} T(s,t) = A(s,t) \quad \text{for } (s,t) \in G_{\text{phys}} \tag{15.148}$$

of the imaginary and absorptive parts of the (invariant) amplitude in the physical domain of the elastic two-particle process $\kappa_1 + \kappa_2 \to \kappa_1 + \kappa_2$ with spinless particles. [Hint: Use the relation

$$\begin{aligned}(2\pi)^4\delta(p_4 + \ldots + p_1)\cdot 2i\operatorname{Im} T(p_4,p_3,p_2,p_1) &= \\ &= \langle 0|\tilde{\phi}^{\text{in}(\bar\kappa_2)}(p_4)\tilde{\phi}^{\text{in}(\bar\kappa_1)}(p_3)(T - T^*)\tilde{\phi}^{\text{in}(\kappa_2)}(p_2)\tilde{\phi}^{\text{in}(\kappa_1)}(p_1)|0\rangle,\end{aligned} \tag{15.149}$$

where $iT \equiv S - 1$, which was established in Exercise 7.38. One must prove that the right hand side of (15.149) is the same as

$$\begin{aligned}(2\pi)^4\delta(p_1 + \ldots + p_4)\delta^+_{m_1}(p_3)\delta^-_{m_1}(p_1)\cdot 2iA(s,t,p_2^2,p_4^2) &= \\ = i(2\pi)^4\delta(p_1 + \ldots + p_4)g(p_3,p_2,p_1) &= i\langle 0|\tilde{\phi}^{(\bar\kappa_1)}(p_3)[\tilde{j}^{(\bar\kappa_2)}(p_4), \tilde{j}^{(\kappa_2)}(p_2)]\tilde{\phi}^{(\kappa_1)}(p_1)|0\rangle\end{aligned}$$

for $p_2^2 = m_2^2 = p_4^2$. To do this, use reduction formulae of type (13.102) for the right hand side of (15.149) as well as the relation

$$A(\tilde{\phi}^{(\bar\kappa_2)}(p_4); \tilde{\phi}^{(\kappa_2)}(p_2)) - R(\tilde{\phi}^{(\bar\kappa_2)}(p_4); \tilde{\phi}^{(\kappa_2)}(p_2)) = i[\tilde{\phi}^{(\bar\kappa_2)}(p_4), \tilde{\phi}^{(\kappa_2)}(p_2)].]$$

Remark. Mandelstam (1958) proposed the so-called ***double dispersion relation*** for the amplitudes of two-particle processes:

$$\begin{aligned}T(s,t,u) = \frac{1}{\pi}\int_{s_0}^{\infty}\frac{a_s(s')}{s'-s}ds' + \frac{1}{\pi}\int_{t_0}^{\infty}\frac{a_t(t')}{t'-t}dt' + \frac{1}{\pi}\int_{u_0}^{\infty}\frac{a_u(u')}{u'-u}du' + \frac{1}{\pi^2}\int_{M_{12}^2}^{\infty}ds'\int_{M_{13}^2}^{\infty}dt' \times \\ \times\frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{M_{13}^2}^{\infty}dt'\int_{M_{14}^2}^{\infty}du'\frac{\rho_{tu}(t',u')}{(t'-t)(u'-u)} + \frac{1}{\pi^2}\int_{M_{14}^2}^{\infty}du'\int_{M_{12}^2}^{\infty}ds'\frac{\rho_{us}(u',s')}{(u'-u)(s'-s)}.\end{aligned} \tag{15.150}$$

A remarkable property of the Mandelstam representation is that the variables s, t, u are treated on an equal footing; as a consequence of this, the amplitudes of the s-, t- and u-channels turn out to be different boundary values of the same analytic function $T(s,t,u)$. The results given in this chapter on analyticity with respect to the variables s, t separately also follow from (15.150). However, the representation (15.150) was not derived from the general principles of quantum field theory (Mandelstam (1959) verified it only for the Feynman diagram of fourth order with respect to the coupling constant) and is therefore a conjecture.

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* The dispersion relations for nucleon-nucleon scattering were proved in any order of perturbation theory of the Yukawa model of the meson-nucleon interactions (see Logunov et al., 1962; [T4]).

CHAPTER 16

Analytic Properties of the Four-Point Green's Function

16.1. Generalized Retarded Functions

A. Generalized retarded products

According to Proposition (13.2) the causal Green's function coincides with the retarded and advanced Green's functions in certain domains of p-space; thus it is possible to go over to the complex momentum domain for the causal function. For a more systematic exploitation of this possibility we introduce the notion of generalized retarded products of fields.

In the S-matrix method (§14.1.D) the generalized retarded radiation operators were introduced by means of two types of formal variational derivatives with respect to the in- and out-fields. Correspondingly in the LSZ formalism we define in addition to the ordinary variational derivative $\delta/\delta\eta(x)$ with respect to the classical source $\eta(x)$, a variational derivative of a new type which we denote by $\delta/\delta\eta(x,+)$; it acts according to the rule

$$\frac{\delta}{\delta\eta(x,+)}C(\eta) = \mathfrak{T}^{-1}(\eta)\frac{\delta}{\delta\eta(x)}[\mathfrak{T}(\eta)C(\eta)\mathfrak{T}(\eta)^{-1}]\mathfrak{T}(\eta) \tag{16.1}$$

(here $\mathfrak{T}(\eta)$ is the generating functional for the T-products of the fields, $C(\eta)$ is an arbitrary formal series with respect to the classical source and with operator-valued "coefficient" generalized functions). We also use the notation $\delta/\delta\eta(x,-)$ for the previous derivative:

$$\frac{\delta}{\delta\eta(x,-)}C(\eta) = \frac{\delta}{\delta\eta(x)}C(\eta). \tag{16.2}$$

The following exercise justifies the use of the term "derivative" for the expression (16.1).

Exercise 16.1. Prove that $\delta/\delta\eta(x,+)$ is subject to the rules of differentiation of type (13.27).

Exercise 16.2. Prove the formula

$$\Re(x;\eta) \equiv -i\mathfrak{T}(\eta)^{-1}\frac{\delta\mathfrak{T}(\eta)}{\delta\eta(x,-)} = -i\frac{\delta\mathfrak{T}(\eta)}{\delta\eta(x,+)}\mathfrak{T}(\eta)^{-1}. \tag{16.3}$$

We give some important properties of variational derivatives.

Exercise 16.3. Prove the following properties (where $\sigma = \pm$):

$$\Big(\frac{\delta}{\delta\eta(x,\sigma)} - \frac{\delta}{\delta\eta(x,-\sigma)}\Big)C(\eta) = i\sigma[\Re(x;\eta),C(\eta)]_{\mp}, \tag{16.4a}$$

$$\frac{\delta}{\delta\eta(x,\sigma)}\,\frac{\delta}{\delta\eta(y,\sigma)} \mp \frac{\delta}{\delta\eta(y,\sigma)}\,\frac{\delta}{\delta\eta(x,\sigma)} = 0, \tag{16.4b}$$

$$\Big(\frac{\delta}{\delta\eta(x,\sigma)}\,\frac{\delta}{\delta\eta(y,-\sigma)} \mp \frac{\delta}{\delta\eta(y,-\sigma)}\,\frac{\delta}{\delta\eta(x,\sigma)}\Big)C(\eta) = i\sigma\Big[\frac{\delta}{\delta\eta(x,\sigma)}\Re(y;\eta),C(\eta)\Big]_{\mp}, \tag{16.4c}$$

$$\frac{\delta}{\delta\eta(x,\sigma)}\Re(y;\eta) = \pm\frac{\delta}{\delta\eta(y,-\sigma)}\Re(x;\eta). \tag{16.4d}$$

[Hint: For the proof of (16.4c) express $\delta/\delta\eta(y,-\sigma)$ in terms of $\delta/\delta\eta(y,\sigma)$ according to (16.4a).]

By a *Steinmann polynomial* we mean an expression of the form

$$\frac{\delta}{\delta\eta(x_n,\sigma_n)}\cdots\frac{\delta}{\delta\eta(x_2,\sigma_2)}\Re(x_1;\eta), \tag{16.5}$$

where the indices $\sigma_n,\ldots,\sigma_2$ take the values $\pm$ (we call them the signs of the differentiations). The operator-valued generalized functions

$$R(\phi_n(x_n,\sigma_n)\ldots\phi_2(x_2,\sigma_2)\phi_1(x_1,\sigma_1)) \equiv \frac{\delta}{\delta\eta(x_n,\sigma_n)}\cdots\frac{\delta}{\delta\eta(x_2,\sigma_2)}\Re(x_1,\eta)\bigg|_{\eta=0} \tag{16.6}$$

are called *generalized retarded products* of Steinmann type (GRP for short), while their vacuum expectation values

$$r(x_n,\sigma_n;\ldots;x_1,\sigma_1) = \langle 0|R(\phi_n(x_n,\sigma_n)\ldots\phi_1(x_1,\sigma_1))|0\rangle, \tag{16.7}$$

are generalized functions in $\mathcal{S}'(\mathrm{M}^n)$, called (Green's) *generalized retarded functions* of Steinmann type (GRF for short). Note that in the formulae (16.6), (16.7) we suppose that

$$\sigma_1 = -\sigma_2. \tag{16.8}$$

The class of GRF's was introduced by Steinmann (1960) (by a different method). Ruelle (1961b) (see also Araki, 1961b) put forward a class of n-point generalized retarded functions, which, in general, is wider than the class of GRF's; these are associated with cells on the plane $\Sigma_n = \{(s_1,\ldots,s_n) \in \in \mathbf{R}^n : s_1+\ldots+s_n = 0\}$. (Cells are non-empty convex open cones in Σ_n, into which Σ_n is, apart from a finite number of planes, decomposed by all possible planes $\sum_{j\in J} s_j = 0$, where J is an arbitrary proper * subset of $\{1,\ldots,n\}$.) The GRF's of Steinmann type correspond to cells that are simplicial cones. For the case $n = 4$ of special interest to us (and, more generally, for $n \le 4$) all cells are simplicial cones and both the Steinmann and Ruelle sets of generalized retarded functions are the same.

Exercise 16.4. Verify that the retarded products $R(\phi_n(x_n)\ldots\phi_2(x_2);\ \phi_1(x_1))$ (13.33) and advanced products $A(\phi_n(x_n)\ldots\phi_2(x_2);\ \phi_1(x_1))$ (13.43) are special cases of GRP's corresponding to the two sets of signs:

$$\text{(a) } \sigma_n = \ldots = \sigma_2 = -\sigma_1 = -, \quad \text{(b) } \sigma_n = \ldots = \sigma_2 = -\sigma_1 = +. \tag{16.9}$$

Exercise 16.5. Prove the relation

$$R(\phi_n(x_n,\sigma_n)\ldots\phi_2(x_2,\sigma_2)\phi_1(x_1,\sigma_1)) = \pm R(\phi_n(x_n,\sigma_n)\ldots\phi_1(x_1,\sigma_1)\phi_2(x_2,\sigma_2)). \tag{16.10}$$

[Hint: Apply (16.4d).]

According to (16.6), the GRP's are expressed in terms of T-products of the fields. In the next exercise we examine some properties of these representations.

Exercise 16.6. (a) Prove that the Steinmann monomial (16.5) is a linear combination of expressions of the form

$$\mathfrak{T}(\eta)^{-1}\frac{\delta^{k_1}\mathfrak{T}(\eta)}{\delta\eta(x_{\pi 1})\ldots\delta\eta(x_{\pi k_1})}\cdots\mathfrak{T}(\eta)^{-1}\frac{\delta^{k_\nu-k_{\nu-1}}\mathfrak{T}(\eta)}{\delta\eta(x_{\pi(k_{\nu-1}+1)})\ldots\delta\eta(x_{\pi n})}, \tag{16.11}$$

where $k_1,\ldots,k_\nu$ are integers such that $1 \le k_1 < k_2 < \ldots \le k_\nu = n$; π is a permutation of the numbers $\{1,\ldots,n\}$. Moreover, the variational first order derivative $\delta\mathfrak{T}/\delta\eta(x_n)$ is obtained here in the form of the rightmost (or second from the left) factor of some of the terms of this decomposition if $\sigma_n = +$ (or $\sigma_n = -$). [Hint: Proceed by induction on n.]

* This means that J is neither the empty set $\emptyset$ nor the entire set $\{1,\ldots,n\}$.

(b) Obtain the further restrictions on the expressions (16.11) involved in the decomposition of the Steinmann monomial (16.5) with $\nu \geq 2$:

$$\sigma_{\min\{\pi 1,\ldots,\pi k_1\}} = -, \quad \sigma_{\min\{\pi(k_{\nu-1}+1),\ldots,\pi n\}} = +. \tag{16.12}$$

[Hint: Prove this by induction using part (a) of this exercise.]

(c) Prove that the GRP (16.6) is a finite linear combination of expressions of the form

$$T(\phi_{\pi 1}\ldots\phi_{\pi k_1})\ldots T(\phi_{\pi(k_{\nu-1}+1)}\cdots\phi_{\pi n}) \tag{16.13}$$

(with the same notation as in (16.11)), where (16.12) holds for $\nu \geq 2$. [Hint: Apply part (b) of this exercise.]

We now establish a connection between the GRF's and the causal Green's functions. We assume that associated with each proper subset $J \subset \{1,\ldots,n\}$ is the sign $\sigma_J = \pm$ according to the rule

$$\sigma_J = \begin{cases} \sigma_{\min J}, & \text{if } 1 \notin J, \\ -\sigma_{\min\{1,\ldots,n\}\setminus J}, & \text{if } 1 \in J. \end{cases} \tag{16.14}$$

Proposition 16.1. *The coincidence properties*

$$\tilde{\tau}(p_n,\ldots,p_1) = (-i)^{n-1}\tilde{r}(p_n,\sigma_n;\ldots;p_1,\sigma_1) \tag{16.15}$$

hold if $p_J \equiv \sum_{j\in J} p_j \notin \overline{V}^{\sigma_J}$ *for all proper subsets* $J \subset \{1,\ldots,n\}$.

■ According to Exercise 16.6, the right hand side of (16.15) is a linear combination of vacuum expectation values of the Fourier transforms of the expressions (16.13). One of the terms corresponding to $\nu = 1$ is clearly $\tilde{\tau}(p_n,\ldots,p_1)$. We claim that in the domain we are considering, the terms with $\nu \geq 2$ provide no contribution. We consider for definiteness the case when the index 1 is missing from the group $J = \{\pi(k_{\nu-1}+1),\ldots,\pi n\}$ of indices of the rightmost factor in (16.13). By the spectrum condition, this term provides no contribution to the right hand side of (16.15) for $p_J \notin \overline{V}^-$, that is, (by (16.12) and (16.14)) for $p_J \notin -\overline{V}^{\sigma_J}$ and hence in our domain as well. The other possible case when the index 1 is missing from the group of indices of the leftmost factor in (16.13) is similarly dealt with. ■

B. SUPPORTS IN x-SPACE

The supports of GRP's and GRF's have the following property.

Proposition 16.2. *The support of the Steinmann monomial* (16.5) (*and of the GRP's* (16.6) *and GRF's* (16.7)) *is contained in the set*

$$\bigcap_{k=2}^{n} \bigcup_{\substack{j<k \\ \sigma_j=-\sigma_k}} \{(x_1,\ldots,x_n) \in \mathbf{M}^n : x_k - x_j \in \overline{V}^{\sigma_k}\}. \tag{16.16}$$

■ We partition the sequence $1,\ldots,n$ into intervals $I_1 = \{1\}$, $I_2 = \{2,\ldots,k_2\},\ldots,$ $I_\lambda = \{k_{\lambda-1}+$ $+1,\ldots,k_\lambda\},\ldots,$ $I_\nu = \{k_{\nu-1}+1,\ldots,n\}$ such that σ_j is constant on each interval and of opposite sign on neighbouring intervals. We suppose for definiteness that $\sigma_1 = -$ and hence $\sigma_2 = +$ (the other case is similar). Then the expression $i\frac{\delta}{\delta\eta(x_n,\sigma_n)}\cdots\frac{\delta}{\delta\eta(x_2,\sigma_2)}\mathfrak{R}(x_1;\eta)$ for ν odd or $i\mathfrak{T}(\eta)\Big\{\frac{\delta}{\delta\eta(x_n,\sigma_n)}\cdots\frac{\delta}{\delta\eta(x_2,\sigma_2)}\times$ $\mathfrak{R}(x_1;\eta)\Big\}\mathfrak{T}(\eta)^{-1}$ for ν even is equal to

$$\prod_{\lambda=1}^{\nu}\prod_{j\in I_\lambda}\frac{\delta}{\delta\xi_\lambda(x_j)}\mathfrak{E}_\nu(\eta;\xi_1,\ldots,\xi_\nu)\Big|_{\xi_1=\ldots=\xi_\nu=0}, \tag{16.17}$$

where $\mathfrak{C}_\nu$ is defined recursively:

$$\begin{aligned}\mathfrak{C}_2(\eta,\xi_1,\xi_2) &= \mathfrak{T}(\eta+\xi_1+\xi_2)\mathfrak{T}(\eta+\xi_2)^{-1},\\ \mathfrak{C}_{\lambda+1}(\eta,\xi_1,\dots,\xi_{\lambda+1}) &= \mathfrak{T}(\eta+\xi_{\lambda+1})^{-1}\mathfrak{C}_\lambda(\eta+\xi_{\lambda+1},\xi_1,\dots,\xi_\lambda)\mathfrak{T}(\eta+\xi_{\lambda+1}) \quad \text{for } \lambda \text{ even},\\ &= \mathfrak{T}(\eta+\xi_{\lambda+1})\mathfrak{C}_\lambda(\eta+\xi_{\lambda+1},\xi_1,\dots,\xi_\lambda)\mathfrak{T}(\eta+\xi_{\lambda+1})^{-1} \quad \text{for } \lambda \text{ odd}.\end{aligned}$$

Thus the statement we are trying to prove says that the support of (16.17) is contained in the set (16.16) which we denote by A_ν.

We proceed by induction on ν. For $\nu = 2$ the assertion follows from (16.4d) and the causality property (13.35e). Assuming the assertion holds for $\mathfrak{C}_\lambda$ for $\lambda < \nu$, we must show that it holds for $\mathfrak{C}_\nu$. It follows from the recurrence relation for $\mathfrak{C}_\nu$ that the support of (16.17) is contained in the set $A_{\nu-1}\times \mathbf{M}^{k_\nu-k_{\nu-1}}$. We suppose for definiteness that ν is odd. Then $\mathfrak{C}_\nu$ can be expressed as a product of $\nu-1$ pairs of operators:

$$\mathfrak{C}_\nu(\eta,\xi_1,\dots,\xi_\nu) = \{\mathfrak{T}(\eta+\xi_\nu)^{-1}\mathfrak{T}(\eta+\xi_{\nu-1}+\xi_\nu)\}\dots\{\mathfrak{T}(\eta+\xi_{\nu-1}+\xi_\nu)^{-1}\mathfrak{T}(\eta+\xi_\nu)\}. \tag{16.18}$$

We claim that this expression does not depend on ξ_ν if

$$\operatorname{supp}\xi_\nu \gtrsim \operatorname{supp}\xi_\lambda \quad \text{for all } \lambda = 2,4,\dots,\nu-1. \tag{16.19}$$

For it follows from the induction hypothesis that the expression (16.17) is concentrated at $x_j - x \in \overline{V}^-$ if j belongs to a group I_μ with odd index $\mu < \nu$, while x belongs to the union of the $\operatorname{supp}\xi_\lambda$ for $\lambda = 2,4,\dots,\nu-1$; if (16.19) holds, then this argument x_j automatically satisfies the condition $\operatorname{supp}\xi_\nu \gtrsim x_j$. Therefore to establish our claim that (16.18) is independent of ξ_ν in the domain (16.19), we merely have to prove its independence in the smaller domain

$$\operatorname{supp}\xi_\nu \gtrsim \operatorname{supp}\xi_\lambda \quad \text{for all } \lambda < \nu. \tag{16.20}$$

But then the independence of ξ_ν of the product of each pair of operators in (16.18) follows from Proposition 13.1 (which is a restatement of the causality condition for $\mathfrak{T}(\eta)$).

It follows from the independence of (16.18) on ξ_ν in the domain (16.19) that the support of (16.17) is concentrated at

$$x_k - x_j \in \overline{V}^-, \quad \text{where } k \in I_\nu,\ j \in I_2 \cup I_4 \cup \dots I_{\nu-1}.$$

In view of the induction hypothesis, this means that the support of (16.18) is concentrated in A_ν, which completes the proof of the induction step (for ν odd; the case for even ν is similar). ∎

Let $\sigma_n,\dots,\sigma_1$ be an arbitrary collection of signs satisfying (16.8) and $\nu_n,\dots,\nu_2$ a collection of numbers from the sequence $1,\dots,n$ such that $\nu_j < j$ and $\sigma_{\nu_j} = -\sigma_j$. We associate with these collections the following set, called an *elementary cone in* $\mathbf{M}^n$:

$$X^{\nu_n\dots\nu_2}_{\sigma_n\dots\sigma_1} = \{(x_1,\dots,x_n)\in \mathbf{M}^n : x_j - x_{\nu_j} \in \overline{V}^{\sigma_j},\ j = 2,\dots,n\}. \tag{16.21}$$

The property of the support of a GRP (or a GRF) expressed by Proposition 16.2 now means that the GRP (16.6) (or the GRF (16.7)) is concentrated in the union

$$X_{\sigma_n\dots\sigma_1} = \cup_{\nu_n\dots\nu_2} X^{\nu_n\dots\nu_2}_{\sigma_n\dots\sigma_1} \tag{16.22}$$

of elementary cones; the cones featuring in (16.22) are said to be *subordinated* to the given GRP (or the given GRF).

The property of the support of a GRF is a reflection of the following deeper result (proved by Bros, 1965): each elementary cone can be associated with a generalized function concentrated in this cone, so that every GRF can be represented as a linear combination of a subset of generalized functions, namely, those that are concentrated in cones subordinated to the given GRF. In §16.2.B we consider this question for the case $n = 4$.

It follows from the property of the support that the Fourier transform of the GRF (16.7) has an analytic continuation to the tube whose base is the interior on the plane

$$\Pi = \{(q_1, \ldots, q_n) \in \mathbf{M}^n : q_1 + \ldots + q_n = 0\} \tag{16.23}$$

of the cone $X^*_{\sigma_n \ldots \sigma_1}$, dual to $X_{\sigma_n \ldots \sigma_1}$.

In the next exercise we describe the cones $X^{\nu_n \ldots \nu_2 *}_{\sigma_n \ldots \sigma_1}$.

Exercise 16.7. For any $j \in \{1, \ldots, n\}$ let N_j be the set of numbers in $\{1, \ldots, n\}$ that are either equal to j or are converted into j under a single or repeated application of the map $\nu : k \to \nu_k$ (where the ν_k are the family of numbers occurring in (16.21)):

$$N_j = \{j\} \cup \nu^{-1}\{j\} \cup \ldots \cup \nu^{-1}(\ldots \nu^{-1}\{j\} \ldots). \tag{16.24}$$

(In particular, $N_1 = \{1, \ldots, n\}$, $N_n = \{n\}$.) Prove that the dual cone of the elementary cone (16.21) has the form

$$X^{\nu_n \ldots \nu_2 *}_{\sigma_n \ldots \sigma_1} = \{(q_1, \ldots, q_n) \in \Pi : \ q_{N_j} \in \overline{V}^{\sigma_j}, \ j = 2, \ldots, n\}. \tag{16.25}$$

[Hint: Make the change of variables $(q_1, \ldots, q_n) \to (r_1, \ldots, r_n)$ so as to obtain the identity

$$\sum_{j=1}^{n} q_j x_j = r_1 x_1 + \sum_{j=2}^{n} r_j (x_j - x_{\nu_j}); \tag{16.26}$$

to do this, one must set

$$q_k = r_k - r_{\nu^{-1}\{k\}}, \ k = 1, \ldots, n. \tag{16.27}$$

Sum these equalities over $k \in N_j$; now use the fact that $N_j = \{j\} \cup \nu^{-1}(N_j)$, to obtain

$$q_{N_j} = r_j. \tag{16.28}$$

Finally, derive (16.25) from (16.26), (16.28).]

Lemma 16.3. *The cone dual to $X_{\sigma_n \ldots \sigma_1}$ is given by*

$$X^*_{\sigma_n \ldots \sigma_1} = \{(q_1, \ldots, q_n) \in \Pi : \ q_{L_j} \in \overline{V}^{\sigma_j} \quad \textit{for } j = 2,, \ldots, n\}, \tag{16.29}$$

where L_j (for $2 \leq j \leq n$) is defined by

$$L_j = \{j\} \cup \{k : j < k \leq n \textit{ and } \sigma_k = -\sigma_j\}. \tag{16.30}$$

■ We temporarily denote (16.29) by P and we begin by showing that $X^*_{\sigma_n \ldots \sigma_1} \subset P$. It follows from (16.22) that $X^*_{\sigma_n \ldots \sigma_1}$ is the intersection of all the dual cones of the cones $X^{\nu_n \ldots \nu_2}_{\sigma_n \ldots \sigma_1}$ subordinated to the given GRP. Consequently $X^*_{\sigma_n \ldots \sigma_1}$ is the set of points $(q_1, \ldots, q_n) \in \Pi$ satisfying all possible conditions

$$q_{N_j} \in \overline{V}^{\sigma_j} \ (j = 2, \ldots, n), \tag{16.31}$$

where N_j is defined (in terms of the given $X^{\nu_n \ldots \nu_2}_{\sigma_n \ldots \sigma_1}$) by (16.24). It is clear that the inclusion $X^*_{\sigma_n \ldots \sigma_1} \subset$ $\subset P$ will be proved if we show that for any $j = 2, \ldots, n$, L_j is one of the sets N_j featuring in (16.31). To this end we define the sequence $\nu_n, \ldots, \nu_2$ by setting $\nu_k = j$ if $j < k \leq n$ and $\sigma_k = -\sigma_j$ and letting $\nu_k = \nu_1$ or ν_2 for the remaining $k \geq 2$, so that the condition $\sigma_{\nu_k} = -\sigma_k$ holds for all $k = 2, \ldots, n$. It is clear that $N_j = L_j$ for such a choice of $\nu_n, \ldots, \nu_2$. This construction completes the proof of the inclusion $X^*_{\sigma_n \ldots \sigma_1} \subset P$.

For the proof of the reverse inclusion we take an arbitrary $(q_1, \ldots, q_n) \in P$. It is then easy to see by induction on $n - k$ that

$$q_k \in \overline{V}^{\sigma_k} \quad \text{for all } k = 2, \ldots, n. \tag{16.32}$$

Now let J be an arbitrary subset of $\{2, \ldots, n\}$; we claim that

$$q_J \in \overline{V}^{\sigma_J}. \tag{16.33}$$

To this end we set $j = \min J$ (so that $\sigma_J = \sigma_j$) and represent the left hand side of (16.33) in the form

$$q_{L_j} + \sum_{k \in L_j \setminus J} (-q_k) + \sum_{\substack{k \in J \setminus \{j\} \\ \sigma_k = \sigma_j}} q_k;$$

here the first term lies in $\overline{V}^{\sigma_j}$ in view of the condition $(q_1, \dots, q_n) \in P$; the other terms lie in $\overline{V}^{\sigma_j}$ by virtue of (16.32). Thus we have proved (16.33). It follows trivially from it that all the conditions (16.31) hold, which proves the inclusion $X^*_{\sigma_n \dots \sigma_1} \supset P$. ∎

Exercise 16.8. Prove that the cone $X^*_{\sigma_n \dots \sigma_1}$ can also be written in the form

$$X^*_{\sigma_n \dots \sigma_1} = \{(q_1, \dots, q_n) \in \Pi : \; q_J \in \overline{V}^{\sigma_J} \quad \text{for all proper subsets } J \subset \{1, \dots, n\}\}. \tag{16.34}$$

[Hint: It has already been proved above that the points of $X^*_{\sigma_n \dots \sigma_1}$ satisfy (16.33) for subsets $J \subset \subset \{2, \dots, n\}$; for the proof of the same formula for subsets J containing 1, use the condition $q_1 + \dots + +q_n = 0$.]

We obtain from Proposition 16.2 and Lemma 16.3 the following result for the Fourier transform of the GRP

$$\int r(x_n, \sigma_n; \dots; x_1, \sigma_1) \exp\Big(i \sum_{j=1}^{n} p_j x_j\Big) dx_1 \dots dx_n = \\ = (2\pi)^4 \delta(p_1 + \dots + p_n) G(p_n, \sigma_n; \dots; p_1, \sigma_1). \tag{16.35}$$

Proposition 16.4. *$G(p_n, \sigma_n; \dots, p_1, \sigma_1)$ is a generalized boundary value (of class $\mathcal{S}'(\Pi)$) of the analytic function $G(p_n, \sigma_n; \dots; p_1, \sigma_1)$ in the tube $\Pi + iQ_{\sigma_n \dots \sigma_1}$, where*

$$Q_{\sigma_n \dots \sigma_1} = \{(q_1, \dots, q_n) \in \Pi : \; q_{L_j} \in V^{\sigma_j}, \; j = 2, \dots, n\} = \tag{16.36a}$$
$$= \{(q_1, \dots, q_n) \in \Pi : \; q_J \in V^{\sigma_J} \textit{ for all proper subsets } J \subset \{1, \dots, n\}\}. \tag{16.36b}$$

For a fixed collection of fields $\phi_1(x_1), \dots, \phi_n(x_n)$ different GRF's (both of Steinmann and Ruelle type) satisfy certain coincidence conditions (to within $\pm$) in domains of p-space. An application of the "edge of the wedge" theorem to these conditions shows that all GRF's in p-space are (to within $\pm$ signs) different boundary values of the same analytic function defined in the so-called primitive domain of holomorphy of the n-point Green's functions; this domain contains a neighbourhood of the point $(k_1, \dots, k_n) = (0, \dots, 0)$ on the complex plane $k_1 + \dots + k_n = 0$ and the union of tubes with bases of the form (16.36b). Below we consider the case $n = 4$ in detail.

16.2. Four-Point Green's Functions

A. Notation

Later in this chapter we shall be considering a fixed quadruple of quantum fields $\phi_1(x_1) \equiv \phi^{(\kappa_1)}(x_1), \dots, \phi_4(x_4) \equiv \phi^{(\kappa_4)}(x_4)$. So as to simplify the notation, we make the assumption (which, in fact, is inessential) that these fields are of bosonic type. The symbol Π (or $C\Pi$) now denotes the plane in $\mathbf{M}^4$ (or $\mathbf{CM}^4$) defined by the equation

$$k_1 + \dots + k_4 = 0. \tag{16.37}$$

Let the numbers j, k, l, n form an arbitrary permutation of the indices 1,2,3,4. Then we can apply the alternative more economical notation for the four-point GRF's:

$$r_n(x) = r(x_j, -; x_k, -; x_l, -; x_n, +), \tag{16.38a}$$
$$a_n(x) = r(x_j, +; x_k, +; x_l, +; x_n, -), \tag{16.38b}$$
$$r_{jn}(x) = r(x_j, +; x_k, -; x_l, -; x_n, +), \tag{16.38c}$$
$$a_{jn}(x) = r(x_j, -; x_k, +; x_l, +; x_n, -), \tag{16.38d}$$

where $x \equiv (x_1, \ldots, x_4) \in \mathbf{M}^4$. The symmetry with respect to j, k, l on the right hand sides of (16.38a) and (16.38b) accounts for the absence of these indices in the notation; similarly, the notation of (16.38c) and (16.38d) takes the symmetry with respect to k, l into account. As a result we have a set of 32 ($= 4+4+4\cdot3+4\cdot3$) four-point GRF's associated with the monomial $\phi_1(x_1)\ldots\phi_4(x_4)$. We denote any of these Green's functions by the symbol Z (which thus takes the "values" r_n, a_n, r_{jn}, a_{jn}).

By Proposition 16.2 we have the following properties of the supports:

$$\operatorname{supp} r_n \subset X_n^r \equiv \{x \in \mathbf{M}^4 : x_j - x_n \in \overline{V}^-, x_k - x_n \in \overline{V}^-, x_l - x_n \in \overline{V}^-\}, \tag{16.39a}$$
$$\operatorname{supp} a_n \subset X_n^a \equiv -X_n^r, \tag{16.39b}$$
$$\operatorname{supp} r_{jn} \subset X_{jkn}^r \cup X_{jln}^r, \tag{16.39c}$$
$$\operatorname{supp} a_{jn} \subset X_{jkn}^a \cup X_{jln}^a, \tag{16.39d}$$

where

$$X_{jkn}^r = \{x \in \mathbf{M}^4 : x_j - x_k \in \overline{V}^+,\ x_k - x_n \in \overline{V}^-,\ x_n - x_l \in \overline{V}^+\}, \tag{16.40a}$$
$$X_{jkn}^a = -X_{jkn}^r = X_{lnk}^r. \tag{16.40b}$$

The Green's functions G^Z in p-space are defined by the Fourier transform

$$\int Z(x_1, \ldots, x_4) e^{i(p_1x_1+\ldots+p_4x_4)} dx_1 \ldots dx_4 = (2\pi)^4 \delta(p_1 + \ldots + p_4) G^Z(p_1, \ldots, p_4). \tag{16.41}$$

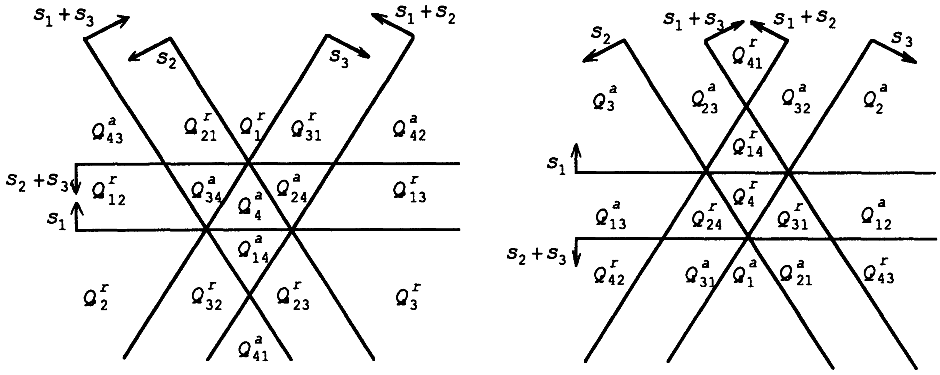
According to Proposition 16.4 $G^Z(p)$ is a boundary value (of class $\mathcal{S}'(\Pi)$) of the analytic function $G^Z(k)$ in the tube

$$\Pi + iQ^Z \subset \mathbf{C}\Pi \tag{16.42}$$

with base of the form

$$Q_m^r = \{q \in \Pi : q_j \in V^-,\ q_k \in V^-,\ q_l \in V^-\}, \tag{16.43a}$$
$$Q_n^a = -Q_n^r, \tag{16.43b}$$
$$Q_{jn}^r = \{q \in \Pi : q_j \in V^+,\ q_j + q_k \in V^-,\ q_j + q_l \in V^-\}, \tag{16.43c}$$
$$Q_{jn}^a = -Q_{jn}^r. \tag{16.43d}$$

To depict these cones graphically, we portray the corresponding cells as the points of the cones with spatial components equal to zero: $\mathbf{q}_1 = \ldots = \mathbf{q}_4 = 0$. We denote the zeroth components by $s_1 = q_1^0, \ldots, s_4 = q_4^0$; they range over the plane $s_1 + \ldots + s_4 = 0$ in $\mathbf{R}^4$. Clearly it suffices to portray the intersections of the cells with the two planes $s_4 = \pm c$ ($c > 0$).



$s_4<0$ $\qquad\qquad\qquad$ $s_4>0$

Exercise 16.9. (a) Prove the relations

$$Q^{r*}_{jn} = \{x \in \mathrm{M}^4 : x_n - x_k \in \overline{V}^+,\ x_n - x_l \in \overline{V}^+,\ x_j + x_n - x_k - x_l \in \overline{V}^+\}, \quad (16.44a)$$

$$Q^{a*}_{kl} = \{x \in \mathrm{M}^4 : x_j - x_l \in \overline{V}^+,\ x_n - x_l \in \overline{V}^+,\ x_j + x_n - x_k - x_l \in \overline{V}^+\}, \quad (16.44b)$$

$$Q^{r*}_{jn} \cap Q^{a*}_{kl} = \{x \in \mathrm{M}^4 : x_j - x_l \in \overline{V}^+,\ x_n - x_l \in \overline{V}^+,\ x_n - x_k \in \overline{V}^+\}, \quad (16.44c)$$

$$Q^r_{jn} + Q^a_{kl} = \{q \in \Pi : q_j \in V^+,\ q_k \in V^-,\ q_j + q_l \in V^-\}. \quad (16.45)$$

[Hint: Equalities (16.44a) and (16.44b) follow immediately from the definitions (16.43c) and (16.43d); for the proof of (16.45) use the fact that the left hand side of this relation is the interior on the plane Π of the dual cone of (16.44c).]

(b) Prove the following formula for the process $\kappa_1 + \kappa_2 \to \bar{\kappa}_3 + \bar{\kappa}_4$ in a domain of pairwise non-collinear momenta of (Bose) particles:*

$$\langle 0|\tilde{\phi}^{\rm out}_4(p_4)\tilde{\phi}^{\rm out}_3(p_3)\tilde{\phi}^{\rm in}_2(p_2)\tilde{\phi}^{\rm in}_1(p_1)|0\rangle =$$

$$= -i\delta^-_{m_1}(p_1)\delta^-_{m_2}(p_2)\delta^+_{m_3}(p_3)\delta^+_{m_4}(p_4)(2\pi)^4\delta(p_1+\ldots+p_4) \lim_{q\to 0,\ q\in Q^a_1} \prod_{j=1}^{4}(p_j^2 - m_j^2)G^a_1(p+iq) \quad (16.46)$$

[Hint: Use the reduction formula (13.125).]

B. Domains of coincidence in p-space

We shall be applying the same stipulations with regard to the mass spectrum as in §15.1.A. We associate with the threshold masses $M_j (j = 1, \ldots, 4)$ and M_{jn} $(1 \leq$ $\leq j < n \leq 4)$ the subsets of $\mathrm{C}\Pi$:

$$\gamma_j = \{k \in \mathrm{C}\Pi :\ k_j^2 \geq M_j^2\}, \quad (16.47a)$$

$$\gamma_{jn} = \{k \in \mathrm{C}\Pi :\ (k_j + k_n)^2 \geq M_{jn}^2\}. \quad (16.47b)$$

We denote their union by γ:

$$\gamma = \Big(\bigcup_{1\leq j\leq 4} \gamma_j\Big) \cup \Big(\bigcup_{1\leq j<n\leq 4} \gamma_{jn}\Big). \quad (16.48)$$

* The restriction to Bose particles is merely due to the fact that the simplified notation for GRF's adopted in this section is well suited to this case.

Proposition 16.5. *The four-point GRF's satisfy the conditions*

$$(p_j^2 - m_j^2)\{G_n^r(p) - G_{jn}^r(p)\} = 0 \quad \text{for } p_j^2 < M_j^2, \tag{16.49a}$$
$$(p_j^2 - m_j^2)\{G_n^a(p) - G_{jn}^a(p)\} = 0 \quad \text{for } p_j^2 < M_j^2, \tag{16.49b}$$
$$((p_j + p_l)^2 - m_{jl}^2)\{G_{jn}^r(p) - G_{lk}^a(p)\} = 0 \quad \text{for } (p_j + p_l)^2 < M_{jl}^2. \tag{16.50}$$

■ It follows from (16.4a) that

$$r_n(x) - r_{jn}(x) = -i\langle[\phi_j(x_j),\ R(x_k, -; x_l, -; x_n, +)]\rangle_0,$$

so that (16.49a) is a corollary of this formula and the spectral conditions. Formula (16.49b) is proved similarly. Formula (16.50) is derived from the equality

$$r_{jn}(x) - a_{lk}(x) = i\langle[R(x_l, -; x_j, +),\ R(x_k, -; x_n, +)]\rangle_0, \tag{16.51}$$

which follows from (16.4c), also by an application of the spectral conditions. ■

The form of the coincidence conditions is simplified if in place of $G^Z(p)$ we introduce the generalized functions $\widehat{G}^Z(p)$ which differ from $G^Z(p)$ by a factor:

$$\widehat{G}^Z(p) = \prod_{j=1}^{4}(p_j^2 - m_j^2) \prod_{1\le j<k\le 4} ((p_j + p_k)^2 - m_{jk}^2)G^Z(p); \tag{16.52}$$

we call them *reduced Green's functions*; (we shall be dealing with these in what follows).

The first four factors correspond to going over to the amputated Green's functions and the last factors compensate the pole singularities of the amplitude. If there is not a bound state in some channel (and hence the mass m_{jk} is missing), then there is no corresponding factor $(p_j + p_k)^2 - m_{jk}^2$ in the definition of the reduced Green's function.

Each coincidence condition (as is obvious from the diagram) relates the GRF's corresponding to adjacent cells (the boundaries of which have a common segment). It follows from these conditions that the $\widehat{G}^Z(p)$ are different generalized boundary values of the same analytic function $\widehat{G}(k)$.

Thus consider the domain

$$\mathcal{O} = \{p \in \mathrm{M}^4 : p_j^2 < M_j^2,\ j = 1, \ldots, 4;\ (p_j + p_k)^2 < M_{jk}^2,\ 1 \le j < k \le 4\}, \tag{16.53}$$

in which all the $\widehat{G}^Z(p)$ are equal. The "edge of the wedge" theorem applied to the pairs $\widehat{G}_n^r(k)$ and $\widehat{G}_n^a(p)$, $\widehat{G}_{jn}^r(k)$ and $\widehat{G}_{jn}^a(k)$ shows that these functions are analytic in some complex neighbourhood $\mathcal{N}(\mathcal{O})$ of $\mathcal{O}$, where the functions in each pair can be analytically continued from one to the other. Next let $\widehat{G}^Z(p)$ and $\widehat{G}^{Z'}(p)$ be a pair of GRF's corresponding to adjacent cells and let $\mathcal{O}^{ZZ'}$ be their domain of coincidence (that is, either $p_j^2 < M_j^2$ for some j or $(p_j + p_k)^2 < M_{jk}^2$ for some j, k). By the "edge of the wedge" theorem again, the functions $\widehat{G}^Z(k)$ and $\widehat{G}^{Z'}(k)$ can be analytically continued from one to the other in the " edge of the wedge" domain $\mathcal{N}(\mathcal{O}^{Z,Z'})$ corresponding to the generalized route $(\Pi + iQ^Z,\ \mathcal{O}^{Z,Z'},\ \Pi + iQ^{Z'})$. Hence it follows in particular that $\widehat{G}^Z(k)$ and $\widehat{G}^{Z'}(k)$ corresponding to adjacent cells and therefore all the functions $\widehat{G}^Z(k)$, coincide in $\mathcal{N}(\mathcal{O})$. Thus they define a single function $\widehat{G}(k)$ which is analytic at least in the domain

$$\bigcup_Z(\Pi + iQ^Z) \cup \bigcup \mathcal{N}(\mathcal{O}^{Z,Z'}) \cup \mathcal{N}(\mathcal{O}), \tag{16.54}$$

where the second union is taken over all adjacent Z, Z'; this is called the primitive domain of analyticity. Thus we have the following result.

Proposition 16.6. *The reduced Green's functions $\widehat{G}^Z(p)$ are different boundary values of the same function $\widehat{G}(k)$, which is analytic in the primitive domain* (16.54).

Taking into account the reduction formulae (see Exercise 16.9b) we see that the amplitudes of the different channels are related by analytic continuation. However in this form, the process of analytic continuation requires passage beyond the mass shell; it is called a "generalized" crossing relation (in contrast to the ordinary crossing relation, see §16.3).

Proposition 16.6 is a translation of the coincidence conditions in the language of analytic functions.

This stage of the study of the analytic properties of Green's functions and scattering amplitudes, which is based essentially on the locality and spectral principles, is sometimes called the linear programme so as to emphasize the linear nature of the relations used. In Appendix J we shall come across non-linear conditions for the purposes of analytic continuation.

Since the domains $\Pi + iQ^Z$, $\mathcal{O}^{Z,Z'}$, $\mathcal{O}^Z$ in (16.54) are starlike (see §5.2.E), the "edge of the wedge" domains $\mathcal{N}(\mathcal{O})$ and $\mathcal{N}(\mathcal{O}^{Z,Z'})$ can also be considered to be starlike (otherwise we could replace, say, $\mathcal{N}(\mathcal{O})$ by $\cup_{0<r\leq 1} r\mathcal{N}(\mathcal{O})$). Then according to Corollary 5.36, the single-sheeted envelope of holomorphy $H(D)$ of (16.54) exists and is starlike.

The problem of holomorphic extension, which the coincidence conditions (16.49) reduce to when taken separately, was considered in §5.2.F. Suppose, for example, that $Z = r_n$, $Z' = r_{jn}$; then by Proposition 5.37, $H(D)$ contains the convex hull of the tubes $\Pi + iQ^r_n$ and $\Pi + iQ^r_{jn}$ with an "analytic cut" along γ_j:

$$H(D) \supset (\Pi + iK^r_{jn})\backslash\gamma_j,$$

where

$$K^r_{jn} = Q^r_n + Q^r_{jn}, \quad K^a_{jn} = Q^a_n + Q^a_{jn}. \tag{16.55}$$

Thus the sets $\mathcal{N}(\mathcal{O}^{Z,Z'})$ in (16.54) corresponding to the coincidence conditions (16.49) can be replaced by the domains of holomorphy $(\Pi + iK^{r,a}_{jn})\backslash\gamma_j$, which are tubes with "analytic cuts" along γ_j. In the next subsection a similar substitution is made for the remaining sets $\mathcal{O}^{Z,Z'}$ (corresponding to the coincidence conditions (16.50)).

Remark. It follows from Lorentz-covariance that the reduced Green's function is analytic in the union of all domains obtained from (16.54) by the action of all possible transformations of the proper complex Lorentz group $L_+(\mathbf{C})$ (or its universal covering $SL(2,C) \times SL(2,C)$); this new domain is also sometimes called the primitive domain of analyticity. The above result is related to the Bargmann-Hall-Wightman theorem 9.1, which is of the same type; only now the proof is much simpler since the domain (16.54) contains the origin, so that Proposition 5.9 is immediately applicable here. In the case of scalar fields, Proposition 5.15 enables us to express $\widehat{G}(k)$ as an analytic function of the scalar products k_ik_j $(1 \leq i \leq j \leq 3)$ or, equivalently, as a function of the invariants $s, t, k_1^2, \ldots, k_4^2$; (as was noted in §9.1.A, the group $L_{(+)}(\mathbf{C})$ is isomorphic to $O_{(+)}(4, \mathbf{C})$).

C. Steinmann identities

By making the substitution $j \leftrightarrow n$, $l \leftrightarrow k$ of the indices in (16.51) we obtain the equality

$$r_{nj}(x) - a_{kl}(x) = i\langle[R(x_k, -; x_n, +),\ R(x_l, -; x_j, +)]\rangle_0,$$

which on comparing with (16.51) leads to the following relation between the GRF's, called the Steinmann identity:

$$r_{jn}(x) - a_{lk}(x) = a_{kl}(x) - r_{nj}(x). \tag{16.56}$$

Hence the reduced Green's functions satisfy the relation

$$\widehat{G}^r_{jn}(p) + \widehat{G}^r_{nj}(p) = \widehat{G}^a_{lk}(p) + \widehat{G}^a_{kl}(p). \tag{16.57}$$

If we go through all the permutations (j, k, l, n) we obtain six (distinct) relations of this sort, where each GRF with two suffixes occurs in just one of these identities. A quadruple of GRF's related by the Steinmann identity is called a (Steinmann) quartet. Accordingly there are six quartets.

In the diagram, the quartets correspond to quadruples of cells in which each of the components $s_1, \ldots, s_4$ has a fixed sign (two components are positive and two are negative).

Each of the GRF's occurring in (16.56) has support in the union of two of the four elementary cones X^r_{jkn}, X^r_{jln}, X^r_{nkj}, X^r_{nlj}, namely: r_{jn} (or a_{kl}) has support in the union of X^r_{jkn} and X^r_{jln} (or X^r_{nlj} and X^r_{jln}). This fact is reflected in the following lemma.

Lemma 16.7. *The Steinmann quartet can be associated with the quadruple of (translation-invariant) generalized functions f_{jkn}, f_{jln}, f_{nkj}, f_{nlj} with supports in the elementary cones X^r_{jkn}, X^r_{jln}, X^r_{nkj}, X^r_{nlj} respectively so that each GRF of a given quartet is represented as the sum of a pair of such generalized functions with supports in the elementary cones subordinated to this GRF.*

■ It suffices to consider any one quartet, for example, r_{14}, r_{41}, a_{23}, a_{32}. We simplify the notation for the corresponding quadruple of elementary cones as follows:

$$X_1 = X^r_{124}, \quad X_2 = X^r_{134}, \quad X_3 = X^r_{421}, \quad X_4 = X^r_{431}.$$

Any pair of these cones has the same intersection:

$$X_i \cap X_j = X_0 \quad \text{for } 1 \le i < j \le 4, \tag{16.58}$$

where

$$X_0 = \{x \in M^4 : x_1 - x_2 \in \overline{V}^+,\ x_4 - x_2 \in \overline{V}^+,\ x_1 - x_3 \in \overline{V}^+,\ x_4 - x_3 \in \overline{V}^+\}.$$

We are looking for a quadruple of generalized functions $f_1, \ldots, f_4$ with supports $\operatorname{supp} f_j \subset X_j$ such that

$$r_{14} = f_1 + f_2, \quad r_{41} = f_3 + f_4, \quad a_{23} = f_2 + f_4, \quad a_{32} = f_1 + f_3. \tag{16.59}$$

Clearly a necessary condition for the solubility of the system is the Steinmann identity. It is also sufficient (since for an arbitrary choice of f_1, three of these equations serve to determine f_2, f_3, f_4 while the fourth equation is satisfied automatically). It remains to ensure that the property of the supports is satisfied. For this purpose we note that the support of r_{14} is contained in the union of the two canonically closed regular sets X_1 and X_2, therefore (according to Proposition A.2) r_{14} can be represented as the sum of a pair of generalized functions f_1 and f_2 with supports in X_1 and X_2 respectively. Assuming f_1 and f_2 to be chosen with the requisite property of the support, we define f_3 by the last of equations (16.59), whence it follows that

$$\operatorname{supp} f_3 \subset X_1 \cup X_3. \tag{16.60a}$$

On the other hand, $f_1 = r_{14}$ outside K_2 and hence, $f_3 = a_{32} - r_{14} = r_{41} - a_{23}$, so that

$$\operatorname{supp} f_3 \subset X_2 \cup X_3 \cup X_4. \tag{16.60b}$$

Now the intersection of the sets on the right hand sides of the relations (16.60) is X_3 by virtue of (16.58); consequently, $\operatorname{supp} f_3 \subset X_3$. Similarly by defining f_4 from the third equation in (16.59) we obtain $\operatorname{supp} f_4 \subset X_4$. ∎

Equalities of type (16.59) are called a "resolution" of the Steinmann identities (since the Steinmann identities follow from them). In p-space these equalities take the form

$$\widehat{G}^r_{14} = h_1 + h_2,\ \widehat{G}^r_{41} = h_3 + h_4,\ \widehat{G}^a_{23} = h_2 + h_4,\ \widehat{G}^a_{32} = h_1 + h_3, \tag{16.61}$$

where $h_j(p)$ is a generalized function on Π continued analytically to the tube $\Pi+$ $+i \operatorname{int} X_j^*$ $(j = 1, \ldots, 4)$. For our quartet, the coincidence conditions of type (16.50) reduce to the two relations:

$$h_1(p) - h_4(p) = 0 \quad \text{for } (p_1 + p_2)^2 < M_{12}^2, \tag{16.62a}$$
$$h_2(p) - h_3(p) = 0 \quad \text{for } (p_1 + p_3)^2 < M_{13}^2. \tag{16.62b}$$

It turns out that the coincidence condition (16.62a) (and similarly (16.62b)) reduces to the problem of holomorphic extension with respect to k considered in §5.2.F. In fact, $h_1(p)$ and $h_4(p)$ are analytically continued into the tubes $\Pi + iQ_1$ and $\Pi + iQ_4$ respectively, where

$$Q_1 = \operatorname{int}_\Pi X_1^* = \{q \in \Pi : q_1 \in V^+,\ q_3 \in V^-,\ q_3 + q_4 \in V^+\},$$
$$Q_4 = \operatorname{int}_\Pi X_4^* = \{q \in \Pi : q_2 \in V^-,\ q_4 \in V^+,\ q_3 + q_4 \in V^-\}.$$

In the new variables $k_1' = k_3 + k_4,\ k_2' = k_1,\ k_3' = -k_3$ these cones are written in the form

$$Q_1 = \{(q_1', q_2', q_3') \in \mathrm{M}^3 :\ q_1' \in V^+,\ q_2' \in V^+,\ q_3' \in V^+\},$$
$$Q_4 = \{(q_1', q_2', q_3') \in \mathrm{M}^3 :\ q_1' \in V^-,\ (q_1' + q_2') \in V^+,\ (q_1' + q_3') \in V^+\},$$

while the domain of coincidence in (16.62a) is $\{(p_1', p_2', p_3') \in \mathrm{M}^3 : (p_1')^2 < M_{12}^2\}$. On the basis of Proposition 5.39, we conclude from this that $h_1(p)$ and $h_4(p)$ have a common analytic continuation to the convex hull of the tubes $\Pi + iQ_1$ and $\Pi + iQ_4$ with an "analytic cut" along $\{k \in \mathbf{C}\Pi : (k_1')^2 \geq M_{12}^2\} = \gamma_{12}$. A similar argument applied to the coincidence condition (16.62b) shows that $h_2(p)$ and $h_3(p)$ have a common analytic continuation to the tube $\Pi + i(Q_2 + Q_3)$ with an analytic cut along γ_{13}.

Exercise 16.10. Prove the relations

$$Q_1 + Q_4 = \operatorname{int}_\Pi X_0^* = Q_2 + Q_3, \tag{16.63a}$$
$$Q^r_{14} + Q^r_{41} = \operatorname{int}_\Pi X_0^* = Q^a_{23} + Q^a_{32}, \tag{16.63b}$$

[Hint: For the proof of (16.63a) use (16.58) and the fact that $Q_j = \operatorname{int}_\Pi X_j^*$; for the proof of the first relation in (16.63b) use (16.44a) and the fact that $Q^r_{14} + Q^r_{41} = \operatorname{int}_\Pi (Q^{r*}_{14} \cap Q^{r*}_{41})^*$; the second relation in (16.63b) is proved similarly.]

As a result, all the generalized functions featuring in (16.61) are analytically continued to the tube $\Pi + i \operatorname{int}_\Pi X_0^* = \Pi + i(Q^r_{14} + Q^r_{41}) = \Pi + i(Q^a_{23} + Q^a_{32})$ with two cuts along γ_{12} and γ_{13}.

A similar derivation clearly holds for any quartet. For $1 \le j < n \le 4$ we introduce the cone

$$\Lambda_{jn} = Q^r_{jn} + Q^r_{nj} = Q^a_{kl} + Q^a_{lk}. \tag{16.64}$$

Then the conditions of coincidence of type (16.50) among the terms of a given Steinmann quartet mean that $\widehat{G}(p)$ is analytically continued to the tube $\Pi + i\Lambda_{jn}$ with two cuts along γ_{jk} and γ_{jl}.

Exercise 16.11. Prove the relations

$$\Pi + iQ^r_{jn} = (\Pi + iQ^r_{jn})\backslash\gamma, \quad (\Pi + iK^{r,a}_{jn})\backslash\gamma_j = (\Pi + iK^{r,a}_{jn})\backslash\gamma, \tag{16.65a}$$

$$(\Pi + i\Lambda_{jn})\backslash(\gamma_{jk} \cup \gamma_{jl}) = (\Pi + i\Lambda_{jn})\backslash\gamma, \tag{16.65b}$$

$$(\Pi + i(Q^r_{jn} + Q^a_{kl}))\backslash\gamma = (\Pi + i(Q^r_{jn} + Q^a_{kl}))\backslash\gamma_{jk}. \tag{16.65c}$$

Thus all the coincidence conditions (16.49) and (16.50) can be expressed in terms of the reduced Green's function (related to the GRF, after passing to generalized boundary values, by (16.52)) as follows.

Theorem 16.7. *The reduced Green's function $\widehat{G}(k)$ (extended to the complex plane) is analytic in the starlike domain*

$$D = (\Pi + i\Xi)\backslash\gamma, \tag{16.66}$$

where Ξ is the union of all the cones (16.55) *and* (16.64)):

$$\Xi = \Big(\bigcup_{j\neq n} K^r_{jn}\Big) \cup \Big(\bigcup_{j\neq n} K^a_{jn}\Big) \cup \Big(\bigcup_{j<n} \Lambda_{jn}\Big). \tag{16.67}$$

D. Analyticity near physical points

A physical point p cannot be found on two cuts γ_{jk} at once; this follows from the stability condition (§15.1.A). In fact, let

$$p_j \in \Gamma^+_{m_j}, \quad p_k \in \Gamma^+_{m_k}, \quad p_l \in \Gamma^-_{m_l}, \quad p_n \in \Gamma^-_{m_n}; \tag{16.68}$$

then $(p_j + p_k)^2 \ge (m_j + m_k)^2 \ge M^2_{jk}$, $(p_j - p_l)^2 \le (m_j - m_l)^2 < M^2_{jl}$, $(p_j - p_n)^2 \le$ $\le (m_j - m_n) < M^2_{jn}$. Thus a point p of the form (16.68) can only be found on one of the cuts (γ_{jk}) of the type we have been considering.

Theorem 16.8. *Let $\mathcal{P}$ be the set of all real points $p \in \Pi$, none of which lies on any of the cuts γ_j $(1 \le j \le 4)$ or belongs to more than one of the cuts γ_{jk} $(1 \le j < k \le 4)$ (in particular, p can be an arbitrary physical point). Then the domain of analyticity of the reduced Green's function $\widehat{G}(k)$ contains the set $\mathcal{K}\backslash\gamma$, where $\mathcal{K}$ is a complex neighbourhood (in $\mathbf{C}\Pi$) of $\mathcal{P}$.*

■ We fix arbitrarily a point $P \in \mathcal{P}$. The case when P does not belong to any of the cuts γ_{jk} is trivial since P is then in $\mathcal{O}$ (16.53) and hence occurs in $H(D)$ together with the complex domain $\mathcal{N}(\mathcal{O})$ (see (16.54)). We therefore carry out the rest of the argument only for a point $P \in \mathcal{P}$ lying in exactly one cut, say, γ_{12}. We suppose further that $P_1 + P_2 \in V^-$ (the case $P_3 + P_4 = -(P_1 + P_2) \in V^-$ is symmetric). We conclude from Theorem 16.7 that the domain of analyticity of $\widehat{G}(k)$ contains the sets $(\Pi + i\Lambda_{14})\backslash\gamma$, $(\Pi + i\Lambda_{23})\backslash\gamma$. Let $\mathcal{R}$ be a sufficiently small complex neighbourhood of P in $\mathbf{C}\Pi$ such that $\mathcal{R}\backslash\gamma = \mathcal{R}\backslash\gamma_{12}$ and let

$$\mathcal{R}^\pm = \{k \in \mathcal{R} : \pm \operatorname{Im}(k_1 + k_2)^2 > 0\}. \tag{16.69}$$

Then the domain of analyticity of $\widehat{G}(k)$ contains four (disconnected) sets of the form

$$(\Pi \pm i\Lambda_{14}) \cap \mathcal{R}^{\pm}. \tag{16.70}$$

We now show that domains $(\Pi \pm i\Lambda_{14}) \cap \mathcal{R}^{\epsilon}$ with the same sign $\epsilon = \pm$ can be joined together in the domain of analyticity. We consider first the pair of domains $(\Pi + i\Lambda_{14}) \cap \mathcal{R}^{+}$ and $(\Pi - i\Lambda_{14}) \cap \mathcal{R}^{+}$. The first contains $(\Pi + iQ^{r}_{14}) \cap \mathcal{R}^{+}$, where the points $k = p + iq \in (\Pi + iQ^{r}_{14}) \cap \mathcal{R}^{+}$ clearly satisfy the relations $p_1 + p_2 \in V^{-}$ (by virtue of the definition of $\mathcal{R}$) and $q_1 + q_2 \in V^{-}$ (by the definition of Q^{r}_{14}); therefore $\mathrm{Im}(k_1 + k_2)^2 > 0$. Thus

$$(\Pi + i\Lambda_{14}) \cap \mathcal{R}^{+} \supset (\Pi + iQ^{r}_{14}) \cap \mathcal{R}. \tag{16.71a}$$

The inclusion

$$(\Pi - i\Lambda_{14}) \cap \mathcal{R}^{+} \supset (\Pi + iQ^{a}_{41}) \cap \mathcal{R} \tag{16.71b}$$

is proved in similar fashion. Here the point P lies in the (real) domain of coincidence of the boundary values of $\widehat{G}(k)$ on the sides iQ^{r}_{14} and iQ^{a}_{41}. For by hypothesis, P satisfies all the restrictions featuring in the coincidence conditions (16.49), (16.50), except one, namely, $(p_1 + p_2)^2 \equiv (p_3 + p_4)^2 < M^2_{12}$. Excluding the condition involving the threshold mass M^2_{12}, we call the remaining conditions admissible. Then it is clear from the diagram in §16.2.A that we can write down a chain of admissible coincidence conditions for the Green's functions $\widehat{G}^{Z}$ which reduce to the equality

$$\widehat{G}^{r}_{14}(p) = \widehat{G}^{a}_{41}(p) \quad \text{for } p \in \mathcal{R} \cap \Pi. \tag{16.72}$$

By the "edge of the wedge" theorem it follows from (16.71), (16.72) that the domain of analyticity of $\widehat{G}(k)$ contains the domain

$$(\Pi + i(Q^{r}_{14} + Q^{a}_{41})) \cap \mathcal{R} = (\Pi + i(Q^{r}_{14} + Q^{a}_{41})) \cap \mathcal{R}^{+} \tag{16.73}$$

(with a neighbourhood $\mathcal{R}$ of P possibly smaller than that originally chosen). As a result we have combined the domains on the left hand side of (16.71) into the domain (16.73) which is also in the domain of analyticity of $\widehat{G}(k)$. If a tube featured instead of $\mathcal{R}^{+}$, then we could again apply the "edge of the wedge" theorem to the triple of domains so obtained and thus derive the analyticity of $\widehat{G}(k)$ in a neighbourhood of the form $\mathcal{R}^{+}$. To replace $\mathcal{R}^{+}$ by a tube, we carry out a suitable change of variables.

We replace k^0_1 by the new variable $z_1 = (k_1 + k_2)^2$ and for the variables z_j $(2 \leq j \leq 12)$ we choose the remaining coordinates on $C\Pi$. Then if the neighbourhood $\mathcal{R}$ of P is chosen sufficiently small, the map $\phi : k \longrightarrow z$ is biholomorphic and takes real points to real points. Under the map ϕ, $\mathcal{R}^{+}$ is taken to the intersection of $\phi(\mathcal{R})$ with the tube $\mathcal{T}^{Y}$ whose base is $Y = \{y \in R^{12} : y_1 > 0\}$. Furthermore by choosing the neighbourhood $\mathcal{R}$ sufficiently small, we can suppose that ϕ differs from a real inhomogeneous linear map by an arbitrarily small amount; therefore the images of the domains $(\Pi \pm i\Lambda_{14}) \cap \mathcal{R}^{+}$, $(\Pi + i(Q^{r}_{14} + Q^{a}_{41})) \cap \mathcal{R}^{+}$ under the map ϕ contain the intersections of $\phi(\mathcal{R}) \cap \mathcal{T}^{Y}$ with the tubes of the form $\mathcal{T}^{K}$, $\mathcal{T}^{-K}$, $\mathcal{T}^{K'}$, where K and K' are open convex cones in R^{12} with K' having a non-intersection with $K \cap Y$, $(-K) \cap Y$. By construction, the function $\widehat{G}(\phi^{-1}(z))$ is analytic in the intersection of $\phi(\mathcal{R})$ with the tubes $\mathcal{T}^{K \cap Y}$, $\mathcal{T}^{(-K) \cap Y}$ and $\mathcal{T}^{K'} = \mathcal{T}^{K' \cap Y}$. By making $\mathcal{R}$ even smaller if necessary, we can use the "edge of the wedge" theorem (or Proposition 5.34) to conclude that $\widehat{G}(\phi^{-1}(z))$ is analytic in the intersection of $\phi(\mathcal{R})$ with the tube $\mathcal{T}^{Y}$, which is the convex hull of the tubes $\mathcal{T}^{K \cap Y}$ and $\mathcal{T}^{(-K) \cap Y}$.

We have shown that $G(k)$ is analytic in $\mathcal{R}^{+}$. Here we have used the "edge of the wedge" theorem. Taking into account the remark to this theorem in §5.1.D and the commutativity of the operations of a regular change of coordinates and passing to the generalized boundary value, we can make the following conclusion (cf. Proposition B.12): the function $\widehat{G}(k)$ has a generalized boundary value in $\mathcal{R} \cap \Pi$ coinciding with (15.72) and, more generally,

$$\lim_{q \to 0, \, \mathrm{Im}\, s > 0} \widehat{G}(p + iq) = \widehat{G}^{Z}(p) \quad \text{in } \mathcal{D}'(\mathcal{R} \cap \Pi) \tag{16.74a}$$

subject to the condition that

$$Q^{Z} \subset \{q \in \Pi : q_1 + q_2 \in V^{-}\}. \tag{16.74b}$$

(In the derivation of (16.72) we saw that every reduced Green's function $\widehat{G}^Z(p)$ is related to $\widehat{Q}^r_{14}(p)$ in $\mathcal{R} \cap \Pi$ by a chain of admissible coincidence conditions.)

A similar argument applied to the other pair of domains in (16.70) shows that $\widehat{G}(k)$ is analytic in $\mathcal{R}^-$. If the fixed point P does not lie on the threshold itself, that is, if $(P_1 + P_2)^2 > M_{12}^2$, then we have proved the theorem for it (since the intersection of a sufficiently small neighbourhood of P with γ_{12} has the form $\mathcal{R}^+ \cup \mathcal{R}^-$).

It remains to consider the case of a threshold point $(P_1 + P_2)^2 = M_{12}^2$. In this case, $\mathcal{R} \backslash \gamma_{12}$ differs from $\mathcal{R}^+ \cup \mathcal{R}^-$ by points $k \in \mathcal{R}$, for which $(k_1 + k_2)^2$ is real and less than M_{12}^2. The real points k satisfying this condition are in the domain $H(D)$ where $\widehat{G}(k)$ is analytic (see the beginning of the proof of this theorem). The fact that the complex points $k \in \mathcal{R}$ satisfying the condition $(k_1 + k_2)^2 < M_{12}^2$ also belong to $H(D)$ follows (after perhaps decreasing $\mathcal{R}$) from Proposition 5.26 (in which the role of f is played by $(k_1 + k_2)^2$ and the boundary point w of $H(D)$ is P). ■

Theorem 16.8 enables us to relate the amplitudes of two-particle processes with amputated Green's functions on the complex mass shell

$$\mathbf{C}\mathfrak{M} = \{k \in \mathbf{C}\Pi : \ k_j^2 = m_j^2, \ j = 1, \ldots, 4\}. \tag{16.75}$$

Let S, T, U be the physical domains of the processes in the s-, t- and u-channels. By Theorem 16.8 there exist complex neighbourhoods $\mathcal{N}(S)$, $\mathcal{N}(T)$, $\mathcal{N}(U)$ of these sets on $\mathbf{C}\mathfrak{M}$ such that the sets

$$S^{\pm} = \mathcal{N}(S) \cap \{k \in \mathbf{C}\mathfrak{M} : \pm \operatorname{Im} s > 0\}, \tag{16.76a}$$
$$T^{\pm} = \mathcal{N}(T) \cap \{k \in \mathbf{C}\mathfrak{M} : \pm \operatorname{Im} t > 0\}, \tag{16.76b}$$
$$U^{\pm} = \mathcal{N}(U) \cap \{k \in \mathbf{C}\mathfrak{M} : \pm \operatorname{Im} u > 0\} \tag{16.76c}$$

are in the domain $H(D)$ in which the reduced Green's function $\widehat{G}(k)$ is analytic. For definiteness, we take a point $P \in S$ such that $P_1 \in V^-$, $P_2 \in V^-$, $P_3 \in V^+$, $P_4 \in V^+$. It then follows from formulae (16.74) that

$$\lim_{q \to 0, \operatorname{Im} s > 0} \prod_{j=1}^{4} (k_j^2 - m_j^2) G(k) = \lim_{q \to 0, q \in Q_1^a} \prod_{j=1}^{4} (k_j^2 - m_j^2) G_1^a(k) \quad \text{in } \mathcal{D}'(\mathcal{R} \cap \Pi), \tag{16.77}$$

where $\mathcal{R}$ is a sufficiently small complex neighbourhood of P (here we have gone over from the reduced Green's function to the amputated Green's function since the factors $(p_j + p_k)^2 - m_{jk}^2$ in (16.52) do not vanish here).

Corollary 16.9. The reduction formula (16.46) (as before, for Bose particles in a domain of pairwise non-collinear momenta) takes the following form in terms of the analytically continued amputated Green's function:

$$\begin{aligned}\langle 0 | \tilde{\phi}_4^{\text{out}}(p_4) \tilde{\phi}_3^{\text{out}}(p_3) \tilde{\phi}_2^{\text{in}}(p_2) \tilde{\phi}_1^{\text{in}}(p_1) | 0 \rangle &= -i \delta_{m_1}^-(p_1) \delta_{m_2}^-(p_2) \times \\ &\times \delta_{m_3}^+(p_3) \delta_{m_4}^+(p_4) (2\pi)^4 \delta(p_1 + \ldots + p_4) \lim_{q \to 0, \operatorname{Im} s > 0} \prod_{j=1}^{4} (k_j^2 - m_j^2) G(k).\end{aligned} \tag{16.78}$$

Here, as in the derivation of (16.74a), we can use any regular change of coordinates in a neighbourhood of p for passing to the generalized boundary value. In particular, we can choose s, t, p_j^2 $(j = 1, \ldots, 4)$ for the first six local coordinates; the factors $\delta_{m_j}^{\pm}(p_j)$ then mean that $k_j^2 = m_j^2$ is fixed, that is, we have transferred to the amputated Green's function on the complex mass shell $\mathbf{C}\mathfrak{M}$; the limit in (16.78) then becomes the limit as $\operatorname{Im} k \to 0$, $k \in S^+$.

In similar fashion the amplitudes of the t- and u-channels are related to the boundary values of the amputated Green's function as $\operatorname{Im} k \to 0,\ k \in T^+$ and $\operatorname{Im} k \to 0,\ k \in U^+$ respectively.

16.3. Crossing Relation

A. Statement of the result

The domains of analyticity $S^\pm$, $T^\pm$, $U^\pm$ of the amputated Green's function on the mass shell form an open subset of $\mathbb{C}\mathfrak{M}$ and the physical amplitudes are (to within a factor) different generalized boundary values of this function. In this situation it only makes sense to talk about a single analytic function if these domains are related to one another in the sense that they are parts of some larger domain of analyticity of the amputated Green's function on the complex mass shell. It turns out that this is in fact true and that the amplitudes of the s-, t- and u-channels are related by analytic continuation on the mass shell. As was remarked in §16.2.B, the crossing relation is involved here.

Information of crossing type is contained in the dispersion relation. If the dispersion relation holds for fixed $t < 0$, then the amplitude $T(s,t)$ is analytic in the complex s-plane with two cuts $(-\infty, s_1]$ and $[s_2 + \infty)$ (where $s_1 < s_2$), where the physical amplitudes T_s and T_u of the s- and u-channels are different boundary values of $T(s,t)$ at the right upper and left lower cuts in the s-plane respectively. Thus there is a path joining T_s and T_u.

In the general case a somewhat weaker property can be proved: the physical amplitude of the s-channel is for any fixed $t < 0$ the boundary value of a function $T(s,t)$ that is analytic in s in the upper half-plane excluding a finite domain which depends on t *

More precisely, we have the following result due to Bros et al. (1965).

Theorem 16.10. *The invariant amplitude of the two-particle process $T(s,t)$ is analytic in a domain Ω_{su} (on the complex mass shell $\mathbb{C}\mathfrak{M}$) which is a complex neighbourhood in $\mathbb{C}\mathfrak{M}$ of the set*

$$\{k \in \mathbb{C}\mathfrak{M} :\ t < 0,\ \operatorname{Im} s > 0,\ |s| > R(t)\}, \tag{16.79}$$

where $R(t)$ is a positive number (continuously dependent on t).

The domain Ω_{su} connects S^+ and U^- in $\mathbb{C}\mathfrak{M}$. A similar consideration (under the substitution $s \leftrightarrow u$) gives a complex neighbourhood Ω_{us} of the set $\{k \in \mathbb{C}\mathfrak{M} :\ t < 0,\ \operatorname{Im} u > 0,\ |u| < R(t)\}$ on the mass shell, in which the amputated Green's function is analytic. By changing the channels we obtain complex neighbourhoods Ω_{st}, Ω_{ts}, Ω_{ut}, Ω_{tu} connecting the corresponding pairs in the set of domains $S^\pm$, $T^\pm$, $U^\pm$. As a result, the physical amplitudes turn out to be different boundary values of a single analytic function defined in the domain in $\mathbb{C}\mathfrak{M}$ that is the union of $S^\pm$, $T^\pm$, $U^\pm$, Ω_{su}, Ω_{us}, Ω_{st}, Ω_{ts}, Ω_{ut}, Ω_{tu}.

The proof of Theorem 16.10 repeats the line of argument in §15.2 in the derivation of the dispersion relations (the differences are largely due to the fact that now we

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* The corresponding integral representation of Cauchy type expressing the amplitude in terms of its values on the boundaries of Ω_{su} and Ω_{us} is called a *quasi-dispersion relation*.

are handling a different object for extending the amplitudes beyond the mass shell, namely, the amputated Green's function). As in §15.2 we confine attention to the case of scalar fields (that is, spinless particles).

By analogy with §15.2.B we begin (see §16.3.B) by fixing the momenta k_1, k_3 so that

$$k_1, k_3 \in \mathbf{M},\ k_1^2 < M_1^2,\ k_3^2 < M_3^2,\ t \equiv (k_1 + k_3)^2 < 0. \tag{16.80}$$

In this way we distinguish in $\mathbf{C}\Pi$ a four-dimensional complex plane L_0 with k_2 as the independent coordinate. It turns out that points with $\operatorname{Im} k_2 \in V^{\pm}$ are in the domain of analyticity of the amputated Green's function. Hence (as in §15.2.C) it follows that for fixed t the amputated Green's function is analytic in s in the upper half-plane for certain non-physical (negative) values of the "squares of the masses"

$$\zeta_j = k_j^2,\ j = 1, \ldots, 4. \tag{16.81}$$

The next stage (§16.3.C) consists in the analytic continuation with respect to the parameters ζ_j. But now (in contrast to §15.2.D) we do not introduce the absorptive part of the amplitude; instead by analogy with §15.2.D, we now fix the momentum

$$k_1 + k_2 = \sigma, \tag{16.82}$$

which is a real timelike vector. We also fix the momentum

$$k_1 + k_3 = \tau, \tag{16.83}$$

which is a spacelike vector. Formulae (16.82), (16.83) distinguish in $\mathbf{C}\Pi$ a four-dimensional complex plane with independent variable k_2. Generally speaking, this plane does not intersect the envelope of holomorphy of the primitive domain of analyticity (since the points satisfying (16.82) for $\sigma^2 \geq M_{12}^2$ lie on the "analytic cut" with respect to γ_{12}). Nevertheless, the intersection of this plane with the boundary of the domain of analyticity (on the side $\operatorname{Im}(k_1 + k_2) \in V^+$) contains a sufficiently extensive manifold. The holomorphic extension (by the JLD method) of the manifolds to the boundary enables us to obtain a domain of analyticity with respect to s, ζ which intersects the mass shell. A symmetric discussion of the u-channel (for fixed $k_1 + k_3$ and $k_1 + k_4$) gives an analogous domain in the variables u, ζ.

Finally in §16.3.D we use the holomorphic extension of the union of the domains (with respect to the variables s, ζ for fixed t) obtained at the second stage to construct the domain Ω_{su} on the mass shell.

B. The case of imaginary "masses"

Taking into account Lorentz-invariance (see the remark in §16.2.B) we express the reduced Green's function in terms of the invariants

$$\widehat{G}(k) = \hat{g}(s, t, \zeta_1, \ldots, \zeta_4). \tag{16.84}$$

The mass shell corresponds to the parameters $\zeta_j = m_j^2$. Outside the mass shell it suffices to assume that just one of these parameters is independent, say $\zeta = \zeta_2 - m_2^2$, by setting

$$\zeta_j = \zeta + m_j^2 \quad (j = 1, \ldots, 4). \tag{16.85}$$

In this case we write $\hat{g}(s,t,\zeta)$ instead of (16.84).

Lemma 16.11. *The reduced Green's function $\hat{g}(s,t,\zeta)$ is analytic in a complex neighbourhood of the set*

$$\{(s,t,\zeta) \in \mathbf{C}^3 : \ t < 0, \ \operatorname{Im} s > 0, \ \zeta < Z(t)\}, \tag{16.86}$$

where $Z(t)$ is a negative continuous function of t.

■ We distinguish the complex plane L_0 in $\mathbf{C}\Pi$ as indicated in (16.80). We verify that the points of the manifold

$$L_0^+ = \{k \in L_0 : \ \operatorname{Im} k_2 \in V^+\}$$

belong to the domain of analyticity $H(D)$ of the reduced Green's function. For this purpose we use the modified continuity principle (§5.2.C). Let L_t be the plane in $\mathbf{C}\Pi$ obtained from L_0 by replacing k_1 and k_3 by $k_1 + ite_0$ and $k_3 + ite_0$, where $t > 0$. Then the manifolds $L_t^+ = \{k \in L_t : \ \operatorname{Im} k_2 \in V^+\}$ lie in the primitive domain of analyticity D (more precisely, in the tube $\Pi + iQ_4^a$). Furthermore, L_0^+ has a non-empty intersection with D since D contains complex neighbourhoods of all points in $\Pi\backslash\gamma$ (see Theorem 16.8). Since $L_t^+ \to L_0^+$ as $t \to +0$, it follows from Theorem 5.27 (with $k = 4$) that $L_0^+ \subset H(D)$.

We suppose additionally that k_1 and k_3 lie in a spacelike plane and are non-collinear. This imposes on ζ_1, ζ_3 and t the conditions

$$\zeta_1 < 0, \ \zeta_3 < 0, \ t < 0, \tag{16.87a}$$

$$|\sqrt{-\zeta_1} - \sqrt{-\zeta_3}| < \sqrt{-t} < \sqrt{-\zeta_1} + \sqrt{-\zeta_3}. \tag{16.87b}$$

We encountered a similar situation in §15.2.B where we proved (see Lemma 15.2) that the image of L_0^+ under a map into the space of Lorentz-invariants contains all points that satisfy (15.64), (15.65) in addition to the conditions (16.87). By imposing the condition (16.85), we obtain the statement of the lemma. ■

C. ANALYTIC CONTINUATION WITH RESPECT TO THE MASS VARIABLES

We now fix a real spacelike vector τ with $t = \tau^2 < 0$, so that equation (16.83) distinguishes in $\mathbf{C}\Pi$ an eight-dimensional plane l with independent variables k_1, k_2. The plane l contains several tubes lying in the domain D (16.66). Thus let the permutation (j,k,l,n) of the indices (1,2,3,4) be such that

$$\text{either } \{j,k\} = \{1,3\}, \quad \text{or } \{j,k\} = \{2,4\}$$

(there are, of course, eight such permutations). Since D contains the sets (16.65c) and γ_{jk} is not intersected by the plane l, the intersections of the cones $\Pi + i(Q_{jn}^r + Q_{kl}^a)$ with l are contained in D. As a result we conclude that $D \cap l$ contains eight cones which (by (16.45)) have the form

$$\{k \in \mathbf{C}\Pi : \ k_1 + k_3 = \tau, \ q_j \in V^+, \ q_j + q_l \in V^-\}. \tag{16.88}$$

None of the variables ζ_j can take the physical value m_j^2 inside such a tube (since $q_1, \ldots, q_4 \in V^+ \cup V^-$). To attain physical values of the masses it is necessary to consider at least a pair of such tubes with opposite signs of the zeroth components of the vectors $q_1, \ldots, q_4$ and these tubes must be joined on the plane (16.83) by part of the domain of analyticity.

Consider for definiteness the two tubes

$$A = \{k \in \mathbf{C}\Pi : \ k_1 + k_3 = \tau, \ q_1 \in V^-, \ q_1 + q_2 \in V^+\}, \tag{16.89a}$$

$$B = \{k \in \mathbf{C}\Pi : \ k_1 + k_3 = \tau, \ q_2 \in V^-, \ q_1 + q_2 \in V^+\} \tag{16.89b}$$

on the plane l corresponding to the permutations (j,k,l,n) in the form (3,1,2,4) or (4,2,1,3). Their convex hull is clearly

$$l^+ = \{k \in \mathbf{C}\Pi : \ k_1 + k_3 = \tau, \ q_1 + q_2 \in V^+\}.$$

By Theorem 16.8, there is a complex neighbourhood $\mathcal{N}$ (on the plane l) of the set

$$\{p \in \Pi : \ p_1 + p_3 = \tau, \ p_j^2 < M_j^2, \ j = 1, \ldots, 4, u < M_{14}^2\} \tag{16.90}$$

such that the domain $\mathcal{N}^+ = \mathcal{N} \cap l^+$, and hence (see Exercise 5.27(d)) its envelope of holomorphy $H(A \cup B \cup \mathcal{N}^+)$ (on the plane l) lie in $H(D)$. By carrying out the holomorphic extension of $A \cup B \cup \mathcal{N}^+$ we can prove the following result.

Lemma 16.12. *The reduced Green's function $\hat{g}(s,t,\zeta)$ is analytic in the intersection of a complex neighbourhood of the set*

$$\{(s,t,\zeta) \in \mathbf{C}^3 : \ t < 0, \ s > s_0(t), \ | \mathop{arg} (\Delta - \zeta)| < \vartheta(t)\} \tag{16.91}$$

with the half-space $\operatorname{Im} s > 0$*; here $s_0(t)$ and $\vartheta(t)$ are continuous functions of t and*

$$\Delta = \min_{j=1,\ldots,4} (M_j^2 - m_j^2). \tag{16.92}$$

If we use the technique of analytic continuation on the boundary (see Exercise 5.30), then it becomes considerably easier to obtain the necessary information concerning $H(A \cup B \cup \mathcal{N}^+)$. We fix the vector τ and a positive timelike vector σ such that

$$\sigma\tau = \tfrac{1}{2}(m_1^2 - m_2^2 - m_3^2 + m_4^2); \tag{16.93a}$$

for definiteness, we take τ along the e_3 axis and σ in the plane of the vectors e_0, e_3 of the Lorentz frame:

$$\tau = (0,0,0,\sqrt{-t}), \quad \sigma = \left(\left(s - \frac{(m_1^2 - m_2^2 - m_3^2 + m_4^2)^2}{4t}\right)^{1/2}, \right.$$
$$\left. 0, 0, \frac{-m_1^2 + m_2^2 + m_3^2 - m_4^2}{2\sqrt{-t}}\right). \tag{16.93b}$$

We define the four-dimensional complex plane l' in $\mathbf{C}\Pi$ by the equations (16.82), (16.83). Let A', B' and $\mathcal{N}'$ be the parts of the tubes (16.89) and the domain $\mathcal{N}$ lying in l'. Since the domain $A \cup B \cup \mathcal{N}^+$ is clearly adjacent to $A' \cup B' \cup \mathcal{N}'$ on the side iQ, where $Q = \{q \in \Pi : \ q_1 + q_3 = 0, \ q_1 + q_2 \in V^+\}$, the same is true (by Exercise 5.30) on going over to the envelope of holomorphy. This means that

$$H(A \cup B \cup \mathcal{N}^+) \supset l^+ \cap \mathcal{L}(H(A' \cup B' \cup \mathcal{N}')), \tag{16.94}$$

where $\mathcal{L}(H(A' \cup B' \cup \mathcal{N}'))$ is a complex neighbourhood of $H(A' \cup B' \cup \mathcal{N}')$ on the plane l.

The problem is reduced to constructing a holomorphic extension of $A' \cup B' \cup \mathcal{N}'$. The only independent variable on the plane l' is k_1. In order to decrease the number of independent components, we impose the extra restriction (which is weaker than (16.85))

$$\zeta_1 - \zeta_3 = m_1^2 - m_3^2, \tag{16.95}$$

which (by the equality $k_1 + k_2 = \tau$) means that the projection of k_1 on τ is fixed:

$$k_1^3 = -k_1\tau/\sqrt{-t} = (-m_1^2 + m_3^2 + t)/2\sqrt{-t}. \tag{16.96}$$

Moreover, the projection of k_2 on τ is automatically fixed from (16.93), (16.96):

$$k_2^3 = -k_2\tau/\sqrt{-t} = (m_2^2 - m_4^2 - t)/2\sqrt{-t}, \tag{16.97}$$

whence it follows that

$$\zeta_2 - \zeta_4 = m_2^2 - m_4^2. \tag{16.98}$$

In what follows, $\tilde{l}, \tilde{A}, \widetilde{B}, \widetilde{\mathcal{N}}$ denote the intersections of the sets $l', A', B', \mathcal{N}'$ with the complex hyperplane (16.96) and the triple $\tilde{k}_j \equiv (k_j^0, k_j^1, k_j^2)$ gives the first three components of the vectors k_j ($j = 1, \dots, 4$). The intersection of $\tilde{l}$ with the mass shell is characterized by the additional equations $k_1^2 = m_1^2$, $k_2^2 = m_2^2$, or, equivalently,

$$(\tilde{k}_1)^2 = \mu_1^2(t), \quad (\tilde{k}_2)^2 = \mu_2^2(t), \tag{16.99}$$

where

$$\mu_1^2(t) = m_1^2 - \frac{(m_1^2 - m_3^2 - t)^2}{4t}, \quad \mu_2^2(t) = m_2^2 - \frac{(m_2^2 - m_4^2 - t)^2}{4t}. \tag{16.100}$$

As the independent variable on $\tilde{l}$ we choose the 3-vector

$$\tilde{r} = \tilde{k}_1 - \frac{\mu_1^2(t) - \mu_2^2(t) + (\sigma^0)^2}{2\sigma^0}\tilde{e}_0, \tag{16.101}$$

in terms of which, the equation $k_1^2 - k_2^2 = m_1^2 - m_2^2$ takes on the simple form:

$$\tilde{r}^0 = 0. \tag{16.102}$$

In terms of the variable $\tilde{r}$, the tubes $\tilde{A}$ and $\widetilde{B}$ have the form $\mathbf{R}^3 - i\widetilde{V}^+$ and $\mathbf{R}^3 + i\widetilde{V}^+$, where $\widetilde{V}^+$ is the upper light cone in three-dimensional space-time and $\widetilde{\mathcal{N}}$ is a complex neighbourhood (in $\tilde{l}$) of the complement (in $\operatorname{Re}\tilde{l}$) of the real closed set

$$\Big\{k \in \operatorname{Re}\tilde{l}:\ \tilde{r} \in \Big(\frac{-\mu_1^2(t) + \mu_2^2(t) + (\sigma_0)^2}{2\sigma_0}\tilde{e}_0 + \overline{\widetilde{V}}{}^+_{\mathcal{M}_1(t)}\Big) \cup \cup\Big(\frac{-\mu_1^2(t) + \mu_2^2(t) + (\sigma_0)^2}{2\sigma_0}\tilde{e}_0 - \overline{\widetilde{V}}{}^+_{\mathcal{M}_2(t)}\Big)\Big\}, \tag{16.103}$$

where

$$\mathcal{M}_1^2(t) = \Big(M_1^2 - \frac{(m_1^2 - m_3^2 - t)^2}{4t}\Big) \wedge \Big(M_3^2 - \frac{(m_3^2 - m_1^2 - t)^2}{4t}\Big), \tag{16.104a}$$

$$\mathcal{M}_2^2(t) = \Big(M_2^2 - \frac{(m_2^2 - m_4^2 - t)^2}{4t}\Big) \wedge \Big(M_4^2 - \frac{(m_4^2 - m_2^2 - t)^2}{4t}\Big). \tag{16.104b}$$

In fact, the condition $(p_1)^2 < \mathcal{M}_1^2(t)$ corresponds to the pair of inequalities $p_1^2 < M_1^2$, $p_3^2 < M_3^2$ in (16.90), whereas the condition $(-\tilde{p}_1 + \tilde{\sigma})^2 < \mathcal{M}_2^2(t)$ replaces the inequalities $p_2^2 < M_2^2$, $p_4^2 < M_4^2$. We suppose further that s is sufficiently large, for example, such that

$$\sigma^0 > \mathcal{M}_1(t) + \mathcal{M}_2(t); \tag{16.105}$$

this enables us to leave out in (16.103) the condition $(2\tilde{p}_1 - \tilde{\sigma})^2 < M_{14}^2 - \frac{1}{4t}(m_1^2 + m_2^2 - m_3^2 - m_4^2)^2$, corresponding to the inequality $u < M_{14}^2$ in (16.90).

It is easy to see that (16.103) is contained in the set

$$\Big\{k \in \operatorname{Re}\tilde{l} : \ \tilde{r} \in \Big(-\frac{|\mu_1^2(t) - \mu_2^2(t)| + (\sigma_0)^2}{2\sigma_0}\tilde{e}_0 + \overline{\tilde{V}}{}^+_{\mathcal{M}(t)}\Big) \cup \cup \Big(\frac{|\mu_1^2(t) - \mu_2^2(t)| + (\sigma_0)^2}{2\sigma_0}\tilde{e}_0 - \overline{\tilde{V}}{}^+_{\mathcal{M}(t)}\Big)\Big\} \tag{16.106}$$

if we set

$$\begin{gathered} \mathcal{M}^2(t) = \Delta + (\mathcal{M}_1^2(t) \vee \mathcal{M}_2^2(t)), \\ \Delta = (\mathcal{M}_1^2(t) - \mu_1^2(t)) \wedge (\mathcal{M}_2^2(t) - \mu_2^2(t)) = \min_{1 \le j \le 4}(M_j^2 - m_j^2). \end{gathered} \tag{16.107}$$

In fact, supposing for definiteness that $\mu_1(t) \ge \mu_2(t)$, we have $\mathcal{M}^2(t) = \mu_1^2(t) + \Delta$, so that $\mathcal{M}(t) \le \mathcal{M}_1(t)$ and hence the first of the sets in (16.103) is contained in the first of the sets in (16.106). To prove that the second of the hyperboloids in (16.103) is contained in (16.106), it suffices to verify that the radius ρ of the ball in which the second of the hyperboloids in (16.103) intersects the plane $\tilde{r}^0 = 0$ does not exceed the radius ρ' of the ball in which the hyperboloids in (16.106) intersect the same plane. It is not difficult to see that the inequality $\rho \le \rho'$ is ensured by the fact that $\mathcal{M}^2(t)$ $(= \mu_1^2(t) + \Delta) \le \mathcal{M}_2^2(t) + \mu_1^2(t) - \mu_2^2(t)$.

If we replace the domain $\tilde{\mathcal{N}}$ by a (generally speaking) smaller one that is a neighbourhood (in $\tilde{l}$) of the complement (in $\operatorname{Re}\tilde{l}$) of the set (16.106), then we arrive at a symmetric problem of JLD type. According to Proposition 5.44,* the corresponding envelope of holomorphy is

$$\{k \in \tilde{l} : (\tilde{r})^2 - (\tilde{r}^1 - v^1)^2 - (\tilde{r}^2 - v^2)^2 \notin [\kappa^2(|v|), \infty) \quad \text{for all } v \in \mathbf{R}^2 \text{ with } |v| \le R(s,t)\}; \tag{16.108}$$

here

$$\begin{gathered} R(s,t) = \frac{1}{2\sigma_0}(|\mu_1^2(t) - \mu_2^2(t)| + (\sigma_0)^2), \\ \kappa(|v|) = 0 \vee (\mathcal{M}(t) - (R^2(s,t) - |v|^2)^{1/2}). \end{gathered}$$

It suffices for our purposes to restrict our attention to points $\tilde{r}$ with $\tilde{r}^0 = \tilde{r}^2 = 0$; such points of the set (16.108) are defined by the condition

$$-(\tilde{r}^1 - v^1)^2 \notin [\kappa^2(|v|) + (v^2)^2, \infty) \quad \text{for } v \in \mathbf{R}^2, \ |v| \le R(s,t),$$

or, equivalently,

$$-(\tilde{r}^1 - x)^2 \notin [\kappa^2(|x|), \infty) \quad \text{for } x \in \mathbf{R}, \ |x| \le R(s,t). \tag{16.109}$$

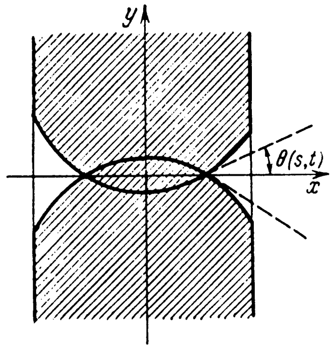
This condition clearly means that

$$\tilde{r}^1 \ne x + iy \tag{16.110}$$

for all $x, y \in \mathbf{R}$, $|x| \le R(s,t)$, $|y| \ge \kappa(|x|)$.

* Although this result was stated there as applied to four-dimensional space-time, it goes over to the case of space-time of any dimension (in particular, equal to three).

The variables x, y range over the (shaded) set, illustrated in the diagram below, bounded by the vertical lines $x = \pm R(s,t)$ and the circular arcs $x^2 + (y \pm \mathcal{M}(t))^2 = R^2(s,t)$.



We shall suppose that s is large enough for the circles to intersect the line $y = 0$. Taking the earlier restriction (16.105) into account, we suppose that $s > s_0(t)$, where $s_0(t)$ is chosen so that the value σ_0 corresponding to it (see (16.93b)), which we denote by $\sigma_0(t)$, satisfies the condition

$$\sigma_0(t) > 2\mathcal{M}(t).$$

Then the domain over which $\tilde{r}^1$ varies in (16.110) contains at least the angle

$$|\arg(\tilde{r}^1 - \rho(s,t))| < \theta(s,t), \tag{16.111}$$

where

$$\rho(s,t) > 0,\ \rho^2(s,t) = R^2(s,t) - \mathcal{M}^2(t) = \left(\frac{\mu_1^2(t) - \mu_2^2(t) + \sigma_0}{2\sigma_0}\right) - \mu_1^2 - \Delta$$

and $\theta(s,t) > 0$ increases as s increases; therefore there exists $\vartheta(t)$ such that $0 < \vartheta(t) < \theta(s,t)$ for $s > s_0(t)$.

As a result we conclude that some holomorphic extension of the domain $\tilde{A} \cup \tilde{B} \cup \tilde{\mathcal{N}}$ contains the points of the form

$$\left\{k \in l' :\ k_1 = \left(\frac{1}{2\sigma_0}(\mu_1^2(t) - \mu_2^2(t) + \sigma_0^2),\ \tilde{r}^1, 0, \frac{m_3^2 - m_1^2 + t}{2\sqrt{-t}}\right)\right\}, \tag{16.112}$$

where $\tilde{r}^1$ ranges through the angle defined in (16.111). It is easy to see that here $(\tilde{r}^1)^2$ ranges over the domain containing the angle $\{z \in \mathbf{C} :\ |\arg(z - \rho^2(s,t))| < \vartheta(t)\}$. Accordingly for the points of the set (16.112), the quantity $\zeta_2 = k_2^2 = k_1^2 + m_2 - m_1^2$ ranges over a domain containing the angle $|\arg(m_2^2 + \Delta - \zeta_2)| < \vartheta(t)$; this proves that $H(A' \cup B' \cup \mathcal{N}')$ contains the points of the form (16.91). This conclusion together with (16.94) completes the proof of Lemma 16.12.

By interchanging the roles of the momenta k_2 and k_4 we can prove the following result in precisely the same way (where $u_0(t)$ is a positive continuous function of t).

Lemma 16.13. *The reduced Green's function $\hat{g}(s,t,\zeta)$ is analytic in the intersection of a complex neighbourhood of the set*

$$\{(s,t,\zeta) \in \mathbf{C}^3 :\ t < 0,\ u > u_0(t),\ |\arg(\Delta - \zeta)| < \vartheta(t)\} \tag{16.113}$$

with the half space $\operatorname{Im} u < 0$.

Since $\vartheta(t)$ has not been specified here, it can be considered to be the same as in the preceding lemma (in other words, we choose for $\vartheta(t)$ the smaller of the two values in these lemmas).

D. PASSAGE TO THE MASS SHELL

Lemmas 16.12 and 16.13 contain information on the analyticity of $\hat{g}(s,t,\zeta)$ on the mass shell ($\zeta_2 = m_2^2$) in the s- and u-channels respectively. But in this form these domains are not joined to each other. On the other hand, for non-physical ζ there is a domain of analyticity in the s-plane joining the positive and negative semi-axes. This suffices to prove Theorem 16.10 (and hence the crossing property).

To this end we carry out a holomorphic extension of the domain so obtained with respect to the variables s, ζ for fixed $t < 0$. For $\operatorname{Im} s > 0$ and for ζ in the domain

$$|\arg(\Delta - \zeta)| < \vartheta(t), \ \zeta \notin (-\infty, Z(t)], \tag{16.114}$$

we go over to the new variables

$$s' = \ln(s - s_0(t)), \ \zeta' = \ln(w + \sqrt{w^2 - 1}), \quad \text{where } w = 2\Big(\frac{\Delta - \zeta}{\Delta - Z(t)}\Big)^{\pi/\vartheta(t)} - 1. \tag{16.115}$$

The map $\zeta \to w$ takes the domain (16.114) into the complex plane with cuts along the semi-axes $(-\infty, -1]$ and $[1, \infty)$; under the map $w \to w + \sqrt{w^2 - 1}$ this new domain is taken into the upper half-plane, which in turn is taken into the semi-axis $0 < \operatorname{Im} \zeta' < \pi$ on going over to the variable ζ'. Note that under the map $\zeta \to \zeta'$, points close to the cut $(-\infty, Z(t)]$ are taken to the part of the neighbourhood of the real line lying in the upper half-plane. We denote by $e^{i\phi}$ $(0 < \phi < \pi)$ the image of the physical point $\zeta = 0$ under the map $\zeta \to \zeta'$.

It follows from Lemmas 16.11 and 16.12 that the domain of analyticity of $\hat{g}(s,t,\zeta)$ in the variables s', ζ' contains the part of the neighbourhood (in $\mathbf{C}^2$) of the tube

$$0 < \operatorname{Im} s' < \pi, \quad \operatorname{Im} \zeta' = 0,$$

lying in the half-space $\operatorname{Im} \zeta' > 0$ and the part of a neighbourhood of the tube

$$\operatorname{Im} s' = 0, \quad 0 < \operatorname{Im} \zeta' < \pi,$$

lying in the half-space $\operatorname{Im} s' > 0$. According to Exercise 5.7, $\hat{g}(s,t,\zeta)$ is also analytic in the tubes

$$0 < \operatorname{Im} s' < \psi, \quad 0 < \operatorname{Im} \zeta' < \pi - \psi$$

for all $0 < \psi < \pi$. Here we restrict attention to some fixed value of $\psi < \pi - \phi$. In terms of the original variables we find that $\hat{g}(s,t,\zeta)$ is analytic in the direct product of the angle $0 < \arg(s - s_0(t)) < \psi$ and the set

$$W = \{\zeta \in \mathbf{C} : \ a(t) < \operatorname{Re} \zeta < b(t), \ |\operatorname{Im} \zeta| < \tfrac{1}{4}\delta(t), \ \zeta \notin (a(t), Z(t))\}, \tag{16.116}$$

which is a rectangle in the ζ-plane with the interval $(a(t), Z(t)]$ removed. Here $\delta(t) >$ > 0 and the numbers $a(t)$ and $b(t)$ are such that $a(t) < Z(t)$, $0 < b(t) < \Delta$.

Similarly we obtain the domain of analyticity in the variables u, ζ using Lemmas 16.11 and 16.13 (instead of 16.12). The following lemma combines both these results.

Lemma 16.14. *For any $t < 0$ the reduced Green's function $\hat{g}(s,t,\zeta)$ is analytic in the union of the domains*

$$\{(s,\zeta) \in \mathbf{C}^2 : 0 < \arg(s - s_0(t)) < \psi,\ \zeta \in W\}, \tag{16.117a}$$

$$\{(s,\zeta) \in \mathbf{C}^2 :\ \pi - \psi < \arg(u_0(t) - u) < \pi,\ \zeta \in W\}. \tag{16.117b}$$

We recall that here $u = -s - t + \sum_{j=1}^{4}(\zeta + m_j^2 - m_2^2)$.

It is convenient to replace the rectangle featuring in (16.116) by a somewhat smaller set, namely, the region between the two symmetrically placed arcs joining the points $a(t)$ and $b(t)$ in $\overline{W}$ and lying (with the end points $a(t)$ and $b(t)$ excluded) in the upper and lower half-planes respectively. We set

$$\zeta'' = \ln(w' + \sqrt{w'^2 - 1}), \quad \text{where } w' = 2\Big(\frac{b(t) - \zeta}{b(t) - Z(t)}\,\frac{\zeta - a(t)}{Z(t) - a(t)}\Big)^{\pi/\vartheta'(t)} - 1;$$

$2\vartheta'(t)$ is the angle between the arcs at the points of intersection $a(t)$ and $b(t)$. As a result of this construction, the points ζ for which $0 < \operatorname{Im}\zeta'' < \pi$ lie in the domain W. The values of ζ'' corresponding to the points $\zeta = 0$ are denoted by $e^{i\phi'}$ $(0 < \phi' < \pi)$.

Now it is not difficult to see that for sufficiently large $c(t) > 0$, the domains (16.117) contain the subdomains

$$\{(s,\zeta) :\ 0 < \arg(s - c(t) - i\delta(t)) < \psi,\ 0 < \operatorname{Im}\zeta'' < \pi\} \tag{16.118a}$$

$$\{(s,\zeta) : \pi - \psi < \arg(s + c(t) - i\delta(t)) < \pi,\ 0 < \operatorname{Im}\zeta'' < \pi\} \tag{16.118b}$$

respectively. We introduce the variable

$$s'' = -(s - i\delta(t))^{-1};$$

then (16.118) can be written in the form

$$\{(s,\zeta) \in \mathbf{C}^2 :\ s'' \in G^{(1)},\ \ 0 < \operatorname{Im}\zeta'' < \pi\}, \tag{16.119a}$$

$$\{(s,\zeta) \in \mathbf{C}^2 :\ s'' \in G^{(2)},\ \ 0 < \operatorname{Im}\zeta'' < \pi\}, \tag{16.119b}$$

where $G^{(1)}$ and $G^{(2)}$ are two domains in the upper half-plane $\operatorname{Im} s'' > 0$ adjacent to the intervals $(-1/c(t), 0)$ and $(0, 1/c(t))$ respectively.

The domains of analyticity (16.119) in the s- and u-channels are still not joined to each other. To join them, we use Lemma 16.11 once more, according to which, $\hat{g}(s,t,\zeta)$ (for fixed $t < 0$) is analytic in any domain of the form

$$\{(s,\zeta) \in \mathbf{C}^2 :\ s'' \in G_r,\ 0 < \operatorname{Im}\zeta'' < y(r)\} \tag{16.119c}$$

for any $r > 0$; here $G_r = \{s'' \in \mathbf{C} :\ |s''| > r,\ \operatorname{Im} s'' > 0\}$, and $y(r)$ is a sufficiently small positive number. It remains to carry out the holomorphic extension uniting all the domains (16.119). Since this union is a semitubular domain, our holomorphic extension has the form

$$\{(s,\zeta) \in \mathbf{C}^2 :\ \operatorname{Im} s'' > 0,\ 0 < \operatorname{Im}\ \zeta'' < Y(s'')\}$$

in view of Proposition 5.35; here $Y(s'')$ is a non-negative superharmonic function in the upper half-plane which (at any rate) for $s'' \in G^{(1)} \cup G^{(2)}$ is equal to π. Hence we can conclude that

$$\lim_{s'' \to 0,\, \operatorname{Im} s'' > 0} Y(s'') = \pi. \tag{16.120}$$

Exercise 16.12. Prove (16.120). [Hint: Apply the inequality (5.39) to the function $-Y(s'')$, supposing that the radius of the disc is equal to R and the centre is at the point $i(R+\epsilon)$ ($\epsilon > 0$). By letting R tend to infinity and ϵ to zero, derive the inequality

$$Y(s'') \geq \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{Im}\frac{Y(x+i\epsilon)}{x-s''+i\epsilon}dx \geq \int_{-1/c}^{1/c} \operatorname{Im}\frac{1}{x-s''}dx.$$

Using this and the fact that $Y(s'') \leq \pi$, obtain (16.120).]

It follows from (16.120) that for any value of $y \in (0,\pi)$ there exists a sufficiently small number $r = r(y) > 0$ such that the domain of analyticity for $\hat{g}(s,t,\zeta)$ contains the domain

$$\{(s.\zeta) \in \mathbf{C}^2 : \; s'' \in G_r, \; 0 < \operatorname{Im}\zeta'' < y\}. \tag{16.121}$$

Here we have chosen $y > \phi'$. By choosing points with $\zeta = m_2^2$ (that is, with $\zeta'' = e^{i\phi'}$) in (16.121) and (16.117) we see that (for any fixed $t < 0$) the domain of analyticity of the reduced Green's function $\hat{g}(s,t,m_2^2)$ (and with it the amputated Green's function) contains the points of the form (16.79). This completes the proof of Theorem 16.10 and also the crossing property.

Appendix J. The Role of Unitarity

J.1. Partial wave decomposition of the scattering amplitude of a two-particle process

Let B be an arbitrary bounded Poincaré-invariant operator in the two-particle subspace $\mathfrak{H}_2^{[\kappa_1,\kappa_2]}$ (of Fock space) corresponding to the elastic process

$$\kappa_1 + \kappa_2 \to \kappa_1 + \kappa_2 \tag{J.1}$$

of two spinless particles. According to §7.3.F, the operator B can be characterized in the domain $s > s_{\text{phys}} \equiv (m_1+m_2)^2$ by the generalized function $B(s,z) \in \mathcal{S}'((s_{\text{phys}},\infty)\times[-1,1])$:

$$\langle 0|A_2(p_2')A_1(p_1')BA_1^*(p_1)A_2^*(p_2)|0\rangle = \\ = (2\pi)^4\delta(p_1+p_2-p_1'-p_2')\delta_{m_1}^+(p_1)\delta_{m_2}^+(p_2)\delta_{m_1}^+(p_1')\delta_{m_2}^+(p_2')B(s,z); \tag{J.2}$$

here the variable

$$z = 1 + t/2k^2(s) \tag{J.3}$$

is the cosine of the scattering angle in the centre-of-mass frame:

$$z = \cos\theta \equiv \mathbf{n}\mathbf{n}';$$

$k(s) \equiv K_{12}(s)$ is the length of the three-dimensional momentum of the particle in this frame. Let

$$N = \begin{cases} 1 & \text{for } \kappa_1 \neq \kappa_2, \\ 2 & \text{for } \kappa_1 = \kappa_2. \end{cases} \tag{J.4}$$

We expand $\frac{k(s)}{4\pi N\sqrt{s}}B(s,z)$ in a series of Legendre polynomials *

$$B(s,z) = \frac{4\pi N\sqrt{s}}{k(s)}\sum_{l=0}^{\infty}(2l+1)B_l(s)P_l(z), \tag{J.5}$$

* This is a particular case of the expansion of a generalized function in Jacobi polynomials (see Appendix I).

where the generalized functions $B_l(s) \in \mathcal{S}'((s_{\text{phys}}, \infty))$ are given by the formula

$$B_l(s) = \frac{k(s)}{8\pi N\sqrt{s}} \int_{[-1,1]} B(s,z)P_l(z)dz.$$

The series (J.5) is called the *partial wave expansion* and the $B_l(s)$ the *partial wave coefficients* (or *amplitudes*) of the operator B.

Proposition J.1. *The partial wave coefficients $B_l(s)$ of the bounded Poincaré-invariant operator B are measurable (essentially) bounded functions of $s \in (s_{\text{phys}}, \infty)$:*

$$|B_l(s)| \leq ||B||, \tag{J.6}$$

where

$$(B^*)_l(s) = \overline{B_l(s)}, \quad (BB')_l(s) = B_l(s)B'_l(s) \tag{J.7}$$

and

$$B_l(s) \geq 0 \quad \text{if } B \geq 0. \tag{J.8}$$

■ Let

$$\Phi = \int A_1^*(p_1)A_2^*(p_2)|0\rangle F(p_1,p_2)d_4p_1d_4p_2,$$

where $F(p_1,p_2) \in \mathcal{S}(\mathbf{M}^2)$ (in the case $\kappa_1 = \kappa_2$ the function $F(p_1,p_2)$ is symmetric in p_1, p_2) Then

$$||\Phi||^2 = N \int_{\Gamma_{m_1}^+ \times \Gamma_{m_2}^+} |F(p_1,p_2)|^2 (dp_1)_{m_1}(dp_2)_{m_2}.$$

We go over from $(p_1,p_2) \in \Gamma_{m_1}^+ \times \Gamma_{m_2}^+$ to new variables $p \in V_{m_1+m_2}^+$, $\mathbf{n} \in S^2$ by setting

$$p = p_1 + p_2, \; \Lambda_p^{-1}p_1 = (E_1(s), k(s)\mathbf{n}), \; F(p_1,p_2) = f(p,\mathbf{n});$$

here Λ_p^{-1} is the Lorentz rotation taking the vector p to $\sqrt{s}e_0$ (see (7.49)). The scalar square of Φ is now equal to

$$||\Phi||^2 = \int |f(p,\mathbf{n})|^2 \frac{Nk(s)}{16\pi^2\sqrt{s}} d_4p\, d\Omega_{\mathbf{n}},$$

where $d\Omega_{\mathbf{n}} \equiv d\cos\theta d\phi$ is the element of area on the sphere S^2. This formula establishes an isomorphism

$$\mathfrak{H}_2^{[\kappa_1,\kappa_2]} \approx \int_{V_{m_1+m_2}^+}^{\oplus} \mathcal{H}_p \frac{Nk(s)}{16\pi^2\sqrt{s}} d_4p \tag{J.9}$$

between $\mathfrak{H}_2^{[\kappa_1,\kappa_2]}$ and the direct integral of the Hilbert spaces $\mathcal{H}_p$, each of which is $\mathcal{L}^2(S^2)$ or (in the case $\kappa_1 = \kappa_2$) $\mathcal{L}_+^2(S^2)$ (the subspace of even functions $f(\mathbf{n}) \equiv f(-\mathbf{n})$ in $\mathcal{L}^2(S^2)$).

The operator B preserves the total momentum of the particles $p = p_1 + p_2$; hence it commutes with any bounded function of the operator p. Hence it follows (according to [N2], §41.2.1) that B decomposes into a direct integral (J.9) of the measurable family $\{B_p\}$ of (essentially) bounded operators in $\mathcal{L}_{(+)}^2(S^2)$, * where

$$(\text{ess})\sup ||B_p|| = ||B||.$$

We use the representation (J.2) to find the explicit form of B_p:

$$\langle \Phi', \, B\Phi \rangle = \int \overline{f'(p,\mathbf{n}')} B(s,\mathbf{n}'\mathbf{n}) f(p,\mathbf{n}) \Big(\frac{k(s)}{16\pi^2\sqrt{s}}\Big)^2 d_4p d\Omega_{\mathbf{n}} d\Omega_{\mathbf{n}'},$$

* This means that all possible matrix elements of the operators B_p are measurable (essentially) bounded functions of p ($(\text{ess})\sup|f(p)|$ is the natural norm on the equivalence classes of such functions).

whence it follows that the kernel of the operator B_p is

$$B_p(\mathbf{n}',\mathbf{n}) = \frac{k(s)}{N\cdot 16\pi^2\sqrt{s}}B(s,\mathbf{n}'\mathbf{n}).$$

On the other hand, the functions $\frac{1}{4\pi}(2l+1)P_l(\mathbf{n}'\mathbf{n})$ are the kernels of mutually orthogonal finite-dimensional projectors in $\mathcal{L}^2(S^2)$. This follows, for example, from the representation

$$\frac{(2l+1)}{4\pi}P_l(\mathbf{n}'\mathbf{n}) = \sum_{m=-l}^{l}\overline{Y_l^m(\mathbf{n}')}Y_l^m(\mathbf{n}) \tag{J.10}$$

([V3],§III.4.2, formulae (2),(5)), where the $Y_l^m(\mathbf{n})$ (for $l\in \overline{Z}_+$, $m\in Z$, $|m|\le l$) are the spherical functions forming an orthonormal basis in $\mathcal{L}^2(S^2)$ and which are defined in spherical polar coordinates by the formula

$$Y_l^m(\mathbf{n}) = (-i)^m\sqrt{\frac{2l+1}{4\pi}}t^l_{m0}(\theta)e^{im\phi} \tag{J.11}$$

(the $t^l_{mn}(\theta)$ are the matrix elements

$$t^l_{mn}(\theta) = \langle m|\mathfrak{D}^{(l)}(\exp(i\theta\tau_1/2))|n\rangle \tag{J.12}$$

of the operator $\mathfrak{D}^{(l)}(\exp(i\theta\tau_1/2))$ of the representation of $SU(2)$ with spin l in the so-called canonical basis; see [V3],§III.3). Therefore (J.5) gives the expansion of the operators B_p in orthogonal projectors, where the coefficients of this expansion are precisely the coefficients $B_l(s)$. The required result now follows trivially from this. ∎

We apply the above result to the amplitude of the process (J.1). For the scattering matrix S in Fock space we denote by iT the part of the operator $S-1$ belonging to the subspace $\mathfrak{H}_2^{[\kappa_1,\kappa_2]}$ (the projector onto which we denote by $E_2^{[\kappa_1,\kappa_2]}$):

$$iT = E_2^{[\kappa_1,\kappa_2]}(S-1)E_2^{[\kappa_1,\kappa_2]}. \tag{J.13}$$

According to the definition of the invariant amplitude we have

$$\begin{aligned}\langle 0|A_2(p_2')A_1(p_1')TA_1^*(p_1)A_2^*(p_2)|0\rangle &= \\ &= (2\pi)^4\delta(p_1+p_2-p_1'-p_2')\delta^+_{m_1}(p_1)\delta^+_{m_2}(p_2)\delta^+_{m_1}(p_1')\delta^+_{m_2}(p_2')T(s,t).\end{aligned} \tag{J.14}$$

By the standard abuse of notation we write $T(s,z)$ in place of $T(s,t(z))$. The partial wave expansion for T has the form

$$T(s,z) = \frac{4\pi N\sqrt{s}}{k(s)}\sum_{l=0}^{\infty}(2l+1)f_l(s)P_l(z). \tag{J.15}$$

According to Proposition J.1, the partial wave amplitudes $f_l(s)$ are bounded measurable functions of s. It follows from the unitarity of the S-matrix that

$$T^*T \le i(T^*-T), \tag{J.16}$$

whence (again by Proposition J.1) we obtain

$$|f_l(s)|^2 \le 2\,\mathrm{Im}\,f_l(s) \tag{J.17}$$

and in particular,

$$|f_l(s)| \le 2, \tag{J.18}$$

$$0 \le \mathrm{Im}\,f_l(s) \le 2. \tag{J.19}$$

The inequalities (J.17) together with the analyticity of the amplitude with respect to t in the Lehmann ellipse (§15.1.C) lead to the following conclusion.

Proposition J.2. (a) *The amplitude $T(s,z)$ of the process* (J.1) *in the physical domain is a locally square-integrable function of the variable $s \in (s_{\rm phys}, \infty)$ which is C^∞-dependent (in the $\mathcal{L}^2(I)$ topology, where I is any compact subinterval of $(s_{\rm phys}, \infty)$) on $z \in [-1,1]$ as a parameter.*

(b) *For any $\tau < 0$ and $s > s_{\min}(\tau)$,* the amplitude $T(s,t)$ of the process* (J.1) *in the physical domain is a locally square-integrable function of the variable $s \in (s_{\min}(\tau), \infty)$ which is C^∞-dependent (in the $\mathcal{L}^2(I)$ topology, where I is an arbitrary compact subinterval of $(s_{\min}(\tau), \infty)$) on $t \in [\tau, 0]$ as a parameter.*

■ According to Theorem 15.1, for any test function $v(s) \in \mathcal{D}((s_{\rm phys}, \infty))$, the function $\int T(s,z) \times$ $\times\ v(s)ds$ is analytic with respect to z in an ellipse (which depends on $v(s)$) with foci at ± 1. On the other hand, we have the Legendre expansion

$$\int T(s,z)v(s)ds = \sum_{l=0}^{\infty}(2l+1)\Big(\int \frac{4\pi N\sqrt{s}}{k(s)} f_l(s)v(s)ds\Big)P_l(z). \tag{J.20}$$

It is well known from the theory of the expansion of analytic functions in an ellipse into Legendre polynomials ([S18],Theorem 9.1.1) that the coefficients of the expansion are bounded by a decreasing exponential in l:

$$\Big|\int \frac{4\pi N\sqrt{s}}{k(s)} f_l(s)v(s)ds\Big| \le c_v \exp(-\chi_v l) \tag{J.21}$$

(where $\chi_v > 0$), and that the series (J.20) is uniformly convergent with respect to z on any compactum in the ellipse of analyticity.

Let I be an arbitrary compact subinterval of $(s_{\rm phys}, \infty)$. We claim that the series (J.20) is uniformly convergent in $\mathcal{L}^2(I)$ with respect to s uniformly in z in some ellipse E with foci at ± 1. To show this, it suffices to estimate the norm $\|\ldots\|$ in $\mathcal{L}^2(I)$ of each term of the series. To this end we choose a non-negative function $v(s) \in \mathcal{D}((s_{\rm phys}, \infty))$ equal to 1 on I; then

$$\Big\|(2l+1)\frac{4\pi N\sqrt{s}}{k(s)} f_l(s)P_l(z)\Big\|^2_{\mathcal{L}^2(I)} \le (2l+1)\int \frac{4\pi N\sqrt{s}}{k(s)} \operatorname{Im} f_l(s)v(s)ds \cdot \sup_{z\in E}|P_l(z)|^2. \tag{J.22}$$

Here if we choose the ellipse E sufficiently small, then by virtue of (J.21) and the inequalities **

$$|P_l(z)| \le P_l\Big(\frac{|z-1|+|z+1|}{2}\Big) \quad \text{for } z \in \mathbf{C}, \tag{J.23}$$

$$P_l(x) \le (x+\sqrt{x^2-1})^l \quad \text{for } x \ge 1, \tag{J.24}$$

the expression (J.22) is bounded above by a quantity of the form $c'\exp(-\chi' l)$. Therefore the series (J.15) converges in the $\mathcal{L}^2(I)$ topology uniformly in $z \in E$, which proves part (a) of Proposition J.2.

For the proof of part (b), instead of z we take t to be the parameter ranging over a sufficiently small complex neighbourhood G of the interval $[\tau, 0]$. We then have an estimate of type (J.22) for the terms of the series (J.12), which proves that the series (J.12) is convergent in the $\mathcal{L}^2(I)$ topology (where I is a compact interval in $(s_{\min}(\tau), \infty)$) uniformly in $t \in G$. ■

In particular, it follows from Proposition J.2 that for the process (J.1) the differential elastic cross section

$$\frac{d\sigma_{\rm el}}{d\cos\theta} = \frac{1}{32\pi s}|T(s,\cos\theta)|^2 \tag{J.25}$$

and the elastic cross section

$$\sigma_{\rm el}(s) = \frac{1}{N}\int_{-1}^{1} \frac{d\sigma_{\rm el}}{d\cos\theta} d\cos\theta \tag{J.26}$$

are well defined and locally integrable functions of s. The total cross section of the interaction of the particles κ_1, κ_2 has the same property; in accordance with the optical theorem, it is given by

$$\sigma_{\rm tot}(s) = \frac{1}{2\sqrt{s}k(s)} \operatorname{Im} T(s,\cos\theta)|_{\cos\theta=1}. \tag{J.27}$$

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* The quantity $s = s_{\min}(\tau) > 0$ is defined from the equation $-4k^2(s) = \tau$.

** See Exercise J.1(c) below concerning these inequalities.

(The factor $1/N$ in (J.26) arises from the fact that elastic scattering of identical particles through an angle θ and through an angle $\pi - \theta$ are one and the same physical event.)

The following important property of the scattering amplitude (more precisely, its imaginary part) is based on the inequalities for Legendre polynomials:

$$|\partial_z^n P_l(z)| \leq \partial_z^n P_l(z)|_{z=1} \quad \text{for } l, n \in \overline{\mathbb{Z}}_+,\ z \in [-1, 1]. \tag{J.28}$$

This follows from the fact that the Legendre polynomials are convex linear combinations of the Chebyshev polynomials $T_k(z) \equiv \cos(k \cos^{-1} z)$:

$$P_l(z) = \sum_{k=0}^{l} 2^{-2l} \frac{(2k)!(2(l-k))!}{(k!)^2((l-k)!)^2} T_{|l-2k|}(z) \tag{J.29}$$

(see [S18],Formula 4.9.4)) and from the corresponding inequalities for the Chebyshev polynomials which we now verify (see the following exercise).

Exercise J.1. (a) Prove the inequalities for the Chebyshev polynomials

$$|T_k(z)| \leq T_k\Big(\frac{|z-1| + |z+1|}{2}\Big) \quad \text{for } z \in \mathbb{C}, \tag{J.30}$$

$$\partial_\chi T_k(\cosh\chi) \leq k T_k(\cosh\chi) \quad \text{for } \chi \geq 0. \tag{J.31}$$

[Hint: (J.30) follows from the trivial relations

$$|T_k(\cos\theta)| \equiv |\cos(k\theta)| \leq \cosh(k \operatorname{Im}\theta) \equiv T_k(\cosh \operatorname{Im}\theta);$$

(J.31) follows from the identity $T_k(\cosh\chi) \equiv \cosh k\chi$.]

(b) Prove the inequalities

$$|\partial_z^n T_k(z)| \leq \partial_\xi^n T_k(\xi)|_{\xi=(|z-1|+|z+1|)/2} \quad \text{for } z \in \mathbb{C}. \tag{J.32}$$

[Hint: The inequality is proved for $n = 0$ in part (a) of this exercise. Now proceed by induction on n using the fact that the Chebyshev polynomial is a convex linear combination of the Chebyshev polynomials

$$\partial_z T_k(z) = \frac{k \sin k\theta}{\sin\theta} = k \sum_{p=0}^{k} T_{|k-2p|}(z),$$

where $\theta \equiv \cos^{-1} z$.]

(c) Prove the following inequality for the Legendre polynomials:

$$|\partial_z^n P_l(z)| \leq \partial_\xi^n P_l(\xi)|_{\xi=(|z-1|+|z+1|)/2} \quad \text{for } z \in \mathbb{C}, \tag{J.33}$$

and also inequality (J.24). [Hint: The inequality (J.33) follows from the fact that the Legendre polynomials are convex linear combinations of Chebyshev polynomials and from the inequalities (J.32). The inequality (J.24) is obtained from the "initial condition" $P_l(1) = 1$ and from the inequality $\partial_\chi P_l(\cosh\chi) \leq l P_l(\cosh\chi)$, which in turn follows from the corresponding inequalities (J.31) for the Chebyshev polynomials.]

Exercise J.2. Let $f(z)$ be an analytic function in a domain $D \subset \mathbb{C}$ containing the interval $[-1, a]$ (where $a > 1$), that has the following series expansion in Legendre polynomials with non-negative coefficients on any interval $[-1, 1]$ (and hence on any ellipse in D with foci at the points ± 1):

$$f(z) = \sum_{l=0}^{\infty} (2l+1) c_l P_l(z), \ c_l \geq 0 \quad \text{for all } l. \tag{J.34}$$

Prove that the function $f(z)$ is analytic in the ellipse $|z-1| + |z+1| < 2a$. [Hint: Let $E = \{z \in \mathbb{C} : |z-1| + |z+1| < 2b\}$, where $b < \infty$ is the maximal ellipse with foci at ± 1 in which $f(z)$ is analytic.

It suffices to prove that $f(z)$ cannot be continued analytically to any complex neighbourhood of the point $z = b$. For this, use the inequality

$$|\partial^n f(z)| \leq \partial^n f(\xi)|_{\xi=(|z-1|+|z+1|)/2}, \; z \in E,$$

which follows from (J.33), (J.34). Hence it follows that the radius of convergence of the Taylor series of $f(z)$ at any point $z \in E$ is not less than the corresponding radius of convergence at the point $\xi = \frac{1}{2}(|z-1| + |z+1|)$.]

The inequalities (J.33) together with (J.19) lead to corresponding inequalities for the imaginary part of the amplitude.

Lemma J.3. *The imaginary part of the amplitude $T(s,t)$ of the elastic process* (J.1) *in the physical domain satisfies the inequality*

$$|\partial_t^n \operatorname{Im} T(s,t)| \leq \partial_t^n \operatorname{Im} T(s,t)|_{t=0} \tag{J.35}$$

for all $n \in \overline{\mathbf{Z}}_+$, $t \in [\tau, 0]$ (where $\tau < 0$ is arbitrary), $s \in (s_{\min}(\tau), \infty)$.

■ It follows from the proof of Proposition J.2(b) that we have the representation

$$\partial_t^n \operatorname{Im} T(s,t) = \frac{4\pi N\sqrt{s}}{k(s)} \sum_{l=0}^{\infty} (2l+1) \operatorname{Im} f_l(s)(2k^2(s))^n \partial_z^n P_l(z)|_{z\equiv z(t)},$$

in which the series converges in $\mathcal{L}^2(I)$ for any $t \in [\tau, 0]$ (where $\tau < 0$ and I is an arbitrary compact subinterval of $(s_{\min}(\tau, \infty))$. Inequality (J.35) follows immediately from (J.19) and (J.33). ■

Remark. Proposition J.2 gives a "local" characterization of the amplitude $T(s,t)$ as a function of s. A similar statement of "global" type can be made. For the moment we confine attention to the following statement for the imaginary part (since the essential details of the proof have already been given in §15.2):* for any $\tau < 0$, $s_0 > s_{\min}(\tau)$ there exists $\nu > 0$ such that $s^{-\nu-1} \operatorname{Im} T(s,t)$ is an absolutely integrable function of $s \in (s_0, \infty)$ for any $t \in (\tau, 0]$:

$$s^{-\nu-1} \operatorname{Im} T(s,t) \in \mathcal{L}^1((s_0, \infty)). \tag{J.36}$$

Thanks to (J.35) for $n = 0$, it clearly suffices to prove (J.36) for $t = 0$ (for some $\nu > 0$, $s_0 > s_{\text{phys}}$). In view of the fact that $\operatorname{Im} T(s,t)|_{t=0}$ is non-negative, the statement we are trying to prove is equivalent to the polynomial boundedness (for $s > s_0$) of the convolution $v(s) * \operatorname{Im} T(s,t)|_{t=0}$, where $v(s)$ is an arbitrary fixed non-negative function in $\mathcal{D}(\mathbf{R})$ with integral 1. But this has already been proved in Lemma 15.6 (here we use the fact that $A_s(s,t)|_{t=0} = \operatorname{Im} T(s,t)|_{t=0}$ for $s > s_{\text{phys}}$ according to Exercise 15.11). We shall see in the next subsection that $s^{-\nu-1} \operatorname{Im} T(s,t)$ (as an element of $\mathcal{L}^1((s_0, \infty))$ with respect to s) depends continuously (and even analytically) on the parameter t, while we shall see in §17.1.A that we can set $\nu = 2$.

It should be further noted that the results of this subsection can be extended without difficulty to the case of the inelastic two-particle process

$$\kappa_1 + \kappa_2 \to \kappa_3 + \kappa_4, \tag{J.37}$$

since the partial wave amplitudes $g_l(s)$ of this process satisfy the estimate of type (J.17)

$$|g_l(s)|^2 \leq 2 \operatorname{Im} f_l(s); \tag{J.38}$$

here, as before, the $f_l(s)$ are the partial wave amplitudes of the elastic process (J.1) (see Sommer, 1967d).

J.2. Analytic continuation of the dispersion relation with respect to t

If we suppose that the dispersion relation holds for the process (J.1) for $t \in (\tau_{\min}, 0]$, we can write it in the form

$$\frac{1}{\nu!} \partial_s^\nu T(s,t) = \frac{1}{\pi} \int \frac{A_s(s',t)ds'}{(s'-s-i0)^{\nu+1}} + \frac{1}{\pi} \int \frac{A_u(u',t)du'}{(u'-u+i0)^{\nu+1}} + \text{ pole terms}; \tag{J.39}$$

*The methods of §16.3 can be used to extend this derivation to all elastic two-particle processes for which the quasi-dispersion relations hold (which are not necessarily dispersion relations as in §15.2).

here $\nu \in \mathbf{Z}_+$ is chosen even and sufficiently large. As we remarked earlier, the absorptive part $A_s(s',t)$ coincides in the physical domain with the imaginary part of the amplitude $T(s',t)$; therefore, according to the results of the preceding subsection, $(s')^{-\nu-1}A_s(s',t)$ is an integrable function of $s' \in [s_0,\infty)$ for sufficiently large $\nu, s_0 > s_{\mathrm{phys}}$. Similarly, $A_u(u',t)$ coincides with the imaginary part of the crossing process in the physical domain of the u-channel and $(u')^{-\nu-1}A_u(u',t)$ is also a non-negative integrable function of $u' \in [u_0,\infty)$ (where $u_0 > s_{\mathrm{phys}}$ is sufficiently large). It turns out that the property (J.35) of the imaginary part of the amplitude enables us to continue analytically the dispersion relation to a domain of complex t.

Lemma J.4. *Let $F_t(\xi)$ be a family of integrable functions of the variable $\xi \in J \equiv (-\infty, s_1] \cup [s_2, +\infty)$ that is C^∞-dependent on the parameter $t \in [-\epsilon, 0]$ in the topology of the spaces $\mathcal{L}^1(J\cap I)$, where I is any interval in* $\mathbf{R}$,* *and satisfying the two conditions:*

a) *the functions $\partial_t^n F_t(\xi)$ satisfy the following inequality for all $n \in \overline{\mathbf{Z}}_+,\ t \in [-\epsilon, 0]$:*

$$|\partial_t^n F_t(\xi)| \le \partial_\tau^n F_\tau(\xi)|_{\tau=0}; \tag{J.40}$$

b) *there exists a function $\chi_t(\xi)$ in the space $C(J)$ (of complex bounded continuous functions of ξ) which is analytically dependent (in the topology of $C(J)$ defined by the* sup *norm) on the complex parameter t in the disc $|t| < \rho$ and is such that all its derivatives with respect to t are non-negative at $t = 0$, $\chi_0(\xi) \equiv \chi_t(\xi)|_{t=0}$ is bounded below by a positive number (say a) and the function $(F_t, \chi_t) \equiv \equiv \int_J F_t(\xi)\chi_t(\xi)d\xi$ is analytic with respect to t in the disc $|t| < \rho$.*

Then $F_t(\xi)$ can be analytically continued (in the topology of $\mathcal{L}^1(J)$) in t to the disc $|t| < \rho$ in the complex plane.

■ By decreasing a and ϵ if necessary, we can assume that

$$\chi_t(\xi) \ge a \quad \text{for } t \in [-\epsilon, 0],\ \xi \in J; \tag{J.41}$$

this is made possible by the fact that for all $t \in [-\epsilon, 0]$

$$\chi_0(\xi) - \chi_t(\xi) \le |t| |\partial_\tau \chi_\tau(\xi)|_{\tau=0}.$$

For the proof of the lemma it suffices to verify the following: for all $n \in \overline{\mathbf{Z}}_+,\ t \in [-\epsilon, 0]$

$$\partial_t^n F_t(\xi) \in \mathcal{L}^1(J); \tag{J.42a}$$

$$\text{the } n\text{th derivative } \partial_t^n F_t(\xi) \text{ of } F_t(\xi) \text{ exists in the } \mathcal{L}^1(J) \text{ topology} \tag{J.42b}$$

(and hence is the same as the n-th derivative $\partial_t^n F_t(\xi)$ of $F_t(\xi)$ in the topology of any space $\mathcal{L}^1(J \cap I)$, where I is a bounded interval in $\mathbf{R}$);

$$\|\partial_t^n F_t\|_{\mathcal{L}^1(J)} \le An!(\rho')^{-n} \quad \text{for } \rho' < \rho, \tag{J.42c}$$

where A is a constant that is independent of n (for example, A can be chosen from the condition $|\partial_t^n (T_t, \chi_t)|_{t=0}| \le \frac{A}{a} n!(\rho')^{-n}$).

We proceed by induction on n. For $n = 0$ the statement is trivial. Assuming the statement holds for $n \le k$, we prove it for $n = k+1$. By the induction hypothesis, the expressions

$$B_{t,\tau}(\xi) = (t-\tau)^{-1}(\partial_t^k F_t(\xi) - \partial_\tau^k F_\tau(\xi))$$

(where $t, \tau \in [-\epsilon, 0],\ t \ne \tau$) are functions in $\mathcal{L}^1(J)$; furthermore, $B_{t,0}(\xi)$ is a non-negative function for $t \in (-\epsilon, 0)$ by condition (a) of the lemma. Therefore for any test function $u(\xi) \in \mathcal{D}(J)$ we have

$$|(\partial_\tau^{k+1} F_\tau|_{\tau=o}, u)| = \lim_{t\to 0} |(B_{t,0}, u)| \le \overline{\lim_{t\to 0}}(B_{t,0}, |u|) \le a^{-1} \sup |u| \overline{\lim_{t\to 0}}(B_{t,0}, \chi_t), \tag{J.43}$$

where

$$\overline{\lim_{t\to 0}}(B_{t,0}, \chi_t) \le \overline{\lim_{t\to 0}} \frac{1}{t}\{(\partial_t^k F_t, \chi_t) - (\partial_\tau^k F_\tau, \chi_\tau)|_{\tau=0}\} + \overline{\lim_{t\to 0}}\Big(\partial_\tau^k F_\tau|_{\tau=0}, \frac{-\chi_t + \chi_0}{t}\Big). \tag{J.44}$$

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* Thus for any $n \in \overline{\mathbf{Z}}_+,\ t \in [-\epsilon, 0]$, the distributions $\partial_t^n F_t(\xi) \in \mathcal{D}'(I)$ are defined and are locally integrable functions of $s \in J$.

By the induction hypothesis, the second term on the right hand side of (J.44) has the limit $-(\partial_\tau^k F_\tau|_{\tau=0}, \partial_\tau \chi_\tau|_{\tau=0}) \leq 0$, therefore

$$\varlimsup_{t\to 0}(B_{t,0},\chi_t) \leq \varlimsup_{t\to 0}\frac{1}{t}\Big\{(\partial_t^k F_t,\chi_t) - (\partial_\tau^k F_\tau,\chi_\tau)|_{\tau=0}\Big\}. \tag{J.45}$$

In view of the facts that F_t is k times differentiable with respect to t in the class $\mathcal{L}^1(J)$ and χ_t is analytic with respect to t in the class $\mathcal{C}(J)$, the function (F_t,χ_t) is k times differentiable with respect to t and satisfies the Leibnitz differentiation rule

$$\partial_t^k(F_t,\chi_t) = \sum_{p=0}^{k}\frac{k!}{p!(k-p)!}(\partial_t^p F_t, \partial_t^{k-p}\chi_t). \tag{J.46}$$

The induction hypothesis also gives us the equality:

$$\partial_t\sum_{p=0}^{k-1}\frac{k!}{p!(k-p)!}(\partial_t^p F_t,\partial_t^{k-p},\chi_t) = \lim_{t\to 0}\sum_{p=0}^{k-1}\frac{k!}{p!(k-p)!}\frac{1}{t}\{(\partial_t^p F_t,\partial_t^{k-p}\chi_t) - (\partial_\tau^p F_\tau,\partial_\tau^{k-p}\chi_\tau)\}|_{\tau=0}.$$

The left hand side here is non-negative by the condition of the lemma, therefore adding this expression to the right hand side of (J.45) does not disturb the inequality:

$$\varlimsup_{t\to 0}(B_{t,0},\chi_t) \leq \varlimsup_{t\to 0}\frac{1}{t}\{\partial_t^k(F_t,\chi_t) - \partial_\tau^k(F_\tau,\chi_\tau)|_{\tau=0}\} = \partial_t^{k+1}(F_t,\chi_t)|_{t=0}.$$

It is now clear from (J.43) that (J.42a) and (J.42c) hold for $n = k+1$.

It remains to prove (J.42b), that is, that

$$B_{t,\tau}(\xi) \to \partial_t^{k+1}F_t \text{ in } \mathcal{L}^1(J) \quad \text{for } \tau \to t.$$

Since by the condition of the lemma

$$B_{t,\tau}(\xi) \to \partial_t^{k+1}F_t \text{ in the topology of } \mathcal{L}^1(J\cap I) \quad \text{for } \tau \to t$$

(where I is any bounded interval in $\mathbf{R}$), it suffices to prove the estimate

$$|B_{t,\tau}(\xi)| \leq b(\xi) \text{ (almost everywhere on } J) \tag{J.47}$$

for all $\tau \neq t$, where $b(\xi)$ is a non-negative function in $\mathcal{L}^1(J)$. For this we set $b(\xi) = \partial_\tau^{k+1}F_\tau(\xi)|_{\tau=0}$; then for all $u \in \mathcal{D}(J)$ we have

$$|(B_{t,\tau},u)| = |t-\tau|^{-1}\left|\int_\tau^t(\partial_{\tau'}^{k+1}F_{\tau'},u)d\tau'\right| \leq (b,|u|),$$

from which (J.47) clearly follows. Thus (J.42b) is proved as well. ∎

Theorem J.5 (Martin). *The amplitude $T(s,t)$ of the elastic process* (J.1) *is analytic with respect to the variables s,t in the complex domain*

$$\{(s,t)\in\mathbf{C}^2 : |t| < \mu^2, |s| > R^2,\ s\notin[M_{12}^2,\infty),\ u\notin[M_{14}^2,\infty)\}, \tag{J.48}$$

where μ, R are positive parameters, and for some $s_0 > s_{\text{phys}}$ and $\nu \in \overline{\mathbf{Z}}_+$

$$s^{-\nu-1}A(s,t)\in\mathcal{L}^1((s_0,\infty)) \quad for\ |t| < \mu^2. \tag{J.49}$$

If the dispersion relation holds for the process (J.1) *in some interval $t \in (\tau_{\min}, 0]$, then it also holds in the domain*

$$\{(s,t)\in\mathbf{C}^2 : |t| < \mu^2,\ s\notin[M_{12}^2,\infty),\ u\notin[M_{14}^2,\infty)\}, \tag{J.50}$$

where μ is a positive parameter.

■ We begin by supposing that the dispersion relation holds for the process (J.1). In order to deal with the absorptive parts of the amplitudes in the physical domain, we choose sufficiently large $s_1, u_1 > s_{\text{phys}}$ and subtract from $T(s,t)$ the contribution

$$\frac{1}{\pi}\int_{M_{12}^2}^{s_1}\frac{A_s(s',t)ds'}{s'-s}+\frac{1}{\pi}\int_{M_{14}^2}^{u_1}\frac{A_u(u',t)du'}{u'-u}+\text{ pole terms }. \tag{J.51}$$

By Theorem 15.4, the absorptive parts $A_s(s',t)$, $A_u(u',t)$ are analytic with respect to t in the Lehmann large ellipses, the intersection of which (for s', u' in the domain of integration in (J.51)) contains a common complex neighbourhood $|t| < \mu'^2$ of the point $t = 0$. It therefore suffices to prove the theorem for $T_1(s,t)$ equidistant from $T(s,t)$ and (J.51); then by (J.39) we have

$$\frac{1}{\nu!}\partial_s^\nu T_1(s,t)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{A_s(s',t)ds'}{(s'-s-i0)^{\nu+1}}+\frac{1}{\pi}\int_{u_1}^{\infty}\frac{A_u(u',t)du'}{(u'-u+i0)^{\nu+1}}. \tag{J.52}$$

We now choose a point s_0 in the interval (s_{phys}, s_1) such that $s_0 > 2(m_1^2+m_2^2) - u_1 + \mu'^2$ (this can be done if s_1, u_1 are chosen sufficiently large). It then follows from the analyticity of the amplitude $F(s,t)$ in the Lehmann small ellipse (that is, from Theorem 15.1) that there exists a positive parameter $\mu < \mu'$ such that the left hand side of (J.52) is analytic with respect to t in the complex disc $|t| < \mu^2$ when $s = s_0$. We set

$$F_t(\xi) = \theta(\xi - s_1)\xi^{-\nu-1}A_s(\xi,t) + \theta(-\xi - u_1)|\xi|^{-\nu-1}A_u(-\xi,t),$$
$$\chi_t(\xi) = \Big\{\frac{1}{\pi}\theta(\xi - s_1)\Big(\frac{\xi}{\xi - s_0}\Big)^{\nu+1} + \frac{1}{\pi}\theta(-\xi - u_1)\Big(\frac{\xi}{\xi + 2(m_1^2+m_2^2) - s_0 - t}\Big)^{\nu+1}\Big\}(1-\rho^{-1}t)^{-\nu-1},$$

where ρ is an arbitrary parameter in the interval $(\mu^2,\ s_0 + u_1 - 2(m_1^2+m_2^2))$. It is not difficult to verify that all the conditions of Lemma J.4 hold; consequently the functions $F_t(\xi)$ can be analytically continued (in the $\mathcal{L}^1(\mathbf{R})$ topology) with respect to t to the disc $|t| < \mu^2$. In this case, (J.52) gives an analytic continuation of $\frac{1}{\nu!}\partial_s^\nu T_1(s,t)$ to the domain (J.50). This (together with the analyticity with respect to t of the derivatives $\partial_s^p T_1(s,t)|_{s=s_0}$ in the disc $|t| < \mu^2$) implies that the amplitude $T(s,t)$ is analytic and satisfies the dispersion relation in the domain (J.50).

In the general case the theorem is proved by an application of the quasi-dispersion relation (see §16.3.A) rather than the dispersion relation; here s_1, u_1, R^2 are chosen sufficiently large at the outset and $T_1(s,t)$ is defined as the distance between $T(s,t)$ and the contributions of the quasi-dispersion integrals over the circle $|s'| = R^2$ and the intervals $R^2 < s' < s_1$, $2(m_1^2+m_2^2) + R^2 - t < u' < u_1$. Formula (J.52) then holds as before. The rest of the argument remains unchanged. ■

In the case of quasi-dispersion relations, the method of proof of the theorem does not give information concerning the size of μ; however, for processes for which the dispersion relations have been proved, it can be used for the calculation of the parameter μ. By choosing s_1, u_1, s_0 in optimal fashion, Sommer (1967a) showed, for example, that for the strong-interaction processes $\pi\pi$, πK, KK, $K\overline{K}$ the parameter μ is defined by the threshold mass in the t-channel:

$$\mu = M_{13} = 2m. \tag{J.53}$$

For processes of type πN, $\pi\Lambda$, the method indicated gives an underestimate for μ; however, Bessis and Glaser (1967) showed that in this case μ is defined by (J.53); for this they used the non-physical value $s_0 \in ((m_1 - m_2)^2,\ (m_1 + m_2)^2)$ (as a preliminary, acting in the spirit of §16.3, they carried out an auxiliary explicit holomorphic extension of the domain of analyticity of $T(s,t)$ with respect to the two variables s,t).

Corollary J.6. For $s > s_{\text{phys}}$, the amplitude $T(s,t)$ of the elastic process (J.1) and its absorptive part $A_s(s,t)$ are analytic with respect to t in the *Martin small*

$$|t + 4k^2(s)| + |t| < 4k^2(s)y(s) \tag{J.54}$$

and *large*

$$|t + 4k^2(s)| + |t| < 4k^2(s)Y(s) \tag{J.55}$$

ellipses; here

$$y(s) = (1 + \mu^2/4k^2(s))^{1/2}, \tag{J.56}$$

$$Y(s) = 1 + \mu^2/2k^2(s). \tag{J.57}$$

In fact, the absorptive part $A_s(s,t)$ is analytic with respect to t in the Lehmann large ellipse and (by what was proved above) can be continued analytically to points $\tau \in (0, \mu^2)$; furthermore, it has non-negative coefficients $\operatorname{Im} f_l(s)$ for the partial wave expansion. By virtue of the properties of series in Legendre polynomials with non-negative coefficients (see Exercise J.2), the partial wave expansion for $A_s(s,t)$ (smoothed with respect to s with an arbitrary non-negative test function $v(s)$ with arbitrarily small support in a neighbourhood of the fixed point $s_0 \in (s_{\text{phys}}, \infty)$) converges in the smallest ellipse with foci at $t_{\min} = -4k^2(s_0)$, $t_{\max} = 0$ containing the point $\tau \in (0, \mu^2)$; this proves the analyticity of $A_s(s,t)$ with respect to t in the large Martin ellipse. Bearing this in mind, estimates of type (J.22) used in the proof of Proposition J.2 show that the amplitude $T(s,t)$ is analytic with respect to t in the Martin small ellipse.

Exercise J.3. (a) Prove the following inequality for Legendre polynomials:

$$P_l(z) \geq a(2l+1)^{-1/2}(z + \sqrt{z^2-1})^l \geq a'(2l+1)^{-1/2}(1 + \sqrt{2(z-1)})^l \quad \text{for } z \geq 1,$$

where a, a' are positive constants. [Hint: It follows from (J.29) for $z = \cosh\chi$, $\chi > 0$, that

$$P_l(\cosh\chi) \geq 2^{-2l}\frac{(2l)!}{l!}\cosh(l\chi) \geq 2^{-(2l+1)}\frac{(2l)!}{l!}(z + \sqrt{z^2-1})^l;$$

now use Stirling's formula

$$n! = \sqrt{2\pi}\exp\left\{\left(n + \tfrac{1}{2}\right)\ln n - n\right\}(1 + O(n^{-1})) \quad \text{as } n \to \infty.]$$

(b) Prove that for any $\tau \in (0, \mu^2)$, $s_0 > s_{\text{phys}}$ there exists a constant $b > 0$ such that

$$\int_{s_0}^{\infty} \sqrt{2l+1} \cdot \operatorname{Im} f_l(s) s^{-\nu-1}\left(1 + 2\sqrt{\frac{\tau}{s}}\right)^l ds \leq b \tag{J.58}$$

for all $l \in \overline{\mathbb{Z}}_+$. [Hint: By Theorem J.5, $s^{-\nu-1}A_s(s,\tau) \in \mathcal{L}^1((s_0,\infty))$; substitute into this the expansion of $A_s(s,t)$ in Legendre polynomials and use the fact that all the terms of the series are non-negative and also part (a) of the exercise.]

Remark. The amplitude of any (possibly inelastic) process (J.37) for $s > s_{\text{phys}}$ is also analytic with respect to t in the corresponding Martin small ellipse

$$|t - t_{\min}(s)| + |t - t_{\max}(s)| \leq 4K_{12}(s)K_{34}(s)\sqrt{1 + \mu^2/4K_{12}^2(s)}, \tag{J.59}$$

where the parameter $\mu \equiv \mu_{12}$ is the same as in Martin's theorem for the elastic process (J.1). As in the case of an elastic process, this statement is an immediate corollary of the estimate (J.38) for the partial wave amplitudes of the process (J.37).

CHAPTER 17

Consequences for High-Energy Elementary Processes

17.1. Restrictions on the Behaviour of Cross Sections at High Energies

A. FROISSART BOUND

The analytic properties of the amplitudes combined with the unitarity condition of the S-matrix lead to a number of restrictions on the behaviour of the cross sections of high-energy elementary processes (more precisely, as the energy of the colliding particles tends to infinity). One of the first estimates obtained in this way was the *Froissart bound* for the total cross section of the interaction of two particles: *

$$\sigma_{\text{tot}}(s) \leq \text{const} \cdot \ln^2 s. \tag{17.1}$$

For a more general statement of this result ** we must introduce the total cross section averaged over an interval of values of s:

$$\hat{\sigma}_{\text{tot}}(s) = \frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} \sigma_{\text{tot}}(s')ds', \tag{17.2}$$

where $s > s_0 > s_{\text{phys}} + \Delta/2$ and Δ is an (arbitrarily) fixed positive parameter. In what follows, the sign $\hat{}$ above other physical quantities denotes a similar averaging with respect to s.

Exercise 17.1. Prove that $\hat{\sigma}_{\text{el}}(s)$ and $\hat{\sigma}_{\text{tot}}(s)$ (for $s > s_0$) are continuous functions and are positive if $T(s,t) \not\equiv 0$. [Hint: The continuity of $\hat{\sigma}_{\text{el}}(s)$ and $\hat{\sigma}_{\text{tot}}(s)$ is a consequence of the fact that by Proposition J.2, $\sigma_{\text{el}}(s)$ and $\sigma_{\text{tot}}(s)$ are locally integrable functions. The positivity of $\hat{\sigma}_{\text{el}}(s)$ and $\hat{\sigma}_{\text{tot}}(s)$ is proved by the method of contradiction: if $\hat{\sigma}_{\text{el}}(s') = 0$ at some point $s' > s_0$, then $T(s,t) = 0$ on some non-empty open set of points $(s,t) \in \mathbf{R}^2$; now use the generalized uniqueness theorem B.10.]

Proposition 17.1. *The following upper estimates hold for the interaction total cross section and the differential cross section of the elastic scattering of a pair of particles* κ_1, κ_2*:*

$$\hat{\sigma}_{\text{tot}}(s) \leq \text{const} \cdot \ln^2 s, \tag{17.3}$$

$$\frac{\widehat{d\sigma_{\text{el}}}}{d\cos\theta} \leq \text{const} \cdot \sqrt{s}\ln^3 s/\sin\theta \quad \textit{for } 0 < \theta < \pi, \tag{17.4}$$

$$\left.\frac{\widehat{d\sigma_{\text{el}}}}{d\cos\theta}\right|_{\cos\theta=1} \leq \text{const} \cdot s\ln^4 s. \tag{17.5}$$

* The high-energy asymptotics $\sigma_{\text{tot}}(s) \approx \text{const} \cdot \ln^2 s$ had been obtained earlier by Heisenberg (1952) from quasi-classical considerations.

** It is relevant to recall once more (see the footnote † on p. 549) that the results set out here are based on the axioms of the LSZ formalism or the S-matrix method; the particles are assumed to be spinless.

■ In the physical domain of the elastic process $\kappa_1 + \kappa_2 \to \kappa_1 + \kappa_2$ we have the expansion

$$\operatorname{Im} T(s,z)|_{z=1} = \frac{4\pi N\sqrt{s}}{k(s)} \sum_{l=0}^{\infty}(2l+1)\operatorname{Im} f_l(s) \tag{17.6}$$

(see (J.15)). It follows from (J.58) that

$$\operatorname{Im} \hat{f}_l(s) \le b_1(2l+1)^{-1/2}(s+\Delta/2)^{\nu+1}\Big(1+2\sqrt{\frac{\tau}{s+\Delta/2}}\Big)^{-l} \tag{17.7}$$

for all $l \in \mathbf{Z}_+$, $s > s_0$ (here $\tau \in (0,\mu^2)$ is fixed and μ is the parameter featuring in Theorem J.5). Let $a > (\nu + 3/4)/2\sqrt{\tau}$; we divide the terms of the series (17.6) into two groups: for $l \le a\sqrt{s}\ln s$ where we use the estimate (J.19) (and the fact that in the case of identical particles $f_l(s) = 0$ for odd l), and for $l > a\sqrt{s}\ln s$ for which (17.7) is applicable. Then

$$\operatorname{Im}\widehat{T}(s,z)|_{z=1} \le \frac{8\pi\sqrt{s+\Delta/2}}{k(s-\Delta/2)}(1+a\sqrt{s}\ln s)^2 + b' s^{\nu+1}\int_{a\sqrt{s}\ln s}^{\infty} \sqrt{l}\Big(1+2\sqrt{\frac{\tau}{s+\Delta/2}}\Big)^{-l} dl. \tag{17.8}$$

The first term on the right hand side of (17.8) behaves like $16\pi a^2 s\ln^2 s$ as $s \to \infty$. The second term behaves like $s\ln^2 s \cdot o(1)$, therefore

$$\operatorname{Im}\widehat{T}(s,0) \le \text{const}\cdot s\ln^2 s, \tag{17.9}$$

whence (17.3) follows. En route we obtain

$$\varlimsup_{s\to\infty} \hat{\sigma}_{\text{tot}}(s)/\ln^2 s \le (\nu+3/4)^2 4\pi\mu^{-2}. \tag{17.10}$$

To estimate the differential cross section we use the partial wave expansion (J.15) and the inequality (J.17):

$$\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2}\Big|\frac{k(s')}{4\pi N\sqrt{s'}}T(s',z)\Big|^2 ds' \le 2\Big(\sum_{l=0}^{\infty}(2l+1)(\operatorname{Im}\hat{f}_l(s))^{1/2}|P_l(z)|\Big)^2. \tag{17.11}$$

If we replace $|P_l(z)|$ by 1 here and estimate the series so obtained by the same method as that used in deriving (17.6), then we obtain

$$\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2}|T(s',z)|^2 ds' \le \text{const}\cdot s^2\ln^4 s \quad \text{for all } z \in [-1,1], \tag{17.12}$$

whence (17.5) follows (for $z = 1$). On the other hand, if we apply the inequality

$$|P_l(z)| < \sqrt{2}|\pi l\sqrt{1-z^2}|^{-1/2} \quad \text{for } z \in [-1,1] \tag{17.13}$$

([S18], Theorem 7.3.3), then in similar fashion we deduce from (17.11) that

$$\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2}|T(s',z)|^2 ds' \le \text{const}\cdot s^{3/2}\ln^3 s\cdot(1-z^2)^{-1/2} \quad \text{for } z \in [-1,1], \tag{17.14}$$

which proves (17.4). ■

Clearly the standard statement (17.1) of the Froissart estimate is somewhat stronger than the relation (17.3) proved above. It follows from (17.3) if we suppose further that the cross section does not oscillate too violently on any interval of energy of fixed length, for example, in the following sense:

$$\Big\{\inf_{s-\Delta/2<s'<s+\Delta/2}\sigma_{\text{tot}}(s')\Big\} \Big/ \Big\{\sup_{s-\Delta/2<s'<s+\Delta/2}\sigma_{\text{tot}}(s')\Big\} > \text{const} > 0. \tag{17.15}$$

It should be noted however, that the formulation of (17.3) in terms of the averaged cross section $\hat{\sigma}_{\text{tot}}(s)$ is completely adequate for the phenomenology , since in an actual experiment one measures quantities (in the present instance, cross sections) that are averaged over some energy interval (the length of which depends on the resolving capabilities of the equipment and, needless to say, increases with s).

In the derivation of the estimate (17.1) one ordinarily makes the following "technical" assumption concerning the behaviour of the absorptive part of the amplitude at "non-physical" points:

$$A_s(s,t)|_{t=\mu^2} \leq \text{const} \cdot s^{\nu-\epsilon}, \quad s > s_{\text{phys}}, \tag{17.16a}$$

in which one chooses

$$\nu = 2, \tag{17.16b}$$

and $\epsilon > 0$ is arbitrarily small (this "pointwise" restriction does not contradict Corollary 17.3, but it does not follow from it either). Hence one obtains (17.1) and also

$$\overline{\lim_{s\to\infty}}\, \sigma_{\text{tot}}(s)/\ln^2 s \leq 4\pi\mu^{-2} \tag{17.17}$$

(see Lukaszuk and Martin, 1967; Singh and Roy, 1970).

It follows trivially from (17.3) that

$$\hat{\sigma}_{\text{el}}(s) \leq \text{const} \cdot \ln^2 s. \tag{17.18}$$

As is obvious from the derivation of (17.12), the same estimate holds if we take a fixed $t < 0$:

$$\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} |T(s',t)|^2 ds' \leq \text{const} \cdot s^2 \ln^4 s. \tag{17.19}$$

This enables us to sharpen the global characterization of the amplitude in terms of the variable s (see the remark in §J.1).

Corollary 17.2. For any $z \in [-1,1]$ we have

$$s^{-2}T(s,z) \in \mathcal{L}^2((s_0,\infty)); \tag{17.20a}$$

similarly, for any $t < 0$,

$$s^{-2}T(s,t) \in \mathcal{L}^2((s_1,\infty)); \tag{17.20b}$$

where $s_1 > s_{\min}(t)$.

Corollary 17.3. At most two subtractions suffice in the dispersion (or quasi-dispersion) relation for fixed $t < 0$ (that is, $\nu \leq 2$).*

In fact, arguing as in the proof of Theorem J.5, we subtract from $T(s,t)$ the contribution of the (quasi-)dispersion integral along the finite non-physical intervals (and round the circle in the case of a quasi-dispersion relation). Clearly the resultant quantity $T_1(s,t)$ also satisfies (17.20b); it also satisfies (J.52) which can now be written in the form

$$T_1(s,t) = \frac{1}{\pi}\int_{s_1}^{\infty}\left(\frac{s}{s'}\right)^2 \frac{A_s(s',t)ds'}{s'-s-i0} + \frac{1}{\pi}\int_{u_2}^{\infty}\left(\frac{u}{u'}\right)^2 \frac{A_u(u',t)du'}{u'-u+i0} + p_t(s), \tag{17.21}$$

where $p_t(s)$ is a polynomial in s of degree $\leq \nu - 1$. The convolutions in (17.21) are well defined, since by virtue of (17.20b), $(s')^{-2}A_s(s',t) \in \mathcal{L}^2((s_1,\infty))$ and hence

$$\int_{s_1}^{\infty} (s')^{-2}A_s(s',t)(s'-s-i0)^{-1}ds' \in \mathcal{L}^2(\mathbf{R}).$$

* This is also true for complex t in the disc $|t| < \mu^2$.

Therefore the first term on the right hand side of (17.21) also satisfies (17.20b); so does the second term in (17.21) for similar reasons. It now follows from the condition $s^{-2}p_l(s) \in \mathcal{L}^2((s_1,\infty))$ that the degree of the polynomial $p_l(s)$ in s does not exceed unity; this proves that ν can be taken to be equal to two.

The mechanism for obtaining the Froissart bound has a transparent physical meaning. By way of comparison we consider the scattering of a pointlike classical particle by an immovable impermeable ball of radius r. Then the total cross section of the scattering σ is πr^2. On the other hand, scattering clearly only occurs if the angular momentum L of the particle does not exceed $L_{\max} = kr$, where k is the momentum of the particle. Thus the radius of the interaction which determines the total cross section is simply related to the maximum angular momentum under the scattering. The situation is somewhat similar in the collision of relativistic quantum particles at high energy (say, in the centre-of-mass frame). We suppose for simplicity that the asymptotic behaviour of the total cross section as $s \to \infty$ is defined by the Froissart bound:

$$c_1 \ln^2 s \le \hat{\sigma}_{\mathrm{tot}}(s) \le c_2 \ln^2 s, \tag{17.22}$$

where c_1 and c_2 are positive constants (in this case, we say that the Froissart bound is saturated). As is clear from the proof of the Froissart bound, the terms of the partial wave expansion with $l > l_{\mathrm{eff}}(s) = a\sqrt{s}\ln s$ give a contribution that is negligible by comparison with (17.22); the main contribution of $\pi k^{-2}(s) l^2_{\mathrm{eff}}(s)$ to the Froissart bound is given by the terms with $l < l_{\mathrm{eff}}(s)$. Thus in accordance with the classical picture, it is natural in this case to define the effective radius of the interaction from the condition

$$l_{\mathrm{eff}}(s) = k(s) r_{\mathrm{eff}}(s). \tag{17.23}$$

We can keep the equality (17.23) as the definition in the general case; but now instead of the asymptotic equality $l_{\mathrm{eff}}(s) = \mathrm{const}\cdot\sqrt{s}\ln s$, we can only assert that

$$l_{\mathrm{eff}}(s) \le \mathrm{const}\cdot\sqrt{s}\ln s. \tag{17.24}$$

The notion of effective angular momentum together with the estimate (17.24) is clearly useful for obtaining the various asymptotic relations. We shall use them in the proof of the following result.

Proposition 17.4. *If the condition*

$$\hat{\sigma}_{\mathrm{tot}}(s) \ge b s^{-k} \quad (\textit{ where } s > s_0) \tag{17.25}$$

holds for some $b > 0$, $k > 0$, then the cross section of the interaction of a pair of particles satisfies the relations

$$(\hat{\sigma}_{\mathrm{tot}}(s))^2 \le \mathrm{const}\cdot \ln^2 s \cdot \hat{\sigma}_{\mathrm{el}}(s), \tag{17.26}$$

$$\frac{\widehat{d\sigma_{\mathrm{el}}}}{d\cos\theta} \le \mathrm{const}\cdot\sqrt{s}\ln s\cdot\hat{\sigma}_{\mathrm{el}}(s)/\sin\theta \quad \textit{for } 0<\theta<\pi, \tag{17.27}$$

$$\left.\frac{\widehat{d\sigma_{\mathrm{el}}}}{d\cos\theta}\right|_{\cos\theta=1} \le \mathrm{const}\cdot s\ln^2 s\cdot\hat{\sigma}_{\mathrm{el}}(s). \tag{17.28}$$

■ It follows from the Cauchy-Bunyakovsky-Schwarz inequality that

$$\Big(\sum_{l\le a\sqrt{s}\ln s}(2l+1)\,\mathrm{Im}\,\hat{f}_l(s)\Big)^2 \le \tfrac{1}{2}\Big(\sum_{l\le a\sqrt{s}\ln s}(2l+1)\Big)\Big(\sum_{l=0}^{\infty}(2l+1)\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2}|f_l(s')|^2 ds'\Big). \tag{17.29}$$

According to the estimates made in the derivation of the Froissart bound, for any $n > 0$ and sufficiently large a, the addition of the terms with $l > a\sqrt{s}\ln s$ to the sum on the left hand side of (17.29) changes it by a quantity that is arbitrarily small (as $s \to \infty$) by comparison with s^{-n-1}. As a result we obtain

$$(\hat{\sigma}_{\text{tot}}(s) + o(s^{-n}))^2 \le \text{const} \cdot \ln^2 s \cdot \hat{\sigma}_{\text{el}}(s).$$

For $n \ge k$ inequality (17.26) clearly follows from this and from (17.25). Note that it also follows from (17.25), (17.26) that

$$\hat{\sigma}_{\text{el}}(s) \ge b' s^{-k'} \tag{17.30}$$

for some $b' > 0$, $k' > 0$. A similar application of the Cauchy-Bunyakovsky-Schwarz inequality to the partial wave expansion of the amplitude leads to (17.27), (17.28), where the terms that are small by comparison with s^{-n} (where n is sufficiently large) are negligible by virtue of (17.30); here, as in the derivation of (17.4), the inequality (17.13) for the Legendre polynomials is used in the derivation of (17.27). ∎

In view of (17.18), Proposition 17.4 generally improves the Froissart estimate with the exception of the case when the Froissart bound for the total cross section is saturated, that is, when

$$\varliminf_{s\to\infty} \hat{\sigma}_{\text{tot}}(s)/\ln^2 s > 0; \tag{17.31a}$$

in this case, thanks to (17.26), we also have

$$\varliminf_{s\to\infty} \hat{\sigma}_{\text{el}}(s)/\ln^2 s > 0, \tag{17.31b}$$

and the estimates (17.26)–(17.28) become (17.3)–(17.5).

Remark. Condition (17.25) entering into Proposition 17.4 is a fairly natural assumption. As a motivation for it we give a related result dealing with lower estimates of cross sections. Jin and Martin (1964) established a lower estimate of the following type for the forward scattering amplitude:*

$$\varlimsup_{s\to\infty} s^{k_0}\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} |T(s',t)|^2 ds'|_{t=0} > 0 \tag{17.32a}$$

for some k_0. (If the dispersion relation holds for the given process, then we can take $k_0 = 2$; the general case is based on the quasi-dispersion relation.) Hence and from the inequality

$$\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} |T(s',t)|^2 ds'|_{t=0} + o(s^{-n}) \le \text{const} \cdot s^2 \ln^2 s \cdot \hat{\sigma}_{\text{el}}(s)$$

(for any $n > 0$) which arises in the course of the proof of (17.28) (and does not use (17.25)), we have the estimate of type

$$\varlimsup_{s\to\infty} s^{k'} \hat{\sigma}_{\text{el}}(s) > 0 \tag{17.32b}$$

and also

$$\varlimsup_{s\to\infty} s^{k'} \hat{\sigma}_{\text{tot}}(s) > 0, \tag{17.32c}$$

This means that an inequality of type (17.25) holds at least on some sequence of points $s_n \to \infty$.

Corollary 17.5. (a) The relations

$$\varlimsup_{s\to\infty} (s\ln^2 s)^{-1} \frac{\widehat{\delta\sigma_{\text{el}}}}{d\cos\theta}\bigg|_{\cos\theta=1} \le \text{const} \cdot \varlimsup_{s\to\infty} \hat{\sigma}_{\text{el}}(s) \tag{17.33a}$$

holds, where const is a positive constant.

* Here, of course, we exclude the trivial case $T(s,t) \equiv 0$.

(b) If $\hat{\sigma}_{\text{el}}(s)$ satisfies an inequality of type (17.30) (for some b', $k' > 0$), then

$$\varliminf_{s\to\infty} (s\ln^2 s)^{-1} \frac{\widehat{d\sigma_{\text{el}}}}{d\cos\theta}\bigg|_{\cos\theta=1} \leq \text{const}\cdot \varliminf_{s\to\infty} \hat{\sigma}_{\text{el}}(s) \tag{17.33b}$$

(where again const$\in \mathbf{R}_+$).

B. Comparison of the cross sections of the interaction of a particle and an antiparticle with the same target

Because of the crossing property, the asymptotic behaviours of the cross sections of the interaction of the pairs of particles $\kappa_1 + \kappa_2$ and $\kappa_1 + \bar{\kappa}_2$ (where in the second case one of the particles is replaced by the antiparticle) turn out to be interconnected.

In particular, under certain extra conditions, the difference between the total cross sections of the interactions of the pairs $\kappa_1 + \kappa_2$ and $\kappa_1 + \bar{\kappa}_2$ tend to zero as $s \to \infty$ (results of this kind bear the general name of *Pomeranchuk theorem* type results; see, for example, Proposition 17.10 and Corollary 17.11 below).

First we prove a more general inequality. Here we shall agree to use the superfix $(+)$ or $(-)$ to distinguish the quantities relating to the pairs of particles $\kappa_1 + \kappa_2$ and $\kappa_1 + \bar{\kappa}_2$ (for example, $\sigma_{\text{tot}}^{(+)}(s)$, $\sigma_{\text{tot}}^{(-)}(s)$). We shall also use the notation

$$c^{(\pm)} = \varliminf_{s\to\infty} \hat{\sigma}_{\text{tot}}^{(\pm)}(s), \quad C^{(\pm)} = \varlimsup_{s\to\infty} \hat{\sigma}_{\text{tot}}^{(\pm)}(s). \tag{17.34}$$

Lemma 17.6. *If $C^{(-)} \leq c^{(+)}$ and $C^{(-)} < \infty$, then*

$$\varliminf_{s\to\infty} (s\ln^2 s)^{-1} \frac{\widehat{d\sigma_{\text{el}}^{(-)}}}{d\cos\theta}\bigg|_{\cos\theta=1} \geq \frac{1}{32\pi^3}(c^{(+)} - C^{(-)})^2. \tag{17.35}$$

■ Since (17.35) is trivial for $C^{(-)} = c^{(+)}$, we may assume that for $C^{(-)} < c^{(+)}$. Let $v(s)$ be any fixed non-negative function in $\mathcal{D}(\mathbf{R})$ with integral 1. We claim that for any b in the interval $0 < b < c^{(+)} - C^{(-)}$ and for sufficiently large s, we have

$$|v(s) * (s^{-2}T^{(-)}(s,0))| \geq \frac{1}{\pi} b s^{-1} \ln s. \tag{17.36}$$

Clearly, this will suffice to prove (17.35).

We use the (quasi-)dispersion relation for the amplitude $T^{(-)}(s,0)$. Clearly the contribution of the bounded parts of the integration in the (quasi-)dispersion integral has order $O(s^{-1})$ as $s \to \infty$, therefore it suffices to prove (17.36) for the distance $T_1^{(-)}(s,0)$ between $T^{(-)}(s,0)$ and this contribution. The dispersion relation for $T_1^{(-)}(s,0)$ takes the form (17.21), in which the positions of the s and u channels need to be interchanged. Since $p_0(s) \equiv p_t(s)|_{t=0}$ is a polynomial in s of degree $\leq \nu - 1 = 1$ (see Corollary 17.3), this polynomial is not essential for an estimation of type (17.36); we subtract this from $T_1^{(-)}(s,0)$ but still denote the new quantity by $T_1^{(-)}(s,0)$. We choose the parameters s_1 and u_1 in (17.21) large enough for the inequalities

$$(u')^{-1}\hat{A}_s(u',0) \geq c' \quad \text{for } u' \geq u_1, \tag{17.37a}$$

$$(s')^{-1}\hat{A}_u(s',0) \leq C^{(-)} + \epsilon \quad \text{for } s' \geq s_1, \tag{17.37b}$$

where $c' < c^{(+)}$ and ϵ are positive parameters such that $b \leq c' - C^{(-)} - \epsilon$. Thus for sufficiently large s we have

$$|v(s) * (s^{-2}T_1^{(-)}(s,0))| \geq \frac{1}{\pi} c' \int_{u_1}^{\infty} \frac{du'}{u'(u'+s)} - \frac{1}{\pi}\left|\int_{s_1}^{\infty} V(s')(s')^{-2}A_u(s',0)ds'\right|, \tag{17.38}$$

where

$$V(s) = v(s) * (s + i0)^{-1}. \tag{17.39}$$

The first term on the right hand side of (17.38) is easily calculated. To estimate the second term we need the following property of $V(s)$.

Exercise 17.2. Prove that the function $V(s)$ (17.39) (which is in $\mathcal{S}(\mathbf{R}_\infty)$; see Appendix A.3) satisfies an estimate of the form

$$|V(s)| \leq (1 + s^2)^{-1/2} + a(1 + s^2)^{-1}.$$

Hence for sufficiently large s we have

$$\begin{aligned}
&|v(s) * (s^{-2}T_1^{(-)}(s, 0))| \geq \\
&\geq \frac{1}{\pi}\left\{c' s^{-1} \ln \frac{s + u_1}{u_1} - (C^{(-)} + \epsilon) \int_{s_1}^{\infty} (s')^{-1}[(1 + (s' - s)^2)^{-1/2} + a(1 + (s' - s)^2)^{-1}]ds'\right\} = \\
&\qquad = \frac{s^{-1}}{\pi}\{(c' - C^{(-)} + \epsilon) \ln s + O(1)\} \quad \text{as } s \to \infty.
\end{aligned}$$

Thus the estimate (17.36) is proved. ■

Using Lemma 17.6 we obtain the following result of a qualitative nature.

Proposition 17.7. (a) *If the cross section $\hat{\sigma}_{\text{tot}}^{(+)}(s)$ is bounded below by a positive constant:*

$$\hat{\sigma}_{\text{tot}}^{(+)}(s) > \text{const} > 0, \tag{17.40}$$

then the cross section $\hat{\sigma}_{\text{tot}}^{(-)}(s)$ does not tend to zero as $s \to \infty$, that is,

$$\overline{\lim_{s\to\infty}}\, \hat{\sigma}_{\text{tot}}^{(-)}(s) > 0. \tag{17.41}$$

(b) *If*

$$\hat{\sigma}_{\text{tot}}^{(+)}(s) \to \infty \quad \text{as } s \to \infty, \tag{17.42}$$

then the cross section $\hat{\sigma}_{\text{tot}}^{(-)}(s)$ is not bounded below, that is,

$$\overline{\lim_{s\to\infty}}\, \hat{\sigma}_{\text{tot}}^{(-)}(s) = \infty. \tag{17.43}$$

■ We prove assertion (a) by the method of contradiction. Suppose that $\hat{\sigma}_{\text{tot}}^{(-)}(s) \to 0$ as $s \to \infty$. Then $C^{(-)}=0$ and hence the condition of Lemma 17.6 holds, so that

$$\underline{\lim}_{s\to\infty} (s \ln^2 s)^{-1} \frac{d\widehat{\sigma_{\text{el}}^{(-)}}}{d\cos\theta}\bigg|_{\cos\theta=1} \geq \frac{1}{32\pi^3}(c^{(+)})^2 > 0. \tag{17.44}$$

Hence and from (17.33a) it follows that

$$C^{(-)} \geq \overline{\lim_{s\to\infty}}\, \hat{\sigma}_{\text{el}}^{(-)}(s) > 0.$$

This contradiction completes the proof. Assertion (b) is proved by the same method. ■

The subsequent results require the classical Phragmén-Lindelöf theorem from the theory of functions of a complex variable.

Theorem 17.8 (Phragmén-Lindelöf). (a) *Let $D \subset \mathbf{C}$ be the upper half-plane* $\operatorname{Im} z > 0$ *from which possibly a bounded subset has been removed. Let $f(z)$ be an analytic function in D and continuous in the closure of D such that*

$$|f(z)| \leq a e^{\alpha |z|^p} \quad \textit{for } z \in D \tag{17.45}$$

and

$$|f(z)| \leq A \quad \textit{for } z \in \partial D, \tag{17.46}$$

where a, α, A *are positive numbers and* $p < 1$. *Then*

$$|f(z)| \leq A \quad \textit{for } z \in D. \tag{17.47}$$

(b) *Suppose in addition that the following limits exist:*

$$\lim_{x \to \pm\infty} f(x) = f(\pm\infty); \tag{17.48}$$

then they are equal:

$$f(\pm\infty) \equiv f(\infty); \tag{17.49}$$

furthermore,

$$f(z) \to f(\infty) \quad \textit{for } z \in D,\ |z| \to \infty. \tag{17.50}$$

The proof can be found in [T3],§§5.6.1 and 5.6.4. This theorem will be applied by us to functions $f(z)$ of polynomial growth in D, for which (17.45) holds trivially.

Exercise 17.3. Let $f(z)$ be an analytic function in the domain $D = \{z \in \mathbf{C} : \operatorname{Im} z > 0, |z| > 1\}$ that is continuous and of polynomial growth in its closure and is such that

$$\overline{\lim_{x \to \infty,\, x \in \mathbf{R}}} |f(x)| = A.$$

Prove that

$$\overline{\lim_{|z| \to \infty,\, z \in D}} |f(z)| = A.$$

[Hint: For any $\epsilon > 0$ there exists a sufficiently large $a > 0$ such that $|z(z + ia)^{-1} f(z)| \leq A + \epsilon$ for all $z \in \partial D$. Now use part (a) of Theorem 17.8.]

Lemma 17.9. *The relation*

$$\overline{\lim_{s \to \infty}} (s \ln^2 s)^{-1} \frac{\widehat{d\sigma}_{\text{el}}^{(\pm)}}{d \cos\theta}\bigg|_{\cos\theta = 1} \geq \frac{1}{32\pi^3} \underline{\lim}_{s \to \infty} (\hat{\sigma}_{\text{tot}}^{(+)}(s) - \hat{\sigma}_{\text{tot}}^{(-)}(s))^2 \tag{17.51}$$

holds for at least one of the two values of the symbol $(\pm)$ *on the left hand side.*

■ The proof proceeds along the same lines as in Lemma 17.6. We set

$$b = \underline{\lim}_{s \to \infty} |\hat{\sigma}_{\text{tot}}^{(+)}(s) - \hat{\sigma}_{\text{tot}}^{(-)}(s)|;$$

inequality (17.51) is trivial for $b = 0$, therefore we suppose that $0 < b \leq \infty$. Then for sufficiently large s, the function $\hat{\sigma}_{\text{tot}}^{(+)}(s) - \hat{\sigma}_{\text{tot}}^{(-)}(s)$ (which is continuous) is of constant sign; we suppose for definiteness that it is positive. Hence for any b' in the interval $0 < b' < b$ there exists a sufficiently large s_1 such that

$$\hat{A}_s(s') - \hat{A}_u(s') \geq s'b' \quad \text{for } s' \geq s_1. \tag{17.52}$$

Let $v(s)$ be an arbitrary fixed non-negative even function in $D(R)$ with integral 1 and let B be the larger of the two quantities

$$\overline{\lim_{s \to \infty}} |(s \ln s)^{-1} (v(s) * s^{-2} T^{(\pm)}(s, 0))|.$$

We claim that

$$B \geq \pi^{-1} b'. \tag{17.53}$$

Clearly this will suffice to prove the lemma. The case $B = \infty$ is trivial, therefore we suppose that $B < \infty$.

Consider the function

$$f(s) = (s\ln s)^{-1}\Big\{v(s) * \int_{s_1}^{\infty}\Big(\frac{A_s(s',0)}{(s')^2(s'-s-i0)} + \frac{A_u(s',0)}{(s')^2(s'+s+i0)}\Big)ds'\Big\}. \tag{17.54}$$

Clearly it can be continued analytically to the upper half-plane $\operatorname{Im} s > 0$ and is polynomially bounded and continuous for $\operatorname{Im} s \geq 0$, $|s| \geq 1$. The argument in the proof of Lemma 17.6 shows that the difference between $f(s)$ and $(s\ln s)^{-1}\{v(s) * (s^{-2}T^{(+)}(s,0))\}$ tends to zero as $s \to \infty$. Similarly, the difference between $\overline{f(2m_1^2+2m_2^2-s)} \equiv \overline{f(u)}$ and $(s\ln s)^{-1}\{v(s) * (s^{-2}T^{(-)}(s,0))\}$ tends to zero as $s \to +\infty$. It follows from the definition of B that

$$\overline{\lim_{s\to\infty,\, s\in\mathbf{R}}} |f(s)| = B,$$

therefore by Exercise 17.3,

$$\overline{\lim_{|s|\to\infty,\, \operatorname{Im} s>0}} |f(s)| = B. \tag{17.55}$$

On the other hand, it clearly follows from (17.54) that

$$|f(i\tau)| \geq \frac{1}{\pi}(\tau\ln\tau)^{-1}\operatorname{Im}\int v(s)\Big\{\int_{s_1}^{\infty}\Big(\frac{A_s(s',0)}{(s')^2(s'-s-i\tau)} + \frac{A_u(s',0)}{(s')^2(s'-s+i\tau)}\Big)ds'\Big\}ds =$$
$$= \frac{1}{\pi}(\ln\tau)^{-1}\int v(s)\Big\{\int_{s_1}^{\infty}\frac{A_s(s',0)-A_u(s',0)}{(s')^2((s'-s)^2+\tau^2)}ds'\Big\}ds$$

for all $\tau > 0$, which together with (17.52) implies that

$$\overline{\lim_{\tau\to+\infty}} |f(i\tau)| \geq \pi^{-1}b'.$$

Hence and from (17.55) the required inequality (17.53) follows. ■

The next result of Pomeranchuk theorem type follows trivially from Lemma 17.9.

Proposition 17.10. *If*

$$(s\ln^2 s)^{-1}\frac{d\widehat{\sigma}_{\mathrm{el}}^{(\pm)}}{d\cos\theta}\bigg|_{\cos\theta=1} \to 0 \quad \text{as } s\to\infty \tag{17.56}$$

for both values of the symbol $(\pm)$, *then*

$$\underline{\lim}_{s\to\infty} |\hat{\sigma}_{\mathrm{tot}}^{(+)}(s) - \hat{\sigma}_{\mathrm{tot}}^{(-)}(s)| = 0. \tag{17.57}$$

If it is further supposed that the (finite or infinite) limit

$$\lim_{s\to\infty} |\hat{\sigma}_{\mathrm{tot}}^{(+)}(s) - \hat{\sigma}_{\mathrm{tot}}^{(-)}(s)| \tag{17.58}$$

exists, then this limit is equal to zero.

Inequality (17.33a) enables us to replace (17.56) by the corresponding condition in terms of the cross sections $\hat{\sigma}_{\mathrm{el}}^{(\pm)}(s)$; in this way we obtain another version of the statement.

Corollary 17.11. If both the elastic cross sections $\hat{\sigma}_{\mathrm{el}}^{(\pm)}(s)$ tend to zero as $s \to \infty$, then (17.57) holds; in particular, if in addition, the (finite or infinite) limit (17.58) exists, then it is equal to zero.

17.2. Inclusive Processes

A. Physical Characteristics of Inclusive Processes

As the collision energy of a pair of elementary particles increases, the role played by the processes of creation of particles

$$\kappa_1 + \kappa_2 \to \kappa_3 + \ldots + \kappa_n \tag{17.59}$$

becomes appreciable. The number of possible channels sharply increases, therefore the theoretical and experimental investigation of each channel separately becomes altogether a more tedious problem. In this connection it is interesting to study reactions in which one or several particles of a specified type are detected in the final state, for example,

$$\kappa_1 + \kappa_2 \to \kappa_3 + \ldots, \tag{17.60}$$

$$\kappa_1 + \kappa_2 \to \kappa_3 + \kappa_4 + \ldots \tag{17.61}$$

and so on, where the dots denote any admissible complex of particles. The collection of all reactions (17.59) with a fixed subcomplex of detectable particles in the complex $\{\kappa_3, \ldots, \kappa_n\}$ of outgoing particles is called an *inclusive process* (whereas the fixed multiparticle channel (17.59) is called an *exclusive process*). Inclusive processes are simpler to analyse theoretically and experimentally and still provide interesting information concerning the interaction of the particles.

Even the first experimental investigations of inclusive processes carried out in 1968 on the Serpukhov accelerator (Bushnin et al., 1969) led to the discovery of the new phenomenon of scale invariance of cross section; this consisted in the fact that the relative transverse cross sections of the formation of hadrons at high energies are universal functions of the reduced momentum $k(s)/k_{\max}(s)$ which are practically independent of the collision energy of the particles. At about the same time, experiments were carried out on the SLAC at Stanford on deeply inelastic scattering of electrons by protons; the corresponding cross sections of the deeply inelastic processes also uncovered scale-invariant behaviour of the structure functions (Blum et al., 1969; Breidenbach et al.,1969; a theoretical analysis has been given by Bjorken, 1969). The subsequent study of other inclusive processes led to the discovery of a considerable number of laws of behaviour and, in particular, the universal nature of scale invariance emerged.

In this section we briefly consider certain applications of the machinery developed above to inclusive processes. The fundamental quantities that characterize both the inclusive and the exclusive processes are the cross sections. We denote by $\kappa^{(1)}, \ldots, \kappa^{(r)}$ the different possible types of particles created from the collision of particles κ_1 and κ_2. Then an arbitrary exclusive process

$$\kappa_1 + \kappa_2 \to N_1\kappa^{(1)} + \ldots + N_r\kappa^{(r)} \equiv \mathbf{N}\kappa \tag{17.62}$$

can be identified with the multi-index $\mathbf{N} \equiv (N_1, \ldots, N_r)$, where $N_1, \ldots, N_r \in \overline{\mathbf{Z}}_+$; here

$$|\mathbf{N}| \equiv N_1 + \ldots + N_r$$

is the total number of particles in the final state. The process (17.62) will also be written in the form $\kappa_1\kappa_2 \to \mathbf{N}$ for brevity; if there are no selection rules for the quantum numbers, then it becomes "open" when

$$s_{\text{phys},\mathbf{N}} = (m_1 + m_2) \vee \Big(\sum_{\rho=1}^{r} N_\rho m^{(\rho)}\Big), \tag{17.63}$$

where $m^{(\rho)}$ is the mass of the particle of type $\kappa^{(\rho)}$. We denote an element of volume of the momentum space of the outgoing particles by

$$d\Xi_{\mathbf{N}} = \prod_{\rho=1}^{r} \prod_{j=1}^{N_r} (dp_{\rho j})_{m(\rho)}, \tag{17.64}$$

where $p_{\rho j}$ is the momentum of the jth particle in the rth group of particles of type $\kappa^{(\rho)}$.

According to formulae (7.175), the differential and total cross sections of the exclusive process (17.62) have the form

$$\frac{d\sigma_{\kappa_1\kappa_2\to N}}{d\Xi_N} = \frac{1}{4\sqrt{s}k(s)}(2\pi)^4\delta\Big(p_1 + p_2 - \sum_{\rho=1}^{r}\prod_{j=1}^{N_r} p_{\rho j}\Big)|T(p_N|p_2,p_1)|^2, \tag{17.65}$$

$$\sigma_{\kappa_1\kappa_2\to N}(s) = \frac{1}{N!}\int \frac{d\sigma_{\kappa_1\kappa_2\to N}}{d\Xi_N} d\Xi_N. \tag{17.66}$$

We also define the differential cross section with respect to the momenta of some subcomplex of the outgoing particles. For $\mathbf{K} \leq \mathbf{N}$ we can represent (17.64) in the form

$$d\Xi_{\mathbf{N}} = d\Xi_{\mathbf{K}} \otimes d\Xi_{\mathbf{N}-\mathbf{K}}; \tag{17.67a}$$

by definition we then set

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{K}}} = \frac{1}{(\mathbf{N}-\mathbf{K})!}\int \frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{N}}} d\Xi_{\mathbf{N}-\mathbf{K}}. \tag{17.67b}$$

The differential cross sections are symmetric with respect to the momenta of particles of the same type, therefore it is convenient to give the generating functional for them*

$$G(\{h\};p_1,p_2) = \sum_{\mathbf{N}} \frac{1}{\mathbf{N}!}\int \frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\xi_{\mathbf{N}}}\Big(\prod_{\rho=1}^{r}\prod_{j=1}^{N_r} h^{(\rho)}(p_{\rho j})\Big) d\Xi_{\mathbf{N}}, \tag{17.68}$$

where $h \equiv \{h^{(\rho)}\}$ is a collection of functions $h^{(\rho)}(p)$ on $\Gamma^+_{m^{(\rho)}}$ (for example, of class $\mathcal{O}_M(\mathbf{R}^3)$ with respect to $\mathbf{p}$); here

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{N}}} = \frac{\delta^{|\mathbf{N}|}G}{(\delta h)^{\mathbf{N}}}\bigg|_{h=0} \equiv \frac{\delta^{|\mathbf{N}|}G}{\prod_\rho \prod_j \delta h^{(\rho)}(p_{\rho j})}\bigg|_{h=0}.$$

In this notation, the inclusive process

$$\kappa_1 + \kappa_2 \to \mathbf{K}\kappa + \ldots \tag{17.69}$$

(or simply $\kappa_1\kappa_2 \to \mathbf{K}\ldots$) is a collection of exclusive processes $\kappa_1\kappa_2 \to \mathbf{N}$ where $\mathbf{N} \geq \mathbf{K}$ (this implies that $N_\rho \geq K_\rho$ for all $\rho = 1,\ldots,r$); $\kappa_1, \kappa_2, \mathbf{K}$ are fixed. In the notation of (17.69), all the products of the reaction are divided into "detectable" particles and "all the rest" (we call the inclusive processes "one-particle", "two-particle"' and

*When $s \equiv (p_1+p_2)^2$ varies over a bounded interval, say $s \in (s_0, s')$, the number of open exclusive channels is finite (it does not exceed $(\sqrt{s'}/\min_\rho m^{(\rho)})^r$), therefore the number of non-zero terms of the series (17.68) is finite here.

so on, depending on the number of detectable particles). The differential and total inclusive cross sections are defined as follows:

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}} = \sum_{\mathbf{N}\geq\mathbf{K}} \frac{\mathbf{K}!(\mathbf{N}-\mathbf{K})!}{\mathbf{N}!}\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{K}}}, \tag{17.70}$$

$$\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots} = \frac{1}{\mathbf{K}!}\int \frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}}d\Xi_{\mathbf{K}}, \tag{17.71a}$$

that is,

$$\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots} = \sum_{\mathbf{N}\geq\mathbf{K}} \sigma_{\kappa_1\kappa_2\to\mathbf{N}}. \tag{17.71b}$$

Thus the total inclusive cross section is the part of the total cross section of the interaction of the particles κ_1, κ_2 that is equal to the sum of the total cross sections of the exclusive processes (17.62) containing the fixed subcomplex $\mathbf{K}\kappa$ of "detectable" particles. In particular, the total cross section of the interaction of the particles κ_1 and κ_2 corresponds to the case $\mathbf{K} = 0$ (when what interests us is merely the fact of an interaction of colliding particles with any non-trivial outcome):

$$\sigma_{\text{tot}} \equiv \sigma_{\kappa_1\kappa_2\to\dots} = \sum_{\mathbf{N}\in(\overline{\mathbf{Z}}_+)^r} \sigma_{\kappa_1\kappa_2\to\mathbf{N}}. \tag{17.72}$$

The relations

$$w(\mathbf{N}) = \hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{N}}/\hat{\sigma}_{\text{tot}}, \tag{17.73}$$

$$w(\mathbf{N}|\mathbf{K}\dots) = \hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{N}}/\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots} \tag{17.74}$$

give the respective probabilities of the specific exclusive channel $\kappa_1\kappa_2 \to \mathbf{N}$ in the "0-particle" inclusive process $\kappa_1\kappa_2 \to \dots$ and the "$|\mathbf{K}|$-particle" inclusive process $\kappa_1\kappa_2 \to \mathbf{K}\dots$. By $\langle f(\mathbf{N})\rangle$ we mean the average calculated via the probability (17.73):

$$\langle f(\mathbf{N})\rangle = \sum_{\mathbf{N}} f(\mathbf{N})w(\mathbf{N}). \tag{17.75}$$

In particular, the quantity $\langle N_\rho\rangle$ is the average multiplicity of particles of type $\kappa^{(\rho)}$ in the inclusive process $\kappa_1 + \kappa_2 \to \dots$.

According to the next exercise, the totality of all the inclusive cross sections contains the same information as the totality of all the exclusive cross sections.

Exercise 17.4. Derive the expression for the exclusive cross sections in terms of the inclusive ones:

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{N}}} = \sum_{\mathbf{0}\leq\mathbf{K}-\mathbf{N}\leq\mathbf{1}} (-1)^{|\mathbf{K}-\mathbf{N}|}\frac{\mathbf{N}!(\mathbf{K}-\mathbf{N})!}{\mathbf{K}!}\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{N}}} \equiv \tag{17.76a}$$

$$\equiv \sum_{\mathbf{0}\leq\mathbf{K}-\mathbf{N}\leq\mathbf{1}} \frac{(-1)^{|\mathbf{K}-\mathbf{N}|}\mathbf{N}!}{\mathbf{K}!}\int\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}}d\Xi_{\mathbf{K}-\mathbf{N}} \tag{17.76b}$$

(where $\mathbf{0} \equiv (0,\dots,0)$, $\mathbf{1} \equiv (1,\dots,1)$).

Whereas the total inclusive cross sections have a simple meaning (in accordance with (17.71b)) the differential inclusive cross sections (17.70) are partly artificial quantities,* so that it is a fairly complicated matter to deal with the laws of conservation of additive physical quantities (for example, charges, momenta) in terms of them. Therefore in the phenomenology, the related notion of ***differential inclusive spectrum*** (also called ***distribution function*** of the inclusive process) is more commonly used; this is equal to the sum of the relative differential cross sections of the corresponding exclusive processes:

$$\frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}} = \sum_{\mathbf{N}\geq\mathbf{K}} \frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{K}}}. \tag{17.77}$$

The total inclusive spectrum is

$$\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots} = \frac{1}{\mathbf{K}!}\int \frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}} d\Xi_{\mathbf{K}}; \tag{17.78a}$$

$$\hat{\tau}_{\kappa_1\kappa_2\to\mathbf{K}\dots} = \left\langle \frac{\mathbf{N}!}{(\mathbf{N}-\mathbf{K})!\mathbf{K}!} \right\rangle \hat{\sigma}_{\text{tot}}. \tag{17.78b}$$

Exercise 17.5. (a) Let $Q^{(\rho)}$ be the electric charge ** of a particle of type $\kappa^{(\rho)}$ $(\rho = 1,\dots,r)$. Prove the "summation law" with respect to the charge for inclusive processes

$$Q_1 + Q_2 = \sum_{\rho=1}^{r} Q^{(\rho)}\langle N_\rho\rangle = \sum_{\rho=1}^{r} Q^{(\rho)}\hat{\tau}_{\kappa_1\kappa_2\to\kappa^{(\rho)}\dots}/\hat{\sigma}_{\text{tot}}. \tag{17.79}$$

(b) Prove the following " sum rules" with respect to the momenta for inclusive processes:

$$(p_1^\mu + p_2^\mu)\sigma_{\text{tot}} = \sum_{\rho=1}^{r} p_\rho^\mu d\tau_{\kappa_1\kappa_2\to\kappa^{(\rho)}\dots} \tag{17.80}$$

$$(p_1^\mu + p_2^\mu)(p_1^\nu + p_2^\nu)\sigma_{\text{tot}} = \sum_{\rho=1}^{r}\left\{\int p_\rho^\mu p_\rho^\nu d\tau_{\kappa_1\kappa_2\to\kappa^{(\rho)}\dots} + 2\int p_\rho^\mu p_\rho^\nu d\tau_{\kappa_1\kappa_2\to\kappa^{(\rho)}\kappa^{(\rho)}\dots}\right\} + \\ + \sum_{\rho\neq\rho'}\int p_\rho^\mu p_{\rho'}^\nu d\tau_{\kappa_1\kappa_2\to\kappa^{(\rho)}\kappa^{(\rho')}\dots}. \tag{17.81}$$

The totality of all inclusive spectra contains the same information as the totality of all inclusive (or all exclusive) spectra. In fact, the connection between the inclusive spectra and exclusive cross sections can be written in terms of the generating functional (17.68) in the form of the relations

$$\frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{\delta\Xi_{\mathbf{K}}} = \mathbf{K}!\frac{\delta^{|\mathbf{K}|}G}{(\delta h)^{\mathbf{K}}}\bigg|_{h=1}, \tag{17.82}$$

$$G\{h\} = \sum_{\mathbf{K}}\int\Big[\prod_{\rho}\prod_{j}(h^{(\rho)}(p_{\rho j}) - 1\Big]d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}, \tag{17.83}$$

whence

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{N}}} = \sum_{\mathbf{K}}(-1)^{|\mathbf{K}-\mathbf{N}|}\frac{\mathbf{N}!d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}} \equiv \sum_{\mathbf{K}}\frac{(-1)^{|\mathbf{K}-\mathbf{N}|}\mathbf{N}!}{(\mathbf{K}-\mathbf{N})!\mathbf{K}!}\int\frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\Xi_{\mathbf{K}}}d\Xi_{\mathbf{K}-\mathbf{N}}. \tag{17.84}$$

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* It should be borne in mind that in the literature, the (total or differential) inclusive cross section is taken to mean what we shall be calling below the (total or differential) inclusive spectrum.

** Instead of the electric charge we can take any other additive conserved charge.

B. ANALYTIC PROPERTIES OF DIFFERENTIAL CROSS SECTIONS WITH RESPECT TO ANGULAR VARIABLES

Using the reduction formula (13.103) we can write the amplitude of the exclusive process $\kappa_1\kappa_2 \to \mathbf{N}$ in the form

$$(2\pi)^4\delta\Big(p_1 + p_2 - \sum_{\rho=1}^{r}\sum_{j=1}^{N_r} p_{\rho j}\Big)\mathcal{T}(p_{\mathbf{N}}|p_1, p_2) = \delta^+_{m_1}(p_1)\delta^+_{m_2}(p_2)\times$$
$$\times\Big\{(p_1^2 - m_1^2)(p_2^2 - m_2^2)\Big\langle 0\Big|\Big(\prod_{\rho=1}^{r}\prod_{j=1}^{N_r}\tilde{\phi}^{\mathrm{in}(\kappa^{(\rho)})}(p_{\rho j})\Big)A(\tilde{\phi}_2(-p_2);\,\tilde{\phi}_1(-p_1))\Big|0\Big\rangle\Big\}. \quad (17.85)$$

In the centre-of-mass frame, the vectors p_1 and p_2 are defined on the mass shell by the length $k(s)$ of their three-dimensional parts and the vector $\mathbf{n} \equiv \mathbf{n}_{12}$ on the sphere S^2 (see (7.85)–(7.87)). Using the JLD representation for the advanced commutator, we find by the same method as that in §13.1.C that the expression in the curly brackets in (17.85) can be continued analytically with respect to $\mathbf{n}$ to the domain

$$\{\mathbf{n} \in \mathbf{C}S^2 :\ |\operatorname{Re}\mathbf{n}| < x_{12}(s)\} \quad (17.86)$$

on the complex sphere $\mathbf{C}S^2$, where $x_{12}(s)$ is defined by (15.4). Since this domain is invariant with respect to complex conjugation $\mathbf{n} \to \bar{\mathbf{n}}$, the differential cross section of the process (17.62) enjoys the same property.

We conclude as a result that the differential cross section $d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}/d\Xi_{\mathbf{N}}$ of an arbitrary exclusive process is analytic with respect to $\mathbf{n}$ in the complex domain (17.86). We fix some particle κ_3 in the final state and denote by $z = \cos\theta$ the cosine of the angle between the momenta of the particles κ_1, κ_3 in the centre-of-mass frame. In writing down the differential cross section with respect to $\cos\theta$, defined by the formula

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\cos\theta} = N_{\kappa_3}\int\delta\Big(\frac{\mathbf{p}_1\mathbf{p}_3}{|\mathbf{p}_1|\,|\mathbf{p}_3|} - \cos\theta\Big)\frac{1}{\mathbf{N}!}\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\Xi_{\mathbf{N}}}d\Xi_{\mathbf{N}}, \quad (17.87)$$

it is convenient to specify the coordinate frame even further by choosing the unit vector $\mathbf{e}_3$ along $\mathbf{p}_3$:

$$\mathbf{p}_3 = |\mathbf{p}_3|\mathbf{e}_3. \quad (17.88)$$

Then

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\cos\theta} = \frac{N_{\kappa_3}}{4\mathbf{N}!\sqrt{s}k(s)}\int|\mathcal{T}(p_N|p_1,p_2)|^2 d\lambda_{s,N}, \quad (17.89)$$

where

$$d\lambda_{s,N} = (2\pi)^4\delta\Big(s - \sum_{\rho=1}^{r}\sum_{j=1}^{N_r}p^0_{\rho j}\Big)\delta\Big(\sum_{\rho=1}^{r}\sum_{j=1}^{N_r}\mathbf{p}_{\rho j}\Big)\Big|_{\mathbf{p}_3=|\mathbf{p}_3|\mathbf{e}}\frac{|\mathbf{p}_3|^2 d|\mathbf{p}_3|}{(2\pi)^2 2p_3^0}\prod{}'(dp_{\rho j})_{m^{(\rho)}}$$
$$(17.90)$$

(under the $\prod'$ sign we have the elements of volume of momentum space of all the final particles apart from the selected κ_3). According to what we have said above, the (relative) differential cross section (17.89) is analytic with respect to $\mathbf{n}$ in the domain (17.89). On the other hand, it only depends on $\mathbf{n}$ via $\cos\theta$. Hence it follows (see the similar argument in the derivation of Theorem 15.1) that the quantity (17.89) is analytic with respect to z in the Lehmann small ellipse

$$|z+1|+|z-1|<2x_{12}(s). \tag{17.91}$$

The same property of analyticity is enjoyed by the inclusive (relative differential) cross sections

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\cos\theta}=K_{\kappa_3}\int\delta\Big(\frac{\mathbf{p}_1\mathbf{p}_3}{|\mathbf{p}_1|\,|\mathbf{p}_3|}-\cos\theta\Big)\frac{1}{\mathbf{K}!}\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\Xi_{\mathbf{K}}}d\Xi_{\mathbf{K}} \tag{17.92}$$

and the inclusive spectra

$$\frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\cos\theta}=K_{\kappa_3}\int\delta\Big(\frac{\mathbf{p}_1\mathbf{p}_3}{|\mathbf{p}_1|\,|\mathbf{p}_3|}-\cos\theta\Big)\frac{1}{\mathbf{K}!}\frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\Xi_{\mathbf{K}}}d\Xi_{\mathbf{K}} \tag{17.93}$$

(since all these quantities are linear combinations of quantities of type (17.89)).

To extend the domain of analyticity further, we use the unitarity of the S-matrix, from which (in the centre-of-mass frame) it follows that

$$2\operatorname{Im}T(s,z)|_{z=\mathbf{nn}'}=\sum_{\mathbf{N}}\frac{1}{\mathbf{N}!}\int\overline{T(p_{\mathbf{N}}|s,\mathbf{n}')}T(p_{\mathbf{N}}|s,\mathbf{n})d\Lambda_{s,N}, \tag{17.94}$$

where

$$d\Lambda_{s,N}=(2\pi)^4\delta\Big(\sqrt{s}-\sum_{\rho=1}^{r}\sum_{j=1}^{N_r}p^0_{\rho j}\Big)\delta\Big(\sum_{\rho=1}^{r}\sum_{j=1}^{N_r}\mathbf{p}_{\rho j}\Big)d\Xi_{\mathbf{N}}, \tag{17.95}$$

$$T(p_N|s,\mathbf{n})\equiv T(p_N|p_2,p_1)|_{\mathbf{p}_1+\mathbf{p}_2=0}. \tag{17.96}$$

If we substitute into (17.94) the expansion of the amplitude in a series of spherical functions (where $N\equiv N_{\text{in}}$ is defined by (J.4))

$$T(p_N|s,\mathbf{n})=4\pi\Big(\frac{N_{\text{in}}\sqrt{s}}{k(s)}\Big)^{1/2}\sum_{l,m}\sqrt{2l+1}T_{lm}(p_n;s)Y_l^m(\mathbf{n}) \tag{17.97}$$

and take (J.10) into account, we obtain

$$2\operatorname{Im}f_l(s)=\sum_{\mathbf{N}}\frac{1}{\mathbf{N}!}\sum_{m=-l}^{l}\int|T_{lm}(p_N;s)|^2d\Lambda_{s,N}; \tag{17.98}$$

hence we have the inequality

$$\sum_{m=-l}^{l}\int|T_{lm}(p_N;s)|^2d\Lambda_{s,N}\le 2\mathrm{N}!\operatorname{Im}f_l(s). \tag{17.99a}$$

Note that $\sum_{m=-l}^{l}|T_{lm}(p_N;s)|^2$ is an $O_+(3)$-invariant function of the momenta, therefore the integral in (17.99a) can be replaced by the corresponding integral in the system (17.88); as a result we obtain

$$\int\sum_{m=-l}^{l}|T_{lm}(p_N;s)|^2d\lambda_{s,N}\le 2\mathrm{N}!\operatorname{Im}f_l(s). \tag{17.99b}$$

Hence and from the estimate (J.58) we have the following result.

Proposition 17.12. *The amplitude $T(p_N|p_1, p_2)$ of the process (17.62) in the centre-of-mass frame is an analytic function of the vector $\mathbf{n} \equiv \mathbf{p}_1/|\mathbf{p}_1|$ in the domain*

$$\{\mathbf{n} \in \mathbf{C}S^2 : |\operatorname{Re}\mathbf{n}| < y(s)\} \tag{17.100}$$

on the complex sphere $\mathbf{C}S^2$ (where $y(s)$ is defined by (J.56)).

■ Each term of the series (17.97) is analytic with respect to $\mathbf{n}$ on $\mathbf{C}S^2$ (in fact, $Y_l^m(n)$ is a polynomial in $\mathbf{n}$ restricted to $\mathbf{C}S^2$). Furthermore, we have the estimate

$$\sum_{m=-l}^{l} |Y_l^m(\mathbf{n})|^2 \leq \frac{2l+1}{4\pi}(|\operatorname{Re}\mathbf{n}| + |\operatorname{Im}\mathbf{n}|)^{2l} \quad \text{for } \mathbf{n} \in \mathbf{C}S^2. \tag{17.101}$$

Exercise 17.6. Prove (17.101). [Hint: Use (J.10) and the estimate (J.23) for the Legendre polynomials.]

We define for an arbitrary non-negative $v(s)$ in $\mathcal{D}((s_{\mathrm{phys},\mathbf{N}}, \infty))$ the seminorm $\|F\|_v$ for functions of the variables $p_\mathbf{N}$, s:

$$\|F\|_v^2 = \int |F(p_\mathbf{N};\ s)|^2 d\Lambda_{s,\mathbf{N}} v(s) ds.$$

The terms of the series (17.97) then have the estimate:

$$\sum_{l,m} \sqrt{2l+1}\|T_{lm}(p_\mathbf{N};\ s)\|_v |Y_l^m(\mathbf{n})| \leq \left(\frac{2\mathbf{N}!}{4\pi}\right)^{1/2} \sum_l (2l+1) \int \operatorname{Im} f_l(s) v(s) ds (|\operatorname{Re}\mathbf{n}| + |\operatorname{Im}\mathbf{n}|)^l.$$

It clearly follows from (J.58) that this series converges if $\mathbf{n}$ is in the intersection of the domains (17.100) for all $s \in \operatorname{supp} v$. ■

There is an analogous result for the differential cross sections.

Proposition 17.13. *The relative differential cross section $d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}/d\cos\theta$ of the process (17.62) and also the inclusive relative differential cross sections and the spectra*

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\cos\theta}, \quad \frac{d\tau_{\kappa_1\kappa_2\to\mathbf{K}\ldots}}{d\cos\theta}$$

are analytic with respect to the variables $z \equiv \cos\theta$ (where θ is the angle between the momenta of the particles κ_1, κ_3 in the centre-of-mass frame) in the Martin small ellipse

$$|z+1| + |z-1| < 2y(s). \tag{17.102}$$

■ As we have already noted, $d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}/d\cos\theta$ depends on $\mathbf{n}$ only in terms of $\cos\theta$, therefore it suffices to prove that it can be analytically continued with respect to $\mathbf{n}$ to the domain (17.100). For this we substitute (17.97) into (17.89):

$$\frac{d\sigma_{\kappa_1\kappa_2\to\mathbf{N}}}{d\cos\theta} = \frac{4\pi^2 N_{\mathrm{in}} N_{\kappa_3}}{\mathbf{N}!k^2(s)} \sum_{l',m'} \sum_{l,m} \sqrt{2l'+1}\sqrt{2l+1} \times$$

$$\times \int \overline{T_{l'm'}(p_\mathbf{N};\ s)} T_{lm}(p_\mathbf{N};\ s) \overline{Y_{l'}^{m'}(\mathbf{n})} Y_l^m(\mathbf{n}) d\Lambda_{s,\mathbf{N}}, \tag{17.103}$$

and use the Cauchy-Bunyakovsky-Schwarz inequality:

$$\sum_{l',m'} \sum_{l,m} \sqrt{2l'+1}\sqrt{2l+1} \int |T_{l'm'}(p_\mathbf{N};\ s)| \cdot |T_{lm}(p_\mathbf{N};\ s)| \cdot |Y_{l'}^{m'}(\mathbf{n})| \cdot |Y_l^m(\mathbf{n})| d\Lambda_{s,\mathbf{N}} \leq$$

$$\leq \frac{1}{4\pi} \Big[\sum_l (2l+1)(|\operatorname{Re}\mathbf{n}| + |\operatorname{Im}\mathbf{n}|)^l \Big(\sum_m \int |T_{lm}(p_\mathbf{N};\ s)|^2 d\lambda_{s,\mathbf{N}}\Big)^{1/2}\Big]^2. \tag{17.104}$$

It now follows from (17.99b) and the estimate (J.58) that the series in (17.103) (after integrating with respect to s with an arbitrary non-negative function $v(s) \in \mathcal{D}((s_{\mathrm{phys},\mathbf{N}}, \infty))$) converges if $\mathbf{n}$ lies in the domains (17.100) for all $s \in \operatorname{supp} v$. ■

C. ASYMPTOTIC ESTIMATES

It follows from the analyticity of the amplitude with respect to $\mathbf{n}$ that for large s the main contribution in the expansion (17.97) is in fact given by the terms with $l \le l_{\mathrm{eff}}(s)$ ($\le a\sqrt{s}\ln s$). As in §17.1.A, we use this consideration for asymptotic upper estimates of the exclusive and inclusive differential cross sections in the physical domain as $s \to \infty$.

Proposition 17.14. *If the exclusive and inclusive cross sections* $\sigma_{\kappa_1\kappa_2\to\mathbf{N}}$, $\sigma_{\kappa_1\kappa_2\to\mathbf{K}\dots}$ *have a lower estimate of the form*

$$\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{N}}(s) \ge bs^{-k}, \quad \textit{where } s \ge s_1, \tag{17.105}$$

$$\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots}(s) \ge bs^{-k}, \quad \textit{where } s \ge s_1, \tag{17.106}$$

(where b, k, s_1 are positive), then for any two values of $\delta = 0$ or $\delta = 1$ we have the relations

$$\frac{\widehat{d\sigma}_{\kappa_1\kappa_2\to\mathbf{N}}}{d\cos\theta} \le \mathrm{const}\frac{s\ln^2 s}{(\sqrt{s}\ln s\cdot\sin\theta)^{\delta/2}}\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{N}}, \tag{17.107}$$

$$\frac{\widehat{d\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\cos\theta} \le \mathrm{const}\frac{s\ln^2 s}{(\sqrt{s}\ln s\cdot\sin\theta)^{\delta/2}}\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots}, \tag{17.108}$$

$$\frac{\widehat{d\tau}_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\cos\theta} \le \mathrm{const}\frac{s\ln^2 s}{(\sqrt{s}\ln s\cdot\sin\theta)^{\delta/2}}\hat{\tau}_{\kappa_1\kappa_2\to\mathbf{K}\dots}. \tag{17.109}$$

■ These relations are proved by the same method, therefore we shall merely prove (17.108). We make use of (17.103), (17.104):

$$\frac{\widehat{d\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots}}{d\cos\theta} \le \sum_{\mathbf{N}}{}' \frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} \frac{\pi N_{\mathrm{in}}K_{\kappa_3}}{\mathbf{N}!k^2(s')}\Big[\sum_l (2l+1)\Big(\sum_m \int |T_{lm}(p_{\mathbf{N}};\, s')|^2 d\lambda_{s',\mathbf{N}}\Big)^{1/2}\Big]^2 ds'; \tag{17.110}$$

here $\sum'_{\mathbf{N}}$ denotes summation over all channels satisfying the condition $\mathbf{N} \ge \mathbf{K}$. We denote the right hand side of this inequality by $I(s)$ and we denote by $I_{\le}(s)$ (or $I_{>}(s)$) the expression obtained from the right hand side of (17.110) by replacing $\sum_l$ by the summation over $l \le a\sqrt{s}\ln s$ (or over $l > a\sqrt{s}\ln s$) only. It is clear that $I(s) \le (\sqrt{I_{\le}(s)} + \sqrt{I_{>}(s)})^2$. By the Cauchy-Bunyakovsky-Schwarz inequality we have

$$I_{\le}(s) \le \sum_{\mathbf{N}}{}' \frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} \frac{\pi N_{\mathrm{in}}K_{\kappa_3}}{\mathbf{N}!k^2(s')}\Big(\sum_{l\le a\sqrt{s}\ln s}(2l+1)\Big)\sum_{l,m}(2l+1)|T_{lm}(p_{\mathbf{N}};\, s')|^2 d\lambda_{s',\mathbf{N}}ds' \le$$
$$\le \mathrm{const}\cdot s\ln^2 s\cdot\hat{\sigma}_{\kappa_1\kappa_2\to\mathbf{K}\dots}. \tag{17.111}$$

To estimate $I_{>}(s)$ we use (17.99b) as well as the fact that the number of open channels does not exceed $(\sqrt{s}/\min m^{(\rho)})^r$:

$$I_{>}(s) \le \sum_{\mathbf{N}}{}' \frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2} \frac{2\pi N_{\mathrm{in}}K_{\kappa_3}}{k^2(s')}\Big(\sum_{l>a\sqrt{s}\ln s}(2l+1)\sqrt{\operatorname{Im} f_l(s')}\Big)^2 ds' \le$$
$$\le \mathrm{const}\cdot s^{r/2}\Big(\sum_{l>a\sqrt{s}\ln s}(2l+1)\sqrt{\operatorname{Im}\hat{f}_l(s)}\Big)^2. \tag{17.112}$$

As in the derivation of the Froissart bound (see Proposition 17.1), we use the estimate (J.58) to show that for any $n > 0$ and sufficiently large a, the quantity $I_>(s)$ is infinitesimally small by comparison with s^{-n} as $s \to \infty$; by virtue of (17.106), it can be neglected (for a suitable choice of a) in comparison with the right hand side of (17.111) as $s \to \infty$. Thus we have proved (17.108) for $\delta = 0$.

To prove (17.108) for $\delta = 1$, we note that the left hand side of (17.103) only depends on $\mathbf{n}$ in terms of $\cos\theta$, that is, in terms of the projection of the vector $\mathbf{n}$ onto $\mathbf{e}_3$ (in the frame (17.88)). Therefore equality still holds if on the right hand side of (17.103) we replace $\mathbf{n}$ by $R\mathbf{n}$ (where R is an element of the group $O_+(2)$ of rotations in the plane orthogonal to $\mathbf{e}_3$) and average over R. As a result, only the terms with $m' = m$ remain in (17.103). To majorize the terms of the series so obtained, we use the inequality

$$|Y_l^m(\mathbf{n})| \le \left(\frac{3(2l+1)}{2\pi^3|\sin\theta|}\right)^{1/4} \quad \text{for } \mathbf{N} \in S^2, \tag{17.113}$$

which follows from (J.10) and (17.13):

$$\begin{aligned}|Y_l^m(\mathbf{n})|^2 &= \frac{2l+1}{8\pi^2}\int_0^{2\pi} P_l(\cos^2\theta + \sin^2\theta\cos\phi)e^{-im\phi}d\phi \le \\ &\le \frac{1}{8\pi^2}\sqrt{\frac{6(2l+1)}{\pi}}\int_0^{2\pi}(1-(\cos^2\theta+\sin^2\theta\cos\phi)^2)^{-1/4}d\phi \le \\ &\le \frac{1}{8\pi^2}\sqrt{\frac{6(2l+1)}{\pi(\sin\theta)}}\int_0^{2\pi}(1-|\cos\phi|)^{1/4}d\phi;\end{aligned}$$

here

$$\int_0^{2\pi}(1-|\cos\phi|)^{-1/4}d\phi \le 4\int_0^{\pi/2}\left(\frac{2\phi}{\pi}\right)^{-1/2}d\phi = 4\pi.$$

By applying the Cauchy-Bunyakovsky-Schwarz inequality we then obtain the relation

$$\frac{\widehat{d\sigma}_{\kappa_1\kappa_2\to\mathrm{K}\dots}}{d\cos\theta} \le \sum_{\mathrm{N}}{}'\frac{1}{\Delta}\int_{s-\Delta/2}^{s+\Delta/2}\frac{4\pi\sqrt{3}N_{\mathrm{in}}K_{\kappa_3}}{\mathrm{N}!k^2(s')\sqrt{\sin\theta}}\Big[\sum_l(2l+1)^{3/4}\sum_m(|T_{lm}(p_{\mathrm{N}};\ s')|^2d\lambda_{s',\mathrm{N}})^{1/2}\Big]^2 ds',$$

analogous to (17.110). The same reasoning as in the case $\delta = 0$ then leads to the inequality (17.108) for $\delta = 1$. ■

The estimates that we have obtained have a general character and do not depend on any model assumptions. In particular, the bounds of type

$$\frac{\widehat{d\tau}_{\kappa_1\kappa_2\to\kappa_3\dots}}{d\cos\theta} \le \text{const}\cdot\langle N_{\kappa_3}\rangle\frac{s^{3/4}\ln^{7/2}s}{\sqrt{\sin\theta}} \tag{17.114}$$

for the inclusive spectra (following from (17.109) and the Froissart bound (17.3)) are of interest for the phenomenology as an indication of the extent to which the scale invariance of inclusive spectra can be broken without contradicting the general principles.

Commentary on the Bibliography and References

Chapter 1. For a further acquaintance with normed linear spaces (§1.1) and their applications see [K2, D6]. There is a very thorough account of the theory of LCS's (§1.2) in the book [S2]; concerning Fréchet spaces (§1.3) see [Y1, R1,Vol. 1, R4]. The theory of countably normed spaces was originally presented in [G5, G6]. See [A4, R1] concerning unbounded operators in Hilbert space and spectral theory (§1.4). The monographs [D4, D5, N2] are mathematical texts on the theory of C^*-algebras and von Neumann algebras (§1.5) (see also [R1, R3, S1, E6, B13]). We have not given due attention to general Banach algebras (especially with involution) (they occur naturally in quantum theory with an indefinite metric); there is a thorough account of them in the books [R3, N2]. See Araki and Woods (1968), Araki et al. (1971), Araki (1975), Connes (1973,1976) concerning the classification of von Neumann algebras. Of those relating to the contents of this chapter (and also Ch. 8,10), topological (non-abelian) *-algebras are worthy of attention (Vasil'ev, 1970; Powers, 1971,1974; Lassner, 1972; Lassner and Timmermann, 1972); in Ch. 8 we come across the special Borchers algebra (Borchers, 1961,1972; Yngvason, 1973).

Chapter 2. A systematic account of the theory of generalized functions can be found in the monographs by Schwartz [S3], Gel'fand et al. [G4, G5] and Vladimorov [V5]. Generalized functions as linear functionals on smooth functions (with compact support) were first introduced by Sobolev (1936). Schwartz (1945) (see also [S3]) developed Sobolev's ideas and drawing upon the general theory of topological vector spaces created the modern theory of distributions and generalized functions as linear functionals over the spaces $\mathcal{D}$ and $\mathcal{S}$. An equivalent definition of generalized functions in $\mathcal{S}'$ as functionals over the classes $C(p,q;m)$ was used by Bogolubov (see, for example, [B10] §§18,19). Mikusiński [M9] developed an approach to generalized functions as classes of fundamental sequences. Bargmann (1967) found an interesting isomorphism between the space of generalized functions $\mathcal{S}'$ and a certain subspace of entire analytic functions. The remark concerning the impossibility of giving a natural definition of the product of two arbitrary generalized functions (§2.3.A) is due to Schwartz (1954); he is also responsible for the notions of multiplicator (§2.3.A), Fourier transforms of generalized functions (§2.5.A), convolutes (§2.5.C) and generalized functions of integrable type (§2.5.D) (called by him distributions of class $\mathcal{D}'_{L_1}$); all these are to be found in [S3]. The problem of division by a polynomial (§2.3.B) for the case of generalized functions of one variable was solved by Schwartz [S3] and for several variables by Malgrange (1953), Ehrenpreis (1954) and Hörmander (1958). The kernel theorem (§2.4.A) for the spaces $\mathcal{D}$ and $\mathcal{S}$ was proved by Schwartz (1952a). Schwartz (1953/54) also introduced the notion of a nuclear map. The general definition of nuclear spaces was given by Grothendieck (1955). An accessible account of these questions can be found in the book [G6], also in [R1] (Appendix to §V.3). In the subsection on the convolution (§2.5.E) we have drawn considerably on the book [V5] (in addition to [S3]), although we have used a slightly different definition of the convolution. The reader can find another account of multiplication of generalized functions different from that in §2.6.C in [R1] (§IX.10). See Bogolubov and Parasyuk (1957), Hepp (1966b), Speer [S13] with regard to multiplication of causal functions in quantum field theory. Vector-valued and operator-valued generalized functions (§2.7) were introduced by Wightman and Gårding (1952, published in 1965); see also Schwartz (1957, 1959) and [S16], §3.1. Gel'fand and Kostyuchenko (1955) developed an approach to generalized eigenvectors (different from that set out in §2.7.C) based on the notion of rigged (or nested) Hilbert space (see also [G6]). The existence of a complete system of generalized eigenfunctions can also be stated in terms of a pair of Hilbert spaces (see Berezanskii, 1959; Kac, 1960). The papers by Böhm (see his lecture (1967) and the references to earlier papers therein), Grossmann (1965, 1966, 1967) and Antoine (1969) are devoted to the various applications of the theory of rigged Hilbert spaces in quantum theory.

Appendix A. With regard to §A.2, see the book [M3],Ch.1. The notion of regular set is due to Whitney (1934) (although the definition we have used is not identical to his); see also [V9], §§3.5, 3.8 where a result analogous to Corollary A.2 is given.

Appendix B. The classical theory of the Laplace transform is set out in the monograph [W4]. The approach to the Laplace transform set out here is due to Schwartz (1952b) and Lions (1952/53) who were responsible for the results in §B.1. The results of §§B.2–B.4 were obtained by Vladimirov (1968; 1973b; [V4], §26) and Bros et al. (1967); see also [S16] §2.3 and the Epstein lectures (1966), §2.4. Vladimirov (1969a, 1972) obtained an integral representation of Cauchy-Bochner type for functions that are analytic in tubular domains and have less than exponential growth (in particular, for the Laplace transform of generalized functions with support in a pointed cone).The method of regularization or "subtractions") in dispersion relations is set out in §B.5 for the first time. In §B.6, results of Osterwalder and Schrader (1973; see also Osterwalder, 1973a) are given. See also the article by Zinov'ev (1979) concerning the inverse Laplace transform.

Appendix C. For the construction of HGF's of a single real variable (§C.2) by the method of "analytic regularization" (going back to Riesz, 1949) we have followed the book [G4] (which, more generally, serves as a reference on HGF's). HGF's of complex variables are considered in the appendix to [G4]. HGF's and associated HGF's related to quadratic forms ([G4], §III.2) are of interest for applications to quantum field theory.

Chapter 3. See [W6, V3, G3, N1, S16], also Bargmann (1962) concerning finite-dimensional representations of the rotation group and the Lorentz group (§3.1). See [V3], Bargmann and Todorov (1977) concerning representations of class 1 (in the space of symmetric tensors) of the rotation group $SO(n)$. A detailed treatment of the homomorphism $L_+^\uparrow \to SL(2,C)$ can be found in MacFarlane (1962). The general form of Lorentz-invariant generalized functions of a single 4-vector (§3.2) was found by Methée (1954) (see also 1955, 1957); there is also a brief account in Gårding and Lions (1959). In §3.3.B we have followed Appendix C of the paper by Oksak and Todorov (1969). Lemma 3.9 and the general form of the rotation-invariant generalized function of a single vector (Example 1 in §3.4.1) was suggested by Schwartz (1954/55). See Oksak (1976) for the proof of Propositions 3.1—3.14. Example 2 in §3.4.1 was obtained by Hepp (1964b), who also justified (for the scalar case) the method of transferring to the rest frame of a variable vector in V^+ (§3.4.C).

Appendix D. The book by Pontryagin [P2] is the fundamental text in the theory of continuous groups. An interesting and fresh account (in the form of lectures in its original version) is given in the book by Zhelobenko [Z2]; the survey by Gürsey (1964), which is smaller in volume, is very good. Of the general text books addressed specifically to physicists, we can recommend [W5, L3, H1], Michel (1970), [B2, E5]. The book [G10] is devoted to semisimple Lie algebras; here (§3.5) a proof of Theorem D.1 (Cartan-Levi-Mal'tsev) can be found. See [P2] and also the survey by Gürsey mentioned above for a proof of Lie's theorem (§D.5). A thorough treatment of the representations of the rotation and Lorentz groups can be found in the monographs [G3, N1]. There are a number of monographs of handbook type [Z3, K6, K7] on infinite-dimensional representations of Lie groups; see also [V3, Z3], Naimark (1964). The book [N3] is devoted to the application of compact groups to the description of symmetries of elementary particles (see the lectures by Bogolubov, 1967 for an elementary account of these applications). These groups retain their significance even for a modern systematization of elementary particles and their interactions (see, for example, the surveys by Langacker, 1981, Slansky, 1981). Co-adjoint representations (§7.D) have found interesting applications (Kirillov, 1962; [K6], §15) in representation theory and in quantum theory (in the so-called geometric quantization approach, initiated by Segal, 1960; see also Souriau, 1966, Kostant, 1966, 1970, [W6]).

Chapter 4. The integral representation considered here was originally put forward by Jost and Lehmann (1957) for the "symmetric case" (§4.3.C); the general case was considered by Dyson (1958b). In our account we have followed Dyson's approach taking into account the arguments of Wightman (1960b); in §4.2.C we have used §31 of [V4]. A simple derivation of the "symmetric" JLD representation can be found in the paper by Vladimirov and Zav'yalov (1980). There are other derivations of the JLD representation partially or wholly based on the theory of analytic functions of several complex variables ([V4], §32; Bros et al., 1966). In the article by Vladimirov and Zharinov (1970) the JLD type representation is generalized to the case when instead of light cones $V^\pm$, cones $\pm C$ are taken, where C is an arbitrary pointed cone in $\mathbf{R}^n$. Along these lines one can justify the representation of Nakanishi type (1961) for the amplitude of relativistic two-particle scattering in the approximation of plane diagrams (Zharinov, 1971). Generalizations of the JLD representation for classes of distribution that are broader than the Schwartz space $S'(\mathbf{R}^n)$ can be found in the works of Jaffe (1966, 1967), Lazur and Khimich (1977a), and B.I. Zav'yalov (1981). Another generalization of the JLD representation can be found in Sénéor (1969). The JLD representation was presented as an effective means of obtaining analytic properties of scattering amplitudes (see Part V concerning this). It has found further applications in the work of Bogolubov et al. (1972) for the analysis of the automodel behaviour of the structure functions of deeply inelastic scattering of an electron by a proton using the causality and spectral principles (see also B.I. Zav'yalov, 1973; Vitsorek et al., 1973), as well as for the analysis of the singularities of the commutators of fields on the light cone (see B.I. Zav'yalov, 1973, 1974, 1977, Brünning and Stichel, 1974; Vladimirov and B.I. Zav'yalov, 1979, 1980, 1982). The result of Exercise 4.17 was proved by another method by Bogolubov and Vladimirov (1958–1959); it follows from the representation (4.136) that the generalized functions $h_{\pm}(p)$ transform according to a finite-dimensional representation of the Lorentz group (or are "finitely-covariant" in the terminology of Bros et al., 1967). If the variables p and x are interchanged, this result has a quantum field theoretic interpretation: even without the hypothesis on Lorentz-covariance of the fields, it follows from the spectral and locality axioms in the Wightman theory that the two-point Wightman function is automatically "finitely-covariant". This result was subsequently generalized to n-point functions (Streater, 1962a; Bros et al., 1967; Bogolubov and Vladimirov, 1971). See Streater (1972) concerning the application of the result to field theoretic models.

Chapter 5. The systematic accounts in Bochner and Martin [B6], Vladimirov [V4] and Shabat [S8] are devoted to the classical results of complex analysis. The monograph [V4] is of interest to us because of its orientation towards field theory. Other methods of complex analysis can be found in [G1, W1]. The "edge of the wedge" theorem (§5.1.D) was obtained by Bogolubov (1956); the first complete proof of it appeared in the monograph [B9] (Appendix A, Theorem 1). The name "edge of the wedge" theorem was suggested by Bremermann et al. (1958); a proof of the theorem is given in this paper but under stronger hypotheses concerning the continuity of the boundary values. In the original version of the theorem, only the case of "oppositely placed cones" Ω_1 and $\Omega_2 = -\Omega_1$ was considered (here, the "edge of the wedge" domain is a complete complex neighbourhood of the real domain of coincidence $\mathcal{O}$). Epstein (1960) extended the theorem to the case of "inclined cones" Ω_1 and Ω_2 (when $\Omega_1+\Omega_2 \neq \mathbf{R}^n$). In our proof we have used an idea due to Epstein (1966) (which, as Epstein points out, goes back to Malgrange and Zerner) of reducing Theorem 5.12 to Lemma 5.11. Other proofs of the "edge of the wedge" theorem can be found in the works of Vladimirov (1962; [V4], §27) and in the book [S16],§3.5; here the ideas of Dyson, 1958a, and Glaser are used. Concerning further generalizations, see Browder (1963), Martineau (1964), Borchers (1964), Araki (1963c), Holm and Nagel (1968). Proposition 5.15 is an analogue of the theorem of Hall and Wightman (1957) on Lorentz-invariant analytic functions in the future tube $T_n^+ = \mathbf{M}^n + i(V^+)^n$; this theorem was subsequently generalized by Hepp (1963c) (instead of our "technical" condition (5.35), Hepp requires I-saturatedness of the domain D, which restricts the applicability of the proposition). In the same article, Hepp proves Proposition 5.16 (for a more general situation). The characterization of domains of holomorphy (§5.2.A) are due to Cartan and Thullen (1932); the connection between holomorphic convexity and pseudo-convexity * (§5.2.B) was established by Oka (1953) and Bremermann (1954b, 1956). The modified continuity principle (§5.2.C) is a somewhat different form of the so-called "strong theorem on the disc" by Bremermann ([V4], §17.3) in which instead of real analyticity with respect to the deformation parameter t, he requires that the deformations with respect to the parameter t reduce to translation of the "disc" by a vector $\lambda(t)b$ lying on a one-dimensional complex line. Proposition 5.34 (Bochner) and 5.36 (Hartogs), as well as Corollary 5.38 relate to the very first results on envelopes of holomorphy (more precisely, holomorphic extensions, see [B6]; the interpretation in terms of envelopes of holomorphy was given later; see [B6], §§21.1,21.2 in connection with Propositions 5.34, 5.36; an alternative proof of Corollary 5.38 is given in Epstein (1966), §2.3). Proposition 5.35 was proved by Bremermann (1954a). More complete information on methods of constructing envelopes of holomorphy can be found in the book [V4] (§§21,27,33 and elsewhere). In connection with holomorphic extensions, we note the theorem due to Vladimirov (1960; [V4], §28) on the C-convex hull (it is applied, in particular, for the proof of the result due to Vladimirov and Petrina mentioned in §9.1.E); an analogous result for the special case of light cones, the " diamond theorem", was subsequently obtained by Borchers (1961). With regard to the applications of complex analysis in quantum field theory, see the papers by Bogolubov

*Note that our definitions of holomorphic convexity and pseudo-convexity differ from the generally accepted ones, but they are, of course, essentially equivalent.

and Vladimirov (1958, 1960), Wightman (1960b, 1960/61), Epstein (1966), and Vladimirov (1973a).

Chapter 6. The algebraic formulation of quantum theory (§6.1) was put forward by von Neumann (1936) and (in the language of C^*-algebras) by Segal (1947, 1959a,b). The requirement that the observables be generated by a C^*-algebra (with closure with respect to the algebraic operations and convergence in norm) is a very strong non-intuitive conjecture. Therefore in the axiomatic accounts proper, which set as their aim the construction of a physical theory on the basis of elementary empirically justified hypotheses, this conjecture splits into several propositions arranged in decreasing order of motivation ([V1, M1], Sherman, 1956; Plymen, 1968). The connection between the C^*-algebra formulation and the more general statistical approach has been studied by Jauch and Piron (1963), Davies (1970), Davies and Lewis (1970), Edwards (1969,1971), and Kholevo (1976), [K5]. The algebraic formulation was comprehensively analysed in the context of local quantum field theory by Haag (1958, 1959a), Araki (1961/62), Haag and Schroer (1962) and especially by Haag and Kastler (1964) (see also the surveys by Wightman, 1964, Robinson, 1965, Guenin, 1966b, Haag, 1970, Polivanov et al. 1973). Haag and Kastler (1964) introduced the notion of physical equivalence of representations (which is the same as the concept of weak equivalence due to Fell (1960). The first to draw attention to the existence of superselection rules in elementary particle physics was Wick et al. (1952) (see Hagedorn, 1959 concerning their role in the theory of symmetries). The concept of a sector as a set of pure states associated with an irreducible representation of the algebra of observables is also presented in the above paper by Haag and Kastler. The definition (6.4) of the transition probability between pure states was proposed by Roberts and Roepstorff (1969) (see Uhlmann, 1976 concerning the transition probability for arbitrary states). The superselection rule (§6.2) has been discussed by Wightman, (Wightman and Barut, 1959; Wightman, 1964; [S16], §1.1), Kharatyan (1968, 1973), Sushko and Khoruzhii (1970). See the survey by T.S. Todorov (1975) concerning superselection rules in elementary particle physics. See Strocchi (1976a) concerning the connection between superselection rules and the infrared problem in quantum chromodynamics. The papers by Doplicher et al. (1969, 1971, 1974), Khoruzhii (1975, 1975–1976) are devoted to a general analysis of superselection rules in the local theory. In the section on symmetry (§6.3) essential use is made of the work of Roberts and Roepstorff (1969) as well as that of Kadison (1965). A geometric proof of (Wigner's) Theorem 6.8 on symmetries can be found in the book [W5]; it was later modified by Bargmann (1964); an elementary proof was also put forward by Wick (1966). The algebraic proof of Wigner's theorem (§6.3.B) was the first to be published (another algebraic proof for a more general situation can be found in [B13]; see Theorem 3.2.8 there). Wigner's theorem was extended by Brach et al. (1975) to pseudo-Hilbert spaces with an indefinite metric (our proof is applicable to this case). In connection with Proposition 6.13 due to Bargmann (1954), see also Parthasarathy (1969), Simms (1971), and de Swart (1974). The uniqueness theorem 6.14 was obtained by von Neumann (1931) (see also Jost, 1974 in this connection). Our proof (§6.4.C) follows [E6], §3.1.3. Our construction of the C^*-algebras of the CCR's (§6.4.C) was proposed by Segal (1961, 1962, [S6]). The criterion for the unitary realization of linear symplectic transformations was given by Shale (1962) (see also [B4] §4.3). Concerning further results for systems of CCR's with an infinite number of degrees of freedom, see Araki and Woods (1963), Loupias and Miracle-Sole (1966), Chaiken (1967, 1968), Manuceau (1968a,b), Manuceau and Verbeure (1968), Zav'yalov and Sushko (1969, 1973), Araki (1971), Kholevo (1971, 1972), [E6], Carey et al. (1982).

Chapter 7. The material on the Poincaré group and its Lie algebra (§7.1) is to be found in Wightman (1960a), Joos (1962), Michel (1964), and [S8, I5]. The formulation of the relativistic invariance principle (§7.2.A) goes back to Wigner (1939; see also [W5]). The postulates of the spectrum and the existence and uniqueness of the vacuum (§7.2.B) were first clearly stated by Wightman (1956) in a set of axioms for quantum field theory. See Borchers (1966) concerning energy and momentum as observables. The role of the condition of the uniqueness of the vacuum and its connection with the cluster property was pointed out in the paper by Hepp et al. (1961); it has also been discussed by Reeh and Schlieder (1962), Jordan and Sudarshan (1962). Our proof of Proposition 7.1 is close to that of Jost ([J3], §III.6) which does not in fact use the locality axiom. A discussion of the question of the existence and role of the vacuum state can be found in Borchers (1962, 1965b) and Borchers et al. (1963). The classification of the irreducible representations of the proper Poincaré group was obtained by Wigner (1939; see also 1964). An abstract formulation of the method of induced representations, which was in fact used by Wigner (§7.2.C) was given by Mackey (1952, 1958) (see also Bruhat, 1956); the books [M2, M8] are devoted to a detailed account of the method. There is a comprehensive survey on unitary representations of the Poincaré group in the paper by Joos (1962) cited above; see also

Yu.M. Shirokov (1957–1959; 1960a,b), Wightman and Barut (1959), Wightman (1960a), Moussa and Stora (1965), Guillot and Petit (1966). With regard to physical applications of representations of class (d) (with $p^2 < 0$) see, for example, Joos (1965). The properties of the spectral decomposition of translation operators in the relativistic theory were studied by Wightman (1961). A manifestly covariant realization of the "physical" irreducible representations in terms of spin-tensors of wave functions has been given by Joos (1962) in the massive case and Zwanziger (1965) in the massless case; (this formalism is adopted in many papers dealing with arbitrary spin; see, for example, Barut et al., 1963; Hepp, 1964a; Weinberg, 1964a,b; [S16]; Cohen-Tannoudji et al., 1968; Mutze, 1975). In connection with the interesting question, not touched upon in this chapter, on the decomposition of a direct product of irreducible representations of the Poincaré group, see Chou Huan Chow and Shirokov (1958), Lomont (1960), Shirokov (1961), Jacob and Wick (1959), Wick (1962), Moussa and Stora (1965) (there are additional references in this article), Schaaf (1970). We note also that in recent years, so-called supersymmetric models of field theory have been extensively studied. Here one uses the new concept of invariance with respect to supergroups (and superalgebras) obtained from the Poincaré group (and its Lie algebra) by adding generators of fermionic type; see the surveys by Ogievetskii and Mezinchesku (1975), Wess (1976), Salam and Strathdee (1978), [S17] concerning this.

The method of second quantization (§§7.3.A, 7.3.B) was proposed by Fock (1932). It became the object of special mathematical examination in the papers by Cook (1953), Segal (1956, 1958), Berezin [B4]; see also [R1], §IX.7. In the papers by Berezin (1967, 1971) there is a study of properties of the expansions of bounded operators in Fock space in a series of normal products of creation and annihilation operators. Concerning the choice of phases in the operations of spatial reflection and time reversal (§7.3.D) see Feinberg and Weinberg (1959), Grawert et al. (1959), Wigner (1964), Parthasarathy (1969). A mathematical examination of the connection between charge conjugation and other discrete symmetries with the gradient transformations relating to the preservation of baryonic and leptonic charges (on the basis of the theory of group extensions) is given in Michel (1964). The fact that time reversal is realized by an anti-unitary operator was first noted by Wigner (see [W5]). Wigner (1960a,b) also investigated the normal form of anti-unitary operators and their phenomenological properties.

Concerning the discrete symmetries (P, T, CP) and the breaking of them in elementary particle physics see T.D. Lee and Yang (1956), Lee et al. (1957), [K1], T.D. Lee (1966, 1967), [L4], Okun' (1966), Arbusov and Filippov (1967), Wick (1967), Kleinknecht (1974), [G7, O1], Chau (1983). The principles underlying the quantum theory of scattering (§7.3.E) are set out in the papers by Haag (1959b), Brenig and Haag (1959), Hepp (1966a), Araki (1969), Goldberger and Watson [G9], Iagolnitzer [I1]. Proposition 7.6 was proved by Hepp (1964b, 1966a). As to a concrete choice of standard covariants in the expansion (7.200) which provide covariant expansions that are free of so-called kinematic singularities, see Williams, 1963, Hepp, 1964a, Mahoux and Martin, 1968, Cohen-Tannoudji et al., 1968, Guertin, 1971.

Appendix E. Dirac's theory is set out in many books on relativistic quantum theory (see, for example, [B10, K3, S5, B5]). A mathematical treatment of Clifford algebras can be found in [A2] (§5.4) as well as in Rachevskii (1955, 1958). Corollary E.2 is known as Pauli's lemma (1936). The discrete transformations of the Dirac spinors are considered in many monographs (in particular, [M7, O1]); details on this question are set out in a series of articles by Winogradzki (1957–1959) (where, however, the "Euclidean" definition of the γ-matrix for which $[\gamma^\lambda, \gamma^\mu]_+ = \delta^{\lambda\mu}$ is used). In the survey by Joos (1962) the Dirac equation and the four-component spinor functions are examined from the point of view of representations of the Poincaré group.

Chapter 8. The contents of this and the following chapters overlap to a large extent with the monographs [S16, J3] written by the founders of the axiomatic method of quantum field theory. In its original version, the algebraic approach to local quantum field theory was stated in terms of von Neumann algebras of bounded operators: Haag (1959a, b), Araki (1961/62), Haag and Schroer (1962) (see, in particular, their discussion of the independence of the axioms), and Wightman (1964). The formulation in terms of nets of abstract C^*-algebras (§8.1.A) is due to Haag and Kastler (1964) (in this connection, see Kastler, 1964 and the lectures by Robinson, 1966 and Haag, 1970). The clarification of the physical meaning of the locality condition (§8.1.B) goes back to Bohr and Rosenfeld (1933, 1950) (in this connection, see Benioff and Ekstein, 1977). The condition of the positivity of the energy in this approach was given by Doplicher (1965). In the papers by Doplicher et al. (1969), the structure of the algebra of local observables in the field algebra was studied on the basis of the principle of gauge invariance (of the first kind); in addition, the highly non-trivial converse problem of the recovery of the field algebra was outlined. A fundamental publication on the Wightman formalism is the paper by Wightman and Gårding (1965), which in fact is according to a paper begun in 1952 and partially set out in articles and lectures by Wightman (1956, 1959, 1963). Wightman (1956) originally used as the test functions of the x-variables the Schwartz space $\mathcal{D}(\mathrm{M})$ of functions with compact support; however, since the generality attained by this is not all that interesting, the more convenient space of type $\mathcal{S}$ (which is invariant on going over from the x- to the p-variables) was subsequently adopted (after Schmidt and Baumann, 1956). In addition to the monographs [S16, J3] there are a number of surveys on the Wightman formalism (Araki, 1961a; Wightman, 1963, 1976; Todorov, 1964, 1965; Hepp, 1966a; Streater, 1975). The conditions of the essential self-adjointness of the field operators under which, one can establish a correspondence with the local von Neumann algebras (§8.2.B) were given by Borchers and Zimmermann (1964), Gachok (1965, 1966a, b), Berezanskii (1966). See Araki (1963b), Doplicher et al. (1969), Khoruzhii (1969, 1970), Bisognano and Wichmann (1975) concerning the duality condition. In connection with §§8.2.C, 8.2.D we note that there are a number of results on the coincidence and difference of local von Neumann field algebras corresponding to different regions of space-time (see Borchers, 1962; Wightman, 1964); it was concluded on this basis that local von Neumann algebras are of infinite type (Kadison, 1963; Guenin and Misra, 1963; Wightman, 1964). The separating property of the vacuum vector for local field algebras (see Propositions 8.3, 8.4) was pointed out by Burgoyne (1958) and by Jost (1961). The spectral representations for two-point Green's functions (closely related to the two-point Wightman functions §8.3.B) were obtained by Kamefuchi and Umezawa (1951), Källén (1952) and Lehmann (1954). In connection with the covariant representations for two-point Wightman functions, see also Källén (1960, 1968/69), Lovitch and Tomozawa (1962), Steinmann (1963b). The Källén-Lehmann representation finds applications, in particular, in perturbation theory. As an example, we point out that summation of several Feynman diagrams under the spectral representation sign avoids the appearance of non-physical singularities in the Green's functions (see Bogolubov et al., 1959). The remark on the non-existence of a non-trivial Wightman field with well-defined values at every point (Exercise 8.8) is due to Wightman (1964) (see also Wizimirski, 1966). The result due to Borchers (1964) is of interest in this connection; according to this, it is sufficient to smooth the quantum fields $\phi(x)$ only with respect to the time coordinate x^0 (the spatial vector $\mathbf{x}$ can be regarded as a parameter). The reconstruction theorem (§8.3.C) was given in Wightman's original paper (1956); it enabled all the information on the quantum field model to be contained in the set of all the Wightman functions. The algebra of test functions used in the reconstruction process was introduced by Borchers (1962); its mathematical structure is examined in detail in the papers by Uhlmann (1962), Morena (1963a, b), Lassner and Uhlmann (1968), Borchers (1965a, b, 1972), Wyss (1972), Yngvason (1973), Borchers and Yngvason (1975), Hegerfeldt (1975), Brünning (1978). Wyss (1973) proposed a generalization of the Wightman formalism starting from a (multiplicatively) positive definite Hermitian form on the Borchers algebra instead of a (multiplicatively) positive linear functional. The standard account of free scalar and Dirac fields (§§§8.4.A–8.4.C) can be found in any textbook on quantum field theory (see, in particular, [B11, S8, B8, K3]). A compact survey of the results from the point of view of the Lagrangian formalism is given in Ch. 2 of the monograph [J3]. Equations for fields with higher spins have been studied in papers by Duffin (1938), Kemmer (1939), Belinfante (1939), Fierz and Pauli (1939), Gårding (1944), Bargmann and Wigner (1948), Gel'fand and Yaglom (1948a, b) (see also [U1, G3, N1]; Harish-Chandra, 1947; Foldy, 1956; Wigner, 1963; Weinberg, 1964a, b; Parke and Jehle, 1965). The difficulties of field theories with high spins have been discussed by Wightman (1968, 1973). Proposition 8.8 (according to which the two-point function completely defines the free field in the generalized Wightman theory) was proved by Federbush and Johnson (1960) and by Jost and Schroer (Jost, 1961); there is a further discussion in Greenberg (1959) and Gachok (1961); the zero-mass case was proved by Pohlmeyer (1969). A simple illustration of the phenomenon of spontaneous breaking of internal symmetries by the example of the transformations (8.88) of a free massless scalar field was considered by Streater (1965a). The net of local von Neumann algebras corresponding to a free scalar field was investigated by Araki (1963b; 1964a, b), Kadison (1963), Guenin and Misra (1963), Kastler (1965), Langerholc and Schroer (1965), Dell'Antonio (1968), Dadashyan and Khoruzhii (1978, 1981). In particular, Araki proved the duality property (see also Osterwalder, 1973b). Generalized free fields were introduced by Greenberg (1961) (these are in fact integrals of free fields with respect to the mass; see Licht, 1963). Conditions on the support of a field in p-space under which this field is a generalized free field were first stated by Dell' Antonio (1961a).

Appendix F. The summary of the most important singular functions of quantum field theory is borrowed from the monograph [B10].

Appendix G. The analogous covariant representations are considered in the paper by Oksak and Todorov (1969).

Chapter 9. Hall and Wightman (1957) gave a representation for Lorentz-invariant holomorphic functions in the future tube $\mathcal{T}_n^+$ (and in $\mathcal{T}_n^-$). This paper contains the first proof to be published of Bargmann's theorem 9.1 for the scalar case. Wightman (1960/61) and Jost (1961) generalized the theorem to the case of covariant functions transforming according to an arbitrary finite-dimensional representation of the Lorentz group. Various generalizations of Hall and Wightman's theorem can be found in Hepp (1963a, c, 1964a), Minkowski et al. (1965), Seiler (1966). The real points of the extended tube (§9.1.C) were described by Jost (1957). A description of the extended tube can be found in Wightman (1960/61). The single-valuedness of the Wightman functions in the symmetrized tube (§9.1.D) was proved by Tomozawa (1963b) and Ruelle ([J3], Appendix II); this fact also follows from a result of Tomazawa on the simple connectedness of the symmetrized tube. In the case $n = 2$, the symmetrized tube $\mathcal{T}_2^S$ is the same as the extended tube $\mathcal{T}_2$. This is no longer the case for $n \geq 3$. The envelope of holomorphy of $\mathcal{T}_3^S$ was found by Källén and Wightman (1958) (see Ruelle, 1961a). Integral representations for the three-point Wightman function were put forward by Källén and Toll (1960) and Streater (1960a,b). In connection with the description of the extended tubes for $n = 4$ and $n = 5$, see the articles by Manoharan (1962), Möller (1962), and Brünning (1971). In the general case, the envelope of holomorphy of the symmetrized tubes $\mathcal{T}_n^S$ has not been explicitly written down. The global nature of locality (Proposition 9.12) was stated as a conjecture by Wightman (1959) and proved by Wightman (1960/61) jointly with Jost and Steinmann (see also Tomazawa, 1963a; [S16], §4.1). A stronger result (without the assumption of Lorentz-covariance) was obtained in the papers by Vladimirov (1960) and Petrina (1961). The result was further strengthened by Pohlmeyer (1968) (see also Borchers and Pohlmeyer, 1968); it turns out that in order that strict locality should hold, it suffices to assume (within the framework of the concept of an almost local field introduced by Streater, 1964) that the matrix elements of the commutators of the fields decrease more rapidly than $\exp(-\epsilon|\mathbf{x} - \mathbf{y}|^p)$ $(p > 1)$ at spatial infinity. The connection between non-renormalizability (in the traditional meaning) and non-locality enters into the ideology of Lagrangian perturbation theory (see [B10], §32). Schroer (1964) gave a graphic illustration of this by means of a simple model. One of the approaches to renormalizable theories consists in choosing classes of generalized functions that are compatible with the microcausality condition (Meiman, 1964; Jaffe, 1966, 1967; Efimov, 1968, [E3]; M.A. Solov'ev, 1971, 1980b,c). Another approach concentrates attention (in the spirit of the "reconstruction theorem") on Wightman functions in p-space (Bardakci and Schroer, 1966; Güttinger, 1966; Khoruzhii, 1967, 1968/69, Ioffa and Fainberg, 1969a,b; J.G. Taylor, 1971; Constantinescu, 1971, V.Ya. Fainberg, 1972; Bümmerstede and Lücke, 1974; Lazur and Khimich, 1977b; V.Ya. Fainberg and M.A. Solov'ev, 1978). These papers extend the Wightman formalism to non-renormalizable theories, although they differ in the restrictions placed on the admissible growth of the Wightman functions in p-space and their treatment of locality. The definition of the class of non-renormalizable theories of pre-exponential growth in p-space, given in the remark in §9.1.E, goes back to Schroer (1964) who called such theories non-renormalizable of the first kind. Without getting involved in a discussion of the numerous attempts to go beyond the framework of the local theory, we note that several of them relate to generalizations or modifications of the group of invariants itself (see, for example, Snyder, 1947a,b; Ingraham, 1962a,b; Blokhintsev and Kolerov, 1964; Kadyshevsky, 1972, 1978, 1979; see also the reports in [I3]).

The TCP-theorem in the standard Lagrangian theory first arose in connection with the problem of spin and statistics: Schwinger (1951), Lüders (1954, 1957, 1961), Pauli (1955) (see also the history of the question in the paper by Jost (1960) and the book [S16], Ch.4). The proof of the TCP-theorem and its generalization in the Wightman approach (§§9.2.A, 9.2.B) is due to Jost (1957). The connection between TCP-invariance and weak local commutativity was subsequently argued by Dyson (1958a). According to Jost (1963), any TCP-invariant relativistic S-matrix can, at least formally, be associated with a weakly local quantum ("interpolating") field. The possibilities of subsequent experimental verification of TCP-invariance in elementary particle physics are discussed by Lee et al. (1957), Shirokov (1962), Okun' (1966), Bilen'kii (1966), Gourdin (1967), Poth (1978); see also [K1, O2]. The example of the propagator $K^0\overline{K^0}$ shows that the consequences of TCP-invariance for this case are in fact the same as the consequences for mere Lorentz-invariance (Lipshutz, 1966). Equivalence classes of local fields (§9.2.C) were introduced by Borchers (1960) (in this connection see the lectures by Wightman, 1963 and the articles by Acharya, 1962 and Araki, 1963a). Examples of mutually local fields with the same S-matrix were given by Kamefuchi et al. (1961) (and in the papers cited therein). Concerning other criteria for the triviality of the S-matrix, see Bardakci and Sudarshan (1961), Greenberg and Licht (1963), Acharya (1963), and Aks (1965). They show that in a local theory there cannot be a non-trivial S-matrix admitting only a finite number of inelastic processes. In the algebraic approach, Epstein (1967) proved the TCP-invariance of the S-matrix under an extra hypothesis on the structure of one-particle states which exclude infinite degeneracy of the mass with respect to the spin (similar conditions were imposed earlier by Haag and Swieca, 1965, and Araki and Haag, 1967).

The theorem on the connection between spin and statistics of free fields with arbitrary spin was proved by Fierz (1939) and Pauli (1940). Proofs of Theorem 9.19 based on general postulates have been given in the papers by Lüders and Zumino (1958), Burgoyne (1958) and Dell' Antonio (1961b). The connection between the theorem on spin and statistics and the cluster property is discussed by Froissart and Taylor (1967). Commutation relations between different fields have been discussed within the framework of the ordinary theory by Lüders and Zumino (1958) and within the framework of the axiomatic approach (Theorem 9.20) by Araki (1961c). The Klein transformation used in these papers was introduced by Klein (1938) in another context. In our proof of Araki's Theorem 9.20 (§§9.3.B–9.3.D) we have followed Jost ([J3], §V.3.C).

The existence of inequivalent representations of the CCR's with an infinite number of degrees of freedom was discovered by Segal (1950, not published) who constructed an example of a unitarily unrealizable (that is, improper) linear canonical transformation of the creation and annihilation operators. Following this, van Hove (1952) gave an example of an improper inhomogeneous canonical transformation (more precisely, an example of a displaced Fock transformation). Since the example of van Hove (which is a model of interaction of a field with a classical source) has a direct physical meaning, it became clear from this that the choice of the representation of the CCR's defines the interaction. Haag's theorem (1955) (§9.4.A) is an even more striking demonstration of the value of non-Fock representations for the Hamiltonian approach in quantum field theory (see [W5], Araki, 1960b concerning this). The generalized Haag theorem (§9.4.B) was given in the Hall and Wightman formulation (1957). There is a physical interpretation of Haag's theorem in the Wightman lectures (1967a,b) (see also [S16], §4.5). Fabri et al. (1967) presented a theorem on the impossibility of describing a "broken symmetry" by a unitary operator depending on time (see [G13] concerning this). In connection with the fact that the Fock representations do not exhaust the physically interesting representations of the CCR's and CAR's, there arose the non-trivial problem of the classification of representations ([F2]; Gårding and Wightman, 1954a,b; Wightman and Schweber, 1955; Segal, 1958; Gel'fand [G6],§IV.3; Weidlich, 1963; Araki and Wyss, 1964; Chaiken, 1967, 1968; Zav'yalov and Sushko, 1969; Golodets, 1969; Powers and Størmer, 1970; Araki, 1971). An example of the appearance of a non-Fock representation in statistical mechanics was given by Bogolubov [B12] (see also Haag, 1962); it was shown that a physical representation in superconductivity theory is constructed from a Fock representation by means of an (improper) linear transformation that mixes up the creation and annihilation operators.

Transferring (by means of the so-called "Wick rotation") from Minkowski space-time to four-dimensional Euclidean space has been widely used for the Feynman integrals. Schwinger (1958, 1959) and Nakano (1959) proposed that the Euclidean formulation (§9.5.A) should be considered as an independent approach to quantum field theory. Symanzik (1964, 1966, 1969a,b) and later Nelson (1973a–d) established a connection between the Euclidean formulation and the theory of random (Markov) fields. The characteristic properties of the Schwinger functions (§9.5.B) and the reconstruction theorem (§9.5.C) appeared in papers by Osterwalder and Schrader (1973) and Osterwalder (1973a). In the papers by Hegerfeldt (1974) Fröhlich (1974), and Osterwalder and Schrader (1975) alternative sets of axioms of Euclidean fields were put forward which suffice for the reconstruction of quantum fields. The Euclidean formulation has achieved wide application in constructive quantum field theory (see [S11, G8] for details concerning this). Concerning conformally invariant Euclidean fields, see Todorov (1973, 1976), Mack (1975), [T5] and the literature cited therein.

Appendix H. Even though the theoretical possibilities of statistics more general than the Bose and Fermi ones were discussed even in the 30's, it was only in the 50's that the effective machinery of parafields was created. Wigner (1950) considered the example of the parafield oscillator. The quantum theory of free parafields (§H.1) was developed by Green (1953). Parastatistics have been investigated by D.V. Volkov (1959,1960), Chernikov (1962), Kamefuchi and Takahashi (1962), Kamefuchi and Strathdee (1963), Galindo and Yndurain (1963), Messiah and Greenberg (1964), Greenberg and Messiah (1964,1965), Greenberg (1966), Govorkov (1967), Ohnuki and Kamefuchi [O3]. An axiomatic approach of Wightman type to parastatistics and the connection between spin and parastatistics (§H.2) was discussed by Dell'Antonio et al. (1964). The equivalence of locality and paralocality for free parafields was proved by Araki et al. (1966). Parastatistics have been studied within the framework of the algebraic approach in the papers by Drühl et al. (1970), Doplicher and Roberts (1972), Doplicher et al. (1971,1974), and Roberts (1976a). New types of generalized statistics and their connection with representations of Lie (super)algebras have been pointed out in the papers by Ganchev and Palev (1980) and Palev (1982) (see also the literature cited there).

In connection with the well known problem of the statistics of quarks arising within the framework of the hadron model of Gell-Mann and Zweig (see [C5]), Greenberg (1964) proposed the employment of parafermi-statistics of order $p = 3$ for quarks. This conjecture enabled one to explain the property of total symmetry of the spin-unitary wave function of baryons, which corresponded to the possibility of arranging up to three spin quark states in the same quantum level. However, a consistent solution of the problem of the statistics of quarks was given by Bogolubov, Struminskii and Tavkhelidze (Bogolubov et al., 1965) and independently by Han and Nambu (1965), who suggested a new quantum number for quarks, subsequently called colour. With respect to the new quantum number which takes (for each type of quark with specified electric charge, isospin and strangeness) three values, all observable hadrons are neutral (or "colourless") (see Tavzhelidze, 1965). The introduction of coloured quarks as fundamental particles subject to Fermi-Dirac statistics has allowed a number of problems of principle in elementary particle theory to be solved, and the colour concept has been set in the foundations of quantum chromodynamics (QCD), which is a gauge theory of strong interactions (Nambu, 1966b) in which the hypothesis of the confinement of quarks finds a natural place (see, for example, Bender, 1981; a discussion of the possibility of quarks escaping can be found in the paper by Arbusov et al., 1983). According to the paper by Govorkov (1982,1983), and Bogolubov et al. (1983), the maximal gauge symmetry admitting a theory of local parafields is the special orthogonal group $O_+(3)$ (in contrast to quantum chromodynamics with unitary gauge group $SU(3)$, this theory possesses three gauge bosons instead of eight). This conclusion implies that the conjecture of the parafermi-statistics of quarks is inequivalent to the theory of coloured fermi-quarks. Further discussions on the relation between the theory of parafields and the theory of coloured quarks can be found in Greenberg and Nelson (1977), Govorkov (1977,1979,1982) (in Govorkov's survey, 1983, there is a more thorough discussion of the literature).

Appendix I. For further information on the elementary representations of $SL(2,C)$ (§I.1) and complex semisimple Lie algebras, see [G2], Zhelobenko (1968), Kostant (1975) (and also [K7,Z3,Z4]). The first example of an infinite-dimensional irreducible representation of $SL(2,C)$ was constructed by Majorana (1932) who suggested the equation for ICF's, the spectrum of the solutions of which contains no negative timelike momenta (but in return, contains spacelike momenta as well, so that the spectrum condition is violated; subsequently Dirac (1971,1972; [D2]) constructed equations for the ICF's with spectrum in V^+). It became clear from Majorana's example and later publications by Gel'fand and Yaglom (1948a,b) that there is no connection between spin and statistics for ICF's (see Ruelle, 1967; Komar and Slad', 1969 with regard to the equations for classical ICF's). Second quantization of free ICF's was considered by Feldman and Matthews (1966,1967); these fields either have infinite degeneracy of mass with respect to the spin or violate locality. Abers et al. (1967) constructed examples of quantum ICF's violating the connection between spin and statistics and the CPT theorem, but here the spectrum contains spacelike momenta, so that the spectrum condition fails. A simple example of the breakdown of the connection between spin and statistics for an ICF (transforming according to a reducible representation of the Lorentz group) with the spectrum condition holding was given by Streater (1967b). A covariant representation for a two-point function of type (I.22) was proposed by Todorov and Zaikov (1969) and proved in the paper by Oksak and Todorov (1969), where the expansion (I.31) was derived and Theorem I.3 was proved. Iverson and Mack (1971) considered the case of zero mass when additional covariant structures arise.

Proposition I.4 was proved by Oksak and Todorov (1971). In the paper by Grodsky and Streater (1968) a very general result was obtained on the infinite degeneracy of the mass with respect to spin for ICF's (in this connection see also Abarbanel and Frishman, 1968; Bierter and Bitar, 1969). Oksak and Todorov (1971) extended the theorem of Grodsky and Streater to the case of local nonrenormalizable theories of pre-exponential growth in p-space (Theorem I.5). In §I.4, the examples from the paper by Oksak and Todorov (1968) are given in somewhat more general form. See the paper by Slad' (1971) on CPT-invariance of classical ICF's. We also note certain applications of ICF's for the algebraic formulation of the dynamical structure of quantum systems (Barut and Kleinert, 1967; Nambu, 1967; Delburgo et al., 1967), and for an alternative formulation of the Bethe-Salpeter equation (Kyriakopoulos, 1968; Budini, 1968; Fronsdal, 1968; Itzykson et al. 1970). Surveys of the theory of ICF's can be found in Nambu (1966a), Ruegg et al. (1967), Stoyanov and Todorov (1968) (here, in particular, the Majorana equation is developed), Streater (1968), and Takabayashi (1969).

Chapter 10. Quantization in a space with an indefinite metric was proposed in the 40's (Dirac, 1942; Pauli, 1943). Pauli (1950) noted the fact that the connection between spin and statistics can break down (the example in §10.1.B). The mathematical study of operators in spaces with an indefinite metric began alongside this (Pontryagin, 1944; Iokhvidov and Krein, 1956,1959; Langer, 1962,1963). An exposition of the mathematical theory * can be found in the lectures by Krein (1965) and in [B7]. The indefinite metric formalism was noted within the axiomatic scheme in an earlier article by Wightman and Gårding (published in 1965) and, as before, attracted attention chiefly in connection with the specific problems of the Lorentz-covariant canonical quantization of gauge fields (Strocchi, 1966–1968,1970,1978; Lowenstein, 1971; Ferrari and Picasso, 1971; Nakanishi, 1972; Strocchi and Wightman, 1974; Ferrari et al., 1974,1977). In this connection it was explained that the translation-invariance of the auxiliary (Hilbert) scalar product and the uniqueness (to within a factor) of the translation-invariant vector are an exception rather than a rule, in that they are absent in physically interesting models with infrared singularities of the Green's functions (this was pointed out by Ferrari, 1974; Strocchi, 1976b; Morchio and Strocchi, 1980). Self-contained accounts of the approach are given in the papers by Strocchi and Wightman (1974), Strocchi (1978), Morchio and Strocchi (1980), which we have in large measure followed in §10.1.B. The major changes introduced by us are in the construction of the physical states (Axiom PW.IX,§10.1.C); here we have systematically used the principle of gauge invariance and the construction of states by the method of integrating the vacuum functional over the gauge group. In §10.1.D a connection is set up with the traditional (less general) method based on isolating the "physical" subspace in the pseudo-Hilbert space. The drawbacks of the pseudo-Wightman formalism relate to the fact that the analogue of Wightman's reconstruction theory requires additional topological conditions (see Scheibe, 1960; Strocchi, 1976b; Yngvason, 1977; Morchio and Strocchi, 1980). Field theories in an indefinite metric admit a formulation in terms of *-algebras of bounded operators (analogous to the C^*-algebra formulation of field theory in a positive metric); see Dadashyan and Khoruzhii (1983) concerning this. See the papers of Zav'yalov (1973,1975), Zav'yalov and Medvedev (1974) concerning the applications of the indefinite metric in the theory of renormalizations.

The model of the dipole ghost field (§10.2.A) was introduced by Ferrari (1974) in the context of theories with an indefinite metric. As we saw, the algebra of observables is trivial in this model. Narnhofer and Thirring (1978) illustrated its widening interpretation with this example of the "physical content" of the model (meaning by this the behaviour of the system including its interaction with other systems). The gradient model (§10.2.A) was used by Zwanziger for the confirmation of certain hypotheses of quantum electrodynamics (QED for short). Among the first works laying down the foundations of QED, the article by Jordan and Pauli (1928) on the quantization of a free electromagnetic field can be singled out. The weak form of the gauge condition requiring that the equation $\partial_\mu A^{\mu(-)}\Phi = 0$ should hold for physical vectors and enabling one to exclude longitudinal and scalar photons in the covariant gauge of QED (in the so-called Gupta-Bleuler gauge) was given by Gupta (1950) and Bleuler (1950). Detailed accounts of QED are contained in the monographs [A3, B10, B5, J2]. Lagrangians with auxiliary fields fixing the class of gauges occur, in particular, in Nakanishi (1966,1972–1974), Lautrup (1967), and Symanzik (1971). The axiomatic approach to local QED was analysed in the paper by Strocchi and Wightman (1974) (see also Strocchi, 1978); one way or another, the majority of later articles depend on it. Strocchi and Wightman proved the superselection rules with respect to electric charge for local observables. Another important result obtained along these lines and related to Wigner's result (1971) is in connection with the generalized Ising model and Elithur's theorem (1975) (according to which, local gauge invariance cannot be spontaneously broken in the statistical physics of gauge-invariant field models on a lattice (Wilson, 1974) without introducing gauge-fixing terms). Thus Maison and Zwanziger (1975) proved that in the pseudo-Hilbert space of QED there are no localized state vectors with zero electric charge satisfying the supplementary Gupta-Bleuler condition which isolates the physical subspace (see also Zwanziger, 1976); Ferrari et al. (1977) extended this conclusion to arbitrary local gauges in QED (and Strocchi, 1976b extended it to

* We use "pseudo-Hilbert space" (§10.1.A) as an equivalent of the mathematical term "J-space".

quantum chromodynamics). In our account of the principles of QED (§10.2.B), the main emphasis is on the algebraic formulation of local and global invariances (the roles of which are essentially different as are the roles of the small and large gauge groups in the more general context of §10.1.C). In this way a certain compactness is achieved since the results mentioned above now, in essence, automatically follow from our definitions of observables and physical quantities. (It is assumed that global gauge invariance is not spontaneously broken.) Here we are to a certain extent oriented to the situation realized in the gradient model of §10.2.A (Zwanziger (1978) with good reason called this model a "soluble lesson" for QED). It should be noted that every exposition of this concrete model (of QED) is of a hypothetical nature, or so it would seem, since (as mentioned in the text) the question of the mathematical existence (or non-existence) of four-dimensional QED is still an open one. Earlier versions are reflected in the lectures by Todorov (1979), Mintchev and Todorov (1981,1985). A somewhat similar point of view has been adopted by d'Emilio and Mintchev (1982a–c), who demonstrated that the use of just the physical quantities (defined on the basis of the principles of local invariance and global covariance) enables one to simplify the derivation of the standard (perturbative) results on the infrared divergences in the electrodynamical S-matrix. In their paper, the role of the operator of the creation of an electron is played by a "physical" non-local charged field of type (10.105a) under a special choice of the generalized function $F(k,p)$ (namely, $F^{\mu}(k,p) = p^{\mu}/(pk)$); a similar "physical" charged field (choosing $F^{\mu}(k,p) = p^{\mu}/(pk - i0)$) was introduced in the paper by Ogima and Hata (1979). A motivation of this choice of the "photon cloud" of the electron can be found in Kibble (1968). Non-local charged objects of type (10.90a) which are invariant with respect to arbitrary gauge transformations of the second kind with vanishing phase at infinity were introduced by Dirac (1955), Mandelstam (1962), Bialynicki-Birula (1976). Steinmann (1976) undertook the formulation of QED in terms of local observables only. The case of a free electromagnetic field in different gauges (from the point of view of the indefinite-metric formalism was examined in detail by Strocchi and Wightman in the paper mentioned above, as well as by Bongaarts (1977,1982) (details relating to Lorentz gauges are discussed in Nakanishi, 1966; Rideau, 1975). The role of the conformal invariance of the Maxwell equations has been discussed in many publications (starting from 1909). Conformal invariance of QED has been considered by Mack and Todorov (1973), Christ (1974), Mintchev et al. (1976) and in a number of other works (see [T5]). The precise definition of electric charge (as an integral of the charge densities) and the work of Strocchi and Wightman are discussed in Orzalesi (1970), Requardt (1976), Reeh and Requardt (1980). There is a discussion of Gauss's law in Streater (1974a), Roberts (1976b), Strocchi (1976b,1981) (in Strocchi (1981) there is also a survey of earlier work). The work of Bloch and Nordsieck (1937), Yennie et al. (1961) explaining the structure of the infrared divergences of the S-matrix elements in the perturbation theory of QED, laid down the foundations for the subsequent discussions of the infrared problem and the construction of physical states; Chung (1965), L.D. Solov'ev (1965,1973), Kibble (1968), Kulish and Faddeev (1970), Fröhlich (1973), Pron'ko and L.D. Solov'ev (1974), Appelquist and Carazzone (1975), Zwanziger (1975,1978b,1979a,b), Haller (1978), Gervais and Zwanziger (1980), d'Emilio and Mintchev (1982a–c), Berger and Szymanowski (1982). In a number of works (Fröhlich et al., 1979a,b; Buchholz, 1982) the possibility is discussed of the spontaneous breaking of Lorentz invariance in the charged sectors. Here they make the oversimplifying assumption that the algebra of observables is generated by the electromagnetic field strength tensor (on the contrary, a sufficiently extensive algebra of observables is a safeguard against such symmetry breaking). The geometric point of view on the vector-potential of the electromagnetic field as a synchronizer of the phases of electrically charged fields at neighbouring points goes back to Weyl (1918,1929). The Yang-Mills theory (see Yang and Mills, 1954; Utiyama, 1956) extended this to the non-abelian case. With regard to the quantization of non-abelian gauge theories, practically all the results relating to local covariant gauges are obtained by the method of perturbations and are therefore outside the scope of our exposition. A very full and systematic account of these questions is set out in [S12, K9, I4]. The formulation of gauge theories in the Dirac approach [D2] to dynamical systems with connections is given by Faddeev (1969) (see also [H2] and the lectures by Todorov, 1979, where there is a more complete list of references). The local operator formalism of renormalizations of gauge theories is set out in the paper by Kugo and Ojima (1979). One can learn about the constructive approach to gauge theories from the Fröhlich lectures (1979) and [S7]. The geometry of (classical) Euclidean Yang-Mills fields is set out in [A5]. Concerning the applications of fibre spaces to gauge theories, see Daniel and Viallet (1980), Dorn (1981), Babilon and Viallet (1981) and [M10]. A survey of the application of holomorphic geometry to gauge fields is given by Manin (1981). With regard to another interesting question in the classical theory of gauge fields, namely the description of monopole solutions, see 't Hooft (1974), Polyakov (1974b,1975), Coleman (1977), [J1, M11], Ward (1981), Hitchin (1982), Rossi (1982) and the references cited therein.

The connection between continuous symmetry groups and local conserved currents (§10.3.A) goes back to the classical Lagrangian field theory (where it is established by Noether's theorem; see, for example, [B10], §2). The quantum field "Noether's theorem" in the framework of the theory of renormalizations is considered by a number of authors (Lowenstein, 1971; Lam, 1972a,b); in accordance with this, the Ward-Takahashi identities were derived in gauge theories by J.C. Taylor (1971) and A.A. Slavnov (1972). The properties of non-translation-invariant currents (corresponding to a wider class of symmetries than the internal ones) have been considered by Ferrari (1973), Gal-Ezer and Reeh (1974,1975), Garber and Reeh (1976). The field-theoretic mechanisms of spontaneous symmetry breaking were originally worked out in statistical physics (see Bogolubov, 1958, 1961, [B12] and the literature cited therein) and later carried over to quantum field theory (§10.3.B) in the papers by Nambu (1960), Nambu and Yona-Lasinio (1961), Goldstone (1961). In the first versions of Goldstone's theorem (see Goldstone et al., 1962; Kastler et al., 1966) only the absence of a mass gap (rather than the presence of a massless particle) was proved under spontaneous symmetry breaking. The proof of Theorem 10.9 was given by Ezawa and Swieca (1967) by making intensive use of the JLD representation (and hence, locality) (their ideas have been used in our proof). The proof of Theorem 10.8 was given in the lectures by Swieca (1970). The connection between the various treatments of the concept of spontaneous symmetry breaking is established in Coleman's theorem (Coleman, 1966; Dell' Antonio, 1967; Landau and Wichmann, 1970; in this connection see also L.J. Landau, 1970; Kraus and Landau, 1972; Gal-Ezer and Reeh, 1974,1975). For a further discussion of Goldstone's theorem and its generalizations see Streater (1965a,b), Strocchi (1966), Fabri et al. (1967), Guenin and Velo (1967), Vasil'ev (1966, 1973), Bulinskii (1969), Reeh (1971), Ferrari (1973), Joos and Weimar (1976), Fubini (1982), [V2, G13]. The Higgs mechanism was proposed in the papers by Higgs (1964a,b; 1966), Englert and Brout (1964), and Guralnik et al. (1964). Concerning renormalizations and the physical consequences of these theories, see the papers by B.W. Lee and Zinn-Justin (1972–1973), Nakanishi (1973), Weinberg (1973), Abers and Lee (1973), B.W. Lee (1974), Coleman (1975a), Creutz and Tudron (1978). The results of Ferrari (1974) and Strocchi (1977) are presented in a somewhat different form in §10.3.C. For further details on the Higgs mechanism see Nakanishi (1973), Streater (1978), De-Angelis et al. (1978), Brydges et al. (1979), and Pervushin (1979,1980). The paper by Fröhlich et al. (1980) is devoted to uncovering the gauge independent content of the Higgs mechanism; the Salam-Weinberg model of the electro-weak interactions is set out from this point of view in the survey by Mintchev and Todorov (1985) (see also [T1]). One must also mention the result of Swieca (1976) (a complete proof of which can be found in Buchholz and Fredenhagen, 1979) according to which, the Maxwell equations $\partial^\lambda \mathcal{F}_{\lambda\mu} = \mathcal{J}_\mu$ together with the hypothesis on the presence of a mass gap (and the massivity of all particles including the "photon") imply that the total charge is identically zero. This conclusion provides a very general explanation of the masslessness of photons in QED on the one hand, and the screening of the charge in the Higgs model on the other.

Chapter 11. The two-dimensional model in §11.1 (in particular, the simple dependence of the fermionic current on the free scalar massless fields) attracted attention in the 30's in connection with an idea of a neutrino theory of light. Further interest was later stimulated by the study of the infrared problem. This model and, related to it, the two-dimensional gradient model were examined in the indefinite metric formalism (§11.1.A) in the papers by Schroer (1963), Tarski (1964), Hadjiivanov and Stoyanov (1979), and also in the lectures by Wightman (1967a). Morchio and Strocchi (1980) turned their attention to certain niceties in this construction, namely, the non-uniqueness of the vacuum and gauge invariance (here they considered only a single charge Q; the second "topological" charge Q' arises in splitting the field ϕ into right and left components). The invariant construction of the field in physical Hilbert space (§11.1.B) was proposed in the paper by Oksak (1981). The algebra of observables (generated by the current) in the physical representation of the fields ϕ^R, ϕ^L decomposes into a direct integral of irreducible representations; these are the so-called displaced Fock representations for the current, constructed earlier by Streater and Wild (1970) (a discussion of such representations in a more general setting can be found in the paper by Basarab-Horwath et al., 1979).

The (massless) Thirring model (Thirring, 1958; Thirring and Wess, 1964) was proposed as a neutral explicitly soluble example in field theory. Glaser (1958) suggested a treatment of the Thirring model based on non-physical Hilbert space with energy of variable sign. In this connection, see Pradhan, 1958; Scarf, 1959a,b; Scarf and Wess, 1962; [B4]; Astakhov et al., 1967; Ruijsenaars, 1982). The conservation of vector and pseudo-vector currents and the commutator between the currents and the Thirring field is illustrated in the approach of Johnson (1961); in this way it is possible to construct all the Wightman functions for the Thirring model (Klaiber, 1964,1968) and to write down a closed expression for the Thirring field in terms of the free fermionic and free scalar fields (Leutwyler, 1965). Our account of the Thirring model (§11.2) is based on the papers by Klaiber (1968) (where the technique of infrared cut-off is developed), Swieca (1977) (where bosonization formulae of Skyrme-Mandelstam type in Hilbert space with an indefinite metric are used) and Oksak (1981) (where bosonization of fermions in physical Hilbert space is considered). In connection with the bosonization formulae (which we first encountered in §11.1.C by way of the example of a free massless spinor field) we note that the possibility of constructing fermions from bosons was first pointed out by Skyrme (1961) in the study of a two-dimensional quantum model which subsequently acquired the name of the sine-Gordon model (SG for short). A more definite result was obtained by Coleman (1975b) who proved the equivalence of the SG model and the massive Thirring model (MT for short). Mandelstam (1975) put forward explicit formulae expressing the MT in terms of the SG field (concerning the mathematical meaning to be attached to such models, see Pogrebkov and Sushko, 1975,1976; Sushko, 1978). The bosonization in the (massless) Thirring model is completely analogous, although it is complicated by the extra infrared difficulties (in this connection we note the paper by Dell' Antonio et al., 1972 and also Streater and Wild, 1970, Streater, 1974b, where an intermediate stage of the bosonization of the Thirring fields is carried out in terms of the free current and artificially appended operators of creation of charges). As was remarked in §11.1.C, ϑ-representations for a free massless spinor field are obtained from the Fock representation by the transformation (11.96), they being unitarily equivalent to the Fock representation if and only if the numbers $\zeta^{R,L}/\sqrt{2\pi}$ are integers. Transformations of this last type (treated as gauge transformations) were also introduced by Rothe and Swieca (1977), Krasnikov et al. (1980) in another context (for the Schwinger model). The Thirring model in the Hamiltonian formalism was constructed in the papers by Volovich and Sushko (1971), Arefyeva et al. (1975) and Pogrebkov (1973).

The two-dimensional massless QED (§11.3) put forward by Schwinger (1962) serves as a model of the confinement of charged particles. Schwinger (1963) found an explicit form for the Green's functions. The formal operator-valued solution of the Schwinger model was put forward by Thirring and Wess (1964). An exhaustive analysis of the Schwinger model in different gauges (including the Lorentz and Coulomb gauges) is given in the paper by Lowenstein and Swieca (1971) which we have used in §11.3. The Schwinger model in an arbitrary covariant gauge is studied in the paper by Capri and Ferrari (1981). Concerning the Hamiltonian approach to the Schwinger model, see the papers by Casher et al. (1974), Danilov et al. (1980,1982), Ito (1980). The Schwinger model has been studied by the method of functional integration in Rothe and Swieca (1978), Krasnikov et al. (1980), Kulikov (1983) ; considerable attention is given in these papers to the anomalies of the pseudo-vector current (see Adler, 1970 concerning the anomalies of the axial current in four-dimensional QED). Other explicitly soluble models are the Federbush model (see Federbush, 1961a,b; Wightman,1967a; Ruijsenaars, 1982), the Liouville model and its generalizations, namely, the supersymmetric Liouville model and the Toda model (Dzhordzhadze et al., 1979; Leznov and Saveliev, 1979; Gitzwiller, 1980–1981; Curtright and Thorn, 1982; Braaten et al., 1982; D'Hoker and Jackiw, 1982; Leznov and Fedoseev, 1983; Leznov and Khrushchev, 1983; Olive and Turok, 1983; Mansfield, 1983; Gervais and Neveu, 1983). The so-called completely integrable two-dimensional quantum models, into which the MT, SG and chiral field models fall, form a wider class. The MT model was originally studied by Berezin and Sushko (1965) who diagonalized the Hamiltonian and constructed the S-matrix in a non-physical representation. The S-matrix for the models indicated above was constructed in subsequent papers (concerning these models, see Dashen et al., 1975; Faddeev and Korepin, 1975; Fröhlich, 1975,1976a,b; Fröhlich and Park, 1977; Zamolodchikov, 1977; Razumov, 1977; Karowski and Weisz, 1978; Berg and Weisz, 1978; Berg et al., 1979; Korepin, 1979; see also the surveys by Faddeev and Korepin, 1978; Faddeev, 1979; Izergin and Korepin, 1982). Considerable progress in the study of general two-dimensional models has been achieved in the constructive approach; see [S11], Fröhlich (1978), [G8] concerning this.

Chapter 12. The main results of this chapter are due to Haag (1958) and Ruelle (1962). Ruelle got rid of the "technical" assumptions in Haag's original proof and, in particular, proved the strengthened property (Proposition 12.4). Other versions of the cluster property have been proved by Araki (1960a), Jost and Hepp (1962), Araki et al. (1962). Lemma 12.10 was also proved by Ruelle; our proof is given in Jost (1966). Ruelle's proof of the Poincaré-invariance of the S-matrix (Theorem 12.1, part (b)) was amplified in Streater's paper (1967a) and this is the one that we have used. Accounts of the Haag-Ruelle theory can also be found in [J3] (Ch.VI), [R1] (§XI.16). Buchholz (1975,1977) extended the Haag-Ruelle theory to massless particles; Streater (1964) to almost local fields; Iofa and Fainberg (1969b) (see also Fainberg, 1972; Steinmann, 1970; Bümmerstede and Lücke, 1974; M.A. Solov'ev, 1980a) to non-renormalizable theories. Scattering theories in terms of local algebras were given by Araki and Haag (1967) (see the survey by Araki, 1969 in this connection). They showed that the theory of scattering could in principle be stated in terms of observables alone (without using operators of fermionic type or varying charges), although the formulation would be more cumbersome.

Chapter 13. The T-products of quantum fields and their vacuum expectation values, the causal Green's functions (§13.1), were introduced by Dyson (1949) and Schwinger (1949). Causal Green's functions were studied by Zimmermann (1954a–d). As was noted in §13.1.A, the "naïve" definitions of type (13.1) are, generally speaking, beset with problems of ill-definition of the same nature as in the ultraviolet divergences in perturbation theory. The "axiomatic" approach to T-products set out in §13.1.A was suggested by Bogolubov (1955) as a method of constructing renormalization schemes (see [B10], Ch.IV,V; these ideas were also developed in papers by Stepanov, 1963,1965; Hepp 1966b,1971; Zimmermann, 1970; Epstein and Glaser, 1971,1973; Shirkov, 1976). Retarded products (§13.1.B) were introduced by Lehmann et al., (1957) (see also Polkinghorne, 1956,1957). The connection between retarded functions and Wightman functions was discussed by Steinmann (1960,1963a,b,1968). The original statement of the approach called the LSZ formalism (Lehmann et al., 1955,1957) was included in the asymptotic condition (13.65) as an axiom. Several modifications of the approach were developed in the papers by Nishijima (1957–1959). Formulae (13.67) which play an important role in the LSZ formalism were first written down by Yang and Feldman (1950) and Källén (1950) who obtained them by integrating the Heisenberg equations of motion. In his work on the connection between the LSZ and Wightman formalisms, Hepp (1963b,1965a,1966a) showed that the asymptotic condition as well as the Yang-Feldman equations and the reduction formulae can be derived from the Wightman formalism under additional hypotheses (§13.1.C). Hepp's results are set out in §13.2. Hepp (1964c) also proved the cluster property for the S-matrix elements under spacelike separation of the arguments. The structure of the one-particle singularities of the Green's functions and the S-matrix elements has been studied by Zimmermann (1959,1960), Streater (1962b), and Hepp (1965b); see Symanzik (1960), Bros (1970), Lassalle (1974), Bros and Lassalle (1975) with regard to the multiparticle structure. Glaser et al., (1957) restated the LSZ theory in terms of r-functions (see also Steinmann, 1968,1972). Perturbation theory in the LSZ formalism was considered by Nishijima (1960), Muraskin and Nishijima (1961), Fried (1962), and Steinmann [S15]. The problem of bound states was discussed by Zimmermann (1958), Nishijima (1958), and Baumann (1958); in particular, Zimmermann showed that every bound state with zero spin can be described by a local scalar field. The formulation of quantum electrodynamics in the LSZ formalism is due to Nakanishi (1974). A survey of the LSZ technique and its applications is given in [B1].

Chapter 14. Heisenberg's S-matrix method (1943) in relativistic quantum theory was the precursor and even the initiative of the axiomatic S-matrix method (or the microcausal S-matrix method) set out in Ch. 14. However, this approach enabled one to obtain the dynamical relations between the S-matrix elements only after introducing variational derivatives with respect to the classical fields into the theory (see Bogolubov, 1952 and the survey by Bogolubov, 1958), in terms of which the microcausality condition can be stated (Bogolubov, 1955; Bogolubov and Shirkov, 1955; [B10]). For this purpose Stueckelberg and Rivier (1950), and Stueckelberg (1951) turned their attention to the role of causality, but without proposing a completed formulation. By combining the causality and spectrum properties it is possible to prove the dispersion relations for pion-nucleon elastic scattering (Bogolubov, 1956; [B9]). In the course of the proof of the dispersion relations, the S-matrix approach was formulated as an independent line (see [B9]) and received further development in the work of Medvedev, Polivanov, Sukhanov and other authors. One of the basic results in this direction is the emergence of the dynamical role of the causality condition, which is written in the form (14.53) and treated as an equation of motion (Medvedev and Polivanov, 1961,1964; Medvedev, 1961,1964). The S-matrix formulation can be formally reduced to the Lagrangian or Hamiltonian forms (Sukhanov, 1965; Medvedev, 1966; Medvedev et al, 1971); the difference between them arises from the difference between the Wick and Dyson T-products (see Sukhanov (1961,1964), Medvedev and Sukhanov (1975) for a discussion of this question; in the paper by Medvedev et al. (1968a), the notion of a quasi-normal product is introduced and its connection with the Wick product is examined). Medvedev and Polivanov (1967a) give a discussion of the difficulties in connection with the adiabatic hypothesis (arising, in particular, from the non-uniqueness of the restriction of generalized functions in p-space to the mass shell) and constructive recipes are given. (In connection with this problem, it was necessary to extend the class of admissible test functions in §14.1.B.) The problem of renormalizations in the matrix method has been studied (Medvedev et al., 1968b,1972; see the survey by Medvedev and Polivanov, 1969), as have the specific singularities of the quantization of interaction models with derivatives (Medvedev and Polivanov, 1967b; Medvedev et al., 1968c). The connection with the LSZ formalism is discussed in the papers by Fainberg (1961), Sukhanov (1964), Medvedev (1965), Medvedev et al. (1972). It was shown in particular (Medvedev, 1965), that a field defined by equations (14.54), (14.67) is local. The connection with the algebraic approach has been examined by D.A. Slavnov (1977), Il'in and Slavnov (1978). Quantum electrodynamics in the S-matrix approach has been considered by Pugh (1965–1966), L.D. Solov'ev (1973), Pron'ko (1974), Bazhanov et al. (1979). The S-matrix method of perturbations, as well as the question of the allowable number of derivatives in quasi-local operators for accordance with the LSZ theory has been considered by Medvedev and Polivanov (1961,1964) and Sukhanov (1966). Other regularized forms of the fundamental equations have been worked out in the papers by Pugh (1963,1965–1966), Fainberg (1964), Rohrlich (1964), Rohrlich and Wilner (1966), Rohrlich and Wray (1966), Chen et al. (1966), Chen (1967), Wray (1968) (there is a synopsis in [B8],§17.4). The important concept of radiation operator was introduced in [B9](§2). A definition is given in [B8] (§3.3.6) of radiation operators (of Steinmann type) analogous to the Steinmann monomials in the LSZ theory. The construction and properties of radiation operators (of Ruelle type) have been considered by Medvedev et al. (1977a,b) and Pavlov (1978b). One can learn about the S-matrix method from the surveys by Medvedev and Polivanov (1964), Polivanov (1965), Medvedev et al. (1972), Zav'yalov [Z1](§1.3). In our account of the S-matrix approach some of the standard details have been changed. The expansion (14.21) in normal products is to be regarded as tentative; the variational derivatives in the asymptotic fields are to be treated as formal. We have allowed the case of arbitrary spin, and the original causality condition is stated in the form (14.31) (instead of (14.40d)).

Chapter 15. The origins of the study of the analytic properties of relativistic amplitudes were actually laid down in the paper by Gell-Mann et al. (1954), where the dispersion relation for two-particle elastic scattering was put forward (see also Goldberger, 1955; Goldberger et al., 1955). The dispersion relations for πN-scattering were first proved within the framework of quantum field theory by Bogolubov (1956); a complete proof was given in the monograph [B9]. Other proofs were given by Symanzik (1957) (for forward scattering) and Bremermann et al. (1958). The scheme of the proof set out in [B9] (see §15.2.A) was essentially retained in the later papers, although it was open to improvements by an application of the JLD representation. Lehmann (1958) proved the analyticity with respect to t of the scattering amplitude (§15.1.C) and the absorptive part (§15.2.D) in the physical domain. Vladimirov and Logunov (1959) obtained more general results on the analytic properties of the absorptive part; in particular, they proved the dispersion relations for a number of virtual scattering processes where one or two particles are outside the mass shell (in this connection, see also Logunov and Solov'ev, 1959; Oehme and Taylor, 1959). Apart from this, Hepp (1964b) proved (see §§15.2.C,15.2.D) that the JLD representation can be used to avoid certain a priori "technical" assumptions on the polynomial boundedness (with respect to s) of the amplitude $T(s,t,\zeta_2,\zeta_4)$ in the physical domain which had been made earlier. After the proof for the case of πN-scattering, dispersion relations were obtained for a number of elastic processes of Compton scattering type for nucleons (Bogolubov and Shirkov, 1957), for (inelastic processes of) photogeneration of pions by nucleons (Logunov et al., 1957; Chew et al., 1957) as well as for processes involving strange particles (Okubo, 1958; Polivanov, 1958; Jin, 1959; Todorov, 1960). Logunov (1958) obtained the dispersion relation for the inelastic processes $2 \to 3$; in particular, a domain of parameters is given for the double Compton effect $p + \gamma \to p + 2\gamma$ in which the unobservable region of energies is absent in the dispersion integral. A detailed derivation and generalizations are given in the papers by Logunov and Tavkhelidze (1958) and Logunov et al. (1958). This line of approach was further developed in papers by Logunov et al. (1977b,1979), where they proved the dispersion relations for an elastic process with two massless particles (for example, $2\gamma + N \to 2\gamma + N$) and the so-called "generalized optical theorem" stated by Müller (1970); it connects one of the jumps of the amplitude of $3 \to 3$ with the inclusive spectrum of the inclusive process $\kappa_1 + \kappa_2 \to \kappa_3 + \ldots$. Müller's approach has also been developed in the papers by Stapp (1971) and Cahill and Stapp (1972). The problem of the spectral representation of three-point functions has also been studied in connection with the dispersion relations (Nambu, 1958; Oehme, 1958; Deser et al., 1959). Jost (1958) constructed a counterexample showing the impossibility of obtaining, in general, a spectral representation for the vertices on the basis of the locality and spectrum conditions alone. Therefore a dispersion relation for a vertex corresponding to a real process of disintegration of a particle into two particles was proposed (where the masses of the decaying particles are regarded as free energy variables); see Goldberger and Treiman (1958) (the decay $\pi \to \mu + \gamma$), Todorov and Khrustalev (1959) (decay processes of the K-meson and hyperon into two hadrons: $K \to 2\pi$, $Y \to N + \pi$). See Lazur and Khimich (1981) concerning the derivation of dispersion relations in the class of non-renormalizable theories of pre-exponential growth in p-space.

Apart from analyticity, which is only derivable from the general principles of quantum field theory, the analytic properties of amplitudes in perturbation theory are of some interest. An effective method of investigating the analyticity of Feynman diagrams (the method of majorization) which gives information on the analytic properties of the amplitudes of a given process at all orders, was proposed by Nambu (1957) and Symanzik (1958). It received further development and application in papers by Nakanishi (1959,1962–1964), Wu (1961), Logunov et al. (1962), Lu-I-Chen and Todorov (1963), Mestvirishvili and Todorov (1963), and Chow (1966). As was noted on p. 573, it has been possible to prove by this means the "perturbative" dispersion relations for certain processes for which the dispersion relations have not been proved from general principles. Another method concentrates on the topology of the singularities of the concrete diagram (see the papers by L.D. Landau (1959), Polkinghorne and Screaton (1960), Cutkosky (1960), Fowler (1963), Petrina (1963), Federbush (1965), Westwater (1967), Risk (1968), Ponzano et al. (1969), Golubeva (1976)). For a survey of analytic properties in perturbation theory, see [E2, P1, H6, T4].

Surveys of the dispersion relations and their applications can be found in Lehmann (1959), Goldberger (1960), Omnès (1960), Sommer (1970); see also [H5, S10, G9, B1, B9]. Certain applications are touched upon in Appendix J and Ch. 17. Of the other applications, we note the dispersion sum rules for strong interactions (L.D. Solov'ev, 1966; de Alfaro et al., 1966). The synthesis of the analyticity of the amplitude with Regge theory (Regge, 1959; Chew and Frautschi, 1961; Froissart, 1961; Gribov, 1962a,b) is reflected in the finite energy sum rules (Logunov et al., 1967a; Igi and Matsuda, 1967; Gatto, 1967), in the model of Veneziano (1968) and its generalizations. For an acquaintance with these themes, see the surveys by Mandelstam (1970), Hara (1972), Veneziano (1974), [D1, S9].

There is another S-matrix approach (different from the method of the microcausal S-matrix set out in Ch.14) based on the idea of analyticity: Chew [C3, C4], Pham (1967), Stapp (1968), Chandler (1968–1969), Chandler and Stapp (1969), Coster and Stapp (1970), Chew and Rosenzweig (1978), Iagolnitzer [I1]. The fields in this approach are subject to ostracism, so that it is no longer a theory of fields but rather a "pure" S-matrix theory. This approach is called either the analytic or the macrocausal S-matrix method, depending on the emphasis placed on the initial standpoint.

Chapter 16. Polkinghorne (1956,1957) drew attention to the role of GRF's. The study of GRF's was in fact begun in the paper by Jost (1958) on the three-point Green's function, in which effective use is made of the complete set of the six GRF's. The class of GRP's and GRF's considered by us (§16.1) was introduced by Steinmann (1960,1963a) by another method. Our construction is based on an analogy with the construction of generalized retarded operators (§14.1.D) (proposed in [B8],§3.3.6). A complete set of GRF's (for $n \geq 5$) was constructed by Ruelle (1961b) and Araki (1961a,b) (see also Araki and Burgoyne, 1960). In §16.1 we have used the lectures by Bros (1965) and Epstein (1966) along with the papers by Steinmann (1960,1963a). See the papers by Medvedev et al. (1977a,b) and Pavlov (1978b) for a further acquaintance with the properties of GRP's and GRF's; (in the latter, retarded radiation operators are considered; their properties are entirely similar to those of GRP's). The results of §§16.2,16.3 were obtained by Bros et al. (1964,1965) (see also the lectures by Epstein, 1966). Epstein et al. (1969) introduced results on the analyticity of the processes $2 \to 2$ from the axioms of the algebraic approach. Results on the analyticity of the amplitudes of $2 \to 2$ in two variables (s,t) were also obtained by Mandelstam (1960) and Lehmann (1966). The method of GRF's was used by Bros et al. (1972) for the study of the analytic properties of n-functions. Logunov et al. (1977) developed a technique for the (forward) $3 \to 3$ process enabling one to prove the dispersion relation. More general results have been obtained by Muzafarov and Pavlov (1978), Pavlov (1978a), Logunov et al. (1979b), Medvedev et al. (1982).

Appendix J. The main result (Theorem J.5) was obtained by Martin (1966b). The parameter μ featuring in this theorem has been refined for concrete processes by Sommer (1967a) and Bessis and Glaser (1967). A generalization of Martin's result to inelastic two-particle processes was made by Sommer (1967c,d) and for particles with spin in papers by Sommer (1967b), Mahoux and Martin (1968,1970), Bell (1969), Ezhela and Mestvirishvili (1971). See Martin [M4], Sommer (1970),and Logunov et al. (1972) for a further acquaintance with related questions.

Chapter 17. The relations (17.3)–(17.5) were obtained by Froissart (1961) under the hypothesis that the Mandelstam representation is valid. Martin's theorem (§J.2) was used to derive these estimates from the axioms of quantum field theory. (Greenberg and Low (1961) remarked that analyticity of the amplitude with respect to t in an ellipse of type (J.54) suffices for this to hold; from the analyticity in the Lehmann ellipse they obtained a weaker bound. On the other hand, Kinoshita et al. (1964) obtained from the representation of Mandelstam a stronger bound than (17.4) which in return is more singular with respect to $\sin\theta$.) Corollary 17.3 on the number of subtractions in the dispersion relation was obtained by Jin and Martin (1964b) and Martin (1966a). Concerning generalized estimates for the case of scattering of particles with spin, see Hara (1964), Cornille (1964), Mahoux and Martin (1968,1970), Bell (1969), Ezhela and Mestvirishvili (1971) (see also Ezhela (1980) concerning the technique of estimation). Inequality (17.26) was obtained by Martin (1963b); inequalities (17.27), (17.28) were obtained by Logunov et al. (1968). See Jin and Martin (1964a), Sugavara (1965), and Vernov (1973b) in connection with the inequalities (17.32). Concerning lower bounds for the scattering amplitude at a non-zero angle, see Cerulus and Martin (1964), Kinoshita (1964), Martin (1965a). The use of positivity enables one to obtain numerical estimates for the scattering amplitude; see Common and Yndurain (1970), Auberson et al. (1974–1975), Lopez and Mennessier (1977). Martin (1963a,b), Macdowell and Martin (1964), Bessis (1966) obtained certain restrictions for the differential peak. Martin (1965a) obtained a lower bound on the form factor as $t \to -\infty$ (in this connection, see also Volkov et al., 1969; Baluni and Broadhurst, 1977). Jaffe (1966) extended Martin's result to the class of local non-renormalizable theories. In its original form, Pomeranchuk's theorem (1958) contained the hypothesis that the total cross sections tended to constants as $s \to \infty$. Sugawara and Kanazawa (1961), Wainberg (1961) proved the theorem under weaker hypotheses, while Meiman (1962) suggested a flexible formulation using "standard" functions for the asymptotics of the cross sections as $s \to \infty$. Proposition 17.10 (of Pomeranchuk theorem type) was proved by Martin (1965b). Proposition 17.7 and a number of other asymptotic theorems were proved by Volkov et al. (1970). Theorems of Pomeranchuk type were extended to differential cross sections in papers by Logunov et al. (1963,1966,1972) (where the method of "standard" functions was applied) and van Hove (1963). Vernov (1970) obtained a relation between the amplitudes $T^{(\pm)}(s,0)$ in the form of some functional of their ratio. A number of other asymptotic theorems for scattering amplitudes were obtained in the papers by Jin and Macdowell (1965), Kinoshita (1965), Eden (1966), Vernov (1967,1973a,b), Gervais and Yndurain (1967–1968), Baluni et al. (1970), Cornille (1970), Fisher et al. (1976). There are surveys on the various questions touched upon in §17.1 by Eden (1964), Logunov et al. (1966,1972,1978), Epstein (1968), Martin [M4], Martin and Chung [M5]. A discussion of the asymptotic theorems within the framework of various models can be found, for example, in the papers by van Hove (1964), [E1], Mandelstam (1970), Shirkov (1970), Hara (1972), Tyurin and Khrustalev (1975), Solov'ev and Shchelkachev (1975), [D1, S9].

The study of inclusive processes was put forward by Logunov, Mestvirishvili and Nguyen Van Hieu (Logunov et al., 1967b) and Bjorken (1968) (who drew attention to the importance of analyzing deeply inelastic lepton-hadron processes). The term "inclusive process" (and also "exclusive process") was introduced by Feynman (1969). In the paper by Logunov et al. (1967b) the total differential cross section of processes with one isolated final particle was examined and high energy upper estimates were obtained for this cross section within the framework of the axiomatic approach. In addition, the analyticity of the cross section of the process with respect to the angular variable $z = \cos\theta$ was established. (The analyticity with respect to $\cos\theta$ in the Lehmann ellipse for the "exclusive" process $2 \to 3$ was proved earlier by Ascoli and Minguzzi, 1960). High energy estimates of a somewhat different type (with singularities not only at $\cos\theta = \pm 1$ but also at $\cos\theta = 0$) were also obtained by Tiktopoulos and Treiman (1968a,b). After the experimental discovery of scale invariance in inclusive processes, a number of models explaining this phenomenon were proposed; among them were the conjecture of limiting fragmentation (Benecke et al., 1969) and the parton model (Feynman, 1969, [F1, C5], and Altarelli, 1982). The methods of quantum field theory found a new effective application (see, in particular, Brandt, 1970b; Gross and Wess, 1970; Preparata, 1972; Wilson and Zimmermann, 1972). A field-theoretic analysis of the automodel principle in deeply inelastic lepton-hadron processes (see the survey by Matveev et al., 1971) was given by Bogolubov et al. (1972) based on the JLD representation. In subsequent articles by Vitsorek et al. (1973), B.I. Zav'yalov (1973,1977), Brünning and Stichel (1974), Smirnov (1978), Vladimirov and Zav'yalov (1980, 1982), very general classes of asymptotics compatible with the general principles (locality and spectrum condition) were constructed. Conformally-invariant quantum field theory received a new impetus (Mack and Salam, 1969; Bulware et al., 1970; Migdal, 1971; Todorov, 1972; Mack and Todorov, 1973; Rühl, 1973; Polyakov, 1974a, Schroer et al., 1975; Fradkin and Palchik, 1976; there is an extensive bibliography on conformal invariance in the monograph [T5] and in the lectures by Petkova et al., 1983). Only the most general field-theoretic results on inclusive processes not based on any model hypotheses are reflected in §17.2. Propositions 17.13 and 17.14 were obtained in the paper by Logunov et al. (1967b), and Propositions 17.12 in the papers by Ezhela et al. (1971), and Ezhela (1971). In the papers by Ezhela et al. (1971,1973) differential inclusive cross sections in the two angular variables (θ, ϕ) are studied and the conjectures of pionization, scale invariance and limiting fragmentation are analyzed (from the point of view of analyticity and unitarity); bounds are obtained for the moments of inclusive distributions with respect to the energy of the detectable particles at a fixed scattering angle. Results in this direction were also obtained by Tiktopoulos and Traiman (1972). Under the extra hypothesis that the domain of analyticity of the inclusive differential cross section with respect to $\cos\theta$ in the inclusive process $\kappa_1 + \kappa_2 \to \kappa_3 + \ldots$ is determined by the t-channel threshold, Logunov et al. (1974,1978) obtained stronger upper bounds as well a lower limit for the rate of decrease as the transverse momentum increases (in this connection see also the paper by Arkhipov and Savrin, 1977). We note that differential cross sections modelled by separate diagrams (Rcheulishvili and Samokhin, 1975), or by sums of diagrams of ladder type (D'yakonov and Rochev, 1977) have a domain of analyticity of this type. A similar analyticity requirement for an elastic two-particle process (weaker than the analyticity from the Mandelstam representation) is used in the paper by Logunov et al. (1979a) for modelling the experimentally observed lower limits of the differential cross sections of forward scattering of hadrons for large s. The asymptotic connection between the processes of deeply inelastic scattering and inclusive annihilation have been studied in the paper by Petrov (1977). A more detailed discussion of the results on inclusive processes (including those obtained by field-theoretic methods) can be found in the surveys by Logunov et al. (1978, 1983). For surveys on the applications of quantum chromodynamics (in particular, the methods of the renorm-group and operator decompositions) to inclusive processes, see Politzer (1974), Crewther (1976), Dokshitzer et al., (1980), Müller (1981), Veneziano (1982), Radyushkin (1983) and J. C. Taylor (1983).

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