Differential Geometric Methods in Theoretical Physics: Physics and Geometry

Ling-Lie Chau
Werner Nahm

Springer

Differential Geometric Methods in Theoretical Physics

Physics and Geometry

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Differential Geometric Methods in Theoretical Physics

Physics and Geometry

Edited by

Ling-Lie Chau and Werner Nahm

University of California, Davis
Davis, California

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Volume 245—DIFFERENTIAL GEOMETRIC METHODS IN THEORETICAL PHYSICS: Physics and Geometry
edited by Ling-Lie Chau and Werner Nahm

PREFACE

After several decades of reduced contact, the interaction between physicists and mathematicians in the front-line research of both fields recently became deep and fruitful again. Many of the leading specialists of both fields became involved in this development. This process even led to the discovery of previously unsuspected connections between various subfields of physics and mathematics. In mathematics this concerns in particular knots von Neumann algebras, Kac-Moody algebras, integrable non-linear partial differential equations, and differential geometry in low dimensions, most importantly in three and four dimensional spaces. In physics it concerns gravity, string theory, integrable classical and quantum field theories, solitons and the statistical mechanics of surfaces. New discoveries in these fields are made at a rapid pace.

This conference brought together active researchers in these areas, reporting their results and discussing with other participants to further develop thoughts in future new directions. The conference was attended by 80 participants from 15 nations. These proceedings document the program and the talks at the conference. This conference was preceded by a two-week summer school. Ten lecturers gave extended lectures on related topics.The proceedings of the school will also be published in the NATO-ASI volume by Plenum.

The Editors

ACKNOWLEDGMENTS

We would like to thank the many people who have made the conference a success. Furthermore, we appreciate the excellent talks. The active participation of everyone present made the conference lively and stimulating. All of this made our efforts worthwhile.

The organization of the conference would have been impossible without the excellent service provided by the staff members at the Physics Department of the University of California at Davis (UCD): Mary Schenck, Tom Mezzanares, Nilda Muniz, Madelin Cameron, Shanti Chitradurgam, and Lynn Rabena. The running of the School and Conference could not have gone so smoothly without the help of many participants, in particular Evan Fletcher, Andrew Parkes and Itaru Yamanaka. It is also our pleasure to thank Mr. and Mrs. Parson and their staff, who have made the Granlibakken Conference Center in Tahoe City an excellent place to have enjoyable meetings.

Of course, none of this would have happened without the financial support of our sponsors: NATO, UCD (the Physics Department and the Dean's Office of Letters and Science). We would like to thank Robert Shelton, Chair of the Physics Department, and Giovanni Venturi of NATO Scientific Affairs for their encouragement and generous support.

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INTRODUCTION TO THE 18TH CONFERENCE ON 'DIFFERENTIAL GEOMETRICAL METHODS IN THEORETICAL PHYSICS'

Konrad Bleuler

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With greatest pleasure and with deep thankfulness towards our community of mathematicians and physicists I am looking back over all these years of beautiful and friendly meetings within the framework of our long lasting series. They were held in different countries and attended by scientists of different background and different ideas: There was, however, one general, invariant and common aim, namely a fruitful and open-minded exchange between representatives of the two sciences, mainly within the framework of recent developments with its far-reaching achievements in both domains. Although this exchange should rather be looked upon as a renewal of important contacts, which existed (after a longer period of separation) already in the course of earlier periods, connected to decisive renewals, slight, perhaps subconscious, feelings of a deep-rooted separation, or perhaps, a droplet of fear in between the two camps, may occasionally still be felt. In this connection it should be emphasized that these contacts became due to the latest breathtaking developments - in particular through the impact and the far-reaching consequences of the 'Strings' - not just important but absolutely 'vital' for future research. In connection with our search for a new and deeper basis it may be remembered that the ultimate scope of theoretical physics has always been the discovery of natural (abstract but natural and, in a way, simple) mathematical structures hidden far behand empirical facts. This calls, however for a long way of successive steps, mainly with the help of intermediate phenomenological theories. It should, however, be emphasized that the beauty and simplicity of really basic physical laws is revealed only when expressed in terms of the corresponding (partly pre-existing and partly to be discovered) mathematical concepts. In other words: Existing structures of mathematics are, in a way, re-discovered through empirical evidence, whereas - on the other side - insights and practical needs in physical research lead in some cases to the creation of new and basic mathematical domains. These far-reaching relations between abstract thoughts and facts of nature was looked at by some of our greatest scientists (e.g. Einstein, Pauli, Dirac, Wigner, a.s.o.)as an unexpected, even super-human wonder, or, in different terms, a great mystery. These, as many of you might say, old-fashioned, even forgotten viewpoints were, in a way, revived in the course of our meetings and became, in fact, a leading aim.

The main themes of our discussions were, therefore, those which naturally called for this exchange and came up in the course of our series, i.e. since '71: Starting with General Relativity (G.R.), i.e. a classical domain of common interest since its foundation, we went on with - extensively studied - 'Geometric Quantisation" in all its different forms, leaving us, however, with the insight that *no* canonically invariant and unique 'Quantisation-scheme' for an arbitrary classical system does (nor should!) exist. The next important topic - basic for recent developments in physics - was 'Supersymmetry', i.e.'Super-Lie-algebras','Superspaces', a.s.o., which - suggested, or better, founded by physicists, i.e. Wess and Zumino - represents by now (according to a remark made before) a new and important domain of mathematics, which now stands in its own right. A similar, even more impressive situation stems from the advent of *non*-Abelian (and non-local!) Gauge-theory: It was, originally, based on a most natural and intuitive geometrical idea taken over (in '29) by H. Weyl from G.R. suggesting a gauge theoretical interpretation of the Maxwellian field. Enlarged (in '54) by Yang-Mills to the *non*--Abelian case, it led to an englobing formulation of all basic fields in physics as well as (in ideal cases) to a satisfactory field-quantisation, a fact which points to a deep relation between geometry and the quantum theory. In our present-day situation, non-Abelian Gauge appears to be *the* underlying principle of all basic laws in physics, i.e. Gravitation (G.R.) Electroweak (GWS) and String (QCD) interactions. On the other side it appears appealing to observe that very much the same principle constitutes also an important tool in pure mathematics. In other words: A geometric principle which embraces the entire realm of known empirical data (leading, through QCD, to a complete theoretical basis for Hadrons *and* Nuclei as well as, through GWS, to a most satisfactory half-phenomenological description of electroweak processes) constitutes, at the same time, the major ingredient for prooving, according Donaldson, the special (unexpected!) topological properties of R(4). We thus find unexpectedly a certain number of 'mysterious' relations between apparently separated domains.

Up to this point, mathematical structures with their most successful and intuitive viewpoints are still under control or are governed through the overwhelming and most extended realm of empirical facts: The choice of the various Gauge groups as well as the numerical values of certain parameters are entirely due to a comparison to experimental data. This situation has, however, drastically changed, even inverted, with the advent of Strings. Mathematical arguments, i.e. simplicity, inner logics and, as a main point, consistency (cp. the longstanding problem of quantizing Gravitation) prevail against direct comparisons with physical facts which, for the time being, seem to loose their basic importance. This situation aroused, of course, a widespread criticism. We should, however, not forget that we have to deal, in fact, with a kind of an intermediate stage towards a deeper understanding of nature, and that the few (partial) successes as well as the various failures aroused an enormous interest for further and deeper research! This 'tantalizing' situation led, in fact, to a far-reaching and fruitful, to a large extent mathematical,research: Well-known examples are the revival of 'Braidgroups' in their connections to 'Knots and Links', which, in turn, led to the really basic topological concept of 'Jones' Polynomials'. At the same time, these groups are intimately related to the fundamental 'Yang-Baxter' equations, which lead, in turn, to the connection of these various concepts to Statistical Mechanics and, finally, as a last, decisive step, to the central problem, namely conformal String theory, i.e. the origin of these important developments. In short, a large number of unexpected relations between apparently independent mathematical domains was, again, revealed just through various valuable attempts for reaching, with the help of the 'Strings', a deeper and consistent understanding of physical laws.

Another major consequence of the string attempt - in relation to the (enforced!) introduction of extreme dimensions, i.e. the Planck length - represents the renewal of an old question related to the very basis of physics, namely the structure of space-time in *small* dimensions. Many of us may have forgotten that B. Riemann wrote, as a final sentence of this famous thesis on 'his' geometry, more than 100 years ago: While this (i.e. his) geometry might be natural and suitable for *large* dimensions, a different (e.g. discrete) geometry should be considered for extremely small spaces. This, in fact, most impressive and, in a way, unexpected hypothesis within the second part of this sentence was - after such a long interval of time - remembered, or better, revived by A. Connes when introducing his basic 'non-commutative Geometry'. Although its general and consistent use as a new basis for physics altogether is far from being accomplished, it appears most appealing to realize its connection and general relation to 2-dim. QFT according K. Osterwalder et al on one side and D. Kastler on the other (cp. Com.Math.Phys.118, 1-14 (1988) and 121, 345-45o (1989)). This 'discovery' of another 'unexpected' relation between so far definitely separated domains enlarging the analogous situation about 'Knots and Links' a.s.o. (mentioned above) might represent a hint as to the existence of a far deeper theoretical viewpoint - so far hidden - hopefully explaining a.o. all these strange and unexpected relations. We are thus in a most interesting, even breath-taking period of research in which again, i.e. after the advent of Relativity and Quantum Theory the very basis of physics is under critical discussion and seriously questioned. I think that this fact endows this week's discussion with a special interest and with a far--reaching scientific aim.

The central problem is - among others - to relate these far--reaching, mainly mathematical, research on one side with the really enormously anlarged realm of empirical evidence on the other. The ancient and characteristic polarisation, which, at the same time, represents the real interest and driving force in research, appears by now in a most acute form.

I should, however, not end without conveying, in the name of all of us, the heartiest thanks to our organizers, Ling-Lie Chau and Werner Nahm, for organizing the 18th meeting at this wonderful spot, and I should not forget to thank, at the same time, the organizers of all these former meetings in France, Poland, Italy, Israel and in Germany.

BAXTERIZATION

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In statistical mechanics one is interested in the globally observable properties of a system with an enormous number of degrees of freedom. One approach to the analysis of such systems is to define simplistic mathematical models which can be calculated exactly. The first example is the Ising model, solved by Onsager in 1944. A state of the system is defined by a configuration of + and - signs (spins) at the vertices of a square lattice in the plane. Each edge of the lattice is to be thought of as an interaction and contributes an energy $\mathrm{E}(\sigma, \sigma')$ to the total energy where σ, σ' are the spins at the ends of the edge. Thus in the Ising model the total energy of a state σ is

$$\sum_{i,j} k_1 \sigma_{1,j} \sigma_{i+1,j} + k_2 \sigma_{i,j} \sigma_{i,j+1} \quad .$$

The partition function is then defined (for any system) to be

$$Z = \sum_{\sigma} \exp(-\mathrm{E}(\sigma)/kT) \quad .$$

Note that Z depends on some parameters (k_1, k_2, kT). We will adopt the convention of summing up this dependence of Z (in any model) by a single "parameter" called the "spectral parameter" λ. Thus $Z = Z(\lambda)$.

The partition function does not make sense for an infinite system so we consider rectangular approximations to the system with N vertices and let Z_N be the corresponding partition function (the boundary conditions need to be treated in some way, e.g., periodic). Then the free energy per site is defined as $F(\lambda) = \lim_{N \to \infty} \frac{1}{N} \log(Z_n)$, where the union of the appproximating rectangles is the whole lattice. One would like to calculate $F(\lambda)$ explicitly as a function of λ. This is called "solving" a model.

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A BAXTER STRATEGY

The most successful technique in solving models is the method of *transfer matrices.* One concocts a matrix $T(\lambda)$ (depending on the horizontal dimension m of the approximating rectangle) such that trace $(T(\lambda)^n)$ is the partition function for the $m \times n$ rectangle. The largest eigenvalue of $T(\lambda)$ then gives the limit of $F(\lambda)$ as $n \to \infty$. Unfortunately the size of $T(\lambda)$ grows exponentially with m so diagonalization is not straightforward.

Baxter's idea was to consider systems for which the transfer matrices commute among themselves for different values of λ, thus $T(\lambda)T(\lambda') = T(\lambda')T(\lambda)$. Then the $T(\lambda)'s$ have a common eigenvector v and $T(\lambda)v = f(\lambda)v$. By a detailed analysis of the symmetries and analytic dependence of the matrix entries of $T(\lambda)$ on λ, the Baxter program is then able to nail down a finite list of possible functions $f(\lambda)$, in particular the largest. By an act of faith this $f_{\max}(\lambda)$ does actually occur and the model is solved.

But how is one to find systems for which the transfer matrices commute? It is natural to factorize the matrix $T(\lambda)$ as $R_m(\lambda)R_{m-1}(\lambda)...R_1(\lambda)$ where each R_i corresponds to adding an energy contribution at the ith position. Then one can ask the question: is there a *local* condition, involving the R_i/s, which guarantees that the $T(\lambda)$'s commute? Suppose that for each λ and λ' there is a third λ'' such that (letting $R_i(\lambda') = R_i'$ etc.)

(YBE) $$R_i R'_{i+1} R''_i = R''_{i+1} R'_i R_{i+1}$$

and

$$R_i R'_j = R'_j R_i \quad \text{for} \quad |i-j| \geq 2.$$

Then an easy computation (see [Ba]) shows that

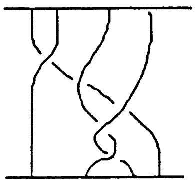
$$\{R_m R''_m (R'_m)^{-1}\} T T' = T' T \{R_1^{-1} (R''_1)^{-1} R^1_1\}.$$

Thus up to boundary terms the transfer matrices commute. It is to be hoped, and indeed true in a host of examples, that the boundary conditions can be defined so that these terms disappear and we have commuting transfer matrices.

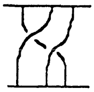
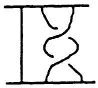
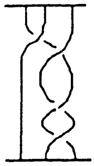
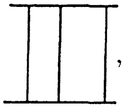
Using this method Baxter defined and solved a model (the "8 vertex model") that generalized the "six vertex method" solved using a Bethe ansatz technique by Lieb. The theory of "quantum groups" was designed by Jimbo and Drinfeld to produce examples of $R_i's$ satisfying YBE. They certainly find such examples for any simple Lie algebra $\mathbf{g}$ and any finite dimensional representation of $\mathbf{g}$ (see [Ji] and [Dr]). Note that there is a free parameter q in these $R_i(\lambda)$ solutions.

THE BRAID GROUPS

A geometric braid on n strings is a way to tying two horizontal sticks together with the strings so that the tangent vector to each string always has a non-zero vertical component. Thus



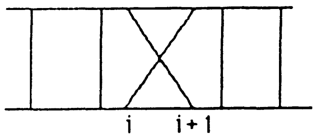
is an example of a 4-string braid. Braids form a *group* with concatenation as operation.

Thus if $\alpha =$  and $\beta =$  , then $\alpha\beta =$  . The identity element is obviously the braid  , and the inverse of a braid is obtained by reflecting it in a mirror and turning it upside down.

Thus for each n we have a group B_n which has a natural surjection onto the symmetric group S_n given by permutation of the end points.

If we define σ_i, for $1 \leq i < n$ as the braid

σ_1:



then the following relations are easy to see

(Br)
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq Z.$$

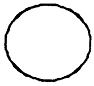
E. Artin proved that the braid group is actually equal to the group presented on $\sigma_1, \ldots, \sigma_{n-1}$ with relations (Br). Thus to define a representation of the braid group it suffices to find matrices σ_i satisfying (Br).

The similarity between (Br) and (YBE) is striking and one could look for representations of B_n using the $R_i(\lambda)'s$ of solvable models. The actual way to get rid of the spectral parameter to get a nontrivial representation may be subtle. Typically one will have to let the λ's go to infinity in some controlled way. But in the cases of models obtained from quantum groups this always works so we get, for every simple Lie algebra $\mathbf{g}$, and every finite dimensional representation of it, a representation of B_n, for every n, depending on a parameter q.

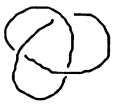
KNOTS AND LINKS

A knot is a smooth non-self intersecting curve in S^3. Two knots are equivalent if there is a diffeomorphism of S^3 taking one onto the other. Knots always admit projections onto $\mathcal{R}^2$ whose singularities are at worst double points so one may always draw a picture which adequately, but highly non-uniquely, represents the knot.

e.g.



the unknot



trefoil

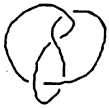
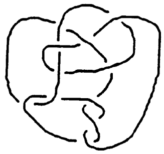


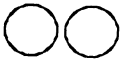
figure 8



Conway knot

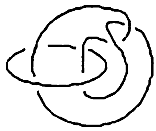
Links are "knots with several components",

e.g.



unlink on
2 components

Hopf
Link



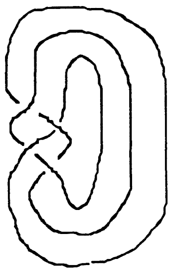
Whitehead
Link

Borromean
rings

It is in general difficult to tell, from looking at pictures of knots, whether they are the same or not. What one needs are invariants, i.e., entities associated with the knot in 3-space rather than just the picture. Most invariants are obtained by looking at the topological space obtained when the knot is removed from 3-space and doing some algebraic topology. The simplest such invariant is the order of the homology mod N of the canonical 2-fold cover of the complement of the knot. It was one of the first invariants used in knot theory. Another well known invariant, discovered in 1928 by J. Alexander [A], is the Alexander polynomial $\Delta_L(t)$ which is obtained from the homology of the infinite cyclic cover of the knot. For the trefoil $\Delta = t^{-1} - 1 + t$ and for the Conway knot $\Delta = 1$.

Given a braid $\alpha \in B_n$, it may be turned into an oriented link by "closing" it as below

$\alpha =$ 

closure $= \hat{\alpha}$ 

It is a result of Alexander that any oriented link may be obtained in this way. So it was natural to look for invariants of links by studying representations of the braid groups. In particular one may look at representations coming from solvable models as described above. Since the partition function is given by the trace, it is natural to take the trace of the matrix given by the braid. The remarkable result is that, if the trace is slightly modified (corresponding to special boundary conditions), all models coming from quantum groups give polynomial invariants (in the variable q) of links in this way. This was proved by Rosso and Reshetikhin. Thus to every simple Lie algebra $\mathfrak{g}$ and every finite dimensional representation of it there is a knot polynomial.

The simplest example is for sl_2 in its 2-dimensional representation. The corresponding polynomial is called $V_L(q)$, and the corresponding model is a 6-vertex model. As examples, $V_{\text{trefoil}} = q + q^3 - q^4$, $V_{\text{fig.8}} = q^{-2} - q^{-1} + 1 - q + q^2$, $V_{\text{conway}} =$ complicated. The n-dimensional representations of sl_n combine to form a 2-variable polynomial P_L known as the HOMFLY polynomial and the sympletic and orthogonal algebras (in their defining representations) combine to form the Kauffman polynomial F_L.

It is indeed a bizarre way to obtain knot invariants: we took a link, turned it into a braid, used some infinite limit in the spectral parameter of a solvable model to obtain a braid group representation and then took a modified trace of the braid to get our invariant! It must be said that this process is not very well understood at this stage. It is possible to avoid the use of braids by using the statistical mechanical model directly on an arbitrary link diagram and calculating its partition function. In the infinite limit of the spectral parameter one obtains an invariant. But the role of solvability and the reason for the infinite limit remain obscure. (See also [W].)

BAXTERIZATION

In a small attempt to understand the correspondence between solvable models and knot theory I propose the following procedure: take a knot invariant, turn it (if possible) into a coherent sequence of braid group representations, and then Baxterize (if possible) by inserting a spectral parameter so that YBE is satisfied and so that the original braid group representation is the infinite limit in the spectral parameter of the Baxterized version.

The simplest way to Baxterize the $\sigma_i's$ is what I call *self-Baxterization* where $R_i(\lambda)$ actually belongs to the algebra generated by σ_i. I know of many cases where this is possible, the first two I shall give are *universal Baxterizations*

1) Hecke case

If the $\sigma_i's$ satisfy a quadratic equation it is possible to renormalize them so that

$$\sigma_i + \sigma_i^{-1} = x1$$

Then we simply put

$$R_i(\lambda) = e^{\lambda}\sigma_i + e^{-\lambda}\sigma_i^{-1}$$

Equation (YBE) follows immediately.

BIRMAN-MURAKAMI-WENZL CASE (see [BW])

Suppose that $\sigma_i's$ have the following property for some numbers x and a:

$$\text{(define } E_i = \frac{1}{x}(\sigma_i + \sigma_i^{-1}) - 1\text{)}$$

$$\text{(BMW)} \qquad \begin{aligned} &E_i^2 = (a + a^{-1} - x)^{-1}E_i \\ &E_i\sigma_{i-1}^{\pm 1}E_i = a^{\mp 1}E_i, E_i\sigma_{i+1}^{\pm 1}E_i = a^{\mp 1}E_i \\ &E_i\sigma_{i\mp 1}\sigma_i = E_iE_{i\pm 1} \end{aligned}$$

(the reader should understand these relations by representing E_i by a "braid like" object $\overset{\cup}{\cap}$, an idea due to Kauffman). Then one may show with some pain that

$$R_i(\lambda) = (e^{\lambda} - 1)k\sigma_i + x(k + k^{-1})1 + (e^{-\lambda} - 1)k^{-1}\sigma_i^{-1}$$

satisfies (YBE). The result can also be deduced with a different kind of pain, from Jimbo's paper [Ji].

Thus whenever a braid group representation factors through the Hecke or BMW algebras it can be Baxterized.

ABELIAN BAXTERIZATION

Abelian Baxterization is when the $R_i(\lambda)'s$ commute among themselves for different values of λ. Self-Baxterization is necessarily abelian Non-abelian. Baxterization occurs in quantum group theory. For the adjoint representation of sl_3, the $R_i(\lambda)'s$ do not commute. I will return to this later.

To see an example of abelian Baxterization which is not self-Baxterization I turn to the main example of this talk (also seen in [KMM]), where we see that the simplest of all knot invariants, the homology of the double cover mod N, can be Baxterized.

Let D_n be the algebra on generators $u_1, u_2, ..., u_{n-1}$ with relations $u_l^N = 1, u_l u_{l+1} = \omega u_{l+1} u_l (\omega = -e^{2\pi/N}), u_k u_l = u_l u_k$ if $|l-k| \geq 2$. Then it is easy to show that if we define $\sigma_k = \sum_{j=0}^{N-1} \omega^{j^2} u_k^j$, that the braid group relations are satisfied. It is equally easy to see that the trace on D_n defined by $tr(\omega) = 0$ if ω is a non-scalar product of $u_l's$, has the right property (Markov) to define a link invariant. In [GJ] it is shown that the absolute value of this invariant is, at least for N odd, the order of the homology of the 2-fold branched cover mod N. One can attempt to Baxterize these representations but if one looks in [FZ] one finds that the models already exist. The Fateev Zamolodchikov model has transfer matrices

$$R_k(\lambda) = \sum_{j=0}^{N-1} \left[\prod_{a=0}^{j-1} \frac{\sin(\pi a/N + \lambda/2N)}{\sin(\pi(a+1)/N - \lambda/2N)} \right] u_k^j$$

Taking the limit $\lambda \longrightarrow i\infty$ we obtain σ_k as above.

If N is a prime (or 6) it is easy to see that $R_k(\lambda)$ is in the algebra generated by σ_k, so we have self-Baxterization. For other values of N this is not true but the Baxterization is clearly abelian.

FURTHER EXAMPLES

There are many known knot invariants that can be turned into traces of braid group representations. The number of representations of the fundamental group of the complement into a finite group is one due to Kuperberg. But the one that is most tempting to try at this stage is the homology of higher branched covers mod N. These may be obtained from the Burau representation as in [GJ]. To develop the theory in an analogous way to the mod 2 case one may proceed as follows (suppose N is an odd prime).

Let F be a quadratic extension of $\mathcal{Z}/\mathrm{N}\mathcal{Z}$ with involution $*$ and let $S \in \mathcal{F}$ be any element with $S \neq S^* = S^{-1}$, write $\mathrm{Im(a)} = \frac{a-a^*}{s-s^*}$ for $a \in \mathcal{F}$. Then to replace the algebra D_n previously defined we let $D_{n,s}$ be the (complex) algebra on generators $u_l^a, a \in \mathcal{F}, l = 1, ..., n-1$ with relations $(\omega = e^{2\pi i/N})$

$$\begin{aligned} u_l^a u_l^b &= \omega^{(s+s^{-1}) Im(ab^*)} u_l^{a+b} \\ u_l^a u_{l+1}^b &= \omega^{2 Im(\bar{a}b)} u_{l+1}^b u_l^a \\ u_k^a u_l^b &= u_l^b u_k^a \text{ for } |k-l| \geq 2 \end{aligned}$$

Then it may be deduced for [GJ] or shown directly that if $\sigma_i = \sum_{a \in \mathcal{F}} \omega^{a^* a} u_i^a$, then the braid relations are satisfied.

We conjecture that these representations can be Baxterized.

CONCLUDING REMARKS

Besides the probability of new solvable models, what are the lessons to be learned from Baxterization? I believe the biggest is that YBE is in some sense more natural than the braid relations. Let me give two reasons for thinking this, based on our examples.

1) In the Fateev Zamolodchikov model one may calculate the structure of two algebras: one the algebra generated by $R_1(\lambda), R_2(\lambda), \ldots, R_n(\lambda)$, the other the algebra generated by $\sigma_1, \sigma_2, \ldots, \sigma_n$. When N is a prime (or 6) these algebras are the same. In the other cases the *first* algebra is natural and simple to describe: it is the algebra of fixed points for the involution on D_n given by $u_i \to u_i^{-1}$. The *second* algebras seem quite unpleasant.

2) In the theory of quantum groups one may consider the commutant of the representations of $u_h(\mathbf{g})$ in the tensor powers of an irreducible representation. It is known that, unlike in the classical limit where it is only true for sl_n, the braid group generates the commutant for all representations of sl_2 and the smallest representations of the A, B, C, D series. On the other hand this is not true for the adjoint representation of sl_3. But it is precisely in this case that the $R_i(\lambda)$ matrices generate more than the braid group. One is naturally led to the question: is the commutant of $u_n(\mathbf{g})$ always equal to the algebra generated by the $R_i(\lambda)$?

REFERENCES

[A] J. Alexander.

[Ba] R. Baxter, "Exactly solved models in statistical mechanics", New York Academic Press, 1982.

[BW] J. Birman, H. Wenzl "Braids, link polynomials and a new algebra", Trans. AMS (to appear).

[Dr] V. Drinfeld "Quantum groups", Proc. ICM, 1986.

[FZ] V. Fateev and A. Zamolodchikov, "Self-dual solutions of the star-triangle relations in $\mathcal{Z}_N$ models", Phys. Lett. **92A**, 37-39 (1982).

[GJ] D. Goldschmidt and V. Jones "Metaplectic link invariants", Geometric Dedicata (to appear).

[Ji] M. Jimbo, "Quantum R-matrix for the generalized Toda system", Comm. Math. Phys. **102**, 537-547 (1986).

[KMM] T. Kobayashi, H. Murakami, J. Murakami, "Cyclotomic invariants for links", Proc. Japan Acad., 235-238 (1988).

[W] E. Witten, "Gauge theories an integrable lattice models", IAS Preprint, 1989.

Geometric Classification of Commutative Algebras of Ordinary Differential Operators*

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Dedicated to the memory of Professor Michio Kuga

1. The purpose of this paper is to give a geometric classification of all commutative algebras consisting of linear ordinary differential operators whose coefficients are scalar-valued functions.

The problem of determining commuting ordinary differential operators has a long history in mathematics – it started in 1879. Since then it has been studied by many people in various contexts and from different motivations. The so-called rank one case (see Section 2 for definition) was essentially worked out in the 1920's and the most general theorem in the rank one case was obtained by Krichever and Mumford in 1970's. We will present here a complete solution of this problem which is valid for all ranks and generalizes naturally the theorem of Krichever in terms of the geometry of vector bundles on algebraic curves.

Let B be a commutative algebra of linear ordinary differential operators with scalar-valued functions as coefficients. We say two operators P and Q *commute* with one another if the operator product $P \cdot Q$ coincides with $Q \cdot P$. We impose the following conditions on B.

(B-1) B is a $\mathbb{C}$-algebra with the identity operator $1 \in B$.

(B-2) B has an operator P of order $n > 0$ such that its leading coefficient is 1 and the other coefficients have power series expansion at one point. In other words, B has a *monic* operator which is regular at a certain point.

Since we can always add the identity operator to a given commutative algebra, (B-1) is not a restriction for B . The second condition (B-2)

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looks serious. For example, $\mathbb{C}[x\frac{d}{dx}]$ is a commutative algebra of ordinary differential operators but it does not satisfy (B-2). However, if we change the coordinate x by $u = \log x$, then the operator $x\frac{d}{dx}$ becomes

$$x\frac{d}{dx} = x \cdot \frac{du}{dx} \cdot \frac{d}{du} = \frac{d}{du},$$

which is monic. It means that the notion of monicness depends on the choice of a coordinate. In general, if we use the coordinate u defined by

$$u = \int f(x)^{-\frac{1}{n}} dx ,$$

then $f(x)\left(\frac{d}{dx}\right)^n$ becomes a monic n-th order differential operator in u. Thus it is not a severe condition to require the existence of a monic operator in B. We can simply change the coordinate to make one of the operators in B monic. Similarly, if an operator is given in a practical situation, we can find a point at which all the coefficients have power series expansion. Therefore, (B-2) is not a strong restriction, either.

Let us choose a coordinate x so that the operator P is monic and is regular at $x = 0$. Thus every coefficient of P is an element of the ring $\mathbb{C}[[x]]$ of formal power series in x. We identify two commutative algebras B_1 and B_2 if there is an invertible function $f = f(x) \in \mathbb{C}[[x]]$ such that

$$B_1 = f \cdot B_2 \cdot f^{-1} . \tag{1}$$

In this paper, we present a bijective correspondence, or a *dictionary*, between the set of commutative algebras of ordinary differential operators satisfying (B-1) and (B-2), and a moduli space of certain geometric data consisting of an algebraic curve, a smooth point on it, a vector bundle on the curve, a local covering of the curve defined on a neighborhood of the point and a local trivialization of the bundle near the given point.

Professor Michio Kuga has worked in the area of mathematics where number theory, algebraic geometry and the theory of differential equations meet together [**Ku**]. After I finished my work on the Schottky problem [**M1-2**], I spent one year at Stony Brook with him. I really enjoyed discussing mathematics and exchanging mathematical dreams with him. Around that time, I started to think about the higher rank analogue of the Krichever construction and the classification problem of the commuting differential operators.

When the final answer to this problem was obtained, I was truly surprised by the deep interplay between algebraic geometry and the theory of differential equations [M3]. Today, I wish I could tell him this result, and ask him about its possible relation to number theory.

2. In order to give the precise statement of the main theorem, let us define the moduli space $\mathcal{M}_r^+(0,-1)$ of quintuples $(C, p, \mathcal{F}, \pi, \phi)$ which is formed by the following objects:

- C is an irreducible complete algebraic curve, i.e. a compact Riemann surface with (or without) singularities;
- p is a smooth point of C;
- $\mathcal{F}$ is a torsion free sheaf of $\mathcal{O}_C$-modules of rank r such that $H^0(C,\mathcal{F}) = H^1(C,\mathcal{F}) = 0$. If C is a non-singular curve of genus g, then $\mathcal{F}$ is a vector bundle on C of degree $r(g-1)$ with no non-trivial holomorphic sections;
- $\pi : U_0 \to U_p$ is an r-sheeted covering ramified only at $p = \pi(0) \in U_p$, where U_0 is an open disk of $\mathbb{C}$ with center at the origin 0, and $U_p \subset C$ is an open neighborhood of $p \in C$;
- $\phi : \mathcal{F}|_{U_p} \xrightarrow{\sim} \pi_*\mathcal{O}_{U_0}(-1)$ is an $\mathcal{O}_{U_p}$-module isomorphism, where $\mathcal{F}|_{U_p}$ is the restriction of the vector bundle $\mathcal{F}$ on U_p and $\pi_*\mathcal{O}_{U_0}(-1)$ is the direct image by the covering map π of the twisted line bundle $\mathcal{O}_{U_0}(-1)$ on U_0 defined by the divisor $\{0\}$ of U_0.

Since $\pi_*\mathcal{O}_{U_0}(-1)$ is a trivial vector bundle of rank r on U_p, we can say that ϕ gives a local trivialization of $\mathcal{F}$ on U_p.

We say that two *quintets* $(C, p, \mathcal{F}, \pi_1, \phi_1)$ and $(C, p, \mathcal{F}, \pi_2, \phi_2)$ are *isomorphic* if there is an automorphism $h : U_0 \xrightarrow{\sim} U_0$ and a line bundle automorphism $\psi : \mathcal{O}_{U_0}(-1) \xrightarrow{\sim} \mathcal{O}_{U_0}(-1)$ such that $\pi_1 = \pi_2 \circ h$ and $\phi_1 = \phi_2 \circ \pi_*(\psi)$. Let us denote the set of all isomorphism classes of quintets by $\mathcal{M}_r^+(0,-1)$. Now we can state the main theorem.

MAIN THEOREM.

Let $\mathcal{B}_r$ be the set of all commutative algebras B of rank r consisting of ordinary differential operators with scalar-valued function as coefficients satisfying (B-1) *and* (B-2), *where the rank of B is defined by*

$$\operatorname{rank} B = G.C.D.\{\operatorname{ord} Q \mid Q \in B\}\ .$$

Then there is a natural bijection

$$\mu_r : \mathcal{B}_r \xrightarrow{\sim} \mathcal{M}_r^+(0,-1)$$

for every $r > 0$.

In other words, there is a bijective correspondence between the following objects:

Analytic Object. A commutative algebra of linear ordinary differential operators of rank r satisfying (B-1) and (B-2). Two of such algebras are identified by (1).

Geometric Object. An isomorphism class of a quintet $(C, p, \mathcal{F}, \pi, \phi)$ consisting of a curve, a point, a vector bundle, a local covering and a local trivialization.

REMARK: In the case of rank one, the isomorphism relations of the quintets almost completely wipe out the information of π and ϕ. Therefore, $\mathcal{M}_1^+(0, -1)$ is exactly the same as the moduli space of the data $(C, p, \mathcal{L}, v)$, where $\mathcal{L}$ is a generic line bundle of degree $g-1$ and v is a non-zero tangent vector of C at p. Thus our theorem reduces directly to the theorem of Krichever and Mumford [**K**], [**Mum**].

Verdier [**V**] obtained a different classification of commutative rings of ordinary differential operators. But since he uses the parabolic structure and the connections of the vector bundles, and since he does not incorporate the identification of (1), his result has a rather different flavor from the point of view of the work of Krichever and Mumford.

3. Since the key ideas of the theorem emerged from the earlier works of many mathematicians, let us sketch the history of this problem. We will concentrate on the old part of the history here, because an extensive review of recent works on this subject can be found in [**PW**].

Since differential operators do not usually commute, it is natural to ask what happens if they do. As far as I know, the first person who asked this question and tried to give a systematic answer was G. Wallenberg. In his paper [**W**] of 1903, Wallenberg studied the classification problem of pairs of commuting ordinary differential operators. He mentioned that this problem did not seem to be studied before, even in the fundamental work of G. Floquet [**F**]. But he credited Floquet that the case of two operators of order one has been worked out in [**F**].

Floquet's 1879 paper [**F**] is 130-page long and seems to be considered as one of the standard references of ordinary differential equations in his time. Among other things, he established the Euclid division algorithm for ordinary differential operators.

Let P and Q be ordinary differential operators of order m and n, respectively. If $m \geq n$ and the leading coefficient of Q is invertible, then there are operators P' of order $m-n$ and R of order less than n such that

$$P = P'Q + R\,.$$

Thus the system

$$\begin{cases} P\psi = 0 \\ Q\psi = 0 \end{cases}$$

is equivalent with

$$\begin{cases} Q\psi = 0 \\ R\psi = 0 \,, \end{cases}$$

which has lower order than the first one. From the modern point of view, we can say that Floquet studied $\mathcal{D}$-modules. In fact, he proved the following: let D_K be the set of all ordinary differential operators with coefficients in the field $K = \mathbb{C}((x))$ of formal Laurent resies in x. Then every D_K-module is generated by a single element. Since he was interested in the non-commutative nature of the ring D_K, he did not ask the question of Wallenberg.

It is easy to determine all commuting operators of order one. Wallenberg started from this point. In [**W**], he gave a complete determination of commuting operators P and Q when

1. $\operatorname{ord} P = \operatorname{ord} Q = 2$, and
2. $\operatorname{ord} P = 1$ and $\operatorname{ord} Q = n > 0$.

So far still everything is easy. Then he studied the case of ord $P = 2$ and ord $Q = 3$, and noticed that the Weierstrass elliptic function appears in the coefficients of these operators (cf. Eq. (2)). He dealt with a few more examples such as order 2 and 5, but did not obtain any general theorem. Maybe Wallenberg's attempt was too much ahead of time. Indeed, the first general result toward the classification problem was only established 20 years later.

In 1923, Burchnall and Chaundy [**BC**] started the systematic investigation on this problem, without knowing the previous work of Wallenberg. One of their remarkable theorems asserts

FIRST THEOREM OF BURCHNALL-CHAUNDY. *Let P and Q be linear ordinary differential operators of positive order. If P and Q commute, then they satisfy a non-trivial polynomial relation $f(P,Q) \equiv 0$, where $f(\lambda,\mu) \in \mathbb{C}[\lambda,\mu]$ is a non-zero polynomial in two variables.*

Let us illustrate why this is true. So let P be a second order operator and Q be of order three. As we observed in Section 1, we can assume that P is monic. Note that every operator which commutes with a monic operator of positive order must have a constant leading coefficient. Thus the leading coefficient of Q is a constant. Therefore, we can choose a constant $c_1 \in \mathbb{C}$ such that $Q^2 - c_1 P^3$ has order less than 6. Since $Q^2 - c_1 P^3$ commutes with P, its leading coefficient is again a constant.

So we can find another constant c_2 so that $Q^2 - c_1P^3 - c_2PQ$ becomes an operator of order less than 5. We can continue this procedure until we obtain an operator

$$R = Q^2 - c_1P^3 - c_2PQ - c_3P^2 - c_4Q - c_5P$$

of order less than 2. If $\operatorname{ord} R < 1$, then R must be a constant because it commutes with P. If $\operatorname{ord} R = 1$, then we can choose constants c_6 and c_7 such that $P - c_6R^2 - c_7R$ becomes a constant. In any case, we have obtained a non-trivial polynomial relation among P and Q ! This argument works in the general situation and gives a proof of the above theorem. (Note that this argument does not work for the quasi-commuting pairs. See Section 7.)

Thus a pair of commuting operators produces a plane algebraic curve. Let us work out an explicit example due to Wallenberg. We choose

$$\begin{cases} P = \left(\frac{d}{dx}\right)^2 - 2u(x) \\ Q = \left(\frac{d}{dx}\right)^3 - 3u(x)\frac{d}{dx} - \frac{3}{2}u'(x) , \end{cases} \tag{2}$$

where $u'(x)$ is the derivative of $u(x)$ with respect to x. It is an easy calculation to show that P and Q commute if and only if u satisfies

$$u''' = 12uu' .$$

We can integrate this equation twice to obtain

$$(u')^2 = 4u^3 - g_2u - g_3 , \tag{3}$$

where g_2 and g_3 are the constants of integration. Therefore, P and Q of (2) commute if and only if their coefficients are given by the Weierstrass elliptic functions and their degenerations, since g_2 and g_3 in (3) are arbitrary. If u satisfies (3), then P and Q satisfy a polynomial relation

$$Q^2 = P^3 - \frac{g_2}{4}P - \frac{g_3}{4} ,$$

which defines a plane cubic curve. It was already known to Wallenberg that every commuting pair of operators of order 2 and 3 can be reduced to this example by a simple transformation. In this sense this example is a universal one and the appearance of the elliptic function is essential.

When P and Q commute, the simultaneous eigenvalue problem

$$\begin{cases} P\psi = \lambda\psi \\ Q\psi = \mu\psi \end{cases} \tag{4}$$

makes sense. Since

$$0 = f(P,Q)\psi = f(\lambda,\mu)\psi \ ,$$

the eigenvalues λ and μ must satisfy the same polynomial relation. Therefore, the plane curve we obtained is a *spectral curve*. Moreover, a simple linear algebra argument shows that if $f(\lambda,\mu) = 0$, then there exists a non-trivial solution of (4). If one uses the technique of asymptotic expansion of ψ, then it is not so hard to see that the dimension of the simultaneous eigenspace is equal to the greatest common divisor of the order of P and Q. Thus, we can construct a vector bundle of rank $r = G.C.D.(\mathrm{ord}\,P, \mathrm{ord}\,Q)$ on the plane curve $\{(\lambda,\mu) \in \mathbb{C}^2 \mid f(\lambda,\mu) = 0\}$ whose fiber at (λ,μ) is the corresponding simultaneous eigenspace. There is a functorial way of extending this eigenspace bundle to the point at infinity of the spectral curve. Therefore, we obtain a compact Riemann surface by attaching a point at infinity to the plane curve and a globally defined vector bundle on it. In other words, we have constructed the data $(C, p, \mathcal{F})$ from the commutative algebra $B = \mathbb{C}[P,Q]$ generated by P, Q and 1, where C is a compact Riemann surface which may have singularities, p is the point at infinity and $\mathcal{F}$ is a vector bundle of rank $r = G.C.D.(\mathrm{ord}\,P\,, \mathrm{ord}\,Q) = \mathrm{rank}\,B$. In our example of (2), C is an elliptic curve, p is the point at infinity and $\mathcal{F}$ is a line bundle.

Let us denote by $\mathcal{M}_1$ the moduli space of all data $(C, p, \mathcal{L}, v)$ consisting of an algebraic curve of arbitrary genus g, a smooth point $p \in C$, a line bundle $\mathcal{L}$ on C of degree $g-1$ which has no non-trivial global holomorphic sections, and a non-zero tangent vector $v \in T_pC$. (The tangent vector comes from the monicness assumption of the operator.) The second amazing theorem due to Burchnall and Chaundy states the following.

SECOND THEOREM OF BURCHNALL-CHAUNDY. *Let $\mathcal{B}_1$ be the set of commutative algebras of ordinary differential operators of rank 1 satisfying* (B-1), (B-2). *Then there is a canonical bijection between $\mathcal{B}_1$ and $\mathcal{M}_1$;*

$$\beta_1 : \mathcal{B}_1 \xrightarrow{\sim} \mathcal{M}_1 \ .$$

These theorems were forgotten for more than half a century. Meantime, in the middle of 1970's, Krichever [**K**] re-discovered these theorems without knowing the earlier work of Burchnall and Chaundy. A complete proof of these theorems using the modern language of algebraic geometry was given by Mumford [**Mum**].

Here comes a natural question: which kind of geometric data correspond to $\mathcal{B}_r$ for $r \geq 2$?

An obvious candidate is the moduli space $\mathcal{M}_r$ of the data $(C, p, \mathcal{F}, v)$, where $\mathcal{F}$ is now a vector bundle of rank r and degree $r(g-1)$ such that $H^0(C, \mathcal{F}) = 0$. But it does not work, because $\mathcal{B}_r$ is far larger than $\mathcal{M}_r$.

4. In order to prove the Main Theorem, we use the original technique of Burchnall and Chaundy as well as its modernized version due to Mumford. We also need another machinery, which is believed to be rather modern, but is actually very old: that is the theory of pseudo-differential operators.

Let us to go back to the history once again. I. Schur, inspired by the 1903 paper of Wallenberg, proved the following theorem in 1905. (In the same year he obtained also the famous Schur's lemma.)

SCHUR'S OTHER LEMMA. *Let B_P be the set of all linear ordinary differential operators which commute with a given operator P of order $n > 0$. Then B_P is a commutative algebra.*

The theorem is rather surprising, because it does not hold for the case of matrices and partial differential operators. Schur's proof is the following. As before, we can assume that P is monic. The first thing he did is to define the n-th root of P. He showed that there is a unique monic pseudo-differential operator

$$L = \frac{d}{dx} + a_0 + a_1\left(\frac{d}{dx}\right)^{-1} + a_2\left(\frac{d}{dx}\right)^{-2} + \cdots \tag{5}$$

of order one such that $L^n = P$. (He credited S. Pincherle [**P**], which appeared in 1897, for the technique of pseudo-differential calculus. The idea of fractional powers of operators is due to Schur himself.) Let $Q \in B_P$ be an arbitrary operator of order, say m. Since Q commutes with P, its leading coefficient is a constant. Therefore, there is a constant $c_0 \in \mathbb{C}$ such that $Q - c_0 L^m$ has order $m-1$ or less. Since L commutes with P, $Q - c_0 L^m$ also commutes with P. Therefore, its leading coefficient is again a constant, and hence there is $c_1 \in \mathbb{C}$ so that $Q - c_0 L^m - c_1 L^{m-1}$ is of order $m-2$ or lower. By continuing this process infinitely many times, he obtained

$$Q = \sum_{\ell=0}^{\infty} c_\ell L^{m-\ell} . \tag{6}$$

Therefore,

$$B_P \subset \mathbb{C}((L^{-1})) , \tag{7}$$

where $\mathbb{C}((z))$ denotes the set of all formal Laurent series in z with finite poles at $z = 0$. But since $\mathbb{C}((L^{-1}))$ is commutative, so is B_P!

Let us consider the abstract version of Schur's argument. Since Q is a differential operator, m of (6) is always positive. Thus we have a subalgebra

$$A \subset \mathbb{C}((z)) \tag{8}$$

satisfying the condition

$$A \cap \mathbb{C}[[z]] = \mathbb{C}\,. \tag{9}$$

A commutative algebra B satisfying (B-1) and (B-2) produces a pair (A, L), where L is a monic pseudo-differential operator of order 1 obtained by taking the n-th root of P, and A is a subalgebra of $\mathbb{C}((z))$ with (9) obtained by replacing L^{-1} in (6) by z.

Now the question: how can we go back from an abstract algebra A satisfying (8) and (9) to the algebra B of ordinary differential operators?

An easy answer is this: simply replace z by L^{-1}. But then how do we know that the resulting algebra consists of only *differential* operators? In other words, how can we find an operator L so that the algebra

$$B = \{a(L^{-1}) \mid a(z) \in A\}$$

obtained by replacing z by L^{-1} in A contains only differential operators? In order to answer this question, we need another idea.

5. In their fundamental work [**GD**] on completely integral systems appeared in 1975 and 76, Gel'fand and Dikii introduced negative and fractional powers of operators in this field. But actually the same technique had been known to Pincherle and Schur for more than 70 years before. Thus many of these modern theories on this subject have deep historical background, even though it was not recognized until recently. Then M. Sato [**Sa**] discovered a remarkable correspondence between an infinite dimensional Grassmannian and the pseudo-differential operators of Schur and Gel'fand-Dikii.

We have already encountered the set $\mathbb{C}((z))$ of all formal Laurent series. Let $V = \mathbb{C}((z))$ and $V^{(\nu)} = \mathbb{C}[[z]] \cdot z^{-\nu}$. Then the filtration

$$\cdots \supset V^{(\nu+1)} \supset V^{(\nu)} \supset V^{(\nu-1)} \supset \cdots$$

defines a topology in V. We define the Grassmannian $G(\mu, \nu)$ of index μ and level ν as the set of all closed vector subspaces W of V such that both $W \cap V^{(\nu)}$ and $V/(W + V^{(\nu)})$ have finite dimensions and

$$\dim(W \cap V^{(\nu)}) - \dim\left(V/(W + V^{(\nu)})\right) = \mu\,.$$

This is indeed the Grassmannian of Fredholm operators of index μ. The big-cell of the index 0 Grassmannian is defined by

$$G^+(0,\nu) = \{W \in G(0,\nu) \mid W \cap V^{(\nu)} = V/(W + V^{(\nu)}) = 0\} .$$

In 1981, Sato found a bijective correspondence between the Grassmannian $G(0,0)$ and the set of monic pseudo-differential operators

$$S = 1 + s_1(x)\left(\frac{d}{dx}\right)^{-1} + s_2(x)\left(\frac{d}{dx}\right)^{-2} + \cdots \tag{10}$$

of order zero which have certain type of singularities in their coefficients $s_j(x)$. In particular, if $s_j(x)$'s are all regular at $x = 0$, i.e. $s_j(x) \in \mathbb{C}[[x]]$ for all $j \geq 1$, then S of (10) corresponds bijectively to a point of the big-cell $G^+(0,0)$. Unfortunately this correspondence is not canonical, because it depends on the choice of a basis for V. The corresondence becomes completely canonical if one uses $G^+(0,-1)$ instead of $G^+(0,0)$. Thus one obtains the following.

THEOREM 1. *Let Γ_m be the set of all monic zeroth order pseudo-differential operators of the form (10) such that all the coefficients are in $\mathbb{C}[[x]]$. Then there is a canonical bijection*

$$\Gamma_m \xrightarrow{\sim} G^+(0,-1) .$$

We can choose an invertible function $f \in \mathbb{C}[[x]]$ so that Schur's operator L of (5) becomes

$$f \cdot L \cdot f^{-1} = \frac{d}{dx} + 0 + u_1(x)\left(\frac{d}{dx}\right)^{-1} + u_2(x)\left(\frac{d}{dx}\right)^{-2} + \cdots .$$

Then there is an operator $S \in \Gamma_m$ such that $S \cdot \frac{d}{dx} \cdot S^{-1}$ is equal to the above expression. Therefore, if we start with $f \cdot B \cdot f^{-1}$ instead of B (and which we identify anyway by (1)), then we have an operator $P \in B$ such that $S^{-1} \cdot P \cdot S = \left(\frac{d}{dx}\right)^n$. It is easy to see that every operator commuting with $\left(\frac{d}{dx}\right)^n$ for some $n \geq 1$ has only constant coefficients. Therefore, we have

$$A = S^{-1}BS \subset \mathbb{C}((\partial^{-1})) = \mathbb{C}((z)) ,$$

where $\partial = \frac{d}{dx}$ and we identify $z = \partial^{-1}$. Note that "replacing z^{-1} by L" simply means that changing ∂ by $S\partial S^{-1} = L$. Thus the operator $S \in \Gamma_m$ connects the algebra B and the abstract algebra $A \subset \mathbb{C}((z))$. So we can replace the pair (A, L) by another pair (A, W), where $W \in G^+(0, -1)$ is the point of the Grassmannian corresponding to S. We call (A, W) a *Schur pair*.

6. A connection between the Grassmannian and the geometric data consisting of pointed algebraic curves and vector bundles on them was discovered by G. Segal and G. Wilson [**SW**] in 1983. They dealt with the Grassmannian from the point of view of loop groups. Hence they used the Hilbert space $H^{(r)}$ of $\mathbb{C}^r$-valued square integrable functions on a circle in order to obtain an extended moduli space of curves and vector bundles of rank r. Then they defined an isomorphism $H^{(r)} \xrightarrow{\sim} H^{(1)}$ and brought the higher rank situation to the single Grassmannian, which is the analytic version of Sato's $G(0,0)$. However, the isomorphism between $H^{(r)}$ and $H^{(1)}$ is not canonical.

In order to find a canonical bijection between the data of the Grassmannians and the geometric data of pointed curves and vector bundles, we use Schur pairs. We call (A, W) a Schur pair of rank r, index μ and level ν if

1. $W \in G(\mu, \nu)$ and
2. $\mathbb{C} \subset A \subset V$ is a subalgebra such that $AW \subset W$ by multiplication of $V = \mathbb{C}((z))$ and

$$r = \operatorname{rank} A = G.C.D.\{\text{pole order of } a(z) \text{ at } z = 0 \mid a(z) \in A\}.$$

Let $\mathcal{S}_r(\mu, \nu)$ denote the set of all Schur pairs of rank r, index μ and level ν.

The geometric counterpart of the set of Schur pairs is the moduli space $\mathcal{M}_r(\mu, \nu)$ of quintets $(C, p, \mathcal{F}, \pi, \phi)$, where C, p and π are as in Section 2, $\mathcal{F}$ is now an arbitrary torsion free sheaf of rank r $\mathcal{O}_C$-modules such that

$$\dim H^0(C, \mathcal{F}) - \dim H^1(C, \mathcal{F}) = \mu\ ,$$

and

$$\phi : \mathcal{F}\big|_{U_p} \xrightarrow{\sim} \pi_* \mathcal{O}_{U_0}(\nu)$$

is an $\mathcal{O}_{U_p}$-module isomorphism. Note that in the case of rank one, π gives a local coordinate on U_p and hence our quintet becomes the quintuple of Segal-Wilson.

We have the following theorem.

THEOREM 2. *There is a canonical bijection*

$$\chi_{r,\mu,\nu} : \mathcal{M}_r(\mu, \nu) \xrightarrow{\sim} \mathcal{S}_r(\mu, \nu) \tag{11}$$

for every $r > 0$, $\mu \in \mathbb{Z}$ and $\nu \in \mathbb{Z}$. Moreover, there are the following canonical isomorphism under this correspondence:

$$H^0(C, \mathcal{F}) \simeq W \cap V^{(\nu)} \quad \text{and}$$
$$H^1(C, \mathcal{F}) \simeq V/(W + V^{(\nu)}) .$$

Roughly speaking, $\chi_{r,\mu,\nu}(C, p, \mathcal{F}, \pi, \phi) = (A, W)$ is defined by

$$A = \text{ holomorphic functions on } C \setminus \{p\}, \text{ and}$$
$$W = \text{ holomorphic sections of } \mathcal{F}\big|_{C\setminus\{p\}} .$$

In [**M3**], this bijection is obtained as a fully faithful functor between anti-equivalent categories.

As the final step to prove the main theorem, we need the following

THEOREM 3. *Let $(A, W) \in \mathcal{S}_r(0, -1)$ be a Schur pair of rank r, index 0 and level -1 such that W belongs to the big-cell $G^+(0, -1)$ of the Grassmannian, and let $S \in \Gamma_m$ be the pseudo-differential operator corresponding to the point W by the Sato correspondence (Theorem 1). Then $B = S \cdot A \cdot S^{-1}$ is a rank r commutative algebra of ordinary differential operators, where we regard A as a subring of $\mathbb{C}\big((\partial^{-1})\big)$ by the identification $z = \partial^{-1}$.*

Let D denote the set of all linear ordinary differential operators with coefficients in $\mathbb{C}\big[[x]\big]$. Then for a given algebra A with (8) and (9), the condition $S \cdot A \cdot S^{-1} \subset D$ gives a system of nonlinear ordinary differential equations among the coefficients of S which was studied in [**M2**, Section 1]. Theorem 3 tells us that this system has a regular solution if and only if there is a point $W \in G^+(0, -1)$ such that $AW \subset W$.

Schur showed that every $B \in \mathcal{B}_r$ gives a pair (A, L) and hence, by Theorem 1, the Schur pair $(A, W) \in \mathcal{S}_r(0, -1)$. Now Theorem 3 tells us how to go back from (A, W) to B. On the other hand, we have a bijection $\chi_{r,\mu,\nu}$ of (11) by Theorem 2. Combining these two constructions, we obtain the main theorem.

One of the most important motivations of studying commuting ordinary differential operator is their relation with completely integrable systems, and this point of view of the problem is entirely missing in the classical works I mentioned above. For the recent deep works on commuting operators and completely integrable systems, I refer to two books [N] and [**NMPZ**].

A detailed discussion of the topics of this paper is given in [**M3**].

7. As we saw in Section 3, one of the key steps of our classification theorem is the Burchnall-Chaundy theory: $[P,\ Q] = 0$ implies a polynomial relation. Being motivated by the recent theory of two dimensional quantum gravity, a natural question arises here: what happens is we have quasi-commutativity $[P,\ Q] \equiv 1$?

Certainly, the argument of Section 3 does not apply.

THEOREM 4. *Let P and Q be linear ordinary differential operators satisfying the quasi-commutativity condition*

$$[P,\ Q] = c = \textit{constant.}$$

If $c \neq 0$, then the operators cannot satisfy any non-trivial relation of the form

$$\sum_{i,j}^{\text{finite}} c_{ij} P^i Q^j = 0$$

with constant coefficients.

PROOF: Let

$$f(X,Y) = \sum_{i,j}^{\text{finite}} c_{ij} X^i Y^j$$

be a polynomial such that $f(P,Q) \equiv 0$ in this order. If this equation is non-trivial with respect to Q, then we can choose such a polynomial with the lowest degree in Q. Note that $[P,\ f(P,Q)] = 0$, i.e.

$$\sum_{i,j}^{\text{finite}} c_{ij} c j P^i Q^{j-1} = 0\ .$$

But this equation has a lower degree in Q, hence $f(X,Y)$ does not depend on Y.

This is the point where the quantum and the non-commutative geometry start, but we have to stop this article here.

References

[BC] J.L. Burchnall and T.W. Chaundy: Commutative ordinary differential operators, Proc. London Math. Soc. Ser. 2, **21** (1923) 420-440; Proc. Royal Soc. London Ser. A, **118** (1928) 557–583.

[F] G. Floquet: Sur la théorie des équations différentielles linéaires, Ann. Sci. de l'École Norm. Supér., **8** (1879) Suppl., 3–132.

[GD] I.M. Gel'fand and L.A. Dikii: Asymptotic behavior of the resolvent of Sturm-Liouville equations and the algebra of the Korteweg-de Vries equations, Russ. Math. Surveys, **30** (1975) 77–113; Fractional powers of operators and Hamiltonian systems, Func. Anal. Appl. **10** (1976) 259–273.

[K] I.M. Krichever: Methods of algebraic geometry in the theory of nonlinear equations, Russ. Math. Surveys **32** (1977) 185–214.

[Ku] M. Kuga: Galois' Dream, (translated by S. Addington from the Japanese text), Birkhäuser, to appear.

[M1] M. Mulase: Algebraic geometry of soliton equations, Proc. Japan Acad. Ser. A, **59** (1983) 285–288.

[M2] _________: Cohomological structure in soliton equations and jacobian varieties, J. Diff. Geom. **19** (1984) 403–430.

[M3] _________: Category of vector bundles on algebraic curves and infinite dimensional Grassmannians, Intern. Journ. Math. **1** (1990) to appear.

[Mum] D. Mumford: An algebro-geometric constructions of commuting operators and of solutions to the Toda lattice equations, Korteweg-de Vries equations and related non-linear equations, In Proc. Internat. Symp. on Alg. Geom., Kyoto 1977, Kinokuniya Publ. (1978) 115–153.

[N] S.P. Novikov, ed.: Integrable systems, London Math. Soc. Lec. Notes Ser. **60**, Cambridge Univ. Press (1981).

[NMPZ] S.P. Novikov, S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov: Theory of solitons — The inverse scattering method, Consultants Bureau, New York (1984).

[P] S. Pincherle: Mémoire sur le calcul fonctionnel distributif, Math. Ann. **49** (1897) 325–382.

[PW] E. Previato and G. Wilson: Vector bundles over curves and solutions of the KP equations, Proc. Symp. Pure Math. **49** (1989) 553–569.

[Sa] M. Sato: Soliton equations as dynamical systems on an infinite dimensional Grassmannian manifold, Kokyuroku, Res. Inst. Math. Sci., Kyoto Univ. **439** (1981) 30–46.

[SN] M. Sato and M. Noumi: Soliton equations and universal Grassmann manifold (in Japanese), Sophia Univ. Lec. Notes Ser. in Math. **18** (1984).

[Sc] I. Schur: Über vertauschbare lineare Differentialausdrücke, Sitzungsber. der Berliner Math. Gesel. 4 (1905) 2–8.

[SW] G.B. Segal and G. Wilson: Loop groups and equations of KdV type, Publ. Math. I.H.E.S. **61** (1985) 5–65.

[V] J.-L. Verdier: Equations differentielles algébriques, Séminaire de lÉcole Normale Supérieure 1979-82, Birkhäuser (1983) 215–236.

[W] G. Wallenberg: Über die Vertauschbarkeit homogener linearer Differentialausdrücke, Archiv der Math. u. Phys., Drittle Reihe **4** (1903) 252–268.

GEOMETRICAL ASPECTS OF SOLVABLE TWO DIMENSIONAL MODELS

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ABSTRACT

It was noted that there is a connection between the non-linear two-dimensional (2D) models and the scalar curvature r, i.e. when $r = -2$ the equations of motion of the Liouville and sine-Gordon models were obtained. Further, solutions of various classical nonlinear 2D models can be obtained from the condition that the appropriate curvature two form $\Omega = 0$, which suggtests that these models are closely related. This relation is explored further in the classical version by obtaining the equations of motion from the evolution equations, the infinite number of conserved quantities, and the common central charge. The Poisson brackets of the solvable 2D models are specified by the Virasoro algebra.

SCALAR CURVATURE

Models of field theory are very useful laboratories for understanding the dynamics of fields in a relatively simple manner. But there are very many such models and it will be very useful to find a common origin for them. the bosonic models, Liouville, sine-Gordon, and non-linear σ models in two dimensions, were previously obtained from a common Lagrangian.[1] In order to minimize the complexity of these various models, we limit ourselves to two dimensions. Let g_{AB} $(A,\ B\ =\ 0,\ 1)$ be the metric on a two-dimensional manifold, where the indices 0 and 1 refer to time and space, respectively. We parametrize the metric as

$$g_{AB} = \begin{bmatrix} P & Q \\ Q & R \end{bmatrix}, \qquad g = PR - Q^2, \tag{1}$$

where P, Q, and R are functions of x and t. The Riemann curvature tensor is given by

$$\begin{aligned} R_{ABCD} = \frac{1}{2}\Bigg[&\frac{\partial^2 g_{AD}}{\partial x^B \partial x^C} + \frac{\partial^2 g_{AC}}{\partial x^A \partial x^D} \\ &- \frac{\partial^2 g_{BD}}{\partial x^A \partial x^C} - \frac{\partial^2 g_{AC}}{\partial x^B \partial x^D}\Bigg] \\ &+ \left(\Gamma_{FAD}\Gamma^F_{BC} - \Gamma_{FAC}\Gamma^F_{BD}\right), \end{aligned} \tag{2}$$

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where Γ is the affine connection given by

$$\Gamma^A_{BC} = \frac{1}{g} g^{AE} \left[\frac{\partial g_{EB}}{\partial x^C} + \frac{\partial g_{EC}}{\partial x^B} - \frac{\partial g_{BC}}{\partial x^E} \right]. \tag{3}$$

In two dimensions there is only one independent component of the Riemann tensor that must be proportional to the scalar curvature r. So we have[2]

$$r = \frac{2}{g} R_{0101}. \tag{4}$$

The scalar curvature r can then be written after some algebraic manipulations as

$$r = -\frac{1}{g^{1/2}} \left[\left[\frac{P_x - Q_t}{g^{1/2}} \right]_x + \left[\frac{R_t - Q_x}{g^{1/2}} \right]_t \right]$$
$$+ \frac{1}{2g^2} \begin{bmatrix} P & Q & R \\ P_x & Q_x & R_x \\ P_t & Q_t & R_t \end{bmatrix}, \tag{5}$$

where the last term is a determinant and the subscripts denote partial derivatives. In the models discussed here, the determinant term does not contribute, in which case

$$r = -g^{-\frac{1}{2}} \partial_\alpha (g^{-\frac{1}{2}} \partial_\beta g \, g^{-1}_{\alpha\beta})$$

We first obtain the well-known Liouville and sine-Gordon models.[3] We note that for

$$g_{AB} = \begin{bmatrix} e^\rho & o \\ 0 & e^\rho \end{bmatrix} \tag{6}$$

the Liouville equation

$$\frac{1}{2}(\rho_{xx} + \rho_u) = e^\rho \tag{7}$$

results from Eqs. (5) and (6) for $r = -2$. For the metric

$$g_{AB} = \begin{bmatrix} 1 & \cos\alpha \\ \cos\alpha & 1 \end{bmatrix}, \tag{8}$$

one obtains, from Eqs. (5) and (8) when $r = -2$,

$$\alpha_{tx} = \sin\alpha, \tag{9}$$

Recently it has been pointed out that many equations in physics have a geometrical integrability origin.[4] It is expected that the present approach will provide an alternative method for gaining a deeper understanding of these nonlinear equations.

There is, therefore, a connection between the non-linear two-dimensional models and the scalar curvature r. This method of generating equations of motion is limited because the metric g_{AB} in two dimensions has three independent components. On the other hand, the curvature two-form Ω obtained from the affine connection Γ has in general six independent parameters, so when one takes $\Omega = 0$, additional equations of motion such as the Korteweg-de Vries (KdV) equation and modified Korteweg-de Vries (MKdV) equation were obtained.[5] We are interested in obtaining equations of motion from $\Omega = 0$, because the information we gain in the process may be useful in obtaining their solutions.

The popular method of solution[6] for certain nonlinear differential equations is to set up the linear scattering problem in the x variable, choose the time dependence of eigenfunctions, solve at $t = 0$, and then determine the solution at later times from the scattering data. The two first-order equations are

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_x = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv -\Gamma^R_{S1} V, \tag{10}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}_t = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \equiv -\Gamma^R_{S0} V, \tag{11}$$

where η is the eigenvalue, all the quantities are functions of x and t, and the subscripts are partial derivatives.

From $V_{xt} = V_{tx}$ and the requirement η is time independent, $\eta_t = 0$, we obtain

$$\begin{aligned} -A_x + qC - rB &= 0, \\ q_t - B_x + 2\eta B - 2Aq &= 0, \\ r_t - C_x - 2\eta C + 2Ar &= 0. \end{aligned} \tag{12}$$

One expands A, B, and C in terms of η and solves the equations. Equations (10) and (11) can be written as

$$\begin{aligned} &\frac{\partial V^R}{\partial x^m} + \Gamma^R_{Sm} V^s = 0, \quad m = 0, 1, \\ &R, S = 1, 2, \quad x^0 = t \quad x^1 = x, \end{aligned} \tag{13}$$

that is, vanishing of the covariant derivative of V, where Γ^R_{Sm} are components of the affine connection. The $\Gamma^R_S = \Gamma^R_{Sm} dx^m$ is the one-form with values in SL(2,R).

The curvature two-form is

$$\Omega^R_S = d\Gamma^R_S + \Gamma^R_T \wedge \Gamma^T_S, \quad \Gamma = \theta_\alpha X_\alpha, \quad \alpha = 1, 2, 3, \tag{14}$$

where

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It was shown from (10), (11), and (14),

$$\begin{aligned}\Omega =& dx \wedge dt\,\{(-A_x + qC - rB)X_1 \\ &+ (q_t - B_x + 2\eta B - 2qA)X_2 \;. \\ &+(r_t - C_x - 2\eta C + 2rA)X_3\}\end{aligned} \tag{15}$$

Therefore, the condition $\Omega = 0$ is equivalent to the requirement that the eigenvalue of η is time independent as given in Eq. (12). The following choices of the components of the affine connections yield the equations of motion of various models considered here. There does not seem to be a systematic way of formulating a given nonlinear equation into the condition $\Omega = 0$. The structure of the coefficients of X_1, X_2, and X_3 are quite restrictive.

(a) MKdV: $u_t + 6u^2u_x + u_{xxx} = 0$,

$$\begin{aligned}-\Gamma^R_{S1} &= \begin{pmatrix} \eta & u \\ -u & -\eta \end{pmatrix}, \\ -\Gamma^R_{S0} &= \begin{pmatrix} -4\eta^3 - 2\eta u^2 & -u_{xx} - 2\eta u_x - 4\eta^2 u - 2u^3 \\ u_{xx} - 2\eta u_x + 4\eta^2 u + 2u^3 & 4\eta^3 + 2\eta u^2 \end{pmatrix}.\end{aligned} \tag{16}$$

(b) NLS [nonlinear Schrödinger equation]: $iu_t + u_{xx} + 2|u|^2 u = 0$,

$$\begin{aligned}-\Gamma^R_{S1} &= \begin{pmatrix} \eta & u \\ -u^* & -\eta \end{pmatrix}, \\ -\Gamma^R_{S0} &= \begin{pmatrix} 2i\eta^2 + i|u|^2 & iu_x + 2i\eta u \\ iu^*_x - 2i\eta u^* & -2i\eta - i|u|^2 \end{pmatrix}.\end{aligned} \tag{17}$$

(c) KdV: $u_t + 6uu_x + u_{xxx} = 0$,

$$\begin{aligned}-\Gamma^R_{S1} &= \begin{pmatrix} \eta & u \\ -1 & -\eta \end{pmatrix}, \\ -\Gamma^R_{S0} &= \begin{pmatrix} -4\eta^3 - 2\eta u - u_x & -u_{xx} - 2\eta u_x - 4\eta^2 u - 2u^2 \\ 4\eta^2 + 2u & 4\eta^3 + 2\eta u + u_x \end{pmatrix}.\end{aligned} \tag{18}$$

SOLUTIONS OF THE NONLINEAR MODELS

In order to solve the equations for the models (a) and (b), one notes[7] that all the terms are of odd powers in the amplitude u, or q of Γ^R_{S1} because $q = u$ or $q \sim u_x$. This means that if we write the solution q in terms of sech (hyperbolic secant) the equations are expressible in terms of odd powers of sech with the aid of $\tanh^2\theta = 1 - \mathrm{sech}^2\theta$. One then needs to match the coefficients of the odd powers of sech to satisfy the equations of motions. The sign of the argument θ is arbitrary.

Equations (10) and (11) suggest $v_t = e^{\pm\eta x}$ and $v_i = e^{\pm a_i t}(i = 1, 2)$, respectively, where a_1 is the real part of the constant term A_c of A. So a reasonable choice for the argument is $\theta = -\eta x - a_t$. The imaginary part of A_c will be put in an exponential factor that multiplies the sech. The argument is 2θ because of the definition of the constant part of A,

$$A_c = a_1 + ia_2. \tag{19}$$

We thus start with a form of solution

$$q = N\mathrm{sech}2\theta e^{2ia_2t}, \tag{20}$$

$$\theta = -\eta x - a_1 t. \tag{21}$$

(a) MKdV: From (16), (19), (20), and (21), we get

$$\begin{gathered} a_1 = -4\eta^3, \qquad a_2 = 0, \\ \theta = -\eta x + 4\eta^3 t, \\ u = q = N\mathrm{sech}(-2\eta \mathrm{x} + 8\eta^3 \mathrm{t}), \end{gathered} \tag{22}$$

and upon substitution in the MKdV equation, obtain the solution

$$u = 2\eta\mathrm{sech}(-2\eta\mathrm{x} + 8\eta^3\mathrm{t}). \tag{23}$$

(b) NLS: From (17), (19), (20), and (21), we get

$$a_1 = 0, \qquad a_2 = 2\eta^2,$$

and

$$u = q = 2\eta\mathrm{sech}(-2\eta\mathrm{x})e^{4i\eta^2 t} \tag{24}$$

as a solution.

(c) KdV: This equation does not consist of odd powers of u but a solution can be obtained from a solution of $MKdV$.

From (18), (20) and (21)

$$a_1 = -4\eta^3, \qquad a_2 = 0, \tag{25}$$

$$\theta = -\eta x + 4\eta^3 t,$$

and we obtain the MKdV solution

$$q = 2\eta\mathrm{sech}(-2\eta\mathrm{x} + 8\eta^3\mathrm{t}). \tag{26}$$

The Miura transformation[8]

$$u = q^2 + iq_x \tag{27}$$

gives a solution u of the KdV equation

$$u = 4\eta^2(\text{sech}^2 2\theta + i\text{sech}2\theta \tanh 2\theta). \tag{28}$$

SCALAR CURVATURE AND GEOMETRY

The notion of scalar curvature and curvature two forms in two dimensions is useful in discussing nonlinear equations. Conformal field theory on Riemann surfaces plays an important role in understanding the dynamics of string theory. It is therefore interesting to discuss the relation of scalar curvature r to the geometry.

To set the notation, the element of length ds for a Minkowski metric g_{AB} is given by

$$ds^2 = g_{AB}dx^A dx^B = dt^2 - dx^2 \qquad g_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$x^o = t, \qquad x^1 = x$$

On the light cone $x_+ = t + x \qquad x_- = t - x$ and

$$ds^2 = dx_+ dx_-.$$

In complex coordinates,

$$z = t + ix \qquad \bar{z} = t - ix, \tag{29}$$

$$ds^2 = g_{MN}dz^M d\bar{z}^N = \frac{1}{2}(dz^2 + d\bar{z}^2), \qquad g_{AB} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$z^o = z, \qquad z' = \bar{z}$$

$$\partial_t = \partial_z + \partial_{\bar{z}} \qquad \partial_z = \frac{1}{2}(\partial_t - i\partial_x), \tag{30}$$

$$\partial_x = i(\partial_z - \partial_{\bar{z}}) \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_t + i\partial_z).$$

If two metric tensors g_{ij} and $\bar{g}_{ij}$ in spaces V_n and $\bar{V}_n$ in n dimensions are related by $\bar{g}_{ij} = e^{2\sigma}g_{ij}$ then V_n and $\bar{V}_n$ are conformal spaces and the transformation is a conformal transformation.[1] In V_2, every metric is conformal to a constant metric or any V_2 can be mapped conformally on a flat space[9] S_2.

In two dimensions, we parametrize the metric as $g_{AB} = \begin{pmatrix} P & Q \\ Q & R \end{pmatrix}$ and obtain the scalar curvature r given in (5). The square of the element of length is

$$ds^2 = Pdt^2 + 2Qdtdx + Rdx^2. \tag{31}$$

Equations of motion of various models are obtained from a given g_{AB} and r. We rewrite ds^2 and r in the z coordinates with the aid of (29) and (30), and get the following relations.

$$ds^2 = P'dz^2 + 2Q'dzd\bar{z} + R'd\bar{z}^2 \\ = \lambda|dz + \mu d\bar{z}|^2 \tag{32}$$

$$P' = \frac{1}{4}(P - R - 2iQ) = \lambda\mu^*, \qquad \lambda = \frac{1}{4}(P + R + 2\sqrt{g}) > 0$$

$$R' = \frac{1}{4}(P - R + 2iQ) = \lambda\mu, \qquad |\mu| < 1$$

$$Q' = \frac{1}{4}(P + R) = \frac{1}{2}\lambda(1 + |\mu|^2),$$

$$g_{AB} = \begin{pmatrix} P' & Q' \\ Q' & R' \end{pmatrix}, \qquad g' = P'R' - Q'^2 = -\frac{g}{4}, \tag{33}$$

and

$$\mathrm{r} = -\frac{1}{\sqrt{g'}}\left[\left(\frac{R'_z - Q'_{\bar{z}}}{\sqrt{g'}}\right)_z + \left(\frac{P'_{\bar{z}} - Q'_z}{\sqrt{g'}}\right)_{\bar{z}}\right], \tag{34}$$

up to a determinant that is suppressed. Equation (34) can be obtained from (5) by making the substitutions $t \to z$, $\mathrm{x} \to \bar{z}$, $\mathrm{g} \to \mathrm{g}'$ etc..

We obtain ds^2 and r in the w coordinates by defining

$$d\mathrm{w} = dz + \mu d\bar{z}, \tag{35}$$

$$d\bar{\mathrm{w}} = d\bar{z} + \mu^* dz,$$

or

$$dz = n(d\mathrm{w} - \mu d\bar{\mathrm{w}}), \qquad n = (2 - |\mu|^2)^{-1}$$

$$d\bar{z} = n(d\bar{\mathrm{w}} - \mu^* d\mathrm{w}),$$

From (32) and (35) one gets

$$ds^2 = \lambda d\mathrm{w} d\bar{\mathrm{w}}, \tag{36}$$

Also one has from (35)

$$\partial_{\mathrm{w}} = n(\partial_z - \mu^*\partial_{\bar{z}}) \qquad \partial_z = \partial_{\mathrm{w}} + \mu^*\partial_{\bar{\mathrm{w}}}$$

$$\text{or} \tag{37}$$

$$\partial_{\bar{\mathrm{w}}} = n(\partial_{\bar{z}} - \mu\partial_z) \qquad \partial_{\bar{z}} = \partial_{\bar{\mathrm{w}}} + \mu\partial_{\mathrm{w}}$$

The r in the w coordinates can be obtained from that in x coordinates by substituting $t \to \mathrm{w}$, $\mathrm{x} \to \bar{\mathrm{w}}$, $P = R = 0$, $Q \to \frac{\lambda}{2}$

$$\sqrt{g} \to \sqrt{g_\lambda} = \frac{i}{2}\lambda,$$

$$\mathrm{r} = -\frac{2}{\lambda}\left[\left(\frac{\lambda\mathrm{w}}{\lambda}\right)_{\bar{\mathrm{w}}} + \left(\frac{\lambda\bar{\mathrm{w}}}{\lambda}\right)_{\mathrm{w}}\right] = -\frac{4}{\lambda}(\ell n\lambda)_{\mathrm{w}\bar{\mathrm{w}}}. \tag{38}$$

There is no suppression of a determinant term in (38) because of the relation P=R=0.

Let us put

$$ds^2 = \lambda dz d\bar{z} \qquad ds'^2 = (\lambda e^{2\sigma}) dz d\bar{z} \tag{39}$$

From (36), (38) and (39), one obtains r and r' for λ and $\lambda e^{2\sigma}$, respectively

$$r = -\frac{4}{\lambda}(\ell n \lambda)_{z\bar{z}} \tag{40}$$

$$r' = e^{-2\sigma}\left(r - \frac{8}{\lambda}\sigma_{z\bar{z}}\right) \tag{41}$$

We consider the relation of r to the geometry. Riemann surface for any algebraic function is topologically a sphere with g handles, where g is the genus of the surface. How is sign of r related to geometry? If (string) world sheet is a compact Riemann surface of genus g, the (topological invariant) Euler characteristic on manifold M is defined as [10]

$$\chi = 2 - 2g$$

$$\chi(M) = \frac{1}{4\pi}\int d^2z\sqrt{|\mathrm{g}|}\mathrm{r}.$$

Recall (39) and (41)

$$\mathrm{r}' = e^{-2\sigma}(\mathrm{r} - \frac{8}{\lambda}\sigma_{z\bar{z}})$$

and

$$|\mathrm{g}'| = \frac{\lambda^2}{4}e^{4\sigma} \qquad |\mathrm{g}| = \frac{\lambda^2}{4} \tag{42}$$

Let us normalize σ by requiring[11]

$$\int d^2z\sqrt{|\mathrm{g}'|}\mathrm{e}^{-2\sigma} = 4\pi \qquad \text{or} \qquad \int d^2z\frac{\lambda}{2} = 4\pi$$

then

$$\chi = \frac{1}{4\pi}\int d^2z\sqrt{|\mathrm{g}|}\mathrm{r} = \mathrm{r}\frac{1}{4\pi}\int d^2z\frac{\lambda}{2} = \mathrm{r} \tag{43}$$

Let M be a compact Riemann surface without boundary or window, then the uniformization theorem[12] states there are three distinct compact Riemann surfaces up to holomorphic equivalence(a) sphere $g = 0$ (b) torus $g = 1$ (c) upper half plane $g > 1$.

We find that[13] $r = \chi = 2$, for $g = 0$; $r = \chi = 0$, for g=1; and $r = \chi = -2$ for $g = 2$.

POISSON BRACKETS FROM VIRASORO ALGEBRA

The Poisson brackets of various solvable two-dimensional models are specified by the Virasoro algebra. As a result, their equations of motion result from appropriate evolution equations. These models share an infinite number of conserved quantities and the same central charge, and are related by suitable changes of dynamical variables.

Conformal field theory in two dimensions (2D) plays an important role in string theory and solvable 2D models. We have noted that the solution of various classical nonlinear 2D models can be obtained from the condition that the appropriate curvature two-form $\Omega = 0$. This suggests that these models are closely related among themselves; here we further explore the close relations among the models. The condition for solvability is the existence of an infinite number of conserved quantities. We focus our attention on these conserved quantities and the central charge c in the classical version of the Virasoro algebra which specifies the Poisson brackets. We begin with the Poisson bracket of the Virasoro generators L_n and obtain the Poisson bracket of their Fourier transform $u(x)$. The KdV equation for the amplitude $u(x)$ then follows from the evolution equation given a suitable Hamiltonian.[14,15] The infinite set of commuting conserved quantities involved (which include the Hamiltonian) and the central charge are obtained.

The Virasoro algebra for classical systems is expressed in terms of the Poisson bracket,

$$i\{Ln, Lm\} = (n-m)L_{n+m} + \frac{1}{12}c(n^3 - n)\delta_{n+m,0}, \tag{44}$$

where L_n are the Virasoro generators and c is the central charge. We introduce the field $u(x)$, when the time dependence is suppressed,

$$u(x) = 2\hbar \sum_{n=-\infty}^{\infty} L_n e^{-inx} - \frac{1}{4}, \tag{45}$$

where the scale factor $2\hbar$ is a parameter chosen to yield the appropriate commutation relation in the quantized field theory, and the constant $\frac{1}{4}$ is included to remove the δ' term from the Poisson bracket of $u(x)$ that results from (44). The periodic boundary condition $u(t, x + 2\pi) = u(t, x)$ is imposed throughout, and it is assumed that $u(x)$ has continuous x derivatives of any order. From (44) and (45), we obtain,

$$\{u(x), u(y)\} = 2\pi\hbar\left[-\delta'''(x-y) + 2u(x)\delta'(x-y) + 2u(y)\delta'(x-y)\right], \tag{46}$$

where the primes on the δ function are derivatives with respect to x, provided the central charge is chosen to be

$$c = 3/\hbar. \tag{47}$$

Observe the formal anology of Eq. (46) with the quantum commutator of stress-energy tensors, which also represent the Virasoro algebra in the context of conformal field theory. Periodic boundary condition $u(t, x + 2\pi) = u(t, x)$ is imposed throughout.

The infinite set of conserved quantities Hn of the KdV model are given by[14–17]

$$Hn = \frac{1}{4\pi\hbar}\int_0^{2\pi} dx w_{2n+1}(u) \qquad n \geq 0, \tag{48}$$

where

$$\begin{aligned} w_1 &= u \\ w_{n+1} &= -\sum_{r=1}^{n-1} w_r w_{n-r} - w_n'. \end{aligned} \tag{49}$$

The w_2 to w_5 that result from (49) are

$$\begin{aligned} w_2 &= -u' \\ w_3 &= u'' - u^2 \\ w_4 &= -u''' + 4uu' \\ w_5 &= u'''' - 4uu'' - 3u'^2 + 2u^3 \end{aligned} \tag{50}$$

It is shown that[14-17]

$$\{Hn, Hm\} = 0 \tag{51}$$

Indeed, the KdV equation follows from the evolution equation

$$u_t = \{u, H\}, \tag{52}$$

provided H_1 is picked as the Hamiltonian so that

$$u_t = \left\{ u, \frac{1}{4\pi\hbar} \int dx - u^2 \right\} = u_{xxx} - 6u(x)u_x(x). \tag{53}$$

We pause here and remark on the physical significance of the conserved properties of the KdV equation. The equation of motion of the KdV Lagrangian[18]

$$\mathcal{L} = \frac{1}{2}\phi_x\phi_t - \phi_x^3 - \frac{1}{2}\phi_{xx}^2$$

yields

$$\phi_{xt} - 6\phi_x\phi_{xx} - \phi_{xxxx} = 0$$

or for $u = \phi_x$

$$u_t - 6uu_x - u_{xxx} = 0 \ .$$

This is another form of the KdV equation that follows from (53) upon $u \to -u$. The Hamiltonian density for $\mathcal{L}$ is

$$H = \phi_x^3 + \frac{1}{2}\phi_{xx}^2 = u^3 + \frac{1}{2}u_x^2$$

Therefore the integral corresponds to H_2 of (48)

$$H_2 = \frac{1}{4\pi\hbar}\int dx w_5 = \frac{1}{2\pi\hbar}\int dx(u^3 + \frac{1}{2}u_x^2) \ .$$

The KdV equation describes shallow water waves, where the amplitude u is proportional to the square root of the depth as seen from the solution (28). The H_2 expresses the conservation of energy of shallow water waves.

Write the KdV equation (53) as a conservation equation of density T and flux X i.e.

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0,$$

where

$$T = u, \qquad X = -u_{xx} + 3u^2.$$

If T and X are integrable for $-\infty < x < \infty$, then $\frac{d}{dt}\int_{-\infty}^{\infty} u\, dx = -X|_{-\infty}^{\infty} = 0$ or $H_0 =$ const where u is a solution of the KdV equation. T may be regarded as the density of the flux so $H_0 =$ const expresses the conservation of mass. Similarly, $u(u_t - u_{xxx} + 6uu_x) = 0$ can be written as a conservation equation where $T = \frac{1}{2}u^2$, $X = 2u^3 - uu_{xx} + \frac{1}{2}u_x^2$, and $\int_{-\infty}^{\infty} u^2 dx =$ const or $H_1 =$ const which expresses the conservation of momentum of the water waves described by the KdV equation.[17] The KdV equation is invariant under the scale transformation $u \rightarrow \lambda^2 U$, $t \rightarrow T/\lambda^3$, $x \rightarrow X/\lambda$, $\varphi \rightarrow \lambda\phi$, as this symmetry holds for the evolution equation (52) and the Poisson bracket (46).

To obtain the other two dimensional nonlinear models one can proceed to transform amplitudes from $u(x)$ to $p(x)$ via $u(x) = p^2(x) + p_x(x)$, where the subscript denotes derivative with respect to the x variable, and obtain the Poisson bracket of $p(x)$. As a consequence, the modified Korteweg de Vries (MKdV) equation, sine Gordon (S-G) equation, and Liouville (L) equation are obtained from the evolution equation for a suitable Hamiltonian.[19] When $v(x) = p^2 + ip_x$ is formed, another form of the KdV equation follows. The conserved quantities and c are shared by the KdV, MKdV, SG and L models. Finally, the additional transformation $p(x) = \psi_x(x) + \psi^+(x)$ is made, and the nonlinear Schrödinger (NLS) equation is obtained from the evolution equation.[20] The Virasoro algebra employed above is not a degeneracy symmetry, i.e. its generators do not represent charges that commute with the Hamiltonian. Instead, they constitute the dynamical variable (amplitude) of the system, and the algebra serves to specify its dynamics through their nontrivial commutators with the Hamiltonian.[21]

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REFERENCES

1. H. Bohr, B. Hou, and S. Saito, Nuovo Cimento A 84, 237 (1984); K. Tanaka, ibid. 93, 63 (1986).
2. S. Weinberg, Gravitation and Cosmology (Wiley, New York, 1972).
3. N.K. Mishra and K. Tanaka, Phys. Rev. D 36, 3722 (1987); L. Dolan, ibid. 15, 2337 (1977).
4. L.L. Chau, in proceeding of Seminare de Mathematique, Montreal, 1985 (unpublished).
5. M. Crampin, F.A.E. Pirani and D.C. Robinson, Lett. Math. Phys. 2, 15 (1977).
6. M.J. Ablowitz, D.J. Kaup, A.C. Newell, and H. Segur, Phys. Rev. Lett. 31, 125 (1973).
7. K. Tanaka, J. Math. Phys. 30, 172 (1989).
8. R.M. Miura, J. Math. Phys. 9, 1202 (1968).

9. L.P. Eisenhart, Riemannian Geometry, (Princeton University Press, Princeton, 1949).

10. M. Green, J. Schwarz and E. Witten, Superstring Theory, (Cambridge University Press, 1986).

11. O. Alvarez, Nucl. Phys. B216, 125 (1983).

12. W. Abikoff, The Real Analytic Theory of Techmüller Space (Springer-Verlag New York, 1980).

13. K. Tanaka, New Theories in Physics, World Scientific, Singapore, 1988.

14. J.L. Gervais, Phys. Lett. 160B, 277 (1985); J.L. Gervais, ibid. 160B, 279 (1985).

15. R. Sasaki and I. Yamanaka, in Proceedings of the "Conformal Field Theory and Solvable Lattice Models", Nagoya University, 1987.

16. B.A. Dubrovin, V.B. Matveev and S.P. Novikov, Usp. Math. Nank 31, 55 (1976).

17. C.S. Gardner, J. Math. Phys. 12, 1548 (1971).

18. P.G. Drazin and R.S. Johnson, "Solitons: an introduction" (Cambridge University Press, Cambridge, England, 1989).

19. K. Tanaka, The Ohio State University preprint.

20. M. Omote et al, Phys. Rev. 35, 2423 (1987).

21. C. Zachos, private communication.

EXPLICIT SOLITON-GENERATING BÄCKLUND TRANSFORMATIONS

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Some 2-d nonlinear differential equations (NLDE) can be regarded as the integrability condition for certain linear systems[1] possessing a free spectral parameter λ:

$$\partial_x\phi(\lambda) = U(\lambda)\phi(\lambda), \qquad \partial_t\phi(\lambda) = V(\lambda)\phi(\lambda), \tag{1}$$

where ϕ, U, V are all square matrix functions of x, t and λ, and U, V are rational functions of λ. Thus the compatibility condition for Eq. (1),

$$U_t - V_x + [U,V] = 0 \qquad \text{for all } \lambda, \tag{2}$$

corresponds exactly to the NLDE under consideration. A very general and powerful method to derive new solutions from a given seed solution of such NLDE is the Riemann problem technique[2-4] and a particular special case of this method provides a systematic scheme for constructing purely solitonic solutions[3]. Here we briefly describe some results of this scheme and indicate to what physically interesting models it has been applied.

Given a solution $u^o(x,t)$ of the NLDE [and hence the associated $U^o(\lambda)$, $V^o(\lambda)$], one can compute the corresponding wave function $\phi^o(x,t;\lambda)$ from Eq. (1). Let ψ_1 and $\psi_2 \equiv \psi_1^{-1}$ be two matrix functions parametrized by

$$\psi_1 = 1 - \frac{\lambda_1-\mu_1}{\lambda-\mu_1} P(x,t), \qquad \psi_2 = 1 + \frac{\lambda_1-\mu_1}{\lambda-\lambda_1} P(x,t) \tag{3}$$

where λ_1 and μ_1 are two suitably chosen points in the complex λ-plane, and P is an undetermined projection matrix, with $P^2 = P$. Then we tentatively define $U(\lambda)$ and $V(\lambda)$ by

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$$U \equiv \psi_2 U^o \psi_1 + (\partial_x \psi_2)\psi_1, \qquad V \equiv \psi_2 V^o \psi_1 + (\partial_t \psi_2)\psi_1. \tag{4}$$

Since we wish to extract a new solution u(x,t) from so defined $U(\lambda)$ and $V(\lambda)$, they are required to possess the right λ-structure so that Eq. (2) will give exactly the desired NLDE. For example, we must at least demand that the RHS of Eqs. (4) be free of pole singularities at λ_1 and μ_1. This immediately leads to a constraint on the projection matrix P,

$$\partial_x P = (1-P)U^o(\mu_1)P - P\, U^o(\lambda_1)(1-P), \tag{5}$$

[plus a parallel equation involving ∂_t and V^o; in the following such parallel equations will be omitted but understood]. Eq. (5) can be explicitly solved in terms of ϕ^o. In the case of 2x2 matrix, for example, P can be written as

$$P(x,t) = \phi^o(x,t;\mu_1)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(\gamma,\delta)\ \phi^o(x,t;\lambda_1)^{-1}/\text{trace}, \tag{6}$$

where (α,β) and (γ,δ) are two suitably chosen constant vectors.

After removing the apparent pole terms from the RHS of Eq. (4), we obtain

$$U(\lambda) = U^o(\lambda)+(\lambda_1-\mu_1)\left\{P\left[\frac{U^o(\lambda)-U^o(\lambda_1)}{\lambda-\lambda_1}\right](1-P)-(1-P)\left[\frac{U^o(\lambda)-U^o(\mu_1)}{\lambda-\mu_1}\right]P\right\}, \tag{7}$$

which becomes an explicit Bäcklund transformation (BT) between the seed solution $u^o(x,t)$ and the new solution u(x,t), provided the RHS of Eq. (7) has the correct λ-structure that a $U(\lambda)$ is supposed to have. Thus u is now explicitly calculable in terms of u^o and ϕ^o, and, in addition, the wave function ϕ corresponding to the new solution u is merely algebraically related to the old wave function ϕ^o by

$$\phi(\lambda) = \psi_2(\lambda)\phi^o(\lambda). \tag{8}$$

When the correct boundary behavior is also implemented, the new solution u will have one more "soliton" in its physical contents than the seed solution u^o. Thus Eq. (7) becomes an explicit "adding-one-soliton" BT. One can also use Eqs. (5) and (7) to derive a more familiar (but non-explicit) BT if one so wishes. The above constructive procedure can be iterated to algebraically generate a sequence of solutions with ever increasing soliton numbers. Quite a few well-known 2-d NLDE's, including K-dV, mK-dV, NLS, s-G and Liouville eqs., can be treated in this manner.

The 2-d principal chiral field equation is another important system in physics that can be treated in this way. The linear system in this case can be chosen to be[5]

$$(\lambda-1)\partial_\xi\phi = (g^{-1}\partial_\xi g)\phi, \qquad -(\lambda+1)\partial_\eta\phi = (g^{-1}\partial_\eta g)\phi. \tag{9}$$

Eq. (9) fits nicely into the soliton-generating scheme described above, and one can easily write down an explicit BT [Eq. (7)] for the principal chiral fields, as well as derive the more familiar but non-explicit BT[6]:

$$\begin{aligned} g^{-1}\partial_\xi\, g - g^{o-1}\partial_\xi\, g^o &= k\ \ \partial_\xi(g^{o-1}\, g), \\ g^{-1}\partial_\eta\, g - g^{o-1}\partial_\eta\, g^o &= -k\ \partial_\eta(g^{o-1}\, g). \end{aligned} \tag{10}$$

Basically the same method is applicable to the principal chiral model with a Wess-Zumino term, and to the super-symmetric principal chiral model, with or without a Wess-Zumino term[7], because they all have a linear system of the same type as Eq. (9).

Now the self-dual Yang-Mills (SDYM) equation in J-formulation has an associated linear system[8,4]

$$(\lambda\partial_{\bar{z}}-\partial_y)\phi = (J^{-1}\partial_y J)\phi \ , \qquad (-\lambda\partial_{\bar{y}}-\partial_z)\phi = (J^{-1}\partial_z J)\phi, \tag{11}$$

which has some similarity in structure with Eq. (9), but differs from it by having four instead of two variables in the eqs. However, two of the four degrees of freedom do not actually enter Eq. (11) and hence only become free parameters of the system. In particular, the poles λ_1 and μ_1 of Eq. (3) must now depend on these two parameters. Except for this complication, the same procedure for constructing soliton solutions goes through for the SDYM eq., resulting the algebraic generation of one-monopole[4] and one-instanton[9] solutions from the vacuum solutions. It is also possible to derive a more conventional-looking but non-explicit BT[9]:

$$\begin{aligned} J^{-1}\partial_y J - J^{o-1}\partial_y J^o &= \partial_{\bar{z}}[e^{i\alpha}(J^{o-1}J-\beta)], \\ J^{-1}\partial_z J - J^{o-1}\partial_z J^o &= -\partial_{\bar{y}}[e^{i\alpha}(J^{o-1}J-\beta)], \end{aligned} \tag{12}$$

where

$$2\beta \equiv |\mu_1| - \frac{1}{|\mu_1|} \ , \qquad e^{i\alpha} \equiv \mu_1/|\mu_1|. \tag{13}$$

Finally, the most interesting case among all integrable systems of the type we have been discussing here is the supersymmetric Yang-Mills (SSYM) eq. in 4 ordinary plus N super dimensions, since it contains as a subset the full Yang-Mills eq.[10] when N = 3,4. The SSYM linear system in J-formulation can be written as[11]

$$\begin{aligned} (\lambda D_1^s - D_2^s)\Phi(\lambda,\bar{\lambda}) &= (J^{-1}D_2^s J)\Phi(\lambda,\bar{\lambda}), \\ (\bar{\lambda}D_{\dot{1}t} - D_{\dot{2}t})\Phi(\lambda,\bar{\lambda}) &= (J^{-1}D_{\dot{2}t}J)\Phi(\lambda,\bar{\lambda}), \end{aligned} \tag{14}$$

which is again a bit similar in structure to Eq. (11), with, however, a major difference. Eq. (14) is now a 2-parameter instead of 1-parameter linear system. After a lot of trials and errors, we now feel that the special Riemann problem technique as discussed in this paper is probably not quite capable of dealing with this 2-parameter linear system[12]. It seems likely that here one needs a complete reformulation of the Riemann problem for two complex variables.

Notice that there is no such difficulty for the self-dual SSYM eq., because in this special case Eq. (14) is reduced to a one-parameter linear system:

$$\begin{aligned} &(\lambda D_1^s - D_2^s)\Phi(\lambda) = (J^{-1}\, D_2^s J)\Phi(\lambda), \\ &D_{\dot{1}t}\,\Phi(\lambda) = 0, \qquad -D_{\dot{2}t}\,\Phi(\lambda) = (J^{-1}\, D_{\dot{2}t}J)\Phi(\lambda). \end{aligned} \tag{15}$$

The soliton-generating procedure is again applicable, resulting, for example, the construction of a super one-instanton solution[9] from the vacuum solution.

In conclusion, we have seen that the special Riemann problem technique provides a direct and unified method to generate soliton solutions for many nonlinear equations, which are characterized by an associated 1-parameter linear system. However, the same method would probably not work for 2-parameter linear systems.

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References

1. P. D. Lax, Comm. Pure Appl. Math. 21, 467 (1968); V. E. Zakharov and P. B. Shabat, Sov. Phys. JETP 34, 62 (1972); M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Phys. Rev. Lett. 31, 125 (1973); Stud. Appl. Math. 53, 249 (1974)

2. M. F. Atiyah and R. S. Ward, Comm. Math. Phys. 55, 117 (1977); E. Corrigan, D. B. Fairlie, P. Goddard and R. G. Yates, Comm. Math. Phys. 58, 223 (1978); V. E. Zakharov and A. V. Mikhailov, Sov. Phys. JETP 74, 1953 (1978); V. E. Zakharov and P. B. Shabat, Funct. Anal. Appl. 13, 166 (1979)

3. S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, "Theory of Solitons", Consultants Bureau, New York 1984

4. P. Forgacs, Z. Horvath and L. Palla, Phys. Rev. D23, 1876 (1981); Nucl. Phys. B229, 77 (1983)

5. V. E. Zakharov and A. V. Mikhailov, Sov. Phys. JETP 47, 1017 (1978); L.-L. Chau Wang, in Proceedings of Guanzhou Conference on Theoretical Particle Physics, Jan. 1980, Canton

6. K. Pohlmeyer, Comm. Math. Phys. 46, 207 (1978); L. V. Cheredrik, Teor. Mat. Fiz. 38, 120 (1979); A. T. Ogielsky, M. K. Prasad, A. Sinha and L.-L. Chau Wang, Phys. Lett. 91B, 387 (1980)

7. T. L. Curtright and C. K. Zachos, Phys. Rev. D21, 411 (1980); Z. Popowicz and L.-L. Chau Wang, Phys. Lett. 98B, 253 (1981); G. Bhattacharya and S. Rajeev, Nucl. Phys. B246, 157 (1984); K.-C. Chou and Y.-B. Dai, Commun. Theor. Phys. (China) 3, 767 (1984); H. J. de Vega, preprints LPTHE 85/23 and 85/53; M. C. B. Abdalla, Phys. Lett. B152, 215 (1985); S. J. Gates, C. M. Hull and M. Rocek, Nucl. Phys. B248, 157 (1984); T. L. Curtright and C. K. Zachos, Phys. Rev. Lett. 53, 1799 (1984); P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, Nucl. Phys. B253, 701 (1985); E. Abdalla and M. C. B. Abdalla, Phys. Lett. B152, 59 (1985); L.-L. Chau and H. C. Yen, Phys. Lett. B177, 368 (1986)

8. A. Belavin and V. E. Zakharov, Phys. Lett. 73B, 53 (1978); K. Pohlmeyer, Comm. Math. Phys. 72, 37 (1980); G. Sartori, Nuovo Cimento 56A, 73 (1980); L.-L. Chau, M. K. Prasad and A. Sinha, Phys. Rev. D23, 2321 (1981); ibid D24, 1574 (1981)

9. L.-L. Chau, J. C. Shaw and H. C. Yen, Int. J. Mod. Phys. A4, 2715 (1989)

10. E. Witten, Phys. Lett. 77B, 394 (1978); M. Sohnius, Nucl. Phys. B136, 461 (1978); L.-L. Chau, M.-L. Ge and C.-S. Lim, Phys. Rev. D33, 1056 (1986).

11. I. V. Volovich, Phys. Lett. 129B, 429 (1983); C. Devchand, Nucl. Phys. B238 333 (1984); L.-L. Chau, M.-L. Ge and Z. Popowicz, Phys. Rev. Lett. 52, 1940 (1984)

12. J. Harnad and M. Jacques, J. Math. Phys. 27, 2394 (1986); J. Tafel, J. Math. Phys. 28, 240 (1987); N. Suzuki, Comm. Math. Phys. 113, 155 (1987); C. Devchand, preprint THEP 88/13.

INTEGRABILITY CONDITIONS:

RECENT RESULTS IN THE THEORY OF INTEGRABLE MODELS

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I INTRODUCTION

This paper reports various results achieved recently in the theory of integrable models. These are summarised in the Fig.1! At the Chester meeting [1] two of the authors were concerned [1] with the local Riemann-Hilbert problem (double-lined box in the centre of Fig.1), its limit as a *non-local* Riemann-Hilbert problem used to solve classical integrable models in 2+1 dimensions (two space and one time dimensions) [2,3], and the connection of this Riemann-Hilbert problem with Ueno's [4] Riemann-Hilbert problem associated with the representation of the algebra $gl(\infty)$ in terms of $\mathbb{Z}\otimes\mathbb{Z}$ matrices ($\mathbb{Z}$ the integers) and the solution of the K-P equations in 2+1. We were also concerned [1] with the construction of the integrable models in 1+1 dimensions from the loop algebras $\hat{g} = g\otimes[\lambda,\lambda^{-1}]$ where g is a simple finite dimensional Lie algebra and $\lambda \in \mathbb{C}$. Extensions to super-Lie algebras and super-integrable models in 1+1 were also sketched [1,5,6].

The paper identified three particular 'centres' of classical integrability in 1+1 dimensions:- (i) the Lax pair $v_x = Uv$, $v_t = Vv$; $U,V \in g = sl(N,\mathbb{C})$; (ii) the loop algebras and the KSA (Kostant-Symes-Adler [5,7]) theorem; and (iii) the Sklyanin bracket $\{T(\lambda)\overset{\otimes}{,}T(\mu)\} = [T(\lambda)\otimes T(\mu),r(\lambda,\mu)]$ (where T is a monodromy matrix, $\lambda,\mu \in \mathbb{C}$ and $r(\lambda,\mu)$ is the classical 'little r-matrix' [8,9,10]).

As befits a meeting on 'Geometry and Physics' the present paper is concerned to trace a connection between one piece of "real physics" and another piece of "real physics" via a significant part of the *mathematics* (i.e. mathematical physics) now surrounding the integrable models: this mathematics is that summarised in the Fig.1. The first piece of real physics (on which we can scarcely dwell here) concerned the propagation of intense (gigawatt cm^{-2} at peak) short (n sec.) optical pulses in a resonant medium [11-15] and it was where the interest of one of the authors (RKB) in solitons [16] (as well as much of that subject [15]) actually began. The second piece of real physics is much more recent and concerns solitons as thermal excitations in real systems - particularly in the ferromagnetic $CsNiF_3$ and the antiferromagnetic TMMC [17], in biological molecules like DNA [18,19], and in high-T_c superconductivity [20]. These references since 1982 form the particular 'EXPERIMENTS' output from Fig.1, and the calculations to this end form the route from top left to top right on that Figure (there are of course very many other applications of soliton theory to "real" physical problems now - some are in, for example, [12] - but we cannot touch on these here). This paper will be concerned primarily with the route top left to top right on Fig.1 mentioned: the idea of the paper is to show how the classical r-matrix and the quantum R-matrix for classical and quantum integrable models in 1+1 dimensions respectively, may be used to evaluate the classical and quantum partition functions

$$Z = \mathrm{Tr} \int \mathcal{D}\Pi\mathcal{D}\Phi \; \exp S[\Phi] \tag{1}$$

in which $S[\Phi]$ is the appropriate classical action. The 'EXPERIMENTS' output follows eventually from the calculation of this functional integral.

Functional integrals (1), quantum or classical, are taken over a symplectic manifold $\mathbb{M} = \mathbb{R}^{2\infty}$ with closed differential 2-form $\mathcal{B}$: $d\mathcal{B} = 0$. Canonical variables $\Pi(x,\tau)$, $\Phi(x,\tau)$ satisfy a bracket $\{\Pi,\Phi\} = \delta(x-x')$ and, in the quantum cases, $S[\Phi]$ is the Wick rotated classical action taken in Hamiltonian form [21,22]

$$S[\Phi] = \int_0^\beta d\tau \left[i\int_{-L}^{L} dx\Pi(x,\tau)\Phi_{,\tau}(x,\tau) - H[\Phi]\right] \tag{2}$$

in which $H[\Phi]$ is the classical Hamiltonian ($i=\sqrt{-1}$ and $\Phi_{,\tau} \equiv \partial\Phi/\partial\tau$). Integration (on $\mathbb{R} \otimes \mathbb{R}$) is over the space-time torus $\{(x,\tau) \in \mathbb{R}^2$: $-L \leqslant x < +L$, $0 \leqslant \tau < \beta\}$ and $\beta^{-1} = T =$ temperature: the Tr in (1) implies periodic boundary conditions (b.c.s) in β while periodic b.c.s in x are assumed so that, eventually $L \to \infty$ *in finite density thermodynamic limit* (see below). Note that, despite the formal Wick rotation, $(x,\tau) \in \mathbb{R}^2$ and classical trajectories evolve in $\mathbb{M}$ under H in real time τ. Note also that $\hbar = 1$ in the classical action (2). In classical limit $\hbar \to 0$ this action becomes [22]

$$S[\Phi] = \int_0^\beta -H[\Phi]d\tau = -\beta H[\Phi] \; . \tag{3}$$

If $H[\Phi]$ is quadratic in the canonical momenta $\Pi(x,\tau)$, the action (2) with $\hbar \neq 0$ can be reduced to the usual Lagrangian action by integration on $\mathcal{D}\Pi$. The measure in (1) is then the Feynman measure [23]. The symplectic measure $\mathcal{D}\Pi\mathcal{D}\Phi$ in (1) is simpler than the Feynman measure [21,22]. It is the same in both the classical and quantum cases, i.e. whether $S[\Phi]$ is (3) or (2). The symplectic measure *must* be used with (3) in the classical cases - Feynman's measure for the partition function exploits the Wick rotation and is interpreted differently [23].

By evaluating (1) for an integrable model in 1+1 dimensions we find the free energy density $\lim_{L\to\infty} FL^{-1} = \lim_{L\to\infty} (-\beta^{-1}L^{-1}\ln Z)$: F is the Helmholtz free energy. We shall be primarily concerned in this paper with the sine-Gordon (s-G) model with classical Hamiltonian

$$H[\Phi] = \gamma_0^{-1} \int dx\left[\tfrac{1}{2}\gamma_0^2\Pi^2 + \tfrac{1}{2}\Phi_x^2 + m^2(1-\cos\Phi)\right] \tag{4}$$

(in which $\gamma_0 > 0$, $\gamma_0 \in \mathbb{R}$ is a coupling constant; $m > 0$, $m \in \mathbb{R}$ is a mass and $\{\Pi,\Phi\} = \delta(x-x')$) and with its free energy density $\lim_{L\to\infty} FL^{-1}$, as well as with the sinh-Gordon (sinh-G) model

$$H[\Phi] = \gamma_0^{-1} \int dx \left[\tfrac{1}{2}\gamma_0^2\Pi^2 + \tfrac{1}{2}\Phi_x^2 + m^2(\cosh\Phi - 1)\right] \tag{5}$$

(where γ_0, m > 0) and *its* free energy density. These (classical) models have equations of motion

$$\Phi_{xx} - \Phi_{tt} = m^2 \sin \Phi \tag{6}$$

$$\Phi_{xx} - \Phi_{tt} = m^2 \sinh \Phi \tag{7}$$

respectively. They are covariant and integrable in 1+1, while by canonical transformation $\Pi \to \gamma_0^{-\frac{1}{2}}\Pi$, $\Phi \to \gamma_0^{\frac{1}{2}}\Phi$, analytical continuation in γ_0 ($\gamma_0 \to -\gamma_0$), and inverse canonical transformation, s-G (sinh-G) becomes sinh-G (s-G). Other relevant classical integrable models are the nonlinear Schrödinger (NLS) models

$$-i\Phi_t = \Phi_{xx} - c\Phi^*\Phi^2 \tag{8}$$

with immediate continuation in the coupling constant c : c > 0 is 'repulsive' NLS, c < 0 is 'attractive'. Moreover, the methods sketched in this paper to calculate quantum and classical Z apply to all of the integrable models in 1+1 with finite density thermodynamic limits [19]. They apply to models with classical Lax pairs $v_x = Uv$, $v_t = Vv$ and $U,V \in sl(2,\mathbb{C})$ - which includes s-G, sinh-G, both NLS models, the Landau-Lifshitz model, the Heisenberg ferromagnet, for example; but they apply also to, for example, the so-called Toda lattice [24] where $U,V \in g = sl(N,\mathbb{C})(N>2)$ and more generally (the Toda lattice of [24], to be distinguished from Toda's lattice [15,18,19], is a sequence of quantum integrable models involving arbitrarily many coupled fields: the beginnings of the classical Toda lattice were found in [25] (and refer Chapter III in [15]). But because there were apparently no Bäcklund transformations we thought (in 1976-1977) that the system could not be integrable [15,25]. The equation referred to in [26] is in [25]). Note in any case that U,V involve a spectral parameter $\zeta \in \mathbb{C}$, and $U,V \in \hat{g}$, $\hat{g}$ the loop algebra $\hat{g} = g \otimes [\zeta,\zeta^{-1}]$ - see the connections Loop Algebras → Integrable Model (which can then be taken back up to Lax Pair) in Fig1.

The Fig.1 as presented now mildly extends that Figure in [1]. New features included are:- (i) the 'route' Lax pair → compatibility (zero curvature) → monodromy matrix $T \in G$ (→ Sklyanin bracket) → (TrT) → Hamiltonians → Integrable Model

$$U_t - V_x + [U,V] = 0 \quad . \tag{9}$$

This route, already well known [27], is now included to stress the fundamental significance of classical 'complete integrability', in the sense of Liouville-Arnold [28,29], to all of the mathematical structure exposed in Fig.(1). The second new route is (ii) The route Loop Algebras → (quantum deformation) → $R\Delta = \bar{\Delta}R$, Hopf algebra → (Dual) → Hopf Algebra $\Delta T = T\otimes T$, Quantum Groups → Quantum Inverse Method $R\ T\otimes T = T\otimes T\ R$ or Bethe Ansatz ↔ R-matrix, Yang-Baxter Relation

$$(R\otimes I)(I\otimes R)(R\otimes I) = (I\otimes R)(R\otimes I)(I\otimes R) \quad . \tag{10}$$

The third new route is (iii) The two routes Oriented *3-manifold* M, Covariant (invariant) theory

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\ (A_\wedge\, dA + \tfrac{2}{3}A_\wedge\, A_\wedge\, A) \tag{11}$$

= integral of Chern-Simmons 3-form → Knot (link) Polynomials and also → Conformal Field Theory. These new routes (iii) are traced back to the braid groups in, for example, [30] and are treated in [31]. They are included in Fig.1 only because this 3-manifold theory must relate to the

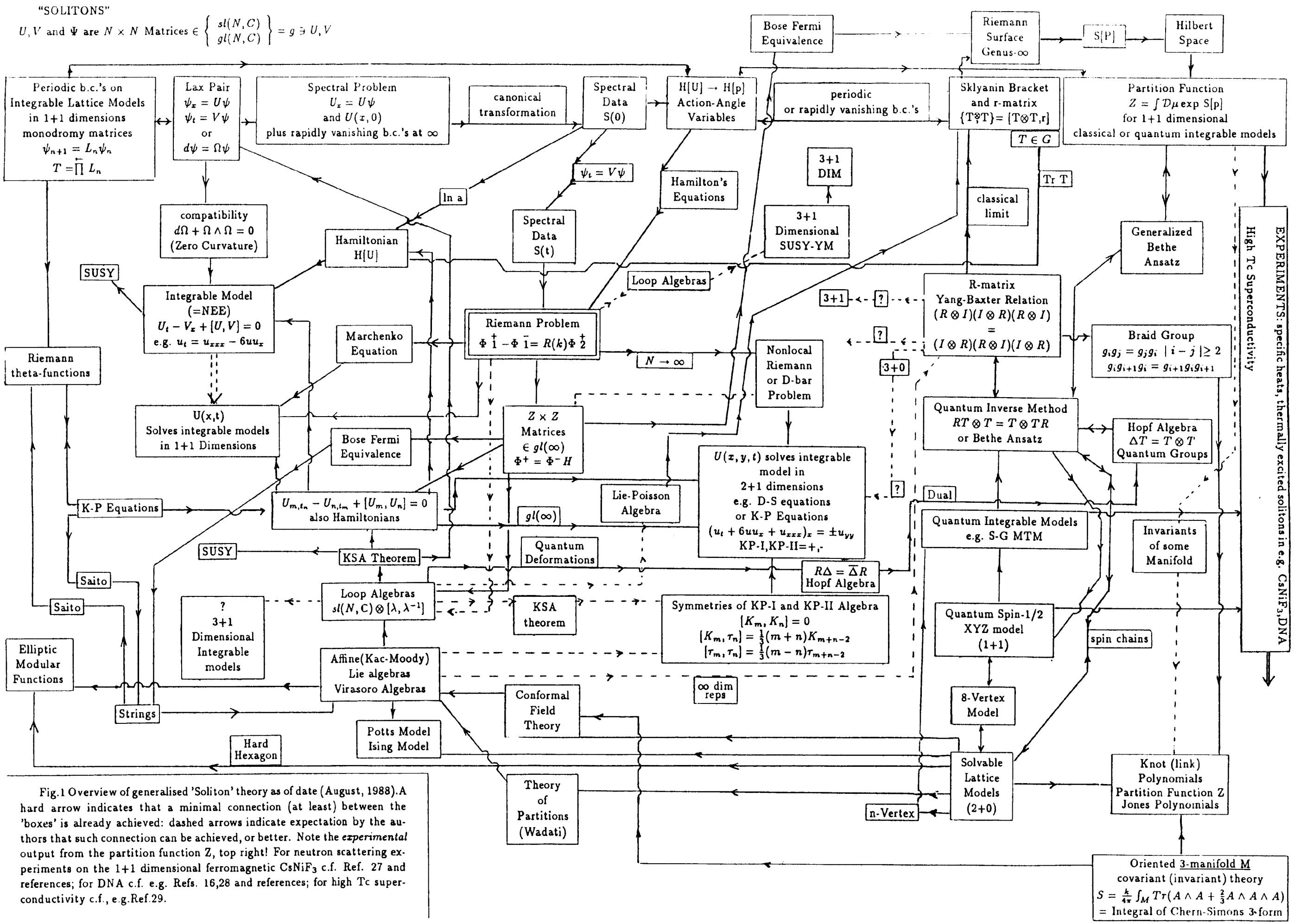

Fig.1 Overview of generalised 'Soliton' theory as of date (August, 1988). A hard arrow indicates that a minimal connection (at least) between the 'boxes' is already achieved: dashed arrows indicate expectation by the authors that such connection can be achieved, or better. Note the *experimental* output from the partition function Z, top right! For neutron scattering experiments on the 1+1 dimensional ferromagnetic CsNiF$_3$ c.f. Ref. 27 and references; for DNA c.f. e.g. Refs. 16,28 and references; for high Tc superconductivity c.f., e.g. Ref.29.

integrable models in 2+1 as well as to the solvable lattice models in 3+0 - but these theories have still to be found.

In this paper we retrace routes (i) and (ii). But we are primarily concerned with the route Periodic b.c.s (top left in Fig.1) → H[U] → H [p], Action-Angle Variables → Partition Functions

$$Z = \int \mathcal{D}\mu \exp S[p] \tag{12}$$

for 1+1 dimensional classical or quantum integrable models ↔ generalised Bethe Ansatz ↔ Quantum Inverse Method. Implicitly we are also concerned with $\mathbb{Z}\otimes\mathbb{Z}$ Matrices $\in$ gl(∞) → Bose-Fermi Equivalence → Riemann Surface Genus ∞ → S[p] → Hilbert Space → Partition Function with Z given in (12). It is primarily from (12) rather than (1) that the output 'EXPERIMENTS' derives, and our strategy of calculation is to transform (1) to (12), expressed in terms of action-angle variables with proper measure $\mathcal{D}\mu$. The strategy is to 'throw' all of the actual difficulty of the calculation into $\mathcal{D}\mu$ and the problem then is to calculate $\mathcal{D}\mu$. This way, with $\mathcal{D}\mu$ determined, the actual functional integrals for Z, quantum or classical, become wholly innocent - all ordering problems are eliminated for example - and the role of Z as a functional integral becomes simply to *minimise* the free energy F. The calculation of $\mathcal{D}\mu$ for classical $\lim_{L\to\infty} F/L$ is concerned with classical r-matrix theory: the calculation of $\mathcal{D}\mu$ for quantum $\lim_{L\to\infty} F/L$ is concerned with R-matrix theory: and the two measures are different. It is plain from Fig.1 that these objects, R and r, must enter into the calculations. Note (Fig.1) how the quantum integrable models in 1+1 relate to the solvable models in 2+0 and the n-vertex models (IRF and SOS models [32,33]). The connection with Jones's paper [34] is evident. Note, for example, how, by following across to Conformal Field Theories and Virasaro Algebras and Strings, Saito [35] has been able to sketch a connection through Mulase's work [36] to the integrable classical K-P equations in 2+1 dimensions. We believe these connections need still to be thoroughly worked out however.

II SOME "REAL PHYSICS"

The remarks in §I must be enough on the Fig.1 as such. We develop the mathematical physics of the paper in §§III and IV, so in this §I we add a few (very brief) further remarks on the "real physics" we have already mentioned.

The 'EXPERIMENTS' output from expressions (12) for Z is referenced in e.g. [17-20]. In [20] and references, Hubbard type lattice models are introduced as models of high-T_c superconductivity. In [20], in the continuum and scaling limits, two integrable quantum s-G models emerge (in the charge and spin sectors respectively). Bose-Fermi equivalence of s-G to the Massive Thirring Model (MTM - see Fig.1 and §IV) is exploited to see superconducting fluctuations in a massless TM correlation function at high temperatures (~1000°K). In [17] the classical s-G model is derived as a model of the ferromagnetic $CsNiF_3$ in a transverse magnetic field at temperatures $3°K \leqslant \beta^{-1} \leqslant 15°K$. The thermal excitations which determine the neutron scattering cross-sections are investigated. The low temperature asymptotic expansion for the free energy of s-G (given by continuation from our equation (76) in §IV below) can provide specific heats for this model. Similar theory applies to the anti-ferromagnetic TMMC [17]. In e.g. [18,19] and references, incomplete calculations of the number of solitons thermally excited in the biological molecule DNA at body temperatures (310°K) are given. An inference from the results of §IV of this paper is that the so-called 'Davydov soliton' proposed to explain some features of the infra-red spectrum of the α-helix of biological proteins breaks up at body temperatures [37]. This is all very real physics and the reader can consult these papers to find it.

For the earlier physics of 1968-1973 (optical pulse propagation in a resonant medium) we should perhaps only remind the reader that such a medium without level degeneracy [12,15] is modelled semi-classically [11-16] by 2-level atoms coupled to Maxwell's c-number electromagnetic equations. The 2-level atom has an SU(2) representation: SU(2) is a double cover of SO(3): SO(3) is the group of transformations of the rigid pendulum and its group manifold is the Bloch Sphere [11,16]. An optical pulse of "area" θ [11,12], a θ-pulse, tips the Bloch vector through polar angle θ, and a 2π-pulse tips it by 2π taking the initial atomic state (spin-down = atomic ground state) via spin-up (excited state) back to spin-down. No energy is absorbed, though there is actually a Berry phase of π from the double cover which is SU(2) - compare Venturi [38]. Consequently, a resonant dielectric of 2-level atoms traversed by the 2π-pulse becomes transparent to that pulse and this is the very real physics of 'self-induced transparency (SIT)' [11-13,16].

For the present brief discussion the equations of motion for SIT (the RMB and SIT equations [11-13,16]) are simply those of a set of rigid pendulums coupled by linear springs and these are the equations (6) of the s-G model, for suitable m^2 ($\hbar = c = 1$). It is relevant to the §III of the present paper that the classical s-G model has kink, antikink and (kink-antikink pair) breather solutions [11,12,15] while there is "radiation", harmonic type solutions $\Phi = \epsilon \cos(\omega t - kx + \delta)$ which for small enough amplitude $\epsilon \in \mathbb{R}$ linearise the s-G model to the Klein-Gordon (K-G) model

$$\Phi_{xx} - \Phi_{tt} = m^2\Phi \tag{13}$$

with dispersion

$$\omega(k) = (m^2 + k^2)^{1/2} \quad . \tag{14}$$

We call these harmonic solutions 'phonons' later in this paper.

The K-G model also plays an important role in the real physics of $CsNiF_3$ in that it is crucial to the calculation of both quantum and classical Z for s-G and sinh-G. This becomes very apparent in §IV below, notably in equation (76). In the classical calculation of (12) for s-G, Z_{KG}, the partition function of the K-G equation (13), *renormalises* Z for s-G [39].

We now turn to the mathematical physics of this paper.

III TWO SIGNIFICANT 'ROUTES' IN MATHEMATICAL PHYSICS

These are the routes (i) and (ii) itemised in §I.

(i) Flat connection is the Sklyanin bracket

The Lax pair $v_x = Uv$, $v_t = Vv$ ($U,V \in \hat{g} = g \otimes [\zeta, \zeta^{-1}]$ can be expressed [15] in terms of the 1-form

$$\Omega = U dx + V dt \tag{15}$$

and $dv = \Omega v$. Compatibility of this pair of linear equations in the column vector v requires the Poincaré integrability condition $d^2 v = 0$ so

$$\Theta \equiv d\Omega - \Omega \wedge \Omega = 0 \tag{16}$$

which for isospectral flows $\zeta_t = 0$ [15] is the Nonlinear Evolution Equation (NEE) which is (9). Conversely the NEE (9) is integrable (can be solved), since an inverse method is available [3], if $v_x = Uv$, $v_t = Vv$.

But (16) is the condition that a certain curvature $\Theta = 0$. Note [15] that $\Theta = 0$ is invariant under arbitrary gauge transformations $v \to v' = Bv$, $\det B = 1$, for which $\Omega \to \Omega' = dBB^{-1} + B\Omega B^{-1}$. Evidently $\Theta \to \Theta' = B\Theta B^{-1}$ and $\Theta = 0$ iff $\Theta' = 0$. The NEE (9) is $[\partial/\partial x - U, \partial/\partial t - V] = 0$ and U,V are coefficients of a flat connection (in the trivial vector bundle $\mathbb{R}^2 \times \mathbb{C}^N$ with $v(x,t,\zeta)$ taking values in the fibre $\mathbb{C}^N$ [27]). Along a curve $\gamma \in \mathbb{R} \times \mathbb{R}$ $(x_0,t_0) \to (x,t)$ one usefully defines the space and time ordered transport coefficient

$$\Omega_\gamma = \exp\left(\int_\gamma \Omega\right) \tag{17}$$

with the evident property $\Omega_{\gamma_1 + \gamma_2} = \Omega_{\gamma_2}\Omega_{\gamma_1}$. If

$$L_n = I + \int_{\gamma_n} \Omega \equiv I + \int_{\gamma_n} (Udx + Vdt) \tag{18}$$

and

$$\Omega_N = \prod_n L_n = L_N . L_{N-1} . \cdots . L_1 , \tag{19}$$

then $\Omega_\gamma = \lim \Omega_N$ as the partitions become dense. Then

$$v \to v_\gamma = \Omega_\gamma v \tag{20}$$

is parallel transport of v, while under gauge transformation

$$\Omega'_\gamma = B(x,t)\ \Omega_\gamma\ B^{-1}(x_0,t_0) . \tag{21}$$

Vanishing curvature (flat connection) means Ω_γ depends only on the end points (x_0,t_0) and (x,t), i.e. $v(x,t) = \Omega_\gamma\ v(x_0,t_0)$ satisfies $dv = \Omega v$ at (x,t). The existence of an infinity of *conservation laws* for the NEE (9) follows by taking a closed loop for γ so that $\Omega_\gamma = I$. We did this rather differently from $\Theta = 0$ in [15,40,41] developing the theory of prolongation structures. But for the present purposes let us here follow [27] - so take the closed curve γ and for it the square box in $\mathbb{R}^2$ formed by the straight lines $(-L,t_0) \to (L,t_0) \to (L,t_1 > t_0) \to (-L,t_1) \to (-L,t_0)$. Thus at fixed $t = t_0$ make parallel transport along x: $x = -L$ to $x = +L$ and define, under periodic b.c.s in x of period 2L, the monodromy matrix

$$T_L\ (\zeta,t_0) = \exp \int_{-L}^{L} U(x,t_0,\zeta)\ dx \quad ; \tag{22}$$

$U \in \hat{g} = sl\ (N,\mathbb{C}) \otimes [\zeta,\zeta^{-1}]$ (say) and U is an NxN matrix containing up to N(N-1) independent fields $\Phi(x,t)$ [3]. If $S_\pm = \exp \int_{t_0}^{t_1} U(\pm L,t,\zeta)dt$, periodic b.c.s in x mean $S_+ = S_-$ ($\equiv$ S). Since the flat connection means

$$S_-^{-1}\ T_L^{-1}(t_1)\ S_+\ T_L(t_0) = I \quad , \tag{23}$$

periodicity in x means

$$T_L(t_1) = S\ T_L(t_0)\ S^{-1} \quad , \tag{24}$$

and $\Delta_L \equiv TrT_L(t_1) = TrT_L(t_0)$ is independent of time t. Then $\Delta_L = \Delta_L(\zeta)$ depending only on the spectral parameter $\zeta \in \mathbb{C}$.

Evidently (22) means for $U \in g$ that the monodromy matrix $T_L \in G$, the group G with algebra g. Then since $U \in \hat{g}$, the loop algebra $\hat{g} = g \otimes [\zeta,\zeta^{-1}]$, $T_L = T_L(\zeta,t) \in \hat{G}$ the loop group with algebra $\hat{g}$. The monodromy matrix (22), defined through (18) and (19) (and see below) is defined on a symplectic manifold. This is implicit in the statement [15] that the spectral transform (ST) is a canonical transform (Fig.1) (in Fig.1 the Lax pair is $\Psi_x = U\Psi$, $\Psi_t = V\Psi$ and the ST derives from $\Psi_x = U\Psi$: Ψ is an N x N matrix made up of N independent N-column vectors v). The ST is $\Psi_x = U\Psi$ with initial data $U(x,0)$ at $t = 0 \rightarrow$ spectral data $S(0)$ at $t = 0$. Then $\Psi_t = V\Psi$ gives $S(t)$ at $t > 0$ [15] and this can be inverted [3] via the local R-H problem with or without Marchenko equation [3,15] to $U(x,t)$ solving the NEE (9) as shown in Fig.1. This is the classical inverse (or ST) method [15] and this is canonical [15]. (The conservation laws follow [15,40,41].) Still it seems $T_L(\zeta)$ is defined on a symplectic manifold because the loop algebras lead via the KSA theorem [1,5,7] to the evolution equations

$$U_{m,t_n} - U_{n,t_m} + [U_m,U_n] = 0 \tag{25}$$

where U_m, $U_n \in \hat{g} = g \otimes [\zeta,\zeta^{-1}]$, which contain (9) as particular examples. Results (25) are demonstrated [1,5,7] by appeal to the observation of Kirillov [7] that the co-adjoint orbits of the group G (loop group $\hat{G}$ [5,7]) in the dual g^* (dual $\hat{g}^*$) of the algebra g (loop algebra $\hat{g}$) define a symplectic manifold.

It follows that there is a bracket, the Kirillov bracket [5,7] and that this can be replaced by a usual Poisson bracket [7], $\{...\}$ = P.B. on the symplectic manifold $\mathbb{M} = \mathbb{R}^{2\infty}$ for the *elements* of $T_L(\zeta)$. It was the discovery of Sklyanin [42] that the N^2 possible brackets between these elements could be written in the form

$$\{T_L(\lambda) \overset{\otimes}{,} T_L(\mu)\} = [T_L(\lambda) \otimes T_L(\mu),\ r(\lambda,\mu)] \tag{26}$$

where $\otimes$ is Kronecker product, [...] is the N^2 x N^2 matrix commutator, and the N^2 x N^2 matrix $r(\lambda,\mu)$, $\in \hat{g} \otimes \hat{g}$, is the 'little r-matrix' [8,10,27]. Typically $r(\lambda,\mu)$ depends on $\lambda-\mu$, λ, $\mu \in \mathbb{C}$. The matrix trace of (26) means, with $\Delta_L(\lambda) \equiv \mathrm{Tr}T_L(\lambda)$, that

$$\{\Delta_L(\lambda),\ \Delta_L(\mu)\} = 0 \quad . \tag{27}$$

There is a continuous infinity of independent constants in involution (commuting under the P.B.) and the NEE ('flow') (9) is a continuously infinite dimensional completely integrable (Hamiltonian) system. The quantity $\ln\Delta(\zeta)$, also constant, expanded about some point $\zeta_0 \in \mathbb{C}$, is a generator of Hamiltonians which can be identified with those found [7] from the loop algebras. Thus, for integrable field models in 1+1, all of the classical models are completely integrable Hamiltonian systems and this fact underlies all of the structure of Fig.1. Moreover, the situation is essentially the same in 2+1 dimensions, for example the K-P, Davey-Stewartson and 3-wave interaction equations [3] are completely integrable [43].

However, we must take notice of boundary conditions (b.c.s). Complete integrability of the models in 1+1 was first proved for $x \in \mathbb{R}$ with vanishing b.c.s at $\pm\infty$. It remains true (of course) under periodic b.c.s (as (27) shows). However, nothing is proved in 2+1 dimensions except for vanishing b.c.s at infinity [43] and no extension of the Sklyanin bracket (26) is known. In the calculation of Z for models integrable in 1+1 we take a *finite density* thermodynamic limit under periodic b.c.s of period 2L letting $2L \rightarrow \infty$. Our method for Z connects action-angle variables under these periodic b.c.s with the well known ones [15,27] under vanishing b.c.s at $\pm\infty$. Although these latter are calculated at *zero* density [15,27], the connection can still be established at finite density to $O(L^{-1})$ as $L \rightarrow \infty$. It is this relationship between action-angle variables under periodic b.c.s with those under vanishing b.c.s at $\pm\infty$ which establishes the proper form of the measure $\mathcal{D}\mu$. No comparable analysis for integrable models in 2+1 dimensions is found yet.

(ii) Deformed loop algebras are dual to the quantum groups $\hat{G}_q$

This is the route Loop Algebras → (Quantum Deformation) → $R\Delta = \bar{\Delta}R$, Hopf Algebra → (Dual) → Hopf Algebra $\Delta T = T \otimes T$ → Quantum Inverse Method $RT \otimes T = T \otimes TR$ in Fig. 1. The final condition is

$$R(\lambda,\mu)\ \hat{T}_L(\lambda) \otimes \hat{T}_L(\mu) = \hat{T}_L(\mu) \otimes \hat{T}_L(\lambda)\ R(\lambda,\mu) \tag{28}$$

($\lambda,\mu \in \mathbb{C}$) in which $\hat{T}_L(\lambda)$ is the appropriate extension of $T_L(\lambda)$ to the quantum case (it is the quantum monodromy matrix [8,10]). $R(\lambda,\mu)$ is the quantum R-matrix and the $\hat{T}_L(\mu)$ lie in the 'quantum group' $\hat{G}_q$ (which is a semi-group): $\hat{G}_q$ is a *deformation* of the group $G \ni T_L(\mu)$, it is an algebra with group-like 'structure factors' $R(\lambda,\mu)$ and 'commutation relations', for the elements of $\hat{T}_L(\lambda)$ for the quantum integrable model, given by (28). The matrix trace of (28) means, with $\hat{\Delta}_L(\lambda) \equiv \mathrm{Tr}\ \hat{T}_L(\lambda)$, that

$$[\Delta_L(\lambda),\ \Delta_L(\mu)] = 0\ , \tag{29}$$

quantum integrability conditions, and $\ln\Delta_L(\zeta)$, $\zeta \in \mathbb{C}$ is a generator of ('operator') Hamiltonians $\hat{H}$.

Following eg. Jimbo [44] and Drinfeld [45], consider the algebra $g = sl(2)$ and 'deform' it with parameter $q \equiv e^t \neq 1 (t\neq 0)$. In Cartan-Weyl basis $sl(2) \to sl_q(2)$, i.e. $[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$ goes to

$$\begin{aligned}[h,e] = 2e,\ [e,f] = -2f,\ [e,f] &= (k^2 - k^{-2})/(q - q^{-1}) \\ &= (\sinh ht)/(\sinh t)\end{aligned} \tag{30}$$

where $k^2 = q^h$: $t \to 0$ ($q \to 1$) takes $sl_q(2)$ to $sl(2)$. Other useful relations are $kek^{-1} = qe$, $kfk^{-1} = q^{-1}f$ ($t \neq 0$). The two-dimensional representation of $sl(2)$, $h = \sigma_3$, $e = \sigma_+$, $f = \sigma_-$ has the corresponding representation of $sl_q(2)$, $h = \sigma_3$, $e = \sigma_+$, $f = \sigma_-$ with $k^2 = q^{\sigma_3} = \mathrm{diag}\ (q,q^{-1})$, $k^2 - k^{-2} = \mathrm{diag}\ (q - q^{-1}, q^{-1} - q) = (q - q^{-1})\sigma_3$.

The deformed algebra $sl_q(2)$ is a Hopf algebra H (a quasi-triangular Hopf algebra as defined in [46]): for our purposes the feature of a Hopf algebra is co-multiplication [46]. Two spins S_1,S_2 add in quantum mechanics as $S = S_1 \otimes 1 + 1 \otimes S_2$ acting in the Hilbert space $|S_1,m_1\rangle |S_2,m_2\rangle$. A corresponding representation independent formula for $a \in sl(2)$ is $a \to \Delta(a) = a \otimes 1 + 1 \otimes a$ with multiplicative property $\Delta(ab) = \Delta(a)\Delta(b)$. So for $sl_q(2)$ set the co-multiplications

$$\begin{aligned}&\Delta(h) = h \otimes 1 + 1 \otimes h, \\ &\Delta(e) = e \otimes k + k^{-1} \otimes e,\ \Delta(f) = f \otimes k + k^{-1} \otimes f\ .\end{aligned} \tag{31}$$

From $\Delta(h)$ one finds $\Delta(k) = k \otimes k$, while $\Delta(e)$ and $\Delta(f)$ are chosen to induce an automorphism of g_q. Thus

$$\Delta(e)\Delta(h) = eh \otimes k + e \otimes kh + k^{-1}h \otimes e + k^{-1} \otimes eh$$

$$\Delta(h)\Delta(e) = he \otimes k + hk^{-1} \otimes e + e \otimes kh + k^{-1} \otimes he$$

so

$$[\Delta(h),\ \Delta(e)] = [e,h] \otimes k + k^{-1} \otimes [e,h] = 2\Delta(e); \tag{32}$$

similarly $[\Delta(h), \Delta(f)] = -2\Delta(f)$, while $[\Delta(e), \Delta(f)] = [e,f] \otimes k^2 + k^{-2} \otimes [e,f] = (\sinh t)^{-1} [\sinh ht \otimes k^2 + k^{-2} \otimes \sinh ht] = ½(\sinh t)^{-1} [k^2 \otimes k^2 - k^{-2} \otimes k^{-2}] = (\sinh \Delta(h)t)/(\sinh t)$!

Now observe that apart from (31) there must be the second co-multiplication

$$\bar{\Delta}(h) = h \otimes 1 + 1 \otimes h$$

$$\bar{\Delta}(e) = e \otimes k^{-1} + k \otimes e,\ \bar{\Delta}(f) = f \otimes k^{-1} + k \otimes f \tag{33}$$

inducing the automorphism $[\bar{\Delta}h, \bar{\Delta}(e)] = 2\bar{\Delta}(e)$ etc. of $g_q = sl_q(2)$. Both co-multiplications (31) and (33) are *associative*. From the abstract notion of addition of spins we need

$$\begin{aligned}\Delta_3(a) &= a \otimes 1 \otimes 1 + 1 \otimes a \otimes 1 + 1 \otimes 1 \otimes a\\ &= (\mathrm{id} \otimes \Delta) \circ \Delta a \ \underline{\text{or}}\ (\Delta \otimes \mathrm{id}) \circ \Delta a\end{aligned} \tag{34}$$

(see [46]) and also

$$(\mathrm{id} \otimes \bar{\Delta}) \circ \bar{\Delta} a = (\bar{\Delta} \otimes \mathrm{id}) \circ \bar{\Delta}(a) \equiv \bar{\Delta}_3(a)\ . \tag{35}$$

The two distinct laws of quantum addition (co-mulitplication) map a quantum algebra to itself and become identical when $q \to 1$. We must expect them to be equivalent and therefore there must be a relation between them. This relation is

$$R\Delta(a) = \bar{\Delta}(a)R,\ a \in g_q,\ R \in g \otimes g \tag{36}$$

(g, R form the quasi-triangular Hopf algebra [46]). If (36) is true for all $a \in g$, it is a very overdetermined system and R satisfies the Yang-Baxter relation [8,10]

$$R_{12}\, R_{13}\, R_{23} = R_{23}\, R_{13}\, R_{12} \in g \otimes g \otimes g\ . \tag{37}$$

The indices label the 'spaces': $R_{12} \in g_1 \otimes g_2$ which is extended to $R_{12} \otimes I \in g_1 \otimes g_2 \otimes g_3$, etc. Proofs of the Yang-Baxter relation in e.g. [8,10] are 'physical'. Proof in e.g. [46] computes $\mathrm{id} \otimes \tau \circ \Delta R$ in two ways using $(\Delta \otimes \mathrm{id})R = R_{12}\, R_{23}$, $(\mathrm{id} \otimes \Delta)R = R_{13}\, R_{12}$ ($\tau\ u \otimes v = v \otimes u$, the 'twist' map [46]). The Yang-Baxter relation (37) is Onsager's star triangle relation [8,10,33] and is also the condition for a factorisable S-matrix [8,10,33]. Thus R in (36) is indeed the quantum R-matrix. So far it has no spectral parameter.

We specifically interpret $sl(2)$ as $sl(2,\mathbb{C})$ with its deformation $sl_q(2,\mathbb{C})$. We can add extra roots e_0, f_0, $h_0 = \lambda f_\Phi$, $\lambda^{-1} e_\Phi$, h (Φ the highest root, $\lambda \in \mathbb{C}$). This is equivalent to using the loop algebra $\hat{g} = g \otimes [\lambda,\lambda^{-1}]$. Then $\hat{g}$ deforms to $\hat{g}_q = g_q \otimes [\lambda,\lambda^{-1}]$, an infinite dimensional Hopf algebra, and the spectral parameter $\lambda \in \mathbb{C}$ enters throughout all of the argument so far. In particular, it enters the Yang-Baxter relation (37). Conveniently we take it in the form

$$R_{12}(\lambda)\ R_{13}(\lambda+\mu)\ R_{23}(\mu) = R_{23}(\mu)\ R_{13}(\lambda+\mu)\ R_{12}(\lambda) \tag{38}$$

where $\lambda,\mu \in \mathbb{C}$. In general $R_{12} = R_{12}(\lambda,\mu)$ etc., but typically $R_{12}(\lambda,\mu) = R_{12}(\lambda-\mu)$ for the quantum models: the $\lambda + \mu$ in (38) is a conservaton of momentum (parametised by λ,μ).

The Hopf algebra $H = sl_q(2)$ has a dual K. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in K$, dual to H, we can note that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (e) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{39}$$

so that $h(e) = 1$, $a(e) = c(e) = d(e) = 0$. Similarly $a(k) = q^{½}$, $d(k) = q^{-½}$, $c(k) = b(k) = 0$ and, by defining the product $ab(e)$

$$\begin{aligned} ab(e) = (a \otimes b)\Delta e &= (a \otimes b)\ (e \otimes k + k^{-1} \otimes e) \\ &= a(e)b(k) + a(k^{-1})b(e) \\ &= 0 + q^{-½} = q^{-½} \end{aligned} \tag{40}$$

and similarly for $ba(e)$, which proves to be $q^{½}$, one has

$$ab(e) = q^{-1}ba(e) \ . \tag{41}$$

This is one of the equations found by Manin [41] for the 'group' G_q of the 'quantum plane'.

At this point we would like to make plain that these very *simple* and explicit computations which have taken us from equation (30) onwards to (40) were all explained to us by R Sasaki. Dr Sasaki is currently at the University of Durham, UK, and he is also a co-author of [24].

Manin's 'quantum plane' $Rq(2|0)$ is the set (x,y) such that

$$xy = q^{-1}yx \ . \tag{42}$$

There is the dual set $(\xi,\eta) \in Rq(0|2)$ such that $\xi\eta = -q\eta\xi$, $\xi^2 = \eta^2 = 0$. If $\begin{bmatrix} x' \\ y' \end{bmatrix} = M\begin{bmatrix} x \\ y \end{bmatrix} \in Rq(2|0)$ (where a,b,c,d in M commute with x,y but not, in general, with each other) the coefficients of x^2, y^2 and xy derived from (42), $x'y' = q^{-1}y'x'$, require

$$ac = q^{-1}ca,\ bd = q^{-1}db,\ ad-da = q^{-1}cb - qbc \ . \tag{43a}$$

If $\begin{bmatrix} x'' \\ y'' \end{bmatrix} = M^T\begin{bmatrix} x \\ y \end{bmatrix}$ also (M^T is the transpose of M), then

$$ab = q^{-1}ba,\ cd = q^{-1}dc,\ bc = cb \tag{43b}$$

(and $q \neq -1$). We call the six equations (43a,b) Manin's G_q relations. It is easily checked that the M's are multiplicative. If $q = 1$, $M \in G = Gl(2)$. So if $q \neq 1$ (and $q \neq -1$) $M \in G_q$ the 'deformed' or 'quantum group' $Gl_q(2)$. If we define

$$\det_q M = ad - q^{-1}bc = da - qcb = 1 \ , \tag{44}$$

$G_q = Sl_q(2)$. It is easily checked that $\det_q MM' = \det_q M.\det_q M'$. There is an inverse

$$M^{-1} = (\det_q M)^{-1} \begin{bmatrix} d & -qb \\ -q^{-1}c & a \end{bmatrix} \tag{45}$$

since $M^{-1}M = 1$. But $M^{-1} \in G_{q^{-1}}$ not G_q. So $M \in G_q = K$ form a semi-group, not a group.

The 'language' of Hopf algebras is co-multiplication (K is a Hopf algebra). We *define* $\Delta M = M \otimes M$, i.e. $\Delta(a) = a \otimes a + b \otimes c$, $\Delta(b) = a \otimes b + b \otimes d$, etc. ($\otimes$ is the symbol of co-multiplication here). Then we find $\Delta(a)\Delta(b) = q^{-1}\Delta(b)\Delta(a)$, $\Delta(a)\Delta(c) = q^{-1}\Delta(c)\Delta(a)$, etc. and this is an automorphism of the G_q relations.

These results all extend to arbitrary $G_q = Sl_q(N)$. An obvious extension of (42) when $N = 3$ is $(x,y,z).S_i.(x,y,z)^T = 0$ where

$$S_1 = \begin{bmatrix} 0 & \sqrt{q} & 0 \\ -1\sqrt{q} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ etc.} \tag{46}$$

and this extends to arbitrary $N > 3$. Note the essentially 'trivial' way in which $G = Gl(N)$ deforms to $G_q = Gl_q(N)$. However, except that q could be any power of q, other deformations have to be trivial.

The R-matrix, which arises at (36) from the two co-multiplications from $g_q = sl_q(2)$ introduced at (30), proves to be

$$R = \begin{bmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & (q^{-1}-q) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{bmatrix} . \tag{47}$$

With this same R-matrix all of the G_q relations (42) can be written

$$RM' \, M'' = M'' \, M'R \tag{48}$$

where $M' = M \otimes I$, $M'' = I \otimes M$, I is the 2×2 unit matrix and $\otimes$ is Kronecker product: M' derives from M in the space '1', M" from M in '2'. Evidently $M \in G_q$ the quantum group through (48) and *associativity* proves to be guaranteed under the Yang-Baxter relation (37): the indices label the spaces again, as noted by (37).

If we now use prime to indicate the *order* in the elements of $M \in G_q$, the G_q relations (48) can be written $RM \otimes M' = M' \otimes MR$. Since $M \in G$ deforms to satisfy these G_q relations while $G_q = K$, the dual of H, and H is the deformation G_q of the algebra g of G, we can deform the monodromy matrix $T_L \in G$ (argument of (i) above, 'Flat connection') to $\hat{T}_L \in G_q$. This induces a quantisation (non commutative elements of $\hat{T}_L$) on T_L. Then since $T_L \in G \to T_L(\lambda) \in \hat{G}$, the loop group with loop algebra $\hat{g} = g \otimes [\lambda,\lambda^{-1}]$, we can deform $T_L(\lambda)$ to $\hat{T}_L(\lambda)$ satisfying the quantum loop group, $\hat{G}_q$, relations and these would be

$$R(\lambda,\mu) \; \hat{T}_N(\lambda) \otimes \hat{T}_N(\mu) = \hat{T}_N(\mu) \; \hat{T}_N(\lambda) \; R(\lambda,\mu) \; . \tag{49}$$

We explain the label N, not L, in one moment. Otherwise (49) is exactly the result (28) looked for. It imposes 'natural' quantisation on the elements of $\hat{T}_N(\lambda)$ $(=T_L(\lambda))$ and it implies the quantum integrability condition $[\hat{\Delta}_N(\lambda), \hat{\Delta}_N(\mu)] = 0$ (which is (29): $\hat{\Delta}_N(\lambda) = \mathrm{Tr}\ \hat{T}_N(\lambda)$). Then $\ln\hat{\Delta}_N(\lambda)$ is a generator of operator $\hat{H}$'s.

The quantum group relation (49) is that found directly in [8] for particular quantum integrable models (this is the 'quantum inverse method' [8,10]). From the local lattice transition matrix $\hat{L}_n(\lambda)$ associated with the spectral problem $v_{n+1} = \hat{L}_n(\lambda)\ v_n$ (deriving from, and reducing to, the spectral problem $v_x = Uv$ for classical *fields*) one derives exactly (49) with $\hat{T}_N(\lambda) = \overleftarrow{\prod_n} L_n(\lambda) = \hat{L}_N(\lambda).L_{N-1}(\lambda). \cdots .\hat{L}_{-N}(\lambda)$ (refer back to Ω_N (19) and T_L (22)). The $\hat{L}_n(\lambda)$ depend (linearly) on quantised quantities $\Phi_n(t)$ [10] deriving from quantised fields $\Phi(x,t)$ with the appropriate commutation relations. Periodic b.c.s are assumed on the lattice, there are 2N lattice points in a period, there is a lattice spacing a_0 (say), and $2L = (2N + 1)a_0$. This explains the switch of label $N \to L$ in $\hat{T}_L \to \hat{T}_N$. Evidently (49) gives the quantum commutation relations for the elements of $\hat{T}(\lambda)$ corresponding to the particular quantum model. In fact the R-matrix $R(\lambda,\mu)$ (which can be written $R(\lambda-\mu)$ and satisfies the Yang-Baxter relations (38) as explained) determines one whole 'hierarchy' of integrable quantum models whose Hamiltonians $\hat{H}$ are found from $\ln\hat{\Delta}(\lambda)$. We now see that the 'natural' quantisation of the integrable quantum models coincides with the 'quantisation' which is the appropriate quantum group with relations (49).

We find the argument to the 'quantum groups' beautiful - and certainly remarkable (the quantum inverse method came first after all [8,10]). The actual remarks made in this III are all we can say on it here and these are partly heuristic. We hope they are at least sufficient to show the elegance of the structure. Our next task is to turn, in the IV next, to the main point of the paper and use R-matrix theory to compute the quantum partition functions Z for integrable models defined by (1). Our strategy is to transform (1) to (12) so that we can use R to calculate a quantum measure μ - as was explained already in I. To do this we need explicit forms of R. Thus before going to IV we look at three particular R-matrices. Despite the elegance of the quantum groups, rather more direct methods can be effective in finding R's. We show this here by deriving R's for g = SU(N) from a simple ansatz. Our second example introduces the G = sl(2,$\mathbb{C}$) R-matrices for the s-G and sinh-G models and derives their corresponding r-matrices by classical limit. We use these R- and r-matrices in the IV.

We take the Yang-Baxter relations (37) in indexed form

$$R^{\alpha\beta}_{j\ell}(\lambda)\ R^{k\gamma}_{\alpha p}(\lambda+\mu)\ R^{mq}_{\beta\gamma}(\mu)\ =\ R^{\beta\gamma}_{\ell p}\ R^{\alpha q}_{j\gamma}(\lambda+\mu)\ R^{km}_{\alpha\beta}(\lambda) \tag{50}$$

In (50) $R_{12}(\lambda-\mu) \in \hat{g}_1 \otimes \hat{g}_2$ in (37) becomes $R^{\alpha\beta}_{j\ell}(\lambda)$ in which $(j\alpha)$ determines 'blocks' in the matrix R (from the space '2') and $(\ell\beta)$ determines elements of the 'block' (from the space '1'). If g = SU(3) i.e. $\hat{g}$ = SU(3) $\otimes$ $[\lambda,\lambda^{-1}]$, then a solution of (50) can be found by making the ansatz [48]

$$R^{km}_{j\ell} = \delta_{jm}\ \delta_{k\ell} + (f(\lambda) - \tfrac{1}{3})\ \delta_{jk}\ \delta_{\ell m}\ . \tag{51}$$

After some considerable algebra one finds many terms cancel and the solution is [48]

$$f(\lambda) = (\eta - 3\lambda)/3\eta\ ;\ \eta \in \mathbb{R},\ \lambda \in \mathbb{C}\ . \tag{52}$$

From (49) by expanding $\ln\hat{\Delta}(\lambda)$ about $\lambda = 0$ one finds from the coefficient of λ an integrable lattice model which is the spin chain model with Hamiltonian

$$\hat{H} = \pm(2\eta)^{-1} \sum_n (4\vec{S}_n.\vec{S}_{n+1}) + 4(\vec{S}_n.\vec{S}_{n+1})^2 \tag{53}$$

where

$$[S_n^{\alpha},\ S_m^{\beta}] = i\ \epsilon^{\alpha\beta\gamma}\ \delta_{mn}\ S_n^{\gamma} \tag{54}$$

(and $\frac{1}{2}\eta \in \mathbb{R}$ is the coupling constant). Obviously the analysis extends immediately to g = SU(N), while for SU(2), $R(\lambda) = (\eta - 2\lambda)\ (2\eta)^{-1}\ I + \frac{1}{2}\vec{\sigma} \times \vec{\sigma}$, $\vec{\sigma}$ the Pauli matrices. Then $\hat{H}$ corresponding to (53) is the lattice quantum Heisenberg ferromagnet or anti-ferromagnet.

In IV we work on the classical s-G and sinh-G models. For quantum s-G, in terms of 'bare' rapidities α_0, α_0'

$$R(\alpha_0 - \alpha_0') = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{55}$$

where

$$a = i(\sin \tfrac{1}{4}\gamma_0)/[\sinh \tfrac{1}{2}(\alpha_0 - \alpha_0' + i\tfrac{1}{4}\gamma_0)]$$

$$b = [\sinh \tfrac{1}{2}(\alpha_0 - \alpha_0')]/[\sinh \tfrac{1}{2}(\alpha_0 - \alpha_0' + i\tfrac{1}{4}\gamma_0)] \tag{56}$$

in which γ_0 is the 'bare' coupling constant of s-G (compare the classical (4)). As $\gamma_0 \to 0$ with commutators $[\ldots] \to i\{\ldots\}$ and $\hat{T}_N(\lambda) \to T_N(\lambda)$ one easily finds from (49) with R-matrix (55) that

$$\{T_N \overset{\otimes}{,} T_{N'}\} = i^{-1}[S, T_N \otimes T_N'] + [T_N \otimes T_N', r] \tag{57}$$

where

$$r(\sigma_0-\sigma_0') = \frac{\gamma_0}{16}[I \otimes I - \sigma_3 \otimes \sigma_3]/[\sinh(\sigma_0-\sigma_0')]$$
$$- \frac{\gamma_0}{16}[\sigma_1 \otimes \sigma_2 + \sigma_2 \otimes \sigma_1] \cosh(\sigma_0-\sigma_0')/[\sinh(\sigma_0-\sigma_0')] \tag{58}$$

in which $\sigma_0 = i^{-1} \ln\lambda_0 = \tfrac{1}{2}\alpha_0$, $\alpha_0' = i^{-1}\ln\lambda_0' = \tfrac{1}{2}\alpha_0'$. However, one can see that S is a permutation matrix and the particular quantum R which is (55) with (56) differs from the R's considered so far by a factor S: such factors change nothing in R-matrix theory [8,10] except some details (e.g. the Yang-Baxter relation (37) becomes $(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$ (in $\hat{g} \otimes \hat{g} \otimes \hat{g}$) - see Fig.1 - by incorporating such a permutation matrix [8,10]. We can therefore ignore the term in S in (57) to see that

$$\{T_N \otimes T_{N'}\} = [T_N \otimes T_{N'}, r] \tag{59}$$

in which the r-matrix is given by (58). This is exactly the Sklyanin bracket (26) here regained as a classical limit from the quantum groups. This indicates the consistency of the classical and quantum theories - route (i), 'Flat connection' above and this route (ii), 'Deformed loop algebras'. Strictly $\gamma_0 \to 0$ in *semi*-classical limit: r of order γ_0 emerges from (I - R) as $\gamma_0 \to 0$ so

$$R = I + r + O(\gamma_0^2) \tag{60}$$

as $\gamma_0 \to 0$. It then follows from the Yang-Baxter relation (37) that

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0 \tag{61}$$

the 'classical Yang-Baxter relation'.

We now have enough machinery to evaluate the functional integrals (1) and (12) for the s-G and sinh-G models. The r-matrix for sinh-G derives from (58) for s-G by continuation $\gamma_0 \to -\gamma_0$.

IV EVALUATION OF THE PARTITION FUNCTIONS

We have pointed out that our methods for evaluating quantum or classical Z make the canonical transformation from expression (1) for Z to expression (12): in the quantum case $S[\Phi]$ is given by (2); in the classical case it is (3). In transforming to (12) all of the difficulties in calculating Z are thrown into computing the measure $\mathcal{D}\mu$. Since $S[\Phi]$ and $S(p)$ are *classical* actions it might appear that only the classical r-matrix can figure in the calculations. Apparently this remains true of expression (1) since in the quantum case $S[\Phi]$ is the classical action (2) - so it would be interesting to know how the quantum integrability theory from the R-matrix enters the calculation. But by transforming to (12) we find we can use R to compute $\mathcal{D}\mu$ in the quantum case and r to compute $\mathcal{D}\mu$ in the classical case and the two measures are not the same even though the transformation (1) to (12) is canonical.

This 'trick' carries with it a further consequence. In the quantum case $S[\Phi]$ is given by (2). Since Z is to be real and positive if $F = -\beta^{-1} \ln Z$ is to be real, it is natural to *remove* the $i\int_0^\beta d\tau\, \Pi(x,\tau)\Phi_{,\tau}(x,\tau)$ for each label x by setting it equal to $2\pi i m(x)$ where $m(x)$, labelled by x, is an integer. This step becomes more illuminating in terms of action-angle variables.

It is convenient to work first with the sinh-G model which is simpler than the s-G model: the classical sinh-G, although integrable, has no solitons; the s-G does have solitons (kinks, antikinks and breathers, §I). For sinh-G $H[\Phi]$ is given by (5). Under canonical transformation, with Φ defined for $x \in \mathbb{R}$ with vanishing b.c.s at $\pm\infty$, $H[\Phi] \to H[p]$ given by

$$H[p] = \int_{-\infty}^{\infty} \omega(k)\, P(k)\, dk \tag{62}$$

$$\omega(k) = (m^2 + k^2)^{1/2} \tag{63}$$

and

$$\{P(k),\, Q(k')\} = \delta(k - k') \quad . \tag{64}$$

The phase spaces are $0 \leqslant P(k) < \infty$, $0 \leqslant Q(k) < 2\pi$ and the motion is confined to a continuously infinite dimensional torus.

For the evaluation of (12) the action $S[p]$ in the classical case is $-\beta H[p]$. For (12) in the quantum case $S[p]$ corresponding to (2) discretizes to

$$S[p] = \int_0^\beta d\tau \left\{ i\sum_{n=-N}^{+N} P_n\, Q_{n,\tau} - H[p] \right\} \tag{65}$$

with

$$H[p] = \sum_{n=-N}^{+N} \omega(\tilde{k}_n)\, P_n \; . \tag{66}$$

The modes $\tilde{k}_n$ are the allowed modes i.e. those contributing to the measure $\mathscr{D}\mu$ as we explain shortly. The modes $\tilde{k}_n$ are dense in the thermodynamic limit $L \to \infty$ and $P_n \leftrightarrow P(\tilde{k}_n)d\tilde{k}_n$ i.e. $P_n \leftrightarrow P(\tilde{k})d\tilde{k}$ where $\tilde{k}$ is dense (and is the limit of $\tilde{k}_n$): $\{P_m, Q_m\} = \delta_{nm}$ and $0 \leqslant Q_m < 2\pi$ so we can think of P_n as $O(1)$ - see below. The measure $\mathscr{D}\mu$ is

$$\mathscr{D}\mu = \lim_{N\to\infty} \prod_{n=-N}^{+N} \frac{dP_n dQ_n}{2\pi}, \tag{67}$$

and the normalisation by 2π is actually the usual normalisation by h: $\hbar = 1 \Rightarrow h = 2\pi$. Note that as a matter of convenience we overcount on n with $-N \leqslant n \leqslant N$ not $-N \leqslant n < N$, but this introduces no error in the limit $L \to \infty$ (i.e. $N \to \infty$).

The allowed elements $dP_n\; dQ_n/2\pi$ in $\mathscr{D}\mu$ (67) depend on the indices labelling the allowed modes $\tilde{k}_n$: we compute these shortly. To remove the term in i in the action (65) set

$$\int_0^\beta d\tau\, P_n\, Q_{n,\tau} = \oint P_n\, dQ_n = 2\pi m_n \tag{68}$$

with the m_n integers (for each n). The integral of period β is wrapped m_n

times around the torus labelled by n. This is *Bohr* quantisation, but it is not semi-classical quantisation as we shall see.

There are now two natural choices for the integers m_n:

$$m_n = 0,\ 1\ \text{(fermions)} \tag{69a}$$

$$m_n = 0,\ 1,\ 2,\ \ldots\ \text{(bosons)}\ . \tag{69b}$$

We shall make each of these choices working with fermions in the one case and bosons in the other. We shall see that these two choices are entirely equivalent to each other and we are apparently exploiting 'bose-fermi equivalence'. As noted in Fig.1 bose-fermi equivalence is implicit from the connection **Z** x **Z** Matrices $\in gl(\infty)$ → Bose-Fermi Equivalence → ... → Partition Function eqn. (12): the algebra $gl(\infty)$ has two representations, one in the Lie algebra of free fermion products the other in terms of vertex operators on a bosonic space [49]. However, there are obviously very many other choices for a real free energy. One of these is integers other than the two cases (69a,b) of course, but we do not know how to handle these other cases. Choices (69) fit naturally into the theory as we shall see and in fact quantisation is *still* imposed as 'quantum group' quantisation. We can find explicit 2-body S-matrix phase shifts Δ_f, Δ_b for these two choices, and these phase shifts determine the measures $\mathcal{D}\mu$. Consequently there are two different quantum measures $\mathcal{D}\mu_f$ for fermions and $\mathcal{D}\mu_b$ for bosons, and there is yet a third measure $\mathcal{D}\mu_c$ for the classical case. We shall see that condition (69a) (fermions) is already implicit in the quantum inverse method as it is so far used [8,10]. It is also implicit in the method of quantum Bethe ansatz as it is used [50]. As used the QIM and quantum BA are equivalent but the QIM exposes the R-matrix-quantum-group structure. A new feature in *our* analysis is the choice (69b), description in terms of bosons.

Notice that conditions (69) mean that the functional integral (12) is reduced to the classical form for which $S[p] = -\beta H[p]$ (indeed the functional integral (1) is reduced to the classical form with action $S[\Phi]$ given by (3)) but the extra quantum conditions (69a) or (69b) apply in the quantum case. It is possible to impose these conditions as such when there are action-angle variables i.e. the classical model is completely integrable: if we impose the conditions $\int d\tau \Pi \Phi_{,\tau} = 2\pi m(x)$ first mentioned, complete integrability is not required but the meaning is obscure - there is in general no torus to integrate around.

In order to calculate the classical measure $\mathcal{D}\mu_c$ for sinh-G one route is the following: from the bare r-matrix, which is (58) with $\gamma_0 \to -\gamma_0$, construct a classical integrable *lattice* with the same r-matrix. This can be done [51] and the idea is due to Izergin and Korepin [52]. For this sinh-G lattice we can find action-angle variables under periodic b.c.s of period 2N [51]. One can then take the classical thermodynamic limit $N \to \infty$ at finite density. If a_0 is the lattice spacing $2L = (2N + 1)a_0$ so the limit is $L \to \infty$ at finite density. At $L < \infty$ one can establish a connection, to $O(L^{-1})$, between the action-angle variables under periodic b.c.s and the action-angle variables under vanishing b.c.s at $\pm\infty$ [51]. A rather more direct method, the 'Poisson bracket method', which starts from the Sklyanin bracket (59), is described in [53]. In either method the connection depends on

$$\lim_{N\to\infty} Z^{-\frac{1}{2}(N+1)\sigma_3} X^{-1}\ T_n(\lambda)\ X\ Z^{-\frac{1}{2}N\sigma_3} = S(\lambda)\ , \tag{70a}$$

$$S(\lambda) \equiv \begin{bmatrix} a(\lambda), & b(\lambda) \\ b^*(\lambda), & a^*(\lambda) \end{bmatrix}\ . \tag{70b}$$

In (70a) Z depends on λ (and is a term in the monodromy matrix L_n for the sinh-Gordon lattice [51,53]): $X \equiv 2^{-1/2}(I + i\,\sigma_1)$ and the σ_i are Pauli matrices. The scattering matrix $S(\lambda)$, (70b), is the usual scattering matrix of the spectral transform (ST) method (I) and $a(\lambda)$ is a constant of the motion: thus $\ln a(\lambda)$ generates H[p] for sinh-G (Fig.1). In terms of $S(\lambda)$ the Sklyanin bracket (59) becomes

$$\{S(\lambda) \overset{\otimes}{,} S(\mu)\} = S(\lambda) \otimes S(\mu)\, r_+(\lambda,\mu) - r_-(\lambda,\mu)\, S(\mu) \otimes S(\lambda) \tag{71}$$

where $r_\pm(\lambda,\mu)$ are found from $r(\lambda,\mu)$.

By either method the result one finds is the condition on the allowed modes [51,53,54]

$$\tilde{k}_n = k_n - L^{-1} \sum_{m\neq n} \Delta_c(\tilde{k}_n, \tilde{k}_m)\, P_m + O(L^{-1}) \tag{72}$$

in which $k_n = 2\pi q_n L^{-1}$, q_n an integer different for each n, and where as before $P_m \leftrightarrow P(\tilde{k}_m)d\tilde{k}_m$ (i.e. $P(\tilde{k})d\tilde{k}$) as $L \to \infty$. In (72) and what follows L replaces 2L (i.e. the underlying period of periodic b.c.s is L not 2L). The phase shift $\Delta_c(k,k')$ in (72) is a classical phonon-phonon phase shift (recall sinh-G only has harmonic type, i.e. phonon, solutions, I)

$$\Delta_c(k,k') = -\tfrac{1}{4}\, \gamma_0 m^2\, [k\omega(k') - k'\omega(k)]^{-1} . \tag{73}$$

Notice the important feature of the thermodynamic limit now achieved: since $P_m = O(1)$ and $P_m \leftrightarrow P(\tilde{k})d\tilde{k}$ as $L \to \infty$, then since $d\tilde{k} = O(L^{-1})$, $P(\tilde{k}) = O(L)$ as $L \to \infty$. This allows us to define a finite particle density $\rho(\tilde{k}) = L^{-1}P(\tilde{k})$ for $L \to \infty$ [54].

Notice that everything derives from the r-matrix for sinh-G, including the classical phase shifts Δ_c. Thus the r-matrix determines everything for the functional integral (12) in the classical case. If we note that H[p] is discretized to (66) for the sinh-G model and the $\tilde{k}_n$ are determined by (72) with (73) we may see that the functional integral (12) can be evaluated by iterating (72) [10,54]. It is remarkable that the resultant iterated series can be put in the form [54]

$$\begin{aligned}\lim_{L\to\infty} FL^{-1} = {} & (2\pi\beta)^{-1} \int_{-\infty}^{\infty} dk\, \ln(\beta\omega(k)) - (2\pi\beta)^{-2} \int_{-\infty}^{\infty} dq\, [\omega(q)]^{-1} \int_{-\infty}^{\infty} dk \\ & \times \Delta_c(k,q)\, d(\ln\omega(k))/dk + (2\pi\beta)^{-3} \int_{-\infty}^{\infty} dp [\omega(p)]^{-1} \int_{-\infty}^{\infty} dq \\ & \times \Delta_c(q,p) \frac{\partial}{\partial q} \left([\omega(q)]^{-1} \int_{-\infty}^{\infty} dk\, \Delta_c(k,q)\, d(\ln\omega(k)/dk) \right. \\ & -\tfrac{1}{2}(2\pi\beta)^{-3} \int_{-\infty}^{\infty} dq [\omega(q)]^{-2} \left[\int_{-\infty}^{\infty} dk \Delta_c(k,q)\, d(\ln\omega(k)/dk) \right]^2 \\ & + \ldots\ldots \quad . \end{aligned} \tag{74}$$

And although this expansion is no better than strictly asymptotic (in the parameter β^{-1}: it is a *low temperature* expansion) it is formally *summed* to, i.e. is the iteration of,

$$\lim_{L\to\infty} FL^{-1} = (2\pi\beta)^{-1} \int_{-\infty}^{\infty} dk\, \ln(\beta\epsilon(k))$$

$$\epsilon(k) = \omega(k) + (2\pi\beta)^{-1} \int_{-\infty}^{\infty} dk' (d\Delta_c(k,k')/dk) \ln(\beta\epsilon(k')) \tag{75}$$

and $\omega(k)$ is the dispersion relation (63). In both (74) and (75) L replaces 2L as noted at (72). In this way the evaluation of the functional integral (12) in the classical case for sinh-G, for which $S[p] = -\beta H[p]$, is reduced to solving a nonlinear integral equation. That solution has the asymptotic expansion (74). Moreover, all the integrals in (74) can be done and the result is the asymptotic expansion [54]

$$\lim_{L\to\infty} FL^{-1} = m\beta^{-1}\left[\frac{1}{4}(M\beta)^{-1} - \frac{1}{8}(M\beta)^{-2} + \frac{3}{16}(M\beta)^{-3} - \frac{53}{128}(M\beta)^{-4} + \,..\right] + F_{KG} \tag{76}$$

where $F_{KG} = \beta^{-1}a_o^{-1}(\ln\pi\beta e^{-1}a_o^{-1} + ½ma_o)$ and $M = 8m\gamma_o^{-1}$. To get F_{KG}, the free energy density of K-G equation (13), in this form we must use the dispersion relation (14) with a cut-off $2\pi a_o^{-1}$ on the first integral of the iteration which is (72). It may be identified as 2π over the lattice spacing as the lattice theory shows. F_{KG} diverges classically, a classical u.v. divergence, as $a_o \to 0$. We can regain (76) directly from the functional integral (1) with $S[\Phi]$ given by (3) for sinh-G. The method, the transfer integral method, makes no appeal to the complete integrability of sinh-G. Thus the TIM serves as a check on the evaluation of (12) instead of (1) (a remarkable check in our view for e.g. (76) is only asymptotic and methods of matched asymptotics are needed to find it by TIM [54]).

This check also serves to check the quantum theory. The quantum cases are more complicated because sinh-G is relativistic. In these cases one uses the R-matrix (55), with $\gamma_o \to -\gamma_o$, to derive the condition for quantum pseudo-particles

$$\tilde{k}_n = k_n - \sum_{m\neq n} \Delta(\alpha_{on} - \alpha_{om}) \tag{77}$$

where $\tilde{k}_n = m_o'\sinh\alpha_{on}$ and $m_o' = ¼\, m_o^2 \sin ⅛\, \gamma_o$ is the mass of a pseudo-particle (m_o is the unrenormalised sinh-G mass m):

$$\Delta(\alpha_{on} - \alpha_{om}) = i\ln S(\alpha_{on} - \alpha_{om}) \tag{78}$$

and

$$S(\alpha_o - \alpha_o') = \frac{\sinh ½(\alpha_o - \alpha_o' - ¼i\gamma_o)}{\sinh ½(\alpha_o' - \alpha_o - ¼i\gamma_o)} \tag{79}$$

deriving from b given in expression (56). One now works with *fermions* to fill the Dirac sea so that phonons become excitations (and holes) from this Dirac sea. We are not yet able to work directly with bosons in this renormalisation procedure.

The (renormalised) result is [54]

$$\tilde{k}_n = k_n - \sum_{m\neq n} \Delta_f(\tilde{k}_n, \tilde{k}_m) \tag{80}$$

where

$$\Delta_f(k, k') = -2\tan^{-1}[m^2\sin ⅛\gamma_o''\, [k\omega(k') - k'\omega(k)]^{-1}] \tag{81}$$

and γ_o'' is renormalised: $\gamma_o'' = \gamma_o[1+\gamma_o/8\pi]^{-1}$. It is important that (81) is the *smooth branch* $-2\pi < \Delta_f < 0$. One can now see that (80) is exactly (72) with Δ_f replacing Δ_c and with $P_m = 0, 1$. This is the condition (69a) (fermions)!

For bosons one again uses (72) with $P_n = 0, 1, 2, \ldots$ and with Δ_b replacing Δ_c: Δ_b derives from Δ_f by

$$\Delta_b(k, k') = \Delta_f(k, k') + 2\pi\theta(k' - k) \tag{82}$$

where $\theta(k)$ is the unit step. Thus Δ_b is no longer the smooth branch and $\Delta_b(k - k') \to 0$ as $k - k' \to \pm\infty$. Note that with $\Delta(k, k') = i\ln S(k, k')$, $S(k = k') = -1$ (fermions) and $= +1$ (bosons) as expected.

By using (72) with $P_n = 0, 1$ in the form (80) (fermions) and in the form $P_n = 0, 1, 2, ..$ with (82) (bosons) we find that $Z = \int \mathcal{D}\mu \exp S[p]$, equation (12), in which $S[p] = -\beta H[p]$ and $H[p]$ discretizes to (66), can be reduced to the two systems of integral equations [54]

$$\lim_{L\to\infty} FL^{-1} = \mu\,\bar{n} - (2\pi\beta)^{-1} \int_{-\infty}^{\infty} \ln(1 + e^{-\beta\bar{\epsilon}(k)})dk,$$

$$\bar{\epsilon}(k) = \omega(k) - \mu - (2\pi\beta)^{-1} \int_{-\infty}^{\infty} (d\Delta_f(k, k')/dk) \times$$

$$\times \ln(1 + e^{-\beta\bar{\epsilon}(k')})dk' \tag{83}$$

and

$$\lim_{L\to\infty} FL^{-1} = (2\pi\beta)^{-1} \int_{-\infty}^{\infty} \ln(1 - e^{-\beta\epsilon(k)})dk,$$

$$\epsilon(k) = \omega(k) + (2\pi\beta)^{-1} \int_{-\infty}^{\infty} (d\Delta_b(k, k')/dk) \ln(1 - e^{-\beta\epsilon(k')})dk' . \tag{84}$$

In the fermion case we introduce a chemical potential μ and evaluate $\int \mathcal{D}\mu_f \exp -\beta[H[p]-\mu N]$ where $N = \sum_{n=-N}^{N} P_n$ (then $\bar{n} = \lim_{L\to\infty} 2N/(2L)$ – and $2L \to L$). Since (82) means $d\Delta_b/dk = d\Delta_f/dk - 2\pi\delta(k' - k)$, (83) with $\mu = 0$, becomes (84) when $\ln (1 + e^{-\beta\bar{\epsilon}(k)}) = -\ln (1 - e^{-\beta\epsilon(k)})$. Thus (84) and (83) are 'bose-fermi equivalent' (refer back to Fig.1 where bose-fermi equivalence traces back to the representations of $gl(\infty)$). The fermions result (83) was found in [55] for the quantum (normally ordered) repulsive NLS model (8) for which, however, $\omega(k) = k^2$ and $\Delta_f = -2\tan^{-1}(c/k - k')$. Evidently there is the bose-fermi equivalent form of this result. Finally note that the classical results (75) follow in the classical limit $\gamma_0 \to 0$ on the *bose* results (84): evidently Δ_b with its jump of 2π at $k = k'$ becomes Δ_c expression (73) with a pole at $k = k'$ in semi-classical limit $\gamma_0 \to 0$: $\sin \frac{1}{8}\gamma_0" \to \frac{1}{8}\gamma_0 > 0$ in semi-classical limit. The classical repulsive NLS model results follow in a similar way [54].

There is a similar argument which evaluates (12) for the classical and quantum s-G models. Although quantum s-G is bose-fermi equivalent to the quantum Massive Thirring model (§I), like for sinh-G we are so far only able to evaluate (12) for quantum s-G by renormalising in terms of fermions, namely renormalisation is MTM renormalisation. A very complete argument for $\lim_{L\to\infty} FL^{-1}$ is then given in terms of fermions in [53]. However the functional integral is not evaluated there: one uses instead our so-called generalised Bethe ansatz method [10,39,54] (and see Fig.1) which defines an entropy S and a free-energy $F = E - \beta^{-1}S$ (E is the energy H); then one minimises F. Results concide with the functional integral methods quantum or classical and it is plain that once $\mathcal{D}\mu$ is determined by (72) quantum or classical the role of the functional integral is simply to minimise F – a point already mentioned in §I.

In the functional integral methods for the s-G model, quantum or classical , there are subtilties about the action S[p]. In the classical case $S[p] = -\beta H[p]$ and [39]

$$H[p] = 2\sum_{i=0}^{N_k} (M^2 + p_i^2) + \int_{-\infty}^{\infty} \omega(k)\, P(k)dk \tag{85}$$

comparing with (62). This Hamiltonian has only kink and antikink contributions ($M = 8m\gamma_0^{-1}$ is the classical kink = antikink mass) and *phonons*: there are no breather contributions (e.g. [10,15]) and §I). We find this is a consequence of the classical finite density thermodynamic limit [10,53,56] in which classical particles with two degrees of freedom, the breathers, are involved. We find this way [39] that the classical functional integral (12) reduces to

$$\lim_{L\to\infty} FL^{-1} = (2\pi\beta)^{-1}\int_{-\infty}^{\infty}\omega(x)\ \ln(\beta\epsilon(x))dx - 2(2\pi\beta)^{-1}\int_{-\infty}^{\infty}E(x)\ e^{-\beta\tilde{E}(x)}dx \tag{86}$$

with

$$\epsilon(x) = \omega(x) + (2\pi\beta)^{-1}\int_{-\infty}^{\infty}(d\Delta/dx)\ \ln(\beta\epsilon(x'))dx' + 2(2\pi\beta)^{-1}\int_{-\infty}^{\infty}(d\Delta_k/dx)\ e^{-\beta\tilde{E}(x')}dx'$$

$$\tilde{E}(x) = E(x) - 2(2\pi\beta)^{-1}\int_{-\infty}^{\infty}(d\Delta_{kk}/dx)\ e^{-\beta\tilde{E}(x')}dx' - (2\pi\beta)^{-1}\int_{-\infty}^{\infty}(d\Delta_k/dx)\ \ln(\beta\epsilon(x'))dx' \tag{87}$$

where $\omega(x) = m\cosh x$ and $E(x) = M\cosh x$, rapidities are used, and the phase shifts Δ_{kk}, Δ_k (k is 'kink' = 'antikink') and Δ are given in [39,53]. In particular with $k = m\sinh x$, etc., $\Delta(k, k') = \frac{1}{4}\gamma_0 m^2[k\omega(k') - k'\omega(k)]^{-1}$ and is the analytic continuation in γ_0 of the phonon-phonon phase shift for sinh-G, equation (73).

The quantum functional integral for s-G uses the 'renormalised' classical H[p]

$$H[p] = 2\sum_{i=1}^{N_k}(M^2 + p_i{}^2)^{\frac{1}{2}} + \sum_{\ell=1}^{N_b}\sum_{\ell'=1}^{N_b'-1}(4M^2\sin^2\theta_{\ell,\ell'} + \hat{p}_\ell{}^2)^{\frac{1}{2}} \tag{88}$$

in which $\hat{p}_\ell = 2\pi\ell/L$ is a translational momentum and $\theta_{\ell,\ell'} = \sin(\ell'\gamma_0''/16)$, $\gamma_0'' = \gamma_0/[1 - \gamma_0/8\pi]$: M involves the renormalised mass M of s-G [53]. It is Bohr quantisation analogous to (69) which puts $\hat{p}_\ell = 2\pi\ell/L$ and $\theta_{\ell,\ell'}$ as quoted. The classical breather masses are $2M\sin\theta_\ell$ for each breather ℓ and $0 \leqslant \theta_\ell < \frac{1}{2}\pi$: the θ_ℓ are internal canonical momenta (more precisely the $4\gamma_0{}^{-1}\theta_\ell$ are, with conjugate Φ_ℓ in $0 \leqslant \Phi_\ell < 8\pi$: $\{4\gamma_0{}^{-1}\theta_\ell, \Phi_m\} = \delta_{\ell m}$ [10,15,22]). Thus corresponding to (69)

$$\int_0^\beta\left[4(\gamma_0'')^{-1}\theta_\ell\,\Phi_{\ell,\tau}\right]d\tau = \oint 4(\gamma_0'')^{-1}\theta_\ell\,d\Phi_\ell = 2\pi\ell' \tag{89}$$

i.e. $\theta_{\ell,\ell'} = \ell'\gamma_0''/16$. It is interesting that the 'classical' action $-\beta H[p]$ (88) needs γ_0'' not γ_0. We use classical s-G lattice theory to show no phonons contribute to (88) and that breathers do [51,53]. By using the *renormalised* phase shift equations analogous to (72) (and given in [53]. This explains why the renormalised γ_0'' enters (88)) we can expect that the functional integral (12) will reduce to (details are not complete)

$$\lim_{L\to\infty} FL^{-1} = \mu_s\bar{n}_s + \sum_\ell\mu_\ell\bar{n}_\ell - (2\pi\beta)^{-1}\sum_{\ell'=1}^{N_b'-1}\int_{-\infty}^{\infty}d\alpha_\ell\,M_\ell\cosh\alpha_\ell\,\ln(1 + e^{-\beta\epsilon_\ell}) - 2(2\pi\beta)^{-1}\int_{-\infty}^{\infty}d\alpha_s\,M_s\cosh\alpha_s\,\ln(1 + e^{-\beta\epsilon_s}) \tag{90}$$

in agreement with the generalised BA method [53]. In (90) $M_\ell = 2M\sin(\ell\gamma_0''/16)$, $M_s = M$, a renormalised kink (antikink) mass, $\epsilon_\ell = \epsilon_\ell(\alpha_\ell)$, $\epsilon_s = \epsilon_s(\alpha_s)$, and μ_ℓ, μ_s are chemical potentials for each 'particle'. As in (88) the number $N_b' = [8\pi\gamma_0''^{-1}]$ (= integral part) and $N_b' = [8\pi\gamma_0{}^{-1}]-1 = N_b-1$.

The energies ϵ_ℓ, ϵ_s satisfy

$$\epsilon_\ell = m_\ell \cosh \alpha_\ell - \mu_\ell - (2\pi\beta)^{-1} \sum_{\ell'=1}^{N_b'-1} \int_{-\infty}^{\infty} d\alpha_{\ell'} \frac{\partial \Delta}{\partial \alpha_\ell}{}_{\ell\ell'} \ln(1 + e^{-\beta\epsilon_{\ell'}})$$
$$- 2(2\pi\beta)^{-1} \int_{-\infty}^{\infty} d\alpha_s \frac{d\Delta_{\ell s}}{\partial \alpha_\ell} \ln(1 + e^{-\beta\epsilon_s})$$

$$\epsilon_s = m_s \cosh \alpha_s - \mu_s + (2\pi\beta)^{-1} \sum_{\ell'=1}^{N_b'-1} \int d\alpha_{\ell'} \frac{\partial \Delta}{\partial \alpha_s}{}_{\ell' s} \ln(1 + e^{-\beta\epsilon_{\ell'}})$$
$$- 2(2\pi\beta)^{-1} \int d\alpha'_s \frac{\partial \Delta_{ss'}}{\partial \alpha_s} \ln(1 + e^{-\beta\epsilon_{s'}}) \ . \qquad (91)$$

The phase shifts are given in [53]. The same result has been reached in several different ways by other people (and ourselves) using quantum Bethe ansatz. One has used the equivalence of the spin -½ XYZ model in continuum limit to the quantum MTM, fermi-bose equivalent to quantum s-G [57]. These results illustrate yet again the 'total connectivity' which forms the Fig.1. In classical limit (90) with (91) reduces to the results for classical s-G (86) with (87) [57,58]. The iteration of this classical system gives an asymptotic expansion which is the analytical continuation in γ_0 (found by the Transfer Integral Method) of the low temperature asymptotic expansion (76) for sinh-G [39,53,56]. This checks all of the classical calculations from (12) for s-G and checks the quantum calculations from (12) also. It checks r- and R-matrix theory and the 'quantum groups'!

REFERENCES

1. R K Bullough and S Olafsson, in: "Proc. 17th Intl. Conference on Diff. Geometrical Methods in Theoretical Physics" (Chester Meeting, August 1988) A Solomon ed., World Scientific, Singapore. To appear 1989.
2. Zhuhan Jiang and R K Bullough, J. Phys. A: Math. Gen. 20:L429 (1987).
3. P J Caudrey, Physica 6D:51 (1982); in: "Soliton Theory a Survey of Results" A P Fordy ed., Manchester UP, Manchester (1989) and references.
4. Kimio Ueno in: "Vertex Operators in Mathematics and Physics" Math. Sciences Research Institute Publications #3 J Lepowsky, S Mandelstam and I Singer eds, Springer-Verlag, New York (1984) pp.291-302.
5. R K Bullough, Yu-Zhong Chen, S Olafsson and J Timonen in: "Proc. of the Oléron Colloquium" P Lochak, M Balabane and C Sulem eds, Springer Lecture Notes in Mathematics, Springer-Verlag, Heidelberg (1989). To appear.
6. S Olafsson, J Phys A: Math. Gen. 22:157 (1989).
7. S Olafsson and R K Bullough, and references. To be published (1989).
8. P P Kulish and E K Sklyanin in Proc. Tvärminne Symp., Finland, 1981, J Hietarinta and C Montonen eds, Springer-Verlag, Heidelberg (1982).
9. A G Reyman and M A Semenov-Tian-Shansky, Commun. Maths. Phys. 105:461 (1986).
10. R K Bullough, in: "Solitons" M Lakshmanan ed., Springer Series in Nonlinear Dynamics, Springer-Verlag, Heidelberg (1988) pp.250-281.
11. J C Eilbeck, J D Gibbon, P J Caudrey and R K Bullough, J Phys. A 6:1337 (1973).
12. R K Bullough, 'Solitons' in: "Interaction of Radiation with Condensed Matter. Vol. I" IAEA-SMR-20/51, Intl. Atomic Energy Agency, Vienna (1977) pp.381-469.
13. R K Bullough, P W Kitchenside, P M Jack and R Saunders, Physica Scripta 20:364 (1979).
14. P J Caudrey, J D Gibbon, J C Eilbeck and R K Bullough, Phys. Rev. Letts 30:237 (1973).

15. "Solitons", Springer Topics in Current Physics 17, R K Bullough and R J Caudrey eds, Springer-Verlag, Heidelberg (1980).
16. R K Bullough and F Ahmad, Phys. Rev. Letts 27:330 (1971).
17. J Timonen and R K Bullough, Phys. Letts 82A:183 (1981).
18. R K Bullough, D J Pilling, Yi Cheng, Yu-Zhong Chen and J Timonen, Journal de Physique, Colloque C3. Suppl. au n°3, 50:C3-41 (1989).
19. R K Bullough, J Timonen, Yu-Zhong Chen, Yi Cheng and M Stirland in: "Proc. Intl. Workshop on Nonlinear Evolution Equations: integrability and spectral methods" (Como, Italy, July 1988) A P Fordy and A Degasperis eds, Manchester UP, Manchester (1989).
20. G T Zimanyi, S A Kivelson and A Luther, Phys. Rev. Letts 60:2089 (1988).
21. C Itzykson and J B Zuber, "Quantum Field Theory", McGraw-Hill, New York (1980).
22. R K Bullough, in: "Nonlinear Phenomena in Physics" F Claro ed., Springer-Verlag, Heidelberg (1989) pp.70-102.
23. R P Feynman and A R Hibbs, "Quantum Mechanics and Path Integrals", McGraw-Hill, New York (1965).
24. H W Braden, E Corrigan, P E Dorey and R Sasaki, "Aspects of Perturbed Conformal Field Theory, Affine Toda Field Theory and Exact S-Matrices". Paper this meeting.
25. R K Dodd and R K Bullough, Proc. Roy. Soc. London A 352:481 (1977).
26. P Christe, "S-Matrices of the Tricritical Ising Model and Toda System". Paper this meeting.
27. L D Faddeev and L A Takhtajan, "Hamiltonian Methods in the Theory of Solitons", Springer-Verlag, Heidelberg (1987).
28. J Liouville, Journal de Math. 20:137 (1855).
29. V I Arnold, "Mathematical Methods of Classical Mechanics", Springer-Verlag, Heidelberg (1978).
30. E Witten, in: "IXth Intl. Congress on Math. Phys." B Simon, A Truman and I M Davies eds, Adam Hilger, Bristol (1989).
31. Yong-Shi Wu, "Topological Chern-Simons Gauge Theories and New Hierarchies of Knot/Link Polynomials". Paper this meeting.
32. R J Baxter, "Exactly Solved Models in Statistical Mechanics", Academic Press, New York (1982).
33. M Wadati and Y Akutsu, in: "Solitons" Springer Series in Nonlinear Dynamics, M Lakshmanan ed. Springer-Verlag, Heidelberg (1988) pp.282-306.
34. Vaugh Jones, "Baxterization". Paper this meeting.
35. S Saito, Phys. Rev. D 36:1819 (1987).
36. M Mulase, J. Diff. Geom. 19:403 (1984).
37. R K Bullough, Yu-Zhong Chen and J Timonen. To be published.
38. G Venturi, "Quantum Gravity and the Berry Phase". Paper this meeting.
39. J Timonen, M Stirland, D J Pilling and R K Bullough, Phys. Rev. Letts 56:2233 (1986).
40. R Sasaki and R K Bullough in: "Nonlinear Evolution Equations and Dynamical Systems" Lecture Notes in Physics 120, M Boiti, F Pempinelli and G Soliani eds, Springer-Verlag, Heidelberg (1980) pp.315-337.
41. R Sasaki and R K Bullough, Proc. Roy. Soc. London A, 376:401 (1981).
42. E Sklyanin, LOMI Preprint E-3-1979 (1979).
43. Z Jiang, "Integrable Systems and Integrability", PhD Thesis, University of Mancheseter (Feb. 1987).
44. M Jimbo, Lett. Math. Phys. 10:63 (1985).
45. V G Drinfeld, Soviet Math. Dokl. 27 (No.1):68 (1983); "Quantum Groups" in: Proc. Intl. Cong. of Mathematicians, Berkeley (1987) pp.789-820.
46. S Majid, "Quasitriangular Hopf Algebras and the Yang-Baxter Equations", Int. J. Mod. Phys. A (1989). To appear.
47. Yu I Manin, Ann. Inst. Fourier, Grenoble 37, 4:191 (1987).

48. I Paczek, "Statistical Mechanics of Some Integrable Quantum Spin Systems in One Dimension", PhD Thesis, University of Manchester. To be submitted (1990).
49. M Jimbo and T Miwa, "Integrable Systems and Infinite Lie Algebras" in: "Integrable Systems in Statistical Mechanics", G M D'Ariano, A Montorsi and M G Rasetti eds, World Scientific, Singapore (1985).
50. H B Thacker, Rev. Mod. Phys. 53:253 (1981).
51. Yi Cheng, "Theory of Integrable Lattices", PhD Thesis, University of Manchester (Jan. 1987).
52. A G Izergin and V E Korepin, Lett. Math. Phys. 5:199 (1981).
53. Yu-Zhong Chen, "Statistical Mechanics of the Classical and Quantum 1+1 Dimensional Integrable Models", PhD Thesis, University of Manchester (July 1989).
54. R K Bullough, D J Pilling and J Timonen, J. Phys. A: Math. Gen. 19:L955 (1986).
55. C N Yang and C P Yang, J. Math. Phys. 10:1115 (1969).
56. J Timonen, Yu-Zhong Chen and R K Bullough, Nucl. Phys. B (Proc. Suppl.) 5A:58 (1988).
57. Nu Nu Chen, M D Johnson and M Fowler, Phys. Rev. Letts 56: 507 (1986); 56: 1427 (E) (1986).
58. J Timonen, R K Bullough and D J Pilling, Phys. Rev. B 34:6525 (1986).

NONLINEAR DIFFERENTIAL EQUATIONS IN PHYSICS AND THEIR GEOMETRICAL INTEGRABILITY PROPERTIES

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INTRODUCTION

The attempt of this line of research is to try to treat Yang-Mills, and gravitational fields as nonlinear systems, and try to see how much they possess the geometrical integrability properties that have been the guiding force in many two-dimension nonlinear systems. Though the study so far has been quite formal and mathematical, the ultimate goal we have in mind is for particle physics: to solve the full Yang Mills and gravitational fields, and to formulate new ways to quantize the fields.

Recently linear systems and conservation laws have been constructed for the extended conformal supergravity theories,[1] which have been shown to be the consequences of light-like integrability in curved extended superspace.[2] This completes the picture of a unifying description of equations of motion from the point of view of geometrical integrability and marks possibly the beginning of a new chapter of formulating quantum field theories using these understandings.

The generic structure of geometrical integrability properties can be summarized in Fig. 1.

The heart of the matter is first to find linear systems with parameters. The linear systems are usually of the form

$$\nabla_X \psi(X,Y) = 0, \qquad \nabla_Y \psi(X,Y) = 0,$$

where ∇_X, ∇_Y are some generalized covariant derivatives in some generalized geometrical spaces; e.g. (complexified) space-time space plus complex

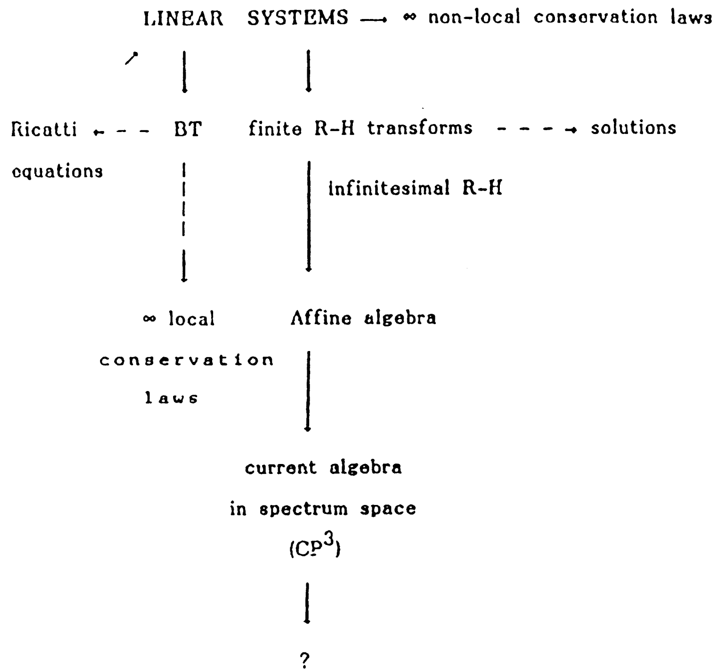

Fig. 1. Some Generic Structures of integrable Non-linear systems

parameters;[5–12] (complexified) space-time with superspace extensions plus[13–14] complex parameters;[15–19] loop spaces; noncommutative geometrical spaces; etc. The integrability of ψ requires $[\nabla_X, \nabla_Y\} = 0$, i.e. curvatureless. The equations of motion or the original nonlinear systems then follow from this generalized curvatureless condition. It is from these linear systems with parameters that powerful methods can be used to generate new solutions. The general theme of these methods are gauge transformations of ψ, and through its analyticity properties in the complex parameters that the so-called nonlocal conservation laws can be derived and that the original nonlinear differential equations are converted into integral equations in the complex-parameter space, i.e., the finite Riemann-Hilbert (RH) transformations. The RH transforms satisfy group properties and their infinitesimal elements form the Affine Kac-Moody algebra, which can be viewed as the result of the current algebra in the complex parameter space. Another branch of development is to derive parametric Bianchi-Bäcklund transformation (BT) from a special kind of finite RH transformation. Such BT's historically were found by guesswork and now we have a more general understanding of their origin. From the BT's with parameters, local conservations can be derived, and another nonlinear system, the Ricatti equations can be constructed. The Ricatti equations then can be shown to share the same linear systems.

The beautiful and surprising thing is that so many equations of motions in physics possess these geometrical integrability properties when the proper formulations and proper extended spaces are found. The two dimensional systems include chiral models: principal; symmetric-space; superized; chiral models with Wess-Zumino term and its superized version; the Ernest equations (which are reduced systems of stationary axially symmetric Einstein equations and also static axially symmetric self-dual Yang-Mills equations); Sine-Gordon equations, KdV equations, Liouville equations in the SL(2C) formulations. All the developments in the chart have been done for these two-dimensional systems. The four dimensional models includes self-dual Yang-Mills; supersymmetric Yang-Mills; and the conformal supergravity equations. All of them have been shown to have infinite nonlocal conservation laws and linear systems, from which integrability properties of varied degrees as listed in Fig. 1 have been derived. For my reviews and many other reports on related developments see Refs. [7,18,28].

The important extended space that has emerged from such studies is the extended superspace. Irrespect to whether Nature has manifesting representation of superspaces as supersymmetrical particle states or actual superspaces, superspace has already demonstrated its richness as a useful framework to look at our physical equations and may turn out to be an essential part of physical description as complex number is.

Because of limitation in space, I shall only list the recent work done with my collaborators:

I. Supergravity Theories

I.a. $D = 4$: It has been shown that light-like integrability conditions for $n \geq 5$, 6, 7, 8 lead to conformal supergravity equations of motion.[(2)]

I.b. $D = 4$: Linear systems have been constructed from all ($n = 1, ..., 8$) the light-like integrability conditions.[1)] These linear systems help to solve the light-like constraints and thus equation of motion for $n = 5, 6, 7, 8$; and helps to solve the light-like constraints for $n = 1, 2, 3, 4$ for off-shell formulation.

I.c. $D = 10$, $n = 1$: It has been shown that light-like integrability constraints lead to equations (Poincare) of motion only if an additional algebraic constraint is imposed.[(3)] Thus the light-like integrability constraints can allow an off-shell formulation of the theory.

I.d. $D = 10$, $n = 1$: Linear systems and conservation laws can be constructed for the light-like integrability conditions,[4] and thus useful for the off-shell formulation of the $D = 10$, $n = 1$ supergravity theory. In the construction of the linear systems and conservation laws, it is essential to use the bi-spinor representation for the light-like vectors.

II. Supersymmetric Yang-Mills Theories

In addition to the similar developments[15–18] as mentioned in section I for $D = 4$, supergravity theories, our recent new addition is the construction of linear systems, and an infinite number of nonlocal conservation laws using the bi-spinor representation[19] for any light-like vector in $D = 6$ and 10. These will be certainly useful for constructing new solutions in $D = 6$ and 10, and then in $D = 4$ by dimensional reduction.

III. Progresses Made For The $D = 4$, Self-Dual Yang-Mills Equation

III.a. Permutability property has been shown to be true for the Chau-Prasad-Sinha Bäcklund transformations (BT).[20]

III.b. The sequence, Parametric BT $\rightarrow$ Riccati $\rightarrow$ linear systems, has been constructed for the self-dual Yang-Mills equations.[21]

III.c. A generalized Bäcklund transformation, which is capable of generating instanton solutions has been constructed for the (supersymmetric) self-dual Yang-Mills equations.[22]

IV. The $D = 2$ Theories

IV.a. The Ernst equations which are reduced non-linear systems of static and axially symmetric Einstein, or Yang-Mills equations: linear systems, infinite-nonlocal conservation laws, finite Riemann-Hilbert transforms, and infinitesmal RH transform $\Rightarrow$ Kac-Moody algebra; Bäcklund transformations, etc. have been thoroughly discussed.[23]

IV.b. All the integrability properties as listed in IV.a. have been constructed for the super-chiral equations with Wess-Zumino term.[24]

IV.c. A general gauge covariant formulation, as well as all the integrability properties have been constructed for general symmetric-space chiral fields.[25]

V. General Integrability Discussions:

V.a. A unifying derivation of BT has been given from the point of view of infinite Riemann-Hilbert transformation.[26]

V.b. A general discussion of Kac-Moody algebra has been made from the point of view of infinite Riemann-Hilbert transformation.[27]

Concluding Remarks

Now we are ready to move forward in two fronts: first, finding solutions to the full Yang-Mills equations. The essential new feature in the search for classical solutions for the full Yang-Mills and supergravity equations is the use of superspace, and to develop two-complex-variable Riemann-Hilbert transform, contrasting to the one-complex-variables Riemann-Hilbert transform used in two-dimensional systems and the self-dual Yang-Mills systems. And second, quantizing the super-Yang-Mills and supergravity fields from these new points of view.[28]

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REFERENCES

1. Chau, L.-L., "Linear Systems and Conservation Laws of Gravitational Fields in Four Plus Extended Superspace", Phys. Lett. B202 (1988) 238.
2. Chau, L.-L. and Lim, C.-S., Phys. Rev. Lett. 56, 294 (1986).
3. Chau, L.-L. and Milewski, B., "Light-Like Integrability and Supergravity Equations of Motion in $D = 10$, $N = 1$ Superspace", UCD-87-05-R*; and Chau L.-L. and Lim, C.-S. to appear in Mod. Phys A. 1989.
4. Chau, L.L. and B. Milewski "Linear Systems and Conservation Laws in D=10, N=1 Supergravity", to appear in Phys. Lett.
5. For general discussions on such equations, see, for example, G.L. Lamb, Jr., "Elements of Soliton Theory" (Wiley, New York, 1980); M.J. Ablowitz and H. Segur, "Solitons and the Inverse Scattering Transform" (SIAM, Philadelphia, 1981); F. Calogero and A. Degasperis, "Spectral Transform and Solitons, I" (North-Holland, Amsterdam, 1982).
6. Faddeev, L.D. and N. Yu. Reshetikhin, Annals of Physics 167 (1986) 227; E.K. Sklyanin, L.A. Takhtadzhyan, and L.D. Faddeev, Teoreticheskaya i Matematcheskaya Fiziks, Vol. 40 (1979) 194.

7. Chau, L.-L., in Proceedings of Nonlinear Phenomena, Mexico, 1982, *Lecture Notes in Physics* **189** (Springer-Verlag, New York, 1983).

8. Novikov, S., S.V. Manakov, L.P. Pitaevskii and V.E. Zakharov, "Theory of Solitons: The Inverse Scattering Method" (Consultants Bureau, New York, 1984).

9. Yang, C.N., Phys. Rev. Lett. 38 (1977) 1377. S. Ward, Phys. Lett. 61A (1977) 81. Y. Brihaye, D.B. Fairlie, J. Nuyts, R.F. Yates, JMP 19 (1978) 2528.

10. Atiyah, M.F. and R.S. Ward, Commun. Math. Phys. 55 (1977) 117.

11. Atiyah, M.F., V.G. Drinfeld, N.J. Hitchin and Yu.I. Manin, Phys. Lett. A 65 (1978) 185.

12. Corrigan, E.F., D.B. Fairlie, R.G. Yates and P. Goddard, Commun. Math. Phys. 58 (1978) 223.

13. Salam, A. and J. Strathdee, Phys. Lett. 51B, 353 (1974). For review see J. Scherk, in *Recent Development in Gravitation* (Plenum, New York, 1979); B. Zumino, in *Proceedings of the NATO Advanced Study Institute on Recent Developments in Gravity, Cargése, 1978* (Plenum, New York, 1979), p. 405; J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton Univ. Press, Princeton, N.J., 1983).

14. Witten, E., Phys. Lett. 77B, 394 (1978); M. Sohnuis, Nucl. Phys. B136, 461 (1978).

15. Volovich, I.V., Phys. Lett. 129B, 429 (1983).

16. Devchan, C., Nucl. Phys. B238, 333 (1984).

17. Harnard, J., Hurtubise, J., and Shnider, S., Supersymmetric Yang Mills Equations and Supertwistors, *Annals of Physics* (to appear).

18. Chau, L.-L., Ge, M.-L., Popowicz, Z., Phys. Rev. Lett. 52, 1940 (1984); Chau, L.-L., Ge, M.-L., Lim, C.-S., Phys. Rev. D33, 1056 (1986). "Vertex Operators in Mathematics and Physics", Proceedings of the 1983 Berkeley Workshop, Eds. Lepoulsky, Mandelstam, Singer.

19. Chau, L.-L. and Milewski, B., Phys. Lett. 198 (1987) 356.

20. Chau, L.-L. and Chinea, F.J., Lett. in Math. Phys. 12, 189 (1986); Chau, L.-L., Yen, H.-C., Chen, H.-H., Chinea, F.J., Lee, C-R., Shaw, J.-C., Mod. Phys. Lett. 1, 285 (1986).

21. Chau, L.-L. and Yen, H.-C., J. Math. Phys. 28, 1167 (1986).

22. Chau, L.L., J.C. Shaw, and H.C. Yen, Int. J. of Mod. Phys. 4 (1989) 2715.

23. Chau, L.-L., Chou, K.-C., Hou, B.-Y., Song, X.-C., Phys. Rev. 34D, 1814 (1986).

24. Chau, L.-L. and Yen, H.C., Phys. Lett. B 177, 368 (1986).

25. Chau, L.-L., Hou, B.-Y., Phys. Lett. 145B, 347 (1984); Chau, L.-L., Hou, B.-Y., Song, X.-C., Phys. Lett. 151B, 421 (1985).

26. Chau, L.-L., Ge, M.-L., "Finite Riemann-Hilbert Transformation and A Unifying Derivation of Bäcklund Transformation", UCD-87- (1987).

27. Chau, L.-L., Ge, M.-L., Yen H.-C., J. of Math. Phys. 30 (1989) 166.

28. For a recent review on the field: see Chau, L.L., "Geometrical Integrability and Equations of Motion in Physics: A Unifying View," talk given at the Workshop held at Nankai Institute of Mathematics, Nankai University Tianjin, China.

INTEGRABILITY OFF CRITICALITY AND QUANTUM INTEGRABLE SYSTEMS*,**

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ABSTRACT: Using the equivalence between the Virasoro algebra and the Hamiltonian structure of integrable systems of the Korteweg-de Vries type, it is shown that in conformal field theories, there are three infinite families of commuting conserved integrals of normal ordered polynomials of the energy-momentum tensor. For minimal conformal models, there are three distinguished relevant degenerate scalar fields and it turns out that the perturbation with one of these fields preserves one infinite family of conserved commuting integrals. The relation with Toda systems and the supersymmetric extension are briefly discussed.

1. INTRODUCTION

One of the most important development in theoretical physics over the last ten years is certainly the formulation of two dimensional conformal field theory and its application to critical phenomena [1]. However a complete understanding of critical phenomena must also include an understanding of the vicinity of the critical point (where conformal invariance no longer holds) that is

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the determination of the universal properties of the scaling region. An important step in this direction has been made by Zamolodchikov who proposed to study the off critical region by perturbing conformal field theories with suitable relevant perturbations [2,3]. A remarkable offshoot of this approach was the observation that there exists particular perturbations which appear to preserve the complete integrability of the conformal field theory [3]. The aim of the present contribution (based on [4,5]) is to try to clarify this result and to make contact with quantum integrable systems of the Korteweg-de Vries (KdV) type and their Toda field relatives.

2. INTEGRABILITY AT CRITICALITY

In local field theory the invariance under a scale transformation, when supplemented by the basic invariance properties under translation and rotation, implies conformal invariance. For two dimensional conformal field theory, the energy-momentum tensor splits into a holomorphic and a antiholomorphic part: with $z=x_1+ix_2$, $\bar{z}=x_1-ix_2$, then $T_{zz}=T(z)$, $T_{\bar{z}\bar{z}}=\bar{T}(\bar{z})$ and the mixed term $T_{\bar{z}z}$ vanishes. Their modes $L_n = \oint dz\, z^{n+1}\, T(z)$ (where the countour circulates once around the origin in the positive direction and where a factor $(2\pi i)^{-1}$ in front of the integral is omitted) satisfy the Virasoro algebra:

$$[L_n,L_m] = (n-m)\, L_{n+m} + (c/12)(n^3-n)\delta_{n+m,0} \;, \tag{1}$$

(and similarly for $\bar{L}_n$, with $[L_n, \bar{L}_m] = 0$). The central charge c is the basic parameter characterizing the theory.

In any conformal field theory one can find an infinite number of conserved quantities, that is expressions whose $\bar{z}$ derivative vanishes. In fact the $\bar{z}$ derivative of any normal ordered polynomial in T and its derivative vanishes since $\partial_{\bar{z}}T=0$. However the integrability property in field theory entails more than just the existence of an infinite number of conserved quantities. An infinite number of them must also commute among themselves. Below we will display three distinct infinite families of commuting conserved quantities. This makes any conformal field theory at least three times more integrable than an ordinary integrable field theory! To establish this result, one must first review the relation between the KdV equation and the Virasoro algebra.

The KdV equation,

$$\partial_t u = -u''' + 6uu' \tag{2}$$

(where $u = u(x,t)$ and the prime indicates space derivative) is the prototype of completely integrable systems. It is related to the Virasoro algebra via its second Hamiltonian structure, which is defined by

$$\partial_t u = \{u, H\}, \qquad H = \frac{1}{2} \int u^2 \, dx \quad , \tag{3}$$

and

$$\{u(x), u(y)\} = [-\partial^3 + 4u\partial + 2u'] \; \delta(x-y) \quad . \tag{4}$$

$(\partial = \partial_x)$. The Poisson bracket (4) is just the global form of the Poisson bracket realization of the Virasoro algebra. Indeed, let $x \in [0,2\pi]$ and Fourier expand $u(x)$ as

$$u(x) = (6/c) \sum L_n e^{-inx} - 1/4 \quad . \tag{5}$$

Then the substitution of (5) into (4), with a rescaling of the Poisson bracket, yields exactly (1) with [,] replaced by i{,} [6].

The integrability of (2) implies the existence of an infinite number of conserved quantities H_n in involution, i.e. $\partial_t H_n = 0$ and $\{H_n, H_m\} = 0$. These conserved quantities are integrals of polynomials in u and its derivative, homogeneous with respect to the grading defined by deg $(\partial^k u) = 2+k$; non trivial conservation laws exist for each non zero odd value of the grading. It must be stressed that these H_n, after the Fourier decomposition (5), are polynomials in the L_n's and that they "commute" with respect to the Virasoro bracket. Therefore if we can obtain the quantum form of this result we would end up with an infinite number of integrals over normal ordered polynomials in T and its derivative which commute with respect to the Virasoro algebra or equivalently, the operator product expansion

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T'(w)}{z-w} + \dots \qquad (6)$$

For this one must first quantize the KdV equation in such a way that the integrability property is maintained. In view of the present application, the quantization must also preserves the (second) Hamiltonian character of the equation. It is thus natural to define the quantum KdV (qKdV) equation canonically as [4]

$$\partial_t T = [T, H_q] \quad , \qquad \text{with} \quad H_q = \oint dz\,(TT) \qquad (7)$$

where ( ) denotes normal ordering, i.e.

$$(AB)(z) = \oint \frac{dx}{x-z}\, A(x)B(z) \quad . \qquad (8)$$

With (6) this yields

$$\partial_t T = \frac{(1-c)}{6} T''' - 3(TT)' \quad . \qquad (9)$$

This is the qKdV equation. (This form was given implicitly in [7]). It is conjectured to be completely integrable. Assuming the validity of the conjecture it follows that there exists an infinite number of commuting integrals H_{2k+1}, where the subscript indicates the conformal dimension. Therefore the relation between the qKdV equation and the Virasoro algebra yields a method for constructing an infinite number of commuting integrals. (Unfortunately no recursive procedure has been found for generating these integrals.) In the qKdV context these integrals are conserved because their time derivative is zero but in the context of conformal field theory they are said to be conserved because their $\bar{z}$ derivative vanishes.

As this point it is natural to ask whether there are other integrable Hamiltonian systems which can be formulated by means of the Poisson bracket (4). In fact two other equations of this type are known. These are [8] the Gibbon-Sawada-Kotera (GSK) equation

$$\partial_t u = u''''' - 5uu''' - 5u'u'' + 5u^2u' \ , \quad \text{with} \quad H_G = \frac{1}{6}\int [u^3 + 3(u')^2]dx \qquad (10)$$

and the Kupershmidt (K) equation

$$\partial_t u = u''''' - 20uu''' - 50u'u'' + 80u^2u', \text{ with } \quad H_K = \frac{1}{6}\int[16u^3+3(u')^2]dx. \quad (11)$$

These equations can be quantized as in the KdV case. The appropriate Hamiltonian is

$$H = \oint dz\ [(T(TT)) + a(T'T')] \qquad (12)$$

with

$$a \equiv a_{\pm} = \frac{\left[543-51c\pm\sqrt{1-c}\sqrt{25-c}\right]}{192} \qquad (13)$$

where a_+ (a_-) applies for the qGKS (qK) equation respectively. The explicit form of these equations is

$$\partial_t T = \frac{[42-9c-a(18-20c)]}{120}T''''' + \frac{[8a-7-c]}{4}(TT)''' - 5a(T'T')' - 5(T(TT)). \quad (14)$$

A subtle point here concerns the appropriate choice of the values of a. Indeed the only constraint imposed by the classical limit is that as $c \to \infty$, a_+/a_- must equal 16. We will comment on this in section 4. Again it is conjectured that these equations are completely integrable so that there exists conservation laws $P_n^{(i)}$ (i=G,K) for every $n=6k\pm1>0$, as in the classical case. This gives us two new and distinct infinite families of commuting integrals constructed out of the energy-momentum tensor . Therefore in any conformal theory there exists three infinite families of commuting conserved integrals.

3. INTEGRABILITY OFF CRITICALITY

Consider the class of minimal unitary models whose central charge is $c = 1 - 6/p(p+1)$, $p = 3,4,...$ These models contain a finite number of primary fields and their (holomorphic) conformal dimension is given by the Kac formula [1]

$$h_{n,m} = \frac{[n(p+1) - mp]^2 - 1}{4p(p+1)} \quad . \tag{15}$$

(The same formula applies for the antihomorphic conformal dimension $\bar{h}$). Now let us consider, following Zamolodchikov, the perturbation of such a minimal model by a term of the form $\lambda\int\Phi(z,\bar{z})dzd\bar{z}$, where Φ is a scalar ($h=\bar{h}$) relevant ($h<1$) field. Such a perturbation will generically spoils the integrability property present at $\lambda=0$. However Zamolodchikov [3] found that if Φ is one of the following degenerate fields

$$\text{(i)} \;\; \Phi_{(1,3)}, \;\; \text{(ii)} \;\; \Phi_{(1,2)}, \;\; \text{(iii)} \;\; \Phi_{(2,1)},$$

then the perturbed model is still integrable. In fact each of these perturbations preserves one of the infinite family of conserved integrals, i.e. the corresponding integrals remain conserved despite the fact that $\partial_{\bar{z}} T \neq 0$ when $\lambda \neq 0$! For the cases (i), (ii), and (iii) these are respectively the qKdV, the qGSK and the qK conservation laws. The above three operators are characterized by the fact that they are the most relevant fields in their appropriate operator product subalgebra. The existence of these integrable off critical models is remarkable and their analysis should be enormously facilitated by this integrability property.

4. DIGRESSION ON TODA FIELDS:

In this section (which is based mainly on [9,10]) we try to justify the relation between the above integrable off critical models and the equations of the KdV type, via Toda fields. The basic idea here is to assume that (extended) conformal field theories can be effectively described by (extended) quantum Liouville type systems (i.e. Toda field equations associated with simple Lie algebras) with their Hamiltonian formulated in terms of the Dotsenko-Fateev screening operators. A perturbed model would then be described by the system obtained by adding to the unperturbed Hamiltonian density, the precise vertex operator which represents the perturbating field. When this operator is such that the resulting theory corresponds exactly to a Toda model associated with an affine Lie algebra, then the integrability property is preserved ! Let us see how this works in the case where there is no additional symmetry.

In the above discussion there was an underlying Feigin-Fuchs representation

$$T = i\alpha_0\phi'' - (1/2)(\phi'\phi') \tag{16}$$

where $\phi(z)\phi(w) = -\ln(z-w)$. In this representation, a vertex operator $V_\beta = (\exp(i\beta\phi))$ has dimension $\beta^2/2 - \alpha_0\beta$. Therefore $V_{\alpha\pm}$ (where $\alpha\pm = \alpha_0 \pm (\alpha_0{}^2+2)^{1/2}$) has dimension one so that $[T,Q(\alpha\pm)] = 0$,where $Q(\beta) = \oint dz V_\beta$. Therefore for the Hamiltonian of the associated Liouville theory one can choose either $Q(\alpha_+)$ or $Q(\alpha_-)$. Consider now the (1,3) perturbed model. The vertex operator representation of $\phi_{(1,3)}$ is $V_{-\alpha_-}$. Therefore for the Hamiltonian of the perturbed theory one takes $Q(\alpha_-)+ Q(-\alpha_-)$. (The relative coefficient of these two terms is actually irrelevant). The resulting model is the quantum sine-Gordon equation which is integrable. It has the same polynomial conservation laws as the qKdV equation when T is replaced by (16) [8]. (As far as the qKdV conservation laws are concerned, one can replace α_- by α_+ without modifying the results. The latter case would correspond to a (3,1) perturbed model but notice that this perturbation is not relevant). Similarly $\phi_{(1,2)}$ can be represented by $V_{-\alpha_-/2}$ and the Hamiltonian of the perturbed model would then be $Q(\alpha_-)+2Q(-\alpha_-/2)$. The resulting system is the quantum Bullough-Dodd equation, again an integrable system. This equation could also be written in terms of α_+ and this possibility is associated with a (2,1) perturbation since $V_{-\alpha_+/2}$ has dimension $h_{(2,1)}$! Now since the sine-Gordon and the Bullough-Dodd equations are the only scalar affine Toda systems, the present picture suggests that the perturbations considered by Zamolodchikov are the only one which can lead to integrable systems for generic values of p.

Remark: The values of $a_\pm$ in eqs.(12) and (13) are uniquely fixed by the requirements that $[H,Q(-\alpha_\pm/2)=0$.

5. SUPERSYMMETRIC EXTENSION:

It has been verified in [5] that the perturbation of the superconformal minimal models with the primary degenerate superfield $\Phi_{(1,3)}$ yields an integrable model whose conservation laws are exactly those of the supersymmetric qKdV equation. The latter is the only known Hamiltonian supersymmetric integrable equation whose Poisson bracket is the classical super Virasoro algebra. It is naturally related to the supersymmetric sine-Gordon equation and the latter is the only Toda system, among those associated with affine superalgebras, which is supersymmetric [11].

REFERENCES

1. C. Itzykson, H. Saleur and J. B. Zuber, "Conformal invariance and applications to statistical mechanics", World Scientific 1988.
2. A. B. Zamolodchikov, JEPT Lett. 43 (1986) 730; see also A. W. Ludwig and J. L. Cardy, Nucl. Phys. B285 (1987) 687.
3. A.B. Zamolodchikov, JETP Letters 46 (1987) 160; Int. J. Mod. Phys. A 3 (1988) 743; "Integrable field theory from conformal field theory", preprint 01/89.
4. B.A. Kupershmidt and P. Mathieu, "Quantum Koteweg-de Vries like equations and perturbed conformal field theories", UTSI/Laval preprint.
5. P. Mathieu, "Integrability of perturbed superconformal minimal models", Laval preprint.
6. J.L. Gervais, Phys. Lett. B 160 (1985) 277. B.A. Kupershmidt, Phys. Lett. A 109 (1985) 417.
7. R. Sasaki and I. Yamanaka, Adv. Stu. Pur. Math. 16 (1988) 271.
8. A.P. Fordy and J. Gibbons, J. Math. Phys. 21 (1980) 2508; 22 (1981) 1170; Comm. Math. Phys. 77 (1981) 21.
9. T. Eguchi and S. K. Yang, Kyoto preprint RIFP 797.
10. T. J Hollowood and P. Mansfield, Oxford preprint 89-17P.
11. M. A. Olshanetsky, Comm. Math. Phys. 88 (1983) 63.

QUANTIZATION OF THE CHIRAL SOLITONIC BAG MODEL

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ABSTRACT

A consistent quantization scheme for the two flavor chiral solitonic bag model with unequal quark masses is developed employing a propagator formulation.

The low energy properties of hadrons, and their interactions, must be consequences of quantum chromodynamics (QCD), since, as is generally believed, this is the fundamental theory of strong interactions. However, since as of today, the QCD could not have been solved exactly, for computational purposes at least, it is highly desirable to have an effective theory of QCD which would involve only the physically observed hadrons, and is able to explain the low energy properties of them. In the absence of a method which enables us to derive the correct effective lagrangian from the QCD, we must content ourselves with guessing an effective lagrangian with guidance from phenomenology, and whatever hints we can gather from QCD itself. In this quest, a 28 year old idea of Skyrme [1] has lately received much attention. What Skyrme had proposed was that the nonlinear chiral model of pions contain soliton solutions which could be identified with nucleons, provided that some additional terms are included in the lagrangian to stabilize the topological soliton. The hint needed from QCD to take this model more seriously, came sometime later [2], when Witten has shown that, in the large N analysis of two dimensional QCD, the baryon masses have a dependence on the coupling constant exactly like that of soliton, namely it is inversely proportional with the coupling constant.

This development triggered a lot of activity, with the hope that perhaps with further modifications, this solitonic chiral model of pions could be the correct effective QCD lagrangian [3]. It was scon noticed [4] that the modifications needed are those terms in the lagrangian which would reflect the anomaly structure of the underlying theory, the QCD. Another modification was to include the other known low lying hadrons, vector and axial vector mesons, into this scheme [5].

With these modifications, we had a more or less complete low energy description of strong interactions, in accord with the general principles deduced from QCD, and with reasonable agreement with the low energy data.

However, a later unsuccessful attempt [6] to compute the neutron-proton mass difference in this framework, made it clear that the predictive power of the model was

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limited to those problems which do not require the crucial short distance information that baryons are made out of quarks; since the solitonic baryons in this framework do not remember that they are made out of quarks. That the neutron is heavier than the proton is probably a consequence of the crucial property of the underlying QCD, that is, the d quark is heavier than the u quark, forces us towards a complete effective theory which contains some crucial short distance information that the baryon is made out of quarks, with all their peculiar properties.

This necessarily leads us to the so-called chiral solitonic bag model, an intuitively appealing framework that the quarks and skyrmion play complementary roles in the baryon. The quarks keep the skyrmion from collapsing, while the skyrmion keeps the quarks confined. This model is an extension of Chodes-Thorn version [7] of the MIT bag model [8]. The lagrangian of the external pion field is that of Skyrme model. A surface coupling of the quark and pion fields is introduced to restore chiral invariance, which was broken in original MIT bag model.

The two flavor chiral solitonic bag model for which we propose a quantization scheme in this paper, is defined by

$$\mathcal{L} = \mathcal{L}_q\theta(R-r) + \mathcal{L}_m\theta(r-R) + \mathcal{L}_B\delta_B \tag{1}$$

where

$$\begin{aligned}
\mathcal{L}_q &= \bar{\psi}(i\gamma^\mu\partial_\mu - M)\psi \\
\mathcal{L}_m &= \frac{F_\pi^2}{16}\mathrm{tr}(\partial_\mu U^\dagger\partial^\mu U) + \frac{1}{32a^2}\mathrm{tr}(U^\dagger\partial_\mu U, U^\dagger\partial_\nu U)^2 \\
&\quad + \frac{m_\pi^2 F_\pi^2}{8(m_u+m_d)}\mathrm{tr}(M(U+U^\dagger-2I)) \\
\mathcal{L}_B &= -\frac{1}{2}(\bar{\psi}_L U\psi_R + \bar{\psi}_R U^\dagger\psi_L) \\
M &= \mathrm{diag}(m_u, m_d).
\end{aligned} \tag{2}$$

The meson phase is described by the static classical field configuration $U_s = e^{i\vec{\tau}.\hat{x}F(r)}$ with $F(r)$ determined by minimizing the static energy and by imposing the boundary conditions $F(0)=\pi, F(\infty)=0$. The quark phase is described by the quantum field operator $\psi(\vec{x},t)$.

The standard method [9] to excite the solitonic baryon degrees of freedom, that is, to construct the low-lying quantum states above the semiclassical ground state, is to make the substitution

$$\begin{aligned}
U(\vec{x},t) &= A(t)U_s(\vec{x})A^\dagger(t) \\
\psi(\vec{x},t) &= A(t)\psi_0(\vec{x},t)
\end{aligned} \tag{3}$$

that is, to quantize the rotational zero modes associated with the collective variables $A(t)$. Here $U_s(\vec{x})$ and $\psi_0(\vec{x},t)$ are the fields in the rotating (body-fixed) frame. Upon substituting (3) into (2), we get

$$L = L_0 + \lambda_m \, \mathrm{tr}(\dot{A}^\dagger \dot{A}) + \frac{i}{2} X^a \int d^3x \bar{\psi}_0 \gamma^0 \tau^a \psi_0 - \frac{1}{2} \Delta m R^{3,a} \int d^3x \bar{\psi}_0 \tau^a \psi_0 \tag{4}$$

where

$$X^a = \mathrm{tr}(\tau^a A^\dagger \dot{A}) \tag{5}$$

$$R^{ab} = -\frac{1}{2} \mathrm{tr}(A^\dagger \tau^a A \tau^b)$$

In (4), λ_m is the moment of inertia of the meson phase, associated with the collective rotations, and is given by

$$\lambda_m = \frac{2\pi F_\pi^2}{3} \int_R^\infty dr \; r^2 \sin^2 F(r) \left\{ 1 + \frac{4}{(aF_\pi)^2} \left[\left(\frac{dF}{dr} \right)^2 + \frac{\sin^2 F(r)}{r^2} \right] \right\} \tag{6}$$

Notice that, since the mesonic lagrangian is at least quadratic in time derivatives, the approximation of the rotation frame meson field with the Skyrme solution $U_s(\vec{x})$ is consistent.

In order to determine the collective field dependence of the lagrangian (4) completely, we need to resort to the known solutions for the chiral hedgehog quark states in the equal mass case [10]. To make sensible use of these solutions in the framework of perturbation theory, we need to know the equation of motion for the rotating frame field ψ_0. This differs, however, from the lab frame equations by A-dependent terms. Thus (4) does not reflect the full A-dependence; there is further A-dependence buried in ψ_0's.

Subjecting the lab. frame field equation, $(i\gamma^\mu \partial_\mu - M)\psi = 0$, the transformation (3), we get

$$\left(i\gamma^\mu \partial_\mu - m_0 + i\gamma^0 A^\dagger \dot{A} + \frac{1}{2} \Delta m A^\dagger \tau_3 A \right) \psi_0(\vec{x}, t) = 0 \tag{7}$$

subject to the boundary condition on the bag surface

$$-i\hat{x}.\vec{\gamma}\psi_0(\vec{x}, t)|_{Bag} = e^{i\gamma_5 \hat{x}.\vec{\tau} F(r)} \psi_0(\vec{x}, t)|_{Bag} \tag{8}$$

Once this equation is at our disposal, its stationary state solution $\psi_0(\vec{x}, t) = \psi_0(\vec{x}) e^{-iwt}$ can be related to the symmetric case chiral hedgehog quark state solutions $\chi_0(\vec{x})$, which satisfy the equation

$$\left(w\gamma_0 + i\vec{\gamma}\vec{\nabla} - m_0 \right) \chi_0(\vec{x}) = 0 \tag{9}$$

together with the boundary condition (8). $\chi_0(\vec{x})$ is given as [10]

$$\chi_0(\vec{x}) = \frac{N}{\sqrt{4\pi}} \begin{pmatrix} i\sqrt{\frac{E+m_0}{E}} j_0(kr) \mid 0 > \\ -\sqrt{\frac{E-m_0}{E}} j_1(kr)(\vec{\sigma}.\hat{x}) \mid 0 > \end{pmatrix} \tag{10}$$

and has some useful properties

$$\bar{\chi}_0 \tau^a \chi_0 = \chi_0^\dagger \tau^a \chi_0 = 0 \tag{11}$$

The relation between ψ_0 and χ_0 is obtained by comparing the equations (7) and (9),

$$\psi_0(\vec{x}) = \chi_0(\vec{x}) - \int d^3y S_B(\vec{x}, \vec{y}; \omega) \Big[i\gamma^0 A^\dagger \dot{A} + \frac{1}{2} \Delta m_q A^\dagger \tau_3 A \Big] \psi_0(\vec{y}) \tag{12}$$

Here $S_B(\vec{x}, \vec{y}; \omega)$ is the bag propagator defined by

$$\Big(\omega\gamma_0 + i\vec{\gamma}.\vec{\nabla} - m_0\Big) S_B(\vec{x}, \vec{y}; \omega) = \delta^3(\vec{x} - \vec{y}) \tag{13}$$

$$(\exp(i\gamma_5 \vec{\tau}.\hat{x} F) + i\vec{\gamma}.\hat{x}) S_B|_{Bag} = 0$$

(12) can be solved perturbatively to any order desired. Since Δm_q is small, it is consistent to solve it to first order in Δm. Furthermore the collective rotations are adiabatic; thus the rotational velocity $\tau^a A^\dagger \dot{A}$ is also small. Therefore we will solve (12) to first order in the perturbation sense. To this order, the rotating frame field ψ_0 is given in terms of the symmetric hedgehog quark solutions χ_0 as

$$\psi_0(\vec{x}) = \chi_0(\vec{x}) - \int d^3y S_B(\vec{x}, \vec{y}; \omega) \Big[i\gamma^0 A^\dagger \dot{A} + \frac{1}{2} \Delta m_q A^\dagger \tau_3 A \Big] \chi_0(\vec{y}) \tag{14}$$

Substituting (14) in (4), and retaining up to quadratic terms in rotational velocity, X^a, (since the mesonic part is already quadratic in X) and making use of (11), we get the complete A-field dependence of the Lagrangian, to first order in Δm:

$$L = L_0 - \frac{1}{2} \Lambda^{ab} X^a X^b - \frac{i}{4} \Delta m R^{3b} C^{ba} X^a \tag{15}$$

where

$$\Lambda^{ab} = \lambda m \delta^{ab} + \frac{1}{2} \int d^3x d^3y [\bar{\chi}_0(\vec{x}) \tau^a \gamma_0 S_B(\vec{x}, \vec{y}; \omega) \gamma_0 \tau^b \chi_0(\vec{y}) + h.c.] \tag{16}$$

$$C^{ba} = \int d^3x d^3y \Big\{ \chi_0^\dagger(\vec{x}) [\tau^a S_B(\vec{x}, \vec{y}; \omega) \tau^b + \tau^b \gamma_0 S_B(\vec{x}, \vec{y}; \omega) \gamma_0 \tau^a] \chi_0(\vec{y}) + h.c. \Big\} cr$$

The Hamiltonian can now be easily constructed, by taking into account the constraint $A^\dagger A = I$,

$$H = -L_0 - \frac{1}{2} \Lambda^{ab} X^a X^b \tag{17}$$

This is consistent with the fact that, for Lagrangians containing terms linear in velocity, Hamiltonian is quadratic.

The spin and isospin operator can be computed in the usual manner, applying the Noether theorem to the transformations $\delta_r A = -iAr$ and $\delta_l A = ilA$ (with $r, l = \epsilon^a \tau^a/2$) respectively:

$$-S^a = i\Lambda^{ab}X^b - \frac{1}{4}\Delta m C^{ba}R^{3b} \tag{18}$$

$$I^a = R^{ab}S^b$$

By using (18), the Hamiltonian can finally be expressed in terms of spin and isospin operators with further neglect of the terms quadratic in Δm_q

$$H = -L_0 - \frac{1}{2}(\Lambda^{-1})^{ab}S^aS^b - \frac{1}{4}\Delta m(RC\Lambda^{-1}R^{-1})^{3a}I^a \tag{19}$$

The computation of the last term which accounts for the mass splitting among the members of baryon multiplets (in addition to the usually negligibly small electromagnetic contributions [6]) requires the knowledge of S_B. To compute S_B we employ the multiple reflection expansion method [11]. Supported by the claims in literature [12], we will retain only the first reflection term in the expansion

$$S_B(\vec{x}, \vec{y}, \omega) = S^0(\vec{x}, \vec{y}, \omega) + R^2 \int d\Omega_\alpha S^0(\vec{x}, \vec{\alpha}, \omega) K_\alpha S^0(\vec{\alpha}, \vec{y}, \omega) + \cdots \tag{20}$$

where

$$K_\alpha = e^{i\gamma_5 \hat{n}_\alpha . \vec{\tau} F(r)} + i\hat{n}_\alpha . \vec{\gamma} \tag{21}$$

Here S^0 is the usual Dirac propagator. It is expanded in partial waves employing the two component spherical harmonics ϕ_{jlm},

$$S^0(\vec{x}, \vec{y}, \omega) = \sum_{jll'm} S^0_{jll'}(r, r'; \omega)\phi_{jlm}(\Omega)\phi^\dagger_{jl'm}(\Omega') \tag{22}$$

where

$$S^0_{jll'}(r, r'; \omega) = -ik[\delta_{ll'}(\rho_3\omega + m_0) + k(l' - l)\rho_2]f_l(kr)f_{l'}(kr) \tag{23}$$

$$f_l(kr) = j_l(kr)\theta(r' - r) + h^{(1)}_l(kr)\theta(r - r').$$

Although S^0 is diagonal in the flavor space, the same is not true for the first and higher reflection terms. A lengthy analysis, however, shows that both Λ and C matrices are diagonal in flavor space (although the following numerical analysis is carried out to first order only, this diagonality property holds to all orders in multiple reflection expansion). This is

$$\Lambda^{ab} = \delta^{ab}(\lambda_m + \lambda_q) \tag{24}$$

$$C^{ba} = \delta^{ab}C$$

where

$$\lambda_q = \frac{1}{2}\int d^3x d^3y [\bar{\chi}_0(\vec{x})\gamma_0 S_B(\vec{x}, \vec{y}; \omega)\gamma_0 \chi_0(\vec{y}) + h.c.] \tag{25}$$

$$C = \int d^3xd^3y\{\chi_0^\dagger(\vec{x})[S_B(\vec{x},\vec{y};\omega) + \gamma_0 S_B(\vec{x},\vec{y};\omega)\gamma_0]\chi_0(\vec{y}) + h.c.\}$$

Thus the Hamiltonian can be rewritten as

$$H = -L_0 - \frac{\vec{S}^2}{2(\lambda_m + \lambda_q)} - \Delta m_q \frac{C}{4(\lambda_m + \lambda_q)} I^3 \tag{26}$$

After a lengthy calculation the λ_q and C are found as

$$\begin{aligned}
C = &- \frac{4\tilde{R}}{aF_\pi \nu^4 j_0^2(\nu)[\xi[\nu(1+\omega_1^2) - 2\omega_1] + \mu\omega_1]} \int_o^\nu dy\ y^2 \\
&\left\{ \int_o^y dx\ x^2 \Big((\xi+\mu)^2 j_o^2(x) j_o(y) n_o(y) - (\xi-\mu)^2 j_1^2(x) j_1(y) n_1(y) \Big) \right. \\
&+ \int_y^\nu dx\ x^2 \Big((\xi+\mu)^2 j_o(x) n_o(x) j_o^2(y) - (\xi-\mu)^2 j_1(x) n_1(x) j_1^2(y) \Big) \\
&+ \nu \int_o^\nu dx x^2 \Big[\cos F[(\xi+\mu) j_o^2(x) j_o^2(y) \Big((\xi+\mu)^2 \Re(h_o^2(\nu)\Big) - \nu^2\ \Re(h_1^2(\nu)) \Big) \\
&\qquad + [(\xi+\mu) j_1^2(x) j_1^2(y) \Big((\xi+\mu)^2\ \Re(h_1^2(\nu)\Big) - \nu^2\ \Re(h_o^2(\nu))) \Big] \\
&\qquad \left. - \frac{2}{3}\, \nu\ \sin F\ \Re\Big(h_o(\nu) h_1(\nu)\Big) \Big((\xi+\mu)^2 j_o^2(x) j_o^2(y) + (\xi-\mu)^2 j_1^2(x) j_1^2(y) \Big) \Big] \right\}
\end{aligned} \tag{27}$$

$$\begin{aligned}
\lambda_q = &- \frac{\tilde{R}}{aF_\pi \nu^4 j_o^2(\nu)[\xi[\nu(1+\omega_1^2) - 2\omega_1] + \mu\omega_1]} \int_o^\nu dy\ y^2 \\
&\left\{ \int_o^y dx\ x^2 \Big[(\xi+\mu)^2 j_o^2(x) j_o(y) n_o(y) + (\xi-\mu)^2 j_1^2(x) j_1(y) n_1(y) \right. \\
&\qquad + \nu^2 (j_1^2(x) j_o(y) n_o(y) + j_o^2(x) j_1(y) n_1(y)) \Big] \\
&+ \int_y^\nu dx\ x^2 \Big[(\xi+\mu)^2 j_o(x) n_o(x) j_o^2(y) + (\xi-\mu)^2 j_1(x) n_1(x) j_1^2(y) \\
&\qquad + \nu^2 (j_1(x) n_1(x) j_o^2(y) + j_o(x) n_o(x) j_1^2) \Big) \Big] \\
&+ \nu \int_o^\nu dx\ x^2 \Big[\cos F[(\xi+\mu) j_o^2(x) j_o^2(y) \Big((\xi+\mu)^2\ \Re(h_o^2(\nu)\Big) - \nu^2\ \Re(h_1^2(\nu)) \Big) \\
&\qquad + [(\xi-\mu) j_1^2(x) j_1^2(y) \Big((\xi+\mu)^2\ \Re(h_1^2(\nu)\Big) - \nu^2\ \Re(h_o^2(\nu)) \Big) \\
&\qquad - 2\nu^2 j_o^2(x) j_1^2(y) \Big((\xi-\mu)\ \Re(h_1^2(\nu)) - (\xi+\mu)\ \Re(h_o^2(\nu)) \Big) \Big] \\
&\qquad \left. - \frac{2}{3}\, \nu\ \sin F\ \Re\Big(h_o(\nu) h_1(\nu)\Big) \Big((\xi+\mu)^2 j_o^2(x) j_o^2(y) - (\xi-\mu)^2 j_1^2(x) j_1^2(y) \Big) \Big] \right\}
\end{aligned}$$

where

$$\mu = m_0 R = \frac{m_0}{aF_\pi}\tilde{R}, \quad \nu = kR, \quad \xi = ER \quad \omega_1 = \frac{j_1(\nu)}{j_0(\nu)} \tag{28}$$

We have evaluated the radial integrals in (27) by using the numerical solutions of the equation satisfied by Skyrme profile F(r)

$$(\frac{1}{4}\tilde{r}^2 + 2\ \sin^2 F)F'' + \frac{1}{2}\tilde{r}F' + (\sin\ 2F)F'^2 - \frac{1}{4}\ \sin 2F - \frac{\sin^2 F \sin 2F}{\tilde{r}^2} = 0 \tag{29}$$

with the boundary conditions, $F(0) = \pi, F(\infty) = 0$, and $\tilde{r} = aF_\pi r$. Taking $\Delta m_q =$

3.8 MeV, $\mu = 0.5$ and $a = 5.45$, we have plotted $\Delta m_q C/4(\lambda_m + \lambda_q)$ as a function of the bag radius R in Fig. 1. Notice that apart from some negligibly small fluctuations around $R \sim 0.2$ fm (which is probably due to the fact that we truncate our expansion at the first reflection order), the graph for $C/4\lambda_{tot}$ goes to zero smoothly for $R \to 0$, a gratifying result which lends support on the consistency of our quantization scheme.

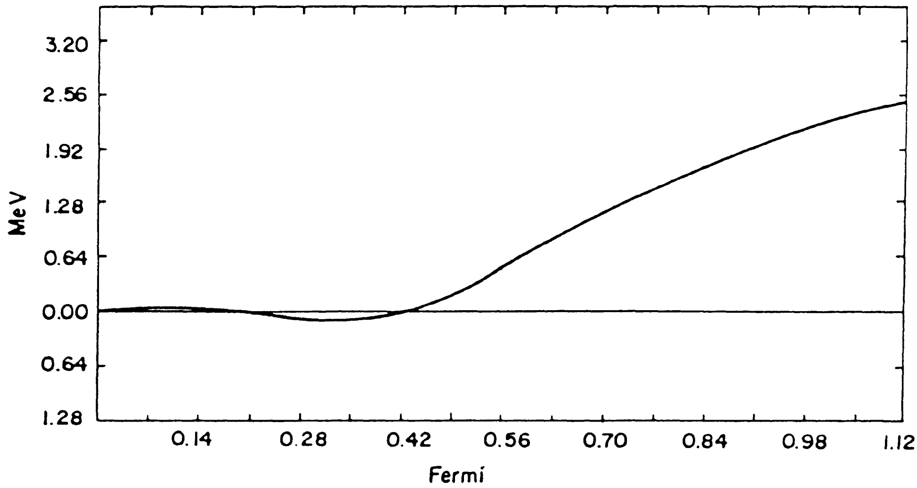

Fig. 1. The $\Delta m_q \frac{C}{4(\lambda_m+\lambda_q)}$ as a function of the bag radius.

The simplified expression for the spin carried by the quark sector can readily be read off from 18,

$$-S^a_{(q)} = i\lambda_q \operatorname{tr}(\tau^a A^\dagger \dot{A}) - \frac{1}{4}\Delta m C \; R^{3a} \tag{30}$$

That there exist a Δm dependent term part in $S^a_{(q)}$ is quite natural, follows from the fact that the solutions for the well-defined spin-isospin states (baryons) are no more solutions to the Laplace equation [9] on the three sphere.

REFERENCES

[1] T. Skyrme, Proc. R. Soc. **A260**, 127 (1961).
D. Finkelstein, J. Rubinstein, J. Math. Phys. **9**, 1762 (1968).
J. Williams, J. Math. Phys. **11**, 2611 (1970).

[2] E. Witten, Nucl. Phys. **B160**, 57 (1979).

[3] N.K. Pak, C.H. Tze, Ann. Phys. (N.Y.), **117**, 169 (1979).
R. Balachandran, et al., Phys. Rev. **D27**, 1133 (1983).

[4] E. Witten, Nucl. Phys. **B223**, 442 (1983).

[5] E. Witten, Nucl. Phys. **B223**, 433 (1983).
N.K. Pak, R. Rossi, Nucl. Phys. **B250**, 279 (1985).
Ö. Kaymakcalan, et al., Phys. Rev. **D30**, 594 (1984).

[6] M. Durgut, N.K. Pak, Phys. Lett. **159B**, 357 (1985); ibid.(E) **162B**, 405 (1985).

[7] A. Chodos, C. Thorn, Phys. Rev. **D12**, 2733 (1975).

[8] A. Chodos, R. Jaffe, K. Johnson, C. Thorn, U. Weisskopf, Phys. Rev. **D9**, 38, 71 (1974).

[9] G. Adkins, C. Nappi, and E. Witten, Nucl. Phys. **B228**, 552 (1983).

[10] P.J. Mulders, Phys. Rev. **D30**, 1073 (1984).

[11] T. Hanson, and R.J. Jaffe, Phys. Rev., **D28**, 882 (1983).
J. Goldstone, and R. Jaffe, Phys. Rev. Lett. **51**, 69 (1984).

[12] A. Goldhaber, and R.J. Jaffe, Phys. Lett. **131B**, 445 (1983).

STRUCTURE OF SUPERSELECTION SECTORS IN LOW-DIMENSIONAL QUANTUM FIELD THEORY *

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INTRODUCTION

The basic principles of relativistic quantum field theory, i.e. locality as an incorporation of Einstein causality and the spectrum condition as a formulation of stability, which are extremely restrictive in 4 dimensional space time admit a richer structure in low dimensions (2 and 3). So particle statistics is in general described by a representation of the braid group instead of the usual Bose and Fermi statistics, statistical weights in cross sections are given in terms of link invariants, and the notion of a group of internal symmetries is generalized to "quantized symmetries", as quantum groups or Ocneanu's quantized groups.

These phenomena have already been observed to some extent in 2d conformal field theory[2,3]and in 3d topological field theory[4]. But these are only special or limit cases of a general theory which shows that generic quantum field theories in 2 and 3 dimensions exhibit all the mentioned structural properties.

THE ALGEBRAIC FRAMEWORK

The analysis is based on the theory of superselection sectros which after pioneering work of Borchers[5] has been developed by Doplicher, Haag and Roberts[6]. The framework is the formulation of quantum field theory as a Haag-Kastler net of observable algebras[7]. So to each closed double cone O one assigns the algebra $\mathcal{A}(O)$ of all bounded absrvables which can be measured within O. These algebras are von Neumann algebras, typically isomorphic to the unique hyperfinite type III_1 factor[8]. If $O_1 \subset O_2$, $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$, and the C*- algebra

$$\mathcal{A} = \overline{\bigcup_O \mathcal{A}(O)} \tag{1}$$

is called the algebra of quasilocal observables. For arbitrary regions G one defines $\mathcal{A}(G)$ to be the C*-algebra which is generated by all algebras $\mathcal{A}(O)$ with double cones $O \subset G$.

Now locality requires

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2)' \tag{2}$$

(the commutant of $\mathcal{A}(O_2)$) provided O_1 is contained in the spacelike complement O_2' of O_2. In addition we assume covariance under a group P of space time transformations including the translations, i.e. there is a representation $L \to \alpha_L$ of P by automorphisms of $\mathcal{A}$ such that

$$\alpha_L(\mathcal{A}(O)) = \mathcal{A}(LO). \tag{3}$$

* Based on joint work with K.H. Rehren and B. Schroer[1]

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Of special interest, as first emphasized by Borchers[9], are the positive energy representations (PER) of $\mathcal{A}$. These are representations π of $\mathcal{A}$ by bounded operators in some Hilbert space H_π together with a strongly continuous unitary representation U_π of the translation group which implements the translations in $\mathcal{A}$,

$$U_\pi(x)\,\pi(A)\,U_\pi(x)^{-1} = \pi\alpha_x(A), \quad A \in \mathcal{A}, \tag{4}$$

and satisfies the spectrum condition

$$\begin{aligned} spP \subset \overline{V}_+ = \{p \in \Re^d \mid p_o \geq |\vec{p}|\}, \\ P = (P_o, \vec{P}), \;\; U_\pi(x) = e^{ixP} \end{aligned} \tag{5}$$

A superselection sector may now be defined as the equivalence class of an irreducible PER. In this generality the problem of superselection sectors has been solved only in special cases including the free massive field and chiral theories of the energy momentum tensor (Virasoro algebra) or current algebras (Kac Moody algebras).

LOCALLY GENERATED SECTORS

The starting point of the analysis of Doplicher, Haag and Roberts (DHR) is the selection of a more restricted class of sectors, the "locally generated" sectors. These are PER which become unitarily equivalent to a distinguished vacuum representation π_o after restriction to the algebra $\mathcal{A}(O')$ of the spacelike complement of some double cone O. The local algebras in the vacuum representation π_o are assumed to be maximal in the sense that

$$\pi_o(\mathcal{A}(O'))' = \pi(\mathcal{A}(O)), \tag{6}$$

a property called Haag duality. According to Bisognano and Wichmann[10], in a theory generated by Wightman fields one can always enlarge the original Haag-Kastler net such that duality is satisfied. Roberts has shown[11] that in more than 2 dimensions the locally generated sectors are locally generated also with respect to the enlarged net.

Examples for locally generated sectors are sectors which can be generated by applying a field to the vacuum which is relatively local to the observables. This class contains the conformally covariant sectors in conformal field theory[12]. There are also counter examples, namely soliton sectors in 2 dimensions - here a general theory is still missing - and sectors with gauge charges which violate the DHR selection criterion because of Gauss' law - in the massive case the DHR theory can be generalized[13] (see below), in the massless case the theory is still incomplete (for an approach see[14]).

The first step in the DHR analysis is to identify the representation spaces H_π and H_{π_o} by exploiting the unitary equivalence between π and π_o on $\mathcal{A}(O')$ such that

$$\pi(A) = \pi_o(A), \quad A \in \mathcal{A}(O'). \tag{7}$$

One then shows that $\pi(A) \subset \pi_o(\mathcal{A})$ and that $\rho : \pi_o(A) \to \pi(A)$ defines an endomorphism of $\pi_o(\mathcal{A})$.

In the following we identify $\pi_o(\mathcal{A})$ and $\mathcal{A}$ by dropping the symbol π_o. The locally generated sectors are now encoded into "localized transportable morphisms", i.e. endomorphisms ρ of $\mathcal{A}$ which act like the identity on the algebra $\mathcal{A}(O')$ for some double cone O and are equivalent to their translates,

$$\rho^{(x)} = \alpha_x \circ \rho \circ \alpha_{-x} = adV_\rho^{(x)} \circ \rho \tag{8}$$

with $V_\rho^{(x)} = U_{\pi_o}(x)\,U_\pi(x)^{-1}$. Because of duality $V_\rho^{(x)}$ is localized in all neighborhoods of

$$O_\gamma = \bigcup_t (O + \gamma(t)) \tag{9}$$

for any path γ from 0 to x.

The main advantage of the description of representations by endomorphisms is a natural possibility of composing representations,

$$[\rho_1] \times [\rho_2] := [\rho_1 \rho_2] \tag{10}$$

This composition law resembles the tensor product of group (or quantum group) representations; physically it corresponds to the additivity of charges. In $d \geq 3$ space time dimensions Doplicher and Roberts recently proved[15] that this structure is indeed the dual of a compact group which can be identified with the group of internal symmetries. It is an outstanding problem whether their result can be generalized to $d = 2$ dimensions to yield something similar to a quantum group.

BRAID GROUPS AND LINK INVARIANTS

Let us now look at the implications of locality for this product structure. Let ρ_i be localized transportable endomorphisms which are localized in $O_i, i = 1, 2$ such that O_1 is spacelike separated from O_2. Then

$$\rho_1 \rho_2 = \rho_2 \rho_1. \tag{11}$$

To verify (11) take $A \in \mathcal{A}(O)$ for some O. Choose x and y such that $O_1 + x$ and $O_2 + y$ are spacelike to O and such that for some paths γ_1 from 0 to x and γ_2 from 0 to y the region $(O_1)\gamma_1$ is spacelike to $(O_2)\gamma_2$. Then

$$\rho_1 \rho_2(A) = adV_{\rho_1}^{(x)^*} V_{\rho_2}^{(y)^*}(A) = adV_{\rho_2}^{(y)^*} V_{\rho_1}^{(x)^*}(A) = \rho_2 \rho_1(A) \tag{12}$$

The commutativity of morphisms with spacelike separated supports can be used to show that the composition law for sectors is commutative in general (for O_1 not necessarily spacelike to O_2). Choose x and y such that $O_1 + x \subset (O_2 + y)'$. Then

$$\rho_2 \rho_1 = adV^* \circ \rho_2^y \rho_1^x = adV^* \circ \rho_1^x \rho_2^y = adV^* V' \circ \rho_1 \rho_2 \tag{13}$$

with $V = V_{\rho_2}^{(y)} \rho_2(V_{\rho_1}^{(x)})$, $V' = V_{\rho_1}^{(x)} \rho_1(V_{\rho_2}^{(y)})$. The operator

$$\varepsilon(\rho_1, \rho_2) = V^* V' \tag{14}$$

is called the statistics operator. It turns out to be locally constant in x and y. In $d \geq 3$ dimensions it therefore is independent of x and y (provided $O_1 + x \subset (O_2 + y)'$). In $d = 2$ dimensions we get two values according to whether $O_1 + x$ is in the right or left spacelike complement of $O_2 + y$.

In the special case $\rho_1 = \rho_2 = \rho$ the operator

$$\varepsilon_\rho = \varepsilon(\rho, \rho) \tag{15}$$

is in the commutant of $\rho^2(\mathcal{A})$. Moreover one has the relation

$$\varepsilon_\rho \rho(\varepsilon_\rho) \varepsilon_\rho = \rho(\varepsilon_\rho) \varepsilon_\rho\, \rho(\varepsilon_\rho) \tag{16}$$

Proof: Let ρ be localized in O and let U be a unitary in $\mathcal{A}$ such that $adU \circ \rho$ is localized in $\hat{O}$ in the right spacelike complement of O. We have $\varepsilon_\rho = \rho(U^{-1})U$ and $adU \circ \rho(\varepsilon_\rho) = \varepsilon_\rho$ since $\varepsilon_\rho \in \rho^2(\mathcal{A})' \subset \mathcal{A}(O)$. Hence,

$$\varepsilon_\rho\, \rho(\varepsilon_\rho)\varepsilon_\rho = \varepsilon_\rho\, \rho^2(U^{-1})U = \rho^2(U^{-1})\varepsilon_\rho\, U$$
$$\rho(\varepsilon_\rho)\varepsilon_\rho\, \rho(\varepsilon_\rho) = \rho^2(U^{-1})U\rho(\varepsilon_\rho) = \rho^2(U^{-1})(ad\, U \circ \rho)(\varepsilon_\rho)U$$

which implies (16).

The statistics operator ε_ρ together with the endomorphism ρ now generates a unitary representation of the braid group B_∞. Namely, as well known the braid group has a presentation by generators σ_i, $i \in N$ which satisfy the relations

$$\sigma_i \sigma_{i+1}\, \sigma_i = \sigma_{i+1} \sigma_i\, \sigma_{i+1} \tag{17}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2. \tag{18}$$

We set

$$\varepsilon^{(\rho)}(\sigma_i) = \rho^{i-1}(\varepsilon_\rho), \quad i \in N. \tag{19}$$

Then $\varepsilon^{(\rho)}$ extends to a representation of B_∞ since relation (17) is implied by (11) and (18) by the fact that $\varepsilon_\rho \in \rho^2(\mathcal{A})'$. In $d \geq 3$ dimensions one finds in addition the relation

$$\varepsilon_\rho^2 = 1 \tag{20}$$

which means that $\varepsilon^{(\rho)}$ is a unitary representation of the permutation group.

The representation $\varepsilon^{(\rho)}$ can be investigated by using a so-called left inverse ϕ of ρ. This is a positive mapping from $\mathcal{A}$ to $\mathcal{A}$ with $\phi \circ \rho = id$ such that $\rho \circ \phi$ is a conditional expectation from $\mathcal{A}$ to $\rho(\mathcal{A})$. Since $\varepsilon_\rho \in \rho^2(\mathcal{A})'$ we have $\phi(\varepsilon_\rho) \in \rho(\mathcal{A})'$. Thus, for an irreducible endomorphism ρ, $\lambda_\rho = \phi(\varepsilon_\rho)$ is a complex number called the statistics parameter. $d(\rho) = |\lambda_\rho|^{-1}$ is the "statistical dimension" and $\omega(\rho) = \lambda_\rho d(\rho)$ the "statistics phase" of ρ. In the case $d(\rho) < \infty$ (for some left inverse ϕ of ρ) ϕ is the only left inverse of ρ.

We may now iterate ϕ and obtain a function of positive type on B_∞,

$$\varphi(b) = lim_{n\to\infty}\ \phi^n(\varepsilon^{(\rho)}(b)) \tag{21}$$

ϕ satisfies the relation

$$\phi(\varepsilon_\rho\, \rho(\varepsilon^{(\rho)}(b)) = \lambda_\rho\, \varepsilon^{(\rho)}(b) \tag{22}$$

since $\rho \circ \phi$ is a conditional expectation, hence φ has the property

$$\varphi(\sigma_1 b) = \lambda_\rho \varphi(b) \tag{23}$$

where b is a word in $\sigma_i, i \geq 2$. Moreover, one can show that φ is a trace,

$$\varphi(b_1 b_2) = \varphi(b_2 b_1) \tag{24}$$

hence φ is a Markov trace. So

$$\tilde{\varphi}(b) = d(\rho)^{-(n-1)}\, \omega(\rho)^{n_- - n_+}\, \varphi(b), \quad b \in B_n \tag{25}$$

where $n_\pm$ counts the number of factors $\sigma_i^{\pm 1}$ in b as a word in $\sigma_i, = 1, ..n-1$, is a link invariant.

The link invariant $\tilde{\varphi}$ which is associated to the superselection sector of ρ has been derived by algebraic methods. However, the crucial algebraic property which has been exploited was commutativity of observables which are localized in spacelike separated regions, so it is the causal structure of space time which finally leads to the occurence of link invariants. It would be very interesting to understand the occurence of link invariants in a more direct way.

In special cases the Markov trace φ and the braid group representation $\varepsilon^{(\rho)}$ can be completely determined. The simplest case is present when ρ^2 is irreducible. Then ε_ρ is a multiple of 1 and ρ is an automorphism of $\mathcal{A}$, hence $\phi = \rho^{-1}$. $\varepsilon^{(\rho)}$ is in this case a one dimensional representation, and $\varphi = \varepsilon^{(\rho)}$. Examples are the so-called anyon sectors which are well known in 2 dimensional models (see e.g.[15]) and have been shown to occur also in 3 dimensions[16,17].

In the next simplest case ρ^2 is a direct sum of two irreducible representations. Then ε_ρ has two different eigenvalues, and $\varepsilon^{(\rho)}(B_\infty)$ generates a representation of the Hecke algebra. These representations have been completely analysed by Doplicher, Haag and Roberts[6] in the case $\varepsilon_\rho^2 = 1$ (result $d(\rho) \in N \cup \{\infty\}$, $\omega(\rho) = \pm 1$) and by Jones[18], Ocneanu and Wenzl[19] in the general case. There the result can be parametrized by two integers $q \geq 3$ and $2 \leq d \leq q-2$. For the eigenvalues λ_1 and λ_2 of ε_ρ and the statistics parameter λ_ρ one finds the following restrictions:

$$\lambda_1 = -e^{\pm \frac{2\pi i}{q}} \lambda_2$$

$$\lambda_\rho = \alpha\lambda_1 + (1-\alpha)\lambda_2, \quad \alpha = \frac{sin\frac{d-1}{q}\pi}{2sin\frac{d}{q}\pi sin\frac{\pi}{q}} \tag{26}$$

If ρ^2 decomposes into more than two subrepresentations the possible braid group representations have not been determined so far. However, if $d(\rho) < \infty$ ("finite statistics") one can show that ρ^2 is a direct sum of at most $d(\rho)^2$ irreducible subrepresentations, and that the possible values of $d(\rho)$ are contained in the set

$$\{2cos\frac{\pi}{q}, \quad q = 3, 4, ...\} \cup (2, \infty). \tag{27}$$

These are the possbile norm of matrices with integer coefficients and the square roots of the possible indices of inclusions of von Neumann algebras of type II_1 according to the work of Jones[18].

The connection with the norms of integer matrices comes from the fact that $d(\rho)$ is the Frobenius eigenvalue of the "fusion matrix" associated to ρ (see below). The connection with the Jones theory of indices and towers of algebras can be made explicit in the following way. In the case of finite statistics there exists a conjugate $\bar{\rho}$ of ρ, i.e. a localized transportable morphism such that $\bar{\rho}\rho \supseteq id$. There are isometries $R, \bar{R} \in \mathcal{A}$ which intertwine $\bar{\rho}\rho$ and $\rho\bar{\rho}$, respectively, with the vacuum representation such that

$$\bar{\rho}(\bar{R}^*)R = \lambda_\rho = \bar{R}^*\rho(R) \tag{28}$$

Let E and F be the projections on the vacuum subrepresentations of $\bar{\rho}\rho$ and $\rho\bar{\rho}$, respectively, i.e. $E = RR^*, \quad F = \bar{R}\bar{R}^*$. Then

$$E\bar{\rho}(F)E = d(\rho)^{-2}E, \quad \bar{\rho}(F)E\bar{\rho}(F) = d(\rho)^{-2}\bar{\rho}(F). \tag{29}$$

We now define

$$E_{2i-1} = (\bar{\rho}\rho)^{i-1}(E), \quad E_{2i} = (\bar{\rho}\rho)^{i-1}\bar{\rho}(F), \quad i \in N \tag{30}$$

and obtain a sequence of projections satisfying the Lieb-Temperley-Jones relations

$$E_nE_{n\pm1}E_n = d(\rho)^{-2}E_n, \quad E_nE_m = E_mE_n, \quad |n-m| \geq 2. \tag{31}$$

As observed by Longo[20], a corresponding Jones tunnel is given by

$$\cdots \subset \rho\bar{\rho}\rho(\mathcal{A}) \subset \rho\bar{\rho}(\mathcal{A}) \subset \rho(\mathcal{A}) \subset \mathcal{A}. \tag{32}$$

Actually, Longo has shown[20] that the index (suitably generalized to infinite factors) of the inclusion $\rho(\mathcal{A}(O)) \subset \mathcal{A}(O)$ is $d(\rho)^2$.

EXCHANGE ALGEBRAS and R-MATRICES

In order to evaluate the braid group representation explicitly it is convenient to choose one representative morphism ρ_α in each equivalence class of irreducible morphisms. The representation $\rho_\alpha\rho$ can be decomposed

$$\rho_\alpha\rho \simeq \oplus_\beta N_{\alpha\beta}\rho_\beta, \tag{33}$$

$N_{\alpha\beta}$ denoting the multiplicity of ρ_β in $\rho_\alpha\rho$. The matrix $N = (N_{\alpha\beta})$ is called the fusion matrix of ρ. It satisfies the eigenvalue equation

$$\sum_\beta N_{\alpha\beta}\, d(\rho_\beta) = d(\rho)d(\rho_\alpha) \tag{34}$$

which shows that $d(\rho)$ is a Frobenius eigenvalue of N.

Passing from the equivalence classes to the representations we find intertwiners T from ρ_β to $\rho_\alpha\rho$, i.e. operators $T \in \mathcal{A}$ with

$$\rho_\alpha \rho(A) T = T\, \rho_\beta(A), \quad A \in \mathcal{A} \tag{35}$$

Adopting the notation of[14] we denote the linear space of all intertwiners from ρ_1 to ρ_2 by (ρ_2, ρ_1). For $T_1, T_2 \in (\rho_\alpha \rho, \rho_\beta)$ we have

$$T_1^* T_2 \in (\rho_\beta, \rho_\beta) = \rho_\beta(\mathcal{A})' = C. \tag{36}$$

Thus, $(\rho_\alpha \rho, \rho_\beta)$ gets the structure of a Hilbert space, and $dim(\rho_\alpha \rho, \rho_\beta) = N_{\alpha\beta}$.

One can now construct the intertwiner spaces $(\rho_\alpha \rho^n, \rho_\beta)$ out of these elementary spaces. One finds

$$\begin{aligned} (\rho_\alpha \rho^n, \rho_\beta) &= \sum_{\gamma_1, \cdots \gamma_{n-1}} (\rho_\alpha \rho, \rho_{\gamma_1})(\rho_{\gamma_1}\rho, \rho_{\gamma_2}) \cdots (\rho_{\gamma_{n-1}}\rho, \rho_\beta) \\ &\simeq \bigoplus_{\gamma_1, \cdots \gamma_{n-1}} (\rho_\alpha \rho, \rho_{\gamma_1}) \otimes (\rho_{\gamma_1}\rho, \rho_{\gamma_2}) \otimes \cdots \otimes (\rho_{\gamma_{n-1}}\rho, \rho_\beta) \end{aligned} \tag{37}$$

where product and sum in the first line are defined in the sence of subsets of the algebra $\mathcal{A}$, and where isomorphy is to be understood in the sense of Hilbert spaces.

Using the decomposition (37) we can exhibit an orthonormal basis in this space by choosing an orthonormal basis

$$T_{\alpha\beta}^{(i)}, \; i = 1, \cdots N_{\alpha\beta} \tag{38}$$

in each Hilbert space $(\rho_\alpha \rho, \rho_\beta)$. Let G denote the graph with vertices α and $N_{\alpha\beta}$ edges e from α to β, and set

$$T_e = T_{\alpha\beta}^{(i)} \tag{39}$$

if e is the $i-th$ edge from α to β. Now let $Path_{\alpha\beta}^{(n)}$ denote the set of all paths $\xi = (e_1, \cdots e_n)$ of length n from α to β, and set

$$T_\xi = T_{e_1} \cdots T_{e_n}. \tag{40}$$

Then $\{T_\xi, \xi \in Path_{\alpha\beta}^{(n)}\}$ is a basis of $(\rho_\alpha \rho^n, \rho_\beta)$.

For $b \in B_n, \rho_\alpha(\varepsilon^{(\rho)}(b))$ is an intertwiner from $\rho_\alpha \rho^n$ to $\rho_\alpha \rho^n$, hence by left multiplication we obtain a unitary representation of B_n on the Hilbert space $(\rho_\alpha \rho^n, \rho_\beta)$. In terms of the above basis we get

$$\rho_\alpha(\varepsilon^{(\rho)}(b)) T_\xi = \sum_{\xi' \in Path_{\alpha\beta}^{(n)}} R_{\xi\xi'}(b) T_{\xi'}, \tag{41}$$

where $b \to (R_{\xi\xi'}(b))$ is a unitary matrix representation of B_n with dimension $(N^n)_{\alpha\beta}$. In this way one finds in the general framework the R-matrices previously encountered in conformal field theory.

To make the necessary identification we construct an abstract version of the exchange fields originally introduced by Rehren and Schroer[2] in conformal field theory.

Let $H = \oplus H_\alpha$, where $H_\alpha \simeq H_o$ for all α, and let $\mathcal{A}$ be represented on H by

$$A \to \oplus_\alpha \rho_\alpha(A). \tag{42}$$

For each edge e from α to β we define operators $\{e, A\} : H_\alpha \to H_\beta, A \in \mathcal{A}$ by

$$\{e, A\}\phi = T_e^* \rho_\alpha(A)\phi \tag{43}$$

$\{e, A\}$ is called the exchange field. $\{e, A\}$ is said to be localized in O if

$$\rho(B) A = AB, \quad B \in \mathcal{A}(O') \tag{44}$$

Now let $\{e,A\}$ be localized in O, $\{\hat{e},\hat{A}\}$ in $\hat{O}$ in the right spacelike complement of O, and let the source of e be the range of $\hat{e}$. Then one obtains the commutation relation ("exchange algebra")

$$\{e,A\}\{\hat{e},\hat{A}\} = \sum_{f,\hat{f}} R_{(\hat{e},e)(f,\hat{f})}(\sigma_1)\{\hat{f},\hat{A}\}\{f,A\} \tag{45}$$

where the sum extends over all edges $f,\hat{f}$ such that $(f,\hat{f})$ is a path of length 2 with the same source and range as $(\hat{e},e)$.

In conformal field theory one now can perform a scaling limit. One obtains the local exchange fields satisfying again the exchange algebra relation (45). For more details I refer to the contribution of B. Schroer to these proceedings.

3 DIMENSIONAL QUANTUM FIELD THEORY

In the last part of this contribution I want to describe how braid group statistics occurs in 3 dimensional quantum field theory[22,1,II]. The locally generated sectors in 3 dimensions always have permutation group statistics. However, sectors carrying a gauge charge violate the DHR selection criterion. There is a general theorem by Buchholz and myself[13] which states that to each irreducible PER π with isolated mass shell there exists a vacuum representation π_o such that

$$\pi_{|\mathcal{A}(S')} \simeq \pi_{o|\mathcal{A}(S')} \tag{46}$$

for every spacelike cone S. Here, a spacelike cone S is a set of the form

$$S = a + \bigcup_{\lambda>0} \lambda O, \quad a \in \Re^d \tag{47}$$

where O is some closed double cone in the spacelike complement of the origin.

This localization property is weaker than the DHR criterion of being locally generated, and it might be considered as an abstract version of the heuristic construction of charged states in gauge theories by sending a line of electric flux to infinity.

Starting from the localization property (46) one now may identify as in the DHR theory H_π with H_{π_o} such that π and π_o coincide on $\mathcal{A}(S')$, and again $\rho : \pi_o(A) \to \pi(A)$ is a monomorphism from $\pi_o(\mathcal{A})$ into $B(H_{\pi_o})$, hence we may identify $\pi_o(\mathcal{A})$ and $\mathcal{A}$ by dropping the symbol π_o. However, ρ is in general not an endomorphism of $\mathcal{A}$. To understand what is going on we exploit the existence of unitary intertwiners which move the localization cone S of ρ. Let $\hat{\rho} = adV \circ \rho$ be localized in the spacelike cone $\hat{S}$. Then V commutes with $\mathcal{A}(S') \bigcap \mathcal{A}(\hat{S}')$. According to the Bisognano-Wichmann-Theorem[10] we have (under additional assumptions)

$$\mathcal{A}(W')' = \mathcal{A}(W)'' \tag{48}$$

for all wedge regions W where a wedge region is the image of the set $\{x \in \Re^d | |x^o| < x^1\}$ under a Poincaré transformation.

In our general framework we may use (48) as an additional assumption replacing the assumption of Haag duality (6). Now, let S_1, S_2 be spacelike cones with $S_1 \subset S_2'$. Then there exists some wedge W with $S_1 \subset W \subset S_2'$. Hence we obtain the following property (called essential duality[11])

$$\mathcal{A}(S_1')' \subset \mathcal{A}(W')' = \mathcal{A}(W)'' \subset \mathcal{A}(S_2')'' \tag{49}$$

In order to define the composition of representations we make the following construction[13]. Let r be some spacelike vector with $r^2 = -1$. Let $\mathcal{S}(r)$ denote the set of all spacelike cones containing the direction of r in its interior,

$$\mathcal{S}(r) = \{S \text{ spacelike cone} | S + r \subset \mathring{S}\}. \tag{50}$$

$\mathcal{S}(r)$ is an inductive system with respect to inclusion, hence

$$B^r = \overline{\bigcup_{S \in S(r)} A(S')''} \qquad (51)$$

is a C^*-algebra. One finds the following

Theorem[13]: (i) ρ is weakly continuous on $A(S')$ for all spacelike cones S, hence ρ has a unique extension ρ^r to B^r which is weakly continuous on $A(S')''$ for all $S \in S(r)$.

(ii) If the localization region S of ρ is spacelike to r (i.e. to some $S_1 \in S(r)$) then ρ^r is an endomorphism of B^r.

Using this theorem we obtain a composition law for representations

$$\rho_1 \rho_2 := \rho_1^r \rho_2 \qquad (52)$$

where r is spacelike to the localization cone of ρ_2. The product $\rho_1\rho_2$ does not depend on the choice of r. The composition law is associative, and $\rho_1\rho_2 \simeq \rho_2\rho_1$ with the statistics operator $\varepsilon(\rho_1, \rho_2)$ as unitary intertwiner,

$$\varepsilon(\rho_1, \rho_2) = V_2^{-1} \rho_1^r(V_2) \qquad (53)$$

Here, (assuming that ρ_1 and ρ_2 are localized in S) V_2 is a unitary intertwiner which moves the localization cone of ρ_2 into $\hat{S} \subset S'$. The spacelike direction r is chosen spacelike to S and $\hat{S}$ such that by a positively oriented rotation we have the order $S, \hat{S}, r$. If we choose instead a direction r' between S and $\hat{S}$ we obtain an in general different intertwiner from $\rho_1\rho_2$ to $\rho_2\rho_1$

$$\varepsilon(\rho_2, \rho_1)^{-1} = V_2^{-1} \rho^{r'}(V_2). \qquad (54)$$

The possible nontriviality of the operator (the "monodromy operator")

$$\varepsilon_M := \varepsilon(\rho_2, \rho_1)\varepsilon(\rho_1, \rho_2) = \rho_1^{r'}(V_2)^{-1} \rho_1^r(V_2) \qquad (55)$$

is responsible for the occurence of braid group statistics in 3 space time dimensions.

SUM RULE FOR SPINS IN 3 DIMENSIONS

In the final section of this paper I want to describe the implication of Poincaré covariance for the analysis of 3d Quantum Field Theory. We may restrict ourselves to the rotation symmetry. The rotations $\varphi \in [0, 2\pi)$ are represented by automorphisms α_φ of A such that

$$\alpha_\varphi(A(O)) = A(\varphi O) \qquad (56)$$

and on B^r such that

$$\alpha_\varphi(B^r) = B^{\varphi r} \qquad (57)$$

In the representation ρ we have a unitary representation U_ρ of the covering group $\Re$ of the 2 dimensional rotation group such that

$$U_\rho(\varphi)\rho^r(B)U_\rho(\varphi)^{-1} = \rho^{\varphi r}\alpha_{\varphi \bmod 2\pi}(B) \qquad (58)$$

Especially $U_\rho(2\pi) \in \rho(A)'$ and for ρ irreducible

$$U_\rho(2\pi) = e^{2\pi i S_\rho}, \quad S_\rho \in [0, 1). \qquad (59)$$

In the composed representation $\rho_1\rho_2$ we find for small φ

$$U_{\rho_1\rho_2}(\varphi) = U_{\rho_1}(\varphi)\rho_1^r(U_o(\varphi)^{-1}U_{\rho_2}(\varphi)) \qquad (60)$$

where - provided ρ_1 and ρ_2 are localized in $S \subset \{0\}'$ - r is chosen in the spacelike complement of $\bigcup_{0 \le \lambda \le 1} (\lambda\varphi)S$. This fixes the representation $U_{\rho_1\rho_2}$ for all φ. Let us now compute $U_{\rho_1\rho_2}(2\pi)$:

$$\begin{aligned} U_{\rho_1\rho_2}(2\pi) &= U_{\rho_1\rho_2}(\pi)^2 \\ &= U_{\rho_1}(\pi)\rho_1^r(U_o(\pi)^{-1}U_{\rho_2}(\pi))U_{\rho_1}(\pi)\rho_1^r(U_o(\pi)^{-1}U_{\rho_2}(\pi)) \\ &= U_{\rho_1}(2\pi)\rho_1^{\pi r}(U_o(2\pi)^{-1}U_{\rho_2}(\pi)U_o(\pi))\rho_1^r(U_o(\pi)^{-1}U_{\rho_2}(\pi)) \end{aligned} \tag{61}$$

Now $U_o(2\pi) = 1$ and $U_o(\pi)^{-1}U_{\rho_2}(\pi)$ is a unitary intertwiner which maps the localization cone S of ρ_2 into $\pi S \subset S'$. Hence, by (55)

$$U_{\rho_1\rho_2}(2\pi) = U_{\rho_1}(2\pi)\rho_1^{\pi r}(U_{\rho_2}(2\pi))\varepsilon_M \tag{62}$$

The eigenvalues of ε_M are given in terms of the statistics phases $\omega(\rho_\alpha)\omega(\rho_1)^{-1}\omega(\rho_2)^{-1}$ for each $\rho_\alpha \subseteq \rho_1\rho_2$. Thus one obtains the sum rule for spins[1,II]

$$\frac{e^{2\pi i S_\alpha}}{e^{2\pi i(S_1+S_2)}} = \frac{\omega(\rho_\alpha)}{\omega(\rho_1)\omega(\rho_2)} \tag{63}$$

This relation has already been observed in models with one dimensional braid group representations[16,17] ("anyons"). A general proof has also been found independently by Fröhlich, Gabbiani and Marchetti[22].

If one applies the sum rule to the case $\rho_1 = \rho, \rho_2 = \bar{\rho}, \rho_\alpha = \mathrm{id}$ one finds the following connection between spin and statistics

$$e^{2\pi i S_\rho} = \pm\omega(\rho) \tag{64}$$

The usual spin statistics theorem which in its most general form is due to Buchholz and Epstein[23] would determine the sign in (64). A generalization of the Buchholz-Epstein-Theorem to $d = 3$ dimensions has still to be worked out.

REFERENCES

1. K. Fredenhagen, K.H. Rehren, B. Schroer: Superselection sectors with braid group statistics and exchange algebras. I General Theory (to appear in Comm. Math. Phys.), II (to be published).
2. K.H. Rehren, B. Schroer: Einstein causality and Artin braids, Nucl. Phys. B 312, 3 (1989).
3. J. Fröhlich: Statistics of fields, the Yang Baxter equation and the theory of knots and links, Proceedings of the 1987 Cargèse School.
4. E. Witten: Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121, 351 (1989).
5. H.-J. Borchers: On the vacuum state in quantum field theory, Comm. Math. Phys. 1, 57 (1965).
6. S. Doplicher, R. Haag, J. Roberts: Local observables and particle statistics, Comm. Math. Phys. 23, 199 (1971) and 35, 49 (1974).
7. R. Haag, D. Kastler: An algebraic approach to field theory, J. Math. Phys. 5, 848 (1964).
8. D. Buchholz, C. D'Antoni, K. Fredenhagen: The universal structure of local algebras, Comm. Math. Phys. 111, 123-135 (1987).
9. H.-J. Borchers: Local rings and the connection of spin with statistics, Comm. Math. Phys. 1, 281 (1965).
10. J. Bisognano, E. Wichmann: On the duality condition for a hermitean scalar field, J. Math. Phys. 16, 985 (1975).
11. J. Roberts: Spontaneously broken gauge symetries and superselection rules, in Proceedings of the Camerino School 1974 (ed. G. Gallavotti).
12. D. Buchholz, G. Mack, J. Todorov: The current algebra on the circle as a germ of local field theories, DESY 88-126 (Preprint).

13. D. Buchholz, K. Fredenhagen: Locality and the Structure of particle states, Comm. Math. Phys. 84, 1 (1982).
14. D. Buchholz: The physical state space of quantum electrodynamics, Comm. Math. Phys. 85, 49 (1982).
15. S. Doplicher, J. Roberts: C*-algebras and duality for compact groups: why there is a compact group of internal gauge symmetries in particle physics, in Proceedings Marseille 1986 (eds. M. Mebkhout, R. Séneor) World Scientific 1987.
16. R.F. Streater, J.F. Wilde: Fermion states of a boson field, Nucl. Phys. B 24, 561 (1970).
17. F. Wilczek, Phys. Rev. Lett. 48, 1144 (1982), 49, 957 (1982). F. Wilczek, A. Zee, Phys. Rev. Lett. 51, 2250 (1982). Y.S. Wu, Phys. Rev. Lett. 53, 111 (1984).
18. J. Fröhlich, P.A. Marchetti: Quantum field theories of vortices and anyons, Comm. Math. Phys. 121, 177 (1989).
19. V. Jones: Invent. Math. 72, (1983), Ann. Math. 126, 335 (1987).
20. A. Ocneanu (unpublished), H. Wenzl, Invent. Math. 92, 349 (1988).
21. R. Longo: Index of subfactors and statistics of quantum fields (to appear in Comm. Math. Phys.).
22. J. Fröhlich, F. Gabbiani, P.A. Marchetti: Superselection sectors and statistics in three dimensional local quantum theory. ETH -TH/89-22 (Preprint).
23. D. Buchholz, H. Epstein: Spin and statistics of quantum topological charges, Fizika 17, 329 (1985).

CYCLIC COHOMOLOGY, SUPERSYMMETRY AND KMS STATES

THE KMS STATES AS GENERALIZED ELLIPTIC OPERATORS

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Let me begin this talk at a conference devoted to the interface between physics and geometry by stressing the essential identity of geometry and analysis. The distinction is one of methods. The common theme is manifolds and the pseudodifferential operators attached to their vector bundles.

There are strong indications that one of the main avenues of future mathematical development will lead to generalize geometry-analysis in a double respect :

(i) the passage from finite to infinite dimension

(ii) the more radical passage from usual to "non-commutative manifolds" (C^∞ replaced by a non-commutative algebra).

Step (i) is already there classically (e.g. loop spaces). Step (ii) is "*quantization*" in a broad sense, as purported by Alain Connes' "non commutative geometry", the actual frame of my talk [1] [2], at the same time non-commutative analysis (cf. Fredholm modules) and differential geometry/topology (cf. Chern character and cyclic cohomology). This theory has two versions corresponding to finite, resp. infinite dimension (in non commutative paraphrase !). The first is concerned with *cyclic cohomology* and p-*summable Fredholm modules* [1], the second and more recent with *entire cyclic cohomology and* θ-*summable Fredholm modules* [2][2a].

My specific purpose is to convince you that the *KMS states* (physically : temperature equilibrium states, mathematically the "generalized traces" of the second era of Von Neumann algebras) are appropriate substitutes for the classical elliptic operators in the generalizations (i) and (or) (ii) above. This is the moral I want to draw from the fact [3] that *graded* KMS-*functionals of supersymmetric dynamical* C^*-*systems yield entire cyclic cocycles*. Before going into the matter (and rather than displaying gory technicalities[1]) I shall endeavour to put this result with its claimed interpretation into the broad perspective of Alain Connes' doctrine.

The general philosophy is that *non-commutative (associative) algebras* - technically C^*-algebras -*are "non-commutative spaces"*. This thought arose from the recognition that function algebras are a general model of commutative algebras (Gelfand) : specifically : each commutative

1 The proofs in [3] and [4] are, to my taste, too much of a verification. A more conceptual proof would be desirable.

C^*-algebra A is isomorphic to $C_0(X)$, the algebra of continuous functions on its "spectrum" X (in the absence of a unit, one has to take the functions vanishing at ∞ ; for a unital algebra A, the spectrum X is compact).[2] Now, if you suppress the sole commutativity axiom from a system of seven axioms, you pass from model $C_0(X)$ to general C^*-algebras: therefore it is tempting to consider the latter as "non commutative (= quantum) spaces" -whatever this means.

Well, nowadays it means a lot : the "quantum version" of spaces has progressed in four successive stages of growing structure, viz.

1) measure theory
2) topology (topological K-theory)
3) smooth structures
4) riemannian ($spin^c$) structures

as shown in the table below displaying the quantum substitutes of the classical items. A deep parallelism repeatedly appears as the miraculous fact that quantum proofs are already present in classical proofs which, phrased in the language of algebras, appear not to require commutativity !

Classical	**Quantum**
1° **Measure theory**	
Bounded measures	States. Hilbert space representations
$C_0(X)$, X locally compact with countable basis	Separable C^*-algebras.
$(L^\infty(X,\mu)$	Hyperfinite von Neumann algebras completely classified by means of - traces (type I and II) - KMS states (type III)
if a group acts : ergodic theory (topological or measure theoretic)	C^*- and W^*-systems (obtained from actions of groups or more generally Ocneanu paragroups).
2° **Topological K-theory**	
$K^0(X)$, $K^1(X)$ (X locally compact) classification of vector bundles	$K_1(A)$ $K_0(A)$ (A a C^*-algebra) classification of finite projective modules
Bott periodicity	
3° **Smooth structures**	
Finite, resp. ∞-dimensional manifolds	Cyclic, resp. entire cyclic cohomology of non commutative algebras
Elliptic operators	Fredholm modules, p-summable, resp. θ-summable.
Duality with K-theory	
4° **Riemannian ($Spin^c$) structures**	
$Spin^c$ manifolds	C^*-algebras with "Dirac-Fredholm modules"
Dirac operator	

2 This Gelfand *"structure theory"* is the culmination of the *spectral theorem asserting that commuting operators on Hilbert space are described by functions on their common spectrum (almost equivalent, knowing that* C^**-algebras are always realizable by operators on Hilbert space).*

We now comment this table in detail -as a long preamble to our subject proper .

1. Measure theory. The quantum version of measure theory is nowadays completed through a full classification of the hyperfinite factors [5] [6] [7] [8][3].

In his pioneering work with Murray, von Neumann had showed (reduction theory) that "rings of operators" [4] decompose into "factors"[5], and had classified the latter according to the range of their trace on projections, viz. :

- $\mathbb{N}$ for the "I_s" - the only "factors" deserving this name, since obtained as $\mathcal{B}(\mathcal{H}_1) \otimes \mathbf{1}_{\mathcal{H}_2}$ through tensorial factorization of Hilbert space
- $[0,1]$, resp. $[0,\infty]$, for the "II_{1_s}", resp. the "II_{∞}"$_s$
- $\{0,\infty\}$ for the "III_s" for which the trace is thus inefficient, a circumstance which caused Murray and von Neumann to stop investigating the "III_s", after the construction of a few examples. The "III_s" were long deemed pathological and uninteresting, until algebraic field theorists found them everywhere present in physics (as local algebras of wedges in the vacuum representation of relativistic field theories ; and as weak closures of the whole algebra in temperature situations -relativistic or not [9], [10] [11] [12]).

The type III-deadlock was overcome after Tomita's breakthrough [13] with the advent of the Tomita - Takesaki theory [14], this occurring in parallel with the recognition of the basic role of KMS states in equilibrium statistical mechanics [15]. Armed with the KMS concept (the proper substitute of the trace for the "III_s") Alain Connes then completely elucidated their classification in the hyperfinite case [5] [6] [7]. The last remaining step -uniqueness of the hyperfinite III_1 (the one in physics !)- was effected by Haagerup [8] (see also [16], and [17] [17a] for the relation to physics).

We define briefly the all-important KMS concept. Let (A,α) be a C^*-dynamical-system consisting of a C^*-algebra A with a one-parameter group α_t of automorphisms[6]. A state φ of A (=norm **1** positive linear functional) is a *KMS-state to the inverse temperature* β whenever one has [16]

$$(1) \qquad \varphi(ba) = \varphi(a\alpha_{i\beta}(b)) \qquad \begin{cases} a,b \in A \\ b \text{ analytic for } \alpha \end{cases}$$

This *KMS(Kubo, Martin, Schwinger)-condition* (reflecting itself as the fact that the two point function $\varphi\,(a\alpha_t(b))$ is analytic in the strip $0 \le \mathrm{Im} z \le 1$) contains in fact the same information as the *Gibbs Ansatz* :

$$(2) \qquad \varphi(a) = \mathrm{Tr}\,(e^{-\beta H}a) \;/\; \mathrm{Tr} e^{-\beta H} ,$$

but in a form freed from the unphysical constraints of (2) (system in a box, artificially discretized energy spectrum !) : indeed (1) makes sense without the need of the (conceptually obscure if computationally necessary) "thermodynamic limit", and has a direct physical

3 in contrast "non-commutative ergodic theory" (the study of actions of groups on algebas) is still largely open.

4 now called von Neumann or W^*-algebras

5 W^*-algebras with a trivial center (and accordingly a unique normal trace)

6 If A is represented on a Hilbert space with implementation of the dynamics by a hamiltonian H, one has $\alpha_t(a) = e^{iHt}\, a\, e^{-iHt}$

interpretation (in fact characterization) in physical terms (stability w.r.t. local perturbations [18] or "passivity" [19] - an abstract version of the second principle of thermodynamics). The one-parameter automorphism group appearing in (1) is time development (dynamics) for the (gauge invariant) observables, and a mixture of dynamics and gauge measured by the chemical potential for the "fields" -the latter fact evolving from the first in a purely algebraic way [20].

We note that the Gibbs state (2) is of course KMS for α_t as in footnote 6 for the inverse temperature β, as revealed by a two-line verification.

Note that this sketch of "temperature algebraic field theory" (treated at length in [21]) does not touch the "vacuum theory" (Doplicher- Haag-Roberts theory of superselection sectors, cf. quotations in [22]), of recent renewed interest[7]).

To conclude this paragraph, let us mention that the basic appearance of KMS states in von Neumann algebra theory differs technically from that in quantum statistical mechanics. Every von Neumann algebra is automatically equipped by any state φ faithful on its positive cone with a *modular one parameter group* α_t^φ for which the state is KMS (with the conventional choice $\beta = -1$). And the α_t^φ for the different φ differ from each other by cocycles (non commutative Radon-Nicodym derivatives) with values in inner automorphisms. In contrast to this, physical systems are given as pairs -dynamical systems- (A,α) of a C^*-algebra of observables plus a time-development automorphism group α, equilibrium states φ of temperature $(k\beta)^{-1}$ being β-KMS for α (so that $\alpha_{-\beta t}$ is a modular automorphism in the above sense for the weak closure A_φ'' of A in the representation φ generates - A_φ'' is however not a basic entity of the physical system, but rather an attribute of the system plus the state.

2) <u>Topological C^*-K theory</u> We will only mention that the general theory (including Bott periodicity) naturally evolves from phrasing the usual proofs in terms of modules over the algebra rather than in spatial terms. The non commutative generalization arises from the observation (Serre-Swan) that the modules of sections of vector bundles over X are exactly the finite projective modules over $C_0(X)$.

Strikingly enough, the classical proofs [6] when phrased in algebraic language turn to be largely independent of the commutativity axiom. C^*-K theory (including the bivariant KK-theory of Kasparov [26] [29]) is now an ample body of knowledge [24].

3) <u>Quantum smooth structures</u> We now sketch Alain Connes "non commutative differential geometry" [1] [2]. Connes discovered cyclic cohomology whilst recognizing the "Fredholm modules" as the non commutative substitutes for elliptic operators[8]. The classical situation is as follows : an elliptic operator $P : C^\infty(E^\circ) \to C^\infty(E^1)$, E°, E^1 vector bundles over an n-dimensional smooth manifold M, becomes (via extension to appropriate Sobolev-completions) a Fredholm operator $P : H^\circ \to H^1$ with

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7 The DHR theory of superselection sectors is presently successfully applied to 2 and 3 dimensional field theories [22] and Schoer's talk at this conference.

8 The starting point was [21] and [22].

quasi inverse $Q = H^1 \to H^\circ$. Introducing the direct sum $H = H^\circ \oplus H'$ (graded with grading involution $\varepsilon = \mathbf{1}_{H^0} \oplus - \mathbf{1}_{H^1}$), P and Q are subsumed by[9]

(3) $$F = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$$

whilst each $a \in A = C^\infty(M)$ acts on H as

(4) $$(a) = \begin{pmatrix} a^0 & 0 \\ 0 & a^1 \end{pmatrix}$$

(a^i the multiplication by a on $C^\infty(H^i)$). This situation entails the following facts : with $L^p(H)$ the p^{th} Schatten ideal (set of linear operators B on H with $(B^*B)^{p/2}$ trace class), we have

(5) $$(F^2-\mathbb{1}).(a) \in L^1(H), \quad a \in A ,$$

whilst

(6) $$[F,(a)] \in L^p(H) \quad , a \in A , \quad p > n .$$

Together with the Hölder inequality, this allows Alain Connes to define the *p^{th} character of F* as the (p+1)-linear form :

(7) $$\varphi^{(p)} (a_0, a_1, \dots, a_p) = C_n \operatorname{Tr} \{\varepsilon(a_0) [F,(a_1)] \dots [F,a_p]\}, \quad a_0, \dots , a_p \in A$$

(non vanishing only for p even). And Connes makes the key observation that, *if* $F^2 = \mathbf{1}$[10], *$\varphi^{(p)}$ is a Hochschild cocycle* :

(8) $$\varphi \circ b = 0$$

moreover cyclic in the sense

(9) $$\varphi \circ \lambda = 0.$$

Here b is the *Hochschild boundary* [11]

(10) $$b(a_0 \otimes a_1 \otimes \dots \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i a_0 \otimes \dots \otimes a_i \; a_{i+1} \otimes \dots \otimes a_{p+1} - (-1)^p a_p a_0 \otimes a_1 \dots \otimes a_{p+1} ,$$

whilst λ is the *cyclic permuter*

(11) $$\lambda(a_0 \otimes a_1 \otimes \dots \otimes a_p) = (-1)^p ap \otimes a_0 \otimes \dots a_{p-1} .$$

As it is well known, we have $b^2 = 0$. The corresponding cohomology is the *Hochschild cohomology* of A (with values in the dual of A as an A-bimodule). The fact had however remained unnoticed that *the cyclic Hochshild cochains are a subcomplex of the Hochschild complex,* thereby defining *the cyclic cohomology* of $A = C^\infty(M)$.
So far for the classical situation.

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9 2 x 2 matrices with operators entries corresponding to the decomposition $H = H^\circ \oplus H^1$.

10 relatively mild restriction in view cf (5)

11 recall that (p+1) - linear forms are the same as linear forms on $A^{\otimes(p+1)}$

Alain Connes' capital observation is now that (next occurrence of the "non commutative miracle" !), the results (8) and (9) in fact follow from (3), (4), (6) and $F^2 = \mathbf{1}$ **without the need that** A **be commutative** This suggests to postulate (3) through (5), defining *p-summable Fredholm modules* of arbitrary complex algebras A as graded Hilbert spaces $H = H_0 \oplus H_1$ carrying a graded representation $a \to (a)$ by bounded operators plus an odd bounded operator F with the properties (5) and (6). *This warrants the existence of the character* (7), *which, in the case* $F^2 = \mathbf{1}$, *is again a cyclic Hochschild cocycle in the sense* (8), (9).

The fact that cyclic cochains are again a subcomplex of the Hochschild complex now yields the definition of the cyclic cohomology for arbitrary complex algebras.

At this point, we make two remarks :

(i) It is obvious from their definition that the p-summable Fredholm modules are the non-commutative substitutes for elliptic operators.

(ii) We are in fact moving towards "non commutative analysis" as suggested by the following observation : defining

$$\delta a = i[F,(a)] \quad , a \in A , \tag{12}$$

yields a derivation $\delta : A \to B(X)$, moreover of vanishing square if $F^2 = 1$. Viewing $\mathrm{Tr}(\varepsilon\cdot)$ as a kind of a Leibnitz integration symbol[12] $\int$, the character (7) now looks like a "multiple integral" :

$$\int (a_0)\ \delta a_1 \ldots \delta a_n . \tag{7a}$$

However, we should note that what we have here is a quantum version of *finite dimensional* manifolds : indeed, condition $p > n$ in the classical case a priori forbids the existence of p-summable Fredholm modules for classical infinite dimensional manifolds. We shall in fact need a modification of what precedes (viz. θ-summable Fredholm modules and entire cyclic cohomology [2], see below) in order to define the "infinite dimensional quantum smooth structures". Before proceeding to this, we need however to develop a few further aspects of cyclic cohomology.

First, the above heuristic remark (ii) can be given a precise meaning in the following way. To each complex algebra A we associate its *differential envelope* $\Omega(A)$ [1], defined as [13]

$$\Omega(A) = \mathcal{F}/I , \tag{13}$$

quotient of the free algebra $\mathcal{F}$ generated by the symbols a, da, $a \in A$, through the ideal I corresponding to the relations

$$\begin{cases} \alpha \cdot a \dot{+} \beta \cdot b \dot{-} (\alpha a + \beta b) = 0 \\ a \cdot b \dot{-} (ab) = 0 \\ \alpha \cdot da \dot{+} \beta \cdot db \dot{-} d(\alpha a + \beta b) = 0 \\ da \cdot b \dot{+} a \cdot db \dot{-} d(ab) = 0 \end{cases} \tag{14}$$

12 Think of quantum statistical mechanics, where Tr replaces the classical $\iint dpdq$

13 see also [27] for a $\mathbb{Z}/2$ graded version

(operations marked with a dot are in $\mathcal{F}$; the two first relations aim at having A a subalgebra of $\Omega(A)$; the two others at having d a derivation : $A \to \Omega(A)$). $\Omega(A)$ is properly defined by (13) as a complex algebra generated by elements a and b, a, b $\in$ A, with the rules (14). One wishes however a constructive picture, which one easily obtains as follows : it is clear that "words" in $\Omega(A)$, consisting of arbitrary products of symbols of the type a_i and da_k , may be reordered using the last relation (14), so as to bring all symbols da_k to the right of the symbols a_i : $\Omega(A)$ is thus seen to be linearly generated by elements of the form

$$(15) \qquad \begin{cases} a_0\, da_1 \dots da_n \\ \\ da_1 \dots da_n \end{cases} , a_0,\dots,a_n \in A,\ n \in \mathbb{N} ,$$

this making it intuitive that one has

$$(16) \qquad \begin{cases} \Omega(A) = \sum\limits_{p=0}^{\infty} \Omega(A)^p \\ \Omega(A)^\circ = A \\ \Omega(A)^p = A^{\otimes p+1} \oplus A^{\otimes p} \qquad , n \geq 1 , \end{cases}$$

(or else, adding a formal unit $\tilde{\mathbb{1}}$

$$(16a) \qquad \begin{cases} \tilde{\Omega}(A) = \mathbb{C}\, \tilde{\mathbb{1}} + \Omega(A) = \sum\limits_{p=0}^{\infty} \tilde{\Omega}(A)^p \\ \tilde{\Omega}(A)^\circ = \tilde{A} = \mathbb{C}\, \tilde{\mathbb{1}} \oplus A \\ \Omega(A)^p = \tilde{A} \otimes A^{\otimes p} \qquad , p \geq 1 \end{cases}$$

$\Omega(A)$ is so far specified as a vector space. In order to define the product it clearly suffices (apart from obvious rules) to state the rule

$$(17) \qquad (\tilde{a}_0\, da_1 \dots da_p)a_{p+1} = \tilde{a}_0\, a_1\, da_2 \dots da_n + \sum_{i=1}^{p} \tilde{a}_0\, da_1 \dots (a_i a_{i+p}) \dots da_{p+1}$$

obtained by repeated application of the derivation rule, last line of (14). It is intuitive (and easy to show [23]) that (17) (together with obvious rules) makes $\Omega(A)$ an associative complex algebra fulfilling (14). If we then define d on $\Omega(A)$ as

$$(18) \qquad d(a_0 + \lambda\, \mathbb{1})da_1 \dots da_p = da_0\, da_1 \dots da_p,\ a_0,\ a_1,\ \dots, a_p \in A$$

we make $\Omega(A)$ a differential algebra, universal in the sense that each homomorphism $\varphi : A \to \Omega_1$ (as algebras) into a given differential algebra (Ω_1, δ_1) factors through $\Omega(A)$

(19)

$$\begin{tikzcd} \Omega(A) \arrow[dr, "\bar{\varphi}"] & \\ A \arrow[u, hook] \arrow[r, "\varphi"'] & \Omega_1 \end{tikzcd}$$

with $\bar{\varphi} : \Omega(A) \to \Omega$ a homomorphism of differential algebras. The particular case $\Omega_1 = B(H)$, $\delta_1 = \delta$ as given by (12), and (a) = $\varphi(a)$, $a \in A$, proves the above claim that the character (7), pull back by the homomorphism $\bar{\varphi} : (\Omega(A),d) \to (B(H),\delta)$ of the graded trace[14] $Tr \circ \varepsilon$ of $B(H)$ equipped with the obvious $\mathbb{Z}/2$ grading), is itself a graded trace of $\Omega(A)$. This fact is general : the *cyclic cocycles* φ *of* A, *are, via the correspondence*

(20) $$\varphi(a_0,a_1,\dots,a_p) = \phi(a_0 da,\ da_2 \dots da_p), \quad a_1, \dots \ ap \in A,$$

one-to-one with the graded traces ϕ *of* $\Omega(A)$[15]

The second feature which we need is the fact, discovered by Alain Connes whilst developing homological algebraic aspects of cyclic cohomology [1], that the above cyclic Hochschild cocycles ("geometrically" interpretable as the graded traces of $\Omega(A)$, as we just saw) are in fact part of a more general set of cocycles, neither Hochschild not cyclic, but vanishing under the coboundary $\varphi \to \varphi \circ \Delta$, with

(21) $$\Delta = b + B \quad ,$$

where b is the previously defined Hochschild coboundary (10) whilst B is given by :

(22) $$B = B_o\, A \ ,$$

with A the "cyclicizer"

(23) $$A = \sum_{k=0}^{p-1} \lambda^k \qquad \text{on } A^{\otimes p} \quad ,$$

and B_0 given by

(24) $$B_0(a_0 \otimes \dots \otimes a_p) = \mathbb{1} \otimes a_0 \otimes \dots \otimes a_p + (-1)^p \ a_0 \otimes a_1 \otimes \dots \otimes \mathbb{1}$$

(the algebra A is now assumed unital with unit $\mathbb{1}$ -distinct from the formal unit $\tilde{\mathbb{1}}$ added above to a general A). One checks the relations

(25) $$b^2 = B^2 = bB + Bb = 0 \quad ,$$

entailing that Δ is a boundary :

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14 a graded trace ϕ of a $\mathbb{Z}/2$-graded algebra $\mathcal{A}$ is a linear form of $\mathcal{A}$ vanishing on all graded commutators $[a,b] = ab - (1)^{\partial a \partial b}\, ba$, $a,b \in \mathcal{A}$ of respective grades ∂a and ∂b.

15 This is the amount to which the "alternation" of the classical differential forms persists in the non commutative generalisation : $\Omega(A)$ is not anticommutative, as was the case for the De Rham algebra, but has alternation *under appropriate forms* (the cyclic cocycles).

(26) $$\Delta^2 = 0.$$

The general cocycles are within the bicomplex $\mathbb{C}$ with entries

(27) $$C^{p,q} = C^{p-q} \quad ,p\ ,q \in \mathbb{N},\quad p \geq q,$$

(C^p the set of (p+1)-linear forms on A), a p-cocycle having a finite number of components on the p^{th} antidiagonal of $\mathbb{C}$. The bicomplex $\mathbb{C}$ yields cyclic cohomology as the cohomology of its associated total complex. Each cohomology class contains one cyclic Hochschild cocycle of the previously considered type, this causing the latter to "carry" cyclic cohomology. The shift S along the diagonal of $\mathbb{C}$ is the "*Connes periodicity operator*", and cyclic cohomology can be "divided" by S", so as to yield *periodic cyclic cohomology* (of period 2) called by Connes *de Rham cohomology* since reducing to the latter in the classical case $A = C^\infty(M)$. The operator S was originally discovered by Connes through the consideration of the *Chern character* arising as follows : with e a projection in the (stabilized) algebra A, ePe is (by virtue of (6)) a Fredholm operator, moreover (with an appropriate choice of the constants C_{2n}), of index given by

(28) $$\text{Index } ePe = C_{2n}\ \varphi^{2m}(e,e, \dots ,e).$$

Since this holds for all m with $2m \geq p$, the l.h.s. is independent of m, suggesting the existence of a relation between the φ^{2m} : in fact the latter build a "S hierarchy" : $\varphi^{2m+1} = S\ \varphi^{2m}$, and represent the same class of periodic cohomology.

Assuming now that A is a Banach algebra, we are ready to describe Connes "quantum smooth structures" for the infinite dimensional case. The substitutes of elliptic operators are now the *θ-summable Fredholm modules* specified as follows : as previously, one has a graded Hilbert space $H = H^o \oplus H^1$ carrying a graded representation $a \to (a) \in B(H)$ of the algebra A ; and there is moreover an (unbounded) selfadjoint operator $D = D^*$ of odd grade, fulfilling $[\Delta,(a)] \in B(H)$, $a \in A$, plus the (high temperature) condition :

(29) $$e^{-\beta D^2} \in L^1(H)\ .$$

To each such θ-summable module, Connes associates a character [2], now a cocycle for *the entire cyclic cohomology*[16] , an enlargement of periodic cyclic cohomology which encompasses the "infinite dimensional case". The corresponding complex has period 2 :

(30) $$C^{even} \xrightarrow{\Delta} C^{odd} \xrightarrow{\Delta} C^{even}$$

with cochains[17]

(31) $$C^{even} = \{ (\phi_{2p})_{p\in\mathbb{N}}\ ,\ \phi_{2p} \in C^{2p}\ ;\ \sum_{n\in\mathbb{N}} \frac{(2p)!}{p!}\ ||\phi_{2p}||\ z^p \text{ is entire}\}$$

(32) $$C^{odd} = \{ (\phi_{2p+1})_{p\in\mathbb{N}}\ ,\ \phi_{2p+1} \in C^{2p+1}\ ;\ \sum_{n\in\mathbb{N}} \frac{(2p+1)!}{p!}\ ||\phi_{2p}||\ z^p \text{ is entire}\}$$

16 There is an injection of periodic cyclic cohomology into entire cyclic cohomology

17 The growth conditions of the entire function type motivate the name entire cyclic cohomology.

We have no place to describe Connes' characters of the θ-summable modules [2] (see [28] for an introduction). The latter are *"normalized" entire cyclic cocycles* (a generalization of the previous notion of cyclic Hochschild cocycles, now geometrically interpretable as traces of the *Cuntz differential envelope* $q\,A$ (cf. [29], [2], [32]). Let us just mention that in order to construct his character, Connes had to resort to taking a formal square root so-to-speak *"enforcing supersymmetry"*, which lead him to conjecture a deep relationship between cyclic cohomology, supersymmetry, and the KMS structure [30]. This is in line with the fact, displayed by Jaffe, Lesniewski and Weitsman [31], that the supersymmetric Wess-Zumino model (recognized by Witten as the Dirac operator of loop space) yields a remarkable θ-summable module (here pertaining to loop space as an infinite dimensional classical manifold).

This example has led Jaffe et al. to propose an alternative interesting version of the character of a θ-summable Fredholm module [3] (however not normalized in Connes' sense, hence without a geometrical interpretation in terms of the Cuntz envelope)[18]. We do not describe the Jaffe et al. character at this point, because it will appear as a special (suggestive) case of the entire cyclic cocycles attached in [3] to graded KMS functionals. Before coming to the latter, we need one more remark : there is a natural generalization (spelled out in [27], [32]) of cyclic (or, for that matter, entire cyclic) cohomology to $\mathbb{Z}/2$ *graded (Banach) algebras* ($A = A^\circ \oplus A'$ with $A^i\, A^j \subset A^{i+j \bmod 2}$). The latter is obtained by inserting, in the definition formulae of b and λ, the sign factors characteristic of the $\mathbb{Z}/2$ graded frame. The new formulae are

$$(33) \qquad b\,(a_0 \otimes a_1 \otimes \ldots \otimes a_{p+1}) = \sum_{i=1}^{p} (-1)^i\, a_0 \otimes \ldots \otimes a_i\, a_{i+1} \otimes \ldots \otimes a_{p+1}$$

$$- (-1)^{p+\partial a_{p+1} \sum_{k=0}^{p} \partial a_k}\, a_{p+1} a_0 \otimes a_1 \otimes \ldots \otimes a_p$$

$$(34) \qquad \lambda(a_0 \otimes a_1 \otimes \ldots \otimes a_p) = (-1)^{p+\partial a_{p+1} \sum_{k=0}^{p} \partial a_k}\, a_p \otimes a_0 \otimes \ldots \otimes a_{p-1}$$

and we have as above (21), (22) and (23).

We are now ready for

A. *Definition*. Let $A = A^\circ \oplus A'$ be a $\mathbb{Z}/2$ graded C^* algebra, with α_t a continuous one parameter-group of automorphisms. The dynamical system (A,α) is called *supersymmetric* whenever

(i) α preserves the $\mathbb{Z}/2$-grading :

$$(35) \qquad \alpha_t(A^i) \subset A^i \quad , \quad i = 1,\, 2, \qquad t \in R$$

(ii) the infinitesimal generator[19] of α :

$$(36) \qquad D = \frac{d}{dt}\Big|_{t=0} \alpha_t$$

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18 This poses a problem of interpretation. One would wish to understand better the relationship between the Connes and the Jaffe et al character.

19 D is an even derivation of A as a result of (35)

is the square of an odd derivation δ of A :

(37) $\quad D = \delta^2$

This definition aims at capturing the essence of supersymmetry.

B. *Definition.* With (A,α) a supersymmetric C^*-dynamical system and D,δ as in A, a bounded linear form ϕ of A is *graded* β*-KMS*, $\beta \in \mathbb{R}$, whenever one has $\varphi \circ \delta = 0$ and

(39) $$\varphi(ba) = (-1)^{\partial a \partial b}\, \varphi(a\, \alpha_{i\beta}(b)), \quad a,\ b \in A,\ b \text{ analytic for } \alpha$$

Note that the restriction of φ to the bosonic part is β-KMS in the usual sense (1), hence may be a state of A°.

C. *Theorem Let* φ *be a graded* β*-KMS linear form of a supersymmetric* C^**-dynamical system* (A,α), *with* δ *as in* A. *Then, defining, for homogeneous and* α*-analytic* $a_0, a_1, \ldots, a_n \in A$

(40) $$\varphi^\beta(a_0, a_1, \ldots, a_n) =$$

$$\beta^{\frac{n}{2}} i^n (-1)^{n-1+\partial a_n \sum_{k=0}^{n-1} \partial a_k}\, \varphi\left(a_0 \int_{I_\beta^n} \alpha_{it_1}(\delta a_1) \ldots \alpha_{it_2}(\delta a_2) \ldots \alpha_{it_n}(\delta a_n)\, dt\right),$$

where $I_\beta^n = \{ t \in (t_1, \ldots t_n)\ ;\ 0 \le t_1 \le \ldots \le t_n \le \beta\}$, *yields an entire cyclic cocycle* of A.

The Jaffe et al. character of the θ-summable module (H,D) [3] is the special case obtained from $\varphi^\beta = \mathrm{Tr}(\varepsilon e^{-\beta D^2}\,\bullet)$ (special case of a type I or II flavour).

We conclude with a few words about our claim that KMS states generalize elliptic operators in the sense of generalizations (i) and (ii) considered in the outset. This is because KMS states appear here as generalizations of (the characters of) θ-summable Fredholm modules, themselves substitutes of elliptic operators. Note that this type of generalization is already expected to occur in the infinite dimensional classical case. Indeed, most elliptic operators on an infinite-dimensional manifold will, unlike the Dirac operator on loop space, be obtained through a "thermodynamic limit" spoiling formal properties.

We conclude in stressing the challenging apparent need of a supersymmetric frame for establishing the basic relationship we find between KMS states and entire cyclic cohomology.

REFERENCES

[1]. A. Connes, Pub. Math. IHES **62** 257 (1985)
[2]. A. Connes, K-Theory **1** 519 (1988)
[2a]. A. Connes, Pacific. J. Math. **137** 225 (1989)
[3]. D. Kastler, CMP **121** 345 (1989)
[4]. A. Jaffe, A. Lesniewski, K. Osterwalder CMP **118** 1 (1988)

[5]. A. Connes, Ann. Sc. Ec. Norm. Sup. **6** 133 (1973)
[6]. A. Connes, Ann. Math. **104** 73 (1976)
[7]. A. Connes, Proc. Symp. Pur. Math. **38** 43 (1982)
[8]. U. Haagerup, Acta Math. **104** 73 (1976)
[9]. H. Araki, J. Math. Phys. **4** 1343 (1963)
[10]. N. Hugenholtz, CMP, **6** 189 (1967)
[11]. W. Driessler, CMP. **44** 133 (1975)
[12]. R. Longo, Proc. Symp. Pur. Math. **38** 551 (1982)
[13]. M. Tomita, V^{th} Funct. Anal. Symp. Math. Soc. Japan Sendai (1967)
[14]. M. Takesaki, Springer Lecture Notes Math. **128** (1970)
[15]. R. Haag, N. Hugenholtz ; M. Winnink CMP **5** 215 (1967)
[16]. M. Takesaki, Acta Math. **131** 249 (1973)
[17]. D. Kastler Symposia Math. XX 49 (1976)
[17a]. D. Kastler, Proc. Symp. Pur. Math. **38** 467 (1982)
[18]. R. Haag, D. Kastler, E.B. Trych-Pohlmeyer, CMP **38** 173 (1974)
[19]. W. Push, S. Woronowiz, CMP **58** 273 (1978)
[20]. H. Araki, R. Haag, D. Kastler, M. Takesaki, CMP **53** 97 (1977)
[21]. O. Bratteli, D. Robinson. Vol. II, texts and monographies in Physics Springer (1981)
[22]. K. Fredenhagen, K.H. Rehren, B. Schroer, Inst. Theor. Elementarteilchen Preprint Berlin (1988)
[23]. M. Atiyah K-theory Benjamin (1907)
[24]. B. Blackadar, K-theory for operator algebras, Springer MSRI series, Springer 1986.
[25]. M. Atiyah, Proc. Int. Conf. Funct. Anal. Tokyo Univ. Press (1970) Mat.**44** 57 (1980)
[26]. G. Kasparov. Izv. Akad. Nauk. SSSR Ser. Mat. **44** 57 (1980)
[27]. D. Kastler, Travaux en cours 30, Ed. Hermann, Paris (1988)
[28]. D. Kastler, CPT Preprint 88/P. 21-28 Marseille (1988)
[29]. J. Cuntz, K-Theory **1** 31 (1987)
[30]. A. Connes, Private communication
[31]. A. Jaffe, A. Lesniewski, A. Weitsman CMP **112** 75 (1987)
[32]. D. Kastler, K-theory **1** 485 (1989)
[33]. A. Jaffe, A. Lesniewski, K. Osterwalder, Preprint HUTMP, 88/B-229 Harvard
[34]. A. Jaffe, A. Lesniewski, M. Wisniowski HUTMP 88/B-231, Harvard.

GRAPH GEOMETRY, QUANTIZED GROUPS AND NON-AMENABLE SUBFACTORS

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ABSTRACT

The symmetry structure of the quantized space are discussed. The associated structures of the quantized group are also elaborated. The quantized group concept shows that graphs, connections, interwines, harmonic analysis, etc., are part of structures of the symmetries of the non commutative space. All these can be viewed as continuation of John Van Neuman's program of using operator algebras in the description of quantum mechanics and field theory.

NEW KINEMATICS (STATISTICS AND SYMMETRY) IN LOW-DIMENSIONAL QFT WITH APPLICATIONS TO CONFORMAL QFT_2*

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INTRODUCTION

The concepts of field commutation relation, particle statistics and the origin of internal symmetries always have been considered as fundamental in quantum field theory. In the standard Lagrangian approach one usually starts with fields having Fermi- or Bose-statistics which carry an internal compact Lie-group symmetry (i.e. a subgroup of SU_n for large enough n). These fields are then "coupled" in order to implement the idea of local interactions in accordance with the Einstein causality of observables. For the most interesting cases of strictly renormalizable interactions (pocessing dimensionless coupling constants) as e.g. "local gauge theories", there is essentially no non-perturbative analytic result.
Much of the recent progress in low-dimensional QFT notable in the classification of conformal QFT_2 is best understood as a generalization of kinematics: braid group statistics (instead of the Fermion-Boson alternative) and a related "quantized" symmetry. In order to bring out this aspect as clearly as possible, we need a framework which, unlike the Lagrangian- and path-integral approach, is capable of problematizing the issues of statistics and symmetry. Algebraic quantum field theory as originally formulated by Haag and Kastler[1] had been adapted by Doplicher, Haag and Roberts[2] (using previos results of Borchers[3]) precisely for this purpose. An important additional step relevant for low-dimensional ($d \leq 3$) QFT was carried out by Buchholz and Fredenhagen[4]. The present notes are based on an extension of this framework by Fredenhagen, Rehren and myself[5]. A closely connected paper (which has a close relation to Jones work[6]) is that by Longo[7]. Various accounts of methods and results of algebraic QFT have been reviewed in conference reports[8]. An additional motivation for the algebraic approach was coming from works on conformal QFT_2 using the more standard formulation in terms of correlation function[9,10].

Since the methods of algebraic quantum field theory are subtle and only known to some experts (there exists no textbook or extensive review), a good physical motivation may be helpful. Recently there has been a growing desire to understand a new type of phase transition in which the correlation length (= inverse mass of particles) stays finite but the system aquires new quantum numbers which are not describable in term of Lie-group concepts. Accompanying the change of symmetry is a change of statistics. The (quasi-) particles in the new phase are "plektons" i.e. have a non-abelian braid group statistics or in more special cases "anyons" obeying abelian braid group statistics[42]. The former case is especially interesting since all known cases of plekton statistics are quantized whereas for *anyons* the statistical phase is a priori not quantized. The plekton case is more interesting than anyons

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since in addition to phases there are experimentally measurable weights. Anyonic sectors belong (as abelian B_∞^- statistics) to an abelian symmetry group. As an illustration consider the usual Coulomb-gas which is equivalent to the quantum Sine-Gordon system. It has been known for some time that this system has a series of phase transitions in the (Coleman) parameter region $4\pi < \beta^2 < 8\pi$ at quantized values of β^2 which accumulate at 8π.[11] Since these quantized values are precisely those at which the corresponding 6-vertex model permits the $RSOS$-solutions whose conformal limit[12] are the conformal minimal model for $C_{Virasoro} < 1$, the naive expextation would be that some of the plektonic statistics is already there before one take the massless limit at the special values of the coupling constant. As will be explained later, plektonic statistics is "dual" to a new symmetry concept which in the present case is expected to be a quantized symmetry which has similar properties to the original $U(1)$ invariances of the Sine-Gordon model before the symmetry and (statistics) changing phase transition (the new symmetry is expected to belong to a family which is mapped into itself under "quantized symmetry Fourier-Plancherel transform"). Analogous problems in $d = 3$ QFT arose in recent discussions of field theoretic descriptions of high T_c-superconductivity. These problems are extremely elusive from the Lagrangian approach, but, as we will see, algebraic QFT is much more powerful[13].

The already achieved successes of algebraic quantum field theory are more easily understood if one compares its method with the better known algebraic method of studying Verma moduls of the Virasoro algebra or the affine Kac-Moody (current) algebra. This method works with non- local generators L_n resp. J_n^i which are Fourier-coefficients of local fields. The algebra generated by the energy momentum tensor resp. current vector corresponding to finite intervalls on the light-cone form an Einstein causal net of (type III_1) von Neumann algebras[14] (on the light cone Einstein causality reduces to commutativity for non-overlapping intervalls). Algebraic QFT uses this causality property in an *essential way* and converts it into statistics of positive energy representation sectors. In fact it is a classification theory of possible superselection sectors and their braid group B_∞ statistics. Each possible statistics situation is expected to have precisely one realization in conformal QFT on the light cone. The details of the algebra, i.e. the value of the Virasoro charge C and the dimensional trajectories of the fields creating the various highest weight sectors are calculated (rather than assumed) in terms of the statistics representation theory. "Dual" to the statistics is the symmetry. In the case of P_∞-permutation group statistics the symmetry is always described in terms of a compact Lie group; this is the content of an important theorem of Doplicher and Roberts[15]. However, if, as in the case at hand, the B_∞-braid group statistics comes into play, the composition structure of positive energy sectors is described by a "quantized symmetry" (not a quantum group[16] as used in the recent literature). As will be made clear in the sequel, the new symmetry is best expressed in terms of a new tensor-calculus built on the universal hyperfinite II_1 Murray von Neumann factor R.[17] More explicitly the tensor spaces are a certain subcategory of $R - R$ bimodulus ("correspondences" in the terminology of A. Connes[18]). The fusion rules of this tensor-calculus are finite, although the tensor spaces themselves are infinite-dimensional with finite von Neumann dimensions. The mysterious "q-dimensions" of the finite-dimensional q-deformation "Quantum Group" approach⁻ are *true dimensions* in the new framework which harmonizes with their physical measurability.

It is amazing to what extend geometrical concepts still asses themselves: the prophetic terminology "continuous geometry"of Murray and von Neumann is more than appropriate. Following Ocneanu[19] we will call the symmetry going with the $R - R$ bimodule tensor-calculus "quantized symmetry" and the group- like object "quantized group". The word "quantum" here is used in its original physical meaning "discrete"or "rigid" given by Planck, Heisenberg and Schrödinger. In "Quantum Group" on the other hand the word quantum has the opposite meaning, referring to a q-deformation theory in which $\hbar$ in $q = e^{2\pi i \hbar}$ measures the distance from a group[16]. Quantized symmetries are related to certain Jones inclusions[19] and the conversion of inclusions of algebras into symmetries(the smaller one being considered as the fixed point algebra of some group-like object or the dual of such an object acting on the full algebra) is reminiscend of Galois' construction of finite groups from inclusions of (commutative)fields. Lie group symmetries are contained in the $R - R$ bimodule tensor-calculus as a special case. Only in this special case the bimodule tensor formalism may be converted (by absorbing R into the observable algebra) into the standard C- valued finite dimensional tensor formalism of unitary representation of compact Lie groups (see section II).

As in $d \geq 4$-dimensional QFT where the positive energy sectors of observable algebras are classified by compact Lie-groups[15], there are many theories (i.e. observable algebras) which realize a given superselection structure (i.e. for $d \geq 4$ many theories with a given Lie-group internal symmetry with Fermion- or Boson-statistics). One needs an additional principle to fix the theory. In the standard approach based on P_∞ statistics one starts with a free (linear) equation of motion for point-like fields. According to a well-known theorem[48] the internal symmetry together with the spacelike fermionic or bosonic commutation relation allow for only one free field solution of the free field equation. These fields then may be coupled to study perturbation interacting.
Postulating conformal covariance in QFT_2 is similar to study perturbative interactions. The theory decomposes into the light cone contributions and there can be no interactions for the associated one-dimensional QFT. In fact, there are even arguments that conformal QFT can (and should) be viewed as zero mass plektons, i.e. limits of free field theories with non-abelian braid group statistics. These plektons may also exist in integrable models. Integrability, i.e. the existence of factorizable purely elastic S matrix consistent with a given superselection structure (i.e. a prescribed plektonic statistics) constitutes another principle by which realizations of the new kinematics may determine a model uniquely. This and related problems will be briefly discussed at the end of these notes. It seems that ideas in QFT always return with a certain recurrance cycle. At the beginning of the 70's the concept of (what is nowadays called) conformal blocks appeared as a resolution[26] of Einstein causality paradox and the content of what are the algebraic structures which are behind the Knishnik-Zamolodchikov equations[51] surfaced as the nonabelian generalization of the Thirring model[52]. Recent algebraic QFT *directly relates* to these algebraic traditions of the 70's, however, without the rich analytic results of the 80's the confidence in the feasability of a classification program of the New Kinematics would be considerably lower. Another example of the recurrance phenomenon is the reappearance of *infinite component fields*, this time as the carriers of a $R - R$-bimodule tensor calculus related to the new symmetry.

Roughly speaking algebraic QFT can only incorporate ideas which have a sufficiently general conceptual physical basis (i.e. do not depend on analytic details) and the good message we offer here is that all model independent analytic observations in conformal QFT_2 do pass this test.

I. BRAID GROUP STATISTICS AS A CONSEQUENCE OF EINSTEIN CAUSALITY

The mathematical framework of algebraic QFT is that of Haag and Kastler: the observable algebra $\mathcal{A}$, a C^*-algebra, is defined in terms of a net of von Neumann algebras:

$$O \to \mathcal{A}(O), \quad \mathcal{A} = \overline{\cup \mathcal{A}(O)} \tag{1}$$

The smallest natural (Poincaré covariant, causally closed) net is obtained by taking for the O's the family of double cones in Minkowksi space-time

More general space-time regions are then obtained by approximations from double cones. The two *physical pillars* on which the theory is founded are Einstein causality and the physical spectrum condition (which since Dirac's "filling of the Fermi see" is considered an implementation of stability). The former is the requirement for the commutator:

$$[\mathcal{A}(O_1), \mathcal{A}(O_2)] = 0 \text{ for spacelike regions}: O_2 \times O_1 \tag{2a}$$

or the concept of commutants of von Neumann algebras and causal complement for space-time regions:

$$\mathcal{A}(O_1) \subset \mathcal{A}(O_2)' \text{ for} O_2 \subset O_1'$$

Its strengthened form is Haag duality:

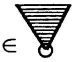
$$\mathcal{A}(O')' = \mathcal{A}(O) \tag{2b}$$

a property which can be derived for causal nets generated by a finite number of point-like (=Wightman-) fields.
The physical spectrum condition for $\mathcal{A}$ means that there exists a unitarily implementable translation automorphism:

$$\alpha_a(\mathcal{A}(O)) = \mathcal{A}(O+a) = U_o(a)\mathcal{A}(O)U_o(-a) \tag{3}$$

where $U_o(a) = e^{iP_{(o)}\cdot a}$ is a (continuous) group of unitary operators with a "physical" energy momentum spectrum:

$$spec\, P_{(o)} \in \tag{4}$$



and a unique (the unicity is no loss of physical generality!) invariant vacuum state $|o>$ at the tip of the forward light-cone in momentum space.
A representation π of $\mathcal{A}$ (*-homomorphism into a von Neumann algebra $\pi(\mathcal{A})$ obeying Einstein causality but generally not Haag duality) is called a PER (positive energy representation) if the translation is unitarily implemented with (4) (without an invariant state).

The main problem of algebraic QFT, as formulated by Borchers, is the classification of all PER. There exists only a partial solution of this program:
(i) DHR solution[2] (Doplicher, Haag, Roberts)
Assume the DHR localization:
For each π there exists a (sufficiently large) O_π with

$$\pi \restriction \mathcal{A}(O') \simeq \pi_o \restriction \mathcal{A}(O') \tag{5}$$

the equivalence being unitary equivalence.
(ii) BF solution[4] (Buchholz, Fredenhagen)
Assume the absence of zero mass particles (spectral gap assumption) and $d \geq 3$. Then the following BF "stringlike" localization is a consequence

$$\begin{aligned} \pi \restriction \mathcal{A}(S') \simeq \pi_o \restriction \mathcal{A}(S') \quad & \text{for all spacelike cones } S \\ S = a + \cup_{\lambda > o} \lambda O, \quad & S' \text{ causal complement of } S \end{aligned} \tag{6}$$

The core (center) of such spacelike cones are semi-infinite strings.
The unitary equivalence permits to use the same Hilbert space H_o for all PER representation π selected by the above principles. In the DHR case the use of Haag duality yields:

$$\pi(\mathcal{A}) \subset \pi_o(\mathcal{A}) = \mathcal{A} \text{ and an endomorphism of } \mathcal{A},\ \rho\colon \pi_o\mathcal{A} \to \pi(\mathcal{A}) \subset \mathcal{A}. \tag{7}$$

The endomorphism ρ is localizable ($\rho = id \ in \ O'$) and transportable by the translations:

$$\rho^{(a)} = \alpha_a \ o \ \rho \ o \ \alpha_{-a} = ad\ V_\rho^{(a)} \ o \ \rho \tag{8}$$

with

$$V_{\rho}^{(a)} = U_o(a)U_{\pi}^{-1}(a) \tag{9}$$

The construction of string-localized endomorphism in the BF case (ii) is considerably more subtle and for a reciew we refer to Fredenhagen's contribution[20]. Since our main illustration will be conformal QFT_2, we may restrict our consideration to the case (i) thanks to the following theorem by Buchholz, Mack and Todorov[21]:
Theorem: The DHR localization (i) is fulfilled for any conformal QFT_2.

The DHR localization generally excludes local gauge theories[4]. In fact it is very difficult to give an intrinsic characterization of what one means by local gauge theories. In the concluding remark, we will give some formal arguments, based on Witten's[22] recent path integral observations concerning "topological field theories" that the superselection structure obtained for the $d = 3$ BF classification theory may very well be the observable structure hidden behind the more formal local gauge principle.
The following theorem is crucial for the development of the new kinematics i.e. the classification of statistics and internal symmetry.
Theorem: [DHR]
(i) For $loc\rho_i \subset O_i$and$O_2 \times O_1$ we have

$$\rho_2\rho_1 = \rho_1\rho_2 \tag{10}$$

(ii) For arbitrary $loc\rho_i$ there exists a "topologically invariant " statistics operator $\varepsilon(\rho_1\rho_2)$ with

$$\rho_2\rho_1\varepsilon(\rho_1,\rho_2) = \varepsilon(\rho_1,\rho_2)\rho_1\rho_2 \tag{11}$$

Comments: The first property yields an abelian composition structure for the "sectors" i.e. the equivalence classes $[\rho]$ (ρ modulus *inner automorphism* of $\mathcal{A}$, often called *charge transporters*). Using the language of *fusion algebra* or *hypergroups*[23] which will be explained in the next section, this property means that QFT leads to an abelian hypergroup. The representation sectors of compact Lie-groups are special examples of abelian hypergroups. The second property (ii) says that there is a *natural* intertwiner (the meaning of the word natural will become clearer in the sequel). It goes with an explicit formula:

$$\varepsilon(\rho_1,\rho_2) = V^*V' \tag{12}$$

$$V = V_{\rho_2}^{(y)}\rho_2(V_{\rho_1}^{(x)}),\ \ V' = V_{\rho_1}^{(x)}\rho_1(V_{\rho_2}^{(y)}) \tag{13}$$

Here, $V_{\rho_1}^{(x)}$ and $V_{\rho_2}^{(y)}$ are charge transporters as in (9) which transport the endomorphisms ρ_1, ρ_2 to spacelike separated "spectator"-endomorphisms $\rho_1^{(x)}$ and $\rho_2^{(y)}$ i.e. with $loc\rho_2^{(y)} \times loc\rho_1^{(x)}$. The topological invariance of ε means that this natural transporter does not depend on the individual spectator endomorphisms but only on their equivalence class which for $d = 2$ consists of two elements, 2×1 and 1×2.
For $\rho_1 = \rho_2 = \rho$ the operator $\varepsilon_\rho = \varepsilon(\rho,\rho)$ is in the commutant of $\rho^2(\mathcal{A})$ and obeys the relation (for an elegant derivation see[20])

$$\varepsilon_\rho\rho(\varepsilon_\rho)\varepsilon_\rho = \rho(\varepsilon_\rho)\varepsilon_\rho\rho(\varepsilon_\rho) \tag{14}$$

Via the definition

$$\varepsilon^{(\rho)}(\sigma_i) = \rho^{i-1}(\varepsilon_\rho) \quad i = 1, 2, \cdots \tag{15}$$

one finally obtains a unitary representation of the Artin braid group B_∞. More generally applying different ρ_i's to $\varepsilon(\rho_1, \rho_2)$ one obtains a "coloured" braid groupoid. In cases of only one spectator class ($d \geq 3$ for DHR and $d \geq 4$ for BF) one obtains the additional relation $\varepsilon_\rho^2 = 1$ i.e. a unitary representation of the permutation group P_∞. In this case Doplicher and Roberts have[15] "categorized" the obtained sector and intertwiner structure and proven that this category is identical to a suitably defined C^* representation category of compact Lie groups (they had to develop a theory different from the "classical" Tanaka-Krein theory). Something similar can certainly be done for the B_∞ case, but in these notes we will stay more concrete.
A further important tool in the kinematical classification of QFT is the existence of left inverses of ρ [2,5,7].
Theorem: For each localized morphism there exists a unique "standard" left inverse ϕ. If the endomorphism is irreducible, the standard left inverse is the only possible left inverse.

A left inverse ϕ of ρ is a positive map $\mathcal{A} \rightarrow \mathcal{A}$ such that $\phi \cdot \rho = id$ and $\varepsilon = \rho \cdot \phi$ is a conditional expectation from $\mathcal{A}$ to $\rho(\mathcal{A})$. The meaning of standard is best understood in terms of a certain extremality property within the convex set of all left inverses[7]. For an irreducible endomorphism ρ the application of Schur's lemma yields the statistics parameter λ_ρ

$$\phi(\varepsilon_\rho) = \lambda_\rho \cdot 1 \tag{16}$$

and its iteration $\rho = lim_{n\rightarrow\infty}\phi^n$ leads to a Markov trace (i.e. a tracial state) on B_∞. For the case of two channels, i.e. if ρ^2 decomposes into two irreducible subrepresentations, the analysis of classifying all Markov states on unitary B_∞ representations constitutes a generalization of the P_∞ DHR analysis and has been carried out by Jones, Ocneanu and Wenzl[24]. The statistical dimension $d_\rho^{-1} = |\lambda_\rho|$ and the phase $\omega_\rho = d_\rho \lambda_\rho$ are characterized in terms of two integers d and q[5]. From this representation $B_\infty(2; d, q)$ one may form the representations corresponding to composite irreducible ρ_α which appear in the powers ρ^n by "strand formation" or "fusion" ("cabling" plus reduction). The number of eigenvalues appearing in $\varepsilon_{\rho_\alpha}$ is generally larger than two, but they obey special relations since ρ_α can be viewed as composite. From a QFT point of view the interesting problem is the classification of 3 and higher channel situation which cannot be obtained in this way. According to my best knowledge there have been no systematic attempts in this direction.
From what has been said up to now, the reader already may have obtained the impression that the statistics analysis is related to the Jones index theory. A theorem which brings out this relation in beautiful clarity is the following[7]
Theorem: (Longo): Let $loc\rho \subset O$, then

$$\text{Index } \ (\mathcal{A}(O) : \rho(\mathcal{A}(O)) = d_\rho^2 \tag{17}$$

On the left hand side the appropriate generalized (Jones restricted his analysis to II_1 factors) Jones index appears. This is (in analogy to the Atiyah-Singer index theorem) the analytic side, since the Jones index measures the size of $\mathcal{A}$ relative to the size of the image of the map ρ in terms of Murray-von Neumann-dimensions. On the other side (the topological side) appears the statistical dimension which orginated from the *quantum geometric concept* of Einstein causality (perhaps the deepest principle of 20th century physics). A notable difference with the Atiyah Singer index is the multiplicativity of Jones' index.
Inclusion is (hidden) *symmetry par excellence*. In fact, groups were invented by Galois in order to understand the position of (commutative) fields with respect to their Galois

extensions. For a physicist symmetry means tensor calculus. So one must convert the QFT "tunnel"

$$\mathcal{A} \supset \rho(\mathcal{A}) \supset \rho^2(\mathcal{A}) \supset \cdots \tag{18}$$

into a tower of finite dimensional algebras

$$M_n \; = \; \rho^n[\mathcal{A})' \cap \mathcal{A} \;\; , \;\; \mathcal{C} \subset \;\; M_1 \subset \;\; M_2 \cdots \tag{19}$$

In this way one generates a hyperfinite II_1 inclusion:

$$\rho(M) \subset \;\; M \;\; , \;\; M \; = \; R \; = \; lim_{n \to \infty} \, M_n \tag{20}$$

An important step in establishing this fact is the recognition that the previously introduced iterated left inverse φ defines a tracial state on the M_n's. For the important special case that this inclusion has a certain periodicity property which corresponds to $out \subset \rho^n$ (all models in conformal QFT_2 which have been studied up to now have this property of a certain power of ρ containing an outer automorphism) there are known methods[24]. A particularly well studied tower is the one which Jones affiliates with an inclusion $\rho(\mathcal{A}) \subset \mathcal{A}$. Again one first considers the QFT tunnel

$$\mathcal{A} \;\; \supset \;\; \rho(\mathcal{A}) \;\; \supset \;\; \rho\bar{\rho}(\mathcal{A}) \;\; \supset \;\; \rho \, \bar{\rho} \, \rho(\mathcal{A}) \;\; \supset \cdots \tag{21}$$

and checks that there exists a affiliated Jones algebra . Here, $\bar{\rho}$ is the conjugate (irreducible) endomorphism i.e. $\rho \, \bar{\rho} \; \simeq \; id \, \oplus \cdots$, $\bar{\rho} \, \rho \simeq \; id \, \oplus \cdots$ which according to the general theory exists in any QFT for finite statistical parameter d_ρ. In fact, $\phi(\mathcal{A}) \; = \; R^* \bar{\rho}(\mathcal{A}) R$ when R is the $(\rho \; \bar{\rho} \; , id)$ intertwiner. Longo recently showed that the endomorphism $\gamma \; = \; \rho\bar{\rho}$ can be alway constructed for any properly infinite conclusion (i.e. not necessarily a QFT inclusion). Similar to the previous case by forming relative commutants one may obtain a II_1 inclusion $\rho(M) \; \subset \; M$ with $M \; \supset \; \rho(M) \; \supset \; \gamma(M) \; \supset \; \cdots$, the Jones tunnel being the mirror image of the Jones tower. In this case there exists a rather developped theory for inclusions of finite depth[25]. The methods and concepts can be nicely related to the geometrical language of graph theory with parallel transport. The reader may consult Ocneanu's contribution[19]. The reconstruction of quantized symmetries in QFT requires to study slightly different graphs from Ocneanu's. They are directed graphs related to the fusion rules (hypergroup) generated by ρ^n rather than the undirected graphs going with the $(\rho \; \bar{\rho})^n$ and $(\rho \; \bar{\rho} \;)^n \; \rho$ Jones tunnel. Related to this difference is the fact that one wants to read the inclusion $\rho(\mathcal{A}) \; \subset \; \mathcal{A}$ as the fixed point part of another inclusion $\rho(\mathcal{F}) \subset \mathcal{F}$.An illustration in the II_1 setting would be to study the fixed points of:

$$R \; \subset \; M_d(R) \tag{22}$$

under the action of the representation of a "group-like object $\mathcal{K}$" i.e.

$$R^{\mathcal{K}} \; \subset \; M_d^{\mathcal{K}}(R) \tag{23}$$

Here, $M_d(r)$ denotes a matrix amplification of R explained in the next section. The classical example is:

$$R^G \subset (R \otimes \mathrm{Mat_n}(C))^G \tag{24}$$

where, if one takes the standard model $R = \otimes^\infty \mathrm{Mat_n}(C)$ (explained in the next section) the action of G (an irreducible subgroup of $SU(n)$) is the adjoint action on each factor $Mat_n(C)$. This inclusion is related to what Takesaki calls "Roberts actions" and it was referred to by Ocneanu as the McKay inclusion (for further remarks see next section). Let us finally explain a crucial theorem of algebraic QFT. This theorem relates the statistic phases ω_ρ to "spin phases". Here, we have to define what we mean by spin phase. In $d = 3$ the relevant spin concept is the one by Wigner (i.e. angular particle spin). Whereas in $d = 4$ the covering of the Lorentz group (or its rotation subgroups) is two-fold in $d = 3$ we have the infinite covering $\widetilde{SO(2,1)}, \widetilde{SO(2)}$. In this case the spin phase is $(-1)^{2j} = e^{2\pi i j}$. The covering situation for the two-dimensional conformal group is similar, in this case there are two phases corresponding to each light cone and the $j_\pm$ are the eigenvalues of the "conformal hamiltonian" $L_{o,\pm}$ which are always positive. Finally the "spin phase" in massive QFT_2 does not refer to the covering group but rather to the Lorentz-transformation properties of localized fields:

$$U(\Lambda)\, \psi_j(x)\, U^+(\Lambda) = e^{\chi j}\, \psi_j\, (\Lambda x)$$

with

$$\Lambda = \begin{pmatrix} cosh\chi, & sinh\chi \\ sinh\chi, & cosh\chi \end{pmatrix}$$

In that case the analytic continuation of the correlation functions leads to the complex Lorentz-group C^* of which j describes the nontrivial covering. The expected spin-statistics theorem is the statement that

$$(-1)^{2j_\rho} = \omega_\rho \tag{25}$$

In this strong from this has only been proven in conformal QFT_2 whereas in the other cases it was derived moduls a $\pm$ sign[20]. Another important (and related) formula is the statistical phase composition law, which, combined with the spin statistics theorem, implies a composition law for spin phases. Take first the anyonic case for a Z_n group. In this case the ε_ρ operator has only one eigenvalue say ω_ρ with $\omega_\rho^n = 1$. According to the general theory, the composite phases $\omega_m, m = 2, 3\cdots$ $(\omega_1 \equiv \omega_\rho)$ are obtained from the center C_m of B_m:

$$\omega_{c_m} = \frac{\omega_m}{\omega_1^m}, \quad \omega_{c_m} = \omega_1^{(m-1)m} \tag{26}$$

The first formula is general, whereas the second one uses the abelian nature of the model. The result is

$$j_m = \text{phase } \omega_m = \frac{m(m-1)}{n} + \frac{m}{n} = \frac{m^2}{n} \quad \text{mod } Z \tag{27}$$

The formula without the mod Z is the well known "Coulombgas"-formula. It had been observed first as the spectrum of the (compact) conformal hamiltonian for the expomential Bose (resp. massless Thirring) field by Swieca and myself in 1974[26]. Let us call (for obvious reasons) the relation (27) without the mod Z the *bound state realization* of the abelian spin-statistics theorem. The general spin-statistics theory only yields the bound state realization mod Z. However, in conformal QFT_2 which is the zero mass limit of massive anyons (section III, formula (66)) one expects only one realization of a given anyonic statistics by positive definite Wightman functions and that is the well-known exponential Bose-field $e^{\alpha\varphi}(\varphi =$ massless Bose field) with the strength α determined by the statistics. the same situation pertains for the plektonic 2-channel case. The Markov trace analysis only determines the statistics up to an anyonic factor which is then fixed by the periodicity of ρ. The arguments of the Hecke-Jones 2-channel statistics have been given in a paper of Rehren and myself[9]. One obtains the Kac-like formula mod Z:

$$\dot{\jmath}_m = h_m = \frac{(m-1)^2(q-1) - 2(m-1)}{4\,q} \qquad \text{mod } Z\,, \quad q \geq 4 \tag{28}$$

A similar formula can easily be derived for the general 2-channel situation. The point we would like to stress, is that a formula of this kind has nothing to do with the property of conformal invariance and rather holds for arbitrary interacting $d \leq 3$ theories as long as they realize the $B_\infty(2_i\ n,\ q)$ braid group representation and the fields carry no other internal quantum number (i.e. are *minimal* model realizations of the prescribed statistics). Victor Kac's formula without the mod Z seems to be always realized in conformal QFT_2 and more generally "free" plektonic theories. The reader may have realized that the formula (27) only contains part of the complete Kac-formula. For the Jones-Hecke 2- channel statistics the complete formula for minimal QFT's is:

$$\dot{\jmath}_{k,l} = \frac{(l\,q\ -\ k(q-1))^2\ -1}{4\,q(q-1)} \qquad \text{mod } Z \tag{29}$$

In conformal QFT_2 it corresponds to the BPZ "rectangle" - and can be derived within our statistics approach by the principle of *maximal extension* (soliton completeness). This is the principle which was used in special cases first by Kadanoff in connection with the Kramers-Wannier duality found in Ising-like models. From the point of view of algebraic QFT it is the problem of whether the model allows a nontrivial extension of the endomorphism ρ by another endomorphism ρ'. It can be shown that the irreducibility of the composite morphism $\rho'\rho$ requires an abelian braid group statistics between them. The application of the KWK duality principle to conformal minimal 2-channel model on the light cone gives precisely the BPZ rectangle. For the Jones-Hecke case this has been explicitly demonstrated in [9]. It is known that the KKW duality is related to the duality concept (Fourier-Plancherel transform) of the internal symmetry. Among Lie-groups, only abelian groups are autodual. Therefore one expects for anyonic statistics (in which case the sector-structure is described by an abelian group) no nontrivial KKW extension. However, plektonic statistics is related to a much larger framework of "quantized symmetries" i.e. related to a new bimodule tensor calculus for which the Plancherel-Fourier transformation technique is still in its infancy[19]. One scenario for a really profound understanding of the relation between the KWK duality principle and the BPZ rectangles would be that by Fourier-transform one obtains e.g. the quantized group related to one side of the rectangle from the quantized group belonging to the other side. Such a result would liberate the relation between the two structures from conformal covariance, and attribute a very fundamental significance to the (complete) Kac-formula.

Duality properties of the quantized symmetry are (as duality properties of abelian groups) of course only kinematical prerequisites for obtaining completely KWK selfdual situations in QFT. In massive $d = 2$ QFT there exists a theorem that order and disorder fields cannot carry superselecton sectors simultaneously[27]. However, conformal theories on the light cone as well as massive $d = 3$ theories allow for unbroken autodual models.

In order to obtain an easily managable QFT formalism, we now define the reduced field bundle and derive the exchange algebra. Choose a "frame" Δ_{red} in the observable algebra $\mathcal{A}$

i.e. a system of intertwiners T_e between a *fixed* set of irreducible endomorphism ρ_α, one for every class $[\rho_\alpha]$ by defining

$$\rho_\alpha \, \rho_\beta \, T_e \; = \; T_e \, \rho_\gamma \tag{30}$$

$e \; = \; (\rho_\alpha \, \rho_\beta \, \rho_\gamma)$: charge induction -(by ρ_β) reduction-edge.
The multiplicity (fusion) matrices $N_{(\beta)}$

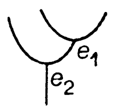
$$[\rho_\alpha \, \rho_\beta] \; = \; \sum_\gamma \; N_{(\beta)\alpha}{}^{\gamma} \, [\rho_\gamma] \tag{31}$$

suggests to interprete e as an $N \cdots$ component vector. The corresponding intertwiners T_e form an algebraic Hilbert space in $\mathcal{A}$, since according to Schur's lemma

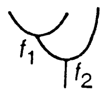
$$T_e^* \, T_e' \; = : (T_e, T_e') \cdot 1, \qquad (T_e, T_e') \; = \; < e, e' > \tag{32}$$

For three endomorphisms $\rho_\alpha \; \rho_\beta \; \rho_\gamma$ one can either perform the charge induction-restriction procedure from the right, in which case one obtains intertwining operators of the form:

$$T_{e_1 \cdot e_2} := \; T_{e_1} \; T_{e_2} \quad : \tag{33}$$



or perform the induction-reduction from the left hand side leading to:

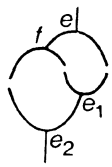


$$S_{f_1 \cdot f_2} := \; \rho_{s(f_2)}(T_{f_1}) T_{f_2} \quad : \tag{34}$$

where $s(f_2)$ means source (starting point) of the edge, the notation range $r(e)$ being also used. This process can be extended to longer path's and with the first method one obtains the T basis and in the second the S basis.

The graphical pictures for these basis are identical to those used by Brustein[28] in his categorical approach to the problem of characterizing the QFT unitary representation of B_∞ in terms of numerical matrices. The T and S basis can be related by numerical coefficients C and their conjugates D:

$$C_{e_1 \cdot e_2 \, ,(f,e)} \quad = \quad (S_{f \cdot e}, T_{e_1 \cdot e_2}) \quad = \tag{35}$$



The picture of conjugates is obtained by an upside-down mirroring.
Define the physical state space H_{red} as:

$$H_{red} \; = \; \oplus_\alpha \, H_{\rho_\alpha} \quad , \quad H_{\rho_\alpha} \; = \; \{\rho_\alpha, H_o\}. \tag{36}$$

and the action of the reduced field algebra $\mathcal{F}_{red}$ consisting of pairs $\{e, A\}$ with $A \in \mathcal{A}$ as:

$$\{e, A\} \; \{\rho_\alpha{}' \; , \; \psi\} \; = \; \delta_{s(e),\alpha'} \{\rho_{r(e)}, \; T_e^* \rho_{s(e)}(A) \psi\} \tag{37}$$

From this one easily derives the "operator product expansion" (or rather its pure algebraic aspect) as

$$\{e_2, A_2\}\,\{e_1, A_1\} = \sum_{f,e} D_{e_1 \cdot e_2 ; (f,e)}\,\}\{e, T_f^* \rho_{e_1}(A_2) A_1\} \tag{38}$$

Here, ρ_e is the ρ which "induces" i.e. the ρ in the middle (arrow) of the edge or in the above used intertwiner-pictures, the line emerging left upward. Analogously to the C-D coefficients, one can introduce the R-matrices in path space as the numerical matrix obtained by evaluating the $\varepsilon^{(\rho)}(b)$ braid group operators in the T (or S) basis[20]. Furthermore, the localization concept for the ρ's can be suitably generalized to elements of $\mathcal{F}_{red}$. One finally obtains the exchange algebra relations ($F_i = \{e_i, A_i\}$):

$$F_2 \cdot F_1 = \sum R_{e_1 \cdot e_2, e_2' \cdot e_1'}(\pm)\; F_1' F_2' \tag{39}$$

where on the right-hand side the prime only refers to the e_i and not the A_i. The + relation holds for $locF_1 \times locF_2$, whereas the $-$ applies to the opposite localization class. The associativity of the reduced field bundle formalism reproduces the well-known Artin relation, symbolically RRR = RRR as the star-triangle relation for matrices and yields the "pentagonal" relations RD = DRR which have the well-known fusion pictures. *We will not follow the standard terminology and use the name Yang-Baxter for the Artin relation.* The latter belong to the *new kinematics*, whereas Yang-Baxter relations (depending nontrivially on an additional variable) are much deeper and constitute already a part of dynamics. More on this isssue can be found in the last section.
The field-bundle construction is *bosonization par excellence.* All n-point function of the charge creating fields with their nontrivial statistics are represented in terms of intertwiners (which are bosonic since they belong to the observable algebra) and endomorphisms applied to the observable algebra. Symbolically:

$$< o|\; F_n \cdots F_2 F_1 |o> \; = \; < o|T^{\,s}, \rho(T)^{\,s} \text{ and } \rho(A_i)^s|o>$$

The reader may for himself compute e.g. the 4-point function using the previous composition laws of the field bundle acting on states. He may want to see correlation functions of point like fields. There are sufficiently general conditions (as e.g. asymptotic scale invariance) under which point like limits may be derived in the net $\mathcal{A}$[29]. Bosonization is not just a property of the massive Thirring model but a generic property of all field theories in any dimension. The speciality of the Thirring model and related model is that it can be described in terms of Bose fields obeying a simple equation of motion. The general statement is one of the main results of the DHR theory and it is older than the Sine-Gordon-Thirring equivalence theorem. I do not know any other example in theoretical physics, where a deep theory over such a long time has been so (almost)completely ignored[30]. Having said this, we should also add the remark that the specification of observable algebras and the study of their endomorphism is not a very practical tool. It shares the apparent inaccessability with the non-perturbative path integral approach. The strategy advocated here, however, is not to compute endomorphisms but rather classify all statistics and (quantized) symmetries allowed by Einstein causality and then to explicitly construct new sophisticated free fields carrying the new kinematics. From the path-integral approach these new free fields are highly non-perturbative, yet they do not pose any obstacle in the framework of algebraic QFT. They carry their observable algebras in the same sense as ordinary old fashioned permutation group statistics free field give rise to observable algebra. So instead of doing representation theory of new algebras (Virasoro, Kac-Moody and perhaps higher dimensional generalizations) one constructs new free field theory in the old fashioned spirit of QFT. However, even though one does not use representation theory of algebras for computational purposes, one needs the full power of the modern development in the theory of von Neumann algebra including the Jones theory in order to *find the computational tools* for the construction of the new field. In the last section we will analyse conformal QFT_2 from this point of view.

II. R-R BIMODULS CALCULUS AND NEW SYMMETRY

After the DHR work and before the announcement of DR theorem[15], there have been attempts to systematize the composition structure of DHR sectors (fusion rules) in terms of a mathematical object called'"hypergroup"[23]. More recently these concepts were reviewed and extended to the setting of R-R bimodules by Sunder[31]. Although this approach was not used in the proof of the DR theorem (which was based on Cuntz algebra techniques) the R-R bimoduls hypergroups together with more recent ideas of Ocneanu[19] turn out to be essential in the setting of plektonic statistics. Before we relate this to the symmetry problem in QFT a brief review of the main ideas is necessary.
Definition: A (discrete) hypergroup is a set G with a function $\mathcal{G}\times\mathcal{G}\times\mathcal{G} \rightarrow N \ (= 0,1,2,\cdots)$ denoted by $(\alpha,\beta,\gamma) \rightarrow <\gamma,\beta\otimes\alpha>$ that satisfies the following conditions:
(i) (local finiteness): for all α,β in $\mathcal{G}$, $<\gamma,\beta\otimes\alpha> \neq 0$ for only finitely many γ
(ii) (associativity): for all α,β and γ in $\mathcal{G}$

$$\sum_{\lambda} <\kappa,\gamma \otimes \lambda><\lambda,\beta \otimes \alpha> = \sum_{\lambda} <\kappa,\lambda \otimes \alpha \quad \lambda,\gamma \otimes \beta>$$

(iii) (identity): there exists an element $\underline{1}$ in $\mathcal{G}$ such that, for all $\alpha,\beta,\in\mathcal{G}$

$$<\beta,\ \alpha \otimes \underline{1}> = <\beta,\ \underline{1} \otimes \alpha> = \delta_{\alpha\beta}$$

(iiii) (contragredient) there exists a self-map of $\mathcal{G}$, $\alpha \mapsto \alpha^{\#}$ with

$$<\gamma,\beta \otimes \alpha> = <\beta,\gamma \otimes \alpha^{\#}>$$

for all $\alpha,\beta,\gamma,\in\mathcal{G}$.
It is clear what one means by an abelian hypergroup. A selfconjugate hypergroup is one in which every element is cyclic of order two: $<\underline{1},\alpha \otimes \alpha> = 1$.
It is easy to check that the duals of compact Lie groups have the structure of a hypergroup. In this case $\mathcal{G}$ is the collection of equivalence classes of irreducible representations π with $<\pi_3,\ \pi_2 \otimes \pi_1>$ denoting the multiplicity with which π_3 occurs in the tensor product and $\pi^{\#}$ the contragredient representation. These hypergroups are abelian and e.g. in the case of $SU(2)$, selfconjugate.
Theorem:[31] Each finite hypergroup admits a unique "dimension function" $\alpha \rightarrow d_\alpha$ from $\mathcal{G}$ to the non-negative numbers with the property

$$d_\beta\, d_\alpha = \sum_{\gamma} <\gamma,\beta \otimes \alpha>\ d_\gamma \tag{40}$$

The proof uses an auxilary matrix Λ with strictly positive entries and the associated unique Perron-Frobenius vector has the d_α's as its entries. For finite hypergroups it is furthermore easy to see that $\mathcal{G}$ with a cardinality n corresponds to n "fusion matrices" $(N_1,\cdots N_n)$ $N_i \in Mat_n(\mathcal{N})$ with $N_1 = 1$ and

$$N_i\, N_j = \sum_{k} N_i\,(k,j)\, N_k \tag{41}$$

The Perron-Frobenius vector of Λ is also a Perron-Frobenius vector of the N's with eigenvalues d_α.
It is helpful to picture a hypergroup $\mathcal{G}$ as a graph with the α's as vertices and the edges between vertices are arrows representing $< \gamma, \beta \otimes \alpha >$:
The number of arrows is given by the value of the γ bracket.
Not every graph features as the graph of a $\mathcal{G}$. For example, the D_{2n+1} and the E_7 Coxeter graphs cannot be completed (say by adding bigger edges) to a hypergroup graph[32,19], since they violate the existence of a unique dimension function. On the other hand graphs which are e.g. fusion graphs of the vector representation of $SU(n)$ (best pictured as simplices belonging to a Weyl chamber) retain the hypergroup structure even after they have been converted by a cut off (simplices which have equally sized boundary edges) into a finite graph.
As a result of Einstein causality the hypergroups belonging to the localized endomorphisms of observable algebras are always abelian. This property the new "plektonic kinematics " shares with the compact Lie-group representation theory. For QFT we only need R-R bimodule hypergroups[31] which we will define in the sequel. As a prerequisite one needs a good understanding of a particular von Neumann algebra: the unique hyperfinite II_1 factor R. A physicists model of R can be obtained in the following way. Consider the tensor product $lim_{n\to\infty} \otimes^n \; Mat_2(\mathcal{C})$ with the tracial state:

$$\varphi(a_1 \otimes a_2 \cdots) = \prod_i tr\, a_i \tag{42}$$

The state-space obtained by the GNS construction defines a representation space H_φ of the simple C^* tensor product algebra of $\otimes^\infty \; Mat_2(\mathcal{C})$. The state ρ has the property of being tracial, ie.e

$$\varphi(a^*a) \geq 0, \;\; \varphi(ab) = \varphi(ba) \tag{43}$$

Physically such tracial states on infinite tensor product matrix algebras correspond to temperature states of the Heisenberg chain in the limit $T \to \infty$. In the φ-state space one obviously obtains a left regular representation of the φ-closed von Neumann-algebra generated by $\otimes^\infty \; Mat_2(\mathcal{C})$ called R. It turns out that the commutant of the left regular representation R is the right regular representation i.e. we have the factor property:

$$(R_l)' = R_r \;\; i.e. \;\; R \cap R' = \mathcal{C} \tag{44}$$

It is a highly nontrivial consequence of the von Neumann algebra theory that every factor with a finite trace which was generated as an inductive limit of finite dimensional algebras is necessarily equal to the above physicists model of R. The above construction also provides the simplest example of a R-R bimodule. Clearly the Hilbert space $L^2(R,\varphi)$ obtained in the above construction from $R = \otimes^\infty \; Mat_2 \; (\mathcal{C})$ by closing with the φ-state, is a trivial R-R bimodule by defining $(a;b)\xi = a\xi b \; , \xi \in L^2(R,\varphi)$.

Note that on the right, the algebra R appears in the opposite order, i.e. as $R^{opp} : b_2 b_1 \xi = \xi b_1 b_2$. It is a classical result of Murray and von Neumann that every left R-module may be written in terms of the above standard module and a projection p in $L^2(R)$ as (we omit the symbol ρ since the normalized tracial state on R is unique):

$$H = L^2(R) \oplus \cdots \oplus L^2(R) \;\; \oplus \; p \, L^2(R) \tag{45}$$

This left module also has a natural right module structure: one considers the left module as a column and acts with the identical representation of R from the right. We obtained the normal form of an R-R bimodule with $dim_R H = n-1+tr\ p = d \leq n$ and $dim H_R = 1$, where the module dimensions introduced by Murray and von Neumann are Hilbert space dimensions. Let us now construct irreducible inclusions of R into itself. Define a matrix "amplification" of R as

$$p\,Mat_n(R)\,p \ \ , \ \ p = \text{projection in } Mat_n(R) \text{ with } dim\,p = \frac{d}{n} \leq 1 \tag{46}$$

This is again the hyperfinite II_1 factor, but this time in the form of a $Mat_d(R)$ model. Assume now that a faithful unital irreducible homomorphism

$$\rho(R) \ : \ R \rightarrow Mat_d(R)$$

is given ($tr\ \rho(1) = d$). Such an ρ leads to an irreducible Jones inclusion with an index = d. Two such homomorphisms ρ_i are equivalent if there exists a partial isometry given by a rectangular matrix u in $pM_{n,m}q$ with $\alpha_2(x) = u^*\ \alpha_1(x)u$ for all $x \in R$. They necessarily have the same d's and the corresponding Jones inclusion are conjugate. Homomorphisms given in terms of matrix-amplifications of R are called co-finite if d is finite. The smallest possible matrix size for a given d is $[d]$, the smallest integer bigger than d. The natural irreducible R-R bimodule associated with the co-finite irreducible homomorphism ρ (or the corresponding Jones inclusion) is now:

$$H = p\,Mat_n(R)q \ \ , \ \ dim\,p = \frac{d}{n}\ , \ \ dim\,q = \frac{1}{n}$$

which by $q = e_{11}$ (the 1-1 matrix unit) and an adjustment of p may be brought into the standard column form (45). Explicit specifications of nontrivial co-finite morphisms is fairly involved and requires the techniques of commuting squares or equivalent technical tools. Let us for the time being only consider the formal algebraic aspects.
It is easy to see, that the above fundamental R-R bimodule H contains the same information as the irreducible morphism ρ. Since the left action on H is through R-valued matrices, the situation resembles the action of matrix groups. However, to compute the homomorphism ρ which e.g. corresponds to the action of the vector representation of a group in R-R bimodule formalism is not an easy matter.
The ρ which corresponds to the tensor calculus of the vector representation of e.g. $G = SU(2)$ can be calculated[33] from the fixed point inclusion (24).
Here, the action of G on R is the adjoint action of $SU(2)$ on each factor of the standard model of R. Using Pauli matrices the generators for R, the calculation of the fixed points clearly amounts to the restriction of products of generators to invariant generators which are formed contracting $SU(2)$ invariant tensors (δ and ε) with Pauli matrices on different sites of a one dimensional chain. The averaging on the right hand side follows a similar statistical mechanism consideration. The invariant part R^G is again isomorphic to R and in order to convert the inclusion into the homomorphism ρ, one has to explicitly use this isomorphism.

We will not persue this problem of constructing concrete $R-R$ bimodulues attached to tensor representations of groups any further. But we would like to emphasize the importance of statistical-mechanics-inspired methods by which one can compute explicitly morphisma into amplifications which represent the known plektonic (abelian) hypergroups (e.g. using the relative tunnels (19) of QFT).

One also would like to understand better the type of obstruction which prevents abelian hypergroups to yield natural braid group representations (i.e. representations which are generated by applying the morphism ρ to the basic exchange intertwiner). A well-known illustration is the D_{2n} Coxeter graph which cannot be used for R matrices[9].

The original inclusion of which (24) was the fixed point part is the diagonal inclusion:

$$R \subset R \otimes \; Mat_n(C) \tag{47}$$

is easy to describe in $R - R$ bimodule language. Take:

$$H = \begin{pmatrix} L^2(R) \\ L^2(R) \end{pmatrix} \tag{48}$$

with the obvious R-valued inner product $\xi, \eta)_R$ defined on the dense set H_o of left and right R-bounded vectors. Clearly this $R - R$ bimodules may be described in terms of an R-valued Cuntz algebra basis:

$$\psi_i^* \psi_k = \delta_{ik} \underline{1} \;, \; \sum_{i=1}^{2} \psi_i \psi_i^* = 1. \tag{49}$$

With the left action of $Mat_2(R)$ on the basis being

$$U\psi_i = \sum_l \psi_l U_{li}, \quad U \epsilon Mat_2(R) \tag{50}$$

one finds that the restriction of $(Mat_2(R), R)$ to $(U_2(R), U_1(R))$ leaves the Cuntz-relations invariant. Calling this Cuntz algebra $\mathcal{O}_d(R)$, one would arrive at the famous Doplicher Roberts[15] result, if the R could be absorbed into $\mathcal{A}$ and in this manner obtain the Cuntz algebra $\mathcal{O}_d(C)$ with the C valued inner product, i.e. the algebra which results from the algebraization of an ordinary finite dimensional tensor calculus. The semicanonical basis $R - R$ bimodule basis are quite different from ordinary vector-space basis) of the $\mathcal{O}_d(R)$ formalism is the traded for a canonical basis of the usual tensor formalism. The possibility of doing so appears very plausible if one looks at the field theoretical origin of R from sequences of relative commutants in the observable algebra. R is obtained as the tracial closure of finite string (observable) algebra generated by $R_\xi T_{\xi'}^*$ where T_ξ are intertwiners (affiliated with the edge-path ξ) in the observable algebra. The only problem is whether the limit elements in the tracial topology can also be considered as belonging to the III_1 von Neumann algebra $\mathcal{A}(\mathcal{O})$ with OF sufficiently big, or whether the limit elements lead to an enlargement. Perhaps Longos'[7] methods will resolve this problem and lead to a rigorous argument. In any case, the formal argument gives the correct DR formula in the case of P_∞ statistics[15]. However, for the nonabelian braid group statistics the non-integer d in $Mat_d(R)$ does not permit the removal R. In this case there is simply no possibility of trading the $R - R$ bimodule formalism for an ordinary Hopf-Kac-von Neumann algebra acting on finite-dimensional tensor spaces. In fact it is easy to check that the $R - R$ bimodule formalism goes over into a hopf-Kac-von Neumann algebra if one *replaces the* R which appears in $\otimes_R$ and the co-unit *by the complex numbers.*[34]. The tensor formalism becomes indentical to the Reshetikhin tensor formalism[34], in fact it is the Cuntz algebraization of that finite-dimensional Hilbert-space formalism.[15] This trick, however, works only for the representation theory of compact Lie-groups which have integer dimension functions and not for more general hypergroups. In fact, already for the most simple of all plektonic models, namely the Ising model with the abelian hypergroups (dimension function: $1, \sqrt{2}, 1$):

$$[\rho_\sigma^2] = [id] + [\rho_\psi], [\rho_\psi^2] = [id], [\rho_\sigma \rho_\psi] = [\rho_\sigma] \tag{51}$$

there is no possibility of passing from the $R - R$ bimodule calculus to a $Mat_2(C)$- ordinary tensor calculus. Only if one gives up the physical positivity and allows more general Hopf algebras which are not von Neumann algebras (non- semi simple with singular radicals) one can formally do it. In this case th edimensional function has nothing to do with von Neumann dimensionas and becomes a "q-dimension". The indefinite metric of the Reshetikhin formalism manifest itself in the property that the tracial state is given in terms of a complex matrix which in ***only*** a positive tracial state on the ***commutant of the tensor-products*** but not on the quantum group itself. The appearance of indefinite metric in a generalization of a Wigner symmetry is intolerable. In other words, if one want to use B_∞ not only for knot-theory but also for quantum physics, the $R-R$ bimodule calculus is the correct way of incorporating plektonic hypergroups. We will leave the setting up of $O_d(R)$ Cuntz algebra, in which the defining relations (49) are modified by the $Mat_n(R)$ projector p as well as the study of the amplification homomorphism $R \underset{\rho}{\longrightarrow} Mat_d(R)$ for a separate treatment.

Although the $R-R$ bimodules are infinite dimensional, they have a finite number of generating elements over the observable algebra $\mathcal{A}$. However, as a result of the non-removability of R from the formalism, the fields are expected to be some sort of infinite component fields, i.e. they carry a R lattice aspect in addition to the space-time labeling.

The Clebsch-Gordan coefficients of the bimodule co-product induced by the homomorphism ρ on the semi-canonical basis should be indentical to the [q]-deformed Clebsch Gordan coefficients at the relevant q-values on the unit circle. However, in the $R-R$ bimodule formalism the "$RSOS$ cutoff" (i.e. the finiteness of the superselection sectors) is buildt-in and does not have to be imposed from the outside.

The perfect harmony with quantum principles raises the hope that cross-product constructions may also be helpful to handle indefinite metric problems elsewhere.

Besides the intrinsic logic of algebraic QFT one of the original motivation for being interested in this $R-R$ bimodule formalism was the desire to understand the enigmatic "Russion Coulomb-gas" in terms of a new symmetry concept[11] in QFT. In order to be successful on this point, one still needs a more explicit form of the bimodule formalism.

III. CONFORMAL QFT_2 AND FREE PLEKTRONS

The requirement of conformal covariance leads to a subtle change in the definition of localized endomorphisms. The physical reason for this is precisely the same as for the curious observation which twenty years ago was termed the "Einstein Causality Paradox of Conformal QFT".

In 2-dimensiona conformal QFT the invariance group of the vacuum is the tensor-product of two Möbious-groups M to the two light cones with coordinates $s_\pm = t \pm x$:

$$M = \frac{SL(2,R)}{Z_2} \tag{52}$$

acting fractionally on each light cone. The observation was that the formal global transformation law involving the parabolic subgroup of special conformal transformation $x \longrightarrow \frac{x-bx^2}{\sigma(b,x)}$, e.g. of a scalar field:

$$U(b)A(x)U^+(b) = \left(\frac{1}{\sigma(b,x)}\right) \tag{53}$$

for noncanonical values of d_A is in contradiction to causality. The resolution given in 1974 by Swieca and myself[26] was precisely the conformal block decomposition theory which 10 years later was rediscovered by other methods together with a rich supply of nontrivial illustrations.[46] Written in modern terminology the solution of the paradox was the decomposition formula:

$$A(x_+, x_-) = \sum_e A_{e_+}(x_+)A_{e_-}(x_-) \tag{54}$$

Here the A_e are fields on the light cone with charge induction reduction edges between the eigenspaces of the centers of the universal covering $\tilde{M}$-representation which are generated by unitary operators $Z(Z_{\pm})$:

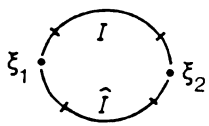
$$Z = \sum_{\alpha} e^{2\pi i h_{\rho\alpha}} P_{\rho_\alpha}, \; Z = e^{2\pi i L_o} \tag{55}$$

In the last formula we used the Virasoro notation. The global covariance laws for the A_e's are very simple. Apart from a subtle phase factor,[26] they coincide with the form of the naive formula (7).

Algebraic QFT starts from the observable algebra on each light-cone. The incorporation of the M-automorphism requires the compactification of the light cone to $R_c = S^1$. This, however, is in apparent conflict (an extension of the previous paradox) with the distinction between right and left which was so crucial for the B_∞ versus P_∞ statistics. The resolution is reminiscend of Fredenhagen's presentation of the $d = 3$ case.[20]. Instead of considering the map $\rho : \pi_o(\mathcal{A}) \to \pi(\mathcal{A})$ in $\mathcal{H}_o$, one defines the endomorphism on the directed net obtained by removing one point ξ

$$\mathcal{A}_\xi = \cup_{O \not\ni \xi} \mathcal{A}(O) \tag{56}$$

The O's are in this case simply intervalls on the light cone, and the point ξ corresponds to the point "at infinity". The endomorphisms ρ^ξ defined on $\mathcal{A}_\xi$ have all the desired properties. Using two different subalgebras $\mathcal{A}_{\xi_i}$ in $B(\mathcal{H}_o)$ and computing the monodromy operator ϵ_M as illustrated in the following picture: (by transporting ρ with $loc\rho \in I$ to $\hat{\rho}$ with $loc\ \hat{\rho}\epsilon\hat{I}$ first on the right-hand side



(in $\mathcal{A}_{\xi_1}$) and then returning on the left-hand side (in $\mathcal{A}_{\xi_2}$)), one obtains:

$$\varepsilon_M = \rho^{\xi_2}(V)^{-1}\rho^{\xi_1}(V) \text{with } V \epsilon \mathcal{A}_{\xi_1} \cap \mathcal{A}_{\xi_2} \tag{57}$$

There is no reason why $\varepsilon_M = 1$, as it is for the P_∞ statistics. The representations π and their compositons as well as the unitary representation of the covering of the Möbiusgroup in each sector do not depend on the spectator point ξ (which is shifted by conformal automorphism).

Since the theory is scale-invariant, the relation between the local von Neumann algebras (for which one has all the recently developed powerful theorems) and the covariant point-like fields which generate these algebras (for which one has the better calculation techniques of QFT) is particularly simple.[29] Take an element F from the reduced field bundle such that it relates the vacuum to the lowest. L_o state ψ_α in the ρ_α sector:

$$\langle \psi_\alpha | F | 0 \rangle \neq 0. \tag{58}$$

Then, if loc F contains the origin, a covariant primary field at the origin may be defined by a scaling limit:

$$F(0) = \lim_{\lambda \rho 0} \lambda^{-d_\alpha} U_{\rho_\alpha}(\lambda) F U^+_{\rho_\alpha}(\lambda) \tag{59}$$

So the main question is: can one compute a unique set of positive definite conformally covariant correlation functions which are consistent with a given unitary braid group representation? For $B_\infty(2,d,q)$ the investigations on the 4-point functions suggest strongly that the answer is positive if one uses the Kac formula solution (i.e. without the mod Z) of the Spin-Statistic theorem. In that case the solution of the Riemann Hilbert-problem with the monodromy given by the spin and statics theorem turns out to be unique: the short distance singularities given by the spin and statics theorem turns out to be unique: the short distance singularities assigned to the 4-point function (by lifting the mod Z restriction in (29)) only allow for one solution. For the Jones B_∞ representation this follows from the known results, for the general $B_\infty(2,d,q)$ case with $d > 2$ this observation and the explicit 4-point-function for the KWK duality extended theory can be found in recent work of Lima-Santos and myself[35]. As in the case $d = 2$ the KWK duality combines the $B_\infty(2,d,q)$ and the $B_\infty(3,d,q \pm 1)$ representations. The c-value belonging to this family can be computed and turns out to be:

$$c = (d-1)\left[1 - \frac{d(d+1)}{q(q+1)}\right]. \tag{60}$$

This illustrates our view of the problem: the algebra with its unspecified structure constant is only used as a setting for Einstein causality and the statistics analysis; all algebras at unphysical (infinite statistics) values of their structure constants are dismissed.

A very intersting problem is the classification of all local 2d fields which can be formed from one KWK-complete exchange algebra i.e. the reversed problem to (54). Here the only published results are based on modular invariance.[36,38] Algebraic QFT attacks this problem directly and works for any value of c allowed by the statistics analysis. We will present the results elsewhere.

It is very intersting to compute the temperature partition functions of this class of models.[35] Algebraic QFT provides a very pretty technique: the KMS temperature formalism.[37] The application of this formalism to the temperature correlation function

$$\langle \rho_\alpha | F_e(x) F^*_{\hat{e}}(y) | \rho_\alpha \rangle_r := \frac{1}{Z} tr|_{H_{\rho_\alpha}} e^{2\pi i r L_0} F_e(x) F^*_{\hat{e}}(y) \tag{61}$$

($\hat{e}$ = conjugate charge-induction-restriciton edge) gives a quasi-periodicity in matrix form. Introducing the compact coordinate $x - y = \tan \pi\alpha$, the quasi-periodicity for $\alpha \to \alpha + 1$ is given in terms of a diagonal matrix $\tilde{T}$ whereas the $\alpha \to \alpha + r(r = i\beta)$ quasi-periodicity is expressed in terms of a non-diagonal matrix $\hat{S}$ obtained from the R-matrices which occur in the 4-point function. $\hat{S}$ and $\hat{T}$ have the same eigenvalues and the two quasi-periodicity relations together with the boundary contition:

$$\langle \rho_\alpha | F_e(x) F^*_{\hat{e}}(y) | \rho_\alpha \rangle_r \underset{\beta \to \infty}{\longrightarrow} 4 - \text{point function} \tag{62}$$

leads to a unique solution for the temperature correlation functions (61). By the same space-time limiting procedure as that for the 4-point function one can then obtain the one point function $\langle \rho_\alpha | T | \rho_\alpha \rangle$ of the energy-momentum tensor which is the logarithmic (with respect to r) derivative of the partition function Z_{ρ_α} of the conformal theory on one light cone. Their modular matrices S and T follows from the two-point function matrices $\hat{S}$ and $\hat{T}$. The following observations[35] on the result of the QFT-KMS calculations are interesting:

1) The functions Z_{ρ_α} only agree with the Rocha-Caridy characters for $C_{\text{Virasuro}} < 1$. Above this value one obtains a finite set of Z_{ρ_α} (assuming the finiteness of the statistics dimension) which transform among each other under modular transformations and allow the formation of invariant partition functions in the $d = 2$ theory. As in the calculation of the 4-point functions, no a priori knowledge on the observable algebra (how it contains the energy momentum tensor algebra) is necessary. The observation of Cardy[38] on the appearance of infinite sums of the Rocha Caridy characters in the decomposition of $c_{\text{Virasoro}} \geq 1$ theories is, although mathematically correct, physically not applicable.[39] The statistics analysis selects

the "good" values of c at which the statistics dimension is finite and the formal Verma module representation theory can be related to QFT. Indeed, the form of the QFT characters Z_{ρ_α} which carry the statistics for $c > 1$ is very similar to the $c < 1$ characters. They are less simple than the Rocha Caridy expression for $c > 1$, but they retain the finiteness with respect to the modular decomposition of the partition function.

2) The matrices S and T from a matrix representation of the modular algebra. For the selfconjugate (Jones-algebra) statistics situation, the nondiagonal matrix fulfills the Verlinde property: it is an orthogonal matrix which diagonalizes the complete set of fusion matrices N. In the algebraic QFT approach this is understood as originating from the KWK duality. The matrix T represents charge measurers and the matrix S charge transporters. This property can be followed back to the very fundamental level of duality property of a general nature, whereas the observations on modular invariance appears as some special analytic feature in conformal QFT_2 and does not easily reveal that it is the manifestation of a general duality principle which by far transends the importance of conformal QFT_2.

Conformal QFT_2 is the simplest *explicit* analytic realization of x-space exchange algebras. It shares with all zero mass theories the conceptual difficulty that the state space cannot be constructed in terms of particle states. This makes the positive definite Hilbert-space structure a very subtle issue, a fact well known from model discussions. The best strategy is to approach the problem as a limit of free massive theories. Here the conceptual problems are simpler, the reason being the validity of (Haag-Ruelle) scattering theory is a very powerful tool derivable from the first principles. Applied to the exchange algebra, scattering theory yields a formula[13] for the inner product of e.g. in-states:

$$\langle p_1', \dots p_n', \xi' | p_1 \dots p_n, \xi \rangle = \sum_{\sigma, \pi \epsilon p_n} R^{(\beta)}_{\xi' \xi}(b_\sigma^{-1} b_\pi) \prod_i \langle p'_{\sigma(i)} | p_{\pi(i)} \rangle \tag{63}$$

Here (as already in the exchange algebra) the groupoid structure of B_n over P_n has been used. ξ', ξ is a pair of path's of n subsequent charge-induction-reduction edges starting at the vacuum and ending at the sector ρ_β (such a pair is often called a closed (algebraic) string). The positivity of the inner product is a consequence of the tracial state property of QFT Markov traces and the positivity of the one particle spaces. It is easy to translate this inner product into an exchange algebra formalism for creation and annihilation operators[41] in momentum space (with an additional $\delta(p' - p)$ contribution for the relation involving one creation and one annihilation operator with the conjugate charge edge):

$$a^*(p, e), a(p, \hat{e}), \quad \hat{e} \text{ conjugate to } e.$$

The crucial question now is: can one introduce localizable operators fulfilling x-space exchange algebra relations, such that

$$\langle 0 | A(x_i, e_1') \dots A(x_n, e_n') | p_1 \dots p_n, \xi \rangle \ , \ e : \text{ edges} \tag{64}$$

have all the properties one expects from "free" fields?

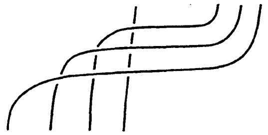
This question is also non-trivial for anyons in which case the R's are just $U(1)$ phases.[42] In that case it has been known for a long time that the p-space anyonic commutation properties can be fulfilled in ordinary Fock space[43] by starting, e.g. with fermions for particles and antiparticles: $a(p), a^+(p), b(p), b^+(b)$ defining

$$\hat{a}_\alpha(p) =: a(p) e^{i\alpha \int_p^\infty n(p) dp} : \tag{65}$$

where $n(p)$ is the number operator density in momentum space. In this case the answer to the above question is affirmative and the explicit free anyone formula in x space is always of the form (even if one would have started with Bosons):

$$A_\alpha(x) =: \mathrm{lin}(a, a^+, b, b^+) e^{\mathrm{bil}(a, a^+, b, b^+)} : \tag{66}$$

with known momentum space kernels depending on α. Probably the first such free anyonic model was the Federbush model.[44] The application of the scattering theory to this model gives a constant S-matrix which is precisely the contribution from the "mirror braid". This is the desired property



expected for the wave functions (64): the necessary crossing of particles with different velocities in comparing asymptotic contributions has to be taken into account in defining the word "free". The Federbush model is decribed in terms of two species of fermions. To obtain anyons in terms of just one field we form the local limit of a free field disorder operator:

$$\mu(x) =: \exp \text{ bilin.}(a, a^+, b, b^+) : \tag{67}$$

(which performs "half-space" U(1)-rotation $e^{i\alpha}$ on a free spinor ψ and therefore has the Bogoliubov Valatin form (67)) together with the ψ, and thus obtain the explicit formula[45] for the anyonic operator (66) in terms of just one set of particle-antiparticle creation and annihilation operator. An additional simplification can be reached if one uses the $\hat{a}$ operators as the reference operators.

For plektons, the Hilbertspace defined by (63) is non-trivially embedded in the tensor-product-Fock-space and momentum-space Klein-transformations of the type (65) do not exist. The main obstacle in generalizing the x-space construction to plektrons is the appearance of the edges and path's of edges which cause an algebraic complication. A closed form of the result in terms of an exponential expression can only be expected if one converts the path-formalism into a $R - R$ bimodule tensor formalism. This remains to be done.

Note that the massive QFT_2 cannot fulfill the KWK duality between charge measurers and charge transporters. This is the main content of the Köberle-Marino-Swieca theorem which states that for $d = 2$ one needs the vanishing of the gap in order that order *and* disorder create sectors obeying the KWK duality (this is not so for $d = 3$). Therefore in case of massive plektons one expects that for $m \to 0$ the resulting light-cone theories have two $B_\infty(2, d, q)$ representations instead of one. The BPZ rectangular situation would come about in the zero mass limit in which the Hilbert-space structure is completely changed. This picture is certainly consistent with the known massive Ising model which only has two sectors and can be described in terms of ordinary fermionic statistics. The plektonic aspect only emerges in the $m \to 0$ limit where the Fermionic situation of two sectors together with the Jones plektonic statistics on 3 sectors form the $2x3$ BPZ rectangle. By analogy one would expect the free massive q=3 Jones-Hecke plektons to yield the 3×4 BPZ rectangle, the longer line always representing the number of charge measurers.

Massive theories do not permit an infinite statistical dimension. Hence restricting conformal QFT_2 only to those which can be obtained by massive ones in the limit $m \to 0$ (other conformal theories either do not exist or are physically unintersting), the *classification of conformal* QFT_2 *is in principle solved:* they are classified by superselection sectors with B_∞ statistics. *Every superselection structure allowed by the general principles has precisely one conformal realization.* The remaining classification problem is of a general kinematical nature and has nothing to do with conformal invariance. The various conformal algebras (Virasoro, Kac-Moody etc.) algebras are not only Einstein causal, but they also fulfill Huygen's principle which for $d = 2$ eliminates interactions so that the new kinematics can be studied in its simplest analytic realization. Such algebras should be expected to emerge as the observable algebras of free massless plektons or anyons.

In algebraic QFT one only uses observable algebras and endomorphism for structural reasons, all practical calculations are either done on the level of free plektons or on the level of classifying factorizable S-matrices which are allowed by a given spin-statistics situation. This means that one looks for unitary solutions of the *true* energy (or rapidity) dependent *Yang-Baxter-equation* which approach asymptotically for $\theta \to \infty$ given B_∞ representation.

At this conference some very startling and puzzling observations on integrable models as massive extension of conformal QFT_2 were presented. It would be interesting to apply the methods of algebraic QFT to this situation. In general for a given set of superselection rules and braid group statistics one would expect many QFT, however, if one restricts to free plektons or plektons with a given S-matrix (having the correct $\theta \to \infty$ asymptotics) one expects a unique QFT.

More interesting problems with direct physical applications are those in $d = 3$[41]. The existence of free x-space plektons in the sense of a constant S-matrix.

$$S = R \text{ (mirror braid).} \tag{68}$$

would open truly spectacular possibilities of constructing integrable models. They would violate the Coleman-Mandula No-Go Theorem which is based on analytic properties of the S-matrix coming from ordinary P_∞ statistics fields.

CONCLUDING REMARKS

The mathematical sophistication of algebraic QFT may have an attractive appeal. But this aspect is in common with other theoretical Ansätze as string theory, whose mathematical entertainment value is also undeniably high. The unique charm of algebraic QFT consists in the proximity of the abstract and concrete. Consider for exampel the issue of high T_c superconductivity. In the old BCS theory with the $U(1)$ symmetry and Fermi statistics, the relation between the critical temperature T_{cr} and the gap $\triangle_o$ at $T = 0$:

$$T_{cr} = c\triangle_o \tag{69}$$

contains a universal constant c. For the hypothetical case of a SU(d) (instead of U(1)) broken symmetry a simple modification of the BCS argument reveals the appearance of an additional weight factor d^2. Our consideration of plektonic statistics and quantized symmetry suggest to modify the relation (69) by a Jones factor d_ρ^2 where ρ belongs to a Jones-Hecke B_∞ statistics, following the observation that plektonic modifications in other observable quantities, (e.g. cross sections) are by statistical dimensions or link invariants at those places where in case of a SU(d) model d^2 or higher traces appear.

Whereas a phase transition which leads from the U(1) many-body Schrödinger situation to a SU(d) situation appears inconceivable, a transmutation of the U(1) situation into the autodual quantized symmetry related with the $B_\infty(2,2,q)$ representation is perfectly palatable (e.g. the Coulomb-gas phase transition mentioned in the introduction). This would lead to the exciting possibility of measuring Jones indices (squares of statistical weights) in the "Jones gap" between the BCS value and the value for T_c which is four times bigger (the smallest one which can be obtained by nonabelian groups). Such enhancement factors within the generalized Hartree-Fock BCS-like approximation cannot result from anyons which only lead to changes of statistical phases. Statistical phases should be very important in the explanation of magnetic properties of high T_c superconductors and the fractional Hall effect.

One enters a new world of plektonic statistical mechanics and judging on the importance of Fermions and Bosons, I find the possibilities created by the new kinematics truley breathtaking. It should be clear that the new kinematics is very important. If none of the new possibilities turn out to be realized in nature one should understand why nature does not realize braid group statistics in two dimensional layers.

Algebraic QFT casts an interesting light on 3d gauge theories. Already Buchholz and Fredenhagen speculated that their string fields constructed in algebraic QFT may be the intrinsic description of "nonabelian gauge theories". As became clear through recent investigations, 3-d string theories allow for a realization of the new kinematics. Is the new kinematics perhaps characteristic for the observable content of 3-d nonabelian gauge theories? A strong additional support comes[46] from Witten's recent observations[22] on topological field theories (these are path integrals representing the pure superselection structures $\triangle_{red}$ without the space-time aspects). Topological field theories are defined in terms of Chern-Simons Lagrangians using nonabelian gauge fields.

Looking only at the applications given in these notes, the reader may obtain the impression that even though algebraic QFT creates a rich world of new kinematics for low dimensional theories, it leaves everything as it used to be in 4d QFT's. This may not be so. Although the structure of superselection sectors (with the exception of the zero mass "infraparticle" issue) have been settled by the DR and BF theories, there is the fascinating and important issue of spontaneous symmetry breaking on which algebraic QFT casts a new light[49]. The broken theories may be "quantized" in the same sense as the plektonic statistics is quantized and they should be approached by new algebraic concepts rather than Higgs deformation theory.

Finally, there is the observation of Roberts[50] that the issue of superselection structure is equivalent to operator 1-cohomology and that operator 2-cohomology also may be related to deep physical concepts. In this context we would like to add the formal remark that spacelike surfaces may also be braided, but the resulting braid group does not fulfill the Artin "hexagonal" relation but rather an octagonal equation. Fields having a surface-like localization and a "hyperplektonic" exchange algebra certainly do not belong to PER's and the usual spin statistic situations, but composition of these fields could be "conventional" PER fields. The reader may guess what object would be a candidate for such a non-measurable (i.e. non-PER) "hyperplekton".[11]

It should be amply clear by now that whereas the Lagrangian approach is basically a deformation approach working with such concepts as: local gauge fields, the BRST formalism, a deformation picture (coset-spaces) of broken symmetries and (perhaps) Quantum Group deformations, algebraic QFT on the other hand is based on: algebraic crossed product construction (instead of classical differential geometry and topology), observable new kinematics (e.g. plektons related to Jones inclusion), a new bimodule tensor calsulus, and(Haag-Ruelle or LSZ) scattering theory. Algebraic QFT tries to discover new islands (e.g. free fields) without perturbation or deformation theory. The latter is only used to explore the infinitesimal vicinity around these new islands. It is very powerful on classification problems but still weak if it comes to questions of the physical content of *specific* Lagrangian field theories. It also gives rise to perplexing philosophical questions like: why does Einstein causality in QFT, a principle governing the temporal-spacial order, lead to a classification of (quantized) internal symmetries?

In a conference like this one, usually the successful marriage between geometry and physics is eulogized. My heretic opinion on this issue is that this marriage in this decade (as in the previous one) was very one-sided. For mathematicians it has brought a rich harvest: the Donaldson theory, Witten's Morse theory and Floer's theory, to name just a few results. Physicists on the other hand went completely empty-handed. Starting with the instanton gas and ending with the more recent string theory, none of these proposals worked in physics and we are left at the end of this decade only with some flying dutchman which do not find a physical landing place.

In fact the best marriage between physics and mathematics was the old one in which von Neumann, Weyl, Dirac , Wigner and Bargman played such a prominent role. It is to be hoped, that with the legacy of von Neumann refined by new beautiful results (of Connes, Jones, Ocneanu, Longo, Popa, Pimsner, Takesaki, Wenzl, and others), a new marriage may also bring success to physics.

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References

1 R. Haag and D. Kastler, J. Math. Phys. 5, 848 (1964).

2 S. Doplicher, R. Haag and J. Roberts, Comm. Math. Phys. 23, 199 (1971) and 35, 49 (1974).

3 H.J. Borchers, Comm. Math. Phys. **121**, 251 (1965).

4 D. Buchholz and K. Fredenhagen, Comm. Math. Phys. **84**, 1 (1982).

5 K. Fredenhagen, K.H. Rehren and B. Schroer: Superselection sectors with braid group statistics and exchange algebras I General Theory (to appear in Comm. Math. Phys.), II (to be published).

6 V. Jones, Invent. Math. **72** (1983), Ann. Math. **126**, 335 (1987).

7 R. Longo: Index of subfactors and statistics of quantum field I (to appear in Comm. Math. Phys.), II, University of Rome preprint 1989.

8 For an early accoutn of the recent development see B. Schroer "Algebraic Aspects of Non-Perturbative Quantum Field Theories", Como 1987 in Differential Geometric Methods in Theoretical Physics, ed, K. Bleuler and M. Werner.

9 K.H. Rehren and B. Schroer, Nucl. Phys. **312**, 3 (1989) and references therein.

10 J. Fröhlich: Statistics of fields the Yang-Baxter equation and the theory of knots and links, Proceedings of the 1987 Cargèse School.

11 This illustration is taken from B. Schroer: New Concepts and Results in Nonperturbative Quantum field theory, in Differential Geometric Methods in Theoretical Physics, Chester 1988, Editor Allan I. Solomon, where also the appropriate references can be found.

12 For a recent reference: M. Karowski, Yang-Baxter-Algebra-Integrable Systems – Conformal Quantum Field Theories, ETH-TH/89-13.

13 B. Schroer: High T_c Superconductivity and a New Symmetry Concept in Low dimensional QFT, FU Berlin preprint, April 1989.

14 D. Buchholz, C. D'Antoni and K. Fredenhagen, Comm. Math. Phys. **111**, 123-135 (1987).

15 S. Doplicher and J. Roberts: C^* algebras and duality for compact groups: why there is a compact group of internal gauge symmetry in particle physics, Proceedings Marseille 1986 (eds. M. Mebkhout and R. Séneor) World Scientific 1987.

16 V.G: Drinfeld: Quantum groups, Proceedings ICM (1986).
M. Jombo. Lett. in Math. Phys. **11**, 247 (1986).

17 J. von Neumann, Ann. Math. **50** (1949), 401-485 and references to previous publications.

18 A. Connes, Correspondences, hand-written notes.
S. Popa, Correspondences, INCREST preprint 1986.

19 A. Ocneanu: Quantized Groups, String Algebras and Galois Theory for Algebras, Dept. of Mathematics Research Report, Pennsylvania State University, State College 1988.
Talk given at this conference.

20 K. Fredenhagen, talk given at this conference.

21 D. Buchholz, G. Mack and J. Todorov: The current algebra on the circle as a germ of local field theories, DESY 88-126 (preprint).

22 E. Witten, Comm. Math. Phys. **121**, 351 (1989).

23 J.R. McMullen and J.F. Price: Reversible hypergroups, Rendi. del Sem. Mat. e. fis. de Milano XL VII (1977), 67-85,
K.A. Ross: Hypergroups and centers of measure algebras, 1st. Naz. di Alta Mat. Symposia, Math. Vol XXII (1977), 189-203.

24 A. Ocneanu (unpublished), H. Wenzl, Invent. Math. **92**, 349 (1988).

25 S. Popa: Classification of subfactors: the finite depth case, preprint 1989.

26 B. Schroer and J.A. Swieca, Phys. Rev. **D10** (1974) 480. See also B. Schroer, J.A. Swieca and A.H. Völkel, Phys. Rev. **D11** (1975) 1509.

27 R. Köberle and E.C. Marino, Phys. Lett. **B126**, 6, 475 (1983).

28 R. Brustein, lecture given at this conference.

29 K. Fredenhagen and J. Hertel, Comm. Math. Phys. **80**, 555 (1989).

30 Even in recent attempts on $d = 3$ Lagrangian bosonization the authors are unaware that the conceptual problems had been solved by DHR.

31 V.S. Sunder: II_1 factors, their bimodules and hypergroups, University of Bangalore, Indian Statistical Institute preprint (1989).
Sunders fundamental conjecture is that every hypergroup has precisely one R-R bimodule realization.

32 This observation Ocneanu attributes to V. Pasquirer, see ref. 19.

33 A. Wassermann, talk given at the U.S. - U.K. joint conference on Operator Algebras at Warwick in July 1987, University of Liverpool preprint.

34 Yu. Reshetikhin: Quantized Universal Enveloping Algebra, The Yang-Baxter Equations and Invariants of Links I, II, Lomi preprint 1988.

35 A. Lima Santos and B. Schroer, in preparation as a FU preprint.

36 A. Capelli, C. Itzykson and H.B. Zuber, Nucl. Phys. **B280**, 445 (1987).

37 For local light-cone fields this formalism has been used in ref. 21.

38 J. Cardy, Nucl. Phys. **B270**, [FS 16], (1986) 186.

39 I thank H. Saleur for directing my attention to this point.

40 K. Fredenhagen: Structural Aspects of Gauge Theories in the Algebraic Framework of Quantum Field Theory, University of Freiburg preprint 1982, THEP 82/9.

41 Many of the recent (and yet unpublished) results on the description of plektons in particular for $d = 3$ have been obtained in collaboration of K. Fredenhagen, K.H. Rehren, R. Schrader and myself.

42 Anyons in configuration-space or path-space formulation have a long history:
J.M. Leinaas and J. Myrheim, Il Nuovo Cimento **37B**, 1 (1977),
F. Wilszek, Phys. Rev. Lett. **48**, 1144 (1982),
F. Wilszek and A. Zee, Phys. Rev. Lett. 51, 2250 (1983) **49**, 975 (1982),
Y.S. Wu, Phys. Rev. **53**, 111 (1984),
J. Fröhlich and P.A. Marchetti, Comm. Math. Phys. **121**, 177 (1989).
All of these treatments are in configuration- or path-space. No operator construction has been presented in $d = 3$ dimensions.
For an interesting account including the histroy of abelian braid group statistics in $d = 2$ dimensions (including Klaibers 1967 work) see
J. A. Swieca: Fields WIth Generalized Statistics: An Exercise in Order- and Disorder in two Dimensional Systems, PUC preprint 1980, published in the 1980 Karpacz Winter School proceedings.

43 B. Schroer and J.A. Swieca, Nucl. Phys. **B121**, 505 (1977).

44 B. Schroer, T.T. Truong and P. Weisz, Ann. Phys. **102**, 156 (1976).

45 The Bogolinkov-Valatin part of the anyonic operators appears at many places: B. Schroer, Nucl. Phys. **295** [FS 21], (1988) 586, formula (2.22) and references.

46 This was discussed by J. Fröhlich at the 1989 Ringberg meeting.

47 A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).

48 R.F. Streater and A.S. Wightman: PCT, Spin And Statistics And All That, Theorem 4-15, W.A. Benjamin, Inc. 1964.

49 J.E. Roberts: Spontaneously Broken Gauge Symmetries and Superselection Rules, Proceedings of the International School of Mathematical Physics, Camerino 1974, University di Camerino 1976.

50 J.E. Roberts, Comm. Math. Phys. **51**, 107 (1976).

51 V.G. Knishnik and A.B. Zamolodchikov, Nucl. Phys. **B247**, 83 (1984).

52 B. Schroer in: A Trip to Scalingland, Lectures given at the V Brasilian Symposium on Theoretical Physics, Rio de Janeiro, January 1974, Vol. I, Editor Erasmo Ferreira, R. Dashen and V. Frishman, Phys. Rev. **D11**, 2781 (1973). In both papers, the algebraic version of the K-Z equation appears for the first time.
The old papers on string theory also were much more algebraic and there is a lot in common with the beginnings of conformal QFT_2:
K. Bardakci and M.B. Halperin, Phys. Rev. **D3**, 2493 (1971) ,
M.B. Halperin, Phys. Rev. **D4**, 2398 (1971).

INFINITE INDEX EMBEDDINGS

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ABSTRACT

Infinite Index Embeddings are discussed in the framework of quantized space and its quantized symmetry groups.

NON-COMPACT CURRENT ALGEBRAS AND HETEROTIC SUPERSTRING VACUA

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Introduction

It has been shown that a four dimensional finite and consistent superstring theory [1] can be constructed by combining any 2-dimensional superconformal field theory with a central charge c=9 with a superstring propagating in flat 4-dimensional Minkowski space-time. Each distinct superconformal field theory corresponds to a possible vacuum state of the full string theory. For N=1 supersymmetry to emerge in this vacuum in 4-dimensions, an N=2 worldsheet supersymmetry is necessary [2]. Gepner has provided [3] a procedure for the construction of a heterotic superstring that satisfies the required properties including modular invariance by using the above ingredients.

In this talk we exhibit the relevance of *non-compact* super Kac-Moody symmetries to superstring theories. Our discussion is based on a recent publication [4] in which new classes of unitary conformal and superconformal theories based on cosets of affine non-compact current algebras are suggested. Unitarity restrictions and the structure of the modules labelled by the primary states are discussed in general terms as well as in detail for SU(1,1)/U(1), SU(N,M)/SU(N)xSU(M)xU(1) and SL(2,C)/SU(2). Large classes of new N=2 superconformal theories are classified and their central charges computed. This gives the non-compact counterpart of the Kazama-Suzuki models. It is shown that compact group and non-compact group Kazama-Suzuki models can be rewritten as coset models of the form (GxH)/H, where H is a maximal compact subgroup of G and G/H is Kählerian. This reveals new symmetry structures that are useful in computations. It is shown that, in applications to heterotic superstring model building in 4-dimensions, a $c \leq 9$ compact space can be

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constructed only from two non-compact N=2 super affine cosets based on $SU(1,1)_{-\hat{k}}$ with $\hat{k} \geq 3$ and $SU(2,1)_{-\hat{k}}$ with $\hat{k} \geq 9$. After performing a GSO projection and heterotic conversion *à la* Gepner, the massless spectrum of the c=9 $SU(1,1)_{-3}$ case and the number of families is analysed in detail.

Non-Compact Current Algebra and Unitarity

Let us consider a non-compact Kac-Moody algebra associated with a non-compact group G

$$[\hat{J}_{nA}, \hat{J}_{mB}] = if^{C}_{AB}\,\hat{J}_{n+m,C} - n\frac{\hat{k}}{2}\eta_{AB}\,\delta_{n+m}, \tag{1}$$

where η_{AB} is taken proportional to the Killing metric and is defined by

$$g_{AB} = f^{D}_{AC} f^{C}_{BD} = g\eta_{AB}. \tag{2}$$

note that we have inserted an i in the definition of the structure constants and a minus in the definition of $\hat{k}$. It is useful to first consider a basis in which $\hat{J}^{\dagger}_{nA} = \hat{J}_{-n,A}$, with real f^{C}_{AB}, η_{AB} and $\hat{k}$. We can choose conventional normalizations of generators such that $\eta_{AB} = diag\ (1,1,\cdots,-1,-1,\cdots)$, with the positive entries associated with the compact generators and the negative entries associated with the non-compact ones. Then g which is equal to the eigenvalue of the quadratic Casimir operator in the adjoint representation is also equal to the Coxeter number. With our definitions compact and non-compact groups have identical Coxeter numbers. For example, for SU(N,M), $g = N + M$.

The compact generators form a maximal compact subalgebra which we will denote as H. When we specialize the index A to this subalgebra we will use the low case letter a while for the remaning coset generators we will use the greek letter α. Thus in this basis we have

$$\begin{aligned}
[\hat{J}_{na}, \hat{J}_{mb}] &= if^{c}_{ab}\hat{J}_{n+m,c} - n\frac{\hat{k}}{2}\delta_{ab}\delta_{n+m}, \\
[\hat{J}_{na}, \hat{J}_{m\beta}] &= if^{\gamma}_{a\beta}\hat{J}_{n+m,\gamma}, \\
[\hat{J}_{n\alpha}, \hat{J}_{m\beta}] &= if^{c}_{\alpha\beta}\hat{J}_{n+m,c} + if^{\gamma}_{\alpha\beta}\hat{J}_{n+m,\gamma} + n\frac{\hat{k}}{2}\delta_{\alpha\beta}\delta_{n+m}
\end{aligned} \tag{3}$$

In order to see what sort of coset we can admit in the construction of a unitary theory, it is useful to consider the large $\hat{k}$ limit. Renormalizing each current by a factor of $\sqrt{\hat{k}}$ and taking a large $\hat{k}$, we see that the first term in (1) becomes negligible and the Kac-Moody algebra tends to an algebra of oscillators accociated with $U(1)$ currents. For positive $\hat{k}$, we see that the oscillators associated with H have negative norm while those accociated with G/H have positive norm. Thus, we see that in the limit $\hat{k} = \infty$ the full Hilbert space is simply the Fock space of these oscillators. A unitary Fock space will be possible by throwing away the negative norm oscillators, namely by considering the coset space $(U(1))^{dim\,G}/(U(1))^{dim\,H}$. In the finite $\hat{k}$ case the non-abelian nature of the algebra is not negligible, but it is intuitively evident that we have a chance of avoiding negative norm states provided we consider the coset G/H, where H is the *maximal* compact subgroup of G. The infinite $\hat{k}$ limit of the conformal theory based on this coset will reduce to that of $(U(1))^{dim\,G}/(U(1))^{dim\,H}$, which is evidently unitary. There is no other coset whose large $\hat{k}$ limit will be unitary without any further constraints.

The energy-momentum tensor in the conformal theory is now given by [1,5,6,9]

$$\hat{T}_G(z) = \eta^{AB}\hat{J}_A(z)\hat{J}_B(z)/(-\hat{k}+g) \tag{4}$$

and the central charge c_G is given by

$$\hat{c}_G = \hat{k}\; dimG/(\hat{k}-g). \tag{5}$$

We see that as far as the structure of the energy momentum tensor in (4-5) is concerned, the difference between a compact group and a non-compact group amounts to inserting the appropriate metric η_{AB} and changing the sign of $\hat{k}$. Of course other than this, the main difference is that the unitary Hilbert spaces are very different in the two cases. Indeed, as emphasized above, the Hilbert space of the non-compact Kac-Moody algebra is certainly non-unitary in the limit of infinite $\hat{k}$ (if there are any unitary representations of non-compact Kac-Moody algebras they must not have a $\hat{k} \to \infty$ limit). Therefore we shall explore the coset G/H, with H the maximal compact subgroup, which could have a unitary Hilbert space as argued above.

The G/H conformal theory is constructed according to the GKO scheme which produces an energy-momentum tensor $T_{G/H}$, whose property is that it *commutes with the currents in H* and hence also commutes with $\hat{T}_H$. This property allows one to relabel the states of the Kac-Moody theory based on G in terms of the quantum numbers of the Kac-Moody theory based on H and the conformal theory based on $T_{G/H}$.

$$|G - state> \rightarrow |H - state> \otimes |\hat{T}_{G/H} - state> . \tag{6a}$$

The expectation is that although the G-states may not provide a unitary representation of the G Kac-Moody algebra, the $\hat{T}_{G/H}$ states could provide a unitary representation for the G/H conformal theory. Again, this is easy to see in the infinite $\hat{k}$ limit in which the G-states correspond to the non-unitary Fock space of all the oscillators, the H-states are those constructed only from the negative norm oscillators while the $\hat{T}_{G/H}$-states are constructed from only the positive norm oscillators (and hence unitary).

At arbitrary values of $\hat{k}$, we label the primary states at the *base* of the G-module by a representation R of the group G and decompose it into representations r of the subgroup H,

$$R \to \sum \oplus r, \tag{6b}$$

and compute the dimension $h_r(R)$ of the $T_{G/H}$ state (eigenvalue of $L_0^{G/H}$

$$h_r(R) = \frac{c_2(R)}{-\hat{k}+g} - \frac{c_2(r)}{-\hat{k}+h}. \tag{7a}$$

where $c_2(R)$, $c_2(r)$ are the eigenvalues of the quadratic Casimir operators for G and H respectively. Furthermore, g, h are the coxeter numbers for the groups G and H.

There are G/H primary states that emerge at the excited states of the G-module. The dimensions of these states are also computed following similar decompositions of G-states into H-states times G/H states. If the level of the G-state above the base is labelled by the integer l_G, then the new primary states that emerge at this level have dimensions $h = h_r(R) + l_G$, with $h_r(R)$ as above, but now the representations r that emerge at this level have to be computed.

The central charge of the G/H theory is given in the GKO scheme as

$$c_{G/H} = \frac{\hat{k}\ dimG}{\hat{k}-g} - \frac{\hat{k}\ dimH}{\hat{k}-h.} \tag{7b}$$

With this information at hand we may impose unitarity conditions at arbitrary values of $\hat{k}$ as follows. We first require that at the *base* all G/H states have positive norm. Since the compact subgroup $|H - states>$ have positive norm *at the base*, we must require that the $|G - states>$ also have positive norm *at the base*. This means that the representation R must be unitary.

At higher levels above the base the H-currents that behave like negative norm oscillators (for $n \geq 1$) produce negative norm states in the G-module or H- module. In order to determine whether the representation for the G/H theory is unitary and also determine the zero norm structure of the module we need to analyse (i) the multiplicity and norm of each new primary state $|h>_{G/H}^{(0)}$ and (ii) the norm of its descendants $|h>_{G/H}^{(l_{G/H})}$. The second part is easy, since the unitarity of the module is completely determined by the conditions

$$c_{G/H} \geq 0, \qquad h \geq 0, \tag{8}$$

which follow from the positive norm of the state $\hat{L}_{-n}^{G/H}\,|h>_{G/H}^{(0)}$, i.e. from demanding

$2nh + (n^3 - n)c_{G/H} > 0$ for n=1 and n sufficiently large. We apply this condition to the states at the base, which yield

$$\frac{\hat{k}\ dimG}{\hat{k}-g} - \frac{\hat{k}\ dimH}{\hat{k}-h} \geq 0, \qquad -\frac{c_2(R)}{\hat{k}-g} + \frac{c_2(r)}{\hat{k}-h} \geq 0, \tag{9}$$

for every r in R. These were applied to SU(1,1)/U(1), SU(N,M)/SU(NxSU(M)xU(1) and SL(2,C)/SU(2) [4]. Below we illustrate how the values of $\hat{k}$ and the representations R get restricted.

These unitarity conditions are necessary but it has not been demonstrated that they are sufficient in all cases. The SU(1,1) case has been studied independently by Dixon, Lykken and Peskin [8] who systematically analysed the norm of excited G-states. Demanding positive norm G/H states at all levels they did not find any new conditions on the representation R or the value of $\hat{k}$, other than the ones produced by our simple argument above. Thus we shall conjecture that the above necessary conditions are also sufficient conditions for unitarity of the entire G/H module. To check this conjecture a detailed investigation of SU(2,1) has been undertaken.

Examples: SU(1,1) AND SU(2,1)

To illustrate the unitarity conditions we first consider SU(1,1), whose Kac-Moody algebra can be written in the form

$$[\hat{J}^0_n, \hat{J}^0_m] = -\delta_{n+m}\, nk/2 \qquad [\hat{J}^0_n, \hat{J}^\pm_m] = \pm\hat{J}^\pm_{n+m} \qquad [\hat{J}^+_n, \hat{J}^-_m] = -2\hat{J}^0_{n+m} + nk\delta_{n+m}\,. \tag{10a}$$

At the base, a representation $|R>=|jm>$ of G=SU(1,1) may be labelled by the real eigenvalue m of $\hat{J}^0_0$ and the complex number j that parametrizes the eigenvalue of the Casimir operator $c_2(R) = j(j+1)$. We must consider only unitary representations of SU(1,1) which are

(1) The principal series, $j = (-1+i\rho)/2$, $-\infty < \rho < \infty$ and $-\infty < m < \infty$,

(2) The supplementary series, $j = (-1+\sigma)/2$, $0 < |\sigma| < 1$, and $-\infty < m < -\frac{1}{2}(1+|\sigma|)$ or $-\frac{1}{2}(1-|\sigma|) < m < \frac{1}{2}(1-|\sigma|)$ or $\frac{1}{2}(1+|\sigma|) < m < \infty$,

(3) The positive discrete series, $m = j+1, j+2, j+3, \cdots$,

(4) The negative discrete series, $m = -(j+1), -(j+2), -(j+3), \cdots$.

In the discrete series $j+1 \geq 0$ is real, but not necessarily quantized. However, we will see that the GSO projection will require it to be half-integer. In the region $-1 \leq$

$j \leq 0$ there are both discrete and supplementary series representations. They are distinguished from each other since $|m| \neq j+1$ in the supplementary series (actually the end points $j = -1, 0$ are excluded from the supplementary series.) Similarly, for $j = -1/2$ there are discrete, supplementary and principal series representations; in the discrete series $|m|$ is half-integer, but not so in the supplementary or principal series. These special points are important in the application to superstrings (see later).

The dimension $h_m(j)$ of the G/H states that emerge at the base are

$$h_m = \frac{-j(j+1)}{\hat{k}-2} + \frac{m^2}{\hat{k}}, \tag{11}$$

and the central charge is

$$c_{G/H} = \frac{3\hat{k}}{\hat{k}-2} - 1.$$

Applying now the unitarity conditions discussed above we find

$$\frac{3\hat{k}}{\hat{k}-2} - 1 > 0, \qquad \frac{-j(j+1)}{\hat{k}-2} + \frac{m^2}{\hat{k}} \geq 0. \tag{12}$$

Thus we find that $\hat{k} > 2$ and no conditions on ρ or σ of the principal or supplementary series since $-j(j+1)$ is positive in these cases. For both discrete series there is a further condition on j, namely,

$$(j+1) < \hat{k}/2 \qquad \hat{k} > 2. \tag{13}$$

Note that $j, \hat{k}$ are not quantized in terms of integers, unlike compact affine algebras. This is the full set of conditions derived in [8] from an iterative analysis of all higher levels, while in our approach the conditions are obtained directly from data at the base.

We apply a similar analysis for SU(N,M). The coset is

$$G/H = \frac{SU(N,M)_{-\hat{k}}}{SU(N)_{-\hat{k}} \times SU(M)_{-\hat{k}} \times U(1)}$$

and the central charge is then

$$
\begin{aligned}
c_{G/H} &= \frac{\hat{k}((N+M)^2-1)}{\hat{k}-N-M} - \frac{\hat{k}(N^2-1)}{\hat{k}-N} - \frac{\hat{k}(M^2-1)}{\hat{k}-M} - 1 \\
&= \frac{(\hat{k}^2-1)MN(2\hat{k}-N-M)}{(\hat{k}-N-M)(\hat{k}-N)(\hat{k}-M)}.
\end{aligned}
\tag{14}
$$

of the several possible ranges of $\hat{k}$ that produce a positive central charge, only

$$
\hat{k} > N + M \tag{15}
$$

is consistent with the positivity of the dimension h as well. The unitarity conditions and the permissible representations of SU(N,M) are complicated and cannot be given here for lack of space. The details may be found in ref.[4]. Here we only record the SU(2,1) case, for which the states are labelled as $|a,\tilde{b};jm;y>$ where $a,\tilde{b}$ are labels related to the quadratic and cubic Casimir operators, j,m label the SU(2) subgroup and y labels the hypercharge (which is normalized to $diag(\frac{1}{3},\frac{1}{3},-\frac{2}{3})$ in the the 3-dimensional fundamental representation). For SU(2,1)/SU(2)xU(1) the dimension of the primary states that appear at the base of the G-module are

$$
h_{jmy}(a,\tilde{b}) = -\frac{C_2(a,\tilde{b})}{\hat{k}-3} + \frac{j(j+1)}{\hat{k}-2} + \frac{3y^2}{4\hat{k}}
$$

where the numerical factors in the last term are due to the normalization of the hypercharge generator defined above. The quadratic Casimir operator is given by $C_2(a,\tilde{b}) = \frac{1}{3}[(a^2+\tilde{b}(\tilde{b}+a-3)]$. There are unitary discrete series as well as continuous representations for SU(1,1) for various ranges of the parameters $a,\tilde{b},j,y$ which have been analyzed in [4] and in greater detail in [12]. In physical applications in string theory it is the discrete series that are relevant (see later) for which $a,\tilde{b}$ are positive integers ($a \geq 0, \tilde{b} \geq P \geq 1$). In the positive discrete series, for $P=1$, we have $j=(n+a)/2, y=n+(a+2\tilde{b})/3$, with $n=0,1,2,\cdots$. For $P \geq 2$, we have $|n-a|/2 \leq j \leq (n+a)/2$ and $y=n+(a+2\tilde{b})/3$ with $n=0,1,2,\cdots$. In the negative discrete series y has the opposite sign. For the unitarity of the affine module, the conditions discussed above restrict further the allowed representations to those discrete series which satisfy [4]

$$
\hat{k} > 3, \qquad 0 \leq a \leq \hat{k}-2, \qquad (1-\lambda_a)\hat{k} \leq a+2\tilde{b} \leq (1+\lambda_a)\hat{k}, \tag{16}
$$

where $\lambda_a = \sqrt{1-[a(a+2)/\hat{k}(\hat{k}-2)]}$. This result will be useful in determining the number of families.

We see that the conditions on the allowed representations are quite complicated, but that the general effect of unitarity is to limit the size of the representation by limiting the possible values of the integers $a, \tilde{b}$. The effect is similar for the discrete representations of SU(N,M) [4].

Superconformal Symmetry and the Coset $G_{-\hat{k}} \otimes H_{g-h}/H_{-\hat{k}+g-h}$

Kazama and Suzuki [7] have constructed a class of N=2 superconformal theories by using coset structures of *compact* super current algebras. We have generalized this construction to non-compact current algebras. Furthermore, for both compact and non-compact cases we have re-written these theories in terms of a simpler coset structure of the form GxH/H that sheds more insight into the structure and symmetries of these conformal theories and also helps in computations.

In addition to the bosonic currents $\hat{J}_A$ of the previous section we now include free fermions Ψ_A, with $A = (a, \alpha)$, that commute with $\hat{J}_A$. It is useful to construct a purely fermionic current $j_A^G = (i/(\hat{k} - g)) f_{AB}^C \eta^{BD} \Psi_D \Psi_C$. This current satisfies commutation rules identical to those of $\hat{J}_A$ except that the level $\hat{k}$ is replaced by g, the Coxeter number of the group G. The energy-momentum tensor and supercurrent density are then given by

$$T_G(z) = \frac{\eta^{AB}}{-\hat{k}}(\hat{J}_A \hat{J}_B - \Psi_A \partial \Psi_B) \qquad G_G(z) = \frac{2\eta^{AB}}{-\hat{k}}(\hat{J}_A + j_A^G/3)\Psi_B. \tag{17}$$

which differs from the compact case of [7] only by changing the sign of $\hat{k}$ and inserting the metric η^{AB} in the appropriate places. The central charge for this superconformal system is

$$c_G = (\frac{1}{2} + \frac{\hat{k}}{(\hat{k} - g)}) dimG. \tag{18}$$

Another way of representing the energy-momentum tensor of free fermions is to construct the noncompact currents of $SO(dimH, dimG/H)_1$ as bilinears in the fermi fields $j_{AB} = (2/(\hat{k} - g))\Psi_A \Psi_B$ and then taking a Sugawara form as a normal ordered product of these currents (contracting the indices by using η^{AB}). There is still another way of representing the energy momentum tensor of a free fermion, namely

$$T_{G\Psi} = \frac{\eta_{AB}}{2g} j_A^G j_B^G = \frac{\eta_{AB}}{k} \Psi_A \partial \Psi_B, \tag{19}$$

where in the first expression the normal ordering is with respect to the currents and in the second expression the normal ordering is with respect to the free fermion oscillators. Such identities follow from Wick's theorem for normal ordered fields. Using this form we can rewrite the energy momentum tensor as the Sugawara theory for

$$G_{-\hat{k}} \otimes G_g . \tag{20}$$

We now select the currents $\hat{J}_a$ and fermions Ψ_a that belong to the subgroup H. In addition, from the fermions Ψ_α in the coset G/H we consruct a current K_a in the subgroup

$$j_a^H(z) = \frac{i}{k} f_{abc} \Psi^b \Psi^c \qquad K_a(z) = \frac{-i}{k} f_{a\beta\gamma} \Psi^\beta \Psi^\gamma \tag{21}$$

where we have followed the convention of structure constants given earlier. The currents H and K commute with each other. The level of the current j_a^H is h, the Coxeter number for the subgroup H. The sum of these two currents is $j_a^G = j_a^H + K_a$, which has level g. This allows us to compute the level of the K-current as $g - h$. Next we define another H-current $\tilde{J}_a = \hat{J}_a + K_a$, whose level is $(\hat{k} + g - h)$. Since the two parts of this current both commute with Ψ_a the total current $\tilde{J}_a$ also commutes with the fermions Ψ_a in the subgroup H. We then construct the N=1 superconformal system for the subgroup H from the currents $\tilde{J}_a$ and fermions Ψ_a, as in (17)

$$T_H(z) = \frac{-1}{k}(\tilde{J}_a \tilde{J}_a - \Psi_a \partial \Psi_a) \qquad G_H(z) = \frac{-2}{k}(\tilde{J}_a + j_a^H/3)\Psi_a, \tag{22}$$

Just as above this system may be reconstructed from the Sugawara theory based on the affine algebras $H_{-\hat{k}+g-h} \otimes H_h$. In many applications the subgroup H is in the form of direct products $H = \prod_i \otimes H^i$. Going through the same reasoning as above we deduce that (22) should be replaced by the sum over all the subgroups contained in H, and furthermore that this expression would be associated with the currents of the affine system whose levels are indicated below

$$\prod_i \otimes (H^i_{-\hat{k}+g-h_i} \otimes H^i_{h_i}), \tag{23}$$

where h_i is the Coxeter number associated with the subgroup H^i.

The N=1 superconformal G/H theory is now obtained from $T = T_G - T_H$, $G = G_G - G_H$. From the above remarks we deduce that this energy momentum tensor can be reconstructed by using the GKO scheme

$$\frac{G_{-\hat{k}} \otimes G_g}{\prod_i (H^i_{-\hat{k}+g-h_i} \otimes H^i_{h_i})}. \tag{24}$$

We now further rewrite this expression as follows. We first notice that the fermions in H, Ψ_a, completely drop out in the expressions for T, G. This implies that they are used as a cruch to display the N=1 structure but that it should be possible to find a reformulation in which they are totally absent. As pointed out by Kazama and Suzuki the fermionic part of T may be rewritten as an $SO(dimG/H)_1$ Sugawara theory, without using the fermions in H. There is another even more useful form which displays the symmetries of the theory and is helpful in explicit computations as well as bosonizations of the theory. It is simpler to consider the cases for which the torsion vanishes $f_{\alpha\beta\gamma} = 0$, and hence we will concentrate on this case. Proper modifications are possible to include non-vanishing torsion. We now note that using the current K by applying Wick's theorem. For *zero torsion* we find

$$T_K = \frac{1}{g} K_a K_a = -\frac{\Psi_\alpha \partial \Psi_\alpha}{k}, \tag{25}$$

This allows us to state that

$$\begin{aligned} \frac{\eta^{AB}}{2g} j_A^G j_B^G &= \frac{1}{2h} j_a^H j_a^H + \frac{1}{g} K_a K_a, \\ \frac{\eta^{AB}}{k} \Psi_A \partial \Psi_B &= \frac{1}{k} \Psi_a \partial \Psi_a - \frac{1}{k} \Psi_\alpha \partial \Psi_\alpha . \end{aligned} \tag{26}$$

which is equivalent to

$$G_g = \prod_i H^i_{g-h_i} \otimes H^i_{h_i}. \tag{27}$$

In these manipulations we have used the identity $g dim(G/H) = 2 \sum_i (g - h_i) dim H^i$, which is valid when the torsion vanishes.

Inserting this form into (24) we see that the factors $\prod_i H^i_{h_i}$ cancel, which is equivalent to the statement that the fermions in H drop out from (24). This allows to rewrite the energy momentum tensor in the form of GKO theory associated with

$$\frac{G_{-\hat{k}} \otimes \prod_i H^i_{g-h_i}}{\prod_i H^i_{-\hat{k}+g-h_i}}, \tag{28}$$

It is possible to exhibit further symmetries by further rewriting this theory in the form

$$\frac{G_{-\hat{k}}}{\prod_i H^i_{-\hat{k}}} \otimes \prod_i \frac{(H^i_{-\hat{k}} \otimes H^i_{g-h_i})}{H^i_{-\hat{k}+g-h_i}}, \tag{29}$$

which corresponds to the following expression for the energy momentum tensor

$$T = \hat{T}_{G/H} + \sum_i T^i_{H^i \times H^i / H^i}. \tag{30}$$

In (30) all the pieces on the right hand side commute with each other and generate separate conformal transformations on the Hilbert space of each separate piece. The first piece is constructed purely from the currents $\hat{J}_A$ and is identical to the theory studied in section 2. An important point conveyed by (30) is that the full Hilbert space of a Kazama-Suzuki model or its non-compact generalization is simply the direct product of the Hilbert spaces associated with each commuting part of the energy momentum tensor.

We have used conformal embeddings to rewrite the energy momentum tensor of the free G/H fermions:

$$SO(dimG/H)_1 = \prod_i H^i_{g-h_i} = \frac{G_g}{\prod_i H^i_{h_i}} = \frac{SO(dimG)_1}{SO(dimH)_1} \tag{31}$$

These identities apply equally to compact as well as noncompact groups G, provided the torsion of G/H is zero. The G, H pairs to which these conformal embeddings apply can be classified We list here a complete list of conformal embeddings that apply when G is simple, non-compact, H=maximal compact group and G/H is Kählerian and torsion free. All of the following models are equivalent to free fermions.

$$
\begin{aligned}
&\frac{SU(n,m)_{n+m}}{SU(n)_n \times SU(m)_m \times U(1)} &&= SU(n)_m \times SU(m)_n \times U(1)_{n+m} &&= SO(2nm)_1,\\
&\frac{SO(n,2)_n}{SO(n)_{n-2} \times SO(2)} &&= SO(n)_2 \times SO(2)_n &&= SO(2n)_1\\
&\frac{SO(2n)^*_{2n-2}}{SU(n)_n \times U(1)} &&= SU(n)_{n-2} \times U(1)_{2n-2} &&= SO(n(n-1))_1\\
&\frac{Sp(2n)^*_{2n+2}}{SU(n)_n \times U(1)} &&= SU(n)_{n+2} \times U(1)_{2n+2} &&= SO(n(n+1))_1\\
&\frac{(E^*_6)_{12}}{SO(10)_8 \times U(1)} &&= SO(10)_4 \times U(1)_{12} &&= SO(32)_1\\
&\frac{(E^*_7)_{18}}{(E_6)_{12} \times U(1)} &&= (E_6)_6 \times U(1)_{18} &&= SO(54)_1
\end{aligned}
\tag{32}
$$

where the groups denoted by a "*" are the non-compact versions that have the maximal compact subgroup indicated. This list is a subset of the cases listed in [17]. We have thus found new symmetry structures in the Kazama- Suzuki models as well as in their non-compact generalizations.

As in [7] N=2 superconformal systems may now be constructed with appropriate conditions on the torsion $f_{\alpha\beta\gamma}$. Even if the torsion is non-zero one can get N=2 systems provided G/H is Kählerian and the torsion corresponds to the structure constants of a semi-simple Lie group. The simplest case is when the torsion vanishes and the space G/H is Kählerian. Then the expression for the supercurrent $G(z)$ splits into two parts $G = G_+ + G_-$ which form the N=2 supercurrents. For example, in the $SU(N,M)$ case $G_+ = (1/(k-N-M))(J_-)^\alpha_b(\Psi_+)^b_\alpha$, $G_- = (1/(k-N-M))(J_+)^a_\beta(\Psi_-)^\beta_a$, with $a = 1,2,\cdots,N$ and $\alpha = 1,2,\cdots M$. The $U(1)$ current $J(z)$ is given by

$$
J(z) = (\hat{k}Y_\Psi(z) + g\hat{Y}(z))/(\hat{k}-g), \tag{33}
$$

where Y_Ψ is a $U(1)$ fermion bilinear current constructed as $Y_\Psi = \Psi_\alpha\Psi_{\bar{\alpha}}/(\hat{k}-g)$. The normalization of $Y_\Psi, \hat{Y}$ are such that they respectively assign charge +1 to $\Psi_\alpha, \hat{J}_\alpha$ and -1 to $\Psi_{\bar{\alpha}}, \hat{J}_{\bar{\alpha}}$ and commute with the rest of the currents in H. To give some examples, consider SU(1,1), then $Y_\Psi, \hat{Y}$ are associated with the third component of isospin $\tau_3/2$ under which $\Psi^\pm, \hat{J}^\pm$ have isospin ± 1; for SU(2,1) they are associated with hypercharge=diag (1/3,1/3,-2/3); for SU(N,M), they are associated with the traceless diagonal matrix $diag(M/(N+M),\cdots,-N/(N+M),\cdots)$ that has N identical entries followed by M identical entries.

The central charge of the N=2 superconformal system can be calculated from GxH/H structure by computing the central charge of each factor. Using the group theoretical identity mentioned above for the case of zero torsion we find the overall central charge

$$
c = \frac{3\hat{k}dim(G/H)}{2(\hat{k}-g)}, \tag{34}
$$

which applies to all G/H that are Kählerian and torsion free. A classification of all such cases that involves a non-compact group G is given below

$$
\begin{array}{lll}
1) & G/H = SU(n,m)/SU(n)\times SU(m)\times U(1) & c = 3\hat{k}mn/(\hat{k}-m-n), \\
2a) & G/H = SO(n,2)/SO(n)\times SO(2) & c = 3\hat{k}n/(\hat{k}-n), n>2, \\
2b) & G/H = SO(2,1)/SO(2) & c = 3\hat{k}/(\hat{k}-2), \\
3) & G/H = SO(2n)^*/SU(n)\times U(1) & c = 3\hat{k}n(n-1)/2(\hat{k}-2n+2), \\
4) & G/H = Sp(2n)^*/SU(n)\times U(1) & c = 3\hat{k}n(n+1)/2(\hat{k}-2n-2), \\
5) & G/H = E_6^*/SO(10)\times U(1) & c = 48\hat{k}/(\hat{k}-12), \\
6) & G/H = E_7^*/E_6\times U(1) & c = 81\hat{k}/(\hat{k}-18),
\end{array}
\tag{35}
$$

where the groups denoted by a "*" are the non-compact versions that have the maximal compact subgroup indicated. A first condition for unitarity is $c \geq 0$ which demands $\hat{k} \geq g$ for all of the above cases whichs is consistent with the previous section. In comparison to ref.[7] we see that our results differ by replacing $-\hat{k}$ by $\hat{k}$. We have thus obtained the non-compact counterpart of the classification of ref.[7].

In applying some of these schemes to superstring compactification we are interested in those cases with $c \leq 9$. Applying this condition we find that the only allowed noncompact affine algebras are $SU(1,1)_{-\hat{k}}$ with $\hat{k} \geq 3$ and $SU(2,1)_{-\hat{k}}$ with $\hat{k} \geq 9$. All other cases that are allowed can be rewritten in terms of these or their products with themselves or with other compact affine algebras. Thus the relevant GKO schemes are

$$
\frac{SU(1,1)_{-\hat{k}} \otimes U(1)_2}{U(1)_{-\hat{k}+2}}; \qquad \frac{SU(2,1)_{-\hat{k}} \otimes SU(2)_1 \otimes U(1)_3}{SU(2)_{-\hat{k}+1} \otimes U(1)_{-\hat{k}+3}}. \tag{36}
$$

with the indicated ranges of $\hat{k}$.

Unitarity in N=2 Superconformal Theory

The N=2 superconformal algebra contains two global operators that commute with each other $L_0 = \oint z dz T(z)$ and $J_0 = \oint dz J(z)$. Their eigenvalues h, q label the spectrum. These eigenvalues satisfy inequalities that follow from unitarity. Since we have fermions in the theory there are antiperiodic or periodic boundary conditions that may be applied to them, which are refered to as NS and R conditions respectively [18]. In these sectors the inequalities are

$$
\begin{array}{ll}
h \geq \frac{1}{2}|q| & NS\ sector \\
h \geq \frac{c}{24} & R\ \ sector
\end{array}
\tag{37}
$$

To construct the states $|h, q>$ we follow the GKO schemes GxH/H described above. Just as in the previous section, the coset states can be constructed and their dimension h computed. For SU(1,1) we find

$$h = \frac{-\hat{j}(\hat{j}+1) + (\hat{m}+m_\Psi)^2}{\hat{k}-2} + \frac{m_\Psi^2}{2} + \mathit{integer},$$

$$q = \frac{\hat{k}m_\Psi + 2\hat{m}}{\hat{k}-2}. \tag{38}$$

The added integer arises at higher levels above the base. The quantum number $m_\Psi = 0, \pm1, \pm\frac{1}{2}$ corresponds to the allowed representations in the factor of H associated with the fermions. The integer values of m_Ψ corespond to the NS sector and the half-integer values to the R- sector. Checking the above unitarity conditions in every sector, for all possible values of $\hat{j}, \hat{m}, m_\Psi$ and the *integer* that appears at the higher levels, we find the identical conditions on $\hat{k}, j$ that we found in the previous sections.

For the SU(2,1) case we follow a similar route. In this case the affine algebra H=SU(2)xU(1) associated with the fermions are labelled by the quantum numbers $|j_\Psi m_\Psi; y_\Psi>$. The allowed values for these quantum numbers are deduced by conformally embedding $SU(2)_1 \times U(1)_3 = SO(4)_1$ since these currents describe the 4 free fermions in the coset SU(2,1)/SU(2)xU(1). We find then

$$\begin{array}{lll} NS\ sector & |j_\Psi = 0, y_\Psi = 0>, & |j_\Psi = 1/2, y_\Psi = \pm1>, \\ R\ \ sector & |j_\Psi = 1/2, y_\Psi = 0>, & |j_\Psi = 0, y_\Psi = \pm1>, \end{array} \tag{39}$$

The eigenvalues of the states $|h,q>$ can now be computed, in either the R or NS sectors, according to the coset schemes.

$$\begin{aligned} h = & \frac{1}{\hat{k}-3}\Big(-C(a,\tilde{b}) + j_T(j_T+1) + \frac{3}{4}(\hat{y}+y_\Psi)^2\Big) \\ & \qquad + \frac{1}{3}\Big(j_\Psi(j_\Psi+1) + \frac{3}{4}y_\Psi^2\Big) + \mathit{integer} \\ q \ = & \frac{1}{\hat{k}-3}(\hat{k}y_\Psi + 3\hat{y}). \end{aligned} \tag{40}$$

where $j_T = \hat{j}$ for $j_\Psi = 0$ and $j_T = \hat{j} \pm 1/2$ when $j_\Psi = 1/2$ which arise as a result of addition of angular momentum in the direct product of states $|a,\tilde{b};\hat{j}m;\hat{y}> \otimes|j_\Psi m_\Psi; y_\Psi>$. The N=2 unitarity conditions are then seen to be consistent with the previous sections.

GSO Projection, Heterotic Strings and Massless States in 4-d

In order to construct a consistent heterotic string in 4-dimensions we follow Gepner's prescription. This first requires a c=9 superconformal theory that is used to describe the propagation of the left as well as the right movers of the superstring in the extra compactified dimensions. We have a provided such theories in the previous sections. Here we will concentrate on the $c = 9$ $SU(1,1)_{-3}$ theory as an example.

In the lightcone gauge formulation of the heterotic string, the left movers are described by two free transverse bosons ($x^I(z)$, $I = 1,2$), plus two free transverse fermions ($\chi^I(z)$, $I = 1,2$), plus the SU(1,1)/U(1) c=9 N=2 superconformal theory. This gives a total central charge for the left moving Virasoro algebra $c_L = 2 + 2(1/2) + 9 = 12$, which is the required number to construct a consistent space-time super-Poincaré algebra. The right movers are described by two free transverse bosons ($\tilde{x}^I(\bar{z})$, $I = 1,2$), plus the current algebra Sugawara theories for $SO(10)_1 \times (E_8)_1$, and the SU(1,1) c=9 superconformal theory that parallels the left moving one. The central charge for the right moving sector is then $c_R = 2+5+8+9 = 24$ as required in the heterotic string for the consistency of the Lorentz algebra in the right moving sector.

The two left-moving free fermions χ^I can be reformulated as a current algebra Sugawara theory for $SO(2)_1 = U(1)_2$ as described in section 3. This affine algebra has primary fields for only the scalar, vector, spinor representations of SO(2)=U(1) which may be labelled by a quantum number m_χ that is allowed to take the values $m_\chi = 0, \pm 1$ in the NS sector (scalar, vector) and $m_\chi = \pm 1/2$ in the R sector (spinors).

For the heterotic string a physical state is constructed by taking direct products of states $|state>_L \times |state>_R$ from the left (L) and right (R) moving sectors, with the condition that the mass of the left-state is equal to the mass of the right-state $M_L^2 = M_R^2$ [1,18]. There are also Gepner's matching conditions for modular invariance [3], as will be discussed below. The mass formula for any state in the left-moving sector is given by

$$\frac{M_L^2}{8} = l_x + (\frac{m_\chi^2}{2} + l_\chi) + (h(j, \hat{m}, m_\Psi) + l) - \frac{1}{2} \tag{41}$$

where $l_x = 0, 1, 2, 3, \cdots$ is the eigenvalue of the level operator $N_\alpha = \sum \alpha_{-n}^I \alpha_n^I$ for the free bosons, $l_\chi = 0, 1, 2, 3 \cdots$ is the level of the SO(2)=U(1) Sugawara theory that describes the fermions χ, $h+l$ which is contributed by the SU(1,1) theory is obtained from (38)

$$h = -j(j+1) + (\hat{m} + m_\Psi)^2 + \frac{m_\Psi^2}{2} \tag{42}$$

and l is the integer that appears in (38). The last term of $-\frac{1}{2}$ is the vacuum energy which is the same in the NS or R sectors with our definition of quantum numbers and the mass operator.

For the consistency of spin-statistics and supersymmetry (GSO/Gepner projection: q=odd integer), it is required that m_χ, m_Ψ both take on integer values or both take on half-integer values and that

$$m_\chi + 3m_\Psi + 2\hat{m} = \pm 1, \pm 3, \pm 5, \cdots. \tag{43}$$

This implies that $2\hat{m}$ is an integer in all cases, and this in turn demands $2(\hat{j}+1)$ to be an integer if $\hat{j}$ describes the discrete series of SU(1,1) Recall that in the discrete series we must further require the unitarity condition $0 \le \hat{j}+1 \le \hat{k}/2$, which for $\hat{k}=3$ allows only

$$\hat{j} = -1, -\frac{1}{2}, 0, \frac{1}{2}. \tag{44}$$

As discussed in section 2b, the limiting case of $\hat{j} = +1/2$ produces zero norm states when $l_G = 1$ and such states should be discarded, however we may keep the primary states at $l_G = 0$. We emphasize that if it were not for the generalized GSO projection (43) we would obtain an infinite number of massless states since $\hat{j}, \hat{m}$ could take continuous values instead of the few values allowed in (44). Note that $\hat{j} = -1, 0$ coincide with the $\sigma \to \pm 1$ limit of the supplementary series for which $\hat{j} = \frac{1}{2}(-1+\sigma)$. Although these points are excluded from the principal series, the neighboring values of σ will give states whose masses will be infinitesimally close to those of the discrete series with $\hat{j} = -1, 0$. Similarly, $\hat{j} = -\frac{1}{2}$ coincides with the $\sigma or \rho \to 0$ limit of the supplementary or principal series respectively. However, in this case the $2\hat{m}$ values are odd for the discrete series and even for the supplementary or principal series, and this is sufficient to eliminate the non-discrete series for these limiting values since (43) cannot be satisfied for them.

With these restrictions we look for the massless states in the left-moving sector. We see that the principal and supplementary series representations can produce only massive states since either (43) is not satisfied or $\hat{j}(\hat{j}+1)$ which appears in (38) is always positive (the limiting points mentioned above are not part of the supplementary series). There remains to consider the positive and negative discrete series with the $\hat{j}$ values listed in (44). We find that the massless states are just the following

$$\begin{aligned}
&|scalar, A_{\mp}, q=\mp 1>=|l_x=0; m_\chi=0=l_\chi; (h=1/2, l=0), A_{\mp}, q=\mp 1>,\\
&|vector_{\pm}, B, q=0>=|l_x=0; m_\chi=\pm 1, l_\chi=0; (h=0, l=0), B, q=0>,\\
&|spinor_{\pm}, A_{\mp}, q=\pm 1/2>=|l_x=0; m_\chi=\pm\frac{1}{2}, l_\chi=0; (h=3/8, l=0), A_{\mp}, q=\pm 1/2>,\\
&|spinor_{\pm}, B, q=\mp 3/2>=|l_x=0; m_\chi=\pm\frac{1}{2}, l_\chi=0; (h=3/8, l=0), B, q=\mp 3/2>,
\end{aligned} \tag{45}$$

where the $\pm$ labels on the vector and spinor states refer to the sign of their helicities (rotations in the 2 transverse dimensions) given by m_χ, and the $A_{\mp}, B$ labels refer to the different ways in which the c=9 theory can satisfy the required values of (h, l) after taking into account the GSO, spin-statistics restrictions and unitarity conditions of (43-44). The number of ways that is possible to construct the required values of $h = 1/2$ (the $A_{\mp}$ labels) will be related to the number of families, while the number of ways of constructing $h = 0$ (the B labels) will be related to the possible candidates for graviton. Only one of the possible values of B must be included in a modular invariant to insure a theory with a single graviton. In terms of the quantum numbers $(\hat{j}, \hat{m}, m_\Psi, l; q)$ all the allowed A, B labels can be given as follows.

$$
\begin{array}{ll}
h+l=1/2 & A_\mp = 1_\mp \sim (\hat{j}=-1/2, \hat{m}=\mp 1/2, l=0); \quad (m_\Psi=0 \to q=\mp 1), \\
 & A_\mp = 2_\mp \sim (\hat{j}=0, \hat{m}=\pm 1, l=0); \quad (m_\Psi=\mp 1 \to q=\mp 1), \\
h+l=0 & B=1 \sim (\hat{j}=-1, \hat{m}=0, l=0); \quad (m_\Psi=0 \to q=0), \\
 & B=2 \sim (\hat{j}=1/2, \hat{m}=\mp 3/2, l=0); \quad (m_\Psi=\pm 1 \to q=0), \\
h+l=3/8 & A_\mp = 1_\mp \sim (\hat{j}=-1/2, \hat{m}=\mp 1/2, l=0); \quad (m_\Psi=\pm 1/2 \to q=\pm 1/2), \\
 & A_\mp = 2_\mp \sim (\hat{j}=0, \hat{m}=\pm 1, l=0); \quad (m_\Psi=\mp 1/2 \to q=\pm 1/2), \\
 & B=1 \sim (\hat{j}=-1, \hat{m}=0, l=0); \quad (m_\Psi=\mp 1/2 \to q=\mp 3/2) \\
 & B=2 \sim (\hat{j}=1/2, \hat{m}=\mp 3/2, l=0); \quad (m_\Psi=\pm 1/2 \to q=\mp 3/2).
\end{array}
\tag{46}
$$

The opposite values of $\hat{m}$ are in different representations of SU(1,1) since for positive $\hat{m}$ we have the positive discrete series and for negative $\hat{m}$ we have the negative discrete series.

We see that two charged scalar states are naturally associated with two left handed fermionic states (A=1,2), while the oppositely charged scalar states are associated with the fermions of the opposite charge and chirality, which are interpreted as the anti-particles of the first set. When combined with the right-moving states below they lead to supersymmetric families of quarks and leptons in the **27** dimensional representation of E_6. Similarly, the two helicity components of the vector state are associated with the two helicity components of the spinors; these will lead to a single gauge vector supermultiplet and a single supergravity multiplet, as required, provided we take either $B=1$ or $B=2$. The N=1 space-time supersymmetry charge [3] of the left sector is the symmetry generator responsible for this outcome of supersymmetry multiplets.

We now turn to the right moving sector. The mass formula for any state is

$$
\frac{M_R^2}{8} = \tilde{l}_x + (h+l)_{SO(10)} + (h+l)_{E_8} + (\tilde{h}+\tilde{l}) - 1, \tag{47}
$$

The -1 at the end is the vacuum energy, which in our formalism is identical in the NS and R sectors. $\tilde{l}_x$ is the eigenvalue of the level operator $N_{\tilde{\alpha}} = \sum \tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n$ of the right moving free bosons. The l added to h refer to the level l_G for $G=SO(10), E_8$, and correspond to the sum of the moments of the currents that are applied on the highest weight states $|h>$. For example for $l_G=1$, we have $J^G_{-1}|h>$, where J^G_{-1} is in the adjoint representation of G. For $SO(10)_1$ unitarity allows only the highest weight states $|h>$ that correspond to singlet=**1**, vector=**10**, spinor=**16** and antispinor=$\bar{\mathbf{16}}$ representations. For these $l_G = 0$ states, we have the eigenvalues of the Virasoro operator L_0^G that gives the dimensions $h(\mathbf{1}) = 0, h(\mathbf{10}) = 1/2, h(\mathbf{16}) = h(\bar{\mathbf{16}}) = 5/8$. For (h=0,l=1) case in SO(10) the state would be in the adjoint representation **45** with (h+l)=1. Thus the **45** of SO(10) is a higher level state in the scalar=**1** module of $SO(10)_1$. Similarly, for E_8 the only relevant state for our purposes is the (h=0,l=1) which is in the adjoint representation **248** of E_8 and satisfies (h+l)=1. Finally $\tilde{h}, \tilde{l}$ describe the dimensions of the states associated with the right moving c=9 sector which parallels the one in the left moving sector.

Accordingly, the massless states in the right-moving sector are

$$
\begin{aligned}
&|vector, \tilde{B}, \tilde{q} = 0 >= |\tilde{l}_x = 1; \mathbf{1}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 0, \tilde{l} = 0), \tilde{B}, \tilde{q} = 0 > \\
&|(\mathbf{1}, \mathbf{248}), \tilde{B}, \tilde{q} = 0 >= |\tilde{l}_x = 0; \mathbf{1}_{SO(10)}; \mathbf{248}_{E_8}; (\tilde{h} = 0, \tilde{l} = 0), \tilde{B}, \tilde{q} = 0 > \\
&|(\mathbf{45}, \mathbf{1}), \tilde{B}, \tilde{q} = 0 >= |\tilde{l}_x = 0; \mathbf{45}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 0, \tilde{l} = 0), \tilde{B}, \tilde{q} = 0 > \\
&|(\mathbf{1}, \mathbf{1}), \tilde{B}, \tilde{q} = 0 >= |\tilde{l}_x = 0; \mathbf{1}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 0, \tilde{l} = 1), \tilde{B}, \tilde{q} = 0 > \\
&|(\mathbf{1}, \mathbf{1}), \tilde{A}_\mp, \tilde{q} = \pm 2 >= |\tilde{l}_x = 0; \mathbf{1}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 1, \tilde{l} = 0), \tilde{A}_\mp, \tilde{q} = \pm 2 > \\
&|(\mathbf{10}, \mathbf{1}), \tilde{A}_\mp, \tilde{q} = \mp 1 >= |\tilde{l}_x = 0; \mathbf{10}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 1/2, \tilde{l} = 0), \tilde{A}_\mp, \tilde{q} = \mp 1 > \\
&|(\mathbf{16}, \mathbf{1}), \tilde{A}_-, \tilde{q} = +1/2 >= |\tilde{l}_x = 0; \mathbf{16}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 3/8, \tilde{l} = 0), \tilde{A}_-, \tilde{q} = +1/2 > \\
&|(\mathbf{16}, \mathbf{1}), \tilde{B}, \tilde{q} = -3/2 >= |\tilde{l}_x = 0; \mathbf{16}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 3/8, \tilde{l} = 0), \tilde{B}, \tilde{q} = -3/2 > \\
&|(\bar{\mathbf{16}}, \mathbf{1}), \tilde{A}_+, \tilde{q} = -1/2 >= |\tilde{l}_x = 0; \bar{\mathbf{16}}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 3/8, \tilde{l} = 0), \tilde{A}_+, \tilde{q} = -1/2 > \\
&|(\bar{\mathbf{16}}, \mathbf{1}), \tilde{B}, \tilde{q} = +3/2 >= |\tilde{l}_x = 0; \bar{\mathbf{16}}_{SO(10)}; \mathbf{1}_{E_8}; (\tilde{h} = 3/8, \tilde{l} = 0), \tilde{B}, \tilde{q} = +3/2 >
\end{aligned}
\tag{48}
$$

where one notices that the values allowed for $(\tilde{h}, \tilde{l} = 0)$=(0,0),(1/2,0),(3/8,0) and the corresponding values of $\tilde{q}$ are identical to those of the left sector as listed in (46) except for $\tilde{j}, \tilde{m}, \tilde{m}_\Psi$ replacing $(\hat{j}, \hat{m}, m_\Psi)$. Thus, $\tilde{A}_\mp, \tilde{B}$ on the right side describe the identical $SU(1,1)$ states as $A_\mp, B$ on the left side. Furthermore, there is an additional allowed value of $(\tilde{h} = 1, \tilde{l} = 0)$ which is associated with the states $(\tilde{A}_\mp, \tilde{q} = \pm 2) = [(\tilde{j} = -1/2, \tilde{m} = \mp 1/2), (\tilde{m}_\Psi = \pm 1 \rightarrow \tilde{q} = \pm 2)]$ or $[(\tilde{j} = 0, \tilde{m} = \pm 1), (\tilde{m}_\Psi = 0 \rightarrow \tilde{q} = \pm 2)]$ which again correspond to the same $SU(1,1)$ states associated with the label $A_\mp = \tilde{A}_\mp$. In constructing the states (48) one other restriction due to modular invariance was observed [3]. Namely, the **16** was put in correspondance with the $spinor_+$ of the left side and therefore allowed to have only one sign of the charge $\tilde{q}$. The remaining correpondances required by Gepner's modular invariance turn out to be automatically obeyed since there are no other choices of states that satisfy the required values of $(\tilde{h}, \tilde{l})$.

We can now take direct products of left-moving states with right-moving states to make physical states. Modular invariance must be obeyed in this construction. The characters of our c=9 theory have labels $\chi^{\hat{j}}_{M,m_\Phi}$, with $M = \hat{m} + m_\Psi$. Under modular transformations each one of these labels transforms with a separate unitary matrix. The modular invariants are of the form $\sum A_{\hat{j}\tilde{j}} B_{M\tilde{M}} C_{m_\Phi \tilde{m}_\Phi} \chi^{\hat{j}}_{M,m_\Phi} \bar{\chi}^{\tilde{j}}_{\tilde{M},\tilde{m}_\Phi}$. $A_{\hat{j}\tilde{j}}$ must be such that either $\hat{j} = \tilde{j} = -1$ or $\hat{j} = \tilde{j} = -1/2$ but not both values of j's are included in the sum in order to insure a single graviton. Thus, determining all the coefficients $A_{\hat{j}\tilde{j}}, B_{M\tilde{M}}, C_{m_\Phi \tilde{m}_\Phi}$ is equivalent to classifiying all phenomelogically relevant heterotic superstring vacua. Clearly this is an enormous job. Unfortunately, at this stage we do not know the classification of all possible such coefficients, so we cannot determine all possible models.

Here we will be more modest and for the purpose of illustration we will make some assumptions which we will qualify below. Our only defense for doing so is that what we list below are the simplest possibilities. Thus, as far as the *massless sector* is concerned we will assume without justification that, after projecting out one of the values for B, we have a left-right diagonal invariant $A_{\hat{j}\tilde{j}} = N_{\hat{j}} \delta_{\hat{j}\tilde{j}}, B_{M\tilde{M}} = \delta_{M\tilde{M}}$, $C_{m_\Phi \tilde{m}_\Phi} = \delta_{m_\Phi \tilde{m}_\Phi}$. We shall not make any attempts on guessing what the structure of $A_{\hat{j}\tilde{j}}$ might be for the values of $\hat{j}, \tilde{j}$ that contribute only in the heavy sector. Thus, we associate left and right states as follows:

$$|\alpha, A_- >_L \times |\beta, \tilde{A}_- >_R , \qquad |\alpha, A_+ >_L \times |\beta, \tilde{A}_+ >_R , \qquad |\alpha, B >_L \times |\beta, \tilde{B} >_R , \tag{49}$$

where the A, B labels can have the following possibilities

$$\begin{aligned} &B = \tilde{B} = 1 \ (\hat{j} = -1\ ) \qquad && A = \tilde{A} = 1,2 \quad (\hat{j} = -\frac{1}{2}, 0) \\ &B = \tilde{B} = 2 \ (\hat{j} = 1/2) \qquad && A = \tilde{A} = 1,2 \quad (\hat{j} = -\frac{1}{2}, 0) \end{aligned} \tag{50}$$

To insure a single graviton, in the first line we have considered all states except those associated with $\hat{j} = 1/2$ and in the second line we kept all states except those with $\hat{j} = -1$. The actual number of families is determined by modular invariance which may require either or both values of $A = \tilde{A} = 1,2$ to be present. Thus, if $A_{\hat{j}\tilde{j}} = N_{\hat{j}} \delta_{\hat{j}\tilde{j}}$, then the number of families is $N_{-\frac{1}{2}} + N_0$. If the supplementary series with $|\sigma| \to 1$ is included in the modular invariant then there would be *massive* states whose mass$=\frac{1}{4}(1 - \sigma^2)$ and $\hat{j} = \frac{1}{2}(-1 + \sigma)$ quantum numbers would come infinitesimally close to those of the the A=2 (j=0), B=1 (j=-1) states in (50) and then there would be no mass gap for that case. If $N_0 = 0$ for the second line of (50) there would be no quantum numbers at the limiting values of the supplementary series and hence more likely to have a mass gap. It is not known at this stage whether there are modular invariants that selectively may include or exclude the supplementary series at the limiting values of $\hat{j}$ that are relevant.

The $SO(10) \times E_8 \times U(1)$ quantum numbers of the first term in (49) are

$$A_- = \tilde{A}_- = 1,2 \qquad (\mathbf{1},\mathbf{1},2),\ (\mathbf{10},\mathbf{1},-\mathbf{1}),\ (\mathbf{16},\mathbf{1},\mathbf{1/2}) \tag{51}$$

where the U(1) charge is simply given by $\tilde{q}$. We see that these states form chiral supermultiplets including complex scalars and left-handed fermions. Note that the second term in (50) produces exactly the anti-particles of the first term. Similarly, the $B = \tilde{B} = 1$ *or* 2 states form a supergravity multiplet (if we take the *vector* state on the right side) and also a gauge supermultiplet whose $SO(10) \times E_8 \times U(1)$ quantum numbers are

$$(\mathbf{1},\mathbf{248},0),\ (\mathbf{45},\mathbf{1},0),\ (\mathbf{1},\mathbf{1},0),\ (\mathbf{16},\mathbf{1},-3/2),\ (\bar{\mathbf{16}},\mathbf{1},3/2). \tag{52}$$

As expected the gauge symmetry is seen to actually correspond to $E_6 \times E_8$ since the gauge supermultiplet and chiral supermultiplet respectively form the adjoint **(78,1)+(1,248)** and fundamental **(27,1)** representations of the larger symmetry. There are no $E_6 \times E_8$ singlets in our theory. This phenomenologically favored situation arises as a consequence of building the $c = 9$ theory irreducibly.

Depending on the type of modular invariant that one chooses, anti-families which end up as $(\bar{\mathbf{27}},\mathbf{1})$, could also emerge together with families. Those would correspond to states of the form $|\alpha, A_- >_L \otimes |\beta, \tilde{A}_+ >_R$, where in the R-states $\tilde{A}_+$ has been used instead of $\tilde{A}_-$. If our completely L-R symmetric example , which we used for illustration purposes, is a correct modular invariant, anti-families do not occur. Gepner has conjectured that all N=2, c=9 theories must be Calabi-Yau manifolds. Assuming that this conjecture is correct, if there are no anti- families it must be that the Hodge numbers are given as $h^{2,1} = 0,\ h^{1,1} \neq 0$, since there must be at least one $h^{1,1}$ form corresponding to the Kähler form. The number of families is then identified with $h^{1,1} = N_{-\frac{1}{2}} + N_0$. There are examples of heterotic superstring vacua in 4-d in which anti-families do not arise [20]. If modular invariance is achieved with a diagonal invariant of the type considered above it is then impossible for E_6 singlet representations to arise since we cannot make SO(10) singlets that are neutral under the U(1) (the $\tilde{q} \neq 0$ except for the B-states). The absence of such singlet representations is phenomenologically favored.

In conclusion we have developed the technology for non-compact current algebras and have shown their relevance to string compactification and the construction of heterotic string vacua. We have identified $SU(1,1)_{-\hat{k}}$, with $\hat{k} \geq 3$, and $SU(2,1)_{-\hat{k}}$, with $\hat{k} \geq 9$ as the only phenomenologically relevant cases. If these are used at the minimum allowed values of $\hat{k}$ they produce c=9 N=2 superconformal theories in an *irreducible* way. It is also possible to apply them at the higher allowed values of $\hat{k}$ in combination with other $c < 9$ superconformal theories provided the total c=9. The main problem that requires more investigation is the classification of modular invariants for both SU(1,1) and SU(2,1). Since SU(1,1) is intimately connected to SU(2) it may be that some of the techniques that worked for the A,D,E classification [19] may be extended to the present case.

Applying a similar analysis to $SU(2,1)_{-9}$ we find five possibilities for the quantum numbers $A, \tilde{A}$ of massless states which are obtained just by analysing the massless scalar states. The details of similar investigations for SU(2,1) will be published elsewhere [21].

In addition to the possibility of constructing models with families and anti-families through various combinations of modular invariants for $SU(1,1)_{-\hat{k}}$ or $SU(2,1)_{-\hat{k}}$ in the above formalism, there also remains the possibility of constructing modular invariants that may differ from Gepner's scheme and thus arrive at other acceptable heterotic string vacua in 4-dimensions. Such other possibilities also require more knowledge of modular transformation properties of non-compact characters and a classification of modular invariant forms. These questions will be further analysed in future publications.

References

(1) M.B.Green and J.H.Schwarz, Nucl. Phys. B255 (1985) 92.
D.J.Gross, J.Harvey, E.J.Martinec, and R.Rhom, Nucl.Phys. B256 (1985) 253.

(2) T.Banks, L.J.Dixon, D.Friedan, and E.Martinec, Nucl.Phys. B299 (1988) 613.

(3) D. Gepner, Nucl. Phys. B296 (1987) 757, and Phys. Lett 199B (1987) 380.

(4) I. Bars, USC preprint, USC-89/HEP02, May 1989.

(5) D. Gepner and E. Witten, Nucl.Phys. B278 (1986) 493.

(6) M.B.Halpern, Phys.Rev. D4 (1971) 2398.
P. Goddard, A. Kent and D. Olive, Phys. Lett. 152B (1985) 88, and Comm. Math. Phys. 103 (1986) 105.

(7) Y. Kazama and H. Suzuki, preprint UT-Komaba 88-8 (1988).

(8) L.Dixon, M.E.Peskin and J. Lykken, SLAC-PUB-4884 (1989).

(9) P. Goddard and D. Olive, Int. J. Mod. Phys. A, 1 (1986) 303.
D. Nemeshansky and S. Yankieolowicz, Phys.Rev. Lett. 54 (1985) 620.

(10) V. Bargmann, Ann. Math. 48 (1947) 568 and Comm. Pure & Appl. Math. 14 (1961) 187.

(11) F. Gursey and S. Orfanidis, Phys.Rev. D7 (1973) 2414.

(12) I. Bars and Z.J. Teng, to be published.

(13) I. Bars, Lect. Applied Math. 21 (1983) 17.

(14) M. Günaydin and C. Saclioglu, Comm. Math. Phys. 87 (1982) 159.

(15) A.M. Perelemov and V.S. Popov, Sov. J. Nucl. Phys. 3 (1966) 676 and 5 (1967) 489.

(16) M.A. Naimark, *Linear Representations of the Lorentz Group*, Pergamon Press, 1964.
I.M.Gelfand, M.I.Graev and N.Y.Vilenkin, *Generalized Functions*, Vol.5, N.Y. Academic Press 1968.
I.Bars and F.Gursey, J. Math. Phys. 13 (1972) 131.

(17) P. Goddard, W. Nahm and D. Olive, Phys. Lett. 160B (1985) 111.

(18) M. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Cambridge Univ. Press (1987).

(19) A. Cappelli, C.Itzykson and J.B. Zuber, Nucl. Phys. B280 (1987) 445.
D.Gepner and Z. Qiu, Nucl.Phys. B285 (1987) 423.

(20 C. A. Lutken and G. Ross, Phys. Lett. B213 (1988) 152.

(21) I. Bars, to be published.

ASPECTS OF PERTURBED CONFORMAL FIELD THEORY, AFFINE TODA FIELD THEORY AND EXACT S-MATRICES

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1. INTRODUCTION

Recently, Zamolodchikov has suggested a way of exploring the properties of the Ising model in a background magnetic field, *i.e.* away from criticality[1]. The critical temperature is fixed so that unperturbed the theory is described by a $c = \frac{1}{2}$ conformal field theory. The Hamiltonian is then perturbed by the addition of a non conformally invariant piece depending upon the background magnetic field. An obvious question to ask concerns the nature of this perturbed two-dimensional field theory. Via a sequence of ingenious arguments Zamolodchikov suggests the hypothesis that the perturbed theory is actually an integrable theory of eight massive bosons. Following this development, our aim is to try to discover what this integrable theory is by conjecturing a candidate (inspired by comments of Hollowood and Mansfield[2]), and testing Zamolodchikov's arguments. On the way, we have discovered[3] (and sometimes rediscovered) a number of interesting facts concerning affine Toda field theory and two-dimensional factorisable S-matrices. It seems to us that this by itself warrants further investigation. Much of this material has also been found by Christe and Mussardo[4], whose work has considerable overlap with ours.

We begin by outlining Zamolodchikov's strategy. The unperturbed $c = \frac{1}{2}$ conformal field theory has field content I, σ and ϵ, with conformal weights $(0,0)$, $(\frac{1}{16}, \frac{1}{16})$ and $(\frac{1}{2}, \frac{1}{2})$, (together with their descendants), with respect to the two sets of mutually commuting sets of Virasoro generators L_n and $\bar{L}_n$. The Virasoro generators correspond to the moments of the holomorphic and anti-holomorphic parts of the energy-momentum tensor

$$T(z) = \sum_{-\infty}^{\infty} L_n z^{-n-2} \qquad \bar{T}(\bar{z}) = \sum_{-\infty}^{\infty} \bar{L}_n \bar{z}^{-n-2},$$

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in turn $(2,0)$ and $(0,2)$ descendants of I. There are many 'conserved' quantities $P_s^{(k)}$ (or $P_{-s}^{(k)}$)

$$P_s^{(k)} = \oint dz\, T_{s+1}^{(k)} \tag{1.1}$$

in the sense that $\partial_{\bar{z}} P_s^{(k)} \equiv 0$. Indeed, in this case all the quantities $T_{s+1}^{(k)}$ belong to the holomorphic part of the conformal family of I. The conserved quantities are labelled by their spin s (or $-s$ for the anti-holomorphic part) and k labels their multiplicity. Any quantitiy $T_{s+1}^{(k)}$ which is itself a derivative with respect to z is to be regarded as trivial in this context since it will not lead to an interesting conserved charge.

In the perturbed theory, where the Hamiltonian is given by

$$H = H_{CFT} + \lambda\sigma(z, \bar{z}), \tag{1.2}$$

most of the conserved quantities will cease to be so, but not all. The idea is to seek out those densities $T_{s+1}(z, \bar{z})$ which at $\lambda = 0$ belong to the previous set, and which satisfy

$$\partial_{\bar{z}} T_{s+1} = \lambda R_s = \partial_z S_{s-1} \tag{1.3}$$

for some operator S_{s-1}. From (1.2), λ is assigned conformal dimensions $(\frac{15}{16}, \frac{15}{16})$ and thus R_s has dimension $(s + \frac{1}{16}, \frac{1}{16})$ and therefore belongs to the conformal family of σ. Comparing dimensions of the spaces of operators of spin $s+1$ in the conformal family of I modulo z-derivatives with the space of operators of spin s in the conformal family of σ modulo z-derivatives, it is found that for $s = 1, 7, 11, 13, 17, 19$ the dimension of the first space exceeds by one the dimension of the second. In these cases, the mapping defined by $\partial_{\bar{z}}$ (eq(1.3)) followed by projection onto the second space (*i.e.* removing the z-derivative pieces) certainly has non trivial kernel. In other words, there will be operators S_{s-1} for at least these spins, and hence conserved quantities. Zamolodchikov further suggests that a complete set of conserved quantities will correspond to those whose spins are the integers coprime to 30, the first six of which are those given above. A more interesting observation perhaps is that these integers are also the exponents of the Lie algebra E_8 modulo 30 (and 30 is the Coxeter number of E_8)[5]. That there is a coset construction of the $c = \frac{1}{2}$ conformal theory based on E_8 ($E_8^{(1)} \times E_8^{(1)}/E_8^{(2)}$) is well known[6]. The remarks about conserved quantities makes it plausible that the perturbation λ is somehow probing the coset structure.

The next stage in Zamolodchikov's argument is to explore a theory with the above set of conserved charges (is it uniquely determined?) using techniques from S-matrix theory to find a minimal solution to the bootstrap equation consistent with the conjectured conserved quantities. The outcome is a theory of eight self-conjugate massive scalar bosons. In the next section we briefly review the main features of the S-matrix and bootstrap ideas to be employed.

2. FACTORISABLE S-MATRICES[1,7]

We shall be brief in this section and assume in the first instance that the interesting theories contain self-conjugate particles only, with distinct masses. (This assumption is too strong for most of the Toda theories we consider later but is sufficient for the present discussion. For E_8, E_7 and D_{even} the particles are self-conjugate, but only in the first two cases are the particles distinguished by mass alone. For all other simply laced cases, some particles appear in conjugate pairs.) In particular, there is no reflection and multiparticle S-matrix elements are products of two particle S-matrices. Thus, the two particle S-matrix is

$$|a(\theta_1), b(\theta_2)\rangle_{in} = S_{ab}(\theta)\,|a(\theta_1), b(\theta_2)\rangle_{out} \tag{2.1}$$

where θ_i is the rapidity, *i.e.*

$$p_i = m_i(\cosh\theta_i, \sinh\theta_i) \qquad \theta = \theta_1 - \theta_2. \tag{2.2}$$

The S-matrix is

(*i*) unitary, which in this case means that for each element

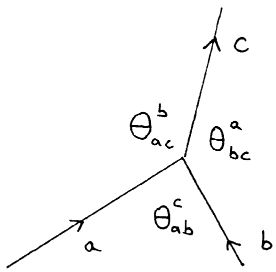
$$S(\theta)S^{\dagger}(\theta) \equiv S(\theta)S(-\theta) = 1; \tag{2.3}$$

(*ii*) crossing symmetric, which in this case requires

$$S(\theta) = S(i\pi - \theta), \tag{2.4}$$

and is therefore $2i\pi$ periodic taking (2.3) and (2.4) together.

(*iii*) Additionally, particles may occur as bound state poles in S-matrix elements for $0 \le \theta \le i\pi$, with positive residue. Thus, particle c may occur as a bound state in the process $ab \to ab$ if there is a simple pole at $\theta = \theta_b - \theta_a = i\theta_{ab}^c$ in the S_{ab} S-matrix element (with a suitable residue). In that case (see figure)



we have,

$$\theta_a = \theta_c - i\left(\pi - \theta_{ac}^b\right) \qquad \theta_b = \theta_c + i\,(\pi - \theta_{bc}^a)\,, \tag{2.5}$$

and

$$\theta_b - \theta_a = i\left(2\pi - \theta_{ac}^b - \theta_{bc}^a\right) = i\theta_{ab}^c.$$

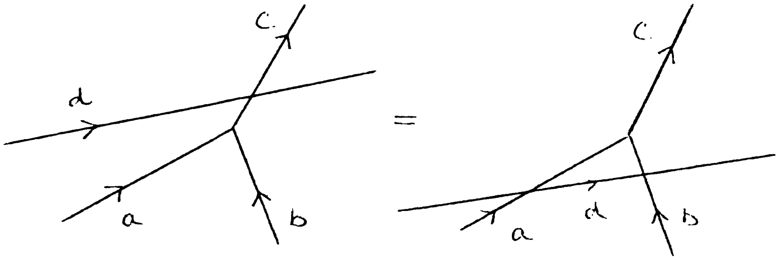
On the other hand,

$$\begin{aligned} m_c^2 = p_c^2 = (p_a + p_b)^2 &= m_a^2 + m_b^2 + 2m_a m_b \cosh(\theta_b - \theta_a) \\ &= m_a^2 + m_b^2 + 2m_a m_b \cos\theta_{ab}^c, \end{aligned} \tag{2.6}$$

and the bound state masses correspond to a particular rapidity value. Indeed, in view of (2.6) the three masses m_a, m_b, m_c form the sides of a triangle Δ_{abc}

with internal angles $\bar\theta^c_{ab}, \bar\theta^a_{bc}, \bar\theta^b_{ca}$ $(\equiv \pi - \theta^c_{ab}, \pi - \theta^a_{bc}, \pi - \theta^b_{ca})$. Equally, particle a or b may occur as a physical bound state in the processes $bc \to bc$ or $ca \to ca$, respectively, although attention must be paid to the sign of the residue.

(*iv*) The bootstrap property represents a non trivial consistency requirement on the scattering of three particles, implied by the existence of bound states which are themselves possible asymptotic states. It is indicated by the diagram



and expressed by the formula

$$S_{dc}(\theta) = S_{da}(\theta - i\bar\theta^b_{ac}) S_{db}(\theta + i\bar\theta^a_{bc}), \tag{2.7}$$

where $\theta = \theta_c - \theta_d$.

(*v*) The conserved quantities mentioned in the previous section must be consistent with the S-matrix. Since they are not in general scalars, this is a non trivial requirement. Thus, when two particles bind to form a third we would expect (using an analytic continuation argument), that if

$$P_s \left|a(\theta)\right\rangle = q^a_s e^{s\theta} \left|a(\theta)\right\rangle,$$

then

$$q^a_s e^{-is\bar\theta^b_{ac}} + q^b_s e^{is\bar\theta^a_{bc}} = q^c_s \tag{2.8}$$

for all possible fusing channels. There is a similar set of equations for the companion quantities bearing the opposite spins P_{-s}, but with no extra content.

Actually, eqs(2.8) are very strong in the sense that given a conjecture for the spins of the conserved quantities the possible fusing angles are highly constrained and hence also the possible particle masses. With a few extra assumptions, properties (i)-(iii) and in particular (iv) and (v) uniquely determine the 'minimal' model consistent with the set of conserved charges. Zamolodchikov's starting point assumes the existence of two particles 1 and 2 say, each containing itself and the other one as a possible 'fusion'. Provided the mass ratio m_2/m_1 is the 'golden ratio' (*i.e.* $2\cos\pi/5 \approx 1.618$) the fusion angles are consistent with the conserved quantities suggested in the previous section. However, the manifestly crossing symmetric unitary S-matrix

$$S_{11}(\theta) = \left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{3}{10}\right), \tag{2.9}$$

where

$$(\alpha) = \frac{\sinh\left(\frac{\theta}{2} + i\pi\alpha\right)}{\sinh\left(\frac{\theta}{2} - i\pi\alpha\right)},$$

does not yet satisfy the first of the bootstrap conditions (eq(2.7) with $a = b = c = d = 1$), to wit

$$S_{11}(\theta) = S_{11}\left(\theta - \frac{i\pi}{3}\right) S_{11}\left(\theta + \frac{i\pi}{3}\right).$$

However, this is remedied by the addition of one more pole at $\theta = i\pi/15$, with its crossed channel partner at $14\pi/15$ and an overall $-$ sign. Thus, instead of (2.9) we are forced to set

$$S_{11}(\theta) = -\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)\left(\frac{1}{5}\right)\left(\frac{3}{10}\right)\left(\frac{1}{30}\right)\left(\frac{14}{30}\right), \tag{2.10}$$

and recognise the need for a third particle 3 for which $m_3/m_1 = 2\cos\pi/30 \approx 1.989$. The next step uses the bootstrap to set up candidates for $S_{12}(\theta)$, $S_{13}(\theta)$ and so on, introducing extra particles as necessary to render the matrix elements consistent. Remarkably, the procedure closes on a 'minimal' solution containing eight particles in the computed mass ratios (to m_1):

1.618 1.989 2.405 2.956 3.218 3.891 4.783.

The question is, what is really going on and can we understand this minimal solution field theoretically? Perhaps there is no field theory for which the minimal solution occurs as a perturbative solution.

In the next section we shall describe a simple field theory which is certainly related to the S-matrix Zamolodchikov describes and which might be used as a framework within which to explore further. In any case, it seems to us that the connection with Toda theory is by itself interesting and revives interest in work begun many years ago.

3. AFFINE TODA FIELD THEORY

Integrable Toda theory associated to each Lie algebra Dynkin diagram has been much discussed in the past[8] and its quantum theory is known to be conformal[9]. It is natural then, when considering a perturbed conformal field theory, to seek a perturbed Toda theory, perturbed in such a way as to preserve integrability. It is also known that the conserved quantities of these theories do indeed occur at spins corresponding to the exponents of the underlying Lie algebra whose simple roots define the theory[10]. These considerations and the remarks in the previous two sections concerning E_8 suggest that the scalar field theory we seek might be affine Toda theory[11] (*i.e.* Toda theory with an additional exponential term depending on the extra root), whose Lagrangian after a field shift could be taken to be

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a - \frac{m^2}{\beta^2}\sum_{i=0}^{r} n_i\, e^{\beta\alpha_i\cdot\phi} \tag{3.1}$$

where $\sum_0^r n_i\alpha_i = 0$ and the r-component simple roots are selected from any one of the affine Dynkin diagrams (of rank $r+1$) in Kac' list[5]. The roots α_i are normalised so that the long roots have length $\sqrt{2}$, and the parameter β keeps track of the overall scale of the roots (classically it is irrelevant). We have organised (3.1) in such a way that the minimum of the potential (for real β) occurs at $\phi^a = 0, \quad a = 1, \dots, r$ and the potential may be expanded perturbatively about the origin. Thus

$$V(\phi) = \frac{m^2}{\beta^2} \sum_0^r n_i + \frac{m^2}{2} \sum_0^r n_i \alpha_i^a \alpha_i^b \phi^a \phi^b + \frac{m^2 \beta}{3!} \sum_0^r \alpha_i^a \alpha_i^b \alpha_i^c \phi^a \phi^b \phi^c + \dots, \tag{3.2}$$

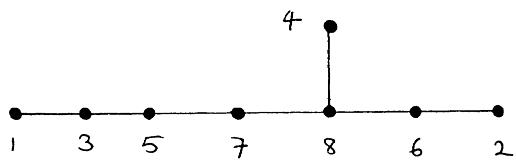
and, for any choice of roots we may compute the 'classical' masses and couplings. The first surprise is that for $e_8^{(1)}$, the affine diagram associated with E_8, the classical masses are in precisely the ratio discovered by Zamolodchikov. In other words, diagonalising the $(\text{mass})^2$ matrix

$$(M^2)_{ab} = m^2 \sum_0^r n_i \alpha_i^a \alpha_i^b \tag{3.3}$$

yields (in ascending mass order), the eigenvalues m_i^2/m^2:

$$\begin{gathered} 4\sqrt{3} \sin \pi/30 \sin \pi/5 \\ 16\sqrt{3} \sin \pi/30 \sin \pi/5 \cos^2 \pi/5 \\ 16\sqrt{3} \sin \pi/30 \sin \pi/5 \cos^2 \pi/30 \\ 64\sqrt{3} \sin \pi/30 \sin \pi/5 \cos^2 \pi/5 \cos^2 7\pi/30 \\ 4\sqrt{3} \sin 11\pi/30 \sin \pi/5 \\ 4\sqrt{3} \sin 7\pi/30 \sin 2\pi/5 \\ 4\sqrt{3} \sin 13\pi/30 \sin 2\pi/5 \\ 256\sqrt{3} \sin \pi/30 \sin \pi/5 \cos^2 2\pi/15 \cos^4 \pi/5. \end{gathered} \tag{3.4}$$

An even greater surprise is that the 8-vector $\mathbf{m} = (m_1, \dots, m_8)$ is an eigenvector of the E_8 Cartan matrix corresponding to the smallest eigenvalue $2 \cos \pi/30$. This enables us to assign the masses unambiguously to the vertices of the Dynkin diagram for E_8:



(This is also in ascending order in dimension of the first eight representations of E_8.)* For the other affine theories the masses fit into a similar pattern. However, we have no satisfactory explanation for these facts.

★ These masses are also numerically the same as the 'dimensions' of Ocneanu's operators associated with[12] E_8.

Once the mass matrix is diagonalised, the couplings of mass eigenstates are computed (but not by hand!), and are found to be zero except for the entries in the table at the end of this paper. The top entry in each row in the upper half of the table labels the bound state particle in ascending mass (and indicates the sign of the coupling), the lower entry is the fusion angle as a multiple of $\pi/30$. The magnitude of the couplings is given by the general formula

$$|c^{abc}| = \frac{2}{\sqrt{h}} m_a m_b \sin \theta_{ab}^c \tag{3.5}$$

where h is the Coxeter number (30 for E_8). Actually, the same formula is applicable to all simply laced Toda theories (*i.e.* corresponding to the $a^{(1)}, d^{(1)}$ and $e^{(1)}$ series) and, with slight modification, to the non-simply laced cases as well. We will not discuss the signs of the couplings any further.

Using the masses and couplings we can conjecture a 'minimal' S-matrix (the part ignoring β) consistent with the bootstrap programme outlined in section (2). The result for E_8 is presented in the lower half of the table containing the couplings. The notation is quite condensed since:

$$[\alpha] = \left(\frac{\alpha}{60}\right)\left(\frac{30-\alpha}{60}\right), \tag{3.6}$$

a manifestly unitary and crossing symmetric building block for the S-matrix elements. We also note,

$$[-\alpha] = [\alpha]^{-1} \qquad [30-\alpha] = [\alpha] = [\alpha + 60].$$

This S-matrix is the one obtained by Zamolodchikov[1], although he did not reveal the full table.

There are a number of interesting remarks to be made. First of all, the positions of the bound state poles are fixed by the bootstrap and for the conjectured S-matrix to actually follow from a Lagrangian like that of (3.1) there must be miracles in perturbation theory. For example, we have checked to one loop order that masses all renormalise *in the same way* so that their ratios are unaffected by the coupling constant β. To do this requires the evaluation of propagator bubbles and the result (again valid for each of the a, d, e series) is

$$\frac{\Delta m_i^2}{m_i^2} = \frac{\beta^2}{4h} \cot\left(\frac{\pi}{h}\right) \qquad i = 1, \ldots, r \tag{3.7}$$

an interesting universal formula. Obviously a proof to all orders is highly desirable.

Secondly, since the masses (and wave functions) renormalise we do not expect the couplings computed from the S-matrix to coincide with those of the bare Lagrangian. Moreover, the 'minimal' S-matrix requires adjustment to take into account the β dependence (as $\beta \to 0$ we expect the S-matrix to be unity!). In spite of this, the basic couplings in the Lagrangian appear to have topological significance in the sense that when a coupling is zero in the Lagrangian (3.1) it will not appear in the full S-matrix. Again, for this to be the case, any couplings that can be generated at higher order (e.g. a triangle vertex graph) must cancel against each other or with higher order terms in the Lagrangian. To check this for the $e_8^{(1)}$ case is a formidable task, but we have done 'random' checks on other, simpler theories in the a, d, e series. Similarly, it is not obvious from the Lagrangian that production processes cannot occur (they are forbidden in the S-matrix by the requirement of consistency with the conserved quantities). However, we have found it to be the case wherever we have checked, at least for trees and to one loop. It is as though the higher order part of the Lagrangian is there for the express purpose of cancelling the unwanted processes, the basic structure being dictated by the masses and the three point coupling terms[13].

Finally, we may conjecture (and then attempt to verify at least to order β^4) the coupling constant dependence of the S-matrix along the lines proposed by Arinshtein et al[14]. Suppose we take the $S_{11}^{(0)}(\theta)$ matrix element from the $e_8^{(1)}$ table:

$$S_{11}^{(0)}(\theta) = -[20][12][2],$$

and modify it by adding a factor whose sole purpose is to remove the poles and zeroes as $\beta \to 0$. It must satisfy the same bootstrap and consistency conditions as $S^{(0)}$ and avoid introducing extra poles in the physical strip. We would guess that factor to be something like

$$[-20+b][-12+b][-2+b], \tag{3.8}$$

where $b = \beta^2 \sum_0^\infty b_n \beta^{2n}$. However, (3.8) fails to satisfy the bootstrap (and anyway the residue signs are wrong). To correct it needs a further factor. We set,

$$\begin{aligned} S_{11}^{(\beta)}(\theta) &= S_{11}^{(0)}(\theta)[-20+b][-12+b][-2+b][-b] \\ &\equiv S_{11}^{(0)}(\theta) F_{11}(\beta) \end{aligned} \tag{3.9}$$

which does work satisfactorily (note, $[0] = -1$). Moreover, the rest of the bootstrap appears consistent with this ansatz. Note too, at $b = 2$ the S-matrix is again unity and, further, (3.9) is invariant under the transformation $b \to 2 - b$. It is tempting to conjecture that $b \to 2$ corresponds to the large β limit of the theory. A natural assumption is to take the function b to be given by an expression of the form

$$b(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1+\beta^2/4\pi}, \tag{3.10}$$

as suggested by earlier work. It satisfies

$$b(4\pi/\beta) = 2 - b(\beta),$$

displaying the symmetry between strong and weak coupling limits emphasised in the context of non-affine Toda theory by Hollowood and Mansfield[2] (who appear to have set $\hbar = 4\pi$ rather than unity). Whether (3.10) is a sensible conjecture requires checking in perturbation theory. A first non trivial check is to examine tree diagrams, this gives the small β behaviour of $S^{(\beta)}$ and verifies (3.9) and (3.10). Similar conjectures for a, d, e series agree to this order with (3.10). (An explanation for (3.5)?)

Actually, in any of the a, d or e theories with self-conjugate particles we would define similarly

$$[\alpha] = \left(\frac{\alpha}{2h}\right)\left(\frac{h-\alpha}{2h}\right), \tag{3.11}$$

where h is the appropriate Coxeter number and analogous statements hold concerning the factor $F(\beta)$.

TABLE I E_8 COUPLINGS AND S-MATRICES

1^* 2 3^* 20 12 2	1 2^* 3 4^* 24 18 14 8	1^* 2 4 5 29 21 13 3	2^* 3 4^* 5^* 6 25 21 17 11 7	3 4^* 6 7^* 28 22 14 4	4 5 7 25 19 9	5^* 6 8 27 23 5	7 8 26 16
	1^* 2 4 5 6^* 24 20 14 8 2	1 3 6 25 19 9	1^* 2 7 27 23 5	2 6 26 16	2^* 3 5 7 8 29 25 19 13 3	4 6 7^* 8 27 21 17 11	6 7 28 22
		2 3 5 6 7 22 20 14 12 4	1 5 26 16	1 3 4 7 8 29 23 21 13 5	2 3 6 8 26 24 18 8	3 5 28 22	5 6 8^* 27 25 17
−[20] [12] [2]			1^* 4^* 6 7^* 8^* 26 20 16 12 2	1^* 3 5^* 8 27 23 19 9	1 4 28 22	2 4^* 7 8^* 28 24 18 14	4^* 5 7^* 29 25 21
[24] [18] [14] [8]	−[24] [20] [14] [8] [2] $[12]^2$			4^* 5 8^* 22 20 12	1 2 7^* 27 25 17	1^* 3 6^* 29 25 21	3 4 5^* 8^* 28 26 24 18
[29] [21] [13] [3] $[11]^2$	[25] [19] [9] [15] $[7]^2$ $[13]^2$	−[22] $[20]^3$ [14] $[12]^3$ [4] $[28]^2$			3 6^* 8^* 24 20 14	1 2 5^* 8^* 28 26 22 16	2 3 6^* 7^* 29 27 23 21
[25] [21] [17] [11] [7] [15]	[27] [23] [5] [15] $[19]^2$ $[9]^2$ $[13]^2$	[26] $[16]^3$ $[20]^2$ $[12]^2$ $[6]^2$ $[8]^2$	−[26] $[20]^3$ $[16]^3$ $[12]^3$ [2] $[6]^2$ $[8]^2$			2^* 4 7^* 26 24 20	1 2 4^* 6^* 8 29 27 25 23 19
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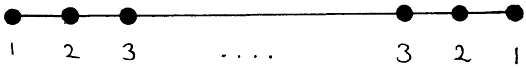
If one examines the S-matrix elements carefully, it is clear that the pole structure is very complex, many of the bound state poles being hidden in a multipole structure. We have not analysed these carefully, but it is expected that all these singularities are explicable in terms of singularities of Feynman loop diagrams in perturbation theory(15). We give some examples in the next section, not for $e_8^{(1)}$ but for $d_4^{(1)}$ which is rather simpler to understand.

The detailed structure of the $e_7^{(1)}$ theory has been elucidated by Christe and Mussardo(4).

4. OTHER THEORIES: BRIEF REMARKS

In this section we shall briefly remark upon other affine Toda field theories(3). The simplest in many ways is the $a^{(1)}$ series. There, the exponents are the set of integers $1, 2, 3, \ldots, n$ and the masses are $2\sin \pi k/(n+1)$. For n even every particle occurs with its conjugate partner, for n odd there is one self-conjugate particle (the heaviest). The particles are associated with the Dynkin diagram in mass order as before:



The couplings are given by the nice formula:

$$\begin{aligned} c^{abc} &= 0 \quad \text{if} \quad a+b+c \neq 0 \mod n+1 \\ &= \frac{1}{\sqrt{n+1}}(1-\omega^a)(1-\omega^b)(1-\omega^c) \\ &\quad\ \text{if} \quad a+b+c = 0 \mod n+1 \end{aligned}$$

where ω is an $(n+1)$th root of unity. The fusing angles are given by

$$\theta_{ab}^c = \frac{a+b}{n+1}\pi, \quad a+b+c = n+1; \quad \theta_{ab}^c = 2\pi - \frac{a+b}{n+1}\pi, \quad a+b+c = 2(n+1).$$

The associated minimal S-matrix is that of Köberle and Swieca(16) with a coupling dependence conjectured by Arinshtein et al(14). We shall not reproduce it here.

The $d_n^{(1)}$ theories are more subtle. The exponents are $1, 3, 5, \ldots, 2n-3, n-1$ with a set of masses

$$\sqrt{2},\ \sqrt{2},\ 2\sqrt{2}\sin\frac{k\pi}{2(n-1)} \qquad k = 1, 2, \ldots, n-2. \tag{4.1}$$

For n even, each particle is self-conjugate, for n odd the pair with equal mass are conjugates. Occasionally, for $n = 1 \mod 3$, there is another particle mass degenerate with the pair. However, it is self-conjugate. Attached to the Dynkin diagram the degenerate pair correspond loosely speaking to the spinor, conjugate-spinor representations of the algebra. For even n there is also a degenerate pair of conserved charges with spin $n-1$. Each of these theories has a minimal S-matrix associated with it, but we list only $d_4^{(1)}$ to illustrate certain general features. There, the masses are $\sqrt{2}$ for each of the three light particles (l, l', l'') and $\sqrt{6}$ for the heavy particle h. The light particles are distinguished by the conserved charges of spin 3. The three non-zero couplings are:

$$c^{ll'l''} = \sqrt{2}, \quad c^{llh} = \frac{-1}{\sqrt{2}}, \quad c^{hhh} = \frac{1}{\sqrt{2}}$$

The minimal S-matrix has the form

$$S_{ll'}(\theta) = [4]$$

where l, l' are any two different light particles, and

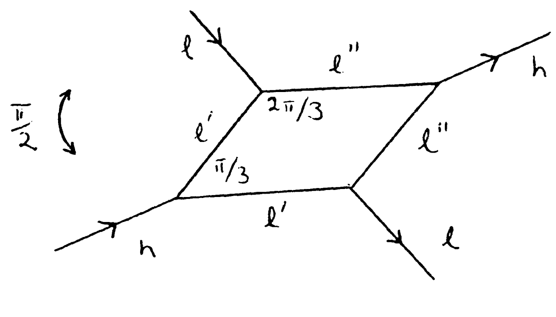
$$\begin{aligned} S_{ll}(\theta) &= -[4] \\ S_{lh}(\theta) &= [1][3] \\ S_{hh}(\theta) &= -[4]^3 \end{aligned} \tag{4.2}$$

where we have used the same notation as before (3.11), with $h = 6$. The β dependence is derived consistently from the ansatz

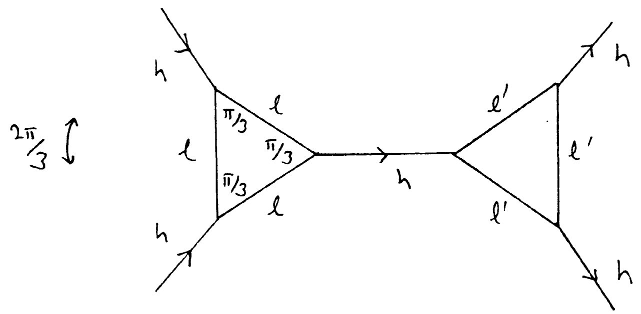
$$F_{ll'}(\beta) = [-4 + b] \tag{4.3}$$

the bootstrap and the conjecture (3.10).

In this case it is relatively easy to check the claim about mass renormalisation (eq(3.7)) and the absence of production processes, at least to order β^2. It is also easy to identify the perturbative Feynman diagrams responsible for the second and third order poles in expressions (4.2). The idea, originally exploited by Coleman and Thun[17], is to identify diagrams which can be drawn as geometrical figures with internal lines contracted or on-shell. For example, the double pole in S_{lh} at $\theta = i\pi/2$ occurs in the box diagram



where the internal angles sum to 2π and hence corresponds to a planar figure with the internal light particles on their mass shells. The triple pole in S_{hh} arises differently, from a pair of vertex corrections.



Each triangle graph is singular (a simple pole) at $\theta = 2\pi i/3$, as evidenced by the internal angles summing to π when the light particles are on shell.

Obviously, the pole structure of the $e_8^{(1)}$ theory is more complex. Nevertheless, the ideas are identical although it will require some ingenuity to spot the relevant diagrams (where are the twelfth order poles in $S_{88}(\theta)$, for example?). We are confident this will work although we have not checked all possibilities explicitly.

The $d_n^{(1)}$ theories are interesting for another reason which we mention briefly. Recall the Sine-Gordon theory has a spectrum of solitons (a pair) and breathers[18]. In terms of the Lagrangian parameters,

$$V(\phi) = \frac{\alpha}{\kappa^2} \cos \kappa\phi$$

the soliton mass is given by

$$m_{soliton} \equiv m = \frac{8\sqrt{\alpha}}{\kappa^2}$$

and the spectrum of breathers by

$$m_k = 2m \sin\left(\frac{k}{16}\bar{\kappa}^2\right) \qquad \bar{\kappa}^2 = \frac{\kappa^2}{1 - \kappa^2/8\pi} \tag{4.4}$$

where $k = 1, 2, \ldots < 8\pi/\bar{\kappa}^2$. For certain values of κ (actually just before a new breather enters the spectrum), there is known to be no reflection in the Sine-Gordon S-matrix. This occurs just at the moment when $8\pi/\bar{\kappa}^2$ reaches the next integer. At that point, $\bar{\kappa}^2 = 8\pi/N$ say, and the breather masses are

$$2m \sin\left(\frac{k\pi}{2N}\right) \qquad k = 1, 2, \ldots, N-1$$

i.e. precisely those of the $d_{N+1}^{(1)}$ theory (including the soliton masses). Moreover, the S-matrix is essentially identical to the 'minimal' $d_{N+1}^{(1)}$ S-matrix. For example, $\kappa^2 = 2\pi$ corresponds to $\bar{\kappa}^2 = 8\pi/3$ and a spectrum of three light and one heavy particles in the mass ratio 1 to $\sqrt{3}$. A comparison of the Faddeev-Korepin[19] S-matrix (and its extension to breathers[20]), with the expression (4.2) reveals the identity, apart from signs in some of the $l - l'$ matrix elements which are not in any case fixed by the bootstrap. Remarkably, it seems that tuning the Sine-Gordon coupling constant κ

yields successive members of a tower of $d_n^{(1)}$ minimal theories. We strongly suspect this is related to perturbed $c = 1$ conformal field theory. We note that the coset models $d_n^{(1)} \times d_n^{(1)}/d_n^{(2)}$ all have $c = 1$, and that perturbations of $c = 1$ theories have been identified with Sine-Gordon theory[21]. As with the Ising-$e_8^{(1)}$ relationship there remains the problem of understanding the affine Toda coupling constant dependence. How are two theories related if their minimal (or in the case of Sine-Gordon, actual) S-matrices happen to agree?

COMMENTS

The major problem in relating this work to perturbed conformal field theory lies in understanding the rôle of the coupling constant β. To obtain a value of the central charge c small enough to lie in one of the discrete series of coset models Hollowood and Mansfield were forced to assume an analytic continuation to imaginary β. The same conclusion has been reached by Eguchi and Yang[22], from a different starting point. It is not however clear that such a choice for β would make sense, from the straightforward Lagrangian field theory point of view, to compute an S-matrix. For example, the bootstrap proposed by Zamolodchikov provides a minimal S-matrix identical with the one we have described. However, an imaginary coupling in the postulated extra factor $F(\beta)$ would appear to alter the physical pole structure. Thus, we are not really able to confirm that the theory described by eq(3.1) actually coincides with Zamolodchikov's conjecture for any value of the coupling despite the fact that it shares the same minimal S-matrix.

On the other hand, although we were motivated to study this problem by the work on conformal field theories, the results we have obtained are independently interesting. For example, although we have not mentioned it in this article so far, the a, d, e series is rather special, here as elsewhere [3,23], in the following sense. Firstly, each of the Toda theories corresponding to a non-simply laced algebra is obtainable as a truncation of a theory in the a, d, e series. Each of the simply-laced theories enjoys a minimal S-matrix in the sense we have described but the S-matrix does not respect the truncation. In fact, the non-simply laced theories do not have minimal S-matrices; for example, the masses fail to renormalise universally. It seems that the degeneracies present in most of the a, d, e theories are actually important; if they are removed by truncation the nice features tend to disappear. In the non-simply laced theories the coupling constant dependence must play a more significant role in the bootstrap. There are many questions for the future besides elucidating the connection with conformal field theory.

ACKNOWLEDGEMENTS

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REFERENCES

1. A. B. Zamolodchikov, Integrable Field Theory from Conformal Field Theory, Proceedings of the Taniguchi Symposium, Kyoto (1988)

   —, RAL preprint 89-001.

   —, JETP Letters 46 (1987) 161

   —, Int. J. Mod. Phys. A3 (1988) 743
2. T. J. Hollowood and P. Mansfield, Phys. Letts. B226 (1989) 73
3. H. W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, Durham preprint UDCPT 89-23, to appear in Physics Letters B.
4. P. Christe and G. Mussardo, Santa Barbara preprint (revised) UCSBTH 89-19
5. S. Helgason, Differential geometry, Lie groups, and symmetric spaces (Academic Press Inc. 1978)
6. P. Bowcock and P. Goddard, Nucl. Phys B285 (1987)651
7. A. B. Zamolodchikov, Al. B. Zamolodchikov, Ann. Phys. 120 (1979) 253, and references therein.
8. A. N. Leznov and M. V. Saveliev, Group methods for the integration of nonlinear dynamical systems (Moscow Nauka 1985)
9. J-L. Gervais and A. Neveu, Nucl. Phys. B224 (1983) 329

   P. Mansfield, Nucl. Phys B222 (1983) 419

   E. Braaten, T. Curtright, G. Ghandour and C. Thorn, Phys. Letts. B125 (1983) 301

10 G. Wilson, Ergod. Th. and Dynam. Sys. 1 (1981) 361

11. A. V. Mikhailov, M. A. Olshanetsky and A. M. Perelomov, Comm. Math. Phys. 79 (1981) 473

    D. I. Olive and N. Turok, Nucl. Phys. B215[FS7] (1983) 470
12. A. Ocneanu, Lectures to this conference.
13. C. J. Goebel, Prog. Theor. Phys. Suppl. 86 (1986) 261
14. A. E. Arinshtein, V. A. Fateev and A. B. Zamolodchikov, Phys. Letts. B87 (1979) 389
15. R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, The Analytic S-matrix, (Cambridge University Press 1966)
16. R. Köberle and J. A. Swieca, Phys. Letts. B86 (1979) 209
17. S. Coleman and H. Thun, Comm. Math. Phys. 61 (1978) 31
18. S. Coleman, Aspects of Symmetry, (Cambridge University Press 1985)
19. L. D. Faddeev and V. E. Korepin, Theor. Mat. Fiz. 25 (1975) 147
20. M. Karowski and H.-J. Thun, Nucl. Phys. B130 (1977) 295
21. H. Yoshii, Phys. Letts. B223 (1989) 353
22. T. Eguchi and S-K. Yang, Phys. Letts. B224 (1989) 373
23. A. Cappelli, C. Itzykson and J-B Zuber, Comm. Math. Phys. 113 (1987) 1

    V. I. Arnold, Catastrophe Theory (Springer-Verlag 1984)

CODES, LATTICES, AND CONFORMAL FIELD THEORY

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ABSTRACT

Analogies and connections between Codes, Lattices, and Conformal field theories are discussed and elaborated.

TITLE OF LECTURE SERIES: TOPICS IN CONFORMAL FIELD THEORY

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91191 Gif/Yvette Cedex, France

The following topics and there inter-relations are

1– Continuous phase transition;

2– Conformal invariance at T_c;

3– Hamiltonian (operator) formalism;

4– Irreducible modules;

5– Modular invariance – classification of universality classes;

6– ADE connection.

discussed and elaborated.

FEIGIN-FUCHS REPRESENTATION OF STRING FUNCTIONS

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ABSTRACT

The null space structure of the parafermionic theory is studied. A Feigin-Fuchs representation with two bosons on a Lorenzian lattice is used. The null states are constructed as contour integrals of screening operators. The characters or the string functions for the parafermionic theory are derived from the null states.

COULOMB-GAS CONSTRUCTION ON HIGHER-GENUS RIEMANN SURFACES

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1. Introduction

The Coulomb-gas construction provides a powerful tool for computing correlation functions in the minimal series of two-dimensional conformal field theories. The basic idea is to replace the irreducible highest-weight representations of the Virasoro algebra with the simpler Fock space of a free scalar field. It has been known for some time [1] that on such Fock spaces one can construct representations of the Virasoro algebra with arbitrary central charge.

In [2], Feigin and Fuchs showed how to recover the minimal-model representations from these Fock spaces. Fateev and Dotsenko [3] then developed the Coulomb-gas approach, introducing screening operators to compute multipoint correlators on the plane. However, the detailed relation between their work and that of Feigin and Fuchs was unclear. More recently, Felder [4], using results from [2] and [5], explained this relation using a BRST-like construction. As a result of his analysis, he was able to find a Coulomb-gas construction for the correlation functions on the torus. Similar expressions were also found in [6].

In this talk we will extend these results to higher-genus surfaces. Our approach will be to define the higher-genus correlators by sewing amplitudes on spheres and tori [7]. We will then show how to write these expressions in terms of a higher-genus scalar field theory [8][9][10].

2. Coulomb-Gas Representation

Before we begin, let us first review the Coulomb-gas representation for the minimal models. We start with a single scalar field, described by the action

$$S = \frac{1}{4\pi}\int d^2x\,\sqrt{h}\,(\,h^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\;-\;i\alpha_0 R\phi\,)\;+\;B.T. \tag{1}$$

Here $h_{\mu\nu}$ is the metric on a Riemann surface of genus g, and R is its curvature, which satisfies

$$\frac{1}{8\pi}\int d^2x\sqrt{h}R \;=\; (1-g)\,. \tag{2}$$

Differential Geometric Methods in Theoretical Physics
Edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990

$B.T.$ denotes a boundary term which is necessary to define winding sectors on surfaces with $g > 1$ [11].

The action (1) is not quite the action for a free scalar field. It differs by the term proportional to R. This term changes the central charge of the theory from one to

$$c = 1 - 24\alpha_0^2 . \tag{3}$$

With $\alpha_0^2 = (p' - p)^2/4pp'$, equation (3) reproduces the central charges of the minimal models.

The primary fields in this theory are the vertex operators $V_\alpha \equiv \exp i\alpha\phi$. Because of the curvature term, correlators of such fields will vanish unless

$$\sum_j \alpha_j - 2\alpha_0(1-g) = 0 . \tag{4}$$

If we think of the vertex operator $V_\alpha(z)$ as inserting a "charge" α at the point z, we see that (4) ensures charge neutrality on a genus-g surface. The curvature term in (1) gives rise to the "background charge" $-2\alpha_0(1-g)$.

In the remainder of this section we will focus our attention on the simplest Riemann surface, the sphere S^2. On this surface, the neutrality condition reduces to

$$\sum_j \alpha_j - 2\alpha_0 = 0 , \tag{5}$$

which implies that the operator conjugate to V_α is $V_{2\alpha_0-\alpha}$. The condition (5) has a simple interpretation, which is most easily seen by choosing $h_{\mu\nu}$ to be the singular metric on S^2 that is flat everywhere except at one point, "the point at infinity." This metric gives rise to a delta-function singularity in R, and the exponential of the curvature term in (1) becomes a vertex operator at infinity. The charge of this vertex operator must be cancelled by the other fields in the correlation function.

Let us now suppose that the field ϕ is compactified on a circle of radius $\sqrt{pp'}$, so ϕ is identified with $\phi + 2\pi\sqrt{pp'}$. The Hilbert space of the theory then decomposes into a direct sum over products of Fock spaces $F_\alpha \otimes \bar{F}_{\bar{\alpha}}$. For simplicity, we restrict our discussion to the holomorphic factor F_α, and consider just the holomorphic contributions to the correlation functions, called conformal blocks.

The Fock space F_α is constructed from the state $|\alpha\rangle \equiv \exp i\alpha\phi(0)|0\rangle$ by arbitrary applications of the oscillators

$$a_n = \oint \frac{dz}{2\pi} z^n \partial_z \phi ,$$

with $n < 0$. It is not hard to show that the state $|\alpha\rangle$ satisfies the conditions $a_0|\alpha\rangle = \alpha|\alpha\rangle$, $a_n|\alpha\rangle = 0$, $n > 0$. Furthermore, each Fock space F_α spans a (reducible) representation of the Virasoro algebra, where the Virasoro generators are given by

$$L_m = \frac{1}{2}\sum_{k=1}^{\infty} a_{m-k}a_k - \alpha_0(m+1)a_m + \frac{1}{4}\alpha_0^2\,\delta_{m,0}\,.$$

The states in F_α are graded by their eigenvalues under L_0. Since $[L_0, a_n] = -na_n$, the state $|\alpha\rangle$ is the unique eigenvector with the minimal eigenvalue $h_\alpha = \alpha(\alpha - 2\alpha_0)$. Setting

$$\alpha_{n',n} = \left(\frac{1-n'}{2}\right)\alpha_- + \left(\frac{1-n}{2}\right)\alpha_+\,, \tag{6}$$

where $\alpha_+ = \sqrt{p'/p}$, $\alpha_- = -1/\alpha_+$ and $\alpha_0^2 = (p'-p)^2/4pp'$, one finds

$$h_{n',n} = \frac{(n'p - np')^2 - (p-p')^2}{4pp'}\,. \tag{7}$$

For $1 \le n' \le p'-1$ and $1 \le n \le p-1$, these are precisely the dimensions of the primary fields in the minimal models. However, it is important to note that the Fock spaces $F_{\alpha_{n',n}} \equiv F_{n',n}$ are *not* the corresponding irreducible highest-weight representations. The irreducible representations are found by restricting one's attention to the subspace of $F_{n',n}$ generated by the L_{-m}, and then modding out by all the singular vectors that appear.

In Refs. [2] and [4], this procedure was made precise. It was shown that there exist BRST-like operators

$$\begin{aligned} Q_{2j} &: F_{n',n+2jp} \to F_{n',2jp-n} \\ Q_{2j+1} &: F_{n',2jp-n} \to F_{n',n-(2j+1)p} \end{aligned} \tag{8}$$

with $j \in \mathbf{Z}$, which commute with the Virasoro algebra, and satisfy $Q_jQ_{j+1} = 0$. It was also shown that the irreducible highest-weight representation of dimension $h_{n',n}$ is given by

$$\mathcal{H}_{n',n} = H^0 \equiv \mathrm{Ker}Q_0/\mathrm{Im}Q_{-1}\,,$$

which is the cohomology in the middle degree of the complex:

$$\cdots\ F_{n',2p-n} \xrightarrow{Q_{-1}} F_{n',n} \xrightarrow{Q_0} F_{n',-n}\ \cdots$$

The cohomology in all other degrees is zero. Thus, if we define

$$\mathcal{F}_{n',n} \;=\; \oplus_j F_{n',n+2jp} \qquad\qquad \mathcal{F}_{n',-n} \;=\; \oplus_j F_{n',-n+2jp}$$

we have

$$\mathcal{F}_{n',n} \;\cong\; H^0 \,\oplus\, \mathcal{F}_{n',-n} \tag{9}$$

as an isomorphism of Virasoro modules.

To compute the correlation functions, it is helpful to introduce the notion of chiral vertex operators. A chiral vertex operator $\phi^{l'l}_{n'n,m'm}$ is a primary field which maps $\mathcal{H}_{m'm}$ to $\mathcal{H}_{l'l}$

$$\phi^{l'l}_{n'n,m'm} : \mathcal{H}_{m'm} \;\rightarrow\; \mathcal{H}_{l'l}\ ,$$

normalized so that its matrix element between the highest-weight states is one. In [4] it is shown that $\phi^{l'l}_{n'n,m'm}$ may be represented by a BRST-invariant operator $V^{r',r}_{n',n}$ which maps the Fock space $F_{m',m}$ to the Fock space $F_{l',l}$

$$V^{r',r}_{n',n} : F_{m',m} \;\rightarrow\; F_{l',l}\ .$$

We call this operator a screened vertex operator. It is given by

$$\begin{aligned} V^{r',r}_{n',n}(z) \;&=\; \oint du_1 \ldots du_{r'}\, dv_1 \ldots dv_r\; V_{\alpha_{n',n}}(z) \\ &\qquad \times\; V_{\alpha_-}(u_1)\ldots V_{\alpha_-}(u_{r'})\, V_{\alpha_+}(v_1)\ldots V_{\alpha_+}(v_r) \\ &\equiv\; \oint\; V_{\alpha_{n',n}}(z)\, V^{r'}_{\alpha_-}\, V^{r}_{\alpha_+}\ , \end{aligned} \tag{10}$$

where $r' = \frac{1}{2}(l'+1-m'-n')$ and $r = \frac{1}{2}(l+1-m-n)$. In (10), the operators V_{α_-} and V_{α_+} are dimension-one screening operators which ensure that the screened vertex operator has the correct charge, without changing the conformal properties of $V_{\alpha_{n',n}}$. The contours begin and end at the point z, as described in [4]. By deforming the contours, one can show that the correlators computed with (10) agree with those computed by Fateev and Dotsenko [3].

3. Higher-Genus Surfaces

We shall now extend these results to higher genus. We proceed by induction, and define the higher-genus correlators by sewing correlators on surfaces of lower genus. Essentially, one sews two correlators as follows: Consider two Riemann surfaces Σ_1 and Σ_2 of genus g_1 and g_2, respectively. Remove points p_1 and p_2 from Σ_1 and Σ_2, and choose coordinates z_1 and z_2 which map neighborhoods of

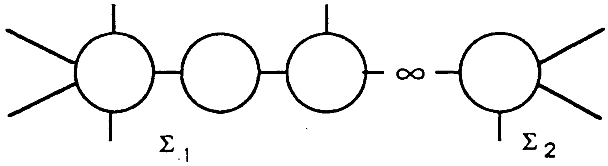

Figure 1: A higher-genus correlator is defined by sewing correlators of lower genus.

p_1 and p_2 to the complex plane. For each $t \in \mathbf{C}$, with $|t| < 1$, construct a new surface $\Sigma = \Sigma_1 \infty \Sigma_2$ of genus $g_1 + g_2$ by identifying points on Σ_1 and Σ_2 which satisfy $z_1 = t/z_2$. A correlator $\langle \psi_1 \dots \psi_n \rangle_\Sigma$ on the sewn surface is then defined by [7]

$$\sum_{i,j,k} t^{-\Delta_{i,k}} \langle \psi_1 \dots \psi_m \, \mathcal{L}_{-i}\chi_k(p_1) \rangle_{\Sigma_1} \, \langle \mathcal{L}_{-j}\widetilde{\chi}_k(p_2) \, \psi_{m+1} \cdots \psi_n \rangle_{\Sigma_2} \, \mathcal{M}^{-1}_{k,ij} \; . \tag{11}$$

In (11), we have suppressed the antiholomorphic dependence of the correlator. The sum over $\mathcal{L}_{-i}\chi_k$ includes the full set of primary fields and their Virasoro descendants. The $\mathcal{L}_{-j}\widetilde{\chi}_k$ are the corresponding conjugate fields, and $\Delta_{i,k}$ is the dimension of $\mathcal{L}_{-i}\chi_k$. The matrix $\mathcal{M}_{k,ij}$ is the matrix of two-point functions on the sphere. This procedure is illustrated schematically in Figure 1.

To isolate a particular block in the sewn correlator, we choose blocks in the original correlators, and then restrict the sum over k to a single field χ. In what follows, we will represent this block in terms of a free scalar field, whose correlation functions are defined by the functional integral

$$\langle \psi_1 \dots \psi_n \rangle_\Sigma = \int \mathcal{D}\phi \; e^{-S} \; \psi_1 \dots \psi_n \; , \tag{12}$$

where $\phi : \Sigma = \Sigma_1 \infty \Sigma_2 \to S^1$. We will make use of the fact that (12) can also be represented by sewing, where the sum runs over all *Fock*-space descendents,

$$\sum_{i,j,k} t^{-\Delta_{i,k}} \langle \psi_1 \dots \psi_m \mathcal{A}_{-i}\chi_k(p_1) \rangle_{\Sigma_1} \, \langle \mathcal{A}_{-j}\widetilde{\chi}_k(p_2) \psi_{m+1} \cdots \psi_n \rangle_{\Sigma_2} \, \mathcal{M}^{-1}_{ij} \; . \tag{13}$$

The sum over k runs over all fields χ_k which are highest weight with respect to the Fock algebra, and $\mathcal{A}_{-i}\chi_k$ denotes an arbitrary Fock descendent. It is instructive to check that (13) is consistent with charge-conservation (4) on the surfaces Σ, Σ_1 and Σ_2.

We now have what we need to write the minimal-model conformal blocks in terms of a scalar field. We start with (11) and restrict the sum over k to a fixed field χ of dimension $h_{n',n}$. By induction, we assume that we already

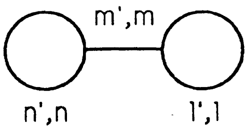

Figure 2: A genus-two block.

have scalar-field representations for the correlators $\langle \psi_1 \cdots \psi_m \mathcal{A}_{-i}\chi(p_1)\rangle_{\Sigma_1}$ and $\langle \mathcal{A}_{-j}\widetilde{\chi}(p_2)\psi_{m+1}\cdots\psi_n\rangle_{\Sigma_2}$, in which the fields χ and $\widetilde{\chi}$ are represented by screened vertex operators

$$
\begin{aligned}
V^{r',r}_{n',n}(p_1) &= \oint V_{\alpha_{n',n}}(p_1)\, V^{r'}_{\alpha_-}\, V^{r}_{\alpha_+} \\
V^{\widetilde{r}',\widetilde{r}}_{p'-n',p-n}(p_2) &= \oint V_{\alpha_{p'-n',p-n}}(p_2)\, V^{\widetilde{r}'}_{\alpha_-}\, V^{\widetilde{r}}_{\alpha_+}\ .
\end{aligned}
\tag{14}
$$

However, these operators differ from those of section 2 because the points p_1 and p_2 do not appear on the surface Σ. The vertex operators $V_{\alpha_{n',n}}(p_1)$ and $V_{\alpha_{p'-n',p-n}}(p_2)$ are missing as well, even though their screening operators are still present. This implies that the contours in (14) must be modified to run *around* p_1 and p_2. The vertex operators with the closed contours are related to those with the standard contours by a change of normalization. In a similar fashion, the fields ψ_i are also written in terms of screened vertex operators (or their Virasoro descendents). The screening operators associated with these fields are chosen to have the same contours that they had on the original surfaces Σ_1 and Σ_2.

With this said, we are free to represent (11) as follows:

$$
\begin{aligned}
\sum_{i,j} \oint t^{-\Delta_i} &\langle \psi_1 \cdots \psi_m\, V^{r'}_{\alpha_-}\, V^{r}_{\alpha_+}\, \mathcal{L}_{-i} V_{\alpha_{n',n}}(p_1)\rangle_{\Sigma_1} \\
&\times\ \langle \mathcal{L}_{-j} V_{\alpha_{p'-n',p-n}}(p_2)\, V^{\widetilde{r}'}_{\alpha_-}\, V^{\widetilde{r}}_{\alpha_+}\, \psi_{m+1}\cdots\psi_n\rangle_{\Sigma_2}\, \mathcal{M}^{-1}_{ij}\ .
\end{aligned}
\tag{15}
$$

The correlators on Σ_1 and Σ_2 are known by induction. Of course, this expression for the correlator is not particularly natural for a scalar field theory. We would rather sum over all Fock descendents,

$$
\begin{aligned}
\sum_{i,j} \oint t^{-\Delta_i} &\langle \psi_1 \cdots \psi_m\, V^{r'}_{\alpha_-}\, V^{r}_{\alpha_+}\, \mathcal{A}_{-i} V_{\alpha_{n',n}}(p_1)\rangle_{\Sigma_1} \\
&\times\ \langle \mathcal{A}_{-j} V_{\alpha_{p'-n',p-n}}(p_2)\, V^{\widetilde{r}'}_{\alpha_-}\, V^{\widetilde{r}}_{\alpha_+}\, \psi_{m+1}\cdots\psi_n\rangle_{\Sigma_2}\, \mathcal{M}^{-1}_{ij}\ ,
\end{aligned}
\tag{16}
$$

in which case we can replace the sum by a functional integral.

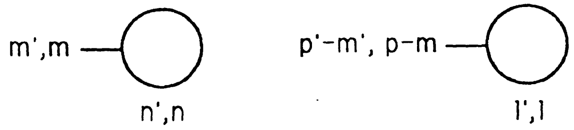

Figure 3: The genus-two block of Figure 2 is built out of two genus-one blocks.

In [8] it was shown that this is indeed possible; the BRST symmetry ensures that the extra states in $\{\mathcal{A}_{-i}V_{\alpha_{n',n}}\}$ do not contribute to the sum. To see this, let us choose a splitting for the Fock space $F_{n',n}$ and its conjugate $F_{p'-n',p-n}$:

$$\begin{aligned} \mathrm{Ker}Q_0 &= \mathrm{Im}Q_{-1} \oplus h \\ F_{n',n} &= \mathrm{Im}Q_{-1} \oplus h \oplus C \\ \mathrm{Ker}\widetilde{Q}_0 &= \mathrm{Im}\widetilde{Q}_1 \oplus \widetilde{h} \\ F_{p'-n',p-n} &= \mathrm{Im}\widetilde{Q}_{-1} \oplus \widetilde{h} \oplus \widetilde{C}\ , \end{aligned} \tag{17}$$

where h and $\widetilde{h}$ correspond to irreducible representations of the minimal models, while C, $\widetilde{C}$, $\mathrm{Im}Q_{-1}$ and $\mathrm{Im}\widetilde{Q}_{-1}$ contain the spurious states. Now, the matrix elements of $\mathrm{Im}Q_{-1}$ with $\mathrm{Ker}\widetilde{Q}_0$ and $\mathrm{Im}\widetilde{Q}_{-1}$ with $\mathrm{Ker}Q_0$ are automatically zero. Therefore, we can choose the splitting (17) so that $\mathcal{M}$ is of the form

$$\mathcal{M} = \begin{array}{c} \\ h \\ C \\ \mathrm{Im}Q_{-1} \end{array} \begin{array}{c} \begin{array}{ccc} \widetilde{h} & \widetilde{C} & \mathrm{Im}\widetilde{Q}_{-1} \end{array} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{pmatrix} \end{array}, \tag{18}$$

where * denotes a nonzero element. Inverting, we find

$$\mathcal{M}^{-1} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix} . \tag{19}$$

All the fields ψ in (16) are BRST-invariant, so the states in $\mathrm{Im}Q_{-1}$ and $\mathrm{Im}\widetilde{Q}_{-1}$ decouple from the sum. The only nonvanishing contributions come from the elements of h and $\widetilde{h}$, which are precisely the Virasoro descendents in (15). Therefore, extending the sum over the full Fock space does not change the correlator – none of the extra states actually propagate across the neck. This allows us to write (15) in terms of a functional integral, and gives a representation of the minimal model in terms of a free scalar field.

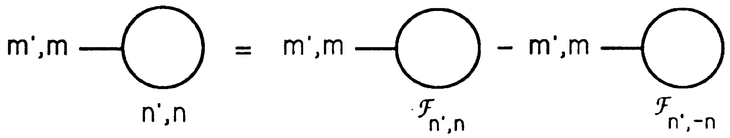

Figure 4: A genus-one block can be written as the difference of two blocks in a scalar theory.

Let us now illustrate this procedure by sewing two genus-one blocks to find the genus-two blocks for the minimal-model partition function. (The generalization to higher genus and to other correlation functions is straightforward.) For concreteness, we consider the block shown in Figure 2. This block is obtained by sewing the two genus-one blocks of Figure 3. From previous work [4][6], we know that each of these blocks has a representation in terms of two scalar blocks, as shown in Figure 4. Note that the entire Fock spaces $\mathcal{F}_{n',n} = \oplus_j F_{n',n+2jp}$ and $\mathcal{F}_{n',-n} = \oplus_j F_{n',-n+2jp}$ propagate around the loops.

The next step is to write the genus-two block in terms of a scalar field on a genus-two surface. The resulting expression involves four scalar blocks, as shown in Figure 5. Each of the scalar blocks is given by a correlator of screening operators on a genus-two surface

$$\langle V_{\alpha_{p'-1,p-1}}(z)\, V_{\alpha_-}(u_1)\dots V_{\alpha_-}(u_{p'-2})\, V_{\alpha_+}(v_1)\dots V_{\alpha_+}(v_{p-2})\rangle\,. \tag{20}$$

The total number of α_- and α_+ screening operators is $p'-2$ and $p-2$, respectively. The total charge of the fields in this correlator is therefore $\alpha_{p'-1,p-1} + (p'-2)\alpha_- + (p-2)\alpha_+ = -2\alpha_0$, which precisely cancels the genus-two background charge.

The correlator (20) can be evaluated using the functional-integral techniques described in [11]. The block shown in Figure 6 is given by

$$\begin{aligned} Z_{k\tilde{k}} \;=\; & \vartheta\begin{bmatrix} \frac{k}{N} & \frac{\tilde{k}}{N} \\ 0 & 0 \end{bmatrix}\left(2(p-p')(\Delta - z) - 2p\sum_{i=1}^{p'-2} u_i + 2p'\sum_{j=1}^{p-2} v_j \,\middle|\, N\Omega\right) \\ & \times \prod_{i=1}^{p'-2} E(z,u_j)^{2\alpha_-\alpha_{p'-1,p-1}} \prod_{j=1}^{p-2} E(z,v_j)^{2\alpha_+\alpha_{p'-1,p-1}} \\ & \times \prod_{i\neq j} E(u_i,u_j)^{2p'/p} \prod_{i\neq j} E(v_i,v_j)^{2p/p'} \prod_{i=1}^{p-2}\prod_{j=1}^{p'-2} E(u_i,v_j)^{-2} \\ & \times \sigma(z)^{-4\alpha_0\alpha_{p'-1,p-1}} \prod_{i=1}^{p'-2} \sigma(u_i)^{-4\alpha_0\alpha_-} \prod_{j=1}^{p-2} \sigma(v_j)^{-4\alpha_0\alpha_+}\,. \end{aligned} \tag{21}$$

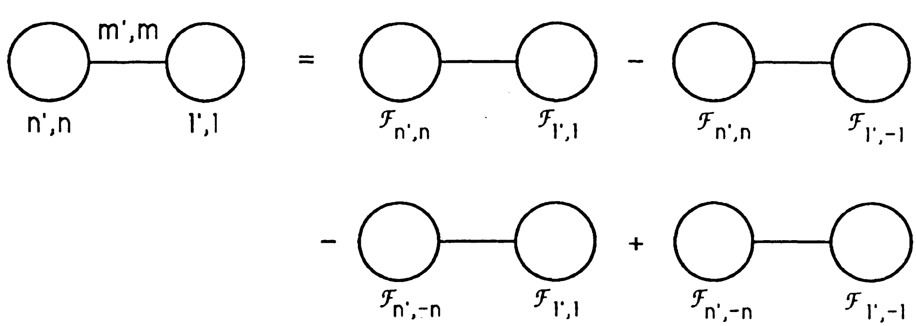

Figure 5: A genus-two block can be written as the sum of four blocks of a scalar theory. In the figure, we have suppressed the phases, screening operators and one insertion of $e^{2i\alpha_0\phi}$.

where momenta k and k' run around the lobes of the genus-two surface. In (21), Δ is the Riemann class, Ω is the period matrix, E is the prime form and σ gives the contraction of the vertex operators with the curvature term in the action (see [11] for details). The theta function is defined by

$$\vartheta\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}\left(\mathbf{X}\middle|\Omega\right) = \sum_{\mathbf{n}\in\mathbf{Z}\times\mathbf{Z}} e^{i\pi(\mathbf{n}+\mathbf{a})^t\cdot\Omega\cdot(\mathbf{n}+\mathbf{a})} e^{2\pi i\mathbf{X}\cdot(\mathbf{n}+\mathbf{a})} , \tag{22}$$

where the a and b cycles are those induced by the a and b cycles on the two sewn tori. As explained in [8], we have dropped overall factors which cancel in ratios of blocks.

Combining these results, we find that the genus-two minimal-model block of Figure 2 is given by

$$\begin{aligned} &\oint Z_{k_{n',n}k_{l',l}} - e^{2\pi i\theta_1}\oint Z_{k_{n',-n}k_{l',l}} \\ &\quad - e^{2\pi i\theta_2}\mathcal{N}\oint Z_{k_{n',n}k_{l',-l}} + e^{2\pi i(\theta_1+\theta_2)}\mathcal{N}\oint Z_{k_{n',-n}k_{l',-l}} , \end{aligned} \tag{23}$$

where the phases follow from [4], and $k_{n',n} = (n'p - np') + (p' - p) \bmod N$. The normalization $\mathcal{N}$ arises when converting the open contours to closed contours, as discussed above. It appears in (23) because the various terms have different numbers of screening operators associated with $V_{\alpha_{p'-m',p-m}}$.

Equation (23) gives one block of the genus-two minimal-model partition function, in terms of a scalar field. Similar expressions can be found for other correlators and other surfaces. In each case, the expressions degenerate correctly. For example, if one degenerates (23) by pinching the coordinate t in moduli space, one recovers the original genus-one amplitudes after applying the appropriate formulae for the degeneration of ϑ, E, σ, and Δ [11]. Of course, this is guaranteed by the sewing procedure and by the fact that the spurious states decouple.

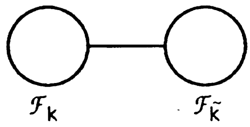

Figure 6: The scalar block of equation (21).

4. Modular Covariance

The higher-genus correlators we have constructed are automatically modular-covariant because they are built out of modular-covariant amplitudes on the sphere and the torus [7]. It is a nontrivial check, however, to verify the modular covariance of the genus-two blocks we have just found. The modular group at genus two is generated by Dehn twists about the cycles $a_1, a_2, b_1, b_2, {a_1}^{-1}a_2$ [12]. The resulting transformations on the homology are given by

$$D_{a_1} = \begin{array}{c c} & \begin{array}{cccc} a_1 & a_2 & b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ b_1 \\ b_2 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$D_{a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad D_{{a_1}^{-1}a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix} \tag{24}$$

$$D_{b_1} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad D_{b_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

These matrices generate $Sp(4, \mathbf{Z})$. Under a modular transformation $U \in Sp(4, \mathbf{Z})$, the period matrix and Riemann class transform as follows [11],

$$\Omega \rightarrow \Omega(C\Omega + D)^{-1} \qquad U \equiv \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

$$\Delta \rightarrow \Delta(C\Delta + D)^{-1} + \frac{1}{2}\begin{pmatrix} \mathrm{diag}\,(CD^t) \\ \mathrm{diag}\,(AB^t) \end{pmatrix} ,$$

while E and σ transform as

$$\begin{aligned} E(z,w) &\rightarrow \exp[i\pi(w-z)\,(C\Omega+D)^{-1}\,C\,(w-z)]\,E(z,w) \\ \sigma(z) &\rightarrow \exp[\frac{i\pi}{g-1}(\Delta-(g-1)z)(C\Omega+D)^{-1}\,C\,(\Delta-(g-1)z)]\,\sigma(z)\,. \end{aligned} \tag{25}$$

Substituting these transformations into the expression for the theta functions (22), we find

$$\vartheta\begin{bmatrix} \frac{k_1}{N} & \frac{k_2}{N} \\ 0 & 0 \end{bmatrix}\left(z+2(p-p')\Delta\middle|\Omega\right)$$

$$\begin{aligned} D_{a_j}: \quad &\rightarrow \exp\left[i\frac{\pi}{N}(k_j+p-p')^2 - i\frac{\pi}{N}(p-p')^2\right] \\ &\quad\times\ \vartheta\begin{bmatrix} \frac{k_1}{N} & \frac{k_2}{N} \\ 0 & 0 \end{bmatrix}\left(z+2(p-p')\Delta\middle|\Omega\right) \end{aligned}$$

$$\begin{aligned} D_{a_1{}^{-1}a_2}: \quad &\rightarrow \exp\left[i\frac{\pi}{N}(k_1-k_2)^2 - \frac{2\pi}{N}i(p-p')(k_1+k_2)\right] \\ &\quad\times\ \vartheta\begin{bmatrix} \frac{k_1}{N} & \frac{k_2}{N} \\ 0 & 0 \end{bmatrix}\left(z+2(p-p')\Delta\middle|\Omega\right) \end{aligned}$$

$$\begin{aligned} D_{a_1}{}^{-1}D_{b_1}D_{a_1}{}^{-1}: \quad &\rightarrow \left(\frac{-i\Omega_{11}}{N}\right)^{1/2}\sum_{l=1}^{N}\exp -2\pi i k_1 l/N \\ &\quad\times\ \vartheta\begin{bmatrix} \frac{l}{N} & \frac{k_2}{N} \\ 0 & 0 \end{bmatrix}\left(z+2(p-p')\Delta\middle|\Omega\right) \\ &\quad\times\ \exp\Bigg[i\pi[z+2(p-p')\Delta]\begin{pmatrix} \frac{1}{\Omega_{11}} & 0 \\ 0 & 0 \end{pmatrix} \\ &\qquad\times\ [z+2(p-p')\Delta]\Bigg]\,, \end{aligned} \tag{26}$$

and similarly for $D_{a_2}{}^{-1}D_{b_2}D_{a_2}{}^{-1}$.

Using these results, it is not hard to show that blocks of the form (23) transform correctly under D_{a_1} and D_{a_2}. These twists do not change the contours for the screening operators. Other twists, however, do change the contours. To prove covariance, it is necessary to relate the new contours to the old. This is a difficult problem, which was solved in [13] for the case of genus one. For higher genus, the problem is even more difficult, and we do not know the general solution.

For certain models, it is possible to avoid this problem by writing (23) with the contour integrals replaced by surface integrals, as in [3], [6] and [14]. For example, the models with $p' = 2$ have the correct periodicity to give sensible surface integrals. Following [6] and [15], for this restricted class of models, we can write the partition function as

$$\int d^2u_1 \dots d^2u_{p'-2} d^2v_1 \dots d^2v_{p-2} \sum_{k=1}^{N} \sum_{\tilde{k}=1}^{N} |Z_{k\tilde{k}} - Z_{w(k)\tilde{k}} - Z_{kw(\tilde{k})} + Z_{w(k)w(\tilde{k})}|^2 , \tag{27}$$

where

$$w(k) = (ap + bp')[k - (p' - p)] + (p' - p) \qquad \mathrm{mod}\ 2pp' ,$$

with a and b integers satisfying $ap - bp' = 1$. The only terms in the sum that contribute are those with $k = k_{n',n}$ or $k_{n',-n}$ and $\tilde{k} = k_{l',l}$ or $k_{l',-l}$, where $1 \le n', l' \le p' - 1$, $1 \le n, l \le p - 1$. It is now straightforward to check that (27) is modular invariant under all the above modular transformations. Note that this requires the cancellation of a number of nontrivial phases. A similar construction works for the case $p = 2$ as well.

5. Conclusions

In this talk we have described how to extend the Coulomb-gas construction to higher-genus surfaces. Our prescription ensures that when the surface is pinched, amplitudes degenerate correctly, splitting into a sum over lower-genus amplitudes. However, it is difficult to directly verify modular invariance because of the complicated higher-genus contours.

The discussion presented here is incomplete. As the number of screening charges associated with internal states increases, the resulting contours become quite complex and we have not found a general expression for them [16]. Even if we were able to find the general contours, the integrals would be difficult to evaluate explicitly.

There have recently been a number of applications of the Coulomb-gas technique to conformal theories with larger chiral algebras. In [17] and [18], for example, the representations of the W_n algebras are constructed. A number of groups have worked out the construction of zero- and one-loop correlators for WZW theories and parafermionic theories [19]. It appears that our treatment can be extended to these models as well. Work along these lines is in progress.

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References

[1] D.B. Fairle, unpublished; A. Chodos and C. Thorn, Nucl. Phys. B72 (1974) 59.

[2] B.L. Feigin and D.B. Fuchs, Funct. Anal. and Appl., 13, N4 (1979) 91; Funct. Anal. and Appl., 16, N2 (1982) 47; *Representations of the Virasoro Algebra.*

[3] V. Dotsenko and V.A. Fateev, Nucl. Phys. B224 (1984) 312; Nucl. Phys. B251 (1985) 691.

[4] G. Felder, Nucl. Phys. B317 (1989) 215.

[5] C. Thorn, Nucl. Phys. B248 (1984) 551.

[6] J. Bagger, D. Nemeschansky, and J.B. Zuber, Phys. Lett. B216 (1988) 320;
T. Jayaraman and K.S. Narain, ICTP preprint (1988);
O. Foda and B. Nienhuis, Utrecht preprint (1988).

[7] H. Sonoda, Nucl. Phys. B311 (1988) 401; Nucl. Phys. B311 (1988) 417.

[8] J. Bagger and M. Goulian, Harvard preprint (1989).

[9] G. Felder and A. Silvotti, ETH preprint (1989).

[10] See also A. Morozov and A Gerasimov, Phys. Lett. B225 (1989) 87;
O. Foda, Nijmegan preprint (1989);
M. Frau, A. Lerda, J. McCarthy and S. Sciuto, Phys. Lett. B228 (1989) 205.

[11] E. Verlinde and H. Verlinde, Nucl. Phys. B288 (1987) 357.

[12] L. Alvarez-Gaumé, G. Moore, C. Vafa, Comm. Math. Phys. 106 (1986) 1.

[13] G. Felder and R. Silvotti, ETH preprint (1988).

[14] V. Dotsenko, Kyoto preprint (1986).

[15] A. Capelli, C. Itzykson, and J.-B. Zuber, Nucl. Phys. B280(1987) 445.

[16] For recent progress in this direction, see [9].

[17] V. Fateev and A. Zamolodchikov, Nucl. Phys. B280 (1987) 644.

[18] V. Fateev and S. Lykanov, Int. J. Mod. Phys. A3 (1988) 507.

[19] M. Wakimoto, Commun. Math. Phys. 104 (1986) 605;
A. Zamolodchikov, Montreal seminar (1988);
V. Dotsenko, Harvard seminar (1989);
A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, ITEP preprint (1989);
B. L. Feigin and E. Frenkel, Moscow preprint (1989);
D. Nemeschansky, Phys. Lett. B224 (1989) 121; USC preprint (1989);
J. Distler and Z. Qiu, Cornell preprint (1989);
D. Bernard and G. Felder, ETH preprint (1989);
K. Ito, UT-Komaba preprint (1989).

CONFORMAL ALGEBRAS AND NON-LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT

The method of Hamiltonian reduction used by Drinfeld and Sokolov in the theory of integrable non-linear differential equations, is applied to two dimensional field theories. We show that conformal symmetries can be obtained from Kac-Moody Lie algebras by introducing appropriate gauge choices. We also present R-matrix generalizations of the underlying (Gelfand-Dickey) algebraic structures and outline possible extensions of the main results to higher dimensions. Finally, we discuss the significance of these ideas in theories of quantum gravity.

INTRODUCTION

Recent advances in the theory of integrable non-linear differential equations have made transparent the important role that infinite dimensional algebras play in many areas of theoretical physics. It is well known that integrable equations admit a Lax pair (Hamiltonian formulation) which makes it possible to apply the inverse scattering method and obtain an infinite number of conserved quantities, construct explicit solutions and show the existence of soliton configurations in the theory (see for instance [1] and references therein). Gauge equivalence is (perhaps) the most important physical concept in the theory of integrable nonlinear differential equations, since it provides a systematic way to associate equations of the Korteweg-deVries type with any simple Lie algebra as well as study general questions of algebraic and geometric character connected with the method of inverse scattering [2]. Furthermore it helps to uncover some (otherwise) hidden relations between conformal and Kac-Moody Lie algebras which become relevant in two dimensional induced quantum gravity [3]. The purpose of the present work is to abstract the algebraic structures used in the theory of integrable non-linear differential equations and show that conformal symmetries of 2-dim field theory can be obtained from centrally extended current algebras by choosing appropriate gauges.

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Although most of the results we discuss here have become quite standard by now (see for instance [4,5]), their derivation will be presented in a way that incorporates naturally the Yang-Baxter equation and suggests generalizations to higher dimensions. The main tool in our disposal will be the Adler-Manin residue formula [6] from the calculus of formal pseudodifferential operators $X = \ldots + \partial^{-2}x_{-2}(z) + \partial^{-1}x_{-1}(z) + x_o(z) + \ldots + \partial^m x_m(z)$ that depend on a complex variable z. Let us introduce some notation which will be helpful for what follows:

$$\text{res } X = x_{-1}(z);\ X_+ = \sum_{0}^{m>0} \partial^i\ x_i(z);\ X_- = \sum_{-\infty}^{-1} \partial^i\ x_i(z). \tag{1}$$

Then the Adler-Manin formula

$$<X,\ Y> = \int \text{res } (XY) \tag{2}$$

defines a pairing in the space of formal pseudodifferential operators, which generalizes the (smooth) in-out duality of primary conformal fields of weight λ and $1-\lambda$ [7]. We also introduce a linear operator R so that

$$RX_+ = X_+\ ;\ RX_- = -\ X_-\ . \tag{3}$$

Clearly it satisfies the conditions

$$R^2 = 1\ ;\ R + R^+ = 0\ , \tag{4}$$

since $<RX,\ Y> + <X,\ RY> = 0$ for all operators X, Y. Moreover the "chirality" operators

$$R_\pm := \frac{1}{2}\ (1 \pm R) \tag{5}$$

project into the + and - components of X respectively, ie $R_\pm\ X = X_\pm$.

The second Hamiltonian structure of integrable non-linear differential equations of the KdV type is described by the Lie Poisson bracket

$$\Big\{f[u],\ g[u]\Big\}_L = \frac{1}{2}\ \Big[<R(X_fL),\ X_gL> - <R(LX_f),\ LX_g>\Big] \tag{6}$$

which is associated with any given Lax operator $L = \partial^n + u_{n-1}(z)\partial^{n-1} + \ldots + u_o(z)$. Here X_f (and similarly X_g) represents the formal sum

$$X_f = \sum_{i=1}^{n} \partial^{-i}\ x_i\quad ;\quad x_i = \frac{\delta f[u]}{\delta u_{i-1}}\ . \tag{7}$$

The bracket (6) is called Gelfand-Dickey bracket (of the second kind) and introduces a Lie algebra structure on the space of all functionals $f[u_o, \ldots, u_{n-1}]$ [8]. Antisymmetry of (6) is a consequence of the relation $R + R^+ = 0$. Verifying the Jacobi identity is not an easy task, but as it turns out the proof relies on the following identity:

$$[RX, RY] + [X, Y] = R\Big([RX, Y] + [X, RY]\Big). \tag{8}$$

This is trivially satisfied by the linear operator R given by eq. (3).

Before we proceed further a remark is in order. The condition (8) has a natural geometric interpretation in the theory of (integrable) complex structures: the operator $J := iR$ squares to -1 and (8) states that the Nijenhuis tensor $N_J(X, Y)$ vanishes [1]. Moreover, the identity (8) is equivalent to the (modified) classical Yang-Baxter equation associated with the algebra of formal pseudodifferential operators [9]. This suggests that the Gelfand-Dickey bracket of the second kind can be generalized, provided that other solution of eq. (8) (which are different from (3)) are found. We shall return to this point later.

.THE METHOD OF HAMILTONIAN REDUCTION

It is a remarkable consequence of the hidden gauge symmetries in the theory of integrable non-linear differential equations that the bracket (6) can be obtained from the Lie-Poisson bracket of Kac-Moody algebras by Hamiltonian reduction [2]. To be more precise let us consider the (affine) space of first order linear differential operators (ie covariant derivatives)

$$D_q = \partial + q(z), \tag{9}$$

where the potential function q takes values in the Lie algebra of the general linear group GL(n) (represented by non-singular nxn matrices). The gauge group $\mathscr{G}$ with values in GL(n) acts on D_q by conjugation and its orbits are identified with gauge equivalent connections

$$q^g(z) = g^{-1}(z)q(z)g(z) + g^{-1}(z)g'(z) \tag{10}$$

for all $g \in \mathscr{G}$. Next we (partially) fix a gauge by choosing elements of the form

$$q(z) = \begin{pmatrix} q_{11}(z) & q_{12}(z) & \cdots & q_{1n}(z) \\ 1 & q_{22}(z) & \cdots & q_{2n}(z) \\ 0 & 1 & \ddots & \\ \vdots & & \ddots & \ddots \\ 0 & 0 \cdots & 1 & q_{nn}(z) \end{pmatrix} . \tag{11}$$

The subgroup $\mathscr{G}_+ \subset \mathscr{G}$ with values in the strictly upper triangular matrices acts on the configurations (11) by means of gauge transformations. It is always possible to find $g \in \mathscr{G}_+$ so that $g^{-1}(z)\, D_q\, g(z) = \partial + u(z)$, where u(z) has the special form

[1] I thank M. Bowick for bringing this point to my attention.

$$u(z) = \begin{bmatrix} 0 & 0 \; \ldots\ldots & 0 & -u_o(z) \\ 1 & 0 & 0 & -u_1(z) \\ 0 & 1 & \vdots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 \; \ldots & 1 & -u_{n-1}(z) \end{bmatrix} \qquad . \tag{12}$$

In other words, u(z) labels the space of gauge equivalence classes of the connections (11) with respect to the action of $\mathscr{G}_+$. Equivalently, the gauge equivalence classes (12) may be parametrized by the n-th order (scalar) Lax operators

$$L = \partial^n + u_{n-1}(z)\, \partial^{n-1} + \ldots + u_o(z) \quad . \tag{13}$$

The gauge fixing procedure we have just outlined can be implemented in the algebra of functionals over the space of gauge connections q. In particular, for any two such functionals f, g, we introduce the Lie-Poisson bracket

$$\Big\{f[q],\; g[q]\Big\}_{D_q} = \frac{1}{2}\Big[<R(X_f D_q),\; X_g D_q> - <R(D_q X_f),\; D_q X_g>\Big] \; . \tag{14}$$

Here $X_f = \partial^{-1} \frac{\delta f}{\delta q}$ is matrix valued, $< \; , \; >$ is defined as in eq. (2) (combined together with the trace in color space) and R is the projection operator (3) composed with the unit nxn matrix in GL(n). Notice that the bracket between the coordinate functionals q is given by

$$\{q^a(z),\; q^b(z')\}_{D_q} = f^{ab}_c\; q^c(z)\delta(z-z')-\mathrm{Tr}(T^aT^b)\partial_z\delta(z-z'), \tag{15}$$

where $\{T^a\}$ are the generators of the underlying symmetry algebra. Equation (15) describes the commutation relations of the centrally extended algebra of $\mathscr{G}$. (Arbitrary values of the central charge ~ k will arise if we choose $D_q = k\,\partial + q(z)$). The main result then reads as follows: for q(z) of the form (11), the D_q-bracket between $\mathscr{G}_+$-invariant functionals $\tilde{f}[q]:=f[u]$ and $\tilde{g}[q]:=g[u]$ is equal to the Gelfand-Dickey bracket (6). This clarifies the group theoretical origin of the second Hamiltonian structure in the theory of integrable non-linear differential equations.

To appreciate fully the significance of the result above in 2-dim conformal field theory, we consider the simplest example: n = 2 and Trq(z) = 0. In this case the relevant Lie algebra is SL(2) and the (sub)group of upper triangular matrices generated by $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ is the semi-direct product group $\mathbb{R} \circledS \mathbb{R}_+$ [2]. The space of gauge equivalence classes (12) is parametrized by the Hill operators $L = \partial^2$

[2] $\mathbb{R} \circledS \mathbb{R}_+$-current algebras have also been considered in the context of group theoretical approaches to the canonical quantization of reparametrized theories in 1 + 1 dimensions [10].

$+ u_o(z)$ and the commutation relations between the coordinate functionals u_o (computed either with respect to the D_q or L-bracket) are found to be

$$\{u_o(z), u_o(z')\} = (u_o(z) + u_o(z'))\partial_z\delta(z-z') + \frac{1}{2}\partial_z^3\,\delta(z-z') \,. \qquad (16)$$

Therefore, u_o generates the Virasoro algebra with non-zero central charge. (Once more, arbitrary values of the central charge will arise if we choose to work with $k\partial + q(z)$ in general).

More generally, Hamiltonian reduction for the group SL(n) yields an infinite dimensional algebra generated by $u_{n-2}(z)$, ..., $u_o(z)$ ($u_{n-1}(z) = 0$, since $\mathrm{Tr}q(z) = 0$). As before, $u_{n-2}(z)$ satisfy the commutation relations of the Virasoro algebra with central charge $c = n^3 - n$ and $u_{n-3}(z)$, ..., $u_o(z)$ correspond to conformal fields with spin 3, ..., n respectively. We point out that for $n \geq 3$ the determining relations of the resulting algebras are quadratic in nature and describe higher spin operator algebras of 2-dim conformal field theory. The simplest non-trivial example is provided by Zamolodchikov's spin-3 operator algebra which is associated with the Lie algebra of SL(3) [11]. The explicit form of the commutation relations is quite complicated and we refer the reader to the published work [4,5] for further details. However, it is worth mentioning that the Hamiltonian reduction procedure provides a geometric interpretation of the (modified) Sugawara construction for higher spin conformal fields in terms of the generators of Kac-Moody algebras. Of course the resulting expressions have to be normal ordered when considering extended conformal algebras as symmetries of 2-dim quantum field theories.

It is quite clear by now that the scheme we have adopted here can be used for the classification of various conformal algebras. Recall that simple Lie algebras can be obtained from the general linear algebra by imposing appropriate conditions on its matrix elements. In turn, this will restrict the operator content of conformal symmetries that emerge by reduction, in a way which is consistent with the Jacobi identity. As a result, unitary representations of these extended conformal algebras correspond to rational conformal field theories whose properties are determined by simple Lie algebras. In this fashion one hopes to understand better the vacuum structure of string theory as well as construct a periodic table of all types of criticality in 2-dim statistical mechanics, parallel to the Cartan classification of algebras.

REMARKS ON THE UNDERLYING STRUCTURES

It is interesting to know that there is a geometric way to obtain conformal symmetries from Kac-Moody Lie algebras by applying gauge fixing techniques. As we have already noted this is a consequence of the hidden gauge symmetries in the Hamiltonian formulation of integrable non-linear differential equations. Undoubtedly, it would be even more interesting to understand the origin of gauge equivalence principles in the theory of integrable and conformal structures. It seems that the geometry of self-dual Yang-Mills fields in 4 dimensions could help to explain many intricate facts of 2-dim physics that are puzzling otherwise. (This point of view has also been expressed by Atiyah in various occasions.) It is intriguing that the self-duality condition

$$F(A) = \pm {}^{*}F(A) \tag{17}$$

is itself an integrable non-linear differential equation (though in 4-dim) which can be formulated as an inverse scattering problem [12]. This equation plays a "master" role in the different types of integrability we encounter in 2 dimensions, since the Toda field equations and many others can be obtained from it by dimensional reduction (see for instance [13] and references therein). However, a complete classification of the different ansatz for the gauge connections A that effectively reduce eq. (17) to all possible integrable differential equations in 2 dimensions is not available yet. Nevertheless, we think that inquiries along these lines could clarify the role of gauge symmetries in conformal and integrable theories. Certain results in this general direction have already been obtained by stepping out of the "flatland" and considering Chern-Simons topological field theories in 2+1 dimensions [14].

At this point we would like to comment on the role of the (modified) Yang-Baxter equation in the algebra of formal pseudodifferential operators. The following solutions of eq. (8) are known:

$$\text{(i)} \qquad RX = X_+ + X_- := X \tag{18a}$$

$$\text{(ii)} \qquad RX = X_+ - X_- \quad . \tag{18b}$$

The first one, being the identify, is trivial and not interesting at all because the right hand side of eq. (6) vanishes. This can be easily verified by noting that the residue of the commutator of any two (formal) pseudodifferential operators is a total derivative. The second solution gives rise to the (conventional) Gelfand-Dickey bracket of the second kind which has already been appreciated. However, there are no examples (at least so far) of integrable differential equations whose Hamiltonian description requires linear operators R other than (18b). It would be interesting to examine eq. (8) in the framework of Hamiltonian reduction, since other solutions of it (if they exist) could lead to more exotic operator algebras for the generating fields u.

Regardless the existence of other solutions for eq. (8), the choice (3) for the (classical) Lie-Poisson brackets (6) and (14) may be used for developing the quantum Hamiltonian reduction in the spirit of quantum inverse scattering methods. For this purpose it is more advantageous to pass to the tensor form notation for the current algebra (15) and construct first the quantum Kac-Moody group associated with it [15]. Then the quantum Virasoro (and other conformal) groups could follow by appropriate gauge choices. In this framework the (quantum) Yang-Baxter equation plays the role of the Jacobi identity. A detailed progress report will be given elsewhere [16].

APPLICATIONS TO QUANTUM GRAVITY

In the rest of the paper we discuss briefly the significance of all these ideas in theories of quantum gravity (other than strings). Perhaps the most important application in 2 dimensions is Polyakov's theory of induced quantum gravity [3]. In the light-cone chiral gauge the 2-dim metric assumes the form

$$ds^2 = dzd\bar{z} + h(z,\bar{z})dz^2 \tag{19}$$

and the gravitational action $S \propto \int R \frac{1}{\Box} R$ exhibits a remarkable symmetry generated by the Beltrami fields $h(z,\bar{z})$. Using the classical equation of motion $\partial^3_{\bar{z}} h(z,\bar{z}) = 0$, the following decomposition (in powers of $\bar{z}$) is obtained:

$$h(z,\bar{z}) = J^{(+)}(z) - 2\,\bar{z}\,J^{(o)}(z) + \bar{z}^{\,2} J^{(-)}(z) \; . \tag{20}$$

It turns out that the currents $J^{(\pm)}$, $J^{(o)}$ generate the Kac-Moody algebra for the group SL(2), which allows one to derive differential equations for the correlation functions of the theory. Since $J^{(-)}(z)$ is the curvature $R[h]$ of the metric (19), it is possible to normalize $J^{(-)}(z) = 1$ by an appropriate Weyl scaling. This provides a field theoretic justification for the choice (11), when the group SL(2) is considered. Then the stress-energy tensor of induced gravity is given by the (modified) Sugawara expression that follows from the Hamiltonian reduction procedure. Of course appropriate normal orderings have to be introduced in quantization.

More generally, it is possible to implement the gauge fixing procedure (11), (12) in gauged Wess-Zumino-Witten models for the group SL(n) (and others) and derive effective actions for extended induced quantum gravity theories [17]. In this (abstract) setting the Virasoro algebra is replaced by extended conformal symmetries that depend on the groups we choose. However, the geometric interpretation of higher spin fields (and Toda equations[3]) is still lacking; it is an interesting problem for further study.

Another area of gravitational physics where conformal algebras and integrable non-linear differential equations make their appearance is reduced gravity. In this case one seeks solutions of (vacuum) Einstein equations $R_{\mu\nu} = 0$ with two commuting Killing vector fields, which effectively reduce the problem into a 2-dim one. Then, the 4-dim metric becomes block diagonal (see for instance [19])

$$ds^2 = h_{AB}\, dx^A dx^B + \lambda_{MN} dx^M dx^N; \quad A,B = 0,\ 1\ ; \quad M,N = 2,3 \tag{21}$$

and both fields h, λ depend only on two coordinates which are determined by the spacelike or timelike character of the Killing vector fields. It turns out that the gravitational field equations can be formulated as an inverse scattering problem and the constraints of general relativity form two (independent) copies of the Virasoro algebra (16), with zero central charge. Also the gauge group of the SL(2) Kac-Moody algebra (known as the Geroch group) acts on the space of all metrics (21) and transforms one solution with two Killing vector fields into another (see [20] for a recent exposition). It is quite useful to apply techniques of conformal field theory and soliton physics in favor of quantizing the reduced sector of Einstein's gravity. This way we hope to arrive at a

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[3]For the description of higher spin algebras in terms of Toda field equations, see [18].

non-perturbative R-matrix formulation of "quantum geometries". More details will be presented in [21].

Finally, we comment on possible generalizations of the method of Hamiltonian reduction to symmetry algebras in higher dimensions. From the point of view of gravitational physics, the most appealing feature of Hamiltonian reduction is that diffeomorphism algebras can be obtained from centrally extended current algebras by gauge fixing. For this reason it is important to know whether there are hidden relations between the symmetries of gauge and reparametrized theories in space-time dimensions $d > 2$. Although many of our considerations depended a lot on specific properties of infinite dimensional algebras in $1 + 1$ dimensions, recent advanced in the theory of pseudodifferential operators with more than one variable [22] suggest strongly that the main results we have obtained might be valid in all dimensions. In particular, the Adler-Manin residue formula (2) which is the main tool in the Hamiltonian formulation of Gelfand-Dickey algebras, has been generalized by Wodzicki in the framework of spectral geometry. To be more precise, one defines

$$\operatorname{res} X := \frac{d}{dt} \left(\operatorname{ord} Y \cdot \operatorname{Res}_{s=-1} \zeta\,(s;\; Y + tX)\right)\Big|_{t=0} \tag{22}$$

for any arbitrary pseudodifferential operator X in higher dimensions. Here Y is an (arbitrary) elliptic pseudodifferential operator admitting complex powers and of the order strictly bigger than that of X, while $\zeta(s;\, Y)$ is the zeta function of Y. To appreciate the significance of this definition, recall that there is no invariant decomposition $X = X_+ + X_-$ into differential and formal parts for the algebra of operators with more than one variable. It follows that $\operatorname{Res}_{s=-1} \zeta(s;\, Y)$ behaves much like a trace functional and the definition (22) is in fact independent of Y. The most remarkable property of res X is its uniqueness. Moreover, (22) reduces to the Adler-Manin prescription when applied to pseudodifferential operators in one variable.

We think that appropriate use of (22) for defining Lie Poisson brackets analogous to (6) and (14), will help extending our results to higher dimensions. If this expectation materializes, the quantization program of gravity in 2+1 and 3+1 dimensions will be put into a new prospective. The existence of hidden gauge symmetries in reparametrized field theories (and possibly string theory) is a challenging question.

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REFERENCES

[1] L. D. Faddeev, L. A. Takhtajan: "Hamiltonian Methods in the Theory of Solitons" (Springer-Verlag, 1987).
[2] V. G. Drinfeld, V. V. Sokolov: Journ. Sov. Math. 30 (1985) 1975.
[3] A. M. Polyakov: Mod. Phys. Lett. A2 (1987) 893.
V. G. Knizhnik, A. M. Polyakov, A. B. Zamolodchikov: Mod. Phys. Lett. A3 (1988) 819.

[4] V. A. Fateev, S. L. Lykyanov: Int. Jour. Mod. Phys. A3 (1988) 507.
T. G. Khovanova: Funct. Anal. Appl. 20 (1986) 332.
P. Mathieu: Phys. Lett. B208 (1988) 101.
[5] I. Bakas: Phys. Lett. B213 (1988) 313; Phys. Lett. B219 (1989) 283; Commun. Math. Phys. (in press).
D. Smit: Utrecht preprint (1988)
A. A. Belavin: in the proceedings of the Taniguchi symposium, Kyoto 1988 (to appear).
[6] M. Adler: Invent. Math. 50 (1979) 219.
Yu. I. Manin: Journ. Sov. Math. 11 (1979) 1.
[7] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov: Nucl. Phys. B241 (1984) 333.
[8] L. A. Dickey: Commun. Math. Phys. 87 (1982) 127.
I. M. Gelfand, I. Dorfman: Funct. Anal. Appl. 15 (1981) 173.
[9] M. A. Semenov-Tian-Shansky: Funct. Anal. Appl. 17 (1983) 259; Publ. RIMS, Kyoto Univ. 21 (1985) 1237.
[10] C. J. Isham: in "Relativity, Groups and Topology", ed. B. S. DeWitt, R. Stora (North-Holland, 1984).
J. R. Klauder: Phys. Rev. D2 (1970) 272.
[11] A. B. Zamolodchikov, V. A. Fateev: Nucl. Phys. B280[FS18] (1987) 644.
[12] A. A. Belavin, V. E. Zakharov: Phys. Lett. B73 (1978) 53.
[13] R. S. Ward: Phil. Trans. Roy. Soc. Lond. A315 (1985) 451.
[14] E. Witten: Princeton preprint (1988).
[15] N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky: LOMI preprint (1989).
[16] I. Bakas, M. J. Bowick: work in progress (to appear).
[17] A. Alekseev, S. Shatashvili: LOMI preprint (1988).
M. Bershadsky, H. Ooguri: Princeton preprint (1989).
Y. Matsuo: Chicago preprint (1989).
[18] A. Bilal, J.-L. Gervais: Phys. Lett. B206 (1988) 412.
[19] V. A. Belinskii, V. E. Sakharov: Sov. Phys. JETP 48 (1978) 985; Sov. Phys. JETP 50 (1979) 1.
[20] P. Breitenlohner, D. Maison: Ann. Inst. Henri Poinc. 46 (1987) 215.
[21] I. Bakas: in preparation.
[22] M. Wodzicki: in "K-Theory, Arithmetic and Geometry", Lect. notes in Math., Vol. 1289, ed. Yu. I. Manin (Springer Verlag, 1987).
A. Connes: Commun. Math. Phys. 117 (1988) 673.

S MATRICES OF THE TRICRITICAL ISING MODEL AND TODA SYSTEMS

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1 Introduction

After the excellent discussion of the subject presented at this conference by E. Corrigan, it would be redundant to repeat in detail the different steps that make it possible to simplify the difficult problem of the determination of the S matrices in certain integrable off-critical systems. Instead, I will only briefly review the different tools, those developed by Zamolodchikov [1], and the arguments that connect the subject to Toda field theories [2, 3, 4], in an effort to add new comments about each of them. The Tricritical Ising Model (TIM), that will be the standard example, is discussed in more detail in ref.[5], written with G. Mussardo. The interesting examples of the D_n Toda systems are discussed in [6] and in the talk of E. Corrigan at this conference. In a second part, I will discuss in more detail certain selected features of Toda systems. The rich multipole structure and the connection of the exact solution for the S functions to the perturbative expansion is discussed. Finally I will comment on the relation between unitary and non-unitary off-critical integrable models that seems very closely related. The content of this talk represents the results obtained in collaboration with Giuseppe Mussardo.

2 Brief review and comments

2.1 Zamolodchikov's counting argument

The Zamolodchikov's counting argument [1] provides a sufficient condition to detect the pres-

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ence of conserved quantities when a model at criticality is perturbed by a relevant operator. If the dimension of the space $\hat{\Lambda}_{n+1}$ of quasiprimary operators of scaling dimension $n+1$ in the conformal block of the identity is greater than the dimension of the corresponding space $\hat{\phi}_n$ of quasiprimary operators at level n in the conformal block of the perturbing operator ϕ, then there must exist elements $T_{n+1} \in \hat{\Lambda}_{n+1}$ and $Q_{n-1} \in \hat{\phi}_{n-1}$ such that

$$\partial_{\bar{z}} T_{n+1} = \partial_z Q_{n-1} \tag{2.1}$$

As an example, Table 1 gives the dimensions of the spaces of quasiprimary operators in the case of the TIM perturbed by the magnetic operator $\phi_{(12)}$ with conformal dimensions (1/10, 1/10).

Table 1. Dimensions of the spaces $\hat{\Lambda}_{s+1}$ and $\hat{\phi}_{(12)s}$. In bold are the spins of the conserved currents found by the counting argument for the TIM.

s	**1**	2	3	4	5	6	**7**	8	**9**
$\hat{\Lambda}_{s+1}$	1	0	1	0	2	0	3	1	4
$\hat{\phi}_{(12)s}$	0	0	1	1	1	2	2	3	3
s	10	**11**	12	**13**	14	15	16	17	18
$\hat{\Lambda}_{s+1}$	2	6	3	9	6	12	9	18	14
$\hat{\phi}_{(12)s}$	5	5	8	8	11	13	17	19	25

The spins of the first conserved currents are Coxeter exponents of the E_7 Lie algebra (up to $s = 17$). As for the Ising model, where the E_8 Coxeter exponents were found, this strongly suggests, that the conserved spins off criticality are the E_7 Coxeter exponents modulo the Coxeter number (18 for E_7). A rigorous proof of that reasonable assumption does not exist. A perturbation of a critical model with a relevant operator does not mean that an infinite number of conserved charges do exist. The counting argument always provides a mean to show the existence of the ones with the lowest spins. However, for certain relevant operators, we can argue in favour of an infinite number. Recently a connection was found between the conservation laws of the quantum Korteweg de Vries, of the quantum Gibbon-Sawada-Kotera, and of the Kuperschmidt equations and the conservation laws of minimal models perturbed by the operators ϕ_{13}, ϕ_{12}, and ϕ_{21} respectively [7]. Other strong arguments follow from the connection to Toda systems [2, 3, 4, 5, 6]. This provides arguments in the cases of perturbations with other operators, such as ϕ_{15} or ϕ_{51} in ref.[4]. This will be commented a little bit more later.

2.2 Computation of two particle S-matrices

On the assumption of the conservation of a given spectrum of conserved charges, we can try to find a consistent minimal solution for the S-matrices. However, it is first important to argue from case to case on the non existence of massless particle that could still exist off criticality. An example was proposed by J.L. Cardy [8] in the TIM when perturbed with the subleading magnetization operator ϕ_{21} of dimension (7/16,7/16). The TIM can be written as a special case of the Blume-Emery-Griffiths model [9] where the spins take values $0, \pm 1$ (0 stands for the vacancies). At zero magnetic field, the tricritical point is the end point of a first order critical line in the phase space given by the temperature and the crystal field[1]. Taking a magnetic field into account, this line appears as the junction of two critical surfaces ("wings") [10]. The perturbation of the tricritical point with the subleading magnetization operator corresponds to switch on a magnetic field and to trigger temperature and crystal field in such a way that the system remains on the boundaries of the wings. Depending on the sign of the perturbation, this amounts to suppress one of the initial values ± 1. As the coupling increases, the system flows down to the Ising model with only two possible states per site. In this case, we expect massless excitation all the way from the TIM down to the Ising model. This example is interesting because in general the sign of the perturbation matters and we can expect the system to be purely massive for at least one choice of the sign.

In the case of perturbation with the magnetic field, such massless excitation are not expected. The system flows down to $c = 0$ (Virasoro central charge) as the coupling increases. Moreover, in the case of the Ising model and of the TIM, the mass spectrum deduced from the exact S-matrices is (at least partially) confirmed by computer simulations [11, 12].

Assuming that we have an infinite set of conserved charges and a purely massive theory, we can now list the system of equations that determine the S-matrices. For the sake of simplicity, we also assume that the mass spectrum is non degenerate. This situation occurs in the TIM[2]. I introduce the velocity θ to characterize a particle with momentum $\vec{p}$.

$$\vec{p} = m(\cosh(\theta), \sinh(\theta)) \tag{2.2}$$

As a consequence of the factorization of the multi-particle S-matrices, it is sufficient to determine the two particle ones $\hat{S}_{AB}(\theta)$, that describe the process involving the two particles with masses m_A and m_B ($\theta = \theta_A - \theta_B$). Crossing symmetry and unitarity read

$$\begin{aligned} \hat{S}_{AB}(\theta) &= \hat{S}_{AB}(i\pi - \theta) \\ \hat{S}_{AB}(\theta)\hat{S}_{AB}(-\theta) &= 1 \end{aligned} \tag{2.3}$$

[1] Beyond this point, the line continues and represents a continuous phase transition

The consistency condition (bootstrap) with the three particle S-matrices is

$$\hat{S}_{CD}(\theta) = \hat{S}_{AC}(\theta - i\bar{u}_{AD}^{B})\hat{S}_{BC}(\theta - i\bar{u}_{BD}^{A}) \tag{2.4}$$

where $\bar{u}_{AB}^{C} = \pi - u_{AB}^{C}$ and iu_{AB}^{C} is the position of the first order pole in $\hat{S}_{AB}$ that correspond to the bound states with mass m_C. This process is illustrated in Figure 1.

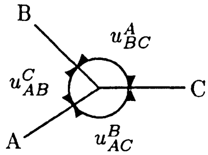

Figure 1: Three particle coupling.

Finally, the current conservation reads

$$q_s^A \exp(-i\bar{u}_{AC}^{B}) + q_s^B \exp(\bar{u}_{BC}^{A}) = q_s^C \tag{2.5}$$

A more detailed discussion can be found in the talks of E. Corrigan and G. Mussardo at this conference.

2.3 Connection to affine Toda Field Theories

The connection of models like Ising or TIM to affine Toda models is far from obvious. Starting from statistical lattice systems does not provide any clue to recognize the structure of the Lie algebra E_7 and E_8 in these both models. The first connection comes from the observation, that the conformal field theories, that describe them at criticality, can be obtained by a coset construction ($(\hat{E}_8)_1 \otimes (\hat{E}_8)_1/(\hat{E}_8)_2$ and $(\hat{E}_7)_1 \otimes (\hat{E}_7)_1/(\hat{E}_7)_2$ respectively)[3]. Recently, Eguchi and Yang [2], and independently Leclair [3] established a direct link between perturbed systems and affine Toda systems with imaginary coupling constants. This latter affine algebra correspond to the one that enter in the coset construction of the model at criticality. The absence of solitons off criticality and the exact computation of the S matrix is more in favour of a connection with affine Toda systems with a given real coupling constant β. For Ising and TIM we have

$$\beta = -\sqrt{\frac{h^{\mathcal{G}}+1}{h^{\mathcal{G}}+2}} \tag{2.6}$$

where $h^{\mathcal{G}}$ is the Coxeter number of E_8 and E_7 respectively. In a recent paper Hollowood and Mansfield [4] consider Toda systems that renormalize to a minimal conformal field theory after an analytic continuation in the real coupling constant to imaginary values. It may be

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[2] In his talk, E. Corrigan discussed a degenerate case.

[3] As usual $(\hat{G})_k$ denotes the affine Lie algebra G at level k.

not surprising that with such a continuation, only a sector of the Toda model with imaginary coupling is obtained. They also briefly discuss the extension to the affine Toda models and obtain similar results as the other authors.

3 Selected features of affine Toda systems

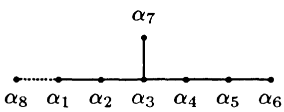
3.1 Area formula for the three particle couplings

The Lagrangian for a general affine Toda system is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2}\sum_{i=1}^{r+1} n_i \exp(\beta \alpha_i^j \phi^j), \qquad \sum_{i=1}^{r+1} n_i \alpha_i = 0 \tag{3.1}$$

where α_i[4] are the linearly dependent roots of an affine Lie Algebra. $-\alpha_{r+1}$ is always the maximal root, even for non simply-laced algebras. This maximal root is just the highest weight of the adjoint representation. Table 2 gives all information about the coefficients n_i and the Coxeter exponents for E_7.

Table 2. Coxeter exponents and coefficients n_i for E_7.



$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_7 = -\alpha_8$
Exponents: $1, 5, 7, 9, 11, 13, 17$

The roots are chosen to diagonalize the mass matrix[5]

$$m^2 \sum_k n_k \alpha_k^A \alpha_k^B = m_A^2 \delta^{AB} \tag{3.2}$$

I do not discuss the question of the renormalization. I want just to stress, that the ratios of the masses and β do not renormalize. The three particle couplings of the model are important, because they determine the odd order poles (i.e. bound states) in the S matrix.

$$f_{ABC} = m^2 \beta \sum_k n_k \alpha_k^A \alpha_k^B \alpha_k^C \tag{3.3}$$

It is interesting to notice that these couplings are non zero if they are permitted by cinematic

[4]The square of the long root is normalized to two.

[5]In case of mass degeneracy, it is simpler to organize particles with the same mass in complex conjugate pairs. The related Z_2 symmetry exchanges the two particle.

reasons and not forbidden by discrete internal symmetries of the affine Dynkin diagram (such as Z_2 for E_7). The following was also found by E. Corrigan and his collaborators. The non zero f_{ABC} for simply laced Lie algebras are given by

$$|f_{ABC}| = \frac{4\beta}{\sqrt{h^{\mathcal{G}}}}\mathcal{A}_{ABC} \tag{3.4}$$

where $\mathcal{A}_{ABC}$ is the area of the triangle whose boundaries have the lengths m_A, m_B, and m_C. A similar formula holds for non-simply laced algebras (at least classically). Here, we only consider the case where $-\alpha_{r+1}$ is the maximal roots, but it is also possible to extend the Lie algebra with other roots. We have to distinguish l strong particle, one for each long root and s weak ones, one for each short root. Let ν the ratio of the lengths of the long root with the short one ($\nu = \sqrt{2}$ for B_n, C_n, F_4, and $\sqrt{3}$ for G_2). As soon as A or B or C represents a strong particle, and if the coupling is not forbidden by symmetry or cinematic reasons, then Eq.(3.4) applies. If all three particles are weak, then we must take the following formula

$$|f_{ABC}| = \left(\frac{\nu^2 - 1}{\nu}\right)\frac{4\beta}{\sqrt{h^{\mathcal{G}}}}\mathcal{A}_{ABC} \tag{3.5}$$

These interesting formulas do not seem to be known in the literature. We conjecture, that similar geometric formulas must hold for higher-order couplings.

3.2 Structure of the S functions

With the knowledge of the mass spectrum, the possible three particle couplings, the conserved spins, and the implementation of the fundamental constraints acting on the two particle $\hat{S}_{AB}$ functions, we found out that $\hat{S}_{AB}$ decomposes into a product of two terms

$$\hat{S}_{AB}(\theta) = Z_{AB}(\theta;\beta)S_{AB}(\theta) \tag{3.6}$$

The S_{AB} form the so-called "minimal solution". This part contains all the physical information on the system. Single poles with positive residues correspond to the existence of a bound state. Higher order poles correspond to multiloop effects. Bound states are also the consequence of higher order poles and are produced by diagrams like the one in Figure 2.

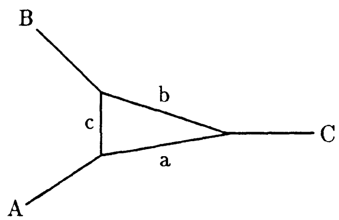


Figure 2: Higher-order three particle coupling.

As an example, we obtained for the $\hat{E}_7$ the mass spectrum given in Table 3. We obtained the following two particles function S_{11} for the lightest particle m_1

$$S_{11}(\theta) = -\frac{\tanh(\frac{\theta}{2} + i\frac{2\pi}{9})\tanh(\frac{\theta}{2} + i\frac{4\pi}{9})}{\tanh(\frac{\theta}{2} - i\frac{2\pi}{9})\tanh(\frac{\theta}{2} - i\frac{4\pi}{9})} \tag{3.7}$$

Table 3: Mass spectrum of E_7 Toda system. $M^2 = 8m^2\sin^2(\frac{\pi}{9})$.

m_1	$=$	M
m_2	$=$	$2M\cos(\frac{\pi}{18})$
m_3	$=$	$M/2\sin(\frac{\pi}{18})$
m_4	$=$	$4M\cos(\frac{\pi}{18})\cos(\frac{\pi}{9})$
m_5	$=$	$4M\cos(\frac{\pi}{18})\sin(\frac{2\pi}{9})$
m_6	$=$	$2M\sin(\frac{2\pi}{9})$
m_7	$=$	$2M\cos(\frac{\pi}{9})$

The pole of S_{11} with positive residues are located at $\theta = i\pi/9$ and $i5\pi/9$. They correspond to the bound states with masses m_2 and m_6 respectively. I will not display here the full set of minimal S_{AB} functions, that can be found in [5]. For this example, we found the origin of all higher-order poles. This requires to generalize the calculations of S. Coleman and H.J. Thun [13] to higher-order loops. Example of diagrams that give rise to poles of orders three and four are pictured in Figure 3. The third order pole example illustrates why odd order

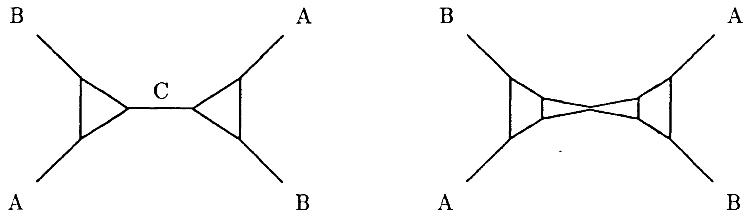


Figure 3: Third and fourth order pole in S_{AB}.

poles are responsible for bound states as in Figure 2.

It is important to verify, that all higher-order poles can be understood at the level of a graph expansion (like the bound state in Figure 2). On the basis of a given mass spectrum, it is sometimes possible to find a minimal solution compatible with unitarity, crossing, and bootstrap. This can be done without having the knowledge of any Lagrangian. However, there are examples where some higher-order poles cannot be understood by the computation of the singular behaviour of a particular graph. This means, that we do not know any interactive process, that would explain such a pole structure. As expected, this is not the case for the S_{AB} functions of our example. It is also important to verify, at least at one loop level, that there is no particle production in this system.

Z_{AB} is the function of the coupling constant, that connects the minimal solution to the exact solution $\hat{S}_{AB}$. This can be seen as a deformation of $1/S_{AB}$. These factors have to satisfy the same constraints as the functions S_{AB}. However, the zeroes of Z must be located in the physical strip and have to match with the poles of S as β goes to zero. In that limit, we recover a free massive theory and $\hat{S}_{AB} = 1$ for any (A, B). As a consequence, the poles of Z, that must not be located in the physical strip, have to match the zeroes of the minimal solution in this limit. Moreover, at the level of perturbation expansion, the higher order poles must appear at the correct order of power in β (a loop correspond to a term proportional to β^2). At last, new poles in the physical strip, that would correspond to new bound states (i.e. new particles), are strictly forbidden. This strongly restrict the possible deformation of $1/S_{AB}$. We have constructed such Z functions for all $\hat{S}$ functions in the $\hat{E}_7$ case. In particular, we obtained for $\hat{S}_{11}$ the following Z_{11} function

$$Z_{11}(\theta) = -\frac{\tanh(\frac{\theta}{2} - i\frac{2\pi}{9} - i\frac{\pi b}{2}) \tanh(\frac{\theta}{2} - i\frac{4\pi}{9} - i\frac{\pi b}{2}) \tanh(\frac{\theta}{2} - i\frac{\pi b}{2})}{\tanh(\frac{\theta}{2} + i\frac{2\pi}{9} + i\frac{\pi b}{2}) \tanh(\frac{\theta}{2} + i\frac{4\pi}{9} + i\frac{\pi b}{2}) \tanh(\frac{\theta}{2} + i\frac{\pi b}{2})} \tag{3.8}$$

Here b is a function of β. Perturbatively, we obtain

$$\pi b = \frac{\beta^2}{2h^{\mathcal{G}}} + ... \tag{3.9}$$

Arguments based on the general structure, higher-order calculations, the anomaly of the energy-momentum tensor, together with analogies suggested in ref.[14], suggest that the exact formula is

$$\pi b = \frac{\beta^2}{2h^{\mathcal{G}}} \left(1 + \frac{\beta^2}{4\pi}\right)^{-1} \tag{3.10}$$

This implies the interesting "duality" property

$$\hat{S}(\beta) = \hat{S}(\frac{4\pi}{\beta}) \tag{3.11}$$

We do not have a rigorous proof of this property. However, as a function of b, the $\hat{S}$ functions are always invariant under the map

$$b \rightarrow \frac{2}{h^{\mathcal{G}}} - b \tag{3.12}$$

This seems to be a general property of affine Toda systems.

3.3 About off-critical non-unitary models

I would like now to show the importance of the Z factor because it can contribute to a major change of interpretation of the minimal solution. The simplest example is provided by the Lee-Yang model. Recently, the minimal solution was proposed by J.L. Cardy and G. Mussardo [15]. Off criticality, the model has only one massive particle with a three particle coupling with imaginary coupling constant. The unique S function of the model is

$$S(\theta) = \frac{\tanh(\frac{\theta}{2} + i\frac{\pi}{3})}{\tanh(\frac{\theta}{2} - i\frac{\pi}{3})} \tag{3.13}$$

A physical pole, now with negative residue, is located at $2i\pi/3$. The negative sign accounts for an imaginary coupling constant.

However, S is also the minimal solution for another model, that is unitary. The Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{\beta^2}\left(2e^{\beta\phi} + e^{-2\beta\phi}\right) \tag{3.14}$$

The classical equation of motion is known as the Bullough-Dodd equation. In this case, the Z function is very important to restore the positivity of the residue and to obtain a real coupling constant. In ref.[14], the following Z function was found

$$Z(\theta) = \frac{\tanh(\frac{\theta}{2} - i\frac{\pi}{3} + i\frac{\pi b}{2})\tanh(\frac{\theta}{2} - i\frac{\pi b}{2})}{\tanh(\frac{\theta}{2} + i\frac{\pi}{3} - i\frac{\pi b}{2})\tanh(\frac{\theta}{2} + i\frac{\pi b}{2})} \tag{3.15}$$

In this case, β^2 depends on b as follows

$$\pi b = \frac{\beta^2}{6}\left(1 + \frac{\beta^2}{4\pi}\right)^{-1} \tag{3.16}$$

It is easy to verify, that Eq.(3.11) is satisfied.

A generalization of the solution for the Lee-Yang model was recently proposed in ref.[16]. It is not very difficult to find Z functions, that match with the ones of a Toda-like theory with all couplings real. A possibility consists in taking the affine A_n models, where particles are organized in complex conjugate pairs with the same mass (for n even) and to take the fields ϕ real analytic to remove the mass degeneracy.

However, if we do not try to match with a given Lagrangian, it is perhaps possible to find other Z functions, only compatible with crossing, unitarity, and bootstrap in such a way, that some residues for bound states remain negative. This would imply the presence of some imaginary couplings. The non-unitary theories seems more intricated than the unitary ones.

Acknowledgements

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References

[1] A.B. Zamolodchikov, *Integrable field theory form conformal field theory*, Proc. of the Taniguchi Symposium, Kyoto 1988, to appear in *Advanced Studies in Pure Mathematics.*

[2] T. Eguchi, S.K. Yang, *Deformation of Conformal Field Theories and Soliton Equations*, Kyoto preprint RIFP-797.

[3] A. LeClair, *Restricted Sine-Gordon Theory and the Minimal Conformal Series*, Princeton preprint PUPT-1124.

[4] T.J. Hollowood, P. Mansfield, *Rational Conformal Field Theory at, and away from, Criticality as Toda Field Theories*, Oxford preprint 89-17P.

[5] P. Christe, G. Mussardo, *Integrable Systems away from Criticality: the Toda Field Theory and S Matrix of the Tricritical Ising Model*, Santa Barbara preprint UCSBTH-89-19.

[6] H.W. Braden, E. Corrigan, P.E. Dorey, and R. Sasaki, *Extended Toda Field Theory and Exact S-Matrices*, Durham preprint UDCPT-89-23.

[7] B.A. Kuperschmidt, P. Mathieu, *Quantum Korteweg-de Vries like Equations and Perturbed Conformal Field Theories*, Phys. Lett. B (to appear);
P. Mathieu, *Integrability of Perturbed Superconformal Minimal Models*, Québec preprint.

[8] John L. Cardy, private communication.

[9] M. Blume, V.J. Emery, and R.B. Griffiths, Phys. Rev. A4(1971)1071.

[10] R.B. Griffiths, Phys. Rev. B7(1973)545.

[11] V. Rittenberg, private communication.

[12] H. Saleur, M. Henkel, *Remarks on the Mass Spectrum of non Critical Coset Models from Toda Theories*, Saclay preprint.

[13] S. Coleman and H.J. Thun, Comm.Math.Phys. 61(1978)31.

[14] A.E. Arinshtein, V.A. Fateyev, and A.B. Zamolodchikov Phys.Lett. B87(1979)389.

[15] J.L. Cardy, G. Mussardo, *S-matrix of the Yang-Lee Edge Singularity in Two Dimensions*, Santa Barbara preprint UCSBTH-89-11, Phys.Lett. B (to appear).

[16] P.G.O. Freund, T.R. Klassen, and E. Melzer, *S-matrices for perturbations of certain conformal field theories*, Chicago preprint EFI 89-29.

QUANTUM GROUPS, BRAIDING MATRICES and COSET MODELS

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Abstract

We discuss a few results on quantum groups in the context of rational conformal field theory with underlying affine Lie algebras. A vertex-height correspondence - a well-known procedure in solvable lattice models - is introduced in the WZW theory. This leads to a new definition of chiral vertex operator in which the zero mode is given by the q-Clebsch Gordan coefficients. Braiding matrices of coset models are found to factorize into those of the WZW theories. We briefly discuss the construction of the generators of the universal enveloping algebra in Toda field theories.

By now, there are two apparently distinct places in theoretical physics in which the quantum group *i.e.* the q- deformation of the universal enveloping algebra [1,2,3] provides a relevant structure. The one is physics of quantum spin chains (or equivalently the classical statistical mechanics) associated with the trigonometric solutions[4] of the Yang-Baxter equation[5,6]. This is the original place in which the quantum group structure was found. The underlying system can be viewed as a lattice-regulated quantum field theory. The other appears as monodromy properties[7,8] of a class of rational conformal field theories with underlying affine Lie algebras. The most concrete realization can be found in the WZW theory. It is well-known that conformal field theories predict finite size corrections to the macroscopic quantities of the system in the large volumes. On the other hand, there are in fact computations of finite size corrections based on Bethe ansatz in solvable lattice models, which agree with conformal field theory predictions. It is, therefore, not inconceivable to suspect that these two applications of the quantum groups are in fact related: the latter being obtained as the continuum limit of the former. I will not try to pursue this program in this talk. What I will discuss instead is a few results[9] which are inspired and transplanted from the structure of the solvable lattice models, but which have their own rationales solely in the context of rational conformal field theories. First, I will give a brief account on the construction of the guantum group generators in the generalized Toda system (GTS) as quantum field theory[10]. Then, without a logical connection, I turn to the discussion of the monodromy properties of WZW theory. I take the differential equation of Knizhinik and Zamolodchikov as a starting point and present a few attempts mentioned. The monodromy properties of the coset theories can be obtained from those of the WZW theory. Finally, I will briefly discuss a perspective for the more logical connection as well as the construction of universal enveloping algebra in Toda field theories[11]. Further references on the subjects discussed here can be found in [9,10].

1)° In a class of vertex models where fundamental variables are arrows taking the value $1 \sim N$, Yang-Baxter relation takes the form

$$
\begin{aligned}
X_j \, X'_{j+1} \, X''_j &= X''_{j+1} \, X'_j \, X_{j+1} \ , \\
X_i \, X'_j &= X'_j \, X_i \ .
\end{aligned} \tag{1}
$$

Here, the operators $X_j \equiv X_j(u)$, $X'_j \equiv X_j(u')$, and $X''_j \equiv X_j(u'' = u' - u)$ act nontrivially on the arrows in the j th column and the $j+1$ th column $(-M \leq j \leq M)$ in a fixed row and take them to the ones in the next row. (See Fig.1). One-parameter family of the matrix elements labeled by u represents local Boltzman weights of the corresponding arrow configuration around a vertex. Here, we have drawn a row (fixed time slice) 45 degrees tilted to the x axis. The resultant time flow is to the southwest.

One way to characterize GTS is that the above infinite dimensional algebraic relation reduces to a finite dimensional algebra (or, to be more accurate, its completion) In A_n case $(N = n+1)$, eq. (1) reduces to $X_j = \rho(1 + y(u)U_j)$ with

$$U_jU_{j+1}U_j - U_j = U_{j+1}U_jU_{j+1} - U_{j+1} \ , \tag{2}$$

$$U_j^2 = 2\cosh\lambda U_j \ , \tag{3}$$

$$U_iU_j = U_jU_i \ , \ \text{for } i \neq j \pm 1 \ . \tag{4}$$

Here, λ is a coupling (or parameter) of the theory and ρ is an arbitrary constant. We also introduce $q \equiv -e^{\lambda}$, and $\Delta \equiv -\cosh\lambda$. The regions $-1 < \Delta < 1$ and $\Delta < -1$ represent respectively critical and antiferromagnetic regimes. In order not to introduce further relationship among U_j's, we have adopted here fixed boundary condition: $\mu'_{-M} = \mu_{M+1} =$ the highest value.

Up to a trivial rescaling, Equations (2), (3), (4) are defining relations of the Hecke algebra $\cup_m H_m$. The function $y(u)$ sets a momentum-rapidity relation of the system :

$$y(u) \equiv \frac{q(\zeta-1)}{q^2\zeta-1} \equiv e^{ip} \quad . \tag{5}$$

Here, we introduced $\zeta \equiv e^{-2u}$. For $\Delta < -1$, u imaginary (real) corresponds to real (imaginary) p i.e. Minkowski (Euclidean) field theory. For $-1 < \Delta < 1$, set $\lambda = i\mu$ with μ real, and $u = i\mu/2 - \alpha/2$. Real α corresponds to $-(\pi-\mu) < p < (\pi-\mu)$. The solution to eq. (1) in A_n case is expressible as

$$U_j = \sum_{\alpha,\beta=1,\alpha\neq\beta}^{N} E^j_{\alpha,\beta}\otimes E^{j+1}_{\beta,\alpha} - q^{-1}\sum_{\alpha,\beta=1,\alpha>\beta}^{N} E^j_{\alpha,\alpha}\otimes E^{j+1}_{\beta,\beta} - q\sum_{\alpha,\beta=1,\alpha<\beta}^{N} E^j_{\alpha,\alpha}\otimes E^{j+1}_{\beta,\beta} \quad . \tag{6}$$

It is worthwhile to note that $\lim_{\rho=iq^{-1},u\to+\infty} X_j(u) = i(U_j+1/q) \equiv \sigma^j$, safely provides an operator obeying Braid relation :

$$\sigma^j\sigma^{j+1}\sigma^j = \sigma^{j+1}\sigma^j\sigma^{j+1} \quad , \tag{7}$$

$$\sigma^i\sigma^j = \sigma^j\sigma^i \quad , \quad i \neq j\pm 1 \quad . \tag{8}$$

It is well known that a sequence of mutually commuting conserved charges is generated by one-parameter family of commuting transfer matrices. In the current formalism, the transfer matrix is simply

$$T(\zeta) = \lim_{M\to+\infty} X_{-M}\cdots X_M \quad . \tag{9}$$

The Hamiltonian can be defined to be the lowest nontrivial term in the expansion of $T(\zeta)$ around $\zeta = 1$. It consists of the nearest neighbour interations only. In A_n case,

$$H = \sum_j \mathcal{H}_j \equiv \frac{1}{q-q^{-1}}\sum_j U_j \quad . \tag{10}$$

This identifies the generators of the Hecke algebra with the Hamitonian density. The higher order terms tell us

$$\sum_j U_j \ ,\sum_j U_jU_{j+1} \ ,\cdots,\sum_j U_jU_{j+1}\cdots U_{j+\ell},\ \cdots \quad , \tag{11}$$

are a set of bases for the conserved charges.

Another useful bases for the sequence of conserved charges are

$$Q_n = \oint \frac{d\zeta}{2\pi i}\zeta^{n-1}\mathcal{S}T(\zeta) \ , n\in\mathcal{Z} \quad . \tag{12}$$

Here, $\mathcal{S}$ is the shift operator which shifts the arrow labeling by one unit. The boost operator $\mathcal{L}_0 = \sum_j j\mathcal{H}_j$ acts on Q_n as

$$[\mathcal{L}_0, Q_n] = nQ_n \quad . \tag{13}$$

Let us now examine the symmetries of the system. For simplicity, we restrict ourselves to A_n case. An inspection on expression (6) immediately tells that there are rather obvious n abelian generators which commute with Hamiltonian and the transfer matrix :

$$h_\ell = \sum_j h_\ell^j , \quad 1 \leq \ell \leq n , \tag{14}$$

$$\exp h_\ell^j \equiv \mathrm{diag}\left(1, \cdots, \frac{1}{q}, q, 1, \cdots\right)^j . \tag{15}$$

What is much less obvious is that, for an arbitrary value of q, we can construct conserved operators which are analogs of the step operators of $SU(n+1)$. Let

$$\begin{aligned} e^j_{\ell,\ell+1} &\equiv \exp\left(\sum_{i=-\infty}^{j-1} h^i_\ell\right) E^j_{\ell,\ell+1} = f^j_\ell , \\ e^j_{\ell+1,\ell} &\equiv \exp\left(-\sum_{i=j+1}^{\infty} h^i_\ell\right) E^j_{\ell+1,\ell} = e^j_\ell . \end{aligned} \tag{16}$$

The generators corresponding to simple roots are $e_\ell = \sum_j e^j_\ell$ and $f_\ell = \sum_j f^j_\ell$. The choice of the exponential factors is crucial in proving vanishing commutator with the Hamiltonian and the transfer matrix:

$$\begin{aligned} [\, H, e_\ell \,] &= [\, H, f_\ell \,] = 0 \\ [\, T(\zeta), e_\ell \,] &= [\, T(\zeta), f_\ell \,] = 0 . \end{aligned} \tag{17}$$

At $\Delta = 1$, e_ℓ's and f_ℓ's become the ordinary group generators of $SU(n+1)$. At $\Delta = 0, n = 1$, eq. (16) reduces to the familiar Jordan -Wigner transformation for the Pauli spin operators. The generators h_ℓ, e_ℓ, and f_ℓ constructed above are the Chevally generators of $\mathcal{U}(A_n)$ and form an algebra

$$\begin{aligned} \exp(h_\ell/2)\, e_{\ell'} \exp(-h_\ell/2) &= q^{a_{\ell,\ell'}/2} e_{\ell'} \\ \exp(h_\ell/2)\, f_{\ell'} \exp(-h_\ell/2) &= q^{-a_{\ell,\ell'}/2} f_{\ell'} \\ [e_\ell, f_{\ell'}] &= \delta_{\ell,\ell'} \frac{\exp h_\ell - \exp -h_\ell}{q - q^{-1}} . \end{aligned} \tag{18}$$

Here, the matrix $a_{\ell,\ell'}$ is a Cartan matrix for A_n. The remaining generators of $\mathcal{U}(g)$ are given recursively by $t_{\ell,\ell'} = t_{\ell,\ell''} t_{\ell'',\ell'} - q t_{\ell'',\ell'} t_{\ell,\ell''}$ $(\ell \leq \ell'' \leq \ell')$ with $t_{\ell,\ell+1} \equiv e_\ell$ and $t_{\ell,\ell-1} \equiv f_\ell$. This choice is dictated by the q-analog of the Chevally relation.

We have presented here the quantum group structure of GTS in the vertex representation. The transition to the path representation is made through the q-Clebsch Gordan coefficients as an intertwiner. This is a special case of the well-known vertex-height correspondence. We will see the same structure in the WZW theory.

2)° Monodromy properties of Wess-Zumino-Witten model: Let $\hat{g}$ be a simple affine Lie algebra and g_α^Λ be a primary field of the WZW theory transforming in the representation labelled by the highest weight Λ. Here, a weight index α, which is generally denoted by a set of integers, labels the components of the primary fields. The object we study is the n-point correlation function

$$\mathcal{G}_n \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} (z_1, z_2, \cdots z_n; \bar{z}_1, \bar{z}_2, \cdots \bar{z}_n) \equiv \langle g_{\alpha_1}^{\Lambda_1}(z_1, \bar{z}_1)\; g_{\alpha_2}^{\Lambda_2}(z_2, \bar{z}_2) \cdots\; g_{\alpha_n}^{\Lambda_n}(z_n, \bar{z}_n) \rangle \quad , \tag{19}$$

obeying the differential equation of Knizhnik and Zamolodchikov[12]:

$$\left(\frac{\partial}{\partial z_i} + \sum_{j(\neq i)} \frac{2}{(k + \tilde{h})\theta^2} \frac{1}{z_i - z_j} t_i^a \otimes t_j^a \right)_{\alpha_1, \alpha_2, \cdots, \alpha_n}^{\alpha_1', \alpha_2', \cdots, \alpha_n'} \mathcal{G}_n \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1' \alpha_2' \cdots \alpha_n' \end{pmatrix} = 0 \tag{20}$$

Here, k and $\tilde{h}$ are the level of the affine Lie algebra and the dual Coxeter number respectively, and θ^2 is defined through $f_{ac}^d f_{bd}^c = -\tilde{h}\theta^2 g_{ab}$. The correlator is originally defined in the region $|z_1| < |z_2| < \cdots < |z_n|$ and analytically continued to the other regions. The differential equation is, therefore, defined over the domain $X_n \equiv \{(z_1, \cdots z_n) \in \mathcal{C}_n \; ; z_i \neq z_j \text{ if } i \neq j\}$. The fundamental group of X_n is the pure braid group with n strands. From now on, we suppress the dependence on the $\bar{z}_i$'s. Eq. (20) can be written as

$$(d + \omega)_{\alpha_1, \alpha_2, \cdots, \alpha_n}^{\alpha_1', \alpha_2', \cdots, \alpha_n'} \mathcal{G}_n \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1' \alpha_2' \cdots \alpha_n' \end{pmatrix} (z_1, z_2, \cdots, z_n) = 0 \quad . \tag{21}$$

by introducing a one-form $\omega = \sum_{1 \leq i < j \leq n} \frac{2}{(k+\tilde{h})\theta^2} t_i^a \otimes t_j^a d\log(z_i - z_j)$, taking the value in $End\left(V^{\Lambda_1} \otimes \cdots V^{\Lambda_n}\right)$. The solution can formally be written as

$$\mathcal{G}_n \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} (z_1, z_2, \cdots z_n) = \left(\mathcal{P} \exp \left(- \int_{\mathbf{z}_0}^{\mathbf{z}} \omega \right) \mathcal{G}_n \right) \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} (z_{10}, z_{20}, \cdots z_{n0}) \tag{22}$$

with respect to the one at base point $\mathbf{z}_0 = (z_{10}, z_{20}, \cdots z_{n0})$ and the symbol $\mathcal{P}$ implies the path ordering of the exponential. We introduce $q \equiv \exp(-\frac{\pi i}{k + \tilde{h}})$.

The monodromy of the conformal block around the singularity $z_i = z_{i+1}$ is defined to be $\mathcal{M}^{(i)} = \mathcal{P} \exp\left(- \oint_{\gamma_0^{(i)}} \omega \right)$. Here $\gamma_0^{(i)}$ denotes a closed path which starts and ends at base point $\mathbf{z}_0$ and goes around the line $z_i = z_{i+1}$. The braiding matrix $\sigma^{(i)}$ is defined to be the square root of $\mathcal{M}^{(i)}$ times the permutation matrix. It is straightforward to evaluate the eigenvalues of a *particular* $\sigma^{(i)}$ from the above expression. But such evaluation does not provide a conceptual explanation of the coincidence noted by many people. What we would like to explain is that *all* elements of $\sigma^{(i)}$'s are, up to a similarity transformation, equal to the vertex Boltzman weight of the corresponding

GTS realizing the quantum group. For definiteness, we consider the case A_{N-1} in which all Λ's are in the fundamental representation. Here, we only give essential logical points. For a full explanation, see in ref.[8,9].

The first step is to show that $\sigma^{(i)}$'s form a representation of Hecke algebra H_n: $(\sigma_i - q)(\sigma_i + q^{-1}) = 0$. For that purpose, unambiguous bases for the conformal blocks must first be determined. The ordinary definition of chiral vertex operator[7] provides the path bases which qualify this. In this bases, all elements σ_r's are related to σ_1 simply by a similarity transformation given by the fusion matrix[13]. These arguments are essentially due to Kohno[8]. The second step is the point we already discussed before: the $\check{R}$ matrix or vertex Boltzman weight in the integrable lattice models at infinite spectral parameter in general provides a representation of the braid group B_n through $End\left(V^{\Lambda_1} \otimes \cdots V^{\Lambda_n}\right)$. In the A_{N-1} case, it also provides a representation of Hecke algebra. The third step, which is due to Wenzl[14], is that any finite dimensional representation of Hecke algebra can be obtained through this procedure.

The above argument is sufficient to tell us that there exists a similarity transformation $\mathcal{S}$ which brings all elements of the braid group into the $\check{R}$ matrix of the quantum group:

$$\sigma^{(i)\alpha_i}_{\alpha'_i}|_{\alpha'_{i+1}}{}^{\alpha_{i+1}} (\mathcal{S}\mathcal{G}_n) \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} = \mathcal{S}\check{R}(u \to \infty)_{\alpha'_i}{}^{(i)\ \alpha_i}|_{\alpha'_{i+1}}{}^{\alpha_{i+1}} (\mathcal{G}_n) \begin{pmatrix} \Lambda_1 \Lambda_2 \cdots \Lambda_n \\ \alpha_1 \alpha_2 \cdots \alpha_n \end{pmatrix} . \tag{23}$$

On the other hand, the existence of the operator formalism of WZW theory guarantees the factorization of n-point correlator into four point blocks, and we may write

$$g^{\Lambda_i}_{\alpha_i}(z_i) g^{\Lambda_{i+1}}_{\alpha_{i+1}}(z_{i+1}) = \sigma^{(i)\alpha'_i}_{\alpha_i}|_{\alpha_{i+1}}{}^{\alpha'_{i+1}} g^{\Lambda_{i+1}}_{\alpha'_i}(z_{i+1}) g^{\Lambda_i}_{\alpha'_{i+1}}(z_i) . \tag{24}$$

We have seen that the product of the primary fields of WZW theory carrying weight indices of the classical Lie algebra has turned out to be the representation space (of the vertex type) for the Hecke algebra and therefore for quantum group. The intertwiner to the path representation is given by q- Clebsch-Gordan coefficients.

Let the composed operator in eq. (24) be acting on a direct sum of the irreducible $\hat{g}$ modules $\mathcal{H}_d$'s with the highest weight Λ_d's. The above discussion tells us that , in accordance with the decomposition of the tensor module ,

$$\mathcal{H}_a \otimes \mathcal{H}_i = \sum_b \mathcal{H}_b, \quad \mathcal{H}_b \otimes \mathcal{H}_{i+1} = \sum_d \mathcal{H}_d \quad , \tag{25}$$

it is legitimate to expand eq. (24) as

$$g^{(\Lambda_i)}_{\alpha_i}(z_i) g^{(\Lambda_{i+1})}_{\alpha_{i+1}}(z_{i+1}) = \sum_{a,b,d,} \mathcal{V}^{(\Lambda_i)}_{a,b}(z_i) \mathcal{V}^{(\Lambda_{i+1})}_{b,d}(z_{i+1}) \mathcal{S}_{\alpha_i,\alpha_{i+1}}{}^{\alpha'_i,\alpha'_{i+1}} \Phi^{(\Lambda_i)}_{a,b}{}_{\alpha'_i} \Phi^{(\Lambda_{i+1})}_{b,d}{}_{\alpha'_{i+1}} c_b^{(a,d,\Lambda_i,\Lambda_{i+1})} . \tag{26}$$

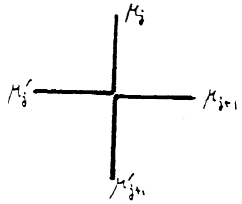

Fig. 1

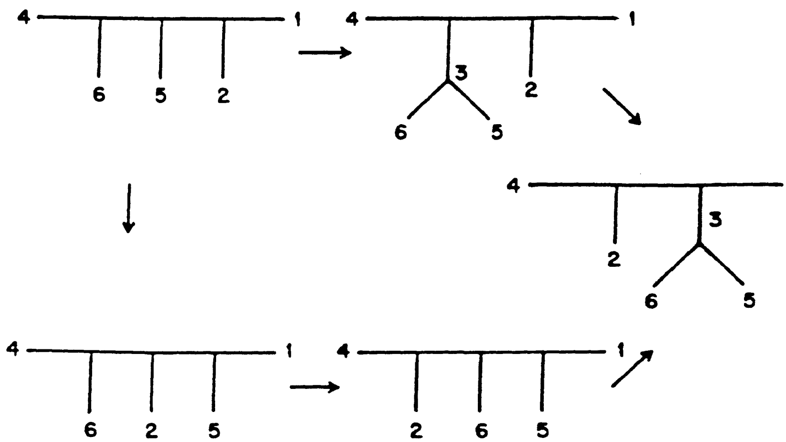

Fig. 2

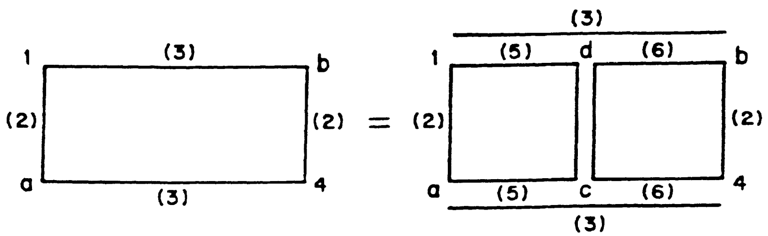

Fig. 3

Here, we have introduced an operator $\mathcal{V}_{a,b}^{(\Lambda_i)}(z_i)$ which might be called a q-version of the chiral vertex operator discussed before. This operator carries over the normalization of the ordinary chiral vertex operator, but has no reference to the weight indices. A set of coefficients $\Phi_{a,b}^{(\Lambda_i)}\ {}_{\alpha_i} =< a, M_i \mid b, M_i - \alpha_i ; \Lambda_i, \alpha_i >$ is a q-version of the Clebsch-Gordan coefficients. They serve as basis vectors of flat sections (zero mode part) of the trivial vector bundle over X_n. The coefficient $c_b^{(a,d,\Lambda_i,\Lambda_{i+1})}$ reflects representation dependent normalization of the chiral vertex operators. [1]

Eq. (26) permits us to translate eqs. (23),(24), into the exchange algebra of the q-vertex operators :

$$\mathcal{V}_{a,b}^{(\Lambda_i)}(z_i)\mathcal{V}_{b,d}^{(\Lambda_{i+1})}(z_{i+1}) = \sum_{b'} B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a, d \\ b, b' \end{bmatrix} \mathcal{V}_{a,b'}^{(\Lambda_{i+1})}(z_{i+1})\mathcal{V}_{b',d}^{(\Lambda_i)}(z_i) \ ,$$

$$B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a, d \\ b, b' \end{bmatrix} = q^{-3/2}(c_b^{(a,d,\Lambda_i,\Lambda_{i+1})})^{-1} W^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a, b' \\ b, d \end{bmatrix}(u\to\infty)\, c_{b'}^{(a,d,\Lambda_i,\Lambda_{i+1})} \quad (27)$$

The braiding matrix $B^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a, d \\ b, b' \end{bmatrix}$ is, up to the diagonal similarity transformation due to the normalization of the vertex operator, given by the face Boltzmann weight $W^{(\Lambda_i,\Lambda_{i+1})}\begin{bmatrix} a, b' \\ b, d \end{bmatrix}(u\to\infty)$ of the corresponding GTS at infinite spectral parameter. In the case where all representations Λ_i's are in the fundamental representation, the explicit answer for the Boltzman weight can be extracted from ref. [15]. In the simplest case $A_1^{(1)}$ in which Λ_i's are in the spin 1/2 representation, the answer $B^{(1/2,1/2)}\begin{bmatrix} j_0, & J \\ j, & j' \end{bmatrix}$ agrees with the result by Tsuchiya and Kanie [7] :

$$\begin{aligned} & q^{-3/2}\begin{pmatrix} q^{-2j_0}/[2j_0+1], & c_{j_0+1/2}^{-1}c_{j_0-1/2}\,q\sqrt{[2j_0][2j_0+2]}/[2j_0+1] \\ c_{j_0-1/2}^{-1}c_{j_0+1/2}\,q\sqrt{[2j_0][2j_0+2]}/[2j_0+1], & -q^{2j_0+2}/[2j_0+1] \end{pmatrix} \quad \text{for} \quad 1/2\le j_0=J\le k/2 \\ & -q^{1/2}\delta_{j,j'} \qquad \text{for} \quad J=j_0\pm 1/2 \\ & q^{-3/2}\delta_{j,j'} \qquad \text{for} \quad j_0=J=0,\ k/2 \ . \end{aligned} \quad (28)$$

Here, $c_{j_0+1/2}=\Gamma(-\frac{2j_0+1}{k+2})/\sqrt{\Gamma(-\frac{2j_0+2}{k+2})\Gamma(-\frac{2j_0}{k+2})}$, and $c_{j_0-1/2}=\Gamma(\frac{2j_0+1}{k+2})/\sqrt{\Gamma(\frac{2j_0+2}{k+2})\Gamma(\frac{2j_0}{k+2})}$, and the q-number j is defined by $[j]=\frac{q^j-q^{-j}}{q-q^{-1}}$.

Among the primary fields of of a given RCFT, the most relevant operator i.e. the one carrying the lowest conformal dimension has been given a special meaning: we can regard it as an elementary field out of which the rest of the primary fields is expressed as its composites. The formula (28) given above is for those cases in which the external primary fields are the most relevant ones. The braiding matrices for the arbitrary primary fields can be obtained from those for the most relevant ones.

[1] To determine this, one usually has to solve the connection problem of the attendant differential equation. See [7].

This is certainly true for WZW and the coset $\frac{G(k)\otimes G(1)}{G(k+1)}$. All primary fields can then be obtained as repeated products of the most relevant ones. The problem now is to find braiding matrices for these "composite" (higher spin) fields. The relevant matrices satisfying the braid relation can be obtained from the (trigonometric) solution of the Yang-Baxter equation for the higher representations by sending the spectral parameter to infinity. There exists a well-known procedure called fusion procedure in the integrable lattice models which generates these solutions from the fundamental ones[16]. We will not review it here. The solutions essentially consist of product of R matrices with prescribed shifts of the spectral parameter. The product begins and ends with projectors. The identical procedure, modulo the problem of the phase, can be implemented solely in the context of RCFT. In Fig. 2, we indicate how the procedure goes through. Take, for instance, a five point block and regard that $\mathbf{1}, \mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}$, are most relevant fields. We obtain

$$B^{(\mathbf{2},\mathbf{3})}\begin{bmatrix} \mathbf{1}, \mathbf{4} \\ a, b \end{bmatrix} = \sum_{c,d} F^{-1\ (\mathbf{5},\mathbf{6})}\begin{bmatrix} a, \mathbf{4} \\ c, \mathbf{3} \end{bmatrix} B^{(\mathbf{2},\mathbf{5})}\begin{bmatrix} \mathbf{1}, c \\ a, d \end{bmatrix} B^{(\mathbf{2},\mathbf{6})}\begin{bmatrix} d, \mathbf{4} \\ c, b \end{bmatrix} F^{(\mathbf{5},\mathbf{6})}\begin{bmatrix} \mathbf{1}, b \\ d, \mathbf{3} \end{bmatrix}. \tag{29}$$

We represent this equation by Fig. 3. The fusion matrix F in eq. (29) play the role of projection and inclusion operators. They are the 6-j symbols of the underlying quantum group.

3)° Monodromy properties of coset models: Coset models form an interesting class of RCFT. Here we would like to show how the braiding properties of the G/H coset models can be obtained from the ones of the G-WZW theory and the ones of H-WZW theory. For definiteness, we consider a coset $G/H = \frac{A^{(1)}_{N-1}(k)\otimes A^{(1)}_{N-1}(1)}{A^{(1)}_{N-1}(k+1)}$ with diagonal embedding. The arguments $k, 1$, and $k+1$ refer respectively to the levels of the Kac-Moody algebras.

The factorization formula of the characters by GKO implies that , for a given primary state in the coset theory under consideration and a primary state in H theory, one can find a unique state in G theory. This state, however, is not necessarily primary: the diagonal embedding of the H theory into G theory is in the sense of Kac-Moody modules. A point worth making here is that the monodromy properties of the conformal block is insensitive to this ambiguity as conformal dimensions for the primary fields and the ones for their descendents differ only by integers. The formula we give below should be understood in this sense. One can also show the factorization of fusion coefficients, starting from the Verlinde's formula[17].

Therefore, the q-vertex operator of the G theory introduced in eq. (26) naturally factorizes into the one of the H theory and the one of the coset theory. We may, therefore, write

$$\mathcal{V}_{\mathbf{a}_1,\mathbf{a}_2;\ \mathbf{b}_1,\mathbf{b}_2}^{(\Lambda,1)}(z) = \sum_{a,b,\lambda} V_{a,b}^{(\lambda)}(z)\, P_{\alpha,\beta}^{(\sigma)}(z) \quad . \tag{30}$$

Here, α, β and σ respectively denote triplets of integers $\alpha = \begin{pmatrix} \mathbf{a}_1,\mathbf{a}_2 \\ a \end{pmatrix}$, $\beta = \begin{pmatrix} \mathbf{b}_1,\mathbf{b}_2 \\ b \end{pmatrix}$, and $\sigma = \begin{pmatrix} \Lambda,1 \\ \lambda \end{pmatrix}$.

Let the exchange algebra of the coset theory be

$$P_{\alpha,\ \beta}^{(\sigma_1)}(z_1) P_{\beta,\ \delta}^{(\sigma_2)}(z_2) = \sum_{\beta'} B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha,\delta \\ \beta,\beta' \end{bmatrix} P_{\alpha,\ \beta'}^{(\sigma_2)}(z_2) P_{\beta',\ \delta}^{(\sigma_1)}(z_1) \quad . \tag{31}$$

Here, $B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha,\delta \\ \beta,\beta' \end{bmatrix}$ is a braiding matrix for the coset theory which we would like to express in terms of the one $\mathcal{B}_G^{(\Lambda_1,1),(\Lambda_2,1)} \begin{bmatrix} \mathbf{a}_1,\mathbf{a}_2;\mathbf{d}_1,\mathbf{d}_2 \\ \mathbf{b}_1,\mathbf{b}_2;\mathbf{b}_1',\mathbf{b}_2' \end{bmatrix}$ for the G theory and the one $B_H^{(\lambda)} \begin{bmatrix} a,d \\ b,b' \end{bmatrix}$ for the H theory.

Start from the exchange algebra of G theory (*cf* eq. (27)). We apply the factorization property (eq. (30)) to the right hand side. As for the left hand side, we first use the factorization property and subsequently the exchange algebra of the H theory and the one of the coset theory *i.e.* eq. (31). Taking matrix elements with respect to an arbitrary state in the coset theory, and subsequently in H theory, we conclude

$$\begin{aligned} &\sum_{\mathbf{b}_1'',\mathbf{b}_2''} \mathcal{B}_G^{(\Lambda_1 1,\Lambda_2 1)} \begin{bmatrix} \mathbf{a}_1,\mathbf{a}_2;\mathbf{d}_1,\mathbf{d}_2 \\ \mathbf{b}_1,\mathbf{b}_2;\mathbf{b}_1'',\mathbf{b}_2'' \end{bmatrix} \delta_{b',b''}\theta\left(b' \subseteq (\mathbf{b}_1'',\mathbf{b}_2'')\right) \\ &= \sum_{b\subseteq \mathbf{b}_1'',\mathbf{b}_2''} B_H^{(\lambda_1,\lambda_2)} \begin{bmatrix} a,d \\ b,b' \end{bmatrix} B_{G/H}^{(\sigma_1,\sigma_2)} \begin{bmatrix} \alpha,\delta \\ \beta,\beta'' \end{bmatrix} \quad . \end{aligned} \tag{32}$$

The braiding matrices for the coset models are given by solving the above factorized formula. The symbol $\theta(b' \subseteq (b_1'', b_2''))$ refers to an embedding restriction.

It is straightforward to apply the formula eq. (32) to the minimal discrete series described by the coset $\frac{A_1^{(1)}(k)\otimes A_1^{(1)}(1)}{A_1^{(1)}(k+1)}$. The most relevant primary field is labeled by $\sigma = \begin{pmatrix} \frac{1}{2},0 \\ \frac{1}{2} \end{pmatrix}$. Let the incoming and outgoing primary states of the G theory be $(\mathbf{a}_1, \mathbf{a}_2) = (\mathrm{spin}j_i, 0)$ and $(\mathbf{d}_1, \mathbf{d}_2) = (\mathrm{spin}j_f, 0)$ respectively. Likewise, $(\mathbf{b}_1, \mathbf{b}_2) = (j = j_i \pm 1/2,\ 0)$, $(\mathbf{b}_1'', \mathbf{b}_2'') = (j'' = j_i \pm 1/2,\ 0)$. The H-primary fields a, d, b, b'' are diaginal embeddings of the above ones. Since both $\mathcal{B}_G$ and B_H are known from the WZW model (eq. (28)), the above formula (eq. (32)) determines the braiding matrices of the most relevant field for the arbitrary incoming and outgoing primary states in the minimal series. It is instructive to compare this result with the one from the Coulomb gas approach in the case of the Ising model. We denote the braiding matrices $B^{(\sigma,\sigma)}_{G/H}\left[\begin{array}{c}\alpha,\delta\\ \beta,\beta''\end{array}\right]$ by $\begin{array}{ccccc} & (\sigma) & \alpha & (\sigma) & \\ \beta & & \diamond & & \beta'' \\ & (\sigma) & \delta & (\sigma) & \end{array}$ for $1=\begin{pmatrix}0,0\\0\end{pmatrix}$, $\sigma=\begin{pmatrix}\frac{1}{2},0\\ \frac{1}{2}\end{pmatrix}$, and $\epsilon=\begin{pmatrix}0,0\\1\end{pmatrix}$. We obtain

$$\begin{array}{ccc} & \sigma & \\ \beta & \diamond & \beta'' \\ & \sigma & \end{array} = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-\pi i/8}, & \frac{\sqrt{6}}{4}e^{-5\pi i/8} \\ \sqrt{6}\,e^{-5\pi i/8}, & \frac{1}{\sqrt{2}}e^{-\pi i/8} \end{pmatrix}$$

$$\begin{array}{ccc} & 1 & \\ \sigma & \diamond & \sigma \\ & 1 & \end{array} = \begin{array}{ccc} & \epsilon & \\ \sigma & \diamond & \sigma \\ & \epsilon & \end{array} = e^{\pi i/8}, \quad \begin{array}{ccc} & 1 & \\ \sigma & \diamond & \sigma \\ & \epsilon & \end{array} = \begin{array}{ccc} & \epsilon & \\ \sigma & \diamond & \sigma \\ & 1 & \end{array} = e^{-3\pi i/8} \tag{33}$$

This agrees with the the result from the Coulomb gas approach[18] except for the $-\sqrt{3}$ factors in $\begin{array}{ccc} & \sigma & \\ \beta & \diamond & \beta'' \\ & \sigma & \end{array}$ for $(\beta,\beta'')=(1,\epsilon),(\epsilon,1)$. These factors can be attributed to the normalization of the bases different from the one employed here. The braiding matrices for general external primary fields of coset models can be obtained from the one given in eq. (32) by the fusion procedure described in eq. (29) for the WZW theory.

4)° So far, our discussion on quantum groups consists of the two diconnected parts which one can relate only by an analogy. Let me suggest here a more direct connection between the braiding matrices of WZW theory and $\check{R}$ matrix of the GTS. Our main proposal eq. (26) is partly suggested by drawing an analogy to the well-known vertex-height correspondence in a class of integrable lattice models. It would be desirable if the exchange algebra eq.(27) follows directly from the structure of the lattice correlation functions. The braid relation appearing in WZW conformal blocks is nothing but the Yang-Baxter relation in the infinite momentum frame. Moreover, the notion of monodromy is already in the lattice correlation functions despite the absence of the complex z-plane[19]. One can presumably deduce monodromy structure of the conformal block, by studying lattice correlation function in the analytic rapidity plane and taking a continuum limit in the end. Some of the techniques developed in [20] appear to be relevant.

Let us finally present a result[11] in Toda field theories which is motivated from the form of the quantum group generators (eq. (16)) in GTS. This expression bears a striking resemblance to Mandelstam's soliton operator in sine-Gordon theory. In fact, the following expression turns out to be a density for the universal enveloping algebra which is a symmetry of $su(2)$ Toda field theories:

$$\Psi(x) =: \exp[\gamma \int_{-\infty}^{x} d\xi p(\xi) + C \int_{-\infty}^{x} \partial_\xi \phi(\xi)] :$$

$$\tilde{\Psi}(y) =: \exp[\gamma' \int_{y}^{+\infty} d\xi p(\xi) + C' \int_{y}^{+\infty} \partial_\xi \phi(\xi)] : \quad . \tag{34}$$

Here, γ, γ', C, and C' are constants fixed by the requirements of symmetry generators and statistics. For details, see ref.[11].

In this talk, I discussed the deformation of symmetry generators due to the change of the coupling constant which is the only parameter (except the spectral parameter) in the trigonometric solutions of the Yang-Baxter relation. In the general elliptic case, there is another deformation or distortion due to the modulus. Physically, this is a mass parameter and one realization of such deformation is the deformation of conformal field theory which has received much attention now. Integer eigenvalue structure of the corner transfer matrix(CTM) in the eight-vertex model and its relatives is the statement that the CTM spectrum stays invariant under the both of the deformations. The relevant operator algebra is the noncritical Virasoro algebra proposed a while ago[21]. All these developments together with the quantum group seem to point towards a single appealing and coherent framework of operator algebras based on noncommutative geometry.

References

[1] V. G. Drinfeld, Soviet Math. Dokl.32, 254 (1985)

[2] M. Jimbo, Lett. Math. Phys. 10, 63 (1985)

[3] E. Sklyanin, *Func. Anal. and Appl.*, **16**(1982) 263 ; P. Kulish, and N. Reshetikhin , *J. Soviet Math.* **23**, (1983) 2435

[4] M. Jimbo, Commun. Math. Phys. 102, 537 (1986).

[5] C. N. Yang, Phys. Rev. Lett 19, 1312 (1967).

[6] R. J. Baxter, Ann. Phys. (N. Y.) 70, 193 (1972).

[7] A. Tsuchiya and Y. Kanie, *Lett. Math. Phys.* **13** (1987) 303 ; *Adv. Stud. Pure Math.* **16** (1988) 297

[8] T. Kohno, *Contemp. Math.* **78** (1988) 339; *Ann. Inst. Fourier* **37** (1987) 139; Nagoya preprint, November 1988

[9] H. Itoyama and A. Sevrin, ITP-SB-89-33, to appear in Int. Journ. Mod. Phys. A

[10] H. Itoyama, ITP-SB-88-82, to appear in Phys. Lett. A

[11] H. Itoyama and P. Moxhay, ITP-SB-89-57

[12] V .G. Knizhnik and A. B. Zamolodchikov, *Nucl. Phys.* **B247** (1984) 83

[13] G. Moore and N. Seiberg, *Phys. Lett.* **212B** (1988) 441

[14] H. Wenzl, *Invent. Math.* **92** (1988) 349

[15] M. Jimbo, T. Miwa, and M. Okado, *Commun. Math. Phys.***116** (1988) 507

[16] R. P. Kulish, N. Yu. Reshetikhin, E. K. Sklyanin *Lett. Math. Phys.* **5** (1981) 393

[17] E. Verlinde, *Nucl. Phys.*, **B300** [FS22] (1988) 360

[18] L. Alvarez-Gaume, G. Gomez and G. Sierra, *Phys. Lett.* **220B** (1989) 142; Cern TH 5267/88

[19] See, for example, B. M. McCoy, J. H. Perk and T. T. Wu *Phys. Rev. Lett.* **46** (1981) 757.

[20] H. Itoyama and H. B. Thacker, Nucl. Phys. B320[FS](1989)541; Nucl. Phys.B (Proc. Suppl.) 5A (1988)9.

[21] H. Itoyama and H. B. Thacker, Phys. Rev. Lett. 58, 1395 (1987).

GAUGED WZW MODELS AND THE COSET CONSTRUCTION OF CONFORMAL FIELD THEORIES*

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ABSTRACT

The gauged Wess-Zumino-Witten models are shown to be conformal field theories providing field theoretic realizations for the Goddard-Kent- Olive coset construction G/H. In particular the conformal charge of the model coincides with the one obtained in the GKO construction. The physical spectrum of the gauged WZW model is analyzed in the BRST framework. We discuss how the physical states implement unitary representations of the coset Virasoro algebra. When H is the Cartan subalgebra of G the BRST invariant current operators (the physical currents) are shown to be the parafermionic currents.

INTRODUCTION

The study of conformal field theories is important since they form the basic building blocks of string theories and describe two-dimensional statistical systems near second order phase transitions. A wide class of conformal field theories can be classified as G/H coset models using the Goddard, Kent and Olive (GKO) algebraic construction.[1-3]

The GKO construction [1] relies on the relationship between affine Kac-Moody algebras and Virasoro algebras. Given a finite, compact Lie algebra G one can define an affine Kac-Moody algebra $\hat{G}$

$$[J_m^a, J_n^b] = i f^{abc} J_{m+n}^c + km\delta^{ab}\delta_{m+n,0} \tag{1}$$

where f^{abc} are the structure constants of G and k is a real number called the central charge. A highest weight irreducible, unitary representation of (1) is built up from a primary state $|\phi >$ which belongs to a finite dimensional irreducible representation of G,

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$$
\begin{aligned}
J_n^a|\phi> &= 0 \qquad n>0 \\
J_0^a|\phi> &= t_R^a|\phi>
\end{aligned} \tag{2}
$$

by applying creation operators J_{-n}^a, $n>0$. The requirement of unitarity imposes constraints on k and the allowed representations R of the primary states $|\phi>$. In fact k has to be an integer and the highest weight μ of the representation R must satisfy $k \geq \psi \cdot \mu$, where ψ is the highest root of G normalized to $\psi^2 = 2$.

Given $\hat{G}$ one can use the Sugawara-Sommerfield construction to express the energy-momentum tensor in terms of currents. The energy-momentum modes L_n^G are given by

$$
L_n^G = \frac{1}{2k+c_G}\sum_{a=1}^{dimG}\sum_{m=-\infty}^{\infty} :J_{n-m}^a J_m^a: \tag{3}
$$

where c_G is the quadratic Casimir of the adjoint representation of G. One can verify that L_n^G satisfy a Virasoro algebra with central charge [f1]

$$
c(G,k) = \frac{2k\,dimG}{2k+c_G} \tag{4}
$$

Goddard, Kent and Olive[1] generalized the above construction by considering two algebras H, G with $H \subset G$. They noted that the difference

$$
L_n^{GKO} = L_n^G - L_n^H \tag{5}
$$

satisfies a Virasoro algebra with central charge $c = c(G,k) - c(H,k)$. Moreover L_n^{GKO} commute with $\hat{H}$

$$
[L_n^{GKO}, J_m^i] = 0 \qquad i = 1,\dots,dimH \tag{6}
$$

In this construction a unitary highest weight representation of $\hat{G}$ provides a highest weight representation of T^{GKO}, which is automatically unitary.

The above construction is purely algebraic. Recently there has been interest in finding a field theoretic realization of the GKO construction.[4-8] Apart from giving a new perspective on calculations within the GKO framework, a Lagrangian description would provide an intuitively simpler starting point for generalizations, for example, the inclusion of relevant operators which perturb the system away from the critical point. Here we will examine the gauged WZW model as a field theoretic candidate for realizing the GKO construction. We will mainly follow the point of view presented in ref.[8].

PATH INTEGRAL FORMULATION OF THE GAUGED WZW MODEL

We begin with a level k WZW action on the group manifold G [9,10]

$$
kI(g) = \frac{k}{8\pi}\int d^2x\, tr(\partial_\mu g \partial^\mu g^{-1}) + \frac{k}{12\pi}\int tr(dgg^{-1})^3 \tag{7}
$$

This action is invariant under the transformation

$$
g(z,\bar{z}) \rightarrow \bar{\Omega}(\bar{z}) g(z,\bar{z}) \Omega(z) \tag{8}
$$

[f1] Here we have assumed that G is simple. The Sugawara-Sommerfield construction also applies to an abelian G and can be easily generalized for a semi-simple G.

where $z = exp[ix_+], \bar{z} = exp[ix_-]$ and $x_\pm$ are the light-cone coordinates defined by $x_\pm = x_1 \pm x_2$. Ω, $\bar{\Omega}$ are group valued functions of z and $\bar{z}$ respectively. The generators of this symmetry are the currents

$$\begin{aligned} J(z) &= J^a(z)t^a = kg^{-1}\partial_z g \\ \bar{J}(\bar{z}) &= \bar{J}^a(\bar{z})t^a = -k\partial_{\bar{z}} g g^{-1} \end{aligned}$$

which satisfy two commuting copies of Kac-Moody algebra with central charge k. The spectrum of this model provides a realization of the representation space (unitary) of a $\hat{G}_L \times \hat{G}_R$ Kac-Moody algebra.[10]

We now gauge the anomaly-free vector subgroup H of the global $G_L \times G_R$ symmetry of the action. In terms of light-cone variables the gauged action is

$$kI(g,A) = kI(g) + \frac{k}{4\pi} tr \int d^2x\{A_+\partial_- g g^{-1} - A_- g^{-1}\partial_+ g + A_+ g A_- g^{-1} - A_- A_+\} \quad (9)$$

where the light-cone components of the gauge field, $A_\pm$, belong to the adjoint representation of H. The gauge fields are non-propagating, there is no corresponding kinetic energy term, and therefore they act as Lagrange multipliers which classically set the H currents to zero. Quantum mechanically the theory is defined by the generating functional

$$Z = \int [dg][dA_+][dA_-]exp[-kI(g,A)] \quad (10)$$

We would like to find out whether (10) defines a conformal field theory and if so what is the corresponding conformal charge.

In order to evaluate the functional integral over A we perform a change of variables

$$A_- = \partial_-\tilde{h}\tilde{h}^{-1}; \; A_+ = \partial_+ h h^{-1} \quad (11)$$

where h, $\tilde{h}$ are independent group elements of H.

Using the Polyakov-Wiegmann identity[11]

$$I(gh) = I(g) + I(h) - \frac{1}{4\pi}\int d^2x tr[g^{-1}\partial_+ g \partial_- h h^{-1}] \quad (12)$$

we can rewrite the action in terms of the new variables as

$$I(g,A) = I(h^{-1}g\tilde{h}) - I(h^{-1}\tilde{h}) \quad (13)$$

This expression of the action exhibits explicitly the original invariance under vector gauge transformations which are now defined as

$$\begin{aligned} g &\to \lambda g \lambda^{-1} \\ h &\to \lambda h \\ \tilde{h} &\to \lambda \tilde{h} \end{aligned} \quad (14)$$

The change of variables in (11) also introduces a Jacobian factor in the measure

$$[dA_+][dA_-] \to [dh][d\tilde{h}]detD_+ detD_- \quad (15)$$

where $detD_\pm = det\{\partial_\pm - [A_\pm, \quad]\}$ and $[dh]$, $[d\tilde{h}]$ denote Haar measures.

The Jacobian factor can be represented in terms of Grassmann variables which are the ghost and antighost fields ($c^i, \bar{c}^i, b^i_\pm$, $i = 1, ..., dimH$) taking values in the adjoint representation of H. The ghosts c and $\bar{c}$ have conformal weight zero, while $b_\pm$ have conformal weight one.

$$\begin{aligned} detD_+ detD_- &= \int [db_+][db_-][dc][d\bar{c}]exp[-tr\int d^2x(b_+D_-c + b_-D_+\bar{c})] \\ &= exp[c_H I(h^{-1}\tilde{h})]\int [db_+][db_-][dc][d\bar{c}]exp[-tr\int d^2x(b_+\partial_-c + b_-\partial_+\bar{c})] \end{aligned} \quad (16)$$

The last step was evaluated[11] by making a chiral rotation on the ghost, antighost fields in order to get rid of the gauge interaction in the action. Because of anomalies the measure is not invariant under this rotation. As a result there appears an extra factor in terms of a WZW action with a level c_H, which is the quadratic Casimir of the adjoint representation of H. A vector gauge invariant regulator has been used in (16).

So far we have kept intact the initial vector gauge invariance. In order to avoid a contribution from an infinite gauge volume in (10) we have to fix the gauge. A convenient choice is $\tilde{h} = 1$ (equivalently $A_- = 0$). Combining the previous results and the invariance of the Haar measure under group rotations the generating functional in (10) takes the following form [f2]

$$\begin{aligned} Z &= \int [dg][dh][db_+][db_-][dc][d\bar{c}] exp[-kI(g)] exp[(k + c_H)I(h)] \\ &\quad \times exp[-tr \int d^2x(b_+\partial_- c + b_-\partial_+\bar{c})] \end{aligned} \tag{17}$$

The gauging of the original WZW model of level k generated an extra WZW sector over the group manifold H with negative level $-(k+c_H)$ and a ghost sector. These sectors give rise to negative norm states and ghosts, which could in principle spoil the unitarity of the theory. We will come back to this point later.

The partition function in (17) factorizes into three decoupled sectors, each of them conformally invariant. Therefore the theory defined by (10) is conformally invariant. We can easily now compute the conformal charge. It is enough to focus on the holomorphic (or antiholomorphic) sector only.

The holomorphic stress-tensor corresponding to (17) is

$$\begin{aligned} T(z) &= \sum_{a=1}^{dimG} \frac{:J^a(z)J^a(z):}{2k + c_G} - \sum_{i=1}^{dimH} \frac{:\tilde{J}^i(z)\tilde{J}^i(z):}{2k + c_H} - \sum_{i=1}^{dimH} :b^i_z\partial_z c^i: \\ &= T^G + \tilde{T}^H + T^{gh} \end{aligned} \tag{18}$$

It is straightforward to compute the following OPE

$$\begin{aligned} T^G(z)T^G(w) &= \frac{c(G,k)}{2(z-w)^4} + \frac{2T^G(w)}{(z-w)^2} + \frac{\partial_w T^G(w)}{z-w} + r.t. \\ \tilde{T}^H(z)\tilde{T}^H(w) &= \frac{c(H, -k - c_H)}{2(z-w)^4} + \frac{2\tilde{T}^H(w)}{(z-w)^2} + \frac{\partial_w \tilde{T}^H(w)}{z-w} + r.t. \\ T^{gh}(z)T^{gh}(w) &= \frac{-2d_H}{2(z-w)^4} + \frac{2T^{gh}(w)}{(z-w)^2} + \frac{\partial_w T^{gh}(w)}{z-w} + r.t. \end{aligned} \tag{19}$$

where $r.t.$ denotes regular terms. The conformal charge of the WZW sector with level k is

$$c(G,k) = \frac{2kd_G}{2k + c_G} \tag{20}$$

Therefore the total conformal charge of the theory is

$$\begin{aligned} c^{tot} &= \frac{2kd_G}{2k + c_G} + \frac{2(-k - c_H)d_H}{2(-k - c_H) + c_H} - 2d_H \\ &= \frac{2kd_G}{2k + c_G} - \frac{2kd_H}{2k + c_H} \end{aligned} \tag{21}$$

[f2]For simplicity we have assumed that the index of embedding of the Lie algebra of H in G is one.

It is interesting to note that gauging of the original WZW action lowers its conformal charge. In a sense this is to be expected since the conformal charge is a measure of the degrees of freedom present in the corresponding theory[12] and by gauging we introduce constraints into the theory which eliminate some of the degrees of the freedom.

The conformal charge c^{tot} of the gauged WZW model coincides with the one obtained in the GKO construction. However in order to compare our model with the GKO analysis further issues have to be understood. For example we find that the energy-momentum tensor of our theory does not agree with the one given by the GKO construction. This is, of course, to be expected since the quantization of the gauged WZW model contains extra degrees of freedom (negative norm states, ghosts) than the ones appearing in the GKO construction. In fact

$$\begin{aligned} T &= T^G - T^H + T' \\ &= T^{GKO} + T' \end{aligned} \tag{22}$$

where

$$T' = \sum_{i=1}^{dimH} \{ \frac{:J^i J^i:}{2k + c_H} - \frac{:\tilde{J}^i \tilde{J}^i:}{2k + c_H} - :b^i_z \partial_z c^i: \} \tag{23}$$

The modes of T' satisfy a Virasoro algebra with zero conformal charge and moreover T^{GKO} and T' commute with each other,

$$T^{GKO}(z) T'(w) = r.t. \tag{24}$$

We have found that T decomposes into two commuting pieces, one with conformal charge c^{GKO} and the other with $c' = 0$. Similarly, the representations of T split into a direct product of representations of T^{GKO} and representations of T'. In order to make contact with the GKO construction we should eventually prove that T' representations are trivial. Moreover the physical spectrum of our theory should provide unitary representations of T^{GKO}. In the algebraic GKO construction one gets automatically unitary representations of T^{GKO} given a unitary highest weight representation of the Kac-Moody algebra $\hat{G}$. However in our case the unitarity of the physical spectrum is not obvious, because of the presence of ghost states and negative norm states.

The connection with the GKO analysis is clarified once we specify the physical Hilbert space for our theory. This is done in the BRST framework.

BRST QUANTIZATION

We found that the partition function in (17) factorized into three sectors, each of them conformally invariant. Although these sectors seem to be decoupled at the Lagrangian level, in fact there are constraint conditions which couple the states of the three sectors. A natural framework for discussing constraints and identifying the physical states is the BRST formalism. The initial vector gauge invariance of our theory is retained as a BRST invariance of the gauged fixed action in (17). The corresponding BRST charge operator Q (holomorphic part) for the gauged WZW model is [f3]

$$Q = \sum_{n=-\infty}^{\infty} :c^i_{-n}(J^i_n + \tilde{J}^i_n + \frac{1}{2} J^i_{gh,n}): \tag{25}$$

[f3] We can similarly construct a second BRST charge $\bar{Q}$ built out of the antiholomorphic fields.

where $J^i_{gh,n} = -if^{ijk}\sum_{-\infty}^{\infty} : b^j_{-m}c^k_{n+m} :$ and

$$\begin{aligned} c^i(z) &= \sum_{n=-\infty}^{\infty} c^i_n z^{-n} \qquad\qquad b^i_z(z) = \sum_{n=-\infty}^{\infty} b^i_n z^{-n-1} \\ J^i(z) &= \sum_{n=-\infty}^{\infty} J^i_n z^{-n-1} \qquad\quad \tilde{J}^i(z) = \sum_{n=-\infty}^{\infty} \tilde{J}^i_n z^{-n-1} \end{aligned} \tag{26}$$

The nilpotency of Q is more easily verified if we write Q in the following form

$$\begin{aligned} Q &= \oint \frac{dz}{2\pi i}[: c^i(z)(J^i(z) + \tilde{J}^i(z) + \frac{1}{2}J^i_{gh}(z)) :] \\ &\equiv \oint \frac{dz}{2\pi i}Q(z) \end{aligned} \tag{27}$$

where the integration is along a closed contour around the origin of the complex plane. We then find

$$\begin{aligned} Q^2 &= \frac{1}{2}\{Q,Q\} \\ &= \frac{1}{2}\oint \frac{dw}{2\pi i}\oint \frac{dz}{2\pi i}Q(z)Q(w) \\ &= 0 \end{aligned} \tag{28}$$

where the z-contour surrounds w. In this calculation it is sufficient to pick up the pole singularities of the OPE of the integrand as $z \to w$. We find that all the singularities cancel out and therefore $Q^2 = 0$.

The existence of a nilpotent BRST operator Q is closely related to the fact that

$$J^{i,tot} = J^i + \tilde{J}^i + J^i_{gh} \tag{29}$$

is a first class constraint operator, namely it satisfies a Kac-Moody algebra with zero central charge.

Given the BRST charge operator Q we can define the physical space as the cohomology of Q. The physical subspace is the collection of states satisfying

$$Q|phys>= 0 \tag{30}$$

where the BRST trivial states $(Q|\chi>)$ have been moded away.

Let us first specify the extended Fock space of the theory and then suitably restrict it to the physical subspace. Since the partition function factorizes into three decoupled sectors the Fock space factorizes similarly as

$$\mathcal{F} = \mathcal{F}_k \times \mathcal{F}_{-(k+c_H)} \times \mathcal{F}_{gh} \tag{31}$$

where $\mathcal{F}_k$ denotes the Fock space of the WZW sector with level k, $\mathcal{F}_{-(k+c_H)}$ denotes the Fock space of the WZW sector with negative level $-(k+c_H)$ and $\mathcal{F}_{gh}$ denotes the ghost Fock space.

The sector $\mathcal{F}_k$ has already been investigated in detail.[10] Its spectrum provides a realization of the highest weight, unitary representations of the Kac-Moody algebra $\hat{G}$ described at the beginning.

The sector $\mathcal{F}_{-(k+c_H)}$ is analogously defined. One can build the full sector by applying the creation modes $\tilde{J}^i_{-n}$ on the primary states $|\tilde{\phi}>$ which satisfy

$$\begin{aligned} \tilde{J}^i_n|\tilde{\phi}> &= 0 \qquad n > 0 \qquad i = 1, ..., dimH \\ \tilde{J}^i_0|\tilde{\phi}> &= t^i_{\tilde{R}}|\tilde{\phi}> \end{aligned} \tag{32}$$

Thus $|\tilde{\phi}>$ again provides a representation of the finite Lie algebra of the zero modes $\tilde{J}^i_0$. There is an important difference however with the previous sector. Since the level is negative the states generated from $|\tilde{\phi}>$ by the application of the creation operators $\tilde{J}^i_{-n}$ provide a non-unitary representation of the Kac-Moody algebra, due to the existence of negative norm states. The representations $\tilde{R}$ of the primary states are no longer restricted by unitarity and there is an infinite number of primary fieds. As a result the energy spectrum of this sector is not bounded from below, since $c_{\tilde{R}}$ can be arbitrarily large,

$$\tilde{L}_0|\tilde{\phi}>= -\sum_{i=1}^{dimH}\frac{\tilde{J}^i_0\tilde{J}^i_0}{2k+c_H}|\tilde{\phi}>= -\frac{c_{\tilde{R}}}{2k+c_H}|\tilde{\phi}> \tag{33}$$

As we shall see, this problem is resolved when we consider the spectrum of the physical subspace.

The sector $\mathcal{F}_{gh}$ is built up from the ghost vacuum by acting with the creation modes b^i_{-n}, c^i_{-n}. However the vacuum is degenerate and the ghost Fock space splits into disjoint sectors. The degeneracy appears in the following way. The ghost modes satisfy

$$\{c^i_n, c^j_m\} = \{b^i_n, b^j_m\} = 0 \qquad \{b^i_n, c^j_m\} = \delta_{nm}\delta^{ij} \tag{34}$$

The zero modes satisfy a Clifford algebra. They also commute with the ghost energy-momentum tensor which in terms of modes is given by

$$L_{gh,n} = \sum_{i=1}^{dimH}\sum_{m=-\infty}^{\infty}(n-m) : b^i_m c^i_{n-m} : \tag{35}$$

This implies that the ground state forms an irreducible representation of a Clifford algebra with a 2^{dimH} degeneracy. Each of these states is characterized uniquely by the ghost number, which is the eigenvalue of the operator

$$N = \sum_{i=1}^{dimH}[\frac{1}{2}(c^i_0 b^i_0 - b^i_0 c^i_0) + \sum_{n=1}^{\infty}(c^i_{-n}b^i_n - b^i_{-n}c^i_n)] \tag{36}$$

Among the 2^{dimH} degenerate states, there is a unique one which is $SL(2,C)$ invariant and this must be taken as the physical ghost vacuum $|0>_{gh}$. It satisfies

$$\begin{aligned} c^i_n|0>_{gh} &= 0 \qquad n \geq 1 \\ b^i_n|0>_{gh} &= 0 \qquad n \geq 0 \end{aligned} \tag{37}$$

and is characterized by ghost number $-dimH/2$. The ghost Fock space built on $|0>_{gh}$ is annihilated by the zero antighost mode b^i_0. As a result every BRST invariant state $|s>$ in $\mathcal{F}$ is a singlet under global H-rotations, since

$$\{Q, b^i_0\}|s>= (J^i_0 + \tilde{J}^i_0 + J^i_{gh,0})|s>= 0 \tag{38}$$

The full Fock space $\mathcal{F}$ can be built up from a "ground" state $|v>$ which is a product of primary states of the three sectors

$$|v>= |\phi>_G \otimes|\tilde{\phi}>_H \otimes|0>_{gh} \tag{39}$$

by acting with the creation operators J^a_{-n}, $\tilde{J}^i_{-n}$, b^i_{-n}, c^i_{-n}. Let us now impose the requirement that $|v>$ be a physical state, so that

$$Q|v>= 0 \;\Rightarrow\; (J^i_0 + \tilde{J}^i_0)[|\phi>_G \otimes|\tilde{\phi}>_H] = 0 \tag{40}$$

since $J^i_{0,gh}|0>_{gh}=0$. Equation (40) restricts $|\phi>_G \otimes|\tilde{\phi}>_H$ to be a singlet under global H rotations. This condition constrains the possible representations $\tilde{R}$ as follows. The primary state $|\phi>_G$ can be decomposed into H-representations R' as

$$|\phi>_G=\sum\oplus R' \tag{41}$$

In order that (40) be satisfied the representation $\tilde{R}$ must be related to R' such that

$$R'\otimes\tilde{R}\supset I \tag{42}$$

where I denotes the singlet representation. This implies that the allowed $\tilde{R}$ representations are the conjugate representations of those that already appear in the decomposition (41). Since only a finite set of representations R' appear in (41), the allowed representations $\tilde{R}$ of the primary states of $\mathcal{F}_{-(k+c_H)}$ are restricted to a finite set. As a consequence of (42) the spectrum of the theory is bounded from below since

$$\begin{aligned} L_0'|v> &= 0 \\ L_0^{tot}|v> &= L_0^{GKO}|v> \end{aligned} \tag{43}$$

Beginning with a physical ground state $|v>$, one can build other physical states by applying physical creation operators Φ to $|v>$, where physical operators are those that commute with Q, i.e.

$$[Q,\Phi]=0 \tag{44}$$

and are not BRST exact operators. In our case Φ is a polynomial expression of $J^a_{-n},\tilde{J}^i_{-n},b^i_{-n}$ and c^i_{-n} for $n>0$.

In particular the energy-momentum tensor T^{tot}, as well as T^{GKO} and T' commute with Q.

$$[Q,L_n^{GKO}] = [Q,L_n'] = 0 \qquad \text{for any } n \tag{45}$$

Moreover T' is a BRST exact operator, namely it can be written as

$$T'(z)=\{Q,X(z)\} \tag{46}$$

where $X(z)$ is given by

$$X(z)=\frac{1}{2k+c_H}\sum_{i=1}^{dimH}:b^i(z)[J^i(z)-\tilde{J}^i(z)]: \tag{47}$$

As a result the creation modes of T' generate BRST trivial (null) states and they decouple from the physical subspace. Therefore we proved that the representations of T' are trivial and on the physical subspace $KerQ/ImQ$

$$T|phys>=T^{GKO}|phys> \tag{48}$$

Going back to the unitarity issue we mentioned before we would like to show that the full physical spectrum of the gauged WZW model provides unitary representations of the GKO Virasoro algebra (decoupling of negative norm states and ghosts). This is addressed in the next section. We will first restrict to the case where H is abelian and then we will comment on the non-abelian case.

Let us now focus on the modes of the H-operators $J^i, \tilde{J}^i, b^i$ and c^i. Their BRST transformations are given by

$$
\begin{aligned}
\{Q, b^i_{-n}\} &= J^{i,tot}_{-n} \\
\{Q, c^i_{-n}\} &= -\frac{i}{2} f^{ijk} \sum_{l=-\infty}^{\infty} c^j_l c^k_{-n-l} \\
[Q, J^i_{-n}] &= -i f^{ijk} \sum_{l=-\infty}^{\infty} c^j_{-l} J^k_{l-n} + nkc^i_{-n} \\
[Q, \tilde{J}^i_{-n}] &= -i f^{ijk} \sum_{l=-\infty}^{\infty} c^j_{-l} \tilde{J}^k_{l-n} - n(k + c_H) c^i_{-n}
\end{aligned} \tag{49}
$$

The BRST transformation of a single mode generates an expression which is quadratic in modes, due to the non-abelian structure. Consider first the case where H is abelian. The special case where the H algebra is the Cartan subalgebra of G describes an important class of coset models, the parafermionic models.[3] The operators $J^{i,tot}_{-n}, J^i_{-n}, b^i_{-n}$ and c^i_{-n} , where $i = 1, ..., r$ with r the rank of the group G, arrange themselves in a "quartet" of unphysical modes, as described by Kugo and Ojima.[13] It is straightforward then, following their method, to show that the states generated by the above modes decouple from the physical spectrum. Defining

$$
\begin{aligned}
\chi^i_n &= \frac{J^i_n}{\sqrt{|n|k}} \qquad \beta^i_n = \frac{J^{i,tot}_n}{\sqrt{|n|k}} \qquad\qquad n \geq 1 \\
\gamma^i_n &= i\sqrt{|n|k}\, c^i_n \qquad \bar{\gamma}^i_n = \frac{b^i_n}{\sqrt{|n|k}}
\end{aligned} \tag{50}
$$

we obtain the following commutation relations

$$
\begin{aligned}
[Q, \chi^i_n] &= i\gamma^i_n \qquad \{Q, \gamma^i_n\} = 0 \\
\{Q, \bar{\gamma}^i_n\} &= \beta^i_n \qquad [Q, \beta^i_n] = 0
\end{aligned} \tag{51}
$$

and for the matrix of (anti)commutators,

$$
\eta^{ij}_{nm} \equiv [a^i_n, (a^j_m)^\dagger]_\pm = \begin{array}{c c} & \begin{array}{cccc} \chi^j_{-m} & \beta^j_{-m} & -\gamma^j_{-m} & \bar{\gamma}^j_{-m} \end{array} \\ \begin{array}{c} \chi^i_n \\ \beta^i_n \\ \gamma^i_n \\ \bar{\gamma}^i_n \end{array} & \begin{pmatrix} \delta^{ij}\delta^{nm} & \delta^{ij}\delta^{nm} & 0 & 0 \\ \delta^{ij}\delta^{nm} & 0 & 0 & 0 \\ 0 & 0 & 0 & i\delta^{ij}\delta nm \\ 0 & 0 & -i\delta^{ij}\delta^{nm} & 0 \end{pmatrix} \end{array}
$$

where a^i_n is any of the $\chi^i_n, \beta^i_n, \gamma^i_n$ or $\bar{\gamma}^i_n$.

With the help of the inverse of the matrix η we can express the projection operator $P^{(N)}$ onto the N-unphysical mode sector as

$$
\begin{aligned}
P^{(N)} = \frac{1}{N} \sum_{i=1}^{r} \sum_{n \geq 1}^{\infty} \; & (\beta^i_{-n} P^{(N-1)} \chi^i_n + \chi^i_{-n} P^{(N-1)} \beta^i_n - \beta^i_{-n} P^{(N-1)} \beta^i_n \\
& - i\gamma^i_{-n} P^{(N-1)} \bar{\gamma}^i_n - i\bar{\gamma}^i_{-n} P^{(N-1)} \gamma^i_n)
\end{aligned} \tag{52}
$$

where $P^{(0)}$ is the physical projection operator not including any H-modes. $P^{(0)}$ is constructed out of modes of physical operators Ψ [satisfying eq.(44)] acting on the H-singlet primary states. At this point we do not need an explicit expression for $P^{(0)}$. The only relevant thing is that Ψ commutes with the quartet modes. Later we will see that Ψ's are related to the parafermionic operators.

The projection operator $P^{(N)}$ can be rewritten as

$$\begin{aligned} P^{(N)} &= \{Q, R^{(N)}\} \qquad for \;\; N \geq 1 \\ R^{(N)} &= \frac{1}{N}\sum_{i=1}^{r}\sum_{n\geq 1}^{\infty}(\bar{\gamma}^i_{-n}P^{(N-1)}\chi^i_n + \chi^i_{-n}P^{(N-1)}\bar{\gamma}^i_n - \beta^i_{-n}P^{(N-1)}\bar{\gamma}^i_n) \end{aligned} \tag{53}$$

From this it follows that any state belonging to the sector with N unphysical modes ($N \geq 1$) is a BRST trivial state. It has zero norm and it is orthogonal to any BRST invariant state.

Using orthogonality and completeness properties of the projection operators we find that any state annihilated by Q can be written as

$$\begin{aligned} |\psi> &= \sum_{N=0}^{\infty} P^{(N)}|\psi>= P^{(0)}|\psi> + \sum_{N=1}^{\infty}\{Q, R^{(N)}\}|\psi> \\ &= P^{(0)}|\psi> + Q|\zeta> \end{aligned} \tag{54}$$

where $|\zeta>= \sum_{N=1}^{\infty} R^{(N)}|\psi>$. Therefore we proved that the H-degrees of freedom decouple, among them the negative norm states and ghosts, and the physical subspace ($KerQ/ImQ$) remains unitary. The original gauged WZW model, where an abelian subgroup H has been gauged, is equivalent to the G/H conformal theory constructed by GKO. As we mentioned before, one example is a theory describing parafermions. [3]

Unfortunately it is difficult to generalize the above proof for H non-abelian. The main difficulty is that the BRST operator Q now contains cubic terms, and the BRST transformation of a single mode generates an expression quadratic in modes, which complicates the decoupling mechanism. As a result we get an operator expression for the matrix η. The best we can do is to try to invert it in a perturbative fashion. For example, we can introduce a gauge coupling g and try to invert η in powers of g. As a result the projection operator $P^{(N)}$ involves an infinite expansion in terms of the coupling constant, with the leading term in the expansion coinciding with the abelian case. Since we lack a closed form for the projection operator $P^{(N)}$, we cannot easily reproduce an expression like eq.(53), which would guarantee the decoupling of the unphysical modes.

We can instead give an alternate proof of decoupling based purely on the fact that T' is a BRST exact operator, eq.(46), without deriving explicit expressions for $P^{(N)}$. In particular

$$L_0' = \{Q, X\} \tag{55}$$

where $X \equiv \oint \frac{dz}{2\pi i} zX(z)$. However L_0' is precisely the operator which counts the grade of the unphysical states since

$$\begin{aligned} [L_0', J^i_{-n}] &= nJ^i_{-n} \\ [L_0', \tilde{J}^i_{-n}] &= n\tilde{J}^i_{-n} \\ [L_0', c^i_{-n}] &= nc^i_{-n} \\ [L_0', b^i_{-n}] &= nb^i_{-n} \end{aligned} \tag{56}$$

Let us now assume that a BRST invariant state $|\psi>$ has a nonzero grade m

$$L_0'|\psi>= m|\psi> \tag{57}$$

From this it follows that $|\psi>$ is a BRST trivial state since

$$|\psi>= \frac{1}{m}QX|\psi> \tag{58}$$

All the H-secondary states generated from the BRST invariant ground state $|v>$ by the action of the unphysical modes J^i_{-n}, $\tilde{J}^i_{-n}$, c^i_{-n}, b^i_{-n} are of this form [see eqs.(40), (43)] and they therefore decouple from the physical spectrum. The operator X plays the role of a homotopy operator.[14] We see from eqs.(57),(58) that all cohomologically nontrivial states are eigenstates of L_0' with eigenvalue zero. The BRST ground state $|v>$ and the states obtained by the action of the physical operators Ψ on $|v>$ obviously satisfy this condition. These are exactly the states appearing in the GKO construction. However, in the non-abelian case, one may also have isolated secondary H-states which satisfy

$$L_0'|\chi>=0 \tag{59}$$

and might be BRST invariant. This possibility exists since we can start from H-primaries $|v>$ which are not BRST invariant and satisfy

$$L_0'|v>=(\frac{c_{R'}}{2k+c_H}-\frac{c_{\tilde{R}}}{2k+c_H})|v>=\lambda|v> \tag{60}$$

with λ a negative integer and build H-secondary states which satisfy (59). The decoupling of these states, which does not automatically follow from the above analysis, needs further investigation. This complication does not arise in the abelian case where one can easily show that the only BRST invariant states satisfying (59) are the states built from the BRST invariant $|v>$ by the action of the physical operators Ψ.

An interesting problem is to identify the true physical creation operators and possibly construct their corresponding chiral algebra. After the decoupling of the H-modes we are left only with the currents J^α, where $\alpha = dimH+1,...,dimG$. However these are not physical operators since they transform under H-rotations and therefore are not Q-invariant. Instead we can try to isolate from J^α its BRST invariant projection. This can be done as follows. Let us choose a parametrization of the finite Lie algebra G such that the generators t^i and t^α are orthogonal, namely

$$tr(t^i t^\alpha)=0 \tag{61}$$

where $i=1,...,dimH$ and $\alpha = dimH+1,...,dimG$. Now we can factorize J^α as

$$\begin{aligned} J^\alpha(z) &= :tr(t^\alpha h'^{-1}(z)t^\beta h'(z)):\Psi^\beta(z) \\ &\equiv D^{\alpha\beta}(z)\Psi^\beta(z) \end{aligned} \tag{62}$$

where

$$J^i(z)h'(w)=\frac{h'(w)}{z-w}t^i+r.t. \tag{63}$$

The field $h'(w)$ transforms as the holomorphic part of a primary field of the $\hat{H}$ Kac-Moody algebra. We can easily check that

$$J^i(z)D^{\alpha\beta}(w)=if^{i\alpha\gamma}\frac{D^{\gamma\beta}(w)}{z-w}+r.t \tag{64}$$

Comparing eq.(64) with

$$J^i(z)J^\alpha(w)=if^{i\alpha\gamma}\frac{J^\gamma(w)}{z-w}+r.t. \tag{65}$$

we conclude that $\Psi^\beta(z)$ has a regular OPE with $J^i(z)$ and therefore it commutes with Q. Thus $\Psi(z)$ are physical operators which generate physical states by acting on the H-singlet ground state $|v>$. The operators $\Psi(z)$ provide a generalization of the parafermionic currents for the case of H non-abelian. When H is the

Cartan subalgebra of G, expressions (62) and (63) reduce trivially to the well known expressions for parafermionic currents and they provide the usual OPE rules for parafermions.[3] We can explicitly demonstrate this by choosing $h'(z) = exp[i\frac{\omega^i(z)t^i}{\sqrt{k}}]$, where ω^i are free fields satisfying

$$< \omega^i(z)\omega^j(w) >= -\delta^{ij}\ln(z-w) \tag{66}$$

and $i = 1, ..., r$ with r the rank of the group G. Then equations (62) and (63) reduce to the usual expressions for parafermionic currents (in the Cartan-Weyl basis)[3]

$$\begin{aligned} J^i(z) &= i\sqrt{k}\partial_z\omega^i(z) \\ J^{\vec{\alpha}}(z) &= :exp[i\frac{\vec{\omega}(z)\cdot\vec{\alpha}}{\sqrt{k}}]:\Psi^{\vec{\alpha}}(z) \end{aligned} \tag{67}$$

where $\vec{\alpha}$'s correspond to the roots of G.

So far we have focused on the holomorphic sector only. We can extend all the above results independently to the antiholomorphic sector. Eventually we must combine these two sectors. There are constraints in the theory which tell us how to construct the full physical Hilbert space by linking the two sectors together. The simplest way to obtain these constraints is to consider the theory on a cylinder (space variable x_2 is compactified). In this case the parametrization (11) for $A_\pm$ has to be modified by the inclusion of a new parameter A_0 corresponding to the holonomy of A around the nontrivial cycle. The equations of motion corresponding to A_0 introduce new constraints relating the zero modes of the holomorphic and antiholomorphic H-currents[5]

$$J_0^i = \bar{J}_0^i \tag{68}$$

Once this constraint is taken care of, the physical quantities of the theory such as the partition function and correlators will be monodromy invariant. On higher genus surfaces this constraint is crucial for modular invariance.[6]

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REFERENCES

1. P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88; Comm. Math. Phys. 103 (1986) 105; P. Goddard and D. Olive, Int.J.Mod.Phys.A1 (1986) 303.

2. D. Friedan, Z. Qiu, and S. Shenker,Phys.Rev.Lett.52 (1984) 1575; Phys.Lett. B151 (1985) 37; V.A. Fateev and A.B. Zamolodchikov, Nucl.Phys. B280 (1987) 644; A.B. Zamolodchikov, Theor.Math.Phys. 65 (1986) 1205; J. Bagger, D. Nemeschansky, and S. Yankielowicz, Phys.Rev.Lett.60 (1988) 389; F. Ravanini, Mod.Phys.Lett.A3 (1988) 397; M.R. Douglas, CALT-68-1453 (1987); D. Kastor, E. Martinec, and Z. Qiu, Phys. Lett. B200 (1988) 434; P. Goddard and A. Schwimmer, Phys.Lett. B206 (1988) 62; P. Bowcock and P. Goddard, Nucl. Phys. B305 [FS23] (1988) 685; M.B. Halpern, LBL-26297 / UCB-PTH-88/30 preprint and references therein.

3. V.A. Fateev and A.B. Zamolodchikov, Sov.Phys. JETP 62 (1985) 215; ibid. 63 (1986) 913; D. Gepner and Z. Qiu, Nucl.Phys. B285 [FS19] (1987) 423; D. Gepner, Nucl.Phys. B290 (1987) 10; E.B. Kiritsis, Mod. Phys. Lett. A4 (1989) 437; G. Dunne, I. Halliday and P. Suranyi MIT-CTP 1663 (1988).

4. A. Kent, Phys.Lett.173B (1986) 413; E. Guadagnini, M. Martellini and M. Mintchev, Phys. Lett. B194 (1987) 69; K. Bardakçi, E. Rabinovici, and B. Säring, Nucl.Phys.B299 (1988) 151.

5. W. Nahm, UCD-88-02 preprint; P. Bowcock, Nucl.Phys.B316 (1989) 80.

6. K. Gawedzki and A. Kupiainen, Phys.Lett.B215 (1988) 119; Nucl. Phys. B320 (1989) 625.

7. D. Karabali, Q-Han Park, H.J. Schnitzer and Z. Yang, Phys.Lett.B216 (1989) 307; H.J. Schnitzer, Brandeis preprint BRX TH-254 (1988).

8. D. Karabali and H.J. Schnitzer, Brandeis preprint BRX TH-267 (1989), to appear in Nucl.Phys.B.

9. E. Witten, Comm.Math.Phys.92 (1984) 455.

10. V. Knizhnik and A.B. Zamolodchikov, Nucl.Phys.B247 (1984) 83; D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493.

11. A.M.Polyakov and P.B.Wiegmann,Phys.Lett. 141B (1984) 223.

12. W. Nahm, in *Superstrings, supergravity and unified theories*, eds. G. Furlan *et al.* (World Scientific, 1986); M.K. Fung, Mod. Phys. Lett. A4 (1989) 675; A.M. Polyakov, Les Houches Lectures 1989.

13. T. Kugo and I. Ojima, Suppl.Prog.Theor.Phys. 66 (1979) 1; M. Kato and K. Ogawa, Nucl.Phys. B212 (1983) 443.

14. H. Aratyn, these proceedings.

FLAT CONNECTION, CONFORMAL FIELD THEORY AND QUANTUM GROUP

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INTRODUCTION

Recently many people[1] are discussing general framework of rational conformal field theories (RCFT). There, one of the important concept is a connection matrix of conformal blocks. Once connection matrices are given for the conformal blocks, which are made from chiral vertex operators of a given chiral algebra by sandwiching them with SL_2 invariant vacuum, then we can construct the physical correlation functions invariant under monodromy transformations and determine operator product expansion coefficients in principle.

The connection matrices satisfy some polynomial equations[2] such as pentagon identity for example. Some people are trying to classify RCFT by solving the polynomial equations for connection matrices. The connection matrices, however, have only global information of conformal blocks. Moreover it is unclear how one can study dynamical or physical properties of the conformal blocks if playing only with the connection matrices.

On the other hand, we know that the conformal block satisfies a certain differential equation which is the result of null state structure of the representation.[3,4] The differential equation contains both global and local information of the conformal block.

In the present note we will study the differential equations for conformal blocks in general framework. Especially we would like to point out the importance of *flat connection*. It may give a new and more tractable tool for classification of RCFT.

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Indeed for a special case we can completely classify the flat connections and hence CFT. Another advantage of using flat connection in writing down a differential equation is that a monodromy property of conformal block can be easily determined. In some case we can see quantum group structure for connection matrices without giving their explicit forms.

DIFFERENTIAL EQUATIONS FOR CONFORMAL BLOCKS

It is known that four-point function is the first nontrivial multi-point function in CFT. Let us assume a four-point function ψ satisfy the following first order linear differential equation:

$$\left(\frac{\partial}{\partial z_i} - \omega_i\right) \psi(z_1, z_2, z_3, z_4) = 0 \qquad i = 1,2,3,4 \tag{1}$$

where ψ is a n-column vector of functions and ω_i's are $n \times n$ matrices of functions. In general, the number of fundamental solutions of (1), which are conformal blocks, is not greater than n.

Here one might wonder if differential equation for conformal block is always written in a first order form like (1). In the minimal model case, for instance, a correlation function (let us denote it by f) which contains primary field $\phi_{h_{(r,s)}}(z)$ is known to satisfy a differential equation of order rs with respect to z. Even in this case we can define a rs-column vector ψ consists of $f, \frac{d}{dz}f, \cdots$, and $\left(\frac{d}{dz}\right)^{rs-1} f$ so that ψ satisfies first order equation of the form (1). With similar tricks we can see that for every known differential equation of four-point function we can rewrite it in first order form (1). Hence (1) is most general form.

Now let us consider requirements for ω_i. Among them two important things are (A) SL_2 invariance and (B) integrability.

(A) *SL_2 invariance*: Since we are considering CFT on a sphere, SL_2 invariance is one of necessary conditions. If we denote conformal dimension of each vertex operator by Δ_i ($i = 1,2,3,4$), then SL_2 invariance of ψ requires the following relations:

$$\sum_{i=1}^{4} \frac{\partial}{\partial z_i} \psi = 0 \tag{2a}$$

$$\sum_{i=1}^{4} (z_i \frac{\partial}{\partial z_i} + \Delta_i) \psi = 0 \tag{2b}$$

$$\sum_{i=1}^{4} ({z_i}^2 \frac{\partial}{\partial z_i} + 2 z_i \Delta_i) \psi = 0 \tag{2c}$$

Using eq. (1), we can derive the conditions for ω_i:

$$\sum_{i=1}^{4} \omega_i \psi = 0 \tag{3a}$$

$$\sum_{i=1}^{4}(\Delta_i + z_i\omega_i)\psi = 0 \tag{3b}$$

$$\sum_{i=1}^{4}(2z_i\Delta_i + z_i{}^2\omega_i)\psi = 0 \tag{3c}$$

(B) *Integrability*: Eq. (1) consists of four equations. So they have to be consistent with each other. This is achieved if differential operators $\frac{\partial}{\partial z_i} - \omega_i$ $(i = 1,2,3,4)$ commute with each other. Thus we have another set of conditions:

$$\frac{\partial \omega_j}{\partial z_i} - \frac{\partial \omega_i}{\partial z_j} + [\,\omega_i, \omega_j\,] = 0 \tag{4}$$

If you regard ω_i as a connection, (4) tells ω_i has zero curvature. It is very natural that a flat connection governs conformal block since analytic continuation of conformal block function does not depend on small (infinitesimal) deformation of continuation path. We will see later how the flat connection actually determine the connection matrix.

Besides the above conditions (A) and (B), we need several physical requirements to obtain differential equations corresponding to some RCFT. Unitarity, boundary condition at infinity and gauge choice are some of examples. Classification of RCFT based on this direction is now under investigation. Here we describe a remarkable trial of classification using a certain ansatz.

Let us assume ω_i has following form:

$$\omega_i = \sum_{j \neq i} r_{ij}(z_{ij}) \tag{5}$$

where $z_{ij} = z_i - z_j$ and $r_{ij}(z_{ij})$ is a matrix of functions only depending on z_{ij} satisfying $r_{ij}(z_{ij}) = -r_{ji}(z_{ji})$. Then the integrability condition (4) becomes

$$[\,r_{ij}(z_{ij}), r_{ik}(z_{ik})\,] + [\,r_{ij}(z_{ij}), r_{jk}(z_{jk})\,] + [\,r_{ik}(z_{ik}), r_{jk}(z_{jk})\,] = 0 \tag{6}$$

This equation is nothing but *classical Yang-Baxter equation*[5] *(CYBE)!* Fortunately there exists a classification of its solutions by Belavin and Drinfel'd[6]; a) rational, b) trigonometric and c) elliptic solutions. It can be easily seen that among them only rational solution satisfies SL_2 invariance. The rational solution has a form

$$r_{ij}(z_{ij}) = \frac{\lambda}{2}\frac{1}{z_{ij}}\sum_a \rho_i(T^a) \otimes \rho_j(T^a) \tag{7}$$

where T^a's are the orthonormal basis of arbitrary simple Lie algebra and ρ_i is a certain representation. Normalization constant λ can be fixed by another requirement such as unitarity. Substituting this expression into eq. (1), we obtain Knizhnik-Zamolodchikov[4](KZ) equation.* This means the WZW model is a unique CFT

* Conversely, the fact that the integrability condition of KZ equation coincides with CYBE is first noted by Kohno[7].

under the ansatz (5). Although this ansatz does not include other interesting classes of CFT such as minimal model, the above discussion sufficiently exhibit powerfulness of this framework toward the classification of CFT.

MONODROMY PROPERTIES OF CONFORMAL BLOCKS

As mentioned before, an advantage of writing the differential equation in the form of eq. (1) is that it enables us to write down a formal result of analytic continuation of solution. Let γ be a path from a point $\{z_i\}$ to another point $\{z_i'\}$ in $C^4\backslash\Delta$ (we denote Δ as a subspace of C^4 in which any two coordinates z_i and z_j coincide). Then the analytic continuation $\psi_\gamma(z_i')$ of $\psi(z_i)$ along γ is expressed by a path-ordered integral:

$$\psi_\gamma(z_i') = P_\gamma \exp\Big(\int_{\{z_i\}}^{\{z_i'\}} \sum_j \omega_j dz_j\Big)\psi(z_i) \tag{8}$$

From this expression we can extract the property of connection matrix. Before doing so, let us consider gauge degrees of freedom of the differential equation.

Eq. (1) has a gauge invariance (we should rather say covariance) under the transformation

$$\begin{aligned} \psi &\to \psi' = \mathcal{U}\psi \\ \omega &\to \omega' = \mathcal{U}\omega\mathcal{U}^{-1} + (d\mathcal{U})\mathcal{U}^{-1} \end{aligned} \tag{9}$$

where $\mathcal{U}(z_1, z_2, z_3, z_4)$ is a gauge transformation matrix and $\omega = \sum_i \omega_i dz_i$. What is more important is a subclass of gauge transformation which leaves ω invariant. The condition that $\mathcal{U}$ leaves ω invariant is

$$d\mathcal{U} + [\mathcal{U}, \omega] = 0 \tag{10}$$

Then $\mathcal{U}\psi$ satisfies the same differential equation as ψ does, *i.e.* eq. (1), and can be expressed by a linear combination of fundamental solutions of (1). Defining a matrix $\Psi = (\psi^{(1)}, \psi^{(2)}, \cdots, \psi^{(n)})$ which consists of fundamental solutions $\psi^{(i)}$, we can summarize the action of $\mathcal{U}$ in the following form:

$$\mathcal{U}\Psi = \Psi U \tag{11}$$

where U is a constant matrix.‡ We call $\mathcal{U}$ that satisfies (10) *monodromy operator*§ and U *connection (monodromy) matrix.*

Now let us consider the braiding operation for which we take a path γ such that $z_i' = z_{p(i)}$ where $p(i)$ is a permutation. The analytic continuation operator along γ

$$T_\gamma = P_\gamma \exp\Big(\int_{\{z_i\}}^{\{z_{p(i)}\}} \omega\ \Big) \tag{12}$$

‡ We can easily see $d(\Psi^{-1}\mathcal{U}\Psi) = 0$ by using the relations (10), $d\Psi = \omega\Psi$, and $d\Psi^{-1} = -\Psi^{-1}\omega$.

§ Here the term *monodromy* is used somewhat extensively.

becomes a matrix of functions of z_i. Generally T_γ does not satisfy (10) unless $p(i) = i$. Instead,

$$dT_\gamma = \omega(z_{p(i)})T_\gamma - T_\gamma\omega(z_i) \tag{13}$$

is held. If we find a constant matrix $\mathcal{A}$ which satisfies $\mathcal{A}\omega(z_{p(i)})\mathcal{A}^{-1} = \omega(z_i)$, then $\mathcal{A}T_\gamma$ becomes a monodromy operator.

We take the WZW model for the illustration of above construction. In this case ω_i is given by eqs. (5) and (7), and ψ is a function of four vectors v_i of Lie algebra on which the representation $\rho_i(T^a)$ acts and four complex coordinates z_i. Taking an appropriate basis ψ becomes a column vector. Now a braiding operator, say $\mathcal{B}_{23}$, is constructed as follows: First we obtain an analytic continuation operator T_γ with an appropriate braiding path γ for z_2 and z_3

$$\psi_\gamma(v_1 \otimes v_2 \otimes v_3 \otimes v_4; z_1, z_3, z_2, z_4) = T_\gamma\psi(v_1 \otimes v_2 \otimes v_3 \otimes v_4; z_1, z_2, z_3, z_4) \tag{14}$$

where

$$T_\gamma = P_\gamma \exp\left(\int_{\{z_1,z_2,z_3,z_4\}}^{\{z_1,z_3,z_2,z_4\}} \omega\ \right) \tag{15}$$

Second we find a constant matrix which turns $\omega(z_1, z_3, z_2, z_4)$ back to $\omega(z_1, z_2, z_3, z_4)$; the answer is permutation operator $\mathcal{P}_{23}$ of $v_2 \otimes v_3$. Furthermore in this case we can explicitly perform the path-ordered integral and so the final result becomes

$$\mathcal{B}_{23} = \mathcal{P}_{23}e^{i\pi\lambda\Omega_{23}} \tag{16}$$

where $\Omega_{ij} = \frac{1}{2}\sum_a \rho_i(T^a) \otimes \rho_j(T^a)$.

We define a connection matrix $B_{i\,i+1}$ corresponding to braid operation $\mathcal{B}_{i\,i+1}$ according to the general story:

$$\mathcal{B}_{i\,i+1}\Psi = \Psi B_{i\,i+1} \tag{17}$$

Although the determination of B itself is generally a hard task, much of its important properties can be obtained from the properties of $\mathcal{B}$ *via* eq. (17). Indeed B and $\mathcal{B}$ have common properties such as eigenvalues and braid relation. The eigenvalues of $\mathcal{B}_{ij}$ are easily determined as $\epsilon^k_{ij}\, q^{c_k - c_i - c_j}$, where c_i is Casimir invariant evaluated on the representation ρ_i and $q = e^{i\pi\lambda}$. Index k runs over the representations which appear in the decomposition $\rho_i \otimes \rho_j = \oplus_k \rho_k$, and ϵ^k_{ij} is either $+1$ or -1 depending on whether ρ_k is constructed symmetrically or anti-symmetrically from ρ_i and ρ_j.

In this line of argument, Kohno[7] showed that the braiding matrices satisfy Hecke algebra if we choose SL_n for Lie algebra and n-dimentional fundamental representation for every ρ_i. Furthermore we can show[8,9] for the case of SU_2 that any connection matrices can be expressed by the $6j$-symbols[10] of *quantum* SU_2 and the normalization factors for the conformal blocks. They completely agree with the explicit evaluation of the connection matrices by Tsuchiya and Kanie.[11]

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REFERENCES

1. See, for example, references listed in G. Moore and N. Seiberg, Classical and quantum conformal field theory, *Commun. Math. Phys.* 123:177 (1989).
2. G. Moore and N. Seiberg, Polynomial equations for rational conformal field theories, *Phys. Lett.* B212:451 (1988).
3. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetry in two-dimensional quantum field theory, *Nucl. Phys.* B241:333 (1984).
4. V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess-Zumino model in two dimensions, *Nucl. Phys.* B247:83 (1984)
5. See, for example, L. Faddeev, Integrable models in (1+1)-dimentional quantum field theory, *in:*"Recent Advances in Field Theory and Statistical Mechanics" (Les Houches 1982), J.-B. Zuber and R. Stora, eds., Elsevier Science Pub., (1984).
6. A. A. Belavin and V. G. Drinfel'd, Solutions of the classical Yang-Baxter equation for simple Lie algebras, *Funkts. Anal. i Ego Prilozhen.* 16:1 (1982).
7. T. Kohno, Hecke algebra representations of braid groups and classical Yang-Baxter equations, *Adv. Stud. in Pure Math.* 16:255 (1988).
8. T. Kohno, Quantized universal enveloping algebras and monodromy of braid groups, Nagoya Univ. preprint (1989).
9. M. Kato and P. Valtancoli, unpublished.
10. See, for example, A. N. Kirillov and N. Yu. Reshetikhin, Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, LOMI preprint E-9-88 (1988).
11. A. Tsuchiya and Y. Kanie, Vertex operators in conformal field theory on P^1 and monodromy representations of braid group, *Adv. Stud. in Pure Math.* 16:297 (1988).

CLASSICAL AND QUANTUM CALABI–YAU MANIFOLDS

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1. Introduction

It has been clear for some time now that not all compactification schemes used to construct consistent Heterotic String vacua are independent. This is particularly obvious in the case of Calabi–Yau manifold compactifications and the $c = 9$, $N = 2$ minimal superconformal tensor models. In two case studies [1] Gepner presented compelling evidence for the existence of a deep relation between exact models and Calabi–Yau manifolds (CYs). Gepner showed that not only do these two models have the same massless spectrum and exhibit the same discrete symmetries, but that the fields also transform in the same way under discrete symmetry transformations. Although somewhat indirect, this evidence lead Gepner to conjecture that all exact $N = 2$ superconformal models correspond to Calabi–Yau manifolds.

Recently this conjecture received some support through the work of Martinec and Greene, Vafa and Warner [2,3]. Using Zamolodchikov's ideas [4] on the relation between the $N = 0(1)$ (super–)conformal discrete series and Landau–Ginzburg type theories applied to the $N = 2$ superconformal discrete series as in [2,5,6] they were able to give a canonical construction of Calabi-Yau manifolds corresponding to the sub–class of Gepner's models explicitly constructed in [7]. This construction shows that the Landau–Ginzburg theory is the crucial element that was missing to proceed in a natural way from exact models to CYs and vice versa.

In the first part of this discussion I review the considerations of [2,3] and their extensions discussed in [8]. First I will refine the comparison of the exact models and the manifolds by computing the number of generations and anti–generations of the manifolds by a combination of the Lefshetz Hyperplane Theorem and the resolution of the singularities that generically occur in these spaces. Then I extend the identification to certain complete intersection Calabi–Yau manifolds involving products of weighted projective spaces (weighted CICYs). The subclass of weighted CICYs that is considered here corresponds to solvable tensor models with affine D–invariants. Recently we have computed the spectrum of all $c = 9$, $N = 2$ minimal tensor models with the various modular invariants [9]. This result will serve as the necessary input which allows a systematic comparison of Calabi–Yau manifolds with the exact models. The cohomology groups of the Calabi–Yau spaces agree with the number of generations and anti–generations of the exact theories.

After having found agreement in the massless particle spectrum for these rather different constructions the next question is obviously how the couplings compare. A number of non-renormalization arguments show that the couplings between the generations agree in both types of models [10] and in fact all explicit calculations done so far [11] confirm these conclusions. The couplings of the anti-generations however can receive corrections; therefore it is of interest to compute their interactions.

In the second part of this discussion I review the computation [12] of the Yukawa couplings for the anti-generations for the three-generation manifold I constructed in reference [13] and compare the results with those of the corresponding exact model. It will be seen that the two sets of couplings are genuinely different.

2. RG Flow Fixed Points in the Diagonal Affine Series

Consider a set of chiral superfields ϕ_i on the worldsheet. The most general action for an $N=2$ supersymmetric theory takes the form

$$S=\int d^2zd^4\theta K(\phi_i,\bar{\phi}_i)+\int d^2zd^2\theta W(\phi_i)+c.c. \tag{1}$$

The first term in this action involving the Kähler potential K is called the D–term, the second one involving the superpotential W is the so–called F–term. The superpotential W is a holomorphic function of the superfields. It is assumed that the non-renormalization theorems of $N=2$ supersymmetric theories hold, implying that the F–term does not get renormalized whereas the D–term does receive corrections. Since the only relevant operators appear in the F–term, this implies that the universality class is determined only by the superpotential.

At the fixed point the theory must be scale invariant implying that the superpotential must be a function of homogeneity one (since the measure is also a function of homogeneity one). By following Zamolodchikov's idea of comparing OPEs in the exact theory with those dictated by the superpotential, it is possible to establish relations between the A–D–E classification in the $N=2$ discrete exact series and $N=2$ Landau–Ginzburg potentials corresponding to modality zero catastrophes.

Tensoring r solvable models together simply corresponds to a superpotential with a number of fields ϕ_i and ψ_i (in the case of the non–diagonal invariants). In the case of the quintic in $\mathbb{P}_4$ the most symmetric polynomial is $p=z_1^5+z_2^5+z_3^5+z_4^5+z_5^5$. Therefore the exact model which should correspond to this Calabi–Yau manifold is the product of five level three $N=2$ theories. In a sense this example is too trivial and does not generalize easily to other examples. The reason for this is that except for very few examples [7] the levels of the various factors are different, meaning that there is no obvious way in which the polynomial(s) could ever be homogeneous. The crucial insight of Martinec and Vafa and Warner was to notice that if one would go beyond the class of CICYs it would be possible to interpret the superpotential as a quasi–homogeneous polynomial of an algebraic variety. The idea is most easily discussed in terms of the models with four or five factors. Consider an exact model $k_1\cdots k_5$ and choose the diagonal invariant in each of the five factors. Then the superpotential defining the universality class of the associated Landau–Ginzburg theory is given by

$$W=\phi_1^{k_1+2}+\phi_2^{k_2+2}+\cdots+\phi_5^{k_5+2} \tag{2}$$

If one assigns the weight $\frac{1}{k_i+2}$ to the i'th field ϕ_i the superpotential has degree one and it is possible to compare the exact model with the variety defined by a polynomial of degree one in a weighted projective space

$$(k_1 \cdots k_5)_{A_{k_1+1}\otimes\cdots\otimes A_{k_5+1}} \sim \mathbb{P}_{(\frac{1}{k_1+2},\frac{1}{k_2+2},\ldots,\frac{1}{k_5+2})}[1]. \tag{3}$$

To get the correct normalization for topological calculations one should in fact multiply through with the smallest common multiple of the denominators, so that the greatest common denominator of the weights is one (for details on weighted projective spaces see [14]). (In the case of four factors one has to augment a trivial theory to the model and repeat the construction for the five coordinates). By calculating the Euler number for these manifolds and comparing them with the results for the exact models Martinec and Greene, Vafa and Warner found agreement for many examples. Their calculations were restricted to the case of diagonal affine invariants. Before extending their results to more general models I will discuss how to calculate the complete spectrum (generations and anti-generations) for the manifolds at hand without having to resort to spectral sequences.

The first step is to calculate the Euler number. As for CICYs the Euler number of weighted CICYs is easy to calculate as long as they are smooth. Along the lines of [13,15] one calculates the third Chern classes of weighted CICYs and then uses Poincaré duality to lift the computation to the ambient space. By multiplying the third Chern class of the manifold with the first Chern class of the normal bundle it is then easy to read off the coefficient of the monomial with the highest power in the generators. Dividing through by the product of the weights therefore gives the Euler characteristic. For the CICYs in one weighted projective space $\mathbb{P}_{(l_1,l_2,\ldots,l_5)}[l]$ this leads to the simple formula

$$\chi_s = -\frac{l}{l_1\cdots l_5}\left[\sum_{i=1}^{5} l_i^2\,\hat{l}_i + \sum_{\substack{i<j<k \\ i,j,k\in\{1,\ldots,5\}}}{}^{'}\; 2l_il_jl_k\right] \tag{4}$$

where $l = \sum_{i=1}^{5} l_i$ and $\hat{l}_i = l - l_i$. This however gives the correct Euler number for the exact model only for smooth manifolds. In general weighted projective spaces have singularities and if these are sufficiently severe they will descend down to the embedded Calabi–Yau manifold. Whenever such a situation occurs relation (3) will be understood to hold as an equivalence between the exact model and the CY with all singularities resolved, i.e. the blowing up procedure will be implicitly assumed whenever necessary.

In fact one can do better than to just compare the Euler number of the two models. It is easy in these models to compute the Hodge diamond in a canonical manner as will be shown now.

The way to proceed is to first compute the number of (1,1)–forms of the Calabi–Yau manifold and then use the Euler number to determine the only other independent Hodge number $h^{(2,1)} = \dim H^{(2,1)}(M,\mathbb{Z})$. One of the (1,1)–forms clearly is present in all these manifolds: it is the pullback of the Kähler form of the weighted projective space. Whenever the CY is smooth this is the only one and therefore one finds $h^{(1,1)} = 1$ in those cases. If there are singularities however the blow–up procedure introduces additional (1,1)–forms. The situation for isolated singular points has been discussed in great detail by Roan and Yau [16].

I now briefly discuss the technique for blowing up curves [8]. Consider the action of a $\mathbb{Z}_n$, $n \leq l_i$, on a weighted CICY

$$\mathbb{Z}_n \ni g : \mathbb{P}_{(l_1,\ldots,l_5)}[d] \longrightarrow \mathbb{P}_{(l_1,\ldots,l_5)}[d] \tag{5}$$

leaving a curve invariant. In the three–dimensional CY the normal bundle of this curve has fibres $\mathbb{C}_2$ and therefore this discrete group induces an action on the fibres that can be written as

$$\mathbb{Z}_n \ni g : \mathbb{C}_2 \longrightarrow \mathbb{C}_2 \quad : g = \begin{pmatrix} \alpha^{mq} & 0 \\ 0 & \alpha^m \end{pmatrix} \tag{6}$$

for some $0 \leq m \leq n$, where $\alpha^n = 1, \alpha \neq 1$. The singularity is then defined to be of type (n, q). The action has an isolated singularity which needs to be resolved. To proceed I use some techniques pioneered by F.Hirzebruch in his dissertation [17] (for an earlier application of these techniques see [13]). The essential point is that the singularity of $\mathbb{C}_2/\mathbb{Z}_n$ can be described as the singular set of the surface

$$\mathbb{C}_3 \supset S = \{(z_1, z_2, z_3) \in \mathbb{C}_3 | z_3^n = z_1 z_2^{n-q}\} \tag{7}$$

and therefore can be resolved by a Hirzebruch–Jung tree with a valuation that is determined by the type (n, q) through the method of continued fractions. The tree–structure dictates the plumbing process which replaces the singularity and also determines the additional generators of the cohomology in the following way. First, the valuation of the trees determine the intersection numbers of the projective curves involved in the blow–up. Furthermore the number of $\mathbb{P}_1$'s necessary to resolve the singularity is precisely the number of steps needed in the evaluation of $\frac{n}{q} = [b_1, .., b_s]$. An example should make it clear how to proceed in general.

Consider e.g. the model

$$\left(4^2 \cdot 10^2\right)_{A_5^2 \otimes A_{11} \otimes D_7} \sim \mathbb{P}_{(1,2,2,2,5)}[12] \tag{8}$$

whose singularity set consists of a $\mathbb{Z}_2$–curve $C = \mathbb{P}_2[6]$ with Euler number $\chi_C = -18$ and a $\mathbb{Z}_5$–point. With $\chi_s = -157\frac{4}{5}$ one finds for the Euler number of the smooth CY

$$\chi = \chi_s + \sum \left(\chi(p_i) n_i - \frac{\chi(p_i)}{n_i}\right) + \sum \left(\chi(C_i) n_i - \frac{\chi(C_i)}{n_i}\right) = -180. \tag{9}$$

The action of the $\mathbb{Z}_5$ on the normal bundle is of type $(n, q) = (2, 1)$. Expanding n/q as a continued fraction therefore leads to the trivial tree (point) with the valuation –2 at the vertex. The blow–up therefore is of type S^2 and therefore contributes one generator to H^2. The blow–up of the $\mathbb{Z}_5$–point contributes two generators and therefore one finds a 4–dimensional second cohomology group for H^2, in agreement with the exact result [9].

Putting the results of Roan and Yau for the blow–up of points together with the prescription just given allows for the computation of the spectrum of all models discussed in [2,3] as well as for more general manifolds involving the Kazama–Suzuki models [18]. The results of the calculations for the manifolds agree with those of the exact models, lending further support for the identification proposed in [2,3]. In the next section it will be seen that these techniques can also be applied to more general spaces involving products of weighted projective spaces.

3. RG Flow Fixed Points in the D-Series

Consider the following class of weighted CICYs

$$\begin{matrix} \mathbb{P}_{(l_1,l_2,l_3,l_4,l_5)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix}, \tag{10}$$

where $l = l_2 + l_3 + l_4 + l_5$. It will be assumed that the $\mathbb{N} \ni l_i > 0$ are such that $l/l_i \in \mathbb{N}$ for $i = 2, \ldots 5$. (This condition can be relaxed.) Obviously the manifolds defined in (10) have vanishing first Chern class and complex dimension three and therefore define CYs of the type we are interested in. Using again the techniques from above one finds for the Euler numbers of these configurations

$$\chi_s = -\frac{2l}{l_1 \cdots l_5} \left[l_1^3 + \sum_{i=2}^{5} l_i^2 \hat{l}_i + \sum_{\substack{i<j<k \\ i,j,k \in \{2,..,5\}}} 2 l_i l_j l_k \right] \tag{11}$$

where $\hat{l}_i = l - l_i$. As in the case of one weighted projective space this formula does not take blow–ups into account.

Assuming $l_1 = l_2$ one can associate an exactly solvable model to each of these manifolds in the following way. Starting from the weights l_i of the weighted projective space we define levels as follows

$$k_1 = k_2 = \frac{2l}{l_1} - 2, \quad k_i = \frac{l}{l_i} - 2, \quad i = 3, 4, 5 \tag{12}$$

The $k_1^2 \cdot k_3 \cdot k_4 \cdot k_5$ model then is a tensor model with central charge $c = 2\frac{3k_1}{k_1+2} + \sum_{i=3}^{5} \frac{3k_i}{k_i+2} = 9$. The point in the moduli space of each of these configurations which we wish to identify with this exact model is defined by the following choice of polynomials

$$\begin{aligned} p_1 &= y_1 z_1^2 + y_2 z_2^2 && (13) \\ p_2 &= \sum_{i=1}^{5} y_i^{l/l_i} , && (14) \end{aligned}$$

where the $y_i's$ denote the coordinates of the first factor and the z_j denote the coordinates of the second factor. From this it is clear that we should identify these weighted CICYs with the models $k_1^2 \cdot k_3 \cdot k_4 \cdot k_5$ in which the left and right sectors are glued together with the affine invariants $D_{\frac{l}{l_i}+1}$ in the first two factors and the diagonal affine invariant in the last factors. (The generalization of the above divisibility criterion for the weights leads to non–Fermat polynomial type for p_2, allowing for A–D–E and more generally for Kazama–Suzuki type models).

As mentioned above however weighted CICYs most often are not smooth but have singularities. The set of singularities generically consists of a set of points and curves, either of which may be empty. In those circumstances the relation

$$\begin{matrix} \mathbb{P}_{(l_1,l_1,l_3,l_4,l_5)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix} \sim \left(2\left(\frac{l}{l_1} - 1\right)\right)^2 \cdot \prod_{i=3}^{5} \left(\frac{l}{l_i} - 2\right) \tag{15}$$

should be understood to hold as an equivalence between the exact model and the CY with all the singularities *resolved*, i.e. the blowing up procedure is implicitly assumed whenever necessary. An example is furnished by the manifold

$$\begin{matrix} \mathbb{P}_{(3,3,2,3,4)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} 3 & 12 \\ 2 & 0 \end{bmatrix} \tag{16}$$

for which $k_1 = 6 = k_2$, $k_3 = 4$, $k_4 = 2$ and $k_5 = 1$. The model $(6^2 \cdot 4 \cdot 2 \cdot 1)_{D_5^2 \otimes A_5 \otimes A_3 \otimes A_2}$ has 55 generations and 7 anti–generations [9]. The singular weighted CICY has Euler number $\chi_s = -62\frac{1}{3}$. The weighted projective space has three types of singularities: a $\mathbb{Z}_2$–singular curve, a $\mathbb{Z}_3$–singular surface and a $\mathbb{Z}_4$–singular point. The singular point does not descend down to the CY manifold. The $\mathbb{Z}_2$–singular curve however induces on the Calabi–Yau manifold three singular curves $C = \mathbb{P}_1[3] \times \mathbb{P}_1$ with Euler number $\chi_C = 6$.

The $\mathbb{Z}_2$–singular surface induces the singular curve $C' = \begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_1 \end{matrix} \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ with Euler number $\chi_{C'} = -16$ and therefore the resolution of the singularities leads to a smooth manifold with Euler characteristic

$$\chi = -62\frac{1}{3} - \frac{\chi_C}{2} + 2\chi_C - \frac{\chi_{C'}}{3} + 3\chi_{C'} - = -96 \tag{17}$$

which agrees again with the exact result. Applying the techniques described above for the blow–up of these curves one finds that the $\mathbb{Z}_2$–curve introduces 3 new (1,1)–forms whereas the $\mathbb{Z}_3$–curve contributes 2 additional generators to H^2. Adding the two generators coming from the Kähler forms of the ambient space gives the total dimension seven for the second cohomology group, as found in the exact model.

In summary what has been discussed in this first part is the transition from a diagonal Calabi–Yau of the type discussed in [2,3] to a non–diagonal Calabi–Yau. In a nutshell this step can be summarized in the following diagram

$$\begin{array}{ccc}
\mathbb{P}_{(\frac{l_1}{2},\frac{l_2}{2},l_3,l_4,l_5)}[l] & \longrightarrow & \left(2\left(\frac{l}{l_1}-1\right)\right)^2_{A^2_{\frac{2l}{l_1}-1}} \cdot \prod_{i=3}^5 \left(\frac{l}{l_i}-2\right)_{A_{\frac{l}{l_i}-1}} \\
\Big\downarrow & & \Big\downarrow \\
\begin{matrix} \mathbb{P}_{(l_1,l_1,l_2,l_3,l_4)} \\ \mathbb{P}_{(1,1)} \end{matrix} \begin{bmatrix} l_1 & l \\ 2 & 0 \end{bmatrix} & \longrightarrow & \left(2\left(\frac{l}{l_1}-1\right)\right)^2_{D^2_{\frac{l}{l_1}+1}} \cdot \prod_{i=3}^5 \left(\frac{l}{l_i}-2\right)_{A_{\frac{l}{l_i}-1}}
\end{array}$$

The horizontal arrows describe the LG–type identifications of exact models and weighted CICYs. The vertical arrows describe the replacement of the diagonal invariants by the D–invariants in the first two factors.

In fact there is a deeper geometrical link between these two models. Consider the zero–set of the two polynomials p_1, p_2 in more detail. For a generic point in $\mathbb{P}_{(\frac{l_1}{2},\frac{l_1}{2},l_3,l_4,l_5)}$ the polynomial constraint $p_1 = 0$ has two solutions on $\mathbb{P}_1$, i.e. generically the manifold is a double cover of $\mathbb{P}_{(\frac{l_1}{2},\frac{l_1}{2},l_3,l_4,l_5)}[l]$. If however either $x_1 = 0$ or $x_2 = 0$ then there is just one solution. Furthermore if $x_1 = 0 = x_2$ then there is one full $\mathbb{P}_1$ for each point on the curve $\mathbb{P}_{(l_3,l_4,l_5)}$. The curve described by setting $x_1 = 0 = x_2$ however is precisely the curve that can be found in the diagonal manifold by setting the first two coordinates equal to zero.

4. Yukawa Couplings

4.1. The Manifold

Define K_0 as a particular point of the moduli space of the deformation class

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}^{h^{(1,1)}=8}_{\chi=-54} . \tag{18}$$

All non-singular members of this deformation class have the same spectrum. From $\chi = 2(h^{(1,1)} - h^{(2,1)})$, where $h^{(1,1)}$ =dim $H^{(1,1)}(K_0)$ and $h^{(2,1)}$ =dim $H^{(2,1)}(K_0)$, it is then easy to see that this manifold has $(\chi, h^{(1,1)}, h^{(2,1)}) = (-54, 8, 35)$. The manifold to be investigated in the following is defined as the intersection of the zero set of the polynomials [13]

$$p^1 = \sum_{A=0}^{3} y_A^3 \,, \quad p^2 = \sum_{A=0}^{2} x_A^3 y_A. \tag{19}$$

With 35 generations and 8 anti-generations these manifolds are clearly phenomenologically uninteresting and one needs some mechanism to reduce the number of fields. The standard procedure is to consider orbit manifolds. Consider the action

$$g : \mathbb{P}_2 \times \mathbb{P}_3 \longrightarrow \mathbb{P}_2 \times \mathbb{P}_3 \tag{20}$$

defined by

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Call this group $\mathbb{Z}_3^r$. Clearly the manifold K_0 is invariant under this group and it is easy to check that in fact this action induces a fixed point free action on K_0. Therefore the orbit manifold

$$K_{orb} = K_0/\mathbb{Z}_3^r \tag{22}$$

defines a smooth Calabi–Yau manifold with Euler number -18. The Hodge numbers are easily found to be $h^{(1,1)} = 4, h^{(2,1)} = 13$.

There is another $\mathbb{Z}_3^s$ that can be modded out. Consider the following action

$$g : \mathbb{P}_2 \times \mathbb{P}_3 \longrightarrow \mathbb{P}_2 \times \mathbb{P}_3 \tag{23}$$

defined by

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^2 \end{pmatrix} \oplus id. \tag{24}$$

Clearly this transformation defines an automorphism group of the manifold defined by the polynomials (19). However it does not act fixed point free. The fixed point sets in $\mathbb{P}_2 \times \mathbb{P}_3$ are

$$(1,0,0) \times \mathbb{P}_3, \quad (0,1,0) \times \mathbb{P}_3, \quad (0,0,1) \times \mathbb{P}_3. \tag{25}$$

On the Calabi–Yau manifold K_0 embedded in the ambient space as the zero set of the polynomials p^1, p^2 these describe the sets

$$\begin{aligned} (1,0,0) \times \{y_1^3 + y_2^3 + y_3^3 = 0\} &= (1,0,0) \times \mathbb{P}_2[3] =: T_1 \\ (0,1,0) \times \{y_0^3 + y_2^3 + y_3^3 = 0\} &= (0,1,0) \times \mathbb{P}_2[3] =: T_2 \\ (0,0,1) \times \{y_0^3 + y_1^3 + y_3^3 = 0\} &= (0,0,1) \times \mathbb{P}_2[3] =: T_3 \end{aligned} \tag{26}$$

i.e. there are three invariant tori on K_0 and therefore $K_s = K_0/\mathbb{Z}_3^s$ is singular. This space has $(h^{(1,1)}, h^{(2,1)}) = (8, 17)$.

As described in [13] blowing up the singular tori introduces 6 new (1,1)–forms. In addition to the (1,1)–forms the blow–up also introduces 6 new (2,1)–forms thereby defining a manifold with $h^{(2,1)} = 23$ and $h^{(1,1)} = 14$, while leaving the Euler number unchanged. The blow–up of these tori makes a new smooth manifold which is simply connected and which is still a Calabi–Yau space [13]. Modding out the fixed point free discrete group $\mathbb{Z}_3^r$ leads to a Calabi–Yau manifold K with $h^{(1,1)} = 6$ and $h^{(2,1)} = 9$.

4.2 Symmetries and their representations

In this section the automorphism group of K will be described. The fundamental group of K is $\mathbb{Z}_3$. Given the symmetry group D_0 of the covering manifold K_0, all one needs to check is whether an element $f \in D_0$ satisfies

$$fgf^{-1} = g' \tag{27}$$

for $g, g' \in \mathbb{Z}_3$. The first step then is to consider the symmetry group D_0. This group is given by a product of five groups $D_0 = \mathbb{Z}_9^3 \times \mathbb{Z}_3 \times S_3$. The cyclic groups are generated by maps

$$g : \mathbb{P}_2 \times \mathbb{P}_3 \longrightarrow \mathbb{P}_2 \times \mathbb{P}_3$$

defined by

$$g = \begin{pmatrix} \alpha^{-m_1} & 0 & 0 \\ 0 & \alpha^{-m_2} & 0 \\ 0 & 0 & \alpha^{-m_3} \end{pmatrix} \oplus \begin{pmatrix} \alpha^{3m_1} & 0 & 0 & 0 \\ 0 & \alpha^{3m_2} & 0 & 0 \\ 0 & 0 & \alpha^{3m_3} & 0 \\ 0 & 0 & 0 & \alpha^{3m_4} \end{pmatrix} \tag{28}$$

where $\alpha^9 = 1$, $\alpha \neq 1$ and $(m_1, m_2, m_3) \in \mathbb{Z}$ mod 9, $m_4 \in \mathbb{Z}$ mod 3. S_3 is the symmetric group on 3 elements, i.e. the diagonal on $(x_0, x_1, x_2, y_0, y_1, y_2)$.

The symmetry of K_0 then is given by a finite group of order $3 \cdot 9^3 \cdot 6/9 = 1458$ whose generators are denoted by $(d_1, d_2, d_3, d_4, \pi_1, \pi_2)$ where π_1 is the cyclic generator. D_0 contains the freely acting group $\mathbb{Z}_3^r$ of cyclic permutations on 3 elements and the scaling subgroup $\mathbb{Z}_3^s$ which leaves three tori on K_0 fixed. Going through the analysis just described, it is easy to determine the discrete symmetry group of K to be given by $\mathbb{Z}_3 \times \mathbb{Z}_2$, where $\mathbb{Z}_2 = S_3/\mathbb{Z}_3^r$ and $\mathbb{Z}_3$ is the scaling symmetry generated by d_4.

To determine the physical fields we need to know their transformation properties i.e. we want to know what the characters of the group are on $H^{(1,1)}(K)$. To achieve this the Lefshetz Hyperplane Theorem (LHT) and the Lefshetz Fixed Point Theorem (LFT) will be extremely useful. Let

$$\Gamma^{(p,q)} : G \longrightarrow Hom(H^{(p,q)}(K))$$

be the representation of G on the (p,q)-forms. Then the LFT states that for each $g \in G$

$$\sum_{p,q=0}^{n} (-1)^{p+q} \chi^{p,q}(g) = \sum_{m_g} e(m_g), \tag{29}$$

where m_g is the fixed point set of g, $e(m_g)$ is its Euler number and $\chi^{(p,q)}(g) := Tr_{H(p,q)}(K)\Gamma^{(p,q)}(g)$ denotes the character of the representation $\Gamma^{(p,q)}(g)$.

In the following analysis only the untwisted sector of the $\overline{\mathbf{27}}$ of K will be considered. Since this part is just given by the (1,1)–cohomology of K_{orb}, we need to study the representation theory of the discrete symmetry group of K. Consider first the factor S_3. Since there are three conjugacy classes of S_3, $\mathcal{C}_1 = \{1\}$, $\mathcal{C}_2 = \{\pi_1, \pi_1^2\}$, and $\mathcal{C}_3 = \{\pi_2, \pi_1\pi_2, \pi_2\pi_1\}$, there are three irreducible representations, the sum of the squares of the dimensions of which is the order of the group. This leads to two one–dimensional representations and one two–dimensional representation.

Next consider the factor $\mathbb{Z}_3$. Since the group is abelian, the $\overline{27}$ decomposes into 1-dimensional irreducible representations, i.e. the characters actually define the representations. Denote those (1,1)-forms that are invariant under π_1 by $\bar{l}_i$, those that transform as α by $\bar{Q}_i$ and those that transform like α^2 by $\bar{q}_i$. Putting everything together leads to the transformation of the physical fields given in table 1.

Table 1. *Transformations of (1,1)-forms under* $\mathbb{Z}_3 \times S_3$.

	$d_4 \in \mathbb{Z}_3$	$\pi_1 \in S_3$	$\pi_2 \in S_3$
$\bar{l}_1$	1	1	1
$\bar{l}_2$	1	1	1
$\bar{l}_3$	α	1	-1
$\bar{l}_4$	α^2	1	-1
$\bar{Q}_1$	α	α	$\rightarrow \bar{q}_1$
$\bar{Q}_2$	α^2	α	$\rightarrow \bar{q}_2$
$\bar{q}_1$	α	α^2	$\rightarrow \bar{Q}_1$
$\bar{q}_2$	α^2	α^2	$\rightarrow \bar{Q}_2$

4.3 Yukawa Couplings of the Manifold

The Yukawa couplings of the $\overline{27}$ fields on a Calabi–Yau manifolds are based on the intersection cubic associated to the manifold. This intersection cubic generalizes the intersection matrix of complex surfaces which is perhaps the most important tool in the analysis of the structure of these manifolds. The same is true for 3 complex–dimensional manifolds: a classification theorem by Wall [19] states that 6 real–dimensional manifolds are classified up to diffeomorphisms by their intersection cubics and their first Pontrjagin number. The computation of the Yukawa couplings starts off by first calculating the intersection cubic on the manifold K_0.

Consider the embeddings

$$K_0 \stackrel{j}{\hookrightarrow} \mathbb{P}_2 \times \mathbb{P}_3[3] \stackrel{1\times i}{\hookrightarrow} \mathbb{P}_2 \times \mathbb{P}_3 \tag{30}$$

and define the forms

$$h_1 = j^*(H_2)\ ; \quad h_2 = j^*(i^*(H_3)). \tag{31}$$

The other six generators that complete the basis of $H^2(K_0)$ come from the cubic surface and are most easily described in terms of their homological counterparts. Fortunately the cubic surface in $\mathbb{P}_3$ is one of the most thoroughly studied manifolds in algebraic geometry. There are 7 homology classes on this surface, one of them being the class associated with h_2. However there are not only 7 lines to be found in the cubic, as one might expect, but there are actually 27 (complex) lines embedded in this space. This can be shown to be true for a generic cubic [20]. These lines are easily constructed explicitly for the Fermat cubic. They are given by the following equations in $\mathbb{P}_3$ which identically satisfy the Fermat cubic:

$$
\begin{aligned}
L^1_{i,j} &= \{y_0 + \alpha^i y_1 = 0 = y_2 + \alpha^j y_3\} \\
L^2_{i,j} &= \{y_0 + \alpha^i y_2 = 0 = y_1 + \alpha^j y_3\} \\
L^3_{i,j} &= \{y_0 + \alpha^i y_3 = 0 = y_1 + \alpha^j y_2\}.
\end{aligned} \tag{32}
$$

To find a homology basis for the cubic among these 27 lines we have to find 6 lines that do not intersect with each other. It can be checked that the following subset $\{E_i\}$ of lines satisfies these conditions:

$$
\{E_1, E_2, E_3, E_4, E_5, E_6\} = \{L^2_{00}, L^2_{12}, L^2_{21}, L^3_{01}, L^3_{10}, L^3_{22}\}. \tag{33}
$$

The second homology group therefore decomposes naturally into three parts: the two Kähler forms, and the six lines E_i. With $E_i \cap E_j = -\delta_{ij}$ and $h_2 \cap E_i = 3$ on the cubic, it is straightforward to compute the intersection cubic. Only couplings between different sectors can be non–zero. The following results are found:

Kähler $\times$ *Kähler* $\times$ *Kähler* – sector:

$$
(h_1, h_1, h_2) = 3, \quad (h_1, h_2, h_2) = 9 \tag{34}
$$

Kähler $\times$*Kähler* $\times$ *exceptional* – sector:

$$
(h_1, h_1, e_i) = 1, \quad (h_1, h_2, e_i) = 3, \tag{35}
$$

Kähler $\times$ *exceptional* $\times$*exceptional* – sector:

$$
(h_1, e_i, e_j) = -3\delta^{ij}. \tag{36}
$$

The next step is to relate the physical fields to the basis chosen above. To achieve this, one needs first to check the transformation behaviour of the basis under discrete symmetries. This can easily be done by explicitly transforming the lines above. Using this information as input it is straightforward to find the physical fields in terms of the basis used above

$$
\begin{aligned}
\bar{l}_1 &= aH_1 \\
\bar{l}_2 &= bH_2 \\
\bar{l}_3 &= c[H_2 + E_1 - 2E_2 + \alpha(2-\alpha)E_3 + \alpha(2\alpha-1)E_4 + E_5 - 2E_6] \\
\bar{l}_4 &= d[H_2 + E_1 - 2E_2 + \alpha(2\alpha-1)E_3 + \alpha(2-\alpha)E_4 + E_5 - 2E_6] \\
\bar{Q}_1 &= e[H_2 - 2E_1 + \alpha(2-\alpha)E_2 + E_3 + E_4 - 2E_5 + \alpha(2\alpha-1)E_6] \\
\bar{Q}_2 &= f[H_2 + \alpha(2\alpha-1)E_1 + E_2 - 2E_3 - 2E_4 + \alpha(2-\alpha)E_5 + E_6] \\
\bar{q}_1 &= e[-\alpha H_2 - (2\alpha^2-1)E_1 - \alpha E_2 + 2\alpha E_3 + 2\alpha E_4 - (2-\alpha^2)E_5 - \alpha E_6] \\
\bar{q}_2 &= f[-\alpha H_2 + 2\alpha E_1 - (2-\alpha^2)E_2 - \alpha E_3 - \alpha E_4 + 2\alpha E_5 - (2\alpha^2-1)E_6].
\end{aligned} \tag{37}
$$

This representation for the physical fields together with the intersection number above leads to the following nonvanishing couplings for the four invariant fields $\bar{l}_i$:

$$
\mu_{112} = 3a^2 b, \quad \mu_{122} = 9ab^2, \quad \mu_{134} = -81acd. \tag{38}
$$

All other interactions (except permutations of the above) vanish.

The physical significance of these results is the following. As they stand these couplings represent perturbative results in the α'–expansion of the σ model as well as in the string perturbation theory. However there exist a number of strong non–renormalization theorems. Whereas the D–terms are corrected, the superpotential is not renormalized to all finite orders in the string tension and is also not renormalized by string loops [10]. Therefore the couplings are correct to every finite order. However they can be corrected by nonperturbative effects such as instantons. In fact the manifold above has instantons embedded in it and therefore one would in general expect the couplings to receive additional contributions from those, thereby generating new non–vanishing couplings. Rather than add up all instanton contributions to check this, I will compare the result above with the 3–point function of exact model.

4.4 The Exact Model

In order to compare the Yukawas in the Calabi–Yau construction of section 2 with the three–point function in the exact model, I will quickly review some of the general structure of conformal field theory and apply it to Gepner's exact model of my manifold.

Recall that the manifest gauge symmetry in Gepner's construction is $E_8 \times SO(10)$. The heterotic partition function is given by [21]

$$Z_{het} = (4\mathrm{d} - \mathrm{part}) \times \sum_{\lambda \in \{0,v,s,\bar{s}\}} B_\lambda^{SO(2)}(\tau) B_{\bar{\lambda}}^{E_8 \times SO(10)}(\bar{\tau}) Z_{int}(\tau, \bar{\tau}). \tag{39}$$

The $SO(10)$ current algebra picks up one of the superconformal $U(1)'s$ of the internal tensor products of the $N = 2$ discrete series to build up the E_6 representations. Consider e.g. the matter fields. The **27** decomposes under the $SO(10) \times U(1)$ as

$$\mathbf{27} = \mathbf{10}_{-1} \oplus \mathbf{16}_{1/2} \oplus \mathbf{1}_2. \tag{40}$$

To these decompositions one associates vertex operators depending on the choice of picture for the 4–dimensional part. Using the free fermions λ^a to construct the $SO(10)$ current algebra, a space–time scalar which transforms in the **10** of the $SO(10)$ can be written in the form

$$V_{-1}^a(z, \bar{z}) = e^{-\rho(z)} \mathcal{O}(z, \bar{z}) \lambda^a, \tag{41}$$

where $\mathcal{O}(z, \bar{z})$ is an operator in the internal theory with conformal weights $\Delta = 1/2 = \bar{\Delta}$ and $U(1)$ charges $Q = -1 = \bar{Q}$ and ρ is the bosonized ghost. Also of interest here are the spacetime spinors. The simplest picture for these is the $-1/2$ picture in which the operators have the form

$$V_{-1/2}^{\alpha}{}^{\cdot}(z, \bar{z}) = e^{-\rho(z)} S^\alpha \mathcal{O}(z, \bar{z}) R^{\,\cdot}, \tag{42}$$

where S^α is the spin field of ψ^μ, $\mathcal{O}$ is the purely internal part and R is the vertex operator of a primary field of the right moving $SO(10)$ current algebra, depending on a particular representation, represented by the dot $(\cdot \in \{\mathrm{singlet}, \mathrm{vector}, \mathrm{spinor}, \mathrm{anti-spinor}\})$.

In [1] Gepner found an exact model exhibiting many of the properties of the three–generation manifold discussed above. The internal sector here is described by a product of one level 1 factor and three factors of level 16. The invariants chosen are the diagonal one in the first factor and the exceptional one for the other three factors:

$$\begin{matrix} \mathbb{P}_2 \\ \mathbb{P}_3 \end{matrix} \begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}^{h^{(1,1)}=8}_{\chi=-54} \quad \sim \quad (1 \cdot 16^3)_{A_2 \otimes E_7^3} \tag{43}$$

This tensor model has the same number of generations and anti–generations, and the same discrete symmetries (with the particular choice of polynomials discussed above). Furthermore the massless fields transform under discrete symmetries in exactly the same way in the exact and the manifold model. This is strong evidence for an intimate relation between the two, and in the following I wish to investigate the relation further by comparing the Yukawa couplings of the two. The couplings in the exact model are easily calculated using the fact that the primary fields in the $N = 2$ discrete series are essentially given in terms of primary fields of the $SU(2)$ current algebra (modulo trivial scalar fields). Recall that the $N = 2$ superconformal theory at level k has primary fields

$$\Phi^{l,q,s}_{\bar{l},\bar{q},\bar{s}} = \phi^{l,q-s}_{\bar{l},\bar{q}-\bar{s}} \; e^{i(\alpha_{qs}\varphi+\alpha_{\bar{q}\bar{s}}\bar{\varphi})}, \tag{44}$$

where $\phi^{l,m}_{\bar{l},\bar{m}}$ are parafermionic primary fields, φ is a scalar field and

$$\alpha_{qs} = \frac{-q+\frac{s}{2}(k+2)}{\sqrt{k(k+2)}}. \tag{45}$$

Here $0 \leq l \leq k$, $q = q \bmod 2(k+2)$ and $s = s \bmod 4$. In the following the notation

$$\Phi^{l,q,s}_{\bar{l},\bar{q},\bar{s}} = \begin{bmatrix} l & q & s \\ \bar{l} & \bar{q} & \bar{s} \end{bmatrix} \tag{46}$$

will be adopted. The eight anti–generations combine to four anti–generations which are invariant under the discrete symmetries to be modded out later. If they are represented by the spinor in the singlet of the $SO(10)$ then they are given by

$$\overline{L}_1 = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} 6 & 7 & 1 \\ 10 & 10 & 0 \end{bmatrix}^3 \tag{47}$$

$$\overline{L}_2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 5 & 1 \\ 12 & 12 & 0 \end{bmatrix}^3 \tag{48}$$

$$\overline{L}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 2 & 3 & 1 \\ 8 & 8 & 0 \end{bmatrix} \begin{bmatrix} 8 & 9 & 1 \\ 14 & 14 & 0 \end{bmatrix}^2 + permutations \right\} \tag{49}$$

$$\overline{L}_4 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \left\{ \begin{bmatrix} 2 & 3 & 1 \\ 8 & 8 & 0 \end{bmatrix}^2 \begin{bmatrix} 8 & 9 & 1 \\ 14 & 14 & 0 \end{bmatrix} + permutations \right\}. \tag{50}$$

4.5. Yukawa Couplings of the Exact Model

As discussed above one needs in fact to calculate the three–point function of either the one spacetime scalar in the vector representation of $SO(10)$ and two spacetime spinors in the spinor representation or alternatively the coupling of one spacetime scalar in the $SO(10)$–singlet representation and two spacetime spinors in the vector representation. In the following the former is chosen. To get from the spacetime spinors in the $SO(10)$ singlet written above to the spacetime scalar and the vector of the $SO(10)$, one has to act with the supersymmetry charges on the fields. The super charges Q^α have an internal part with weights and charges $(\Delta, Q; \bar{\Delta}, \bar{Q}) = (3/8, 3/2; 0, 0)$. The only field that has such quantum numbers is

$$\Sigma(z) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}^r \tag{51}$$

with an analogous field in the antiholomorphic sector. Applying these operators to the $N = 2$ superconformal primary fields simply shifts the charge and sector quantum number by one.

Without actually calculating the couplings it is easy check which couplings have to be zero purely because of the symmetries of the theory. In an exact model of r $N = 2$–superconformal factors the gauge group is enlarged from the familiar E_6 of Calabi–Yau compactification to $E_6 \times U(1)^{r-1}$ because each factor contributes one $U(1)$. The diagonal $U(1)$ goes together with the manifest $SO(10)$ to build up E_6 leaving $(r-1)$ $U(1)$s. Tracing the $U(1)$ quantum numbers of the fields it is easy to see that the only couplings that can possibly be non–zero are

$$\mu_{112} \neq 0, \quad \mu_{134} \neq 0. \tag{52}$$

These two couplings in fact are non–zero, their values being of order unity; their precise values are not important for the following. (For rather explicit discussions of couplings in exact models see reference [11]).

The comparison of the exact result with the Calabi–Yau couplings shows that there is no non–singular transformation of the fields that would transform one set of couplings into the other, meaning that the two models are physically distinct. Therefore we have the surprising result that instead of lifting zeros in the couplings, instantons create them (at least in some circumstances).

Interpreting the couplings of the exact model as the intersection cubic of some geometric model (along the lines of an LG interpretation discussed in the first part) it is therefore clear that the two models are not diffeomorphic, describing inequivalent physical and mathematical structures.

ACKNOWLEDGEMENT

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REFERENCES

1. D.Gepner, Phys.Lett.**199B**(1987)380; *String Theory on Calabi-Yau manifolds: The Three-Generation Case*, PUPT preprint.

2. E.J.Martinec, Phys.Lett. **217B**(1989)431.

3. B.R.Greene, C.Vafa and N.Warner, *"Calabi-Yau Manifolds and Renormalization Group Flows"*, HUTP-88/A047, preprint.

4. A.B.Zamolodchikov, Sov.J.Phys. **44**(1986)529 [Yad.Fiz. **44** (1986)821].

5. D.A.Kastor, E.J.Martinec and S.H.Shenker, Nucl.Phys. **B316**(1989)590.

6. C.Vafa and N.Warner, Phys.Lett. **218B**(1989)51.

7. M.Lynker and R.Schimmrigk, Phys.Lett.**208B**(1988)216; *ibid* **215B**(1988)681; C.A.Lütken and G.G.Ross, Phys.Lett **213B**(1988)512.

8. R.Schimmrigk, *"Heterotic RG Flow Fixed Points with Non-Diagonal Affine Invariants"*, UTTG-13-89 preprint.

9. M.Lynker and R.Schimmrigk, preprint in preparation.

10. E.Witten, Nucl.Phys. **B268**(1986)79.
M.Dine and N.Seiberg, Phys.Rev.Lett. **57**(1986)2625.

11. J.Distler and B.R.Greene, Nucl.Phys. **B309**(1988)295;
D.Gepner, Nucl.Phys. **B311**(1988)191;
G.Sotkov and M.Stanishkov, Phys.Lett. **215B**(1988)674.
S.Cordes and Y.Kikuchi, *"Correlation Functions and Selection Rules in Minimal N=2 String Compactification"*, CPT-TAMU-92/88 preprint;
B.R.Greene, C.A.Lütken and G.G.Ross, *"Couplings in the Heterotic Superconformal Three Generation Model"*, HUTP-88/A062, Nordita-89/8P preprint.

12. R.Schimmrigk, *"Heterotic (2,2)-Vacua: Manifold Theory and Exact Results"*, UTTG-32-89 preprint.

13. R.Schimmrigk, Phys.Lett. **193B**(1987)175

14. I.Dolgachev, in: Group actions and vector fields, ed. J.Carrell, Springer LNM, Vol. 956 (Springer, Berlin);
A.Dimca, J. Reine Ang. Math. **366**(1986)184.

15. P.Candelas, A.Dale, C.A.Lütken and R.Schimmrigk, Nucl.Phys. **B298** (1988)493.

16. S.-S.Roan and S.-T.Yau, Acta Math. Sinica, (New Series) **3**(1987)25 6.

17. F.Hirzebruch, Math.Ann. **126**(1953)1.

18. Y.Kazama and H.Suzuki, Phys.Lett **216B**(1989)112; Nucl Phys. **B321**(1989)232.

19. C.T.C.Wall, Inv.Math. **1**(1966)355.

20. R.Hartshorne, Algebraic Geometry (Springer, Berlin, 1977)
P.Griffiths and J.Harris, Principles of Algebraic Geometry (Wiley-Interscience, New York, 1978)

21. D.Gepner, Nucl.Phys. **B296**(1988)757.

CONFORMAL FIELD THEORIES AND CATEGORY THEORY

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Abstract: A relation between Conformal Field Theories and braided monoidal categories is established. The coherence theorem for these categories implies the existence of some fundamental relations obeyed by crossing matrices of Conformal Field Theories.

1. INTRODUCTION.

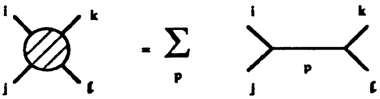
The problem of classification of Rational Conformal Field Theories in two dimensions (RCFT) has received a lot of attention lately, a sample can be found in [1-5]. . In the algebraic approach to this problem a key role is played by the crossing matrices[6],[7]. The crossing matrices obey some constraints that are associated with transformations of graphs. The constraints are similar in form to those introduced by Mac Lane in category theory[8]. The purpose of this talk is to explain the nature of the constraints on crossing matrices of RCFT, and their relation to Mac Lane's theory [9]. This will then allow us to apply Mac Lane's results (in a recently extended form [10]) to determine a generating set of these constraints. Most of the Content of this talk was obtained in the mathematical Physics seminar at Tel Aviv University in June and July 1988.

RCFT's are characterized by the central charge c of the left and right Virasoro algebras, the finite set of primary fields and their conformal weights $(h_i, h_{\bar{i}})$ and by the structure constants C_{IJK} of the operator product algebra (O.P.A.). Factorization of the n-point function leads to the construction of conformal blocks[6]. (For simplicity we consider minimal theories only and do not distinguish between fields and their conjugates to avoid complicated notation.). An important example is the 4-point function. It can be written as

$$\langle \phi_i(z_1,\bar{z}_1)\phi_j(z_2,\bar{z}_2)\phi_k(z_3,\bar{z}_3)\phi_l(z_4,\bar{z}_4)\rangle \sim \sum_p c_{ijp}c_{pkl} I \begin{bmatrix} jk \\ il \end{bmatrix}_p (z) \ \times \text{"c.c."} \tag{1}$$

where by "c.c." we mean the corresponding expression with the bar variables and z is the cross ratio $z = \frac{z_1-z_2}{z_1-z_3}\frac{z_4-z_2}{z_4-z_3}$.

The I's are the holomorphic conformal blocks of the 4-point function. The blocks span a vector space of singular functions with singularities at $z = 0, 1, \infty$. Equation (1) can be described pictorially by

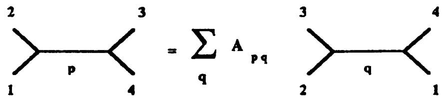


We see that we can associate to each holomorphic (antiholomorphic) conformal block of the 4-point function a 4-leg labeled graph. Similarly we can associate labeled graphs to the blocks of the n-point function. The legs of the graphs usually correspond to some primary fields while the internal edges correspond to summation of descendants of a given conformal family.

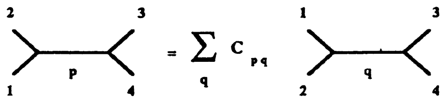
Crossing matrices appear in the following way: The completeness of the s- and t-channel blocks leads to the relation[6],[7],[4]

$$c_{12p}c_{p34} I \begin{bmatrix} 23 \\ 14 \end{bmatrix}_p (z) \ = \sum_q A \begin{bmatrix} 23 \\ 14 \end{bmatrix}_{pq} \ c_{23q}c_{q14} I \begin{bmatrix} 34 \\ 21 \end{bmatrix}_q (1-z) \tag{2}$$

which can be described pictorially in terms of an operation on graphs



Another crossing matrix is the one described pictorially by



The matrices C are diagonal matrices whose entries depend on the h_i's

$$C \begin{bmatrix} 23 \\ 14 \end{bmatrix}_{pq} \ = e^{i\pi(h_p-h_1-h_2)}\delta_{pq} \tag{3}$$

All other crossing matrices can be expressed in terms of A,C,C^{-1}.

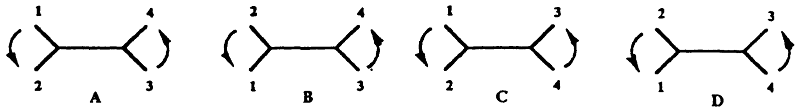
The constraints on the crossing matrices arise when one performs a sequence of crossing moves inside n-point functions. In general it is possible to reach the same conformal block by performing different sequences of moves. The demand that the n-point function is well defined in a way compatible with the operator product expansion leads to constraints on the crossing matrices However it turns out that all the constraints are generated by a finite set of basic constraints.

2. MOVES ON GRAPHS

A convenient way of studying properties of the crossing transformations is through the definitions of moves on graphs. To explain this we first need some definitions.

A graph consists of two sets, E, its set of edges (or one dimensional objects) and N, its set of nodes (zero dimensional objects). Furthermore, to each edge, e, is associated a subset $\{n_1, n_2\}$ of N called its boundary nodes. We say that n_1 and n_2 are incident to e. We will consider graphs where every node is incident to either three edges (the internal nodes) or to one edge (the external nodes). This means that we are considering graphs similar to the Feynman diagrams for a ϕ^3 theory. External edges are called legs.

We can define a cyclic order on a graph by choosing a cyclic order at each (internal) node. Let us consider the basic 4-leg graph ${}^{2}_{1}{>}\!\!-\!\!{<}^{3}_{4}$. There are four corresponding ordered graphs.



There is a natural group action on graphs. The group G=S(Legs(${>}\!\!-\!\!{<}$)) of all one to one transformations of the finite set Legs(${>}\!\!-\!\!{<}$) onto itself. If we label the graph as above then G is isomorphic to S_4. Consider the graphs ${}^{2}_{1}{>}\!\!-\!\!{<}^{3}_{4}$ ${}^{4}_{1}{>}\!\!-\!\!{<}^{3}_{2}$ and ${}^{1}_{3}{>}\!\!-\!\!{<}^{4}_{2}$. These are three unordered graphs and the group G acts transitively on this three element set. Each of the unordered graphs gives rise to four cyclically ordered graphs. The group G acts transitively on the twelve element set of ordered graphs.

If ${>}\!\!-\!\!{<}$ is an internal subgraph of some unordered graph X_0, then we have two graphs Y_0 and Z_0 associated to X_0 by crossing, and G acts transitively on this three element set as before. Let us choose a cyclic order on X_0. We denote the corresponding cyclically ordered graph by X. Suppose that ${>}\!\!-\!\!{<}$ is a 4-legged subgraph of X and $b \in G({>}\!\!-\!\!{<})$. We let bX be the cyclically ordered graph given by cyclically ordering the nodes of bX_0 as follows: Each node of bX_0 other than the two nodes of ${>}\!\!-\!\!{<}$ comes from a unique node of X_0. On each such node we put the old cyclic ordering as in X. At the nodes corresponding to those of the crossing subgraph, ${>}\!\!-\!\!{<}$, we put the cyclic ordering given by the action of G as described above.

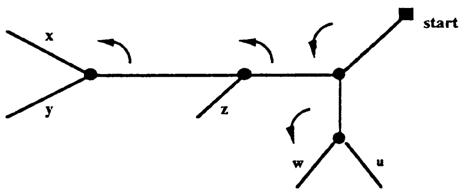
We can now define a sequence of moves on cyclically ordered graphs: For a cyclically ordered graph, X, pick a 4-legged subgraph, ${>}\!\!-\!\!{<}$. Then pick an element $b \in G({>}\!\!-\!\!{<})$ and move to the graph bX. We can also define restricted moves on cyclically ordered graphs. A cyclic order on the nodes of ${>}\!\!-\!\!{<}$ induces a cyclic order on the legs of ${>}\!\!-\!\!{<}$. If X is a cyclically ordered graph and ${>}\!\!-\!\!{<}$ some 4-legged subgraph then ${>}\!\!-\!\!{<}$ is cyclically ordered and therefore determines an element $a \in$G(${>}\!\!-\!\!{<}$). Apply this preferred element a to X and move it to aX. Then choose another 4-legged subgraph etc.

The mathematical problem that we are faced with is then to describe the relations among the moves. That is describe those sequences of moves that lead from X back to the same graph X.

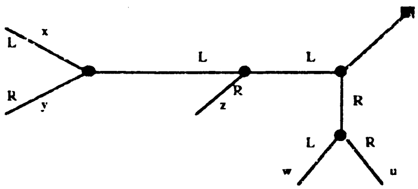
3. ROOTED TREES, TENSOR PRODUCTS AND COHERENCE

In this section we show that a cyclically ordered rooted tree is a scheme for forming tensor products. We also show that moves on a rooted tree correspond to associativity and commutativity operators in the sense of Mac Lane, and hence we can apply the coherence theorem of Mac Lane and its generalization[10] to determine all the relations among moves on cyclically ordered rooted trees.

Suppose that we are given a rooted tree. We can regard the cyclic order on the nodes as a "sorting procedure" on the remaining legs. Start from the root (the marked leg). Mark the first edge encountered in the cyclic order by L (for left) and the second edge by R (for right). The interpretation is the following: All the legs connected to the edge marked L will be placed to the left of all the legs attached to the edge marked R. If the edge marked L is not a leg, continue to its other joining node and repeat the procedure of marking the first edge encountered in the cyclic order by L and the second by R. After that do the same for the edge marked R if its not a leg. Continue until all the edges are marked. Each interior node can be regarded as a tensor product and hence each interior edge as a parenthesis system. For example the following rooted tree

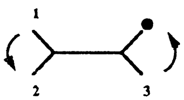


is marked as

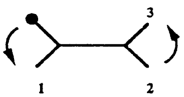


and corresponds to the tensor product $((x \otimes y) \otimes z) \otimes (w \otimes u)$.

Now consider the 4-legged graph



where we have marked the leg 4. It corresponds to $(1 \otimes 2) \otimes 3$. The operator a=(1234) applied to the graph yields



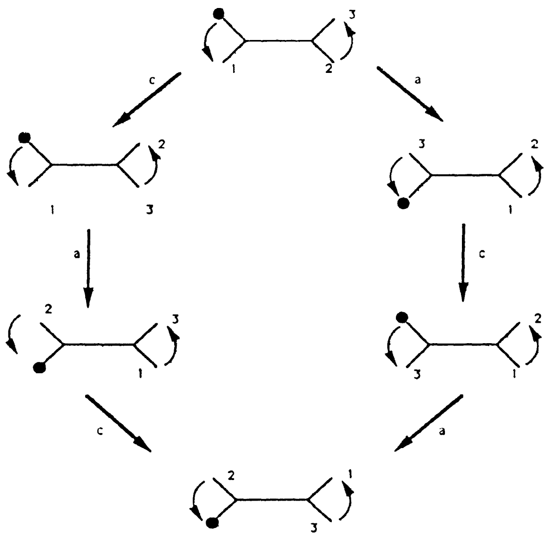
which corresponds to $1 \otimes (2 \otimes 3)$. Thus a is the "formal associativity operator".

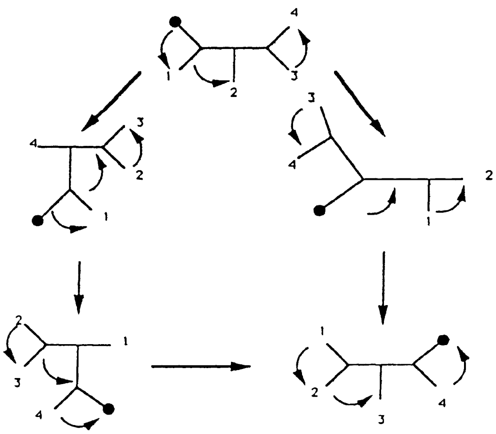
If $>\!\!-\!\!-\!\!<$ is an interior subgraph of a rooted tree, X, one and only one of its legs will lead to the root, since X is a tree, and we may consider this leg of $>\!\!-\!\!-\!\!<$ as marked. Hence every restricted move on a cyclically ordered rooted tree is an instance of associativity in the terminology of ref.[8]. We can therefore read the relations between restricted moves on rooted trees from Mac Lane's paper. They are generated by relations of two types:

1) naturality, which asserts that moves associated to disjoint subtrees commute.

2) a pentagon identity which relates the two paths going from $1 \otimes (2 \otimes (3 \otimes 4))$ to $((1 \otimes 2) \otimes 3) \otimes 4$, One in two moves and the other in three moves.

We can also describe a formal commutativity operator in terms of moves on graphs. On any rooted tree there exists a unique transposition in $G(>\!\!-\!\!<)$, the transposition of the two legs not joined to the marked edge. This transposition clearly corresponds to an instance of commutativity, c, in the tensor product interpretation. We have to distinguish between the cases, $c^2 = 1$ and $c^2 \neq 1$. The former was considered by Mac Lane and the latter by Joyal and Street. In the case that $c^2 = 1$ we have (in addition to naturality for commutativity) one hexagon identity involving associativity and commutativity. In the case $c^2 \neq 1$ there is one more hexagon identity involving c^{-1} instead of c. In terms of our graphs the pentagon and hexagon identities are:





The mathematical framework in which these identities were found is that of monoidal categories or more generally braided monoidal categories. The conditions on the operators a and c ensuring the existence of a unique natural transformation between any two n-fold tensor products are those encoded by the pentagon and hexagon identities.

4. MOVES ON LABELED GRAPHS

Let the group G act on the set P. Let fun(P,L) denote the set of all functions from P to L, i.e. the set of all "labels" of elements of P by elements of L. Then the group G acts on fun(P,L) by the rule $(bf)(p)=f(b^{-1}p)$. Let F be a finite set, a vector bundle E→F is a rule which associates a vector space E_f to each f∈F. Suppose now that we are given an action of G on F. Then an action of G on E (consistent with the given action on F) is a rule which associates to each b ∈G and to each f ∈F a linear transformation $b : E_f \rightarrow E_{bf}$. The identity of G gives the identity transformation for each E_f and $E_f \rightarrow E_{bf} \rightarrow E_{cbf}$ equals $E_f \rightarrow E_{cbf}$. Suppose that all the E_f have the same dimension and have been identified with a fixed vector space V. Then $b : E_f \rightarrow E_{bf}$ can be identified with a linear transformation $A(f,b) : V \rightarrow V$. The consistency condition that we have a group action the becomes

$$A(bf,c)A(f,b) = A(f,cb) \tag{4}$$

If V has a preferred basis the A's become matrices and Eq.(4) becomes a matrix equation.

Now consider a cyclically ordered graph X in which we have labelled all the edges from a finite set L. Let $\succ\!\!\!-\!\!\!\prec$ be a 4-legged sub diagram of X and $b \in G(\succ\!\!\!-\!\!\!\prec)$. Then each edge, e, of the graph bX other than the new central connecting edge corresponds to an edge of X, and hence carries a label. Thus we label all these edges according to our standard rule, $(bh)(e) = h(b^{-1}e)$. For the central edge we proceed as follows: We consider the vector bundle over the space of labels on the legs of $\succ\!\!\!-\!\!\!\prec$ whose fiber is C^L.

We assume that we are given an action of G on this vector bundle. In other words, to each $b \in G(\succ\!\!\!-\!\!\!\prec)$ and to each label f on the legs of $\succ\!\!\!-\!\!\!\prec$ we are given a matrix $A(f,b)_{pq}$ $p,q \in L$ satisfying Eq.(4).

Let $(bX)_q$ denote the graph bX with the label q on the new central edge with all the other central edges labelled as above. Then we move X into $\sum_q A(f,b)_{qp}(bX)_q$. We have moved a labelled graph X into a linear combination of labelled graphs. Thus our moves are on the space of linear combinations of labelled graphs.

Notice that after a finite number of steps the sum is over the various internal labels, but not over the labels of the external legs. So we once again obtain consistency relations, since we get from one labelled graph to a sum of others with the same external legs by several routes, and the two sums over the internal labels must be equal. This imposes a set of constraints on the matrices A, above and beyond Eq.(4). These constraints were discussed previously.

5. BACK TO CFT'S

After a long digression we return to the crossing matrices of RCFT. Equations (2),(3) and their pictorial representation show that the crossing matrices A,C are nothing but matrix representation of the associativity operator a, and the commutativity operator c. They obey the consistency condition Eq.(4). A special case of this equation is the expression for the braiding matrix B in terms of A and C, B=CAC. The finite set of labels in this case is the set of conformal families and the vector bundle is span by the collection of conformal blocks.

Note however that in the case of RCFT $c^2 \neq 1$, since it follows from Eq.(3) that C^2 is a scalar matrix consisting of a non trivial phase factor. Hence the relevant coherence theorem is the one of Joyal and Street. For other physical systems such as the factorizable S-matrix theory[11] it seems that the Mac Lane picture is the relevant one.

We have to justify the fact that the same matrices A and C represent the associativity and commutativity operators inside higher order graphs. For this we need to know that the conformal blocks with descendants as external legs transform with the same matrices A and C. In addition we need to know that we can isolate a conformal block of the 4-point function inside a higher order functions and operate on it with the usual operations. This is a non-trivial fact since in general the blocks of the n-point function do not contain any of the 4-point function blocks as factors. Nevertheless This point can be proved by using the operator product expansion in the n-point function and by summing and desumming over descendants[12]. The first point can be proved by using the linear relation between blocks with external descendants and external primaries[6].

Now the identification of crossing operations on conformal blocks with our general formalism of moves on graphs is complete. The coherence theorem implies that the crossing matrices have to satisfy a generating set of equations which reads as follows:

$$\sum_p A\begin{bmatrix} 3 & 4 \\ r & 5 \end{bmatrix}_{sp} A\begin{bmatrix} 2 & p \\ 1 & 5 \end{bmatrix}_{rq} A\begin{bmatrix} 2 & 3 \\ q & 4 \end{bmatrix}_{pt} = A\begin{bmatrix} 2 & 3 \\ 1 & s \end{bmatrix}_{rt} A\begin{bmatrix} t & 4 \\ 1 & 5 \end{bmatrix}_{sq} \tag{5}$$

and

$$\sum_q A\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}_{pq} C\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}_{qq} A\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}_{qr} = C\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}_{pp} A\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}_{pr} C\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}_{rr} \tag{6}$$

and the corresponding equation with C^{-1}.

Equations (5) and (6) are a highly over-determined set of coupled algebraic equations. These equations are not regular matrix equations because the indices and the names of the matrices mix. To the best of our knowledge there is no systematic approach to search for solutions of these equations, although it has been checked for some RCFT's that crossing matrices do obey these equations.

References

1. E. Verlinde, Nucl. Phys. **B300** (1988) 360.
2. K. H. Rehren and B. Schroer, FU preprint.
3. G. Moore and N. Seiberg, Phys. Lett. **B212** (1988) 451.
4. R. Brustein, S. Yankielowicz and J.B. Zuber, Nucl. Phys. **B313** 321.
5. G. Moore and N. Seiberg, Comm. Math. Phys. 123, 117
6. A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333.
7. V. S. Dotsenko and V. Fateev, Nucl. Phys. **B240** (1984) 312.
8. S. Mac Lane, Rice Univ. Studies **49** (1963) 28.;
   see also G. M. Kelly. J. of Alg. **1** (1964) 397 and J. D. Stasheff, Trans. Am. Math. Soc. **108** (1963) 275.

9. R. Brustein, Y. Neeman and S. Sternberg, to appear in Isr. J. of Math.
10. A. Joyal and R. Street, Macquarie Math. Reports, 1986;
P. Freyd and D. Yetter, Advances in Mathematics.
11. E. Corrigan, these proceedings.
12. S. Yankielowicz, Private communication;
R. Brustein, Ph. D. Thesis, Tel Aviv Univ. (1988), Unpublished

QUANTUM BÄCKLUND TRANSFORMATIONS AND CONFORMAL ALGEBRAS

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ABSTRACT

A quantum Bäcklund transformation can provide a mapping from one realization of su(1,1), and any of its quantum deformations or Virasoro algebra extensions, to another. Nontrivial realizations can be related to trivial ones. Since the quantum transformation is not necessarily unitary, the two realizations can have different hermiticity properties when acting on arbitrary states. These general features are illustrated in detail using the Liouville field theory in 1+1 dimensions, and a simple potential model.

Bäcklund transformations provide maps between various fields, say $\Phi \longleftrightarrow \varphi$, and hence relate functionals of one field to those of another, $Q[\Phi] \longleftrightarrow q[\varphi]$. In particular, if Q is a conserved charge then the Bäcklund transformation provides a natural re-expression of the symmetry of the Φ system in terms of φ, and vice versa. All this is more or less well-known for classical field theories. For example, see [1] and references therein.

In this talk I wish to briefly explore these ideas in the context of quantum theories. First, I quickly summarize the use of the Bäcklund transformation for quantum fields, emphasizing the Liouville theory in two-spacetime dimensions as an interesting example. Then I consider some of the effects of quantum Bäcklund transformations on conformal symmetry. My comments are based on research carried out intermittently during the last few years with various collaborators, most often with Dr. Ghassan Ghandour and most recently with my student Thomas McCarty. Unfortunately, many aspects of this research are still incomplete. Accordingly I will limit most of my discussion of conformal transformations to a simple potential model. Even so, this elementary example permits some interesting quantum effects to be explained.

The canonical example of a Bäcklund transformation is the standard map relating the Liouville field in two spacetime dimensions, $\Phi(\sigma,\tau)$, to a free pseudoscalar field, $\varphi(\sigma,\tau)$. The transformation is defined by a pair of first-order nonlinear equations.

$$\partial_\sigma \varphi - e^{\Phi} \sinh\varphi = \partial_\tau \Phi \equiv \Pi\ , \qquad \partial_\sigma \Phi + e^{\Phi} \cosh\varphi = \partial_\tau \varphi \equiv \pi\ . \tag{1}$$

The integrability conditions for this pair are the well-known Liouville and free field wave equations.

$$(\partial_\tau^2 - \partial_\sigma^2)\,\Phi + e^{2\Phi} = 0\,, \qquad\qquad (\partial_\tau^2 - \partial_\sigma^2)\,\varphi = 0\,. \tag{2}$$

There are several good reasons for referring to this system as *the* canonical example of a Bäcklund transformation. For instance, it *is* a canonical transformation. A generating functional for the transformation at any fixed τ is given by

$$F[\Phi,\varphi] = \int d\sigma \left[\, \Phi\, \partial_\sigma \varphi - e^{\Phi} \sinh\varphi \,\right]. \tag{3}$$

Canonical momenta are determined from $F[\Phi,\varphi]$ in the standard fashion through functional derivatives at fixed τ.

$$\Pi = \delta F/\delta\Phi\,, \qquad\qquad \pi = -\,\delta F/\delta\varphi\,. \tag{4}$$

Evaluation of these two derivatives reproduces the original pair of equations in (1).

For the classical field theory, this is essentially the whole story. It is now possible to solve for $\exp(\Phi)$, thence Π, and thus obtain expressions for functionals of the Liouville field variables in terms of free field variables, $G[\Phi,\Pi] = g[\varphi,\pi]$, at any fixed τ. For the quantum theory, however, all this is essentially just a starting point.

One way to proceed with the quantum theory would be to use operator methods to convert the classical expressions for $g[\varphi,\pi]$ into well-defined free field operator expressions in such a way that they have the locality and conformal transformation properties that would be expected to hold given the original $G[\Phi,\Pi]$ as equal-time expressions involving canonical, interacting fields. This is a fairly complicated procedure, but it can be carried through [2].

Another, equivalent way to proceed is to use the Schrödinger equal-time functional formalism [3]. This is perhaps a more natural approach in view of the fundamental functional derivative relations in (4). However, this approach requires further knowledge of the proper quantum interpretation of the classical generating functional for the Bäcklund transformation, and especially any quantum corrections to it. Of course, a first step towards this proper interpretation is simply to exponentiate the classical generating functional, up to factors of i and $\hbar$, as we have all learned to do since childhood for other well-known cases such as the action. But then what?

Evidently, such an exponential of $F[\Phi,\varphi]$ acts on a tensor product of two different functional spaces: one for Φ, the other for φ. So a natural way to think of $\exp(iF[\Phi,\varphi])$ is as a transformation functional from one space to the other. With that interpretation in mind, an obvious next step is to determine the invariants of this transformation functional. Thus we are led to consider the effects of various symmetry transformations on $\exp(iF[\Phi,\varphi])$. Now, a priori there are two ways to carry out such symmetry transformations, by acting either on Φ or on φ. Clearly, if possible we should compare the effects of these two types of symmetry transformations.

In particular, we may consider the effect of time translations on $\exp(iF[\Phi,\varphi])$. We may do this by comparing the effect on $\exp(iF[\Phi,\varphi])$ of the Hamiltonian for Φ with that for φ, where each Hamiltonian is expressed in the Schrödinger functional formalism. That is,

$$H[\Phi] = \int d\sigma \left[\, \left(\frac{\delta}{\delta\Phi}\right)^2 + (\partial_\sigma \Phi)^2 + e^{2\Phi} \,\right], \tag{5}$$

$$h[\varphi] = \int d\sigma \left[\, \left(\frac{\delta}{\delta\varphi}\right)^2 + (\partial_\sigma \varphi)^2 \,\right]. \tag{6}$$

Now, it is remarkable for the Liouville theory that both these Hamiltonians have the same effect on $\exp(iF[\Phi,\varphi])$.

$$H[\Phi]\, \exp(iF[\Phi,\varphi]) = h[\varphi]\, \exp(iF[\Phi,\varphi])\,. \tag{7}$$

In this sense, the energy is an invariant of the transformation functional.

This invariance property immediately allows the construction of energy eigenfunctionals for the Liouville theory in terms of those for the free field, essentially as Schrödinger wave functional transforms. Explicitly,

$$\Psi_E[\Phi] = \int \mathscr{D}\varphi \exp(iF[\Phi,\varphi])\, \psi_E[\varphi] \,. \tag{8}$$

Using equation (7) and functionally integrating by parts, one immediately checks that $H[\Phi]\,\Psi_E[\Phi] = E\,\Psi_E[\Phi]$ if and only if $h[\varphi]\,\psi_E[\varphi] = E\,\psi_E[\varphi]$. (N.B. the integration in (8) is a functional integral over all field configurations at a fixed time, and is *not* a path integral.)

Ghandour [4] has elucidated the above concepts by investigating the analogue of (7) in the context of various quantum mechanical models. In general, beginning with a classical canonical transformation, equation (7) does not hold unless quantum corrections are added to the generating functional. For the Liouville theory, it is fortunately not necessary to make quantum modifications to the generating functional. Of course, in making such a statement I have been somewhat cavalier regarding ultraviolet divergences. A more precise statement is that renormalizations *are* required, even for the Liouville case, but the form of the generating functional is unchanged from the classical form.

Renormalization effects for the functional transform may be seen by perturbatively calculating various matrix elements and correlation functions for the Liouville theory by using functional methods. For example, consider

$$\langle\Psi_1|\exp(\alpha\Phi)|\Psi_2\rangle = \int \mathscr{D}\Phi\ \Psi_1^*[\Phi] \exp(\alpha\Phi)\ \Psi_2[\Phi] \,. \tag{9}$$

Again, the RHS is a functional integral over all field configurations at fixed time (*not* a path integral). Replacing Ψ's on the RHS using (8), and taking the usual free field Gaussians, it is straightforward to expand $\exp(iF[\Phi,\varphi])$ and obtain a series of Gaussian functional integrals. Evaluating these, using a momentum cut–off, it is possible to reproduce the perturbation theory results obtained earlier by operator methods [5]. Details of these calculations will appear elsewhere [6].

In principle, one might also hope to apply nonperturbative (lattice?) methods to evaluate the functional integrals in (9). This remains an open problem.

Actually, there are two related important points that should be clarified about (7) and (8). Viewed as a defining equation for $T[\Phi,\varphi] \equiv \exp(iF[\Phi,\varphi])$, (7) obviously does *not* have a unique solution. Any functional of $H[\Phi]$, or $h[\varphi]$, acting on $\exp(iF[\Phi,\varphi])$ also gives a solution to the equation $H[\Phi]\,T[\Phi,\varphi] = h[\varphi]\,T[\Phi,\varphi]$. This ambiguity carries over to (8). The RHS of that equation could be multiplied by an energy–dependent normalization and phase. Thus, $\Psi_E[\Phi]$ as given by (8) is not necessarily normalized the same as $\psi_E[\varphi]$. That is, the transformation generated by the transformation functional $\exp(iF[\Phi,\varphi])$ is not a unitary transformation, but rather only a similarity transformation. I will say more about this later. For now, I remark in passing that such ambiguities must be taken into account in order to obtain the correct results for matrix elements such as in (9). This is straightforwardly done.

Let us return to consider the effects of other symmetries on the quantum Bäcklund transformation, i.e. on the transformation functional $\exp(iF[\Phi,\varphi])$. Quite generally, the action of any functional in terms of Φ is re–expressed in terms of a corresponding action in terms of φ, and vice versa, simply by acting on $\exp(iF[\Phi,\varphi])$. Schematically,

$$Q[\Phi] \exp(iF[\Phi,\varphi]) = q[\varphi] \exp(iF[\Phi,\varphi]) \,. \tag{10}$$

Although my experience is that the results are simple only for genuine symmetries of either the Φ or φ theories, a more systematic analysis of the general problem has not been given. For now, let us consider only the obvious symmetries. In particular, consider all the charges in either the su(1,1) which includes the Hamiltonian, or the full extension of that su(1,1) to the Virasoro algebra.

The conformal properties of the transformation functional for the Liouville theory, $\exp(iF[\Phi,\varphi])$, or the wave functional in (8), are actually more fully understood by exponentiating and considering the conformal group, rather than just the algebra. (This point of view has also been emphasized by Jackiw [7] in a more general context for two dimensional field theories.) In particular, this facilitates handling the ultraviolet properties of the theory. Unfortunately I must postpone a discussion of the conformal group structure for the Liouville theory. The analysis is unfinished [6]. In this talk, it will suffice to consider the conformal transformation properties of a simple quantum mechanical system, some of whose properties are similar to those of the Liouville theory.

For example, consider the well–known first–order "angular" realization of su(1,1) and its extension to the Virasoro algebra for a free quantum particle moving on a circle.

$$\ell_n = e^{in\theta}\,(i\,\partial_\theta - n/2 + \beta + i\,\gamma\,n\,)\,. \tag{11}$$

This realization contains two parameters, β and γ. For any choice of these parameters, the ℓ_n's realize the centerless algebra

$$[\ell_n,\ell_m] = (n-m)\,\ell_{n+m}\,. \tag{12}$$

If β and γ are both real parameters, the ℓ_n's also obey the hermiticity conditions

$${\ell_n}^{\dagger} = \ell_{-n}\,, \tag{13}$$

when acting on arbitrary periodic functions of θ.

The well–known quadratic Casimir invariant for the su(1,1) subalgebra generated by $\ell_{\pm1}$ and ℓ_0 is

$$C[\theta] = \ell_{+1}\,\ell_{-1} - \ell_0\,(\ell_0 + 1) = \gamma^2 + 1/4\,. \tag{14}$$

Note that this is independent of β. More importantly, for real γ as required by the hermiticity properties in (13), note that $C[\theta] \geq 1/4$.

Next, consider the slightly more unusual, second–order realization of su(1,1) [8] in terms of the "radial" variable r.

$$L_0 = -\,\partial_r^2 - \alpha/r^2 + r^2/16\,, \tag{15}$$

$$L_{\pm1} = -\,\partial_r^2 - \alpha/r^2 - r^2/16 \mp \tfrac{1}{2}\,(\,r\,\partial_r + 1/2\,)\,. \tag{16}$$

This realization contains a single parameter α. The quadratic Casimir is now

$$C[r] = L_{+1}\,L_{-1} - L_0\,(L_0 + 1) = (3 + 4\alpha)/16\,. \tag{17}$$

In contrast to the previous θ realization, note that $C[r] \leq 1/4$ for real $\alpha \leq 1/4$.

The angular and radial realizations of su(1,1), while quite different in appearance, are very directly related. There exists a quantum Bäcklund transformation which interchanges the two realizations. Of course, as easily anticipated for the case of single particle models in one space dimension as opposed to two–dimensional field theories, this quantum Bäcklund transformation is a fairly simple canonical transformation relating r and θ. Nevertheless, there are several parallels with the Liouville $\longleftrightarrow$ free field Bäcklund transformation discussed above. For example, it is useful to think of this $r \longleftrightarrow \theta$ transformation as a map from an interacting theory for a particle moving in an r–dependent potential, with Hamiltonian given by $H = L_0$, to a free theory for a particle moving on the circle, with Hamiltonian given by $h = (-)\,\ell_0$.

For the interacting r–theory, note that $\alpha > 1/4$ corresponds to an attractive radial potential which is strong enough that a particle would "fall into the origin" [9]. This simple physical constraint therefore requires $\alpha \lesssim 1/4$, and hence the su(1,1) Casimir for the r–theory must also be $\lesssim 1/4$, as mentioned previously. Therefore, for $\alpha \lesssim 1/4$ we see immediately a possible conflict dictated by the physics of the θ– and r–theories. The hermiticity properties of (13) imposed on the results in (14) exclude the allowed physical range for the Casimir of the r–theory. Fortunately, this conflict is avoided because the transformation between the r– and θ–theories is a similarity but *not* a unitary transformation, as I explain further below.

A generator of the transformation between r– and θ–theories is given by

$$F(r,\theta) = \beta\,\theta - \tfrac{i}{2}\,\lambda\,\ln(r) + \tfrac{1}{8}\,r^2\cot(\theta/2) + \tfrac{i}{2}\,(\lambda+1)\,\ln(\sin(\theta/2))\,, \tag{18}$$

where λ is related to the parameter α. In fact, there are two possible choices for λ.

$$\lambda_\pm(\alpha) = 1 \pm \sqrt{1 - 4\alpha}\ . \tag{19}$$

These values are determined by the requirement that

$$L_0\, e^{iF(r,\theta)} = \ell_0\, e^{iF(r,\theta)}\,, \tag{20}$$

or equivalently, that

$$(\partial_r F)^2 - i\,\partial_r^2 F - \alpha/r^2 + r^2/16 = -\,\partial_\theta F + \beta\,. \tag{21}$$

This immediately leads to $(2-\lambda)\lambda = 4\alpha$, for which we obtain

$$L_0\, e^{iF(r,\theta)} = \ell_0\, e^{iF(r,\theta)} = \frac{1}{16\ \sin^2(\theta/2)}\left[\,r^2 - 2i\,(1+\lambda)\,\sin\theta\,\right] e^{iF(r,\theta)}\,. \tag{22}$$

However, (20) does not determine the parameter β. One way to fix β is to consider the relation between r– and θ–dependent wave functions.

As in the Liouville field theory case, in this quantum mechanical example such a relation between wave functions is given by a "nonlinear" Fourier transform, i.e. an integral transform whose kernel is the transformation function exp(iF).

$$\Psi_E(r) = \mathscr{N}_E \int_0^{2\pi} d\theta\, e^{iF(r,\theta)}\,\psi_E(\theta)\,, \tag{23}$$

where $\mathscr{N}_E$ is an energy–dependent normalization as discussed above. The transformation is easily seen to be invertible. For real β and λ

$$\psi_E(\theta) = \mathscr{N}_E \int_0^{\infty} dr\, e^{-iF^*(r,\theta)}\,\Psi_E(r)\,. \tag{24}$$

Another statement of this is

$$L_0\, e^{-iF^*(r,\theta)} = (2\beta - \ell_0)\, e^{-iF^*(r,\theta)}\,. \tag{25}$$

At first sight, this may seem a bit odd but it is consistent with (20), (23), and (24). Upon integrating by parts, (20) and (23) give

$$L_0\,\Psi_E(r) = \mathcal{N}_E \int_0^{2\pi} d\theta\, e^{iF(r,\theta)} \left[(2\beta - \ell_0)\, \psi_E(\theta) \right]$$

$$+ i\, \mathcal{N}_E \left[e^{iF(r,2\pi)}\, \psi_E(2\pi) - e^{iF(r,0)}\, \psi_E(0) \right] . \tag{26}$$

The action of L_0 on r–dependent wave functions is thereby given in terms of the corresponding action of ℓ_0 on θ–dependent wave functions. In particular, if

$$e^{iF(r,2\pi)}\, \psi_E(2\pi) = e^{iF(r,0)}\, \psi_E(0) , \tag{27}$$

then an eigenfunction of ℓ_0 is mapped onto an eigenfunction of L_0.

It is straightforward to determine the spectrum of L_0, say by the usual method of series solutions. Imposing boundary conditions $\Psi(r) \sim r^{\lambda/2}$ as $r \to 0$ and $\Psi(r) \sim \exp(-r^2/8)$ as $r \to \infty$, we find

$$\sigma(L_0) = \{ \tfrac{1}{4}\, (\lambda+1) + n \mid n \in \mathbb{Z}_+ \} . \tag{28}$$

For example, the simplest case is $n = 0$, which corresponds to the "lowest weight" eigenfunction of L_0 . The corresponding eigenfunction and eigenvalue equations are

$$\Psi_0(r) = r^{\lambda/2}\, e^{-r^2/8} , \quad L_0\, \Psi_0(r) = \tfrac{1}{4}\,(1+\lambda)\, \Psi_0(r) , \quad L_1\, \Psi_0(r) = 0 . \tag{29}$$

Now to map this wave function onto a single–valued eigenfunction of ℓ_0, i.e. one of

$$\psi_n(\theta) = e^{in\theta} , \; n \in \mathbb{Z} , \tag{30}$$

it is necessary to choose $\beta = (1+\lambda)/4 + N$. For simplicity, we take $N = 0$.

Admittedly, I have not been very careful about the branch points in exp(iF), or the validity of (27). In this short talk, I am content to leave such matters as an exercise for those interested. Here it will suffice to consider the simplest possibility to understand what is going on. Let us transform $\Psi_0(r)$ and see if a single–valued function of θ is the result.

$$\int_0^\infty dr\, e^{-iF^*(r,\theta)}\, \Psi_0(r) =$$

$$= \int_0^\infty dr\, e^{-i\beta\theta}\, r^\lambda\, (\sin(\theta/2))^{-(\lambda+1)/2} \exp\left[-\tfrac{i}{8} r^2 \left[\cot(\theta/2) - i \right] \right]$$

$$= e^{-i\beta\theta} \int_0^\infty dr\, r^\lambda\, (\sin(\theta/2))^{-(\lambda+1)/2} \exp\left[-\tfrac{i}{8} r^2 \frac{e^{-i\theta/2}}{\sin(\theta/2)} \right]$$

$$= e^{-i\beta\theta}\, e^{i(\lambda+1)\theta/4} \int_0^{\infty\, e^{-i\theta/4}} du\, u^\lambda\, \exp\left[-\tfrac{i}{8} u^2 \right] \tag{31}$$

where I have changed variables to $u = r\, e^{-i\theta/4}/\sqrt{\sin(\theta/2)}$. Thus we see that all fractional powers of $\exp(i\theta)$ are trivially eliminated from the result, for arbitrary λ, if $\beta = (\lambda+1)/4 + N$, where $N \in \mathbb{Z}$.

So, let us choose

$$\beta = (1+\lambda)/4 \, , \tag{32}$$

and move on to consider the effects of the other su(1,1) generators acting on the transformation functional.

By using the right–most result in (22) it is tedious but straightforward to show that

$$L_{\pm 1}\, e^{iF(r,\theta)} = \ell_{\mp 1}\, e^{iF(r,\theta)} \tag{33}$$

if and only if

$$\gamma = i\,(\lambda - 1)/4 = \pm \tfrac{1}{4}\, i \sqrt{1 - 4\alpha} \, . \tag{34}$$

Thus β, γ, and λ are expressed in terms of α, and we have only one free parameter in the expressions for the conformal charges and the generating function. In particular, note that

$$\beta = \tfrac{1}{2} - i\,\gamma \, . \tag{35}$$

Remarkably, the result for γ is purely imaginary for $\alpha \leq 1/4$, and results in the equality of the r– and θ–Casimirs. Collecting a variety of results for the Casimir, we have

$$\begin{aligned} C[\theta] = C[r] &= L_{-1}\, L_{+1} + L_0\,(1 - L_0) \\ &= \gamma^2 + 1/4 \qquad\qquad = \beta\,(1-\beta) \\ &= (3 + 4\alpha)/16 \qquad = (3-\lambda)(1+\lambda)/16 \, . \end{aligned} \tag{36}$$

Note how the $\pm$ subscripts on the LHS of (33) become $\mp$ on the RHS. This is necessary to be consistent with the su(1,1) commutators, given (20). It is also necessary so that both the r– and θ–Casimirs have the same effect on exp(iF). That is,

$$\begin{aligned} &L_{+1}\, L_{-1}\, e^{iF} = L_{+1}\, \ell_{+1}\, e^{iF} \\ &= \ell_{+1}\, L_{+1}\, e^{iF} = \ell_{+1}\, \ell_{-1}\, e^{iF} \, . \end{aligned} \tag{37}$$

The second line in (37) follows because the L's and ℓ's trivially commute.

Now let us go back to the problem mentioned earlier about the incompatibility of the Casimirs in the r– and θ–realizations. The transformation functional, exp(iF), sidesteps the problem by selecting a purely imaginary value for γ which is manifestly incompatible with the hermiticity properties stated in (13). At the same time, however, the transformation preserves all commutation relations and other operator identities. Obviously, this is possible because the transformation is not unitary when acting on arbitrary states, but only a similarity transformation. The $r \longmapsto \theta$ map differs from a unitary transformation through the presence of L_0–dependent factors. We discussed this possibility above in pointing out the ambiguities in exp(iF). I have made some further general remarks about this structure in the Appendix. Although the final form of the θ–dependent Virasoro generators does not obey the general hermiticity constraints given in (13), evidently there is no problem with unitarity in the model since the space of θ–dependent functions has been restricted. In particular, only $n \geq 0$ is allowed for the single–valued functions in (30) or else the conditions on the spectrum in (28) would fail.

The situation here is *very* reminiscent of what occurs in two–dimensional field theory when the "Coulomb gas" is mapped onto the conformal series with central charge less than one [10]. This similarity will be discussed elsewhere.

Next, consider the action of the full Virasoro algebra on the generating functional. This is easy to do for potential models, at least formally, because the full algebra can be realized in terms of rational functions of the su(1,1) generators [11]. Thus for $n \geq 0$

$$L_n = (L_0 + n\beta)\,\frac{\Gamma(L_0 + \beta)}{\Gamma(L_0 + \beta + n)}\,(L_1)^n\,, \tag{38}$$

$$L_{-n} = (L_0 - n\beta)\,\frac{\Gamma(L_0 + 1 - \beta - n)}{\Gamma(L_0 + 1 - \beta)}\,(L_{-1})^n\,. \tag{39}$$

Due to the structure of the Casimir, $C = \beta(1-\beta)$, we may interchange $\beta \longleftrightarrow 1-\beta$ in these expressions. This will be useful below.

Acting with L_n as given in (38) on the transformation functional gives

$$\begin{aligned}
L_n\, e^{iF} &= (L_0 + n\beta)\,\frac{\Gamma(L_0 + \beta)}{\Gamma(L_0 + \beta + n)}\,(L_1)^n\, e^{iF} \\
&= (L_0 + n\beta)\,\frac{\Gamma(L_0 + \beta)}{\Gamma(L_0 + \beta + n)}\,(\ell_{-1})^n\, e^{iF} \\
&= (\ell_{-1})^n\,(L_0 + n\beta)\,\frac{\Gamma(L_0 + \beta)}{\Gamma(L_0 + \beta + n)}\, e^{iF} \\
&= (\ell_{-1})^n\,(\ell_0 + n\beta)\,\frac{\Gamma(\ell_0 + \beta)}{\Gamma(\ell_0 + \beta + n)}\, e^{iF} \\
&= (\ell_0 - n + n\beta)\,\frac{\Gamma(\ell_0 - n + \beta)}{\Gamma(\ell_0 + \beta)}\,(\ell_{-1})^n\, e^{iF}\,.
\end{aligned} \tag{40}$$

That is, for $n \geq 0$,

$$L_n\, e^{iF} = \ell_{-n}\, e^{iF}\,, \tag{41}$$

$$\ell_{-n} = (\ell_0 - n + n\beta)\,\frac{\Gamma(\ell_0 + \beta - n)}{\Gamma(\ell_0 + \beta)}\,(\ell_{-1})^n\,. \tag{42}$$

A similar calculation beginning with (39) gives

$$L_{-n}\, e^{iF} = \ell_n\, e^{iF}\,, \tag{43}$$

$$\ell_n = (\ell_0 + n - n\beta)\,\frac{\Gamma(\ell_0 + 1 - \beta)}{\Gamma(\ell_0 + 1 - \beta + n)}\,(\ell_1)^n\,. \tag{44}$$

Of course, it should be noted that these expressions for ℓ_n and ℓ_{-n} reduce to (11) upon substituting the explicit forms for ℓ_0 and $\ell_{\pm 1}$. Also note that it is again necessary (recall (37)) that the $r \longleftrightarrow \theta$ transformation functional changes the sign of the subscript on all the Virasoro generators to be consistent with the commutator algebra.

In my view, the results in (41) and (43) are an especially nice feature of the $r \mapsto \theta$ map. The transformation functional naturally *defines* an extension of the second–order radial realization of su(1,1) to include the full Virasoro algebra. This follows immediately from the usual extension of the angular realization.

As a final point, we consider the quantum deformation [12] of su(1,1) acting on exp(iF). This can be realized in a manner similar to the Virasoro algebra in (38) and (39). Define

$$\mathscr{L}_{\pm 1} = \left[\frac{q(L_0 \pm \beta)}{(L_0 \pm \beta)} \frac{q(L_0 \pm 1 \mp \beta)}{(L_0 \pm 1 \mp \beta)} \right]^{1/2} L_{\pm 1} , \tag{45}$$

where $q(x) \equiv (q^x - q^{-x})/(q - q^{-1})$. These realize the "deformed" commutation relations

$$[\mathscr{L}_{+1}, \mathscr{L}_{-1}] = q(2L_0), \qquad [\mathscr{L}_{\pm 1}, L_0] = \pm\, \mathscr{L}_{\pm 1} . \tag{46}$$

Since the construction in (45) realizes the algebra (46) for any realization of su(1,1) with Casimir $C = \beta(1-\beta)$, it also may be used to define $\not{\ell}_{\pm 1}$ in terms of the ℓ's. Following steps similar to those in (40), we then easily obtain the result

$$\mathscr{L}_{\pm 1}\, e^{iF} = \not{\ell}_{\mp 1}\, e^{iF} . \tag{47}$$

In conclusion, allow me to advertise a bit. The above ideas are being pursued in the context of quantum field theory, particularly the Liouville theory, and in non–relativistic many body systems [6]. In the latter context, quantum Bäcklund transformations may be useful to relate models with different statistics, and perhaps with different central charges. (See [13] for a clear discussion of many body systems with pure $1/r^2$ two–body potentials.)

APPENDIX:

Consider the problem of relating two realizations of an arbitrary Lie algebra, including those which are infinite dimensional. Following tradition, I label the maximal commuting set of generators by H_i , $i=1...m$, where m is the rank of the algebra. Then

$$[H_i , H_j] = 0 . \tag{A1}$$

I also assume this Cartan subalgebra is hermitian, $H_i = H_i^\dagger$, and label the other generators by E_α , where α is a root of the algebra. As is conventional in the Cartan–Weyl basis, we have

$$E_\alpha^\dagger = E_{-\alpha} , \tag{A2}$$

$$[H_i , E_\alpha] = \alpha_i\, E_\alpha , \tag{A3}$$

$$[E_\alpha , E_{-\alpha}] = \alpha_i\, H_i , \tag{A4}$$

and if $\alpha \neq -\beta$,

$$[E_\alpha , E_\beta] = N_{\alpha\beta}\, E_{\alpha+\beta} . \tag{A5}$$

Now consider another realization of this same algebra and for clarity denote the generators by h_i and e_α . Assume that these obey the same commutation relations and hermiticity properties. Next denote the eigenstates of the Cartan subalgebra as $|\eta\rangle$ and distinguish the two realizations with subscripts. Thus

$$H_i\,|\eta\rangle_H = \eta_i\,|\eta\rangle_H\,, \qquad\qquad h_i\,|\eta\rangle_h = \eta_i\,|\eta\rangle_h\,. \tag{A6}$$

Finally, assume that the eigenvalues are identical for each realization. Then it is trivial to construct a formal unitary transformation which connects the two realizations.

$$U = \int d\eta\,|\eta\rangle_H\,{}_h\langle\eta|\,, \qquad\qquad U^\dagger = \int d\eta\,|\eta\rangle_h\,{}_H\langle\eta|\,, \tag{A7}$$

where the normalizations ${}_H\langle\bar\eta|\eta\rangle_H = \delta(\bar\eta-\eta) = {}_h\langle\bar\eta|\eta\rangle_h$ give

$$|\eta\rangle_H = U\,|\eta\rangle_h\,, \qquad\qquad |\eta\rangle_h = U^\dagger\,|\eta\rangle_H\,, \tag{A8}$$

and of course, $U\,U^\dagger = U^\dagger\,U = 1$. It follows that all the generators of the two realizations are also related by U.

$$H_i = U\,h_i\,U^\dagger = \int d\eta\,|\eta\rangle_H\,\eta_i\,{}_H\langle\eta|\,, \tag{A9}$$

$$E_\alpha = U\,e_\alpha\,U^\dagger = \int d\eta\int d\bar\eta\,|\eta\rangle_H\,{}_h\langle\eta|e_\alpha|\bar\eta\rangle_h\,{}_H\langle\bar\eta|\,. \tag{A10}$$

Next, extend the construction to include similarity transformations which differ from the unitary transformation through the action of a nonsingular operator which involves only the generators of the Cartan subalgebra. Thus

$$S = N(H)\,U = U\,N(h)\,, \qquad\qquad S^{-1} = \frac{1}{N(h)}\,U^\dagger = U^\dagger\,\frac{1}{N(H)}\,. \tag{A11}$$

The various equalities follow from the explicit form of U and the equality of H and h eigenvalues. It further follows from the explicit form of S that the Cartan subalgebra generators are again directly related by the similarity transformation, since it reduces to the unitary transformation when acting on H or h.

$$S\,h_i\,S^{-1} = U\,N(h)\,h_i\,\frac{1}{N(h)}\,U^\dagger = U\,h_i\,U^\dagger = H_i\,. \tag{A12}$$

On the other hand, when acting on the noncommuting generators, the similarity transformation gives a more interesting effect. Namely

$$\begin{aligned}\bar E_\alpha &\equiv S\,e_\alpha\,S^{-1} = N(H)\,U\,e_\alpha\,U^\dagger\,\frac{1}{N(H)} = N(H)\,E_\alpha\,\frac{1}{N(H)}\\ &= E_\alpha\,N(H+\alpha)\,\frac{1}{N(H)} = N(H)\,\frac{1}{N(H-\alpha)}\,E_\alpha\,.\end{aligned} \tag{A13}$$

In the second line, I used the commutator (A3). Now it is clear that the $\bar E_\alpha$ obey the same commutation relations as the E_α. However, in general they do not have the same hermiticity properties. Instead, we have

$$\begin{aligned}\bar E_\alpha^\dagger &= \frac{1}{N(H)}^\dagger\,E_\alpha^\dagger\,N(H)^\dagger = \frac{1}{N(H)}^\dagger\,E_{-\alpha}\,N(H)^\dagger\\ &= E_{-\alpha}\,\frac{1}{N(H-\alpha)}^\dagger\,N(H)^\dagger = \frac{1}{N(H)}^\dagger\,N(H+\alpha)^\dagger\,E_{-\alpha}\,.\end{aligned} \tag{A14}$$

Alternatively,

$$\bar E_\alpha^\dagger = \frac{1}{|N(H)|^2}\,\bar E_{-\alpha}\,|N(H)|^2 = \left|\frac{N(H+\alpha)}{N(H)}\right|^2\bar E_{-\alpha} = \bar E_{-\alpha}\left|\frac{N(H)}{N(H-\alpha)}\right|^2. \tag{A15}$$

Therefore $\tilde{E}_{\alpha}^{\dagger} = \tilde{E}_{-\alpha}$ if and only if $|N(H)| = |N(H+\alpha)|$ for all roots α . That is, if we write $N(H) = R(H) \exp(i\Phi(H))$ where R and Φ are real functions, then $\tilde{E}_{\alpha}^{\dagger} = \tilde{E}_{-\alpha}$ if and only if $R(H) = \pm R(H+\alpha)$.

The point to emphasize here, in reference to the discussion in the main text, is that quantum Bäcklund transformations in general involve factors $N(H)$ which are *not* simple phases.

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REFERENCES

[1] C. Rogers and W.F. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, Inc., New York, 1982.

[2] E. Braaten, T. Curtright, and C. Thorn, Phys. Lett. **118B** (1982) 115; Ann. Phys. (NY) **147** (1983) 365.

[3] T. Curtright and G. Ghandour, *unpublished.*

[4] G. Ghandour, Phys. Rev. **D35** (1987) 1289.

[5] E. Braaten, T. Curtright, G. Ghandour, and C. Thorn, Phys. Rev. Lett. **51** (1983) 19; Ann. Phys. (NY) (1983) .

[6] T. Curtright, G. Ghandour, and T. McCarty, *in preparation.*

[7] R. Jackiw, MIT 1989 preprint, CTP # 1703.

[8] L. Biedenharn, J. Nuyts, and N Straumann, Ann. Inst. H. Poincaré **3** (1965) 13; J. Lanik, Nucl. Phys. **B2** (1967) 263. Also see Chapter 18 in B.G. Wybourne, *Classical Groups for Physicists*, John Wiley & Sons, Inc., New York, 1974.

[9] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics; Nonrelativistic Theory*, 3rd Edition, Pergamon Press, Inc., New York, 1977. See especially §18 and §35.

[10] V.S. Dotsenko and V.A. Fateev, Nucl. Phys. **B240** (1984) 312, *ibid* **B251** (1985) 691, Phys. Lett. **154B** (1985) 291.

[11] D.B. Fairlie, J. Nuyts, and C.K. Zachos, Phys. Lett. **B202** (1988) 320.

[12] V.G. Drinfel'd, *Proc. Int. Congress of Mathematicians*, Berkeley (1986) 798; M. Jimbo, Lett. Math. Phys. **10** (1985) 63; S.L. Woronowicz, Publ. R.I.M.S., Kyoto Univ. **23** (1987) 117.

[13] A. Polychronakos, Florida preprint UFTP–88–5.

A COSET-CONSTRUCTION FOR INTEGRABLE HIERARCHIES [1]

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ABSTRACT

The principal realization of the basic representation of an affine Kac-Moody algebra can be applied to construct soliton solutions of hierarchies of partial differential equations, among them the KdV- and KP-equations.
In this construction, called orbit-construction, the equations of the hierarchy itself arise in Hirota bilinear form. Using the Goddard-Kent-Olive coset-construction of conformal field theory, we show that the equations are generated by a pair of commuting c=1/2 Virasoro algebras in the KdV-case. We end with a brief discussion of other cases.

INTRODUCTION

We briefly recall the basics of the orbit-construction, which for example can be found in [1].
Let $L(\Lambda)$ be an integrable highest weight module over an affine Kac-Moody algebra, and let $v_\Lambda \in L(\Lambda)$ be its highest weight vector. On $L(\Lambda)$ there is a notion of orthogonality by means of a (unique) positive definite Hermitian contravariant form H such that $H(v_\Lambda, v_\Lambda) = 1$. This form determines, in the obvious way, such a form on the tensor product $L(\Lambda) \otimes L(\Lambda)$, so that we have an orthogonal direct sum of modules :

$$L(\Lambda) \otimes L(\Lambda) = L_{high} \oplus L_{low} \tag{1}$$

where L_{high}, the highest component, is the submodule generated by $v_\Lambda \otimes v_\Lambda$, and L_{low} is its orthocomplement. L_{high} is of course isomorphic to $L(2\Lambda)$.

By taking exponentials of the generators, we can construct an infinite-dimensional group of automorphisms of $L(\Lambda)$, called the Kac-Moody group G_Λ. The G_Λ-orbit of the highest weight vector v_Λ is denoted by O_Λ. The orbit-construction of soliton-solutions is based on the trivial fact that

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$$\tau \in O_\Lambda \Longrightarrow \tau \otimes \tau \in L_{high} \tag{2}$$

Elements of O_Λ are called τ-functions. Namely , the diagonal part of L_{high} , i.e. elements of the form $\tau \otimes \tau$ in L_{high} , can be described as a subset of the whole tensor product by an infinite number of equations :

$$\tau \otimes \tau \in L_{high} \Longleftrightarrow H(\tau \otimes \tau, x) = 0 \quad \forall\, x \in L_{low} \tag{3}$$

For many cases , in explicit realizations of the representations these equations turn out to be the partial differential equations of an integrable hierarchy , in Hirota bilinear form. For example , the principal realization of the basic representation of A_∞ and $A_1^{(1)}$ give rise to the KP-hierarchy and the KdV-hierarchy respectively.
Soliton solutions can then be generated by exponentiating the vertexoperators of the principal realization.

Here we shall only be interested in the Hirota bilinear equations and show that , for $A_1^{(1)}$, the complete set of equations is generated by two commuting $c = 1/2$ Virasoro algebras. To our knowledge , this is the first direct way to really construct all Hirota equations starting from a single one.
The basic idea is , using the GKO coset-construction of conformal field theory [2,3] , to precisely analyse the structure of the tensorproduct. A detailed paper , with more results , will be published elsewhere [4].

APPLYING THE COSET-CONSTRUCTION

Consider the principal realization of the basic representation $L(\Lambda_0)$ (i.e. the singlet at level $k = 1$) of the affine Kac-Moody algebra $A_1^{(1)}$. Therein , the principal Heisenberg subalgebra of $A_1^{(1)}$ is represented by differential and multiplication operators on a polynomial space in an infinite number of variables. Furthermore , $L(\Lambda_0)$ turns out to be irreducible with respect to this Heisenberg algebra [1] , so that

$$L(\Lambda_0) = \mathbf{C}[x] \tag{4}$$

where x stands for $(x_1, x_3, x_5, \ldots)$.
The other generators of $A_1^{(1)}$ are represented by complicated expressions in x and $\partial/\partial x$ (vertexoperators).

In the tensor product , we have two sets of variables , $x^{(1)}$ and $x^{(2)}$, and obviously

$$\mathbf{C}[x^{(1)}] \otimes \mathbf{C}[x^{(2)}] = \mathbf{C}[y] \otimes \mathbf{C}[x] \tag{5}$$

where $x_i = (x_i^{(1)} + x_i^{(2)})/2$ and $y_i = (x_i^{(1)} - x_i^{(2)})/2$.
This equality reflects the reduction of the tensor product with respect to the 'diagonal' Heisenberg subalgebra , and it implies that we must have

$$L_{low} = Hir \otimes \mathbf{C}[x] \tag{6}$$

$$L_{high} = Hir^{\perp} \otimes \mathbf{C}[x] \tag{7}$$

where the elements of the space Hir and $Hir^{\perp}$ are polynomials in y only. Hir is the space of *Hirota polynomials.*
Using the explicit form of the Hermitian form H , it can be shown that (see [1]) the diagonal part $\tau(x^{(1)}) \otimes \tau(x^{(2)})$ of L_{high} is determined by the equations

$$P(\frac{\tilde{\partial}}{\partial y})\tau(x-y)\tau(x+y)|_{y=0} \equiv 0 \quad \forall\, P \in Hir \tag{8}$$

where

$$\frac{\tilde{\partial}}{\partial y} = (\frac{\partial}{\partial y_1}, \frac{1}{3}\frac{\partial}{\partial y_3}, \frac{1}{5}\frac{\partial}{\partial y_5}, \frac{1}{7}\frac{\partial}{\partial y_7}, \ldots)$$

Therefore , to find all equations , we have to determine the space Hir as a subspace of $\mathbf{C}[y]$. This is an easy exercise , if one exploits the coset-construction which naturally reveals the relevant structure of the tensor product.
Thus we have reduced the problem of finding the orbit to the determination, in the tensor product , of the intersection of the coset with respect to the diagonal Heisenberg subalgebra and the representation of the 'diagonal' Kac-Moody algebra with the highest weight.

First of all , the GKO coset-construction tells us that in the tensor product of two representations of an affine Kac-Moody algebra , there acts a Virasoro algebra which commutes with the 'diagonal ' Kac-Moody algebra. The value of the central charge of this Virasoro algebra is trivially determined by the Kac-Moody representations and if it is smaller than 1 , there is a finite reduction with respect to the direct sum of the two afore mentioned algebras [3].
For the case at hand , the central charge is $1+1-3/2 = 1/2$, and we have the branching

$$L(\Lambda_0) \otimes L(\Lambda_0) = [V(\tfrac{1}{2},0) \otimes L(2\Lambda_0)] \oplus [V(\tfrac{1}{2},\tfrac{1}{2}) \otimes L(2\Lambda_1)] \tag{9}$$

where $V(\frac{1}{2},0)$ and $V(\frac{1}{2},\frac{1}{2})$ are the $c = 1/2$ Virasoro representations with highest weight 0 and $1/2$ respectively , and $L(2\Lambda_0)$ and $L(2\Lambda_1)$ are the singlet and triplet representations of $A_1^{(1)}$ at level $k = 2$ respectively.
Furthermore , we can reduce $L(2\Lambda_0)$ and $L(2\Lambda_1)$ with respect to the principal Heisenberg subalgebra. Again , there is a $c = 1/2$ coset Virasoro algebra , which we shall indicate with a prime , and

$$L(2\Lambda_0) = V'(\tfrac{1}{2},\tfrac{1}{16}) \otimes \mathbf{C}[x] \tag{10}$$

$$L(2\Lambda_1) = V'(\tfrac{1}{2},\tfrac{1}{16}) \otimes \mathbf{C}[x] \tag{11}$$

where $V'(\frac{1}{2},\frac{1}{16})$ is the $c = 1/2$ Virasoro representation with highest weight $1/16$.
Of course the Sugawara-construction provides us with explicit formulas for the generators of both Virasoro algebras [3,4].

From equations (5),(9),(10) and (11) we see that the space $\mathbf{C}[y]$ decomposes into representations of the two commuting Virasoro algebras in the following way :

$$\mathbf{C}[y] = [V(\tfrac{1}{2},0) \otimes V'(\tfrac{1}{2},\tfrac{1}{16})] \oplus [V(\tfrac{1}{2},\tfrac{1}{2}) \otimes V'(\tfrac{1}{2},\tfrac{1}{16})] \tag{12}$$

The direct sum corresponds to the even and odd polynomials , where the left-hand part gives the even polynomials , with common highest weight vector 1 , and the right-hand part gives the odd polynomials , with common highest weight vector y_1.
It then follows from equation (10) that the space $Hir^{\perp}$ is generated from the highest weight vector 1 by $V'(\frac{1}{2}, \frac{1}{16})$, and that the rest of $\mathbf{C}[y]$ is Hir.
The structure of $\mathbf{C}[y]$ is picturized in fig.1. Each dot stands for a number of independent polynomials ; this number is indicated next to the dot. Open dots correspond to elements of Hir , closed dots to elements of $Hir^{\perp}$. The Virasoro algebras act in the directions indicated , enlarging the L_0-eigenvalue by unity from one dot to the next.

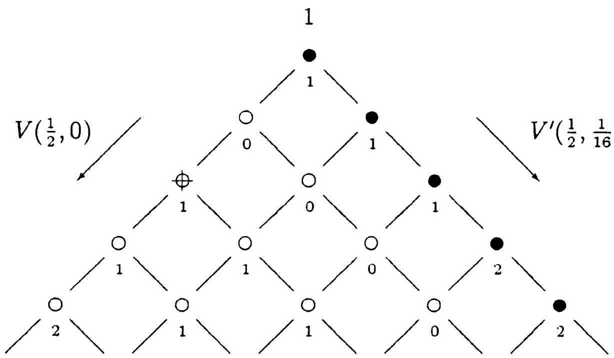

Fig.1a. $\mathbf{C}[y]_{even}$

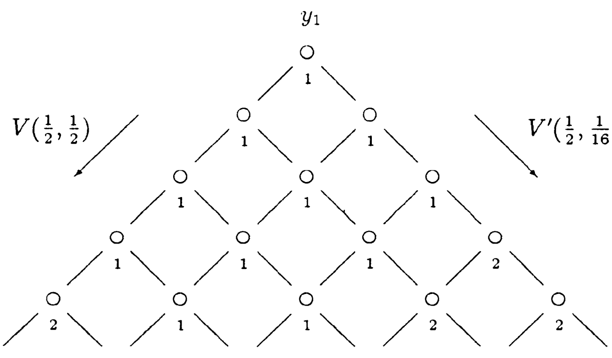

Fig.1b. $\mathbf{C}[y]_{odd}$

From equation (8) it is clear that odd Hirota polynomials give rise to trivial equations. Only the even polynomials give interesting equations of the hierarchy. The simplest one is indicated with a '+' in fig.1a. All other even Hirota polynomials can be generated from this one by the action of the two Virasoro algebras. Together , all nontrivial equations form the KdV-hierarchy in bilinear form. The first equation is the Korteweg-de Vries equation , after the magic substitution

$$u(x) = 2\frac{\partial^2}{\partial {x_1}^2} \log \tau(x). \tag{13}$$

DISCUSSION AND OUTLOOK

In the previous pages we have shown how the GKO coset-construction , applied to the tensor product space of representations of the $A_1^{(1)}$ Kac-Moody algebra , yields a beautiful characterization of a classical hierarchy of integrable equations in terms of definite representations of two commuting $c = 1/2$ Virasoro algebras.
So far we only considered the orbit of the highest weight of the singlet representation $L(\Lambda_0)$ in the principal realization. This naturally raises the question what happens with other representations , realizations and algebras. We have considered various possibilities along these lines. We briefly mention some results which will be discussed in detail elsewhere [4].

· If instead of the space $L(\Lambda_0) \otimes L(\Lambda_0)$ one takes $L(\Lambda_1) \otimes L(\Lambda_1)$, nothing new happens , one recovers again the KdV-hierarchy. It is also possible to analyse the coset for $L(\Lambda_0) \otimes L(\Lambda_1)$ in which case once more we obtain the decomposition (12) , however , in this case the identification of the subspaces Hir and $Hir^{\perp}$ is different. This gives rise to the MKdV-hierarchy [12].

We also considered representations in the homogeneous realization , which is more involved because (4) no longer holds and the two $c = 1/2$ Virasoro algebras are realized differently. The hierarchies one obtains are the Toda-AKNS system and the Kac-van Moerbeke system for $L(\Lambda_0) \otimes L(\Lambda_0)$ and $L(\Lambda_0) \otimes L(\Lambda_1)$ respectively [8,9].

Finally we investigated the problem with higher rank algebras. It is clear that the above analysis can be repeated but involves various apropriate extensions of the Virasoro algebra , i.e. algebras which contain the Virasoro algebra as a subalgebra , like for example the Zamolodchikov W algebras [6,10,11]. The case $A_2^{(1)}$, associated with the Boussinesq-hierarchy , will be discussed in detail in [4].

We conclude with a brief remark concerning another context in which the (extended) Virasoro algebras emerged in the theory of integrable hierarchies , namely through the second Hamiltonian structure [5,7]. We note that in that case the classical hierarchy is related to a *single* (extended) Virasoro algebra , whose central charge c is not fixed [13]. It is an interesting but nontrivial problem to understand the connection of these two structures, i.e. to analyse the precise role of the Hamiltonian structure in the bilinear picture of the hierarchy.

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References

[1] V. G. Kac, Chapter 14 , in : Infinite dimensional Lie algebras, Cambridge University Press , Cambridge (1985).

[2] P. Goddard, A. Kent and D. I. Olive, Phys. Lett. 152B, 88 (1985).

[3] P. Goddard and D. I. Olive, Int. J. Mod. Phys. A1, 303 (1986).

[4] F. A. Bais and K. de Vos , in preparation.

[5] V. A. Fateev and S. L. Lykyanov, Int. J. Mod. Phys. A3, 507 (1988).

[6] A. B. Zamolodchikov, Theor. Math. Phys. 65,1205 (1985).

[7] J. L. Gervais and A. Neveu, Nucl. Phys. B209, 125 (1982).

[8] A. P. E. ten Kroode and M. J. Bergvelt, Lett. Math. Phys. 12, 139 (1986).

[9] M. Kac and P. van Moerbeke, Proc. Natl. Acad. Sci. USA 72, 1627 (1975).

[10] F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. B304, 348 (1988).

[11] F. A. Bais, P. Bouwknegt, K. Schoutens and M. Surridge, Nucl. Phys. B304, 371 (1988).

[12] G. Post, J. Math. Phys. 27, 678 (1986).

[13] P. Mathieu, Phys. Lett. B208, 101 (1988).

AWAY FROM CRITICALITY: SOME RESULTS FROM THE S MATRIX APPROACH

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1 Introduction

In two dimensions, conformal invariance provides a natural way to classify the universality classes for critical phenomena [1]. It is of growing interest to extend the Conformal Field Theories (CFT) away from criticality to reach a complete understanding of the scaling region around the fixed points. Zamolodchikov [2] found that particular deformations of the CFT preserve the integrability of the models and the corresponding massive field theories can be described by a factorizable S-matrix. The S-matrix of the three-state Potts model [3] and the S-matrix of the Ising model in a magnetic field [2] are the first examples of this approach. The hidden E_8 symmetry of Ising model is responsible for the rich structure of the S-matrix in the latter case. Recently, other models have been discussed: the S-matrix of non-unitary Yang-Lee edge singularity [4] and the S-matrix of the thermal perturbation of the tricritical Ising model [5]. P. Christe will discuss the latter model, expecially its hidden E_7 symmetry, in his talk [6].

As many explicit solutions suggest, there is a deep connection between the off critical models and the integrable non-linear systems. This idea is becoming a subject of intense investigation [5, 7, 8, 9, 10, 11, 12]. A particular role is played by the generalized Toda systems constructed by the roots of the Dynkin diagram of the affine Lie algebras [13, 14, 15, 16].

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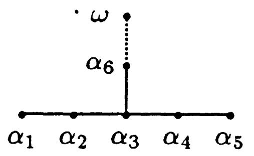
This is obtained by the usual Dynkin diagram adding to it the maximal root ω. In general this operation enlarges the discrete symmetries of the system: for example, in the case of A_n series, there is a Z_{n+1} automorphic symmetry [17]. In the same way, the Dynkin diagram of the affine E_7 algebra presents a Z_2 symmetry that gives a selection rule in the bootstrap equations of the tricritical Ising model [5, 6]. Another interesting example is given by the Toda system associated to the affine E_6 algebra. Table 1 gives all the informations about it. This theory is conjectured to describe the thermal perturbation of the tricritical three-state

Table 1. Coxeter exponents and coefficients n_i for E_6.



$$-\omega = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$$

Exponents: $1, 4, 5, 7, 8, 11$

Table 2. Mass spectrum of E_6 Toda system.

$$\begin{aligned}
m_1 &= m_{\bar{1}} = M \\
m_2 &= m_{\bar{2}} = 2M\cos(\tfrac{\pi}{12}) \\
m_3 &= 2M\cos(\tfrac{\pi}{4}) \\
m_4 &= 2M\cos(\tfrac{\pi}{12})\cos(\tfrac{\pi}{4})
\end{aligned}$$

Potts model [9]. The S-matrix of the corresponding massive theory involves six particles, organized into two doublets of particles with degenerate masses and two singlet particles [5]. The mass spectrum is given in Table 2. In this case the discrete automorphism is given by the permutation group S_3. It plays an important role in the discussion of the model.

The scenario recovers by all these explicit solutions seems very appealing but one may wish to have a more general theory. In particular, it would be very interesting to develop a general formalism that allows to derive the thermodynamics of the models away from criticality knowing their S-matrices. Equivalently, one would like to compute the Green functions off mass-shell.

This paper reports some results obtained separately with J.Cardy and P.Christe. It is organized as follows: in the next section I discuss some properties of factorizable S-matrix and the general problem of solving the bootstrap equations with N particles of distinct masses. In section 3 it is computed the S-matrix associated to the deformation by a relevant operator of the simplest, albeit non-unitary, conformal field theory, namely that of the Yang-Lee edge singularity [4]. The spectrum of massive excitations contains only a single particle. I describe briefly the main features of this model. In section 3, as application of the general principles discussed in section 2, it is given a classification of the S-matrices involving two distinct masses in the bootstrap equations [12].

2 S-matrix of integrable models in (1+1) dimensions

For the theories with an infinite number of conserved currents the S-matrix satisfies several strong constraints [18]. It describes only elastic processes and moreover the general n particle scattering amplitudes factorize into a product of $n(n-1)/2$ elastic 2-particle S-matrices. In (1+1) dimensional space-time a convenient parametrization of the momentum of the particle i is

$$p^0 = m_i \cosh\theta, \qquad\qquad p^1 = m_i \sinh\theta \tag{2.1}$$

where θ is the rapidity. The Mandelstam variable s is given by

$$s = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1 m_2 \cosh\theta_{12} \tag{2.2}$$

($\theta_{12} = \theta_1 - \theta_2$). The two particle S-matrix elements are defined by

$$\mid A_i(\theta_1)A_j(\theta_2) >_{in} = S_{ij}^{kl}(\theta_{12}) \mid A_k(\theta_1)A_l(\theta_2) >_{out} \tag{2.3}$$

As a consequence of Lorentz invariance, S only depends on the difference of the rapidities. If the mass spectrum is degenerate they are non diagonal matrices and the scattering processes implies a redistribution of the momentum among the particles with the same mass. The factorization condition for the three particle S-matrix implies that $S_{kl}^{ij}(\theta_{12})$ must satisfy a cubic relation called star-triangle or Yang-Baxter equations

$$S_{i_1 i_2}^{k_1 k_2}(\theta_{12}) S_{k_1 k_3}^{j_1 j_3}(\theta_{13}) S_{k_2 i_3}^{j_2 k_3}(\theta_{23}) = S_{i_1 i_3}^{k_1 k_3}(\theta_{13}) S_{k_1 k_2}^{j_1 j_2}(\theta_{12}) S_{i_2 k_3}^{k_2 j_3}(\theta_{23}) \tag{2.4}$$

Eqs.(2.4), together with the properties of unitary, analyticity and crossing symmetry, are in many cases sufficient to determine a consistent solution for the S-matrix [2, 3, 4, 5, 17, 18, 19, 20, 21, 22]. Here we discuss the models when the particles have different masses and the theory is invariant under time reversal and parity (charge conjugation is automatic in this case because each particle is self-conjugate). In this case, the S-matrix is diagonal and the star-triangle equations are trivially satisfied. We denote $\hat{S}_{AB}(\theta)$ the scattering amplitudes of the particle A and B (symmetric in A,B). As a function of θ, $\hat{S}$ is a meromorphic function. The two branch cut singularities of $\hat{S}_{AB}$ in the complex plane s (precisely at $s = (m_A + m_B)^2$ and $s = (m_A - m_B)^2$) are mapped onto lines parallel to the real axis passing through $\theta = 0$ and $i\pi$ respectively. It is worth noting that their origin is only kinematic. Crossing symmetry is expressed by the the condition

$$\hat{S}_{AB}(\theta) = \hat{S}_{AB}(i\pi - \theta) \tag{2.5}$$

The unitarity condition reads

$$\hat{S}_{AB}(\theta)\hat{S}_{AB}(-\theta) = 1 \tag{2.6}$$

The bootstrap approach imposes that the bound-state particles, described by the simple poles, must belong to the basic set of massive particles we started with. They are located at $\theta = iu_{AB}^C$ and relate the different masses m_A, m_B, and m_C in the following way

$$m_C^2 - m_A^2 - m_B^2 = 2m_A m_B \cos(u_{AB}^C) \tag{2.7}$$

Eq.(2.5) implies the existence of another pole at $\bar{u}_{AB}^C = \pi - u_{AB}^C$, that corresponds to the same particle but in the cross channel. The bootstrap condition has to be implemented in scattering processes that involve more then two particles. This leads to the equation [2, 21]

$$\hat{S}_{CD}(\theta) = \hat{S}_{AC}(\theta - i\bar{u}_{AD}^B)\hat{S}_{BC}(\theta - i\bar{u}_{BD}^A) \tag{2.8}$$

Up to now, we have not discussed the importance of the conserved currents. These strongly restrict the possible theories. The general discussion can be found in the Zamolodchikov's paper [2]. Here we recall the main equations. In (1+1) dimensions there is no spin, so the conserved currents P_n of degree n can be constructed only in terms of the momentum p. The locality of the conservation law implies

$$P_n \mid A_{a_1}(p_1) \cdots A_{a_k}(p_k) >= (\omega_n^{a_1}(p_1) + \cdots + \omega_n^{a_k}(p_k)) \mid A_{a_1}(p_1) \cdots A_{a_k}(p_k) > \tag{2.9}$$

where $\omega_n^a(p)$ are the eigenvalues of P_n for the one-particle state

$$P_n \mid A_a(p) >= \omega_n^a(p) \mid A_a(p) > \tag{2.10}$$

It can be shown [2] that Lorentz invariance fixes the following form of $\omega_n^a(p)$

$$\omega_n^a(p) = \chi_n^a \, p^n \tag{2.11}$$

where χ_n^a are constants. If we apply eq.(2.9) to the 2-particle state near the bound-state pole

$$\mid A(\theta + i\overline{u}_{AC}^B - \frac{\epsilon}{2})B(\theta + i\overline{u}_{BC}^A + \frac{\epsilon}{2}) > \;\sim \frac{1}{\epsilon} \mid C(\theta) > \tag{2.12}$$

we get the following system of linear equations for the constants χ_n^a

$$\chi_n^A \, (m_A)^n e^{-in\overline{u}_{AC}^B} + \chi_n^B \, (m_B)^n e^{in\overline{u}_{BC}^A} = \chi_n^C \, (m_C)^n \tag{2.13}$$

This linear system is usually overdetermined and the request to have solution with $\chi_n^a \neq 0$ strongly restricts the possible theories. For instance, let us suppose that the particle A can be considered as "bound state" of itself, i.e. it appears as pole singularity of the scattering matrix S_{AA} [2]. In this case, by simmetry, $u_{AA}^A = \frac{2\pi}{3}$ and the eq.(2.13) becomes

$$e^{-\frac{i\pi n}{3}} + e^{+\frac{i\pi n}{3}} = 1 \tag{2.14}$$

The set of allowed values of n is then

$$n = 1, 5 \qquad (mod\, 6) \tag{2.15}$$

In the next section we will see that these are the degrees of the conserved currents of the Yang-Lee edge singularity away from criticality [4]. Applying the eq.(2.13) in the case of tricritical Ising model perturbed with energy operator, one gets the following set of values

$$n = 1, 5, 7, 9, 11, 13, 17 \qquad (mod\, 18) \tag{2.16}$$

i.e. the Coxeter exponents of E_7 modulo the Coxeter number $h = 18$ [5].

It is possible to classify the elastic scattering theories using both the eq.(2.13) and an ansatz of the possible pole structure of the S-matrix. An example of this program is given in section 3 for the case of systems with $N = 2$ particles with distinct masses [12].

3 S-matrix of Yang-Lee edge singularity

The Yang-Lee edge singularity [23, 24] describes the critical behavior of an Ising model in a pure magnetic field ih. For $h > h_c$ the zeros of the partition function are dense on the imaginary h-axis. The Lagrangian of this model is [25]

$$\mathcal{L} = \int [\frac{1}{2}(\partial\phi)^2 - i(h - h_c) - ig\phi^3] d^2x \tag{3.1}$$

At criticality ($h = h_c$), this theory has only one relevant operator, namely the field ϕ itself. Cardy [26] showed that this property is satisfied by the simplest non-trivial CFT with central charge $c = -\frac{22}{5}$, which has only one relevant operator with scaling dimensions $(\Delta, \overline{\Delta}) = (-\frac{1}{5}, -\frac{1}{5})$. This operator corresponds to the position (1,2) or (1,3) in the Kac table. The negative values of c and Δ's indicate that this theory is not unitary, as it is also evident from the explicit factor of i in the Lagrangian (3.1). Let us suppose now that this CFT is perturbed away from criticality by the operator $\phi_{1,2}$. There is a counting argument [2] that allows to determine the lowest degrees of the conserved currents away from criticality. It makes use of the characters of the CFT. The explicit computation for the Yang-Lee model gives [4]

$$n = 1, 5, 7, 11, 13, 17, 19, 23 \tag{3.2}$$

It was conjectured that these are the first values of the series (2.15) and moreover, that the S-matrix describes only one massive excitation A. This statement is supported by the fact that the existence of further bound states usually tends to rule out some of the values of n of eq.(3.2). Another argument for the benefit of this conjecture can be given on the basis of the Lagrangian (3.1): in fact, in the non-relativistic Born approximation, the exchange of the A particle in the t-channel will lead to a *repulsive* potential, because of the imaginary coupling constant. Then, it is not expected no further bound states. The S-matrix of the model is uniquely fixed to be

$$S(\theta) = \tanh(\frac{\theta}{2} + \frac{i\pi}{6}) \coth(\frac{\theta}{2} - \frac{i\pi}{6}) \tag{3.3}$$

It has a pole in the s-channel at $\theta = 2i\pi/3$ but the residue comes out to be negative! That is, the correct sign expected at lowest order in perturbation theory for the Lagrangian (3.1) but the wrong one respect that expected in a unitary theory. Nevertheless, the function (3.3) does satisfy the unitary condition (2.6). The solution of this apparent paradox is related to the non-hermiticity of the Hamiltonian H corresponding to eq.(3.1). We can define an operator C which takes $\phi \to -\phi$ and $H^\dagger = CHC$. The Fock space states of the theory are all eigenstates of C with eigenvalue $(-1)^N$, where N is the particle number. Since H is not hermitian, its left eigenstates $< n_L \mid$ are not the adjoints of the the right eigenstates $\mid n_R >$. Instead we have $< n_L \mid = < n_R \mid C$. The completeness of the eigenstates of H is

$$\sum_n \mid n_R >< n_L \mid = \sum_n \mid n_R >< n_R \mid C = 1 \tag{3.4}$$

Now the unitarity of the S-matrix $SS^\dagger = 1$ follows solely from the fact the in-kets and the out-kets both give a basis for the space, and this does not dependent whether or not H is hermitian. But, when we insert a complete set of states, eq.(3.4), into this equation, each term will pick up a factor $(-1)^N$. This explains why the residue of the pole has a wrong sign. On the other hand, if we choose to make H hermitian by defining a new pseudo-inner product $< a \mid b > \equiv < a \mid C \mid b >$ then some of the physical states will have negative norm.

In conclusion, we have shown that the simplest possible CFT leads, away from criticality, to the simplest possible S-matrix, involving only one particle. The theory is not unitary. In the next section we discuss the case of two-particle systems.

4 Classification of non-degenerate two-particle S-matrices

The general program of classification of the S-matrices gives nice examples for the two-particle systems. In this section I report some results of recent work with P. Christe that will be discussed in more detail in [12]. Let's denote by A the lightest particle and by B the other one. We restrict our attention here to the non-degenerate systems and to the minimal solutions of the S-matrices. The notation

$$a \times b \rightarrow c + \cdots d \tag{4.1}$$

means that in the scattering of the particles a and b there appear the particles $c \cdots d$ as bound states. In the following, I'll use the notation of the ref.[5] for the functions $f_x(\theta)$ entering the S-matrix

$$f_x(\theta) = \frac{\tanh(\frac{\theta}{2} + i\pi x)}{\tanh(\frac{\theta}{2} - i\pi x)} \tag{4.2}$$

Neglecting the complete decoupled process

$$A \times A \rightarrow A, \qquad B \times B \rightarrow B$$

we have to consider the following two cases

1. the particle A couples to itself and to B and the same happens to B, i.e.

$$A \times A \rightarrow A + B, \qquad B \times B \rightarrow A + B \tag{4.3}$$

If we want to have only simple poles there is no ways to close a consistent solution by themselves and in fact they lead to the S-matrix of Ising model in magnetic field [2]. In this case the degrees of the conserved currents are the Coxeter exponents of E_8 modulo the Coxeter number $h = 30$.

If we relax the condition of simple poles, there are other possibilities. As it was explained above, the natural ansatz for S_{AA}

$$S_{AA} = f_{\frac{1}{3}} f_x \tag{4.4}$$

(x has to be fixed) does not work because does not satisfy the bootstrap equation involving the particle A alone. We need at least another function entering (4.4) and to avoid the introduction of new particles it should be at least a double pole. Let us analyze this simple condition. It can be either (i) $f_{\frac{1}{4}}$ or (ii) $(f_y)^2$ for some y.

- (i) in this case the solutions of the bootstrap equations are

$$\begin{aligned} S_{AA} &= f_{\frac{1}{3}} f_{\frac{1}{12}} f_{\frac{1}{4}} \\ S_{AB} &= (f_{\frac{1}{8}})^2 (f_{\frac{5}{24}})^3 f_{\frac{1}{24}} \\ S_{BB} &= (f_{\frac{1}{4}})^3 (f_{\frac{1}{12}})^3 (f_{\frac{1}{3}})^5 \end{aligned} \tag{4.5}$$

  The mass ratio is

$$\frac{m_B}{m_A} = 2\cos(\frac{\pi}{12}) \tag{4.6}$$

  This coincides with the mass ratio of the non-linear sigma model on the group manifold of G_2 [19, 20]. The residues at the poles come out to be positive. To find the allowed degrees of conserved currents we have to consider the processes $A \times A \to A$ (the same for B), $A \times A \to B$ and $B \times B \to A$. From the first one, as usually, it comes $n = 1, 5$ $(mod\, 6)$. From the latter ones

$$n = 1, 4, 5, 7, 8, 11 \qquad (mod\, 12) \tag{4.7}$$

  It is worth to note that these are just the exponents of E_6. The combination of the two equations leaves as possible degrees of the conserved currents $n = 1, 5$ $(mod\, 6)$.

- (ii) In this case we start with

$$S_{AA} = f_{\frac{1}{3}} f_x f_y^2 \tag{4.8}$$

  for some x, y, with $x < 1/4$. But it does not exist a solution of the bootstrap equation for S_{AA} involving the particle A alone.

2. the particle A couples to B but not to itself

$$A \times A \to B, \qquad\qquad A \times A \not\to A \tag{4.9}$$

Then $S_{AA} = f_x$, for some x. If we want to consider unitary theories then $x < 1/4$. The mass ratio is

$$\frac{m_B}{m_A} = 2\cos(\pi x) \tag{4.10}$$

The bootstrap procedure, eq.(2.8), gives

$$S_{AB} = f_{\frac{x}{2}} f_{\frac{3x}{2}} \tag{4.11}$$

Either $f_{\frac{3x}{2}}$ is a double pole (i) or describes another singularity. Since we want to close the bootstrap equation with two particles, this singularity is due to the particle B (ii). Let us analyze the two possibilities.

- (i) in this case $x = 1/6$ and the set of functions is

$$S_{AA} = f_{\frac{1}{6}}, \qquad S_{AB} = f_{\frac{1}{12}} f_{\frac{1}{4}}, \qquad S_{BB} = (f_{\frac{1}{6}})^3 \tag{4.12}$$

the mass ratio is that of the G_2 Toda system

$$\frac{m_B}{m_A} = 2\cos(\frac{\pi}{6}) = \sqrt{3} \tag{4.13}$$

The particle B couples to itself, so the set of allowed degrees is $n = 1,5 \pmod 6$. No further restrictions come from the particle A. It is worth to note that these values of n are just the Coxeter exponents of G_2 modulo the Coxeter number $h = 6$. At the classical level these are the degrees of the conserved currents of the Toda system constructed by the group G_2 [14, 15].

- (ii) in this case the unique rational solution is $x = 1/5$. The S-matrices are

$$S_{AA} = f_{\frac{1}{5}}, \qquad S_{AB} = f_{\frac{1}{10}} f_{\frac{3}{10}} \qquad S_{BB} = f_{\frac{2}{5}} (f_{\frac{1}{5}})^2 \tag{4.14}$$

The mass ratio is

$$\frac{m_B}{m_A} = 2\cos(\frac{\pi}{5}) = \frac{\sqrt{5}+1}{2} \tag{4.15}$$

The simple pole in S_{BB}, relative to the process $B \times B \to A$, has the residue negative. This is another example of non unitary theory. It can be obtained by the A_4/Z_2 Toda field theory, i.e. if we identify the particles of A_4 with the anti-particles and construct by them particle states that are Z_2 even. The allowed degrees of the conserved currents are those of A_4 Toda field theory, i.e. $n \neq 0 \pmod 5$. The general scheme of A_k/Z_2 solutions is analyzed in [12]. Similar results are obtained also in [27].

These possibilities enter the classification of the minimal S-matrices involving two particles.

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References

[1] *Conformal Invariance and applications to statistical mechanics*, C. Itzykson, H. Saleur, and J.B. Zuber, Eds. World Scientific, and references therein.

[2] A.B. Zamolodchikov, *Integrable field theory form conformal field theory*, Proc. of the Taniguchi Symposium, Kyoto 1888, to appear in *Advanced Studies in Pure Mathematics.*

[3] A.B. Zamolodchikov, Int.Journ.Mod.Phys. A3(1988)743.

[4] J.L. Cardy, G. Mussardo, *S-matrix of the Yang-Lee Edge Singularity in Two Dimensions*, UCSBTH-89-11 to appear in Phys.Lett. B.

[5] P. Christe and G. Mussardo *Integrable systems away from criticality: the Toda field theory and S-matrix of the tricritical Ising model* UCSBTH-89-19 submitted to Nucl.Phys.B

[6] P. Christe and G. Mussardo *S-matrices of the tricritical Ising model and Toda systems* to appear in *NATO Conference on Nonlinear Science: Physics and Geometry, Tahoe City 2-8 July 1989.*

[7] H.W. Braden, E. Corrigan, P.E. Dorey, R. Sasaki *Extended Toda Field Theory and exact S-matrices* UDCPT-89-23.

[8] T.J. Hollowood and P. Mansfield, Oxford preprint 89-17P.

[9] T. Eguchi, S.K. Yang, *Deformation of Conformal Field Theories and Soliton Equations*, RIFP-797 preprint.

[10] A. LeClair *Restricted Sine-Gordon Theory and the Minimal Conformal Series* PUPT-1124

[11] M. Henkel and H. Saleur *Remarks on the Mass Spectrum of non Critical Coset Models from Toda Theories*, Saclay preprint.

[12] P. Christe, G. Mussardo *in preparation.*

[13] A.V. Mikhailov, M.A. Olshanetsky, and A.M. Perelomov, Comm.Math.Phys. 79(1981)473.

[14] G. Wilson, Ergod.Th.Dyn.Syst. 1(1981)361.

[15] D.Olive, N. Turok, Nucl.Phys. B257[FS14](1985)277.

[16] S.A. Bulgadaev, Phys.Lett. 96B(1980)151.

[17] A.E. Arinshtein, V.A. Fateyev, and A.B. Zamolodchikov Phys.Lett. B87(1979)389.

[18] A.B.Zamolodchikov, Al.B.Zamolodchikov, Ann.Phys. 120 (1979),253.

[19] E. Ogievetsky, P. Wiegmann, Phys.Lett. 168B (1986),360.

[20] E.I. Ogievetsky J.Phys.G: Nucl.Phys. 12(1986)L105.

[21] M. Karowski, H.J. Thun, Nucl.Phys. B130 (1977),295.

[22] R. Koberle, J.A. Swieca, Phys.Lett. B86 (1979),209.

[23] C.N. Yang and T.D. Lee, Phys.Rev. 87 (1952),404.

[24] T.D. Lee and C.N. Yang, Phys.Rev. 87 (1952),410.

[25] M.E. Fisher Phys.Rev.Lett. 40 (1978), 1610.

[26] J.L. Cardy Phys.Rev.Lett. 54 (1985), 1354.

[27] P.G. Freund, T.R. Klassen, E. Melzer *S-Matrices for Perturbations of Certain Conformal Field Theories* preprint EFI 89-29

NORMAL ORDERED PRODUCTS AND PARAFIELDS IN CONFORMAL QFT$_2$

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Abstract: Firstly, an improved normal ordered product is defined, which allows to write down concisely the derivative terms in operator product expansions. Secondly, it is shown that representations of chiral algebras always allow the construction of relatively local interpolating fields (parafields). If the grading of the representation is integral or half-integral, these fields become local.

Let the ϕ_i form a basis in the vector space of chiral fields of a conformal QFT$_2$. The fields are periodic under the shift $u \to u + 2\pi$ and have the Fourier components

$$\phi_{in} = \oint e^{inu}\phi_i(u), \quad n \in \mathbf{Z} .$$

Usually the normal ordered product $N(\phi\chi)(u)$ is defined to be the local field with Fourier coefficients

$$N(\phi_i\phi_j)_n = \sum_{m<h(j)} \phi_{i,n-m}\,\phi_{jm} + \sum_{m\geq h(j)} \phi_{jm}\,\phi_{i,n-m} ,$$

where $h(j)$ is the scaling dimension of ϕ_j. However, in many respects this definition is awkward. $N(\phi_i\phi_j)(u)$ has components of different dimensions and contains both non-derivative and derivative fields. Since the vector space of local fields is isomorphic to the space of finite energy vectors in the Hilbert space of the vacuum sector of the theory, we can define a projection operator $\mathcal{P}_N$ which projects onto the subspace of non-derivative fields of dimension N. We define

$$\mathcal{N}(\phi_i\phi_j) = \mathcal{P}_{h(i)+h(j)}\,N(\phi_i\phi_j) .$$

The new normal ordered product is commutative, since the commutator $[\phi_i\phi_j]$ only yields fields of dimension $\leq h(i)+h(j)-1$. Moreover,

$$\mathcal{N}(\phi_i, \partial\phi_j) = -\mathcal{N}(\partial\phi_i, \phi_j) \ ,$$

since $\partial N(\phi_i\phi_j) = N(\phi_i, \partial\phi_j) + N(\partial\phi_i, \phi_j)$ and $\mathcal{P}\partial N(\phi_i\phi_j) = 0$. Explicit computation yields

$$\mathcal{N}(\phi_i\phi_j) = N(\phi_i\phi_j) - \sum_{\{k|h(ijk)\geq 1\}} C_{ij}^k \, a^{h(ijk)}(h(i), h(j), h(k)) \binom{i\partial - h(k)}{h(ijk)} \phi_k \ ,$$

where $h(ijk) = h(i) + h(j) - h(k)$ and

$$a_r(\alpha\beta\gamma) = \binom{2\gamma + r - 1}{r}^{-1} \binom{\gamma + \alpha - \beta + r - 1}{r} \ .$$

A similar but more complicated formula exists for the normal ordered product with derivative fields.

We now switch over to the euclidean formulation. The euclidean field ϕ^e corresponding to ϕ_i is given by

$$\phi_i^e(z) = \sum_n z^{n-h(i)} \phi_{ni} \ .$$

One can show that the operator product expansion takes the form

$$\begin{aligned}
&\phi_i^e(z)\phi_j^e(w) \\
&= \sum_{\{k|h(ijk)\geq 1\}} (z-w)^{-h(ijk)} C_{ij}^k \sum_r a_r(h(i), h(j), h(k)) (r!)^{-1} \partial^r \phi_k^e(w) \\
&+ \sum_{\nu=0}^{\infty} \sum_{r=0}^{\infty} \frac{a^r(h(i), h(j), h(i)+h(j)+\nu)}{\nu! r!} \partial^r \mathcal{N}(\phi_j (i\partial)^\nu \phi_i)^e(w) \ .
\end{aligned}$$

The ordered product expansion for the dimension 3 field U in Zamolodchikov's Z_3 algebra[1]

$$\begin{aligned} U(z)U(w) =& \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ &+ \frac{3}{10}\frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15}\frac{\partial^3 T(w)}{z-w} \\ &+ \frac{16}{22+5c}\left(\frac{2\mathcal{N}(TT)(w)}{(z-w)^2} + \frac{\partial\mathcal{N}(TT)(w)}{z-w}\right. \\ &+ regular \ , \\ \mathcal{N}(TT) =& N(TT) - \frac{3}{10}\partial^2 T \ , \end{aligned}$$

now can be derived with very little calculation. Here all fields are euclidean, though this has not been expressed in the notation. The field $T(z)$ is the holomorphic component of the energy momentum tensor, which has dimension 2. The only part of the derivation which does not follow from the previous formulas is the calculation of the structure constants C^T_{UU} and $C^{\mathcal{N}}_{UU}$, but this part is easy, too, since the structure constants C_{ijk} (all indices lowered) are cyclically symmetric.

Such calculations always have been described as very tedious, but with the formulas given above they are easy to do or easy to program.

Now let me come to my second topic. Let us continue to use the euclidean formalism. Suppose that the chiral algebra has its basic representation on $\mathcal{H}$ and another irreducible unitary positive energy representation on a space $\mathcal{H}'$. We will not need the full conformal invariance, but the subgroup $SU(1,1)$ with generators L_{-1}, L_0, L_1 must act on both spaces, with the usual transformations

$$[L_n \phi_i(z)] = z^{-n}(z\partial + (n-1)h(i))\phi_i(z)$$

of the non-derivative chiral fields, with $n = -1, 0, 1$.

We will construct a complete set of parafields which take values in the transformations $\mathcal{H} \to \mathcal{H}'$ and are local relatively to the chiral fields.

First we define the action on the vacuum vector $v \in \mathcal{H}$. For each $w_\alpha \in \mathcal{H}$ with $L_0 w = h(\alpha) w$ we define the action of ψ_α on v by

$$\psi_\alpha v = w_\alpha$$

and by

$$[L_n \psi_\alpha] = (z\partial + (n-1)h(\alpha))\phi_\alpha$$

for $L_{-1}w = 0$. Vectors which satisfy the latter condition thus are mapped into non-derivative parafields.

Now $SU(1,1)$ invariance yields

$$\langle\psi_\beta(z'')\phi_i(z')\psi_\alpha(z)\rangle = C_{\beta i\alpha}(z''-z')^{h(\beta i\alpha)}(z-z'')^{h(\beta\alpha i}(z'-z)^{h(i\alpha\beta)}$$

for $|z''| > |z'| > |z|$. This function has a one-valued analytic continuation to $|z''| > |z| > |z'|$. Since the states ϕ_{in} with $n \geq h(i)$ yield a basis of $\mathcal{H}$ we may define the action of $\psi_\alpha(z)$ on $\mathcal{H}$ by the analytic continuation of the preceding equation. The only point to check is that $\psi_\alpha(z)\phi_{in}v$ vanishes for $n < h(i)$, but this is obvious from the power series expansion of the 3-point function in z'.

Now we have to check that the ψ_α are relatively local with respect to all chiral. This requires that the 4-point function

$$G^4(z_r) = \left\langle\psi_\beta^+(z_3)\phi_i(z_2)\psi_\alpha(z_1)\phi_j(z_0)\right\rangle$$

for $|z_{i+1}| > |z_i|$ has a one-valued analytic continuation in a neighbourhood of $|z_2| = |z_1|$, apart from the point $z_2 = z_1$. Now $SU(1,1)$ invariance allows us to restrict ourselves to $z_3 = \infty$. With $\psi_\beta(0)v = w$ we obtain

$$\langle w|\,\phi_i(z_2)\psi_\alpha(z_1)\phi_j(z_0)\,|v\rangle = (z_2-z_0)^{h(\beta)-h(i)-h(j)-h(\alpha)}F_\pm(z)\ ,$$

where $z = (z_2-z_0)/(z_2-z_1)$. The function $F_+(z)$ is the one obtained by analytic continuation of the 4-point function from the region $|z_2| > |z_1| > |z_0|$, the function $F_-(z)$ the one obtained by analytic continuation from the region $|z_1| > |z_2| > |z_0|$. We want to show that they are equal. Now F_- has a one valued analytic continuation for all of $|z_1| > |z_0|, |z_2|$,apart from the pole at $z_0 = z_2$, since this only involves the exchange ob ordinary chiral fields. This shows that F_- is holomorphic on the complex plane , cut along the negative real axis.

Moreover, F_+ by definition of ψ_α has a one-valued analytic continuation to all of $|z_2| > |z_0|, |z_1|$, and further on to the union of the sets given by $|z_1| < |z_0|$ and $|z_1| < |z_2|$. This show that F_+ is holomorphic on the Riemann sphere, apart from the points $0, 1, \infty$.

Now we use the Laurent expansions of $F_\pm$ given by the operator product expansions of $\phi_i(z_2)\phi_j(z_0)$. For a field ϕ_k occuring in this expansion, the Laurent coefficient in F_+ is proportional to $C_{\beta k\alpha}$, the one in F_- is proportional to $C_{\beta\alpha k}$. According to our definition of ψ_α, the Laurent expansion agree, such that

$$F_+ = F_- \, .$$

As one sees, the argument is fairly simple and straightforward. I haven't seen it in the literature, but it may have been made before.

If the representation of the chiral algebra on $\mathcal{H}'$ is real, one may define $\psi = \psi^+$ for fields such that $\psi(0)v$ belongs to the real subspace of $\mathcal{H}'$. Since $(\psi_r)^+ = (\psi^+)_{-n}$, this is only possible, if the dimension of ψ is integral or half-integral, in other words, if the eigenvalues of L_0 on $\mathcal{H}'$ are of this kind. By $SU(1,1)$ invariance, the singularity of the 4-point function at $z_3 = z_1$ can be read of from the singularity at $z_2 = z_0$. One obtains a power series expansion in $z_3 - z_0$, multiplied by $(z_3 - z_0)^{-h(\alpha)-h(\beta)}$. Thus the parafields ψ constructed in terms of the representation on $\mathcal{H}'$ are ordinary bosonic chiral fields, if the eigenvalues of L_0 on $\mathcal{H}'$ are integral, and fermionic chiral fields, if they are half-integral.

Thus one always can construct larger chiral algebras, if a given chiral algebra has a real representation. There are many well known examples, in particular:

– The fermionic emission vertices of the superstring in 9+1 dimension are ordinary fermi fields, since more generally their dimension is $(d-2)/16$ in d dimensional space time.

– The basic representation of the E_8 at level 1 can be constructed by starting with the basic representation of the $SO(16)$ current algebra and adding the fields in a half spinor representation, among which their are 128 new currents.

– The finite simple group F_1 of order $\simeq 8 \cdot 10^{53}$ can be constructed in the following way. One takes a bosonic field on the Leech torus in 24 dimensions and considers the chiral subalgebra invariant under a point reflection on this torus. This subalgebra has four irreducible unitary positive energy representations, with lowest eigenvalues of $L_{\mathbb{C}}$ equal to $0, 1, 3/2, 2$. All these representations are real. Taking the ones labelled by 0 and 1 together yields back the theory on the torus. Adding the chiral fields obtained from the one labelled by 2 yields a new chiral algebra, which has F_1 as its symmetry group[2] .

REFERENCES

1) A.B. Zamolodchikov, Theor.Math.Phys. 65 (1986) 1205

2) I.B. Frenkel, J. Lepowsky and A. Meurman, Proc.Natl.Acad.Sci. USA 81 (1984) 3256; Vertex Operator Algebras and the Monster, Academic Press, New York, to be published

Chiral Gauge Field Theory in Two Dimensions

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Abstract

Chiral gauge field theory in two space–time dimensions, and in particular, the study of the anomalies which arise in such theories, has been the basis for a number of interesting results in recent years. These results are summarized for Abelian gauge field theory and analogous results are presented for non–Abelian field theory.

I. Introduction

The rise of string theory has brought about a renaissance in two–dimensional quantum field theory. In particular, conformal field theory has been demonstrated to be a natural framework from which string theory can be constructed, and much work has been devoted to elucidating its properties [1]. However, there remain other interesting open questions in two–dimensional field theory which deserve consideration. For example, gauge field theories without kinetic terms have recently been discovered to be very useful for the construction of string models. Tomboulis and Porrati have observed that integration over the gauge degrees of freedom on the world sheet provides a mechanism for symmetry breaking in strings, as well as an equivalence to the Thirring model on an arbitrary genus surface [2]. Freedman and Pilch have shown in detail how the Thirring interaction can be solved on an arbitrary Riemann surface in this way [3].

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But even without string theory as a motivation, two-dimensional field theory has been the source of a number of interesting developments. For example, the early work of Coleman [4] and Mandelstam [5] demonstrated the equivalence between fermions and bosons (and their currents) in two dimensions. It has since been shown that any fermi theory in two dimensions is equivalent to a local bose theory which manifestly preserves all the symmetries of the fermi theory (see, for example, reference [6]).

Quantization of anomalous theories has also been studied in the context of two-dimensional models. It was suggested by Fadeev and Shatashvili [7] that the correct way to achieve a gauge invariant quantization of an anomalous theory is through the addition of bosonic degrees of freedom to the action whose gauge variation cancels the anomalous fermionic contribution (also see the work of Harada and Tsutsui [8]). In fact, the earlier work of Polyakov and Weigmann [9], Witten [10], and Alvarev [11] had already noted the relationship between gauge anomalies in fermionic theories and σ-models with Wess-Zumino terms [12].

Two-dimensional field theory has also provided an example of dynamical mass generation. In a study of the chiral Schwinger model [13], Jackiw and Rajamaran [14] have exploited the ambiguity in the regularization of the fermionic determinant to show the existence of a unitary but gauge noninvariant theory containing a dynamically generated mass for the gauge boson.

Also of note is the work of Rothe [15]. Using heat kernel methods, he calculated the anomaly in a theory which contained a single gauge field coupled to Dirac fermions. This work is significant in that it presents a scheme in which the infinitesimal anomaly can be easily integrated and which admits a relatively simple comparison to the results obtained by Polyakov and Weigmann.

Recently, the analysis of anomalous theories has been extended to general non-Abelian gauge fields coupled to chiral fermions [16]. In this paper, we will present a brief discussion of some of the results obtained in recent years for anomalous two-dimensional gauge field theories. We will first revew the case of an Abelian gauge field coupled to chiral fermions, and then present analogous results for a non-Abelian theory.

II. Abelian Field Theory

We begin by reviewing some of the well known properties of Abelian field theory in two Euclidean[1] space-time dimensions. Consider the theory described by the action

$$S = \int d^2x \; \{\overline{\psi}_R \, i\gamma^\alpha D_\alpha \psi_R \; + \overline{\psi}_L \, i\gamma^\alpha D_\alpha \psi_L\}, \tag{2.1}$$

where $D_\alpha = \partial_\alpha - i\mathcal{A}_\alpha$. Note here that the chiral fermions $\psi_{\substack{R\\L}}$ only couple to the "chiral" sectors of the gauge fields $\mathcal{A}_{\substack{R\\L}} = \mathcal{A}\frac{1}{2}(1 \pm \gamma_5)$. The action is invariant under the infinitesimal vector gauge transformation given by

$$\psi_{\substack{R\\L}} \to \psi'_{\substack{R\\L}} = e^{i\varphi}\psi_{\substack{R\\L}}\,; \tag{2.2a}$$

$$\mathcal{A}_\alpha \to \mathcal{A}'_\alpha = \mathcal{A}_\alpha + \partial_\alpha\varphi\,. \tag{2.2b}$$

In addition, due to the well known relationship in two space–time dimensions, $\gamma_\alpha\gamma_5 = i\epsilon_{\alpha\beta}\gamma^\beta$, the action also possesses an additional axial–vector gauge invariance:

$$\psi_{\substack{R\\L}} \to \psi'_{\substack{R\\L}} = e^{\gamma_5\phi}\psi_{\substack{R\\L}} = e^{\pm\phi}\psi_{\substack{R\\L}}\,; \tag{2.3a}$$

$$\mathcal{A}_\alpha \to \mathcal{A}'_\alpha = \mathcal{A}_\alpha - \epsilon_{\alpha\beta}\partial^\beta\phi\,, \tag{2.3b}$$

Therefore, the action possesses a natural $U(1) \times U(1)$ gauge symmetry[2].

Variation of the action with respect to the gauge field yields the constraint $\overline{\psi}_R\gamma^\alpha\psi_R + \overline{\psi}_L\gamma^\alpha\psi_L = 0$; thus classically, we can choose the "decoupling gauge," $\mathcal{A}_\alpha = 0$. This can be understood in the context of the Hodge–DeRahm theorem which states that an arbitrary one–form can be decomposed into the sum of an exact, a co–exact, and a harmonic form, $\mathcal{A} = d\sigma + *d\rho + \alpha^H$. However, there do not exists any globally defined harmonic one–forms in two flat space–time dimensions, so this decomposition is equivalent to writing[3] $\mathcal{A}_\alpha = \partial_\alpha\sigma - i\epsilon_{\alpha\beta}\partial^\beta\rho$. Hence for infinitesimal φ and ϕ, a vector transformation acting on the gauge field is equivalent to $\sigma \to \sigma + \varphi$ and an axial vector transformation is equivalent to $\rho \to \rho - i\phi$. Therefore, appropriate infinitesimal transformations of the form given above, integrated to finite values, allow us to classically gauge away $\mathcal{A}_\alpha$.

However, quantum anomalies present an obstruction to such a transformation. Fujikawa [17] provided a simple means of understanding the mechanism by which anomalies arise in gauge theories. According to Fujikawa, anomalies are due to the noninvariance of the path integral measure under gauge transformations. If we write the partition function as

$$Z = \int [d\mu]e^{-S}, \tag{2.4}$$

[1] We adopt the following Euclidean space conventions: $\gamma_5 = i\gamma^1\gamma^2$; $\gamma^{\alpha\dagger} = -\gamma^\alpha$; $\gamma_5\psi_{\substack{R\\L}} = \pm\psi_{\substack{R\\L}}$, $\gamma_\alpha\gamma_5 = i\epsilon_{\alpha\beta}\gamma^\beta$.

[2] Equivalently, we could consider a theory with Dirac fermions transforming under $\psi \to e^{(i\varphi+\gamma_5\phi)}\psi$ along with a compensating transformation of the gauge field.

[3] We choose the phase of the ρ term so that we can make the decomposition $\mathcal{A}_\alpha = \partial_\alpha\sigma + \gamma_5\partial\rho$. With this choice, $\mathcal{A}_{\substack{R\\L}\alpha} = \partial_\alpha(\sigma \pm \rho)$

then an infinitesimal gauge transformation leaves the action invariant, but the fermionic measure transforms as

$$[d\mu] \rightarrow [d\mu]' = [d\mu]e^{d\Gamma} \tag{2.5}$$

where $d\Gamma$ is the infinitesimal anomaly. This interpretation provides a means by which the anomaly may be calculated for infinitesimal gauge transformations; this result can then be integrated to finite transformations. The result is a theory with free fermions and massive gauge bosons.

However, there is a problem due to an ambiguity in the regularization of the fermionic determinant which arises in the evaluation of the Jacobian.[4] If we consider a system of Dirac fermions and choose a regularization scheme which preserves vector current conservation, the infinitesimal anomaly is

$$d\Gamma_1 \;=\; \tfrac{1}{2\pi}\int d^2x\, \phi\epsilon^{\alpha\beta}\mathcal{F}_{\alpha\beta} = \tfrac{1}{\pi}\int d^2x \epsilon^{\alpha\beta}\mathcal{A}_\alpha\partial_\beta\phi \tag{2.6}$$

If, however, we choose a regularization scheme which preserves the form of the coupling of the left– and right– handed fermions to the gauge field[5], we find an infinitesimal anomaly

$$d\Gamma_2 = -\tfrac{i}{2\pi}\int d^2x g^{\alpha\beta}\mathcal{A}_\alpha\partial_\beta(i\varphi) \;+\tfrac{1}{2\pi}\int d^2x\epsilon^{\alpha\beta}\mathcal{A}_\alpha\partial_\beta\phi. \tag{2.7}$$

This ambiguity was resolved by the observation of Jackiw and Rajaraman [14] that these two forms of the anomaly differ only by the gauge variation of a "local" counterterm (which is an effective mass term for the gauge field):

$$d\Gamma_2 = d\Gamma_1 + \delta S'[\mathcal{A}] \tag{2.8a}$$

$$where$$

$$S'[\mathcal{A}] = \tfrac{1}{4\pi}\int d^2x g^{\alpha\beta}\mathcal{A}_\alpha\mathcal{A}_\beta. \tag{2.8b}$$

Following the observation of Fadeev and Shatashvili [7] that a consistent quantization can be achieved through the addition of bosonic terms to the fermionic action, we look for a bosonic action whose variation reproduces the fermionic anomaly. After a little algebra, we find that if we demand vector current conservation, then the nonvanishing part of the anomaly given in (2.6) is given by $d\Gamma_1 = \delta S_1[\sigma,\rho] = (\delta_R + \delta_L)S_1[\sigma,\rho]$, where

$$S_1[\sigma,\rho] = -\tfrac{1}{2\pi}\int d^2x\rho\Box\rho\,, \tag{2.9}$$

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[4]This is due to the fact that in two dimensions, vector and axial vector transformations are equivalent for chiral fermions. Compare equations (2.2a) and (2.3a).

[5]Such a scheme is natural in string theory where we demand chiral factorization on the world sheet.

while the anomaly which preserves the left–right couplings given in (2.7) is reproduced by the variation of

$$S_2[\sigma,\rho] = -\tfrac{1}{4\pi}\int d^2x(\sigma\Box\sigma + \rho\Box\rho)\,. \tag{2.10}$$

III. Non-Abelian Field Theory

For completeness, we present a brief summary of the results obtained for the non-Abelian model considered in reference [16]. We begin with the action depending on the parameter r,

$$\begin{aligned} S^{(r)} &= \int d^2x\overline{\psi}i\gamma^\alpha[\partial_\alpha - i\mathcal{A}^{(r)}_{R\alpha}\tfrac{1}{2}(1+\gamma_5) - i\mathcal{A}^{(r)}_{L\alpha}\tfrac{1}{2}(1-\gamma_5)]\psi \\ &= \overline{\psi}i\gamma^\alpha[\partial_\alpha - i\tfrac{1}{2}(g_{\alpha\beta} - i\epsilon_{\alpha\beta})\mathcal{A}^{(r)\beta}_R - i\tfrac{1}{2}(g_{\alpha\beta} + i\epsilon_{\alpha\beta})\mathcal{A}^{(r)\beta}_L]\psi, \end{aligned} \tag{3.1}$$

describing the coupling of the "chiral components" of the Lie Algebra valued gauge field $\mathcal{A}^{(r)}_\alpha = \mathcal{A}^{(r)a}_\alpha\tau^a$ to Dirac fermions. We choose to parameterize $\mathcal{A}^{(r)}_\alpha$ by

$$\mathcal{A}^{(r)}_\alpha = \tfrac{1}{2}(g^{\alpha\beta} - i\epsilon^{\alpha\beta})[i\mathcal{U}_r(\partial_\beta\mathcal{U}_r^{-1})] + \tfrac{1}{2}(g^{\alpha\beta} + i\epsilon^{\alpha\beta})[i\mathcal{V}_r(\partial_\beta\mathcal{V}_r^{-1})], \tag{3.2}$$

with $\mathcal{U}_r = e^{r(i\varphi + \phi)}$ and $\mathcal{V}_r = e^{r(i\varphi - \phi)}$.

With such a parameterization, a general, infinitesimal gauge transformation acting on the fermions of the form $\psi \;\rightarrow\; \psi' \;=\; e^{dr(i\varphi+\gamma_5\phi)}\psi$ leaves the action invariant provided the gauge field undergoes a transformation equivalent to transforming the parameter r by $r \rightarrow r + dr$.[6]

We then follow a procedure similar to that used to arrive at (2.6), we find that if we enforce vector current conservation, the transformation given above produces an anomalous response given by

$$d\Gamma_1 \;=\; \tfrac{dr}{2\pi}\int d^2x\; \epsilon^{\alpha\beta}\mathrm{Tr}\;\phi\mathcal{F}^{(r)}_{\alpha\beta}\,, \tag{3.3}$$

where as usual, $\mathcal{F}^{(r)}_{\alpha\beta} \equiv \partial_\alpha\mathcal{A}^{(r)}_\beta - \partial_\beta\mathcal{A}^{(r)}_\alpha - i[\mathcal{A}^{(r)}_\alpha, \mathcal{A}^{(r)}_\beta]$. Equivalently, we can write this in terms of the component fields, $\mathcal{A}^{(r)}_{\substack{R\\L}}$,

$$\begin{aligned} d\Gamma_1 \;=\; & \tfrac{idr}{2\pi}\int d^2x\,\mathrm{Tr}\,\{[g^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha} - \mathcal{A}^{(r)}_{R\alpha}) - i\epsilon^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha} + \mathcal{A}^{(r)}_{R\alpha})]\partial_\beta\phi\} \\ & + \tfrac{dr}{2\pi}\int d^2x\,\mathrm{Tr}\,(g^{\alpha\beta} - i\epsilon^{\alpha\beta})\phi[\mathcal{A}^{(r)}_{L\alpha}, \mathcal{A}^{(r)}_{R\beta}]. \end{aligned} \tag{3.4}$$

We can also calculate the anomaly in a scheme which restricts the gauge fields to appear only in the form in which they couple to the chiral fermions. This leads to the analog of (2.7):

[6]The usefulness such a paramterization should be obvious; the infinitesimal anomaly arising from the infinitesimal gauge transformation can be integrated to finite transformations simply by integrating over dr.

$$
\begin{aligned}
d\Gamma_2 &= -\tfrac{idr}{4\pi}\int d^2x\,\mathrm{Tr}\,\Big[(g^{\alpha\beta}-i\epsilon^{\alpha\beta})\mathcal{A}^{(r)}_{R\beta}\partial_\alpha(i\varphi+\phi) \\
&\qquad\qquad + (g^{\alpha\beta}+i\epsilon^{\alpha\beta})\mathcal{A}^{(r)}_{L\beta}\partial_\alpha(i\varphi-\phi)\Big] \qquad (3.5a)\\
&= -\tfrac{idr}{4\pi}\int d^2x\,\mathrm{Tr}\,\{[g^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha}+\mathcal{A}^{(r)}_{R\alpha}) - i\,\epsilon^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha}-\mathcal{A}^{(r)}_{R\alpha})]\partial_\beta(i\varphi)\}\\
&\quad +\tfrac{idr}{4\pi}\int d^2x\,\mathrm{Tr}\,\{[g^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha}-\mathcal{A}^{(r)}_{R\alpha}) - i\,\epsilon^{\alpha\beta}(\mathcal{A}^{(r)}_{L\alpha}+\mathcal{A}^{(r)}_{R\alpha})]\partial_\beta\phi\}. \qquad (3.5b)
\end{aligned}
$$

Note that once again, the two forms of the anomaly differ only by the variation of a local term,

$$d\Gamma_2 = d\Gamma_1 + \delta S'[\mathcal{A}] \qquad (3.6a)$$

where

$$S'[\mathcal{A}] = \tfrac{1}{4\pi}\int d^2x g^{\alpha\beta}\mathrm{Tr}\,(\mathcal{A}^{(r)}_\alpha\mathcal{A}^{(r)}_\beta). \qquad (3.6b)$$

Finally, note that we can construct bosonic σ–models which reproduce the infinitesimal anomalies given above. We consider a chiral bosonic field $\mathcal{G}(x)$ which transforms under the $G_R \times G_L$ gauge transformations of the chiral fermions as

$$\delta\mathcal{G} = (\delta_R+\delta_L)\mathcal{G} = -\mathcal{G}dr(i\varphi+\phi)dr + (i\varphi-\phi)\mathcal{G}. \qquad (3.7)$$

Then, for example, the form of the anomaly given in (3.5) is reproduced by the variation of the Wess–Zumino model described by the action[7]

$$
\begin{aligned}
S^{(r)}_1[\mathcal{G},\mathcal{A}^{(r)}_R,\mathcal{A}^{(r)}_L] &= \tfrac{i}{12\pi}\int_N \mathrm{Tr}\,(\mathcal{G}^{-1}d\mathcal{G})^3 + \tfrac{1}{4\pi}\int_\Sigma \mathrm{Tr}\,\{\mathcal{G}^{-1}d\mathcal{G}(\mathcal{A}^{(r)}_R + i*\mathcal{A}^{(r)}_R)\}\\
&+\tfrac{1}{4\pi}\int_\Sigma \mathrm{Tr}\,\{d\mathcal{G}\mathcal{G}^{-1}(\mathcal{A}^{(r)}_L - i*\mathcal{A}^{(r)}_L)\} + \tfrac{1}{8\pi}\int_\Sigma \mathrm{Tr}\,[(\mathcal{G}^{-1}d\mathcal{G})(\mathcal{G}^{-1}*d\mathcal{G})]\\
&-\tfrac{i}{8\pi}\int_\Sigma \mathrm{Tr}\,[(\mathcal{A}^{(r)}_L - i*\mathcal{A}^{(r)}_L)\mathcal{G}(\mathcal{A}^{(r)}_R + i*\mathcal{A}^{(r)}_R)\mathcal{G}^{-1}]\\
&+\tfrac{c}{8\pi}\int_\Sigma \mathrm{Tr}\,[(\mathcal{G}^{-1}*\mathcal{D}^{(r)}\mathcal{G})(\mathcal{G}^{-1}\mathcal{D}^{(r)}\mathcal{G})] \qquad (3.8)
\end{aligned}
$$

where N is a three–dimensional manifold whose boundary is two–dimensional space–time Σ. Here $\mathcal{D}^{(r)}\mathcal{G} = d\mathcal{G} - i\mathcal{A}^{(r)}_L\mathcal{G} + i\mathcal{G}\mathcal{A}^{(r)}_R$ is the gauge covariant derivative, and the constant c is undetermined. However, if we require that the bosonic action depend only on the gauge field combinations $(\mathcal{A}^{(r)}_L - i*\mathcal{A}^{(r)}_L)$ and $(\mathcal{A}^{(r)}_R + i*\mathcal{A}^{(r)}_R)$ in which they couple to t he chiral fermions, we can set $c=0$. Obviously, this differs from the action which would reproduce the anomaly given in (3.4) only by the local term given in (3.6b).

[7]In order to simplify this presentation, we adopt a differential form notation with the following conventions: $\mathcal{A} = \mathcal{A}_\alpha dx^\alpha$; $*\mathcal{A} = \epsilon_{\alpha\beta}\mathcal{A}^\alpha dx^\beta$; $**\mathcal{A} = -\mathcal{A}$; $dx^\alpha dx^\beta = \epsilon^{\alpha\beta}d^2x$; $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta^\gamma_\alpha$.

References

[1] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333;
J. L. Cardy, Nucl. Phys. **B240** (1984) 514;
D. Friedan, Z. Qiu and S. Shenkar, Phys. Rev. Lett. **52** (84) 1575;
D. Friedan and S. Shenkar, Nucl. Phys. **B281** (1987) 509;
R. Dijkgraff, E. Verlinde and H. Verlinde, Comm. Math. Phys. **115** (1988) 649, and references therein.

[2] E.T. Tomboulis, Phys. Lett. **198 B** (1987) 165;
M. Porrati and E.T. Tomboulis, Nucl. Phys. **B315** (1989) 615.

[3] D.Z. Freedman and K. Pilch, Phys. Lett. **213 B** (1988) 331; Preprint CPT#1701 (1989).

[4] S. Coleman, Phys. Rev. **D11** (1975) 2088.

[5] S. Mandelstam, Phys. Rev. **D11** (1975) 3026.

[6] E. Witten, Comm. Math. Phys. **92** (1984) 455.

[7] L. Fadeev, Phys. Lett. **145 B** (1984) 81;
L. Fadeev and S. Shatashvili, Phys. Lett. **167 B**.

[8] K. Harada and I. Tsutsui, Phys. Lett. **183 B** (1987) 311.

[9] A.M. Polyakov and P.B. Wiegmann, Phys. Lett. **131 B** (1983) 121; Phys. Lett. **141 B** (1984) 223.

[10] E. Witten, Nucl. Phys. **B223** (1983) 422.

[11] O. Alvarez, Nucl. Phys. **B238** (1984) 61.

[12] J. Wess and B. Zumino, Phys. Lett. **37 B** (1971) 95.

[13] J. Schwinger, Phys. Rev. **128** (1962) 2425.

[14] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. **54** (1985) 1219.

[15] K. D. Rothe, Nucl. Phys. **B269** (1986) 269.

[16] J. Quackenbush, Preprint UCLA/89/TEP/23 (1989).

[17] K. Fujikawa, Phys. Rev. Lett. **42** (1979) 1195; Phys. Rev. Lett. **44** (1980) 1733; Phys. Rev. **D21** (1980) 2824; Phys. Rev. **D22** (1980) 1499; Phys. Rev. **D23** (1981) 2262; Phys. Rev. **D31** (1985) 341.

MONODROMY PROPERTIES OF CONFORMAL FIELD THEORIES

AND QUANTUM GROUPS

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INTRODUCTION

Recently a growing interest in extracting new results from 2D conformal field theories (CFT) going away from criticality [21,22] has motivated the study of the connection between solvable conformal invariant models and the condition of integrability in 2D, the Yang-Baxter equation.

A particular solution of these conditions is a quantum group [8,20], a q-deformation of a Lie group [12]. So naturally it arises the question if quantum groups can be realized in solvable physical models, like CFT [10]. If this is so, then one can introduce perturbations from the critical point that maintain the integrable structure of a quantum group.

Another issue in which quantum groups may be useful is the classification of rational conformal field theories (RCFT). These are theories with a finite number of primary fields. Important key-points in understanding the classification problem are the fusion algebra [5] and the polynomial equations [3]. The new insight is that both properties can be incorporated in a new hidden quantum symmetry [10].

Quantum groups are highly non linear symmetries that appear in statistical models (XXZ Heisenberg chain [18]) or in those 2d models which are solvable with the Bethe ansatz [12]. They have also nice mathematical properties, related to knot theory [4] and to a new category of quantum sets [17].

I discuss here the link between CFT and quantum groups, given by the monodromy properties of the 4-point correlation function. This is the most important correlation function because it contains the product expansion coefficient c_{ijk} that defines the BPZ algebra of the operator product expansion [1]. Using c_{ijk} one can reconstruct all the correlation functions of the theory by analyticity.

The computation of the 4-point correlation function requires a decomposition in conformal blocks, that are no more well defined on complex plane (the space-time of a 2d CFT) but have monodromy around their singularities [11].

It can be shown that these monodromy properties are related to a hidden quantum group structure, that is responsible for the fusion rules of the operator product expansion [10]. I'll point out that the product expansion coefficient cannot be computed using only quantum groups, so a complete derivation of CFT by this new point of view is lacking [13].

The simplest case in which it is possible to analyze in detail this relation is the case of SU(2)-Kac-Moody conformal field theories, in which the computation based on quantum groups can be made explicit [7].

I write down a formula that may be generalized to the case of a generic group, but that uses explicitly the product expansion coefficient which has not been computed in general. The next step is the case of minimal models, in which I'll try another point of view, more natural, the Feigin-Fuks construction [15]. In this construction the monodromy properties are encoded in the charge screening operators, so it is interesting to see if the generators of a quantum group are realized in them.

The actual computation shows that the algebra of the charge screening operators is very similar to the quantum group algebra but it contains a very subtle modification that may be useful to derive the product expansion coefficient. Work is in progress in this direction.

MONODROMY PROBLEM IN CFT

Integrability in CFT means a finite number of primary fields that close the operator product expansion, in other words a RCFT. In general a RCFT is defined by a symmetry algebra:

$$A \; = \; A_L \otimes A_R \tag{1}$$

A_L and A_R are not necessary isomorphic but for simplicity I treat the symmetric case. The Hilbert space of states can be expressed as a direct su of irreducible representations of A. Each of them is generated by a primary field $\phi_i(z)$, and the other states are obtained acting on it with the creation operators of the algebra A.

A RCFT is characterized by a finite number of primary fields. Another basic property, that belongs to any CFT, is the associativity of the operator product expansion (OPE). The OPE rule is:

$$\phi_i(z)\phi_j(w) \; = \; \sum_p C^p_{ij} \; (z-w)^{-\Delta_i-\Delta_j+\Delta_p} \; \phi_p \; + \; \text{descendants} \tag{2}$$

The coefficients Δ_i are the conformal dimensions of the primary fields $\phi_i(z)$.

The associativity of the OPE means that physical correlation functions of CFT are single-valued and independent of the order of the conformal fields. Using OPE it is possible to decompose physical correlation functions as a sum over intermediate channels of tree level diagrams. Each channel is called conformal block.

Let's consider the 4-point correlation function $G(z,\overline{z})$ in the special points (0, 1, z, ∞). For simplicity the indices of the external lines are suppressed. The crossing symmetry of $G(z,\overline{z})$ means that the s-channel and t-channel decomposition are equivalent:

$$G(z,\overline{z}) \; = \; \sum_{p,p'} X_{pp'} I_p(z)\overline{I_{p'}(z)} \; = \; \sum_{q,q'} \tilde{X}_{qq'}\tilde{I}_q(z)\overline{\tilde{I}_{q'}(z)} \tag{3}$$

The function $I_p(z)$ has its singular points in 0, 1, and ∞. If we continue z analytically along a closed contour around the point 0 or 1, the function $I_p(z)$ is transformed with a matrix g_0 or g_1. These matrices are the generating elements of the monodromy group transformations, because g_∞ is a linear combination of g_0 and g_1, so we have to discuss only two points separately.

The monodromy problem in CFT is to define the coefficients $X_{pp'}$ in order that G(z,$\overline{z}$) is monodromy invariant. In the s-channel g_0 is a diagonal matrix, so the g_0-invariance is satisfied if $X_{pp'}$ is diagonal:

$$X_{pp'} \; = \; \delta_{pp'} X_p \tag{4}$$

The matrix g_1 has a diagonal form for another base of conformal functions (t-channel), that is related to $I_p(z)$ by a matrix F, called fusion matrix:

$$I_p(z) = \sum_q F_{pq} \tilde{I}_q(1-z) \tag{5}$$

The new coefficient of G(z,$\bar{z}$) is:

$$\tilde{X}_{qq'} = \sum_p X_p F_{pq} F_{pq'} \tag{6}$$

and the request of g_1-invariance is satisfied if $\tilde{X}_{qq'}$ is diagonal:

$$\sum_p X_p F_{pq} F_{pq'} = 0 \qquad q \neq q' \tag{7}$$

If the matrix F_{pq} is known, the coefficient X_p can be found easily. The 4-point correlation function can be computed once that the fusion matrix is known. The product expansion coefficient can be related to the fusion matrix if the conformal field theory is scalar, i.e. there is no difference between holomorphic and antiholomorphic part. In fact it can be shown that:

$$X_p = C_{ij}^{p} C_{klp} \tag{8}$$

$$C_{ijk} = \frac{F_{0i}\left[\begin{smallmatrix} j & j \\ k & k \end{smallmatrix}\right]}{F_{i0}\left[\begin{smallmatrix} j & k \\ j & k \end{smallmatrix}\right]} \tag{9}$$

The fusion matrix is related to the s-t duality.

Another matrix can be introduced for s-u duality, the braiding matrix B, given explicitly by:

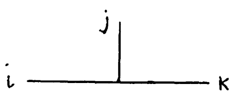
$$\phi_i(z_1)\phi_j(z_2) = \sum_{k,l} B_{i,j}^{k,l}\ \phi_k(z_2)\phi_l(z_1) \tag{10}$$

In the next chapter I will try to explain how to relate the fusion and braiding matrices to quantum groups.

POLYNOMIAL EQUATIONS

Duality is fundamental in 2d in order to have a consistent operator formalism. In terms of conformal blocks it means that the correlation function is independent of the basis of blocks, and hence there exist two basic duality matrices describing the change in the representation content of conformal blocks.

The basic building blocks of the theory are the three-point function:



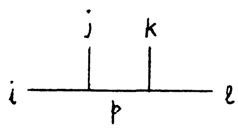
$$= V_{0,ijk} \tag{11}$$

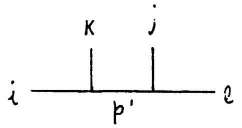
A dual description is obtained introducing an operator language for these blocks. If the fusion rules are respected (that means i, j , k are compatible in the OPE), we can construct a chiral vertex operator $\phi\left(\begin{smallmatrix} & j & \\ i & & k \end{smallmatrix}\right)(z)$ mapping a primary field (of the chiral algebra A_L or A_R) ϕ_k in ϕ_i. $\phi\left(\begin{smallmatrix} & j & \\ i & & k \end{smallmatrix}\right)(z)$ is a primary field of the chiral algebra with respect to the label j. The conformal blocks are just obtained sewing (like tensor product) the space V of chiral vertices.

Let us specialize to the 4-point function. In order to compare duality with the integrability conditions in 2d, it is useful consider B and F as linear operators on the tensor product of the vector space of the chiral vertex operators V.

$$F : V \otimes V \to V \otimes V \tag{12}$$

$$F : \phi\left(\begin{smallmatrix} i_1 & \\ j_1 & p \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} p & \\ j_2 & k_2 \end{smallmatrix}\right)(z) \to \phi\left(\begin{smallmatrix} i_1 & \\ q & k_2 \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} q & \\ j_1 & j_2 \end{smallmatrix}\right)(z) \tag{13}$$

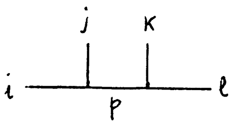


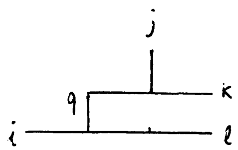
$$= \sum_{p'} B_{pp'}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \tag{14}$$


The difference in their action is hidden in the difference of sewing the indices of the chiral vertex operators:

$$B : V \otimes V \to V \otimes V \tag{15}$$

$$B : \phi\left(\begin{smallmatrix} i_1 & \\ j_1 & p \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} p & \\ j_2 & k_2 \end{smallmatrix}\right)(z) \to \phi\left(\begin{smallmatrix} i_1 & \\ j_2 & q \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} q & \\ j_1 & k_2 \end{smallmatrix}\right)(z) \tag{16}$$



$$= \sum_{q} F_{pq}\left[\begin{smallmatrix} j & k \\ i & l \end{smallmatrix}\right] \tag{17}$$


F must also satisfy a normalization condition when the 4-point function is trivial:

$$F\left(\phi\left(\begin{smallmatrix} i & \\ j & k \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} k & \\ k & 0 \end{smallmatrix}\right)(z)\right) = \phi\left(\begin{smallmatrix} i & \\ i & 0 \end{smallmatrix}\right)(z) \star \phi\left(\begin{smallmatrix} i & \\ j & k \end{smallmatrix}\right)(z) \tag{18}$$

Moore and Seiberg [3] have listed all the constraints that the duality matrices have to obey. Between them, associativity leads to a pentagonal equation for F and an hexagonal one for B which is the Yang-Baxter equation:

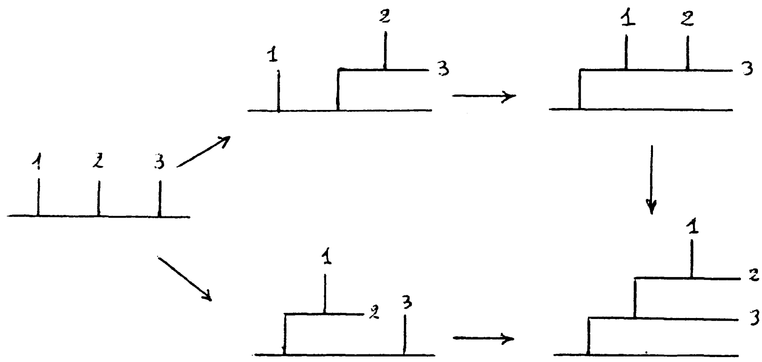
$$F_{23}\, F_{13}\, F_{23} = P_{23}\, F_{13}\, F_{12} \tag{19}$$

$$B_{12}\, B_{23}\, B_{12} = B_{23}\, B_{12}\, B_{23} \tag{20}$$

These formulas, written explicitly in component and diagrammatically, look like:

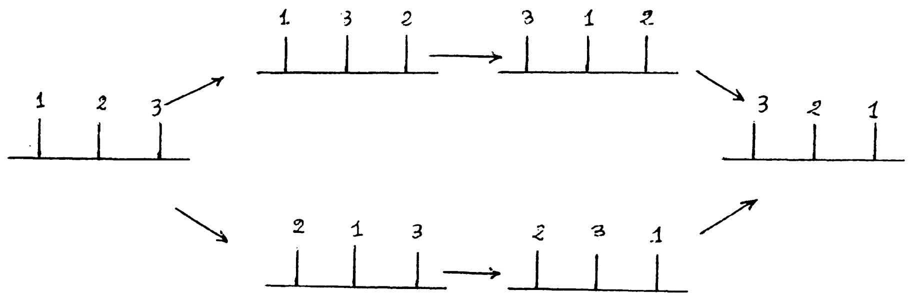
$$\sum_s F_{as}\left[\begin{smallmatrix} j & k \\ i & b \end{smallmatrix}\right] F_{bp}\left[\begin{smallmatrix} s & l \\ i & m \end{smallmatrix}\right] F_{sq}\left[\begin{smallmatrix} k & l \\ j & p \end{smallmatrix}\right] =$$

$$= F_{ap}\left[\begin{smallmatrix} j & q \\ i & m \end{smallmatrix}\right] F_{bq}\left[\begin{smallmatrix} k & l \\ a & m \end{smallmatrix}\right] \tag{21}$$



$$\sum_{c} B_{ic}\left[\begin{matrix} a & j \\ b & p \end{matrix}\right] B_{pd}\left[\begin{matrix} a & k \\ c & l \end{matrix}\right] B_{cc'}\left[\begin{matrix} j & k \\ b & d \end{matrix}\right] =$$

$$= \sum_{p'} B_{pp'}\left[\begin{matrix} j & k \\ i & l \end{matrix}\right] B_{ic'}\left[\begin{matrix} a & k \\ b & p' \end{matrix}\right] B_{p'd}\left[\begin{matrix} a & j \\ c' & l \end{matrix}\right] \tag{22}$$



Combining braiding and fusion operations leads to another consistency condition:

$$B = \Omega_1 \, F \, \Omega_2 \tag{23}$$

where Ω can be computed from the product expansion coefficient. Its action on a chiral vertex operator t is given by:

$$\Omega \, t = \sigma_{jk} \, e^{i\pi\Delta_t} \, t \tag{24}$$

The relation (23) between braiding and fusion matrices in components is:

$$F_{pp'}\left[\begin{matrix} j & k \\ i & l \end{matrix}\right] = e^{[i\pi(\Delta_p + \Delta_{p'} - \Delta_j - \Delta_l)]} \, B_{pp'}\left[\begin{matrix} i & k \\ j & l \end{matrix}\right] \tag{25}$$

Therefore, the only difference between s-u duality and s-t duality is a phase, that can be calculated from the three-point block.

The simplest case is SU(2) kac-Moody conformal field theory in which the fusion rules are similar to the classical spin [2,16]. The braiding and fusion matrices can be represented diagrammatically as a tetrahedron. This, of course, is an index that the possible candidate for the fusion matrix of a SU(2) Kac-Moody conformal field theory is a 6j-symbol of SU(2) quantum group. In general, for more general groups, one can consider the analogous concept of 6j-symbol.

The properties of quantum 6j-symbols are listed in [9]. Between them, one is particular interesting, because it is a pentagonal identity, as we need to solve the constraint of associativity (21):

$$\begin{aligned}\sum_s \left\{ \begin{matrix} j & k & s \\ i & b & a \end{matrix} \right\}_q \left\{ \begin{matrix} s & l & p \\ m & i & b \end{matrix} \right\}_q \left\{ \begin{matrix} k & l & q \\ p & j & s \end{matrix} \right\}_q &= \\ = \quad \left\{ \begin{matrix} j & q & p \\ m & i & a \end{matrix} \right\}_q \left\{ \begin{matrix} k & l & q \\ m & a & b \end{matrix} \right\}_q & \end{aligned} \tag{26}$$

So a natural candidate for the fusion matrix is:

$$F_{pq}\left[\begin{matrix} j & k \\ i & l \end{matrix} \right]. = \left\{ \begin{matrix} j & k & q \\ l & i & p \end{matrix} \right\}_q \tag{27}$$

There is unfortunately a normalization problem, due to the fact that the conformal blocks are not normalized to 1, because of the vertex coupling. In fact if one reconsider the relation between product expansion coefficient and fusion matrix (9),the contribution of the quantum 6j-symbol to c_{IJK} is 1:

$$c_{IJK} \neq \frac{\left\{ \begin{matrix} J & J & I \\ K & K & 0 \end{matrix} \right\}}{\left\{ \begin{matrix} J & K & 0 \\ J & K & I \end{matrix} \right\}} = 1 \tag{28}$$

The required normalization corresponds to a non trivial redefinition of the solution of the polynomial equations (non a simple overall constant).

One way to compute the normalization is to use the explicit solution of the differential equation, the conformal blocks, and calculate the monodromy group directly, as in [7]. This cannot be done in general, but only for the spin 1/2.

Another way is to use directly the link between the product expansion coefficient and the fusion matrix. The only possible combination that preserves the polynomial equation and satisfies the normalization condition is:

$$F_{pq}\left[\begin{matrix} j & k \\ i & l \end{matrix} \right] = \sqrt{\frac{C_{jkq}\, C_{ilq}}{C_{ijp}\, C_{klp}}} \left\{ \begin{matrix} j & k & q \\ l & i & p \end{matrix} \right\}_q \tag{29}$$

The index symmetry property of the product expansion coefficient is crucial for the cancellation.

ALGEBRA OF CHARGE SCREENING OPERATORS

In order to discuss the monodromy properties of the 4-point correlation function in the case of minimal models it is useful to use the representation of Feigin-Fuks [15,19].

The minimal series are labelled by an integer m [1]. The central charge of the Virasoro algebra c and the charge at the infinity λ of the Feigin-Fuks construction are given by:

$$c = 1 - \frac{6}{m(m+1)} \tag{30}$$

$$\lambda = \frac{1}{m+1} \tag{31}$$

The Feigin-Fuks representation is very powerful [11] because it allows computing the correlator in term of vertex operators of a free bosonic theory and charge screening operators, given by:

$$S_{\pm} \;=\; \int_C dz\; J_{\pm}(z) \tag{32}$$

These operators contain a dependence from the contour C.

In order that $S_{\pm}$ have zero conformal dimension, $e_{\pm}$ must be combined to be:

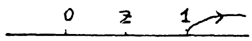
$$h(e_{\pm}) \;=\; 1 \qquad e_{\pm} \;=\; \lambda \;\pm\; \sqrt{\lambda \;+\; \frac{4m}{m+1}} \tag{33}$$

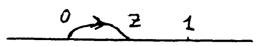
The simplest non trivial case, the 4-point correlator with the insertion of the least non trivial conformal operator ϕ_{12} is given by the integral ($\alpha_i \;=\; \alpha_{n_i m_i}$) :

$$< \phi_{n_1 m_1}(0)\; \phi_{12}(z)\; \phi_{n_3 m_3}(1) \phi_{n_4 m_4}(\infty) > \;=$$

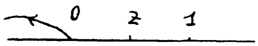
$$= \int_C dt < V_{\alpha_1}(0) V_{\alpha_2}(z) V_{\alpha_3}(1) V_{\alpha_4}(\infty) J_+(t) > \tag{34}$$

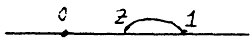
There is only one insertion of charge screening operators in the correlator, and the possible choices of contours are two in correspondence with the number of conformal blocks. Then the conformal blocks can be represented by different choices of the same integral. For example the base in the s-channel is given by the contour:

$I_1(z) \;=$  (35)

$I_2(z) \;=$  (36)

Instead the base in the t-channel is given by:

$\tilde{I}_1(z) \;=$  (37)

$\tilde{I}_2(z) \;=$  (38)

Each contour can be obtained as a combination of the other base.

The monodromy properties of minimal models reduce hence to a discussion of different choices of contours. So it is interesting to see if the generators of a quantum group are realized in the charge screening operators. The actual computation gives [14]:

$$[S_z, S_+] \;=\; \frac{m+1}{m} S_+ \tag{39}$$

$$[S_z, S_-] \;=\; -S_- \tag{40}$$

$$[S_+, S_-] \;=\; \frac{q^{S_z} \;-\; q^{-S_z}}{q \;-\; q^{-1}} e^{-\lambda\phi} \tag{41}$$

This algebra is associative, but it is not closed for the presence of $e^{-\lambda\phi}$. Basically there appear two modifications from the algebra of a quantum group, one in the charge of S_+ and the other one in (41).

These two modifications are consistent between them, because the difference in charge between S_+ and S_- requires an extra term to balance the charge in the second part of (41). These modifications are probably useful for the computation of the product expansion coefficient. Work is in progress in this direction.

REFERENCES

1. Belavin, A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two dimensional quantum field theory, Nucl. Phys. B 241, 33 (1984)
2. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess-Zumino model in two dimensions, Nucl. Phys. B 247, 83 (1984)
3. Moore, G., Seiberg, N.: Polynomial equation for rational conformal field theories, Phys. Lett. B 212, 451 (1988)
4. Witten E.: Quantum field theory and the Jones polynomial. To appear in the Proc. of the IAMP Congress, Swansea, July, 1988
5. Verlinde, E.: Fusion rules and modular transformations in 2d conformal field theory, Nucl. Phys. B 300, 360 (1988)
6. Brustein, R., Yankielowicz, S., Zuber, J.-B.: Factorization and selection rules of operator product algebras in conformal field theory, TAUP-1647-88
7. Tsuchiya, A., Kanie,Y.: Vertex operators in the conformal field theory on P1 and monodromy representations of the braid group: Conformal field theory and solvable lattice models. Adv. Stud. Pure Math. 16, 297 (1988)
8. Drinfeld, V.: Quantum Groups: Proc. at the Intl. Cong. of Math. 1986, 798
9. Kirillov, A.N., Reshetikhin, N.Y.: Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, LOMI preprint E-9-88
10. Alvarez-Gaume' L., Gomez C., Sierra G.: Hidden quantum symmetries in rational conformal field theories, Nucl. Phys. B319 (1989), 155
11. Dotsenko V.S., Fateev V.A., Conformal algebra and multipoint correlation functions in 2d statistical models, Nucl. Phys. B240 (1984) 312; Four point correlation functions and the operator algebra in the 2d conformal invariant theories with the central charge $c < 1$, Nucl. Phys. B251 (1985) 691; Phys. Lett. B 154 91985) 291
12. Faddeev L.D., Reshetikhin N. Yu., Takhtajan L.A.: Quantization of Lie groups and Lie algebras, LOMI preprint E-14-87
13. Valtancoli P., Phd thesis: Simmetrie quantistiche nascoste in teorie di gauge e conformi, Pisa University (1988)
14. Kato M., Valtancoli P., in preparation
15. Feigin B.L., Fuks D.B., Verma modules over the Virasoro algebra, Funktsional'nyi Analiz i ego Prilozheniya 17, N0. 3 (1983), 91-92
16. Gepner D., Witten E.: String theory on a group manifold, Nucl. Phys. B 278 (1986),493
17. Connes A., Cyclic cohomology and non commutative differential geometry, Proc. Intl. cong. Math., Berkeley (1986)
18. Pasquier V., Saleur H.: Symmetries of the XXZ chain and quantum SU(2), to be published in Les Houches
19. Zuber J.B.: Conformal field theories, Coulomb gas picture and integrable models, Lectures given at Les Houches Summer School on particle Physics, (1988)
20. Jimbo M.: A q-difference analogue of U(g) and the Yang-Baxter equation, Lett. Math. Phys. 10 (1985), 63
21. Eguchi T., Yang S.K.: Deformations of conformal field theories and soliton equations, Phys. Lett. B 224 (1989), 373
22. Zamolodchikov A.B., JETP Lett. 43 (1986), 730

MATRIX ELEMENTS OF UNITARY REPRESENTATION OF THE QUANTUM GROUP SU_q (1,1) AND THE BASIC HYPERGEOMETRIC FUNCTIONS

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§0. Introduction In this paper, we will study matrix elements of unitary representations of the quantum group $SU_q(1,1)$. We begin with classification of real forms of the universal quantum enveloping algebra $\mathfrak{U}_q(\mathfrak{sl}(2))$ (§1), and next consider the structure of a topological quantum group $\mathcal{A}$ associated with this algebra (§2). In §3, we "exponentiate" a family of infinite dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}(2))$, and determine the matrix elements in $\mathcal{A}$ in terms of the basic hypergeometric functions in the q-analogue analysis. The differential representation of $\mathfrak{U}_q(\mathfrak{sl}(2))$ in $\mathcal{A}$ provides us a fundamental tool for this procedure. In § 4, introducing the unitary structure of the real form $\mathfrak{U}_q(\mathfrak{su}(1,1))$, we classify series of unitary representations of the quantum group $SU_q(1,1)$. Throughout this paper, $\mathbb{Z}$ denotes the set of integers, $\mathbb{N}$ the set of non-negative integers. $(a;q)_m = \prod_{r=0}^{m-1}(1-aq^r)$ is the q-shifted factorial. For the details of the contents in this paper, the readers should refer to [6].

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§1. Quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$.

The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ is the algebra over $\mathbb{C}$ with a complex parameter q ($q \neq 0, \pm 1$) generated by $k^{\pm 1}$, e, f with the following relations [2,4];

$$(1.1) \qquad kek^{-1} = qe, \quad kfk^{-1} = q^{-1}f, \quad [e,f] = \frac{k^2 - k^{-2}}{q - q^{-1}} .$$

This is a non-commutative and non-cocommutative Hopf algebra with the coproduct $\Delta : \mathcal{U}_q(\mathfrak{sl}(2)) \longrightarrow \mathcal{U}_q(\mathfrak{sl}(2)) \otimes \mathcal{U}_q(\mathfrak{sl}(2))$ given by

$$(1.2) \qquad \Delta(k) = k\otimes k, \ \Delta(e) = e\otimes k + k^{-1}\otimes e, \ \Delta(f) = f\otimes k + k^{-1}\otimes f.$$

The counit $\varepsilon : \mathcal{U}_q(\mathfrak{sl}(2)) \longrightarrow \mathbb{C}$ is defined on the generators as

$$(1.3) \qquad \varepsilon(k) = 1, \quad \varepsilon(e) = \varepsilon(f) = 0 .$$

The antipode $S : \mathcal{U}_q(\mathfrak{sl}(2)) \longrightarrow \mathcal{U}_q(\mathfrak{sl}(2))$ is defined by

$$(1.4) \qquad S(k) = k^{-1}, \quad S(e) = -qe, \quad S(f) = -q^{-1}f .$$

In the sequel we assume that $q^m \neq 1$ for any integer m.

A real form of $\mathcal{U}_q(\mathfrak{sl}(2))$ is regarded as a pair of $\mathcal{U}_q(\mathfrak{sl}(2))$ and a * structure of it. We classify all real forms of $\mathcal{U}_q(\mathfrak{sl}(2))$ [6].

<u>Proposition 1</u>. Real forms of $\mathcal{U}_q(\mathfrak{sl}(2))$ are classified, up to equivalence, as follows:

$\mathcal{U}_q(\mathfrak{su}(2))$ with $-1<q<1$ ($q\neq 0$) (a compact real form). The * structure is

$$(1.5) \qquad k^* = k, \quad e^* = f, \quad f^* = e.$$

$\mathcal{U}_q(\mathfrak{su}(1,1))$ with $-1<q<1$ ($q\neq 0$) (a non-compact real form). The * structure is

$$(1.6) \qquad k^* = k, \quad e^* = -f, \quad f^* = -e.$$

$\mathfrak{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ with $|q|=1$ (a non-compact real form). The $*$ structure is

(1.7) $\qquad k^* = k, \quad e^* = -e, \quad f^* = -f.$

We observe that real Lie algebras $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2,\mathbb{R})$ are equivalent via the Cayley transform in the classical limit $(q \longrightarrow 1)$ while in the quantum level these two real forms are not equivalent.

§2. A topological quantum group associated with $\mathfrak{U}_q(\mathfrak{sl}(2))$.

We first clarify the notion of a topological quantum group. Let $\mathcal{A} = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{U},\mathbb{C})$, the full dual space of $\mathfrak{U} = \mathfrak{U}_q(\mathfrak{sl}(2))$. We introduce the weak $*$ topology in $\mathcal{A}$. Namely, we say that a sequence φ_j converges to φ in $\mathcal{A}$ if $\varphi_j(a)$ converges to $\varphi(a)$ for any $a \in \mathfrak{U}$. With this topology, $\mathcal{A}$ is complete. We also introduce the weak $*$ topology in $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{U}^{\otimes n},\mathbb{C})$. We note that the algebraic tensor product $\mathcal{A}^{\otimes n}$ is dense in $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{U}^{\otimes n},\mathbb{C})$. Hence the topological tensor product $\mathcal{A}^{\hat{\otimes} n} = \mathcal{A} \hat{\otimes}_w \mathcal{A} \hat{\otimes}_w \cdots \hat{\otimes}_w \mathcal{A}$ is identified with $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{U}^{\otimes n},\mathbb{C})$. Then $\mathcal{A}$ is endowed with a structure of Hopf algebra: The multiplication morphism in $\mathcal{A}$,

$\mu_{\mathcal{A}} : \mathcal{A} \hat{\otimes}_w \mathcal{A} \longrightarrow \mathcal{A}$ is defined by

(2.1) $\qquad \mu_{\mathcal{A}}(\Phi)(a) = \Phi(\Delta(a)) \quad$ for $\Phi \in \mathcal{A} \hat{\otimes}_w \mathcal{A}$, $\ a \in \mathfrak{U}$.

With this multiplication, $\mathcal{A}$ is a topological associative algebra (the unit in $\mathcal{A}$ is the counit ε in $\mathfrak{U}$). The coproduct

$\Delta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A} \hat{\otimes}_w \mathcal{A}$ is given by

(2.2) $\qquad \Delta_{\mathcal{A}}(\varphi)(a \otimes b) = \varphi(a \cdot b) \quad$ for $\varphi \in \mathcal{A}$, $a, b \in \mathfrak{U}$

which is continuous and coassociative. The counit $\varepsilon_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathbb{C}$ and the antipode $S_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ are continuous morphisms defined through duality;

$$(2.3)\qquad \varepsilon_{\mathcal{A}}(\varphi) = \varphi(1) \quad (\varphi\in\mathcal{A},\ \text{and } 1 \text{ is the unit of } \mathfrak{U}),$$

$$(2.4)\qquad S_{\mathcal{A}}(\varphi)(a) = \varphi(S(a)) \quad \text{for } \varphi\in\mathcal{A},\ a\in\mathfrak{U}.$$

We refer to $\mathcal{A}$ as a *topological quantum group* associated with $\mathfrak{U}$.

Now we consider the *differential representations* of $\mathfrak{U}$ on the topological quantum group $\mathcal{A}$. For $a\in\mathfrak{U}$, we set a linear operator $\pi_\ell(a)$ (resp. $\pi_r(a)$, $a\in\mathfrak{U}$) in $\mathcal{A}$ by

$$(2.5)\qquad [\pi_\ell(a)\cdot\varphi](b) = \varphi(b\cdot a) \quad (\text{resp. } [\pi_r(a)\varphi](b) = \varphi(a\cdot b)\ )$$

for $\varphi\in\mathcal{A}$, $b\in\mathfrak{U}$. Such linear operators satisfy the following properties (cf. [6]).

<u>Proposition 2</u>.

(i) An operator $\pi_\ell(a)$ (resp. $\pi_r(a), a\in\mathfrak{U}$) is left invariant (resp. right invariant) in the sense that

$$(\mathrm{id}_{\mathcal{A}}\otimes\pi_\ell(a))\circ\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}}\circ\pi_\ell(a)$$

$$(\ \text{resp. } (\pi_r(a)\otimes\mathrm{id}_{\mathcal{A}})\circ\Delta_{\mathcal{A}} = \Delta_{\mathcal{A}}\circ\pi_r(a)\ ).$$

(ii) $\pi_\ell(ab) = \pi_\ell(a)\pi_\ell(b)$

(resp. $\pi_r(ab) = \pi_r(b)\pi_r(a)$) for $a,b\in\mathfrak{U}$.

(iii) Let $\Delta(a) = \sum_\mu a_\mu\otimes a^\mu$ $(a\in\mathfrak{U})$. Then

$$\pi_\ell(a)\varphi\psi = \sum_\mu \pi_\ell(a_\mu)\varphi\cdot\pi_\ell(a^\mu)\psi\ .$$

In particular we have the *twisted derivation rule*;

$$\pi_\ell(k)\varphi\psi = \pi_\ell(k)\varphi\cdot\pi_\ell(k)\psi\ ,$$

$$\pi_\ell(e)\varphi\psi = \pi_\ell(e)\varphi\cdot\pi_\ell(k)\psi + \pi_\ell(k^{-1})\varphi\cdot\pi_\ell(e)\psi\ ,$$

$$\pi_\ell(f)\varphi\psi = \pi_\ell(f)\varphi\cdot\pi_\ell(k)\psi + \pi_\ell(k^{-1})\varphi\cdot\pi_\ell(f)\psi\ ,$$

The same formulas hold for $\pi_r(a)$.

(vi) Let $\Delta_{\mathcal{A}}(\varphi) = \sum_\mu \varphi_\mu\otimes\varphi^\mu$ $(\varphi\in\mathcal{A})$. Then

$$\pi_\ell(a)\varphi = \sum_\mu \varphi^\mu(a)\cdot\varphi_\mu \quad (\text{resp. } \pi_r(a)\varphi = \sum_\mu \varphi_\mu(a)\varphi^\mu\ ).$$

(v) Operators $\pi_\ell(a)$, $\pi_r(a)$ are continuous in the weak * topology.

Thus the topological quantum group $\mathcal{A}$ is a $(\mathfrak{U},\mathfrak{U})$-bi-module via

$$a\cdot\varphi = \pi_\ell(a)\varphi \ , \qquad \varphi\cdot a = \pi_r(a)\varphi \ .$$

Next we see that the coordinate ring $A(SL_q(2))$ of the quantum group $SL_q(2)$ is realized as a Hopf subalgebra of $\mathcal{A}$ [2].

The natural representation of $\mathfrak{U}$ is a two dimesional left $\mathfrak{U}$-module $V_\square = \mathbb{C}\xi_0 \oplus \mathbb{C}\xi_1$ on which the action is prescribed by

$$(2.6) \qquad k\cdot\xi_j = q^{\frac{1}{2}-j}\xi_j \ , \quad e\cdot\xi_j = \delta_{1j}\xi_{j-1} \ , \quad f\cdot\xi_j = \delta_{0j}\xi_{j+1} \qquad (j=0,1).$$

Then $V_\square$ is canonically viewed as a right $\mathcal{A}$-comodule, $R_\square : V_\square \longrightarrow V_\square \otimes \mathcal{A}$. The right coaction $R_\square$ determines the elements x, u, v, $y \in \mathcal{A}$ by the formula

$$(2.7) \qquad R_\square(\xi_0) = \xi_0\otimes x + \xi_1\otimes v \ , \quad R_\square(\xi_1) = \xi_0\otimes u + \xi_1\otimes y \ .$$

Indeed, by putting $X = \begin{pmatrix} x & u \\ v & y \end{pmatrix}$,

we have

$$X(k) = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} , \quad X(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

$$X(f) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} ,$$

and

$$(2.8) \qquad X(ab) = X(a)\cdot X(b) \quad \text{for any } a,b\in\mathfrak{U}.$$

This is equivalent to say that

$$(2.9) \qquad \Delta_{\mathcal{A}}(X) = X\otimes X$$

(the above tensor product "⊗" is an abbreviated notations standing for $\Delta_{\mathcal{A}}(x) = x\otimes x + u\otimes v$, etc.).

The elements $x,u,v,y \in \mathcal{A}$ are called *coordinate elements* on $\mathcal{A}$. Considering the commutant algebra of $V_{\square}^{\otimes 2}$ ([2,4]), we obtain

$$(2.10) \qquad qxu = ux, \quad qxv = vx, \quad quy = yu, \quad qvy = yv, \quad uv = vu,$$

$$\text{and} \quad \det{}_q X = xy - q^{-1}uv = yx - quv = 1_{\mathcal{A}} ,$$

($\det_q X$ is referred to as the quantum determinant of X). The ring $A(SL_q(2))$ is a Hopf algebra generated by the coordinate elements. The counit and the antipode on $A(SL_q(2))$ are described as

$$(2.11) \qquad \varepsilon_{\mathcal{A}}(X) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S_{\mathcal{A}}(X) = X^{-1} = \begin{pmatrix} y & -qu \\ -q^{-1}v & x \end{pmatrix} .$$

From characterization of monomial elements $x^L u^M v^N$ and $u^M v^N y^L$ in $\mathcal{A}$ as linear functionals on $\mathfrak{U}$ [6], we see that a basis of $A(SL_q(2))$ is given by

$$A(SL_q(2)) = \sum_{0\le L,M,N}^{\oplus} \mathbb{C}x^L u^M v^N \oplus \sum_{0<L,\,0\le M,N}^{\oplus} \mathbb{C}u^M v^N y^L .$$

Now we define the set of *formally analytic elements* in $\mathcal{A}$ by

$$A[[SL_q(2)]] = \Sigma x^m v^n \mathbb{C}[[\xi]] + \Sigma x^m u^n \mathbb{C}[[\xi]] + \Sigma \mathbb{C}[[\xi]] u^n y^m + \\ + \Sigma \mathbb{C}[[\xi]] v^n y^m,$$

where the element ξ is given by $\xi = -q^{-1}uv$ [5] and $m,n \in \mathbb{N}$.

<u>Proposition 3</u>. (i) Any element of $A[[SL_q(2)]]$ belongs to $\mathcal{A}$.

(ii) Let $\varphi = \sum_{0\le L,M,N} \varphi_{LMN}\, x^L u^M v^N + \sum_{0<L,\,0\le M,N} \varphi_{-L,MN}\, u^M v^N y^L$

be an element of $A[[SL_q(2)]]$. Then $\varphi = 0$ in $\mathcal{A}$ if and only if all the coefficients of φ vanish.

Hence $A[[SL_q(2)]]$ is a subalgebra of $\mathcal{A}$. One should note that $A[[SL_q(2)]]$ is not a Hopf algebra.

In order to investigate the structure of $A[[SL_q(2)]]$, we introduce a quantum subgroup of $\mathcal{A}$. Let $\mathcal{T} = \mathbb{C}[k^{\pm 1}]$, a Hopf subalgebra of $\mathfrak{U}$, and set $\mathcal{K} = \mathrm{Hom}_{\mathbb{C}}(\mathcal{T},\mathbb{C})$, the full dual space of $\mathcal{T}$. With the weak $*$ topology, $\mathcal{K}$ is a topological Hopf algebra, and we have a surjective Hopf algebra homomorphism

$$\mathcal{A} \longrightarrow \mathcal{K} \quad (\varphi \longmapsto \varphi|_{\mathcal{T}}).$$

We regard $\mathcal{K}$ to correspond to the maximal compact subgroup of $SL_q(2)$. Let us consider *relatively invariant elements* of $\mathcal{A}$ with respect to $\mathcal{K}$ (cf. [5]). We set

$$(2.12) \qquad \mathcal{A}[m,n] = \{\varphi\in\mathcal{A} \mid \pi_\ell(k)\varphi = q^{m/2}\varphi , \ \pi_r(k)\varphi = q^{n/2}\varphi \},$$

where m,n are integers. Then we obtain

<u>Proposition 4</u>. (i) $\mathcal{A}[m,n] = \{0\}$ unless $m\equiv n \bmod 2\mathbb{Z}$.

(ii) $$A[[SL_q(2)]] = \bigoplus_{m\equiv n \bmod 2\mathbb{Z}} \mathcal{A}[m,n],$$

and each component $\mathcal{A}[m,n]$ is given as, for example,

$$\mathcal{A}[m,n] = x^{\frac{1}{2}(m+n)} v^{\frac{1}{2}(m-n)} \mathbb{C}[[\xi]] \qquad \text{for } 0\le m+n,\ n\le m.$$

§3. Exponentiating infinite dimensional representations of $\mathfrak{U}_q(\mathfrak{sl}(2))$.

First we introduce an infinite dimensional left $\mathfrak{U}$ module $V_\ell = \bigoplus_{j\in I_\ell} \mathbb{C}\xi_j$, where ℓ is a complex spin, and the index set I_ℓ is a subset of $\mathbb{Z}$ or $\mathbb{Z}+\frac{1}{2}$. The action of $\mathfrak{U}$ is given on the generators by

$$k\cdot\xi_j = q^{-j}\xi_j ,$$

$$(3.1) \qquad e\cdot\xi_j = q^{\frac{1}{2}-\ell}\frac{1 - q^{2(\ell+j)}}{1 - q^2}\xi_{j-1} ,$$

$$f\cdot\xi_j = q^{\frac{1}{2}-\ell}\frac{1 - q^{2(\ell-j)}}{1 - q^2}\xi_{j+1} .$$

Then we have the elements $w_{ij}^{(\ell)} \in \mathcal{A}$ $(i,j \in I_\ell)$ determeined by the action (3.1);

$$(3.2) \qquad a\cdot\xi_j = \sum_{i\in I_\ell} \xi_i \cdot w_{ij}^{(\ell)}(a) , \quad a \in \mathfrak{U},$$

which corresponds to the exponential of the infinitesimal representation (3.1) of $\mathfrak{U}$. We regard $w_{ij}^{(\ell)}$ as the matrix elements associated with the representation of $SU_q(1,1)$ on V_ℓ. In fact, by putting an infinite matrix with its entry in $\mathcal{A}$

$$(3.3) \qquad W_\ell = (w_{ij}^{(\ell)})_{i,j\in I_\ell} ,$$

we see that

$$(3.4) \qquad W_\ell(ab) = W_\ell(a)\cdot W_\ell(b) \quad \text{for } a,b\in\mathfrak{U},$$

holds as infinite matrices, or

$$(3.5) \qquad \Delta_{\mathcal{A}}(W_\ell) = W_\ell \otimes W_\ell .$$

This corresponds to the multiplicative structure of the quantum group $SU_q(1,1)$. The infinite sum in the right hand side of (3.5) makes sense in the weak * topology of $\mathcal{A}^{\hat{\otimes} 2}$ (see, (3.6) below).

In what follows, we give an explicit formula for the matrix elements by combining infinitesimal methods (the differential representation of $\mathfrak{U}$ on $\mathcal{A}$) and the classical q-analysis [5,6].

From (3.1) and (3.4), we get the formula

$$(3.6) \qquad W_\ell(e^\mu k^\lambda f^\nu) = \sum_{j\in I_\ell} q^{-\lambda\mu-\lambda j+(\mu+\nu)(1/2-\ell)}$$
$$\times \frac{(q^{2(\ell+1+j)};q^2)_\mu}{(q^2;q^2)_1{}^\mu} \times \frac{(q^{2(\ell+1-j-\mu)};q^2)_\nu}{(q^2;q^2)_1{}^\nu} E_{j,j+\mu-\nu} ,$$

where $E_{ij} = (\delta_{i\alpha}\delta_{j\beta})_{\alpha,\beta\in I_\ell}$. Proposition 2 (iv) shows that the action of $\mathfrak{U}$ on $W=W_\ell$ is given by

$$(3.7) \qquad \pi_\ell(a)W = W\cdot W(a) , \quad \pi_r(a)W = W(a)\cdot W \quad (a\in\mathfrak{U}).$$

Hence we have

Lemma 5. $w_{ij}^{(\ell)} \in \mathcal{A}[-2j,-2i]$.

From Proposition 4, we see that the matrix elements $w_{ij}^{(\ell)}$ take the forms, for example,

$$(3.8) \qquad w_{ij}^{(\ell)} = x^{-i-j}v^{i-j}\cdot\varphi_{ij}^{(\ell)}(\xi) \quad \text{for } i+j\le 0,\ j\le i.$$

where $\varphi_{ij}^{(\ell)}(\xi) \in \mathbb{C}[[\xi]]$.

The *Casimir element* C of $\mathfrak{U}$ is, by definition,

$$(3.9) \qquad C = \frac{qk^2 + q^{-1}k^{-2} - 2}{(q - q^{-1})^2} + fe,$$

which is a generator of the center of $\mathfrak{U}$ ([4]). From (3.7) and Proposition 2 (iv), we have

$$(3.10) \qquad \pi_\ell(C)w_{ij}^{(\ell)} = \frac{q^{2\ell+1}+q^{-2\ell-1}-2}{(q - q^{-1})^2}\, w_{ij}^{(\ell)}.$$

Now we consider the action π_ℓ of $\mathfrak{U}$ on $A[[SL_q(2)]]$. First we have

$$\pi_\ell(k)\begin{pmatrix} x^n & u^n \\ v^n & y^n \end{pmatrix} = \begin{pmatrix} x^n & u^n \\ v^n & y^n \end{pmatrix}\begin{pmatrix} q^{n/2} & 0 \\ 0 & q^{-n/2} \end{pmatrix},$$

$$(3.11) \qquad \pi_\ell(e)\begin{pmatrix} x^n & u^n \\ v^n & y^n \end{pmatrix} = q^{-(n-1)/2}\,\frac{1 - q^{2n}}{1 - q^2}\begin{pmatrix} 0 & xu^{n-1} \\ 0 & vy^{n-1} \end{pmatrix},$$

$$\pi_\ell(f)\begin{pmatrix} x^n & u^n \\ v^n & y^n \end{pmatrix} = q^{-(n-1)/2}\,\frac{1 - q^{2n}}{1 - q^2}\begin{pmatrix} x^{n-1}u & 0 \\ v^{n-1}y & 0 \end{pmatrix}.$$

From this together with the twisted derivation rule (Proposition 2 (iii)), we get

Proposition 6. The elements $\varphi_{ij}^{(\ell)}(\xi) \in \mathbb{C}[[\xi]]$ in (3.8) satisfy the following q-difference equations of the second order (cf. [5]),

$$(3.12)\qquad Q_{|i+j|,|i-j|}(\varphi_{ij}^{(\ell)}(\xi)) = \frac{q^{2\ell+1}+q^{-2\ell-1}-2}{(q-q^{-1})^2}\,\varphi_{ij}^{(\ell)}(\xi),$$

where

$$(3.13)\qquad Q_{mn} = \frac{q^2}{(1-q^2)^2\xi}\{(q^{m+n+1}\xi-q^{-m+n-1})\,T_{q^2} - 2\xi+q^{-m+n-1} + q^{-m-n-1}(\xi-1)(T_{q^2})^{-1}\}.$$

and T_{q^2} is the *q-shift operator* $T_{q^2}(\varphi(\xi)) = \varphi(q^2\xi)$.

By substitution of the power series expansion of $\varphi_{ij}^{(\ell)}(\xi)$,

$$\varphi_{ij}^{(\ell)}(\xi) = \sum_{r=0}^{\infty} \varphi_{ij;r}^{(\ell)}\,\xi^r$$

into (3.12), we find

$$\varphi_{ij;r+1}^{(\ell)} = \frac{q^2(1-q^{2\ell-2j+2r+2})(1-q^{-2\ell-2j+2r})}{(1-q^{2r+2})(1-q^{2i-2j+2r+2})}\,\varphi_{ij;r}^{(\ell)}.$$

Furthermore the leading coefficient $\varphi_{ij;0}^{(\ell)}$ is determined by (3.6).

Thus we obtain the main result in this paper.

<u>Proposition 7</u>. The matrix elements $w_{ij}^{(\ell)}$ $(i,j \in I_\ell)$ are given, for example, as follows;

$$(i+j\le 0,\, j\le i)\quad w_{ij}^{(\ell)} = q^{(j-i)(\ell+j)}\,\frac{(q^{2(\ell+1-i)};q^2)_{i-j}}{(q^2;q^2)_{i-j}}\,x^{-i-j}v^{i-j} \times {}_2\varphi_1\left(\begin{matrix} q^{2(\ell-j+1)},\,q^{-2(\ell+j)} \\ q^{2(i-j+1)} \end{matrix};q^2,q^2\xi\right).$$

In other regions, we get the similar formulas. Here

$${}_2\varphi_1\left(\begin{matrix} a,\ b \\ c \end{matrix};q,z\right) = \sum_{r=0}^{\infty} \frac{(a;q)_r(b;q)_r}{(q;q)_r(c;q)_r}\,z^r$$

is the *basic hypergeometric function* [2]. (As for the classical case, see [1].)

§4. Unitary representations of $SU_q(1,1)$.

In this section, we let $0<q<1$. For the real form $\mathfrak{U} = \mathfrak{U}_q(\mathfrak{su}(1,1))$, a left $\mathfrak{U}$ module V is referred to as a unitary $\mathfrak{U}_q(\mathfrak{su}(1,1))$ representation if there exists a positive definite hermitian inner product $\langle\ ,\ \rangle$ on V such that

$$(4.1)\qquad \langle a\xi,\eta\rangle = \langle\xi,a^*\eta\rangle\ ,\quad \xi,\eta\in V,\ a\in\mathfrak{U}.$$

Then we have

<u>Proposition 8</u>. In the two cases of $I_\ell\subset\mathbb{Z}$ and $I_\ell\subset\mathbb{Z}+\frac{1}{2}$, the following representations V_ℓ are irreducible and unitary with respect to the inner product (4.2);

the case of $I_\ell\subset\mathbb{Z}$;

(1) $\ell\in\mathbb{N}$, and $I_\ell = \{\ell+1,\ell+2,\cdots\}$ or $I_\ell = \{-\ell-1,-\ell-2,\cdots\}$,

(2) $\ell = -\frac{1}{2}+i\lambda\quad(0\le\lambda\le\frac{\pi}{2h})$, and $I_\ell = \mathbb{Z}$,

(3) $\ell = -\frac{1}{2}+\frac{\pi i}{2h}+s\quad(s>0)$, and $I_\ell = \mathbb{Z}$,

(4) $-\frac{1}{2}<\ell<0$, and $I_\ell = \mathbb{Z}$,

the case of $I_\ell\subset\mathbb{Z}+\frac{1}{2}$;

(1) $\ell\in\mathbb{N}+\frac{1}{2}$,and $I_\ell = \{\ell+1,\ell+2,\cdots\}$ or $I_\ell = \{-\ell-1,-\ell-2,\cdots\}$,

(2) $\ell = -\frac{1}{2}+i\lambda\quad(0\le\lambda\le\frac{\pi}{2h})$, and $I_\ell = \mathbb{Z}+\frac{1}{2}$,

(3) $\ell = -\frac{1}{2}+\frac{\pi i}{2h}+s\quad(s>0)$, and $I_\ell = \mathbb{Z}+\frac{1}{2}$,

(4) $-\frac{1}{2}<\ell<0$, and $I_\ell = \mathbb{Z}+\frac{1}{2}$,

(1)' $\ell = -\frac{1}{2}$, and $I_\ell = \{\frac{1}{2},\frac{3}{2},\cdots\}$ or $I_\ell = \{-\frac{1}{2},-\frac{3}{2},\cdots\}$,

where $q = e^{-h}$. For each family, we introduce a hermitian inner product on V_ℓ determined by

$$(4.2)\qquad \langle\xi_i,\ \xi_j\rangle = \delta_{ij}c_j$$

with

$$(4.3)\qquad c_{j+1}/c_j = \begin{cases} -\dfrac{1-q^{2(\ell-j)}}{1-q^{2(\ell+j+1)}} & \text{for } (1),(4),(1)'\ , \\ q^{-2j-1} & \text{for } (2), \\ q^{2s-2j-1}\dfrac{1+q^{-2s+2j+1}}{1+q^{2s+2j+1}} & \text{for } (3). \end{cases}$$

Then, with respect to this inner product, the representation V_ℓ are unitary. Further, any irreducible and unitary representation of $\mathfrak{U}$ is isomorphic to one of the above families.

The family of the representations (1) is called *discrete series*, the families (2) and (3) are called *continuous series*, and the family (4) is called *complementary series*, and finally the family (1)' is called *the limit of the discrete series*. Here we should remark that the family (3) vanishes in the classical limit.

Let us take an orthonomal basis $\{\tilde{\xi}_j\}_{j\in I_\ell}$ of V_ℓ defined by

$$(4.4) \qquad \tilde{\xi}_j = \sqrt{c_j}^{\,-1}\, \xi_j .$$

Then the representation matrix of the module V_ℓ with respect to this basis is as follows:

$$(4.5) \qquad \tilde{W}_\ell = (\tilde{w}_{ij}^{(\ell)})_{ij\in I_\ell} \quad \text{with} \quad \tilde{w}_{ij}^{(\ell)} = \sqrt{c_j/\,c_i}\; w_{ij}^{(\ell)} .$$

Let us induce a $*$ structure of $\mathcal{A}$ from the $*$ structure of $\mathfrak{U}_q(\mathfrak{su}(1,1))$ by

$$(4.6) \qquad \varphi^*(a) = \overline{\varphi(S(a)^*)} .$$

Then we have

$$(4.7) \qquad \begin{pmatrix} x^* & v^* \\ u^* & y^* \end{pmatrix} = \begin{pmatrix} y & qu \\ q^{-1}v & x \end{pmatrix} .$$

The coordinate ring of the quantum group $SU_q(1,1)$ is the $*$-Hopf algebra ($A(SL_q(2))$, $*$). The representation matrix $\tilde{W}_\ell$ can be thought of to define the unitary representation of $SU_q(1,1)$ when ℓ takes the values in Proposition 8. In fact, $\tilde{W}_\ell$ is a unitary matrix with respect to the involution (4.6):

<u>Proposition 9</u>. Set $\tilde{W}_\ell^{\,*} = (w_{ji}^{(\ell)*})_{i,j\in I_\ell}$. Then we have

$$\tilde{W}_\ell \cdot \tilde{W}_\ell^* = \tilde{W}_\ell^* \cdot \tilde{W}_\ell = I_{V_\ell},$$

where I_{V_ℓ} is the unit matrix on V_ℓ.

Refereneces

[1] V. Bargmann, Annals of Math. 48, 568-640 (1947).

[2] V.G. Drinfeld, Quantum groups, Proceeding of the International Congress of Mathematics, Berkeley, California, USA, 798-820 (1986).

[3] G. E. Andrews, q-series: Their development and application in analysis , number theory, combinatrics, physics and computer algebra. CBMS Regional Coference Series in Math. 66 AMS, 1986.

[4] M. Jimbo, Lett. in Math. Phys. 10, 63-69 (1985).

[5] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi and K. Ueno, C.R. Acad. Sci. Paris, 307, Serie I,559-564 (1988), preprint (1989), to appear in J. Funct. Anal..

[6] T.Masuda, K. Mimachi, Y. Nakagami, M. Noumi, Y. Saburi and K. Ueno, preprints (1989).

A q-ANALOGUE OF THE LIE SUPERALGEBRA osp(2,1) AND ITS METAPLECTIC REPRESENTATION

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Abstract
A quantum deformation of the lie superalgebra osp(2,1) is presented and its metaplectic representation discussed.

1) $osp(2,1)_q$

There has recently been a great deal of interest in quantum groups or deformations of Lie algebras involving a parameter q [1-3]. I wish to discuss $osp(2,1)_q$, the q-deformation of the super Lie algebra osp(2,1) [4-7] having the su(2) (or sl(2)) generators $H, J_\pm$ as a basis for the even part, the odd generators $V_\pm$ being su(2) spinors and (anti) commutation relations

$$[H, J_\pm] = \pm J_\pm \quad , \quad [J_+, J_-] = 2H \tag{1a}$$

$$[H, V_\pm] = \pm \tfrac{1}{2} V_\pm \quad , \quad \{V_\pm, V_\pm\} = \pm \tfrac{1}{2} J_\pm \quad , \quad [J_\pm, V_\pm] = 0 \quad , \tag{1b}$$

$$[J_\pm, V_\mp] = V_\pm \quad , \quad \{V_+, V_-\} = -\tfrac{1}{2} H \tag{1c}$$

with Casimir operator

$$C_2 = (H+\tfrac{1}{2})^2 + J_-J_+ + \tfrac{1}{2}[V_+, V_-] \quad . \tag{1d}$$

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The q-deformation of the lie algebra su(2) has already been extensively discussed by many authors [8-11]. This is described by the commutation relations

$$[H, J_\pm] = \pm J_\pm \quad , \quad [J_+,J_-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}} \equiv [2H]_q \tag{2}$$

where the parameter $q = e^s$, s real and positive, so that this algebra reduces to the su(2) lie algebra in the $s \to 0$ "classical" limit.

In order to extend (2) to a q-deformation of osp(2,1), we proceed by solving for the relations involving $V_\pm$, under the requirement that these should imply the $su(2)_q$ relations for the even generators. Let us begin by assuming the relations

$$[H,V_\pm] = \pm \tfrac{1}{2} V_\pm \tag{A}$$

and the definition of the even generators $J_\pm$:

$$\{V_\pm,V_\pm\} = \pm \tfrac{1}{2} J_\pm \quad . \tag{B}$$

A and B imply that

$$[H,J_\pm] = \pm J_\pm \quad , \tag{C}$$

$$[J_\pm,V_\pm] = 0 \quad . \tag{D}$$

To proceed, we need to consider the products V_+V_- and V_-V_+. Take

$$V_+V_- = f(H - \tfrac{1}{4}) \quad , \tag{3}$$

where $f(H)$ is an arbitrary function of H.
Now, associativity, $V_-(V_+V_-) = (V_-V_+)V_-$, requires that

$$V_-V_+ = f(H + \tfrac{1}{4}) \quad , \tag{4}$$

since A implies that

$$V_{\pm}f(H) = f(H \mp \tfrac{1}{2})V_{\pm} \quad . \tag{5}$$

Now,

$$\begin{aligned}[J_{\pm},V_{\mp}] &= \pm 4\ [V_{\pm},\{V_+,V_-\}] = \pm 4\ [V_{\pm},\ f(H)+f(H+\tfrac{1}{2})] \\ &= \pm 4\ (\ f(H \mp \tfrac{3}{4}) - f(H \pm \tfrac{1}{4})\ )V_{\pm}\end{aligned} \tag{6}$$

and similarly,

$$\begin{aligned}[J_+,J_-] &= 16\ (V_-(V_-V_+)V_+ - V_+(V_+V_-)V_-) \\ &= 16\ (V_-f(H+\tfrac{1}{4})V_+ - V_+f(H-\tfrac{1}{4})V_-) \\ &= 16\ (f(H+\tfrac{3}{4})f(H+\tfrac{1}{4}) - f(H-\tfrac{3}{4})f(H-\tfrac{1}{4})) \quad .\end{aligned} \tag{7}$$

Requiring the $su(2)_q$ relations

$$[J_+,J_-] = [2H]_q \tag{E}$$

yields the difference equation

$$g(H) - g(H-1) = [2H]_q \tag{8}$$

for

$$g(H) = 16\ f(H+\tfrac{3}{4})f(H+\tfrac{1}{4})\ ; \tag{9}$$

having solution:

$$g(H) = \frac{1}{\left[q - q^{-1}\right]^2}\left\{\ q^{2H+1} + q^{-2H-1}\right\} + \alpha \quad , \quad \alpha = \text{cst.} \tag{10}$$

Using a power series in q^H as an ansatz, we find that (9) is solved by

$$f(H) = \frac{a}{4}\ \frac{q^{2H} + bq^{-2H}}{q - q^{-1}} \quad , \qquad a^2 = 1 = b^2 \quad . \tag{11}$$

Of these four solutions, only the solution with $a = b = -1$,i.e.

$$f(H) = -\tfrac{1}{4}\ [H]_q \tag{12}$$

has the classical limit yielding relations (1c). This choice yields the relations

$$\{V_+,V_-\} = -\tfrac{1}{4}\ \left([H + 1/4]_q + [H - 1/4]_q \right) \ , \qquad \text{F}$$

$$[J_\pm, V_\mp] = \mp \left([H \mp 3/4]_q - [H \pm 1/4]_q \right) V_\pm \ . \qquad \text{G}$$

A-G are the (anti)commutation relations for the quantum supergroup $osp(2,1)_q$; they reduce to the osp(2,1) algebra (1) in the "classical" limit. The Casimir operator for these $osp(2,1)_q$ relations is given by

$$C = [H-\tfrac{1}{2}]_q^2 + J_+J_- + \tfrac{1}{2}\, q^H \left(V_+V_- - V_-V_+q^{-\frac{1}{2}} \right) \ , \tag{13}$$

which clearly reduces to the osp(2,1) Casimir (1d).

Kulish [12] has also considered $osp(2,1)_q$ from the point of view of Yang-Baxter equations. However, the algebra A-G given here appears to be rather different.

2) The metaplectic representation

Let us now consider a specific representation of $osp(2,1)_q$ satisfying the star (hermiticity) conditions appropriate to a noncompact su(1,1) even part:

$$H = H \ , \quad J_\pm^\dagger = -J_\mp \ , \quad V_\pm^\dagger = V_\mp \ ; \tag{14}$$

and having the metaplectic representation of osp(2,1) as its $s \to 0$ limit. Representations of osp(2,1) have been classified by Hughes [7]. One series amenable to a q-extension is his negative discrete series D^-, which satisfies the star conditions (14), and consists of representations labelled by an arbitrary real number l (where $l(l+1)$ is the eigenvalue of the even-part Casimir operator $H^2 + H + J_-J_+$). Any irreducible representation (IR) of osp(2,1) consists of states with only two distinct l values, j and $j+\frac{1}{2}$, say. Each IR acts on a vector space with basis $|l,m\rangle$, m having a maximum ($m \leq -(j+1)$ for any state $|j,m\rangle$ of an IR), but no minimum value. The action of osp(2,1) operators on $|j,m\rangle$ and $|j+\frac{1}{2},m+\frac{1}{2}\rangle$ is given by:

$$
\begin{aligned}
V_+ \, |j,m\rangle &= \tfrac{1}{2}\,(-(j+m+1))^{\frac{1}{2}}\, |j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle \\
V_- \, |j,m\rangle &= \tfrac{1}{2}\,(j-m+1)^{\frac{1}{2}}\, |j+\tfrac{1}{2},m-\tfrac{1}{2}\rangle \\
V_+ \, |j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle &= \tfrac{1}{2}\,(j-m)^{\frac{1}{2}}\, |j,m+1\rangle \\
V_- \, |j+\tfrac{1}{2},m+\tfrac{1}{2}\rangle &= \tfrac{1}{2}\,(-(j+m+1))^{\frac{1}{2}}\, |j,m\rangle \qquad (15) \\
J_+ \, |l,m\rangle &= (-(l-m)(l+m+1))^{\frac{1}{2}}\, |l,m+1\rangle \\
J_- \, |l,m\rangle &= -\,(-(l+m)(l-m+1))^{\frac{1}{2}}\, |l,m-1\rangle \\
H \, |l,m\rangle &= m\, |l,m\rangle
\end{aligned}
$$

where $l=j$ or $j+\frac{1}{2}$ in the last three equations. A simple extension of this representation to a representation of $osp(2,1)_q$ (A-F) exists only for $j = -3/4$. This is a q-extension of the metaplectic representation discussed by Hughes [7]. The states of the representation are $|-\frac{3}{4}\,,\,m\rangle$ and $|-\frac{1}{4}\,,\,m+\frac{1}{2}\rangle$; and the $osp(2,1)_q$ operators act on these states as follows:

$$
\begin{aligned}
V_+ \, |-\tfrac{3}{4}\,,\,m\rangle &= \tfrac{1}{2}\,[-(m+\tfrac{1}{4})]_q^{\frac{1}{2}}\, |-\tfrac{1}{4}\,,\,m+\tfrac{1}{2}\rangle \\
V_- \, |-\tfrac{3}{4}\,,\,m\rangle &= \tfrac{1}{2}\,[\tfrac{1}{4}-m]_q^{\frac{1}{2}}\, |-\tfrac{1}{4}\,,\,m-\tfrac{1}{2}\rangle \\
V_+ \, |-\tfrac{1}{4}\,,\,m+\tfrac{1}{2}\rangle &= \tfrac{1}{2}\,[-(m+\tfrac{3}{4})]_q^{\frac{1}{2}}\, |-\tfrac{3}{4}\,,\,m+1\rangle \\
V_- \, |-\tfrac{1}{4}\,,\,m+\tfrac{1}{2}\rangle &= \tfrac{1}{2}\,[-(m+\tfrac{1}{4})]_q^{\frac{1}{2}}\, |-\tfrac{3}{4}\,,\,m\rangle \\
J_+ \, |-\tfrac{3}{4}\,,\,m\rangle &= \left([m+\tfrac{3}{4}]_q\,[m+\tfrac{1}{4}]_q\right)^{\frac{1}{2}}\, |-\tfrac{3}{4}\,,\,m+1\rangle \qquad (16) \\
J_+ \, |-\tfrac{1}{4}\,,\,m+\tfrac{1}{2}\rangle &= \left([m+\tfrac{3}{4}]_q\,[m+\tfrac{5}{4}]_q\right)^{\frac{1}{2}}\, |-\tfrac{1}{4}\,,\,m+\tfrac{3}{2}\rangle
\end{aligned}
$$

$$J_- \,|-\tfrac{3}{4}\,,\, m> \;=\; -\left[[\, m-\tfrac{3}{4}\,]_q \;\; [\, m-\tfrac{1}{4}\,]_q\right]^{\frac{1}{2}} \;|-\tfrac{3}{4}\,,\, m-1 >$$

$$J_- \,|-\tfrac{1}{4}\,,\, m+\tfrac{1}{2} > \;=\; -\left[[\, m+\tfrac{1}{4}\,]_q \;\; [\, m-\tfrac{1}{4}\,]_q\right]^{\frac{1}{2}} \;|-\tfrac{1}{4}\,,\, m-\tfrac{1}{2} >$$

$$H\,|l,m> \;=\; m\,|l,m>$$

The $s\to 0$ limit of this representation clearly corresponds to the metaplectic representation of osp(2,1) discussed by Hughes. The even part of this representation corresponds to the modification of Jimbo's representation of $su(2)_q$ [8] to a representation of $su(1,1)_q$, specialised to $j = -\frac{3}{4}$.

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References

[1] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Quantization of Lie groups and Lie algebras, LOMI E-14-87.

[2] Yu.I Manin, Quantum Groups and Non-Commutative Geometry, (Montreal, 1988).

[3] V.G. Drinfeld, ICM (Berkeley, 1986) Proceedings.

[4] A. Pais and V. Rittenberg, J. Math. Phys. 16 (1975) 2062.

[5] M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 18 (1977) 146

[6] M. Scheunert, W. Nahm and V. Rittenberg, J. Math. Phys. 18 (1977) 155

[7] J.W.B. Hughes, J. Math. Phys. 22 (1981) 245

[8] M. Jimbo, Lett. Math. Phys. 10 (1985) 63; 11 (1986) 247; Comm. Math. Phys. 102 (1986).

[9] S. Woronowicz, Comm. Math. Phys. 111 (1987) 613.

[10] E.M. Sklyanin, Funkt. Anal. Pril. 16 (1982) 27; 17 (1983) 34.

[11] A.N. Kirillov and N.Yu. Reshetikhin, Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links.

[12] P.P. Kulish, Kyoto preprint, RIMS 615, 1988.

Note added

After this talk was given, T. Curtright pointed out that the ansatz (3) is representaion dependent since it implies that $J_+J_- = [3/4 - H]\ [H - 1/4]$ which is clearly equal to $[h-H][H+h-1]$ for $h = 3/4$. The latter is a general formula for the product J_+J_- of $SU(2)_q$ generators in a representaion whose Casimir operator has eigenvalue with classical limit $(j+\frac{1}{2})^2 = (h-\frac{1}{2})^2$. It is therefore the ansatz (3) which picks out the metaplectic representation discussed in section 2. The generalisation of this ansatz yielding representation-independent relations is currently under investigation. I should like to thank Thomas Curtright for very helpful discussions.

q-DEFORMATION OF $SU(1,1)$ CONFORMAL WARD IDENTITIES AND q-STRINGS

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We define a q-deformation of the $SU(1,1)$ Ward identities of 2d conformally invariant field theory based on the quantum $SU(1,1)$ algebra. The deformation preserves the main properties of the conformal Ward identities, namely that the two and three point functions are completely determined. A connection with a q-deformation of the Veneziano amplitude is revealed.

Quantum groups were discovered in attempts to quantize 1+1 dimensional integrable quantum field theory, and have many important applications in this context. Recently, quantum groups have been related to the representations of the braid group induced by the monodromies of correlation functions in conformally invariant field theory. Loosely speaking, a quantum group is a one-parameter (q) deformation of group theory that preserves some special properties (Hopf algebra). In the limit $q \to 1$, one recovers all the usual concepts and relations. In the work [1], our aim was to provide a q-deformation of conformal field theory itself, and was thus very different from the applications of quantum groups mentioned above.

A natural question presents itself: Why do this? We have no compelling reasons to offer, but mention the following. 1.) It is becoming clear to many researchers in string theory that conformal invariance is perhaps too restrictive. For example, string loop corrections to equations of motion must possess a degree of consistency that is not necessarily a consequence of conformal symmetry since contributions from surfaces of different genus are involved. At present we have no evidence that loop corrections can be organized into the structures described in [1] . 2.) It turns out that some of our results can be interpreted as a peculiar regularization of conformal field theory. This may eventually have practical applications. 3.) There has appeared in the literature a q-deformation of the Veneziano amplitude [2][3]. Whereas the underlying structure of the usual Veneziano amplitude is conformal symmetry, it appears a q-deformation of conformal symmetry provides a partial answer to the meaning of the q-Veneziano amplitude (q-strings) .

It is well known that the conformal symmetry algebra in 1+1 dimensions is the Virasoro algebra [4]. A q-deformation of the Virasoro algebra is unknown, despite the knowledge of q-deformations of Kac-Moody algebras. However, the Virasoro algebra contains the finite subalgebra $SU(1,1)$. The basic idea of our construction is to replace $SU(1,1)$ by its q-analogue $\mathcal{U}_q(SU(1,1))$. The usual conformal Ward identities thus get deformed to q-Ward identities.

In order to describe this in more detail, let me begin by reviewing the usual $SU(1,1)$ Ward identities. For quasiprimary fields $\Phi_i(z_i)$ of weight Δ_i, we have,

$$\sum_{i=1}^{N} L_n^{(i)}\langle\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle = 0, \tag{1}$$

where $L^{(i)}_{n=\pm1,0}$ acts on $\Phi_i(z_i)$. The $L_n^{(i)}$ take the form

$$\begin{aligned} L_0^{(i)} &= z_i\partial_{z_i} + \Delta_i \\ L_{+1}^{(i)} &= z_i^2\partial_{z_i} + 2\Delta_i z_i \\ L_{-1}^{(i)} &= \partial_{z_i}. \end{aligned} \tag{2}$$

For fixed i, the $L_n^{(i)}$ satisfy the relations of the $SU(1,1)$ Lie algebra. The Ward identities (1) may be written as

$$\Delta^{(N)}(L_n)\langle\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle = 0, \tag{3}$$

where $\Delta^{(N)}$ is the comultiplication:

$$\Delta^{(N)}(L_n) = \sum_{j=1}^{N} 1\otimes\cdots\otimes L_n\otimes\cdots\otimes 1. \tag{4}$$

In the above sum, L_n acts on the j-th space in the N-fold tensor product.

The algebra $\mathcal{U}_q$(SU(1,1)) is a deformation of the universal enveloping algebra of $SU(1,1)$ depending on a free parameter q [5][6][7]. It is generated by $\mathcal{K}_0, \mathcal{L}_\pm$ satisfying

$$\begin{aligned} \mathcal{K}_0\mathcal{L}_{\pm1} &= q^{\pm1}\mathcal{L}_{\pm1}\mathcal{K}_0 \\ [\mathcal{L}_{+1},\mathcal{L}_{-1}] &= -\frac{\mathcal{K}_0^2-\mathcal{K}_0^{-2}}{q-q^{-1}}. \end{aligned} \tag{5}$$

As $q\to1$, the usual $SU(1,1)$ relations are recovered if one identifies $\mathcal{K}_0 = q^{\mathcal{L}_0}$. The above relations are distinguished by admitting a comultiplication

$$\begin{aligned} \Delta: \mathcal{U}_q(SU(1,1)) &\to \mathcal{U}_q(SU(1,1))\otimes\mathcal{U}_q(SU(1,1)) \\ \Delta(\mathcal{K}_0) &= \mathcal{K}_0\otimes\mathcal{K}_0 \\ \Delta(\mathcal{L}_{\pm1}) &= \mathcal{L}_{\pm1}\otimes\mathcal{K}_0 + \mathcal{K}_0^{-1}\otimes\mathcal{L}_{\pm1}. \end{aligned} \tag{6}$$

There exists a representation of $\mathcal{U}_q(\mathrm{SU}(1,1))$ as operators on the space F of functions $f(z)$,

$$\begin{aligned}\mathcal{L}_{-1}f(z) &= \frac{1}{z}\frac{f(zq)-f(zq^{-1})}{q-q^{-1}}\\ \mathcal{L}_{+1}f(z) &= z\frac{q^{2\Delta}f(zq)-q^{-2\Delta}f(zq^{-1})}{q-q^{-1}}\\ \mathcal{K}_0 f(z) &= q^{\Delta}f(zq).\end{aligned}\tag{7}$$

$\mathcal{K}_0$ is the dilatation operator, i.e. the q-exponential of the standard $\mathcal{L}_0 = z\partial_z + \Delta$. One may easily check that as $q \to 1$, $\mathcal{L}_n$ reduce to the usual form (2) . An elliptic function version of (7) was found by Sklyanin[8], which up to a redefinition reduces to (7) as the elliptic modulus tends to zero.

Our proposal for the q-deformed Ward identities consists of the equation (3) , where now the comultiplication and L_n are replaced by their counterparts in (6) ,(7) . Namely,

$$\Delta^{(N)}(\mathcal{L}_n)\langle\Phi_1(z_1)\cdots\Phi_N(z_N)\rangle_q = 0.\tag{8}$$

The above equation is meaningful because the q-correlation functions $\langle\prod_{i=1}^{N}\Phi(z_i)\rangle_q$ are elements of the N-fold tensor product $F\otimes\cdots\otimes F$, and $\Delta^{(N)}$ of $\mathcal{L}_n$ defines a representation of $\mathcal{U}_q(\mathrm{SU}(1,1))$ on $F\otimes\cdots\otimes F$.

We found the following results:

1. The q-Ward identities were consistent. We believe this is due to the underlying Hopf algebra structure. The two and three-point functions were completely determined, as in conformal field theory. The solution for the two-point function was

$$< \Phi_1(z_1)\Phi_2(z_2) >_q = \delta_{\Delta_1,\Delta_2}\; z_1^{-2\Delta}\frac{(q^{2\Delta}z_2/z_1;q^2)}{(q^{-2\Delta}z_2/z_1;q^2)}\tag{9}$$

for $|z_1| > |z_2|$. We have used the notation:

$$(x;q^2) = \prod_{n=0}^{\infty}(1-xq^{2n}).\tag{10}$$

The definition (10) and others that will follow are standard to an obscure branch of mathematics called q-analysis [9]. In the limit $q \to 1$ the q-correlation function (9) reduces to the standard result: $< \Phi_1(z_1)\Phi_2(z_2) >_q \to (z_1 - z_2)^{-2\Delta}$. The three-point functions are

$$\begin{aligned}&< \Phi_1(z_1)\Phi_2(z_2)\Phi_3(z_3) >_q = \\ &C_{123}\; z_1^{\gamma_{12}+\gamma_{23}}\; z_2^{\gamma_{23}}\frac{(x_{12}q^{\Delta_1+\Delta_2};q^2)}{(x_{12}q^{\Delta_1+\Delta_2+2\gamma_{12}};q^2)}\frac{(x_{13}q^{-\Delta_1-\Delta_3-2\gamma_{13}};q^2)}{(x_{13}q^{-\Delta_1-\Delta_3};q^2)}\frac{(x_{23}q^{\Delta_2+\Delta_3};q^2)}{(x_{23}q^{\Delta_2+\Delta_3+2\gamma_{23}};q^2)}\end{aligned}\tag{11}$$

with $x_{ij} = z_j/z_i$ and $\gamma_{ij} = \Delta_k - \Delta_i - \Delta_j$. The only unknowns are the structure constants C_{ijk}.

2. There does not appear to be any local operator formalism that is consistent with the above results. More precisely, we found vertex operators that gave the correct two-point

function, but failed to give a correct three-point function. The vertex operators we found were slight modifications of those used in vertex operator representations of quantum affine algebras [10][11][12].

3. We found the following connection to q-strings. String theory began with Veneziano's guess for an amplitude with the integral representation [13]:

$$A(s,t) = \lim_{z_1 \to \infty} \int_0^1 dx z_1^2 \langle V_1(z_1) V_2(1) V_3(x) V_4(0) \rangle, \tag{12}$$

where

$$\langle \prod_{i=1}^{4} V_i(z_i) \rangle = \prod_{i<j} (z_i - z_j)^{\alpha_i \cdot \alpha_j}, \tag{13}$$

and $\alpha_i^2 = 2$ and $s = -(\alpha_1 + \alpha_2)^2$. $t = -(\alpha_2 + \alpha_3)^2$. The integration measure $\int dx z_1^2$ follows from gauge fixing the $SU(1,1)$ invariance. A q-deformation of (13) can be found by looking for a solution of the q-Ward identities that have the correct $q \to 1$ limit. We began with an ansatz with a product form analagous to the solution to the three-point function. It turned out this ansatz was only a solution for $s + t = -4$. Let us take the ansatz to be valid for all s and t. The integration measure must now be specified. Note that the differential operator ∂ is deformed to the operator $\mathcal{L}_{-1}$. We can define a q-calculus based on the symbol $\partial^{(q)}$. A q-integration is defined as the 'inverse' of $\partial^{(q)}$,

$$\int_x^y d_q z \partial^{(q)} f(z) \equiv f(y) - f(x). \tag{14}$$

The only definition satisfying (14) is

$$\int_x^y d_q z f(z) \equiv (q^{-1} - q) \sum_{i=0}^{\infty} q^{2i+1} \Big[y f(y q^{2i+1}) - x f(x q^{2i+1}) \Big]. \tag{15}$$

Putting all of this together, we define a q-deformation of the Veneziano amplitude as

$$\begin{aligned} A_q(s,t) &= \lim_{z_1 \to \infty} \int_0^1 d_q x z_1^2 \langle V_1(z_1) V_2(1) V_3(x/q) V_4(0) \rangle_q \\ &= \int_0^1 d_q x (x/q)^{\alpha_1 \cdot \alpha_2} G_q(x/q | 2; \alpha_2 \cdot \alpha_3). \end{aligned} \tag{16}$$

It can be shown that

$$A_q(s,t) = \frac{\Gamma_{q^2}(-s/2 - 1) \Gamma_{q^2}(-t/2 - 1)}{\Gamma_{q^2}(-s/2 - t/2 - 2)}, \tag{17}$$

where

$$\Gamma_{q^2}(a) \equiv (1 - q^2)^{1-a} \frac{(q^2; q^2)}{(q^{2a}; q^2)}. \tag{18}$$

Equation (17) is the amplitude that has appeared in the liturature. One sees from the q-integration formula that the z coordinate is broken up into a geometric lattice, thereby providing a regularization.

I would like to conclude with a few remarks. I must emphasize that we have deformed only a small part of conformal field theory. It remains to be seen whether the other essential constructions, such as operator product expansion, can also be deformed in a way consistent with what we have done. Because we have only deformed a small part of the structure of conformal field theory, we have not yet found a satisfactory interpretation for what we have done. Recognizing some similarities between our work and Prof. Ueno's talk for example [14], it is suggested that a proper interpretation may be a quantum field theory on a non-commutative space[15][16][17]. Indeed, the Haar measure on the SU(2) non-commutative space is closely related to the definition of q-integration, and the q-analogue of orthogonal polynomials is similar to our solutions of the q-Ward identities.

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REFERENCES

[1] D. Bernard and A. LeClair, Princeton University preprint PUPT-1123, to appear in Physics Letters B.
[2] L. J. Romans, *A New Family of Dual Models ('q-strings')*, Univ. Southern Cal. Preprint USC-88/HEP014.
[3] D. D. Coon and Simon Yu, Phys. Rev. D 10 (1974) 3780, and references therein.
[4] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B241 (1984) 333.
[5] P. P. Kulish and N. Yu. Reshetikhin, J. Soviet Math. 23 (1983) 2435.
[6] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.
[7] V. G. Drinfel'd, Doklady Akad. Nauk. SSSR 283 (1985) 1060.
[8] E. K. Sklyanin, Funct. Anal. Appl. 17 (1983) 34.
[9] G. E. Andrews, *q-series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra,* AMS Regional Conference Series 66 (1986).
[10] V. G. Drinfel'd, Sov. Math. Dokl. 36 (1988) 212.
[11] I. B. Frenkel and N. Jing, *Vertex Representations of Quantum Affine Algebras*, Yale Math. Preprint, (1988)
[12] D. Bernard, Lett. Math. Phys. 17 (1989)
[13] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory I*, Cambridge University Press (1987).
[14] Talk delivered at this conference, and references therein.
[15] S. L. Woronowicz, Comm. Math. Phys. 111 (1987) 613.
[16] Yu. Manin, *Quantum Groups and Non-Commutative Geometry*, in Publication du Centre de Recherches Mathématiques, (1988).
[17] T. Masuda et al., C. R. Acad. Sci. Paris, t.307, Série I, p559,1988.

Q-DEFORMATION OF $sl(2,\mathbb{C})\times Z_N$ AND LINK INVARIANTS*

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1. INTRODUCTION

It is by now well known[1-3] that for each simple Lie algebra or Kac-Moody algebra $\mathfrak{g}$ there is a quantum group $\mathcal{U}_q(\mathfrak{g})$, and that for each representation of this group, a knot invariant of the Jones type can be defined. Thus the Jones polynomial[4] corresponds to the 2×2 matrix representation of $\mathcal{U}_q(sl(2,\mathbb{C}))$ and the Akutsu-Wadati polynomial[5], the 3×3 matrix representation. Interestingly enough, the oldest knot polynomial, the Alexander-Conway polynomial[6], is yet to be identified with a quantum group. On the other hand, it is already known that the Alexander-Conway polynomial is given by a two-state solution of the quantum Yang-Baxter equation. Recently, Kauffman[7] derived the polynomial from a state model, and discovered a bialgebra associated with it. Lee[8] has shown that the polynomial at least is associated with a pseudo Hopf algebra - the representation for the universal $\mathcal{R}$-matrix was given, but the abstract form of the antipode of the algebra was not.

Here we show that the Alexander-Conway polynomial is given by the 2×2 representation of the quantum group: $\mathcal{U}_q(sl(2,\mathbb{C})\times Z_N)$ There seems to be an infinite family of quantum groups obtained by extending $\mathcal{U}_q(sl(2,\mathbb{C}))$ this way: $\mathcal{U}_q(sl(2,\mathbb{C})\times Z_N)$. We also give an explicit representation for $N=3$, which gives the Alexander-Conway type polynomial derived previously by Lee & Couture[9].

A characteristic of an Alexander-Conway type link invariant is that it has as its kernel the set of all split links[9]. This property is not shared by Jones type invariants defined on $\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is a simple Lie algebra. In fact, the standard Markov trace of a braid group representation constructed from the universal $\mathcal{R}$-matrix of $\mathcal{U}_q(sl(2,\mathbb{C})\times Z_N)$

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gives for links a trivial invariant map, of which the kernal is the domain.

Actually, there appears to be a more powerful class of invariants than the Markov trace: the universal $\mathcal{R}$-matrix can be used to map every tangle[10] - a link with one string cut - to the unit element of the Hopf algebra. Furthermore, this map appears to have as an equivalence class all the tangles obtained from cutting a link in all possible ways.

$$\rho: \{T \,|\, \text{equivalence classes of tangles}\} \rightarrow e \in \mathcal{U}_q \tag{1}$$

The **coefficient** of the map, $\rho(T)$, then gives the invariant for the equivalence class of links $[L] \equiv [T]$ to which the closure of the tangle T belongs, by

$$\Gamma[T] = \lambda^{(w-\sigma)/2} \rho(T) \tag{2}$$

where λ is in the centre of the algebra, and σ and w are respectively the writhe number and the loop number of the tangle. A description of the map ρ will be given later. Here we emphasize that $\Gamma[T]$ is a quantity defined on $\mathcal{U}_q$; no representation of $\mathcal{U}_q$ has as yet been mentioned. In general, a tangle with one open string and ℓ-1 links can be defined by ℓ representations, $p_1 \otimes \cdots \otimes p_\ell$ of $\mathcal{U}_q$, one for each string. The link invariant is then given by

$$(p_1 \otimes \cdots \otimes p_\ell)(\Gamma[T]) = \Big[\prod_i (p_i(\lambda))^{(\sigma_i - 2w_i)/4}\Big](p_1 \otimes \cdots \otimes p_\ell)(\rho(T)) \tag{3}$$

2. THE QUANTUM GROUPS $\mathcal{U}_q(s\ell(2,\mathbb{C}) \times Z_N)$

Define $q \equiv t^2 \equiv e^\eta$. Recall that the quantum group $\mathcal{U}_q(s\ell(2,\mathbb{C}))$[2,3] is generated by (H, X_+, X_-) with relations

$$[H, X_\pm] = \pm 2X_\pm \tag{4}$$

$$[X_+, X_-] = (e^{\eta H} - e^{-\eta H})/(q - q^{-1}) \tag{5}$$

comultiplication

$$\Delta(H) = 1 \otimes H + H \otimes 1 \tag{6}$$

$$\Delta(X_\pm) = X_\pm \otimes e^{\eta H/2} + e^{-\eta H/2} \otimes X_\pm \tag{7}$$

and antipode

$$\gamma(H) = -H \tag{8}$$

$$\gamma(X_\pm) = -e^{\eta H/2} X_\pm e^{-\eta H/2} \tag{9}$$

The new quantum group $\mathcal{U}_q(s\ell(2,\mathbb{C}) \times Z_N)$ containes a Hopf algebra $\mathcal{A}$ generated by $(H, h_N, h_N^{-1}, X_+, X_-)$ with relations (4), (6) and (8), and

$$[H, h_N] = 0 \tag{10}$$

$$h_N X_\pm h_N^{-1} = \omega^{\mp 1} X_\pm \tag{11}$$

$$q^{-1} X_+ X_- - \omega q X_- X_+ = (h_N^2 - 1)\, e^{-\eta H} \tag{12}$$

$$\Delta(h_N) = h_N \otimes h_N \tag{13}$$

$$\Delta(X_\pm) = X_\pm \otimes h_N e^{-\eta H/2} + e^{-\eta H/2} \otimes X_\pm \tag{14}$$

$$\gamma(h_N) = h_N^{-1} \tag{15}$$

$$\gamma(X_\pm) = -e^{-\eta H/2} X_\pm e^{\eta H/2} h_N^{-1} \tag{16}$$

where ω is the generator of Z_N valued on $\mathbb{C}$, *i.e.*, $\omega^N = 1$. The counit follows from (13-16):

$$\varepsilon(H) = \varepsilon(X_\pm) = 0, \qquad \varepsilon(h_N) = 1 \tag{17}$$

Note that (13-17) satisfy

$$m(1 \otimes \gamma)\Delta(a) = m(\gamma \otimes 1)\, \Delta(a) = \varepsilon(a), \ \forall\ a \in \mathscr{A} \tag{18}$$

Thus (4), (6), (8), (10-17) defines a Hopf algebra.

From (11) it follows that $(h_N)^N$ commutes with $X_\pm$, and therefore with all of $\mathscr{A}$. Therefore $(h_N)^N$ is central to $\mathscr{A}$. In fact $q^{1-N} h_N$ is the $\mathscr{A}$-valued generator of Z_N.

It is sometimes convenient to express the algebra in terms H exponentiated. Recall that if one defines $k \equiv e^{\eta H/2}$, then the Hopf algebra for $\mathscr{U}_q(s\ell(2,\mathbb{C}))$ is given by

$$k X_\pm k^{-1} = q^{\pm 1} X_\pm; \qquad [X_+, X_-] = (k^2 - k^{-2})/(q - q^{-1})$$

$$\Delta(k) = k \otimes k; \qquad \Delta(X_\pm) = X_\pm \otimes k + k^{-1} \otimes X_\pm;$$

$$\gamma(k) = k^{-1}; \qquad \gamma(X_\pm) = -k X_\pm k^{-1}\ .$$

For $\mathscr{U}_q(s\ell(2) \times Z_N)$, we can similarly define $k_1 \equiv e^{\eta H/2}$, $k_2 \equiv h_N e^{-\eta H/2}$. Then

$$k_1 X_\pm k_1^{-1} = q^{\pm 1} X_\pm; \qquad k_2 X_\pm k_2^{-1} = (\omega q)^{\mp 1} X_\pm \tag{19,20}$$

$$q^{-1} X_+ X_- - \omega q X_- X_+ = k_2^2 - k_1^{-2} \tag{21}$$

$$\Delta(k_i) = k_i \otimes k_i; \qquad \Delta(X_\pm) = X_\pm \otimes k_2 + k_1^{-1} \otimes X_\pm \tag{22,23}$$

$$\gamma(k_i) = k_i^{-1}; \qquad \gamma(X_\pm) = -k_1 X_\pm k_2^{-1} \tag{24,25}$$

$$\varepsilon(k_i) = 1; \qquad \varepsilon(X_\pm) = 0 \tag{26,27}$$

To complete the picture, the skew-antipode, defined by

$$m(1\otimes\gamma_0)\sigma\bullet\Delta(a) = m(\gamma_0\otimes 1)\sigma\bullet\Delta(a) = \varepsilon(a) \tag{28}$$

is given by (note that $\gamma_0=\gamma^{-1}$)

$$\gamma_0(k_i) = k_i^{-1}; \qquad \gamma_0(X_\pm) = -k_2^{-1}X_\pm k_1 \tag{29}$$

We now specialized to the case of N=2. Then (20) and (21) becomes respectively

$$k_2X_\pm k_2^{-1} = -q^{\mp 1}X_\pm \tag{30}$$

$$q^{-1}X_+X_- + qX_-X_+ = (k_2^2-k_1^{-2}) \tag{31}$$

Consider now the limit $\eta\to 0$ of (31), remembering that $(h_2)^2$ is central. If we normalize h_2 such that $q^{-2}h_2^2 \sim 1$, then to $\mathcal{O}(1)$ we have

$$[X_+,X_-]_+ \sim 1 \tag{32}$$

and to $\mathcal{O}(\eta)$ we have

$$[X_+,X_-] \sim H \tag{33}$$

The "$\sim$" sign can be replaced by an equality sign for any representation. The extra relation (32) is indicative of the fact that the new quantum group is more restrictive than $\mathcal{U}_q(s\ell(2,\mathbb{C}))$.

3. QUASITRIANGULAR HOPF ALGEBRA AND THE UNIVERSAL $\mathcal{R}$-MATRIX

Let $\mathcal{C}_+$ and $\mathcal{C}_-$ be the two Borel subalgebras of $\mathcal{A}$, the Hopf algebra of $\mathcal{U}_q(\mathfrak{g})$. In the case $\mathfrak{g}=s\ell(2,\mathbb{C})\times Z_N$, $\mathcal{C}_\pm$ are generated by $(H,h_N^{\pm 1},X_\pm)$. Let $\{e_\sigma\}$ be a basis for $\mathcal{C}_-$ and $\{e^\sigma\}$ a basis for $\mathcal{C}_+$. The two bases are dual if the multiplication for $\{e^\sigma\}$ is the comultiplication for $\{e_\sigma\}$, and the co-multiplication for $\{e^\sigma\}$ is the multiplication of $\{e_\sigma\}$ preceded by transposition. In other words, if

$$m(e_\rho\otimes e_\sigma) \equiv e_\rho e_\sigma = m^\tau_{\rho\sigma}\, e_\tau \tag{34}$$

$$\Delta(e_\rho) = \mu^{\sigma\tau}_\rho e_\sigma\otimes e_\tau \tag{35}$$

Then

$$m(e^\rho\otimes e^\sigma) = \mu^{\rho\sigma}_\tau e^\tau \tag{36}$$

$$\Delta(e^\rho) = m^\rho_{\tau\sigma}\, e^\sigma\otimes e^\tau \tag{37}$$

There exists in $\mathcal{C}_-\otimes\mathcal{C}_+$ an invertible element known as the universal $\mathcal{R}$-matrix

$$\mathcal{R} = \sum_\tau e_\tau\otimes e^\tau \tag{38}$$

that satisfies

$$\sigma \bullet \Delta(a)\mathscr{R} = \mathscr{R}\Delta(a), \qquad \forall\ a \in \mathscr{A} \tag{39}$$

Here σ is the transposition operator. Given $\{e_\sigma\}$, (34) and (35), $\{e^\sigma\}$ can be solved with (36-39) as constraints. Let $\mathscr{R}_{ij} \in \mathscr{A}^{\otimes 3}$, with the e_σ's in the i^{th} and j^{th} spaces. $\mathscr{A}$ is *quasitriangular* if $\mathscr{R}$ satisfies

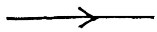
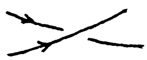
$$(\Delta \otimes 1)\mathscr{R} = R_{13}\mathscr{R}_{23}, \quad (1 \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}R_{12} \tag{40}$$

It is furthermore *triangular* if $\mathscr{R}_{12}\mathscr{R}_{12}=1$. Eqs. (40) take the conventional form, but their power are more fully displayed when expressed as the operator relations

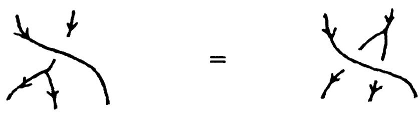
$$(\Delta \otimes 1)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{23}(\Delta \otimes 1), \quad (1 \otimes \Delta)\mathscr{R} = \mathscr{R}_{13}\mathscr{R}_{12}(1 \otimes \Delta) \tag{41}$$

which are true when both sides act on any element in $\mathscr{A}^{\otimes 2}$.

Eqs. (41), when taken in representations, give precisely the fusion-braid- ing relations in conformal field theory[11], where each element in $\mathscr{A}$ is realized as a field (in the sense of field theory for particles) in a particular representation of the algebra. Here we see that *the quasitriangular relation inHopf algebra is the abstract form of fusion-braiding relations; it transcends any specific (set of) representation.* That the two relations have a direct correspondence is more easily visualized by the following dictionary

Hopf Algebra	Conformal Field Theory	Diagrammatics
element	field	
Δ	fusion rule	
$\mathscr{R}$	intertwining of fields	

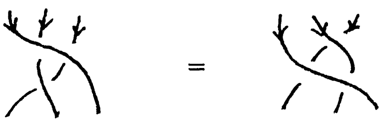
In diagrammatics, the quasitriangular relation (41) is given by



Another consequence of quasitriangularity is the quantum Yang-Baxter relation[12]

$$\mathscr{R}_{12}\mathscr{R}_{13}\mathscr{R}_{23} = \mathscr{R}_{23}\mathscr{R}_{13}\mathscr{R}_{12} \tag{42}$$

which, in diagrammatics, appears as



Many of the most interesting properties concerning $\mathscr{R}$ involve the antipodes γ and γ_0. Three fundamental relations are

$$(i) \qquad \mathscr{R}^{-1} = (1\otimes\gamma)\mathscr{R} = (\gamma\otimes 1)\mathscr{R} \tag{43}$$

$$(ii) \qquad \mathscr{R}^{-1} = (1\otimes\gamma_0^{-1})\mathscr{R} = (\gamma_0^{-1}\otimes 1)\mathscr{R} \tag{44}$$

$$(iii) \qquad \mathscr{R} = (\gamma\otimes\gamma_0)\mathscr{R} = (\gamma_0\otimes\gamma)\mathscr{R} = (\gamma^{-1}\otimes\gamma_0^{-1})\mathscr{R} = (\gamma_0^{-1}\otimes\gamma^{-1})\mathscr{R} \tag{45}$$

Others are obtained by repeated left multiplication of *(iii)* by $1\otimes\gamma$, $\gamma\otimes 1$, $\gamma_0^{-1}\otimes 1$ and $1\otimes\gamma_0^{-1}$. The relation *(i)* follows directly from

$$e^\rho\varepsilon(e_\rho) = e^\rho[m(1\otimes\gamma)\Delta(e_\rho)] = e^\rho[m(\gamma\otimes 1)\Delta(e_\rho)] = e.$$

(ii) follows from $e^\rho\varepsilon(e_\rho) = e^\rho[m(1\otimes\gamma_0)\sigma\bullet\Delta(e_\rho)] = e$, which gives

$$e_\tau e_\sigma\otimes\gamma_0(e^\sigma)e^\tau = (1\otimes\gamma_0)[((1\otimes\gamma_0^{-1})\mathscr{R})\mathscr{R}] = e\otimes e\ ,$$

which, upon left-multiplication by $(1\otimes\gamma_0^{-1})$, gives *(ii)*. *(iii)* is a consequence of *(i)* and *(ii)*.

A few remarks on γ and γ_0. From (43),

$$\mathscr{R}^{-1} = \gamma(e_\sigma)\otimes e^\sigma = e_\sigma\otimes\gamma(e^\sigma).$$

Let $\gamma(e_\sigma)\equiv\gamma_\sigma{}^\rho e_\rho$, $\gamma(e^\sigma)\equiv\tilde{\gamma}^\sigma{}_\rho e^\rho$. Then $\tilde{\gamma}^\sigma{}_\rho=\gamma_\rho{}^\sigma$, which can be viewed as a definition of the matrix $\tilde{\gamma}$. Similarly $\gamma_0(e^\sigma)\equiv(\tilde{\gamma}_0)^\sigma{}_\rho e^\rho=(\gamma_0)_\rho{}^\sigma e^\rho$. It follows from (45) that

$$\mathscr{R} = \gamma(e_\sigma)\otimes\gamma_0^{-1}(e^\sigma) = \gamma_\sigma{}^\rho e_\rho\otimes(\tilde{\gamma}_0^{-1})^\sigma{}_\tau\ e^\tau = (\gamma_0^{-1}\gamma)^\rho_\tau\ e_\rho\otimes e^\tau$$

This does not imply $\gamma_0^{-1}\gamma=1$, however. In general, $\gamma_0^{-1}\gamma\neq 1$.

Define $g_\sigma\equiv\gamma(e_\sigma)$, $f^\sigma\equiv\gamma_0(e^\sigma)$, then useful forms of (43) and (44) are

$$e_\sigma g_\rho\otimes e^\sigma e^\rho = g_\rho e_\sigma\otimes e^\rho e^\sigma = e\otimes e \tag{46}$$

$$e_\sigma e_\rho\otimes f^\rho e^\sigma = e_\sigma e_\rho\otimes e^\rho f^\sigma = e\otimes e \tag{47}$$

Later we shall see that the elements

$$h \equiv e_\rho f^\rho\ , \qquad\qquad \hbar \equiv f^\rho e_\rho \tag{48}$$

are particularly important. From (47), we have

$$(iv) \qquad e_\sigma h e^\sigma = e^\sigma\hbar e_\sigma = e \tag{49}$$

Using (18) and (28) we have, $\forall$ a $\in\mathscr{A}$,

$$(v) \qquad hah^{-1} = \gamma\gamma_0^{-1}(a) \tag{50}$$

(vi) $$\hbar a \hbar^{-1} = \gamma_0 \gamma^{-1}(a) \qquad (51)$$

which, combined with (49) yields

(vii) $$h^{-1} = e_\sigma \gamma \gamma_0^{-1}(e^\sigma) = \gamma_0 \gamma^{-1}(e_\sigma) e^\sigma \qquad (52)$$

(viii) $$\hbar^{-1} = \gamma_0^{-1} \gamma(e^\sigma) e_\sigma = e^\sigma \gamma_0 \gamma^{-1}(e_\sigma) \qquad (53)$$

It follows immediately from (50,51) that $h\hbar$ and $\hbar h$ belong to the centre of $\mathcal{A}$. We have

(ix) $$e^\sigma h g_\sigma = g_\sigma \hbar e^\sigma = h\hbar = \hbar h = \lambda\ . \qquad (54)$$

Since $h\hbar$ is central, for any representation p of $\mathcal{A}$, $\lambda_p \equiv p(\lambda)$ is a c-number, and $p(h\hbar) = \lambda_p$ is proportional to the representation of the identity element. Finally, the following relations are easily derived from *(i-iii, v, vi)*.

(x) $$e_\sigma g_\tau \otimes e^\tau h e^\sigma = g_\sigma e_\tau \otimes e^\tau h e^\sigma = e^\sigma e^\tau \otimes e_\tau h g_\sigma = e^\sigma e^\tau \otimes g_\tau h e_\sigma = e \otimes h \qquad (55)$$

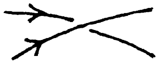
(xi) Same as *(x)*, with $h \to \hbar$ (56)

4. LINK INVARIANTS AND TANGLES

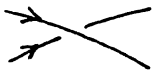
The relations *(i-xi)* plus the quantum Yang-Baxter relation (42) are sufficient for the construction of link invariants from $\mathcal{R}$ for any quasi-triangular Hopf Algebra. First we give the diagrammatical representation of the building blocks.

<u>element/relation</u> | <u>diagrammatics</u> | <u>remark</u>

$\mathcal{R}$



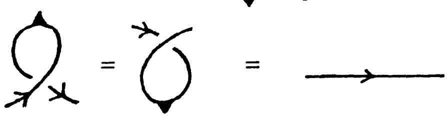
$\sigma\mathcal{R}^{-1}$



h

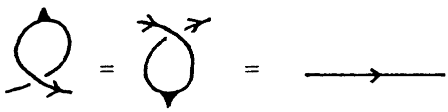
$\hbar$

(iv)



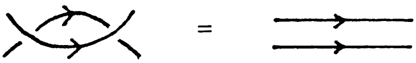
Reidemeister I

(ix)

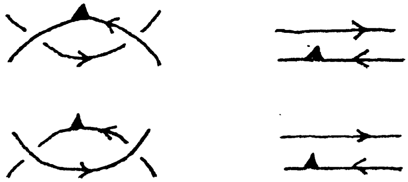


λ Reidemeister I

(i-iii)



Reidemeister IIa

(x)  Reidemeister IIb

Reidemeister IIb

(xi) same as *(x)* with  replaced by 

(42) Yang-Baxter relation Reidemeister III

One sees the important roles played by the elements h and $\hbar$. Generally, either one or the other need to be attached to any line going in the "wrong" direction. In particular, they are essential for the construction of a map that is to be invariant under the Reidemeister move I and the Reidemeister move II_b. The latter is sometimes referred to as crossing symmetry.

We now give the general rule for constructing a link invariant. A link L composed of ℓ unbroken oriented strings is an element in $\mathscr{A}^{\otimes \ell}$, each string occupying one of the spaces. To determine where factors of h or $\hbar$ should be inserted, one must adopt a standard direction of the strings, let it be downward. Then, between two crossings, a section of a string can be arranged such that it points either upwards or downwards. Now imagine that all crossings are "spliced", which is the action of replacing both $\mathscr{R}$ and $\mathscr{R}^{-1}$ by $e \otimes e$. What emerges will be a set of disconnected closed loops. On each counterclockwise (clockwise) loop, attach a factor of h ($\hbar$) to the upward pointing section of the string. The reconnected link now has all the h and $\hbar$ factors in place. At a positive (i.e. $\mathscr{R}$) crossing, a string picks up a factor of $e_\sigma(e^\sigma)$ if it is the upper (lower) member. At a negative (i.e., $\sigma\mathscr{R}^{-1}$) crossing, a string picks up a factor of g_σ (e^σ) if it is the upper (lower) member.

Each string is now an ordered product of factors of e_σ, g_σ, e^σ, h or $\hbar$ so that

$$L = a_1 \otimes a_2 \otimes \ldots \otimes a_\ell \in \mathscr{A}^{\otimes \ell} \in \mathscr{U}_q \tag{57}$$

Because of the h and $\hbar$ insertions, L is invariant under all three Reidemeister moves, it therefore belongs to an ambient isotopic equivalence class. To define a link invariant, we need a map

$$\mathrm{Tr}: \quad \mathscr{U}_q \to \mathbb{C} \tag{58}$$

such that for any representation p of $\mathscr{A}$

$$\mathrm{Tr}_p(a) = \mathrm{Trace}(p(a)) \qquad \forall\ a \in \mathscr{A} \tag{59}$$

It follows that

$$P[L] \equiv \prod_{i=1}^{\ell} \lambda^{(\sigma_i - 2w_i)/4}\, \mathrm{Tr}(a_i) \tag{60}$$

is a link invariant, where w_i is the total number of h's and ħ's attached to, or the loop number of, the i^{th} string, $\lambda = h\hbar$, and σ_i is the total number of positive minus negative crossings on the i^{th} string ($\sum_i \sigma_i$ is the writhe number of the link). It is important to note in the above that P[L] is defined abstractly on $\mathcal{U}_q$. It is clear that an ℓ-component link can be given by a set of possibly ℓ distinct representations, $\{p_1,\ldots,p_\ell\}$. Thus

$$P[L;p_1,\ldots,p_\ell] \equiv \prod_{i=1}^{\ell} \lambda_{p_i}^{(\sigma_i-2w_i)/4} \, \mathrm{Tr}_{p_i}(a_i) \tag{61}$$

We now discuss tangles. A tangle is composed of ℓ strings, only one of which is open in such a way that its two open ends can be closed to form a link without any string being crossed. Thus a tangle is a link with one of its external lines cut. By convention we label the open string as string 1 and arrange it to point downwards. Using the same method described earlier to construct P[L], we have, if the closure of a tangle T is equivalent to L,

$$\Gamma[T\sim L] = \lambda^{(\sigma_1-2w_1)/4} \, a_1' \prod_{i=2}^{\ell} \lambda^{(\sigma_i-2w_i)/4} \, \mathrm{Tr}(a_i) \in \mathcal{A} \tag{62}$$

where $a_1'h$ or $a_1'\hbar = a_1$, as the case may be. For example, the tangle whose closure is the simple Seifert link is

$$\sim \qquad \lambda^{-3/2} g_\sigma e^\tau \, \mathrm{Tr}(e^\sigma g_\tau \hbar)$$

We now make the following propositions.

Proposition 1. Tangles given by (62) are central to $\mathcal{A}$.

Proposition 2. All tangles whose closures are equivalent to a link are equivalent.

Thus (62) defines a link invariant. It follows that if the closure of $T \sim L$, then

$$P[L] = \mathrm{Tr}(h)\Gamma[T\sim L] = \mathrm{Tr}(\hbar)\Gamma[T\sim L] \tag{63}$$

since the tangle can be closed counterclockwise or clockwise. Note that by definition (48) Tr(h)=Tr(ħ). Propositions 1 and 2 together constitute a much stronger statement than (60). A corallory is that any representation of any tangle is proportional to the identity. Eq. (62) is equivalent to the construction given in refs. 9 and 13 in terms of the exchange R-matrix by a partial Markov trace. The simplest tangles are those given in *(iv)* and *(ix)*. Complete proofs of propositions 1 and 2 have not yet been found. A partial proof will be given elsewhere. They are empirically true for the known representations of $\mathcal{U}_q(s\ell(2,\mathbb{C}))$ and $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_N)$. In the latter case, Tr(h) = Tr(ħ)=0, so a nontrivial definition of link invariants is given only via the tangles.

5. REPRESENTATIONS OF $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_N)$

We give representations of $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_2)$ in terms of 2×2 matrices. The 2×2 representation for the generator ω of Z_2 is $h_2 \propto \begin{pmatrix}1&0\\0&-1\end{pmatrix}$. In the following the representations for key elements of $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_2)$ and the exchange R-matrix $R^{ij}_{k\ell} \equiv (e_\sigma)^i_\ell (e^\sigma)^j_k$ are given along with those for $\mathcal{U}_q(s\ell(2,\mathbb{C}))$. It has been shown[8] that these two give the only two possible 2×2 matrix representations of a Hopf algebra. Define $a \equiv \sqrt{t^4-1}$, then

element	$\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_2)$	$\mathcal{U}_q(s\ell(2,\mathbb{C}))$
k_1	$\begin{bmatrix}t&0\\0&t^{-1}\end{bmatrix}$	$\begin{bmatrix}t&0\\0&t^{-1}\end{bmatrix}$
k_2	$\begin{bmatrix}t&0\\0&-t^{3}\end{bmatrix}$	—
X_-	$a\begin{pmatrix}0&0\\1&0\end{pmatrix}$	$a\begin{pmatrix}0&0\\1&0\end{pmatrix}$
X_+	$a\begin{pmatrix}0&1\\0&0\end{pmatrix}$	$a\begin{pmatrix}0&1\\0&0\end{pmatrix}$
e_1	k_1^{-1}	k_1^{-1}
e_2	k_2	k_1
e_3	X_-	X_-
e^1	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$
e^2	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$
e^3	$-t^{-1}X_+$	$-t^{-1}X_+$
h	$t\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$t\begin{bmatrix}1&0\\0&t^4\end{bmatrix}$
$\hbar$	$t^{-3}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$t\begin{bmatrix}t^4&0\\0&1\end{bmatrix}$
λ	t^{-2}	t^{5}
R	$\propto \begin{bmatrix}1&0&0&0\\0&0&t^2&0\\0&t^2&1-t&0\\0&0&0&-t\end{bmatrix}$	$\begin{bmatrix}1&0&0&0\\0&0&t^2&0\\0&t^2&1-t&0\\0&0&0&1\end{bmatrix}$

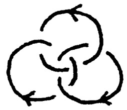
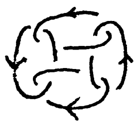
The R-matrices are precisely those known to give the Alexander-Conway and Jones link polynomials, respectively[9,13]. Note that for the Z_2 case, $\mathrm{tr}(h) = \mathrm{tr}(\hbar) = 0$, so link invariants must be computed via tangles. All tangles for both cases computed from (63) reproduce known results computed from the (partial[9], in the case of Alexander-Conway) Markov trace using R. Details will be discussed elsewhere.

The following is a 3×3 matrix representation of $\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_3)$. Let E_{ij} be the matrix with unit element in the (ij) position, $\lambda_i \equiv E_{ii}$ and $\omega = e^{2\pi i/3}$.

element	$\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_3)$	element	$\mathcal{U}_q(s\ell(2,\mathbb{C})\times Z_3)$
ω	$e^{2\pi i/3}$	h_3	$t^4(\lambda_1+\omega\lambda_2+\omega^2\lambda_3)$
k_1	$t^2\lambda_1+\lambda_2+t^{-2}\lambda_3$	k_2	$t^2\lambda_1+\omega t^4\lambda_2+\omega^2 t^6\lambda_3$
X_-	$t^{-1}aE_{21}+tbE_{32}$	X_+	$t^{-1}aE_{12} + tbE_{23}$
e_1	k_1^{-2}	e^1	λ_1
e_2	$k_1^{-1}k_2$	e^2	λ_2
e_3	k_2^2	e^3	λ_3
e_4	$-it^{-1}k_1^{-1}X_-$	e^4	$-it^{-2}aE_{12}$
e_5	$-it^{-1}k_2X_-$	e^5	$-i\omega^2t^{-2}bE_{23}$
e_6	$-t^{-2}X_-^2$	e^6	$\omega t^{-2}abE_{13}$
h	t^4h_3	λ	t^{-8}
$\hbar$	$t^{-12}h_3^2$		

The basis $\{e_\sigma\}$, which has the form $e_{pqr} = k_1^p k_2^q X_-^r$, is infinite, but the representations vanish for all members in the dual basis except those listed above. In the table $a \equiv \sqrt{t^8-1}$, $b \equiv \sqrt{(1+\omega)(\omega t^8-1)}$. The exchange R-matrix given by this representation was previously found by directly solving the Yang-Baxter equation[9]. Link invariants computed from (62) are identical to those computed from the R-matrix using the partial Markov trace[13]. Other properties of the new quantum groups will be given elsewhere.

Finally, we list a few links calculated from (62) for the representations given above. The notation used is that of Rolfsen[14], *i.e.*, K^ℓ_m is the m^{th} ℓ-component link with K crossings. For the Alexander-Conway polynomial (*i.e.*, the $s\ell(2,\mathbb{C})\times Z_2$ case), $\Gamma(K^\ell_m)$ has ℓ-1 factors of $\mathcal{L}_2 \equiv \Gamma(2^2_1) = \Gamma(\text{Seifert link}) = t^{-2}-t^2$. For the $s\ell(2,\mathbb{C})\times Z_3$ case, $\Gamma(K_m^{\ell\geq 2})$ has at least one factor of $\mathcal{L}_3 \equiv \Gamma(2^2_1) = t^{-8}+\omega^2+\omega t^8$, $\omega = e^{2\pi i/3}$.

Rolfsen's Notation K^{ℓ}_{m}		$\mathcal{U}_q(sl(2,\mathbb{C})\times Z_2)$ $\Gamma(K^{\ell}_{m})/\mathcal{L}_2^{\ell-1}$	$\mathcal{U}_q(sl(2,\mathbb{C})\times Z_3)$ $\Gamma(K^{\ell}_{m})/\mathcal{L}_3$
3^1_1		$\mathcal{L}_2^2+1$	$(\mathcal{L}_3^2-\omega^2\mathcal{L}_3-1-4\omega)\mathcal{L}_3^1$
		$\mathcal{L}_2^2+1$	$(\omega\mathcal{L}_3^2-\mathcal{L}_3+1)\mathcal{L}_3^1$
2^2_1		1	1
		-1	ω^2
5^2_1		$\mathcal{L}_2^2$	$\omega\mathcal{L}_3^2+1-\omega$
6^2_3		$-\mathcal{L}_2^2+2$	$\omega^2\mathcal{L}_3^2+2\omega\mathcal{L}_3+3+\omega^2$
6^3_2		$\mathcal{L}_2^2$	$\omega\mathcal{L}_3^3+3\omega$
8^3_3		$\mathcal{L}_2^2-1$	NA
8^4_1		4	NA
8^4_1		$-\mathcal{L}_2^2$	NA

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References

1. V.G. Drinfeld, Proc. Int'l Cong. Math., Berkeley, 1986; p.798; DAN SSSR, 5(1985)1060.
2. M. Jimbo, Lett. Math. Phys. 10(1985)63; Comm. Math. Phys. 102(1986)537.
3. N. Yu. Reshetikhin, LOMI preprint E-4-87 (Steklov Math. Inst. 1988).
4. V.F.R. Jones, Bull. Am. Math. Soc. 12(1985)103.
5. Y. Akutsu and M. Wadati, J. Phys. Soc. Jap. 56(1987)3464.
6. J.W. Alexander, Trans. Math. Soc. 30(1923)275; J.H. Conway, *"Comp. Problems in Abstract Algebra"*, ed. J. Leech (Pergamon, London, 1969) p.329.

7. L.H. Kauffman, in Appendix of *"Knots, Abstract Tensors and the Yang-Baxter Equation"*, preprint 1989.
8. H.C. Lee, *"Hopf Algebra, Complexification of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{C}))$ and Link Invariants"*, Symp. on Fields, Strings and Quantum Gravity, Beijing, 1989 (to be published by Gordon & Breach).
9. H.C. Lee and M. Couture, CRNL preprint TP-88-1118R (1988).
10. I adopt the terminology used by Kauffman, ref. 7.
11. G. Moore and N. Seiberg, Phys. Lett. B212(1988)451; E. Verlinde, Nucl. Phys. B300(1988)360.
12. C.N. Yang, Phys. Rev. Lett. 23(1967)1312; R.J. Baxter, *"Exactly Solved Models in Statistical Mechanics"* (Academic Press, London, 1982).
13. H.C. Lee, M. Couture & N. Schmeing, CRNL preprint TP-88-1125R (1988).
14. D. Rolfsen, *"Knots & Links"*, Math. Lectures Series No. 7 (Publish or Perish Press, 1976).

QUANTUM GROUP DUALITY IN VERTEX MODELS AND OTHER RESULTS IN THE THEORY OF QUASITRIANGULAR HOPF ALGEBRAS

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INTRODUCTION

The Hopf algebra duality between observables and states in the author's non-commutative geometric approach to quantum mechanics on curved spacetimes, is transferred to the context of vertex models. Among the results, for general invertible solution R of the QYBE we obtain from the bialgebra $A(R)$ a quantum group $\check{A}(R)$ and a dual quantum group $\check{U}(R)$, with $\check{U}(R)$ quasitriangular. Previously this was known only for specific examples such as $SL_q(2)$ and $U_q(sl_2)$. The pairing between $\check{A}(R)$ and $\check{U}(R)$ leads to a direct expression for the partition function of associated exactly solvable vertex models in terms of the quantum group structures, as well as to a general variant of an ansatz of Kulish and Reshetikhin. A 5-vertex model is given as a simple example. We also obtain a general category-theoretic rank for the representation theory of quasitriangular Hopf algebras, generalizing the known "quantum dimension" $(q^{2j+1} - q^{-(2j+1)})/(q-q^{-1})$ for the spin j representation of $U_q(su(2))$.

HOPF ALGEBRA DUALITY BETWEEN OBSERVABLES AND STATES

Non-commutative and non-cocommutative Hopf algebras have recently arisen in physics in two contexts: in the author's non-commutative geometric approach to quantum-mechanics on homogeneous spacetimes [1] and in quantum inverse scattering ("quantum groups") e.g. [4]. In the former, there is a simple interpretation of Hopf algebra duality: the dual Hopf algebra described a dual model in which the roles of observables and states (and position and momentum) were interchanged. In this paper, we shall transfer over these duality considerations to the latter examples from inverse scattering. There the Hopf algebras are not precisely the algebra of observables of anything, but nevertheless the analogy will prove fruitful.

The ingredients for the models in [1] are matched pairs of groups. Two groups (G_1, G_2) are a matched pair if there are actions (α, β) of each on the space of the other such that

$$\alpha_{u^{-1}}(st) = \alpha_{u^{-1}}(s)\alpha_{\beta_{s^{-1}}(u)^{-1}}(t), \qquad \alpha_{u^{-1}}(e) = e,$$

$$\beta_{s^{-1}}(uv) = \beta_{s^{-1}}(u)\beta_{\alpha_{u^{-1}}(s)^{-1}}(v), \quad \beta_{s^{-1}}(e) = e$$

(the conventions here are suited for the right actions $\alpha_{u^{-1}}, \beta_{s^{-1}}$, which is the origin of the inverses). Associated to such a matched pair is a Hopf algebra A defined approximately as

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follows. A can be built on functions of compact support on G_1 with values in functions $C(G_2)$ and

$$(FG)(u,s)=\int_{G_1} dvF(v,s)G(v^{-1}u,\alpha_{v^{-1}}(s)),\quad (\Delta F)(u,s,v,t)=\delta_{\beta_{s^{-1}}(u)}(v)F(u,st),$$

$$\epsilon F=\int duF(u,e),\quad \forall F,G\in A,\ u,v\in G_1,\ s,t\in G_2.$$

Here Δ is the coproduct, ϵ the counit and there also a map $S:A\to A$, the antipode[1]. du denotes left invariant Haar measure. δ-functions are defined with respect to this and, technically, should be approximated by smooth bump-functions. A is called a *bicrossproduct* and denoted $C^*(G_1)^\beta \bowtie_\alpha C(G_2)$. It comes equipped with canonical maps $G_1 \hat{\to} A$ and $C(G_2)\hat{\to} A$ given by $\hat{u}=\delta_u$ a δ-function at u and constant on G_2, and $(\hat{q})(u,s)=q(s)$ for $q\in C(G_2)$. The approximation problems can be handled by working with Hopf-von Neumann algebras[1].

In the physical interpretation of these models, A is the algebra of observables of a quantum system. The underlying classical system is that of a particle moving on the position space G_2 along the geodesics of a metric determined by α, namely a natural metric on the orbits in G_2 such that the geodesics are the one-parameter flows of α. Thus the geodesics through $s(0)\in G_2$ take the form

$$s(\tau)=\alpha_{\exp\tau\xi}(s(0)),\quad \tau\in\boldsymbol{R},\xi\in g_1$$

where g_1 is the Lie algebra of G_1. Different choices of α correspond to different metrics in this class. The precise nature of the correspondence is given in [1]. If G_1 is semisimple, the metric is non-degenerate. If G_1 is compact, the metric is positive definite. Elements of g_1 are called *momentum observables* because they label the geodesics. Elements of $C(G_2)$ are the *position observables.* The phase space etc. are discussed in [1].

A and the maps $\hat{}$ correctly quantize this system. This is because the relations in A are

$$\widehat{e^{\tau\xi}}\hat{q}\widehat{e^{-\tau\xi}}=\widehat{\alpha_{e^{\tau\xi}}(q)}$$

where $\alpha_u(q)\in C(G_2)$ is defined by

$$\alpha_u(q)(s)=q(\alpha_{u^{-1}}(s)),\qquad \forall q\in C(G_2),u\in G_1$$

(the Heisenberg commutation relations in an exponentiated and co-ordinate invariant form.). For actual operators on a Hilbert space simply view A in a concrete representation. At the infinitesimal level, define $\hat{\xi}=\frac{d}{d\tau}_{|0}\widehat{e^{\tau\xi}}$. There is also a canonical Hamiltonian. For example, let g_1 be semisimple, $\{e_i\}$ a basis of g_1 and K the inverse matrix of the Killing form, and suppose that α_* is order $\hbar$. Then $H=\frac{1}{2m\hbar}K^{ij}\hat{e}_i\hat{e}_j$ generates to lowest order in $\hbar$ the classical geodesic motion described above.

Note that α corresponds in this framework to the spacetime metric. Regarding β as an auxiliary variable, the matched pair equations therefore play the role of a completely integrated version of "Einstein's equations" for the spacetime in which the particle moves. The physical content of the general solution of the simplest class, $G_1=G_2=\boldsymbol{R}$, is investigated in [1]. The solution space has two free parameters, which we identified[1] roughly with $\hbar$ and G, where G is the gravitational coupling constant.

These toy models, in which both quantum mechanics and gravity-like forces are encoded into a single algebraic structure, are characterized by an interesting property. For each model, there is a dual model. The algebra of observables of the dual model is the dual Hopf algebra (see below for definitions). With suitable functional-analytic details[1] to be filled in, one finds essentially

$$(C^*(G_1)^\beta \bowtie_\alpha C(G_2))^*\cong C^*(G_2)^\alpha \bowtie_\beta C(G_1).\tag{1}$$

Thus the dual model is of the same form, with G_1, G_2 (i.e. position, momentum) interchanged and α,β (in practice $\hbar, \frac{1}{G}$) interchanged. The significance of this is as follows. In the C^* approach to quantum systems that we have adopted, the role of wave functions is played by states. A *state* ϕ is a positive linear map $A \to \boldsymbol{C}$, normalized to $\phi(1) = 1$ if A has a unit. Positive means here that $\phi(F^*F) \geq 0$ for all $F \in A$. If A is viewed concretely as a subalgebra of bounded operators on a Hilbert space H, then it is well known that typical states are of the form

$$\phi(F) = \sum_i t_i < \psi_i|F|\psi_i >, \ t_i \geq 0, \ \sum_i t_i = 1$$

i.e. a statistical combination of particle states determined by vectors $|\psi_i >\in H$. But abstractly, all we need is ϕ as a linear map. The physical observables are the self-adjoint elements of A, the states are the positive elements of A^* and the pairing $\phi(F)$ is the "expectation value of observable F in state ϕ". The significance of the Hopf algebra structure on A is that it restores the symmetry between observables and states. A coalgebra structure on A is essentially equivalent to an algebra structure on A^*. The dual model take the same ingredients A and A^*, but views A^* as the algebra of observables and $A \subseteq A^{**}$ as the space in which the states lie. Roughly speaking, in the dual model the same experimental quantity $\phi(F)$ would be written $F(\phi)$ (or symmetrically, $< F, \phi >$) and interpreted as "the expectation value of observable ϕ in state F". In the first sections of the paper we shall attempt to extend these duality considerations to quantum groups. The physical setting here is very different but the Hopf algebras are nevertheless somewhat self-dual, a map $A \to A^*$ being provided by the "universal R-matrix", $\mathcal{R}$.

Note that the philosophical content of such a self-duality of the system is representation theoretic: for C^* algebras, the states are in one to one correspondence with the C^* algebra representations. The correspondence is via the well-known GNS construction[14], which may be described as follows. Any C^* algebra A can be considered as acting on itself by multiplication (the left regular representation). Every state on A defines a semidefinite inner product on A by $< b|a >= \phi(b^*a)$. Dividing out by the vectors of zero norm and completing, we obtain a Hilbert space H_ϕ on which A still acts. If ϕ is a pure state (not a non-trivial convex linear combination of other states) then the representation of A so obtained is irreducible. In this way the observable-state symmetry above can be understood as a representation theoretic self-duality. To extend the observable-state symmetry to more realistic models one would need to take such a representation-theoretic point of view. This leads to a study of the representation theory of quasitriangular Hopf algebras in section 3, including a general notion of "quantum dimension" introduced in section 4. The work establishes a connection between the properties of internal hom in such categories and the construction of knot and link invariants.

This is the final updated version of the preprint Hopf Algebraic Expression for the Partition Function of Exactly Solvable Vertex Models which now appears as sections 1-2. It includes further results to appear in [2] where detailed proofs may be found.

1 QUANTUM GROUPS $\breve{A}(R)$ AND $\breve{U}(R)$

Quantum groups have previously been used only as a tool to generate solutions R of the Quantum Yang-Baxter Equations (QYBE), e.g. [4]. The solutions then give rise to statistical mechanical lattice models which are exactly solvable e.g. [5]. It may be expected that these Hopf algebras play a much more fundamental role in the resulting statistical mechanical model. In section 2 we point out a direct role of this type, recovering the partition function for certain square-lattice vertex models from the natural pairing between a quantum group and its dual. We also recover in this context a general version of an ansatz of Kulish and Reshetikhin[11].

The first step, which is the main result of this section, is to obtain mutually dual quantum groups, which we call $\breve{A}(R)$ and $\breve{U}(R)$, for general invertible solution R of $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (the matrix QYBE). It is well known that for such every R, there is a bialgebra $A(R)$[6]. These are not yet quantum groups (with antipode): for example, to obtain $SL_q(2)$ one must divide by a further quantum determinant. Such additional relations, and relations needed in the dual[8][6] are thereby generalized to general invertible R in the definition of $\breve{A}(R)$ and $\breve{U}(R)$ below. Moreover, the definitions are such that $\breve{U}(R)$ is quasitriangular. The definitions will be motivated in part by [6] who began the study of the dual of the bialgebra $A(R)$ and also studied the Hopf algebras $SL_q(2)$ and $U_q(sl_2)$ (but not the general $\breve{A}(R)$, $\breve{U}(R)$).

Thus, let $R \in M(n)\otimes M(n)$ be invertible and satisfy the matrix QYBE. The associated bialgebra (or quantum semigroup) $A(R)$, is[6] a free associative algebra modulo relations

$$A(R) = \frac{\{u\}}{Ru_1u_2 - u_2u_1R}, \quad \Delta(u^i{}_j) = \sum_k u^i{}_k \otimes u^k j, \; \epsilon(u^i{}_j) = \delta^i{}_j.$$

Here the n^2 indeterminates $\{u^i{}_j\}^n_{i,j=1}$ are regarded together as $u \in M(n, A(R))$, $u_1 = (u\otimes 1)$, $u_2 = (1\otimes u)$ and the multiplication in the first line takes place in $M(n, A(R))\otimes M(n, A(R))$ (matrix $\otimes$). R itself is regarded as lying in this space as $\boldsymbol{C} \subseteq A(R)$ via the identity. The well known example for $n = 2$,

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \in M(2, M(2, \boldsymbol{C})), \; q \in \boldsymbol{C} - \{0\} \tag{2}$$

gives $A(R) = M_q(2)$. Note that $A(R)$ is only a bialgebra rather than a Hopf algebra (i.e. more precisely a quantum semigroup rather than a quantum group).

For every finite-dimensional bialgebra (or Hopf algebra) A the dual bialgebra (or Hopf algebra) structure is built on A^*, the space of linear maps from A to $\boldsymbol{C}$, defined from the structure on A via

$$< a, hg >=< \Delta a, h\otimes g >, \quad < a\otimes b, \Delta h >=< ab, h >, \quad < a, 1 >= \epsilon(a), \qquad \epsilon(h) =< 1, h > \tag{3}$$

and in the Hopf algebra case,

$$< a, Sh >=< Sa, h > . \tag{4}$$

In these equations $a, b \in A$, $h, g \in A^*$ and the pairing $< \, , \, >$ denotes the evaluation map, $< a, h >\equiv h(a)$, while 1 denotes the relevant identity element. In the infinite dimensional case the correct notion of dual is more intricate and not unique, e.g.[7] A°, built on a certain subset of A^*. For our purposes, that two bialgebras are dual will mean that equations (3) hold ((3),(4) in the Hopf algebra case), and the pairing is non-degenerate. Non-degenerate means that there are no non-zero null elements in A^* or in A ($h \in A^*$ is null if $< a, h >= 0$ for all $a \in A$, and $a \in A$ is null if $< a, h >= 0$ for all $h \in A^*$.)

The dual of $A(R)$ is the quantized universal enveloping/convolution bialgebra, and has been denoted $U(R)$ in [6]. What follows in an explicit construction for general R, not exactly of $U(R)$, but the dual of a Hopf algebra obtained as $A(R)$ modulo null directions. We define first a bialgebra $\tilde{U}(R)$, a free associative algebra modulo relations

$$\tilde{U}(R) = \frac{\{l^+, l^-\}}{l_1^\pm l_2^\pm R - R l_2^\pm l_1^\pm, l_1^- l_2^+ R - R l_2^+ l_1^-}, \quad \Delta(l^{\pm i}{}_j) = \sum_k l^{\pm i}{}_k \otimes l^{\pm k}{}_j, \; \epsilon(l^{\pm i}{}_j) = \delta^i{}_j. \tag{5}$$

Here $l^\pm$ are two sets of n^2 indeterminates. This is arranged so that $A(R)$ and $\tilde{U}(R)$ are paired as in equations (3) by

$$< u, l^+ >= R, \qquad < u, l^- >= \tau(R^{-1}) \tag{6}$$

(cf. [6]). Here τ is the twist map in $M(n, \boldsymbol{C})\otimes M(n, \boldsymbol{C})$ (permutation of the factors) and R^{-1} is the inverse matrix.

Typically, however, the pairing between $A(R)$ and $\tilde{U}(R)$ is degenerate: let $\breve{U}(R)$ denote $\tilde{U}(R)$ modulo the null subspace. One may show that the null subspace is a bi-ideal in the sense of [7, p87] and hence that $\breve{U}(R)$ is a bialgebra. Similarly, let $\breve{A}(R)$ denote $A(R)$ modulo its null subspace. $\breve{A}(R)$ and $\breve{U}(R)$ are then dual in the sense that the pairing between them is then non-degenerate. Equivalently, the pairing defines algebra inclusions $\breve{A}(R) \subseteq \breve{U}(R)^*$ and $\breve{U}(R) \subseteq \breve{A}(R)^*$. One can then show that these inclusions are essentially isomorphisms: $\breve{U}(R)^*$

is technically a bigger space than $\check{A}(R)$, but the difference arises purely from the fact that the underlying vector spaces are not finite dimensional. $\check{A}(R)$ has the character of polynomials in the u while $\check{U}(R)^*$ has the character of formal power-series in the u. (Similarly for $\check{U}(R)$ in relation to $\check{A}(R)^*$.) Technical aspects of the construction are in [3].

THEOREM For every regular invertible R obeying the matrix QYBE, both $\check{A}(R)$ and $\check{U}(R)$ are essentially Hopf algebras and $\check{U}(R)$ is essentially quasitriangular. The antipodes are defined by

$$< Su, l^+ >=< u, Sl^+ >= R^{-1}, \quad < Su, l^- >=< u, Sl^- >= \tau(R) \tag{7}$$

when extended as antialgebra maps, and the universal $\mathcal{R}$ (viewed as a map) is defined by

$$\mathcal{R} : u \mapsto l^+ \tag{8}$$

extended to $\check{A}(R)$ as an antialgebra map. See [3] for details.

Proof This defines $< Su, \ >\in \check{U}(R)^*$, $< \ , Sl^\pm >\in \check{A}(R)^*$ and in practice $Su \in \check{A}(R)$, $Sl^\pm \in \check{U}(R)$ as explained above. For $\check{U}(R)$ to be quasitriangular the universal $\mathcal{R}$ (as a map) is $\check{U}(R)^* \to \check{A}(R)$, i.e. in practice $\check{A}(R) \to \check{U}(R)$ as given. It can be checked that it obeys the various axioms (listed in section 3)[2, 3.2.3].

Note that using equations (6) we have also $< u, \mathcal{R}(u) >\equiv< u\otimes u, \mathcal{R} >= R$ where on the right $\mathcal{R}$ is viewed equivalently as an element of (a completion of) $\check{U}(R)\otimes\check{U}(R)$. The universal $\mathcal{R}$ will be play an important role in applications in the next section.

EXAMPLE Since the above construction of mutually dual Hopf algebras $\check{A}(R)$ and $\check{U}(R)$ works for any invertible solution of the QYBE, it includes quantum $SL(n)$ etc. For the example $n = 2$ with the specific R above, we find that relations generating null directions are

$$l^{+1}{}_2 = 0 = l^{-2}{}_1, \qquad l^{+1}{}_1 l^{-1}{}_1 = 1, \qquad l^{+2}{}_2 l^{-2}{}_2 = 1 \tag{9}$$

(the resulting bialgebra is naturally paired with $M_q(2)$) and a further null relation is

$$\det l^+ = 1 \tag{10}$$

(or $\det l^- = 1$) giving a Hopf algebra, $\check{U}(R) = U_q(sl_2)$. Similarly, in $A(R)$ the relation

$$u^1{}_1 u^2{}_2 - q^{-1} u^1{}_2 u^2{}_1 = 1, \tag{11}$$

generates a null direction, recovering the well known expression $\det_q u$. Dividing by this null space gives a Hopf algebra, $\check{A}(R) = \frac{A(R)}{(\det_q u)-1} = SL_q(2)$. It is known[8] that $U_q(sl_2)$ and $SL_q(2)$ are dual. The antipode by our general construction (7) recovers the well known antipode $Su = \begin{pmatrix} u^2{}_2 & -qu^1{}_2 \\ -q^{-1}u^2{}_1 & u^1{}_1 \end{pmatrix}$. The known "universal R-matrix" for $U_q(sl_2)$ is also recovered from equation (8).

This is not the usual construction of $U_q(sl_2)$, but coincides (with Jimbo's normalizations) under the following identification

$$l^+ \equiv \begin{pmatrix} q^{\frac{H}{2}} & 0 \\ q^{-\frac{1}{2}}(q - q^{-1})X^+ & q^{-\frac{H}{2}} \end{pmatrix}, \quad l^- \equiv \begin{pmatrix} q^{-\frac{H}{2}} & q^{\frac{1}{2}}(q^{-1} - q)X^- \\ 0 & q^{\frac{H}{2}} \end{pmatrix} \tag{12}$$

cf. [6]. The form of this ansatz is dictated by the null relations (9)-(10): $q^{\frac{H}{2}}, X^+, X^-, q^{-\frac{H}{2}}$ are just symbols for the remaining generators in $l^\pm$, suitably scaled. The remaining relations in $\check{U}(R)$ (coming from equations (5),(7)) are then just the usual relations of $U_q(sl_2)$, namely $q^{\frac{H}{2}} X^\pm q^{-\frac{H}{2}} = q^{\pm 1} X^\pm$, $[X^+, X^-] = \frac{q^H - q^{-H}}{q - q^{-1}}$ etc, as observed in [6]. We have assumed that q is not a root of unity, otherwise the null space contains further relations.

Equation (8) can also be pushed backwards: suppose we are given some alternative form of $\breve{U}(R)$, i.e. a quasitriangular Hopf algebra H dual to $\breve{A}(R)$. Then the generators of $\breve{U}(R)$ in terms of H are necessarily $l^+ = \mathcal{R}^{(1)} < u, \mathcal{R}^{(2)} >$ and $l^- =< u, \mathcal{R}^{-1(1)} > \mathcal{R}^{-1(2)}$ where $\mathcal{R} \equiv \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \in H \otimes H$ is the "universal R-matrix" on H. As a simple example, the form of $\mathcal{R}$ in [9] for $U_q(sl_2)$ then correctly returns exactly equations (12). $\mathcal{R}$ is known for all quantum groups $U_q(g)$[4]. Explicit formulae for the case of $U_q(sl_n)$ were recently obtained in [10].

For another example, our formalism gives for $U_q(sl_3)$,

$$l^+ = \begin{pmatrix} q^{\frac{2}{3}H_1+\frac{1}{3}H_2} & 0 & 0 \\ q^{\frac{1}{6}H_1+\frac{1}{3}H_2} q^{-\frac{1}{2}}(q-q^{-1})X_1^+ & q^{-\frac{1}{3}(H_1-H_2)} & 0 \\ q^{\frac{1}{6}(H_1-H_2)} q^{-\frac{1}{2}}(q-q^{-1})(X_1^+X_2^+ - q^{-1}X_2^+X_1^+) & q^{-\frac{1}{3}H_1-\frac{1}{6}H_2} q^{-\frac{1}{2}}(q-q^{-1})X_2^+ & q^{-\frac{1}{3}H_1-\frac{2}{3}H_2} \end{pmatrix}$$

and similarly for l^-. The pattern for $U_q(SL_n)$ is similar: the ${l^+}^i{}_j$ with $i > j$ correspond to the positive roots, the ${l^-}^i{}_j$ with $i < j$ to the negative roots and the diagonals to the cartan subalgebra.

In summary, we have developed in this section dual quantum groups for general invertible R. In previous work[8][6], the relations (9)-(12) and the antipode structures were all thrown in by hand for specific R: we have recovered them (and $\mathcal{R}$) as part of a general quotienting construction.

2 DUALITY IN EXACTLY SOLVABLE VERTEX MODELS

In the context of exactly solvable vertex models, the pairing between the Hopf algebra $\breve{A}(R)$ and its dual Hopf algebra developed above leads in this section to a simple expression for the partition function, and some understanding of an ansatz of [11]. A specific example, a 5-vertex model, (based on $SL_q(2)$) will be given in detail. First we recall the general framework. The statistical mechanical models that will be considered are built on a square lattice of M rows and N columns. Each bond takes values $1 \cdots n$ and the partition function is the sum over all bond states of the products of the Boltzmann weights at each vertex. The weight at a vertex with adjacent bond states $\overset{k}{\underset{l}{i+j}}$ is $R^i{}_j{}^k{}_l$. Such a model is physical if R obeys certain unitarity, symmetry and positivity conditions. It is exactly solvable[5] if R is $R(\lambda)$ i.e. depends on a spectral parameter $\lambda \in \mathbb{C}$ such that $R(\lambda)$ obeys the parametrized Quantum Yang-Baxter Equation $R_{12}(\lambda-\lambda')R_{13}(\lambda)R_{23}(\lambda') = R_{23}(\lambda')R_{13}(\lambda)R_{12}(\lambda-\lambda')$ for all λ, λ'. More generally, $\lambda - \lambda'$ here can be any function $\lambda''(\lambda, \lambda')$. Typically, there exist special points in the parameter space for which the parametrized QYBE are satisfied with $\lambda = \lambda' = \lambda''$. At such special points we write $R(\lambda)$ simply as R. It then obeys the matrix QYBE as in the last section. The first goal will be an expression for the partition function at such a special point. The result will then be adapted for those $R(\lambda)$[4] that can be obtained by the action of a quantum group automorphism α_λ of a fixed R.

Indeed, we are now in a position to observe the following consequence of the formulation above for the pairing between the quantum group $\breve{A}(R)$ and its dual.

PROPOSITION Using equations (3),(6), the pairing between general elements $u^{i_1}{}_{j_1} u^{i_2}{}_{j_2} \cdots u^{i_M}{}_{j_M}$ of $\breve{A}(R)$ and ${l^+}^{k_1}{}_{l_1} {l^+}^{k_2}{}_{l_2} \cdots {l^+}^{k_N}{}_{l_N}$ of $\breve{U}(R)$ is

$$\begin{aligned} &< u^{i_1}{}_{j_1} u^{i_2}{}_{j_2} \cdots u^{i_M}{}_{j_M}, {l^+}^{k_1}{}_{l_1} {l^+}^{k_2}{}_{l_2} \cdots {l^+}^{k_N}{}_{l_N} > \\ &=< u^{i_1}{}_{j_1} \otimes \cdots \otimes u^{i_M}{}_{j_M}, (\Delta^{M-1} {l^+}^{k_1}{}_{l_1}) \cdots (\Delta^{M-1} {l^+}^{k_N}{}_{l_N}) > \\ &=< (\Delta^{N-1} u^{i_1}{}_{j_1}) \otimes \cdots \otimes (\Delta^{N-1} u^{i_M}{}_{j_M}), (\Delta^{M-1} {l^+}^{k_1}{}_{l_1}) \otimes \cdots \otimes (\Delta^{M-1} {l^+}^{k_N}{}_{l_N}) > \\ &=< \begin{matrix} u^{i_1}{}_{m_{11}} \otimes \cdots \otimes u^{m_{1N-1}}{}_{j_1} \\ \otimes u^{i_2}{}_{m_{21}} \otimes \cdots \otimes u^{m_{2M-1}}{}_{j_2} \\ \vdots \\ \otimes u^{i_M}{}_{m_{M1}} \otimes \cdots \otimes u^{m_{MN-1}}{}_{j_M} \end{matrix} , \begin{matrix} {l^+}^{k_1}{}_{n_{11}} \otimes {l^+}^{k_2}{}_{n_{21}} \otimes \cdots \otimes {l^+}^{k_N}{}_{n_{N1}} \\ \otimes {l^+}^{n_{11}}{}_{n_{12}} \otimes {l^+}^{n_{21}}{}_{n_{22}} \otimes \cdots \otimes {l^+}^{n_{N1}}{}_{n_{N2}} \\ \vdots \\ \otimes {l^+}^{n_{1M-1}}{}_{l_1} \otimes {l^+}^{n_{2M-1}}{}_{l_2} \cdots \otimes {l^+}^{n_{NM-1}}{}_{l_N} \end{matrix} > \end{aligned}$$

$$
\begin{aligned}
&= R^{i_1}{}_{m_{11}}{}^{k_1}{}_{n_{11}}\ R^{m_{11}}{}_{m_{12}}{}^{k_2}{}_{n_{21}}\ \cdots\ R^{m_{1N-1}}{}_{j_1}{}^{k_N}{}_{n_{N1}} \\
&\quad R^{i_2}{}_{m_{21}}{}^{n_{11}}{}_{n_{12}} R^{m_{21}}{}_{m_{22}}{}^{n_{21}}{}_{n_{22}} \\
&\quad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\quad R^{i_M}{}_{m_{M1}}{}^{n_{1M-1}}{}_{l_1}\ \cdots\ \cdots\ R^{m_{MN-1}}{}_{j_M}{}^{n_{NM-1}}{}_{l_N} \\
&= Z_R\Big({}_i\overset{k}{\underset{l}{\Box}}{}_j \Big)
\end{aligned}
$$

the partition function for the finite volume square lattice with boundary states $i_1 \cdots i_M$, $l_1 \cdots l_N$, $j_M \cdots j_1$, $k_N \cdots k_1$ as indicated.

The open boundary conditions may be relevant to twisted or open models such as [12] as well as to finite-lattice models such as [13]. More typically, of interest in such models are cyclic boundary conditions. For these let $U = \mathrm{trace}\, u$, $L = \mathrm{trace}\, l^+$. Then

$$< U^M, L^N >= Z_R(\text{cyclic b.c.}). \tag{13}$$

Also, the single row transfer matrix $T^i{}_j$ of interest in the transfer matrix method, and its trace, T, are given by

$$(T^i{}_j)^{k_1k_2\cdots k_N}_{l_1l_2\cdots l_N} =< u^i{}_j, l^{+k_1}{}_{l_1} l^{+k_2}{}_{l_2} \cdots l^{+k_N}{}_{l_N} >, \quad T^{k_1k_2\cdots k_N}_{l_1l_2\cdots l_N} =< U, l^{+k_1}{}_{l_1} l^{+k_2}{}_{l_2} \cdots l^{+k_N}{}_{l_N} > .$$

Of interest for the one-dimensional quantum chain are products of this multi-index matrix T with insertions of other multi-index matrixes ϕ_1, ϕ_2 say. In the special case in which these insertions are of the form $\phi^{k_1k_2\cdots k_N}_{l_1l_2\cdots l_N} =< \Phi, l^{+k_1}{}_{l_1} l^{+k_2}{}_{l_2} \cdots l^{+k_N}{}_{l_N} >$ we have, refining the computation above,

$$< \phi_1\phi_2 >\equiv \mathrm{trace}\, TT\cdots T\phi_1T\cdots T\phi_2T\cdots TT =< UU\cdots U\Phi_1U\cdots U\Phi_2U\cdots UU, L^N > . \tag{14}$$

This shows in direct terms how quantum groups are relevant to lattice vertex models. Note that $\check{A}(R)$ (or $A(R)$) may be thought of heuristically as the abstract algebra of observables of the quantum chain, and L^N as the "vacuum state", corresponding to the fundamental representation T of the quantum chain model. For specific R, suitable $*$-algebra structures are known, and may be expected to exist for all R. Hence it may be possible to develop quantum chains along familiar abstract C^* algebra lines [14], as used for the models in the introductory section.

What follows makes more specific use of the properties of $\check{A}(R)$ and $\check{U}(R)$ found in section 1. Thus, as it stands, the above result only works for special parameter values λ of the Boltzmann weights such that $R(\lambda)$ obeys the matrix QYBE. However, suppose we are given an action α of $\boldsymbol{C}$ on $\check{U}(R)$ that preserves the Hopf algebra structure and $\mathcal{R}$ (i.e. $(\alpha_\lambda\otimes\alpha_\lambda)(\mathcal{R}) = \mathcal{R}$). Then let $R(\lambda) =< u\otimes u, (\alpha_\lambda\otimes \mathrm{id})(\mathcal{R}) >$. Because each α_λ is an automorphism, this necessarily obeys the parametrized QYBE [4]. For $R(\lambda)$ of this form the results above generalize without difficulty. For example,

$$< U^M, \alpha_\lambda(L^N) >= Z_{R(\lambda)}(\text{cyclic b.c.}).$$

Next, of fundamental interest in the method of quantum inverse scattering is a bialgebra $A(\{R(\lambda)\})$ say, defined by generators $\{u(\lambda)\}$ and relations $R(\lambda-\lambda')u_1(\lambda)u_2(\lambda') = u_2(\lambda')u_1(\lambda)R(\lambda-\lambda')$. The following is a variant of equation (8).

PROPOSITION Suppose $R(\lambda)$ can be reconstructed by automorphisms. Then representations of the bialgebra $A(\{R(\lambda)\})$ can be reconstructed from representations of $\check{U}(R)$ via the algebra map $t : A(\{R(\lambda)\}) \to \check{U}(R)$,

$$t(u^i{}_j(\lambda)) \equiv t^i{}_j(\lambda) =< u^i{}_j, \alpha_\lambda(\mathcal{R}^{(1)}) > \mathcal{R}^{(2)}. \tag{15}$$

EXAMPLE For $U_q(sl_2)$ consider the automorphism

$$\alpha_\lambda(X^\pm) = q^{\mp\lambda} X^\pm, \qquad \alpha_\lambda(q^{\pm\frac{H}{2}}) = q^{\pm\frac{H}{2}}.$$

The fundamental representation is $< u, H >= \sigma_3$, $< u, X^\pm >= \frac{1}{2}(\sigma_1 \pm \imath\sigma_2)$ and $\mathcal{R}$ is in [9]. We obtain from the construction above a "five vertex model"

$$R(\lambda) = q^{-\lambda-\frac{1}{2}} \begin{pmatrix} q^{\lambda+1} & 0 & 0 & 0 \\ 0 & q^\lambda & q-q^{-1} & 0 \\ 0 & 0 & q^\lambda & 0 \\ 0 & 0 & 0 & q^{\lambda+1} \end{pmatrix}, \qquad t(\lambda) = q^{-\lambda-\frac{1}{2}} \begin{pmatrix} q^{\lambda+\frac{1}{2}+\frac{H}{2}} & (q-q^{-1})X^- \\ 0 & q^{\lambda+\frac{1}{2}-\frac{H}{2}} \end{pmatrix}. \tag{16}$$

A similar map t is known for the six vertex model and is connected with the Bethe ansatz. Specifically, after rescaling the $R(\lambda)$ and $t(\lambda)$ by $q^{\lambda+\frac{1}{2}}$, the 5-vertex model is "half" of the full 6-vertex model and $t(\lambda)$ is "half" of the ansatz of [11]. For comparison, the six vertex model is given by

$$R(\lambda) = \begin{pmatrix} q^{\lambda+1} - q^{-(\lambda+1)} & 0 & 0 & 0 \\ 0 & q^\lambda - q^{-\lambda} & q-q^{-1} & 0 \\ 0 & q-q^{-1} & q^\lambda - q^{-\lambda} & 0 \\ 0 & 0 & 0 & q^{\lambda+1} - q^{-(\lambda+1)} \end{pmatrix}$$

and the ansatz of [11] by

$$t(\lambda) = \begin{pmatrix} q^{\lambda+\frac{1}{2}+\frac{H}{2}} - q^{-(\lambda+\frac{1}{2}+\frac{H}{2})} & (q-q^{-1})X^- \\ (q-q^{-1})X^+ & q^{\lambda+\frac{1}{2}-\frac{H}{2}} - q^{-(\lambda+\frac{1}{2}-\frac{H}{2})} \end{pmatrix}.$$

3 QUASITENSOR CATEGORIES AND RIGIDITY

The characteristic feature of quantum groups in applications is quasitriangularity, i.e. the existence of a universal $\mathcal{R}$. In this section we shall obtain general results in the representation theory of such quasitriangular Hopf algebras. We begin with a clarified definition: a Hopf algebra H is *quasitriangular* if it possesses an invertible element $\mathcal{R} \in H \otimes H$ such that

$$(\Delta\otimes 1)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (1\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{17}$$

$$\tau \circ \Delta h = \mathcal{R}(\Delta h)\mathcal{R}^{-1}, \ \forall h \in H \tag{18}$$

and is *triangular* if, in addition

$$\tau(\mathcal{R}^{-1}) = \mathcal{R}. \tag{19}$$

This definition is due to [4]. It is well known that $\mathcal{R}$ obeys $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$. Similar computations[2, 1.5.1] imply also that

$$(S\otimes 1)(\mathcal{R}) = \mathcal{R}^{-1}, \qquad (1\otimes S)(\mathcal{R}^{-1}) = \mathcal{R} \tag{20}$$

which will play a crucial role in what follows. Also note[4] that in the case of H finite dimensional we can view $\mathcal{R}$ equivalently as an antialgebra and coalgebra map $H^* \to H$ defined by $a \mapsto \mathcal{R}^{(1)} < a, \mathcal{R}^{(2)} >$, as in equation (8).

A monoidal category is one with an associative "tensor product" structure which we denote $\tilde{\otimes}$. It is a tensor category if in addition $\tilde{\otimes}$ is symmetric. More precisely, $\mathcal{C}$ is a monoidal category if there are functorial isomorphisms $\Phi_{X,Y,Z} : X\tilde{\otimes}(Y\tilde{\otimes}Z) \to (X\tilde{\otimes}Y)\tilde{\otimes}Z$ for all objects X, Y, Z in $Ob(\mathcal{C})$, obeying the pentagon coherence identity for the possible compositions of Φ in $X\tilde{\otimes}(Y\tilde{\otimes}(Z\tilde{\otimes}W)) \cong ((X\tilde{\otimes}Y)\tilde{\otimes}Z)\tilde{\otimes}W$. There should also be a compatible unit object $\underline{1}$. A tensor category $\mathcal{C}$ is a monoidal category for which there exist functorial isomorphisms $\Psi_{X,Y} : X\tilde{\otimes}Y \to Y\tilde{\otimes}X$ such that $\Psi_{Y,X} \circ \Psi_{X,Y} = \mathrm{id}$ and the hexagon coherence identity holds. There should also be compatibility with the unit object. See for example [15].

$$\begin{tikzcd} X\tilde{\otimes}(Y\tilde{\otimes}Z) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\tilde{\otimes}\Psi"'] & (X\tilde{\otimes}Y)\tilde{\otimes}Z \arrow[r, "\Psi"] & Z\tilde{\otimes}(X\tilde{\otimes}Y) \arrow[d, "\Phi"] \\ X\tilde{\otimes}(Z\tilde{\otimes}Y) \arrow[r, "\Phi"] & (X\tilde{\otimes}Z)\tilde{\otimes}Y \arrow[r, "\Psi\tilde{\otimes}\mathrm{id}"] & (Z\tilde{\otimes}X)\tilde{\otimes}Y \end{tikzcd} \qquad \begin{tikzcd} (X\tilde{\otimes}Y)\tilde{\otimes}Z \arrow[r, "\Phi^{-1}"] \arrow[d, "\Psi\tilde{\otimes}\mathrm{id}"'] & X\tilde{\otimes}(Y\tilde{\otimes}Z) \arrow[r, "\Psi"] & (Y\tilde{\otimes}Z)\tilde{\otimes}X \arrow[d, "\Phi^{-1}"] \\ (Y\tilde{\otimes}X)\tilde{\otimes}Z \arrow[r, "\Phi^{-1}"] & Y\tilde{\otimes}(X\tilde{\otimes}Z) \arrow[r, "\mathrm{id}\tilde{\otimes}\Psi"] & Y\tilde{\otimes}(Z\tilde{\otimes}X) \end{tikzcd}$$

Figure 1: Two hexagon identities are needed when we drop $\Psi \circ \Psi = \mathrm{id}$

Like representations of groups, the algebra representations of any Hopf algebra H form a monoidal category, ${}_H\mathcal{M}$. The objects are representations (ρ, V) (i.e. H-modules) and the morphisms are the intertwiners. Specifically, if (ρ_1, V_1) and (ρ_2, V_2) are any two representations of the algebra part of H in vector spaces V_1 and V_2 then there is a new representation $(\rho_1\tilde{\otimes}\rho_2, V_1\tilde{\otimes}V_2)$ defined by $V_1\tilde{\otimes}V_2 = V_1\otimes V_2$ and

$$(\rho_1\tilde{\otimes}\rho_2)_h(v_1\otimes v_2) = \rho_{1h_{(1)}}(v_1)\otimes\rho_{2h_{(2)}}(v_2)$$

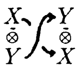
(or more succinctly, $h.(v_1\tilde{\otimes}v_2) = h_{(1)}.v_1\tilde{\otimes}h_{(2)}.v_2$ where the period denotes the relevant action and the $\tilde{}$ is to emphasize that $v_1\tilde{\otimes}v_2$ etc. are to be viewed as elements of the H-module $V_1\tilde{\otimes}V_2$.). It is easy to see that $\tilde{\otimes}$ is associative as a consequence of coassociativity of Δ: Φ is just the associativity isomorphism of vector spaces viewed as an intertwiner. It is also equally easy to see that if H is triangular, then $\tilde{\otimes}$ is symmetric, i.e. ${}_H\mathcal{M}$ is a tensor category: Ψ is given by

$$\Psi_{V_1,V_2} : V_1\tilde{\otimes}V_2 \to V_2\tilde{\otimes}V_1, \quad \Psi(v_1\tilde{\otimes}v_2) = \tau\circ(\rho_1\otimes\rho_2)(\mathcal{R})(v_1\tilde{\otimes}v_2) \tag{21}$$

(i.e. in module notation $\Psi(v_1\tilde{\otimes}v_2) = \mathcal{R}^{(2)}.v_2\tilde{\otimes}\mathcal{R}^{(1)}.v_1$). Here $\tau : V_1\otimes V_2 \to V_2\otimes V_1$ is the twist or permutation map (τ alone is not an intertwiner unless H is cocommutative). A special case, effectively that of Hopf algebras of the form $\tilde{A}(R)$ with R triangular, was shown in [17], which work was discussed in [18][4]. The details for general triangular Hopf algebras are in [2, 7.2.1].

MacLane has shown that once the pentagon and hexagon axioms of a tensor category have been satisfied, then all the other obvious compatibility checks between Φ and Ψ corresponding to different bracketings and orderings also hold. We now generalize these observations to the physically interesting case of H quasitriangular. Thus we define[2, 7.5] a *quasitensor category* $(\mathcal{C}, \tilde{\otimes}, \Phi, \Psi, \underline{1})$ as a monoidal category with functorial isomorphisms $\Psi : X\tilde{\otimes}Y \to Y\tilde{\otimes}X$ as before, obeying now the *two* hexagon identities shown in Figure 1. We do not require any more that $\Psi\circ\Psi = \mathrm{id}$.

MacLane's theorem then has its analog for quasitensor categories: there are no further natural constraints between Φ and Ψ arising from brackets and ordering, once the pentagon and the two hexagons have been imposed. To explain this precisely, note that any map constructed from Φ, Φ^{-1} and Ψ, Ψ^{-1} is of the form $X_1\tilde{\otimes}X_2\cdots\tilde{\otimes}X_N \xrightarrow{\beta} X_{\beta(1)}\tilde{\otimes}X_{\beta(2)}\cdots\tilde{\otimes}X_{\beta(N)}$ where β is an element of the braid group B_N. Here

$\Psi :$  , $\quad \Psi^{-1} :$ 

and Φ and Φ^{-1} are used to suitably position brackets to and from a standard bracketing (say all brackets leftmost). The coherence for quasitensor categories is then that any diagram corresponding to a closed path in B_N, for any N, commutes. The proof is analogous to the tensor category case. Indeed, the coherence assertion for the latter is that any diagram corresponding to a closed path in S_N, the permutation group, commutes[16].

PROPOSITION If H is quasitriangular then ${}_H\mathcal{M}$ is a quasitensor category [2, 7.5].

Another property of vector spaces and group representations that we would also like to generalize is duality. This is expressed in the axioms of rigidity. A rigid tensor category is essentially a tensor category for which there exists an "internal hom" object. More precisely, a tensor category $(\mathcal{C}, \tilde{\otimes}, \Phi, \Psi, \underline{1})$ has an *internal hom* if the contravariant functors $F_{X,Y} = Mor(\ \tilde{\otimes} X, Y)$ (i.e. that send Z to the set $Mor(Z\tilde{\otimes}X, Y)$) are each representable. In this case the representing object in $\mathcal{C}$, the *internal hom*, is denoted $\underline{\mathrm{Hom}}(X, Y)$. Thus for each $X, Y \in Ob(\mathcal{C})$ we need a third object $\underline{\mathrm{Hom}}(X, Y) \in Ob(\mathcal{C})$ such that

$$Mor(Z\tilde{\otimes}X, Y) \cong Mor(Z, \underline{\mathrm{Hom}}(X, Y)) \tag{22}$$

by functorial isomorphisms. Given an internal hom we also define a *duality functor* $^* : \mathcal{C} \to \mathcal{C}$ by $X^* = \underline{\mathrm{Hom}}(X, \underline{1})$. In this case one can show that there are functorial morphisms

$$\underline{\mathrm{Hom}}(X_1, Y_1)\tilde{\otimes}\underline{\mathrm{Hom}}(X_2, Y_2) \xrightarrow{\psi_1} \underline{\mathrm{Hom}}(X_1\tilde{\otimes}X_2, Y_1\tilde{\otimes}Y_2), \quad X \xrightarrow{\psi_2} X^{**}. \tag{23}$$

If these maps are both isomorphisms then $(\mathcal{C}, \tilde{\otimes}, \Phi, \Psi, \underline{1}, \underline{\mathrm{Hom}})$ is said to be *rigid*[15]. These definitions can be adopted unchanged in the quasitensor case[2, 7.5].

THEOREM Let H be a quasitriangular Hopf algebra with bijective antipode. Then its finite dimensional algebra representations $({}_H\mathcal{M}^{\mathrm{f.d.}}, \tilde{\otimes}, \underline{1})$ are a rigid quasitensor category with internal hom defined by

$$\underline{\mathrm{Hom}}(V_1, V_2) = \mathrm{Lin}_k(V_1, V_2).$$

Proof If $f \in Mor(V_3\tilde{\otimes}V_1, V_2)$, the corresponding element in $Mor(V_3, \underline{\mathrm{Hom}}(V_1, V_2))$ is $v_3 \mapsto f(v_3\tilde{\otimes}\ )$ (as for vector spaces). We compute ψ_1, ψ_2. Explicitly, let $f_1 \in \underline{\mathrm{Hom}}(V_1, W_1)$, $f_2 \in \underline{\mathrm{Hom}}(V_2, W_2)$. Then[2, 7.3.7]

$$\psi_1(f_1\tilde{\otimes}f_2)(v_1\tilde{\otimes}v_2) = f_1(\mathcal{R}^{(2)}\mathcal{R}^{-1(2)}.v_1)\tilde{\otimes}\mathcal{R}^{(1)}.(f_2(\mathcal{R}^{-1(1)}.v_2)) \tag{24}$$

and for (say) finite dimensional modules, this map ψ_1 is an isomorphism. To see this, here is the inverse. Given $\phi \in \underline{\mathrm{Hom}}(V_1\tilde{\otimes}V_2, W_1\tilde{\otimes}W_2)$ of the form $\phi = \phi_1\otimes\phi_2$, define $\psi_1^{-1}(\phi) \in \underline{\mathrm{Hom}}(V_1, W_1)\tilde{\otimes}\underline{\mathrm{Hom}}(V_2, W_2)$ by[2, 7.3.7]

$$\psi_1^{-1}(\phi) = \phi_1(\mathcal{R}^{(2)\prime}S\mathcal{R}^{(2)}.\ )\tilde{\otimes}\mathcal{R}^{(1)}.(\phi_2(\mathcal{R}^{(1)\prime}.\ )) \tag{25}$$

which one may check obeys $\psi_1(\psi_1^{-1}(\phi))(v_1\tilde{\otimes}v_2) = \phi(v_1\tilde{\otimes}v_2)$. Similarly the map $V \xrightarrow{\psi_2} V^{**}$ is given explicitly by[2, 7.3.5]

$$\psi_2(v)(f) = (\mathcal{R}^{(2)}.f)(\mathcal{R}^{(1)}.v) = f((S\mathcal{R}^{(2)})\mathcal{R}^{(1)}.v), \ \forall v \in V, f \in V^*. \tag{26}$$

That this ψ_2 is an isomorphism when V is finite dimensional now follows from invertibility of $(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ obtained in the next proposition.

PROPOSITION Let H be a quasitriangular Hopf algebra with bijective antipode and define $\underline{u} = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$. Then $\underline{u}$ is invertible and implements the square of the antipode,

$$\underline{u}^{-1} = \mathcal{R}^{(2)}S^2\mathcal{R}^{(1)}, \quad \underline{u}h\underline{u}^{-1} = S^2(h), \quad \forall h \in H. \tag{27}$$

Moreover, the element $\underline{c} = \underline{u}S(\underline{u})$ is central and $\underline{c} = 1$ if H is triangular. We call $\underline{c}$ the *quantum Casimir element.*

Proof First show using axiom (18) that $(Sh_{(2)})\underline{u}h_{(1)} = \epsilon(h)\underline{u}$ to conclude that $(S^2h)\underline{u} = \underline{u}h$. Also compute that $\mathcal{R}^{(2)}\underline{u}\mathcal{R}^{(1)} = 1$ to conclude that u^{-1} shown is a left inverse. Assuming S^{-2} exists and using equations (20) gives that it is also a right inverse. Noting that $\underline{v} = \mathcal{R}^{(1)}S\mathcal{R}^{(2)}$ has similar properties, $\underline{v}^{-1}h\underline{v} = S^2(h)$, we conclude that $\underline{uv}$ is central. $\underline{v} = S(\underline{u})$ again in virtue of equations (20). Further computation then gives that this central element measures the degree of quasitriangularity as stated. Details are in [2, 7.3.6].

This element $(S\mathcal{R}^{(2)})\mathcal{R}^{(1)}$ arising in equation (26) has independently turned up in the theory of knot invariants[9] where part of the result, equation (27), was also stated. We have denoted it $\underline{u}$ in the proposition because of this connection.

In a rigid tensor category, there is a natural map, *Trace*: $Mor(X,X) \to Mor(\underline{1},\underline{1})$ coming from $Mor(\underline{1},\ )$ applied to

$$\underline{\mathrm{Hom}}(X,X)\xrightarrow{\psi_1^{-1}}\underline{\mathrm{Hom}}(X,\underline{1})\tilde{\otimes}\underline{\mathrm{Hom}}(\underline{1},X)\xrightarrow{ev}\underline{1}.$$

The map *ev* here comes from axiom (22). There is therefore a natural *rank* of X, defined as the *Trace* of id_X[15]. Like Φ and Ψ, $\underline{\mathrm{Hom}}$ in a rigid tensor has remarkable coherence properties. These ensure that[15]

$$rank(X\tilde{\otimes}Y) = rank(X)rank(Y).$$

These definitions can be adopted without change for quasitensor categories. However, in the rigid quasitensor case coherence for $\underline{\mathrm{Hom}}$ does not fare so well: when $\Psi^2 \neq \mathrm{id}$ certain diagrams no longer commute (this seems to be connected with knot theory rather than braids as for coherence of Φ,Ψ). In particular, it turns out[2, 7.5.5] that *rank* is no longer multiplicative. We shall see that fixing this up involves the same ingredients as used by [9] to fix up braid invariants constructed from $\mathcal{R}$ to obtain knot and link invariants. In ${}_H\mathcal{M}$ we identify $Mor(\underline{1},\underline{1}) = \mathbb{C}$ and normalize so that $rank(\underline{1}) = 1$. Using equation (25) we find[2, 7.5.3]

THEOREM Let $(H,\mathcal{R})$ be a quasitriangular Hopf algebra with bijective antipode and $(\rho, V) \in {}_H\mathcal{M}$ (say, finite-dimensional). Then the category-theoretic *rank* is

$$rank_H(V) = \mathrm{trace}_V\,\underline{u}, \qquad \underline{u} = (S\mathcal{R}^{(2)})\mathcal{R}^{(1)} \tag{28}$$

where the trace of $\underline{u}$ is taken in representation ρ.

We can also compute $rank_H(H)$ where H acts on H by multiplication (the left regular representation). This $rank_H(H)$ is an invariant of H *as a quasitriangular Hopf algebra* i.e., if $H_1 \cong H_2$ by an isomorphism that sends $\mathcal{R}$ in H_1 to $\mathcal{R}$ in H_2, then $rank_{H_1}(H_1) = rank_{H_2}(H_2)$. So we define $rank(H) = rank_H(H)$. Actually, this invariance is not a very deep fact in our case: the trace of any combination built out of $\mathcal{R}$ and the Hopf algebra structures would have this property. The special feature of $rank_H(V)$ in the triangular case is that it is multiplicative.

EXAMPLE For V_j the spin j representation of $U_q(su(2))$,

$$\underline{c}|_{V_j} = q^{-4j(j+1)}, \quad rank(V_j) = q^{-2j(j+1)}\left(\frac{q^{2j+1} - q^{-(2j+1)}}{q - q^{-1}}\right). \tag{29}$$

Proof As noted above, the quantity $\underline{u}$ was already studied for $U_q(sl_2)$ in the context of knot invariants[9]. There it was shown that $(S\mathcal{R}^{(2)})\mathcal{R}^{(1)} = q^{-2j(j+1)}q^H$ on V_j. Likewise, it is easy to show that $\mathcal{R}^{(1)}S\mathcal{R}^{(2)} = q^{-2j(j+1)}q^{-H}$ from which $\underline{c}$ follows. For the *rank* let $\{f^{jm}\}$ be a dual basis to the spin j representation $\{e^j_m\}$. Then $rank(V_j) = \sum_m f^{jm}((S\mathcal{R}^{(2)})\mathcal{R}^{(1)}.e^j_m)$. The result now follows from $He^j_m = 2me^j_m$ and an elementary computation.

Now, although $rank(V_j)$ is not multiplicative ($U_q(su(2))$ is not triangular), we shall now see that it fails in an interesting way. Indeed, for $U_q(su(2))$, [9] show that one can adjoin an element $\underline{w}$, the *Weyl element*, such that

$$\underline{w}h\underline{w}^{-1} = T \circ S(h), \quad \underline{u} = (S\underline{w})\underline{w}, \quad S(\underline{u}) = \underline{w}S\underline{w}, \quad \Delta\underline{w} = \mathcal{R}^{-1}(\underline{w}\otimes\underline{w}).$$

Here $T(X^\pm) = X^\mp, T(H) = H$ is a linear antiautomorphism of $U_q(su(2))$. Assuming these facts a computation gives

$$\mathrm{trace}_{V_1\tilde{\otimes}V_2}\underline{u} = \mathrm{trace}_{V_1\otimes V_2}(\underline{v}\otimes\underline{v})B^{-2}, \quad \mathrm{trace}_{V_1\tilde{\otimes}V_2}\underline{c} = \mathrm{trace}_{V_1\otimes V_2}(\underline{c}\otimes\underline{c})B^{-4}.$$

Here $\underline{u},\underline{v},\underline{c}$ are in the relevant representations. $B = \tau \circ R$ where R is $\mathcal{R}$ in the relevant representations, and $\mathrm{trace}\,\underline{u} = \mathrm{trace}\,\underline{v}$. Thus both *rank* and the value of $\underline{c}$ are not multiplicative due to the B^{-2} and B^{-4} factors. This suggests that by combining $rank$ with "$\underline{c}^{-\frac{1}{2}}$" we may be able to obtain a modified $\widetilde{rank}$ that is multiplicative. Note that such Weyl elements $\underline{w}$ and similar formulae may be expected for general quantum groups. For $U_q(su(2))$ we divide through by $\underline{c}^{\frac{1}{2}}$ to obtain

$$\widetilde{rank}(V_j) = \frac{q^{2j+1} - q^{-(2j+1)}}{q - q^{-1}}. \tag{30}$$

Using the decomposition of $V_{j_1}\tilde{\otimes}V_{j_2}$ (it is the same as for $su(2)$ provided q is not a root of unity), it is a short computation to find that this $\widetilde{rank}$ is indeed multiplicative. It has arisen in many contexts concerning $U_q(su(2))$ where it plays a role analogous to dimension. This example also demonstrates the assertion that the problem of fixing up coherence of internal hom in quasitensor categories (in this case fixing up multiplicativity of $rank$) is connected to the problem of constructing knot invariants from braid ones.

Also, the decomposition of the left action of $U_q(su(2))$ on itself (the left regular representation) is known: at least for suitable q it has been shown [19] to have the same multiplicities as for the left regular representation of $SU(2)$, $\overline{U_q(su(2))} \cong \oplus_j (\dim V_j)V_j$ (the direct sum here is a topological one). Using this we can formally compute $rank(H)$ (the $rank$ of the left regular representation) for $H = U_q(su(2))$. We obtain

$$rank(U_q(su(2)) = \sum_j \dim(V_j) rank(V_j) = (1 - q^{-2})^{-1} \sum_{k\in Z} q^{-\frac{1}{2}k^2} = \frac{\theta_3(q^{-\frac{1}{2}})}{1 - q^{-2}}. \tag{31}$$

Here θ_3 is one of the classical Jacobi theta-functions. Note that if G is a finite group and kG its group Hopf algebra, then $rank(kG) = |G|$ (the number of elements in G). In general, $rank(H)$ is not an integer. In the example it diverges as $q \to 1$, as it should.

REFERENCES

[1] S. Majid, Non-commutative-geometric Groups by a Bicrossproduct Construction, Harvard PhD Thesis (1988); S. Majid, Physics for Algebraists: Non-commutative and Non-cocommutative Hopf Algebras by a Bicrossproduct Construction (1987), to appear J. Algebra, **128(2)** (1990); S. Majid, Matched Pairs of Lie Groups Associated to Solutions of the CYBE (1988), to appear Pac. J. Math; S. Majid, Hopf von Neumann Algebra Bicrossproducts, Kac Algebra Bicrossproducts, and the Classical Yang-Baxter Equations (1988), to appear J. Funct. Analysis; S. Majid, Hopf Algebras for Physics at the Planck Scale, J. Class. and Quant. Grav. **5** (1988) 1587-1606.

[2] S. Majid, Quasitriangular Hopf Algebras and Yang-Baxter Equations, to appear Int. J. Mod. Phys. A (1989).

[3] S. Majid, Sequences of Bicrossproduct and Double Crossproduct Hopf Algebras, Proc. Int. Conf. Hopf Algebras, Beer Sheva (1989).

[4] V.G. Drinfeld, Quantum Groups, Proc. ICM, Berkeley (1986), ed. A.M. Gleason, AMS (1987) 798-820.

[5] R.J. Baxter, Exactly Solvable Models in Statistical Mechanics, Academic (1982).

[6] L. D. Faddeev, N.Yu. Reshetikhin, L.A. Takhtajan, Quantization of Lie Groups and Lie Algebras, LOMI preprint (1987).

[7] M.E. Sweedler, Hopf Algebras, Benjamin (1969).

[8] M. Rosso, Comparison des Groupes $SU(2)$ Quantique de Drinfeld et de Woronowicz, CR. Acad. Sci. Paris **304** (1987) 323-326.

[9] A.N. Kirillov, N.Yu. Reshetikhin, Representations of the Algebra $U_q(sl(2))$, q-orthogonal Polynomials and Invariants of Links, LOMI preprint (1988).

[10] N. Burroughs, The Universal R-matrix for $U_q(sl_3)$ and Beyond!, DAMTP preprint (1989); M. Rosso, An Analogue of the P.B.W. Theorem and the Universal R-matrix for $U_hSL(N+1)$, Palaiseau preprint (1989).

[11] P.P. Kulish, N.Yu. Reshetikhin, Quantum Linear Problem for Sine-Gordon Equation and Highest Weight Representations (in Russian), Zap. Nauch. Sem. **101** (1981) 101; see H.J. de Vega, Yang-Baxter Algebras, Integrable Theories and Quantum Groups, LPTHE preprint (1988).

[12] E.K. Sklyanin, Boundary Conditions for the Integrable Quantum Systems, J.Phys. A **21** (1988) 2375-2389.

[13] R.J. Baxter, The Superintegrable Chiral Potts Model, Phys. Lett. A **133** (1988) 185.

[14] I. Segal, Mathematical Problems of Relativistic Physics, AMS (1963); see O. Bratteli, D.W. Robinson, Operator Algebras and Quantum Statistical Mechanics, Vol II, Springer-Verlag (1979).

[15] P. Deligne, J.S. Milne, Tannakian Categories, Springer Lect. Notes Math. **900** (1982).

[16] S. MacLane, Natural Associativity and Commutativity, Rice University Studies **49** (1963) 28-46.

[17] V.V. Lyubashenko, Hopf Algebras and Vector Symmetries, Usp. Math. Nauk. **41** (1986) 185-186.

[18] Yu. Manin, Quantum Groups and Non - commutative Geometry, Montreal Preprint (1988).

[19] T. Masuda, K. Mimachi, Y. Nakagami, M. Noumi, K. Ueno, Representations of Quantum Groups and q-analog of Orthogonal Polynomials, preprint (1988).

PHYSICS AT THE PLANCK LENGTH AND p-ADIC FIELD THEORIES

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There is no doubt that to describe the physics at the Planck scales one needs a new physical theory beyond the standard quantum gravity or string theory. In the last five years we have gotten the custom to think about string theory as a theory describing physics up to the Planck scales. But there are questions which are difficult to understand in string theory. The gravitational collapse and cosmological singularity are such problems. The present string theories do not tell us what to do with these problems. It seems that to understand these problems one needs to understand better what is space-time.

In string theory, as in other modern physical theories, we accept the usual assumption about space-time as a manifold with a metric. This conception has an old tradition and it goes back to Euclid, Riemann and Einstein. However, the string theory deals with very small distances of Planck order, (at least the six extra dimensions have to live on a small manifold of the Planck order). But it looks very questionable that at such extreme small distances the usual geometry could indeed be applicable.

I would like to start my talk from an explanation of why an application of the usual geometry at the Planck scale looks inappropriate.

So the outline of my talk is the following:

1. Motivations of non-Archimedean geometry at Planck scales.
2. Connection between non-Archimedean geometry and p-adic numbers.
3. A simple example of field theory with non-Archimedean space time (p-adic quantum mechanics).
4. p-adic version of general relativity theory.

5. Fluctuations of number field and possible scenario for p-adic cosmology.

1. MOTIVATIONS [1]

Our intuitive understanding of properties of the space is expressed in axioms of elementary geometry. As it is well known a complete list of geometrical axioms has been completed by Hilbert in his "Grundlagen der Geometry".

In this list of axioms there is one which looks like a physical axiom. It is the Archimedean axiom. According to the Archimedean axiom any given large segment on a straight line can be surpassed by successive addition of small segment along the same line.

The Archimedean axiom concerns the process of measurement because it compares two different scales. But as it is well known the gravitational measurement cannot be localized in a sub-Planckian domain. Or in other words, there are principal limitations on field measurements. For example, the following inequality takes place

$$\Delta\Gamma > \ell_{pl}^2/\ell^3, \tag{1}$$

where $\Delta\Gamma$ is the uncertainty in determination of a connection coefficient, ℓ is the length scale, $\ell_{pl} = (\hbar G c^{-3})^{1/2}$ is the Planck length, G is Newton's constant and c is the velocity of the light. These limitations are originated by the fact that on distances of the Planck order the effects of the quantum gravity play a critical role.

There is also an absolute limitation on length measurements

$$\ell \geq \ell_{p\ell},$$

so the Planck length is the smallest possible distance that can in principle be measured.

Let me give a simple explanation of this fact. Indeed, if we want to locate an elementary particle, or string we need to have energy greater than the Planck mass

$$m_p = (\hbar c G^{-1})^{1/2}.$$

But in such case the corresponding gravitational field will have a horizon at the radius

$$r = 2Gm_{pl}c^{-2} = 2\ell_{pl}$$

shielding whatever happens inside the Schwarzchild radius. So, no information of geometry in a sub-Plankian region is available. In some sense a smallest quantum of space and time exists and it is of the order of the Planck scale.

From the above discussion one can conclude that the physical space cannot be described by the usual Eculidian geometry in which (in principle) any given segment can be measured. So there is the physical reason to abandon the Archimedean axiom because due to this axiom in Euclidean geometry one can compare two scales.

We want to construct a new theory based on a non-Archimedean geometry which would be suitable to describe the situation at very small distances (of course at large distances one has to recover the standard description).

How to construct a new theory?

Here let me remind you that there is an analytical description of the geometry.

I mean that usually we use real nubmers to describe a geometrical picture. First of all there is one to one correspondence between segments and real numbers. I mean that the size of a segment is a real number. So, there are two equivalent approaches

$$\boxed{\text{geometry}} \longleftrightarrow \boxed{\text{numbers}}$$

Therefore if we want to abandon standard geometry we have to abandon real numbers.

But what should we use instead of real numbers?

2. CONNECTION BETWEEN NON-ARCHIMEDEAN GEOMETRY AND p-ADIC NUMBERS

We can keep to the following scheme. The ordinary geometry is constructed under the field of real numbers. If we want an essential departure from ordinary geometry and at the same time to have correspondence principle we ought to construct the theory on some field K which has common features with the real field and at the same time on this field it would be possible to construct a non-Archimedean geometry. Let me remind you that a field K is a set together with two operations an addition and a multiplication, such that K is a commutative group under addition and $K-\{0\}$ is a commutative group under multiplication, and the distributive law holds.

The simplest examples of a field are:

i) the field R of real numbers,

ii) the field C of complex numbers,

iii) the field Q of rational numbers.

They are the infinite fields. The simplest example of a finite field is the integers module a prime P (Galoisfield) F_P.

The matter is that any field contains the field of rational numbers Q or a finite Galois field F_P. So, we have only two simplest possibilities: Q or F_p.

$$\begin{array}{ccc} & K & \\ / & & \backslash \\ Q & & F_p \end{array}$$

Let us consider the field of rational numbers Q. To develop the physical theory apparently we need a norm.

Norm on the field K is a map denoted by $\|\ \ \|$ from K to the nonnegative real numbers such that:

i) $\| x + y \| \leq \| x \| + \| y \|$

ii) $\| xy \| = \| x \| \ \| y \|$

iii) $\| x \| = 0$ if and only if $x = 0$.

A basic example of a norm on the rational number field Q is the usual absolute value.

Another example of a norm on the rational is p-adic norm. Let us consider a prime number $p = 2, 3, 5, \ldots$. Any rational number

$$x = M/N$$

can be represented in the form

$$x = p^v \frac{m}{n},$$

where m and n are integers which are not divisible by p (because each integer greater than 1 can be written as a product of primes). Then p-adic norm $|x|_p$ is defined as

$$|x|_p = \left| p^v \frac{m}{n} \right|_p = \frac{1}{p^v}. \tag{2}$$

It is a good exercise to check that eq. (2) indeed defines a norm.

According to the Ostrowski theorem every nontrivial norm on Q is equivalent to the usual absolute value $|\cdot|$ or the p-adic norm $|\cdot|_p$ for some prime p.

The completion of Q with usual absolute value is the field R of real number. The completion of Q with the p-adic norm is the p-adic number field Q_p.

So, if we join the notion of the norm to the diagram on page 4 then we get the following:

$$\begin{array}{ccccc} & & & K & \\ & & / & & \backslash \\ & & & & F_p \\ & Q & & & \\ / & |\cdot| & \backslash|\cdot|_p & & \\ R & & Q_p & & \end{array}$$

Any p-adic number xEQ_p can be represented as the series

$$x = p^v(a_0 + a_1p + a_2p^2 + \ldots), \tag{3}$$

where a_i are integers such that $0 \leq a_i \leq p - 1$.

Eq. (3) is the representation which is similar to the decimal representation of the usual real number

$$x = 10^\mu(a_0 + \frac{1}{10}a_1 + \frac{1}{(10)^2}a_2 + \ldots), \tag{4}$$

xGR^+, and a_i are integers such that $0 \leq a_i \leq p - 1$.

Let me now explain the connection between p-adic number and non-Archimedean geometry. A norm is called non-Archimedean if

$$\| x + y \| \leq \max\ (\| x \|, \| y \|) \tag{5}$$

always holds.

p-adic norm on Q does satisfy Eq. (5). Thus p-adic norm is a non-Archimedean norm on Q.

Our intuition about distance is based of course, on the Archimedean metric. Some properties of non-Archimedean metric $|\ |_p$ seem very strange at first. Let me give you some examples,

i) in non-Archimedean geometry all triangles are isosceles;

ii) any point in a disc is its center;

iii) if we consider two discs then either a disc is contained inside the other one or they do not intersect.

The p-adic number field is the totally disconnected topological space. In contrast to a discrete space where there are holes between different points in the p-adic space there are no holes.

3. p-ADIC QUANTUM MECHANICS [2]

If we would like to use p-adic numbers instead of real numbers then there appear different possibilities to develop quantum mechanics.

I shall discuss an approach with a complex-valued wave function $\psi(x,t)$ depending on the p-adic variables $x \epsilon Q_p, t \epsilon Q_p$. There are two reasons to start from this approach:

1. there is a correspondence principle for this formulation
2. the spectrum has a rich structure.

The usual Schrödinger representation cannot be used as a starting point for construction of p-adic quantum mechanics with complex wave functions because the operators of coordinates and momentum cannot be realized in the usual way as multiplication and differentiation respectively. It is quite remarkable that the exponential form of these operators in the Weyl representation admits a generalization to the case of p-adic numbers. In the same way in p-adic quantum mechanics one cannot start with a quantum Hamiltonian but instead one has to introduce a dynamics in the form of a unitary evolution operator.

So, the formalism for p-adic quantum mechanics is based on a triplet

$$(L_2(Q_p),\ W(Z),\ U(t)), \tag{6}$$

where $L_2(Q_q)$ is the Hilbert space of complex valued square integrable functions on Q_p, $W(Z)$ is the Weyl representation of the commutations relations and $U(t)$ is a time evolution operator.

Let us discuss the two simplest cases, they are a free particle and are harmonic oscillator.

The classical p-adic free particle has the Hamiltonian

$$H = \frac{1}{2m} P^2$$

here $M \epsilon Q_p$, $m \neq 0$ (or $M \epsilon Q$). Hamilton's equations

$$\begin{gathered} p' = 0 \\ q' = \frac{1}{m} p, \\ p(0) = p,\ q(0) = q \end{gathered}$$

have a unique analytical solution for $t \epsilon Q_p$

$$P(t) = p, \quad q(t) = q + \frac{p}{m} t. \tag{7}$$

For the harmonic oscillator the equations of motion

$$p' = -m\omega^2 q$$
$$q' = \frac{1}{m}p,$$
$$p(0) = p, \quad q(0) = q,$$
$$m, \omega \epsilon Q_p, \quad m \neq 0,$$

have the analytical solution which is analogous to the solution over the field of real numbers

$$\begin{pmatrix} q(t) \\ p(t) \end{pmatrix} = T_t \begin{pmatrix} a \\ p \end{pmatrix}, \tag{8}$$

where

$$T_t = \begin{Vmatrix} \cos\omega t & -\frac{1}{m\omega} \sin\omega t \\ -m\omega \sin\omega t & \cos\omega t \end{Vmatrix} \tag{9}$$

In the p-adic quantum mechanics the starting point is the action of unitary operators in the space $L_2(Q_p)$

$$U_q : \psi(x) \to \psi(x+q), \qquad x\epsilon Q_p, \quad q\epsilon Q_p$$
$$V_p : \psi(x) \to \chi(2px)\psi(x), \quad x, \; p\epsilon Q_p,$$

where χ is the additive character on Q_p.

A family of unitary operators

$$W(z) = \chi(-pq)U_p V_q, \qquad z = (q,p)\epsilon Q_p^2,$$

satisfies the Weyl relation

$$W(z)W(z') = \chi(B(z,z,'))W(z+z'), \tag{10}$$

where

$$B(z,z') = qp' - q'p. \tag{11}$$

Let me now explain how one can describe a dynamics in the p-adic quantum mechanics. In the standard quantum mechanics one starts with a quantum Hamiltonian and then one constructs an operator of evolution $U(t)$. From above discussion it is clear that because in the p-adic quantum mechanics we have not canonical coordinates but only the Weyl representation, therefore we have not also the standard Hamiltonian approach and instead we have to construct directly a unitary group $U(t)$.

For the free particle as well as for harmonic oscillator $U(t)$ have to satisfy

$$U(t)W(z) \ \ U(t)^{-1} = W(T_t z), \tag{12}$$

where $T_t z$ is given by eq. (7) for free particle and by eqs. (8), (9) for the harmonic oscillator.

The evolution operator one can write down as an integral operator in x-representation

$$U(t)\psi(x) = \int\limits_{Q_p} K_t(x, -y)\psi(y)dy \tag{13}$$

For the cases of the free particle as well as the harmonic oscillator one can find the corresponding kernels explicitly [9]. For the free particle

$$K_t(x-y) = \lambda_p\left(\frac{t}{4m}\right)\left|\frac{4m}{t}\right|^{\frac{1}{2}}\chi\left(-\frac{m}{t}(x-y)^2\right) \tag{14}$$

the explicit formula for λ_p one can find in Ref. [2], and for the harmonic oscillator

$$K_t(x,y) = \lambda_p\left(\frac{t}{m}\right)\left|\frac{m}{t}\right|_p^{\frac{1}{2}}\chi\left(mw\left(-\frac{x^2+y^2}{H\ tan\ \omega t} + 2\frac{xy}{\sin\omega t}\right)\right). \tag{15}$$

The problem about spectrum of the harmonic oscillator is the problem of expansion of the represenation $U(t)$ over irreducible representation. In the standard quantum mechanics over real numbers the expansion $L_2(R) = \bigoplus\limits_{n=0}^{\infty} \mathcal{H}_n$ over Hermitian polynomials gives the usual spectrum.

In our case the group $G = \left\{t\epsilon Q_p;\ |t|_p \geq \frac{1}{p}\right\}$ is commutative compact group. Therefore $\hat{G}$ is a discrete space and one has

$$U(t) = \sum_\alpha \chi(\alpha t)P_\alpha, \qquad \alpha\epsilon Q_p.$$

It is possible to prove [2] that or $d=0$ either $|\alpha|_p \geq p^2$.

To $\alpha = 0$ it corresponds a subspace $\mathcal{H}_0$ of invariant vectors (vacuum). This subspace has the following dimensions for different p

	$dim\mathcal{H}_0$
$p=2$	2
$p\equiv 1 \pmod 4$	∞
$p \equiv 3 \pmod 4$	1

A subspace of excitations states $\mathcal{H}_2$ which corresponds to the character $\chi(\alpha t)$ has the following dimensions

	$dim\mathcal{H}_\alpha$
$p \equiv 1 \pmod 4$	∞
$p \equiv 3 \pmod 4$	$p+1.$

Let me discuss the correspondence principle between the p-adic quantum mechanic and the standard one. There are two possibilities of the interpretation of the correspondence principle. The first interpretation [2] occur because it is possible to specify such set of rational numbers C_p such that

$$\chi_p(\xi) = e^{2\pi i \xi}, \quad \xi \epsilon C_p. \tag{16}$$

The l.h.s. of eq. (16) is wave function for p-adic free particle and the r.h.s of eq. (16) is the wave function for usual particle. So one can say that it is possible to point out a dense set C_p in Q_p on which the p-adic quantum mechanics gives the same answer as the usual quantum mechanics.

The basic formula for the second interpretation of the correspondence principle is the following adelic formula.

$$\prod_p \chi_p(\xi) = \exp(2\pi i \xi) \tag{17}$$

Here ξ is rational. This formula can be interpreted as the formula expressing the wave function of the free particle over real number in terms of the wave function of the p-adic free particles.

4. p-ADIC VERSION OF GENERAL RELATIVITY THEORY [3]

The p-adic generalization of differential geometry is analogous to the one on usual real manifold, but in the p-adic case one has to consider analytical functions. We will use the following definition of the derivatives in this case. If one has a function $f: Q_p \to Q_p$ and

$$\left| \frac{f(x+\epsilon) - f(x)}{\epsilon} - \frac{\partial f(x)}{\partial x} \right|_p \to 0, \quad |\epsilon|_p \to 0,$$

then $\frac{\partial f}{\partial x}$ is the derivation of the function $f(x)$.

p-adic tensors have the usual transformation rules under the analytical transformation of variables. For example, for the second rank tensor $g_{\mu v}(x)$ on a manifold M^D one has

$$g_{\mu' v'}(x') = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \quad \frac{\partial x^{v}}{\partial x^{v'}} g_{\mu v}(x) \tag{18}$$

Here $x^{\mu} \epsilon Q_p$, $\mu = 1, \ldots D$, $g_{\mu v}(x')$. The $x^{\mu}(x')$ are series with respect to x'^{μ} which are convergent in p-adic norm. Then we can introduce on M^D a p-adic connection $\Gamma^{\lambda}_{\mu v}$ with usual transformation rule and also Riemannian tensor $R^{\sigma}_{\mu v \lambda}$, Ricci tensor $R_{\mu\lambda} = R^{\sigma}_{\mu\sigma\lambda}$ and scalar curvature $R = R_{\mu v} g^{\mu v}$. Note that the inverse tensor $g^{\mu v}$ is defined in p-adic case by the usual formula $g^{\mu v} g_{v\lambda} = \delta^{\mu}_{\lambda}$.

The simplest example of nontrivial p-adic manifold is a sphere S^{n-1}_p in Q^2_p, which is defined by equation

$$x_1^2 + \ldots + \chi_n^2 = \rho^2; \quad x_i, \rho \epsilon Q_p \tag{19}$$

Note that the sphere (19) is a non-compact manifold. The sphere (19) is a space of constant curvature in the sense

$$R_{\mu\nu\lambda\sigma} = K(g_{\mu\lambda} g_{v\sigma} - g_{\mu\sigma} g_{\nu\lambda}),$$

where $K = \frac{1}{\rho}$ is the p-adic curvature of the sphere.

If we adapt the general principle [1] according to which all fundamental physical laws should be invariant with respect to change of these number fields, then the p-adic version of the Einstein gravitational field equations takes the form

$$R_{MN} - \frac{1}{2} g_{MN} R = \chi T_{MN} - \wedge g_{MN}, \tag{20}$$

i.e. it has the same form as in the usual case. See Ref. 3 for possible p-adic solutions of eq. (20).

5. THE FLUCTUATIONS OF NUMBER FIELD AND POSSIBLE SCENARIO FOR p-ADIC COSMOLOGY [4,5]

The usual wave function of the universe with induced 3-matrix h_R is written

$$\psi(h_R) = \int \prod_{\alpha} \left(\chi_R(S_{R,\alpha}) \mathcal{D} g_{R,\alpha} \right)$$

where $\chi_R(S_{R,\alpha}) = \exp 2\pi i S_{R,\alpha}$ is the usual Euclidean action associated with a segment α in the usual space time.

If now we want to take into account the fluctuations of number field then there are two possibilities to do this. In the first case we have in each point to take into account all the possibilities (presumably with equal weight)

$$\psi(h_Q) = N\int \Pi_\alpha \Pi_p \chi_p(S_{p,\alpha})\mathcal{D}g_{\alpha,p}$$
$$= N\int \Pi_p \chi_p(S_p)\mathcal{D}g_p. \tag{21}$$

where χ_p is the additive character, S_p is the p-adic Euclidean action.

The wave function $\psi(h_Q)$ for the second case may be written

$$\psi(h_Q) = \hat{\psi}(h_Q)\,|\Omega(Q)\rangle$$
$$= \Pi_\alpha \hat{\psi}(\Delta h_Q^{(\alpha)})\,|\Omega(Q)\rangle\,. \tag{22}$$

where $|\Omega(Q)\rangle$ is the Frampton-Volovich void with rationals.

As an approximation one has

$$\hat{\psi}(\Delta h_Q^{(\alpha)}) \simeq \chi_p(\Delta h_{p_\alpha}^{(\alpha)})\phi_{p_\alpha}, \tag{23}$$

where ϕ_p acts on rationals as a completion operator with respect of the $|\ |_p$ norm. Together with the operator (23) one can consider the wave function

$$\psi(h_Q) = \sum_{p_1,\ldots} C_{p_1,\ldots}\Pi_\alpha \chi_{p_\alpha}(h_\alpha^4)\phi_{p\alpha} \tag{24}$$

A la third quantization approach one can consider a cubic interaction between operators $t(h)$. It is difficult to specify a unique form of the interaction nevertheless it seems to me that there is reason to expect that the action should be gauge invariant under local change of the completion procedure

$$\delta\psi = \sum_\alpha \delta\lambda_\alpha \phi_{p\alpha}\psi. \tag{25}$$

What is obvious from the above discussion is that the different number fields have interaction. Taking this in mind let us consider the simplest model example of such kind of interaction, i.e.

$$\mathcal{L} = \sum_{p,q,r} V_{pqr}\phi_p\phi_q\phi_r,$$

where V_{pqr} is the interaction kernel. I would like to give an example of the interaction such that for the usual distances ($\ell \gg \ell_{p\ell}$) the real number field R dominates while at

$\ell \simeq \ell_{p\ell}$ all number fields coordinate. To do this let me consider the following vertex:

$$V_{p_{1_R}p_2p_3} = -\frac{1}{3\ln\varsigma(S_0)}\sum_{i=1}^{3}\ln(1-p_i^{-S_0}), \tag{26}$$

where

$$S_0 = 1 + \frac{3\ell_{p\ell}}{\ell_1+\ell_2+\ell_3}$$

Now we can make the following observation about eq. (26). When $|\ell_i| \gg \ell_{p\ell}$ then $S_0 \to 1$ and the minimum of the effective action is achieved at $p=\infty$ (for all other p, $\ln(1-p_i) \neq 0$), therefore only the real number field ($p=\infty$) dominates. For small distances one has taken into account all fields.

CONCLUSION

For conclusion see fig. (1) and fig. (2).

REFERENCES

1. I.V. Volovich, Number theory as the ultimate physical theory, Preprint CERN-TH.4981/87 (1987). I. Ya. Aref'eva and I.V. Volovich "Strings, Gravity and *p*-adic Space-Time" in "Quantum Gravity", Eds. M.A. Markov, V.A. Berezin and V.P. Frolov, World Scientific, p. 409, May 25-29, 1989, Moscow.

2. V.S Vladimirov and I.V. Volovich, *p*-adic Quantum Mechanics, DAN, 302, N2 (1988) 320; Comm. Math. Phys. (1989); Phys. Lett. B217 (1989) 411. V.S. Vladimirov, I.V. Volovich and E. Zelenov, Spectral expansion in *p*-adic quantum mechanics, preprint, SMI-5/89 (to be published in Izvestiya AN of the USSR).

3. I. Ya. Aref'eva, P.H. Frampton, B. Dragovich and I.V. Volvoich, Wave function of the Universe and *p*-adic gravity, preprint (1989).

4. P.H. Frampton and I. Volovich, Primary quantization; creation of the nothing from a void. Preprint IFP-347-UNC (1989).

5. I. Ya. Aref'eva and P.H. Frampton, Beyond Planck Energy to Non-Archimedean Geometry, Preprint (1989).

NON-ARCHIMEDIAN GEOMETRY AND APPLICATIONS TO PARTICLE THEORY

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This talk is devided into the following three sections:

I. Introduction to p-adic numbers and non-Archimedean geometry

II. P-adic strings

III. P-adic gravity and quantum cosmology.

I. INTRODUCTION TO P-ADIC NUMBERS AND NON-ARCHIMEDEAN GEOMETRY.

To begin, I shall give a very elementary introduction to p-adic numbers. I apologize to the cognoscenti but experience suggests that if I start talking about details of p-adic strings all but a small minority of the audience will be unable to follow.

We begin with the group of all integers $Z = 0, \pm 1, \pm 2, \ldots$. As the noted mathematician Kronecker once said, in paraphrase: God gave humans integers, the rest of mathematics is human-made!

These integers are easily extended to the field of rational numbers Q

$$Q = \{M/N\} \qquad M, N, \in \quad Z \tag{1}$$

The completion of Q to the real number field R now needs a little thought. The real irrationals are introduced by making Cauchy sequences $q_1, q_2, \ldots q_n \ldots$ such that they converge to an irrational using as norm the usual absolute value $|q_n - q_{n+1}| \to 0$, $n \to \infty$.

A good norm is one which will allow the resulting number field to be used in analysis, etc. and it can be shown that such a norm must satisfy three conditions:

Differential Geometric Methods in Theoretical Physics
Edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990

$$|x| = 0 \quad \text{iff} \quad x = 0 \tag{2a}$$
$$|xy| = |x|\,|y| \tag{2b}$$
$$|x+y| \leq |x| + |y| \tag{2c}$$

It turns out that there is an infinite number of solutions to (2a) - (2c), one for each prime number p = 2,3,5,7, ..., 137, ... Formally $p = \infty$ is the special case R. The p-adic norm of a rational number $q \in Q$ is defined by

$$|q|_p = \left|\frac{M}{N}\right|_p = \left|\frac{m}{n} p^{\upsilon}\right|_p = p^{-\upsilon} \tag{3}$$

Here m,n are integers prime with respect to p.

A very important point is that the ONLY completions satisfying (2a) - (2c) are R and Q_p (the p-adic field obtained by completing Q using the p-adic norm). This theorem by Ostrowski over fifty years ago is a landmark of number theory. The p-adic numbers were first introduced by Hensel over 100 years ago.

The p-adic norm satisfies a stronger inequality

$$|x + y|_p \leq \max\left(|x|_p, |y|_p\right) \tag{4}$$

This is sometimes called ultrametricity and Eq. (4) characterises non-Archimedean geometry.

Just as geometry is connected to analysis, as eloquently discussed by Kastler at this conference, so there is a one-to-one correspondence between geometry and number fields.

Real R $\rightarrow$ Euclidean geometry

$Q_p \rightarrow$ non-Archimedean geometry

One can carry through arithmetic, analysis, topology, differential geometry, ... with p-adics. For analysis, one uses the quadratic extension $Q_p \rightarrow K_p(\sqrt{\tau})$ counterpart of $R \rightarrow C$ (x + iy). Of course, the details and techniques may differ widely from the conventional case.

The extensive development of p-adic numbers by number theorists is so beautiful and elegent that it would be almost surprising if this is not used in some essential way in physics. I shall describe two applications in theoretical physics:

1) Worldsheet coordinates of the string and superstring. Here p-adics are a regularization procedure and the p-adic string itself is not physical.
2) Spacetime coordinates for lengths below the Planck length 10^{-33} cm and time intervals below the Planck time 10^{-43} sec. Associated with this application are p-adic quantum cosmology and p-adic gravity.

A third application which I shall not address here is p-adic quantum mechanics.

Before entering into applications, a few more elementary comments on p-adics are in order.

Any p-adic number can be expanded as

$$x = p^{\upsilon} \sum_{n=0}^{\infty} a_n p^n \tag{5}$$

with $1 \leq a_o \leq (p - 1)$ and $0 \leq a_n \leq (p - 1)$ otherwise; υ is an integer. Since $p > 1$ this obviously diverges with respect to the usual norm but converges with respect to the p-adic norm. Let us work one example: $p = 2$ and $x = \frac{1}{3}$. We then have

$$\frac{1}{3} = a_0 + a_1.2 + a_2.2^2 + a_3.2^3 + \dots . \tag{6}$$

We deduce that

$$1 - 3a_0 = 0(\text{mod } 2) \tag{7}$$

so that $a_0 = 1$. Then

$$1 - 3a_0 - 6a_1 = 0(\text{mod } 4) \tag{8a}$$

$$1 - 3a_0 - 6a_1 - 12a_2 = 0(\text{mod } 8) \tag{8b}$$

$$1 - 3a_0 - 6a_1 - 12a_2 - 24a_3 = 0(\text{mod } 16) \tag{8c}$$

giving, respectively, $a_1 = 1$, $a_2 = 0$ and $a_3 = 1$. Thus

$$\frac{1}{3} = 1 + 2 + 2^3 + \dots . \tag{9}$$

Now let us give some fundamental properties of non-Archimedean geometry defined by Eq. (4) above:

1) Any triangle is isosceles. Suppose $|x|_p < |y|_p$ then $|y - x|_p \leq |y|_p = |(y - x) + x|_p = |y - x|_p$.

2) Any point in a sphere, including a point on its surface, is the center of the sphere. Let the sphere be

$D_a(r) = \{x \in Q_p: |x - a|_p \leq r\}$

Consider $b \in D_a(r)$ then

$|x - b|_p = |(x - a) + (a - b)|_p \leq r$ .. $D_a(r) = D_b(r)$.

3) In Euclidean geometry, two unequal circles in a plane have three possibilities: they may be non-overlapping, one completely contained in the other, or partially overlapping. In non-Archimedean, geometry, the third possibility of overlapping circles does not occur. The proof is similar to (2) above.

In p-adic analysis, a very important function is the p-adic counterpart of the exponential function, called the additive character $\chi(x)$ satisfying

$$\chi(x + y) = \chi(x)\,\chi(y) \tag{10}$$

We wish χ to map Q_p to C, the complex number. So

$$\chi(x) = \exp(2\pi i x) \tag{11}$$

is itself unacceptable. We define $\{x\}_p$ the p-adic rational part of x by writing

$$x = p^{\upsilon}(a_0 + a_1 \cdot p + a_2 \cdot p^2 + \dots) \tag{12}$$

and defining

$$\{x\} = \begin{cases} 0 & \upsilon \geq 0 \\ p^{\upsilon}(a_0 + a_1 \cdot p + \dots + a_{|\upsilon|-1} p^{|\upsilon|-1}) & \upsilon < 0 \end{cases} \tag{13}$$

This $\{x\}$ is a rational number and we define

$$\chi(x) = \exp(2\pi i \{x\}) \tag{14}$$

$\chi(x)$ is used in p-adic fourier transforms, and will be used in p-adic quantum cosmology below.

Rules for p-adic integration include the following

$$d(x + a) = dx \tag{15a}$$

$$d(ax) = |a|_p \, dx \tag{15b}$$

$$\int_{|x|_p \leq 1} dx = 1 \tag{15c}$$

Here (15c) is a normalization. For a function f(x) mapping $x \in Q_p$ to $f(x) \in C$ we then have

$$\int_{Q_p} f(x)\, dx = \sum_{\gamma = -\infty}^{\infty} \int_{|x|_p = p^{\gamma}} f(x)\, dx \tag{16}$$

Examples of p-adic special functions, defined by such an integral representation are the Gelfand gamma function

$$\Gamma_p(a) = \int_{Q_p} |x|_p^{a-1} \chi_p(x)\, dx = \frac{1 - p^{a-1}}{1 - p^{-a}} \tag{17}$$

and the beta function

$$B_p(a,b) = \int_{Q_p} |x|_p^{a-1} |1 - x|_p^{b-1}\, dx \tag{18}$$

$$= (1 - p^{-1}) \left(\frac{1}{1 - p^{-a}} + \frac{1}{1 - p^{-b}} + \frac{1}{1 - p^{-c}} \right) + (2 - p^{-1}) \tag{19}$$

where c = (1 - a - b). Such a p-adic formula is useful in p-adic strings.

II. P-ADIC STRINGS

Historically, the p-adic string emerged from a re-writing of the four-point bosonic amplitude firstly using p-adic gamma functions which mapped p-adic variables into p-adic variables $Q_p \rightarrow Q_p$. This gave a p-adic amplitude which would necessitate a new quantum mechanics. Use of a different gamma function mapping $Q_p \rightarrow C$ seemed more promising since a complex amplitude can be interpreted with usual quantum mechanics. This latter four point function satisfies an adelic formula that the product over all primes gives the inverse of the Veneziano formula.

For the open string, the extension to the N-particle p-adic amplitude was made, leading to the spacetime field equation of the tachyon. This field equation, found by Okada and me, has much content which remains to be studied - for example, multi-soliton solutions probably exist.

For closed bosonic strings, the same steps have been carried through with relatively small differences including expected factors of 2 in several of the formulas. This led naturally to the closed superstring where the 4-point function for massless external particles has been treated.

For the open trees, the vertex for m tachyons turns out to be

$$V^m = (-1)^{m+1} \prod_{n=2}^{m-2} \left(n - p^{-1}\right) \quad (m \geq 4) \tag{20a}$$

$$V^3 = 1 \tag{20b}$$

The propogator is

$$\Pi(k) = \left(1 - p^{-1}\right) \frac{p^{\alpha(k^2)}}{1 - p^{\alpha(k^2)}} \tag{21}$$

The p-adic string violates causality as can be seen by the poles of $\Pi(k)$ at $\alpha(k^2) = \left(2\pi i N\right)(\ln p)^{-1}$. The p-adic string is not taken seriously as a physical system and is defined only by a set of amplitudes. It can, however, provide a potentially useful tool for studying the usual strings and superstrings.

The adelic formula for the p-adic N-point amplitude $A^{(p)}{}_N$ is

$$A_N \prod_p A_N^{(p)} = 1 \tag{22}$$

For N = 4, Eq. (22) depends on the kinematic region. It is true, with regularization, in a neighborhood of the symmetric point s = t = u. For $N \geq 5$, Eq. (22) again depends on the kinematics.

A p-adic sigma model, with a discretized scale invariance has been shown to lead to the spacetime field equation of the p-adic string tachyon by setting appropriate renormalization group β functions to zero. This follows the logic of the usual string and proves that the p-adic field theory shows internal consistency.

The spacetime field equation for the tachyon is

$$p^{\Box/2}(1 + \phi) = (1 + \phi)^{1/p} \tag{23}$$

This equation is non-local since it contains every power of the d'Alembertian. The shifted field $\psi = \left(1 + \phi\right)$ satisfied a simpler equation and $\eta = 0$ appears as a stationary point of the effective potential and possibly signals a stable vacuum. The equation obtained by putting $\psi = (1 + \phi)$ in Eq. (23) has static soliton and Euclidean instanton solutions of a gaussian type for any spacetime dimension d = 2,3,4, ..., 26, ...; I know of no other field equation with this property!

If multi-soliton solutions of the ψ equaton could be found (it is now 2 years since the one soliton solution was discovered), it could lead to a new string amplitude constructed around a stable vacuum. Such multi-soliton solutions cannot be static, because of a force between the solitons, and the time-dependence is what makes the solution tricky.

It is natural to investigate a p-adic superstring. For the 0(32) open Type I superstring, the group factors preclude summing permutations of external legs and this necessitates using the p-adic counterpart of sign (x), and leads to extra poles, hard to interpret, in the p-adic open superstring. For the closed superstring, we begin with the four-point amplitude for four massless states

$$A_4 = \frac{K_4\,\Gamma(-s/8)\,\Gamma(-t/8)\,\Gamma(u/8)}{\Gamma(1+s/8)\,\Gamma(1+t/8)\,\Gamma(1+u/8)} = C(s^2+t^2+u^2) \tag{24}$$

$$\left[\frac{\Gamma(2-\alpha_s/2)\,\Gamma(1-\alpha_t/2)\,\Gamma(1-\alpha_u/2)}{\Gamma(3-\alpha_{s/2}-\alpha_{t/2})\,\Gamma(2-\alpha_{t/2}-\alpha_{u/2})\,\Gamma(3-\alpha_{u/2}-\alpha_{u/2}-\alpha_{s/2})} + \text{perms}\right] \tag{25}$$

where K is a unique kinematic factor and C is a constant. Eq. (25) is in a Virasoro form where one can exploit the p-adic identity for the quadratic extension $K(\sqrt{p})$ of Q_p

$$\int_{K(\sqrt{p})} dz\, |z|_p^{2A}\, |1-z|_p^{2B} = \Gamma_p(A+1)\,\Gamma_p(B+1)\,\Gamma_p(1-A-B) \tag{26}$$

This leads to the p-adic closed superstring amplitude:

$$A_4^{(p)} = C(s^2+t^2+u^2) \sum_{x=stu} (1-p^{-1}) \left(\frac{p^{(\alpha_x-2)/2}}{1-p^{(\alpha_x-2)/2}} + \frac{p^{(\alpha_x-4)/2}}{1-p^{(\alpha_x-4)/2}}\right) + 3(1-2p^{-1}) \tag{27}$$

From this, and a generalization to A_N, one could hope to compute the effective potential $V(\phi)$ for the dilation field ϕ at the p-adic level. For the superstring itself it is well-known that $V(\phi)$ vanishes at tree level. A non-trivial $V(\phi)$ could signal breaking of supersymmetry.

The fact that the N-dilaton superstring tree amplitude vanishes for all external momenta vanishing follows from $OS_p(2|1)$ invariance of the superconformal theory. One might therefore study p-adic superconformal field theory. For the bosonic case the p-adic sigma model, for closed strings, has the action

$$S_B = \frac{1}{2}B \int_{K(\sqrt{p})} dx\, dy\ \phi(x)\, \frac{1}{|x-y|^4}\, \phi(y) \tag{28}$$

with corresponding Green's function

$$G_B(x,y) = \ln|x-y|\ |\bar{x}-\bar{y}| \tag{29}$$

The naive generalization of S_B to fermions, with a cube in the denominator, vanishes for anticommuting fermions. One suggestion is to put in a parameter k and write

$$S_F = \frac{1}{2}A \int_{K(\sqrt{p})} dx\, dy\ \bar{\psi}(x)\, \frac{1}{|x-ky|^3}\, \psi(y)) \tag{30}$$

Defining

$$K_F\ \phi(x) = A \int_{K(\sqrt{p})} dy\ \frac{1}{|y-kx|_p^3}\, \phi(y) \tag{31}$$

one can write

$$D\,\phi(x,\theta,\overline{\theta}) = \theta\, K_F\, \phi + \frac{d}{d\theta}\phi \tag{32}$$

$$S_F = \frac{1}{2} \int dx\; d\theta\; d\bar{\theta}\;\; D\phi\, D\phi \tag{33}$$

leading to

$$G(x,x^i) = -\ln\left[\, |\, x - x^i \,|\;\; (1 + \frac{g\theta_1\,\theta_2}{|\, x - kx^i\,|^{\,1/2}})\right] \tag{34}$$

where $g = (\mathrm{sign}_p(-1))^{1/2}\ |k|\,(p\ \ln p)^{-1/2}\,(1 - p^{-1})^{-1/2}$.

III. P-ADIC GRAVITY AND QUANTUM COSMOLOGY

As my second application of non-Archimedean geometry, I will discuss quantum cosmology. Recent discussions of baby universes and wormholes emphasize the notion of creation of the universe as a whole. In the newly-developing third quantization approaches to universe creation the Fock space vacuum is the so-called nothing state. An operator $\phi(^3g)$ acts on $|$nothing> to create a 4-manifold of which the present geometry is described by the 3-metric ^3g. The state $|$nothing> from which spacetime is created must, in some sense to be made more explicit below, contain the notion of number fields since the Riemannian spacetime manifold must be able to be coordinatized..

In this context, it seems important to employ the plurality of possible number fields for the following reason. Suppose space is R^3 at all length scales and that Euclidean geometry is valid (the difference between Riemannian and Euclidean geometry is irrelevant for this argument since one may assume all gravitational fields are weak). Then suppose we wish to specify the coordinates x, y, z of some point with nearly infinite accuracy, say to 10^{-40} cm, much less than the Planck length. Such an exercise will be stymied by the Heisenberg uncertainty principle which in quantum gravity dictates that there is a fundamental uncertainty in length of the Planck length 10^{-33} cm. For smaller lengths, R^3 is therefore an inadequate description. Note that one axiom of Euclidean geometry is the Archimedean postulate that "repetition of an element of a straight line can eventually exceed the length of any longer line element". This postulate no longer applies below the Planck length. Bearing in mind the one-to-one correspondence between geometry and number fields e.g. between Euclidean geometry and real numbers, this suggests changing to alternative number fields. The fields Q_p correspond to non-Archimedean geometry, because of the inequality, Eq. (4), and hence might be used at the smallest length scales.

A proposal made by Volovich and me is to define a base state $|$void> from which the $|$nothing> state is obtained by acting with a series of operators in order: G, a functor which creates the group of integers; A(Q) elevates this group to the field of rational numbers; $\phi\ (Q_p)$ completes the rationals Q to the p-adic field Q_p. Thus for the usual case

$$|\Omega(R)\rangle = \phi\,(Q_\infty = R)\;\; A(Q)\, G\, |\Omega\rangle \tag{35a}$$

$$= \phi\,(Q_\infty = R)\;\; |\Omega(Q)\rangle \tag{35b}$$

From the void with rationals $|\Omega(Q)\rangle$ we may create $|\Omega(Q_p)\rangle$, the void with p-adics. The usual wave function of the universe

$$\psi(^3g) = \int \exp(iS)\, Dg = \hat{\psi}(^3g)\, |\Omega(R)\rangle \tag{36}$$

should be generalized to

$$\psi(^3g) = \int \prod_p \chi_p\left(\frac{S}{2\pi}\right) Dg_p \tag{37}$$

In particular we may define

$$\psi_p = \int X_p\left(\frac{S}{2\pi}\right)\, Dg_p \tag{38}$$

where g_p is the p-adic metric.

For a rational 3-metric h^Q one has

$$\psi[h^Q] = \prod_p \hat{\psi}_p[h^Q]\, |\Omega(Q)\rangle \tag{39}$$

Let us, at the level of rational points, subdivide the manifold h^Q into small Planck-sized pieces Δh_i^Q. Then in a semi-classical saddle-point approximation one has

$$|\psi\left(\Delta h_i^Q\right)\rangle = \chi_p\left(\frac{S^i}{2\pi}\right)\, \phi_{p,i}\, |\Omega(Q)\rangle \tag{40}$$

For the interaction of very small manifolds one has therefore

$$\hat{\psi}\left[\Delta h_i^Q\right] \hat{\psi}\left[\Delta h_j^Q\right] \hat{\psi}\left[\Delta h_k^Q\right]$$

$$\sim \chi_{p_i}\left(\frac{S_i}{2\pi}\right)\, \chi_{p_j}\left(\frac{S_j}{2\pi}\right)\, \chi_{p_k}\left(\frac{S_k}{2\pi}\right)\, \phi_{p_i}\phi_{p_j}\phi_{p_k} \tag{41}$$

We have $S_i/2\pi \sim \rho_i^4$ where ρ_i is a typical length scale in units of the Planck scale. We may hence define a coupling

$$V_{pqr}\,(\rho_p,\rho_q,\rho_r) = \chi_p\,(\rho_p{}^4)\; X_q(\rho_q{}^4)\; \chi(\rho_r{}^4) \tag{42}$$

Let us take the $\rho_i = 1$ and note that $\chi_p(1) = 1$ for all p. Let us posit an algebra

$$\chi_p\,(\rho_p^4)\, \chi_q(\rho_q^4) = \Sigma\; \hat{V}_{pqr}\, \chi_r(\rho_r{}^4) \tag{43}$$

setting $\rho_\iota = 1$ leads to the sum role

$$\sum_r \hat{V}_{pqr} = 1 \tag{44}$$

and bear in mind that the V_{pqr} of Eqs. (41) and (42) must be symmetrized $V_{pqr} = \hat{V}_{pqr} + \hat{V}_{qrp} + \hat{V}_{rpq} + \; \dots$.

The question now is how to explain the fact that at large distances only real numbers survive in the spacetime coordinate? Recently, together with Aref 'eva, we have invented one scenario which could explain this observation as follows.

In order to satisfy Eq. (44) we may use the Riemann zeta function to write the ansatz

$$V_{pqr} = - \frac{1}{\ln \zeta(S_o)} \sum_{x = pqr} \ln (1 - p^{-S_o}) \tag{45}$$

for an effective action

$$S = V_{pqr} \phi_p \phi_q \phi_r \tag{46}$$

If S_o is some positive constant $A > 1$ then all V_{pqr} are positive except when $p,q,r \rightarrow \infty$ in which case $V_{pqr} \rightarrow 0$. Now the ϕ_p are positive because $\phi_p^2 = \phi_p$ is an idempotent projection operator so in this case the effective action has a minimum where only the real number fields (recall that $R = Q_\infty!$) contribute. On the other hand, when $S_o \rightarrow \infty$ in Eq. (45) all p,q,r can contribute and, in fact, the couplings will dominate for the lowest prime $p = 2$.

To achieve the required physical picture therefore we can make a simple ansatz

$$S_o = A + N\rho^{-1} \tag{47}$$

where ρ is a typical scale in units of the Planck scale. Now if $\rho >> 1$, real number fields are all that remain while if $\rho << 1$ the number field Q_2 eventually dominates.

Such a spacetime manifold below the Planck length departs radically from the usual one. It would be interesting to compute, for example, scattering amplitudes at and above the Planck energy to find how the physics departs from that of, say, the superstring.

IV. SUMMARY

For the bosonic string (open and closed), p-adic counterparts for the tree amplitudes exist and lead to an interesting effective field theory for the tachyon state. From a p-adic conformal theory the corresponding spacetime field equation is reproduced. The field equation has a simple one soliton solution in a shifted (stable?) vacuum, and may have time-dependent multi-soliton solutions although these are not yet found.

For the closed superstring, a p-adic counterpart is known for the 4-massless particle amplitude. Higher amplitudes and the corresponding superconformal theory still need further work to find the best p-adic counterparts.

In quantum cosmology, we have discussed a formalism of projection operators ϕ_p acting on a void state with rationals $|\Omega(Q)>$ and indicated how a suitably generalized wave function of the universe might lead to a spacetime manifold which at separations large compared to the Planck scale has a geometry of the usual Euclidean/Riemannian type while at scales below the Planck scale non-Archimedean geometry is applicable.

V. ACKNOWLEDGEMENTS

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VI. BIBLIOGRAPHY

A reader wishing to pursue the above topics could begin by consulting the following papers. References to significant other p-adic papers from Moscow, Chicago, and elsewhere are contained therein. On p-adic strings:

P. H. Frampton and Y. Okada, The P-Adic String N-Point Function, Phys. Rev. Lett. 60, 484-486 (1988)

P. H. Frampton and Y. Okada, Effective Scalar Field Theory of P-Adic String, Phys. Rev. D37, 3077-3079 (1988)

P. H. Frampton, Y. Okada and M. R. Ubriaco, On Adelic Formulas for P-Adic Strings, Phys. Lett. 213B, 260-262 (1988)

P. H. Frampton, Y. Okada and M. R. Ubriaco, New P-Adic Strings from Old Dual Models, Phys. Rev. D39, 1152-1157 (1989)

P. H. Frampton and H. Nishino, Theory of P-Adic Closed Strings, Phys. Rev. Lett. 62, 1960-1963 (1989)

On p-adic quantum cosmology:

P. H. Frampton and I. Volovich, Primary Quantization: Creation of the Nothing from a Void, UNC-Chapel Hill Report IFP-347-UNC (1989)

I. Aref 'eva and P. H. Frampton, Beyond Planck Energy to non-Archimedean Geometry, UNC-Chapel Hill Report IFP-360-UNC (1989).

Grossman (Rockefeller):

I also am a fan of using p-adic numbers in physics. You have suggested, following the wisdom of Knonecker who said that God created the integers, that there exists an operator that creates the integers out of the void. However, I would like to suggest that perhaps God created Knots, or finite groups representing the group of the knot, insofar as there is a great variety of knots associated with each integer. One might connect the two approaches one number theoretical and one topological through Galois theory and the work of Ocneanu. Do you care to comment?

Frampton (UNC-Chapel Hill):

I took the admittedly conservative view that the p-adic number fields originate from the field of rational numbers which I assume is applicable at all spacetime scales both very large and very small. The rational number field naturally emerges from the integers which are, of course, even more basic in this viewpoint. I was taught that nothing in mathematics is more fundamental than the integers.

BEYOND CONFORMAL FIELD THEORY

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This is an account of some recent work done with H.S. La [1][2], based ultimately on the work of Fischler and Susskind [3] and Polchinski [4].

1. CREDO

Since as Kastler has remarked, this is something of an ecumenical congress, and prone to heresies and inquisitions as only such congresses can be, I would like first of all to outline my prejudices and motivations rather than launching into a long unmotivated construction project.

String theory begins with the observation that certain mathematical constructions produce quantities which superficially resemble the scattering amplitudes of a relativistic quantum theory. These amplitudes moreover enjoy properties reminiscent of the perturbative unitarity rules for quantum field theory. Pursuing the analogy shows, however, that their short-"distance" behavior is significantly nicer than that of field theory amplitudes. As I will mention later, words like "distance" are not to be taken too seriously until we have some adequate notion of quantum spacetime. For now these words are defined by explicit reference to the picture of perturbing about flat spacetime — certainly not a tenable fundamental approach in any theory of gravity.

Just the same, the *long*-distance structure of perturbative string theory is singular, just as we know it must be if it is to describe scattering: amplitudes must have the usual kinematical poles and so on. Unraveling this desired singularity structure, and eliminating undesired singularities, will be my main project today.

What are the mathematical constructions alluded to above? We introduce an auxiliary structure into the problem, namely a compact 2-surface, and attempt to define a quantum field theory thereon. There are various motivations for this construction, all of them well known to this audience. I wish to stress, however, that we should not take these motivations too seriously. Rather, we should analyze *ab ovo* the utility of each element of the construction. I will touch only very briefly on some points chosen mainly on polemical grounds.

First of all, we need to introduce motions of field theory which make invariant sense on arbitrary compact 2-surfaces; the plane is not good enough. This is because successive orders of quantum-mechanical perturbation theory are all supposed to be given by exactly the same local 2d dynamics applied to increasingly complicated surfaces. Secondly, it is absolutely crucial that the local geometry of each surface must drop out.

One way to arrange this is to ask for a 2d *topological* field theory. This is not useful. In two dimensions, however, there is an alternative approach: we can ask for our field theory to be well-defined given only Σ as a complex manifold. Since all complex structures are the same locally this suits our requirements. Furthermore, one finds that the degenerations of Riemann surfaces correspond precisely to possible *long*-distance singularities in quantum field theory. The *short*-distance singularities have no such analogs. That is why they do not arise in string theory, *if* string theory is defined on shell by conformal field theories. Today we have a large kingdom of conformal field theory knowledge.

To repeat: the auxiliary constructions needed for string scattering amplitudes are 2d conformal field theories defined consistently on surfaces of all topologies. There is *no known role* for (a) conformally non-invariant 2d field theories, or (b) theories or the plane with no consistent extension to all genera, *i.e.* modular non-invariant theories. (Certainly there are extremely interesting conjectures that arbitrary 2d field theories play a role in an offshell continuation, though.)

What is more, all CFT's of interest to string theory are of a special form: they are all tensor products of an arbitrary CFT of central charge $c = 26$ with *one universal* CFT of central charge $c = -26$. The latter system plays a fundamental role in the geometry of string theory, as we will see hinted at in the sequel; it deserves further mathematical attention.[1] The physical idea I am alluding to here is of course called "Becchi-Rouet-Stora-Tyutin symmetry;" it plays a much more fundamental role in string geometry than its usual role as a machine implementing symplectic reduction. In this talk I will unfortunately have to suppress details of how this works, but see [1][2].

Having recited my creed, I will now try to show why and how it must be discarded. I do not know precisely what will replace it, but I will sketch an extremely suggestive calculation from [1][2]. Approximately, however, the new creed goes as follows.

It was a grave error to focus on individual fixed Riemann surfaces above. Returning to Polyakov's original heuristic principle, the local dynamics on Σ are expressed in terms of a 2-metric g and some "matter" (*i.e.* unspecified) field x. A large symmetry (2d general coordinate transformations) acts on both g and x, without prejudice. We choose to focus on g when we fix this gauge symmetry, leading to the *illusion* of a fixed nondynamical Riemann surface with a CFT defined thereon. The requirement of conformal invariance then says that for each *fixed* Σ no extra data about Σ are needed to define 'amplitudes'. But of course, our work is not done at this point. We must also complete our implementation of Polyakov's principle by integrating over conformal classes of Σ. Here *new* divergences occur —not unexpected, since the shape of Σ is itself a dynamical variable. Indeed, true amplitudes in general *cannot* be defined without choosing extra data on Σ; this is a failure of conformal invariance even though each CFT on fixed Σ is well defined!

Thus we have no choice: we must *spoil* explicit conformal invariance to *save* it overall; we must give away this kingdom to enter the next. In addition, we'll see

1 But see [5].

that we cannot even work on a Riemann surface of any one fixed topology. Even though worldsheet gravity is almost trivial, it will lead to topology change just like any other quantum gravity. This is of course much more radical than merely relaxing conformal invariance on one fixed surface. Presumably this is a big hint about the true foundations of string theory.

Let's get specific.

2. SHIFTS

To get our feet on the ground, we consider bosonic strings on flat spacetime. The techniques are general. Indeed it's one of the main points of the general CFT apparatus that the formalism is not tied to specific realizations (like sigma models, current algebra, *etc.*). We will use CFT even though our goal is to modify it. Also, as we will remark, all our constructions make sense only in the more refined setting of "superconformal field theory;" this is indeed the main focus of [1][2], where along the way we show how to avoid the famous 'ambiguity' problem of integrals over superspace.

The problem is very simple. A sigma model based on flat 26-dimensional spacetime is well-known to be conformally invariant on fixed Σ once we properly introduce ghost fields to fix reparametrization invariance. A small disturbance of flat $\mathbf{R}^{26}$ may or may not spoil this invariance. To compute the effect of such a change we can take the original theory's partition function and add to it the correlation function of the operator ψ corresponding to the desired deformation, integrated over Σ. If ψ has conformal weight $(h, \bar{h}) = (1,1)$ then $\langle\psi\rangle_\Sigma$ can be regarded as a (1,1)-form on Σ and the integral is well-defined without further choices.[2] Thus we maintain conformal invariance. As is well known this (1,1) condition is the linearized Einstein equation for the deformed metric on $\mathbf{R}^{26}$, $\Box h_{\mu\nu} = 0$, where the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

Consider now the integral over all Σ. Suppressing the unphysical "tachyon" pole (this happens automatically in a more realistic theory), we find an integral of the form

$$Z = \int \frac{d^2q}{|q|^2}(\cdots)(\cdots) \quad . \tag{1}$$

Here q is a complex variable describing how close Σ is to being degenerate; $q \to 0$ produces the long-distance singularity alluded to above. The ellipses denote the *one-point* functions of *massless* fields on surfaces of genus g_1, g_2, respectively; $g = g_1 + g_2$ is the genus of Σ.

Clearly Z diverges if the "tadpole" amplitudes $(\cdots)$ are non-zero. The latter situation must obtain in an interesting theory, *i.e.* one in which supersymmetry breaks, by whatever means. So we have a problem.

From experience with quantum field theory our duty is now clear. We must *cut off* our q integral at some small value δ, then introduce *counterterms* on the lower-genus surfaces Σ_1, Σ_2 in such a way that the full $Z + Z^{c.t.}$ is δ-independent. Of course we don't want this advice to apply too literally, since the goal is to get away from field theory.

Fortunately, the analogy to field theory dissipates almost at once. For one thing, the counterterms we introduce, interpreted as small changes ψ to the flat background,

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2 Actually a broader class of states than this can be inserted. In the BRST formalism we can insert any ψ of total weight (including b, c) (0,0). See [6].

will all be *finite* — that's good. Of more immediate concern, though, is that our cutoff procedure at first seems to be ill-defined — that's bad. The point is that the choice of a coordinate q on the moduli space $\mathcal{M}_g$ is far from canonical, and without such a choice we don't know what it means to say a surface is "too pinched." We must inevitably *spoil conformal invariance* to get a cutoff. The trick is to do so in a wise way.

Before proceeding, let me give the bottom line: the correct counterterms will be finite, but they will be of slightly the *wrong* dimension, $(h,\bar{h}) \neq (1,1)$. Thus we *give up* conformal invariance at lower genus! To insert such states on $\Sigma_{1,2}$ we will again need a cutoff (=normal-ordering prescription). We now have *two* cutoffs, albeit of radically different-seeming sorts. One cuts off the moduli integral over all Σ; the other, the 2d CFT on a given Σ. As remarked in the 'Credo', however, this dissimilarity is an illusion. In fact, we can choose these cutoffs consistently, in such a way that all dependence on the choice drops out for suitable ψ. Thus we save conformal invariance — and the blessings it brings -- by letting go of it.

To define a cutoff, choose surfaces $\Sigma_{1,2}$ with marked points $P_{1,2}$ and a complex number q. Choose moreover local coordinates $z_{1,2}$ centered on $P_{1,2}$. Now glue Σ_1 to Σ_2 by the usual rule $z_1 = q/z_2$. Promoting $\Sigma_{1,2}$ to *families* of surfaces parametrized by $\vec{m}_{1,2}$ gives coordinates $\vec{m}_1$, $\vec{m}_2$, q for $\mathcal{M}_g$.

Our problem is now that the same $\Sigma_{1,2}$, glued with the same q, will yield a *different* surface $\tilde{\Sigma}$ if different $\tilde{z}_{1,2}$ are chosen. Thus, the choice of cutoff amounts to choosing *two* things: a family $z_{1,2}$ of coordinates on $\Sigma_{1,2}$.

Now we begin to see the point: this *same* extra data $z_{1,2}$ are also just what's needed to insert a general background-shift state ψ, not necessarily of weight (1,1), onto $\Sigma_{1,2}$ respectively. We just cut out the disk $|z_1| = \delta$ and insert ψ there. Equivalently, we can cut out $|z_1| = 1$ and insert $\delta^{L_0+\bar{L}_0}\psi$. What's important is that z_1 is totally independent of anything on side 2, and vice versa.

What exactly happens when we change z_1 to $\tilde{z}_1 = z_1 + \Sigma\epsilon_n z_1^{n+1}$? For one thing the insertion $\langle\psi\rangle_{\Sigma_1}$ changes by

$$\delta Z^{c.t.} = \langle\delta^{L_0+\bar{L}_0}\Big(\sum_n \epsilon_n(L_n - \delta_{n,0})\Big)\psi\rangle_{\Sigma_1} \rightarrow \epsilon_0\,\langle(L_0-1)\psi\rangle_{\Sigma_1}\,, \tag{2}$$

where the limit is for $\delta \rightarrow 0$ and ψ is nearly of weight (1,1). The cutoff changes, too. Remarkably, though, the change is extremely simple [2]: $\vec{m}_1$, $\vec{m}_2$, q change to $\vec{m}_1'$, $\vec{m}_2'$, q', where in particular

$$q' = q(1+\epsilon_0) + \mathcal{O}(q^2). \tag{3}$$

Clearly changing the region of integration region of (1) from $\{|q| > \delta\}$ to $\{|q'| > \delta\}$ changes the integral by

$$\delta Z = \log(1+\epsilon_0)\sum_a \langle\langle\phi_a\rangle\rangle_{\Sigma_1}\,, \langle\langle\phi^a\rangle\rangle_{\Sigma_2}\,, \tag{4}$$

where ϕ_a runs over all massless states. Requiring the cutoff-dependences of (2), (4) to cancel now gives a conditions on ψ of the form

$$(L_0 - 1)\psi = \sum_a \phi_a\,\langle\langle\phi^a\rangle\rangle_{\Sigma_2}\,. \tag{5}$$

The LHS of (5) looks like a free wave operator acting on the shifted background state: *e.g.* it contains $\Box h_{\mu\nu}$ and so on. The RHS looks like a *source* for the wave equation. In fact, the equation (5) comes from a loop-corrected quantum action, as many authors have shown in various cases.

Our point here is that, whatever the interpretation of solutions to (5), it embodies the general precepts of the 'Credo': to obtain true, generalized conformal invariance of the full string system we must *destroy* naive conformal invariance in a very special way, one in which different surfaces *conspire* to cancel the anomaly. That this is possible at all comes from the remarkable geometrical fact (3) about moduli space.

The papers [1][2] were mainly concerned with the superconformal case, and specifically the heterotic string. Exactly the same sort of cutoff emerges as above, where we now choose a superconformal coordinate $\mathbf{z} = (z, \theta)$ near the attachment points. The key observation is that with such a cutoff one has only to perform a moduli integral over a supermanifold with boundary (namely $\{|q| > \delta\}$), thus sidestepping problems with integrals over noncompact supermanifolds.

Of course, the challenge is how to find a nonperturbative implementation of these principles. The answer is quite likely to be totally different in spirit from the CFT-inspired derivation above.

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REFERENCES

[1] H.S. La and P. Nelson, "Unambiguous fermionic string amplitudes," Phys. Rev. Lett. **63** (1989) 24.

[2] H.S. La and P. Nelson, "Effective field equations for fermionic strings," UPR-0391T=BUHEP-89-9.

[3] W. Fischler and L. Susskind, "Dilation tadpoles, string condensates, and scale invariance, I; II," Phys. Lett. **171B** (1986) 383; **173B** (1986) 262.

[4] J. Polchinski, "Factorization of bosonic string amplitudes," Nucl. Phys. **B307** (1988) 61.

[5] I. Frenkel, H. Garland, and G. Zuckerman, Proc. Nat. Acad. Sci USA **83** (1986) 8442.

[6] P. Nelson, "Covariant insertions of general vertex operators," Phys. Rev. Lett. **62** (1989) 993.

HIDDEN SYMMETRIES OF STRINGS AND THEIR RELEVANCE FOR STRING QUANTIZATION

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Abstract

The relation between the Nambu-Goto theory of closed bosonic strings and integrable systems is made precise. The complete set of observable symmetries of the classical Nambu-Goto theory is identified, the Poisson-algebra of the corresponding conserved charges is analyzed and the loop wave equation of the Nambu-Goto theory is interpreted as a representation-condition for the symmetry algebra. Further, the WKB quantization of the conserved charges is discussed and consistency conditions for the exact quantization are given.

For a number of years, my collaborator K.-H. Rehren and I have been making a serious effort to implement a non-perturbative, conceptually simple program for the quantization of the closed bosonic string moving in d(≥3)-dimensional flat Minkowski space. The essential idea behind this program is the identification of the quantum theory of strings with an element of that class of quantum theories which a) reduce to the Nambu-Goto theory in the classical limit, b) have the same observable degrees of freedom as and c) possess all the observable symmetries of the latter theory. It is expected [1], that this leads to a one-parameter class of quantum theories. Thus, at no stage of its implementation, ought the program to become diffuse.
Granted the sensibleness of the program, one immediately has to face the following tasks:

1) construction of all observable symmetries of the (classical) Nambu-Goto theory;

2) determination and investigation of the Poisson algebra formed by these observable symmetries;
3) interpretation of the constraint equations of the Nambu-Goto theory, in particular the loop wave equation, in terms of Casimir operators of the above Poisson algebra;
4) renormalization of the infinitesimal generators ("invariant charges") of the observable symmetries;
5) description of the quantum modifications of the Poisson algebra, identification of a Cartan subalgebra of the corresponding commutator algebra, computation of the spectra of its operator elements;
6) construction of the irreducible representations of the commutator algebra, in particular of those with positive energy.

If the discussion is restricted to those symmetries of the Nambu-Goto theory which commute with the mass operator, the implementation of our program should provide for instance

i) the explanation of the mass degeneracy (including the construction of the entire mass eigenspace from one given corresponding mass eigenstate - cf. the capability of the SO(4) symmetry group for the hydrogen atom);
ii) the identification of an invariant measure for the space of closed loops;
iii) the construction of the ground state as the quantum state of highest symmetry.

In short, our program aims at the development of mathematical concepts and methods with the help of which a quantitative treatment of linearly extended geometric objects endowed with physical properties (e.g. glue balls) by successive approximations within a realistic theory (e.g. QCD) might become possible. Surely, the audience/reader will not fail to realize the distinction, both in conception and immediate objectives, of our approach from the main-stream approach.

I. Results and their relevance

The infinitesimal generators of all observable symmetries of the Nambu-Goto theory commuting with the mass operator form the following algebra

$$\underset{\text{boosts}}{\mathfrak{so}(1,d-1)} \uplus \left\{ \mathbb{M}^d \oplus (\mathfrak{h}_{\mathcal{P}}^+ \oplus \mathfrak{h}_{\mathcal{P}}^-)\right\} .$$

Here the symbols $\mathbb{M}^d$, $\mathfrak{h}_{\mathcal{P}}^+$ and $\mathfrak{h}_{\mathcal{P}}^-$ stand for the linear span of the infinitesimal generators $\mathcal{P}_\lambda$, $\lambda = 0,1,..,d-1$, of rigid translations in Minkowski space, and for the algebra of invariant internal charges for the left - and right - movers (with the components $\mathcal{P}_\lambda$ of the energy-momentum vector being treated as c-numbers), respectively. The invariant charges $\mathfrak{Z}^{\pm}_{\mu_1..\mu_N} \in \mathfrak{h}_{\mathcal{P}}^{\pm}$ are obtained from the eigenvalues of the monodromy matrices of the following systems of ordinary linear reparametrization invariant differential equations [2]

$$X^{\pm\prime}_{(A)}(\tau,\sigma) = u^{\pm}_{\mu}(\tau,\sigma) \quad A^{\mu\alpha}\ T_{\alpha}\ X^{\pm}_{(A)}(\tau,\sigma) \qquad \tau,\sigma \ \varepsilon\ \mathbb{R}^1 x S^1,$$

where the prime denotes (partial) differentiation with respect to σ, the T_α are the infinitesimal generators of the group SL(N,$\mathbb{C}$) N=2,3,.., the $u^{\pm}_{\mu}(\tau,\sigma)$ are linear combinations of the dynamical variables $p_\mu(\tau,\sigma)$ and $x'_\mu(\tau,\sigma)$ involving some parameter M of dimension mass, $\mu=0,1,..,d-1$

$$u^{+}_{\mu}(\tau,\sigma) = p_{\mu}(\tau,\sigma) + \frac{M^2}{\hbar}\ x'_{\mu}(\tau,\sigma) \qquad \text{(left movers)}$$

$$u^{-}_{\mu}(\tau,\sigma) = p_{\mu}(\tau,\sigma) - \frac{M^2}{\hbar}\ x'_{\mu}(\tau,\sigma) \qquad \text{(right movers)}$$

and where the $A^{\mu\alpha}$ are constants both w.r.t. the variable σ and - due to the equations of motion for $u^{\pm}_{\mu}$ - also w.r.t. the variable τ.

The fact that there are no observable symmetries commuting with the mass operator other than those quoted above has been established in ref [3] (for closed bosonic strings in Euclidean flat space). Signals for the existence of these hidden symmetries have also been observed by D. Gross [4].

Contrary to the familiar situation of integrable models over two-dimensional space time $\mathbb{R}^1 x\mathbb{R}^1$, here the elements of the above monodromy matrices do not by themselves provide conserved charges. This is so because now the two-dimensional "space time" has the topology of $S^1 x\mathbb{R}^1$ (or even more complicated topologies), and consequently there doesn't exist an asymptotic region for the variable σ. As a sort of compensation, there now enters an infinite set of isospectral parameters into the relevant differential equations. In the familiar case of ultra-local integrable models allowing for the introduction of a single isospectral parameter λ, the expansion of the trace of the monodromy matrix in powers of λ provides a complete set of commuting observables and some specified elements of the monodromy matrix provide the corresponding shift operators. In the situation at hand, there exist many - in fact, infinitely many - independent isospectral parameters and the expansion coefficients of the traces (of positive integer powers of the logarithm) of the monodromy matrices multiplying definite powers of the isospectral parameters fail to be in involution. Moreover, combinations of elements of the monodromy matrices other than functions of their eigenvalues cease to define sensible dynamical quantities. The latter fact can be inferred from ambiguous, in no way acceptable expressions, for the Poisson brackets of such combinations [5]. Evidently, the strategy developed for the quantization of the familiar two-dimensional integrable models cannot cope with the present situation.

The infinitesimal classical generators of the observable symmetries of the Nambu-Goto theory come in multiplets $\mathfrak{Z}^{\pm}_{\mu_1..\mu_N}$ transforming covariantly under Lorentz transformations. A non-trivial example is provided by

$$\mathfrak{Z}^{+}_{\mu_1..\mu_4} = \frac{2}{3}\,\mathcal{P}_{[\mu_1}\mathfrak{R}_{\mu_2][\mu_3\mu_4]} + \frac{1}{4}\,\mathfrak{R}_{[\mu_1\mu_2]}\mathfrak{R}_{[\mu_3\mu_4]} + \text{cycl. permut.}$$

where

$$\mathfrak{R}_{\mu_1..\mu_N} \doteq \int_0^{2\pi} d\sigma_1 \int_0^{\sigma_1} d\sigma_2 .. \int_0^{\sigma_{N-1}} d\sigma_N \prod_{i=1}^{N} u^{+}_{\mu_i}(\tau,\sigma_i) \ .$$

The sets of all these invariant charges $\mathfrak{h}^{+}_{\mathcal{P}}$ (for left movers) and $\mathfrak{h}^{-}_{\mathcal{P}}$ (for right movers) not only form algebras with respect to the Poisson bracket operation but also with respect to the tensor product operation.

The algebras $\mathfrak{h}^{+}_{\mathcal{P}}$ and $\mathfrak{h}^{-}_{\mathcal{P}}$, with the tensor product as the composition law, are isomorphic, whereas, as Lie algebras, with the Poisson bracket operation as the composition law, they differ by a global factor of -1 for the structure constants.

The elements of the algebras $\mathfrak{h}^{+}_{\mathcal{P}}$ and $\mathfrak{h}^{-}_{\mathcal{P}}$ are subject to certain algebraic constraints. A convenient algebraic basis (w.r.t. the tensor product composition) resolving all algebraic constraints has been constructed [5,6]. The construction includes a particularly efficient algorithm for the computation of arbitrary elements in terms of the elements of this basis. An algebraic basis whose linear span still forms a Lie algebra w.r.t. the Poisson bracket operation does not exist.

Both the Lie algebras $\mathfrak{h}^{\pm}_{\mathcal{P}}$ are graded

$$\mathfrak{h}^{\pm}_{\mathcal{P}} = \bigoplus_{\ell=0}^{\infty} V^{(\ell)}(\mathfrak{h}^{\pm}_{\mathcal{P}}) \ .$$

The gradations of $\mathfrak{h}^{+}_{\mathcal{P}}$ are one-sided, i.e. the degree ℓ assumes non-negative values only. The Poisson bracket operation never decreases the degree ℓ. The tensor product operation maps the elements of the strata $V^{(\ell)}$ and $V^{(k)}$ into the stratum $V^{(\ell+k+1)}$.

The number of linearly independent invariant charges in the stratum $V^{(\ell)}$ which are algebraically independent of the invariant charges contained in strata $V^{(\ell')}$ with $\ell'<\ell$ is given by the difference $n(d,\ell+2)-n(d,\ell+1)$ with

$$n(d,\ell) = \frac{1}{\ell} \sum_{D|\ell} \mu(D) d^{\ell/D} \ .$$

Here the sum extends over all divisors D of ℓ. The symbol $\mu(D)$ denotes the Möbius function

$$\mu(D) = \begin{cases} 1 & \text{if } D=1, \\ (-1)^p & \text{if } D \text{ can be decomposed into exactly } p \\ & \text{different prime factors,} \\ 0 & \text{if some prime factors of } D \text{ are equal.} \end{cases}$$

The Poisson "commutants" of the Lie algebras $\mathfrak{h}^{\pm}_{\mathscr{P}}$: $\{\mathfrak{h}^{\pm}, \mathfrak{h}^{\pm}\}$ are strictly smaller than the Lie algebras $\mathfrak{h}^{\pm}_{\mathscr{P}}$ themselves: in every stratum $V^{(\ell)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$ of odd degree $\ell>1$ there exists exactly one "exceptional" element which cannot be expressed as a linear combination of Poisson brackets involving elements of $\mathfrak{h}^{\pm}_{\mathscr{P}}$ [5,7]. (For d=3, also the basis element of the one-dimensional stratum $V^{(0)}(\mathfrak{h}^{+}_{\mathscr{P}})$ $(V^{(0)}(\mathfrak{h}^{-}_{\mathscr{P}}))$ cannot be expressed in this way.) Thus the Lie algebras $\mathfrak{h}^{\pm}_{\mathscr{P}}$ are not finitely generated. However, the basis elements of the stratum $V^{(1)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$ together with the exceptional elements do generate the entire algebras $\bigoplus_{\ell\geq 1} V^{(\ell)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$.
The algebras $\bigoplus_{\ell\geq 1} V^{(\ell)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$ are (weakly) nilpotent.
For $\mathscr{P}^2=m^2>0$, there exist exactly two independent Casimir elements in the formal closures $\bar{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ of $\mathfrak{h}^{\pm}_{\mathscr{P}}$: one in $\bar{\mathfrak{h}}^{+}_{\mathscr{P}}$ and one in $\bar{\mathfrak{h}}^{-}_{\mathscr{P}}$ [8]. Formally the Casimir elements are given by

$$\mathscr{K}^{+} = \oint_{\tau=\text{const}} d\sigma[(u^{+\mu}(\tau,\sigma)u^{+}_{\mu}(\tau,\sigma))^2]^{\frac{1}{4}} \qquad \text{and}$$

$$\mathscr{K}^{-} = \oint_{\tau=\text{const}} d\sigma[(u^{-\mu}(\tau,\sigma)u^{-}_{\mu}(\tau,\sigma))^2]^{\frac{1}{4}}$$

The reparametrization invariant content of the constraints of the Nambu-Goto theory is precisely reflected by the conditions imposed on the Casimir elements:

$$\mathscr{K}^{+} = \mathscr{K}^{-} = 0 \ .$$

These are restrictions on the physically interesting representations of the commutator algebras $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ the quantum modifications of the Poisson algebras $\mathfrak{h}^{\pm}_{\mathscr{P}}$.
For $\mathscr{P}^2=m^2>0$ a maximal set of invariant charges of $\mathfrak{h}^{\pm}_{\mathscr{P}}$ in involution has been identified [9]. This set is expected to play the role of a Cartan subalgebra when the quantum modifications are included: the states of irreducible positive energy representations of the commutator algebras $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ should be labelled by eigenvalues of the quantum versions of the invariant charges from the maximal set quoted above.

Renormalization of the invariant charges in WKB-approximation [1] introduces one additional real parameter corresponding to counterterms already familiar from the construction [10] of the renormalized loop wave-functional $\psi(\mathscr{C})$. This wave-functional is invariant under the above symmetry transformations - at least in WKB-approximation - and arguments presented in ref. [3] suggest that this is the only invariant state. Thus, $\psi(\mathscr{C})$ is likely to correspond to the ground state of the closed bosonic strings.

II. Consistency conditions for the computation of quantum modifications

The passage from the Poisson algebras $\mathfrak{h}^{\pm}_{\mathscr{P}}$ to the commutator algebras $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ should not be accompanied by the occurence of any truly novel element. The Poisson bracket is to be replaced by $(i\hbar)^{-1}$ times the corresponding commutator. The result of the Poisson bracket operation is identified with the classical limit of the corresponding commutator relation. The right hand side of the commutator relation contains, in addition to this classical term, a polynomial in the (quantum versions of the) invariant charges with coefficients exhibiting positive powers of $\hbar$. The terms of this polynomial arise from operator ordering the "classical" term as well as from the appearance of new contributions. The combined behavior of the degrees of the classical invariant charges under the Poisson bracket and tensor product operation suggests that apart from length units, the invariant charges $\mathfrak{Z}^{\pm}_{\mu_1..\mu_N}$ in the stratum $V^{(\ell)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$ carry dimension: action to the power $(\ell+1)$, and are thus given as multiples of $(\hbar)^{\ell+1}$ (compare, for instance, the "angular momenta" in the strata $V^{(0)}(\mathfrak{h}^{\pm}_{\mathscr{P}})$). This observation implies that in the above polynomial of quantum corrections the coefficients exhibiting the power $(\hbar)^{k}$ can only be accompanied by monomials of operator charges with corresponding "classical" monomials in the stratum $V^{(\ell-k)}$ if ℓ is the degree of the classical term of the commutation relation. Thus instead of a gradation, $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ possesses only a semi-gradation. For the appropriately rescaled operator charges $\hat{\mathfrak{Z}}^{\pm}_{\mu_1..\mu_N}$, Planck's constant completely disappears from the commutation relations. Actually, these modifications are necessary if the aforementioned maximal set of (classical) invariant charges in involution should turn into a Cartan subalgebra for $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$.

A second consistency requirement is the existence of finite-dimensional non-trivial representations of $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ with positive energy. Since all representations of $\hat{\mathfrak{h}}^{\pm}_{\mathscr{P}}$ consist of entire multiplets under the little group (of the Lorentz group), any

finite-dimensional representation of $\hat{\mathfrak{h}}^{+}_{\mathcal{P}}$ is bound to have a maximum spin transfer. In any one of them, all operator charges of sufficiently high spin must be represented by the zero operator, and in addition there must exist a finite subset of operator charges of lower spin such that any operator charge can be represented as a function of these. (For an analogue in the context of the Virasoro algebra, see [11].)
A third important consistency requirement for the computation of the quantum modifications is derived from the non-additive composition law of the invariant charges, combined with their conservation law, by stripping the totally spacelike contour τ=const. - along which the charges are evaluated - across a branching point of the string trajectory.

References

[1] K. Pohlmeyer: "The Invariant Charges of the Nambu-Goto Theory in WKB-Approximation: Renormalization" Commun.Math. Phys. 105, 629 (1986)

[2] _____: "A Group-Theoretical Approach to the Quantization of the Free Relativistic Closed String" Phys.Lett. 119B, 100 (1982)

[3] _____, K.-H. Rehren: "The Invariant Charges of the Nambu-Goto Theory: Their Geometric Origin and Their Completeness" Commun.Math.Phys. 114, 177 (1986)

[4] D.J. Gross: "High-Energy Symmetries in String Theory" Phys.Rev.Lett. 60, 1229 (1988)

[5] K. Pohlmeyer, K.-H. Rehren: "Algebraic Properties of the Invariant Charges of the Nambu-Goto Theory" Commun.Math.Phys. 105, 593 (1986)

[6] K. Pohlmeyer: "Solution of the Constraints for Tensors Generated by Iterated Integrals" Linear Alg.Appl. 118, 11 (1989)

[7] _____: "Description of the Poisson Algebra Formed by the Invariant Charges of the Nambu-Goto Theory of Closed Strings Moving in (1+2)-dimensional Minkowski Space" in preparation

[8] _____: "The Poisson Algebra of the Invariant Charges of the Nambu-Goto Theory: Casimir Elements" Commun.Math.Phys. 114, 351 (1988)

[9] _____, K.-H. Rehren: "The Algebra Formed by the Invariant Charges of the Nambu-Goto Theory: Identification of a Maximal Abelian Subalgebra" Commun.Math.Phys. 114, 55 (1988)

[10] M. Lüscher, K. Symanzik, P. Weisz: "Anomalies of the Free Loop Wave Equation in the WKB-Approximation" Nucl.Phys. B173, 365 (1980)

[11] D.B. Fairlie, J. Nuyts, C.K. Zachos: "Construction of Classical Virasoro Algebras as SU(1,1) Extensions" Phys.Lett. B202, 320 (1988)

HAMILTONIAN FLOWS, SU(∞), SO(∞), USp(∞), AND STRINGS

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Abstract: Based on the infinite-dimensional algebras we have introduced, SU(∞) is identified with general hamiltonian flows in 2-d phase-space, SO(∞) with flows generated by x-p-odd hamiltonians, and USp(∞) with those of hamiltonians of special symmetry. Gauge theories for SU(∞) , SO(∞), and USp(∞) are thus formulated in terms of surface (sheet) coordinates for toroidal phase-space. Spacetime-independent configurations of their gauge fields directly yield the quadratic Schild-Eguchi string action.

This is an eclectic summary of recent observations made with David Fairlie and Paul Fletcher, with whom I introduced new infinite-dimensional algebras involving trigonometric functions in their structure constants[1]. The generators of the algebras we have introduced are indexed by 2-vectors $\mathbf{m} = (m_1, m_2)$. The components of these vectors do not need to be integers to satisfy the Jacobi identities, but we take them to be integral for the sake of interpreting them as Fourier modes:

$$[K_{\mathbf{m}}, K_{\mathbf{n}}] = r \sin(k\, \mathbf{m} \times \mathbf{n})\ K_{\mathbf{m}+\mathbf{n}} + \mathbf{a} \cdot \mathbf{m}\ \delta_{\mathbf{m}+\mathbf{n},0}\ . \tag{1}$$

Here, $\mathbf{m} \times \mathbf{n} = m_1 n_2 - m_2 n_1$, r and k are arbitrary (complex) constants, and $\mathbf{a}$ is an arbitrary 2-vector . The Casimir invariants are

$$\sum_{\mathbf{m}} K_{\mathbf{m}} K_{-\mathbf{m}}\ ,$$

$$\sum_{\mathbf{m},\mathbf{n}} e^{ik\mathbf{m}\times\mathbf{n}}\ K_{\mathbf{m}} K_{\mathbf{n}} K_{-\mathbf{m}-\mathbf{n}}\ , \qquad \dots , \tag{2}$$

$$\sum e^{ik(\mathbf{m}\times\mathbf{n}+\mathbf{m}\times\mathbf{p}+\ldots+\mathbf{m}\times\mathbf{r}+\mathbf{n}\times\mathbf{p}+\ldots+\mathbf{n}\times\mathbf{r}+\ldots+\mathbf{p}\times\mathbf{r})}\ K_{\mathbf{m}} K_{\mathbf{n}} K_{\mathbf{p}} \ldots K_{\mathbf{r}} K_{-\mathbf{m}-\mathbf{n}-\mathbf{p}\ldots-\mathbf{r}}\ .$$

These algebras include as a special case that of SDiff$_0(T^2)$, the infinitesimal area-preserving diffeomorphisms of the torus[2,3]: $r = 1/k$ in the limit $k \to 0$ yields the algebra

$$[L_{\mathbf{m}}, L_{\mathbf{n}}] = (\mathbf{m} \times \mathbf{n}) L_{\mathbf{m}+\mathbf{n}} + \mathbf{a} \cdot \mathbf{m}\ \delta_{\mathbf{m}+\mathbf{n},0}\ . \tag{3}$$

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You may find the supersymmetric extension of our algebra (1) and the observations to follow in Ref.[1]. The representation and character theory of these algebras is an open problem.

The algebra (3) is known to be, in a particular basis optimal for the torus, that of the generic area-preserving (symplectic) reparameterizations of a 2-surface. Taking x and p to be local (commuting) coordinates for the surface, and f and g to be differentiable functions of them, a basis-independent realization for the generators of the centerless algebra is[2]:

$$L_f = \frac{\partial f}{\partial x}\frac{\partial}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial}{\partial x} \qquad \Longrightarrow \tag{4}$$

$$[L_f, L_g] = L_{\{f,g\}} \ , \quad [L_f, g] = \{f,g\} \ , \tag{5}$$

where

$$\{f,g\} \equiv \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} \ , \tag{6}$$

the Poisson bracket of classical phase-space. The generator L_f transforms (x,p) to $(x - \partial f/\partial p \ , \ p + \partial f/\partial x)$. Infinitesimally, this is a canonical transformation[4] generated by f, which preserves the phase-space area element $dxdp$. This element is referred to as a *symplectic form* and the class of transformations that leaves it invariant specifies a symplectic geometry. You may regard it as the flow generated by an arbitrary hamiltonian f. For a small patch of 2-surface, you may expand the functions $f(x,p)$ in any coordinate basis you choose. If the surface is a torus, I shall prefer a globally adequate coordinate system, such as $\exp(inx + imp)$; if it is a sphere, spherical harmonics[3]; if it is a plane, powers[5]; and so on. Nevertheless, for the infinitesimal transformations effected by the algebra generators in a patch, *any* coordinate basis will do, and may be transformed to other ones. (When such transformations are singular, however, a number of generators may be lost, leading to a subalgebra, as noted by Pope and Stelle, and Hoppe[6].)

Choosing the torus basis, $f = -e^{i(m_1x+m_2p)}$ and $g = -e^{i(n_1x+n_2p)}$, $0 \le x,p \le 2\pi$, yields

$$L_f = L_{(m_1,m_2)} = -ie^{i(m_1x+m_2p)}(m_1\partial/\partial p - m_2\partial/\partial x) \ , \tag{7}$$

which obey the centerless algebra in the basis (3). Conversely, given the basis (3), any function $f(x,p)$ can be reconstituted through

$$f(x,p) = -\sum_{m_1,m_2} F(m_1,m_2)e^{i(m_1x+m_2p)} \ , \tag{8}$$

and thus the linear combinations

$$L_f = \sum_{m_1,m_2} F(m_1,m_2)L_{(m_1,m_2)} \tag{9}$$

are seen to obey the Poisson-bracket algebra (5).

We have found a corresponding realization for the torus-basis algebra (1) generators:

$$\begin{aligned} K_{(m_1,m_2)} &= (ir/2)\exp(im_1x + km_2\frac{\partial}{\partial x} + im_2p - km_1\frac{\partial}{\partial p}) \\ &= (ir/2)\exp(im_1x + im_2p)\ \exp(km_2\frac{\partial}{\partial x} - km_1\frac{\partial}{\partial p}) \ , \end{aligned} \tag{10}$$

somewhat analogous to the one-variable realization found by Hoppe[3]. Note the triviality in this realization of the Casimir operators, as the indices of each of their terms sum to zero.

To Fourier-compose this to a basis-independent realization, we first define, as in (9),

$$K_f \equiv \sum_{m_1,m_2} F(m_1,m_2) K_{(m_1,m_2)} \equiv \frac{r}{2i} f(x + ik\frac{\partial}{\partial p}, p - ik\frac{\partial}{\partial x}) \ , \tag{11}$$

where the last side of the equation is a formal expression to evoke (8)/(4): the "normal ordering" of its derivatives is specified in its Fourier-series definition, in which they stand to the right of all coordinates, by virtue of eq. (10).

The analog of the Poisson bracket in this case is the *sine,* or *Moyal, bracket* $\{\{f,g\}\}$. This is the extension of the Poisson bracket $\{f,g\}$ to statistical distributions on phase-space, introduced by Weyl[4] and Moyal[7b], and explored by several authors[7] in an alternative formulation of quantum mechanics, regarded as a deformation of the algebra of classical observables. It is a generalized convolution which reduces to the Poisson bracket as $\hbar$, replaced by $2k$ in our context, is taken to zero:

$$\{\{f,g\}\} = \frac{-r}{4\pi^2 k^2} \int dp' dp'' dx' dx'' \ f(x',p') g(x'',p'') \sin\frac{1}{k}\Big(p(x'-x'')+x(p''-p')+p'x''-p''x'\Big) \ . \tag{12}$$

The argument of the sine above is

$$\frac{1}{k}\det\begin{pmatrix} 1 & p & x \\ 1 & p' & x' \\ 1 & p'' & x'' \end{pmatrix} = \frac{1}{k}\int p \cdot dq \ , \tag{13}$$

i.e. $2/k$ times the area of the equilateral phase-space triangle with vertices at (x,p), (x',p'), and (x'',p''). The antisymmetry of f with g is evident in the determinant. The sine brackets satisfy the Jacobi identities[7d], just as their Fourier components (1) (see the next paragraph) do, and thus determine a Lie algebra. These brackets help reformulate quantum mechanics in terms of Wigner's phase-space distribution[7].

The Fourier transform of the sine bracket results from substitution in (12) of the exponential basis used in (7):

$$\{\{f,g\}\} = \frac{-ir}{8\pi^2 k^2}\int dp' dp'' dx' dx'' \ e^{i(m_1 x' + n_1 x'') + i(m_2 p' + n_2 p'')} \ \times$$

$$\times \left(e^{\frac{i}{k}(p(x'-x'')+x(p''-p')+p'x''-p''x')} - (k \leftrightarrow -k) \right) = -r \sin(k\mathbf{m}\times\mathbf{n}) \ e^{i(m_1+n_1)x + i(m_2+n_2)p} \ . \tag{14}$$

As in (9), it then follows through the linearity of the operators defined in (11), and (1), that these indeed obey the algebra

$$[K_f, K_g] = r \sum_{m_1,m_2,n_1,n_2} F(m_1,m_2) G(n_1,n_2) \sin(k\mathbf{m}\times\mathbf{n}) \ K_{\mathbf{m}+\mathbf{n}} = K_{\{\{f,g\}\}} \ . \tag{15}$$

Our algebra is thus identified with that of sine brackets. *Mutatis mutandis*, you might wish to expand it in alternate bases, such as spherical harmonics, so as to specify the corresponding generalizations of $\mathrm{SDiff}_0(S^2)$, powers for the plane[8], and so on.

Focus now on an interesting centerless family of the algebras (1), namely the *cyclotomic* family: the one for which $k = 2\pi/N$, for integer $N > 2$. In this family, there is an additional $\mathbb{Z} \times \mathbb{Z}$ algebra isomorphism

$$K_{(m_1,m_2)} \longmapsto K_{(m_1,m_2)+(Nt,Nq)} \tag{16}$$

for arbitrary integers t and q. Since the structure constants $\sin\frac{2\pi}{N}(m_1n_2 - n_1m_2)$ are only sensitive to the modulo-N values of the indices, the 2-dimensional integer lattice separates into $N \times N$ cells, each of which may be referred to some fundamental cell, e.g. around the coordinate center of the lattice, by proper N-translations. The fundamental $N \times N$ cell contains N^2 index points, but the operator $K_{(0,0)}$, like its lattice translations $K_{N(t,q)}$, factors out of the algebra: it commutes with all K's and cannot result as a commutator of any two such. Thus the fundamental cell involves only $N^2 - 1$ generators, and there are no more structure constants than those occurring in this cell. In consequence, the infinite-dimensional centerless cyclotomic algebras, with the $K_{N(t,q)}$'s factored out, possess the following finite-dimensional invariant subalgebra of "lattice average" operators $\mathcal{K}$:

$$\mathcal{K}_{(m_1,m_2)} \equiv \sum_{s,v} K_{(m_1+Ns,m_2+Nv)}, \qquad [\mathcal{K}_{\mathbf{m}}, \mathcal{K}_{\mathbf{n}}] = r\sin(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n})\,\mathcal{K}_{\mathbf{m}+\mathbf{n}}, \tag{17}$$

where $\mathbf{m},\mathbf{n},\mathbf{m}+\mathbf{n}$ are indices in the fundamental cell, and an infinite normalization has been absorbed in r.

This (N^2-1)-dimensional ideal specifies, in fact, a basis for SU(N) which may be thought of as a generalization of the Pauli matrices[9]. Consider odd N's first. A basis for SU(N) algebras, for odd N, may be built from two unitary unimodular matrices:

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \qquad h \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad g^N = h^N = \mathbb{1}\ , \tag{18}$$

where ω is a primitive N'th root of unity, i.e. with period no smaller than N, here taken to be $e^{4\pi i/N}$. They obey the identity

$$hg = \omega\ gh. \tag{19}$$

You also encounter these matrices in the context of representations of quantum SU(2)[10]. The complete set of unitary unimodular $N \times N$ matrices

$$J_{(m_1,m_2)} \equiv \omega^{m_1m_2/2}\, g^{m_1} h^{m_2}\ , \tag{20}$$

where

$$J^\dagger_{(m_1,m_2)} = J_{(-m_1,-m_2)}\ ; \qquad \mathrm{Tr}J_{(m_1,m_2)} = 0 \quad \text{except for } \ m_1 = m_2 = 0 \bmod N\ , \tag{21}$$

suffice to span the algebra of SU(N). Like the Pauli matrices, they close under multiplication to just one such, by virtue of (19):

$$J_{\mathbf{m}}J_{\mathbf{n}} = \omega^{\mathbf{n}\times\mathbf{m}/2} J_{\mathbf{m}+\mathbf{n}}\ . \tag{22}$$

They therefore satisfy the algebra

$$[J_{\mathbf{m}}, J_{\mathbf{n}}] = -2i\ \sin(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n})\ J_{\mathbf{m}+\mathbf{n}}\ . \tag{23}$$

Consequently, in this convenient two-index basis with the above simple structure constants, SU(N) describes the algebra (17) of the ideal $\{\mathcal{K}\}$.

For even N, the fundamental matrices in (18) are not unimodular, as their determinant may now be -1 as well. One might choose to modify them to

$$g \equiv \sqrt{\omega}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{N-1} \end{pmatrix}, \quad h \equiv \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad g^N = h^N = -\mathbb{1} , \tag{24}$$

with $\omega = e^{2\pi i/N}$, $\sqrt{\omega} = e^{\pi i/N}$. They again obey (19), and again serve to define the unitary basis

$$J_{(m_1,m_2)} \equiv \omega^{m_1 m_2/2}\, g^{m_1} h^{m_2} , \tag{20$'$}$$

$$J_{\mathbf{m}} J_{\mathbf{n}} = \omega^{\mathbf{n}\times\mathbf{m}/2} J_{\mathbf{m}+\mathbf{n}} . \tag{22$'$}$$

The SU(N) algebra is now

$$[J_{\mathbf{m}}, J_{\mathbf{n}}] = -2i \sin(\frac{\pi}{N}\mathbf{m}\times\mathbf{n})\; J_{\mathbf{m}+\mathbf{n}} . \tag{25}$$

It might appear that the fundamental period be $2N$ instead of N. However, note that, by virtue of the symmetry

$$J_{\mathbf{m}+N(t,q)} = (-1)^{(m_1+1)q+(m_2+1)t} J_{\mathbf{m}} , \tag{26}$$

only indices in the fundamental cell $N \times N$ need be considered. Illustrating this for $N = 2$, the Pauli matrices, may be of use to the reader. Naturally, the algebra (25) also holds for N odd, when $\omega = \exp(2\pi i/N)$ is used in (18). Thus, the ideal (17) amounts to SU(N) for N odd and SU($N/2$) for N even.[2] For example, both $N = 3$ and $N = 6$ yield SU(3), $N = 12$ yields SU(6)'s, etc.

In this basis again, the operators $Q_{(m,n)} \equiv J_{(m,n)} - J_{(n,m)}$ close to a subalgebra of SU(N) with $N(N-1)/2$ generators

$$[Q_{(m,n)}, Q_{(m',n')}] = -2i \sin\frac{2\pi}{N}(mn' - m'n)\; Q_{(m+m',n+n')} + 2i \sin\frac{2\pi}{N}(mm' - nn')\; Q_{(m+n',n+m')}, \tag{27}$$

which is shown by reduction to the Cartan-Weyl basis[11] to amount to SO(N). Alternative SO(N)'s may also be found, such as the subset of the above $Q_{(m,n)}$ with $m + n$ = even together with the operators $J_{(m,n)} + J_{(n,m)}$ with $m + n$ = odd; or else, for even $N = 2M$, $J_{(m,n)} - (-)^n J_{(m,-n)}$. Finally, the subalgebra of SU($2M$): $S_{(m,n)} \equiv J_{(m,n)} - (-)^m J_{(m,-n)}$ is seen to be an USp($2M$).

The 2-index SU(N) basis considered here has a particularly simple large N limit. As N increases, the fundamental $N \times N$ cell covers the entire index lattice; the operators $\mathcal{K}$ supplant the K's and, in turn, since $k \to 0$, the operators L of eq.(3).

[2]Actually, in this case[11], the generators describe SU($N/2$)^4, i.e. four mutually commuting SU($N/2$)'s.

More directly, you immediately see by inspection that, as $N \to \infty$, the SU(N) algebra (23) goes over to the centerless algebra (3) of SDiff$_0(T^2)$ through the identification:

$$\frac{iN}{4\pi} J_{\mathbf{m}} \to L_{\mathbf{m}} \ . \tag{28}$$

An identification of this type was first noted by Hoppe[3] in the context of membrane physics: he connected the infinite N limit of the SU(N) algebra in a special basis to that of SDiff$_0(S^2)$, i.e. the infinitesimal symplectic diffeomorphisms in the sphere basis. A discussion of the group topology of SU(N), or SDiff$_0(T^2)$ versus SDiff$_0(S^2)$, or other 2-dimensional manifolds for that matter[5], exceeds the scope of this type of local analysis; such a discussion has been suggested in Refs.[6], which consider central extensions that are sensitive to global features of the 2-surface.

In view of the SO(N) subalgebras described above, we may also simply identify the SO(∞) subalgebra with the Poisson Bracket subalgebra whose shift potentials f are odd under interchange of x with p — they correspond to hamiltonians which evolve even functions to even ones, and odd to odd ones. Likewise, USp(∞) is generated by shift potentials of the form $\exp(imx)\ \sin(np - m\pi/2)$, i.e. toroidal phase-space hamiltonians odd under $p \longmapsto -p$, $x \longmapsto x + \pi$. (Merely p-odd hamiltonians generate the "sibling" SO(∞).) Saveliev and Vershik[12], and we[11] have initiated a program of systematizing such results in a unified framework common with that of the finite Lie algebras.

Floratos et al.[13] utilized Hoppe's identification to take the limit of SU(N) gauge theory. Their results are immediately reproduced without ambiguity, again by inspection, on the basis of the orthogonality condition dictated by (21) and (22):

$$\mathrm{Tr} J_{\mathbf{m}} J_{\mathbf{n}} = N \delta_{\mathbf{m}+\mathbf{n},0} \quad \to \quad \mathrm{Tr} L_{\mathbf{m}} L_{\mathbf{n}} = -\frac{N^3}{(4\pi)^2} \delta_{\mathbf{m}+\mathbf{n},0} \ . \tag{29}$$

As a result, for a gauge field A_μ in an SU(N) matrix normalization with trace 1, the analog of eq. (9) is

$$A_\mu \equiv A_\mu^{\mathbf{m}} \frac{J_{\mathbf{m}}}{\sqrt{N}} \quad \to \quad \frac{4\pi}{iN^{3/2}} A_\mu^{\mathbf{m}} L_{\mathbf{m}} = \tilde{A}_\mu^{\mathbf{m}} L_{\mathbf{m}} \ , \tag{30}$$

where summation over repeated $\mathbf{m}$'s is implied, and I have defined $\tilde{A}_\mu^{\mathbf{m}} \equiv (4\pi/iN^{3/2}) A_\mu^{\mathbf{m}}$. As $N \to \infty$, the indices $\mathbf{m}$ cover the entire integer lattice, so that I may define

$$a_\mu^{(x,p)} \equiv -\sum_{\mathbf{m}} \tilde{A}_\mu^{\mathbf{m}} e^{i(m_1 x + m_2 p)} \ . \tag{31}$$

By eq. (5),

$$[A_\mu, A_\nu] \quad \to \quad [L_{a_\mu}, L_{a_\nu}] = L_{\{a_\mu, a_\nu\}} \ . \tag{32}$$

Hence, by virtue of the linearity of L in its arguments,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad \to \quad L_{f_{\mu\nu}}$$

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu + \{a_\mu, a_\nu\} \ . \tag{33}$$

The group trace defining the Yang-Mills lagrangian density is then

$$\mathrm{Tr} F_{\mu\nu} F_{\mu\nu} \to -\frac{N^3}{(4\pi)^2} \tilde{F}_{\mu\nu}^{\mathbf{m}} \tilde{F}_{\mu\nu}^{-\mathbf{m}} = \frac{-N^3}{64\pi^4} \int dx dp \sum_{m_1, m_2, n_1, n_2} e^{ix(m_1+m_2)+ip(m_2+n_2)} \tilde{F}_{\mu\nu}^{(m_1,m_2)} \tilde{F}_{\mu\nu}^{(n_1,n_2)}$$

$$= (-N^3/64\pi^4)\int dxdp\ f^{(x,p)}_{\mu\nu} f^{(x,p)}_{\mu\nu}\ . \tag{34}$$

Thus, in the SU(∞) gauge theory, the group indices are surface (torus) coordinates, and the fields are rescaled Fourier transforms of the original SU(N) fields; the group composition rule for them is given by the Poisson bracket, and the trace by surface integration.

Now note an intriguing connection to strings which emerges, for the first time *directly at the level of the action*: for gauge fields independent of x^μ (e.g. vacuum configurations), this lagrangian density reduces to $\{a_\mu, a_\nu\}\{a_\mu, a_\nu\}$, the quadratic Schild-Eguchi action density for strings[14], where the a_μ now serve as string variables, and the surface serves as the world-sheet. This action amounts to the square of the sheet area and it is easily seen that its equations of motion contain those of Nambu's action. Thus, at zero energy, the gauge theory reduces to a string. Whether a superstring follows analogously from the super-Yang-Mills lagrangian is an open question.

The lagrangian (34) with the sine bracket supplanting the Poisson bracket is also a gauge-invariant theory, provided that the gauge transformation also involves the sine instead of the Poisson bracket:

$$\delta a_\mu = \partial_\mu \Lambda - \{\{\Lambda, a_\mu\}\}\ , \tag{35}$$

and hence, by virtue of the Jacobi identity,

$$\delta f_{\mu\nu} = -\{\{\Lambda, f_{\mu\nu}\}\}\ . \tag{36}$$

It then follows that

$$\delta\int dxdp\ f_{\mu\nu} f_{\mu\nu} = -2\int dxdp\ f_{\mu\nu}\{\{\Lambda, f_{\mu\nu}\}\} = 0\ . \tag{37}$$

At the moment, however, it is not clear what physical system is described by the corresponding spacetime-independent lagrangian density $\{\{a_\mu, a_\nu\}\}\{\{a_\mu, a_\nu\}\}$. It is further obscure whether a relation exists between the above theories and the Universal Yang-Mills theory[15].

This compact formulation of SU(∞) gauge theory (and that of its subgroups) ought to be of use in large-N model calculations, or various "master-field" efforts; membrane physics[2,3]; and the exploration of connections between gauge theory and strings, as above.

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REFERENCES

1. D. Fairlie, P. Fletcher, and C. Zachos, Phys.Lett. *218B*:203 (1989);
   D. Fairlie and C. Zachos, Phys.Lett. *224B*:101 (1989);
   C. Zachos and D. Fairlie, ANL-HEP-CP-89-36, to appear in the proceedings of *Strings '89, Texas A&M.*

2. E. Floratos and J. Iliopoulos, Phys.Lett. *201B*:237 (1988); Antoniadis et al., Nucl.Phys. *B300 [FS22]*:549 (1988); T. Arakelyan and G. Savvidy, Phys.Lett. *214B*:350 (1988), and *223B*:41 (1989).
   Earliest work: V. Arnold, Ann.Inst.Fourier *16*:319 (1966); V. Arnold, *Mathematical Methods of Classical Mechanics*, Appendix 2.K, p.339, Springer Verlag, New York, (1978); E. Corrigan and D. Fairlie, and J. Goldstone, (unpublished).

3. J. Hoppe, M.I.T. Ph.D. Thesis (1982); *Constraints Theory and Relativistic Dynamics - Florence 1986*, G. Longhi and L. Lusanna eds., p.267, World Scientific, Singapore, (1987); Phys.Lett. *215B*:706 (1988).

4. H. Weyl, *The Theory of Groups and Quantum Mechanics,* Dover, (1931).

5. D. Fuks, Funct.Anal.&Appl. *19*:305 (1985); I. Bakas, Maryland preprint 89-164, April 1989; K. Case and A. Monge, J.Math.Phys. *30*:1250 (1989).

6. C. Pope and K. Stelle, Texas A&M preprint CTP-TAMU-07/89;
J. Hoppe, Karlsruhe preprint KA-THEP-18-1988.

7. a) E. Wigner, Phys.Rev. *40*:749 (1932);
b) J. Moyal, Proc.Camb.Phil.Soc. *45*:99(1949);
c) G. Baker, Phys.Rev. *109*:2198 (1958);
d) D. Fairlie, Proc.Camb.Phil.Soc. *60*:581 (1964);
e) T. Jordan and E. Sudarshan, Rev.Mod.Phys. *33*:515 (1961);
f) F. Berezin, Sov.Phys.Usp. *23*:763 (1980);
g) I. Bakas and A. Kakas, Class.Quant.Grav. *4*:L67 (1987).

8. C. Zachos, unpublished.

9. G.'t Hooft, Nucl.Phys. *B138*:1 (1978); Comm.Math.Phys. *81*:267 (1981);
T. Eguchi and R. Nakayama, Phys.Lett. *122B*:59 (1983);
J. Patera and H. Zassenhaus, J.Math.Phys. *29*:665 (1988).
Also see: V. Rittenberg and D. Wyler, Nucl.Phys. *B139*:189 (1978).

10. D. Fairlie, to appear in the proceedings of *Strings '89, Texas A&M.*

11. D. Fairlie, P. Fletcher, and C. Zachos, Durham preprint /DTP/89/35.

12. M. Saveliev and A. Vershik, Serpukhov preprint IHEP-89-69, 1989.

13. E. Floratos, J. Iliopoulos, and G. Tiktopoulos, Phys.Lett. *217B*:285 (1989).
Also see: B. de Wit, J. Hoppe, and H. Nicolai, Nucl.Phys. *305 [FS23]*:545 (1988); E. Bergshoeff, E. Sezgin, Y. Tanii, and P. Townsend, Trieste preprint IC/88/186.

14. A. Schild, Phys.Rev. *D16*:1722 (1977); T. Eguchi, Phys.Rev.Lett. *44*:126 (1980).

15. S. Rajeev, Phys.Lett. *209B*:53 (1988).

A GEOMETRIC APPROACH TO THE STRING BRS COHOMOLOGY

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INTRODUCTION

A study of constrained dynamical systems has received increasing attention in recent years. A reason for this interest can undoubtedly be traced back to the impact made in theoretical physics by string theory, where importance of the Batalin-Fradkin-Vilkovisky[1] (BFV) quantization method, centered around nilpotent BRS charge, was recognized very early.

In this talk we supplement the set of the first-class constraints by the subsidiary constraints (gauge-fixing conditions) in a scheme based on the bigger, closed extended constraint algebra. In this framework gauge-fixing leads to a simple parametrization of gauge orbits with gauge-fixing conditions playing the role of coordinates. One of the nice features of the extended constrained system is that it naturally admits a symplectic formulation with canonical coordinates. We find explicitly abelian momentum, conjugated to the gauge-fixing condition, associated with the transport along the gauge orbit. This construction is realized geometrically through the use of vielbeins acting between abelian and non-abelian constraint algebras and satisfying the fundamental Maurer-Cartan structure equations.

We also discuss string theory in terms of the extended Virasoro constraint algebra, where the original Virasoro operators are supplemented by the Fubini-Veneziano (gauge-fixing) coordinate. As we show explicitly, in this context certain fundamental objects in string theory–the Virasoro and vertex operators and generators of the spectrum generating algebra–can be understood in terms of natural abelian operators and vielbeins defining the transition from the abelian to the non-abelian formulation.

In the framework of the BFV extended phase space this formulation leads in a straightforward manner to supersymmetry between symplectic coordinates of opposite Grassmannian parity, identified with ghosts and unphysical modes. The symplectic coordinates form (as it will be illustrated for strings) the Kugo-Ojima (KO) quartets[2,3] leading to the construction of a contracting homotopy operator in a framework analo-

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gous to the supersymmetric quantum mechanics. Consequently our construction yields a simple derivation of the string BRS cohomology.

CONSTRAINTS AND THE SYMPLECTIC COORDINATES

We consider a dynamical system governed by a set Φ of functions $\{\phi_a\}$ defined on the phase space W. Functions ϕ_a constrain dynamics to the submanifold $V \subset W$ defined by conditions $\{\phi_a = 0\}$.

The physical space is now identified with the gauge-equivalence classes defined by the gauge transformations generated by Φ on V. We will deal here with the first class constraints only, meaning that Φ is closed under the Poisson bracket $\{\cdot,\cdot\}$ defined on W. We have therefore clearly $\{\phi_a, \phi_b\}_{|V} = 0$.

The above structure has to be completed by the gauge-fixing conditions (also called the subsidiary constraints) Q^a to enable us to choose one element out of a given equivalence class. This is possible if the gauge-fixing conditions intersect each equivalence class uniquely in one point as ensured by the basic condition $\det ||\{\phi_a, Q^b\}||_{|V} \neq 0$.

We are consequently led to consider the bigger set consisting of the first-class constraints ϕ_a and the associated gauge-fixing conditions Q^a. We will be dealing in this talk with the system, which realizes algebraically the above conditions in the following way

$$\begin{array}{lcl} \{\phi_a, \phi_b\}_{|V} = 0 & \longrightarrow & \{\phi_a, \phi_b\} = U^c_{ab}\phi_c \\ \det ||\{\phi_a, Q^b\}||_{|V} \neq 0 & \longrightarrow & \{\phi_a, Q^b\} = \omega^b_a - T^b_{ac}Q^c \end{array} \tag{1}$$

and where ω^b_a is a central element and U^c_{ab} together with T^c_{ab} are structure constants to be determined. The first equation in (1) describes a conventional Lie algebra of first class constraints while the second equation ensures the closure of the resulting algebra containing ϕ_a and Q^a. We make furthermore the convenient choice that

$$\{Q^a, Q^b\} = 0 \tag{2}$$

Note that eq.(1) imposes the condition $\det ||\omega^b_a|| \neq 0$.

We call the set of relations (1) and (2) the "extended constraint algebra". Its consistency imposes the usual Jacobi identity for the U^c_{ab} structure constants supplemented by two extra Jacobi relations mixing U^c_{ab} together with T^c_{ab}. One of these extra relations states that the matrix $(T_a)^c_b = T^c_{ab}$ is the representation of the Lie algebra defined in (1):

$$[T_a, T_b] = U^c_{ab}T_c \tag{3}$$

The other Jacobi relation shows that the central element ω^b_a is a closed $\binom{1}{1}$-form, using a suitable definition[4] of the coboundary operator, which takes into account the peculiar structure of the extended algebra (1-2). Explicitly, this condition takes the following form

$$(d\omega)^c_{ab} \equiv U^d_{ab}\omega^c_d + T^c_{bd}\omega^d_a - T^c_{ad}\omega^d_b = 0 \tag{4}$$

For simplicity, we set in this talk $\omega^b_a = \delta^b_a$, hence from now on $\{\phi_a, Q^b\} = D^b_a$, with

$$D^b_a \equiv \delta^b_a - T^b_{ac}Q^c \tag{5}$$

being an invertible matrix according to (1).

We will now identify Q^a with the usual position coordinate inside the canonical (= symplectic) coordinate $X_a = (P_a, Q^b)$ such that the original Poisson bracket is reproduced by

$$\{G, F\} \equiv \Omega_{a\,b}\,\frac{\partial G}{\partial X_a}\frac{\partial F}{\partial X_b} \tag{6}$$

where G and F are arbitrary functions on the phase space and $\Omega_{a\,b}$ is the associated symplectic metric

$$\Omega_{a\,b} = \begin{pmatrix} 0 & \delta_a^b \\ -\delta_a^b & 0 \end{pmatrix} \tag{7}$$

Consequently, operators P_a, Q^a, satisfy a Heisenberg algebra:

$$\{P_a, Q^b\} = \delta_a^b \qquad \{P_a, P_b\} = 0 = \{Q^a, Q^b\} \tag{8}$$

Note, that Darboux's theorem asserts that locally one can always find coordinates q^a and p^a with the above metric tensor. Below we will prove Darboux's theorem for the system defined by relations (1-2) with $q^a = Q^a$, in the meantime the unproved existence of P_a does not influence the arguments given below. P_a can be thought of as auxiliary quantity making the following construction more transparent.

In this setting it is obvious that the quantity $\mathcal{D}_a \equiv D_a^b P_b$ will reproduce (1)

$$\{\mathcal{D}_a, Q^b\} = D_a^b \tag{9}$$

$$\{\mathcal{D}_a, \mathcal{D}_b\} = D_a^c \frac{\partial \mathcal{D}_b}{\partial Q^c} - D_b^c \frac{\partial \mathcal{D}_a}{\partial Q^c} = U_{ab}^c \mathcal{D}_c \tag{10}$$

where the last equation follows from Jacobi relations (3-4). Moreover one can define a useful canonical transformation, relating the Heisenberg algebra (8) to the extended algebra (1-2), generated by $\Phi \equiv Q^a T_a$. Here, T_a is an operator defined in terms of the variables (P_a, Q^a) as $T_a \equiv -T_{ab}^c Q^b P_c$. The fundamental quantities transform as follows[4,5]

$$\exp(-\Phi)\ \delta_i^a P_a \exp(\Phi) = e_i^a \mathcal{D}_a \equiv P_i \tag{11}$$

$$\exp(-\Phi)\ \delta_a^i Q^a \exp(\Phi) = E_a^i Q^a \equiv Q^i \tag{12}$$

here P_a transforms non-linearly, giving accordingly rise to the non-abelian quantity $\mathcal{D}_a \equiv P_a + T_a$, while Q^a transforms linearly. Remarkably, E_a^i is inverse to the vielbein e_i^a

$$e_i^b E_a^i = \delta_a^b \qquad e_i^b E_b^j = \delta_i^j \tag{13}$$

Both quantities have been shown[4,6] to depend *only* on variables Q^a and constants T_{ab}^c.

Since the mapping in (11-12) was canonical, the quantities $(P_i,\ Q^i)$ defined there satisfy the dual Heisenberg algebra:

$$\{P_i, Q^j\} = \delta_i^j \qquad \{P_i, P_i\} = 0 = \{Q^i, Q^j\} \tag{14}$$

We observe that

$$\exp(\Phi)\ \delta_a^i P_i \exp(-\Phi) = P_a = E_a^i \mathcal{D}_i \tag{15}$$

Since P_a is self-commuting, we expect that $\mathcal{D}_i \equiv e_i^a P_a$ generates a non-abelian algebra. As verified before[4], $\mathcal{D}_i$ and Q^j generate a non-abelian extended algebra dual to (1-2):

$$\{\mathcal{D}_i, \mathcal{D}_j\} = -U_{ij}^k \mathcal{D}_k \qquad \{\mathcal{D}_i, Q^j\} = \delta_i^j + T_{ik}^j Q^k \tag{16}$$

where $T_{ij}^k = \delta_i^a \delta_j^b \delta_c^k T_{ab}^c$ differs from T_{ab}^c only in having indices $(i, j, k, \ldots)$ rather than $(a, b, c, \ldots)$. To make the duality more transparent it is convenient to introduce an invertible matrix

$$D_j^i = \delta_j^i + T_{jk}^i Q^k \tag{17}$$

which allows an alternative expression[4,6] for the quantity $\mathcal{D}_i$ as $\mathcal{D}_i = D_i^j P_j$.

As noted by us[4,6] the generator Φ is invariant under the mapping between the two sets of abelian variables, (P_a, Q^b) and (P_i, Q^j) and this leads to a duality between the quantities with "a" and "i" indices. This duality can be illustrated by the following diagram showing the connection between the various abelian and non-abelian generators:

$$\begin{array}{ccc} P_a & \xrightarrow{\Phi} & \mathcal{D}_a \\ \Big\uparrow{\scriptstyle E_a^i} & & \Big\downarrow{\scriptstyle e_i^a} \\ \mathcal{D}_i & \xleftarrow{-\Phi} & P_i \end{array} \tag{18}$$

Each link in this diagram connects abelian with non-abelian generators.

DUALITY AND THE STRUCTURE EQUATIONS

Since the Poisson bracket structure, as defined by (6), leads to $\{P_a, f\} = \frac{\partial f}{\partial Q^a}$ and $\{P_i, f\} = \frac{\partial f}{\partial Q^i}$ for any function $f(Q)$ of the coordinates, the canonical mapping (11-12) can be cast in the simple form:

$$Q^i = E_a^i(Q)\, Q^a \qquad ; \qquad \frac{\partial}{\partial Q^i} = \frac{\partial Q^a}{\partial Q^i}\frac{\partial}{\partial Q^a} \tag{19}$$

where we wrote $E_a^i(Q)$ to indicate that the vielbein, which "rotates" the coordinate Q^a depends functionally on Q^a (and the structure constant T_{ab}^c). Using that e_i^a is an inverse vielbein to E_a^i, we obtain easily the dual version of the above equation:

$$Q^a = e_j^a(Q)\, Q^j \qquad ; \qquad \frac{\partial}{\partial Q^a} = \frac{\partial Q^j}{\partial Q^a}\frac{\partial}{\partial Q^j} \tag{20}$$

Comparing with (11-12) we find

$$\frac{\partial Q^a}{\partial Q^i} = \frac{\partial e_j^a}{\partial Q^i}\, Q^j + e_i^a = e_i^b D_b^a \qquad ; \qquad \frac{\partial Q^i}{\partial Q^a} = D_j^i E_a^j \tag{21}$$

Inserting definitions of D-matrices and using some fundamental relations governing the above canonical mapping, like $E_a^i \frac{\partial}{\partial Q^i} = D^b_{\ a}\frac{\partial}{\partial Q^b}$, we arrive at

$$\begin{array}{lcl} D_b^d \frac{\partial e_j^a}{\partial Q^d} = -T_{bc}^a\, e_j^c & \Longrightarrow & \{\mathcal{D}_a, e_i^b\} = -T^b_{\ ac} e_i^c \\ D^c_{\ a}\frac{\partial E_b^i}{\partial Q^c} = T_{ab}^c E_c^i & \Longrightarrow & \{\mathcal{D}_a, E_b^i\} = T_{ab}^c E_c^i \end{array} \tag{22}$$

We call these equations the Maurer-Cartan (MC) structure equations[4,6]. The MC structure equations ensure consistency with the canonical character of the mapping in (11-12) and make possible transition from abelian to non-abelian algebras. There exist the following dual counterparts of MC equations

$$\begin{array}{lcl} D_i^k \frac{\partial E_b^j}{\partial Q^k} = T_{ik}^j E_b^k & \Longrightarrow & \{\mathcal{D}_i, E_b^j\} = T_{ik}^j E_b^k \\ D_i^k \frac{\partial e_j^b}{\partial Q^k} = -T_{ij}^k e_k^b & \Longrightarrow & \{\mathcal{D}_i, e_j^b\} = -T_{ij}^k e_k^b \end{array} \tag{23}$$

with generators of the two dual algebras related to the basic momenta in the following form

$$\mathcal{D}_i = D_i^j P_j = e_i^a P_a \tag{24}$$
$$\mathcal{D}_a = D_a^b P_b = E_a^j P_j \tag{25}$$

Consequently the D-matrices are related to each other through relations like $D_i^j = e_i^a(D^{-1})_a^b E_b^j$, $D_a^b = E_a^i(D^{-1})_i^j e_j^b$.

Let us now go back to the original framework with the first class constraints ϕ_a and coordinates Q^b. Since the vielbeins are expressed entirely in terms of Q's and since $\mathcal{D}_a$ acts on the gauge-fixing conditions exactly as ϕ_a we conclude that the MC equations are valid for the original constraints

$$\{\phi_a, E_b^i\} = T_{ab}^c E_c^i \qquad \{\phi_a, e_i^b\} = -T_{ac}^b e_i^c \tag{26}$$

The MC equations can be understood as a statement that the covariant derivatives of vielbeins are zero. Moreover a metric tensors constructed from the vielbeins as follows:

$$g^{ab} \equiv e_i^a e_j^b \delta^{ij} \qquad g_{ab} \equiv E_a^i E_b^j \delta_{ij} \tag{27}$$

will be also covariantly constant.

We can now present the explicit construction of the canonical coordinates in terms of the original constraints (ϕ_a, Q^b) and vielbeins only. Define, namely $\Phi_i \equiv e_i^a \phi_a$. Then it follows from MC equations that Φ_i together with $Q^i = E_a^i Q^a$ satisfy the Heisenberg algebra

$$\{\Phi_i, Q^j\} = \delta_i^j \qquad \{\Phi_i, \Phi_i\} = 0 = \{Q^i, Q^j\} \tag{28}$$

It is easy to see that the Poisson brackets defined as

$$\{G, F\} \equiv \delta_i^j \left(\frac{\partial G}{\partial \Phi_i} \frac{\partial F}{\partial Q^j} - \frac{\partial G}{\partial Q^j} \frac{\partial F}{\partial \Phi_i} \right) \tag{29}$$

will reproduce relations (1-2). Furthermore we can define the non-abelian quantity $\hat{\phi}_i \equiv D_i^j \Phi_j = D_i^j e_j^a \phi_a$ which together with Q^j will satisfy the non-abelian algebra (16) dual to (1-2). The passage back to the quantities labeled by "a" indices can be performed by acting with matrices $\frac{\partial Q^j}{\partial Q^a}$ as follows

$$\frac{\partial Q^j}{\partial Q^a} \Phi_j = E_a^i D_i^j \Phi_j = E_a^i \phi_i \equiv \Phi_a \tag{30}$$

It follows that Φ_a defined in such a way is equal to $(D^{-1})_a^b \phi_a$ and together with Q^a constitutes another canonical coordinates satisfying

$$\{\Phi_a, Q^b\} = \delta_a^b \qquad \{\Phi_a, \Phi_b\} = 0 = \{Q^a, Q^b\} \tag{31}$$

To summarize, we found two dual canonical coordinates $X_a = (\Phi_a, Q^b)$ and $X_i = (\Phi_i, Q^j)$ defined in terms of the original constraints (ϕ_a, Q^b) as follows

$$\Phi_a = E_a^i \hat{\phi}_i = E_a^i D_i^j \Phi_j = (D^{-1})_a^b \, \phi_b \tag{32}$$

$$\Phi_i = e_i^a \phi_a = e_i^a D_a^b \Phi_b = (D^{-1})_i^j \, \hat{\phi}_j \tag{33}$$

This construction can be considered as a proof of the Darboux's theorem valid for the case of the constraint algebra (1-2) through the explicit construction of the canonical coordinates. The abelian quantities constructed here are clearly defined where the gauge-fixing condition is defined. Especially, if we work with the model with globally defined gauge-fixing condition (free of Gribov's problem) our construction will yield globally defined canonical coordinates.

To illustrate the above discussion we now consider a specific case, the Virasoro algebra.

The generators of this algebra in the classical case are defined in terms of the harmonic oscillators $\{\alpha_n^\mu, \alpha_m^\nu\} = in\delta(n+m)\eta^{\mu\nu}$ and $\alpha_0^\mu = p^\mu/\sqrt{2}$, with p^μ the center-of-mass momentum of the string and $\eta^{\mu\nu} = \mathrm{diag}(-+++\ldots)$.

The convention we use in this section is that the indices $(a,b,c,\ldots)$ from the previous section go over to $(n,m,k,\ldots)$, while the dual indices $(i,j,k,\ldots)$ become $(r,s,t,\ldots)$. Elements of both sets $(n,m,k,\ldots)$ and $(r,s,t,\ldots)$ are arbitrary integers.

The basic quantity defined in terms of the above harmonic oscillators is the Fubini-Veneziano (FV) coordinate:

$$Q_\mu(z) = \sqrt{2}x_\mu - i\frac{1}{\sqrt{2}}p_\mu \ln z + i\sum_{n=1}^{\infty}\left(\frac{\alpha_{\mu\,n} z^{-n}}{n} - \frac{\alpha^\dagger{}_{\mu\,n} z^n}{n}\right) \tag{34}$$

where x^μ is the center-of-mass position of the string. Introducing, furthermore a null-vector k^μ , $k\cdot k = 0$, satisfying the light-cone condition $k\cdot p \neq 0$ we next define

$$Q(z) = \frac{\sqrt{2}}{k\cdot p}\, k^\mu\, Q_\mu(z) \tag{35}$$

We will also need a slightly shifted $Q(z)$ defined as $Q'(z) \equiv Q(z) + i\ln z$. Moreover from FV position variable we can construct the string momenta

$$P_\mu(z) = iz\frac{d}{dz}Q_\mu(z) \qquad ; \qquad P(z) = iz\frac{d}{dz}Q(z) \tag{36}$$

We will also need another null-vector $\bar{p}^\mu$ besides k^μ, to complete description of the light-cone directions

$$\bar{p}^\mu \equiv p^\mu - \frac{1}{2}p^2\frac{k^\mu}{k\cdot p} \tag{37}$$

with the following properties

$$\bar{p}^2 = 0 \quad ; \quad \bar{p}\cdot k = p\cdot k \neq 0 \quad ; \quad \bar{p}^\mu p_\mu = \frac{1}{2}p^2 \tag{38}$$

With this introductory information we can now establish dictionary between the quantities defined above in connection with the abstract extended constraint algebra (1-2) and corresponding quantities associated with the Virasoro algebra

$$\begin{array}{rcl}
\phi_a & \longrightarrow & L_n = \frac{1}{2}\sum_m \alpha_m^\mu \alpha_{n-m}^\nu \eta_{\mu\nu} \\
Q^b & \longrightarrow & \text{modes of the FV coordinate } Q'(z) \\
 & & Q^m = \frac{\sqrt{2}}{k\cdot p}\oint \frac{dz}{2\pi i z}\, z^m\, Q'(z) \\
U_{ab}^c & \longrightarrow & U_{nm}^p = i(n-m)\delta(n+m-p) \\
T_{ab}^c & \longrightarrow & T_{nm}^p = -im\delta(n+m-p) \\
D_a^b & \longrightarrow & D_n^m = D(m-n) = \oint \frac{dz}{2\pi i z}\, z^{n-m}\, P(z) \\
(D^{-1})_a^b & \longrightarrow & (D^{-1})_n^m = (D^{-1})(m-n) = \oint \frac{dz}{2\pi i z}\, z^{n-m}\, P^{-1}(z)
\end{array}$$

$$
\begin{array}{rcl}
P_a & \longrightarrow & P_n = \dfrac{1}{\sqrt{2}} \oint \dfrac{dz}{2\pi i z}\, z^n\, \bar{p}^\mu\, P_\mu(z) \\
D_a = D_a^b P_b & \longrightarrow & D_n = D_n^m P_m = L_n^{long} \\
e_i^a & \longrightarrow & \text{modes of string vertex } V(rk,z) = \exp\left(irQ(z)\right) \\
 & & e_r^n = \oint \dfrac{dz}{2\pi i z}\, z^{-n}\, V(rk,z) \\
g^{ab} = e_i^a e_i^b & \longrightarrow & e_r^n e_{-r}^{-m} = (D^{-1})_n^m \\
g_{ab} = E_a^i E_b^i & \longrightarrow & E_{-n}^r E_m^{-r} = D_n^m \\
Q^i & \longrightarrow & Q^r = E_m^r Q^{-m} \\
P_i & \longrightarrow & P_r = e_r^n \mathcal{D}_n = e_r^n D_n^m P_m \\
\mathcal{D}_i & \longrightarrow & \text{longitudinal part of the photon vertex} \\
 & & \mathcal{D}_r = P_n e_r^n = D_r^s P_s \\
\Phi_i & \longrightarrow & \Phi_r = e_r^n L_n = P_r + e_r^n L_n^{tr} \\
D_j^i & \longrightarrow & D_r^s = \delta(r-s) - i(s-r) Q^{r-s} \\
\hat{\phi}_i & \longrightarrow & \hat{L}_r = D_r^s \Phi_s \\
\Phi_a & \longrightarrow & \Phi_n = E_n^r \hat{L}_r = (D^{-1})_n^m L_n
\end{array}
$$

We now offer few comments on the above dictionary. The extended Virasoro algebra

$$
\begin{aligned}
\{L_n, L_m\} &= U_{nm}^k L_k && (39)\\
\{L_n, Q^m\} &= \delta(n+m) - T_{nk}^{-m} Q^{-k} = D_n^{-m} && (40)\\
\{Q^n, Q^m\} &= 0 && (41)
\end{aligned}
$$

with the structure constants defined above, is the algebraic version of the light-cone quantization relying on the condition $k \cdot p \neq 0$.

It is interesting to note that since position Q^m and momentum P_n are confined to the light-cone directions $\bar{p}^\mu$ and k^μ the non-abelian derivative $\mathcal{D}_n$ reproduces the longitudinal part of the Virasoro operator[7]

$$
\mathcal{D}_n = P_n - T_{np}^m Q^{-p} P_m = \frac{1}{2} \sum_m \alpha_m^\mu \alpha_{n-m}^\nu \Pi_{\mu\nu} \tag{42}
$$

where

$$
\Pi^{\mu\nu} = \frac{1}{k \cdot p} \left(k^\mu \bar{p}^\nu + \bar{p}^\mu k^\nu \right) \tag{43}
$$

is a projection operator onto the longitudinal directions on the k–p plane, since $\Pi_\mu^\nu k_\nu = k_\mu$ and $\Pi_\mu^\nu p_\nu = p_\mu$. Hence we recognize in $\mathcal{D}_n$ the two-dimensional Virasoro operators, generating diffeomorphisms on the k-p plane.

Similar remarks apply to the dual counterpart of D_n, which can be rewritten as difference between the dual Virasoro operator $\hat{L}_r$ and its transverse part

$$
\mathcal{D}_r = \hat{L}_r - \hat{L}_r^{tr} = \hat{L}_r - \frac{1}{2} \sum_{s=-\infty}^{\infty} A^i(r-s) A^i(s) \tag{44}
$$

where the transverse part was written in terms of the transverse DDF operators defined as

$$
A^i(r) \equiv \sum_{n=-\infty}^{\infty} \epsilon^i \cdot \alpha_n\, e_r^n \tag{45}
$$

with the polarization vectors ϵ^i_μ describing the transverse directions orthogonal to k–p plane.

$$k \cdot \epsilon^i = p \cdot \epsilon^i = 0 \quad ; \quad \sum_{i=1}^{d-2} \epsilon^i_\mu \, \epsilon^i_\nu = \epsilon_{\mu\nu} \quad ; \quad \epsilon^i_\mu \epsilon^j_\nu \eta^{\mu\nu} = \delta^{ij} \tag{46}$$

Since $\epsilon^i \cdot k = 0$ the DDF operators commute with all longitudinal operators being functions of the coordinate Q^n, like vielbeins and etc.

The basic properties of the DDF operators involve the relation

$$\{A^i(r), A^j(s)\} = ir\delta^{ij}\delta(r+s) \tag{47}$$

and

$$\{L_n, A^i(r)\} = T^k_{nm}\epsilon^i \cdot \alpha_k e^m_r - T^m_{nk}\epsilon^i \cdot \alpha_m e^k_r = 0 \tag{48}$$

The fact that the DDF operators commute with the Virasoro operators led to their widespread use to define the physical states in the old dual models[8,9,10].

Let us now turn our attention to the pair of dual canonical variables (Φ_r, Q^s) and (Φ_n, Q^m) both satisfying simple Heisenberg algebras corresponding to (28) and (31). It appears that $\Phi_n = E^r_n \hat{L}_r$ is an extension of the "abelian derivative" $P_n = E^r_n \mathcal{D}_r$ to include the transverse directions . This inclusion is not changing the bracket relations with longitudinal quantities depending on Q^m but ensures commutativity with the transverse DDF operators. Note, that exactly the same can be said about $\Phi_r = e^n_r L_n$ being extension of $P_r = e^n_r \mathcal{D}_n$ and its relation to DDF operators.

Let us now consider the Poisson brackets of abelian derivatives with dual Virasoro operators.

$$\{L_n, \Phi_m\} = \{D^k_n \Phi_k, \Phi_m\} = T^k_{nm}\Phi_k \tag{49}$$

$$\{\hat{L}_r, \Phi_s\} = -T^t_{rs}\Phi_t \tag{50}$$

leading to the interesting results:

$$\{L_n, \Phi_{m=0}\} = 0 \quad ; \quad \{\hat{L}_r, \Phi_{s=0}\} = 0 \tag{51}$$

for any n and r. Substituting $E^r_0 = \delta(r)$ and $\mathcal{D}_{r=0} = \frac{1}{4}p^2$ we find therefore that

$$\Phi_{m=0} = D^{-1}(n)L_n = E^{-r}_0 \left(\mathcal{D}_r + \hat{L}^{tr}_r\right) = \frac{1}{4}p^2 + \frac{1}{2}\sum_{s=-\infty}^{\infty} A^i(-s)A^i(s) \tag{52}$$

commutes with all Virasoro operators and is in fact the Casimir operator[11] of the Virasoro algebra. It turns out to be a classical analog of the Brink-Olive's[12] operator E (here shifted by L_0). Similarly $\Phi_{r=0}$ can be easily recognized as

$$\Phi_{r=0} = e^n_{r=0}\, L_n = L_0 \tag{53}$$

due to[6] $e^n_0 = \delta(n)$.

SYMPLECTIC COORDINATES AND KUGO-OJIMA QUARTETS

We show in this section how to extend the above geometric construction to the quantum case and then we turn to investigation of its physical applications. Two related technical issues have to be addressed in this connection; one is how to deal with the normal ordering and another is how to add the ghosts in order to take care of the anomaly terms.

Solution to both technical problems turns out to be provided by study of the quantum BRS operator:

$$
\begin{aligned}
\Omega &= \; :\eta^{-n}\left[(L_n - \delta_{n,0}) - \eta^{-m}T^k_{nm}\mathcal{P}_k\right]: \\
&= \; :\eta^{-n}\left[(L^{tr}_n - \delta_{n,0}) + P_n - Q^{-m}T^k_{nm}P_k - \eta^{-m}T^k_{nm}\mathcal{P}_k\right]:
\end{aligned} \tag{54}
$$

where we have introduced the BFV ghosts $(\eta^n, \mathcal{P}_n)$ and used the Jacobi relation $U^k_{nm} = T^k_{nm} - T^k_{mn}$ to reveal symmetry between the bosonic abelian operators (P_n, Q^n) associated with the longitudinal directions and the ghosts. This hints that the Parisi-Sourlas supersymmetry can be used to provide an understanding of the structure of the BRS cohomology classes via cancellation between the unphysical modes.

The right framework to exhibit this supersymmetry is obtained by rewriting the BRS formalism in terms of symplectic variables, both bosonic as well as Grassmannian. The first step is to transform BFV ghosts to the dual space labeled by $(r, s, \ldots)$ indices by acting on them with vielbeins

$$
\mathcal{P}_r \equiv e^n_r \mathcal{P}_n \qquad ; \qquad \eta^r \equiv E^r_n \eta^{-n} \tag{55}
$$

Here vielbeins satisfy the MC equations defined in terms of structure constants written this time in the quantum mechanical convention (and not classical convention of the last section[6,13]).

Remarkably the symplectic coordinates constitute naturally the KO quartet[2]. Our construction of quartets will inherit duality from the geometric discussion of the previous sections. We will embed our first quartet construction in the dual space[6,13]. Explicitly, the quartet is given by

$$
\begin{aligned}
[\Omega, Q^r] &= \eta^r & \{\Omega, \eta^r\} &= 0 \\
\{\Omega, \mathcal{P}_r\} &\equiv \tilde{\Phi}_r & [\Omega, \tilde{\Phi}_r] &= 0 \\
[\Omega, A^i(r)] &= 0 & (i &= 1, \ldots, D-2)
\end{aligned} \tag{56}
$$

Where we included the transverse DDF operators[8] $A^i(r) \equiv \sum_{n=-\infty}^{\infty} \epsilon^i \cdot \alpha_n e^n_r$, having the same form as in (45). Note, that since ϵ^i is orthogonal to k, normal ordering is unnecessary in the definition of $A^i(r)$. As before the DDF operators commute both with the vielbeins and the Virasoro operators and consequently with the rest of the quartet. One finds[6] easily that the DDF operators satisfy the quantum version of algebra (47):

$$
[A^i(r), A^j(s)] = r\delta^{ij}\delta(r+s) \tag{57}
$$

One finds that $\tilde{\Phi}_r$ is explicitly given by

$$
\tilde{\Phi}_r = (\mathcal{L}_n - \delta_{n,0})\, e^n_r + \mathcal{P}_n \eta^{-m} T^n_{mk} e^k_r \tag{58}
$$

where $\mathcal{L}_n - \delta_{n,0} \equiv \{\Omega, \mathcal{P}_n\}$ are the modified BRS-exact Virasoro operators (Lie derivatives in the mathematical language), which satisfy the centerless algebra for the nilpotent Ω. It is interesting to note that the fermionic members η^r and $\mathcal{P}_r$ appear also in the abelization procedure of Batalin and Fradkin[14,6].

We can introduce more economical expression for $\tilde{\Phi}_r$ by introducing the normal ordering of the first and second term in (58) while taking into account

$$
\begin{aligned}
L_n e^n_r &= e^n_r L_n - T^n_{nk} e^k_r \\
\mathcal{P}_n \eta^{-m} T^n_{mk} e^k_r &= -\eta^{-m}\mathcal{P}_n T^n_{mk} e^k_r + T^n_{nk} e^k_r
\end{aligned} \tag{59}
$$

Clearly, the constants on the right hand sides of the above equations cancel and we arrive at

$$\tilde{\Phi}_r =: \left((L_n - \delta_{n,0}) - \eta^{-m}\mathcal{P}_k T^k_{nm}\right) e^n_r : \tag{60}$$

Remarkably, this differs only by the normal ordering from the classical expression $\Phi_r + \eta^{-s}\{\Phi_r, \mathcal{P}_s\}$ for the "pure-gauge" abelian operator[6].

By definition $\tilde{\Phi}_r$ is also the BRS-exact operator and furthermore we have

$$[\tilde{\Phi}_r, \tilde{\Phi}_s] = \{\Omega, [\tilde{\Phi}_r, \mathcal{P}_s]\} = 0 \tag{61}$$

since $[\tilde{\Phi}_r, \mathcal{P}_s] = -T^n_{mk} e^k_s \mathcal{P}_n e^m_r + T^n_{mk} e^k_s \mathcal{P}_n e^m_r = 0$ due to the MC equations. One can easily show that $[\tilde{\Phi}_r, \eta^s] = 0$ and

$$[\tilde{\Phi}_r, Q^s] = \delta(r+s) \tag{62}$$

Hence the Heisenberg algebra originally introduced in our study of the extended constraint algebra (1-2) has maintained its form under passage to the quantum mechanical framework, with the BFV ghosts and normal ordering, presented in this section.

Similarly we can find the way of generalizing the Heisenberg algebra (31). This will amount to constructing the dual KO quartet. Obviously we have to start with the original gauge-fixing operator Q^n and proceed as follows:

$$\begin{array}{rclr} [\Omega, Q^n] & = & D^{-n}_m \eta^{-m} \equiv \hat{\eta}^n & \{\Omega, \hat{\eta}^n\} = 0 \\ \{\Omega, \hat{\mathcal{P}}_n\} & \equiv & \tilde{\Phi}_n & [\Omega, \tilde{\Phi}_n] = 0 \end{array} \tag{63}$$

where we have defined the variable conjugated to $\hat{\eta}^n$ as

$$\hat{\mathcal{P}}_n \equiv (D^{-1})^k_n \mathcal{P}_k \tag{64}$$

clearly, $\{\hat{\mathcal{P}}_n, \hat{\eta}^m\} = \delta(n+m)$. This definition of $\hat{\mathcal{P}}_n$ leads to the following expression

$$\tilde{\Phi}_n = (D^{-1})^k_n(\mathcal{L}_k - \delta_{k,0}) + T^l_{mn}(D^{-1})^k_l \eta^{-m}\mathcal{P}_k - U^k_{ml}(D^{-1})^l_n \eta^{-m}\mathcal{P}_k \tag{65}$$

Using the similar considerations as in (59) we arrive in the more simple normal ordered expression

$$\tilde{\Phi}_n =: (D^{-1})^k_n(L_k - \delta_{k,0}) + T^l_{mn}(D^{-1})^k_l \eta^{-m}\mathcal{P}_k : \tag{66}$$

where in the first term we recognize the normal ordered version of Φ_n from (32). Hence again our construction yields the ghost and quantum extension of the basic quantities from the first two sections.

It is straightforward to verify that

$$[\tilde{\Phi}_n, \tilde{\Phi}_m] = 0 \quad ; \quad [\tilde{\Phi}_n, \hat{\mathcal{P}}_m] = 0 \quad ; \quad [\tilde{\Phi}_n, \hat{\eta}^m] = 0 \tag{67}$$

and of course $[\tilde{\Phi}_n, Q^m] = \delta(n+m)$. The first of identities in (67) follows from $[\tilde{\Phi}_n, \tilde{\Phi}_m] = \{\Omega, [\tilde{\Phi}_n, \hat{\mathcal{P}}_m]\}$, which is a consequence of nilpotency of Ω and the technical identity

$$U^k_{pl}(D^{-1})^p_m(D^{-1})^l_n + T^l_{pm}(D^{-1})^k_l(D^{-1})^p_n - T^l_{pn}(D^{-1})^k_l(D^{-1})^p_m = 0 \tag{68}$$

For the special case $n = 0$ we obtain from (66)

$$\tilde{\Phi}_{m=0} =: (D^{-1})(k)(L_k - \delta_{k,0}) : \tag{69}$$

which, of course must satisfy $[\tilde{\Phi}_n, \tilde{\Phi}_{m=0}] = 0$. Together with the above technical identity (68) this implies $[L_n, \tilde{\Phi}_{m=0}] = 0$ for the nilpotent BRS operator.

This can be also proved independently. Starting with the basic definition (69) and taking care of the normal ordering one can prove the familiar result[11,12] :

$$[L_n, \tilde{\Phi}_0] = \frac{D-26}{12}(n^3 - n)D^{-1}(-n) \tag{70}$$

Hence for the critical dimension the operator $\tilde{\Phi}_0$ commutes with all Virasoro operators as did the classical operator Φ_0 in (51).

QUARTET MECHANISM AND SUSY QUANTUM MECHANICS

It is now easy to adapt the quartet mechanism using operators introduced above. Our discussion here is inspired by work of M. Henneaux[15]. Clearly, our construction suggests existence of two dual quartets containing $(\tilde{\Phi}_r, Q^r, \eta^r, \mathcal{P}_r)$ and $(\tilde{\Phi}_n, Q^n, \hat{\eta}^n, \hat{\mathcal{P}}_n)$ modes. For the quartet in the dual space labelled by r, we now introduce a "conjugated" BRS charge

$$\Omega^\dagger \equiv Q^r \mathcal{P}_{-r} \tag{71}$$

where "conjugation" is used from the point of view of symplectic geometry and not hermiticity.

Clearly $\Omega^\dagger$ is itself nilpotent and its anticommutator with Ω yields according to (56)

$$\begin{aligned} \{\Omega, \Omega^\dagger\} &= Q^r \tilde{\Phi}_{-r} + \eta^r \mathcal{P}_{-r} \\ &= \,:Q^r \tilde{\Phi}_{-r}: + :\eta^r \mathcal{P}_{-r}: \equiv \mathcal{N} \end{aligned} \tag{72}$$

where the "Hermitian" (in the sense of the above conjugation) operator $\mathcal{N}$ counts the modes of ghosts as well as unphysical modes associated with excitations generated by Q^r and $\tilde{\Phi}_{-r}$. In fact $\mathcal{N}$ is an extension of the conventional ghost number operator to include these extra modes.

Let us now consider the physical state $|\psi\rangle$, which is also an eigenstate of $\mathcal{N}$ with non-zero eigenvalue n:

$$\Omega\,|\psi\rangle = 0 \tag{73}$$

$$\mathcal{N}\,|\psi\rangle = n\,|\psi\rangle \tag{74}$$

A simple calculation shows that $\mathcal{N}\Omega^\dagger\,|\psi\rangle = \Omega^\dagger\mathcal{N}\,|\psi\rangle = n\,\Omega^\dagger\,|\psi\rangle$. This result leads to interpretation of $\Omega^\dagger\,|\psi\rangle$ as a supersymmetric partner of $|\psi\rangle$ with the same n eigenvalue. Hence we have a doublet

$$|\psi\rangle \;\underset{\xleftarrow[\Omega]{}}{\xrightarrow{\Omega^\dagger}}\; \Omega^\dagger\,|\psi\rangle \tag{75}$$

associated with each non-zero eigenvalue of $\mathcal{N}$.

A proverbial "alert reader" has at that point already recognized analogy with the supersymmetric quantum mechanics[16]. Origin of this analogy is the supersymmetric algebra satisfied by Ω, $\Omega^\dagger$ and $\mathcal{N}$:

$$\begin{aligned} \{\Omega, \Omega\} = \{\Omega^\dagger, \Omega^\dagger\} &= 0 \\ [\Omega, \mathcal{N}] = [\Omega^\dagger, \mathcal{N}] &= 0 \\ \{\Omega, \Omega^\dagger\} &= \mathcal{N} \end{aligned} \tag{76}$$

Accordingly Ω and $\Omega^\dagger$ play the role of the supersymmetry generators and $\mathcal{N}$ that of Hamiltonian.

As the elementary supersymmetric quantum mechanics teaches us, we will have pairing of the states corresponding to the non-zero eigenvalues of $\mathcal{N}$ with exception of zero eigenvalue. However, since $\mathcal{N}$ is a BRS-exact operator it follows, that

$$\Omega\,\Omega^{\dagger}\,|\psi\rangle = \mathcal{N}\,|\psi\rangle = n\,|\psi\rangle \tag{77}$$

and for $n \neq 0$ the state $|\psi\rangle$ is BRS-exact. Clearly, $\Omega^{\dagger}$ plays in this construction a role of the contracting homotopy operator. We have therefore understood the essence of the Kugo-Ojima quartet mechanism as the construction of the contracting homotopy. This not only represents an abstract refinement of the existing method but more important provides a practical extension of the cohomology derivation to the models presently known only in the non-abelian version[17].

Consequently, the physical states can be decomposed into two sectors. One will contain the transverse DDF states. Another sector wil be associated with unphysical members of the quartet $(\tilde{\Phi}_r, Q^r, \eta^r, \mathcal{P}_r)$, which according to the above discussion constitute the zero-norm BRS-exact states. This decomposition expresses compactly the essence of the no-ghost theorem[9]:

$$|\mathrm{phys}\rangle = |\mathrm{DDF}\rangle + \Omega\,|\mathrm{something}\rangle \tag{78}$$

Similar construction holds for the dual quartet $(\tilde{\hat{\Phi}}_n, Q^n, \hat{\eta}^n, \hat{\mathcal{P}}_n)$.

Clearly the method works only, where our gauge-fixing condition can be imposed. The case $p = 0$ for which we can not impose $k \cdot p \neq 0$ needs therefore to be considered separately[18].

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REFERENCES

1. E.S. Fradkin and G.A. Vilkovisky, Phys. Lett. 55B:224 (1975), and preprint CERN-TH 2332 (1977); I.A. Batalin and G.A. Vilkovisky, Phys. Lett. 69B:309 (1977); M. Henneaux, Phys. Reports 126:1 (1985).
2. T. Kugo and I. Ojima, Suppl. Progr. Theor. Phys. 66:1 (1979).
3. M. Kato and K. Ogawa, Nucl. Phys. B212:443 (1983).
4. H. Aratyn and R. Ingermanson, Nucl. Phys. B299:507 (1988).
5. H. Aratyn, R. Ingermanson and A.J. Niemi, Phys. Lett. 194B:506 (1987); *ibid* 195B:149 (1987); Nucl. Phys. B307:157 (1988).
6. H. Aratyn and R. Ingermanson, Phys. Rev. Lett. 61:2050 (1988); Int. J. Mod. Phys. **A** (review sect.) to appear.
7. H. Aratyn and R. Ingermanson, Class. Quantum Grav. 5:L213 (1988).
8. E. Del Giudice, P. Di Vecchia and S. Fubini, Ann. Phys.(N.Y.) 70:378 (1972).
9. P. Goddard and C.B. Thorn, Phys.Lett. 40B:235 (1972).

10. R.C. Brower, Phys. Rev. D6:1655 (1972); R.C. Brower and P. Goddard, Nucl. Phys. B40:437 (1972); see also J.H. Schwarz, Phys. Repts. 8:269 (1973).

11. A. Pressley and G. Segal, "Loop Groups" Clarendon, Oxford (1986); H. Neuberger, Phys. Lett. 188B:214 (1987).

12. L. Brink and D. Olive, Nucl. Phys. B56:253 (1973).

13. H. Aratyn, to appear in Class. Quantum Grav. and UIC preprint, UICHEP-Pub-89-9.

14. I.A. Batalin and E.S. Fradkin, Nucl. Phys. B279:514 (1987).

15. M. Henneaux, in: "Quantum Mechanics of Fundamental Systems", C. Teitelboim, ed., Plenum Press, New York (1988); Duality Theorems in BRST Cohomology, Univ. Libre de Bruxelles preprint.

16. E. Witten, Nucl. Phys. B188:513 (1981); B207:253 (1982).

17. D. Karabali, contribution to these proceedings and references therein.

18. M. Henneaux, Phys. Lett. 177B:35 (1986).

PROGRESS IN MULTI-GENUS CALCULATIONS FOR THE SPINNING STRING

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ABSTRACT. We review progress in the calculation of multiloop amplitudes for the closed Neveu-Schwarz-Ramond string in a flat ten dimensional background. When the picture changing operators are placed at the zeroes of a holomorphic one-form then it turns out to be possible to do the sum over spin structures for genus $g \leq 9$ for the vacuum amplitude. The result of this sum is zero pointwise in moduli space and not just a total derivative as might be expected in a more general gauge. We look at the possibility that singularities could arise for certain values of the moduli and also discuss why this gauge choice is so powerful by showing the similarity to "light-cone" diagrams.

INTRODUCTION

The demonstration of the vanishing of the one loop vacuum amplitude in the GSO projected version of the NSR string is well known and was at the root of most claims for finiteness of the perturbation theory [1,2]. In outline what one has to do is evaluate the product of various spinor determinants on the torus. The spinors involved are matter fields ψ^{μ} and the ghost fields β,γ. These spinors have different conformal weights but at genus one this does not affect the determinants and so the ghost determinants are just inverses of the matter determinants. Thus the ghost determinants cancel two of the ten matter determinants. Each determinant is the square root of a theta function so the product of spinor determinants is of the form $\Theta^5\Theta^{-1}$ which reduces to Θ^4. The last step is to do the sum over different spin structures with the relative signs between different terms fixed so as to give the correct spectrum and respect modular invariance. For this one observes that the Riemann identities are nothing other than sums of theta functions to precisely the fourth power and contain the correct combinations of signs. The simplest of these identities is then sufficient to do the spin structure sum and show

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that it vanishes. What has been achieved recently is to show that it is possible to set up the calculation of some multi-loop amplitudes in such a way that many of these simple features of the one loop calculation are retained.

The immediate complication for genus $g > 1$ is that the presence of zero modes and supermoduli forces one to calculate not determinants but correlation functions of $2(g-1)$ picture changing operators [3,4,5]. One is free to place these operator insertions wherever one wishes on the surface. For a generic choice one obtains expressions of the form $\Theta^{n+4}\Theta^{-n}$ but the arguments of the theta functions do not allow simplification to Θ^4. Hence the general expressions cannot be summed using the Riemann identities and indeed there does not yet appear to be a way of doing the spin structure sum in general. Even if the sum could be performed one would not expect to obtain zero because general arguments [5,6,7,8,9,10] show that the answer only needs to be a total derivative on moduli space (but see also [11]). Here we shall describe how a judicious choice of the positioning of the operator insertions leads to some results that are very similar in form to the one loop result. In particular the β,γ contributions often cancel two of the ten spacetime spinors to yield Θ^4 terms which can then be summed using the generalised Riemann identities.

In the next section we describe the gauge choice made and how it leads to great simplifications for $g \leq 9$. In the rest of the article we describe what happens to the gauge choice as the moduli vary i.e. start to consider some global properties of the gauge rather than just looking pointwise in moduli space. The main point is that it is inevitable that for some values of the moduli the insertion points will collide and so it is necessary to consider this limiting case and also the chance of "spurious" singularities. Finally we will give a heuristic reason for why this gauge is so close to the one loop results by looking at the surface in a metric corresponding to the choice of operator insertion points. It will turn out to be a distorted light-cone diagram.

REVIEW OF THE CALCULATIONS PROCEDURES AND RESULTS

This is a very condensed version of what is contained mostly in [12,13]. and the reader is referred to these for fuller details. We assume familiarity with the language of theta functions and prime form as given, for example, in [14,15,16] and with the picture changing formalism as given in [3]. For simplicity we will only look at vacuum diagrams and so only need to consider the even spin structures. After holomorphic splitting with respect to the moduli [17,5] and integration over the supermoduli using the method of [5] the quantity that we have to calculate is the following correlation function on the Riemann surface Σ_g.

$$G(z_a) = \sum_{\delta} \left\langle \xi(x) \prod_{a=1}^{2(g-1)} Y(z_a) \prod_i b(w_i) \right\rangle_{\delta} \tag{1}$$

The picture changing operator is

$$Y \;=\; Y_0 + Y_1 + Y_2 \tag{2}$$

where

$$Y_0 = c\partial\xi \quad , \quad Y_1 = -\tfrac{1}{2}e^{\phi}\psi^{\mu}\partial X_{\mu} \quad , \quad Y_2 = -\tfrac{1}{4}[\partial\eta\, e^{2\phi}b + \partial(\eta\, e^{2\phi}b)] \tag{3}$$

Only Y_1 contains matter fields so we call the terms in G containing only Y_1 the "pure matter" terms, all others we call ghost contributions.

We will also use the following useful definition [18]

$$Z_{\delta,\lambda}(\Sigma_i q_i z_i) := Z_1^{-\frac{1}{2}}\, \Theta_\delta(\Sigma_i q_i \vec{z}_i - Q\vec{\Delta}) \prod_{i<j} E(z_i, z_j)^{q_i q_j} \prod_i \sigma(z_i)^{Q q_i} \tag{4}$$

where $Q = 2\lambda - 1$. The result given in [5] for the correlation functions of the β,γ system may then be written as

$$\left\langle \prod_{i=0}^{n} \xi(x_i) \prod_{j=1}^{n} \eta(y_j) \prod_k e^{q_k\phi}(z_k) \right\rangle_\delta^{\beta\gamma} = \frac{\prod_{j=1}^{n} Z_{\delta,\frac{3}{2}}(-y_j + \Sigma x - \Sigma y + \Sigma qz)}{\prod_{i=0}^{n} Z_{\delta,\frac{3}{2}}(-x_i + \Sigma x - \Sigma y + \Sigma qz)} \tag{5}$$

An important tool is the Riemann identity

$$\sum_\delta \langle\alpha|\delta\rangle\; Z_\delta(x_1)\, Z_\delta(x_2)\, Z_\delta(x_3)\, Z_\delta(x_4) \;=\; 2^g\, Z_\alpha(x_1')\, Z_\alpha(x_2')\, Z_\alpha(x_3')\, Z_\alpha(x_4') \tag{6}$$

where the transformation of arguments is given by

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad . \tag{7}$$

In the more familiar version of the identity the Z_δ are simply replaced by Θ_δ. The terms in the sum of (6) are weighted with phases

$$\langle\alpha|\delta\rangle \;:=\; \exp\{4\pi i(\vec{\delta}_1\cdot\vec{\alpha}_2 - \vec{\delta}_2\cdot\vec{\alpha}_1)\} \;=\; \pm 1 \tag{8}$$

where α is some arbitrary spin structure.

Using the above and the standard results for the correlation functions of the ψ system then the pure matter term at $g = 2$ is

$$G^{11}(z_1, z_2) = \sum_\delta \left\langle \xi(x) Y_1(z_1) Y_1(z_2) \prod b(w_i) \right\rangle_\delta = \sum_\delta \frac{\Theta_\delta(0)^4\, \Theta_\delta(\vec{z}_1 - \vec{z}_2)}{\Theta_\delta(\vec{z}_1 + \vec{z}_2 - 2\vec{\Delta})} \cdots \tag{9}$$

where the omitted terms are all independent of the spin structure. In order to do the spin structure sum the theta function in the denominator must cancel something in the numerator and this leads us to set

$$\vec{z}_1 + \vec{z}_2 = 2\vec{\Delta} \tag{10}$$

The points $\vec{z}_a$ lie in the Jacobian which has complex dimension g and the image of the Riemann surface is only one dimensional so an arbitrary condition on the $\vec{z}_a$ would not be consistent with the fact that the z_a are points on the Riemann surface. However (10) is precisely the condition that the z_a are the zeroes on the Riemann surface of a holomorphic one form on the surface. It is remarkable that the simplest useful condition on the $\vec{z}_a$ should also have such a simple geometric interpretation. Given (10) then G^{11} simplifies as follows (omitting irrelevant terms)

$$G^{11} \approx \sum_{\delta} \Theta_\delta(0)^3 \Theta_\delta(\vec{z}_1 - \vec{z}_2) = 4\Theta_0(\frac{\vec{z}_1 - \vec{z}_2}{2})^4 = 4\Theta_0(-\vec{\Delta} + \vec{z}_1)^4 = 0 \qquad (11)$$

Where we have used the Riemann identity to do the sum, then the condition (10), and finally the Riemann vanishing theorem. The calculation in this form first appeared in [12]. To deal with $g > 2$ we keep the same condition namely that the points z_a lie at the zeroes of an unspecified holomorphic one form Ω. This gives

$$\sum_{a=1}^{2(g-1)} \vec{z}_a = 2\vec{\Delta} \qquad (12)$$

as the condition that we will use in the rest of this article. We also make repeated use of

$$\Theta_0\left(\tfrac{1}{2}[\sum_{a=1}^{g-1} \vec{z}_a - \sum_{a=g}^{2(g-1)} \vec{z}_a]\right) = \Theta_0\left(-\vec{\Delta} + \sum_{a=1}^{g-1} \vec{z}_a\right) = 0 \qquad (13)$$

which follows from the Riemann vanishing theorem. At $g = 3$ and $g = 4$ there are no problems and vanishing of the pure matter contributions follows similarly to the $g = 2$ case. At $g = 5$ there are now ten operator insertions and so a term such as

$$\sum_{\text{even } \delta} \frac{\Theta_\delta(\vec{z}_1 - \vec{z}_2)\Theta_\delta(\vec{z}_3 - \vec{z}_4)\Theta_\delta(\vec{z}_5 - \vec{z}_6)\Theta_\delta(\vec{z}_7 - \vec{z}_8)\Theta_\delta(\vec{z}_9 - \vec{z}_{10})}{\Theta_\delta(\vec{0})} \qquad (14)$$

can arise. This term initially appears to be a problem because there are no theta functions of zero argument in the numerator and so it does not look as though we can do the spin structure sum. However the situation is saved here because the trisecant identity of Fay [15,14] can be used to show that for even spin structures

$$Z_\delta(z_1-z_2)Z_\delta(z_3-z_4) = \tfrac{1}{2}Z_\delta(0)\ [Z_\delta(z_1-z_2+z_3-z_4) - Z_\delta(z_1+z_2-z_3-z_4) + Z_\delta(z_1-z_2-z_3+z_4)] \qquad (15)$$

Hence we can combine two terms in the numerator of (14) to recover a $\Theta_\delta(\vec{0})$ and so cancel the theta function from the ghosts. It is then straightforward to show that all the terms vanish. The cases of $g = 6, 7$ work in a similar way in that use of (15) always allows us to cancel the denominator, do the spin structure sum and obtain an answer of zero.

At $g = 8$ there are 14 insertion points to be divided amongst the 5 theta functions in the numerator and one such splitting is [4,4,4,2,0]. (We are using an obvious notation in which (14) is called a [2,2,2,2,2] split). In fact after repeated use of (15) this is clearly the worst term that can arise. There is still a $\Theta_\delta(\vec{0})$ in the numerator so that the ghost theta functions still cancel, however when we come to do the sum using the Riemann identity it only replaces a sum over the even spin structures with one over the odd spin structures. In this case it turns out that one can argue from their different symmetries under interchange of insertion points that the sums must vanish. One can easily obtain a Riemann identity which contains a sum over only the even spin structures by symmetrizing it with respect to the argument of one of the theta functions (so that the odd terms drop out). This is sufficient to treat the $g =$ 8,9 cases.

At $g = 10$ our luck runs out because terms such as

$$\sum_{\text{even }\delta} \frac{\Theta_\delta(\vec{z}_1 - \vec{z}_2 + \vec{z}_3 - \vec{z}_4)\ldots\Theta_\delta(\vec{z}_{13} - \vec{z}_{14} + \vec{z}_{15} - \vec{z}_{16})\Theta_\delta(\vec{z}_{17} - \vec{z}_{18})}{\Theta_\delta(\vec{0})} \tag{16}$$

arise from Wick contracting the spinor fields in the product of 18 picture changing insertions. We do not yet know of any way to simplify this term (though it is possible that the most general form of Fays' identity [15] could prove useful) and so this term does not even seem to reduce to a form where (6) could be useful. Currently such terms block our attempt to go to higher genus. It is not surprising that such a block should happen eventually because all progress this far has relied on showing that all the terms vanish individually. We have not had to cancel any two or more terms against each other and perhaps is inevitable that we will need to consider such cancellations. Of course we cannot rule out the possibility that, even in the gauge (12), the vacuum amplitude is no longer zero beyond $g = 9$.

To be complete we must also deal with the ghost terms i.e. terms containing Y_0 or Y_2. At genus two this means we have to look at

$$\left\langle \xi(x) Y_0(z_1) Y_2(z_2) \prod_{i=1}^{3} b(w_i) \right\rangle$$

whose β,γ piece is

$$\left\langle \xi(x)\,\partial\xi(z_1)\,\eta(z_2')\,e^{2\phi}(z_2)\right\rangle_\delta^{\beta\gamma} = \frac{1}{Z_{\delta,\frac{3}{2}}(z_0-z_2'+2z_2)}\,\frac{\partial}{\partial z_1}\left\{\frac{Z_{\delta,\frac{3}{2}}(z_0+z_1-2z_2'+2z_2)}{Z_{\delta,\frac{3}{2}}(z_1-z_2'+2z_2)}\right\} \tag{17}$$

(where z_2' will be set equal to z_2 after the derivatives in the definition of Y_2 have been performed). In [19,12] we dealt with such terms by taking x to be a ramification point r_β on the surface. However the role of the $\xi(x)$ insertion is simply to absorb the constant zero mode of the ξ field and so the answer should be independent of this choice. In [13] we managed to show this explicitly by using the trisecant identity and its corollaries [15,14], with the result that

$$\left\langle \xi(x)\,\partial\xi(z_1)\,\eta(z_2')\,e^{2\phi}(z_2)\right\rangle_\delta^{\beta\gamma} = -\frac{Z_{\delta,\frac{3}{2}}(2z_1-2z_2'+2z_2)}{Z_{\delta,\frac{3}{2}}(z_1-z_2'+2z_2)^2} \quad . \tag{18}$$

This two loop simplification is best understood as the generalisation of the following calculation on the plane.

$$\left\langle \xi(z_0)\,\xi(z_1)\,\eta(z_2')\,e^{-2\phi}(z_2)\right\rangle^{\beta\gamma} = \frac{(z_0-z_1)}{(z_0-z_2')\,(z_1-z_2')} = \frac{1}{(z_1-z_2')} - \frac{1}{(z_0-z_2')} \tag{19}$$

which is a sum of a term independent of z_0 and a term independent of z_1. Differentiating with respect to z_0 automatically removes the z_1 dependence. In the gauge of (12) the spin structure dependent terms in (18) then reduce to the same form as the pure matter terms considered previously. There is a slight complication due to the derivatives present in the Y_2 but these turn out to make no difference. Thus the arguments above also show the vanishing of the ghost terms for the two loop vacuum amplitude. To move to higher loops we must deal with terms of the form $Y_0^n Y_2^n$ and apply the same methods. This was completed recently in [20] where the argument given above for the removal of the $\xi(x)$ dependence is extended to an arbitrary ghost correlation function. This puts the ghost terms on the same level as the matter terms, namely zero for $g \leq 9$ with terms arising at $g = 10$ that our methods have not yet been able to deal with.

It is possible to extend all of the above discussion to the amplitudes with N vertex operators in the zero picture. This was done in [13] with the result that we could show vanishing of the amplitudes with $N < 3$ and $g + N \leq 9$, we could also do the spin structure sum for the four point function at two loops and find a reasonably simple final answer. The $g = 2$ case was also dealt with in a similar fashion in [21]. Parts of the above work have also appeared in [22] in which the emphasis was on making the picture changing insertions go towards the ramification points on the surface (see also [23]).

REMARKS ON THE CHOICE OF GAUGE

Of course the idea of picking special places to put the picture changing operators is not new. In response to the ambiguity problem the authors of [8] were led to introduce what they called a "meromorphic slice" and a particular example of such a gauge slice was to place the insertions at the zeroes of a holomorphic one form which varied meromorphically with the Teichmüller parameters for the surface. Their one form however was not completely arbitrary because of conditions on how its zeroes should behave at the boundary of moduli space. We only work at a particular point in moduli space and so will not see any condition on the one form used. Nevertheless our calculations do provide strong support for their arguments that the superstring is finite in this class of gauges, with the added bonus that the calculations are also much easier than in a general gauge.

Seemingly related to these meromorphic slices is an earlier calculation at genus two [19] in which we studied the case where both operators were put at the same ramification point r_β where β is an odd spin structure. This is the same as taking the special case in which the one-form used is $\sum_i \omega(z)\partial_i\Theta_\beta(0)$. Carefully looking at the limit as the two operators simultaneously approached r_β led us conclude that the vacuum amplitude vanished in this gauge. However putting the operators at a ramification point fails the condition in [8] that on degeneration into two $g = 1$ surfaces that the insertions should both go to the node (the ramification points do not generally go to the node).

Meromorphic slices were also used in the hyperelliptic formulation [9,24] at genus two in [25] with similar results, but for $g > 2$ most surfaces are not hyperelliptic and this was our main motivation for sticking to the more general theta function language despite the loss of the pure algebraic character of calculations in the hyperelliptic formulation. An early calculation [26] also used this prescription as a way of simplifying the calculation of the measure but did not attempt the sum over spin structures.

GLOBAL QUESTIONS OVER MODULI SPACE

All of the above was only concerned with what happens at a generic point in moduli space and we cannot immediately conclude that the integral over the whole of moduli space is finite. Such global problems were extensively discussed in [8] and we will only make a few extra simple remarks. As described in [8] the one form used to position the picture changing insertions must vary meromorphically with the moduli of the surface and also satisfy certain conditions at the boundary of moduli space. Hence for certain values of the moduli (on a space of complex co-dimension of at least one) the correlation functions involved could become divergent. This could happen in two ways: the one form gets a double zero and so the insertion points collide leading to divergences from the prime forms in the expression; or there might occur "spurious" singularities from the ghost system when the theta functions in the denominator happen to be zero. The authors of [8] tentatively concluded that such singularities do not occur inside moduli space and so it is interesting to compare this with our calculation.

With the gauge choice (12) the "spurious" singularities will occur in the spin structure δ only when $\Theta_\delta(\vec{0})$ is zero, but the main point of the gauge choice (12) is that the theta functions in the denominator are cancelled by those from the matter fields. This is because $\Theta_\delta(\vec{0}) = 0$ is also the condition that the spinor matter fields should have a zero mode. Indeed it immediately follows from (15) that in this case $Z_\delta(\vec{z}_1 - \vec{z}_2) = 0$ for all $z_1, z_2 \in \Sigma$ and so the matter field correlations will remove any spurious singularity from the ghost system. A similar comment was made for the odd spin structures in [12]. The case of colliding insertions is similar to that considered in [19]. This suggests that divergences do occur but that they have coefficient zero after summing over spin structures. If the points collide then the way to regulate the collision is to consider the limit in which the two points approach each other but we then only get zero and presumably this stays zero in the limit. Perhaps the main point is that we expect the result to be meromorphic with respect to the moduli and so only poles and not delta function singularities can occur, clearly there can be no poles if the measure is zero outside a space of codimension one. Both of these points should be investigated more deeply.

A ONE LOOP EXAMPLE

From the above work the reader might have the impression that simply being able to evaluate the spin structure sum is a good indication that the sum vanishes. To counter this we give a (rather artificial) counterexample at one loop. Consider the two point function with both vertices in the -1 picture. That is we need to evaluate

$$G = \langle \xi(x) Y(z_1) Y(z_2) c V_{-1}(y_1) V_{-1}(y_2) b(w) \rangle_\delta \tag{20}$$

Then a good meromorphic slice would be to construct a one form which is holomorphic except for simple poles at y_i and then to place the z_a at its two zeroes. This gives the condition

$$\vec{z}_1 + \vec{z}_2 - \vec{y}_1 - \vec{y}_2 = 2\vec{\Delta} \tag{21}$$

Using $V_{-1} = e^{-\phi} \epsilon^\mu \psi_\mu e^{ikX}$ it is easy to evaluate (20) and do the spin structure sums. All the sums vanish for the standard reasons except for the term in which the fields contained in the $Y(z_a)$ are contracted into each other. The resulting term is not zero but involves expressions such as $\Theta_\beta(\vec{z}_1 - \vec{y}_1)\Theta_\beta(\vec{z}_1 - \vec{y}_2)\Theta_\beta(\vec{z}_2 - \vec{y}_1)\Theta_\beta(\vec{z}_2 - \vec{y}_2)$ where β is the odd spin structure. Note that the calculation in the zero picture may be obtained from the limit $z_a \to y_a$ and in this limit the answer vanishes, in agreement with the usual one loop calculations. Presumably it is possible to show that it is a total derivative with respect to the modular parameter of the torus.

RELATION TO LIGHTCONE

By this we simply mean the observation that a holomorphic one form Ω defines a metric $|\Omega|^2$ on the surface which is flat everywhere except at the zeroes of the one-form. A simple zero corresponds to a doubly ramified point in this metric and so looks like the interaction point of a Mandelstam lightcone diagram [2,27]. The usefulness of this metric, together with placing the operator insertions at the singularities, was observed in [26] and further studied in [28,29]. For a general one form which should expect the rest of the surface to look like sections of cones rather than the cylinders of the light cone picture. It is interesting to work out how to build surfaces using just these sections of cones and interaction points (which are cones with 4π opening angle). However possibly the main point is that at a fixed modular parameter there will be a one form Ω_{LC} in whose corresponding metric the surface really does look like a light cone diagram [27] and nice properties of the gauge are inherited from this special case. Note that Ω_{LC} will have reality properties defining the coefficients in the expansion over the canonical basis of one forms and so by itself cannot form a meromorphic slice. Due to the ghost anomaly the curvature singularities will act as sources of ghost current of strength $e^{-\phi}$ but at least for the pure matter terms this is precisely cancelled by the e^{ϕ} of the picture changing insertion. Hence it is reasonable that the effect of the different conformal weights of the ghost and matter fields is cancelled out and that, just as for one loop, the ghost terms come out as inverses of the matter terms. Of course light cone gauge is not only a matter of having a flat metric on the surface but also involves gauge-fixing conditions on the matter spinor fields. Work on the exact realisation of the above ideas is in progress.

CONCLUSIONS

By placing the picture changing insertions at the zeroes of a holomorphic one form we were able to do the spin structure sum and so explicitly confirm the nonrenormalisation theorems (with N external massless bosonic vertices) for the case $g + N \leq 9$ pointwise in moduli space. We believe that these results also hold true over all of moduli space but that perhaps this should be more fully investigated. The reason behind why this gauge gives such simple results is probably the connection to the manifestly unitary (and so identically zero) "light-cone" diagrams and light-cone gauge. Perhaps a better understanding of why this gauge choice is so nice would give some indication of whether the barrier at $g = 10$ is a real one or just a technical problem. The calculations at 2 and 3 loops should also serve as a good testing ground for our understanding of concepts such as modular invariance and factorization.

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REFERENCES

1. J. H. Schwarz, *Phys. Rep.* **89C** (1982), 223;
J. Scherk, *Rev. Mod. Phys.* **47** (1975), 123;
E. D'Hoker and D. Phong, *Rev. Mod. Phys.* **60** (1988), 917.

2. J. H. Schwarz, ed., *Superstrings,* Vols. I and II, (World Scientific, Singapore, 1985);
M. B. Green and D. J. Gross, eds., *Unified String Theories,* (World Scientific, Singapore, 1986);
M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory,* Vols. I and II, (Cambridge University Press, 1987);
M. Kaku, *Introduction to Superstrings,* (Springer-Verlag, 1988).
3. D. Friedan, E. Martinec and S. Shenker, *Nucl. Phys.* **B271** (1986), 93.
4. E. Martinec, *Nucl. Phys.* **B281** (1986), 157.
5. E. Verlinde and H. Verlinde, *Phys. Lett.* **B192** (1987), 95.
6. J.J. Atick, J. Rabin and A. Sen, *Nucl. Phys.* **B299** (1988), 279.
7. J.J. Atick, G. Moore and A. Sen, *Nucl. Phys.* **B307** (1988), 221.
8. J.J. Atick, G. Moore and A. Sen, *Nucl. Phys.* **B308** (1988), 1.
9. G. Moore and A. Morozov, *Nucl. Phys.* **B306** (1988), 387.
10. H. La and P. Nelson, "Unambiguous fermionic string amplitudes", BUHEP-89-8,UPR-0381T, Feb 89.
11. M. Martellini and P. Teofilatto, *Phys. Lett.* **B211** (1988), 293.
12. A. Parkes, *Phys. Lett.* **B217** (1989), 458.
13. O. Lechtenfeld and A. Parkes, Nucl. Phys. B to be published. "On covariant multi-loop superstring amplitudes" April 1989.
14. D. Mumford, *Tata Lectures on Theta,* Birkhäuser, Boston 1983.
15. J. Fay, *Theta Functions on Riemann Surfaces,* Springer Notes in Mathematics 352, 1973.
16. H.M. Farkas and I. Kra, *Riemann Surfaces* (Springer, 1980).
17. G. Moore, P. Nelson and J. Polchinski, *Phys. Lett.* **B169** (1986), 47;
G. Moore and P. Nelson, *Nucl. Phys.* **B274** (1986), 509;
E. D'Hoker and D. Phong, *Nucl. Phys.* **B278** (1986), 225, *ibid.* **B292** (1987), 317;
S. Chaudhuri, H. Kawai and S.-H. Tye, Cornell preprint CLNS 86/723 (1986).
18. E. Verlinde and H. Verlinde, *Nucl. Phys.* **B288** (1987), 357.
19. O. Lechtenfeld and A. Parkes, *Phys. Lett.* **B202** (1988), 75.
20. O. Lechtenfeld, "Superconformal ghost correlations on Riemann surfaces", City College preprint CCNY-HEP-89/09.
21. A. Tsuchiya, *Phys. Rev.* **D39** (1989), 1626, "On the non-renormalization theorem in superstring theories at the two loop order",TIT-HEP April, 1989.
22. A. Morozov, *Nucl. Phys.* **B318** (1989), 137;
A. Morozov and A. Perelomov, *Int. J. Mod. Phys.* **A4** (1989), 1773;
A. Morozov, *Nucl. Phys.* **B303** (1988), 343.
23. O. Lechtenfeld, *Nucl. Phys.* **B309** (1988) 361, "On the finiteness of the superstring" Nucl. Phys. B to be published.
24. L. Dixon, D. Friedan, E. Martinec and S.H. Shenker, *Nucl. Phys.* **B282** (1987) 13;

A.B. Zamolodchikov, *Nucl. Phys.* **B285** (1987) 481;
M. Bershadsky and A. Radul, *Int. J. Mod. Phys.* **A2** (1987) 165;
V. Knizhnik, *Comm. Math. Phys.* **112** (1987) 567, *Phys. Lett.* **B196** (1987) 473;
D. Montano, *Nucl. Phys.* **B297** (1987) 125;
D. Lebedev and A. Morozov, *Nucl. Phys.* **B302** (1988) 163.

25. E. Gava, R. Iengo and G. Sotkov, *Phys. Lett.* **B207** (1988) 283;
R. Iengo and C.-J. Zhu, *Phys. Lett.* **B212** (1988) 309, 313;
C.-J. Zhu, Trieste preprint ISAS/88/147 (1988).

26. M. Bershadsky, *Phys. Lett.* **B201** (1988) 67.

27. S.B. Giddings and S.A. Wolpert, *Comm. Math. Phys.* **109** (1987) 177;
E. D'Hoker and S.B. Giddings, *Nucl. Phys.* **B291** (1987) 90;
N. Berkovits, *Nucl. Phys.* **B304** (1988) 537, *Phys. Lett.* **B219** (1989) 278;
R. Tzani, *Phys. Rev.* **D15** (1988) 3112, "The problem of the gauge in Polyakov's theory: Relation between light-cone and conformal gauge" City College preprint CCNY-HEP-89/1.

28. K.-J. Hamada and M. Takao, *Int. J. Mod. Phys.* **A4** (1989) 2283, *Nucl. Phys.* **B313** (1989) 80.

29. E. D'Hoker and D.H. Phong "Functional determinants on mandelstam diagrams", Preprint UCLA/88/TEP/39.

THE MINIMAL SET OF THE GENERATORS OF DEHN TWISTS ON A HIGH GENUS RIEMANN SURFACE

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INTRODUCTION

The computation of correlation function on Riemann surfaces is an important problem in conformal field theories. However, explicit constructions of the correlation functions on $g > 1$ Riemann surfaces exist only for a very few theories which are basically free theories.[1] One of the main difficulties here is that the mapping class groups on arbitrary genus surfaces were not known. Although it has been proven that the mapping class group (MCG) of a surface of genus g is generated by $3g-1$ Dehn twists. However, the minimal set of generators was not known.[2,3,4,5]

In this paper, we define the minimal set of the Dehn twists generating the MCG and their representations.

MAPPING CLASS GROUP

Let M_g denote a closed orientable surface of genus g, $z_1^0, \ldots z_n^0$ denote n fixed points on M_g, $F_n M_g$ be the group of all orientation-preserving homeomorphisms $h : M_g \longrightarrow M_g$ such that for each i, $h(z_i^0) = z_i^0$, and $B_n M_g$ denotes the group of all orientation preserving homeomorphisms $h : M_g \longrightarrow M_g$ such that $h(z_1^0, \ldots, z_n^0) = (z_1^0, \ldots, z_n^0)$ allows permutation among these fixed points. Then the pure mapping class group G_g of M_g is the group of path components of $F_n M_g$, i.e., $\pi_0(F_n M_g,\, id)$ while the (full) mapping class group $G(g,n)$ of M_g is the group of path components of $B_n M_g$, i.e. $\pi_0(B_n M_g,\, id)$.

A well-known theorem for Riemann surfaces states that the MCG of a genus g surface is generated by $3g-1$ Dehn twists: Theorem (Dehn and Lickorish).[2,3]

1. Every orientation-preserving homeomorphism of $M_g \to M_g$ is isotopic to a product of Dehn twists.
2. Let $c_1, \ldots, c_{3g-1}$ be the simple closed curves on M_g which are illustrated in Fig. 1. Then a Dehn twist about an arbitrary simple closed curve on M_g is isotopic to a power product of twists about $c_1, \ldots, c_{3g-1}$.

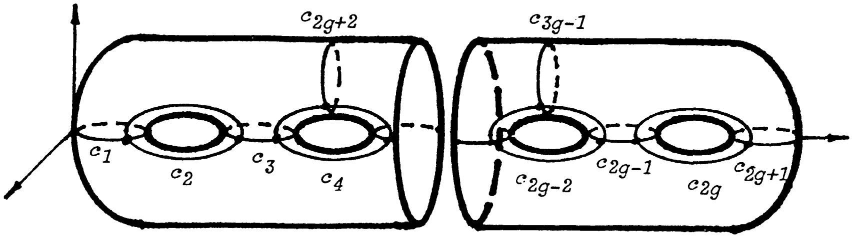

Fig. 1. Dehn twists on a genus g Riemann surface

It is important to note that the Dehn twists cited in the above theorem are not a minimal set of generators. In order to find the minimal set, let us consider the case of a cylinder.

DEHN TWISTS ON A CYLINDER

First of all consider the Dehn twists about any simple closed curves on a cylinder with free ends shown in Fig. 2. The Dehn twist

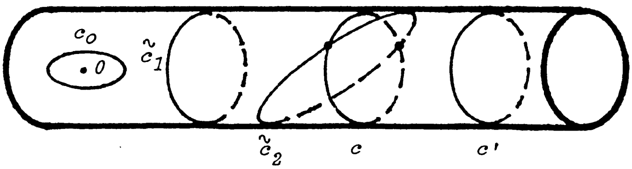

Fig. 2. Dehn twists on a cylinder

$t_c\tilde{c}_1$ of a curve $\tilde{c}_1$ about a curve c with intersection number $|\tilde{c}_1, c| = 0$ is apparently trivial. For the case of the Dehn twist t_c for the curve $\tilde{c}_2$ with intersection number $|c, \tilde{c}_2| = 2$, use the following Lemma:

Lemma. If p and m are simple closed curves on M_g, and h_i: $M_g \to M_g$ is an isotopy with $h_0 = id$ and $h_i(p) = m$, then there is an isotopy between t_p and t_m.

We have $t_c(\tilde{c}_2) \sim t_{c'}(\tilde{c}_2)$ where c' is a curve homotopic to c and $|c', \tilde{c}_2| = 0$. Thus $t_c\tilde{c}_2 \sim \tilde{c}_2$. Since a closed curve c_0 of null homotopic class is homotopic to a point 0. Thus $t_{c_0}(\tilde{c}_1) \sim \tilde{c}_1$.

Moreover, let c be closed curve around the cylinder and b a mother line, with $|c, b| = 1$, as in Fig. 3. Then the Dehn twist t_c for b breaks b at the intersection point p and inserts a copy of c at the break. The resulting curve winds around the cylinder one cycle. But since the two ends of the cylinder are free, thus one can twist back one cylinder continuously and eliminate the winding.

Therefore the Dehn twists on a cylinder are all trivial.

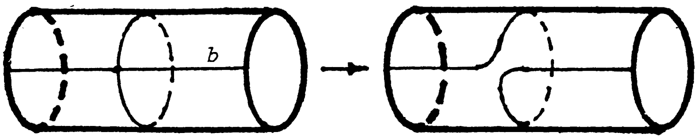

Fig. 3. Dehn twists of a mother line b about a closed curve c around the cylinder.

DEHN TWISTS ON A TORUS

As mentioned above, the Dehn twist on a cylinder with free ends are always trivial. If we glue the two ends of a cylinder to form a torus, then the twist on the cylinder will not be relaxed. Thus Dehn twist on a torus is the most crucial configuration in understanding the action of twists. The nontrivial cases are:

Case I. The Dehn twist $t_c(b)$ of a closed curve b around the hole about the cycle around the torus tube as in Fig. 4. It breaks the curve b at the intersection point p and inserts a copy of c at the break. The resulting curve winds around the torus tube once.

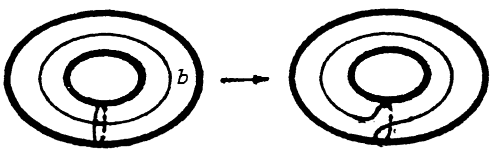

Fig. 4. Dehn twist around a torus tube.

Case II. Dehn twist of a $t_b(c)$. It breaks the curve c at p and inserts a copy of b at the break, resulting in the same curve as $t_c(b)$, i.e., $t_b(c) = t_c(b)$.

Case III. Trivial cases $t_b(b')$, $t_c(c')$ where b and b', c and c' are homotopic respectively as shown in Fig. 5.

Besides the Dehn twists about the closed curves c and c' on a torus are isotopic if and only if c and c' are homotopic.

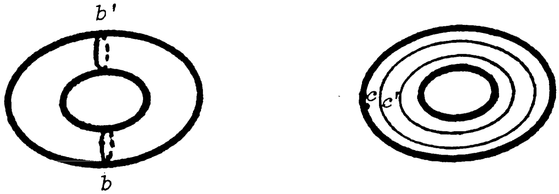

Fig. 5. Dehn twist about homotopic curves c and c' on a torus.

DEHN TWIST ON A TORUS WITH PUNCTURES

The Dehn twists about homotopic curves c and c' on our torus are isotopic even after the torus is punched between these curves at p and p', shown in Fig. 6, namely, although c and c' are non-homotopic due to the punctures between these curves, but the Dehn twists t_c and $t_{c'}$ are still isotopic. Because the curve $\tilde{c}$ after the Dehn twist t_c turns out to be a curve $\bar{c}$ which cut the torus into a connect region. Thus one puncture p can be moved to the neighborhood of p'. Therefore t_c and $t_{c'}$ are isotopic.

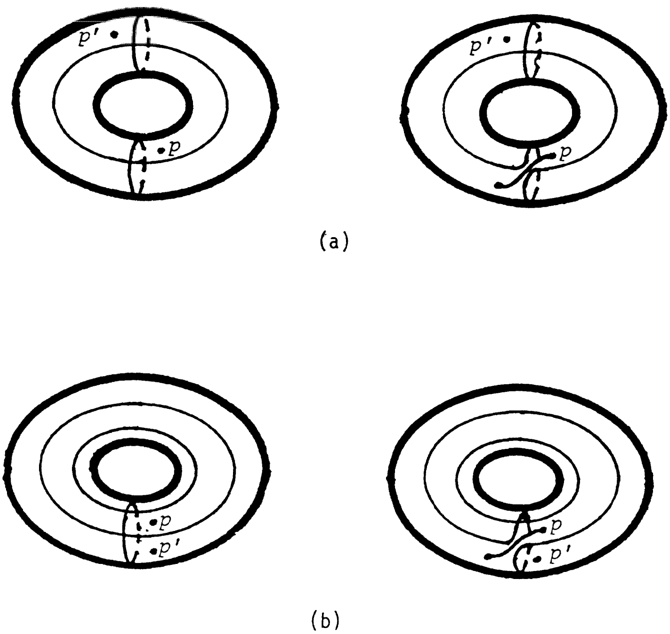

Fig. 6. Dehn twists about curves on a torus with punctures between them.

Now we are on the point to abstract the minimal set of the Dehn twists out of $3g-1$ twist given by Dehn's theorem.

Cut the Riemann surface M_g along the closed curves s_1 and s_2 as in Fig. 7. The section obtained containing c_1 and c_2 and c_3 is a torus with punctures. Then the Dehn twists t_{c_1} and t_{c_3} are isotopic as mentioned above. While s turns a cycle around the torus tube in transforming t_{c_3} to t_{c_1} in three dimensional space shown in Fig. 8.

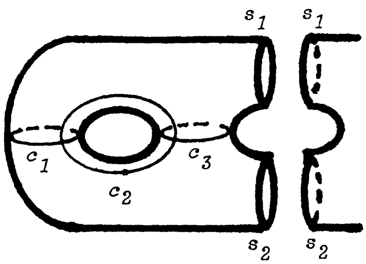

Fig. 7. Cut a torus from the Riemann surface.

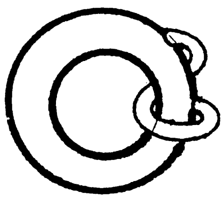

Fig. 8. Winding of a handle in the course of transforming t_{c_3} to t_{c_1}.

The same thing can happen for the section containing c_3, c_4 and c_5, as in Fig. 9. On the resulting torus, it can also be easily seen that t_3, t_5 and $t_{c_{2g+2}}$ are isotopic.

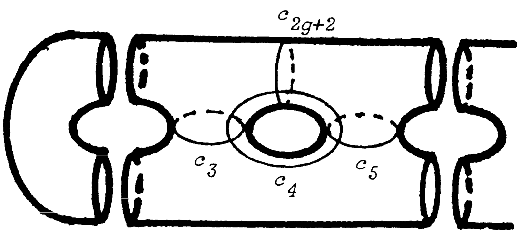

Fig. 9. Torus Section containing c_3, c_4 and c_5.

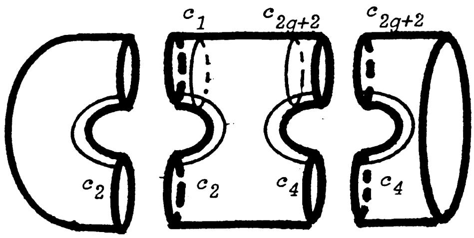

Fig. 10. Action of four twists on the common part of two tori.

Note that t_{c_1}, t_{c_2}, t_{c_4} and $t_{c_{2g+2}}$ act on the common part of two consecutive tori with different parameters.

The Dehn twist t_{c_3} of a curve b winding around the common part of the two consecutive parts of two tori as shown in Fig. 11 is actually isotopic to t_c, or t_{c_3}, and is thus trivial.

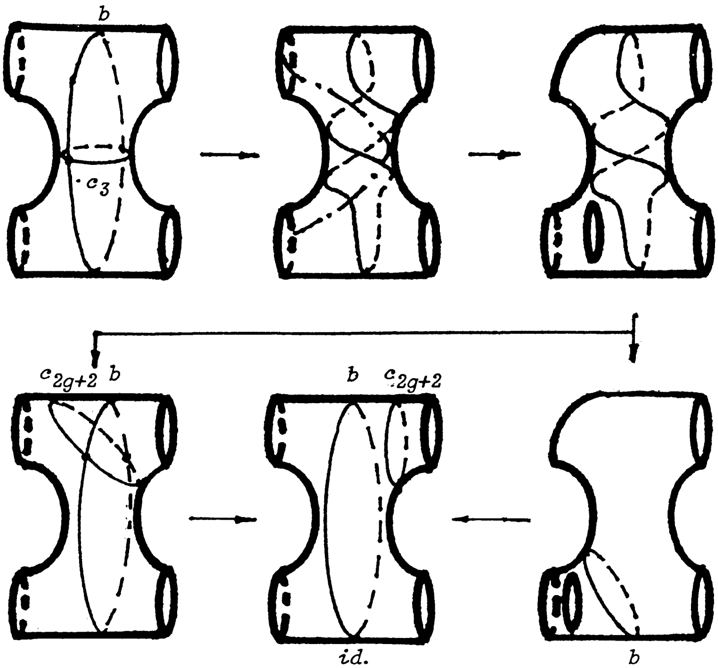

Fig. 11. Action of t_3 is isotopic to t_{c_1}, or t_{c_5}.

Alternatively b can be decomposed into two curves, b_1 and b_2, passing through the same single hole shown in Fig. 12. This curve b is evidently uneffected by t_{c_1}, t_{c_3} and t_{c_5}.

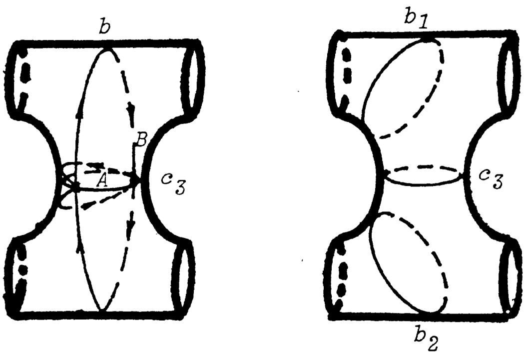

Fig. 12. Decomposition of b into b_1 and b_2.

Therefore we come to the modified Dehn's theorem.

Theorem:

1. Every orientation-preserving homeomorphism of $M_g \to M_g$ is isotopic to a product of Dehn twists.

2. The Riemann surface M_g can be covered by g tori with punctures. Let a_i, b_i $(1 \leq i \leq g)$ be the canonical homology of H_1 (M_g, Z). Then a Dehn twist about an arbitrary simple closed curve on M_g is isotopic to a power product of twists about a_i, b_i $(1 \leq i \leq g)$. Each pair of the Dehn twists t_{a_i}, t_{b_i} act on the homeomorphism of the punched torus they live.

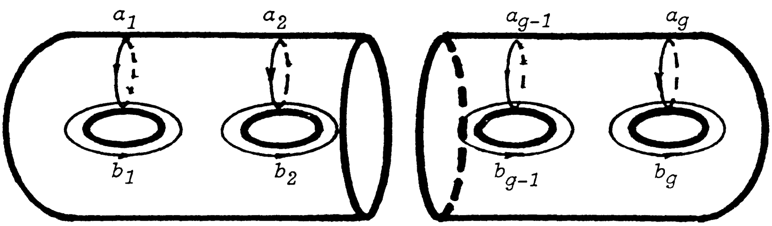

Fig. 13. The minimal set of Dehm twists on a Riemann surface M_g.

Therefore the minimal set of the Dehn twist operators are $2g$ Dehn twists t_{a_i}, t_{b_i} $(1 \leq i \leq g)$. While these twist about the curves linking the two consecutive holes are redundant.

Define the canonical intersection form in terms of a_i, b_i cycles as follows:

$$\begin{aligned} J(a_i, a_i) &= J(b_i, b_j) = 0 \\ J(a_i, b_j) &= -J(b_i, a_j) = \delta_{ij} \end{aligned} \tag{1}$$

or in matrix form so

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

The intersection matrix J is invariant under homeomorphism, so the action $G(g, n)$ on the homology group $H_1(M_g, Z)$ must preserve J.

A representation of nontrivial homeomorphism is provided by their action on the homology of M_g. If curve c generates a nontrivial homology class, then a Dehm twist around c acts nontrivially on the canonical homology basis. We have

$$\begin{aligned} t_{a_i} &: a_j \longrightarrow a_j, \quad b_j \longrightarrow b_j + \delta_{ij} a_i, \\ t_{b_i} &: a_j \longrightarrow a_j + \delta_{ij} b_j, \quad b_j \longrightarrow b_j \end{aligned} \tag{3}$$

or in matrix form as:

$$t_{ai} = \begin{array}{r ccccccc c c}
 & & & & 1 & & & & \vdots & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & \\
 & \vdots & & & \vdots & & & \vdots & \vdots & \\
 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & \\
i \cdots\cdots & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \vdots & 1 \\
 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & \\
 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots & \\
 & \vdots & & & \vdots & \vdots & & \vdots & \vdots & \\
 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \vdots &
\end{array}$$

$$t_{b_i} = \begin{array}{c|c} 1 & \begin{array}{ccccccc} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & \cdots & \cdots & \cdots\cdots i \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array} \\ \hline 0 & 1 \end{array} \tag{4}$$

Thus the matrix representing actions of Dehm twist on $H_1(M_g, Z)$ is nonsingular $2g \times 2g$ integral matrix leaving the sympletic intersection from J invariant, namely an element of $Sp(2g, Z)$. In fact the matrices t_c generate all $Sp(2g, Z)$.[5]

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REFERENCES

1. P. di Francesco, H. Saleux and J.B. Zuber, Nucl. Phys. B290 (1987) 527.
2. M. Dehn, Acta Math., 69 (1938) 135.
3. W.B.R. Lickorish, Ann. Math., 76 (1962) 531; Proc. Camb. Phil. Soc. 60 (1964) 769, 62 (1966) 679.
4. J.S. Birman, Braids, Links and Mapping, Clako Group (Princeton University Press and University of Tokyo Press, 1974).
5. L. Alvarez-Gaumé, G. Moore and C. Vafa, Comm. Math. Phys. 106 (1986) 40.

STRINGS AND TEICHMUELLER SPACE

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INTRODUCTION

The subject of string theory originates from the study of the strong interaction. It starts with Regge poles, duality and Veneziano formalizm. Several important issues in string theory can be illustrated by reviewing the original motivation that lead to a dual resonance model of relativistic strings.

In strings theory we have the following new idea: fundamental objects are one dimensional and they propagate in space-time $\mathbb{R}^{1,D}$. As a string propagates it sweeps out a Lorentzian world sheet Σ^L. However, starting with the Polyakow formalizm physicists incorporate Teichmueller space into the theory by moving between the Euclidean and Lorentzian convention for the world-sheet signature with lack of concern. So, when the theory of Riemann surfaces become an important mathematical tool for string theory one can read "...The string is described by an immersion $X^{\mu}(\sigma^{\alpha})$ of its (compact) two dimensionalworld sheet with coordinates σ^{α} into Euclidean space-time...". In this way all physical argumentations that lead from the dual Veneziano model and Regge trajectories to Lorentzian world sheet have lost their meaning.

In this article we will consider Lorentzian strings and Minkowski space-time $\mathbb{R}^{1,D}$ consequently and we will show that in this case the formalizm of Teichmueller space is natural for string theory.

STRINGS AND RIEMANN SURFACES

Let Σ denote an orientable, connected 2-dimensional manifold which is the underlying manifold of our world sheet Σ^L. Let ξ_{GL+} denote the principal $GL^+(2,\mathbb{R})$ bundle of all

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linear frames over Σ. Any complex structure on Σ (we recall that any almost complex structure on surface is integrable) can be determined by some reduction of the principal ξ_{GL+} bundle to the $GL(1,\mathbb{C})$ group. Since $GL(1,\mathbb{C})=\mathbb{C}^{*}\cong SO(2)\times\mathbb{R}^{+}$ we observe that any complex structure on Σ is equivalent to some conformal structure. Let us fix some of them, say Σ_0. The corresponding Teichmueller space $T(\Sigma_0)$ can be described as a family of concrete complex structures on Σ together with a homotopy class of quasiconformal maps from Σ_0 to a variable Riemann surface Σ_i.

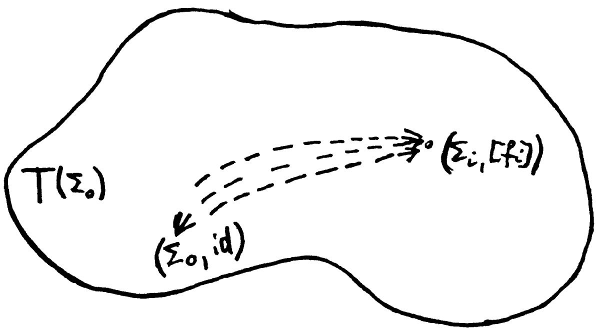

Fig.1

Let us recall that a map $f_i:\Sigma_0\to\Sigma_i$ is quasiconformal if the derivation from conformality $K(p)$ at any differentiable point $p\in\Sigma_0$ is finite.

To find relations between Lorentzian world sheet Σ^L and Riemann surfaces let us first consider the oriented vector space $\mathbb{R}^2$. The space of complex structure on $\mathbb{R}^2$ is the homogeneous space $GL^{+}(2,\mathbb{R})/GL(1,\mathbb{C})\cong$ open unit disc $\Delta=\{z\in\mathbb{C};\ |z|<1\}$. In other words, if we fix one concrete complex structure $z=x+iy$ (it is the same as to fix quadratic form

$Q(x,y)=|z|^2$) then to each $\mu\in\Delta$ we can associate the equivalence class of the quadratic form $Q_\mu(x,y)=|z+\mu\bar{z}|^2$.

If we endow vector space $\mathbb{R}^2$ with concrete Lorentz structure then two different Lorentzian orthonormal frames on $\mathbb{R}^{1,1}$ will determine different elements of $GL^+(2,\mathbb{R})/GL(1,\mathbb{C})$ i.e. different complex structures on $\mathbb{R}^2$. (From Fig.2 we see immediately that frames $(\underline{m},\underline{n})$ and $(\underline{m}',\underline{n}')$ determine unequivalent conformal structures.)

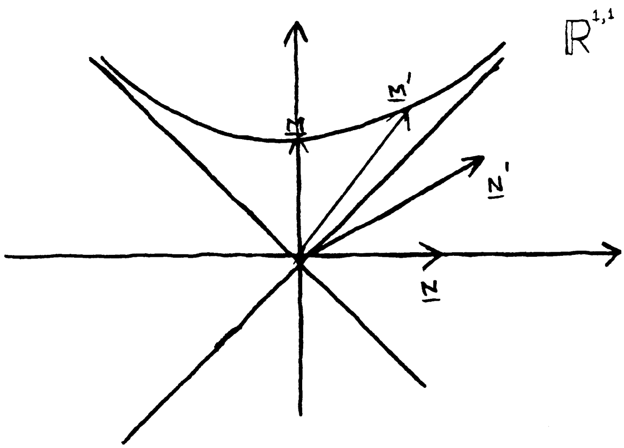

Fig.2

Summerazing we can say that any concrete Lorentzian structure on $\mathbb{R}^2$ determines 1 parameter family of Euclidean (conformal) structures on $\mathbb{R}^2$ or that any fixed time like vector $\underline{m}\in\mathbb{R}^{1,1}$ determine unique, oriented Euclidean structure on $\mathbb{R}^2$.

Coming to string world sheet we have that immersion $X:\Sigma\to\mathbb{R}^{1,D}$ induces Lorentzian structure Σ^L (a.e.). It can be checked [1] that any fixed time-like unit vector $e_0\in\mathbb{R}^{1,D}$ defines (on Σ^L) a field of time-like vectors $\underline{m}(x)$ (almost everywhere). It means that any time-like vector $e_0\in\mathbb{R}^{1,D}$ defines 1-parameter family of conformally unequivalent Riemannian structures Σ_α on Σ given by $\{\Sigma^L+\underline{m}_\alpha(x)\}$ where $\underline{m}_\alpha(x)=\alpha\circ\underline{m}(x)$, $\forall\alpha\in SO(1,1)$. So we observe that to each observer $e_0\in\mathbb{R}^{1,D}$ we have associated some curve $\{\Sigma_\alpha\}$ in the corresponding Teichmueller space. We will call this line P-line and we will see that it is in fact a Teichmueller line.

For any quasiconformal map $f:\Sigma_0 \to \Sigma_1$ we define the global dilatation $K_f = \sup_{x\in\Sigma_0} K_f(x)$. Now in any homotopy class of quasiconformal maps there exsists a map with minimmal global dilatation. It is called Teichmueller map and it is characterized by the existence of canonical quadratic holomorphic differentials q_{Σ_0} on Σ_0 and q_{Σ_1} on Σ_1. They

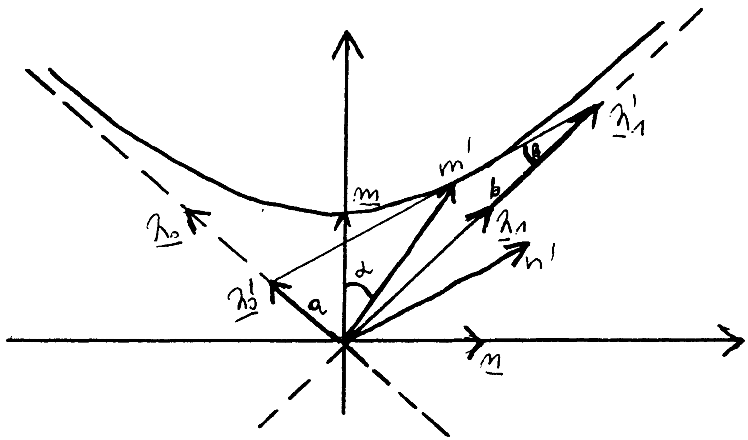

Fig.3

determine pairs of orthogonal, measured foliations on Σ_0 and Σ_1 respectively and the Teichmueller map maps the leaves on Σ_0 into the corresponding leaves on Σ_1 by the so called generalized affine map. In other words if ω is a natural local parameter on Σ_0 introduced by q_{Σ_0}, $\omega=u+iv$ then Teichmueller map from Σ_0 to Σ_1 can be seen as a stretch map (q_{Σ_0}, K)

$$u \longrightarrow u' = K^{-1/2}\cdot u$$

$$K\in[1,\infty)$$

$$v \longrightarrow v' = K^{1/2}\cdot v$$

The Teichmueller geodesic ℓ through (Σ_0, id) is a one parameter family of Riemann surfaces to which Σ_0 is mapped by the (q_{Σ_0}, K) stretch map with fixed q_{Σ_0} and $1 \le K < \infty$.

Now let us notice that different Euclidean structure on $\mathbb{R}^2$ determined by different orthogonal relations $(\underline{m}, \underline{n})$ and $(\underline{m}', n')$ can be introduced also by frames (λ_0, λ_1) and (λ_0', λ_1').
Directions of λ_0 and λ_1 are mutually orthogonal for each Euclidean structure related to a given Lorentzian metric on $\mathbb{R}^2$. The conformal unequivalence is coded in different units in these both directions. If β denotes an angle between $\underline{m}'$ and the direction of λ_1 than the ratio of (λ_0', λ_1') units measured in (λ_0', λ_1') coordinates is

$$\frac{a}{b} = \mathrm{tg}\ \beta \qquad \beta \in (0, \tfrac{\pi}{2})$$

We have such situation in any point of a world sheet Σ^L. More exactly, any time vector $e_0 \in \mathbb{R}^{1,D}$ defines Lorentzian frame field (or equivalently time-like field) almost everywhere on Σ^L. It implies that we have determined fields $\lambda_0(p)$, $\lambda_1(p)$ of light vectors (a.e.). If we introduce local complex coordinates z related to directions $(\lambda_0(p), \lambda_1(p))$, $z=x+iy$ then on Σ_α we have

$$\omega = \omega(z) = u+iv = A\left[\sqrt{\mathrm{tg}\beta}\cdot x + \sqrt{\mathrm{ctg}\beta}\cdot y\right] =$$

$$= \frac{A}{2}\left[\left[\sqrt{\mathrm{tg}\beta} + \sqrt{\mathrm{ctg}\beta}\right]\cdot z + \left[\sqrt{\mathrm{tg}\beta} - \sqrt{\mathrm{ctg}\beta}\right]\bar{z}\right]; \qquad A \in \mathbb{R}^+$$

So the transformation $z \longrightarrow \omega(z)$ is quasiconformal and satisfies $\omega_{\bar{z}} = \frac{\mathrm{tg}\beta - 1}{\mathrm{tg}(\beta+1)}\cdot\omega_z = \mathrm{tg}\alpha\cdot\omega_z$ where $\alpha = \beta - \frac{\pi}{4} \in (-\frac{\pi}{4}, \frac{\pi}{4})$. We see immediately that it is Teichmueller map and that coordinates z and ω are natural parameters of canonical quadratic differentials related to this map. So, our 1-parameter family of a Riemann surfaces determined by some concrete observer e_0 and Lorentzian world sheet Σ^L form a Teichmueller line i.e. infinite Terichmueller geodesic. In this article we will consider only a case where world sheet can be related to a Riemann surface of genus $g \ge 2$ and with finite number of punctures.

Since we are putting a primary relevance to a Lorentz structure on Σ we should know if this fact implies some additional restrictions on the possible Riemannian structures on our world sheet or not. According to the Nash-Green theorem there always exists an isometric embedding of Σ^L into $\mathbb{R}^{k,k}$ with k=50. For physical reasons we assume that our world sheet can be isometrically immersed into a vector space of signature (1,D) i.e. into at most 51 dimensional real vector space. Now let us notice that 51 is exactly the dimension of the Euclidean vector space $\mathbb{R}^{51}$ which admits an isometric embedding of any 2-dimensional Riemannian manifold. So we see that the fact that we put the fundamental relevance to the Lorentz structure does not introduce any additional difficulties into our considerations.

Let us consider a situation when Riemannian structure Σ_0 related to concrete observer $e_0 \in \mathbb{R}^{1,D}$ possessed some concrete properties. The most regular situation would be when $X:\Sigma_0 \longrightarrow \mathbb{R}^{1+D}$ realizes a minimal immersion into $\mathbb{R}^{1+D}$. (We are using the same letter X to denote pointwise the same immersion of Σ_0 into $\mathbb{R}^{1,D}$).

However it is known that although any noncompact Riemannian manifold Σ_0 admits a proper embedding into $\mathbb{R}^k$, k>5 by a harmonic map it is not necessary a conformal one. Moreover there are no compact minimal submanifolds in $\mathbb{R}^n$, n>3. So we see that minimal immersion of Σ_0 into $\mathbb{R}^{1+D}$ is not the case with high probability. The next, also very regular situation appears when an immersion X into $\mathbb{R}^{1+D}$ realizes a minimal immersion into a hypersphere S^D of $\mathbb{R}^{1+D}$. In this case the Gauss map associate to X is harmonic and homothetic and we can prove the following fact [2]

Proposition: If map $X:\Sigma_0 \longrightarrow \mathbb{R}^{1+D}$ realizes minimal immersion of Σ_0 into hypersphere $S^D_{r_0} \longrightarrow \mathbb{R}^{1+D}$ then $X:\Sigma^L \longrightarrow \mathbb{R}^{1+D}$ satisfies wave equation $\frac{\delta^2 X}{\gamma\delta_0^2} - \frac{\delta^2 X}{\gamma\delta_1^2} = 0$ if and only if $X:\Sigma_\alpha \longrightarrow S^D_{r_\alpha}$ is harmonic $\forall \alpha \in (-\frac{\pi}{4}, \frac{\pi}{4})$.

If we meet this case we call our P-line a harmonic line. In this case the quadratic differential canonically associated to P-line (and uniquely determined by a pair of "light" measured foliations, up to a positive constant,) is a Jenkins-Strebel differential [3]. It means that it has closed horizontal trajectories and satisfies:

1. Its critical graph Γ_q (i.e. the set of critical trajectories with their critical endpoints, zeros and simple poles in punctures) is compact.

2. There is a partition of $\Sigma_0 - \Gamma_q$ onto ring domains each of which is swept out by the freely homotopic closed trajectories.

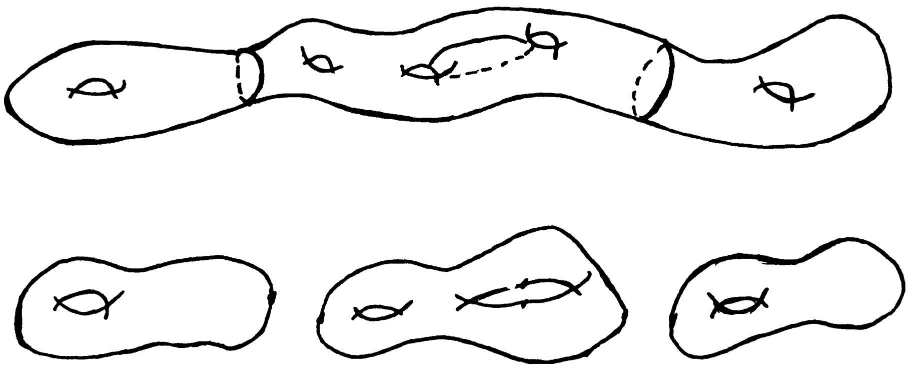

Fig.4

Let $(k,-q)$; $k \in [0,1)$ denote a Strebel ray through the point $(\Sigma_0, id) \in T(\Sigma)$ and let $f_k : \Sigma_0 \longrightarrow \Sigma_k$, $k=k(\alpha)$; $k=K+1/K-1$ be the corresponding Teichmueller map as described above. It is known [4] that the end-point of Strebel ray $(k,-q)$ is given by the punctured model $\tilde{\Sigma}_0$ of Σ_0. Topologically it can be described as pinching a closed curve in each ring domain to a point and then removing the point. So in the boundary of a Strebel ray $(k,-q)$ we obtain one or more Riemann surfaces $S_1 \ldots S_N$ which may be thought to have been obtained by cutting along an admissible system of Jordan curves on Σ_0 and then by gluing a punctured disc to each side of each cut.

We meet the similar situation for Strebel ray $(k,-q)$, $k\in[0,1)$. In [5] we investigate this problem more carefully using the Bers embedding.

So if P-line, harmonic or not, is associated to quadratic differential which is Jenkins-Strebel differential, then any physical object which is related to a Lorentz world sheet Σ^L cannot be stable. It has to be created - what is described by the so called opening procedure for horizontal cylinders of Jenkins-Strebel ray (k,q) and it has to decay - - what is described by the endpoint of Jenkins-Strebel ray $(k,-q)$.

Since for any P-line we have identification of $k\in(-1,1)$ with $tg\alpha$, $\alpha=\beta-\frac{\pi}{4}\in(-\frac{\pi}{4},\frac{\pi}{4})$ and α has well defined physical interpretation, the time orientability of Σ^L guarantees that the notions of "creation" and "decay" are definitely distinguished and well defined. The endpoint of the ray (k,q) can be interpreted as objects which take part in some collision process. Similarly any element S_i of $\tilde{\Sigma}_0$ i.e. any element of decay can take part in some other collision process i.e. be one of the elements of some other opening process. (see example on Fig.5)

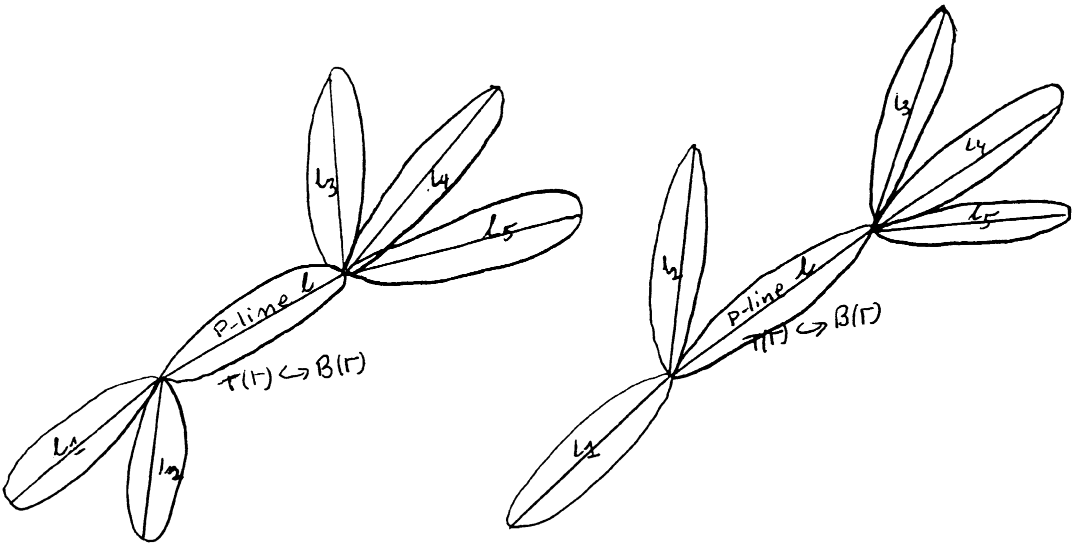

Fig.5

Thus any physical object which can be related to a world sheet with genus $g\geq2$ such that canonical quadratic differential determined by $e_0\in\mathbb{R}^{1,D}$ is a Jenkins-Strebel differential, cannot be stable.

Now let us consider a case when Jenkins-Strebel differential q associated to some harmonic P-line has only one cylinder of each type [6]. Since the quadratic differential with closed horizontal and closed vertical trajectories each determining only one cylinder are dense and any quadratic differential can be approximated by such quadratic differentials, this assumption seems to be not specially unrealistic.

Let α and β be a closed horizontal and vertical trajectory in each of the two homotopy classes respectively. It is known [7] that in this case each component of $\Sigma-\alpha\cup\beta$ is contractible. It implies [8] that the selfmapping h of Σ_0 given by

$$h = T_\alpha \, T_\beta^{-1}$$

is irreducible (here T_α and T_β denote the Dehn twists about α and β respectively), It means that h is absolutely extremal (i.e. hyperbolic) selfmapping of Σ_0. Now, let $h_1=h_2=1$ and $c_1=c_2=c$ be the hights and cirumferences of our ring domains

$$R_1 = \Sigma_0 - \sqrt{q} \qquad \text{and} \qquad R_2 = \Sigma_0 - \sqrt{-q}$$

respectively. It is known [9] that the Teichmueller map from $(\Sigma_0, id)\equiv\tau_0\in T(\Sigma_0)$ to $h^*(\tau_0)$ is quasiformal map whose canonical quadratic differential q_1 is equal to

$$q_1 = \left[1 - \frac{\sqrt{c^2+4} - c}{2} \cdot i \right] \cdot q$$

and its global dilatation K is given by

$$K = \frac{2 + c^2 + c\cdot\sqrt{c^2+4}}{2}$$

On the other hand it is known [10] that K has to be algebraically integer. This last fact implies that only a discrete set O of the elements of the unit time-like hyperboloide $H \rightarrow \mathbb{R}^{1,D}$ can "produce" P-lines related to described above quadratic differentials.

So if we assume that to a world sheet Σ^L we can relate some physical object and if we agree that a Lorentzian structure of this world sheet is important and that Jenkins--Strebel differential associated to P-line has only one horizontal and one vertical cylinder of the same hights and cirumferences then only discrete set of "observers" can observe the same physical rules. In other words we can say that only discrete set of Lorentz transformations can form a symmetry group of such "physical system".

It seems that we have two possibilities. The first one appears when Σ_0 is compact and when introduced above quadratic differential q_1 is the square of an abelian differential i.e. $q_1=\phi^2$. In this case $K^{1/2}$ is an eigenvalue of some integral matrix. Now if only finite number of pieces of physical objects related to strings with compact world sheet is present in the nature then we would have a cellural structure of our space-time. In the second, general case, the dense set O of "observers" is possible.

So we see that already in our starting consideration (there are many open, as well mathematical as physical, questions which should be solved) exists very strong inter dependence between properties of strings world sheets and properties of space-time. And it is very interesting.

REFERENCES

[1] - K.Bugajska "Strings, observer and Teichmueller line" subm.for publ.

[2] - K.Bugajska "Harmonic, homothetic Gauss map of world sheet, harmonic P-line and Nambu-Goto action" subm.for publ.

[3] - J.Hubbard, H.Masur, Acta Math. 142 (1979) 221-274
K.Strebel "Quadratic differentials" Springer Verlag 1984
A.Marden, Bull AMS 3 (1980) 1001-1017

[4] - H.Masur, Ann.of Math. 102 (1975) 205-221

[5] - K.Bugajska "Jenkins-Strebel rays and decay of elementary particles" subm.for publ.

[6] - K.Bugajska "Strings and discrete space-time" subm.for publ.

[7] - I.Kra, Acta Math. 146 (1981) 231-270

[8] - I.Bers, Acta Math. 141 (1978) 73-98

[9] - H.Masur in "Riemann surfaces and Related Topics" Princetot Univ.Press 97 (1980) 417-438

[10] - W.P.Thurston "On geometry and dynamics of diffeomorphisms of surfaces" preprint and see [7]

HOLOMORPHIC DIFFERENTIALS ON PUNCTURED RIEMANN SURFACES

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I. Introduction

The investigation of the dynamics of holomorphic or anti-holomorphic λ-differentials on Riemann surfaces was initiated by the observation, that these differentials enter string theory via the Faddeev-Popov-procedure, albeit as local sections of Grassmann valued line bundles. Furthermore, it is well known that dual line bundles play an important role in the description of modular deformations of complex structures.

As usual in physics, the dynamics of conformal fields was analyzed mainly in terms of local sections living in a certain patch, implying a local operator formalism. Several useful formalisms have been set up to obtain global information via local operators, see e.g. [1,2].

Another powerful approach was initiated by the pioneering work of Krichever and Novikov [3,4], which permits the formulation of a global operator formalism on twice-punctured Riemann surfaces of arbitrary genus (see [5], and various recent issues of Phys. Lett. B), and leads to a considerable clarification of concepts.

However, the local approach is by no means obsolete, because the conceptual simplicity of the global approach has its price in form of certain technical difficulties. Very recently, the bases of holomorphic λ-differentials on twice-punctured surfaces introduced by Krichever and Novikov were generalized to the case of arbitrary many punctures [6,7]. This will be discussed, and further elaborated, in this talk.

II. Holomorphic λ-Differentials on Punctured Riemann Surfaces

In this talk I will denote by $\Omega^{\lambda}_{-D}(X)$ the space of meromorphic λ-differentials ω on a compact Riemann surface X of genus g with the property $\mathrm{ord}_P(\omega) \geq D(P)$ for all points $P \in X$, where D is a divisor on X, and λ takes integer values.

The dimensions of the spaces of holomorphic λ-differentials displayed in Table 1

Differential Geometric Methods in Theoretical Physics
Edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990

are well-known consequences of the Riemann-Roch theorem

$$\dim\Omega^{\lambda}_{-D}(X) = (2\lambda-1)(g-1)-\deg D+\dim\Omega^{1-\lambda}_{D}(X)$$

and the isomorphism $\Omega^{\lambda}(X) \sim \Omega^{0}_{\lambda K}(X)$.

Table 1: dimensions of spaces of holomorphic differentials

$$\dim\Omega^{\lambda}(X) = \begin{cases} (2\lambda-1)(g-1) & \text{if } (\lambda-1)(g-1) > 0 \\ g & \text{if } \lambda = 1 \\ 1 & \text{if } \lambda = 0 \text{ or } g = 1 \\ 0 & \text{if } \lambda(g-1) < 0 \end{cases}$$

These results were exploited in different guises in string theory and conformal field theory. However, until recently such a table was lacking for meromorphic λ-differentials. The gap can be filled in the following fashion [7]:

A point $P \in X$ is called <u>λ-generic with respect to the divisor D</u> if the Wronskian of a basis of $\Omega^{\lambda}_{-D}(X)$ does not vanish in P or $\dim\Omega^{\lambda}_{-D}(X) = 0$.

Due to the compactness of X there are only finitely many non λ-generic points for a fixed divisor D.

Then it is easy to prove the following Lemma [6,7]:

Let P be a λ-generic point with respect to the divisor D. Then for any positive integer γ holds

$$\dim\Omega^{\lambda}_{-D-\gamma P}(X) = \begin{cases} \dim\Omega^{\lambda}_{-D}(X)-\gamma & \text{if } 0 \le \gamma \le \dim\Omega^{\lambda}_{-D}(X) \\ 0 & \text{if } \gamma \ge \dim\Omega^{\lambda}_{-D}(X) \end{cases}$$

For a vanishing divisor the non-generic points are of course the λ-Weierstrass-points. The proof is an immediate generalization of a classical proof of the Weierstrass gap theorem [8] and can be found in [7].

A divisor D will be called λ-generic if those finitely many points where the divisor does not vanish can be enumerated in such a way that

$D = \sum_{j=1}^{N} D(P_j)P_j$ and for all P_j:

If $D(P_j) > 0$ then P_j is λ-generic with respect to $\sum_{k=1}^{j-1} D(P_k)P_k$, and

if $D(P_j) < 0$ then P_j is $(1-\lambda)$-generic with respect to $\sum_{k=1}^{j-1} -D(P_k)P_k$.

Henceforth, the divisor D will be assumed to be λ-generic throughout, with the value of λ determined by the context.

By an alternate application of the Riemann-Roch theorem and the Lemma it is possible to calculate Table 2 [7]:

A) $|(2\lambda-1)(g-1)| > g$ and D is λ-generic:

$$\dim\Omega^{\lambda}_{-D}(X) = \begin{cases} (2\lambda-1)(g-1)-\deg D & \text{if } \deg D \leq (2\lambda-1)(g-1) \\ 0 & \text{if } \deg D \geq (2\lambda-1)(g-1) \end{cases}$$

B) $\lambda = 0$ or $g = 1$ and D is 0-generic:

$$\dim\Omega^{0}_{-D}(X) = \begin{cases} 0 & \text{if } \deg D \geq 1-g \text{ and } \exists P: D(P) > 0 \\ 1 & \text{if } \deg D \geq -g \text{ and } \forall P: D(P) \leq 0 \\ 1-g-\deg D & \text{if } \deg D \leq -g \end{cases}$$

C) $\lambda = 1$ and D is 1-generic:

$$\dim\Omega_{-D}(X) = \begin{cases} g-1-\deg D & \text{if } \deg D \leq g-1 \text{ and } \exists P: D(P) < 0 \\ g-\deg D & \text{if } \deg D \leq g \text{ and } \forall P: D(P) \geq 0 \\ 0 & \text{if } \deg D \geq g \end{cases}$$

Table 2: dimensions of spaces of meromorphic differentials

The proof proceeds by induction on the number N of punctures appearing in the divisor D [7].
$Y = X\backslash\{P_1,...P_N\}$ will denote the punctured surface associated to X and the divisor $D = \sum_{j=1}^{N} D(P_j)P_j$.
An immediate consequence of Table 2 is the Corollary:

A) $|(2\lambda-1)(g-1)| > g$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is λ-generic for every enumeration of points and $\deg D = 2\lambda(g-1)-g$. then there exists an up to multiplication by a constant unique λ-differential ω which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(\omega) = D(P_j)$.

B) $\lambda = 0$ or $g = 1$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is 0-generic for every enumeration of points and $\deg D = -g$, and $\exists P_j$: $D(P_j) > 0$, or if D vanishes, then there exists an up to multiplication by a constant unique function f which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(f) = D(P_j)$.

C) $\lambda = 1$:

If $D = \sum_{j=1}^{N} D(P_j)P_j$ is 1-generic for every enumeration of points and

$\deg D = g-2$ and $\exists P_j$: $D(P_j) < -1$

or

$\deg D = g-2$ and $\exists\{P_j,P_k\}$: $j \neq k$ and $D(P_j) = D(P_k) = -1$

or

$\deg D = g-1$ and $\forall P_j$: $D(P_j) \geq 0$,

then there exists an up to multiplication by a constant unique differential μ which is meromorphic on X and holomorphic on Y and obeys $\mathrm{ord}_{P_j}(\mu) = D(P_j)$.

Once the correct pole orders are identified it is not hard to write down representations of the uniquely determined λ-differentials in terms of prime forms and theta functions. (For an introduction see e.g. [9]). This has been done for the case $N = 2$ already by Bonora et al. [5], and in [10]. Generalization to $N \geq 2$ requires only minor modifications:
If the picture of a divisor D of degree 0 under the usual map from the Picard group Pic(X) to the Jacobian torus Jac(X) of X is again denoted by D (by an obvious abuse of notation, because D now denotes both the divisor and its equivalence class modulo principal divisors), and if $\Delta = \Delta_B + (g-1)B$ is the Riemann divisor class, then a quasiperiodic g/2-differential σ with $\mathrm{div}(\sigma) = 0$ can be constructed [11]:

$$\sigma(P) = \vartheta(P - \sum_{j=1}^{g} S_j + (g-1)B + \Delta_B) \Big/ \prod_{j=1}^{g} E(P,S_j)$$

provided the divisor $E = \sum_{j=1}^{g} S_j$ satisfies $\dim\Omega_{-E}(X) = 0$. B is of course the base point of the Jacobian map for divisors with nonvanishing degree. I usually omit the period matrix in the argument of the theta function.
For $g \geq 2$, the λ-differentials mentioned in the Corollary are in the generic case:

$$\omega_D(P) = \vartheta(P + D - (2\lambda-1)(g-1)B - (2\lambda-1)\Delta_B)\cdot\sigma(P)^{2\lambda-1}\cdot E(P,P_1)^{2\lambda(g-1)-g}\cdot \prod_{j=2}^{N}\left(\frac{E(P,P_j)}{E(P,P_1)}\right)^{D(P_j)}$$

and in the special cases
$\lambda = 0$, $D = 0$: $f_0(P) = 1$
$\lambda = 1$, $D \geq 0$:

$$\mu_D(P) = \vartheta(P + D - g\cdot B - \Delta_B)\cdot\frac{\sigma(P)}{E(P,B)}\cdot E(P,P_1)^{g-1}\cdot\prod_{j=2}^{N}\left(\frac{E(P,P_j)}{E(P,P_1)}\right)^{D(P_j)}$$

The exponents in the above formulas are determined by the required pole orders and the correct transformation behaviour, while the arguments of the theta functions are fixed by the periodicity properties.
In the generic case, the missing g zeros of $\omega_D(P)$ are determined by the Riemann theorem [12]:

$$\sum_{j=1}^{g} Q_j + D = 2\lambda[\Delta_B + (g-1)B]$$

while in the case $\lambda = 1$, $D \geq 0$ the Riemann theorem yields for the missing g-1 zeros

$$\sum_{j=1}^{g-1} Q_j + D = 2[\Delta_B + (g-1)B].$$

In the singular case $g = 1$ all λ-differentials have to be constructed from $\lambda = 0$. Here it is convenient to use Weierstrass σ-functions, like in [3]. For the case of the twice-punctured (super-)torus this was pursued and further elaborated in [13], where the central extension of the corresponding Krichever-Novikov algebra is also displayed.

The case $g = 0$ will be considered in detail in section IV.
If the set of punctures $\Gamma = \{P_1,\ldots P_N\}$ admits only divisors which are λ-generic, and if those divisors, whose degrees and pole orders meet the conditions of the Corollary, are λ-generic for every enumeration of the punctures, then it is possible to write down the Krichever-Novikov basis $B_\lambda(Y)$ of holomorphic λ-differentials on Y which are meromorphic on X. This was done in [6,7].
In the notation introduced in [7] the result is:

A) $|(2\lambda-1)(g-1)| > g$:
$$B_\lambda(Y) = \{\omega_2(\beta): \beta \in \mathbb{Z}\} \cup \left\{ \bigcup_{j=3}^{N} \{\omega_j(\beta): \beta < 0\}\right\}$$

B) $\lambda = 0$:
$$B_0(Y) = \{f_2(\beta); \beta < -g \text{ or } \beta > 0\} \cup \left\{ \bigcup_{j=3}^{N} \{f_j(\beta): \beta < -g\}\right\} \cup \left\{ \bigcup_{j=2}^{N} \{h_j(\beta): -g \le \beta < 0\}\right\} \cup \{1\}$$

C) $\lambda = 1$:
$$B_1(Y) = \{\mu_2(\beta): \beta < -1 \text{ or } \beta \ge g\} \cup \left\{ \bigcup_{j=3}^{N} \{\mu_j(\beta); \beta < -1\}\right\} \cup \{\mu_0(\beta); 0 \le \beta < g\} \cup \left\{ \bigcup_{j=2}^{N} \{\nu_j\}\right\}$$

A simple proof of the completeness of these sets exploits Table 2 [7], and works in the following fashion: The linear independence is easily proved from the mismatch of pole orders. Furthermore, $\Omega^\lambda_{-E}(X) \subset \Omega^\lambda_{-D}(X)$ for $E > D$, and as any meromorphic differential on X which is holomorphic on Y is contained in $\Omega^\lambda_{-D}(X)$ with D chosen small enough, it is sufficient to observe that the $(m+1)$-dimensional space $\Omega^\lambda_{-D}(X)$ with
$$D = [2\lambda(g-1)-g-m-\sum_{j=2}^{N}\beta_j]P_1 + \sum_{j=2}^{N}\beta_j P_j$$
and
A) $|(2\lambda-1)(g-1)| > g$: $\beta_j < 0$, $m+\sum_{j=2}^{N}\beta_j \ge 0$

B) $\lambda = 0$: $\beta_j < -g$, $m+\sum_{j=2}^{N}\beta_j \ge 0$

C) $\lambda = 1$: $\beta_j < -1$, $m+\sum_{j=2}^{N}\beta_j \ge g$

is spanned by exactly $m+1$ differentials in $B_\lambda(Y)$.

III. Internal Time and Global Laurent Expansions on Punctured Riemann Surfaces

Punctured Riemann surfaces carry a natural notion of internal time [3,4,6,7].
To introduce this, it is convenient to switch from
$$\{\mu_0(\beta) = \mu[g-1-\beta,\beta,0,\ldots,0],\ 0 \le \beta < g\}$$
to the usual basis of holomorphic differentials defined by the homology basis chosen:
$$\mu_0(\beta) = \sum_{j=1}^{g} \omega_j \oint_{a_j} \mu_0(\beta)\ , \qquad \oint_{a_j} \omega_k = \delta^j{}_k\ , \qquad \oint_{b_j} \omega_k = \Omega^j{}_k$$
Furthermore, assume all abelian differentials $\nu_2 = \mu[-1,-1,g,0,\ldots,0]$ and $\nu_j = \mu[-1,g,0,\ldots,0,-1,0,\ldots,0]$ normalized to have residue 1 in P_1.

Then every abelian differential of the kind

$$k = \sum_{j=2}^{N} \alpha^j \nu_j$$

defines an internal evolution parameter via

$$\tau(P,P_O) = \int_{P_O}^{P} d\tau$$

$$d\tau = \mathrm{Re}k - \mathrm{Re}\omega_i \cdot \oint_{a_i} \mathrm{Re}k - \mathrm{Im}\omega_i \cdot (\mathrm{Im}\Omega)^{-1}\,{}^i{}_j \cdot [\oint_{b_j} \mathrm{Re}k - \mathrm{Re}\Omega^j{}_m \cdot \oint_{a_m} \mathrm{Re}k]$$

where summation from 1 to g is implied. If all the coefficients α^j and their sum are different from zero the time τ assumes the values $\pm\infty$ in the punctures. This notion of internal time implies, that there is another natural duality between λ-differentials and $(1-\lambda)$-differentials on Y, besides Serre duality [3,4,6]. If S_i denotes that set of indices for which

$\tau(P_j,P_O) = -\infty,\ j \in S_i$

and $C_\tau = \{P: \tau(P,P_O) = \tau\}$, then the dual pairing of a λ-differential ω and a $(1-\lambda)$-differential v is

$$\langle\omega,v\rangle = \frac{1}{2\pi i}\oint_{C_\tau} \omega\cdot v = \sum_{j\in S_i} \mathrm{Res}_{P_j}(\omega\cdot v)$$

To introduce global Laurent expansions on Y it is necessary to orthogonalize the bases of section II. This will be considered for $|(2\lambda-1)(g-1)| > g$:
For simplicity I assume all $\alpha^j > 0$ in the definition of k. Then the only initial point on X is P_1 and the scalar product reduces to the residue of the product in P_1. Like before, ω will be a λ-differential, while v is a $(1-\lambda)$-differential. Then, with a suitable normalization, $\omega_2(\beta)$ and $v_2(-1-\beta)$, $\beta \in \mathbb{Z}$, may serve as a starting point for the construction of orthonormal bases. This starting point is of course just the original Krichever-Novikov basis [3,4] of a twice-punctured surface, which is orthogonal from the very beginning. $B_\lambda(Y)$ can be orthonormalized via:

$$\Omega_2(\beta) = \omega_2(\beta)\ ,\ V_2(\beta) = v_2(-1-\beta)\big/\langle\omega_2(\beta),v_2(-1-\beta)\rangle$$

$$\Omega_j(\beta) = \omega_j(\beta) - \sum_{k=2}^{j-1}\sum_{\gamma} \langle\omega_j(\beta),V_k(\gamma)\rangle\ \Omega_k(\gamma)$$

$$V_j(\beta) = \Big(v_j(-1-\beta) - \sum_{k=2}^{j-1}\sum_{\gamma} V_k(\gamma)\ \langle\Omega_k(\gamma),v_j(-1-\beta)\rangle\Big):$$

$$:\Big(\langle\omega_j(\beta),v_j(-1-\beta)\rangle - \sum_{k=2}^{j-1}\sum_{\gamma} \langle\omega_j(\beta),V_k(\gamma)\rangle\langle\Omega_k(\gamma),v_j(-1-\beta)\rangle\Big)$$

Then the unique decomposition of any meromorphic λ-differential T on X which is holomorphic on Y reads

$$T = \sum_{j=2}^{N}\sum_{\gamma} \langle T,V_j(\gamma)\rangle\ \Omega_j(\gamma)$$

Because of its relevance for the analysis of initial value problems, equations like this are called equal time decomposition in physics.

As T is meromorphic, the sum over γ stops after finitely many terms. However, our experience with harmonic analysis suggests, that the above equation holds pointwise on Y, or in the mean, or in a distributional sense also for suitable classes of holomorphic λ-differentials on Y with essential singularities in some of the punctures.

If, in case $\lambda = 2$, T is interpreted as a classical energy momentum tensor, the modes

$$L_j(\gamma) = \langle T.V_j(\gamma)\rangle$$

reduce for N = 2 and g = 0 to the familiar Virasoro generators.

IV. The Virasoro Algebra for N > 2.

It is well known, that the Virasoro algebra without central extension is the algebra $Vec(S^1)$ of vector fields on the circle. From the work of Krichever and Novikov we learn, however, that it appears in string theory as the Lie algebra of the basis of meromorphic vector fields on S^2 which are holomorphic on the twice punctured sphere, and how it can be generalized to twice-punctured Riemann surfaces of arbitrary genus. The resulting Lie algebra of the basis of meromorphic vector fields holomorphic outside the two punctures was then analyzed qualitatively by calculations near the punctures [3]. This procedure was applied to the case of more punctures in [6]. Finally, of course, the structure coefficients should be calculated explicitly from the formulas of section II.

For technical reasons, I will concentrate here on the case of the N-punctured sphere [6,7,14]. This is interesting in its own, not only as a sphere approximation to the global formalism described here, but also in view of the local formalism [1,2], which comes into play by considering a single patch on Y which contains all the punctures.

Because it is not more complicated, let us consider from the beginning the action on the module of λ-differentials:

$$\mathcal{L}_v\omega = \lambda\cdot\omega\cdot\partial_z v^z + v^z\partial_z\omega \ .$$

Here and in the following equations, v will be a vector.

I will use charts U_j around the punctures, such that $z_j(P_j) = 0$, $z_1(P_j) = c_j$ for $j \geq 2$, and in the overlap regions $z_1 = c_j/(1+c_jz_j)$. The unique λ-differentials of section II are in these coordinates:

$$\omega[-2\lambda-\sum_{j=2}^{N}\beta_j,\beta_2\ldots\beta_N] = \prod_{j=2}^{N}\left(\frac{1}{z_1}-\frac{1}{c_j}\right)^{\beta_j}\cdot\left(\frac{dz_1}{z_1^{\,2}}\right)^{\lambda}$$

and the action of the basis of meromorphic vector fields on the sphere, which are holomorphic outside the N punctures, on the corresponding basis of λ-differentials is explicitly

$$\mathcal{L}_{v_j(\beta)}\omega_j(\gamma) = -(\gamma+\lambda\beta)\cdot\omega_j(\beta+\gamma-1)$$

$j \neq k,\ \beta < 0,\ \gamma < 0$:

$$\mathcal{L}_{v_j(\beta)}\omega_k(\gamma) = -\sum_{n=\beta-1}^{-1}[n+1+(\lambda-1)\beta]\binom{\gamma}{n-\beta+1}\left(\frac{c_k-c_j}{c_j\cdot c_k}\right)^{\beta+\gamma-n-1}\cdot\omega_j(n) - $$
$$-\sum_{n=\gamma-1}^{-1}[\lambda(n+1-\gamma)+\gamma]\binom{\beta}{n-\gamma+1}\left(\frac{c_j-c_k}{c_j\cdot c_k}\right)^{\beta+\gamma-n-1}\cdot\omega_k(n)$$

$k \neq 2,\ \beta \geq 0$:

$$\mathcal{L}_{v_2(\beta)}\omega_k(\gamma) = -\sum_{n=\gamma-1}^{\beta+\gamma-1}[\lambda(n+1-\gamma)+\gamma]\binom{\beta}{n-\gamma+1}\left(\frac{c_2-c_k}{c_2\cdot c_k}\right)^{\beta+\gamma-n-1}\cdot\omega_k(n)$$

$j \neq 2,\ \gamma \geq 0$:

$$\mathcal{L}_{v_j(\beta)}\omega_2(\gamma) = -\sum_{n=\beta-1}^{\beta+\gamma-1}[n+1+(\lambda-1)\beta]\binom{\gamma}{n-\beta+1}\left(\frac{c_2-c_j}{c_2\cdot c_j}\right)^{\beta+\gamma-n-1}\cdot\omega_j(n)$$

If $\beta+\gamma > 0$ part of the last equations must be decomposed further according to $j > 2,\ n \geq 0$:

$$\omega_j(n) = \sum_{m=0}^{n}\binom{n}{m}\left(\frac{c_j-c_2}{c_j\cdot c_2}\right)^{n-m}\cdot\omega_2(m)$$

For a digression on central extensions, see [14].
A closer look at other representations of this algebra and its role in conformal field theory on punctured surfaces remains for future work.

A better handling of punctured Riemann surfaces offers three paths to follow: The conceptually most clear, but maybe also most intricate, path might start off from the "explicit" formulas of section II, considering from the beginning global Riemann surfaces of arbitrary genus.
Another approach would employ the local formalisms already mentioned and, presumably, lead to an intense investigation of the Virasoro algebras on N-punctured spheres.
In a similar manner, holomorphic λ-differentials on N-punctured spheres as dynamical degrees of freedom, and as correlation functions, might be discussed in the framework of a "sewing approach" to higher genus string vertices.
Of course, much of the necessary technology has already been developed in a considerable common effort of many physicists and mathematicians, and much remains to be done.

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References

1. L. Alvarez-Gaume, C. Gomez, G. Moore, and C. Vafa, Nucl. Phys. B **303** (1988) 455
2. H. Sonoda, Nucl. Phys. B **281** (1987) 546
   T. Eguchi and H. Ooguri, Nucl. Phys. B **282** (1987) 308
3. I. M. Krichever and S. P. Novikov, Funkts. Anal. Prilozhen. **21**(2) (1987) 46
4. I. M. Krichever and S. P. Novikov, Funkts. Anal. Prilozhen. **21**(4) (1987) 47
5. L. Bonora, A. Lugo, M. Matone, and J. Russo, Commun. Math. Phys. **123** (1989) 329
6. M. Schlichenmaier: Krichever-Novikov Algebras for More than Two Points, preprint Manusk. Fak. Math. u. Inf. Mannheim 97-1989 (April 1989)
7. R. Dick: Krichever-Novikov-like Bases on Punctured Riemann Surfaces, preprint DESY 89-059 (May 1989), to appear in Lett. Math. Phys.
8. G. Springer: Introduction to Riemann Surfaces, Addison-Wesley, Reading 1957, pp. 272-274
9. D. Mumford: Tata Lectures on Theta I, II, Birkhäuser, Boston 1983
10. S. Klimek and A. Lesniewski: Global Laurent Expansions on Riemann Surfaces, preprint HUTMP B 234 (March 1989)
11. J. D. Fay: Theta Functions on Riemann Surfaces, Lecture Notes in Mathematics 352, Springer, Berlin 1973
12. J. Lewittes, Acta Math. **111** (1964) 37
13. L. Mezincescu, R. I. Nepomechie, and C. K. Zachos, Nucl. Phys. B **315** (1989) 43
14. H.Y. Guo, J. S. Na, J. M. Shen, S. K. Wang, and Q. H. Yu: The algebra of meromorphic vector fields and its realization on the space of meromorphic λ-differentials on Riemann surface (I), preprint AS-ITP-89-10 (May 1989)

ANOMALIES. BRS SYMMETRY AND SUPERCONNECTIONS

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Abstract The anomaly problem in quantum field theory is reviewed, including its resolution in terms of BRS cohomology. The geometric meaning of the BRS symmetry is clarified by the use of superconnections in an appropriate fibre bundle.

1. Introduction

The BRS symmetry was first discovered by Becchi, Rouet and Stora[1] as a symmetry of the Yang-Mills Lagrangian. It was an important tool in establishing the renormalizability of these theories, by allowing a proof of the Ward identities. It was afterwards found to be a symmetry of many subsequently developed models and it continues to play an important role in the investigation of these models, as is witnessed to by a number of talks at this Conference.

An important application of BRS symmetry turned out to be the computation and classification of anomalies in quantum field theories in terms of cohomology classes of appropriate bi-graded algebras.[2] Indeed, this work represents one of the most powerful applications of modern mathematics in elementary particle physics to date. It is an impressive marriage of exact mathematics and important physics. Marriages are rarely perfect, and in this case the mismatch involves the issue of locality: if we assume the anomalies to be local polynomials in the fields and their derivatives we may use algebraic methods to compute and classify them, they cannot, however, be proved to be local by algebraic methods alone.

In most of this work, it seems to us, the question of what the BRS

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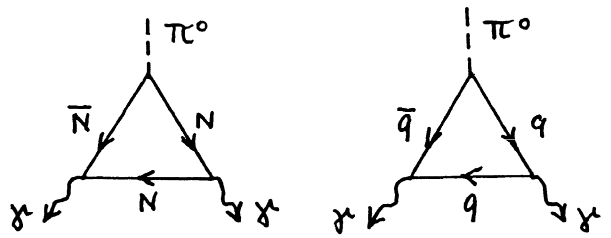
symmetry really is is left open. It is obviously related in some way to the gauge symmetry of the corresponding classical theory, but can this relationship be made precise? Together with F. Hegenbarth, of the Mathematics Department of the University of Potenza, we have attempted to clarify this question. We exhibit a certain graded fibre bundle which carries the degrees of freedom of the quantum field theory. The even part of a superconnection on this bundle represents the usual gauge potential, the odd part the Faddeev-Popov ghost field. An ordinary gauge transformation of the gauge potential induces the BRS transformation of the ghost field. A preliminary version of this work was reported on at the previous meeting in this series[3].

2. The Role of Anomalies in Physics

The first anomaly to be discovered was the chiral triangle anomaly. It was encountered in a calculation of the expected decay rate of the (at that time hypothetical) neutral pi-meson into two gamma rays by a young theoretician at the Institute for Advanced Study in Princeton in 1949[4]. The pion was assumed to decay virtually into its two nucleonic components, which then couple to the photons, as indicated in the first Feynman diagram below.



The calculation yielded for the pion decay amplitude a term proportional to $F \wedge F$, where F is the eloctromagnetic field strength. On the other hand, the amplitude was known to be proportional to the divergence of the axial vector current, which as a Noether current associated with the approximate chiral symmetry of the corresponding classical field theory was supposed to be conserved. This apparent dilemna convinced the theoretician to become an experimentalist. His subsequent career seems to justify this decision, for Jack Steinberger shared the 1988 Nobel prize for Physics.

The calculation of the pion decay amplitude was returned to in 1969 by Adler, and by Bell and Jackiw[5]. At this time the pion was believed to be a composite quark-antiquark state, as shown in the second figure above. By now an experimental value for the pion decay rate was available, which differed from the calculated value by a factor of three. This was Gell-Mann's principal motivation for postulating the SU(3) color symmetry of QCD[6].

In gauge theories the cancellation of anomalies is believed to be a necessary condition for renormalizability, and the compensation of the quark-loop and lepton-loop anomalies in the Weinberg-Salam theory led to the possibility of including hadrons into this theory, and to the prediction of the third generation of quarks[7]. The anomaly cancellation for the E(8)xE(8) gauge group initiated the superstring revival in 1984[8].

3. Translating the Physical Problem into Mathematics

The basic object in quantum field theory is the quantum action funtional $\Gamma(A,\psi)$. To lowest order in Planck's constant $\hbar$ it agrees with the classical action $\Gamma^{(o)}(A,\psi)$. The gauge invariance of the classical theory is expressed by $\delta\Gamma^{(o)} = 0$, where δ is the variation induced by a gauge transformation of the classical fields. We define $\delta\Gamma = \Delta$ to be the anomaly of the theory, and then obviously the quantum theory shares the symmetry of the classical theory only if the anomaly vanishes.

An infinitesimal gauge transformation in a non-Abelian gauge theory is characterized by an element ξ in the Lie algebra of the group of gauge transformations. The anomaly $\Delta = \delta\Gamma$ is linear in ξ, and turns out to be a local functional of A; $\Delta(A,\xi) = \int Q(A,\xi)$, where the integral is over spacetime. A finite renormalization of the quantum action leads to an additional term in Δ of the form $\delta\Gamma^{loc}$, where Γ^{loc} is a local functional of A.

When the action of δ on differential forms on spacetime, which are also antisymmetric multilinear functions of the elements ξ of the Lie algebra, is carefully defined, then δ turns out to be a nilpotent coboundary operator: $\delta^2 = 0$. We thus see that $\delta\Delta = 0$, and, up to finite renormalization, Δ and $\Delta + \delta\Gamma^{loc}$ are to be considered equivalent. The problem of determining the anomalies of a theory thus turns out to be equivalent to computing the cohomology of the space of local functionals. The action of δ referred to above corresponds to the BRS transformation of the fields; this is the way in which the BRS symmetry becomes relevent for the computation of anomalies.

The BRS variation δ combines with the ordinary exterior derivative in the space of differential forms to yield a *total differential* $d + \delta$ on the *bigraded* algebra of forms, where the first grading involves the de Rham degree of the differential form, the second the degree in the ξ's. This latter degree turns out to be equivalent to the *ghost number*. In this way we actually have a *double complex*.

4. The Mathematical Tools

The branch of mathematicas which deals with the computation of such cohomologies is the *Chern-Weil theory of characteristic classes*, developed in the 1950's[9]. The foundation of this theory is the *Weil homomorphism*, which may be formulated as follows. Suppose we are given a principal bundle $P \to M$ with structure group G. For a fixed connection A on P there is an associated curvature $F = dA + [A, A]$. The Weil homomorphism is a homomorphism of the algebra of invariant polynomials in F into the de Rham cohomology algebra $H(M)$. Its great importance stems from the fact that it is independent of the choice of connection. The basic formula is the following. Let A_1 and A_2 be two connections on P. Define $A_t = A_1 + t(A_2 - A_1)$. The curvature associated with A_t is F_t. The fact that the Weil homomorphism is independent of the choice of connection is expressed by the *Chern Transgression Formula*:

$$\mathrm{Tr}\, F_1{}^n - \mathrm{Tr}\, F_2{}^n = n\, d \int_0^1 dt\, \mathrm{Tr}\, (A_1 - A_2)\, F_t{}^{n-1}$$

The elements of the subalgebra of $H(M)$ which are in the image of the Weil homomorphism are the *characteristic classes.*

The application of the above formulae to a *classical* gauge theory would be straightforward, as the corresponce of the gauge potential of a classical field theory to the connection in a principal bundle, whose base manifold is spacetime and whose structure group is related to the gauge group, is well known. However, since the anomalies are a phenomenom of *quantum* field theory, the problem of the quantization of gauge theories must still be dealt with.

5. The Quantization of Gauge Theories

Since classical gauge theories contain unphysical degrees of freedom they correspond to constrained Hamiltonian systems. The BRS quantization of such systems has been treated in an exact mathematical framework by Kostant and Sternberg[10]. In the quantum system the unphysical degrees of freedom correspond to the Faddeev-Popov ghost fields. The observables are polynomials in the gauge fields and their derivatives times antisymmetric products of the ghost fields. The ghost number provides a grading which is in addition to the usual de Rham grading of the differential forms. Denoting the ghost fields by η the BRS transformations of the fields are:

$$\delta A = -d\eta - [A, \eta],$$
$$\delta \eta = -[\eta, \eta].$$

In the algebraic approach the relevent algebra is the *Weil-BRS* algebra, which is a generalization of the classical Weil algebra of the Lie algebra of G. The Weil algebra is generated by the algebraic connections A and the curvatures F, the Weil-BRS algebra involves further generators η and $d\eta$ corresponding to the ghost fields. The algebraic connection on the bigraded Weil-BRS algebra is $A + \eta$, with the associated curvature

$$F = (d+\delta)(A+\eta) + [A+\eta, A+\eta].$$

Because of the BRS property of the fields A and η we have the remarkable property

$$F = \mathrm{F}.$$

This property allows us to generalize the Chern transgression formula to a formula involving the total differential operator $d+\delta$ and the connection $A+\eta$. When this formula is decomposed according to the ghost numbers of its terms it yields the well-known *descent equations* from which the anomalies are actually computed.

6. Geometric Interpretation of the BRS-Transformations

F. Hegenbarth and myself have attempted to give a geometric interpretation for the BRS transformations in Yang-Mills theory. For this purpose the superconnection concept as used by Quillen[11] proves to be useful.

The starting point is a principal G-bundle $P \to M$ and a $\mathbb{Z}_2$-graded vector space V. In our application the grading will be provided by the ghost number. ρ is a representation of G in V, and $E = P \otimes_\rho V$ the associated bundle. The grading of V induces a grading $E = E^0 \oplus E^1$. The adjoint representation of G may be used to construct the graded bundle

$$\mathrm{End}\ E = P \otimes_{\mathrm{ad}} \mathrm{End}\ V = (\mathrm{End}\, E)^0 \oplus (\mathrm{End}\, E)^1.$$

An element $\eta \in (\mathrm{End}\, E)^1$ extends to an endomorphism η: $\Omega^*(M,E) \to \Omega^*(M,E)$ satisfying

$$\eta(\omega \otimes s) = (-1)^k \omega\eta(s)$$

for $\omega \in \Omega^k(M)$.

If $D : \Omega^*(M,E) \to \Omega^*(M,E)$ is a connection on E preserving the grading, then ${}^sD = D + \eta$ is a *superconnection.*

$\Omega^0(M, \mathrm{ad}\, E)$ is the Lie algebra of the gauge group. The representation ρ induces a map ρ^*: $\Omega^0(M, \mathrm{ad}\, P) \to \Omega^0(M, (\mathrm{End}\, E)^0)$ described in [12]. The map δ is the composition $\delta = {}^sD \circ \rho^*$. For T an element of the Lie algebra of the gauge group, ${}^sD + \delta(T)$ is an *infinitesimal gauge transformation* of ${}^sD$. We write $\delta^T({}^sD) = \delta(T)$.

In a local chart $U \in M$ a superconnection may be written as ${}^sD = d + A + \eta$, with $A \in \Omega^1(U, (\mathrm{End}\, V)^0)$ and $\eta \in \Omega^0(U, (\mathrm{End}\, V)^1)$. A is the gauge potential and η the ghost field. We may then write

$$\delta^T({}^sD) = -dT - [A, T] - [\eta, T],$$

$$\delta^T(A) = -dT - [A, T]$$
$$\delta^T(\eta) = \quad -[\eta, T].$$

Identifying End V with Lie G via ρ we may choose in particular $\eta = T$, and the above formulae become, essentially, the BRS transformations of the previous section. This is worked out in greater detail in Ref. [3].

Our conclusion is that the BRS transformation may be interpreted as a gauge transformation of the superconnection. The BRS formalism for the quantum gauge theory thus becomes a staightforward generalization of the Gupta-Bleuler formalism for quantum electrodynamics.

References

1. C. Becchi, A. Rouet and R. Stora, Phys. Lett. B52, 344 (1974).

2. J. Dixon, Proceedings of the 17th International Conference on Differential Geometric Methods in Theoretical Physics, Chester, England August 1988, and unpublished.

   R. Stora, in "Progress in Gauge Field Theories", eds. G. t'Hooft, A. Lehmann, P. K. Mitter, I. M. Singer and R. Stora, Plenum, New York (1984).

   M. F. Atiyah and I. M. Singer, Proceedings of the National Academy of Sciences, USA, Vol. 81, 2597 (1984).

   B. Zumino, in "Relativity, Groups and Topology", eds. B. de Witt and R. Stora, North Holland, Amsterdam (1984).

   L. Bonora and P. Cotta-Ramusino, Comm. Math. Physics 87, 589 (1983).

   M. Dubois-Violette, in "Fields and Geometry", 22nd International Winter School in Karpacz, Poland, ed. A. Jadzyk (1986).

3. F. Hegenbarth and A. C. Hirshfeld, Proceedings of the 17th International Conference on Geometric Differential Methods in Theoretical Physics, Chester, England, July 1988.

4. J. Steinberger, Phys. Rev. 76, 1180 (1949).

5. S. L. Adler, Phys. Rev. 177, 2426 (1969).

   J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).

6. M. Gell-Mann, Acta Phys. Austriaca Suppl. IV, 733 (1972).

7. S. L. Glashow, J. Iliopoulos and L. Maiani, Phys. Rev. D2, 1285 (1970).

8. M. B. Green and J. H. Schwarz, Phys. Lett. 149B, (1984).

9. W. Greub, S. Halperin and R. Vanstone, "Connections, Curvature and Cohomology, Academic Press, New York (1972).

10. B. Kostant and S. Sternberg, Ann. Phys. 176, 49 (1987).

11. D. Quillen, Topology 24, 89 (1985).

12. H. B. Lawson, "The Theory of Gauge Fields in Four Dimensions", CBMS Regional Conference Series in Mathematics, Nr. 58, (1983).

GENERAL COVARIANCE AND STRINGS

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It may be that (super)strings are part of the elusive quantum theory of gravity. If so, one obvious question is whether or not string field theory is generally covariant. Recall that string field theory (for a review see Ref. 1) involves the string creation/annihilation operator $\Phi[X]$, where, for brevity, we omit the ghost coordinates. The action of the free open string is

$$\int DX\ \Phi[X]\ Q\ \Phi[X] \quad , \tag{1}$$

where $Q = Q(X, \delta/\delta X)$ is the so called BRST operator and $\int DX$ integrates over the space of all strings, i.e., "loop space" (note that also for open strings we speak about "loop space"). Since both DX and Q involve the Minkowski metric $\eta_{\mu\nu}$ the action (1) is clearly not generally covariant. Actually, we are interested in covariance not only over the finite dimensional "target space" but over the infinite dimensional "loop space." The most straight forward thing to do is to introduce a background loop space metric and to follow, as closely as possible, the steps of standard general relativity. This program was carried out by J. Greensite and the present author. Here I will list the basic results only, further details can be found in Ref. 2.

1. Notation. In the string coordinate we combine the discrete index $\mu\epsilon\{1, 2, \ldots, D\}$ and the continuous index $\sigma\epsilon[0, \pi]$ into a generalized

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index, for which we will use a generalized summation convention. So we have

$$x^{\mu\sigma} \equiv x^{\mu}(\sigma) \tag{2}$$

and the string as a whole will be denoted by a capital letter X.

2. <u>Tangent Contravariant General Coordinate Transformations</u> (or TC-transformations for short) are coordinate transformations

$$x'^{\mu\sigma} = x'^{\mu\sigma}[X] \quad , \tag{3}$$

where x' is a <u>functional</u> of the whole string X, which obey the TC-condition

$$\left(\frac{\partial}{\partial\sigma_1} x'^{\mu\sigma_1} \right) = \frac{\delta x'^{\mu\sigma_1}}{\delta x^{\nu\sigma_2}} \left(\frac{\partial}{\partial\sigma_2} x^{\nu\sigma_2} \right) \tag{4}$$

with ν and σ_2 on the right-hand side summed/integrated over. Of course, the transformations (3) have to be differentiable and invertible.

3. <u>TC-Metrics (and Vielbeins)</u>

$$g_{\mu\sigma_1\ \nu\sigma_2}[X] \equiv e_{\mu\sigma_1}^{\ \ a\sigma_3}[X]\ \eta_{ab}\ \delta(\sigma_3 - \sigma_4)\ e_{\nu\sigma_2}^{\ \ b\sigma_4}[X] \tag{5}$$

are those which can be transformed flat locally in loop space ($g_{\mu\sigma_1\ \nu\sigma_2}[X_0] = \eta_{\mu\nu}\ \delta(\sigma_1 - \sigma_2)$) by a TC-transformation. This gives conditions on the allowed vielbeins, see Eqs. (22, 23) of Ref. 2. With the preceding definitions and results the action (1) can easily be made TC-invariant. However, in order to have a sensible string theory we still need to verify that $Q^2 = 0$. This gives further constraints.

4. <u>Virasoro Algebra Constraints</u> allow only those TC metrics, which, in suitable coordinates, take the form

$$g_{\mu\sigma_1\ \nu\sigma_2}[X] = G_{\mu\nu}(x^{\lambda\sigma_1})\ \delta(\sigma_1 - \sigma_2) \tag{6a}$$

with $G_{\mu\nu}(x)$ a D-dimensional metric with vanishing Ricci tensor

$$R_{\mu\nu}(x) = 0 \quad . \tag{6b}$$

This completes the construction of the free open string field theory with invariance under loop space general coordinate transformations (3, 4). In Ref. 2 we have indicated how to incorporate open string interactions (presumably free closed strings can also be dealt with).

The restriction (6a) is quite severe and seems to rule out truly "stringy" behaviour of the loop space metric. Note, however, that for any given string X_0, including <u>self intersecting</u> ones, we can still make a TC-transformation so that $g_{\mu\sigma_1\ \nu\sigma_2}[X_0'] = \eta_{\mu\nu}\ \delta(\sigma_1 - \sigma_2)$. Of course, in general relativity one cannot, in general, make the metric $\tilde{g}_{\mu\nu}(x)$ flat along a closed curve. Another way to see the difference is to compare (6b) with the standard non-linear sigma model results[3] of the bosonic string (the tilde indicates that we are using the metric $\tilde{g}_{\mu\nu}(x)$)

$$\tilde{R}_{\mu\nu}(x) + \frac{1}{4\pi T}\,\tilde{R}_{\mu\rho\sigma\tau}(x)\ \tilde{R}_{\nu}^{\rho\sigma\tau}(x) + O(T^{-2}) = 0 \quad , \tag{7}$$

where the full Riemann tensor appears in a two loop calculation and T is the string tension. We conclude that the Polyakov theory of a string fluctuating in a background space with metric $\tilde{g}_{\mu\nu}(x)$, which has to obey (7), is not obviously equivalent to the TC-covariant string field theory. The physical interpretation of the latter is far from clear for the moment.

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REFERENCES

1. C. B. Thorn, "String Field Theory," Phys. Reports <u>175</u>: 1 (1989).
2. J. Greensite and F. R. Klinkhamer, "Approach to General Covariance in String Space of BRST String Field Theory," Phys. Rev. <u>D39</u>: 2317 (1989).
3. D. Friedan, "Nonlinear Models in 2 + ϵ Dimension," Phys. Rev. Lett. <u>45</u>: 1057 (1980).

SPONTANEOUS SYMMETRY BREAKING IN 4-DIMENSIONAL HETEROTIC STRING

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ABSTRACT

The evolution of a 4–dimensional heterotic string is considered in the background of its massless excitations such as graviton, antisymmetric tensor, gauge fields and scalar bosons. The compactified bosonic coordinates are fermionized. The world–sheet supersymmetry requirement enforces Thirring–like four fermion coupling to the background scalar fields. The non–abelian gauge symmetry is exhibited through the Ward identities of the S–matrix elements. The spontaneous symmetry breaking mechanism is exhibited through the broken Ward identities. An effective 4–dimensional action is constructed and the consequence of spontaneous symmetry breaking is envisaged for the effective action.

The string theory[1] offers a promise of unifying all the fundamental forces of nature. One of the marvels of the string theory lies in its rich symmetry structure. It has been argued by Gross[2] that all the string states are gauge particles and most of the string symmetries are spontaneously broken leaving only the familiar local symmetries of the theory. Moreover, the high energy behavior of the scattering amplitudes at Planckian energies reveals many interesting features of the string theory[3]. However, at low energies, energies much smaller than the Planck scale, the string theory is expected to exhibit the salient features of those theories which describe the low energy phenomena adequately. Therefore, all the gauge symmetries, manifest at Planck scale, do not remain unbroken at lower energies. It is now recognized that the Higgs' mechanism plays a cardinal role in the models unifying the fundamental forces. Recently, there have been several attempts to contruct four dimensional string theories following the work of Narain.[4–9] The mechanism of symmetry breaking and the Higgs' phenomena has been envisaged for 4–dimensional heterotic string theory.[10–12]

In this talk, I shall consider the evolution of a 4–dimensional heterotic string in the background of its massless excitations and investigate the phenomena of spontaneous symmetry breaking. The invariance of the action under world–sheet supersymmetry transformations imposes stringent constraints on the coupling of the string to the relevant background fields.

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We consider a 4-dimensional heterotic string model where X^μ are the four space-time coordinates and ψ^μ are their world sheet super partners. The compactified bosonic coordinates are fermionized[13,14] such that the 22 right moving compactified bosonic coordinates give 44 Majorana-Weyl fermions denoted by $\eta^A, A = 1, \cdots 44$. The left moving sector consists of 12 Majorana-Weyl fermions obtained from six compactified bosonic coordinates and six of their super partners collectively denoted by $\chi^a, a = 1, \cdots 18$. These 18 fermions must transform in the adjoint representation of a semi-simple Lie group G in order to facilitate non-linear realization of world-sheet supersymmetry[14] and the choice of G is restricted to $SU(2)^6, SU(3) \otimes SO(5)$ and $SU(4) \otimes SU(2)$ for the 4-dimensional case. In the fermionic formulation, various solutions are obtained by suitable choice of mutually commuting boundary conditions diagonalized in some general complex basis for the fermions consistent with the requirements of modular invariance(9). Let us choose a simple boundary condition where all fermions satisfy Neveau-Schwarz boundary condition. This theory has a tachyonic ground state and several massless encitations; however, the mechanism of spontaneous symmetry breaking can be demonstrated through this simple example. The ground state and the massless states of this theory are as follows:

a) The ground state is a taelyon in the vector representation of $SO(44)$

$$T_A : \eta^A_{1/2} |0> \qquad M^2 = -1/2 \tag{1}$$

b) There are massless gauge bosons in the adjoint representation of $SO(44)$ and G, respectively

$$A^{AB}_\mu : \chi^\mu_{1/2}\eta^A_{1/2}\eta^B_{1/2}|0> \tag{2}$$

$$W^a_\mu : \chi^a_{1/2}\partial X^\mu_R |0> \tag{3}$$

c) Massless scalar bosons in the adjoint representations of both $SO(44)$ and G.

$$\zeta^a_{AB} : \chi^a_{1/2}\eta^A_{1/2}\eta^B_{1/2}|0> \tag{4}$$

d) The usual massless states such as graviton, dilaton and antisymmetric tensor fields are present in addition to the massless states presented above: (2), (3) and (4). The action for the evolution of the 4-dimensional heterotic string in the background of massless fields is[15]

$$\begin{aligned} S &= \frac{1}{2}\int d^2\sigma\Big[G_{\mu\nu}(X)\partial_+X^\mu\partial_-X^\nu + \epsilon^{ab}\partial_aX^\mu\partial_bX^\nu B_{\mu\nu}(X) \\ &+ i\psi^i\partial_-\psi^i + i\psi^i\left(w^{ij}_\mu - S^{ij}_\mu\right)\psi^j\partial_-X^\mu \\ &+ i\eta^A\partial_+\eta^A + i\chi^a\partial_-\chi^a + \eta^A T^m_{AB}\eta^B A^m_\mu(X)\partial_+X^\mu \\ &- \frac{1}{2}\eta^A T^m_{AB}\eta^B F^m_{\mu\nu}(X)\psi^\mu\psi^\nu - if^{abc}\eta^A T^m_{AB}\eta^B\zeta^a_m(X)\chi^b\chi^c \\ &- i\eta^A T^m_{AB}\eta^B D_\mu\zeta^a_m(X)\psi^\mu\chi^a - \frac{i}{2}C^{mnp}\eta^A T^p_{AB}\eta^B\zeta^a_m(X)\zeta^b_n(X)\chi^a\chi^b\Big] \end{aligned} \tag{5}$$

obtained by generalizing the earlier method to construct action for heterotic string in background fields.[16] Our notations are as follows: $G_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$ are the background graviton and antisymmetric fields respectively. $\psi^i = e^i_\mu(X)\psi^\mu$ being the vierbeins and w^{ij}_μ

are the torsion–free connections whereas $S_{\mu\nu\gamma} = \frac{1}{2}(\partial_\mu B_{\mu\gamma} + \partial_{\gamma\mu} + \partial_\gamma B_{\mu\nu})$ is the field strength associated with $B_{\mu\nu}$; $F^m_{\mu\nu}$ are the field strength of the gauge background potential A^m_μ. The structure constants of the group G and $SO(44)$ are denoted by f^{abc} and C^{mnp} respectively; and $D_\mu\zeta^a_m(X) = \partial_\mu\zeta^a_m(X) + C^{mnp}A^n_\mu(X)\zeta^a_p(X), \zeta^a_p(X)$ being the scalar background. We have coupled only the gauge fields $A^m_\mu(X)$ to the stringth for the sake of simplicity in order to illustrate the mechanism of spontaneous symmetry breaking; however, the gauge fields $W^a_\mu(X)$ can be coupled to the string in a similar manner[16]. We set $G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$, in what follow, to study the mechanism of spontaneous symmetry breaking. The action is invariant under following super–symmetry transformations ($G_{\mu\nu} = \eta_{\mu\nu}$ and $B_{\mu\nu} = 0$).

$$\delta X^\mu = \epsilon\psi^\mu \tag{6}$$

$$\delta\psi^\mu = i\epsilon\partial_+ X^\mu \tag{7}$$

$$\delta\eta^A = i\epsilon\eta^B T^m_{BA} A^m_\mu(X)\psi^\mu + i\epsilon\eta^B T^m_{BA}\zeta^a_m(X)\chi^a \tag{8}$$

$$\delta\chi^a = \epsilon f^{abc}\chi^b\chi^c \tag{9}$$

Notice the Thirring–like coupling of the fermions to the scalar background and the coupling corresponds to similar ones for a constant background field considered earlier.[17] However, we note that the world–sheet supersymmetry forces us to include additional terms in the action (5). Our strategies are as follows:

Define the S–matrix generating functional

$$\sum[A,\zeta] = \int d[phase\ space]\, d[ghosts]\, exp(iS_H) \tag{10}$$

where $d[phase\ space]$ is the Hamiltonian phase space measure involving x^μ, ψ^μ, η^A and χ^a and the corresponding canonical momenta P_μ, π^μ, π^A and π^a respectively. The ghosts appear as a consequence of the (1,0) superconformal symmetry. S_H is the Hamiltonian action given by

$$S_H = \int d^2\sigma\left[\dot{X}^\mu P_\mu + \psi^i\dot{\psi}^i + \eta^A\dot{\eta}^A + \chi^a\dot{\chi}^a - H\right] + \int d^2\sigma L_{ghost} \tag{11}$$

and

$$\begin{aligned}
H &= \frac{1}{2}\bar{P}_\mu\bar{P}_\nu\eta^{\mu\nu} + \frac{1}{2}X'^\mu X'^\nu\eta_{\mu\nu} + \Pi^\mu\partial_1\psi^\nu\eta_{\mu\nu} + \Pi^a\partial_1\chi^a - \Pi^A\partial_1\eta^A \\
&+ \frac{1}{2}\eta^A T^m_{AB}\eta^B A^m_\mu X'^\mu + \frac{i}{4}\eta^A T^m_{AB}\eta^B F^m_{\mu\nu}\psi^\mu\psi^\nu + \frac{i}{2}f^{abc}\chi^b\chi^c\zeta^a_m\eta_A T^m_{AB}\eta^B \\
&+ \frac{i}{2}\eta^A T^m_{AB}\eta^B D_\mu\zeta^a_m\psi^\mu\chi^a + \frac{i}{4}C^{mnp}\eta^A T^m_{AB}\eta_B\zeta^a_m\zeta^b_n\chi^a\chi^b \\
\bar{P}_\mu &= P_\mu - \frac{1}{2}\eta T\eta A_\mu
\end{aligned} \tag{12}$$

L_{ghost} is the ghost Lagrangian which has the same form as the L_{ghost} for the heterotic string in the absence of background fields and we do not need its explicit form to derive WI and the effects of spontaneous symmetry breaking. Let us introduce the following infinitesimal generator of a canonical transformation of the form[18]

$$\Phi = \frac{1}{2}\int d\sigma\eta^A T^m_{AB}\eta^B\Lambda^m(X) \tag{13}$$

where $\Lambda(X)$ is an arbitrary function. The variations induced by Φ are

$$\delta_\Phi \eta^A = iT^m_{AB}\eta^B\Lambda^m(X) \tag{14}$$

$$\delta_\Phi P_\mu = \frac{1}{2}\eta^A T^m_{AB}\eta^B\partial_\mu\Lambda^m(X) \tag{15}$$

$$\delta_\Phi X^\mu = \delta_\Phi\psi^\mu = \delta_\Phi\chi^a = \delta_\Phi(ghosts) = 0 \tag{16}$$

The Hamiltonian action satisfies the following relation

$$\delta_\Phi S_H = -\delta_G S_H \tag{17}$$

where $\delta_G S_H$ means that we perform the gauge variations of the background fields $A^m_\mu(X)$ and $\zeta^a_m(X)$ only

$$\delta_G A^m_\mu(X) = \partial_\mu\Lambda^m(X) + C^{mnp}A^n_\mu(X)\Lambda^p(X) \tag{18}$$

$$\delta_G\zeta^a_m(X) = C^{mnp}\zeta^a_n(X)\Lambda^p(X) \tag{19}$$

Now we argue that the path integral phase space measure remains invariant, at least classically, under the canonical transformations (14)–(16). Therefore, the generating functional $\sum[A,\zeta]$ exhibits the following gauge invariance property due to (15).

$$\sum[A,\zeta] = \sum[A+\delta_G A, \zeta+\delta_G\zeta] \tag{20}$$

Consequently,

$$\int dY\left(\frac{\delta\sum}{\delta A^m_\mu(Y)}\delta_G A^m_u(Y) + \frac{\delta\sum}{\delta\zeta^a_m(Y)}\delta_G\zeta^a_m(Y)\right) = 0 \tag{21}$$

Using eq. (18) and (19) in (21), we arrive at

$$<\int d^2\sigma\left[V^m_\mu(X)\left\{\partial_\mu\Lambda^m(X) + C^{mnp}A^n_\mu(X)\Lambda^p(X) + V^a_m(X)C^{mnp}\zeta^a_n(X)\Lambda^p(X)\right\}\right] >= 0 \tag{22}$$

where $V^m_\mu(X)$ and $V^m_a(X)$ are the vertex functions given by

$$\begin{aligned}\frac{\partial L}{\partial A^m_\mu(X)} \equiv V^m_\mu(X) &= -\frac{1}{2}\eta^A T^m_{AB}\eta^B\left(\tilde{P} + X'_\mu\right) + \frac{i}{2}\eta^A T^m_{AB}\eta_B\psi^\mu\psi^\nu\partial_\nu \\ &+ \frac{i}{2}C^{mnp}\eta^A T^p_{AB}\eta^B A^n_\nu(X)\psi^\mu\psi^\nu \\ &+ \frac{i}{2}C^{mnp}\eta^A T^p_{AB}\eta_B\zeta^a_n(X)\chi^a\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial L}{\partial\zeta^a_m(X)} \equiv V^m_a(X) &= \frac{i}{2}f^{abc}\chi^b\chi^c\eta^A T^m_{AB}\eta^B \\ &+ \frac{i}{2}\eta^A T^m_{AB}\eta^B\psi^\mu\chi^a\partial_\mu + \frac{i}{2}C^{pnm}\eta^A T^p_{AB}\eta^B A^n_\mu(X)\psi^\mu\chi^a \\ &+ \frac{i}{2}C^{pmn}\eta^A T^p\eta^B\zeta^b_n(X)\chi^a\chi^b\end{aligned} \tag{24}$$

$<\ldots>$ means expectation value with the measure $\exp(iS_H)d(phase\ space)$. Notice that $\Lambda^m(X)$ appearing in (22) is an arbitrary function, therefore, if we take functional derivative of (22) with respect to $\Lambda^q(y)$ and then set $\Lambda^q(y) = 0$ the relations still hold good.

$$< \int d^2\sigma \Big(V_\mu^m(X) \left[\partial_\mu \delta(x(\sigma) - y) \delta^{mq} + C^{mnq} A_\mu^n \delta(X(\sigma) - y) \right]$$

$$+ V_m^a(X) C^{mnq} \zeta_n^a(X) \delta(x(\sigma) - y) \Big) >= 0 \tag{25}$$

This is the fundamental relation that gives Ward identities for amplitudes involving gauge and scalar bosons. We take appropriate number of functional derivatives of (25) with respect to the gauge and scalar bosons and then set them to their background values.

$$\prod_{i=1}^{N} \prod_{j=1}^{M} \frac{\delta}{\delta A_\mu^{m_i}(X_i)} \frac{\delta}{\delta \zeta^{a_j}(X_j)} < \int d^2\sigma \Big(V_\mu^m(X(\sigma)) \Big[\partial_\mu (X(\sigma) - y) \delta^{mq}$$

$$+ C^{mnq} A_\mu^n(x(\sigma)) \delta(X(\sigma) - y) \Big] + V_m^a(X(\sigma)) C^{mnq} \zeta_n^a(X(\sigma)) \delta(X(\sigma) - y) \Big) >= 0 \tag{26}$$

Here $X_i = X_i(\sigma_i)$. The functional derivative acting on $< \ldots >$ brings down extra vertex functions and therefore, the $N+1$ point amplitudes are related to the lower point amplitudes. We denote the background value of gauge and scalar fields by b.y. which are required to be consistent with conformal invariance.

It is worthwhile to emphasize that the WI presented here are to be considered as the tree level result. Indeed, anomalies might creep in when we carefully compute the Jacobian associated with the fermionic measure under the transformations (14)–(16). This question has been examined by us recently and the results are reported elsewhere.[16]

Now we proceed to discuss the phenomena of spontaneous symmetry breaking in the string theory from our point of view. It is interesting to note that if the scalar background takes a constant value $\zeta_m^a(X) = \beta_m^a$ then we precisely reproduce the Thirring–like four fermion interactions considered by ABK[11] in the content of spontaneous symmetry breaking and Higgs' mechanism in the string theory. However, we encounter additional terms in the action (5) for nontrivial scalar background fields. These terms arise due to the invariance of the action under world–sheet supersymmetry transformations. We may envisage the effects of spontaneous symmetry breaking if we closely examine the two point function for the gauge fields in (26) (modulo complications due to möbius invariance). It is easy to see from (26) that the two point function exhibits a mass term for constant scalar backgrounds and we can interpret it as the analog of the Higgs' mechanism. Indeed, a more elaborate computation[19] reveals that the consistency conditions, satisfied by the gauge and scalar background fields, are obtained from the equations of motion of a four dimensional effective action

$$\begin{aligned} S_{eff} &= \int d^4n \Big[-\frac{1}{4} Tr \left(F_\mu^m(n) \right)^2 + \frac{i}{2} D^\mu \zeta_m^a(n) D_\mu \zeta_m^a(X) \\ &- \frac{4}{3} f^{abc} C^{mnp} \zeta_m^a(X) \zeta_n^b(X) \zeta_p^c(n) \\ &- \frac{1}{16} C^{mqp} C^{pst} \zeta_m^a(X) \zeta_s^a(X) \zeta_t^b(n) \zeta_q^b(X) \Big] \end{aligned} \tag{27}$$

Thus, if the scalar field acquires a non–zero vacuum expectation value, classically the system undergoes spontaneous symmetry breaking and exhibits Higgs' mechanism.

To summarize, we have considered a four dimensional heterotic string theory in the gauge and scalar backgrounds. The generaling functional is constructed in the path integral formalism and Ward identities are derived using the local symmetry properties of the generating functional. The model exhibits spontaneous symmetry breaking phenomena for constant vacuum expectation values of the scalar background fields.

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References

[1] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Vol I and II, Cambridge University Press 1986.

[2] D.J. Gross, Phys. Rev. Lett. 60, 1224 (1988).

[3] I.J. Muzinich and M. Soldate, Phys. Rev. D37, 359 (1988); D. Amati, M. Ciafoloni and G. Veneziano, Phys. Lett. 177B, 81 (1987), CERN–TH 4886; D.J. Gross and P.F. Mende, Phys. Lett. 197B, 129 (1987).

[4] K.S. Narain, Phys. Lett. 109B, 41 (1986); K.S. Narain, M. Sarmadi and E. Witten, Nucl. Phys., B279, (1987).

[5] J.H. Schwarz, Proc. Int. National Conference on Superstrings, Cosmology and Composite Structure, Maryland, 1987, Ed. J. Gates and R.N. Mohapatra, World Scientific Pub. (1988).

[6] W. Lerche, D. Lust and A.N. Schelleken, Nucl. Phys., B287, 477 (1987); N. Seiberg and E. Witten, Nucl. Phys. B276, 272 (1986).

[7] N. Seiberg and E. Witten, Nucl. Phys., B276, 272 (1986).

[8] H. Kawai, D. Lewellen and H.S. Tye, Phys. Rev. D34, 3794 (1986).; Nucl. Phys., B288, 1 (1987).

[9] I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys., B289, 84 (1987).

[10] S. Ferrara, C. Kounnas and M. Porrati, to be published.

[11] I. Antoriadis, C. Bachas and C. Kounnas, Phys. Lett., 200B, 297 (1988).

[12] C. Kounnas and M. Porrati, Nucl. Phys. B310, 355 (1988); D. Chang and A. Kumar, Phys. Rev. D38, 1893 (1988).

[13] P. Goddard, W. Nahm and D. Olive, Phys. Lett. 160B, 111 (1985); P. DiVechhia, V.G. Knizhik, J.L. Petersen and P. Rossi, Nucl. Phys. B253, 701 (1985); P. Goddard and D. Olive, Nucl. Phys. B257, 226 (1985); P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103, 105 (1988).

[14] I. Antoniadis, C. Bachas, C. Kounnas and P. Windey, Phys. Lett., 171B, 51 (1986).

[15] J. Maharana, Phys. Lett., 211B, 431 (1988).

[16] A. Das, J. Maharana and S. Roy, preprint, Fermilab-Pub-89/65-T/UR1106 ER13065-573, Phys. Rev. D (to be published).

[17] Ref. 8, 11 and J. Bagger, D. Nemeschansky, N. Seiberg and S. Yankielowicz, Nucl. Phys., B289, 531 (1987).

[18] For earlier work see G. Veneziano, Phys. Lett., 167B, 388 (1986); J. Maharana and G. Veneziano, Phys. Lett., 109B, 177 (1986); J. Maharana, Nucl. Phys. B286, 126 (1987) ref. 15. Also see M. Evans and B.A. Ovrut, Preprints UPR–0377T, UPR–0392T (1989); M. Porrati, LBL preprint.

[19] J. Maharana, Orsay preprint, LPTHE Orsay 88/29. Also see A. Sen, Phys. Rev. D32, 2102 (1985); Nucl. Phys. 278B, 289 (1986). C.G. Collan, E. Martinec, D. Friedan and M. Perry, Nucl. Phys. 262B, 593 (1985), for the pioneering work on relations between conformal invariance and the equations of motion of background fields.

SUPERGHOST FIELDS IN $N = 2$ SUPERCONFORMAL ALGEBRA

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Abstract

The representation of the $N = 2$ superconformal algebra in terms of BRST super-ghosts is considered. We attempt to apply the Miura transform – or the Feigin-Fuchs contruction – to the ghost field representation of the $N = 2$ superconformal algebra. We find that although it is possible to have such a construction, it is equivalent to the unshifted representation.

Conformal invariance in two dimensions [1] is being studied as one of the main guides in string theory [2]. It has originally been formulated to facilitate the study of asymptotic behaviour of two dimensional statistical systems. Superconformal algebra, especially the $N = 2$ world sheet superconformal invariance [3] is essential in superstring theories because we need the $N = 2$ superconformal invariance in order to achieve spacetime supersymmetry [4]. One other subject whose connection to conformal field theory is becoming clearer is that of systems governed by integrable nonlinear differential equations, such as KdV equations [5]. One of the key results in the theory of integrable systems is the Miura transformation [6], which connects the solutions of KdV equation to those of mKdV equation. In conformal field theory it is better known as the Feigin-Fuchs representation [7] of the Virasoro algebra, or as putting a background charge at infinity in a coulomb gas system [8].

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The Miura transformation has been useful in many different places in conformal field theory and representation theory of conformal algebra which is characterized by the values of c and h. One of the key applications is in the calculation of the Kac determinant [9], the determinant of the matrix formed by inner products of the Fock states spanned by the generators of the conformal algbra. Key steps were taken by Thorn [10] where he used the osciallator representation of the Virasoro algebra to calculate the Kac formula. This was further generalized to the $N = 1$ and the $N = 2$ superconformal algbras [11]. For the representation theory of extended conformal algebra [12], non-abelian generalizations were introduced [13]. For supersymetric extended conformal algebra, some of the problems were pointed out [14].

Let us now briefly review the Miura transformation. A solution u of KdV equation is connected to a solution v of mKdV equation through provided $u = -\partial v - v^2$. For the simplest case, we can have free bosonic fields $\phi(z)$, with the operator product expansion $\phi(z)\phi(z') \sim \ln(z-z')$ such that the stress energy tensor of the system is

$$T_0(z) = \partial\phi(z)\partial\phi(z). \tag{1}$$

It is easy to see that $c = 1$ for the stress energy tensor. A simple way to get the $c \neq 1$ is to add to $T_0(z)$ a piece that is linear in ϕ, and dimensionally we must have $\partial^2\phi$. If we regard the system (1) to be a Coulomb system, then the linear term corresponds to a charge placed at infinity. $T(z) = T_0(z) + i\alpha\partial^2\phi(z)$ gives $c = 1 + 24\alpha^2$. Generalization of the above arguments for the supersymmetric case is straight forward.

As we mentioned earlier the $N = 2$ superconformal algebra plays a key role in compactification of superstrings [15]. Representation theory, is crucial in understanding the structure of the algebra.

Friedan, Martinec and Shenker noted that the ghost sector the $N = 1$ superconformal algebra actually has the $N = 2$ invariance [16]. In this letter we are going to discuss whether it is possible to modify this ghost field representation by adding linear terms in the ghost fields, as we have been doing for the bosonic and fermionic field representations. To say the conclusion first, it is not possible as we mentioned in the previous section. This is in agreement with the fact that ghost contribution to the string amplitudes is background independent. It is not only the geometry of the background that the ghost sector is immune to. As is well demonstrated in the formulation of the Virasoro algebra on a Riemann surface [17], the ghost contribution is independent of the topology [18], the genus of the world sheet.

A point in the superspace is $Z = (z, \theta)$ consisting of one commuting complex variable z and one anticommuting variable θ. We will be considering only analytic functions and the anti-analytic functions involving $Z = (z, \theta)$ can be treated just the same fashion. A covariant derivative is $\mathrm{D} = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial z}$.

The superghost superfields are

$$B(Z) = \beta(z) + \theta b(z), \quad C(Z) = c(z) + \theta\gamma(z). \tag{2}$$

We can expand the ghost fields as $\beta(z) = \sum_n \beta_n z^{-n-3/2}$ and $\gamma(z) = \sum_n \gamma_n z^{-n+1/2}$.

It turns out that we can also represent the $B - C$ system in term of $N = 2$ chiral superfields [16]. The $N = 2$ superconformal algebra in the N-S like sector in its component form is given by

$$\begin{gathered}
[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{4}(n^3 - n)\delta_{n+m}, \\
\{G_r, G_s\} = 2L_{r+s} + c(r^2 - \frac{1}{4})\delta_{r+s}, \quad \{\overline{G}_r, \overline{G}_s\} = -2L_{r+s} - c(r^2 - \frac{1}{4})\delta_{r+s}, \\
\{G_r, \overline{G}_s\} = (r-s)T_{r+s}, \quad [L_n, G_r] = (n/2 - r)G_{n+r}, \quad [T_n, G_r] = -\overline{G}_{n+r}, \\
[T_n, T_m] = \frac{c}{2}n\delta_{n+m}, \quad [L_n, T_m] = -mT_{n+m}.
\end{gathered} \tag{3}$$

In the above equation and throughout this paper we will use k, n, m for integers and r, s for half integers.

The highest weight state is given by

$$L_0|h,t\rangle = h|h,t\rangle, \quad G_r|h,t\rangle = 0, \overline{G}_r|h,t\rangle = 0, \quad T_0|h,t\rangle = t|h,t\rangle. \tag{4}$$

It was shown [11] that for the following oscillator representation of $N = 2$ SCA in terms of ordinary bosons and fermions:

$$\begin{gathered}
L_n = \frac{1}{2}\delta_{ij}\sum_{k=-\infty}^{\infty} a^i_{-k}a^j_{n+k} + \frac{1}{4}\delta^{ij}\sum_{r=-\infty}^{\infty}(n-2r)b^i_{-r}b^j_{n-r} + in\delta^{ij}\beta_i a^j_n, \\
L_0 = \frac{1}{2}(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2) + \delta_{ij}\sum_{k=1}^{\infty} a^i_{-k}a^j_k - \delta^{ij}\sum_{r=1/2}^{\infty} rb^i_{-r}b^j_r, \\
G^1_r = \delta^{ij}\sum_{s=-\infty}^{\infty} b^i_s a^j_{r-s} + 2ir\delta^{ij}\beta_i b^j_r, \\
G^2_r = \varepsilon^{ij}\sum_{s=-\infty}^{\infty} b^i_s a^j_{r-s} - 2ir\varepsilon^{ij}\beta_i b^j_r, \\
T_n = \frac{1}{4}i\varepsilon^{ij}\sum_{r=-\infty}^{\infty} b^i_r b^j_{n-r} + \varepsilon^{ij}\beta_i a^j_n,
\end{gathered} \tag{5}$$

with $\left[a^i_n, a^j_m\right] = n\delta_{i,k}\delta_{n+m,0}$, $\left\{b^i_r, b^j_s\right\} = \delta_{i,j}\delta_{r+s,0}$.

We then have $h = \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_1^2\right)$ and $t = -\alpha_1\beta_2 + \alpha_2\beta_1$, where we have denoted $a_1 = a_0^1$ and $a_2 = a_0^2$. Notice that the linear terms added to the bosonic generator are bosonic ones and are fermionic for the fermionic generator. We can also represent the $N = 2$ SCA by the BRST ghost fields

$$L_m = \sum_k \left[(k - 2m) : c_{m-k} b_k : + (3m/2 - k) : \gamma_{m-k}\beta_k :\right],$$

$$\begin{aligned} G_r &= \sum_k \left[(3m - k) c_{r-k}\beta_k + \gamma_{r-k} b_k\right], \\ \overline{G}_r &= \sum_k \left[(3m - k) c_{r-k}\beta_k - \gamma_{r-k} b_k\right], \\ T_m &= \sum_k \left[3 : \gamma_{m-k}\beta_k : + 2 : b_{m-k} c_k :\right], \end{aligned} \tag{6}$$

with the canonical commutation relation of

$$[\gamma_n, \beta_m] = \delta_{n+m}, \{c_n, b_m\} = \delta_{n+m}.$$

It is well known that the conformal anomaly in this representation is $c = -2$.

The Fock vacuum for the ghost fields.

$$\begin{aligned} b_n|q\rangle &= 0, \quad n > q - 2, \qquad c_n|q\rangle = 0, \ n \geq -q + 2, \\ \beta_n|q\rangle &= 0, \ n > -q - 3/2 \qquad \gamma_n|q\rangle = 0, \ n \geq q + 3/2, \end{aligned} \tag{7}$$

where $q \in Z$ for the N-S sector and $q \in Z + 1/2$ for the R sector. Since we are interested in the N-S sector only put $q = 0$.

Now, one can show, after some computation, that the representation above can be generalized to that with ghost number violating terms, i.e. linear terms in the ghost field. Note that the superghosts b, g are commuting, whereas c and b are anticommuting. So it is natural to add linear terms in b, g in L_m and b, c for G_m.

$$\begin{aligned} T_B &= T_{B(0)} - (f/z)\partial(z^{3/2}\beta) - (g/z)\partial(z^{-1/2}\gamma), \\ T_F &= T_{F(0)} + 2f\sqrt{z}b - 2(g/\sqrt{z})\partial(c/z), \\ \overline{T}_F &= \overline{T}_{F(0)} - 2f\sqrt{z}b - 2(g/\sqrt{z})\partial(c/z), \\ J_B &= J_{B(0)} - 6f\sqrt{z}\beta - 2gz^{-3/2}\gamma - 4fg/z, \end{aligned} \tag{8}$$

where f and g are constants. We can rewrite in superfield formalism

$$T_{gh} = -C\partial B + \frac{1}{2}DCDB - \frac{3}{2}\partial CB + (f/z)D(z^{3/2}B)$$
$$+(g/z)\partial(C/\sqrt{z}) - (g/2z^{5/2})D(\theta C), \tag{9}$$

or in component form.

$$\begin{aligned} L_m &= L_{m(0)} + m(f\beta_m + g\gamma_m), \\ G_r &= G_{r(0)} + 2fb_r + 2grc_r, \\ \overline{G}_r &= \overline{G}_{r(0)} - 2fb_r + 2grc_r, \\ T_m &= T_{m(0)} - 6f\beta_m - 2g\gamma_m - 4gf\delta_{m,0}. \end{aligned} \tag{10}$$

We can actually generalize the above result even further as follows. Suppose we want to add linear terms to the L_n's in the following fashion.

$$L_m = L_{m(0)} + \tilde{f}(m,a)\beta_{m+a} + \tilde{g}(m,b)\gamma_{m+b} + A\delta_{m,0}, \tag{11}$$

where (m,a) and (m,b) are two variable functions in m and a and and in m and b repectively. A priori a and b can be independent. A is a c number which can depend on a, b and m.

If one checks the closure of the $N = 2$ superconformal algebra, one gets the following constraints.

$$\begin{aligned} a &= -b \in Z, \\ \tilde{f}(m,a) &= (m-2a)f_a, \\ \tilde{g}(m,b) &= (m-2a/3)g_a, \end{aligned} \tag{12}$$

and $A = -4f_a g_a a/3$ where f_a and g_a are constants.

Rewriting the ghost representation of the $N = 2$ superconformal algebra fully we have the following key result. Note that a has to be integer to make the ghost oscillator sensible.

$$L_m = L_{m(0)} + \sum_{a=-\infty}^{\infty} f_a(m-2a)\beta_{m+a} + \sum_{a=-\infty}^{\infty} g_a(m-2a/3)\gamma_{m-a} - \frac{4}{3}\sum_{a=-\infty}^{\infty} f_a g_a a\delta_{m,0},$$

$$G_r = G_{r(0)} + 2\sum_{a=-\infty}^{\infty} f_a b_{r+a} + 2\sum_{a=-\infty}^{\infty} g_a(r-a/3)c_{r-a},$$

$$\overline{G}_r = \overline{G}_{r(0)} - 2\sum_{a=-\infty}^{\infty} f_a b_{r+a} + 2\sum_{a=-\infty}^{\infty} g_a(r-a/3)c_{r-a}, \tag{13}$$

$$T_m = T_{m(0)} - 6 \sum_{a=-\infty}^{\infty} f_a \beta_{m+a} - 2 \sum_{a=-\infty}^{\infty} g_a \gamma_{m-a} - 4 \sum_{a=-\infty}^{\infty} f_a g_a \delta_{m,0}.$$

Now note that L_0 does not change under the new representation but T_0 has changed to $T_0 - 6f_a\beta_a - 2g_a\gamma_{-a} - 4f_a g_a$, modifying the eigenvalue of T_0. Note however, that we cannot use the soliton state as the eigenstate anymore because T_0 and $-6f_a\beta_a - 2g_a\gamma_{-a} - 4f_a g_a$ do not commute. So the next question is whether we can find a new eigenstate. The problem here is that the above modification is a trivial one in the sense that it is equivalent to shifting of the ghost algebra itself. I.e.

$$\gamma_m \to \gamma_m + 2f_m, \quad \beta_m \to \beta_m + \frac{2}{3} g_m \tag{14}$$

will give us the representations of the $N = 2$ SCA in terms of bilinear of ghosts.

The next question is whether this shifting of the ghost oscillators changes the Fock vacuum. If we have a highest weight state for the $[\gamma_n, \beta_m] = \delta_{n+m,0}$ algebra, say, then the highest weight state for the shifted ghost algebra is the following coherent state [19]

$$\prod_{m=-\infty}^{\infty} \exp(2f_m \beta_m) \exp(-2g_m \gamma_m/3)|0\rangle. \tag{15}$$

So we have an infinite family of vacua related by the Miura transformation of the ghosts. This is possible since one ot them is hermitian and the other antihermitrian, we can consider a linear combination of the operators to make a creation and an annihilation operator. This is so because $\beta_n^\dagger = -\beta_n$ and $\gamma_n^\dagger = \gamma_n$, we need to form $\alpha_n = \frac{1}{\sqrt{2}}(\gamma_n - \beta_n)$ and $\alpha_n^\dagger = \frac{1}{\sqrt{2}}(\gamma_n + \beta_n)$ such that $\left[\alpha_{-n}, \alpha_n^\dagger\right] = 1$.

We use those to creat a Fock space and see that linear combination of the Fock states give rise to the eigenstates of both L_0 and T_0. Furthermore, if is an eigenstate of $L_0(\gamma, \beta)$, and $T_0(\gamma, \beta)$ then $L_0(\gamma + 2f, \beta + 2g/3)$, and $T_0(\gamma + 2f, \beta + 2g/3)$ have eigenstate

$$\prod_{m=-\infty}^{\infty} \exp(2f_m \beta_m) \exp(-2g_m \gamma_m/3)|h, t\rangle \tag{16}$$

with the same eigenvalues. So a nontrivial modification of the values of h and t is not possible.

One of the main results here is that the values of h and t do not get modified for the ghost sector. This is in agreement with the fact that we can use the same value for cgh always, even when we consider the conformal algebras on a Riemann surfaces. It is expected that the same conclusion holds for the superghosts in the fully $N = 2$ BRST quantization of the $N = 2$ superconformal algebra.

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References

[1] A. Belavin, A.M. Polyakov and A. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.

[2] See for example, P. Ginsparg, Lectures at Les Houches Session XLIX (1988).

[3] M. Ademollo et al., *Nucl. Phys.* **B111** (1976) 77.

[4] T. Banks, L. Dixon, D. Friedan and E. Martinec, *Nucl. Phys.* **B299** (1988) 613.

[5] J.L. Gervais and A. Neveu, *Nucl. Phys.* **B209** (1982) 125; J.L. Gervais, *Phys. Lett* **160B** (1985) 277, 279.

[6] R.M. Miura, *Jour. of Math. Phys.* **9** (1968) 1202.

[7] B.L. Feigin and D.B. Fuchs, *Funk. Anal. Pril.* **16** (1982) 47.

[8] Vl.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** (1984) 312.

[9] V. Kac, in Group Theoretical Method in Physics, W. Beiglbock, A. Bohm and E. Takasugi, ed. (Springer, New York, 1979).

[10] C. Thorn, *Nucl. Phys.* **B248** (1984) 551.

[11] S. Nam, *Phys. Lett* **172B** (1986) 323; Yale Univ. Thesis (1987).

[12] A.B. Zamolodchikov, *Theo. Math. Phys.* **65** (1986) 1205; V.A. Fateev and A.B. Zamoldchokov, *Nucl. Phys.* **B280**[FS18](1987) 644.

[13] V.A. Fateev and S.L. Lykyanov, *Int. Jour. of Mod. Phys.* **A3** (1988) 507.

[14] S. Nam, VPI-IHEP 88/09, to appear in *Int. Jour. of Mod. Phys.* A.

[15] D. Gepner, *Nucl. Phys.* **B311** (1988) 191.

[16] E.J. Martinec and G.M. Sotkov, *Phys. Lett.* **208B** (1988) 249.

[17] I.M. Krichever and S.P. Novikov, *Func. Anal. and its Appl.* **21(2)** (1987) 126.

[18] See for example A. Perelomov, Generalized Coherent States and Their Applications, (Springer-Verlag, Heidelberg, 1986).

TOPOLOGICAL QUANTUM FIELD THEORIES: RELATIONS BETWEEN KNOT THEORY AND FOUR MANIFOLD THEORY

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We discuss Topological Quantum Field Theories related to the Donaldson theory of four manifolds through dimensional reduction. This leads to theories of instantons, magnetic monopoles, vortices as well as other theories. Stochastic quantization offers a unifying picture of relating theories differing in one dimension. We show how the different topological field theories offer different perspectives on knot theory. Finally, a four-dimenensional picture of surfaces in four dimensions is proposed as a four manifold viewpoint on knot theory.

INTRODUCTION

Topological quantum field theories, while motivated by mathematical developments, have the potential for being very interesting from a physical point of view. The recent surge of interest in topological field theories stems from the work of Witten to develop functional integral formulations of the mathematical work of Donaldson on four-manifolds, Floer on three-manifolds, Gromov on pseudo-holomorphic curves and Jones on knots. However, this work has been put in a setting more familiar to physicists through the relation to the supersymmetric formulation of stochastic quantum mechanics. One can also develop

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topological field theories for the moduli space of magnetic monopoles on three manifolds, vortices on Riemann surfaces as well as fractional statistics. The physical relevance for these developments may reside in further understanding of the non-perturbative sector of gauge theories. In particular, many people have speculated that the confining of the QCD vaBuum is related to a condensate of monopoles or vortices; hence further understanding of the topological field theories associated with the moduli space of these configurations should be useful.

Especially suggestive for future development is the relation between the Chern-Simons gauge theory and the invariants of knots called the Jones, HOMFLY and Kaffman polynomials. Since these polynomials are obtained by calculating the expectation value of the Wilson loop, along the knot or link, weighted by the Chern-Simons action, it is natural to expect that further developments will lead to a greater understanding of the Wilson loop as an operator. In particular, one method of calculating the expectation value of the Wilson loop makes use of the Morse theory on a two-dimensional projection of the knot or link; the Wilson loop is evaluated at the critical points of a height function. We shall show how this method of calculation, which leads to new invariants of three-manifolds, can be extended to four-manifold theory, thereby providing the link between the Jones polynomials and Donaldson theory. In particular, the type of four manifolds described are surfaces bounded by the loop of the Wilson loop, embedded in four dimensions. The fact that one can use Donaldson-theory to describe the knots suggests that we investigate further what is special in four dimensions. Since this is the critical dimension for paths intersecting, it may be that one can better understand the gauge theory vacuum in terms of a condensate of knots.

From a more down to earth point of view, the quantum Hall effect, high temperature superconductivity and fractional statistics clearly seem related to abelian Chern-Simons theories. The non-abelian Chern-Simons theory provides an interesting perspective on conformal field theories and integrable models which should have interesting physical consequences. A four-dimensional perspective on these phenomena should also be quite interesting. In particular, a clear understanding of the the relation of experimental phenomena to topological field theories offers the exciting possibility of doing experimental mathematics in a laboratory.

Finally, topological quantum field theories provide a new form of general covariance. The actions for topological theories are constructed by gauge-fixing a metric-independent topological invariant, so any metric dependence will appear only in the gauge-fixing part of the action. All observables, i.e. the expectation values of gauge-invariant operators, are therefore metric independent.

From a mathematical perspective one hopes that the new perspective offered by topological quantum field theories will lead to solutions of unsolved problems in low dimensional topology as well as to a better understanding of the relations between topology in two, three and four dimensions. The Chern-Simons theory has already provided a three-dimensional setting for knot theory. It seems that the Donaldson-theory obtained by gauge fixing the instanton charge provides a four-dimensional setting for framed kinks and the surfaces they bound.

One speculation is that topological quantum field theories will lead to a better understanding of the functional integral, insofar as they are approximately Gaussian. Remember that Wiener measure is in fact the one measure that is well-understood.

Finally, topological quantum field theories provide a better understanding of the meaning of integrability.

THE MEANING OF TOPOLOGICAL QUANTUM FIELD THEORY

The program of topological field theory is due to ideas to Witten and Atiyah and originates with Witten's work on Morse theory. Witten was able to prove the Morse inequalities using a supersymmetric sigma-model quantum mechanics. He furthermore speculated about the possible extension of these ideas to the infinite-dimensional setting of quantum field theory. Based upon these work, Atiyah outlined a program of developments in low-dimensional topology and suggested the existence of quantum field theoretic settings for these developments; quantum field theories which Witten then constructed. The emphasis of this section and the paper in general shall be to provide an original point of view of these developments very close to ideas familiar to physicists. This point of view shows how the Donaldson theory in four dimensions provides a comprehensive viewpoint.

Witten's approach to Morse theory can be summarized by considering a family of Hamiltonians obtained by conjugating the exterior differential, d, on forms in the exterior bundle on a manifold M by $\exp(sh(\phi_i))$ where h is a Morse function on M with local coordinates ϕ_i. Then the Lapacian

$$H_o = dd^* + d^*d$$

becomes the family

$$H_s = d_s d_s^* + d_s^* d_s .$$

$$d_s = \exp(-sh(\phi_i))\ d\ \exp(sh(\phi^i))$$

$$H_s = H_o + s\ \frac{\partial^2 h}{\partial\phi^i \partial\phi^{i*}}[a_i, a_i^*] + s^2 (dh)^2$$

so that in the asymptotic limit $s \to \infty$, one is forced to the critical point $dh=0$. The Hessian counts the number of unstable directions, which when equal to n, restricts the Hamiltonian to harmonic n-forms. The creation operator a_i creates a 1-form. The Hamiltonians can be realized in terms of supersymmetric sigma-models with supercharge

$$Q = \frac{1}{\sqrt{2}}(d + d^*)$$

In this way one can calculate the Euler index of a manifold, for example. One approach to topological field theories is to consider them as generalizations of the above system to a field theoretical setting in order to calculate subtle topological invariants. Atiyah and Segal have tried to axiomatize this approach.

A second approach to topological field theories is to consider the gauge-fixing of a locally exact topological invariant. This leads very quickly to a large number of theories, many of which can be understood as dimensional reductions of the four-dimensional Donaldson theory obtained by gauge-fixing $\int_{M_4} \mathrm{Tr} F \wedge F$. This theory is a theory of the moduli space of instantons on a four-manifold M_4. Since

$$\mathrm{Tr} F \wedge F = d\mathrm{Tr}(AdA + \tfrac{2}{3}A^3) = d\ C.S.$$

the Chern-Simons (C.S.) theory, which has been so rich in structure, can be seen as a trivial dimensional reduction of the Donaldson theory. A less trivial reduction to three-dimensional monopoles is obtained by considering the magnetic monopole charge which is also related to the Chern-Simons form.

$$\int_{M_4} \mathrm{Tr}\ B_i D_i \Phi = \int \delta_\Phi C.S.$$

with

$$\delta_\Phi A_i = D_i \Phi$$

The scalar field Φ is really a time-independent fourth-component of the vector potential, A_o. A further reduction to two-dimensions with a complex scalar $\Phi=A_3+iA_4$ leads to a theory of vortices on a Riemann surface Σ

$$\int_\Sigma \mathrm{Tr}[\epsilon_{ij}(\tfrac{1}{4}F_{ij}[\Phi^4,\Phi]-D_i\Phi^* D_j\Phi)]$$

The theory can be reduced even further to one-dimension with topological charge

$$\int_P \mathrm{Tr} D_A\Big[\Phi_1,[\Phi_2,\Phi_3]\Big]$$

where one integrates along a path P and to zero dimensions

$$\mathrm{Tr}\Big[[\Phi_1,\Phi_2] \quad [\Phi_3,\Phi_4]\Big].$$

It is very curious that if one considers the self-dual conditions $[\Phi_1,\Phi_2]=[\Phi_3,\Phi_4]$, the zero-dimensional theory is a theory of random matrices with action $\mathrm{Tr}[\Phi_1,\Phi_2]^2$.

Our approach to topological field theories is related to Witten's ideas on Morse theory through stochastic quantization. For this approach we consider a general action $S(\phi(x,t))$ where t is a fictitious stochastic time t. The topological action which must be gauge-fixed is

$$I_{TOP}=\int_o^T \frac{\delta S(\phi(x,t))}{\delta t}\,dt=\int_o^T \frac{\delta S}{\delta\phi}\,\dot{\phi}\,dt$$

The first key to this approach to topological field theories is a new form of gauge invariance. In addition to the usual gauge invariance of the second kind that the action S might have, the topological action I_{TOP} is invariant under diffeomorphisms that vanish at the boundary. In particular, since the integrand is necessarily only locally exact, one can obtain non-trivial holonomy associated with all cycles in the interval [0,T].

The second key to our approach is the use of the equations of motion, in this case the Langevin equation, to gauge-fix or choose a slice of the diffeomorphism flow. Such gauge-fixing requires introducing additional fields with a differential s that is like an infinitesimal rotation.

The Langevin equation is a stochastic differential equation for the fictitious-time evolution of ϕ:

$$\dot{\phi} = \frac{\delta S}{\delta \phi} + \eta$$

where η is a stochastic random variable

$$<\eta(t)>=0$$
$$<\eta(t)\eta(t')>=\delta(t-t')$$

In stochastic quantization, the equal-time correlation functions of ϕ, which depend upon η, approach the usual η-independent correlation functions in the infinite t limit, provided the system is ergodic.

$$<\phi(x_1,t) \ldots \phi(x_n,t)>_\eta \xrightarrow{t\to\infty} <\phi(x_1) \ldots \phi(x_n)>$$

This can be seen as a consequence of the Fokker-Planck formulation, which can be proved to be equivalent to the Langevin approach. The Fokker-Planck approach considers the equation for the distribution P_n whose integral gives the partition function

$$H\ P^E_\eta = E\ P^E_\eta$$
$$H = \frac{1}{2}\ \frac{\delta^2}{\delta\phi^2} + \frac{1}{2}\left(\frac{\delta S}{\delta \phi}\right)^2 - \frac{1}{2}\frac{\delta^2 S}{\delta \phi^2}$$

Since the Fokker-Planck Hamiltonian, H, is formally positive-definite, one can consider a spectral resolution

$$P_\eta = \Sigma\ P^E_\eta\ e^{-Et}$$
$$\lim_{t\to\infty} P^o_\eta = \frac{e^{-S}}{Z}$$

where P^o_η is the zero-mode of H, the Gibbs distribution. I must repeat the remark above, that this argument is formal and depends upon the ergodicity of the measure e^{-S}.

In the topological approach to stochastic quantization, one must gauge-fix the I_{TOP} above. The BRST differential s is like an infinitesimal rotation or supersymmetry a la Parisi-Sourlas:

$$s\phi=\psi \qquad s\bar{\psi}=\eta$$
$$s\psi=0 \qquad s\eta=0$$

so $s^2=0$, with ψ, $\bar{\psi}$ conjugate fermionic one forms. The total action, I, is the sum of I_{TOP} and the gauge-fixing piece I_{GF}:

$$I = T_{TOP}+ I_{GF}$$

$$I_{GF}= \int s\ \Sigma\bar{\psi}(\phi -\frac{\delta S}{\delta\phi} - \frac{1}{2}\eta)]$$
$$=\int\eta\ (\phi- \frac{\delta S}{\delta\phi} - \frac{1}{2}\eta)$$
$$+\bar{\psi}\ (\frac{\partial}{\partial t} - \frac{\delta^2 S}{\delta\phi^2})\psi$$

If we integrate out η, we obtain

$$I = \int \frac{1}{2}(\phi)^2+ \frac{1}{2}(\frac{\delta S}{\delta\phi})^2+\bar{\psi}(\frac{\partial}{\partial t} - \frac{\delta^2 S}{\delta\phi^2})\psi$$

The cross-term in $\frac{1}{2}(\phi- \frac{\delta S}{\delta\phi})^2$ cancels I_{TOP} and we are left with the familiar supersymmetric action for stochastic quantization. We note that the action can be put in a Gaussian form a la Nicolai, if we can solve the Langevin equation

$$\frac{1}{2}\eta =\phi - \frac{\delta S}{\delta\phi}$$

so that

$$I = \int \frac{\delta S}{\delta\phi}\ \phi|_{\phi=\phi(n)} + \frac{1}{2}\eta^2$$

The functional integral is now over η. However, as payment for writing the action in such a simple form, we note that the solution $\phi=\phi(n)$ is in general highly non-local; there is generically a moduli space for the solutions.

As a remark, we note that for a manifold M, if we choose the action S equal to $h(\phi^i)$ the height function, we recover Witten's formulation of Morse theory. In this case, we can rescale the action to be forced asymptotically to critical points $\frac{\delta S}{\delta\phi}=0$ and flows between critical points, which are instantons

$$\phi = \frac{\delta S}{\delta \phi}$$

If the action is chosen to be the Chern-Simons action, we obtain a version of Yang-Mills. Moreover, the Langevin flow becomes

$A_i = B_i$, just the self-dual equation in $A_o=0$ gauge. This leads to another reason for trying to understand the Chern-Simons theory for a physicist, i.e. it is the asymptotic limit of a form of Yang-Mills theory.

Finally, we mention as an example $S=\frac{1}{2}\, x^2$, with stochastic action the harmonic oscillator which relaxes asymptotically to the Gaussian ground state.

DONALDSON THEORY

One can view the Donaldson theory of four manifolds as the calculation of polynomial invariants which are gauge invariant correlation function in a topological field theory. This field theory can be obtained by gauge-fixing the topological invariant $\int_{M_4} \mathrm{Tr} F \wedge F$ where M_4 is the given four manifold. As a consequence of the new gauge invariance related to the diffeomorphism group, the usual BRST differential must be modified so that

$$sA_i = -D_i c + \psi_i$$

where $-D_i c$ is the ususal BRST differential of A_i related to an infinitesimal gauge transformation.

Furthermore we have that

$$s\psi_i = D_i \Phi + [c, \psi_i]$$

$$s\Phi = [c, \Phi]$$

One can understand the BRST differential as an infinitesimal rotation of the total curvature of the bundle $\mathfrak{F} = F + \Psi + \Phi$ where each component is a total two-form in space plus the group; i.e. F is a (2,0) form Ψ a (1,1) form and Φ a (0,2) form, where (i,j) indicates (space, group).

The gauge-fixing portion of the action is

$$I_{GF} = \int s \mathrm{Tr} \left\{ \bar{\chi}^{\mu\nu} [F^{\mu\nu} - \tilde{F}^{\mu\nu} + b^{\mu\nu}] + \bar{\Phi}\, D^i \psi^i + \bar{C} \left[(\partial_\mu A_\mu) + b \right] \right\}$$

with

$$s\,\bar{\chi}^{\mu} = b^{\mu\nu}$$

$$s\,\bar{\Phi} = \eta \qquad s\,\bar{\eta}=0$$

$$s\,\bar{c} = b$$

Integrating over $b^{\mu\nu}$ leads to a term $\frac{1}{2}\,(F^{\mu\nu} - \tilde{F}^{\mu\nu})^2$ whose cross term cancels the topological part of the action $TrF\wedge F$. Integrating over b leads to $(\partial_\mu A_\mu)^2$, the usual gauge-fixing term associated with $\xi=1$ gauge.

Note that the metric dependence enters only in the gauge-fixing part of the I_{GF}. This leads to metric independent observables obtained from integrating differential forms related to $TrF\wedge F$ over cycles in the four manifold. These differential forms are obtained by descent equations similar to those found in the theory of anomalies

$$sTr(F\,F) = dTr(F\psi)$$

$$sTr(F\psi) = dTr(F\Phi-\psi\psi)$$

$$sTr(F\Phi-\psi\Psi) = dTr(\psi\Phi)$$

$$str\Psi\Phi = dTr\Phi^2$$

Note that these differential forms can be designated $\Delta_{(i,j)}$ using our previous notation. The Donaldson invariants are obtained by choosing i-cycles C_n^i in the four manifolds and calculating

$$< \prod_n \int_{c_n^i} \Delta_{i,j}>$$

This is metric independent because

$$\delta_g \int_{c_n^i} \Delta_{i,j} = \int_{c_n^i} s\Delta_{i,j}$$

$$= \int_{c_n^i} d\Delta_{i-1,j+1}$$

$$= \int_{\partial c_n^i} \Delta_{i-1,j+1} = 0$$

because $\partial C^i_n = 0$ by definition.

We speculate that there exists an eight-dimensional formulation of Donaldson theory since instantons are related to the Hopf map $S^7 \to S^4$. In this eight-dimensional formulation, the key would be the generalized duality equation

$$F \wedge F = \pm (F \wedge F)^*$$

This seems to be present in the duality between the forms $\Delta_{i,j}$ and $\Delta_{4-i,4-j}$.

DIMENSIONAL REDUCTION

The dimensional reduction of Donaldson theory leads to several other interesting theories. We first consider the reduction to three-manifolds with boundary, like R^3, with independence of the fourth, time direction. In this case, A_o becomes a scalar field ϕ and the topological charge is the magnetic monopole charge. Asmptotically, ϕ approaches a constant, $D_i\phi$ approaches zero as does F_{ij} outside of the core of each magnetic monopole. One gauge-fixes the topological charge by means of the Bogomolny equation

$$B_i = D_i\phi = \delta_\Phi A_i$$

This is a generalization of the flow determined by the Langevin equation; in this case the flow is along the gauge transformation direction determined by ϕ. Similarly, the topological charge has the form

$$\int_{M_3} \mathrm{Tr} B_i D_i \phi = \int_{M_3} \delta_\phi \mathrm{C.S.}$$

The action obtained in this way does not include a potential term for ϕ, so these monopoles are to be considered in the Prasad-Sommerfield limit. In this case, the scalar force balances the gauge and multimonopole configurations are possible. The scattering of slowly moving magnetic monopoles has been considered by Atiyah, Hitchin and Manton in this case and the configuration space shown to be A/G, the space of vector potentials modulo gauge transformations. The semi-classical limit of the above topological field theory is the same limit as that considered for scattering slowly moving monopoles.

The dimensional reduction of Donaldson theory to two-dimensions leads to the study of Riemann surfaces with marked points. On such surfaces, Hitchin has proved that there exist non-trivial solutions, vortices, to the self-dual equations (called the Hitchin equations)

$$F = [\Phi^*, \Phi]$$

$$d_A \Phi = 0$$

Φ is a complex one-form on the Riemann surface obtained from the reduction of A_3+iA_4. It is a theorem that every solution of the Hitchin equations defines a flat complex connection $A+\Phi+\Phi^*$.

The topology field theory of the moduli space vortices differs from the theory for the moduli space of magnetic monopoles in that the former contains a potential term for the scalar field. There is, therefore, explicit symmetry breaking in this theory. However, this symmetry breaking is of an unusual kind insofar as it corresponds to caustic singularities with non-Morse theoretical critical points. This means that the minimum of the potential is not enough to determine the symmetry breaking; there is a degeneracy corresponding to $\det\Phi=0$. Such caustic singularities have an ADE classification and it is intriguing to speculate on the connection between the Hitchin theory and conformal field theories. Note that the Hitchin equations are conformally invariant. The $\det\phi$ acts as a potential whose singularities classify the nature of the possible phases of the theory. For example, the quartic potential $\mathrm{Tr}[\Phi^*,\Phi]^2$ is zero for Φ sitting in the Cartan subalgebra (e.g. $U(1)^{N-1}$) of the Lie algebra G associated with the gauge group (e.g. SU(N)) and $\mathrm{Det}\ \Phi=q^N+$ lower order terms. Under adiabatic interchange, the vortices of the marked points behave like particles with fractional statistics, e.g. 1/N statistics (interchange leads to holonomy $e^{2\pi i/N}$) for the case above. This leads to an intriguing question. Namely, fractional statistics are possible in 2+1 dimensions as we have described above and impossible in 3+1 dimensions. How does the phase associated with interchange of the vortices on Riemann surface disappear as we consider the extension of the vortices into four dimensions? We shall answer this question in the ensuing section.

As a final remark associated with the two-dimensional topological field theory, we note that the dimensional reduction of the three-dimensional Chern-Simons to one dimension is

$$\Phi^* \vec{d}\Phi + A\ [\Phi^*,\Phi]$$

so that the stochastic quantization of this terms leads to

$$\dot{A}_1 = [\Phi^*, \Phi]$$

$$d_A \Phi = 0$$

which are just the Hitchin equations in $A_o=0$ gauge.

One can consider the dimensional reduction of the Donaldson theory to the one-dimensional path, P, determined by a knot or link in three-space. The topological action is

$$\int_P d_A [\Phi_1, [\Phi_2, [\Phi_3]]$$

gauge-fixed by $d_A\Phi_1=[\Phi_2,\Phi_3]$ and cyclic permutations thereof. One might consider Φ_1, Φ_2, and Φ_3 as determining a framing of the knot with three-space mapped into a three-sphere in the group. Note that $[\Phi_1, [\Phi_2,\Phi_3]]$ is a volume 3-form associated with the reduction of the Chern-Simons form to zero dimensions. Finally as we noted earlier, the reduction of Donaldson theory to zero dimensions leads to a random matrix model.

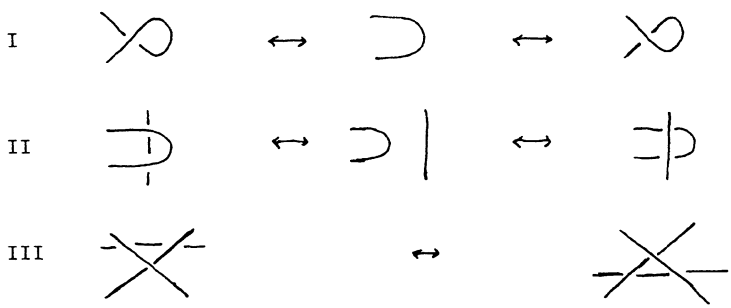
FOUR MANIFOLD THEORY AND KNOT INVARIANTS

Before discussing the relationship of our various viewpoints of four manifolds and Donaldson theory, we first briefly review our present three dimensional understanding of knot invariants that has culminated in Witten's Chern-Simons theory. Witten had claimed that the specialized HOMFLY polynomial for a knot or link in S^3 can be calculated by considering the trace in appropriate representations of the Wilson loop along the knot or link. He has also conjectured a surgery formula which will lead to more general invariants of three manifolds. We therefore consider

$$\int DA\, e^{kC.S.}\, Tr_{R_i} P(\exp\int A)$$

up to gauge fixing factors. He claims that the conformal blocks of a conformal field theory provide a basis for the Hilbert space of states in the Chern-Simons theory. The braiding matrix gives a change of basis in this Hilbert space.

For our purposes, especially the relation to Morse theory, we consider evaluating the functional integral by projecting the knot onto the two-dimensional plane and looking at a time evolution. There are four types of critical points for which the braiding matrix gives a function of the variable q, i.e., one has minima or births, maxima or deaths, and overcrossings and undercrossings. Taking a trace over the product of the values associated with the critical points gives the knot polynomial invariants which is claimed to be equivalent to evaluating the Wilson loop at these critical points. These polynomials are invariant under the three Reidemeister moves:



In considering the relation of Donaldson theory to the Chern-Simons form of the knot invariants, we note that we can relate the Donaldson theory in 4, 3, 2, and 1 dimensions to the Chern-Simons theory through a form of stochastic quantization. Moreover, there is a natural duality or reciprocity between the twistor construction of solutions to instantons in four-dimensions and a matrix problem in zero dimensions, between the Nahm construction of magnetic monopoles in three-dimensions and the Dirac equation in one-dimension. The two-dimensional theory of vortices is selected as special because it is in the middle dimension. Which brings us back to our question of what happens to the fractional statistics phase?

As the vortices move around one another in time, they sweep out surfaces in space. A new type of angle is possible in four dimensions, as pointed out by Polyakov. Namely, since surfaces intersect generically in points in four dimensions, one obtains an instanton charge associated with this intersection

$$N = \int d^2\xi \sqrt{g}\, g^{ab} \partial_a t_{\mu\nu} \partial_b t_{\alpha\beta} \epsilon^{\mu\nu\alpha\beta}$$

and a phase $e^{iN\theta}$ where $t_{\mu\nu}$ is a surface vector defined by

$$t_{\mu\nu} = \epsilon^{ab}\ \partial_a x_\mu \partial_b x_r$$

We are therefore lead to a theory of intersecting surfaces. Moreover the integrable surfaces satisfy $\partial_a(t_{\mu\nu}-t^*_{\mu\nu})=0$. The cancellation between the phase associated with fractional statistics and the instanton phase can be seen from a variety of perspectives. Just as the braid group acts on numbered objects with the braiding matrix having eigenvalues related to the statistics phase in 2+1 dimension, an instanton gives rise to tunneling between numbered vacua whose linear superposition is an eigenvalue of the shift with eigenvalue $e^{i\Theta}$. Equivalently, a Fourier transform of the vortex space yields discrete vortices, while a Fourier transform of the Θ-vacua yields instantons with integer charge. Finally, the monodromy associated with the statistics phase can be interpreted as monodromy associated with the resolution of a singular surface. A more precise analysis will require better understanding of the gauge-fixing problem. One further idea concerning this matter is that the cancellation of the two phases might be simpler in an eight-dimensional formulation looking at

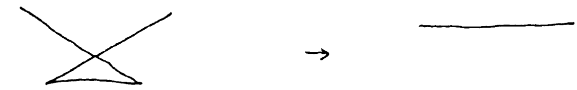
$$F \wedge F = {}^*[\Phi_1,\Phi_2]^2,$$

with the instanton phase matching that coming from the crossing of eigenvalues of the potential on the right-hand side.

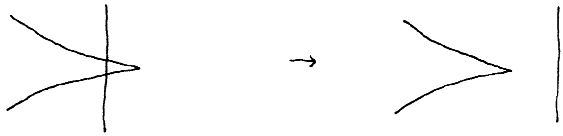
It is important to relate the vortex theory on the Riemann surface to the theory of three-manifolds. This can be done by noting that an arbitrary three manifold can be constructed by either a Heegard splitting or by surgery on tubular neighborhoods of knots and links. The latter approach was crucial to the Chern-Simons theory calculation of invariants of the three manifold. In the Heegard splitting, one considers an appropriately embedded Riemann surface punctured by the previously considered knots and links. One then performs Dehn surgery on this Riemann surface. Louis Crane has pointed out the relevance of the theorem that two three-manifolds are diffeomorphic if their Heegard splittings are invariant under the Suzuki moves: 1) knob twists, 2) handle twists, 3) handle exchanges 4) handle braidings and 5) handle slides. Crane went on to explain that these moves can be related to well-known moves of conformal field theory on a Riemann surface. It is important for our discussion that handle slides can be seen as a higher-dimensional analogue of braiding. However, in general this viewpoint is not as transparent as the four-dimensional one.

From a four-dimensional point of view, everything is much simpler. We consider moves on 2-handles in the four ball attached to framed links in S^3. Instead of the critical points in two-dimensions associated with births, deaths, and crossings, one can consider the birth, and death of handles as well as their slidings. What is remarkable from the four-dimensional point of view is that the moves on critical points display an analogue of the Reidemeister moves

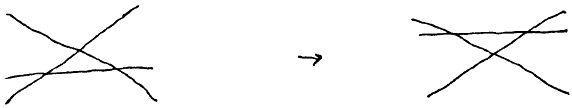
dove tail move



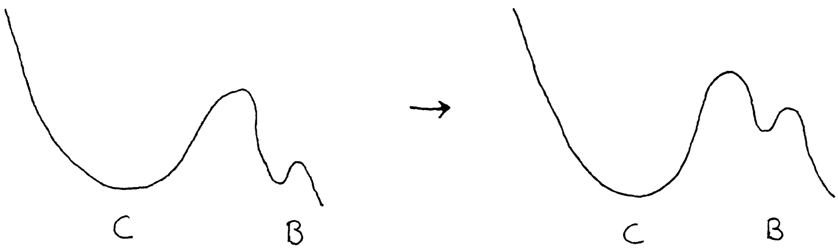
beak move



triangle move



These moves correspond to the movement of critical points that interact only locally. For example, one can understand the beak move in terms of a birth B at the same level at an existing critical pint, C.



The birth B can be moved away since it is a purely phenomena. After it is moved away, there are no longer any crossings. By moves such as these, one can cancel all the critical points in a four manifold except for the two-handles. Then a theorem of Kirby, tells us that two three manifolds are diffeomorphic if and only if the corresponding bounding four-balls with two-handles attached to framed links can be reached by Kirby moves. We postpone further discussion to a paper in preparation.

We conclude with another discussion of three-manifolds based upon the topological theory of magnetic monopoles. It is well known that the Lax pair for spherically symmetric magnetic monopoles,

$$\frac{\partial}{\partial r}\psi = \frac{1}{2}[N^+,N^-]$$

$$\frac{\partial}{\partial r}N^{\pm} = \pm[\psi,N^{\pm}]$$

where

$$\vec{A} = \vec{r}\ x(\vec{T}-r\vec{N})/r^2$$

and

$$\psi = (e\phi - T_3/r)$$

for generators T^a of the group, is equivalent to the Toda equations

$$\theta_j'' = \exp\ (\ \sum_i K_{ji}\theta_i)$$

where K is the Cartan matrix of the group. If we consider S^1 invariant instantons, this leads to an affinization of the group. If we diagonalize K with eigenvalues α_i

$$\exp\ \Sigma i\alpha_i\phi^i$$

looks like a vertex operator. The extension of the Toda equation into two dimensions then can be considered as a conformal field theory with background charges. The Lax pair can be related to a zero curvature condition, which is the constraint found in the Chern-Simons theory. We are then back in a framework similar to that discussed in the Chern-Simons theory, where braiding occurs through a change of basis. However, this is not unexpected since the monopole charge can be related to the Chern-Simons term and the Bogomolny equation to a flow.

REFERENCES

The fundamental mathematical references are:

1. E. Witten, J. Diff. Geom. 17, (1982) 661.
2. S. Donaldson, J. Diff. Geom. 18, (1983) 269; 26 (1988) 397.
D.S. Freed and K.K. Uhlenbeck, Instantons and Four Manifolds,

Springer (New York, 1984).
H.B. Lawson, The Theory of Gauge Fields in Four Dimensions, Amer. Math. Soc. (Providence, R.I., 1986).

3. A. Floer, Bull AMS 16, (1987) 279.

4. V.F.R. Jones, Bull AMS 12 (1985) 103, Ann. Math. 126 (1987) 335.
P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millet and L.H. Kauffman, Topology 26 (1987) 395, On Knots, Princeton, U. Press (Princeton, 1987).

5. M. Gromov, Inven. Math. 82 (1985) 207.

6. N.J. Hitchin, Proc. London Math. Soc. (3) 55 (1987) 59.
The program of topological quantum field theory was explained in

7. M.F. Atiyah, in The Mathematical Heritage of Hermann Weyl, Proc. Symp. Pure Math. 48, ed. R. Wells (Amer. Math. Soc., 1988).

Topological quantum field theories were then constructed, first by Witten

8. E. Witten, Comm. Math. Phys. 117 (1988) 353, 118 (1988) 411, Phys. Lett. 206B (1988) 601.

The gauge fixing procedure, explained in reference 9 lead to many applications.

9. L. Baulieu and I. Singer in Proceedings of the LAPP meeting on Conformal Field Theories, Anneey-le-Vieux, march 14-16 (1988) Nucl. Phys. (in press).

10. L. Baulieu and B. Grossman, Phys. Lett. 212B (1988) 351
For a previous review see
B. Grossman, "Topological Quantum Field Theories," Invited talk at the XVI International Conference in Differential Geometric Methods in Theoretical Physics, Chester England (to be published).

11. The first application, to stochastic quantization, was a natural child of previous approaches suggesting topology.
G. Parisi and N. Sourlas, Phys. Rev. Lett. 43 (1979) 744, Nucl. Phys. B 206 (1982) 321.
H. Nicolai, Phys. Lett. 89 (1980) 341; Nucl. Phys. B176 (1980) 419; Phys. Lett. B117 (1982) 408.
G. Parisi and Y.S. Wu , Sci. Sin (1981) 484.
E. Gozzi, Phys. Rev. D28 (1983) 1922; D30 (1984) 1218
J. Zinn-Justin, Nucl. Phys. B275 [FS17] (1986) 135
S. Alfaro and B. Sakita, Phys. Lett. 121B (1983) 339
Z. Bern, M.B. Halpern, L. Sadun, C. Taubes, Phys. Lett. 165B (1985) 151.

For discussions of the many subtleties involved see the series by
M. Claudson and M.B. Halpern, Nucl. Phys. B 230 (1985) 689.
Phys. Rev. D31 (1985) 3310, Ann. Phys. 166 (1986) 33.
S. Ceccotti and L. Giradello, Ann. Phys. 145 (1983) 81.

Other applications followed:

12. L. Baulieu and B. Grossman, Phys. Lett. 214B (1988) 223.
M.F. Atiyah and W. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton U. press (Princeton, 1987).
13. G. Chapline and B. Grossman, Phys. Lett. 22B (1989).
14. D. Montano and J. Sonnenshein, Nucl. Phys. B 313 (1989) 258.
R. Brooks, D. Montano and J. Sonnenschein, Phys. Lett. 217B. (1988) 91.
D. Montano and J. Sonnenschein, "The Topology of Moduli Space and Quantum Field Theory," SLAC-PUB-4760,
J. Sonnenschein "Topological Quantum Field Theories, Moduli Spaces and Flat Connections", SLAC-PUB-4970.
15. D. Birmington, M. Rakowski and G. Thompson, Phys. Lett. 212B (1988) 187; 714B (198) 381,
F.A. Schaposnik and G. Thompson, "Topological Field Theory and Two-Dimensional Abelian Higgs Instantons," PAR-LP THE-88-51.
16. Y. Yu ,"Topological Yang-Mills Field Theory as Stochastic Quantization of Chern-Simons Gauge Theory" BIHEP-TH-89-3 (Phys. Rev. D. in press).
17. O. Dayi "A General Action for Topological Quantum Field Theories" IC 189147 (Nucl. Phys. B in press).

The quantum field theoretic approach to the Jones polynomial was developed in

18. E. Witten, Comm. Math. Phys. 121 (1989) 351, Gauge Theories and Integrab;e Lattice Models, IASSNS-HEP-89/11.

Connections to Donaldson theory were suggested in reference 13.

19. L. Crane, Topology of 3-Manifolds and Conformal Field Theories, Yale University preprint.

For discussion of the Kirby calculus see
D. Rolfson, Knots and Links (Publish or Perrish, 1976).
20. The model of intersecting surfaces is discussed in A.M. Polyakov, Nucl. Phys. B 268 (1986) 406.

21. The relations of duality and reciprocity can be found in
E. Corrigan and P. Goddard, Ann. Phys. 154 (1984) 253.

22. For discussion of Toda theory see
N. Ganoulis, P. Goddard and D. Olive, Nucl. Phys. B 305 [FS 5] (1982) 601.
J.L. Gervais and A. Neveu, Nucl. Phys. B 224 (1983) 329.

23. The speculated eight-dimensional framework should be related to solutions found in
B. Grossman, T. Kephart and J. Stasheff, Comm. Math. Phys. 96 (1984) 431.

TOPOLOGICAL QUANTUM THEORIES AND REPRESENTATION THEORY

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ABSTRACT: We discuss the relationship between path integrals, geometric quantization and representation theory for a simple quantum theory whose Hilbert space is a group representation. The path integrals involved have interesting cohomological significance and can be evaluated in terms of fixed point formulas to give the Kirillov and Weyl character formulas. The relation to recent work of Witten on Chern-Simons gauge theory is also discussed.

INTRODUCTION

In recent years certain quantum theories have been discovered that have an essentially geometrical or topological nature. These theories have deep connections to index theory and to other areas of active interest in geometry and topology. In this talk we will begin by discussing a simple class of such theories. The physical content of the simplest of these theories is a description of a quantum spin. The mathematical content is that of the representation theory of compact Lie groups. These rather simple quantum systems have a very rich geometrical structure and a proper understanding of this is essential for understanding both the quantization of spin and the more complicated topological quantum theories[1] that have excited recent interest. In the final part of this talk we will see to what extent the simple quantum mechanical systems discussed earlier shed light on Witten's[2] Chern-Simons quantum gauge theory.

The quantum theories that we will be considering are topological quantum theories in the sense that with appropriate choice of boundary conditions their partition functions are indices of elliptic operators. However we wish to correct the wide-spread belief that such quantum theories contain only topological information and no physical degrees of freedom. One of these theories is the supersymmetric quantum mechanics of a Dirac particle coupled to a background electromagnetic field. For a particular choice of boundary conditions all contributions to the partition function except those from the zero modes of the Dirac operator cancel. This does not change the fact that this is a non-trivial theory of great physical importance.

Even when one restricts one's attention to the zero modes of the Dirac operator

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one may find that they carry more structure than just a dimension. In particular they may transform under a group and it is this aspect that will interest us in this paper. The path integral quantization of a quantum spin has often been considered in the physics literature, for references see [3] and the paper [4]. Recently, Stone[5] has considered the quantization of spin from a point of view similar to ours.

QUANTUM MECHANICS AND GROUP REPRESENTATION THEORY

Let us consider what is perhaps the simplest mathematical structure that deserves to be called a quantum theory. The Hilbert space $\mathcal{H} = V_R$ will be the finite dimensional complex vector space corresponding to a unitary representation R of a compact, connected Lie group G. A state of the quantum system will be a vector in $\mathcal{H}$ written

$$|\Psi(\tau)>$$

that depends on the parameter τ, which will have the physical interpretation of time.

The simplest example that we will consider will be for G=SU(2), which has irreducible unitary representations of dimension $n+1$ for every non-negative integer n. Such a representation is said to have "spin" $\frac{n}{2}$ and will describe the dynamics of the spin degrees of freedom of a particle of that spin coupled to a time-varying magnetic field.

The Hamiltonian for this system will be a time dependent Lie algebra element $H(\tau)$ describing the magnetic field acting on the particle and the dynamics of the theory is described by the Schrödinger equation

$$\frac{d}{d\tau}|\Psi(\tau)>= iH(\tau)|\Psi(\tau)>$$

This equation describes the unitary time evolution of a vector in $\mathcal{H}$ and given an initial condition $|\Psi(0)>$ its solution can be written

$$|\Psi(\tau)>= U(\tau)|\Psi(0)>$$

where $U(\tau)$ is, for each value of τ, an element of G acting in the representation $\mathcal{H}$. $U(\tau)$ is often written as the "path-ordered exponential of $H(\tau)$"

$$U(\tau) = Pe^{i\int_0^\tau H(s)ds}$$

If we consider this theory on a fixed time interval T we can define the "partition function" of the theory to be

$$Z = Tr_R(U(T))$$

Z ia a character of G and is the simplest physically relevant quantity in the theory since it is independent of the choice of basis of $\mathcal{H}$.

Solving the quantum theory just requires finding $U(\tau)$, we will try and do this by expressing $U(\tau)$ as a Feynman path integral. This theory is so simple that path integral techniques are clearly not the most efficient way of solving the theory, but the apparatus we will develop generalizes easily to more interesting theories where the Schrödinger formulation is not very useful. Furthermore this is the simplest system in which the notion of geometric quantization works nicely and we will thus be able to explore the relationship between path integral and geometric quantization.

The classical phase space corresponding to this quantum theory will be the orbit under the action of G of the ray corresponding to a highest weight vector in the complex projective space (V). The highest weight that defines V will be denoted λ_V. The state vectors corresponding to rays in this orbit are often called "coherent states". Any element of V can be written as a linear combination of these state vectors. This orbit in $\mathbf{P}$(V) is diffeomorphic to G/G_{λ_V} where G_{λ_V} is the subgroup of G that acts on the highest weight vector by a phase. For a "generic" representation this subgroup will be the maximal torus, denoted T. In what follow we will refer to these orbits as G/T, although for certain representations V what we actually mean is G/G_{λ_V}.

These orbit spaces, which also are often called flag manifolds or co-adjoint orbits (corresponding to two alternate ways of defining them) are Kähler manifolds. Even better, they are projective algebraic varieties with an explicitly given embedding in $\mathbf{P}$(V). The tautological line bundle L over $\mathbf{P}$(V) is the complex line bundle whose fiber above a point p in $\mathbf{P}$(V) is the corresponding complex line. Its restriction to G/T will be denoted L_{λ_V}, it is a holomorphic line bundle and will be of great importance in what follows.

COHERENT STATE PATH INTEGRALS

The most common Feynman path integral is an integral over paths in a configuration space X and is used to construct a quantum theory with Hilbert space $L^2(X)$. This corresponds to a theory with classical mechanical phase space T^*X. In the quantum theory that interests us, the classical phase space M=G/T does not have the structure of a cotangent bundle so the standard sort of path integral does not apply (however see [6] for a discussion of the path integral quantization of spin using a real polarization as in the T^*X case). Various efforts have been made to construct path integrals as integrals over the phase space, and in this case such path integrals go under the name of coherent state path integrals.

Defining a path integral over paths in M seems bound to lead to trouble with the Heisenberg uncertainty principle since one is attempting to specify at each value of time values of both conjugate variables. In this section we will review the standard formalism of coherent state path integrals and see what problems arise. We will deal with the simplest coherent state path integral, that corresponding to the case $H(\tau) = 0$.

The standard treatment of the coherent state path integral is based upon the so-called "resolution of the identity" which expresses the identity operator on the representation space V as

$$1 = \frac{1}{\Gamma} \int_G |g \cdot z_0 >< g \cdot z_0|$$

where $|z_0 >$ is a highest weight vector, $\int_G$ denotes Haar measure on G and

$$\Gamma = \int_G | < g \cdot z_0 | z_0 > |^2$$

is a normalization constant.

This can also be thought of as an integral over M of projection operators

$$1 = \frac{1}{\Gamma'} \int_M |z >< z|$$

Here z labels points in M, there is a phase ambiguity in the definition of $|z>$ but this cancels the phase ambiguity in $<z|$. Γ' is a normalization constant, and the integral over M is defined using the symplectic measure on M.

This identity is used to express the inner product between two highest weight vectors $|z'>$ and $|z''>$ in V as

$$<z''|\prod_{\tau}\frac{1}{\Gamma}(\int_G |g_\tau \cdot z_0><g_\tau \cdot z_0|)|z'>$$

or

$$<z''|\prod_{\tau}\frac{1}{\Gamma'}(\int_M |z_\tau><z_\tau|)|z'>$$

where τ is a variable parametrizing the projection operators which takes a finite number of values. One can for instance define τ to be the finite set

$$\tau = \{0, \Delta\tau, 2\Delta\tau, 3\Delta\tau, \ldots, 1\}$$

where

$$\Delta = 1/N$$

for some integer N. The coherent state path integral representation of this inner product is formally the limit as $N \to \infty$ of this expression.

Clearly this limit exists, since it is independent of N anyway. However the standard interpretation of this product as an integral over paths in M joining the ray defining $|z'>$ to that defining $|z''>$ is problematic. There is no sense in which

$$z_\tau \approx z_{\tau+\Delta\tau}$$

as $\Delta\tau \to 0$ and yet this continuity assumption is often invoked in manipulations of these integrals.

Let us however proceed under the assumption that we are dealing with continuous paths and see how the standard formalism is developed. Also, assume that we are dealing with a representation V such that the orbit of a highest weight vector is the full flag manifold G/T rather than something smaller. Then the "naive" limit as $\Delta\tau \to 0$ of the path integral expression for the inner product between $|z'>=|g_0 \cdot z_0>$ and $|z''>=|g_1 \cdot z_0>$ will be of the form

$$K(g_1, g_0) =<g_1 \cdot z_0|g_0 \cdot z_0>= \frac{1}{\Gamma''}\int_{g_0 \to g_1} e^{\int_0^1 d\tau(\omega + i\theta(\dot{g}))}$$

This is meant to be interpreted as follows. One is integrating over all paths in G from g_0 to g_1 but what is relevant is their projection onto the orbit of the highest weight vector $|z_0>$. θ is a left-invariant 1-form on G, it is the canonical connection 1-form for the tautological line bundle L over the orbit in $\mathbf{P}(V)$. $\omega = \frac{d\theta}{2\pi}$ is the lift to G of the standard symplectic 2-form on $\mathbf{P}(V)$.

The first factor in the integrand is

$$e^{\int_0^1 \omega} = \prod_{\tau} e^{\omega} = \prod_{\tau}\frac{(\omega)^n}{n!}$$

it is meant to be formally interpreted as an infinite product of symplectic volume forms, one for each value of τ, providing a volume form for the loop space. The second factor is

$$e^{i\int_0^1 \theta(\dot{g})}$$

It is just the phase corresponding to parallel transport with respect to the connection θ.

The simplest quantity that one would like to calculate is the partition function

$$Z = Tr_V(1) = \int_G K(g,g) = \dim V = \int_{\Omega G} e^{\oint d\tau[\omega + i\theta(\dot{g}(\tau))]}$$

The integrand is conceptually quite simple, especially when considered in terms of loops on M. Then one is just integrating the holonomy around a loop against the (unfortunately still ill-defined) measure on ΩM that is just the infinite product of the symplectic measures for each τ. The holonomy is

$$hol(C) = e^{2\pi i \int_{S:\partial S = C} \omega}$$

where the exponent here is 2π times the "action", which is well-defined up to an integer ambiguity.

This sort of path integral has several related problems that prevent one from giving it any well-defined meaning. The first is that the illegitimate assumption of the continuity of paths prevents one from keeping track of the normalization of the integral. In Feynman's configuration space version of path integration one has a term

$$e^{-\frac{1}{2}\int |\dot{x}|^2}$$

in the integrand which damps out discontinuous paths. This sort of term is absent here.

A second problem is in how one has taken the trace. One can think of this path integral as a normal Feynman-type integral over a configuration space G (without the necessary damping term) with an integrand that acts as a projection from the full Hilbert space of complex-valued functions $L^2(G)$ to a subspace of equivariant functions, those that are sections of the line bundle L_{λ_V}. One is thus evaluating the trace on the full infinite dimensional induced representation of G on $\Gamma(L_{\lambda_V})$. The representation whose trace we wish to calculate is a finite dimensional subspace of $\Gamma(L_{\lambda_V})$. Thus one is evaluating the wrong path integral and one has to figure out some way of removing the unwanted representations from the trace. This is generally done by trying to push the unwanted representations to infinite energy, for instance by adding a factor

$$e^{-\alpha \int |\dot{x}|^2}$$

and taking the limit $\alpha \to \infty$.

We will see that index theory provides a natural way of cancelling contributions of all but the correct finite dimensional subspace of $\Gamma(L_{\lambda_V})$. Thus we will be looking for a supersymmetric quantum mechanical model with a fermionic path integral that will give the necessary cancellations.

GEOMETRIC QUANTIZATION, BOREL-WEIL-BOTT AND INDEX THEORY

Geometric quantization is a general program for producing a quantum theory associated to a given classical system. The quantization of the flag manifold G/T was one of the inspirations for the geometric quantization program[7] and not surprisingly this is the case where it works most simply. In this case geometric quantization essentially coincides with the Borel-Weil-Bott theorem[8,9]. This theorem states that the representation of G with highest weight λ can be identified with the space of holomorphic sections of a holomorphic line bundle L_λ. Recalling that G is a principal T bundle over G/T and that the weight λ gives a representation of T on $\mathbf{C}$, L_λ is the associated line bundle over G/T given by this representation. Note that the condition of a section being holomorphic picks out a finite dimensional subspace $\Gamma_{hol}(L_\lambda)$ of the infinite dimensional space of sections $\Gamma(L_\lambda)$.

We wish to work here not in the very general context of the theory of geometric quantization, but in the context of index theory which will turn out to be an equivalent point of view[10,11]. What the Borel-Weil-Bott theorem does is construct a map

$$R(T) \rightarrow R(G)$$

from the representation ring of T to the representation ring of G. The highest weight λ_V gives a representation of T and thus an element of R(T) and the Borel-Weil-Bott theorem gives a construction of $V \in R(G)$.

From the point of view of index theory the natural framework for our discussion is that of equivariant K-theory. The definition of the cohomology classes $H^*(M)$ of a manifold is well known to physicists. When a group G acts on M one can define equivariant cohomology classes $H^*_G(M)$, if the action of G is free these reduce to $H^*(M/G)$. The group K(M) is defined in terms of equivalence classes of vector bundles over M and has similar properties to a cohomology group. When G acts on M one can define $K_G(M)$, the equivariant K-theory of M, in terms of equivariant vector bundles on M (see for instance [12]).

In equivariant K-theory we have

$$K_G(pt.) = R(G)$$

and

$$K_T(pt.) = K_G(G/T) = R(T)$$

Just as for a map

$$\pi : M \rightarrow pt.$$

there is a push-forward or integration map π_* in cohomology or in equivariant cohomology, there is an integration map

$$\pi_! : K_G(M) \rightarrow K_G(pt.) = R(G)$$

in equivariant K-theory. If we take M=G/T, this is precisely the map that appears in the Borel-Weil-Bott version of representation theory. It is best described concretely in terms of the index of the Dirac operator on G/T.

If E is a vector bundle representing the class α_E in K(M), then

$$\pi_!(\alpha_E) = index D_E = \dim\ \mathrm{ker} D_E - \dim\ \mathrm{coker} D_E$$

where D_E is the Dirac operator on spinors twisted by E. In the equivariant case $\alpha_E \in K_G(M)$ the kernel and cokernel of the Dirac operator are representation spaces for G and their difference is in R(G). Then

$$\pi_!(\alpha) = [\mathrm{ker} D_\alpha] - [\mathrm{coker} D_\alpha] \in R(G)$$

We want to understand the Borel-Weil-Bott construction of the representation V in these K-theoretic terms. We have seen that V is isomorphic to the space $\Gamma_{hol}(L_{\lambda_V})$ of holomorphic sections of the line bundle L_{λ_V}. In other words

$$V = H^0(G/T; L_{\lambda_V})$$

One can show that for $q > 0$

$$H^{0,q}(G/T; L_{\lambda_V}) = 0$$

Together these facts imply that for the Dolbeault operator

$$\bar{\partial} + \bar{\partial}^* : \Gamma(L_{\lambda_V} \otimes \Lambda^{0,*}) \rightarrow \Gamma(L_{\lambda_V} \otimes \Lambda^{0,*})$$

we have

$$V = index(\bar{\partial} + \bar{\partial}^*) = ker(\bar{\partial} + \bar{\partial}^*) - coker(\bar{\partial} + \bar{\partial}^*)$$

Since G/T is a Kähler manifold, this Dolbeault operator operating on $\Gamma(L_{\lambda_V} \otimes \Lambda^{0,*})$ is identical to the Dirac operator acting on $\Gamma(L_{\lambda_V} \otimes S \otimes (\Lambda^{0,n})^{1/2})$ where S is the spinor bundle and $(\Lambda^{0,n})^{1/2}$ is a square root of the canonical bundle. On G/T, $(\Lambda^{0,n})^{1/2} = L_\delta$ (where δ is half the sum of the positive roots) and the Dirac operator acts as

$$D_{(\lambda_V+\delta)} : \Gamma(S \otimes L_{(\lambda_V+\delta)}) \rightarrow \Gamma(S \otimes L_{\lambda_V+\delta})$$

Finally we see that we have the isomorphism

$$V = index(D_{\lambda_V+\delta})$$

What we have done is shown that the Borel-Weil-Bott construction of a representation of G from a highest weight representation of T is just the integration map

$$\pi_! : K_G(G/T) = R(T) \rightarrow K_G(pt.) = R(G)$$

for the map

$$\pi : G/T \rightarrow pt.$$

An important example of how this works out is that of V=**1**, the trivial representation. Here the Atiyah-Singer index theorem tells us

$$\begin{aligned} dimV = Tr_1(1) &= index(D_\delta) \\ &= \hat{A} \wedge ch(L_\delta)[G/T] \\ &= \tau[G/T] = 1 \end{aligned}$$

where τ is the Todd class. Note that here even the trivial representation involves a non-trivial calculation, and this will be reflected in our path integral calculations by a non-trivial path integral that corresponds to this representation.

We have seen that for a phase space M=G/T quantization is equivalent to integration in K-theory. It turns out that this is also true in other less trivial contexts. While the geometric quantization program has tried with partial success to provide a geometric description of quantization for general symplectic manifolds, thinking of quantization as integration in K-theory gives a conceptually simpler picture when it is applicable.

As another example of this general principle consider the quantization of the harmonic oscillator with phase space M=**C**. This is different than the M=G/T case since **C** is not compact. There is a well-known description of the Hilbert space of the harmonic oscillator in terms of holomorphic sections of the trivial bundle over **C**. The group $\mathbf{C}^*$ of non-zero complex numbers acts on this line bundle and the infinite dimensional Hilbert space decomposes as a sum of one-dimensional representations of

$\mathbf{C}^*$. Thus the Hilbert space of the harmonic oscillator can be thought of as a sum of finite-dimensional spaces of zero modes of a Dirac operator. Note that the $\frac{1}{2}$ that occurs as the ground state energy of the harmonic oscilator has the same origin as the term δ, half the sum of the positive roots, in the G/T case.

The harmonic oscillator thus corresponds to the quantization of $\mathbf{C}^*$, the complexification of U(1). One can also consider the quantization of GL(n,**C**), the complexification of U(n). Here the Hilbert space is $L^2(U(n))$ which could be thought of as the space of holomorphic sections of a certain trivial line bundle over GL(n,**C**). By the Peter-Weyl theorem this Hilbert space contains all the irreducible unitary representations of U(n). In our earlier discussion we restricted attention to one irreducible representation V by looking at not all of $L^2(G)$ but at $\Gamma(L_{\lambda_V})$, which is a subspace of this space satisfying a certain equivariance property. The passage from the complexification of G to G/T is an example of Marsden-Weinstein[13] reduction in symplectic geometry.

We will see later that the principle of quantization as integration in K-theory also seems to apply in the very non-trivial cases of Wess-Zumino-Witten models in conformal field theory and in Witten's Chern-Simons gauge theory. Undoubtedly there are other examples where this principle is valid, the full range of its validity has not yet been investigated.

INDEX THEORY AND SUPERSYMMETRIC QUANTUM MECHANICS

The Atiyah-Singer index theorem tells us that the index of the Dirac operator on M coupled to a vector bundle E is given by $\pi_!(\alpha_E)$. This push-forward map was defined by Atiyah and Singer in terms of embedding M in S^{2N} for some large N and using the Thom isomorphism, for the normal bundle of M to construct an element in $K(S^{2N})$ from the class $\alpha_E \in K(M)$. Bott periodicity implies that $K(S^{2N}) = \mathbf{Z}$ and this integer will be the index.

Remarkably, it turns out that there is an alternate formulation of this topological index map in terms of a certain supersymmetrical quantum mechanical system. The essential idea is that at least formally one can identify $K(M)$ with $K_{S^1}(LM)$ the equivariant K-theory of the free loop space LM of parametrized loops in M with respect to the circle action given by rotation of the loop. The calculation of the index is reformulated in terms of an integration map in the S^1 equivariant cohomology of the loop space. This abstract concept when carried out using differential forms on the loop space is equivalent to the calculation of the partition function in a simple supersymmetrical quantum mechanical system [14].

Following Atiyah's exposition of an idea originally due to Witten[15], the index of the Dirac operator can be expressed as an integral over the loop space ΩM of the equivariantly closed form

$$\mu = e^{\frac{1}{2}(d-i_X)\alpha}$$

Here X is the vector field on LM that generates rotations around the loops, α is the 1-form dual to this vector field (we are assuming that M has a metric and using the induced metric on LM), and i_X is contraction with the vector field X.

For a finite 2n-dimensional manifold $\tilde{M}$ with an S^1 action generated by the vector field X there is a localization formula[16,17] which expresses the integral of an equivariantly closed form in terms of an integral over the submanifold F left fixed by the

S^1 action. For a form μ on $\tilde{M}$ such that $(d - i_X)\mu = 0$,

$$\int_{\tilde{M}} \mu = \int_F \frac{\mu|_F}{\chi(X,N)}$$

In this formula $\chi(X,N)$ is an equivariant Euler characteristic defined as

$$\chi(X,N) = det(J_X - \frac{\Omega}{2\pi i})$$

where N is the normal bundle to F in $\tilde{M}$, $J_X \in \Gamma(EndN)$ is the infinitesimal action of X in N and $\Omega \in \Lambda^2 T^* \tilde{M}_0 \otimes EndN$ is the curvature of an S^1 invariant linear connection on N.

For the special case of $\tilde{M}$ a symplectic manifold and

$$\mu = e^{-H}\frac{\sigma^n}{n!}$$

this formula is due to Duistermaat and Heckman[18]. In this context it states that the stationary phase approximation for $\int_{\tilde{M}} \mu$ is exact.

If we assume that a suitably regularized version of this formula applies to our infinite dimensional case of $\tilde{M} = \Omega M$, Atiyah has shown that one finds

$$\text{index } D = \int_{\Omega M} \mu = \hat{A}(M)$$

giving the standard result for the index of the Dirac operator. In this case the fixed point set is the manifold M itself, and Fourier expansion of vectors in the tangent space to a point loop gives the normal bundle as the infinite direct sum

$$N = (T \otimes \mathbf{C})_1 \oplus (T \otimes \mathbf{C})_2 \oplus \ldots$$

where $(T \otimes \mathbf{C})_p$ is the complexified tangent bundle of M with S^1 acting with rotation number p.

The integrand that occurs here is identical with the integrand in the path integral form of the simple supersymmetric quantum mechanics system where the Lagrangian is written

$$L = \int d\tau \frac{1}{2}(|\frac{dx}{d\tau}|^2 - \psi^i D_\tau \psi^i)$$

In SSQM the partition function is evaluated as

$$Z = \int [dx][d\psi]e^{-L}$$

which can be understood as a way of rewriting the integral expressed through differential forms on loop space in terms of the Riemannian volume form on loop space and a fermionic integration. In the physics literature[14] this integral is evaluated in the stationary phase approximation, which we have seen is exact in this situation.

We will actually need the generalization of this to the case of the Dirac operator coupled to a line bundle L. If A is a connection 1-form for this line bundle then one can formally define a 1-form on LM by

$$\alpha_1 = 2i \oint d\tau A(x(\tau))$$

(the way we have written the connection assumes a choice of section of L, but the final result will be independent of this choice).

The coupling to L simply has the effect of adding α_1 to the 1-form α and the integrand that gives the index in this case will be

$$\mu = e^{\frac{1}{2}(d - i_X)(\alpha + \alpha_1)}$$

Of the two new terms in the exponent, one just gives the holonomy around the loop, the other involves the curvature and is the standard term familiar in QED that couples the spin to the magnetic field. Formally applying the localization formula gives the standard cohomological form of the index theorem. At the fixed point set only the curvature term survives and the equivariant Euler characteristic of the normal bundle is unchanged from the untwisted case.

Given the above, the main point that we would like to make is quite simple. Instead of the standard coherent state path integral and its attendant problems, the appropriate path integral quantization of G/T is given by the SSQM path integral for the index of the Dirac operator twisted by a certain line bundle. Notice that this path integral contains the holonomy term that appears in the coherent state path integral, but it also contains a term

$$e^{-\frac{1}{2}\int |\dot{x}|^2}$$

as well as fermionic variables. The effect of the fermionic variables is to provide for a cancellation of the contribution to the partition function of all states except those corresponding to the zero modes of the Dirac operator. Thus this path integral resolves the problem with the coherent state path integral that we noted before.

In order to avoid problems about exactly how to normalize these integrals (equivalently, how to define the equivariant Euler characteristic of the normal bundle to the point loops in LM), we can compute ratios of path integrals, taking the ratio of the path integral for the representation we want to study with that for the trivial representation. Thus we get the formula for the dimension of a representation

$$dimV = tr(1) = \frac{\int_{\mathcal{O}(\lambda_V + \delta)} e^{\omega}}{\int_{\mathcal{O}(\delta)} e^{\omega'}}$$

Here $O(\lambda_V + \delta)$ and $\mathcal{O}(\delta)$ are the flag manifolds determined by the highest weight λ_V and the weight δ, and ω and ω' are the standard symplectic 2-forms on the two orbits. This orbital integral formula for the dimension of a representation is well known

CHARACTER FORMULAS

We have so far just been discussing the formula for the dimension of the representation, which is an integer index. The equivariant index theorem gives the character of the representation, and our integration formula in this case is a trace formula (see [19] for further discussion of the relation between index theory and the Kirillov formula). The modification that needs to be made is simply that of changing the vector field X on L(G/T) by adding a constant vector field proportional to the vector field on G/T that corresponds to the action of the group element g whose trace we wish to calculate. One gets the formula

$$tr(e^X) = \frac{\int_{\mathcal{O}(\lambda_V + \delta)} e^{if(X) + \omega}}{\int_{\mathcal{O}(\delta)} e^{if(X) + \omega'}}$$

Here $f(X)$ is the moment map corresponding to the action of e^X on G/T. Further applying the fixed-point localization formula to the denominator gives

$$tr(e^X) = det^{-\frac{1}{2}}\left(\frac{e^{ad\frac{X}{2}} - e^{-ad\frac{X}{2}}}{adX}\right)\int_{\mathcal{O}(\lambda_V+\delta)} e^{if(X)+\frac{\omega}{2\pi}}$$

which is the Kirillov character formula. Yet another application of the fixed point formula to the numerator gives a version of the Weyl character formula.

LOOP GROUPS

The representation theory for positive energy unitary representations of the loop group LG can be developed in much the same way as the Borel-Weil-Bott representation theory for G[20]. Here LG/T is an infinite dimensional Kähler manifold that plays the role of G/T in the finite-dimensional case. Actually there is a fibration $LG/T \to LG/G$ of Kähler manifolds with fiber G/T. Restricting attention to a fiber gives back the finite-dimensional picture.

The corresponding quantum theory here is the Wess-Zumino-Witten[21] model of conformal field theory which has received much attention from physicists in recent years. This theory is the simplest example of a field theoretical model that can be most simply understood in terms of the equivariant K-theory framework we have developed.

WITTEN'S CHERN-SIMONS THEORY

Last summer, in a striking paper[2], Witten defined new invariants for links in 3-manifolds using a quantum gauge field theory with action given by the Chern-Simons functional. Witten writes his invariants in terms of a functional integral as

$$Z(C,R,k,N) = \int [dA] W_R(C) e^{2\pi i CS[A]}$$

where C is a link in a 3-manifold M^3, k and N are integers, A is a connection on a the trivial principal SU(N) bundle over M^3, $[dA]$ is the standard physicist's notion of a formal measure on the space of such connections and $W_R(C)$ is the trace of the holonomy around the curve C with respect to the connection A in the SU(N) representation R. CS[A] the Chern-Simons functional of the connection A.

The functionals $W_R(C)$ are well-known to physicists as Wilson loops, they are exactly the sort of objects that appeared as the partition functions for the simple quantum system of the first part of this paper. Thus we have seen that there is a supersymmetric quantum mechanics path integral expression for these quantities.

Restricting attention to the case $C = \emptyset$, we get invariants of the 3-manifold itself and the functional integral over gauge fields Witten uses is very much analogous to the coherent state path integral we discussed at the beginning of this talk. It has similar problems and Witten performs most of his calculations using the associated geometric quantization of the theory rather than the functional integral.

The analogy we wish to point out is most clearly seen if we specialize to the case of a three manifold of the form $M^3 = \Sigma \times S^1$, where Σ is a Riemann surface. The space of connections $\mathcal{A}$ on a principal G bundle over Σ is an infinite dimensional symplectic manifold, given a complex structure on Σ it is Kähler[22]. The group $\mathcal{G}$ of gauge transformations acts on $\mathcal{A}$ and symplectic reduction with respect to this symmetry gives as reduced phase space the moduli space $\mathcal{M}$ of flat connections on Σ. This moduli space is again a Kähler manifold and there is a holomorphic line bundle L over it whose first Chern class is the Kähler 2-form.

A loop in $\mathcal{M}$ corresponds to a connection on M^3 and the Chern-Simons functional of this connection is the holonomy around the loop in the line bundle L. Witten's path integral for this theory is just an integral over loops in $\mathcal{M}$ weighted by the holonomy, exactly as in the G/T coherent state path integral but with G/T replaced by $\mathcal{M}$. Witten points out that his invariant in this case is the dimension of the space of holomorphic sections of the line bundle L^k.

Thus Witten's invariant can be thought of in this case as an integration in K-theory, in particular

$$Z(k,N) = \pi_!(L^k)$$

where

$$\pi : \mathcal{M} \to pt.$$

and it should have an expression as a supersymmetric quantum field theory.

The functional integral that Witten writes down suffers from the same problems as the coherent state path integral for G/T. It is a trace over all sections of L^k, not just the holomorphic ones. In a recent preprint[23], Ramadas, Singer and Weitsman deal with this problem by inserting a term

$$e^{-T\int |\frac{dA}{dt}|^2}$$

into the functional integral and taking the limit $T \to \infty$. It would seem to be preferable to reformulate Witten's functional integral with fermionic variables that would cancel all but the holomorphic sections from the partition function.

Further evidence for the desirability of an index-theory reformulation of Witten's functional integral lies in the problems associated with framings. Witten finds that to get a well defined semi-classical approximation he must add a term involving η invariants to his Chern-Simons action. The effect of this term is to change $k \to k + N$ in the results of his calculations. An index theory reformulation of Witten's invariant leads to this in a natural way since N plays the same role in this case as δ (half the sum of the positive roots) plays in the G/T case. The analogy can most clearly be seen by working through the connection between Witten's invariants and the Wess-Zumino-Witten quantum theory version of loop group representation theory. In the loop group case one finds that one should think of N as half the first Chern class of the tangent bundle of LG/G, just as δ is half the first Chern class of the tangent bundle of G/T.

CONCLUSIONS

In this talk we have tried to explain a new conceptual approach to the problem of quantization through its application to a simple quantum system. This approach seems also the best way to understand the various topological quantum theories that have excited recent interest. These topological quantum theories contain a great deal of structure of physical as well as of mathematical interest. A more detailed exposition of these ideas and their applications is in preparation.

REFERENCES

[1] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988), 353–386.

[2] E. Witten, *Quantum field theory and the Jones polynomial*, Princeton IAS preprint IASSNS-HEP- 88/33 (Sept. 1988).

[3] J. Klauder and B.-S. Skagerstam, "Coherent States:Applications in Physics and Mathematical Physics," World Scientific, Singapore, Philadelphia, 1985.

[4] H. B. Nielsen and D. Rohrlich, *A path integral to quantize spin*, Nucl. Phys. **B299** (1988), 471–483.

[5] M. Stone, *Supersymmetry and the quantum mechanics of spin*, Nucl. Phys. **B314** (1989), 557–586.

[6] A. Alekseev, L. Fadeev and S. Shatashvili, *Quantization of the symplectic orbits of the compact Lie group by means of the functional integral*, Preprint (1988).

[7] B. Kostant, *Quantization and unitary representations*, in "Lectures in Modern Analysis III," Lecture Notes in Mathematics vol. 170, 1970, pp. 86–208.

[8] R. Bott, *Homogeneous vector bundles*, Annals of Mathematics **66** (1957), 203–248.

[9] R. Bott, *On induced representations*, in "The Mathematical Heritage of Hermann Weyl," Vol. 48 Proceedings of Symposia in Pure Mathematics, AMS, 1988, pp. 1–13.

[10] R. Bott, *Homogeneous differential operators*, in "Differential and Combinatorial Topology," S.S. Cairns, ed., 1965, pp. 167–186.

[11] G.B. Segal, *The representation ring of a compact group*, Publ. Math. IHES **34** (1968), 113–128.

[12] G.B. Segal, *Equivariant K-theory*, Publ. Math. IHES **34** (1968), 129–151.

[13] J. Marsden and A. Weinstein, *Reduction of symplectic manifolds with symmetry*, Rep. on Math. Phys. **5** (1974), 121–130.

[14] L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer Index Theorem*, Comm. Math. Phys. **90** (1983), 161-173.

[15] M.F. Atiyah, *Circular symmetry and the stationary phase approximation*, Astérisque **131** (1985), 43–59.

[16] M.F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[17] N. Berline and M. Vergne, *Zeros d'un champ de vecteurs et classes characteristiques équivariantes*, Duke Math. J. **90** (1983), 539–549.

[18] J.J. Duistermaat and G. J. Heckman, *On the variation in the cohomology in the symplectic form of the reduced phase space*, Invent. Math. **69** (1982), 259–268; Invent. Math. **72** (1983), 153–158.

[19] N. Berline and M. Vergne, *The equivariant index and Kirillov's character formula*, Am. J. of Math. **107** (1985), 1159–1190.

[20] A. Pressley and G. Segal, "Loop Groups," Oxford University Press, New York, 1986.

[21] E. Witten, *Non-abelian bosonization in two dimensions*, Comm. Math. Phys. **92** (1984), 455–472.

[22] M.F. Atiyah and R. Bott, *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. Roy. Soc. **A308** (1982), 523–615.

[23] T.R. Ramadas, I.M. Singer and J. Weitsman, *Some comments on Chern-Simons gauge theory*, MIT preprint (1989).

TOPOLOGICAL CHERN-SIMONS GAUGE THEORY AND "NEW" KNOT/LINK POLYNOMIALS

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ABSTRACT

Topological Chern-Simons gauge theory in D=3 is discussed both perturbatively and non-perturbatively. Several topological features have been seen in perturbation theory. Following Witten's non-perturbative approach, we derive the most general skein relation for the expectation values of Wilson loops, by exploring the close relationship with 2-d conformal field theory. For the cases with the fundamental representation of SO(N) and Sp(N), it is shown that the skein relation is sufficient to determine the new hierarchies of topological polynomials for all knots and links dicussed by Kauffman (also by Turaev and by Reshetikhin).

I. INTRODUCTION

A topological Chern-Simons gauge theory[1] is a quantized non-Abelian gauge theory in three dimensions with only the Chern-Simons action:

$$I_{C-S} = -\frac{k}{4\pi}\int_{M^3} \mathrm{tr}\,(A_\wedge dA + 2/3\; A_\wedge A_\wedge A) \tag{1}$$

$$= \int_{M^3} d^3x\, \frac{1}{2}\, \epsilon^{\mu\nu\lambda} \left[A^a_\mu\, \partial_\nu\, A^a_\lambda + \frac{1}{3}\; g f^{abc}\, A^a_\mu\, A^b_\nu\, A^c_\lambda \right] \tag{1'}$$

where $g^2 = 4\pi/k$. This action is natural in D=3 in the sense that the coefficient k or g is dimensionless. In the following, we assume $M^3 = S^3$ (3-sphere); some other interesting cases are $M^3 = S^2 \times S^1$ or $\sum \times R^1$, where $\sum$ is a Riemann surface. The coefficient k should be an integer to maintain invariance under large gauge transformations.[2] The most important feature of the action (1) is that it does not contain a metric,

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so the theory has diffeomorphism invariance without gravitational degrees of freedom. At the classical level this is obvious; what is highly non-trivial is that diffeomorphism invariance is expected to be also true after quantization, even if the usual gauge fixing and Faddeev-Popov procedure (and also some regularizations, see below) always involve a metric. Thus, the quantum gauge theory with the action (1) does not contain any locally propagating degrees of freedom and only gauge-invariant observables are global topological invariants. This theory is an important example of the so-called topological quantum field theory.

Topological Chern-Simon gauge theory is interesting in several aspects. Mathematically it provides an intrinsically 3-dimensional definition for topological invariant polynomials for knots and links such as the well-known Jones[3] or HOMFLY[4] polynomials, all of which are previously defined[5] directly for 2-dimensional projections of knots/links. Physically, there is a very close relationship between the D=3 Chern-Simon theory and conformal field theories in d=2. Later we are going to explore this relationship, following Witten[1], to study the Chern-Simons theory. It is possible to use the relationship in the reverse direction, i.e. to study conformal field theories by using the knowledge of Chern-Simon theory. More importantly, this is a prototype of a metric-independent quantum theory, belonging to a new class of unfamiliar quantum field theories, which might mimic some aspects of string physics above the Planck scale. Also, the Chern-Simons gauge theory, in particular its abelian version, might be relevant to high-T_c superconductivity[6] and other 2-dimensional condensed matter phenomena[7,8].

Here I am going to report on some recent joint works[9,10] with my collaborators. The first work[9] deals with topological features of the quantum Chern-Simons theory seen in perturbation theory. These include the vanishing of the beta function up to three loops, the absence of diffeomorphism anomaly (as exemplified by the absence of terms quadratic in F in the effective action), and the validity of a topological Ward identity by finite renormalization of the coupling constant. In the second work[10], we generalize Witten's result on the skein relation satisfied by the expectation values of Wilson loops as topological invariants. We have been able to derive the skein relation in the most general form explicitly and to show that for the particular cases of the fundamental representation of SO(N) and Sp(N) not studied by Witten, the skein relations are still sufficient to determine the topological invariant polynomial for <u>all</u> knots and links. These "new" knot polynomials coincide with those previously obtained by Turaev[11] and Reshetikhin[12], which in turn coincide with the Kauffman two-variable polynomial[5]. In particular, Reshetikhin[12] has used quantum groups to construct the knot polynomials. The coincidence of our results with his suggests a direct relationship between quantum groups and Chern-Simons gauge theory.

II. REGULARIZATION AND RENORMALIZATION

To quantize (1), gauge-fixing and the Faddeev-Popov procedure leads to

$$I_{g.f.} + I_{gh} = \int_{S^3} \left\{ -\frac{1}{\beta} \operatorname{tr}\left[\partial_\mu A_\mu\right]^2 + \partial_\mu \bar{c}\, D_\mu c \right\} \tag{2}$$

We emphasize that in the gauge fixing we have chosen a (flat) metric $g_{\mu\nu} = \delta_{\mu\nu}$ with S^3 viewed as a compactified version of R^3. If we take $\beta \to \infty$ in the propagator, we are using the Landau gauge, which has the advantage of avoiding potential infra-red divergences. Although it is unavoidable to involve a metric in gauge-fixing, the expectation is that the theory associated with I_{C-S} maintains to be independent of the metric at the quantum level.

Perturbation theory is a traditional method in dealing with quantum field theories. It is interesting to see how many topological features of the theory can be seen in perturbative theory. The Feynman rules are $\delta^{ab}\epsilon_{\mu\nu\lambda}k_\lambda/k^2$ for the A-propagator, $gf^{abc}\epsilon_{\mu\nu\lambda}$ for the AAA-vertex and as usual otherwise. By power counting the theory is renormalizable, so one needs to regularize loop diagrams. We have used two regularizations: 1) Dimensional regularization: First perform all the tensor algebra with $\epsilon_{\mu\nu\lambda}$ and the $\delta_{\mu\nu}$-symbols obtained from the contractions of ϵ-symbols in D=3 (physical dimensions), and then dimensionally extend and evaluate the (scalar) loop-momentum integrals in $\omega = 3-\epsilon$ (regularized dimensions). This hybrid prescription is necessary for maintaining gauge invariance and is similar to the regularization by dimensional reduction used for supersymmetry.[13] 2) F^2-regularization: add a Yang-Mills term action:

$$I_{Y-M} = -\int_{S^3} \frac{1}{2e_0^2} \operatorname{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) \tag{3}$$

which also involves the metric $\delta_{\mu\nu}$. Feynman rules are modified accordingly. The cut-off e_o^2 has (mass) dimension one. With e_o^2 finite, the theory is super-renormalizable and $\mu_o^2 \equiv 4\pi e_o^4/g^2$ behaves like the mass squared of the A-field. With $e_o^2 \to \infty$, the propagating components of the A-field becomes infinitely massive. This regularization is manifestly gauge invariant.

The charge renormalization is $g_r = g\, Z_g Z_A^{-3/2}$, with Z_A and Z_g defined from the anti-symmetric part of the inverse A-propagator and AAA-vertex:

$$\Delta_{\mu\nu}^{-1}(k) \xrightarrow{k\to 0} Z_A\, \epsilon_{\mu\nu\lambda} k_\lambda + Z'_A \left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) + \ldots \tag{4}$$

$$\Gamma^{abc}_{\mu\nu\lambda}(p, q, r) \xrightarrow{p,q,r\to 0} Z_g\, g f^{abc}\, \epsilon_{\mu\nu\lambda} + \ldots \tag{5}$$

Note that it is Z_A, rather than the usual Z'_A, that is the wave function renormalization constant in the present case. Gauge invariance of the effective action under small and large gauge transformations gives

$$Z_g / Z_A = \tilde{Z} / Z_{gh} \tag{6}$$

$$k_r \equiv 4\pi/g_r^2 \equiv (4\pi/g^2)\, Z_A^3 Z_g^{-2} = \text{integer} \tag{7}$$

III. TOPOLOGICAL FEATURES SEEN IN PERTURBATION THEORY

Our main results from perturbation theory are the following.

1) Vanishing of $\beta(g_r)$ up to 3-loops. From (7), g_r should be quantized and, therefore, cannot have a continuous renormalization group flow. Both g and g_r are formally not required to be quantized in perturbation theory. Logically, the perturbative β-function $\beta(g_r)$ can be either identically zero or equal to

zero only when g_r is quantized. What we found is the first possibility, namely $\beta(g_r)$ vanishes for all g_r, at least up to three loops in dimensional regularization.

The proof consists of two parts. First, since both Z_A and Z_g are from the $\epsilon_{\mu\nu\lambda}$-part, one needs to count the number of ϵ-symbols in the AA- and AAA- diagrams. It can be shown graphically that this number is odd (or even) only when there is an even (or odd) number of loops. So for the β-function to vanish, one needs to check only 2-loop diagrams. The second step is to explicitly show that each 2-loop AAA-diagram is finite and the divergent contributions to Z_A from all 2-loop AA-diagrams are exactly summed to zero.

2) Topological Ward identity satisfied by finite renormalization

At 1-loop there is no finite renormalization of g in dimensional regularization at all. However, we have found a finite renormalization in the F^2-regularization. Although g and g_r are not required to be quantized in perturbation theory, consistency with the topological requirement (7) requires the finite renormalization to satisfy

$$4\pi/g_r^2 = 4\pi/g^2 + \text{integer} \tag{8}$$

Indeed we have found that $Z_A = 1 + (7g^2/12\pi)C_2$, $Z_{gh} = 1 - (g^2/6\pi)C_2$ and $\tilde{Z}_g = 1$ at 1-loop in the F^2-regularization. Using (6) we have obtained $Z_g = 1 + (3g^2/4\pi)C_2$. Therefore, the topological Ward identity (8) is satisfied and the integer there is just the quadratic Casimir C_2 of the group G. (For SU(N), $C_2 = N$.) This result coincides with Witten's analysis. The new lesson we learn here is that dimensional and F^2 regularizations give different finite renormalizations, though both satisfy the topological Ward identity (8). Since the bare g is finite in the present case, its finite renormalization is not without physical meaning. There might be different opinions about this result. But it seems to me to have provided an interesting example in which different regularization schemes lead to different renormalized theories from the same bare theory. (Note that our dimensional regularization satisfies all symmetry requirements at the orders we have checked.)

IV. ABSENCE OF DIFFEOMORPHISM ANOMALY

One aspect of our perturbative analysis that is closely relevant to the non-perturbative treatment is the verification of the absence of diffeomorphism anomaly, i.e. the expectation that the effective action is independent of the metric used in the gauge fixing or in the F^2 regularization. This expectation leads to the belief that the expectation values of Wilson loops, as gauge invariant and generally covariant observables in the theory, should be topological invariants, i. e. independent of continuous deformations of the loops.

In path integral formalism, by using a semi-classical method in the weak-coupling limit, Witten has proven[1] that the effective action for flat connections (or pure gauge potentials in D=3, which automatically solve the classical equations of motion) has no metric dependence. His proof involves some results in advanced mathematics.

In our diagrammatic approach we have considered the symmetric part of 1-loop vacuum polarization graphs:

$$\Pi_{\mu\nu}^{(sym)}(k) = \Pi_e\left(k^2\right)\left(k^2\delta_{\mu\nu} - k_\mu k_\nu\right) \tag{9}$$

If $\Pi_e(k^2) \neq 0$ after removing the regulator, then there would be a quadratic term in F, like $\int F_{\mu\nu}\left(D^2\right)^{-\frac{1}{2}} F_{\mu\nu}$, in the effective action. This term depends on the metric and, therefore, would represent a diffeomorphism anomaly in the quantum theory. Our explicit calculation shows that with both dimensional and F^2 regularizations, $\Pi_e\left(k^2\right)$ vanishes after removing the regulator ($\omega \to 3$ or $e_o^2 \to \infty$). This exemplifies the absence of diffeomorphism anomaly.

We emphasize that the term quadratic in F in the effective action which we have examined here is identically zero for flat connections. So whether they are present cannot be inferred from Witten's result.

V. BRIEF REVIEW OF WITTEN'S APPROACH

The arguments in the last section for the absence of diffeomorphism anomaly leads to the expectation that diffeomorphism invariant (or using physicists' language, generally covariant) and gauge invariant observables in the quantum Chern-Simons theory should be topological invariants. The usual n-point A-field functions are not appropriate objects to look at, because of their gauge non-invariance. More appropriate are the Wilson loops defined by

$$W_{R_i}(C_i) = \mathrm{Tr}^{(R_i)} \; P \exp\left\{ \oint_{C_i} A^a{}_\mu \, T^a \, dx^\mu \right\} \tag{10}$$

Here P is the path-ordering operator. To each loop C_i, assign a representation R_i of G. The set $\{C_i\}$ of loops forms a link in S^3 and the expectation value

$$\langle \prod_i W_{R_i}(C_i) \rangle = \int DA \exp\left\{ iI_{C-S} \right\} W_{R_i}(C_i) \; . \; . \; . \; W_{R_n}(C_n) \tag{11}$$

provides us a gauge invariant observable in the theory. Here $\int DA$ denotes the integration over gauge potentials on M modulo gauge transformations and the presence of the factor i in the exponent in (11) is valid both in Minkowskian and in Euclidean signature. (The latter is a feature of a topological action.) Witten has argued[1] that the Chern-Simon theory is exactly soluble in the sense that $\langle \prod_i W_{R_i}(C_i) \rangle$ is calculable. In particular, as a functional of the link $\{C_i\}$, $\langle \prod_i W_{R_i}(C_i) \rangle$ is expected to be a topological invariant of the link and, therefore, provides a 3-dimensional definition for knot/link invariants.

A caution should be addressed here. Namely, to deal with self-linking one needs regularization in the path integral expression (11) for the Wilson loops, which had better respect the diffeomorphism invariance.

One convenient choice is the generalized point-splitting regularization proposed by Tze[14] and Witten[1], in which one thickens each loop C_i into a tiny ribbon with the original C_i as one of its edges. The resulting geometrical object is called a framed knot or link. Different ways of thickening are called as different framings. In mathematical literature, a standard framing is used so that the self-linking (i.e. the linking of the two edges of the tiny ribbon) is always zero. However, in the Chern-Simons theory, what is more convenient is the vertical framing in which the thickening of loops is always done along a fixed direction in 3-spaces, e.g. along the normal direction of a planar projection of the link. The Wilson-loop observable $<\prod_i W_{R_i}(C_i)>$ is framing dependent, but has a simple transformation under the change of framing. Any two framings may differ by t units of twists of the ribbon, t being an integer. Then, for a simple loop C the Wilson loop is changed by a phase:

$$<W_R(C)> \longrightarrow \exp\left\{i2\pi t\ h_R\right\} <W_R(C)> \qquad (12)$$

where $h_R = C_2(R)/(k + g)$, where $C_2(R)$ is the Casimir for the representation R, k the coefficient in (1) and g the Coxeter number of G. The physical interpretation of this result is that the massive "scalar" particle representing the Wilson loop in the representation R is of fractional spin $S = h_R$ and of fractional statistics $\theta = 2\pi h_R$ in this theory.

An astute reader may have noted that the above h_R is nothing but the conformal weight of a primary field in the d=2 WZW model[15,16]. Equation (12) indicates a connection of the D=3 C-S theory with the d=2 WZW model. The discovery of this connection is one of the central points in Witten's original paper[1]. It can be summarized as follows.

Consider a knot (or link) in S^3. An S^2 in S^3 separates the latter into two regions M_1 and M_2 ; the S^2 is their common boundary. Suppose the link intersects with S^2 at points P_i (i=1,...,n). Then the path integral (11) over gauge potentials on M_1 (with the Wilson lines in it) with fixed boundary value of A_μ on S^2 will give rise to a wave-functional

$\psi[A_\mu]$ of A_μ on S^2, i.e. a quantum state $|\psi\rangle$ on S^2 (with punctures P_i).

One of the key-points[1] is that the Hilbert space of states $|\psi\rangle$ obtained in this way can be identified with the space of conformal blocks in d=2 WZW model at level k (with primary fields belonging to representations R_i inserted at P_i). This establishes a very important relationship between the D=3 Chern-Simons theory and d=2 conformal field theories. A similar path integral on M_2 will give rise to a dual quantum state $\langle\chi|$ on S^2. Then the path integral (11) can be written as the inner product

$$\langle\prod_i W_{R_i}(C_i)\rangle = \langle\chi|\psi\rangle \tag{13}$$

This is in fact the composition property of the path integral (11). Though we do not know how to evaluate the inner product, Eq. (13) is sufficient to derive the (skein) relations satisifed by the Wilson-loop observables $\langle\prod_i W_{R_i}(C_i)\rangle$.

Let us assume that all R_i are the same R and the part of the link locating inside M_1 consists of only two Wilson lines, which intersect with S^2 at 4 points: P_1, P_2, P_3 and P_4 associated with $\bar{R}$, R , R and $\bar{R}$. According to the relationship mentioned above, the state $|\psi\rangle$ belongs to the space H of 4-point conformal blocks in the WZW model with level k. The knowledge of the latter[16] tells us that for sufficiently large k,

$$\dim H = s \qquad\qquad \text{if} \quad R\otimes R = \bigoplus_{j=1}^{s} E_j \tag{14}$$

This means that if the direct product R⊗R can be decomposed into a direct sum of s irreducible representations E_j, then the dimension of the Hilbert space of $|\psi\rangle$ on S^2 with punctures $P_1,..,P_4$ is just s. Any (s+1) states $|\psi_1\rangle,...,|\psi_{s+1}\rangle$ must be linearly dependent, so there must be a linear relation among them. Taking the inner product with the same $\langle\chi|$, we will obtain a linear relation among the Wilson-loop observables, or the knot/link invariants, for (s+1) knots/links which differ from each other only by their parts inside M_1. Such a relation is called a skein relation in the mathematical literature of knot theory[5].

Witten has considered the simplest cases, i.e. with G=SU(N) and R=$\underline{N}$. In this case $N \otimes N = \frac{1}{2}N(N+1) \oplus \frac{1}{2}N(N-1)$, so s=2. There is a skein relation involving three links. Choose $|\psi_a\rangle$ (a=+,0,−) to correspond to an over-crossing ( ), no-crossing (↑↑) and under-crossing ( ) inside M_1 respectively. The corresponding elements in the space of conformal blocks differ from each other by the braiding operation B: $|\psi_0\rangle = B|\psi_+\rangle$, $|\psi_-\rangle = B|\psi_+\rangle$. Braiding exchanges two punctures on S^2, thus leading to a change of the Wilson lines inside M_1 as indicated above. From the knowledge of eigenvalues[17] of B in the d=2 WZW model with G=SU(N) and R=$\underline{N}$, Witten has been able to derive the following skein relation with standard framing:

$$q^{N/2}\, V(L_+) - q^{-N/2}\, V(L_-) + \left(q^{-\frac{1}{2}} - q^{\frac{1}{2}}\right) V(L_0) = 0 \qquad (15)$$

(where $q = \exp\{\pi i/(k+N)\}$) for three links L_+, L_0, L_-, which are identical in M_2 but differ from each other by the part in M_1, which is respectively an over-crossing, no-crossing and under-crossing when projected on some plane. This skein relation contains only three terms and is sufficient to determine the invariant V(L) for all knots/links, if for a simple circle V(O) is normalized. V(L) will be a Laurent polynomial of two variables $q^{N/2}$ and $q^{\frac{1}{2}}-q^{-\frac{1}{2}}$, which turns out to be the two-variable HOMFLY polynomial[4]. If N=2, this is just the famous Jones polynomial[3].

VI. GENERAL SKEIN RELATION

In Witten's approach one needs to know the eigenvalues of the braiding matrix B in d=2 WZW model at level k. We have obtained

<u>Theorem:</u> If $R_1 \otimes R_2 = \overset{s}{\underset{j=1}{\oplus}} E_j$, then the eigenvalues of the braiding matrix for 4-point conformal blocks $\langle \phi^{\bar{R}_1}(z_1)\, \phi^{R_1}(z_2)\, \phi^{R_2}(z_3) \phi^{\bar{R}_2}(z_4)\rangle$ are

$$\lambda_j = \epsilon_j \cdot q^{[C_2(R_1)+C_2(R_2)-C_2(E_j)]/2} \qquad (j=1,\ldots,s) \qquad (16)$$

where $\epsilon_j = \mp 1$, depending on whether E_j appears in the direct sum symmetrically or anti-symmetrically. As before,

$$q = \exp\left\{\frac{2\pi i}{k+g}\right\}, \qquad g = \text{Coxeter number of } G \tag{17}$$

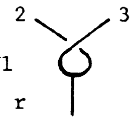
$$C_2(R) = (\Lambda|\Lambda+2\rho)/2, \quad \Lambda\text{: highest weight of } R \tag{18}$$

$$\rho = \text{half sum of positive roots.}$$

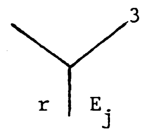
There are two ways to prove this theorem[18]. Here we sketch the one which uses the braiding and fusing matrices. (For the background see, for example, ref. 17.) Suppose $\phi_\alpha^R(z,\bar{z})$ is the primary field belonging to R with weight α. Exchanging z_2 and z_3 with $\bar{z}_2$, $\bar{z}_3$ held fixed in the 4-point function for ϕ, from the operator product expansion

$$\phi^{R_1}_{\alpha_1}(z_2)\phi^{R_2}_{\alpha_2}(z_3) \sim \sum_j \frac{c_j\langle E_j,\alpha_1+\alpha_2|R_1\alpha_1R_2\alpha_2\rangle}{(z_2-z_3)^{h(E_j)-h(R_1)-h(R_2)}}\,\phi^{E_j}_{\alpha_1+\alpha_2}(z_3) \tag{19}$$

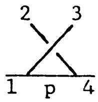
we have



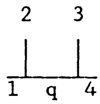
$= \epsilon_j\, e^{-[h(E_j)-h(R_1)-h(R_2)]\pi i}$



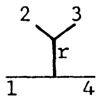
Thus,



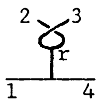
$= \sum_q B_{pq}$



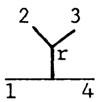
$= \sum_q B_{pq} \sum_r F_{qr}$



$= \sum_r F_{pq}$



$= \sum_r F_{pr}\, \epsilon_r\, q^{\Delta_r}$



where $\Delta_j \equiv \Delta(E_j) = [C_2(R_1)+C_2(R_2)-C_2(E_j)]/2$. Comparing the most right expressions, we see that the fusing matrix (F_{pq}) diagonalizes the braiding matrix (B_{pq}) $(p,q = 1,\ldots,s)$ and the eigenvalues of B are just $\epsilon_j q^{\Delta_j}$ $(j=1,\ldots,s)$, as desired.

From the eigenvalues λ_j, one can immediately write down the characteristic equation for B: $\prod_j^s (B-\lambda_j)|\psi_0\rangle = 0$. Introduce $Z(L_m) = \langle\chi|B^m|\psi_0\rangle = \langle\chi|\psi_m\rangle$ and formally write $Z(L_m) = \hat{B}^m Z(L_0)$. The skein relation satisfied by $Z(L_m)$ is given by

$$\prod_j^s (\hat{B}-\lambda_j) Z(L_0) = 0 \qquad (20)$$

For this to hold all $Z(L_m)$'s have to be in the vertical framing. Assuming R_1, R_2 are the same and changing from the vertical to standard framing,

$$V(L_s) - \sum_{j-1}^{s} \epsilon_j q^{2C_2(R)-C_2(E_j)/2} V(L_{s-1}) + \dots + \prod_{j=1}^{s} \left(-\epsilon_j q^{2C_2(R)-C_2(E_j)/2}\right) V(L_o) = 0 \qquad (21)$$

This is our most general skein relation for (s+1) knots/links. Here L_n denotes a link which contains n over-crossing in a row of two Wilson lines inside M_1 and otherwise all L_m's are identical inside M_2. $V(L_m)$ is the corresponding link invariant in the stadard framing.

VII. SPECIAL CASES OF SKEIN RELATIONS

Now we consider some special examples.

1) G = SU(2), R = $\underline{3}$ (spin-1): s=3

$$V(L_2)-(q-q^3+q^4)\, V(L_1)-(q^4-q^5+q^7)\, V(L_0) + q^8\, V(L_-)=0. \qquad (22)$$

This is the same as that for the Akutzu-Wadati N=3 case[19]. Also when R=$\underline{4}$ (spin-3/2), our skein relations is identical to that for the Akutzu-Wadati N=4 polynomial[16]. We believe that the Akutzu-Wadati polynomials for all N correspond to irreducible representations of G=SU(2), from the point of view of the Chern-Simons gauge theory.

2) G = SO(N), R = $\underline{N}$ (fundamental): s = 3

$$q^{-N} V(L_2) - \left[(1-q^{-(N-1)/2} + q^{-(N+1)/2}\right] V(L_+)$$

$$-\left(1-q^{(N-1)/2}+q^{(N+1)/2}\right) V(L_0)+q^N V(L_-)=0 \qquad (23)$$

This is the same as Turaev[11] and Reshetikhin[12] have obtained.

3) $G = Sp(N)$, $R = \underline{N}$, (fundamental): $s = 3$

$$q^{-(2N+1)}V(L_2)+q^{-N}\left(q^N-q^{-1}+1\right) V(L_+)$$
$$-\,q^N\left(q^{-N}-q+1\right) V(L_0)-q^{2N+1} V(L_-)=0 \qquad (24)$$

also the same as what Turaev[11] and Reshetikhin[12] obtained. Though the skein Relation(24) is different from (23), it can be shown[11] that they lead to essentially the same topological polynomial.

Now the skein relation (23) or (24) contains four terms, it seems insufficient to unknot all knots, since there exist knot diagrams which contain nowhere a part that is two crossings or more in a row with two lines. However, we are going to show that the above skein relation is actually sufficient to determine invariants for all knots.

VIII. DETERMINATION OF KNOT INVARIANTS FROM ORDER-THREE SKEIN RELATIONS

Our skein relation has four terms of the form:

$$-\delta\, V(\;)+\alpha\, V(\;)+\beta\, V(\uparrow\uparrow)+\gamma\, V(\;)=0 \qquad (25)$$

Compared to the skein relation (15) satisfied by the Jones or HOMFLY polynomials ours has one more term, and we call it as of order three.

First we notice that there are different versions of the same skein relation. For example,

$$-\delta V(\;)+\alpha V(\uparrow\uparrow)+\beta V(\;)+\gamma V(\;)=0 \qquad , \qquad (26)$$

More importantly, the fundamental representations of SO(N) and Sp(N) are all real, i.e. $R=\bar{R}$, so we are allowed to braid not only $\phi^R(z_2)$ and $\phi^R(z_3)$, but also $\phi^R(z_1)$ and $\phi^R(z_2)$. The latter braiding leads to

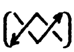
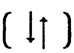
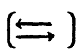
$$-\delta\, V(\;)+\alpha\, V(\downarrow\uparrow)+\beta\, V(\;)+\gamma\, V(\;)=0 \qquad (27)$$

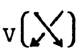
Also the fact $R = \bar{R}$ allows us to reverse the orientation of some or even all components of the link, because reversing the orientation of a loop C_i is just equivalent to changing the representation R to its complex conjugate in the Wilson-loop observable. When an oriented link K is changed to K' in this way, the only change of V(K) is an overall factor due to the change in the writhe number w(k):

$$V(K') = q^{C_2(R)[w(K')-w(k)]} V(K) \tag{28}$$

(Recall that V(K) is defined in the standard framing.)

Combining these knowledges, we are able to derive more skein relations among V(K); e.g.,

δV  $= (\alpha-\beta)\, V$  $+ \beta\, V$ 

$-\delta\, q^{C_2(R)\Delta w}\, V$  $- \gamma\, q^{C_2(R)\Delta w}\, V$  (29)

This relation together with (25) shows that if a knot/link contains two crossings in a row of the same two Wilson lines, regardless of their orientation, we can always reduce the number of crossings.

Then the remaining problem is to unknot knots/links which nowhere contain two crossings in a row of the same two lines. For example, in Rolfsen's knot/link table[20], the knots 8_{18}, 9_{40} and 10_{123} are of this type. By trial-and-error, we find that we can unknot these knots by repeatedly applying the above skein relations to appropriate crossings. At first the number of crossings gets increased, but later it can get decreased, and finally the knot is reduced to those which are easily unknottable. The reason why this works is that by applying the skein relations, over-crossing is changed to under-crossing and vice versa. It is easily understandable that the change in the nature of crossings is always crucial for unknotting a knot! In this way we have been able to unknot all knots/links with no more than 10 crossings, i.e. those present in Rolfsen's table and to show that their associated topological polynomials follow from above skein relations.

We conjectured in ref. 10 that with the above skein relations, the same should be true for any knot/link. Actually this conjecture can be proved[18] by exploring the fact that $R = \bar{R}$, since we are allowed to reverse the orientation of any loop in the link. Thus V(L) for SO(N) or

$$F(L) = \exp\{\pi i \cdot w(L)/2\}\, V(L) \qquad (30)$$

for Sp(N) defines a (regular isotopy) invariant for non-oriented knots/links. From above skein relations we can show that V(L) or F(L) satisfies Kauffman's defining skein relation for his new two-variable polynomial[5].

IX. $\langle W_R(0)\rangle$ AND THE WEYL CHARACTER

As a by-product of our study, we have found[10] an interesting group-theoretical interpretation of the Wilson-loop observable $\langle W\rangle$ for a simple circle: It is nothing but the Weyl character of the representation R:

$$\langle W_R(0)\rangle = \text{Weyl character of } R \qquad (31)$$

We have been able to prove it for any representation R of any G, by starting from Witten's result[1] which involves the surgery of $S^2 \times S^1$ and expresses $\langle W_R(0)\rangle$ in terms of the modular matrix S. Then our proof involves using the Weyl-Kac formula. My feeling is that there should be a more direct, perhaps much deeper way to understand this basic, simple and beautiful fact. What is it? Also, do the Wilson-loop observables for other knots/links have simple group-theoretical interpretation?

X. CONCLUSIONS

The recently found "new" knot polynomials[5,11,12,19] can all be obtained from D=3 Chern-Simons gauge theory as Wilson-loop observables. They can be systematically summarized as follows:

G = SU(2),	R = $\underline{2}$	→	Jones polynomial
SU(N)	$\underline{N}$	→	HOMFLY
SU(2)	$\underline{N}$	→	Akutza-Wadati
SO(N)	$\underline{N}$	→	Kauffman - also Turaev,
Sp(N)	$\underline{N}$	→	Reshetikhin

An interesting problem arises naturally: How to relate the "oldest" Alexander-Conway polynomial to the C-S gauge theory? Also, what is the relationship of quantum groups with the Chern-Simons gauge theory?

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References

1. E. Witten, Commun. Math. Phys. 121 (1989) 351.
2. S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48 (1982) 975.
3. V.F.R. Jones, Bull, Amer. Math. Soc. 12 (1985) 103.
4. P. Freyd, D. Yetter, J. Hoste, W. Lockorish, K, Millet, and A. Ocneanu, Bull. Amer. Math. Soc. 12 (1985) 239.
5. L.H. Kauffman, On Knots, Princeton Univ. Press, 1987 (Appendix).
6. See, e.g., F. Wilczek and A. Zee, Phys. Rev. Lett. 51 (1983) 2250; Y.-S. Wu and A. Zee, Phys. Lett. B147 (1984) 325; I. Dzyaloshinskii A. Polyakov and P. Wiegmann, Phys. Lett. A127 (1988) 112; X. G. Wen and A. Zee, Phys. Rev. Lett. 61 (1988)1025; NSF-ITP-88/150; X. G. Wen, F. Wilczek and A. Zee, Phys. Rev. B39 (1989) 11413 ; P. Wiegmann, Physica S153-155 (1988) 103; Physica Scripta (in press); D. Kheshchenko and P.B. Wiegman, preprint; Y.H. Chen, F. Wilczek, E. Witten and B.I. Halperin, IASSNS-HEP-89/27 and reference therein.
7. For the fractional quantum hall effect see, e.g., S. Girvin and A.H. MacDonald, Phys. Rev. Lett. 58 (1987) 1252; S.-C. Zhang, T. Hannson and S. Kivelson, Phys. Rev. Lett. 62 (1989) 82.
8. For the He^3 superfluid film, see G.E. Volovik and V.M. Yakovenko, Landau Institute preprint, 1989.
9. W. Chen, G.W. Semenoff and Y.-S. Wu, Utah preprint, 1989.
10. K. Yamagishi, M.L. Ge and Y.-S. Wu, Utah reprint HEP-89/1.
11. V.G. Turaev, Inv. Math. 92 (1988) 527.
12. M. Yu. Reshetikhin, LOMI preprints E-4-87, E-17-87.
13. S.J. Gates Jr., M.T. Grisance, R. Rocek, and W. Siegel, "Superspace", (Benjamin/Cummings, 1983).
14. H.C. Tze, Intl. J. Mod. Phys. A3 (1988) and in this Proceedings.
15. E. Witten, Commun. Math. Phys. 92 (1984) 455.
16. V.G. Knizhnik and A.B. Zamolochikov, Nucl. Phys. B247 (1984) 83; D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493
17. G. Moore and N. Sieberg, Phys. Lett. B212 (1988) 451.
18. K. Yamagishi and Y.-S. Wu, in preparation.
19. Y. Akutzu and M. Wadati, J. Phys. Soc. Japan 56 (1987).
20. D. Rolfsen, Knots & Links, Math Lectures Series No. 7 (Publish or Perish Press, 1976).

OBSERVABLES IN TOPOLOGICAL YANG-MILLS THEORY AND THE GRIBOV PROBLEM*

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INTRODUCTION

Topological Yang-Mills theory (TYMT) is one of topological quantum field theories, which were proposed and developed by Witten[1]. Topological invariants obtained from TYMT are Donaldson polynomials[2]. TYMT has BRST symmetry and observables are defined to be BRST cohomology classes. We can prove that correlation functions of observables are independent of a metric on the underlying four dimensional space-time and, therefore, topological invariants. Witten identified these correlation functions with Donaldson polynomials. Thus, Donaldson polynomials appear as BRST cohomology class (observables) of TYMT and we have a path integral representation of Donaldson polynomials.

In Witten's original form, BRST transformation is nilpotent modulo gauge transformation and the action has non-abelian gauge symmetry. If one introduce another ghost to fix this remaining symmetry, the appropriately defined BRST transformation including the additional ghost recovers complete nilpotency[3]. However, we then encounter a subtlety in the non-triviality of BRST cohomology. The point is every BRST cocycle can be represented as BRST coboundary, because BRST transformation is linearly realized[4].

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We can see the algebraic structure of BRST transformation in TYMT from the viewpoint of the Weil algebra structure[5], which is a universal algebraic model of the connection and curvature. Especially, this viewpoint implies that BRST transformation is a kind of the exterior derivative and that Donaldson polynomials are nothing but the second Chern class of this Weil algebra system[5,6,8]. Then it is realized that the superficial triviality of BRST cocycles is related to the local Poincaré lemma which is a general property of the exterior derivative. Hence, the gap between the local and the global Poincaré lemma can make some of BRST cocycles non-trivial. The U(1) gauge theory with a magnetic monopole gives an example of such a gap. In this example the famous Dirac-string singularity invalidates the global Poincaré lemma and implies non-vanishing monopole charge. In our case of TYMT, as we will see below, the singularity due to the Gribov ambiguity plays the role of Dirac-string. Thus, we can claim that the Gribov ploblem is the origin of non-trivial observable in TYMT[7].

BRST TRANSFORMATION AS EXTERIOR DERIVATIVE

The action of TYMT can be derived from BRST gauge fixing procedure. The classical action is zero or a topological invariant (e.g. the second Chern class)[8]. We can make a gauge fixing in three steps[9,10,11] and a ghost field is introduced in each step. First, since the classical action is invariant under an arbitrary deformation of gauge field, we must fix this 'topological symmetry'. We introduce a ghost ψ which is regarded as an infinitesimal deformation of gauge field A. Since the connection one-form A takes values in the Lie algebra, ψ is a Lie algebra valued one-form with ghost number one. Next, we fix 'ghostly symmetry' which is the gauge symmetry for the primary ghost ψ. The secondary ghost ϕ enters here. Since ϕ corresponds to the gauge parameter for ψ, it is a Lie algebra valued function with ghost number two. (Witten's original action is obtained in this stage.) Finally, non-abelian gauge symmetry is fixed by introducing Faddeev-Popov ghost like field ξ. It is a Lie algebra valued function with ghost number one. As a consequence we have three kinds of ghost fields ψ, ϕ and ξ. In this procedure several anti-ghosts and auxiliary fields are also introduced, but they are irrelevant to the following discussion.

The BRST transformations of the gauge field and these ghosts are defined as follows:

$$\begin{aligned} \delta A &= \psi - (d\xi + [A, \xi]) \ , \\ \delta \psi &= -(d\phi + [A, \phi]) + [\psi, \xi] \ , \end{aligned} \tag{1}$$

$$\delta\phi = [\phi, \xi] \ ,$$
$$\delta\xi = -\frac{1}{2}[\xi, \xi] + \phi \ ,$$

where d is the exterior derivative on the space-time. We can easily check

$$\delta^2 = d\delta + \delta d = d^2 = 0 \ . \tag{2}$$

Considering the following combination with the parameter t counting the ghost number (F is the curvature of A) ;

$$\mathcal{A} := A + t\xi \ , \tag{3}$$
$$\mathcal{F} := F + t\psi + t^2\phi \ , \tag{4}$$
$$\Delta := d + t\delta \ , \tag{5}$$

we can see that the transformation law (1) is equivalent to the relations;

$$\Delta\mathcal{A} = \mathcal{F} - \frac{1}{2}[\mathcal{A}, \mathcal{A}] \ , \tag{6}$$
$$\Delta\mathcal{F} = [\mathcal{F}, \mathcal{A}] \ . \tag{7}$$

These are the algebraic relations among the connection $\mathcal{A}$, the curvature $\mathcal{F}$ and the exterior derivative Δ. That is, eq.(6) is the defining relation of the curvature and eq.(7) is the Bianchi identity which is equivalent to the nilpotency of Δ. The graded cochain algebra generated by an odd element $\mathcal{A}$ and an even element $\mathcal{F}$ with the coboundary operator Δ defined by (6) and (7) is called the Weil algebra[12]. In our case the total gradation is given by the sum of the degree of differential forms and the ghost number. It is remarkable that the three step BRST gauge fixing procedure leading to three kinds of ghosts is conformable to the Weil algebra structure with double gradation. Conversely, we can start with assuming the expansion (3)$\sim$(5). Then, decomposing (6) and (7) in the ghost number, we obtain BRST transformation law (1). In this process one can put the condition $\psi = \phi = 0$, consistently. This is what we call 'soul flatness condition' and leads to BRST transformation of the conventional Yang-Mills theory.

The Weil algebra is realized as the algebra of differential forms on a universal G-bundle with a universal connection and, therefore, is a universal algebraic model of the connection and curvature[12]. In this sense BRST transformation of TYMT has a universal nature. In fact we have a geometric realization of the system (3)$\sim$(5) with the relations (6) and (7) in terms of a certain universal bundle for a family of gauge fields[13]. In this realization, BRST transformation is understood as the exterior derivative on the space of all connections modulo gauge transformations. Restricted to the solution space of (anti-)self-dual connections, it is also identified with the exterior derivative on the instanton moduli space. For more details on this geometric realization, see ref 5).

DONALDSON POLYNOMIALS AS THE SECOND CHERN CLASS

The BRST cohomology class which Witten identified with Donaldson polynomials is obtained as the second Chern class of the Weil algebra system $(\mathcal{A}, \mathcal{F}, \Delta)$. Since $\mathcal{F}$ is the curvature of this system, the second Chern class takes the following form;

$$\mathcal{W}_2 = \frac{1}{2} \,\mathrm{Tr}\, \mathcal{F}^2 \,. \tag{8}$$

The Bianchi identity (7) implies that $\mathcal{W}_2$ is a Δ-cocycle;

$$\Delta\, \mathcal{W}_2 = 0 \,. \tag{9}$$

Making the t-expansion or the decomposition with respect to the ghost number;

$$\begin{aligned} \mathcal{W}_2 =& \sum_{k=0}^{4} t^{4-k} W_k[F, \psi, \phi] \\ =& \frac{1}{2} \,\mathrm{Tr}\, F^2 + t \,\mathrm{Tr}\, \psi F + t^2 \,\mathrm{Tr}\, (\frac{1}{2}\psi^2 + F\phi) \\ & + t^3 \,\mathrm{Tr}\, \psi\phi + \frac{t^4}{2} \,\mathrm{Tr}\, \phi^2 \,, \end{aligned} \tag{10}$$

we can see eq.(9) means the following system of equations for W_k's;

$$\begin{aligned} \delta W_0 &= 0 \,, \\ \delta W_1 + dW_0 &= 0 \,, \\ &\vdots \\ \delta W_4 + dW_3 &= 0 \,, \\ dW_4 &= 0 \,. \end{aligned} \tag{11}$$

Note that W_k is a space-time k-form with ghost number $(4-k)$. Since W_k is a BRST-cocycle modulo total derivative, its integration over a k-cycle γ_k (a k-dimensional closed surface in the space-time) defines a BRST cohomology class;

$$\delta \int_{\gamma_k} W_k = 0 \,. \tag{12}$$

Hence, $\int_{\gamma_k} W_k$ is an observable in TYMT. These observables are obtained by integrating the second Chern class $\mathcal{W}_2$ over γ_k ; $\int_{\gamma_k} W_k = \int_{\gamma_k} \mathcal{W}_2$, if the integration of an l-form over γ_k is defined to vanish unless $k = l$. The correlation functions of observables give an expression of Donaldson polynomials in TYMT[1].

If the gauge group is $SU(2)$, the second Chern class is the only generator of the cohomology ring of a classifying space (the base space of a universal bundle). For general $SU(N)$ gauge groups, there are $(N-1)$ generators. Their representatives are the invariant polynomials Tr $\mathcal{F}^n$ with $n = 2, 3, \cdots, N$, which are nothing but the Chern class of higher order. These generators of the cohomology ring are also possible observables and may give a generalization of Donaldson polynomials.

As is well-known, the exterior derivative of the Chern-Simons 3-form gives the second Chern class. In our Weil algebra system, the Chern-Simons 3-form is

$$\begin{aligned}\mathcal{I}_3 &= \mathrm{Tr}\ (\mathcal{A}\Delta\mathcal{A} + \frac{3}{2}\mathcal{A}^3) \\ &= \sum_{k=0}^{3} t^{3-k} I_k[A, \xi, F, \psi, \phi]\ .\end{aligned} \tag{13}$$

Taking the space-time k-form part of the relation $\mathcal{W}_2 = \frac{1}{2}\Delta\mathcal{I}_3$, we have

$$2\ W_k = \delta I_k + dI_{k-1}\ . \tag{14}$$

For example, the explicit form of the 2-form part is

$$2\ W_2 = \mathrm{Tr}\ (2F\phi + \psi^2) = \delta\ [\mathrm{Tr}\ (A\psi + \xi dA)] + d\ [\mathrm{Tr}\ (\xi\psi + A\phi - \xi^2 A)]\ . \tag{15}$$

Hence, we encounter an unpleasant fact that the integration of W_k over γ_k is formally BRST coboundary;

$$2 \int_{\gamma_k} W_k = \delta \int_{\gamma_k} I_k\ . \tag{16}$$

This is a consequence of the local Poincaré lemma for BRST transformation of TYMT. In the relation (14), we note the appearence of the connection field $\mathcal{A} = A + t\xi$ which may not have global meaning. We must examine the global validity of (14), which is the task in the next section.

GHOST FIELDS AS DIFFERENTIAL FORMS ON MODULI SPACE

In evaluating the correlation functions of observables, we have an integration on the instanton moduli space which appears as the zero mode parts of the path integral. The ghost fields are interpreted as differential forms on this moduli space[1]. The ghost number corresponds to the degree of differential forms. Therefore, for the triviality of W_k, eq.(14) must be valid all over the moduli space with the ghost fields regarded as differential forms. The breakdown of the global Poincaré lemma can be detected by the appearence of a singularity in the local Poincaré lemma such as (14). The Dirac-string singularity for a magnetic monopole is a typical example. In this example, the monopole charge is defined as the first Chern number;

$$Q = \frac{1}{2\pi} \int_{S^2} F \ , \tag{17}$$

where F is the field strength of the $U(1)$ gauge theory. The Bianchi identity implies that F is the exterior derivative of a gauge potential A;

$$F = dA \ , \tag{18}$$

which should be compared with (14). The global validity of (18) implies $Q = 0$. In the presence of a magnetic monopole eq.(18), is valid only in a local coordinate patch and a singularity appears if one tries to extend local coordinates to all over the sphere S^2. That is, if we expand the gauge field A in terms of a local basis dx^μ of one forms;

$$A(x) = A_\mu(x) \ dx^\mu \ , \tag{19}$$

there exists a point on S^2 where the component $A_\mu(x)$ diverges.

We will realize the non-triviality of Donaldson polynomials by analogy with this example, or as the appearence of a singularity in the relation (14). To this end we must know the behavior of the ghost fields as differential forms. It can be seen by considering the equation of motiom for ghosts. The three step gauge fixing procedure leads to three gauge fixing conditions;

$$F_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} = 0 \ , \tag{20.a}$$

$$(D_A)_\alpha \psi^\alpha = 0 \ , \tag{20.b}$$

$$\partial_\alpha A^\alpha = 0 \ , \tag{20.c}$$

where D_A is the covariant derivative. Taking BRST transformations of them, we obtain the following equation of motion for ghosts ;

$$(D_A)_\alpha \ \psi_\beta + \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}(D_A)^\gamma \psi^\delta = 0 \ , \tag{21.a}$$

$$(D_A)_\alpha (D_A)^\alpha \ \phi + [\psi^\alpha, \psi_\alpha] = 0 \ , \tag{21.b}$$

$$\partial_\alpha (D_A)^\alpha \xi - \partial_\alpha \psi^\alpha = 0 \ . \tag{21.c}$$

Eq.(21.a) with the condition (20.b) shows that we can take ψ as a basis of one-forms at each point of the moduli space[1]. The other ghosts ξ and ϕ will be expanded in terms of ψ. In fact, Eqs.(21.b,c) give such expansions which explicitly show that ξ and ϕ are a one-form and a two-form on the moduli space. By (21.b), we have an integral representation ;

$$\phi(x) = -\int_M d^4y\sqrt{g}\; G_A(x,y)[\psi_\alpha(y), \psi^\alpha(y)] \;, \tag{22}$$

where $G_A(x,y)$ is the integral kernel for $[(D_A)_\alpha(D_A)^\alpha]^{-1}$. For generic connections we have no zoro modes of the covariant laplacian and (22) is well-defined. In a similar manner we obtain

$$\xi(x) = -\int_M d^4y\sqrt{g}\; K_A(x,y)[A_\beta(y), \psi^\beta(y)] \;, \tag{23}$$

where $K_A(x,y)$ is the kernel for $(\partial_\alpha(D_A)^\alpha)^{-1}$ and we have used (20.b). We notice here that the determinant of the operator $\partial_\alpha(D_A)^\alpha$ is nothing but the Faddeev-Popov determinant for the Lorentz gauge condition (20.c). Therefore, there are points where this determinant vanishes due to the Gribov ambiguity[14,15]. At these points ξ is not well-defined or has a singularity because of the divergence in the kernel $K_A(x,y)$. Thus, the singularity due to the Gribov problem will break the global Poincaré lemma of the form (14). The non-triviality of Donaldson polynomials in TYMT and the Gribov problem in non-abelian gauge theory have the same topological origin.

DISCUSSION

We have seen that the non-triviality of observables in TYMT is related to the Gribov problem. In our consideration, we take the viewpoint that BRST transformation is the exterior derivative on the field configuration space and that the ghosts are differential forms on the moduli space. To be precise, what we have proved is that the Gribov ambiguity is a necessary condition for the non-triviality of observables. A hard analysis seems to be required, if we want to see it is sufficient. We must know the distribution of the zero points of the Faddeev-Popov determinant on the configuration space of the gauge field. Furthermore, we must prove that the intersection of the zero points set and the space of (anti-)self-dual connections is non-empty.

REFERENCES

1. E. Witten, *Commun. Math. Phys.* **117** : 353 (1988).
2. S. Donaldson, *Polynomial invariants for smooth four manifolds.* Oxford preprint.
3. J.M.F. Labastida and M. Percini, *Phys. Lett.* **212B** : 56 (1988).
4. S. Ouvry, R. Stora and P. van Baal, *Phys. Lett.* **220B** : 159 (1989).
5. H. Kanno, *Z. Phys.* **C43** : 477 (1989).
6. D. Birmingham, M. Rakowski and G. Thompson, *Nucl. Phys.* **B315** : 577 (1989).
7. H. Kanno, preprint RIFP-819, June 1989.
8. L. Baulieu and I.M. Singer, *Nucl. Phys.* **B** (Proc. Suppl.) **5B** : 12 (1988).
9. R. Brooks, D. Montano and J. Sonnenschein, *Phys. Lett.* **214B** : 91 (1988).
10. D. Montano and J. Sonnenschein, preprint SLAC-PUB-4760 (to appear in *Nucl. Phys.***B**).
11. J. Sonnenschein, preprint SLAC-PUB-4970, April 1989.
12. M.F. Atiyah and R. Bott, *Topology* **23** : 1 (1984).
13. M.F. Atiyah and I.M. Singer, *Proc. Natl. Acad. Sci. USA* **81** : 2597 (1984).
14. V.N. Gribov, *Nucl. Phys.* **B139** : 1 (1978).
15. I.M. Singer, *Commun. Math. Phys.* **60** : 7 (1978).

LINKING THE GAUSS-BONNET-CHERN THEOREM, ESSENTIAL HOPF MAPS AND MEMBRANE SOLITONS WITH EXOTIC SPIN AND STATISTICS

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ABSTRACT

By way of the Gauss-Bonnet-Chern theorem , we present a higher dimensional extension of Polyakov's regularization of Wilson loops of point solitons. Spacetime paths of extended objects become hyper-ribbons with self-linking, twisting and writhing numbers. Specifically we discuss the exotic spin and statistical phase entanglements of geometric n-membrane solitons of D-dimensional $\mathbf{KP}_1$ σ-models with an added Hopf-Chern-Simons term where (n, D, K) = (0, 3, **C**) , (2, 7, H) , (6, 15, $\mathbf{\Omega}$) . They are uniquely linked to the complex (**C**) and quaternion (**H**) and octonion ($\mathbf{\Omega}$) division algebras.

Two overlapping research areas have recently attracted much attention. They are the topological quantum field theories (TQFT) in D $\geq$ 3 spacetime dimensions [1, 2], inspired by the works of Donaldson and Jones [3], and theories of particles bearing *any* spin and statistics or *anyons* [4] ,fuelled by the quest for a mechanism underlying the fractional quantum Hall effect or high temperature superconductivity.

By relating quantum D=2 conformal and D=3 Chern-Simons field theories, Witten [5] showed a correspondence between the expectation values of the Wilson loops traced by 'colored' *point* sources in spacetime and Jone's polynomials for knots. Notably, to so obtain the fundamental Skein relations, he had to regularize or *frame* the Wilson loops, in addition to doing the standard regularization. It is here that an interesting overlap occurs with the theory of anyons. In fact, such a regularization had been discovered by Polyakov

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[6] in his proof of the D =3 fermi-bose transmutation of baby Skyrmions in their *point-like limit*. Our talk takes off at this intersection, a meeting point of topology, geometry and physics. It is based on work done with Soonkeon Nam.[7] . We defer to this longer article for greater details and a comprehensive list of references as time and space limitations only permit a quick review of some of the highlights here.

We start from the well-studied CP_1 σ- model with a Chern-Simons term [6]

$$A = \int d^3x \left[| D_\mu Z |^2 + \frac{\theta}{8\pi^2} \varepsilon^{\mu\nu\lambda} A_\mu F_{\nu\lambda} + A_\mu J^\mu \right] . \tag{1}$$

$0 \le \theta \le \pi$. The two component complex spinor field $Z^T = (Z_1, Z_2)$ with $|Z|^2 = 1$ lives on S^3. The unit normed field $\mathbf{n}$ is given by the Hopf projection map taking $Z \in S^3$ to $\mathbf{n} = Z^+ \sigma Z \in S^2$. D_μ is the covariant derivative with its holonomic U(1) gauge field $A_\mu = i Z^\dagger \partial_\mu Z$, $F_{\mu\nu}$ being the field strenght. This model admits exact S^2 - solitons. While the third term is the Aharonov-Bohm term, the second term also reads as $S_H = \frac{-1}{2} \int d^3x\ A_\mu J^\mu$ (for $\theta = \pi$) i.e. as an interaction between the field A_μ and the conserved topological current $J_\mu = \frac{1}{8\pi} \varepsilon_{\mu\nu\lambda} \varepsilon_{abc} n^a \partial^\nu n^b \partial^\lambda n^c$. The latter is so normalized that $Q = \int d^2x\ J_0$,the soliton (electric) charge is integral and labels the elements of $\pi_2(S^2) = Z$. The field boundary condition is such that spacetime is $R^3 \cup (\infty) \approx S^3$; the Chern-Simons action is the Hopf invariant for the maps $\mathbf{n} : S^3 \longrightarrow CP_1 \approx S^2$, classified by the generators of $\pi_3(S^2) \approx Z$, the additive group of the integers.

Wilczek and Zee [4] showed that interchanging two Q=1-solitons or equivalently rotating one of them around the other by 2π gives raise to a statistical, alias spin phase factor $e^{i\theta}$ to the wave function. Hence the soliton is an anyon with exotic spin spin $s = \frac{\theta}{2\pi}$ and intermediate statistics This phase originates from the gauge interactions among anyons and corresponds to a mapping with <u>Hopf invariant 1</u> . The latter feature is a key ingredient in our higher dimensional generalizations of the θ-spin and statistics connection.

Polyakov opted for a tractable Wilson loop approach to the large distance behavior of soliton Green functions of system (1). To study the effects induced by the long range Chern-Simons interactions, he approximated the partition function **Z** by

$$\mathbf{Z} = \sum_{(P)}^{\text{all closed paths}} e^{-m L(P)} \langle \exp\left(i \int_P dx^\mu A_\mu \right\rangle . \tag{2}$$

P denotes a Feynman path of a soliton seen as a curve in spacetime R^3, L(P) is the total path length .

The first exponential factor in (2) is the Schwinger action of a free relativistic *point-like* soliton of mass m. The other factor $\Phi(P) = \left\langle \exp(i \oint_P A^\mu dx_\mu) \right\rangle$ where the functional averaging < ...> is w.r.t. the Hopf action embodies the Aharonov-Bohm effect , typical of topologically massive gauge theories. The Chern-Simons-Hopf action induces magnetic flux on electric charges and vice versa. Being Gaussian (and for $\theta = \pi$), this phase is exactly calculable, hence the analytic appeal of the Polyakov approximation. By direct integration of the equation of motion [8], $\Phi(P)$ is given by exponentiating the effective action :

$$\Phi(P) = \frac{1}{N} \exp\left[iS_0 + i\int d^3x \left(\frac{\theta}{4\pi^2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + A_\mu J^\mu \right) \right] . \tag{3}$$

S_0 is the free point particle action and N is a suitable normalization. The conserved current of a Q=1 point source is now $J_\mu(x) = \int d\tau\, \delta^3(x-y(\tau)) \frac{dy_\mu(\tau)}{d\tau}$. From (1) the key equation for J_μ reads

$$J_\mu(x) = - \frac{\theta}{2\pi^2} \varepsilon_{\mu\nu\rho} \partial^\nu A^\rho \tag{4}$$

Upon substitution in (3) with $\theta = \pi$ it yields $\Phi(P) = \exp\left(\frac{i}{2} \int d^3x\, A_\mu J^\mu \right)$. So for the above *given* point current and, in the Lorentz gauge $\partial^\alpha A_\alpha = 0$, A_μ can be solved to give Polyakov's result :

$$\Phi(P) = \exp \{ i\pi\, I_G(P) \} \tag{5}$$

where

$$I_G\, (C_\alpha \to C_\beta) = \frac{1}{4\pi} \oint_{C_\alpha} dx^\mu \oint_{C_\beta} dy^\nu \frac{\varepsilon_{\mu\nu\lambda}(x^\lambda - y^\lambda)}{|x-y|^3} \tag{6}$$

in the limit where the two smooth closed 3-space curves C_α and C_β <u>coincide</u>, namely $C_\alpha = C_\beta = P$, the soliton worldline.

Were C_1 and C_2 in R^3 (or S^3) *disjoint* curves , (6) would just be their Gauss linking coefficient. If Ω (M_2) is the solid angle subtended by C_1 at the point M_2 of C_2, then Stoke's theorem yields $I_G = \frac{1}{4\pi}\int_{C_2} d\,\Omega(M_2)$, which measures the variation of this solid angle divided 4π as M_2 runs along C_2 ; it is the algebraic number of loops of one curve around the other.

There is however an indeterminacy in the analytic result (6) . While its integrand is that of Gauss' invariant, the integration is over one and the same curve. $I_G(P)$, which represents the net action of the Aharonov-Bohm and the Chern-Simons terms in (1) on a charged particle, is thus *undetermined.* Clearly this artifact of the geometric point-limit approximation must be corrected for by a proper definition or regularization of $I_G(P)$.

Polyakov's curing prescription is to trade the delta function in the alternative expression of $\int_P dx^\mu \iint d^2y_\mu \delta(x-y)$ of $4\pi I_G$ for the Gaussian $\delta_\varepsilon(x-y)=(2\pi\varepsilon)^{-\frac{3}{2}}\exp\left(\frac{-|x-y|^2}{\varepsilon}\right)$. He found $I_G(P)_{Reg}$ to be - T(P), the total torsion or twist of the curve P in spacetime with $T(P) = \frac{1}{2\pi}\oint_P d\mathbf{x}\cdot(\mathbf{n}\times\frac{\partial\mathbf{n}}{\partial s}) \equiv \frac{1}{2\pi}\int_P \tau(s)\, ds$. s and $\mathbf{n}$ denote the arc length and the principal normal vector to P at the point $\mathbf{x}(s)$. What then are the geometric underpinnings of this regularization procedure ?

By substituting the Gaussian the dominant contribution of the surface integral comes from an infinitesimal strip Σ_P ; Polyakov [6] effectively turns a spacetime curve into a *ribbon* . Indeed, in 1959, Calugareanu [9] discovered that $I_G(C_\alpha \rightarrow C_\beta)$ is in fact a perfectly well defined entity , a new topological invariant SL , the self–linking number for a simple closed *ribbon*[9]. SL is in fact the linking number of C_β with a twin curve C_α moved an infinitesimally small distance ε along the principle normal vector field to C_β. Being disjoint these two curves can be linked and unlinked, like the strands of a circular supercoiled DNA molecule [10]. Nowadays in knot theory this construction is called the *framing* of a curve C_β . Most significant to our considerations is the existence of the "conservation law": SL = T + W whereby SL, is the algebraic sum of two *differential geometric* characteristics of a closed ribbon, its total torsion or twisting number T and its *writhing number* or *writhe* W. While both T and W, which are metrical properties of the ribbon and its "axis" respectively, can take a continuum of values, their sum SL must be an integer. The writhing number W, better known in topology [11] as the Gauss integral for the map $\phi : S^1 \times S^1 \to S^2$, is the element solid angle or the pullback volume 2–form $d\Omega_2$ of S^2 under ϕ. So Calugareanu's formula reads

$$SL(P) = \frac{1}{4\pi}\int_{P\times P} d\Omega_2 + \frac{1}{2\pi}\int_P \tau\, ds \,. \tag{7}$$

Using the invariance of W under dilatations and let ϕ be the map, $\mathbf{e}(s,u)$ $(\mathbf{e}^2 = 1)$, a local Frenet-Serret frame vector attached to the curve, we get $W = \frac{1}{4\pi}\int_0^L ds \int_0^1 du\, \varepsilon_{abc}\, e^a \partial_s e^b \partial_u e^c$, a,b,c = (1,2,3) and $\partial_s = \partial/\partial s$, $\partial_u = \partial/\partial u$. A conformally invariant action for the frame field $\mathbf{e}$, W is manifestly a Wess-Zumino-Novikov-Witten term and readily identified as a Berry phase upon exponentiation. It is precisely Polyakov's [6] double integral representation (modulo an integer) for the torsion T(P) . From our vantage point, this equivalence,whose importance will soon be clear, is just the equality W = – T (mod Z) . A mondane illustration of the relation W + T = SL for a ribbon is a coiled phone cord. When unstressed with its axis curling like a helix in space, its writhe is large while its twist is small. When stretched with its axis almost straight, its twist is large while its writhe is small.

By now many people [12] have verified that the regularized Hopf (statistical) phase factor is given by the writhe , $\Phi(P) = \exp\{i\pi W(P)\}$. Its alternate form $\Phi(P) = \exp(-i\pi T(P)) \exp(+i\pi n)$ by way of the relation W = - T (Mod Z) is the "spin " phase factor essential to Polyakov 's proof that the 1-solitons in (1) are fermions by obeying a Dirac equation in their point-like limit. Had θ been kept arbitrary, we would have the more general theory of pointlike anyons carrying fractional spin and intermediate statistics [4]. In other words the relation W = -T + SL *is* the mathematical expression of the connection between statistics and spin in the geometric point soliton limit.

In the geometry of 2-surfaces, a form of the celebrated Gauss-Bonnet theorem is $K = 2\pi\chi$. Its telling property , unique in the whole of differential geometry, is the following : Like the formula SL = T + W , it relates entities defined solely in terms of topology such as the Euler characteristics χ of a closed surface M and metrical entities defined purely in terms of distances and angles such as total Gaussian curvature K for M. It is perhaps not too surprising that Fuller [10] showed that the relation SL = T + W is in fact a consequence of the Gauss-Bonnet formula. So we see an interesting role that the latter theorem plays in a fundamental physics principle - the relation between spin and statistics!

Furthermore since White [11] obtained the higher dimensional version of Calugareanu's formula from his formulation of Gauss-Bonnet-Chern theorem for Riemannian manifolds [13], it is natural for us to extend Polyakov's analysis to the D>3 counterparts of the Wilczek-Zee model (1). We first define the generalized Gauss linking number for manifolds.

Generalizing the procedure for linking 3-space curves, consider [11] two continuous maps f(M) and g(N) from two smooth, oriented, non intersecting manifolds M and N, dim(M) = m and dim(N) = n, into R^{m+n+1}. Let S^{m+n} be a unit (m+n)–sphere centered at the origin of R^{m+n+1} and $d\Omega_m$ be the pull–back volume form of S^{m+n} under the map e : $M \times N \rightarrow S^{n+m}$ where to each pair of points $(\mathbf{m},\mathbf{n}) \in M \times N$ we associate the unit vector e in R^{m+n+1} : $e(\mathbf{m},\mathbf{n}) = \frac{g(\mathbf{n})-f(\mathbf{m})}{|g(\mathbf{n})-f(\mathbf{m})|}$. The degree of this map is then the generalized Gauss linking number of M and N, namely

$$L(f(M),g(N)) \equiv L(M,N) = \frac{1}{\Omega_{n+m}} \int_{M \times N} d\Omega_{n+m}. \tag{8}$$

Ω_n $(=2\pi^{(n+1)/2}/\Gamma((n+1)/2))$ is the volume of S^n. We first note that the linking number for <u>even</u> dimensional submanifolds M and N is zero due to the non-commutativity property $L(M, N) = (-1)^{(m-1)(n-1)} L(N, M)$.

Next we specialize White's main theorem : Let $f : M^n \rightarrow R^{D=2n+1}$ be an smooth embedding of a closed oriented differentable manifold into Euclidean (2n+1) space. Let v be an unit vector along the mean curvature vector of M^n . If n is <u>odd</u> (i.e. D=3, 7, 11,15 etc...) then

$$SL(f,f_\varepsilon) = \frac{1}{\Omega_{2n}} \int_{M \times M} d\Omega_{2n} + \frac{1}{\Omega_n} \int_M \tau^* \, dV \tag{9}$$

is the self-linking number of a hyper-ribbon made up of M^n and the same manifold deformed a small distance ε along v, and where the terms on the RHS of (9) are respectively the writhing and twisting numbers, W and T , of the hyper-ribbon. For n <u>even</u> (D=1,5,9,...), both W and T are zero and hence also SL =0 , the situation is uninteresting for the physics of nonintegrable phases.

The universality of the formula SL = W + T mirrors that of Gauss-Bonnet-Chern theorem. It is only natural to expect that for higher dimensional solitons in suitable models White's formula T = -W (mod Z) similarly links their spin and statistical phases . As applied to physics, it could define and relate the twisting and writhing of odd dimensional closed S^3-, S^5-, S^7-... hyper-ribbons, the world volumes of topological S^2 -, S^4-, S^6-membranes solitons in D=7, 11, 15... dimensional spacetime respectively. How to cut down this infinity of choices ? What are the natural D >3 σ-model counterparts of (1) which may admit such solitons with exotic spin and statistics ?

One approach suggests itself. These models should have the three essential ingredients of the CP(1) model [4] : 1) The existence of topological membrane solitons; 2) the presence in the action of an Abelian Chern-Simons term, the Hopf invariant; 3) the associated Hopf mappings $S^{2n-1} \rightarrow S^n$ include ones with <u>Hopf invariant 1</u>. The first two conditions are embodied in the time component of the key equation (4). Its integration over all of space yields the topological charge-magnetic flux coupling which is the very basis of the statistics phenomenon in (2+1) dimensions. As to the third condition, a crucial element in the Wilczek-Zee proof of the fractional spin and statistics for one soliton [4], we have the following property of Hopf mappings in higher dimensions. While for <u>any n even</u> there always exists a map f : $S^{2n-1} \rightarrow S^n$ with only <u>even integer</u> Hopf invariant γ(f), but as for Hopf maps of <u>invariant 1</u>, the celebrated Adams' theorem [14] tells us : If there exists a Hopf map f: $S^D \rightarrow S^{(D+1)/2}$ of Hopf invariant $\gamma(f) = 1$, indeed one with *any* integer γ(f), then D <u>must</u> equal 1, 3, 7 and 15 i.e. m = (D+1)/2 = 1, 2, 4 and 8.

From it follow other relevant theorems[15] . They are

1) R^n admit the structure of a real division algebra if and only if n= 1, 2,4 and 8 . The algebras are the real numbers (**R**), the complex numbers (**C**), the quaternions (**H**) and the octonions (**Ω**).

2) The only parallelizable spheres are S^1 , S^3 and S^7

3) There exists a vector product on R^n if and only if n = 3 or 7 . These products are the familiar Gibbs-Hamilton product of vectors or imaginary quaternions and the Cayley product of imaginary octonions.

4) Among the spheres S^n only S^2 or S^6 admit an almost complex structure.

So Adams' theorem singles out 4 unique sets of spacetime and field topologies for the sought for field theories. These unique four families of Hopf maps , the K = **R, C, H, Ω** Hopf bundles and their various properties are made manifest by the tell-all diagram

$$
\begin{array}{c}
U(1) = SO(2) \\
\| \\
Z_2 = O(1) = S^0 \to \mathbf{S^1} \to \mathbf{S^1/Z_2} = \mathbf{RP(1)} \approx SO(2)/Z_2 \\
\| \\
SO(2) = U(1) = S^1 \to \mathbf{S^3} \to \mathbf{S^2} = \mathbf{CP(1)} \approx SU(2)/U(1) \\
\| \\
SU(2) = Sp(1) = S^3 \to \mathbf{S^7} \to \mathbf{S^4} = \mathbf{HP(1)} \approx Sp(2)/Sp(1)\times Sp(1) \\
\| \\
Spin(8)/Spin(7) = S^7 \to \mathbf{S^{15}}(\approx Spin(9)/Spin(7)) \to \mathbf{S^8} = \mathbf{\Omega P(1)} \approx Spin(9)/Spin(8).
\end{array}
$$

The Hopf maps $f : S^{2n-1} \to S^n$, n=1, 2, 4, 8 ,with Hopf invariant one have found important applications in condensed matter physics and in quantum field theory. Particularly the connection with exotic spin and statistics had long been lurking in the background. Thus, in the D=2 ϕ^4 field theory, the n=1 real Hopf map realizes the 1-kink soliton bearing intermediate spin and obeying exotic statistics [16] . The n=2 complex Hopf map underlies the θ spin and statistics of D=3 CP(1) model with Hopf term. We have shown this pattern to persist in models built on the remaining two Hopf fiberings, namely the D=7 quaternionic HP(1) ($\approx S^4$) and the D=15 octonionic ΩP(1) ($\approx S^8$) σ-models augmented with their respective Hopf invariant term . Modulo possible Skyrme terms needed for soliton stability, their action generalize (1) as

$$
S_{(n)} = \int_M \partial_\mu \vec{N}\cdot\partial^\mu \vec{N}\, d^{2n-1}x + \frac{\theta}{a}\int_M A_{n-1}\wedge dA_{n-1} \tag{10}
$$

$$
n = 4, 8, \quad M = S^7, S^{15}
$$

where $\vec{N}$, $\vec{N}^2=1$, the (n+1)-vector parametrizing S^n . If $K^T = (K_1, K_2)$ such that $K^+K=1$ with $K_1, K_2 \,\varepsilon\, \mathbf{H}, \mathbf{\Omega}$, is a **K**-valued 2-spinor parametrizing S^{2n-1}, the Hopf map is $\vec{N} = \mathrm{Sc}\left(K^\dagger \vec{\gamma} K\right)$ with $K^\dagger = (\overline{K}_1, \overline{K}_2)$, $\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ \overline{e_\mu} & 0 \end{pmatrix}$, $\mu = 0, 1, \ldots m-1$ and $\gamma_m = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, m = 4 and 8. The γ's are just the Dirac matrices of SL(2, K) with γ_m being the analog of γ_5 of the standard D=4 formalism.

The A_{n-1} are composite rank (n-1) Kalb-Ramond fields. They are nonlocal in **N** but local in K . The second term in ( ) is up a multiplicative constant $\theta\, a^{-1}$ the Whitehead-Chern-Simons form of the Hopf invariant $\gamma(\Phi)$ classifying the mappings $\Phi : S^{2n-1} \to S^n$. It also reads

$$S_I = \frac{1}{(n-1)!}\int d^{2n-1}x\; J^{\mu_1\ldots\mu_{n-1}}\, A_{\mu_1\ldots\mu_{n-1}} \tag{11}$$

i.e. as an interaction of the field A_{n-1} with the conserved topological current $J_{n-1} = -\frac{(n-1)!\theta}{4\pi^2}{}^*F_n$, the * means Hodge duality. The sources of J_{n-1} are charged solitonic 2-membranes and 6-membranes respectively, in accordance with the Kleman-Toulouse classification of topological defects [17] .

Since these membrane solitons can be readily shown to have a thin London limit, Polyakov's approximation translates into the Chern-Simons-Kalb-Ramond electrodynamics of Nambu-Goto membranes.To get the statistical phase, we consider [12] the propagation of two pairs of membranes-antimembrane and seek the phase upon adiabatically exchanging the membranes. We then get

$$\left\langle \exp\left(\frac{i}{3!}\oint_{P_1} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\exp\left(\frac{i}{3!}\oint_{P_2} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\right\rangle \tag{12}$$

P_1 and P_2 are S^3 hyper-curves. The functional average $\langle\cdots\rangle$ is taken over the Hopf action (9). As in D=3 case , the resulting phase here is the sum of three phases. The first contribution yields the phase factor $\exp\{2i\,(\pi^2/\theta)\,L\}$ with L being the generalized Gauss' linking coefficient ( ) for two S^3-loops. We get π^2/θ for the statistical phase. The other two phase factors $\Phi(P_i)$ are given by the expectation value of one loop:

$$\Phi(P) = \left\langle \exp\left(\frac{i}{3!}\oint_{P} A^{[\mu\nu\lambda]}dx^\mu\wedge dx^\nu\wedge dx^\lambda\right)\right\rangle . \tag{13}$$

In the London-Nielsen-Olesen limit we use the effective action

$$S = S_0 + \frac{\theta}{(3!)^2 a}\int_{S^7} d^7x\; \varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\, A^{\mu\nu\lambda}\partial^\alpha A^{\beta\gamma\delta} + \frac{1}{3!}\int_{S^7} d^7x\; J_{\mu\nu\lambda}A^{\mu\nu\lambda}. \tag{14}$$

S_0 is the free Nambu-Goto action for a relativistic 2-membrane, $0 \le \theta \le \pi$ and the constant a is yet to be chosen. By direct integration of the equation of motion

$$J_{\mu\nu\lambda} + 2\,\frac{\theta}{3!a}\varepsilon_{\mu\nu\lambda\alpha\beta\gamma\delta}\partial^\alpha A^{\beta\gamma\delta} = 0 \tag{15}$$

with the given current $J^{\mu\nu\lambda}(y) = \int d^3x\, \delta^7(x-y) \frac{\partial(x^\mu,x^\nu,x^\lambda)}{\partial(\tau,\sigma_1,\sigma_2)}$. , in the Lorentz gauge $\partial^\alpha A_{\alpha\beta\gamma} = 0$,

$$i\frac{1}{2\cdot 3\,!}\int d^7x\, J_{\beta\gamma\delta}A^{\beta\gamma\delta}(x) = = i\,\frac{a}{4\theta}\frac{1}{\Omega_6}\int_{S^3} d\Sigma_{\beta\gamma\delta}\int_{S^3} d\Sigma_{\mu\nu\lambda}\frac{\varepsilon^{\mu\nu\lambda\alpha\beta\gamma\delta}(x-y)_\alpha}{|x-y|^7} \tag{16}$$

However just like in the D=3 case , the double integration is over one and the same hypercurve S^3 so the phase (16) is *undetermined.* We thus apply White's result to get the regularized phase

$$\Phi\,(\,P \approx S^3) \;=\; \exp\left(i\,\frac{a}{4\theta}W(P)\right) \tag{17}$$

where $W(P) = \frac{1}{\Omega_6}\int_{S^3\times S^3} d\Omega_6$ is the writhe of the Feynman path P of the Nambu-Goto S^2-membrane, a S^3 hyper-ribon in 7-spacetimes. A similar computation gives the same result for the octonionic case of $P \approx S^7$ in S^{15} -spacetime .

For the value of $a = 4\pi^2$, $\Phi(P)=\exp(\pi^2 iW/\theta)$. By setting $\theta = \pi$, we get the S^3- (S^7-) counterpart of Polyakov's phase factor $\Phi(P) = \exp(-\pi i\; T(P))\exp(\pi i\; n)$, T being the generalized torsion for an S^3-(S^7) ribbon P. We similarly expect this phase factor to embody the thin membrane's spin in a functional integral formalism. Provided that this reasonable conjecture is verified by a construction à la Polyakov of the spin factor for membranes, we will have a 7- (15-) dimensional analog of the D=3 Fermi-Bose transmutation. Since the value of θ is not fixed by the U(1) gauge invariance of the Kalb-Ramond field, for arbitrary values of θ we thus have the possibility of fractional statistics and spin via the relation W = -T (mod Z) for the membranes.

Without probing the complexity of model dependent soliton structure or performing a detailed canonical quantization of the above KP_1 σ-models, the evidence of θ-spin and statistics among the membranes is further supported by the topology of their cobfiguration space of fields.In the Schrödinger picture, the configuration space Γ of finite energy static solutions of our KP(1) σ-model is the *mapping space* of all based preserving smooth soliton maps $\vec{N}(x)$: $x \in S^n \;\rightarrow\; \vec{N}(x) \in S^n$, $n = 2, 4, 8$. The space Γ is an infinite Lie group with connectivity given by

$$\pi_0(\,\Gamma = \{\vec{N}: S^n \rightarrow S^n\}) \approx \pi_n(S^n) \approx Z. \tag{18}$$

So it is divided into an infinite set of pathwise-connected components Γ_α, $\alpha \in Z$, corresponding to the various soliton sectors labelled by the charge Q. Moreover for our membranes, as with Skyrmions and Yang-Mills instantons, each Γ_α has further topological obstructions. According to G.W. Whitehead [18] all Γ_α in Γ have the *same* homotopy type i.e. $\pi_i(\Gamma_\alpha) \approx \pi_i(\Gamma_\beta)$. Of relevance to the question of exotic spin and statistics, for the 1–soliton sector are the key relations

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx Z \ \text{ for } \ (i, n) = (1, 2), (3, 4), (7, 8). \tag{19}$$

by the Whitehead and Hurewicz [19] isomorphisms, the latter stating $\pi_i(\Gamma_\alpha) \approx \pi_{i+n}(S^n)$.

The connections (18) reflect the *multi-valuedness* of Γ_i and imply the possibility of adding to the KP(1) σ-model action a Hopf invariant $\gamma(\vec{N})$, the generator of the torsion free part of $\pi_{i+n}(S^n)$ ($\pi_3(S^2) \approx Z$, $\pi_7(S^4) \approx Z \oplus Z_{12}$ and $\pi_{15}(S^7) \approx Z \oplus Z_{120}$). Generalizing the CP(1) model $\{(i, n) = (1,2)\}$, the nontriviality of these $\pi_i(\Gamma_1)$ implies the possibilities of Aharonov-Bohm effects of a multiply connected configuration space Γ and signals the existence for the membrane solitons of a higher dimensional analog of an θ spin and statistics connection .

In the CP(1) case, the Hopf term induces, upon a 2π rotation P of the Skyrmion or an interchange of two Skyrmions, a projective spin phase factor $\Phi(P) = \exp\{i\theta\} = \exp(i\, 2\pi s)$, s being the soliton spin .The equality $\theta = 2\pi s$ for this process of rotation is a physical realization of the homomorphism:

$$\pi_1(SO(2)) \approx \pi_3(S^2) \approx \pi_1(\,\Gamma_1) \approx Z. \tag{20}$$

It establishes the equality of the kinematically allowed exotic spin to the dynamically induced θ-spin by way of the Hopf term. Yet (20) is but a special case of the Hopf-Whitehead J-homomorphism [20] $\pi_k(SO(n)\,) \approx \pi_{k+n}(S^n)$. Indeed we have the following chain of homomorphisms :

$$\pi_i(\Gamma_1) \approx \pi_i(\Gamma_0) \approx \pi_{i+n}(S^n) \approx \pi_i(SO(n)) \approx Z \tag{21}$$

with (i, n) = (1, 2), (3, 4), (7, 8) . $\pi_3(SO(4)) \approx \pi_7(S^4) \approx Z$, $\pi_7(SO(8)) \approx \pi_{15}(S^8) \approx Z$

It suffices to say the physical interpretation of these topological relations implies a dynamically induced exotic spin and statistics connection for the 2-and 6-membranes.

In conclusion, our work may cast new light on the uncharted problem of the spin and statistics connection for higher dimensional extended objects. It is also a first step both in the bosonic functional integral formulation for spinning extended objects [6,21] and in the study of the θ-vacuum phenomenon in Kaluza-Klein compactification. As their historical developments [22] testify , division algebras stand at the crossroad of several frontier areas of mathematics and physics. Over the last decade, ever more linkages have been discovered between division algebras and basic aspects of unified theories . Examples are field theories with rigid supersymmetries, superstrings and supermembranes in critical dimensions. Here we have uncovered their special relevance in yet another connection , a generalized θ-spin and statistics connection.

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References

1. E. Witten, " Topological Quantum Field Theory", to appear in Comm. Math. Phys.

2. G. T. Horowitz, " Exactly Soluble Diffeomorphism Invariant Theories" NSF-ITP-88-178; G.T. Horowitz and M. Srednicki,"A Quantum Field Theoretic Description of Linking Numbers and Their Generalization", UCSB-TH-89-14.

3. See M. Atiyah, " New Invariants of Three and Four Dimensional Manifolds" to appear in Symposium on Mathematical Heritage of Hermann Weyl, ed. R. Wells et al. (Univ. North Carolina, May 1987) and references therein.

4. F. Wilczek, Phys. Rev. Lett. 48 (1982), 1144; 49 (1982), 957; F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983), 2250.

5. E. Witten, "Quantum Field Theory and The Jones Polynomial", IASSNS-HEP-88/33 (1988), to appear in Proc. of the IAMP Congress,Swansea, 1988.

6. A.M. Polyakov, Mod. Phys. Lett. A3 (1988), 325.

7. C-H. Tze and S. Nam, " Topological Phase Entanglements of Membrane Solitons in Division Algebra σ-models with a Hopf-Term", VPI-IHEP-88-10, to be published in Annals of Physics.

8. R.I. Nepomechie and A. Zee, in "Quantum Field Theory and Quantum Statistics (Fradkin Festschrift)" (I.A. Batalin et al. Ed.), Vol. 2, Adam Hilger, Bristol, 1986.

9. G. Calugareanu, Rev. Math. Pures Appl. 4 (1959), 5; Czech. Math. J. 11 (1961), 588.

10. F. Brock-Fuller, Proc. Nat. Acad. Sci. USA 68 (1971), 815; Proc. Natl.Acad. Sci. USA 75 (1978) 3557; F.H.C. Crick, Proc. Nat. Acad. Sci. USA 73 (1976), 2639.

11. J.H. White, Am. J. Math. 91 (1969), 693.

12. J. Grundberg, T.H. Hansson, A. Karlhede and U. Lindström, "Spin, Statistics and Linked Loops," Stockholm preprint (1988).

13. S.S. Chern, Ann. Math. 43 (1942), 178; 46 (1945).

14. J.F. Adams, Ann. of Math. 72 (1960), 20.

15. M.W. Hirsch, "Differential Topology," Springer-Verlag, Berlin, 1976.

16. See for instance J. Fröhlich, "Statistics of Fields, the Yang-Baxter Equation, and the Theory of Knots and Links", *in* "Nonperturbative Quantum Field Theory, 1987 Cargese" (G. 't Hooft et al. Ed.), Plenum, New York, 1988; "Statistics and Monodromy in Two and Three Dimensional Quantum Field Theory", in "Proceedings of the 1987 Como Conference," (K. Bleuler et al. Eds.).

17. G. Toulouse and M.J. Kleman, J. de Physique Lett. 37 (1976), L-149.

18. G.W. Whitehead, Ann. Math. 47 (1946), 460; J.H.C. Whitehead, Ann. Math. 58 (1953), 418.

19. W. Hurewicz, Proc. Akad. Amsterdam 38 (1935), 112; 39 (1936), 117, 215.

20. See for instance D. Ravenel and A. Zee, Comm. Math. Phys. 98 (1985), 239.

21. T. Jacobson, Boston University Preprint BUHEP-88-12; P. Orland, VPI preprint, VPI-IHEP-88/6; K. Johnson, MIT Preprint CTP# 1656 (1988) and references therein.

22. F. Gürsey, in "Symmetries in Physics (1600-1980)" (M.G. Doncel, A. Hermann, L. Michel and A. Pais Ed.) p. 557, Univ. Autonoma de Barcelona Publications

MODULI SPACES AND TOPOLOGICAL QUANTUM FIELD THEORIES

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Generally covariant field theories have observables which are metric independent. Hence they are global invariants. Recently, a new class of such theories, the so called topological quantum field theories (TQFTs'), were introduced by E.Witten. Originally they were affiliated with Yang-Mills instantons (TYM),[1] sigma models (TSM)[2] , and gravity (TG)[3] . Later on they enveloped other domains of physical systems[5-8] . The main question is obviously whether the TQFT's probe some physical phenomena or are they merely mathematical tools to study topological properties of certain bundles? The answer to this question is two-fold: (i) The observables of the TQFT span the cohomology ring on certain moduli spaces. These moduli spaces may be intimately related to physics. An example familiar to string theorists is the moduli space of Riemann surfaces. Another example is the moduli space of instantons. (ii) The possibility that the TQFT's describe a generally covariant phase of some physical systems[1,8]. In this work we follow the first direction.

The main feature of the TQFT's is the "topological symmetry" which is the largest local symmetry possible for the fields that describe the system. This symmetry is responsible for gauging away any dependence on local properties. The classical action does not play any role and can be taken to be zero or a topological number. The quantum Lagrangian is derived via BRST gauge fixing of the topological symmetry and related "ghost symmetries" [10][11] . The observables of the theory, which are expectation values mainly of the ghost fields, can be expressed as an integral of closed forms on some moduli space. Can one write down a TQFT which corresponds to any given moduli space? In this work we present a general prescription for the building of such a TQFT. Several examples of TQFT's which correspond to interesting moduli spaces are presented. We show how the observables of the theory correspond to cohomologies on the moduli space. The perception that those global invariants are trivial is shown to be incorrect.

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The connection between moduli spaces and physical systems can be described simply in the following way. Assume that a physical system is defined by a set of fields Φ_i on a d dimensional space-time manifold M, and a certain local symmetry G under which the fields Φ_i transform in some representation of the group G . Mathematically, a certain bundle is defined over M. Very often we are interested not in the whole space of possible Φ_i configurations but in a particular subset which can be characterized by

$$\{\Phi_i^0 | F(\Phi_i^0) = 0\}, \tag{1}$$

where $F(\Phi_i)$ is a given functional of the fields Φ_i. The condition (1) can be, for example, the Euler-Lagrange equations of the action describing the physical system. We now perturb a given configuration in this subspace and demand that the perturbation does not take Φ_i out of this subspace, namely:

$$F(\Phi_i^0 + \delta\Phi_i) = 0 \qquad \rightarrow \qquad \frac{\delta F}{\delta \Phi_i}\delta\Phi_i = 0. \tag{2}$$

We want further to mod out from the possible variations, $\delta\Phi_i$, those redundant ones which are the transformation of Φ_i under G. We, therefore, choose a gauge slice by imposing a gauge condition

$$G_{GF}(\Phi_i^0 + \delta\Phi_i) = 0 \qquad \rightarrow \qquad \frac{\delta G_{GF}}{\delta \Phi_i}\delta\Phi_i = 0. \tag{3}$$

As for solutions to the equations (2-3) there are two possibilities (i) No non-trivial solutions, then the Φ_i^0 configurations are isolated. (ii) There are solutions and then these solutions span the moduli space, $\mathcal{M}$, of configurations fulfilling (1) modulo gauge transformations. For finite deformations we want to integrate $\delta\Phi_i$ but there may be obstructions[1,2] to the integration of the infinitesimal deformations. Regarding the solutions of (2-3) as the kernel of an operator $\bar{D}$ acting on $\delta\Phi_i$, then the obstructions are given by the cokernel of this operator. Therefore, the dimension of the moduli space is the number of solutions minus the number of obstructions which is :

$$dim\mathcal{M} = dim(Kernel\bar{D}) - dim(coKernel\bar{D}) = index\bar{D} \tag{4}$$

We demonstrate the statements made above in table 1† for the moduli spaces which are related to various physical systems: (i) Yang Mills instantons in four dimensions[1,11], (ii) flat connections in two dimensions (which is equivalent to the phase-space of the correspondig three dimensional Chern-Simons theory), (iii) flat SO(2,1) connections[12] (which is equivalent to the space of Riemann surfaces with $g > 1$), (iv) World sheet instantons in two dimensions[2], (v) two-torus[4] and (vi) (1,1) forms on Calabi-Yau manifolds.

The basic idea of the use of TQFT to explore the moduli spaces is to formulate a field theory which is invariant under an additional local "topological symmetry"[9-11] of the form $\delta\Phi_i = \Theta_i(x)$ where Θ_i has the same properties as Φ_i under the Lorentz and G transformations but may differ in boundary conditions. The form of the original action is not important as long as it is invariant under the topological symmetry. In general, the Lagrangian is taken to be zero up to topological terms and up to eliminating auxiliary fields. In the case that the configurations, Φ_i^0, are characterized by a topological number which can be expressed as a d dimensional integral,

† The notations in the table follow references:[1,11], [2,4], and[12]

Table 1. *Examples of moduli spaces.*

Configuration	G-Symmetry	Conditions on $\delta\Phi_i$	Moduli Space
A_μ : non-abelian gauge fields in four dim.	non-abelian gauge symmetry	$D_{[\mu}\delta A_{\nu]} + \epsilon_{\mu\nu\rho\sigma}D^{[\rho}\delta A^{\sigma]} = 0$ $D_\mu \delta A^\mu = 0$	Yang Mills instantons
A_α : non-abelian gauge fields in two dim.	non-abelian gauge symmetry	$D_{[\alpha}\delta A_{\beta]} = 0$ $D_\alpha \delta A^\alpha = 0$	non-abelian flat connections
$(e_{\alpha a}, \omega_\alpha)$: world sheet SO(2,1) connections	SO(2,1) gauge symmetry	$\partial_{[\alpha}\delta\omega_{\beta]} + \epsilon_{ab}e^a_{[\alpha}\delta e^b_{\beta]} = 0$ $(\tilde{D}_{[\alpha}\delta e_{\beta]})^a + \epsilon^{ab}e_{b[\alpha}\delta\omega_{\beta]} = 0$ $\tilde{D}^{ab}_\alpha = \delta^{ab}\partial_\alpha + \epsilon^{ab}\omega_\alpha$	Riemann surfaces of $g > 1$
x^i : coordinates on symplectic manifold	world-sheet reparametrization	$D_\alpha \delta x^i + \epsilon_{\alpha\beta}J^i_j D^\beta \delta x^j = 0$ J: complex structure	world sheet instantons
$g_{\alpha\beta}$: metric on torus	world sheet reparametrization	$\partial_z\partial_{\bar z}(g^{z\bar z}\delta g_{z\bar z}) = 0$ $g_{\bar z\bar z} = g_{zz} = 0$	torus
$g_{i\bar j}$: metric on Kahler Manifold	diffeomorphism on Khaler manifold	$\partial_i\partial_{\bar j}(\frac{\delta g}{g}) = 0$ $g = det(g_{i\bar j})$	(1,1) forms Calabi-Yau Manifold

it makes sense to take the later as the action. This will imply some boundary condition on the local parameter of the topological symmetry[11]. Quantization of the TQFT is performed by using the BRST method. $\epsilon\Psi_i$ is now replacing Θ_i where ϵ is an anti-commuting global parameter and Ψ_i is an anti-commuting ghost. The gauge-fixing and Faddev-Popov Lagrangians are derived by BRST variation of a "gauge condition" $\mathcal{Z}^{(1)}$:

$$\mathcal{L}^{(1)}_{(GF+FP)} = \hat{\delta}^{(1)}\mathcal{Z}^{(1)} = \hat{\delta}^{(1)}[\bar{\Psi}F(\Phi_i)] = BF(\Phi_i) - \bar{\Psi}\delta[F(\Phi_i)]. \tag{5}$$

Here $\delta_{BRST} = i\epsilon\hat{\delta}$, $\bar{\Psi}$ is an anti-ghost in a representation of the group G and the Lorentz group such that $\bar{\Psi}F(\Phi_i)$ is a singlet under both groups and B is the associated auxiliary field. The Euler-Lagrange equation for $\bar{\Psi}$ leads to an equation for Ψ_i which is the same as eqn. (2) for $\delta\Phi_i$. The Lagrangian (4) is further invariant under a local "ghost symmetry". The origin of this symmetry is the following: $\mathcal{Z}^{(1)}$ is obviously invariant under the G symmetry, thus transformations that leave Φ_i and $\bar{\Psi}_i$ inert and transform Ψ_i and B in the same way as Φ_i and $\bar{\Psi}$ transform under G, leave (5) invariant. In general, one can replace $\mathcal{Z}^{(1)}$ by $\mathcal{Z}^{(1)'} = \bar{\Psi}(F(\Phi_i) + \alpha B)$ where α is an arbitrary parameter. For $\alpha \neq 0$ the "ghost symmetry" mentioned above is not a symmetry.

However, by adopting the "ghost symmetry" transformation, for the variation of B in $\mathcal{Z}^{(1)'}$ the resulting $\mathcal{L}^{(1)'}$ is invariant again under a "ghost symmetry"[13] We thus use here the $\alpha = 0$ gauge.

To fix the "ghost symmetry" we introduce a commuting "ghost for ghosts" field ϕ and a its anti-ghost $\bar{\phi}$. The BRST gauge fixing Lagrangian now has the following form:

$$\mathcal{L}^{(2)}_{(GF+FP)} = \hat{\delta}^{(2)}\mathcal{Z}^{(2)} = \hat{\delta}^{(2)}[\bar{\phi}G_{GF}(\Psi_i)], \tag{6}$$

where $\hat{\delta}^{(2)}$ is the sum of the $\hat{\delta}^{(1)}$ and the BRST transformations associated with the ghost symmetry. $G_{GF}(\Phi_i)$ is the gauge condition of (3). It is now obvious that the equation of motion of the combined action will require Ψ_i to obey eqn. (3). The equations for Ψ_i are therefore identical to those which define the moduli space (2-3). The condition for having a moduli space $\mathcal{M}$, thus, translate into a condition of having ghost zero modes.

We now describe this construction for the examples of above in table 2.

Table 2. *the TQFT's which correspond to the moduli spaces given in table 1.*

Topological Symmetry	Gauge Fixing	Ghost Symmetry	Equations of Ψ
$\hat{\delta}A_\mu = \psi_\mu$	$\hat{\delta}[\bar{\psi}^{\mu\nu}(F_{\mu\nu} + \tilde{F}_{\mu\nu})]$	$\hat{\delta}\psi_\mu = iD_\mu\phi$ $\hat{\delta}B^{\mu\nu} = i[\bar{\psi}^{\mu\nu},\phi]$	$D_{[\mu}\psi_{\nu]} + \epsilon_{\mu\nu\rho\sigma}D^{[\rho}\psi^{\sigma]} = 0$ $D_\mu\psi^\mu = 0$
$\hat{\delta}A_\alpha = \psi_\alpha$	$\hat{\delta}[\bar{\psi}^{\alpha\beta}(F_{\alpha\beta})]$	$\hat{\delta}\psi_\alpha = iD_\alpha\phi$ $\hat{\delta}B^{\alpha\beta} = i[\bar{\psi}^{\alpha\beta},\phi]$	$D_{[\alpha}\psi_{\beta]} = 0$ $D_\alpha\psi^\alpha = 0$
$\hat{\delta}e_{\alpha a} = \psi_{\alpha a}$ $\hat{\delta}\omega_\alpha = \tilde{\psi}_\alpha$	$\hat{\delta}[\bar{\tilde{\psi}}(\epsilon^{\alpha\beta}\partial_\alpha\omega_\beta + det\ e)]$ $+\epsilon^{\alpha\beta}\bar{\psi}^a D_\alpha e_{\beta a}]$	$\hat{\delta}\psi_\alpha = iD_\alpha\phi$ $\hat{\delta}B = i[\bar{\psi},\phi]$	$\partial_{[\alpha}\tilde{\psi}_{\beta]} + \epsilon_{ab}e^a_{[\alpha}\psi^b_{\beta]} = 0$ $(\tilde{D}_{[\alpha}\psi_{\beta]})^a + \epsilon^{ab}e_{b[\alpha}\tilde{\psi}_{\beta]} = 0$
$\hat{\delta}x^i = \psi^i$	$\hat{\delta}[\bar{\psi}^\alpha_i(D_\alpha x^i + \epsilon_{\alpha\beta}J^i_j D^\beta x^j - B^i_\alpha)]$ $\bar{\psi}^i_\alpha = \epsilon_{\alpha\beta}J^i_j\bar{\psi}^\beta_j$		$D_\alpha\psi^i + \epsilon_{\alpha\beta}J^i_j D^\beta\psi^j = 0$
$\hat{\delta}g_{\alpha\beta} = \psi_{\alpha\beta}$	$\hat{\delta}[\bar{\psi}\sqrt{g}R^{(2)}]$	$\hat{\delta}\psi_{\alpha\beta} = D_{(\alpha}\phi_{\beta)}$ $\hat{\delta}B = \phi^\alpha D_\alpha\bar{\psi}$	$D_\alpha D^\alpha\psi = 0$ $\psi = \psi^\alpha_\alpha$
$\hat{\delta}g_{i\bar{j}} = \psi_{i\bar{j}}$	$\hat{\delta}[\bar{\psi}^{i\bar{j}}\partial_i\partial_{\bar{j}}log\ g]$		$\partial_i\partial_{\bar{j}}(\psi) = 0$ $\psi = \psi_{i\bar{j}}g^{i\bar{j}}$

Several remarks on the TQFT's given in table 2 are in order: (i) The TFC case and its relation to the Chern-Simons theory and to conformal field theory were presented in ref. [5].

(ii) The TFC for the SO(2,1) group was shown to correspond to the space of Reimann surfaces[12,5]

(iii) The combination of the TSM and TG leads to a theory of topological strings[4]. The corresponding target manifold, which has to be a Kahler manifold, can have any number of dimensions. This bosonic string theory is freed from tachyons.

The BRST algebra that we have at the present stage is not nilpotent but rather it is closed up to a G transformation, $\hat{\delta}_G$, with the ghost for ghost ϕ as the parameter of transformation. For example $(\hat{\delta}^{(2)})^2\Phi_i = \hat{\delta}_G\Phi_i$.

So far we considered only configuration which minimize the action. In particular Φ_i^0 and Ψ_i configurations which are solutions to eqn.(2-3). This is justified only if the path integral is dominated by those configurations. As for Φ^0, this is obvious since this was the gauge fixing we used. As for the rest of the fields we can modify the BRST transformations $\hat{\delta} \rightarrow \hat{\delta}' = \kappa\hat{\delta}$ such that the $\mathcal{L} \rightarrow \kappa\mathcal{L}$. It is straightforward to see that correlation functions are also κ independent[5]. Now in the large κ limit it is obvious that the path integral is dominated by the minima of the action.

The correspondence between the TQFT and the related moduli spaces includes the obstruction as well. Recall that the dimension of the moduli space is equal to the index of the operator defined in (2) and (3). In the TQFT the kernel corresponds to the Ψ zero modes. The cokernel is given by the zero modes of $\bar{\Psi}$ and $\hat{\delta}\bar{\phi}$. Thus the number of obstructions is given by the number of the latter zero modes.

$$index\bar{D} = \#(\Psi \text{ zero modes}) - \#(\bar{\Psi}, \hat{\delta}\bar{\phi} \text{ zero modes}) = Dim\mathcal{M} \tag{7}$$

We want to address now the question of the observables of the TQFT's. The correlation functions of BRST invariant operators are independent of arbitrary variations of the metric[1].

$$\delta_{g_{\alpha\beta}} < \mathcal{O} >= \delta_{g_{\alpha\beta}} \int DX\mathcal{O}e^{i\int d^dx\hat{\delta}Z} = \int DXe^{i\int d^dx\hat{\delta}Z}\mathcal{O}\hat{\delta}[\delta_{g_{\alpha\beta}} \int d^dxZ] = 0, \tag{8}$$

where DX is the measure, and $\mathcal{O}$ is an operator which is a BRST scalar and is independent on the metric. We used here the fact that a vev of any BRST transformation is zero.

Due to the BRST symmetry, the fermionic determinant is equal to the bosonic up to a sign[1]. Therefore, in the case of no ghost zero modes ($dim\mathcal{M} = 0$) ,. the partition function is given by $Z = \Sigma_j(-1)^{S_j}$ where the sum is over all isolated $\Phi_i^{0(j)}$ configurations and S_j is the sign of the ratio of determinantes at the (j) configuration. In general, it was shown[1] that an expectation of an operator has the form of an integral over the moduli space of a closed form on this space.

$$< \mathcal{O} >= \int da_1....da_n\ d\psi_1....d\psi_n\ \Omega_{i_1...i_n}\ \psi_{i_1}...\psi_{i_n} = \int_{\mathcal{M}} \Omega, \tag{9}$$

where $da_i, d\psi_i$ denote the bosonic and fermionic zero modes respectively and

$\Omega_{i_1...i_n}da^{i_1}...da^{i_n} = \Omega$ is an n form on $\mathcal{M}$.

The global invariants $I_i^{(i,l)}$, $i = 0, ..., d$ (the pair of superscripts are the degrees of the form on M and $\mathcal{M}$ respectively) obey the following properties:

$$\begin{aligned} I_i^{(i,l)} &= \int_{\gamma_i} W_i^{(i,l)} \\ \hat{\delta}^{(2)}W_i^{(i,l-i)} &= dW_{(i-1)}^{(i-1,l+1-i)} \quad \hat{\delta}^{(2)}W_0^{(0,l)} = 0 \quad dW_d^{(d,l-d)} = 0, \end{aligned} \tag{10}$$

where γ_i is a non-trivial i^{th} homology cycle. In case that $dim\mathcal{M} \neq 0$ there are fermion zero modes for the Ψ system. Therefore the only non-trivial expectation values are of operators which can soak up those zero modes. This condition translates to a requirement on an observable I:

$$< I > = < \Pi_j I_j > \qquad \Sigma l_j = Dim\mathcal{M} \tag{11}$$

In fact, the W_i are mappings from closed forms on M to closed forms on $\mathcal{M}$ and therefore the global invarinats span the cohomology ring on $\mathcal{M}$. It is, thus, clear why they can be sensors only for topological properties on $\mathcal{M}$ but not local ones.

The dimensions[1,5,12,2,4] and global invariants of the previous examples are given in table 3.

Table 3. *Dimension of moduli spaces and global invariants.*

	Dim. of moduli space $(\#\Psi^0 - \#(\bar{\Psi}, \hat{\delta}\bar{\phi})^0)$	**Global Invariants** $I_i = \int_{\gamma_i} W_i$
TYM	$8P - \frac{3}{2}(\chi(M) + \sigma(M))$ for G=SU(2) P-*Pontryagin#* $\chi(M)-$ *Euler #* , $\sigma(M)-$*signature* 8P- 3 *for Euclidean M*	$W_0 = \frac{1}{2}Tr(\phi^2)$ $W_1 = Tr(\phi\psi)$ $W_2 = Tr(\frac{1}{2}\psi^2 + i\phi F)$ $W_3 = iTr(\psi F)$ $W_4 = -\frac{1}{2}Tr(F^2)$
TFC	$(2g-2)DimG$ *for genus g*	$W_0 = \frac{1}{2}Tr(\phi^2)$ $W_1 = Tr(\phi\psi)$ $W_2 = \frac{1}{2}Tr(\psi^2)$
TFC $SO(2,1)$	$(6g-6)$	*same as above*
TSM	$2(2n+1)$ *for* $\Sigma^0 \to CP^n$ 6 *for* $\Sigma^1 \to$ *cubic surface in* CP^3	$W_0 = \Omega\psi^1...\psi^k$ $W_1 = \Omega dx^1\psi^2...\psi^k$ $W_2 = \Omega dx^1 dx^2\psi^3...\psi^k$ $\Omega dx^1...dx^k$ – k-form on M
TG	2	*none*

So far we ignored the necessity to gauge fix the G symmetry prior to any path integral computations. To pick a gauge slice we take the gauge fixing and Faddev-Popov actions of the form: $\mathcal{L}^{(3)}_{(GF+FP)} = \hat{\delta}_T[\bar{c}G_{GF}(\Phi)]$ where $\hat{\delta}_T = \hat{\delta}^{(2)} + \hat{\delta}_G$ and $\bar{c}$ is a new anti-ghost. In general the equation of motion which corresponds to $\bar{c}$ may impose conditions on Ψ_i which are incompatible with eqns. (2-3). However, it turns out that for local symmetries like gauge symmetries and diffeomorphisms (as can be checked in the examples of above) they are compatible. Several other procedures of gauge fixing the G symmetry in the case of TYM were introduce in the past.[10,13][14]

The question is whether the third stage of gauge fixing can alter the previous results. Since the observables in (10) are both $\hat{\delta}^{(2)}$ and gauge invariant, they are also $\hat{\delta}_T$ invariant. However, the issue of triviality[14] under the total BRST cohomology is different for the $\hat{\delta}_T$ and $\hat{\delta}^{(2)}$ operators. To discuss this question we restrict ourselves to the case where the G-Symmetry is a non-abelian gauge symmetry (TYM,TFC). The conclusion , however, will apply also to the rest of the TQFT's. It turns out that all the W_i which were in a non-trivial cohomology class of $\hat{\delta}^{(2)}$ (apart from W_1) can now be written as a sum of an exact form on M and $\mathcal{M}$ for example:

$$
\begin{aligned}
W_0^{(0,4)} &= \hat{\delta}_T Tr(-c\phi + \frac{i}{3}c^3) \\
W_1^{(1,3)} &= \frac{1}{2}[\hat{\delta} Tr(-A\phi - \frac{i}{3}c^2 A + c\psi) + dTr(\frac{1}{3}c^3 - ic\phi)] \\
W_0^{(2,2)} &= \hat{\delta}_T Tr(icDA + A\psi - \frac{i}{3}A^2 c + iAD\phi + iAdc) \\
&\quad + dTr(-iA\phi + ic\psi + \frac{2}{3}Ac^2 - icdc + ic\psi - cD\phi)
\end{aligned}
\tag{12}
$$

Does it mean that the corresponding global invariants are all trivial? To get a better insight on this question we use the interesting geometrical interpretation of the BRST system that was given in[9]. Following the later reference, the BRST transformations of A, ψ, c, ϕ follow from $\tilde{d}\tilde{A} + \frac{1}{2}[\tilde{A}, \tilde{A}] = \tilde{F}$ and the associated Bianchi identity $\tilde{D}\tilde{F} = 0$ where $\tilde{d} = d + \hat{\delta}_T$, $\tilde{A} = A + ic$ and $\tilde{F} = F + \psi + i\phi$. The objects $\tilde{d}$, $\tilde{A}$ and $\tilde{F}$ are the exterior derivative, connection and curvature on the product space $P \times \mathcal{A}/\mathcal{G}$ where P is the principle bundle and $\mathcal{A}/\mathcal{G}$ is the orbit space. In this picture the set of W_i are the $(i, 4-i)$ components of the second Chern class:

$$
Tr(\tilde{F} \wedge \tilde{F}) = \tilde{d}[Tr(\tilde{A} \wedge \tilde{d}\tilde{A} + \frac{2}{3}\tilde{A} \wedge \tilde{A} \wedge \tilde{A})] \tag{13}
$$

Following (10) the BRST variations of the various W_i are given by exterior derivatives on M of W_{i-1} which according to (13) are derivatives of components of a second Chern class. Therefore the BRST variations of the various W_i given in eqn. (10) are also globally valid. On the other hand (12) tells us that the W_i are given by a combination of the exterior derivatives on both M and $\mathcal{A}/\mathcal{G}$ of some functional of the connections over those spaces. The moduli space, hence also the product space, are topologically non-trivial which means that they can not be covered by a single coordinate patch. Thus the statement of equation (12) is that the W_i are only locally trivial. These local properties cannot be simply extended into global properties. This is manifested in eqn. (13). An integral over the product space would not vanish, even though the integrand can locally be expressed as an exterior derivative of a generalized Chern-Simons term (a generalized instanton number). The I_i are therefore BRST invariant but not trivial.

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REFERENCES

1. E. Witten, *Comm. Math. Phys.* **117** (1988) 353.
2. E. Witten, *Comm. Math. Phys.* **118** (1988) 411.
3. E. Witten, *Phys. Lett.* **206B** (1988) 601.
4. D. Montano and J. Sonnenschein, *Nucl. Phys.* **B313** (1989) 258.
5. J. Sonnenschein, SLAC preprint SLAC-PUB-4970, April 1989.
6. J.M.F. Labastida, M. Pernici *Phys. Lett.* **213B** (1988) 319; J. M. F. Labastida, M. Pernici and E. Witten *Nucl. Phys.* **B310** (1988) 611; L.Baulieu and B.Grossman *Phys. Lett.* **212B** (1988) 351, *Phys. Lett.* **214B** (1988) 223; J. Yamron, *Phys. Lett.* **213B** (1988) 325; D. Birmingham, M. Rakowski and G. Thompson, "supersymmetric instantons and topological quantum field theory" Print-88-0557 (ICTP,Trieste), May 1988. 8pp; *Phys. Lett.* **212B** (1988) 187: F.A. Schaposnik, G. Thompson "topological field theory and

two-dimensional abelian Higgs instantons" , PAR-LPTHE-88-51, Dec 1988; A. Iwazaki, "Topological CP^{n-1} Model and Topological Quantum Mechanics", Nishogukusha Univ. # NISHO-2 1988; A. Karlhede and M. Roček, *Phys. Lett.* **212B** (1988) 51, "Topological Quantum Field Theories in Arbitrary Dimensions", SUNY at Stony Brook preprint # ITP-SB-89-04 (January 1989); R. C. Myers and V. Periwal, "New Symmetries in Topological Field Theories", Santa Barbara preprint # NSF-ITP-89-16.

7. R. Brooks, "Topological Invariants and a Gauge Theory of the Super-Poincaré Algebra in Three Dimensions", *Nucl. Phys.* **B320** (1989) 440.
8. R.Brooks "The spectrum of topological quantum field theories on N-manifolds" , SLAC-PUB-4901, Feb 1989 *Nucl. Phys.* **B** in press.
9. L. Baulieu and I. M. Singer, "Topological Yang-Mills Symmetry", LPTHE preprint # LPTHE-88/18, talk presented at the Annecy meeting on Conformal Field Theory and Related Topics, Annecy, France (March 1988)
10. J. Labastida, and M. Pernici *Phys. Lett.* **212B** (1988) 56.
11. R. Brooks, D. Montano and J. Sonnenschein, *Phys. Lett.* **214B** (1988) 91.
12. D. Montano and J. Sonnenschein, "The Topology of moduli space and quantum field theory", SLAC preprint # SLAC-PUB-4760 (October 1988); *Nucl. Phys.* **B**, in press
13. S. Ouvry, R. Stora and P. Van-Baal CERN preprint CERN-TH-5224/88 Nov. 1988.
14. R.Myers "Gauge fixing Topological Yang-Mills" Santa-NSF-ITP-89-45

Knots in Physics

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Introduction

In recent years the interest in knot theory [1] increased drastically. This increase is mainly due to the discovery of new polynomial invariants [2] and their connections to various branches of theoretical and mathematical physics. We begin by scetching some basic facts of knot theory needed to understand these connections. Then we give a short review of theories in which knot theoretical structures appear and outline common features. These theories include a.o. quantum inverse scattering theory, conformal field theory and solvable models in statistical mechanics. We end this work suggesting an extension of knot theoretical aspects to higher genus Riemannian surfaces while employing Krichever-Novikov algebras.

1. Knots, Links and Braids

A knot is defined as an embedding of the circle S^1 into Euclidean 3-space $\mathbb{R}^3$ or the 3-sphere S^3. Similarly, a link is an embedding of a disjoint union of circles into $\mathbb{R}^3$ or S^3. If the circles are oriented, the corresponding knot or link is also called oriented. It is very convenient to describe oriented links as diagrams which are projections of links onto a plane in $\mathbb{R}^3$. The main topic of classical knot theory is to classify these diagrams up to equivalence. Two knots are equivalent (ambient isotopy) according to a theorem by Reidemeister if their diagrams can be deformed into each other by a series of three types of moves, the Reidemeister moves.

An essential tool for knot theory is the theory of braids that was founded by Artin in 1925 (cf. [25] for references). The braid group of n threads (denoted by B_n) is generated by n-1 elements σ_i (i = 1,....,n-1) for which the following relations hold

$$\sigma_i \, \sigma_j = \sigma_j \, \sigma_i \qquad |i - j| \geq 2$$

$$\sigma_i \, \sigma_{i+1} \, \sigma_i = \sigma_{i+1} \, \sigma_i \, \sigma_{i+1} \; .$$

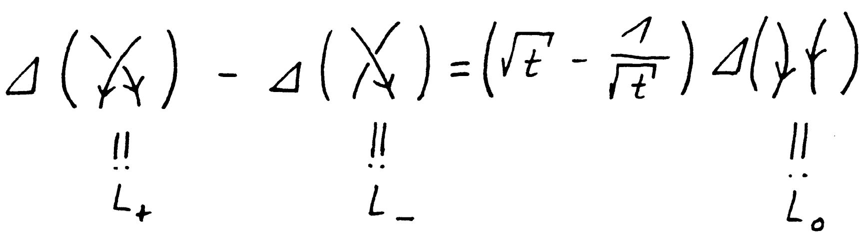
According to a theorem by J.W.Alexander it is possible to represent every knot by the closure of a braid. This representation, however, is far from being unique: many different braids can yield the same knot. A certain theorem by Markov [25] now says that all the braids which represent the same knot can be mutually transformed by so called Markov moves:

$$X\,Y \longrightarrow Y\,X \qquad X,\, Y \in B_n$$

$$X \longrightarrow X\,\sigma_n^{\pm 1} \qquad \sigma_n^{\pm 1} \in B_{n+1}$$

2. Polynomial Invariants

In this section some frequently used polynomial invariants for knots are listed. The first one was given by J.W.Alexander in 1923 [3]. He proved a theorem according to which the polynomials of two ambient isotopic knots are equal up to a multiple of t^n for some integer n. It is given by the trace of the Burau matrix of a corresponding braid. In 1970, J.H.Conway gave a variation of the Alexander polynomial [4] that had a simple recursive definition (called skein relation by Conway). It is in diagrammatic form (for the Alexander polynomial)



These little diagrams are parts of larger diagrams differing only locally as indicated above.

While studying von Neumann algebras V.F.R. Jones found in 1984 a new polynomial which was more sensitive in detecting differences between a knot and its mirror knot (obtained by reversing all the crossings) than the previously known polynomials. The skein relation for the Jones polynomial reads

$$(1/t)\; V(L_+) - t\; V(L_-) = \left(t^{1/2} - t^{-1/2} \right) V(L_0) \; .$$

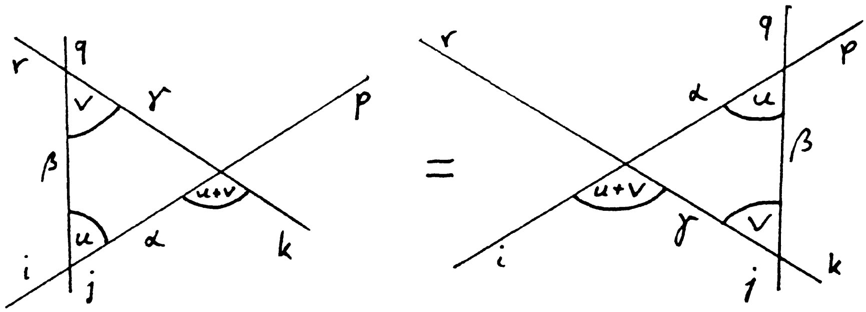
Almost immediately after publication of the Jones polynomial a two variable generalization (commonly denoted as $P(\alpha, z)$) was found by several mathematicians (HOMFLY, Przytycki, Traczyk) [2] who proposed the following skein relation for it:

$$(1/\alpha)\; P(L_+) + \alpha\; P(L_-) = z\; P(L_0) \; .$$

Here $\alpha = \alpha(t)$ and $z = z(t)$ with $t \in \mathbb{C}$. The HOMFLY polynomial contains the Alexander polynomial as well as the Jones polynomial as special cases. In 1985, L.H.Kauffman gave another two variable generalization (denoted $F(\alpha,z)$) of the Jones polynomial which is simultanously an extension of another one variable polynomial $Q(z)$ that had been found shortly before [5]. In the following, we concentrate on the HOMFLY and Kauffman polynomials.

3. Braids in Scattering Theory and Statistical Mechanics

Our first example for the appearance of braid relations in physics is multiple particle scattering theory. In the case of three particles (i, j, k $\longrightarrow$ p, q, r) the three particle S - matrix factorizes into two particle S - matrices. Graphically:



This picture looks similar to the third Reidemeister move [1]. Describing the process (a, b) $\longrightarrow$ (c, d) by a S - matrix $S^{ac}_{bd}(u)$, the 'factorization equation' or 'Yang-Baxter equation' looks as follows:

$$\sum_{\alpha,\beta,\gamma} S^{\beta q}_{\gamma r}(v)\, S^{\alpha p}_{k\gamma}(u+v)\, S^{i\alpha}_{j\beta}(u) = \sum_{\alpha,\beta,\gamma} S^{\alpha p}_{\beta q}(u)\, S^{i\alpha}_{\gamma r}(u+v)\, S^{j\beta}_{k\gamma}(v)$$

The parameter u and v are called 'spectral parameters' and can be physically identified with the rapidities or the scattering angles of the process under consideration. Via a two particle scattering matrix one can define an operator

$$X_i(u) = \sum_{klmp} S^{km}_{lp}(u)\; I^{(1)} \otimes I^{(2)} \otimes \ldots\ldots \otimes E^{(i)}_{pk} \otimes E^{(i+1)}_{ml} \otimes I^{(i+2)} \otimes \ldots. \otimes I^{(n)},$$

where $I^{(j)}$: identity matrices and $(E_{pk})_{ab} = \delta_{pa}\,\delta_{kb}$. Operators defined in this form satisfy the braid relations, i.e. they yield a physical realization of the mathematical braid group generators ! But what is the physical interpretation of these operators $X_i(u)$? A theorem by A.B.Zamolodchikov [6] says that the factorized S - matrices can be viewed as Boltzmann weights of a solvable vertex model in statistical mechanics. The operators $X_i(u)$ are then constituents of the transfer matrix from which, in turn, one calculates the partition function of the model. The same construction is possible for IRF (interaction round a face) models and yields again the braid group relations for the IRF - operators

$X_i(u)$ [7]. Because of the spectral parameter u the operators $X_i(u)$ contain more information than the braid group generators which could be obtained via the identification

$$\lim_{u \to \infty} X_i(u) = \sigma_i \quad .$$

4. Quantum Inverse Scattering Method (QISM) and Link Polynomials

The Yang-Baxter equation also plays an important role also in the quantum theory of inverse scattering. In order to elucidate the intimate connection between these two topics some remarks on the history of the subject are perhaps suited. One root of the origin of the QISM goes back to the solution of the XYZ model (one-dimensional anisotropic Heisenberg chain) by Baxter in 1971 [7a]. At the same time, this is the solution of the two-dimensional eight-vertex model in classical lattice statistical physics. Baxter made use of Onsager's 'star-triangle equation' which can be regarded as the root of the 'Yang-Baxter family' of equations.

The second root of QISM lies in the classical inverse scattering method (CISM) which had been developed since 1967 (for reviews of CISM and QISM with extensive reference lists see [8]) for the solution of non-linear evolution equations like the KdV equation, the non-linear Schrödinger equation (NLS) or the sine-Gordon equation (sG). While studying the quantum version of the sG equation, Faddeev and collaborators recognized that some formulas of the CISM were very similar to formulas that Baxter used in his work. This motivated a development of a quantum version of the CISM which led to a complete solution of the quantum sG equation in 1978. Subsequently the QISM was applied to many other equations and developed to an independent branch of mathematical physics.

In the QISM the YB equation has the following form:

$$R_{12}(u-v)\; R_{13}(u)\; R_{23}(v) = R_{23}(v)\; R_{13}(u)\; R_{12}(u-v)$$

Here the operator R_{ij} is an element of End $(V \otimes V \otimes V)$ of a linear space V (e.g. $\mathbb{C}^2$) given by an operator $\tilde{R} \in$ End $(V \otimes V)$ on the i^{th}, j^{th} factors and the identity on the third, i.e. the canonical embedding of $\tilde{R}$ into $V \otimes V \otimes V$. The key idea of the connection between the QISM and link polynomials is that one can take a R - matrix (solution of YB equation) that comes from a Lie algebra [9] and construct knot polynomials from it (e.g. HOMFLY and Kauffman polynomials) via a representation $\rho\colon \sigma_i \longrightarrow R_i$ of the braid generators [10]. For example, the fundamental vector representation of A_{m-1} gives the following R - matrix:

$$R = -q \sum_{i=1}^{m} E_{ii} \otimes E_{ii} + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} E_{ij} \otimes E_{ji} + (q^{-1} - q) \sum_{\substack{i,j=1 \\ i < j}}^{m} E_{ii} \otimes E_{jj}$$

with $(E_{mn})_{ab} = \delta_{ma}\,\delta_{nb}$.

If one defines a corresponding polynomial $P_m(L)$ of a link L as $P_m(L) \sim tr(\rho(L))$ (where $\rho(L)$ is the braid representation of L in terms of the above mentioned R - matrices replacing braid generators) then it is related to the HOMFLY - Polynomial via the relation

$$P_m(L) = \frac{q^m - q^{-m}}{q - q^{-1}} P(L) \left(\sqrt{-1}\; q^m, \sqrt{-1}\; (q - q^{-1}) \right) .$$

A similar procedure for the algebras B_n, C_n, and D_n leads a.o. to the Kauffman polynomial F(L).

The quantum R - matrix just considered has values in the tensor product U(g) x U(g) where U(g) is the universal enveloping algebra of the finite - dimensional simple Lie algebra g. But this is not yet the most general possibility to get solutions of the Yang-Baxter equation. It is possible to transform U(g) to the 'q - deformed' $U_q(g)$, known as QUEA (quantum universal enveloping algebra) or 'quantum group' [11]. Mathematically, $U_q(g)$ has the structure of a Hopf algebra. For g = sl(2) it has been introduced by Kulish and Reshetikhin in 1981 [12] and in the general case by Jimbo and Drinfeld in 1985 [13].

For example, the commutator relations of $U_q(sl(2))$ are defined by

$$\left[X^{\pm}, H\right] = \mp 2\, X^{\pm} \quad , \quad \left[X^{+}, X^{-}\right] = \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}} .$$

As for the Lie algebra case it is also possible in the quantized case to define a corresponding R - matrix. For the example of $U_q(sl(2))$ the R - matrix is an element of $U_q(sl(2))$ x $U_q(sl(2))$ and defined as

$$R = \exp\left(\frac{h}{4} H \otimes H \right) \sum_{n \geq 0} \frac{(1 - q^{-1})^n}{[n]!} e^n \otimes f^n \; ;$$

where $$q = e^h \; ; \; [n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}$$

$$e = \exp\left(\frac{h}{4} H \right) X^+ \quad ; \quad f = \exp\left(- \frac{h}{4} H \right) X^-$$

Since the so defined R - matrix solves the YB equation it can be used to construct link invariants similar to those discussed in the last section. In their papers [14] Kirillov and Reshetikhin use a.o. q - analogs of the well known Wigner-Racah 6j - symbols to construct polynomials containing a.o. HOMFLY and Kauffman polynomials.

5. CFT and Knot Theory

In recent publications [15] close connections between (rational) conformal field theories (CFT) and quantum groups have been investigated. An example is the possibility to define knot polynomials via CFT or quantum groups. This construction was sketched using quantum groups in section 4. For CFT this relation has been pointed out by Alvarez-Gaume et.al. [16].

Let a rational CFT be given that contains a.o. three primary operators ψ_1, ψ_2, and ψ_3 with conformal dimensions d_1, d_2 and d_3 and fusion rules $\psi_1 \times \psi_1 = \psi_2 + \psi_3$. It is possible to obtain a two-dimensional representation of the Hecke algebra of type A_n, $H_n(q)$, that is characterized through the relations

$$g_i\, g_j = g_j\, g_i \qquad\qquad |i - j| \geq 2$$

$$g_i\, g_{i+1}\, g_i = g_{i+1}\, g_i\, g_{i+1}$$

$$g_i^2 = (q-1)\, g_i + q \quad .$$

(Note that this gives a representation of the braid group B_n because of the first two relations.) It has been shown for example by Ocneanu [2], that a certain trace on this Hecke algebra yields the Jones polynomial. In CFT the g_i are given in terms of a R-matrix that acts on the conformal blocks of the theory. These are (as well as the deformation parameter q) functions of the d_i defined above. Via the skein rule it can be shown that the variables in the HOMFLY polynomial $P(\alpha,z)$ are also determined by the conformal dimensions:

$$\alpha = \alpha(d_1, d_2, d_3) \text{ and } z = q^{1/2} - q^{-1/2} = z(d_2, d_3)$$

In the special case of the minimal models one recovers exactly the Jones polynomial.

A second connection between CFT and quantum groups is provided by the fact that quantum groups are solutions of the polynomial equations of Moore and Seiberg (for details see [17] and references therein). In the classical case the categorial similarities between group theory and CFT lead to the fact that in the special case of SU(2) the fusing matrix F is given by 6j-symbols (Racah coefficients). Correspondingly, in the quantum case $SU(2)_q$ (which is associated to the level k Wess-Zumino-Witten model [17a]) the braiding and fusing matrices are now given by q-analogs of 6j-symbols that were mentioned in the last section.

In close connection to the complex of CFT and quantum groups are results obtained by Witten [18] recovering knot polynomials in terms of Wilson lines (known from QCD) and a special 2+1 - dimensional gauge theory with Chern-Simons term. Lack of space forbids us to digress on this topic (cf. [19] for details).

Two short remarks on this new topic indicate the great impact of Witten's theory : a) Moore and Seiberg conjecture in [20] that all rational CFT's are classified by groups via 2+1 - dimensional Chern-Simons gauge theories,
b) Horowitz shows [21] that Witten's 3-dimensional Chern-Simons gauge theory arises naturally from a <u>four</u> - dimensional topological quantum field theory.

To be complete, we would like to mention that knot and braids also appear in algebraic quantum field theory [22] and in quantum gravity [23].

6. A New Task

While the connection between knot theory and physical theories derived from the Virasoro algebra (the conformal algebra of the $g = 0$ Riemannian surface, the Riemannian sphere) appears fairly well understood, this is not yet achieved for higher genus Riemannian surfaces. A suitable generalization of the Virasoro algebra is the Krichever-Novikov algebra [24] which is the algebra of meromorphic vector fields on a twice punctured Riemann surface of genus g which are holomorphic outside the two punctures.

A natural question about connections between the KN algebra and knot theory arises here. A first hint indicating the existence of a cross relation may be the identification of Artin's braid group as the fundamental group π_1 of the space B_nE^2 , i.e. the space of all unordered n-tuples of distinct points of the Euclidean plane E^2 [25]. Similarly one can associate a braid group to an arbitrary manifold M. In general, there will be more than the two canonical braid group relations. For example, if one replaces E^2 by the two-sphere S^2 there is a third relation

$$\sigma_1 \cdots \sigma_{n-2}\, \sigma_{n-1}^2\, \sigma_{n-2} \cdots \sigma_1 = 1 .$$

Very recently, the KN algebra itself has been generalized from two punctures to n punctures by Dick and Schlichenmaier [26]. So, instead of investigating the original KN algebra it appears natural to include the case of an arbitrary number of punctures, i.e. to look after the connection between the KNDS algebras, mapping class groups and braid groups. Work in this direction is in progress.

References

[1] L.H.Kauffman, On Knots, Princeton University Press, 1987
G.Burde, H.Zieschang, Knots, de Gruyter, 1985

[2] V.F.R. Jones, Bull. AMS 12, 103 (1985)
HOMFLY, Bull. AMS 12, 239 (1985)
Przytycki, Traczyk, Kobe J. Math. 4, 115 (1987)
L.H.Kauffman, An Invariant of Regular Isotopy, preprint, to appear in: Trans. of the AMS

[3] J.W.Alexander, Proc.Nat.Acad. 9, 93 (1923) ; Trans. AMS 30, 275 (1928)

[4] J.H.Conway, An enumeration of knots and links and some of their related properties, in: Computational problems in Abstract Algebra, Proc. Conf. Oxford 1967, Ed. J.Leech

[5] R.D.Brandt, W.B.R.Lickorish, K.C.Millet, Invent Math. 84, 563 (1986)
C.F.Ho, Abstracts of the AMS 6, 300 (1985)

[6] A.B.Zamolodchikov, Comm.Math.Phys. 69, 165 (1979)

[7] R.J.Baxter, Exactly Solved Models in Statistical Mechanics, Academic Press, 1982

[7a] R.J.Baxter, Phys.Rev.Lett. 26, 832 ; 834 (1971); Ann.Phys. 70, 193 (1972)
Ann.Phys. 70, 323 (1972)

[8] L.D.Faddeev, L.A.Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer, 1987
S.Novikov et.al., Theory of Solitons: The Inverse Scattering Method, Consultants Bureau (Plenum Press), 1984
L.D.Faddeev, Integrable Models in 1+1 - Dimensional Quantum Field Theory, in: Les Houches 1982, Elsevier 1984
L.D.Faddeev, Sov.Sci.Rev.Math.Phys. C1, 107 (1980)
E.K.Sklyanin, The Quantum Version of the Inverse Scattering Method, J.Sov.Math. 19, 1546 (1982)
P.P.Kulish, E.K.Sklyanin, Quantum Spectral Transform Method. Recent Developments, in: Springer Lecture Notes in Physics 151

[9] P.P.Kulish, E.K.Sklyanin, Solutions of the Yang-Baxter Equation, J.Sov.Math. 19, 1596 (1982)
P.P.Kulish, N.Yu.Reshetikhin, E.K.Sklyanin, Lett.Math.Phys. 5, 393 (1981)
M.Jimbo, Comm.Math.Phys. 102, 537 (1986)

[10] V.G.Turaev, Invent.Math 92, 527 (1988)

[11] V.G.Drinfeld, Proc.Int.Cong.Math., Berkeley 1986

[12] P.P.Kulish, N.Yu.Reshetikhin, J.Sov.Math. 23, 2435 (1983)

[13] M.Jimbo, Lett.Math.Phys. 10, 63 (1985); 11, 247 (1986)
V.G.Drinfeld, Sov.Math.Dokl. 32, 254 (1985)

[14] N.Yu.Reshetikhin, E-4-87, E-17-87, LOMI preprints, Steklov Inst.
A.N.Kirillov, N.Yu.Reshetikhin, E-9-88, LOMI preprint, Steklov Inst.

[15] G.Moore, N.Reshetikhin, A Comment on Quantum Group Symmetry in Conformal Field Theory, Princeton preprint IASSNS-HEP-89/18 ;
L.Alvarez-Gaume, C.Gomez, G.Sierra, Quantum Group Interpretation of Some Conformal Field Theories, CERN-TH 5267/88

[16] L.Alvarez-Gaume, C.Gomez, G.Sierra, Hidden Quantum Symmetries in Rational Conformal Field Theories, preprint CERN-TH 5129/88

[17] G.Moore, N.Seiberg, Polynomial Equations for Rational Conformal Field Theories, Princeton preprint IASSNS-HEP-88/18 ; Classical and Quantum Conformal Field Theory, Princeton preprint IASSNS-HEP 88/39

[17a] E.Witten, Comm.Math.Phys. 92, 455 (1984)

[18] E.Witten, Quantum Field Theory and the Jones Polynomial, Princeton preprint IASSNS-HEP-88/33 ; Gauge Theories and Integrable Lattice Models, Princeton preprint IASSNS-HEP-89/11

[19] P.Braam, First steps in Jones-Witten theory, preprint, Univ. of Utah

[20] G.Moore, N.Seiberg, Taming the Conformal Zoo, Princeton preprint IASSNS-HEP- 89/6

[21] G.T.Horowitz, Exactly Soluble Diffeomorphism Invariant Theories, preprint NSF-ITP -88-178

[22] K.Fredenhagen, K.H.Rehren, B.Schroer, Superselection Sectors with Braid Group Statistics and Exchange Algebras, I: General Theory, FU Berlin preprint, September 1988 (and references therein);
J.Fröhlich, Statistics of Fields, the Yang-Baxter Equation and the Theory of Knots and Links, preprint, Cargese 1987 Conf.

[23] C.Rovelli, L.Smolin, Phys. Rev. Lett. 61, 1155 (1988)

[24] I.M.Krichever, S.P.Novikov, Funkts.Anal.Prilozhen. 21(2), 46; 21(4), 47 (1987)

[25] J.S.Birman, Braids, Links, and Mapping Class Groups, Annals of Math. Studies 82, Princeton University Press, 1974 ;
Comm. Pure and Appl. Math. 22, 41 ; 213 (1969)

[26] M.Schlichenmaier, Krichever-Novikov Algebras for More Than Two Points, Manusk.Fak.Math.u.Inf. Univ. Mannheim 97-1989 , preprint
R.Dick, Krichever-Novikov-like Bases on Punctured Riemann –Surfaces, DESY preprint 89 - 059, May 1989

SUPERMANIFOLD, SYMPLECTIC STRUCTURE AND GEOMETRIC QUANTIZATION OF BRST SYSTEMS

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ABSTRACT

We introduce a generic, systematical approach to the supergeometric BRST quantization proposed by the authors recently. We investigate the superversions of the symplectic geometry and of the geometric quantization of constrainted system via BRST by means of supermanifold theory with ghost variables. We show the applications to the BRST systems of finite dimensions as well as to the bosonic strings. We also interpret the BRST anomaly in bosonic strings as curvature in certain sense.

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I. INTRODUCTION

Recently, some progresses have been made in the investigation of the symplectic geometry and the geometric quantization of BRST systems. [1-6]. Among others, a generic, systematical approach to this subject has been proposed [6] by means of the theory of supermanifold with pairing (anti) commuting variables of opposite ghost numbers (for short, the U-numbers). In this talk, we will introduce this generic approach and its applications to the BRST systems of finite dimensions and to the bosonic strings. We will also interpret the anomaly of BRST symmetry as curvature in certain sense.

As is well known, for a given BRST system the phase space is always a superspace with pairing opposite ghost number coordinates, even if the original constrainted system is purely bosonic. Therefore, in order to investigate the properties, such as the symplectic structures, of such kind of superphase spaces, we shall first recall some aspects in the theory of supermanifold and make use of them to the case of pairing opposite ghost member variables. We will give the explicit construction of the superversions of the symplectic form and Hamiltonian vector fields on these supermanifolds and show their relation to the BRST symmetry.

As for the geometric quantization[8] , the superversion generalization can be done more or less straightforwardly once the super-symplectic structure is given. We will show the geometric quantization of the BRST systems via prequantization and polarization based upon the super-symplectic structures on the supermanifolds of this kind.

In this talk, we will first introduce briefly the theory of supermanifolds with U-variables and of their symplectic structures. Secondly, we generalize the geometric quantization method to the supermanifold with super-symplectic structure of this kind. Then, we will investigate the applications to the BRST systems of finite dimensions as well as to the bosonic strings[9] . For the case of strings, the BRST invariant superphase space is of infinite dimensions. The application to this case shows that certain infinite dimensional generalization of supermanifold with super-symplectic structure can be done. More rigour description will be given elsewhere. As a byproduct, we will show that the conformal anomaly or the BRST anomaly of the strings can be expressed as the curvature of holomorphic Fock bundle relevant to the BRST invariant symplectic supermanifold rather than the summation of the curvature of the string's Fock bundle and the one of the ghost vacuum bundle. This generalizes the result of the ref[10] . Finally, we will take some brief discussion.

II. SUPERMANIFOLD WITH U-VARIABLES AND SUPERSYMPLECTIC STRUCTURE

There are many nice books and review papers on the theory of supermanifold, as well as interesting talks on this subject at this Conference[11] . For the sake of definiteness and simplicity, we will mainly use some difinitions and results given in [7] and only consider the case of supermanifold with one set of pairing anticommuting variables of U-numbers +1 and −1 corresponding to the ghosts and antighosts respectively. It is straightforward to extend the discussions here to the case of many sets of anticommuting (commuting) variables of various opposite odd (even) U-numbers.

Let $M^{2m;nn}$ be such a supermanifold with commuting $(c-)$ dimensions 2m, anticommuting $(a-)$ dimensions n, n corresponding to ghost and antighost dimensions respectively. Let x^i be the coordinate map from the neighborhood of a point P on M to $R^{2m}(0) \times R^n(-1) \times R^n(1)$. R(U) the real number carrying ghost number U. Let $\mathcal{F}(M)$ be the set of differentiable functions on M. Let $Tp(M)$ be the tangent space of M at P with (local) supervector basis

$$_ie = (\frac{\overrightarrow{\partial}}{\partial x^i})p \quad \text{or} \quad e_i = (\frac{\overleftarrow{\partial}}{\partial x^i})p. \quad _ie = (-)^i e_i. \tag{1}$$

where the index of (-1) is the grassmann parity valued on 0 or 1 for c-variables or a-variables respectively. Every vector (or rank (1.0) tensor) X may be expressed locally by

$$\begin{aligned} X = X^i\ _ie = (-1)^{i(x+i)}{}_ie\ X^i = e_i\ {}^iX. \\ {}^iX = (-1)^{ix}X^i. \end{aligned} \tag{2}$$

Let $T_p^*(M)$ be the cotangent space of M at p dual to $Tp(M)$ with (local) basis

$$e^i =^i e = dx^i. \tag{3}$$

The duality between $_ie$ and e^i reads

$$_ie \times e^j =_i \delta^j. \tag{4}$$

while ${}_i\delta^j = (-1)^i \delta i^j = (-1)^{i+ij} \delta^j i$. Any dual vector (or rank (0.1) tensor) V locally takes form

$$
\begin{aligned}
&V = e^i{}_i V = (-)^{i(i+V)}{}_i V e^i = V_i{}^i e \\
&V_i = (-1)^{i(V+i)}{}_i V.
\end{aligned}
\tag{5}
$$

In general, rank (r, s) tensor and s-form can be defined in a systematic way[7] .

A metric $g = ({}_i g_j)$ on M is a real, c-type, supersymmetric rank (0, 2) tensor. Its inverse $g^{-1} = ({}^i g^j)$ satisfies

$$
{}_i g_j{}^j g^k ={}_i \delta^k = \delta^k i. \qquad {}^i g^j{}_j g_k ={}^i \delta_k = \delta^i_k. \tag{6}
$$

A quadratic form with U=0 may be defined by means of the metric. And its invariance is obvious under the local frame transformation of Osp(2m, n, n).

A super-version of symplectic form on M is a real, c-type, superantisymmetric, nondegenerate and closed two form ω with U=0

$$
d\omega = 0. \tag{7}
$$

Locally,

$$
\omega = \frac{1}{2} dx^i_i \omega_j dx^j \tag{7$'$}
$$

where ${}_i\omega_j$ satisfies

$$
\begin{aligned}
&sdet({}_i\omega_j) \neq 0 \\
&{}_i\omega_j = -(-1)^{i+j+ij}{}_j\omega_i \stackrel{d}{=} -{}_i\,\tilde{\omega}_j. \\
&\omega_{ij,k} + (-)^{i(j+k)}\omega_{jk,i} + (-)^{k(i+j)}\omega_{ki,j} = 0.(\omega_{ij,k} = \omega_{ij}\frac{\overleftarrow{\partial}}{\partial xk})
\end{aligned}
\tag{8}
$$

A vector field X is defined to be Hamiltonian vector field if the Lie superderivative of ω $\omega.r.t$ X satisfies

$$
L_X \omega = 0. \tag{9}
$$

Given $f, g \epsilon \mathcal{F}(M)$, the observables or dynamical variables, there are definite X_f, X_g satisfying the above equation and

$$
\begin{aligned}
& i_{X_f}\omega = df. \qquad U(X_f) = U(f) \\
& L_{X_f} g = X_f g = \{f, g\}_{P.B.} \\
& L X_f X_g = [X_f, X_g] = X_{\{f,g\}_{P.B.}}
\end{aligned} \tag{10}
$$

where [.] is the Lie super-bracket defined by $[X_f, X_g] = X_f X_g - (-)^{fg} X_g X_f$. and $\{.\}_{P.B.}$ the Poisson super-bracket defined by

$$
\{f, g\}_{P.B.} = -\omega(X_f, X_g) \tag{11}
$$

with properties

$$
\begin{aligned}
& \{f, g\}_{P.B.} = -(-)^{fg}\{g, f\}_{P.B.} \\
& \{f, \{g, h\}_{PB}\}_{P.B.} + (-1)^{f(g+h)}\{g, \{h, f\}_{PB}\}_{P.B.} + (-)^{h(f+g)}\{h\{f, g\}_{PB}\}_{P.B.} = 0
\end{aligned} \tag{12}
$$

It is easy to see that for all dynamical variables on M there is a Lie super-algebra in the sense of Poisson super-bracket.

The canonical transformations that leave ω being invariant are generated by the Hamiltonian vector fields X_f. Locally

$$
\begin{aligned}
X_f &= f \frac{\overleftarrow{\partial}}{\partial x^i} {}^i\omega^j \frac{\overrightarrow{\partial}}{\partial xj} \\
\{f, g\}_{P.B.} &= f \frac{\overleftarrow{\partial}}{\partial xi} {}^i\omega^j \frac{\overrightarrow{\partial}}{\partial xj} g = f, i^i\omega^j j, g
\end{aligned} \tag{13}
$$

where $({}^i\omega^j)$ is the inverse of $({}_i\omega_j)$.

As ω is closed locally it can always be expressed as

$$
\omega = d\theta \tag{14}
$$

where θ is called the super-canonical one form or super-symplectic potential. Given a Hamiltonian vector field X, a covariant derivative along X $\omega.r.t$ θ can be defined as

$$\nabla_X = X + \sqrt{-1}\theta(X) \tag{15}$$

and then ω can be expressed as the supercurvature

$$\nabla_X\nabla_Y - (-)^{XY}\nabla_Y\nabla_X - \nabla_{[X,Y]} = -\sqrt{-1}w(X,Y). \tag{16}$$

III. GEOMETRIC QUANTIZATION

It is worthwhile pointing out that once the super-symplectic structure is known the geometric quantization of given system with supermanifold being super-phase space is more or less a straightforward extension of the usual geometric quantization in purely bosonic case. It is obvious that instead of starting with the usual symplectic structure the super-symplectic structure of the super-phase space should be taken as the starting point and then things may be done in parallel with the purely bosonic case. Therefore, we only recall the key points of the usual geometric quantization and keep their superversion extension in our mind.

As is well known, the basic idea of geometric quantization is to find geometrically the corresponding quantum operators for given observables and their operator algebra isomorphic to the one of classical observables in the sense of the Poisson bracket as well as the Hilbert space $\mathcal{H}_F$ being the operators' representation space which should not be well-defined in arbitrary small support of the phase space in order to satisfy the uncertainty principle. Therefore, the geometric quantization approach may be divided into two steps, i.e., the prequantization and the polarization, we will give the basic formulas later. As to quantize a constrainted system, further consideration should be taken such as the reduction of the phase space by the constraints and so on. But, in the BRST approach, instead of the reduction of the phase space we enlarge it by introducing ghosts and anti-ghosts and as a consequence, the constraints are relevant to the BRST cohomology.

Let us first briefly recall the construction of the prequantization line bundle L of usual symplectic manifold (M, ω). The symplectic potential θ should be taken to be its connection 1-form and the symplectic 2-form ω to be the section curvature 2-form. Let ϕ_f^t denote the one parameter subgroup generated by Hamiltonian vector field X_f and ϕ_f^t acts on the section space $\mathcal{H}_p$. The prequantization operator is then defined by

$$0_f\lambda = \sqrt{-1}\frac{d}{dt}(\phi_f^t\lambda)|_{t=0} = (\sqrt{-1}\nabla x_f + f)\lambda, \quad \lambda\epsilon\mathcal{H}p \tag{17}$$

It is easy to verify that $\{0_f\}$ satisfies the operator algebra

$$[0_f, 0_g] = \sqrt{-1}0_{\{f,g\}}, \qquad 0_{f=1} = 1. \tag{18}$$

However, the invariant inner product on $\mathcal{H}p$ against the canonical transformation is well-defined in arbitrary small support so that $\mathcal{H}p$ does not satisfy the uncertainty principle.[8]

In order to complete the geometric quantization, it is necessry to reduce $\mathcal{H}p$ so that the uncertainty principle can be hold. This is the role played by the polarizition. A polarization F is a subspace of $T(M)$ satisfying

$$[F,F] \subset F, \qquad \dim F = \frac{1}{2}\dim M = m.$$
$$\omega(X,Y) = 0, \forall X, Y\epsilon F.$$

Then a product bundle may be constructed by the prequantization bundle and the det bundle of double covering group of $GL(m,R)$ and the quantum Hilbert space $\mathcal{H}_F$ is defined by the subspace of the section space of the product bundle consisting of all covariant constant sections along F.

For given $f\epsilon\mathcal{F}(M)$ and the basis of $F, ({}_ne)$, if

$$[X_f,\ {}_ne] = \sum_m a_n^m\ {}_me. \tag{19}$$

0_f is said that it preserves the polarization F and the quantum operator of f is defined by

$$\hat{f} = 0_f - \frac{1}{2}\sqrt{-1}\sum_m a_m^m. \tag{20}$$

If 0_f does not preserve F but, say, F', then

$$0_f^F = U_{FF'}Q_f^{F'}U_{F'F} \tag{21}$$

where $U_{FF'}$ is a linear map from $\mathcal{H}_{\mathcal{F}'}$ to $\mathcal{H}_F$. In more general situation, the quantum operator of f may be obtained by the BKS kernel approach[8] . We will not be concerned with a situation of this kind here.

As was mentioned at the beginning of this section, for the superversion of the geometric quantization, what should be done are taking all notations and formulas to be the corresponding superextensional ones.

IV. BRST SYSTEMS OF FINITE DIMENSIONS

Let us consider the BRST phase space and geometric quantization of systems with finite dimensional first class constraints, i.e., the BRST systems of finite dimensions. The constraints $K_a(p,q)$ satisfy

$$\begin{aligned} &K_a(p,q) = 0, \qquad U(K_a) = 0, \qquad a = 1,\dots,n \\ &\{K_a, K_b\}_{P.B.} = f^c_{ab} K_c \end{aligned} \tag{22}$$

where (q^r, p_s), $r, s = 1, ..., m = d$, are canonical coordinates, and momentums of the system. By introducing a pair of a-type coordinates, C^a and $\bar{C}_b$, with U number 1 and -1,

$$(C^a)^2 = (\bar{C}_b)^2 = 0, \qquad C^a \bar{C}_b = -\bar{C}_b C^a$$

we can construct a BRST invariant superphase space.

It is easy to prove that, at this time, ${}_i\omega_j$ be constant. The symplectic form and Hamiltonian vector fields take the forms

$$\omega = dp_r \wedge dq^r + d\bar{C}_a \wedge dC^a \tag{23a}$$

$$X_f = f\frac{\overleftarrow{\partial}}{\partial qr}\frac{\overrightarrow{\partial}}{\partial p_r} - f\frac{\overleftarrow{\partial}}{\partial p_r}\frac{\overrightarrow{\partial}}{\partial q^r} + f\frac{\overleftarrow{\partial}}{\partial C^a}\frac{\overrightarrow{\partial}}{\partial \bar{C}_a} + f\frac{\overleftarrow{\partial}}{\partial \bar{C}_a}\frac{\overrightarrow{\partial}}{\partial C^a}, \forall f \epsilon \mathcal{F}(M) \tag{23b}$$

It is not difficult to verify that they satisfy all the required properties. By use of the results discussed above we get the Poisson brackets of coordinates

$$\{q^r, p_s\}_{P.B.} = \delta^r_s, \{\bar{C}_a, C^b\}_{P.B.} = \delta^b_a, \{others\}_{P.B.} = 0 \tag{24}$$

And if we take the constraints K_a and BRST charge Q,

$$Q = C^a K_a + \frac{1}{2} f^c_{ab} \bar{C}_c C^a C^b, \quad U(Q) = 1 \tag{25}$$

to be dynamical variables, some algebraic relations are in hand immediately

$$
\begin{aligned}
&\{Q, \bar{C}_a\}_{P.B.} \overset{def}{=} \tilde{K}_a = K_a + f^c_{ab}\bar{C}cC^b \\
&\{\tilde{K}_a, \tilde{K}_b\}_{P.B.} = f^c_{ab}\tilde{K}_c, \{\tilde{K}_a, \bar{C}_b\}_{P.B.} = f^c_{ab}\bar{C}_c \\
&\{Q, Q\}_{P.B.} = 0, \quad \{Q, \tilde{K}_a\}_{.P.B.} = 0
\end{aligned}
\tag{26}
$$

and

$$
\{Q, C^a\}_{P.B.} + \frac{1}{2}f^a_{bc}C^bC^c = 0 \tag{27}
$$

Obviously, these relations are just the BRST transformations in the sense of Poisson bracket. In addition, from the definition of BRST charge's Hamiltonian vector field X_Q one directly has

$$
\mathcal{L}_{X_Q}\omega = 0 \tag{28}
$$

which explains that the symplectic form (23a) and the corresponding phase space are BRST invariant.

According to the definition of prequantization operator (17), it is easy to prove that all the algebraic relations under Poisson bracket and formulas are still valid in the sense of prequantization operators and of their supercommutators. However, we must make a deeper investigation into the polarization.

Let us consider the Schrödinger polarization. Its polarization vector set is

$$
\{X_qr, X_ca\} = \{\frac{\overrightarrow{\partial}}{\partial pr}, \frac{\overrightarrow{\partial}}{\partial \bar{C}a}\} \tag{29}
$$

One may prove that if constraints satisfy

$$
\left.\frac{\partial^2 Ka}{\partial pr \partial ps}\right|_F = 0 \tag{30}
$$

then all the prequantization operators concerned here preserve this polarization. In such a case we get the quantum operator $\hat{0}^F_f = \hat{f}$ by using formula (20)

$$
\begin{aligned}
&\hat{q}^r = q^r, \hat{p}r = \sqrt{-1}\frac{\overrightarrow{\partial}}{\partial q^r}, \hat{C}^a = C^a, \hat{\bar{C}}^a = \sqrt{-1}\frac{\overrightarrow{\partial}}{\partial C^a}, \\
&\hat{K}_a = \frac{\partial Ka}{\partial pr}\hat{p}_r - \frac{\partial Ka}{\partial pr}pr + K_a, \hat{\tilde{K}}_a = \hat{K}_a + f^b_a cC^c\hat{\bar{C}}_b, \\
&\hat{Q} = C^a\hat{K}_a + \frac{1}{2}f^a_{bc}C^c\hat{\bar{C}}_a.
\end{aligned}
\tag{31}
$$

They satisfy the quantum operator algebra that is isomorphic to the classical Poisson algebra, i.e.

$$\begin{aligned}
&[\hat{q}^r, \hat{p}_s] = \sqrt{-1}\delta^r s, [\hat{C}^a . \hat{\bar{C}}_b] = \sqrt{-1}\delta^a_b, [others] = 0\\
&[\hat{Q}, \hat{\bar{C}}_a] = \hat{\bar{K}}_a\\
&[\hat{\bar{K}}_a, \hat{\bar{K}}_b] = f^c_{ab}\hat{\bar{K}}_c, [\hat{K}_a, \hat{\bar{C}}_b] = f^c_{ab}\hat{\bar{C}}_c\\
&[\hat{Q}, \hat{Q}] = 0 \quad , \quad [\hat{Q}, \hat{\bar{K}}_a] = 0\\
&[\hat{Q}, \hat{C}_a] + \frac{1}{2} f^A_{bc}\hat{C}^b\hat{C}^c = 0.
\end{aligned} \tag{32}$$

This shows that if the constraints satisfy condition (30), then the BRST symmetry has no anomaly.

A simple example is a relativistic point particle moving along a curved line. It is of the first class constraint

$$K = -(p^2 + m^2) = 0$$

and satisfies

$$\frac{\partial^2 K}{\partial q \partial q} = 0$$

Because the Schrödinger representation and momentum representation are only different up to a unitary Fourier transformation, we immediately find that this system has no BRST anomaly.

On the other hand, for BRST invariant quantum system, the conditions that the physical states should satisfy have been changed into the BRST cohomology condition, i.e.,

$$\mathcal{H}_F = \{| \text{ phys } >\} = \{\Psi | \hat{Q}\Psi = 0\}.$$

So far we have discussed the geometric quantization of BRST system with finite dimensional first class constraints. It is clear that the symplectic structure of the superphase space gives the BRST transformation manifestly geometric meanings and plays key roles in the geometric quantization.

V. BOSONIC STRINGS AND BRST ANOMALY AS CURVATURE

Now let us discuss another kind of BRST constrained system, the open bosonic strings. Their constraints are first class but do not satisfy the condition (30), which results in anomalies of the BRST symmetry.

After taking Fourier expansions[9] , the constraints of string take the form

$$L_n = -\frac{1}{2}\sum_{m=-\infty}^{+\infty} \alpha_{n-m}\alpha_m, \quad \alpha_{-n} = \alpha_n^* \tag{33}$$

and satisfy

$$\{L_n, L_m\}_{P.B.} = (n-m)L_{n+m}, \quad n, m \epsilon Z \tag{34}$$

According to BRST procedure, we may introduce an infinite dimensional supermanifold $M = \{(p^o, q^o, \alpha^{m*}, \alpha^m, C^n, \bar{C}^n)|n, m\epsilon Z,\ m > 0\}$, where the coordinates $p^o, q^o, \alpha^{m*}, \alpha^m$ correspond to the expansion modes of string coordinates and $C^n, \bar{C}^n$ to those of ghost and antighost ones. The symplectic form of M is

$$\begin{aligned}
&\omega = dp_o \wedge dq^o + i\sum_{m=1}^{\infty}\frac{1}{m}d\alpha^m \wedge d\alpha_m^* + \sum_{n=-\infty}^{+\infty} d\bar{C}^n \wedge dC_{-n} \\
&\stackrel{d}{=} dx_i^i\omega_j dx^j \\
&X_f \stackrel{d}{=} f\frac{\overleftarrow{\partial}}{\partial x^i}{}^i\omega^j\frac{\overrightarrow{\partial}}{\partial xj}, f\epsilon\mathcal{F}(M)
\end{aligned} \tag{35}$$

The classical BRST charge may be expressed as

$$\begin{aligned}
Q &= \sum_{n=\infty}^{+\infty} C^{-n}L_n + \frac{1}{2}\sum_{n.m=-\infty}^{+\infty}(n-m)\bar{C}_{n+m}C^{-n}C^{-m} \\
&= \sum_{n=-\infty}^{+\infty} C^{-n}L_n + \frac{1}{2}\sum_{n=-\infty}^{+\infty} C^{-n}L_n^{gh}
\end{aligned} \tag{36}$$

which satisfies the nilpotent condition

$$\{Q, Q\}_{P.B.} = 0. \tag{37}$$

It is interesting that all the discussions above about the symplectic structure and the prequantization as well as the polarization can be extended to such an infinite dimensional system. Due to the limit of space, we will concentrate on the effect of polarization. Let the bases of two polarization F_1 and F_2 be $\{\frac{\partial}{\partial q^o}, \frac{1}{\sqrt{m}}\frac{\partial}{\partial \alpha_m^*}, \frac{\partial}{\partial \bar{C}_n}\}$ and $\{\frac{\partial}{\partial q^o}, \frac{1}{\sqrt{m}}\frac{\partial}{\partial \alpha^m}, \frac{\partial}{\partial \bar{C}_n}\}$. We have to deal with two problems. The first, as discussed in [1], the constraints $L_n, n < 0 (n > 0)$ preserves $F_1(F_2)$ but do not preserve $F_2(F_1)$ and L_0 preserves both polarizations. Therefore, if we take F_1, then $\hat{0}^{F_1}_{L_N}(n \leq 0)$ may be obtained directly from formula (20). But to get the expression of $\hat{0}^{F_1}_{L_n}, n > 0$, we have to use both (20) and (21) for a representation transformation. The second problem is about the definition of the vacuum of infinite dimensional ghost oscillators. Since the ghost vacuum is a Dirac sea, its level is of arbitrariness. We define an N level vacuum

$$|gh >_N = C^{-N} C^{-N+1\ldots} \quad , \quad -\infty < N < +\infty \quad , \tag{38}$$

and the creation and annihilation operators

$$\begin{aligned} \hat{0}^{F_1}_{C^m}|gh >_N &= C^m |gh >_N = 0 \quad , \quad m \leq -N \quad , \\ \hat{0}^{F_1}_{\bar{C}^m}|gh >_N &= \frac{\delta}{\delta C^{-m}}|gh >_N = 0 \quad , \quad m \geq N+1. \end{aligned} \tag{39}$$

Having noted two problems above, we may get the quantum operators of $L_n, L_n^{gh}, \tilde{L}_n = L_n + L_n^{gh}$ and BRST charge Q. As presented in [1]

$$\begin{aligned} \hat{0}^{F_1}_{L_0} &= -\frac{1}{2}p_0^2 - \sum_{l>0} l\alpha_l^* \frac{\partial}{\partial \alpha_l^*} + \beta \\ \hat{0}^{F_1}_{L_n} &= -np_0 \frac{\partial}{\partial \alpha_n^*} - \frac{1}{2}\sum_{m=1}^{n-1} m(n-m)\frac{\partial}{\partial \alpha_m^*}\frac{\partial}{\partial \alpha_{n-m}^*} - \sum_{m>n}(m-n)\alpha_m^* \frac{\partial}{\partial \alpha_{m-n}^*} \\ \hat{0}^{F_1}_{L-n} &= p_0\alpha_n^* - \frac{1}{2}\sum_{m=1}^{n-1}\alpha_n^*\alpha_{n-m}^* - \sum_{m>n} m\alpha_{m-n}^* \frac{\partial}{\partial \alpha_m^*}, \quad (n > 0) \end{aligned} \tag{40}$$

As $\hat{0}^{gh}_{L_N}$ preserves both F_1 and F_2, we find

$$\begin{aligned} \hat{0}^{F_1}_{L_n} gh = \hat{0}^{F_2}_{L_n} gh &= \sum_{m=-\infty}^{+\infty}(n-m) : C^m \frac{\partial}{\partial C^{-n-m}} : \\ \hat{0}^{F_i}_{\tilde{L}_n} &= \hat{0}^{F_1}_{L_n} + \hat{0}^{F_1}_{L_n} gh \quad , \quad i = 1,2 \end{aligned} \tag{41}$$

Here : : represents the normal product with respect to N-level vacuum.

Simple calculation shows anomalies of constraint algebra (the Virasoro algebra)

$$
\begin{aligned}
[\hat{0}^{F_1}_{\tilde{L}_n}, \hat{0}^{F_1}_{\tilde{L}_m}] = & \\
& (n-m)\hat{0}^{F_1}_{\tilde{L}_{n+m}} + (-1)^{i+1}\frac{d}{12}(n^3 - \beta n)\delta m + n, 0 \\
& - \frac{1}{6}(13n^3 - (6N^2 + 6N + 1)n)\delta m + n, 0
\end{aligned} \tag{42}
$$

Hence the anomaly cancellation condition of $\hat{0}^{F_1}_{\tilde{L}_n}$ algebra is $d = (-1)^{i+1}26, \beta = (-1)^{i+1}(6N^2 + 6N + 1)/13$. In order to guarantee the vacuum of string to be an $SL(2, C)$ invariant, we have $\beta = (-1)^{i+1}, N = 1$. So we find that F_1, but not F_2, is a physically admissible polarization. In fact, the representation transformation between F_1 and F_2 is not unitary. Their corresponding Hilbert spaces are not equivalent[1] .

Now we calculate the representation of quantum BRST operator at polarization F_1. Q may be rewritten as

$$
\begin{aligned}
& Q = Q_+ + Q_- + Q^{gh}, \\
& Q_+ = \sum_{n=1}^{\infty} C^{-n} L_n \quad , \quad Q_- = \sum_{n=0}^{\infty} C^n L_{-n}, \\
& Q^{gh} = \frac{1}{2} \sum_{m=-\infty}^{+\infty} C^{-m} L^{gh}_m.
\end{aligned} \tag{43}
$$

Both $\hat{0}_{Q-}, \hat{0}_{Q^{gh}}$ preserve F_1 hence (20) may be used directly to get quantum operator. $\hat{0}_{Q+}$ preserves F_2 but not F_1, therefore we have to calculate $\hat{0}^{F_2}_{Q+}$ first then take a representation transformation. Calculation shows

$$
\begin{aligned}
\hat{0}^{F_1}_Q & = \hat{0}^{F_1}_{Q+} + \hat{0}^{F_1}_{Q-} + \hat{0}^{F_1}_{Q^{gh}} \\
& = \sum_{n=1}^{\infty} \hat{0}^{F_1}_{C^{-n}} \hat{0}^{F_1}_{L_N} + \sum_{n=0}^{\infty} \hat{0}^{F_1}_{C^n} \hat{0}^{F_1}_{L-n} + \frac{1}{2} \sum_{m=-\infty}^{+\infty} : \hat{0}^{F_1}_{C^-m} \hat{0}^{F_1}_{L^{gh}_m} :
\end{aligned} \tag{44}
$$

and

$$
[\hat{0}^{F_1}_Q, \hat{0}^{F_1}_Q] = (\frac{d-26}{12}) \sum_{n=1} n^3 \hat{0}^{F_1}_{C^{-n}} \hat{0}^{F_1}_{C^n} - (\frac{d-26}{12} - \beta + 1) \tag{45}
$$

when $d = 26$, and $\beta = 1$, the quantum operator $\hat{0}^{F_1}_a$ is nilpotent and the anomaly in (42) vanishes at $N = 1$.

Now we discuss the geometric meanings of the BRST-anomalies in (42) and (45). For polarization F_1, the space of physical states is

$$\mathcal{H}_{F^1} = \{\Psi | \hat{0}_Q^{F_1} \Psi = 0\} \tag{46a}$$

where

$$\Psi = \Psi(P_o, Z_m)_{i_1 \dots i_n} C^{i_1} \dots C^{i_n} |gh >_1, i_j = -2, -3, \dots \tag{46b}$$

Making a coordinate transformation

$$p_o \to Z_o, Z_o = p_o + \sqrt{-1} q_o, Z_o^* = p_o - \sqrt{-1} q_o. \tag{47}$$

we have

$$F_1 \to F^+ = \{(\frac{\partial}{\partial Z_n^*}, \frac{\partial}{\partial \bar{C}_m}) | n, m \epsilon Z, n \geq 0\}. \tag{48a}$$

and

$$\mathcal{H}_{F_1} \to \mathcal{H}_{F^+} = \{\Psi^+ | \hat{0}_a^{F^+} \Psi^+ = 0\}. \tag{48b}$$

Here

$$\begin{aligned} Z_n^* &= \frac{1}{\sqrt{n}} \alpha_n \quad , \quad Z_n = \frac{1}{\sqrt{n}} \alpha_n^* \\ \Psi^+ &= \Psi(p_o \to Z_o). \end{aligned} \tag{49}$$

The operators now take the form

$$\hat{0}_f^{F^+} = \hat{0}_f^{F_1}(p_o \to \frac{\partial}{\partial Z_o}) \tag{50}$$

It is easy to prove that such a coordinate transformation induces a unitary representation transformation[1] . Therefore, $\mathcal{H}_{F_1}$ and $\mathcal{H}_{F^+}$ describe the same quantum mechanical system and result in equivalent anomalies. Since

$$\frac{\partial}{\partial Z_m^*} \Psi^+ = 0, \quad \frac{\partial}{\partial \bar{C}_n} \Psi^+ = 0, \quad \forall n, m \epsilon Z, m \geq 0 \tag{51}$$

and

$$C^m|gh>_1=0, \quad m\le -1, \quad \frac{\partial}{\partial C^{-m}}|gh>_1=0, \quad m\ge 2$$

$\mathcal{H}_{F+}$ may be viewed as the fiber of a level-1 subbundle of the super holomorphic Fock bundle defined on manifold $G_o = G/H$

$$\begin{array}{ccc} \mathcal{H}_{F+} & \rightarrow & Z_1 \\ & & \downarrow \\ & & G. \end{array}$$

where G is the diffeomorphism group of M and H the subgroup of G generated by L_o. Therefore, similar to ref. [10], $\hat{0}^{F^+}_{\tilde{L}_n}$ may be considered the Toplitz operator $\rho(\tilde{L}_n)$ defined on bundle Z_1

$$\nabla_{X_{\tilde{L}_n}} = \mathcal{L}_{X_{\tilde{L}_n}} + \rho(\tilde{L}_n) \tag{52}$$

The curvature of this bundle is

$$\begin{aligned} F(X_{\tilde{L}_n}, X_{\tilde{L}_m}) &= [\nabla_{X_{\tilde{L}_n}}, \nabla_{X_{\tilde{L}_m}}] - \nabla_{[X_{\tilde{L}_n}, X_{\tilde{L}_m}]} \\ &= [\hat{0}^{F^+}_{\tilde{L}_m}, \hat{0}^{F^+}_{\tilde{L}_n}] - \hat{0}^{F^+}_{\{\tilde{L}m, \tilde{L}n\}P.B.} \\ &= (\frac{d}{12}(n^3-\beta n) - \frac{26}{12}(n^3-n))\delta m+n,o \end{aligned} \tag{53}$$

Hence the BRST anomaly can be interpreted as the curvature of Z_1, bundle and its cancellation condition is that the Z_1 bundle has vanishing curvature.

On the other hand, as was discussed in ref.[1], the BRST charge may be viewed as an exterior differential operator on G. The nilpotence of $\hat{Q}$ means that $\hat{Q}$ can be considered as an exterior differential operator on $\hat{G}$ only at critical dimension, here $\hat{G}$ means that G takes the space $\bar{\mathcal{H}}^{F^+} = \{\bar{\Psi}^+ = \bar{\Psi}^+(Z_o)|gh>_1 |\hat{0}^{F^+}_Q\Psi^+ = 0\} \subset \mathcal{H}^{F^+}$ as the representation space. The violation of the nilpotence is due to that $\bar{\mathcal{H}}^{F^+}$ is not a unitary representation of G but a projective representation. Therefore the equivalence of $\hat{Q}$ to an exterior differential operator holds only under certain conditions.

VI. DISCUSSION

We have discussed the symplectic geometry and geometric quantization on supermanifold. We have found that the symplectic structure and geometric quantization of BRST system can be systematically obtained from the symplectic geometry and geometric quantization on supermanifold with U-variables. As we dealt with the problems in a BRST invariant supermanifold formalism, the geometric meanings of BRST anomaly, such as open bosonic string's, is much more manifest. Similarly, the closed bosonic strings and spinning strings can also be studied in the same way.

However, for chiral anomaly, in gauge theories, we have not explored its geometric meaning in the BRST geometric quantization formalism yet. It is of course a subject of interest. We will leave it for the future investigation.

Moreover, we have only investigated the case $M \overset{loc.}{=} R^n(-1) \times R^{2m}(0) \times R^n(+1)$. In fact, one may apply the similar methods to more complicated supermanifold, for instance, $M \overset{loc.}{=} X_{U=-2}^{U=2} R^{l(U)}(U)$, which corresponds to the case of $TQFT$[12] and antisymmetric rank-2 tensor gauge theory[13] etc. This is also a subject of interest to be investigated as well.

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REFERENCES

[1] Y. Yu and H.Y. Guo, Phys. Lett. 216B(1989)68.

[2] Y. Yu, Phys. Lett. 216B(1989)75.

[3] A. Niemi, CERN preprint (1988).

[4] W. Chen, W.L. Shen and Y. Yu, Phys. Rev. D39(1989)1210.

[5] Y. Yu and S.M. Fei, BIHEP-TH-88-40, to appear in Z.Phys.C.

[6] S.M. Fei, H.Y. Guo and Y. Yu, BIHEP-TH-89-15, AS-ITP-89-08, ibid.

[7] B. DeWitt, Supermanifold (Cambridge University, Cambridge, London New York New Rochelle 1984).

[8] J. Sniatychi, Geometric Quantization and Quantum Mechanics (Springer-Verlag, New York, 1980). N. Woodhouse, Geometric Quantization (Clarendon Press, 1980).

[9] M. Kato and K. Ogawa, Nucl. Phys. B212(1983)443.

[10] M. Bowick and S.G. Rajeev, Nucl. Phys. B293(1987)348 and references therein.

[11] See, for example, papers by D.A. Leites and by R.O. Wells in this Proceedings.

[12] E. Witten, Commun. Math. Phys. 117(1988)353; 118(1988)411; Phys. Lett. 206B(1988)601.

[13] T. Kumura, Prog. Theor. Phys. 64(1980)625; 65(1981)1.

Ambitwistors and Conformal Gravity

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In a landmark paper [13] of the 1970's, Roger Penrose described a remarkable and unexpected connection between Einstein's equations and the theory of complex manifolds. The **twistor correspondence** which he detailed there gives a way of producing all *self-dual* complex solutions of these equations in terms of the global structure of an associated complex manifold called the **twistor space.** Unfortunately, the most interesting solutions from the view-point of physics are not the self-dual ones, but rather those of Lorentz signature. If this had been the end of the story, one might have therefore been tempted to conclude that this correspondence was merely a mathematical curiosity with no bearing on physics. Fortunately, however, the Penrose twistor correspondence is but one aspect of a rather more complicated story, which I will endeavor to recount here. Indeed, it turns out that the general complex solution of the 4-dimensional Einstein equations can also be described in terms of complex deformation theory, albeit in terms of non-reduced complex spaces rather than complex manifolds.

One of the central insights contributed by Penrose was the importance of conformal geometry in the study of Einstein's equations. In fact, the aim of the present article is to describe a twistor correspondence, first discovered at the perturbative level by Robert Baston and Lionel Mason [2], for the following conformally invariant field equations:

$$(\nabla^c\nabla^d + \frac{1}{2}R^{cd})C_{acbd} = 0,$$

$$C^f{}_{ace}\nabla^d(*C)^e{}_{bfd} = (*C)^f{}_{ace}\nabla^d C^e{}_{bfd}.$$

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When the conformal curvature is generic at some point, these conformally invariant equations imply that the metric is conformal to Einstein (in a manner completely determined up to a multiplicative constant); conversely, the Bianchi identities imply that every Einstein metric is a solution of these equations. While full details of the proof can be found in [12], this exposition will include the essence of the argument, and is intended to be self-contained.

It is worth remarking that there seem to be deep connections between the present results and super-symmetry, although I hasten to add that much remains to be elucidated in this regard. In fact, the present correspondence was to a large extent motivated by a theorem of Isenberg *et. al.* [7] concerning Yang-Mills fields in flat space-time; Witten [15] gave an independent argument for this based on super-symmetry. It seems relatively certain that the present results on gravity are similarly related to the perturbative supergravity calculations of Chau and Lin [3], and may perhaps also be related to the ten-dimensional results of Witten [16] by a process of dimensional reduction.

Stylistic Remarks:

- *We will work only with complex space-times, but one may easily reduce to the case of real-analytic space-times of arbitrary signature by introducing an anti-holomorphic involution representing complex conjugation.*
- *The holomorphic tangent bundle of a complex manifold X will simply be denoted by TX, rather than by the more precise $T^{1,0}X$.*
- *If $E \to X$ is a holomorphic vector bundle, $0_E \subset E$ denotes the zero section and $\mathbf{P}E = (E - 0_E)/(\mathbf{C} - 0)$ denotes the associated holomorphic bundle of projective spaces.*

1 Ambitwistors

Suppose that $\mathcal{M}$ is a complex 4-manifold, and let $\mathrm{g} \in \Gamma(\mathcal{M}, \mathcal{O}(\odot^2 T^*\mathcal{M})$ be a holomorphic non-degenerate symmetric 2-tensor on $\mathcal{M}$. We will say that g is a *complex-Riemannian metric* on $\mathcal{M}$, and $(\mathcal{M}, \mathrm{g})$ will be called a *complex space-time*. We can then associate to to $(\mathcal{M}, \mathrm{g})$ a family of curves called *null geodesics* by considering those inextendible connected one-dimensional complex submanifolds $\gamma \subset \mathcal{M}$ for which any non-zero tangent vector field $v \in \Gamma(\gamma, \mathcal{O}(T\gamma))$ satisfies

$$\begin{aligned} \nabla_v v &\propto v \\ \mathrm{g}(v, v) &\equiv 0, \end{aligned}$$

where ∇ denotes the Levi-Civita connection associated with $\mathbf{g}$. Knowing these curves determines the *conformal class* of the complex metric $\mathbf{g}$, since a vector is null iff it is tangent to some null geodesic γ; what is less apparent is that the conformal class conversely determines the set of null geodesics.

To understand this latter fact, let us notice that the projectivized cotangent bundle of $\mathcal{M}$ carries a natural contact structure, by which we mean a line-bundle-valued 1-form $\Theta \in \Gamma(\mathbf{P}T^*\mathcal{M}, \Omega^1(\mathbf{L}))$, such that

$$\Theta \wedge (d\Theta)^{\wedge 3} \neq 0;$$

indeed, Θ is the usual canonical 1-form $p_j dq^j$ defined by

$$\Theta|_{[\phi]} = \pi^*(\phi),$$

where $\pi : \mathbf{P}T^*\mathcal{M} \to \mathcal{M}$ is the canonical projection. Knowing a conformal class on $\mathcal{M}$ amounts to knowing the hypersurface

$$\mathcal{Q} = \{[\phi] \in \mathbf{P}T^*\mathcal{M} | \mathbf{g}^{-1}(\phi, \phi) = 0\}$$

of null covectors. Now the restriction

$$\vartheta = \Theta|_{\mathcal{Q}}$$

is not a contact structure; rather, $\ell := \mathbf{ker}(\vartheta \wedge d\vartheta)$ defines a rank 1 sub-bundle $\ell \subset T\mathcal{Q}$, and this distribution ℓ is then tangent to a holomorphic foliation of $\mathcal{M}$ by complex curves $\hat{\gamma} \subset \mathcal{Q}$. Each leaf of this foliation then projects to a holomorphic curve $\gamma = \pi[\hat{\gamma}]$, and one then checks that each such curve in $\mathcal{M}$ is a null geodesic; indeed, the leaf through $[\phi_a] \in \mathbf{P}T^*\mathcal{M}$ projects to the null geodesic tangent to ϕ^a. This gives a manifestly conformally invariant construction of the null geodesics.

Let $\mathcal{N}$ denote the set of null geodesics of ($\mathcal{M}$,$\mathbf{g}$). Since this is the same as defining $\mathcal{N}$ as the leaf space of the above foliation, this carries a natural topology– namely, let $q : \mathcal{Q} \to \mathcal{N}$ denote the tautological "quotient" map (assigning, to each point of $\mathcal{Q}$, the leaf through it) and equip $\mathcal{N}$ with the quotient topology. If we assume that ($\mathcal{M}$,$\mathbf{g}$) is geodesically convex, $\mathcal{N}$ is then Hausdorff and the foliation tangent to ℓ has trivial holonomy, so that [10] $\mathcal{N}$ has a unique complex structure making q a holomorphic map of maximal rank. The complex manifold $\mathcal{N}$ is then called the *ambitwistor space* of ($\mathcal{M}$, $\mathbf{g}$). Letting $p : \mathcal{Q} \to \mathcal{M}$ denote the restriction to $\mathcal{Q}$ of the canonical projection $\pi : \mathbf{P}T^*\mathcal{M} \to \mathcal{M}$, we have a holomorphic double fibration

$$\begin{tikzcd} & \mathcal{Q} \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathcal{N} & & \mathcal{M} \end{tikzcd}$$

called the *ambitwistor correspondence*, interrelating $\mathcal{M}$ and $\mathcal{N}$. The line-bundle-valued 1-form ϑ descends to $\mathcal{N}$, giving it the structure of a complex contact manifold; i.e. there is a line-bundle-valued 1-form θ on $\mathcal{N}$ with $q^*\theta = \vartheta$, and $\theta \wedge (d\theta)^{\wedge 2} \neq 0$.

Every point $x \in \mathcal{M}$ gives rise to a complex submanifold $Q_x := q[p^{-1}(x)]$ of $\mathcal{N}$ isomorphic to $\mathbf{P}_1 \times \mathbf{P}_1$; we will call this submanifold the *sky* of x, since it exactly represents the set of light rays through x. Since, by construction, ϑ vanishes on any fiber of p, it follows that θ vanishes on every sky; i.e. the skies of $\mathcal{M}$ are *Legendrian submanifolds* [1] of $\mathcal{N}$. Since the restriction of $\mathbf{L} \to \mathbf{P}T^*\mathcal{M}$ to $p^{-1}(x) \cong \mathbf{P}_1 \times \mathbf{P}_1$ is the line bundle $\mathcal{O}(1,1)$ (= the divisor of the diagonal $\mathbf{P}_1 \subset \mathbf{P}_1 \times \mathbf{P}_1$), it follows that the the line-bundle $(\wedge^5 T\mathcal{N})^{1/3}$ in which θ takes its values must restrict to Q_x as $\mathcal{O}(1,1)$, and the fact that Q_x is Legendrian then implies that the normal bundle $\mathbf{N} \to Q_x$ of each sky is isomorphic to $J^1\mathcal{O}(1,1)$. In particular, there is an exact sequence

$$0 \to \Omega^1 \otimes \mathcal{O}(1,1) \to \mathbf{N} \to \mathcal{O}(1,1) \to 0,$$

where $\Omega^1 \cong \mathcal{O}(-2,0) \oplus \mathcal{O}(0,-2)$ is the cotangent bundle of $Q_x \cong \mathbf{P}_1 \times \mathbf{P}_1$, implying that

$$h^m(Q_x, \mathcal{O}(\mathbf{N})) = \begin{cases} 4 & \text{if } m = 0 \\ 0 & \text{otherwise,} \end{cases}$$

and that the 4-parameter family of compact complex submanifolds $\{Q_x \subset \mathcal{N} \mid x \in \mathcal{M}\}$ is therefore complete in the sense of Kodaira [8]. Thus the complex structure of $\mathcal{N}$ encodes the conformal geometry of $(\mathcal{M}, \mathbf{g})$, in the sense that we can recreate $\mathcal{M}$ from $\mathcal{N}$ as a connected component of the the 2-quadrics $\mathbf{P}_1 \times \mathbf{P}_1$ contained in $\mathcal{M}$ provided that $\mathcal{M}$ is Stein– for example, if it is a suitable small complexification of a real-analytic Riemannian manifold, or if it is geodesically convex.

2 Twistors

In the case of the 4-quadric $\mathbf{Q}_4 \subset \mathbf{P}_5$, which is the natural conformal compactification of $(\mathbf{C}^4, \sum_{j=1}^4 (dz^j)^{\otimes 2})$, the corresponding ambitwistor space is given by

$$\mathbf{A} := \{([Z^\alpha], [W_\alpha]) \in \mathbf{P}_3 \times \mathbf{P}_3 | \sum_{\alpha=1}^4 Z^\alpha W_\alpha = 0\}.$$

This is a manifestation of a rather more general phenomenon relating the ambitwistor correspondence to Roger Penrose's classic *twistor correspondence* [13] [14].

Indeed, instead of considering null *curves* in $(\mathcal{M}, \mathbf{g})$, we might choose to look for (totally) null *surfaces*, meaning complex 2-dimensional submanifolds $\Sigma \subset \mathcal{M}$ for which $\mathbf{g}|_\Sigma \equiv 0$. Such submanifolds are necessarily totally geodesic— making them interesting, but also making them rare. In fact, there are typically no such submanifolds at all! Rare though they be, they nonetheless come in two flavors; for if we let

$$* : \bigwedge^2 T\mathcal{M} \to \bigwedge^2 T\mathcal{M}$$

denote the **star-operator** $\frac{1}{2}\epsilon^{ab}{}_{cd}$ of $\mathbf{g}$, then a totally null surface Σ automatically has the property that $\bigwedge^2 T\Sigma$ is an eigenspace of $*$, with eigenvalue ± 1, so that we may label such surfaces as *α-surfaces* and *β-surfaces* in accordance with the sign of the eigenvalue. [1] The condition for the existence of a 3-parameter family of α-surfaces is then that the Weyl curvature C of $(\mathcal{M}, \mathbf{g})$satisfy $C = *C$, where the star-operator treats C as a bundle-valued 2-form; such a space-time is then called *self-dual.* (In terms of curvature spinors [14], our condition becomes $\tilde{\Psi}_{A'B'C'D'} = 0$). For a geodesically convex self-dual space-time, we then have a 3-manifold $\mathcal{P}$ of α-surfaces, called the *twistor space* of $(\mathcal{M}, \mathbf{g})$, and a double fibration

$$\begin{tikzcd} & \mathcal{F} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathcal{P} & & \mathcal{M} \end{tikzcd}$$

called the *twistor correspondence.* The fibers of ν are then $\mathbf{P}_1$'s, while the fibers of μ are complex surfaces.

[1]Note that $*$ is locally determined up to sign, a choice of which is called called a "complex orientation." Changing this "orientation" interchanges our labels of self-duality and anti-self-duality.

Now, quite generally, a null geodesic is contained in at most one α-surface and at most one β-surface. In the case of a self-dual space-time $(\mathcal{M}, \mathbf{g})$, this gives rise to a holomorphic projection

$$\lambda : \mathcal{N} \to \mathcal{P}$$

by sending a null geodesic to the unique α-surface containing it. The fibers of this map are Legendrian submanifolds, and there is therefore a natural induced map

$$\lambda_* : \mathcal{N} \to \mathbf{P}(T^*\mathcal{P})$$

which turns out to be an inclusion. Thus

Proposition 1 *The ambitwistor space $\mathcal{N}$ of a self-dual space-time $(\mathcal{M}, \mathbf{g})$is an open set in the projectivized cotangent bundle of its twistor space $\mathcal{P}$.*

Proof: For more details see [11]. ∎

A particular consequence of this is that the embedding $\mathbf{A} \hookrightarrow \mathbf{P}_3 \times \mathbf{P}_3$ occurring in the conformally flat case has an analogue in the self-dual case. Namely, if $\kappa \to \mathcal{P}$ denotes the canonical line-bundle defined by $\mathcal{O}(\kappa) := \Omega^3_{\mathcal{P}}$, the jet bundle $J^1\kappa \to \mathcal{P}$ is a rank 4 vector bundle, and we have a canonical inclusion $\Omega^1 \otimes \kappa \hookrightarrow J^1\kappa$. This gives rise to an inclusion $\mathbf{P}(T^*\mathcal{M}) \hookrightarrow \mathbf{P}(J^1\kappa)$, since $\mathbf{P}(T^*\mathcal{M}) = \mathbf{P}(\kappa \otimes T^*\mathcal{M})$. Together with Proposition (1), this gives us is defined on $\mathbf{Y}$ except at the zero section of the line bundle, and we may notice that it in fact has a holomorphic continuation to all of $\mathbf{Y}$. In fact, in local coordinates for which the contact form on $\mathcal{N}$ is represented by

$$\vartheta = du + \sum_{j=1}^{2} p_j dq^j,$$

we have

$$\tau = t[(t\frac{\partial}{\partial t} + \sum p_j \frac{\partial}{\partial p_j}) \wedge \frac{\partial}{\partial u} + \sum \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial p_j}],$$

where the local coordinate t on the total space $\mathbf{Y}$ of the line bundle $\mathbf{L} \to \mathbf{X}$ is the fiber coordinate induced by the trivialization of $\mathbf{L}$ for which $\theta \in \Gamma(\mathcal{N}, \Omega^1(\mathbf{L}))$ is represented by ϑ.

Now if $\mathcal{O}_m$ denotes the structure sheaf of the m^{th} infinitesimal neighborhood of the zero section $\mathcal{N} \subset \mathbf{L}$, the exilissic form τ defines a map

$$\begin{array}{rcl} \mathcal{O}_{m-1}/\mathbf{C} & \to & \mathbf{Der}(\mathcal{O}_m) \\ f & \mapsto & \{f, \}, \end{array}$$

and the image of this map is a sheaf of nilpotent Lie algebras that we can exponentiate to give a certain sheaf $\mathcal{G}_m$ of groups of automorphisms of $\mathcal{O}_m$.

A thickening of $\mathcal{N}$ is then said to be of *Poisson type* if its transition functions can be taken to be elements of $\mathcal{G}_m$. Equivalently, a thickening is of Poisson type iff its equivalence class is in the image of

$$H^1(\mathcal{N}, \mathcal{G}_m) \to H^1(\mathcal{N}, \mathbf{Aut}(\mathcal{O}_m)).$$

Elements of the torsor $H^1(\mathcal{N}, \mathcal{G}_m)$ are called *Poisson thickenings.*

A particular case of this construction is that, at the level of first order thickenings, one has a special class of candidates for the extended tangent bundle, given by the image of the map

$$\tau_* : H^1(\mathcal{N}, \mathcal{O}/\mathbf{C}) \to H^1(\mathcal{N}, \mathcal{O}(T\mathcal{N} \otimes \mathbf{L}^*)).$$

In particular, when we throw in the logarithm $log : \mathcal{O}_* \to \mathcal{O}/\mathbf{C}$ we get a natural morphism

$$\tau_*(log)_* : H^1(\mathcal{N}, \mathcal{O}_*) \to H^1(\mathcal{N}, \mathcal{O}(T\mathcal{N} \otimes \mathbf{L}^*)$$

associating an extended tangent bundle to every complex line bundle over $\mathcal{N}$. In fact, there is nothing mysterious about this morphism; if we feed it a line bundle $\mathbf{L}_+$, it spits out the extended tangent bundle

$$[T\mathcal{N} \oplus (\mathbf{L}_- \otimes J^1\mathbf{L}_+)/\underline{\mathbf{C}}]/\mathbf{D}$$

where $\underline{\mathbf{C}}$ is the trivial line sub-bundle spanned by $\theta \in \Gamma(\mathcal{N}, \Omega(\mathbf{L}_+ \otimes \mathbf{L}_-))$, and where $\mathbf{D}$ is included diagonally via $\mathbf{D} \hookrightarrow T\mathcal{N}$ and $\mathbf{D} \overset{d\theta}{\cong} \mathbf{L}_- \otimes \mathbf{D}^* \otimes \mathbf{L}_+ \hookrightarrow \mathbf{L}_- \otimes J^1\mathbf{L}_+/\underline{\mathbf{C}}$. In the particular case where we take $\mathbf{L}_+$ to be the line-bundle whose fiber at a null geodesic is the set of parallel spinor fields $\pi_{A'}$ along the geodesic such that the the tangent field v of the geodesic satisfies $v^{AA'}\pi_{A'} = 0$, this yields the standard extended tangent bundle discovered long ago by the present author [9]. We will now ask for the obstructions to extending this Poisson thickening to higher order. As it turns out, this gives essentially the same answer that would occur if we tried to merely extend as any sort of thickening at all, but eliminates a number of redundancies.

The general theory of torsors now tells us that the obstructions to extending a Poisson thickening from order m to order m+1 is an element of $H^2(\mathcal{N}, \mathcal{O}(L^{-m}))$, and the freedom in so doing is an element of $H^1(\mathcal{N}, \mathcal{O}(L^{-m}))$ should the obstruction vanish. Now these cohomology groups can be calculated by the Penrose transform [4], and it turns out that

$$H^1(\mathcal{N}, \mathcal{O}(L^{-m})) = 0, \ m > 1,$$

whereas

$$H^1(\mathcal{N}, \mathcal{O}(L^{-1})) \cong H^2(\mathcal{N}, \mathcal{O}(L^{-2})).$$

The latter is no coincidence; more detailed calculation shows that the above isomorphism is actually realized by the obstruction map, so that every first order Poisson thickening extends uniquely to third order. We are left with a hierarchy of obstructions in $H^2(\mathcal{N}, \mathcal{O}(L^{-m}))$, $m > 2$. These cohomology groups then corespond by the Penrose transform to the space of symmetric trace-free tensor fields $D_{ab\cdots c}$ on $(\mathcal{M}, \mathbf{g})$of degree m-2 with conformal weight -2 and vanishing divergence:

$$\nabla^a D_{ab\cdots c} = 0.$$

In the next section, we will analyze these obstructions in terms of invariant theory.

4 Conformal Gravity

We will now focus on the following field conformally invariant field equations, which I claim are exactly the ones that will arise from the obstructions we encountered in the last chapter:

$$\begin{aligned} B_{ab} &= 0 && (2)\\ E_{abc} &= 0. && (3) \end{aligned}$$

Here the *Bach tensor* is defined by

$$B_{ab} := (\nabla^c\nabla^d + \frac{1}{2}R^{cd})C_{acbd},$$

where C_{abcd} denotes the Weyl curvature, while

$$\begin{aligned} E_{abc} &:= \tilde{\Psi}_{A'B'C'D'}\nabla^d\Psi_{ABCD} - \Psi_{ABCD}\nabla^d\tilde{\Psi}_{A'B'C'D'} \\ &= \frac{1}{2}[C^f{}_{ace}\nabla^d(*C)^e{}_{bfd} - (*C)^f{}_{ace}\nabla^d C^e{}_{bfd}] \end{aligned}$$

is the so-called *Eastwood-Dighton tensor.* A complex space-time satisfying (2) will be said to be *Bach-flat.* A complex space-time satisfying both (2) and (3) will be called a solution to the *conformal Einstein equations.* The justification for the last bit of terminology is the following result [2]:

Proposition 2 *Suppose that $(\mathcal{M}, \mathbf{g})$is a solution of equations (2) and (3) with algebraically general Weyl curvature. Then there exists a conformal factor α such that $\hat{\mathbf{g}} := \alpha^2\mathbf{g}$ is Einstein, meaning that the Ricci curvature of $\hat{R}$ of $\hat{\mathbf{g}}$ satisfies*

$$\hat{R}_{ab} = \frac{1}{4}\hat{\mathbf{g}}.$$

Now notice that the Bach and Eastwood-Dighton tensors are both symmetric trace-free tensors, and are both conformally invariant, with conformal weight -2, meaning that under

$$\begin{aligned} \mathbf{g} &\mapsto \hat{\mathbf{g}} = \alpha^2 \mathbf{g} \quad \text{we have} \\ B &\mapsto \hat{B} = \alpha^{-2} B \quad \text{and} \\ E &\mapsto \hat{E} = \alpha^{-2} E. \end{aligned}$$

Moreover, they are both invariant under biholomorphisms, meaning that if $\phi : \mathcal{M} \to \hat{\mathcal{M}}$ is a biholomorphism, then these tensors depend upon the metric in such a manner that

$$\begin{aligned} B(\phi^*(\mathbf{g})) &= \phi^*(B(\mathbf{g})) \\ E(\phi^*(\mathbf{g})) &= \pm\phi^*(E(\mathbf{g})), \end{aligned}$$

where the above $\pm$ sign depends upon the choice of sign for the associated star operators $* : \bigwedge^2 \to \bigwedge^2$. Moreover, the value at $x \in \mathcal{M}$ of these tensors is a holomorphic function of the m-jet of $\mathbf{g}$ at x for some $m \in \mathbf{N}$.

This motivates the following definition:

Definition 1 *An* **admissible invariant** *is a symmetric trace-free tensor field $D(\mathbf{g})$ of conformal weight -2, which is invariant under biholomorphisms, and such that, for some $m \in \mathbf{N}$, the value of $D(\mathbf{g})$ at x depends holomorphically on the m-jet of $\mathbf{g}$ at x.*

The interest of this definition lies in the fact that such creatures are, in fact, quite rare:

Proposition 3

1. *Any admissible invariant D_a vanishes.*
2. *Any admissible invariant D_{ab} is a constant multiple of the Bach tensor.*
3. *Any admissible invariant D_{abc} is a constant multiple of the Eastwood-Dighton tensor.*

Proof: Let us consider jets of metrics of the form

$$\mathbf{g}_{ab} = \delta_{ab} + r_{(ab)(cd)}x^c x^d + s_{(ab)(cde)}x^c x^d + t_{(ab)(cdef)}x^c x^d x^f + \cdots + u_{(ab)(cde\cdots f)}x^c x^d \cdots x^f,$$

where δ_{ab} denotes the flat metric on $\mathbf{C}^4$ and x^a are the standard complex coordinates on $\mathbf{C}^4$, since any m-jet can be put in this form by making a coordinate transformation. Thus each component of D is given by an entire holomorphic function $h(r, s, t, \cdots, u)$. But if we make the coordinate transformation $x \mapsto \alpha x$ coupled with the conformal transformation $\mathbf{g} \mapsto \alpha^{-2}\mathbf{g}$ the effect on the above normal form will be

$$\begin{aligned} r &\mapsto \alpha^2 r \\ s &\mapsto \alpha^3 s \end{aligned}$$

$$\begin{aligned} t &\mapsto \alpha^4 t \cdots \\ u &\mapsto \alpha^m u \end{aligned}$$

while the effect on D will be given by

$$D_{\underbrace{ab\cdots c}_{\ell}} \mapsto \alpha^{2+\ell} D_{\underbrace{ab\cdots c}_{\ell}}.$$

Thus our functions h satisfy the mixed homogeneity condition

$$h(\alpha^2 r, \alpha^3 s, \alpha^4 t, \ldots, \alpha^m u) = \alpha^{2+\ell} h(r, s, t, \ldots, u),$$

and, being holomorphic at 0, are actually polynomials.

For $\ell = 1$, we conclude that D_a is linear in s and independent of the other variables. If we now further restrict our class of coordinate systems by assuming that we are working in **geodesic spray coordinates**, and assume, using the the conformal invariance [6], that $\nabla_{(a} \cdots \nabla_b R_{cd)}|_{x=0} = 0$, this says that D_a is linear in the first covariant derivative of the Weyl tensor. Using the natural exponential extension of the action of $SO(4, \mathbf{C} = [SL(2, \mathbf{C}) \times SL(2, \mathbf{C})]/\mathbf{Z}_2$ on the tangent space at the origin, we conclude that $D_a(\mathrm{g})$ would represent an $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$-equivariant map

$$\mathbf{C}_{(ABCD)EE'} \oplus \mathbf{C}_{(A'B'C'D')EE'} \longrightarrow \mathbf{C}_{AA'}$$

and so must vanish, since there are no common irreducible components shared by these two representations.

For $\ell = 2$, we proceed similarly. This time h could be the sum of a linear function of t and a quadratic function of r. We then use conformal normal coordinates to identify r and t with the Weyl curvature and its second covariant derivative. Remembering to use the Bianchi identities for the second derivatives of the Weyl curvature, the homomorphisms of $SL(2, \mathbf{C}) \times SL(2, \mathbf{C})$-modules corresponding to the linear and quadratic pieces of $D_{(ab)}$ would be

$$[\mathbf{C}_{(ABCD)} \oplus \mathbf{C}_{(A'B'C'D')}]^{\otimes 2} \longrightarrow \mathbf{C}_{(AB)(A'B')}$$

and

$$\mathbf{C}_{(AB)(A'B')} \oplus \mathbf{C}_{(ABCDE)F(E'F')} \oplus \mathbf{C}_{(A'B'C'D'E)F(E'F')} \longrightarrow \mathbf{C}_{(AB)(A'B')};$$

but the first vanishes by Schur's lemma, while the second is determined up to scale. Thus the Bach tensor B_{ab} is, up to a constant factor, the only such invariant.

For $\ell = 3$, h must be bilinear in r and s. Comparison of representations tells us this time that our invariant must be of the form

$$c_1 \Psi_{ABCD} \nabla^{DD'} \tilde{\Psi}_{A'B'C'D'} + c_2 \tilde{\Psi}_{A'B'C'D'} \nabla^{DD'} \Psi_{ABCD}.$$

But one then notices that under the conformal change $\mathbf{g} \mapsto \alpha^2 \mathbf{g}$ this becomes

$$\alpha^{-2}(c_1 \Psi_{ABCD} \nabla^{DD'} \tilde{\Psi}_{A'B'C'D'} + c_2 \tilde{\Psi}_{A'B'C'D'} \nabla^{DD'}$$

$$\Psi_{ABCD}) + \alpha^{-2} 2(c_1 + c_2) \Upsilon^{DD'} \Psi_{ABCD} \tilde{\Psi}_{A'B'C'D'},$$

where $\Upsilon = \alpha^{-1} d\alpha$, and so is conformally invariant with weight -2 iff $c_1 = -c_2$. ∎

In fact, the proof actually gives us a bit more than is stated. In particular, even if an admissible invariant $D_{(abc)}$ were defined only as a function of *Bach-flat* metrics, it could still only be a constant multiple of $E_{(abc)}$.

This then leads to the following result:

Theorem 1

1. *There is no obstruction to extending the canonical first-order thickening of $\mathcal{N}$ to fourth order.*
2. *The obstruction to extending the canonical first-order thickening of $\mathcal{N}$ to fifth order is the Bach tensor.*
3. *The obstruction to extending the canonical first-order thickening of $\mathcal{N}$ to sixth order is the Dighton-Eastwood tensor.*

Proof: [Sketch] There are two essential ingredients: one must verify that these obstructions are admissible invariants, and then show that they are non-zero at orders 5 and 6. The former is essentially straight-forward once the obstructions can be shown to only depend on some jet of the metric; however, nailing this down is surprisingly involved [12]. (Of course, the obstruction to *sixth* order extension is only *a priori* defined as an invariant of those metrics with fifth order extension, but, as previously noted, we could weaken our definition of admissibility for $D_{(abc)}$, allowing it to just be defined on the set of *Bach-flat* metrics, without altering the conclusion of theorem 1.) As for the second critical fact– that the obstructions are non-trivial at orders 5 and 6– one may appeal to the perturbative calculations of Baston-Mason [2]. ∎

References

[1] V.I. Arnold, **Mathematical Methods of Classical Mechanics**, Springer-Verlag, 1978

[2] R.J. Baston and L.J. Mason, *Conformal Gravity, the Einstein Equations and Spaces of Complex Null Geodesics*, **Class. Quantum Grav. 4**(1987)815-826

[3] L.L. Chau and C.S. Lim, *Geometrical Constraints and Equations of Motion in Extended Supergravity*,**Phys. Rev. Lett. 56**(1986)294-297

[4] M.G. Eastwood, *The Penrose Transform for Curved Ambitwistor Space*, **Qu. J. Math. Oxford (2)39**(1988)427-441

[5] M.G. Eastwood and C.R. LeBrun, *Thickenings and Supersymmetric Extensions of Complex Manifolds*, **Am. J. Math. 108**(1986)1177-1192

[6] C.R. Graham, *Formal Conformal Normal Form*, to appear.

[7] J. Isenberg, P. Yasskin, and P. Green, *Non-Self-Dual Gauge Fields*, **Phys. Lett. 77B**(1978)462-464

[8] K. Kodaira, *On Complete Systems of Compact Complex Submanifolds of a Complex Manifold*, **Am. J. Math.85**(1962)79-94

[9] C.R. LeBrun, *The First Formal Neighbourhood of Ambitwistor Space for Curved Space-time*, **Lett. Math. Phys. 6**(1982)345-354

[10] C.R. LeBrun, *Spaces of Complex Null Geodesics in Complex-Riemannian Geometry*,**Trans. Am. Math. Soc.278**(1983)209-231

[11] C.R. LeBrun, *Thickenings and Gauge Fields*,**Class. Quantum Gravity 3**, (1986)1039-1059

[12] C.R. LeBrun, *Thickenings and Conformal Gravity*, preprint, 1989.

[13] R. Penrose, *Non-linear Gravitons and Curved Twistor Theory*, **Gen. Rel. Grav. 7**(1976)31-52

[14] R. Penrose and W. Rindler,**Spinors and Space-time, vol. 2**, Cambridge U.P. 1986

[15] E. Witten,*An Interpretation of Classical Yang-Mills Theory*, **Phys. Lett. 77B**(1978)394-398

[16] E. Witten, *Twistor-like transform in ten dimensions*, **Nuc. Phys. B 266**(1986)245-264

TOWARD CLASSIFICATION OF CLASSICAL LIE SUPERALGEBRAS

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INTRODUCTION: How reasonable is to classify simple superalgebras

The purpose of this talk is to list the known simple Z-graded Lie superalgebras of finite growth over the field $\mathbb{C}$ of complex numbers, compare the list with the similar one for Lie algebras, discuss their completeness and make emphasis on the geometric structures preserved by these algebras.

One of the most nontrivial phenomenon is the possibility to realize a vectory Lie superalgebra as a Lie superalgebra of vector fields on *different* supermanifolds, Another one is deformations of Buttin (Schouten, anti-) bracket. This text is a direct continuation of [L1]. Details see in [L2, n31].

We assume that the reader knows the conventional definition of Lie superalgebras and that of cohomology of Lie (super) algebras obtained via Sign Rule, see [L1], [B], [M] and [F].

0.1. To classify algebras with reasonably nice properties is a matter of taste. One of the choices are *simple* algebras. Among such, *Z-graded Lie algebras of finite growth* (ZGLAFG for short) are those that resemble finite-dimensional simple ones most of all and which proved very useful in various branches of mathematics and theoretical physics.

Around 1966 V. Kac classified them **provided they are generated by elements of degree ± 1.** Thus, there is yet no complete classification even of simple ZGLAFGs; however, hopefully we know all of them, cf. Conjecture 0.5. Still, the situation is far from clear. To describe it, recall that a graded algebra A is *simple* if it has no *homogeneous* ideals, i.e. no ideals $I \subset A = \oplus A_i$; such that $I = \oplus I \cap A_i$ and A is of *finite growth* if

i) $\dim A_i < \infty$ for all i

ii) growth $(A) = \overline{\lim}_{n\to\infty} (\ln \dim \oplus_{|i|\leq n} A_i)/ \ln\ n < \infty$

It was clear from the start that simple algebras are rather a prefabricate, the object being the answer to the following

Problem A: *List the results of iterated combinations of the following operations:*

- $g \longrightarrow \delta erg$, the derivation algebra of g (if the grading operator d does not belong to g we denote by δg the Lie superalgebra $g \oplus kd$)

- *$g \to^e g$, a nontrivial central extension of g (we denote by $cg = g \oplus kz$ the trivial central extension of g with 1-dimensional even center generated by z);*

- *deformations (which sometimes lead out of the ZGLAFG class, so what, cf. quantization of the Poisson algebra);*

- *all incompressible Z-regradings (that may also lead out of finite growth class) and gradings with other groups;*

- *forms of these algebras over non closed fields.*

Another major class of nice algebras are maximal solvable and nilpotent subalgebras of simple Lie superalgebras. (So we return again to simple algebras.)

Problem B: *List all maximal solvable and nilpotent subalgebras of simple ZGLAFG's.*

What seems outrageous is the lack of full answer to the above problems. In this work, we will try if not to remedy, then at least to give a precise diagnosis, cf. 0.5.

0.2. Let us describe the known simple ZGLAFG's. To visualize them better we will separate them into several types.

Conjecture (V. Kac, 1076). *All simple Lie algebras over $\mathbb{C}$ are :*

- *finite dimensional (growth 0)*

-*(twisted) loop algebras (growth 1)*

- *vectory algebras, i.e. Lie algebra of vector fields,*[1] *with polynomial coefficients (growth equals the number of indeterminates).*

- *stringy algebra*[2] *$vect^l$* (1) with Laurent polynomials in one variable (the superscript stands for Laurent). This algebra is also called by mathematicians *Witt algebra and by physicists the centerless Virasoro algebra.*

Of the above algebras $vect(1)$ and $vect^l(1)$ are not generated by elements of degree ± 1 with respect to any grading. Thanks to V. Kac's Ph.D. thesis, this conjecture actually states that these are the only exceptions from the general rule (that simple ZGLAFGs are generated by elements of degree ± 1).

One more feature is of great importance in applications: a Lie algebra will be called selfsymmetric[3] if for any root α the multiplicity of $-\alpha$ is the same as that of α.

For superalgebras or for infinite-dimensional (especially vectory) algebras or in characteristic p one should be aware of the fact that roots and weights are considered with respect to a maximal torus (a purely even diagonalizing subalgebra). Cartan subalgebras (nilpotent subalgebras that coincide with their nomalizers) cease to play important roles.

Among selfsymmetric algebras the nicest are those that posses an invariant symmetric bilinear form. They will be called *selfsymmetric algebras with form.*

All these classes are intermixed for superalgebras, and deformations, a we will see further, intertwine these classes.

1 These algebras are usually known under unsuggestive name 'algebras of Cartan type' introduced of lately; this name also confuses with the notion "Cartan subalgebras ".

2 The term coined by imaginative physicists; it means pertaining to string theory; in our setting this means vectory algebra on circle or an algebra derived from such.

3 The term "contragredient" usually applied for these algebras , though cumbersome, is adequate in presence of a symmetric bilinear form; it is unjustified in the absence of such.

0.3. Dealing with superalgebras it sometimes becomes useful to know their definition.

Recall that from the functorial point of view a *Lie (super)algebra* is a (super) manifold $L = (L_{rd}, O_L)$ such that for "any" (say, finitely generated, or of some other appropriate category) commutative superalgebra C the space $L(C) = Hom(Spec\ C,\ L)$, called *the space of C-points of L* is a Lie algebra and the correspondence $C \longrightarrow L(C)$ is a functor in C.

Exercise: What are C-points of a Lie subalgebra L given in conventional definition?
Answer: $(L \oplus C)_{\bar{0}}$.

A *Lie superalgebra homomorphism* in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms $\rho_C : L_1(C) \longrightarrow L_2(C)$ compatible with morphisms of commutative (super)algebras $C \longrightarrow C'$.

In particular, a *representation* of a Lie (super)algebra L in a (super)space V is a homomorphism $L \longrightarrow gl(V)$, i.e. a collection of Lie algebra homomorphisms $L(C) \longrightarrow (gl_{C_{\bar{0}}}(V \oplus C))_{\bar{0}}$.

A representation is *irreducible* (of *general type* of just of *G-type* if there is no invariant subspace, otherwise it is called *irreducible of Q-type* (for reasons to be explained below); such a representation has no invariant *subsuperspace* but *has* an invariant subspace.

In particular, if the adjoint representation of an algebra, whose dimension is not 1 or ε, is irreducible of any type then the algebra is called *simple.*

The usefulness of this definition becomes obvious when we deal with a relative situation (algebras not over k but over a commutative superalgebra C) or when we describe moduli of various kinds: defomrations of algebras , of their representations, etc. which can and often do depend on odd parameters.

Luckily, we can advance step-by-step: first describe geometric points, or k-points, using the naive definition and then calculate appropriate cohomology which describe the tangent space to the supervariety of moduli.

Thus, one should always remember that a Lie superalgebra is not a linear supersapce but a linear supermanifold: otherwise it is impossible to say e.g. what a co-adjoint action is, which baffles those who only learned the naive definition.

0.4. The known classification of simple Z-graded Lie superalgebras of finite growth (SZ-GLSAFGs) is as follows: $\mathbb{C}$ *-points of simple finite-dimensional Lie superalgebras* are classified by Kac [K2]. We can make a final touch and for all but 3 series of superalgebras list all the deforms. Since some of them are with odd parameter, then from "super" point of view we get new superalgebras .

Twisted loop algebra only correspond to outer automorphisms of the target algebra. I had conjectured (1976) that so is the case for superalgebras, too. V. Serganova listed all outer automorphisms of simple finite dimensional Lie superalgebras and the associated twisted loop superalgebras and her results imply that the conjecture should be amended: automorphisms must be considered modulo the connected component of the unit of the group of automorphisms. As shown by J. van de Leur [vdL], for Lie superalgebras with *symmetrizable* Cartan matrix the refined conjecture is true.

Vectory superalgebras are either *Cartan prolongs* or prolongs via a trifle more general construction due to Shchepochkina. Prolongs of simple, almost simple and

Note that for superalgebras the notion of a real structure is more sophisticated than in nonsuper case: there are real and quaternionic forms; several notions of unitarity, etc.

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1. List of classical Lie superalgebras over $\mathbb{C}$

1.1. Some finite-dimensional Lie Superalgebras .

1.1.1. Matrix Lie Superalgebras . 1) The Lie superalgebras $gl(m/n) = \mathrm{Mat}(m/n;\mathbb{C})_L$ and $sl(m/n) = \{X \in gl\ (m/n)$: str $X = 0\}$, where str $\begin{pmatrix} A & B \\ C & D \end{pmatrix} = tr\ A - tr\ D$, are called the *general linear* and *special linear* Lie superalgebras .

The Lie superalgebras $q(n) = \{X \in gl\ (n/n) : [X, J_{2n}] = 0\}$, preserves the complex structure given by the *odd* operator $J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. This Lie superalgebra is called the *(general) queer Lie superalgebra,* and its Lie superalgebra $sq(n) = \{X \in q(n) :$ qtr $X = 0\}$ $\left(\text{where qtr} \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \left(\text{tr } B\right)\left(\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right)\right)$ is called the *special queer subalgebra.* The superalgebras of series gl and q are analogues of the usual $gl(n)$.

2) The algebras preserving nondegenerate forms. (Recall that *supertransposition* is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{st} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$.) The Lie superalgebra

$$osp(m/2n) = \{X \in sl(m/2n) : X^{st}B_{m,2n} + B_{m,2n}X = 0\}$$

preserving the even nondegenerate bilinear form with matrix $B_{m,2n}$, where for $B_{m,2n}$ one can take over $\mathbb{C}$ either

$$B_{m,2n} = \mathrm{diag}(1_m, J_{2n}) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix} \text{ or } B'_{m,2n} = \begin{pmatrix} 0 & 1 & & \\ & & & 0 \\ 1 & 0 & & \\ 0 & & & J_{2n} \end{pmatrix} = \mathrm{diag}\ (\mathrm{ant}(1_m), J_{2n})$$

where $\mathrm{ant}(1_m)$=antidag$(1, \ldots, 1)$ (m-many units), is called the *orthosymplectic Lie superalgebra* . The Lie superalgebra

$$pe(n) = \{X \in gl(n/n) : X^{st}J_{2n} + (-1)^{p(X)}J_{2n}X = 0\},$$

preserving the odd nondegenerate bilinear form with matrix J_{2n}, is called the *peculiar* or, as A. Weil suggested, *periplectic Lie superalgebra* .

Let $spe(n) = \{X \in pe\ (n)$:str$X = 0\}$ be the *special peculiar (periplectic) Lie superalgebra* .

1.1.2. Projectivization: If s is a Lie algebra of scalar matrices, and $g \subset gl(n/n)$ is a Lie superalgebra containing s, then the *projective Lie superalgebra of type g* is $pg = g/s$. Projectivization sometimes leads to new Lie superalgebras : $pgl(n/n), psl(n/n), pq(n), psq(n)$; whereas pgl(p/q)$\cong sl$(p/q) if p $\neq$q.

solvable Lie superalgebras are listed, [LS].Conjectorially, these are all.

Stringy superalgebras constitute a particular case of vectory ones and generalize $vect^L(1)$. They were discovered by physicists who assume now that an elementary particle is not a point but rather a slinky springy string. Over$\mathbb{C}$, these algebras were classified modulo deformations by Kac and van de Leur.

The following table lists distribution of the known classical Lie superalgebras into types with respect to their selfsymmetricity, presence of a Cartan matrix or a bilinear symmetric form.

Table. Selfsymmetricity of Lie superalgebras

A) Selfsymmetric Lie superalgebras

<table>
<tr><td colspan="4">without any bilinear form or Cartan matrix</td></tr>
<tr><td colspan="4">all stringy algebras except $k^L(1/6)$</td></tr>
<tr><td colspan="2">with Cartan matrix</td><td colspan="2">without Cartan matrix</td></tr>
<tr><td>A symmetrizable
with event form</td><td>A non symmetrizable
with odd form</td><td>with even
bilinear form</td><td>with odd
bilinear form</td></tr>
<tr><td>$sl(\mathrm{n/m})$, $m \leq n$;
$psl(\mathrm{n})$, $n > 1$;
$osp(\mathrm{m/2n})$
ag_2; ab_3; $\delta(\alpha)$
$g^{(1)}$ with above g;
(2) or (4)
$(p)sl_{-st}$
$ops(2m/2n)^{(2)}$</td><td>$psq^{(2)}$</td><td>$t^L(1/6)$</td><td>$psq(\mathrm{n})$, $n > 2$
$psq^{(1)}(\mathrm{n})$, $n > 2$
$psq^{(4)}(\mathrm{n})$, $n > 2$</td></tr>
</table>

B) Non-selfsymmetric Lie superalgebras (no Cartan matrix)

<table>
<tr><td colspan="2">without any form</td></tr>
<tr><td colspan="2">spe; vectory and twisted loops there of except ones mentioned below;
$(p)sl^{(2)}_{\pi}; (p)sl^{(2)}_{-sto\Pi}$, $\delta((-1 \pm i\sqrt{3})/2)^{(3)}$</td></tr>
<tr><td>with even bilinear form</td><td>with odd bilinear form</td></tr>
<tr><td>$sh(2n); sh(2n)^{(1)};\ sh(2n)^{(2)}$</td><td>$sh(2n+1),\ sh(2n+1)^{(1)}$</td></tr>
</table>

Deformations, as we will see intertwine all these classes. Very regretably only some examples are known; there is no classification.

I will do my best to describe all the known examples; whenever I can I give a realization of the considered algebra and try to five it a suggestive name. Regretably, I cannot always do it; hence some temporary notations like ag_2, $\delta(\alpha)$, etc.

The details of classification are published in [L2, N22, 31].

0.5. Real forms of finite-dimensional Lie algebras is a classical topic, cf. e.g. [OV]. Same of vectory algebras are described by A. Rudakov [R], same of twisted loops and the only stringy algebra are described by V. Serganova together with similar Lie superalgebras , cf. [S] and the details in [L2], n22.

Statement. *The representation of minimal dimension of a projective Lie superalgebra psl or psq is the adjoint one.*

Thus *psl* and *psq* have no nice matrix representation. Proof due to C. Kac follows from the character formula for irreducible representations.

1.1.3. Exceptional Lie Superalgebras : Let *id* denote the standard (also called identity) $sl(2)$-module in the space of, say, column-vectors. Let *det* be the invariant bilinear form on *id* given by

$$det : (u,v) \to det(u,v) = det \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \qquad (*)$$

Let p:$id \otimes id \to sl(2)$ be determined by the formula

$$p(u,v)(w) = \lambda(det(v,w)u - det(w,u)v) \text{for some } \lambda \in \mathbb{C} \qquad (**)$$

ag_2: Let $\bigcirc$ be the algebra of Caley numbers (octonions), and $g_2 = \delta er\bigcirc$. The form $(x,y) = x\bar{y} + y\bar{x}$, where $x, y \in \bigcirc$, is symmetric nondegenerate and g_2-invariant on $\bigcirc$; and so is its projection onto $\bigcirc^\circ = \bigcirc/R \cdot 1$. Let $L_x(y) = xy$, $R_x(y) = yx$. Then the formula

$$D_{x,y} = [L_x, L_y] + [R_x, R_y] + [L_x, R_y], \text{ where } x, y \in \bigcirc,$$

gives a g_2-invariant map $D : \bigcirc \otimes \bigcirc \to g_2$. Let $D^\circ = D|\bigcirc^\circ$. We set $(ag_2)_{\bar{0}} = sl(2) \oplus g_2$, $(ag_2)_{\bar{1}} = id \otimes \bigcirc^\circ$ and define [.,.]: $S^2(ag_2)_{\bar{1}} \to (ag_2)_{\bar{0}}$ by the formula

$$[x \otimes u, y \otimes v] = (x,y)p(u,v) - det(u,v)D^\circ_{x,y} \text{where } x, y \in \bigcirc^\circ, u, v \in id$$

and where *det* is defined by (*) and *p* by (**) for a certain λ.

ab_3: We set $(ab_3)_{\bar{0}} = sl(2) \oplus o(7)$, $(ab_3)_{\bar{1}} = id\otimes$ spin$_7$. The mapping [.,.]: $S^2(ab_3)_{\bar{1}} \to (ab_3)_{\bar{0}}$ is defined by the formula

$$\Gamma_1 \otimes u_1,\ \Gamma_2 \otimes u_2 = (\Gamma_1, \Gamma_2) \otimes p(u_1, u_2) + \sum_{j,k} det(u_1, u_2)(\Gamma_1, \gamma_j\gamma_k\Gamma_2)(E_{jk} - E_{kj}),$$

Where Γ_1, $\Gamma_2 \in$ $spin_7 \cong \mathbb{C}\ [\gamma_1, \ldots, \gamma_7]$ with $\gamma_i\gamma_j + \gamma_j\gamma_i = \delta_{ij}$ for $\leq i, j, \leq 7$, $(.,.)$ is an invariant form on $spin_7$, and where *det* is defined by (*) and *p* by (**) for a certain λ.

1.1.4. Deformations: $\delta(\alpha)$: This is the deform of the Lie superalgebra $osp(4/2)$. (After we will have found out its realization it would be nice to give an adequate name to this superalgebra.)

We identify $\delta(\alpha)_{\bar{0}}$ with $sl(2) \oplus sl(2) \oplus sl(2) = \{(a_1, a_2, a_3)\}$, and $\delta(\alpha)_{\bar{1}}$ with $id_1 \otimes id_2 \otimes id_3$, where id_j is the identity representation of the j-th copy of $sl(2)$ in the space of column-vectors.

We define [.,.]: $S^2\delta(\alpha)_{\bar{1}} \to \delta(\alpha)_{\bar{0}}$ by the formula

$$[u_1 \otimes u_2 \otimes u_3,\ v_1 \otimes v_2 \otimes v_3] \to \sum_{(i,j,k)=\sigma(1,2,3), \sigma \in S_3} det_i(u_i, v_i)det_j(u_j, v_j)p_k(u_k, v_k)$$

Exercise. The Jacobi identity for the bracket [.,.] holds if and only $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

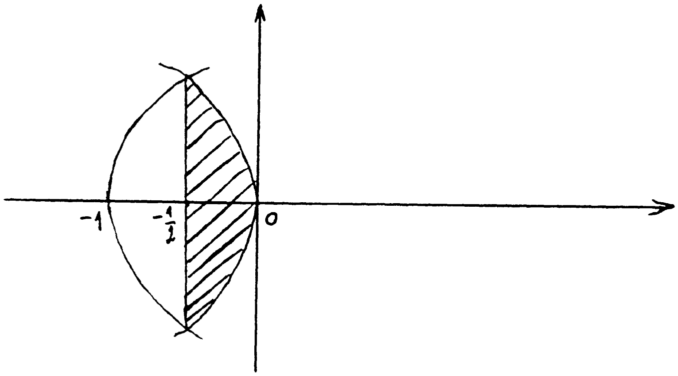
The parameter α belonging to a projective line is $\alpha = \lambda_2/\lambda_1$ (in the affine chart where $\lambda_1 \neq 0$). Now let us calculate the fundamental domain for the S_3-action on the projective line $\mathbb{C} P^1$ whose points are given by triples

$$(\lambda_1 : \lambda_2 : \lambda_3) \text{ with } \lambda_1 + \lambda_2 + \lambda_3 = 0$$

We may put $(\lambda_1 : \lambda_2 : \lambda_3) = (1 : \alpha - 1 - \alpha)$. Now, the elements of S_3 act as follows:

$$\begin{aligned}
S_{12} &:(1 : \alpha : -1-\alpha) \longmapsto (\alpha : 1 : -1-\alpha) = (1 : 1/\alpha : -(1+\alpha)/\alpha)\\
S_{23} &:(1 : \alpha : -1-\alpha) \longmapsto (1 : 1 : -\alpha - \alpha)\\
S_{13} &:(1 : \alpha : -1-\alpha) \longmapsto (-1-\alpha : \alpha : 1) = (1 : -\alpha/(1+\alpha) : -1(1+\alpha))\\
S_{123} &:(1 : \alpha : -1-\alpha) \longmapsto (-1-\alpha : 1 : \alpha) = (1 : 1/(1+\alpha) : \alpha/(1+\alpha))\\
S_{132} &:(1 : \alpha : -1-\alpha) \longmapsto (\alpha : -1-\alpha : 1) = (1 : (-1-\alpha)/\alpha : 1/\alpha)
\end{aligned}$$

Exercise: The closure $\bar{D}$ of the shaded part of the intersection of the circles of radius 1 with centers at 0 and -1 to the right of the straight line Re $\alpha = -\frac{1}{2}$ is the fundamental domain of the above S_3-action.



Problem. Is $\delta(\alpha) \cong \delta(\bar{\alpha})$ for $\alpha \in \bar{D}$ (For $\alpha = (-1 \pm i\sqrt{3})/2$ this is true.)

Deformations of $spe(3)$. The only nontrivial deformation is determined by an odd co-cycle m on $g = spe^{sy}(3)$ whose non-vanishing part is given by the formula

$$m : g_{-1} \otimes g_{-1} \longrightarrow g_1, \; m : \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ c' & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & cc' + c'c \\ 0 & 0 \end{pmatrix}$$

1.1.5. Occasional isomorphisms and simplicity

$gl(m|n) \simeq gl(n|m)$ and consequently $sl(m|n) \simeq sl(n|m)$. In the standard format the isomorphism is given by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D & C \\ B & A \end{pmatrix}$$

In what follows we will assume $m \leq n$, unless otherwise is needed.

$sl(1/2) \simeq osp(2/2)$; the isomorphism is given by the formula.

$$\begin{pmatrix} a+b & x & y \\ z & a & c \\ t & d & b \end{pmatrix} \mapsto \begin{pmatrix} a+b & 0 & x & y \\ 0 & -(a+b) & z & t \\ t & y & a & c \\ -z & -x & d & b \end{pmatrix}$$

As we had already mentioned, $\delta(\alpha) \cong \delta(\alpha')$ if and only if α' is obtained form α by iterations of the following operations: $\alpha \mapsto -(1+\alpha)$ i.e. when $\alpha' = 1/\alpha,\ -(1+\alpha),\ -(1+1/\alpha),\ -\alpha/(1+\alpha)$.

Statement: *Lie superalgebras $sl(m|n)$, for $mn \neq 0$ and $m < n$, $psl(n|n)$ for $n \neq 1$, $osp(m|2n)$ for $mn \neq 0$, $psq(n)$ for $n > 2$, $spe(n)$ for $n > 2$ are simple.*

Exercise: 1) For $\alpha = 0, -1$ the Lie superalgebra $\delta(\alpha)$ is not simple and is simple for $\alpha \neq 0, -1$.

2)In the Lie superalgebras $\delta(-1)$ and $\delta(0)$ there are the ideals $psl(2|2)$ and $sl(2)$ respectively, and we have $\delta(-1)/psl(2|2) \simeq sl(2)$ and $\delta(0)/sl(2) \simeq psl(2|2)$.

1.2. Vectory superalgebras in the standard realization

In applications, Z-graded Lie superalgebras L of polynomial vector fields defined below are usually completed with respect to the topology in which a basis of neighborhoods of 0 is formed by the subspaces $L_i = \{D \in L : \deg D \geq i\}$. The completed algebras are superalgebras of $\delta er\, \mathbb{C}\ [[x,\xi]]$. It is a wonderful and rather unexpected fact that simple vectory superalgebras possess *several* maximal superalgebra of finite co-dimension.

1.2.1. The Lie superalgebras $vect(n/m)$, or for short $ve(n/m)$ is $\delta er\, \mathbb{C}\ [x]$, where $x = (u_1, \dots, u_n, \xi_1 \dots, \xi_m)$ the u_i being even and the ξ_j being odd; it is called *the general vectory superalgebra.*

Remark: Sometimes we will write $vect(x)$ or even $vect(V)$ if $\langle x \rangle = V$ and use similar notations for superalgebras of $vect$ introduced below.

The *divergence* of a field $D = \Sigma f_i \frac{\partial}{\partial u_i} + \Sigma g_j \frac{\partial}{\partial \xi_j}$ is the function (or polynomial, or series)

$$div\ D = \Sigma \frac{\partial f_i}{\partial u_i} + \Sigma (-1)^{p(g_j)} \frac{\partial g_i}{\partial \xi_j}.$$

The Lie superalgebra $sve(n/m) = \{D \in ve(n/m) : dvi\ D = 0\}$ is called the *divergencefree or special vectory superalgebra.* (It may also be described as $\{D \in ve(n/m) : L_D v_x = 0\}$, where v_x is the volume form with constant coefficient in coordinates x and L_D the Lie derivative with respect to D.)

1.2.2. Set $u = (t, p_1 \dots, p_n, q_1 \dots, q_n)$ and let

$$\alpha_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq m} \xi_j d\xi_j \ \omega_0 = d\alpha_1.$$

Sometimes it is more convenient to redenote the ξ and set $\eta_i = \xi_{r+i}$ for $i \leq r =$

$[m/2], \theta = \xi_{2r+1}$ and in place of ω_0 or α_1 take α_1' and $\omega_0' = d\alpha_1'$, where

$$\alpha_1' = dt + \sum_{1\leq i\leq n} (p_i dq_i - q_i dp_i) + \sum_{1\leq j\leq r} (\xi_j d\eta_j + \eta_j d\xi_j) \begin{cases} & \text{if } m = 2r \\ +\theta d\theta & \text{if } m = 2r+1 \end{cases}$$

The forms α_1' and ω_0' correspond to the orthosymplectic form $B'_{m,2n}$ on the space of zeros of α_1 which coincides with the space of zeros of α_1'.

The form α_1 (or α_1') is called *contact*, the form ω_0 (or ω_0') *symplectic.*

Similarly, set $u = q = (q_1, \dots, q_n)$, let $\xi_1, \dots, \xi_n$ and τ be odd. Set

$$\alpha_0 = d\tau + \Sigma(\xi_i dq_i + q_i d\xi_i), \omega_1 = d\alpha_0$$

and call these forms the **odd contact and periplectic,** respectively. The Lie algebra

$$t(2n+1/m) = \{D \in ve(2n+1/m) : L_D\alpha_1 = f_D\alpha_1\},$$

(here $f_D \in ℂ\, [t, p, q, \xi]$ is a polynomial determined by D) is called the **contact superalgebra**, while the Lie superalgebra

$$m(n)\{D \in ve(n/n+1) : L_D \cdot \alpha_0\}, f_D \in ℂ[q, \xi, \tau]$$

is called the **odd contact superalgebra** .

The Lie superalgebra

$$po(2n/m) = \{D \in t(2n+1/m) : L_D\alpha_1 = 0\}$$

is called the **Poisson** superalgebra , while

$$b(n) = \{D \in m(n) : L_D\alpha_0 = 0\}$$

is the **Buttin** superalgebra (in honour of C. Buttin, a student of Schouten, who first proved that Schouten bracket determines a Lie superalgebra structure on its domain.) The Lie superalgebras

$$sm(n) = \{D \in m(n) : div\ D = 0\}, sb(n) = \{D \in b(n) : div\ D = 0\}$$

are called the *divergence-free (or special) odd contact and special Buttin* superalgebras, respectively.

1.2.3. Generating functions: a convenient language for description of t, m and their subalgebras.

For any $f \in \Pi(ℂ\, [t, p, q, \xi]\alpha_1)$, which is usually identified with $ℂ\, [t, p, q, \xi]$ and therefore $(-1)^{p(f)}$ below is understood with respect to the parity in $ℂ\, [t, p, q, \xi]$, set

$$K_f = \Delta(f)\frac{\partial}{\partial t} + H_f + \frac{\partial f}{\partial t}E,$$

where $E = \Sigma y_i \frac{\partial}{\partial y_i}$ (here y are all the coordinates except t) is the Euler operator,

$\triangle(f) = 2f - E(f)$,

$$H_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) + (-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \xi_j},\ f \in\mathbb{C}[p,q,\xi].$$

For any $f \in(\mathbb{C}\ [q,\xi,\tau]\alpha_0)$ which is usually identified with $\mathbb{C}\ [q,\xi,\tau]$ set (with the same reservation of parity as above)

$$M_f = \triangle(f)\frac{\partial}{\partial \tau} + Le_F + \frac{\partial f}{\partial \tau}E,$$

where $E = \Sigma y_i\frac{\partial}{\partial y_i}$ (here are all the coordinates except τ) is the Euler operator, $\triangle(f) = 2f - E(f)$,

$$Le_f = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial \xi_i} - (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial q_i}\right),\ f \in\mathbb{C}[q,\xi].$$

Clearly, $K_f \in t(2n+1/m)$, $M_f \in m(n)$. Indeed:

$$L_{k_f}(\alpha_1) = (-1)^{p(f)}2\frac{\partial f}{\partial t}\alpha_1,\ L_{M_f}(\alpha_0) = -(-1)^{p(f)}2\frac{\partial f}{\partial \tau}\ \alpha_0$$

To the commutators $[K_f,\ K_g]$ or $[M_f,\ M_g]$ there correspond **contact brackets** of the generating functions. Define *Poisson bracket* $\{.,.\}_{P.b.}$ and *Buttin bracket* $\{.,.\}_{B.b.}$ (also known as *Schouten bracket* and dubbed by physicists when they discovered it by *antibracket*) by the formulas

$$\{f,g\}_{P.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial \xi_j}$$

(in the realization with the form ω_0) and

$$\{f,g\}_{B.b.} = \sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial \xi_i} - (-1)^{p(f)}\ \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial q_i}\right).$$

Then the contact brackets take the form, respectively

$$\{f,g\}_{k.b.} = \triangle(f)\frac{\partial q}{\partial t} - \frac{\partial f}{\partial t}\triangle(g) - \{f,g\}_{P.b.}$$

and

$$\{f,g\}_{m.b.} = \triangle(f)\frac{\partial g}{\partial \tau} - (-1)^{p(f)}\ \frac{\partial f}{\partial \tau}\triangle(g) - \{f,g\}_{B.b.}.$$

The Lie superalgebra of **Hamiltonian fields or Hamiltonian superalgebra** is

$$h(2n/m) = \{D \in ve(2n/m) : L_D\omega_0 = 0\}$$

and its odd analogues are

$$le(n)\{D \in ve(n|n) : L_D\omega_1 = 0\} \text{ and } sle(n) = \{D \in le(n) : \text{div } D = 0\}.$$

Exercise.$t(2n+1/m) = \langle K_f : f \in\mathbb{C}[t,p,q,\xi]\rangle$, $m(n) = \langle M_f : f \in\mathbb{C}[\tau,q,\xi]\rangle$, $h(2n/m) = \langle H_f : f \in\mathbb{C}[p,q,\xi]\rangle$, $le(n) = \langle Le_f : f \in\mathbb{C}[q,\xi]\rangle$.

Define the ideals $sle^0(n)$ and $s^0(n)$ from the exact sequences

$$0 \to sle^0(n) \to le(n) \to \mathbb{C}Le_{\xi\dots\xi_n} \to 0,\ 0 \to s^0(n) \to sve(1/n) \to \mathbb{C}\xi\dots\xi_n\frac{\partial}{\partial t} \to 0$$

Remarks: 1) It is obvious that the Lie superalgebras of the series ve, sve, h and po for $n = 0$ are finite-dimensional.

2) A Lie superalgebra of the series h (respectively, le and sle) is a quotient of the Lie superalgebra po (respectively, b and sb) modulo the one-dimensional center z which in realization with generating functions consists of constants. Set

$$spo(m) = \{K_f \in po(0/m) : \int fv_\xi = 0\}, sh(m) = spo(m)/z.$$

Since

$$div\ M_f = (-1)^{p(f)}(2\frac{\partial f}{\partial\tau} - \frac{\partial}{\partial\tau}E(f) + \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i\partial\xi_i} + 2n\frac{\partial f}{\partial\tau} + (\frac{\partial f}{\partial\tau})) = \qquad (1)$$
$$(-1)^{p(f)}(2(n+1)\frac{\partial f}{\partial\tau} + \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i\partial\xi_i},$$

then

$$sm(n) = \left\langle M_f \in m(n) : 2(n+1)\frac{\partial f}{\partial\tau} + \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i\partial\xi_i} = 0\right\rangle.$$

In particular,

$$div\ Le_f = \sum_{i\leq n}\frac{\partial^2 f}{\partial q_i\partial\xi_i}.$$

1.2.4. A digression: The modules of (formal) tensor fields and their dual. There are two functors that to a module over a subalgebra assign a module over an algebra: *ind* and *Coind*. Given a vectory Lie superalgebra $\mathcal{L} = \oplus_{i>-d}\mathcal{L}_i$ and an $\mathcal{L}_0$-module V, consider V as a $\mathcal{L}_>$-module, where $\mathcal{L}_> = \oplus_{i\geq 0}\mathcal{L}_i$ setting $\mathcal{L}_iV = 0$ for $i > 0$. Now, if V is a left module over $\mathcal{L}_0$ and $\mathcal{L}_>$ set

$$ind^{\mathcal{L}}_{\mathcal{L}_>} = U(L)\otimes_{U(\mathcal{L}_>)} V$$

and if V is a right module over $\mathcal{L}_0$ (and $\mathcal{L}_>$) set

$$coind^{\mathcal{L}}_{\mathcal{L}_>}(V) = Hom_{U(L_>)}(U(\mathcal{L}), V).$$

Note that 1) since $U(L)$ is a two sided $U(\mathcal{L}_>)$-module it can formula for (ind) or as a left (in the formula for $coind$) $U(L_>) - m$ 2) any left L_0-module can be naturally considered as a right one or vice versa ($lv = -(-1)^{p(v)p(l)}vl$). If V is a module with highest (lowest) weight X given by its numerical labels with respect to some basis of a maximal torus of L_0, we abbreviate the above notations to $I(X_1, \dots, X_n)$ and $T(X_1, \dots, X_n)$ and if $X_1 = \dots = X_n = \lambda$ we just write $I(\lambda)$ or $T(\lambda)$. The letter T

stands for "tensor". Indeed, given a vector bundle B_M^V over an n-dimensional manifold M^n associated with a representation ρ of $GL(n)$ in a space V, we define the action of the group $Diff(M)$ of diffeomorphisms of M in the space $\Gamma(M, B_M^V)$ of sections of the bundle B_M^V by the formula

$$(gt)(m) = (\rho(J_g(m))(t))(g^{-1}m) \qquad *$$

for $m \in M$, $g \in Diff(M)$, $t \in \Gamma(M, B_M^V)$ and the Jacobi matrix J_g of the transformation g. The elements of $\Gamma(M, B_M^V)$ are called tensor fields of type V; if we confine to jets of the action (*) or consider it locally then the tangent to this action gives us the $vect(n/0)$-module $T(V)$.

1.2.5. Vectory superalgebras in nonstandard realizations. In this subsection we list the natural gradings other than standard ones. These gradings are associated with filtrations first described (for Lie algebras) by Weisfeiler as follows. Let $L = L_{-d} \supset \ldots \supset L_0 \supset \ldots$ be a Lie superalgebra filtered so that L_0 is a maximal subalgebra of finite co-dimension. Let $\mathcal{L}_- = L/L_0$ and $frakL_{-1}$ a non trivial irreducible (L_0/L_1)-submodule of L_-. Set

$$\begin{aligned} &\mathcal{U}_0 = L_0,\ \mathcal{U}_{-1} = L_{-1} \oplus \mathcal{U}_0 \\ &\mathcal{U}_{i-1} = [\mathcal{U}_{-i}, \mathcal{U}_{-1}],\ \mathcal{U}_i = \{u \in \mathcal{U}_{i-1} : [u, \mathcal{U}_{-1}] \subset \mathcal{U}_{i-1} \text{ for } i > 0\} \end{aligned} \qquad (WF)$$

The *Weisfeiler filtration (WF)* is nice because the associated Z-graded Lie algebra $U = \oplus \mathcal{U}_i/\mathcal{U}_{i+1}$ is incompressible.

Let us enumerate all Z-grading in L associated with the Weisfeiler filtrations. The standard realization is labeled by asterisk (*); we note that it corresponds to the case with minimal co-dimension of L_0 (at least in the general case). In the following Table we set shortly deg$x =^0 x$ and suppose that the contact (symplectic) forms are chosen of the form α' and ω' respectively. Note that a grading of the series ve (respectively m or t) induces a grading of the subalgebra of series indicated in parenthesis.

Lie superalgebra	Z-grading
$vect(n/m;r), 0 \leq r \leq m$	${}^0u_i =^0 \xi_j = 1(*)$
	${}^0\xi_j = 0$ for $1 \leq j \leq r$; ${}^0u_i =^0 \xi_{r+j} = 1$ for $j \geq 1$
$t(1/2n;n)$	${}^0t, =^0 \xi_i = 1,\ {}^0\xi_i = 1,\ {}^0\xi_{n+1} = 0$ for $1 \leq i \leq n$
$m(n;r), 0 \leq r \leq n$	${}^0\tau = 2,\ {}^0q_i =^0 \xi_i = 1(*)$
	${}^0\tau =^0 q_i = 1,\ {}^0\xi_i = 0$
	${}^0\tau =^0 q_i = 2,\ {}^0\xi_i = 0$ for $1 \leq i \leq r < n$; ${}^0u_{r+j} =^0 \xi_{r+j} = 1$
$t(2n+1/m;r), 0 \leq r \leq [\frac{m}{2}]$	${}^0t = 2,\ {}^0p_i =^0 q_i =^0 \xi_j = 1$ (*)
	${}^0t, =^0 \xi_i = 2,\ {}^0\xi_{r+1} = 0$ for $1 \leq i \leq r \leq [m/2]$
	${}^0p_i =^0 q_i =^0 \xi_{2r+j} = 1$ for $j \geq 1$

Remark: Experts in string theories have now rediscovered some of exceptional gradings that are responsible for the fact that $vect(1/1)$ and $t(1/2)$ are isomorphic as abstract subalgebra. As was noticed in 1978 by Alexeevsky, Leites and Shchepochkina, these superalgebras are also isomorphic to $m(1)$, cf. [Leites 2].

1.2.6. Deformations of vectory superalgebras: The complete list is unknown yet, we only give some examples.

1) *Quantization* is a physical name for the deformation $po(2n/2m) \to$"gl"$(C[q,\xi])$, where the quotation mark indicate that we are aware of many candidates for the role of gl in the infinite dimensional case. Our choice and the deformation are described as follows. To a generating function $f \in \mathbb{C}\,[q,p,\xi,\eta]$ assign an operator $\hat{f} \in \mathbb{C}\,[q, \frac{\partial}{\partial q}, \xi, \frac{\partial}{\partial \xi}]$. There are many ways to do that, e.g. select a normal form, say a qp form where a polynomial in q,p,ξ,η is arranged so that in every monomial, powers of q and ξ come first followed by powers of p and η and then replace p by $\hat{p}_i = \hbar\frac{\partial}{\partial q_i}, \eta_j$ by $\hat{\eta}_j = \hbar\frac{\partial}{\partial \xi_j}$. (Physicists denote the parameter of the deformation by $\hbar$.) Then:

$$\{f,\hat{g}\}_{P.B.} = [\hat{f},\hat{g}]$$

In particular, passing to quotients we get deformations

$$h(2n/2m) \to pgl(\mathbb{C}[q,\xi])$$

and, for $n = 0$, restricting this co-cycle we get $sh(2n) \to psl(2^{m-1}/2^{m-1})$. For $po(2n/2m-1)$ the restriction of the above construction gives a deformation

$$po(2n/2m-1) \to po(2n/2m-1) \to q(\mathbb{C}[q,\xi])\{\hat{f} \in gl(\mathbb{C}[q,\xi]) : [f,\pi] = 0\}$$

for some odd π such that $\pi^2 = -1$; say $\pi = i(\xi_1 + \frac{\partial}{\partial \xi_1})$.

In particular we get deformations $sh(2m-1) \to psq(2^{m-1})$.

Important remark: The deforms at $\hbar$ and $\hbar'$ are isomorphic but not isomorphic to the initial algebra ($\hbar = 0$).

2) *Quantization of antibracket* is an exceptional deformation of $b(n)$. Kochetev managed to list all the deformations and here are his results. Deformations of $b(n)$ and $sm(n)$. Recall that these superalgebras are contact m-prolongs of $b(n)_0 = pe(n)$ and $sm(n)_0 = s(cpe)(n)$. The contact m-prolongs of $\mathbb{C}\,(ae+bd) \oplus spe(n) = L_0$ (with the same L_{-1} and L_{-2} as for b and sm) will be called the main deformation. Let us describe the main deformation of $b(n)$ and $sm(n)$ differently.

The first algebra preserves $\alpha_0 = d\tau + \Sigma(qd\xi + \xi dq)$, the second one preserves the volume element $v = vol[q/\xi,\tau]$. Since $div(ac+bd) = a+bn$, their deform preserves a subspace

$$\mathbb{C}(a\alpha_0) + (a+bn)v[q/\tau,\xi]$$

Remark. Here we are hinted as to what is a natural parity of v. Set for $\lambda = a/b$,

$$b_\lambda(n) = \{M_f \in m(n) : a\ divM_f = (a+bn)\frac{\partial f}{\partial \tau}.$$

On the space of parameters there are, clearly, three distinguished points

$$\begin{array}{cccc} b_n(n) \cong & sm(n) & b(n) & \\ \hline -n & -1 & 0 & \lambda \in \mathbb{C}P^1 \end{array}$$

To see that $\lambda = -1$ is distinguished, consider $b(n';n)$. Then

$$b_\lambda(n;n)_0 = vect(0/n), \quad b_\lambda(n;n)_{-1} = \Pi(T(\lambda,\ldots,\lambda)).$$

As we know (see [L2]), the $vect(0/n)$-module $T(\lambda)$ is irreducible for any λ except for 0 and -1: in the first case it contains the invariant subspace of constants in the dual second case the invariant subspace of co-dimension ϵ^{n+1} of forms with integral 0. Therefore, there is an exact sequence

$$0 \to b^0_{-1}(n) \to b_{-1}(n) \to \mathbb{C} M_{\xi_1 \dots \xi_n \tau} \to 0.$$

Exercise: Show that $b^0_{-1}(n)$ is a simple algebra.

Since $vect(0/n)$-modules $T(\lambda)$ are non isomorphic for different λ, the algebras $b_\lambda(n)$ are non isomorphic. Hence every $b_\lambda(n)$ can be deformed a long parameter λ, this deformation will be called the *main one.*

Now, look at the formula that determines $b_\lambda(n)$ for $a = n$:

$$div M_f = (1+b)\frac{\partial f}{\partial \tau} \text{ or } 2(n+1)\frac{\partial f}{\partial \tau} + \sum \frac{\partial^2 f}{\partial q_i \partial \xi_i} = (1+b)\frac{\partial f}{\partial \tau}.$$

If $b = 2n+1$, the operator that singles out $b_{n/2n+1}(n)$ does not depend on τ!

As had been noticed by Kochetkov, there are two more distinguished points: $\lambda^{\pm} = -n/(n \pm 2)$. He has found that there is an exact sequence

$$0 \to b^0_{\lambda-}(n) \to b_{\lambda-} \to \mathbb{C} M_{\xi_1 \dots \xi_n} \to 0$$

and the Lie superalgebra $b^0_{\lambda-}(n)$ is simple.

The points $\lambda^{\pm}$ and 0 are distinguished (in addition to the above described property of λ^-) by the following fact: in addition to the main deformation there are deformations at $\lambda^{\pm}$ and 0. These deformations will be called the *exceptional* ones.

Note that for $n = 2$ there is no exceptional deformation at λ^- and $b_{\lambda-}(3) = sm(3)$.

Explicitly the co-cycles c that determine the deformation are given by the following formulas where $q^k = q_1^{k_1} \dots q_n^{k_n}$, $|k| = \Sigma k_i$.

λ	c	$p(c)$
$-n/(n-2)$	$f, \xi_1 \dots \xi_n \mapsto (-1)^{k-1} f$, where $(\Sigma \xi_i \frac{\partial}{\partial \xi_i})(f) = kf$ $\xi_1 \dots \xi_n,\ \xi_1 \dots \xi_n \mapsto -2(n-1)\xi_1 \dots \xi_n$ for even n	$(n+1) \bmod 2$
$-n/(n+2)$	$q^l,\ q^l \mapsto (4 - \lvert k\rvert - \lvert l\rvert) q^{k+l} \xi_1 \dots \xi_n$ $\Sigma(k_i + l_i) q^{k_i + l_i} q_i^{-1} \frac{\partial}{\partial \xi_i}(\xi_n \dots \xi_n)$	$(n+1) \bmod 2$
0	$f,\ g \mapsto (k_f - 1)(k_g - 1) fg$ where $(\Sigma \xi_i \frac{\partial}{\partial \xi_i})(f) = k_f \dot{f}$	1

Theorem: (Yu. Kochetkov) 1) *The co-cycles given in the above Table determine global deformations.*

2) *Let m be the 2-co-cycle corresponding to the main deformation, c the co-cycle described in 1). Then the co-cycle $ac + bm$ extends to a global deformation if and only if $ab = 0$. The deforms of $b_\lambda(n)$ corresponding to exceptional deformations are given by*

the formulas $[X,Y]_{new} = [X,Y]_{old} + tc(X,Y)$ *where* $p(t) = p(c)$.

$$\dim H^2(g_\lambda; g_\lambda) = \begin{cases} 1+\epsilon^{n+1} & \text{for } \lambda = -n/(n\pm 2) \\ 1+\epsilon & \text{for } \lambda = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Deformations of $sb(n)$ are given by the following co-cycles c_1, c_2 and c_3:

$$c_1(f,g) = \{fg, \xi_1 \dots \xi_n\}_{B.b} \text{ for } f,g \in \mathbb{C}[q]$$
$$c_2(f\xi_1 \dots \xi_n\} = (-1)^{k-1}(k-1)f \text{ for } Le_f \in sb(n)$$

where k is the degree of f with respect to odd variables.

$c_3(\xi_1 \dots \xi_n, \xi_1 \dots \xi_n) = 1$ defined for even n only.

Deformations of $sve(0/n)$.

Theorem: *A volume form on the (0/n)-dimensional supermanifold reduces to the following canonical form (by volume preserving automorphisms):* $(a + b\xi \dots \xi_n)v[\xi]$, *where* $p(b) \equiv n$ *mod 2.*

Proof: For $n \equiv 0(2)$ see [Kal], the case $n \cong 1$ is dealt with in literally the same way only the result is over $\mathbb{C}$ $[\tau]$ for an odd τ (that is why it was not considered in [Kal]). □

If n is even and $a \neq 0$ we can, by rescaling, set $b = 1$, therefore all the deforms of $sve(0/2n)$ that preserve $(1 + t\xi_1 \dots \xi_{2n})v[/\xi]$ are isomorphic. Denote them by $s'(2n)$. Denote by $s'(2n+1)$ the above deformation with odd parameter t.

Deformation of $sle(n)$. Denote by $sle_{t'}(n;n)$ the Cartan prolongation of the pair $\mathcal{L}_0 = s'(n)$

$$\mathcal{L}_{-1} = \begin{cases} \Pi(T(-1)/\mathbb{C}(1 + t\xi_1 \dots \xi_n))v_\xi & \text{for } n \equiv 0(2) \\ \Pi(T(-1) \otimes \mathbb{C}[\tau]/\mathbb{C}(1 + t\tau\xi_1 \dots \xi_n))v_\xi & \text{for } n \equiv 1(2) \end{cases}$$

with $t \in \mathbb{C}$. The deformation thus described will be called the *main* one.

Statement: (Yu. Kochetkov). $sle_{t'}(n) \cong sle'_{t'}(n)$ *iftt'* $\neq 0$. *The co-cycle c that determines this deformation is* dc_1, *where* $c_1 \in C^1(sle(n))$ *sends*

$$f(q) \mapsto f(q)\xi_1 \dots \xi_n \text{ for any } f(q) \in \mathbb{C}[q,\xi]$$

(*We identify the elements of* $\Pi(\mathbb{C}[q,\xi])$ *with elements of* $sve(n)$.

Thanks to the statement, we will only consider $t = 1$ and set $sle_{t'}(n) = sle'(n)$. Now, let us describe two more deformations of $sle(n)$. They will be called *exceptional* and are given by the co-cycle

$$c(f, \xi_1 \dots \xi_n) = (-1)^{k-1}(a(k-1) + b(1-l))f$$
$$c(\xi_1 \dots \xi_n, \xi_1 \dots \xi_n) = -2((n-1)a + b)\xi_1 \dots \xi_n$$

where $a,\ b,\ \in \mathbb{C}, k$ (resp.1) is the degree of $f \neq \xi_1 \dots \xi_n$ with respect to odd (even) indeterminates.

Theorem. (Yu. Kochetkov). 1) $sle'(2n)$ *is rigid.*

2) $sl\epsilon'(2n+1)$ has one more deform $sve^{(2)}(2n+1)$ with an odd parameter τ_2; the corresponding co-cycle is dc_1, where

$$c_1(f+\tau_2 f\xi_1\dots\xi_n)=f\xi_1\dots\xi_n,\ c_1(\tau_2 f)=\tau f\xi_1\dots\xi_n$$

3) The construction of heading 2) can be iterated and infinum with odd parameters $\tau_3,\ \tau_4\dots\tau_k,\dots$ thus producing Lie superalgebras $sve^{(k)}(n)$ defined over $\mathbb{C}[\tau,\tau_2,\dots,\tau_k]$.

1.2.7. Three exceptional vectory superalgebras. Three constructions. 1) cve_* Let $g_{-1}=\Pi(T(0)/\langle 1\rangle), g_0=cve(0/n)$. Then $cve(0/3)_*$ is simply exceptional. Besides

$$cve(0/2)_*\cong ve(2/1) \text{ and } cve(0/n)_*\cong \delta ve(0/n)_* \text{ for } n>3.$$

2) cv_*^{mk} and $sve_{a,b}(n)_*^{nk}$: let either
A) $g_0=cve(0/n); g_{-1}=\Pi(T(-1/2))$, the space of half-densities, the forms with constant coefficients being considered as odd, or
B) $g_0=sve(0/n)\oplus\mathbb{C}(ax+bz)$, where x is the Z-grading operator in $sve(0/n)$, z commutes with $sve(0/n)$ and $a,b\in\mathbb{C}; g_{-1}=\Pi(T^0(0)/\langle 1\rangle$, where $T^0(0)=\{f\in T(0):\int fv_\xi=0\}$. Define a nondegenerate form ω with $p(\omega)\equiv n(2)$ on g_{-1} setting

$$\omega(f,g)=\begin{cases}\int fg & \text{in case 1 for } f,g\in\Pi(T(-1/2))\\ \int fgv_\xi & \text{in case 2 for } f,g\in\Pi(T^0(0))\end{cases}$$

Then $cve(0/3)_*^m$ and $sve_{6,-5}(4)^k$ are simply exceptional.

Besides, $cve(0/2)_*^k\simeq t(3/2); sve_{\alpha,\beta}(3)_*^m\cong b_\lambda(3)$ for $\lambda=$. In particular,

$$sve_{0,1}(3)^m\cong sm(3); sve_{3,-2}(3)_*^m.$$

$$cve(0/n)_*^{mk}\equiv vect(0/n)_*^{mk}\oplus\mathbb{C}\cdot z=c(vect(0/n)_*^{mk})$$

$$sve_{a,b}(n)_*^{mk}\equiv sve(0/n)_*^{mk}\oplus\mathbb{C}(ax+bz) \text{ for } n>4 \text{ or } n=4 \text{ and } (a,b)\not\in\mathbb{C}(6,-5)$$

1.2.8. Occasional isomorphisms and simplicity (cf 1.1.5.)

$$sl(1/2)\cong osp(2/2)\simeq vect(0/2)$$

The last isomorphism shows that $h(2/2)$ **has an extra deformation beside the one induced by quantization.**

Nonstandard gradings supply with isomorphisms partially already listed (for exceptional gradings:)

$$ve(1/1;2,1)\cong m(1)\ ve(1/1{:}\,1,-1)\cong t(1/2)$$

Statement. *Lie superalgebras* $vect(m/n)$ for $(m,n)\neq(0,1)$, $sve(m/n)$ for $(m,n)\neq(0,1),(0,2)$ and $m\neq 1$; $sve^0(n)$ for $n\neq 1$, $t(2m+1/n), h$ for $n/3; m(n); sm(n); le(n)$ and $sle^0(n)$ for $n>1$ are simple. Simplicity of deforms and of exceptional algebras was indicated separately above.

1.3. Stringy superalgebras: These superalgebras are particular Lie algebras of vector fields, namely those that preserved a structure on what physicists call a *superstring*, a supermanifold associated with a vector bundle on a circle.

Let φ be an angle parameter on the circle, $t = \exp(i\varphi)$. The stringy superalgebras are algebras of derivations of either the superalgebra $R^l(n) = \mathbb{C}[t^{-1}, t, \xi_1, \dots \xi_n]$ of complex valued functions on the circle expandable into finite Fourier series if the supermanifold in question is $S^{1/n}$ associated with the trivial bundle, or of a similar superalgebra $R^m(n) = \mathbb{C}[t^{-1}, t, \xi_1, \dots, \xi_{n-1}, \sqrt{t}\xi]$ if the supermanifold is $S^{1/n-1/m}$ associated with the Whitney sum of the Moebius bundle and the trivial one. (Since the Whitney sum of two Moebius bundles are isomorphic to the trivial bundle of rank 2, one Moebius sum and suffices.)

1.3.1. Introduce analogs of ve, sve, sve^0 having replaced $= \mathbb{C}[t, \xi_1, \dots, \xi_n]$ by $R^l(n)$:

$$vect^l(n) = der\ R^l(n)$$

$$t^1(n) = \{\mathcal{D} \in vect^l(n) : \mathcal{D}(\alpha) = f_{\mathcal{D}}\alpha_1 \text{ for } \alpha_1 = dt \sum \xi_i d\xi_i \text{ and } f_{\mathcal{D}} \in R^l(n)\}$$

Clearly the formula for K_f is the same as for $t(1/n)$ only with $f \in R^l(n)$.

Exercise: The algebras $vect^m(n), sve^m$ and $sve^{0m}(n)$ obtained by replacing $R(n)$ by $R^m(n)$ are isomorphic to $vect^l(n), sve^l(n)$ and $sve^{0l}(n)$ respectively.

The lift of the contact structure from $S^{1,n}$ to its two-sheeted covering $S^{1/n,m}$, gives a new structure. Indeed, this lift means replacing ξ_n by $\sqrt{t}\theta$, or the form α_1 by the form ${}^m\alpha = dt + \Sigma\xi_i d\xi_i + t\theta d\theta$. As for t, it is convenient to pass to the form:

$${}^m\alpha = \begin{cases} dt + \Sigma_{i \le k}(\xi_i d\eta_i + \eta_i d\xi_i) + t\theta d\theta & \text{if } n = 2k+1 \\ dt + \Sigma_{i \le k}(\xi_i d\eta_i + \eta_i d\xi_i + \zeta d\zeta) + t\theta d\theta & \text{if } n = 2k+2 \end{cases}$$

Now we have two ways for describing vector fields that preserve ${}^m\alpha$:

$$t^m(n) = \{\mathcal{D} \in vect^m(n) = der\ R^m(n) : \mathcal{D}({}^m\alpha) = f_{\mathcal{D}}\alpha_1 \text{ where } f_{\mathcal{D}} \in R^m(n)\} \quad (aut_{R\alpha_1})$$

In which case the fields K_f are given by the same formulas as for t^1 but with generating functions from $R^m(n)$; equivalently we may define

$${}^m t(n) = \{\mathcal{D} \in vect^l(n), : \mathcal{D}\alpha = f^m_{\mathcal{D}}\alpha \text{ where } f_{\mathcal{D}} \in R^l(n)\} \qquad (aut_{R^l}({}^m\alpha))$$

where $\mathcal{D} = \frac{\partial}{\partial t} - \frac{\theta}{2t}\frac{\partial}{\partial \theta}$, ${}^m H_f = (-1)^{p(f)}(\Sigma \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_i} + \frac{1}{t}\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta})$ and $E = \Sigma\zeta_i \frac{\partial}{\partial \zeta_i} + \theta\frac{\partial}{\partial \theta}, \triangle = 2 - E$.

1.3.2. Nonstandards gradings of stringy superalgebras. The Weisfeiler gradings of vectory superalgebras with polynomials coefficients induce incompressible Z-gradings of stringy superalgebras with finite-dimensional fibres. Let us list possible Z-gradings with the above property for $t^m(n)$. In realization $aut_{R^l}(\alpha)$ these Z-gradings are:

$t^m(n)$	$\deg t = 2, \deg\theta = 0, \deg\xi_i = 1$
$t^m(2n; r)$ $1 \le r < n$	$\deg t = \deg\eta_1 = \dots = \deg\eta_r = 2 \ \ \deg\xi_{i+r} = 1$ $\deg\theta = \deg\xi_1 = \dots = \deg\xi_r = 0 \ \ \deg\eta_{i+r} = \deg\zeta = 1$
$t^m(2n+1 : m)$	$\deg t = \deg\eta_1 = \dots = \deg\eta_n$ $\deg\theta = \deg\xi_1 = \dots = \deg\xi_\eta = 0$

1.3.4. Exceptional superalgebras: Nonstandard gradings of $vect^l(1)$ and $t^l(1/2)$ establish an isomorphism between them and the superalgebra

$$m^l(1) = \{\mathcal{D} \in vect^l(1) : \mathcal{D}\alpha_0 = f_{\mathcal{D}}\alpha_0, \text{ where } f_{\mathcal{D}} \in R^l(1), \alpha_0 = d\tau + qd\xi + \xi dq\}$$

which is *the exceptional stringy superalgebra* .

1.3.5. Deformations: The superalgebras $sve(1/n)$, and $sve^0(n)$ do not have deformation that preserve Z-gradings (other deformations may happen; one has to calculate, this is a research problem). The stringy superalgebras $sve^l(n)$ do have deformations. For $\lambda \neq -1$ set

$$sve^l_\lambda(n) = \{\mathcal{D} \in vect^l(n) : div(t^\lambda \mathcal{D}) = 0\}.$$

We will write sve^l for sve^l_0.

Statement: *$vect^l$, $sve^l_\lambda(n)$ for $\lambda \neq 0$ and $n > 1$, $sve^{0l}(n)$ for $n > 1$, $t^m(n)$ and $t^l(n)$ for $n \neq r$ are simple. There is a simple ideal $k^{l0}(4)$ of co-dimension 1 in $t^l(4)$.*

$$0 \to t^{l0}(4) \to t^l(4) \to K_{\xi_1 \ldots \xi_n / t}.$$

To understand the mechanism that brought this exception we have to make a

Disgression: tensor fields on a supercycle with contact structure. Denote by $\mathcal{F}_\lambda$ the $t(2n+1/m)$-module

$$\mathcal{F}_\lambda = \begin{cases} \mathcal{F}\alpha_1^\lambda & \text{if } n = m = 0 \\ \mathcal{F}\alpha_1^{\lambda/2} & \text{otherwise} \end{cases}$$

where $\mathcal{F}$ is the algebra of functions i.e. polynomials, etc.

Denote by $\mathcal{F}^l_\lambda$ the algebra $\mathcal{F}_\lambda$ with Laurent polynomials for elements of $\mathcal{F}$ and set $\mathcal{F}^l_{\lambda,\mu} = t^\mu \mathcal{F}^l_\lambda$.

Examples: 1) The $t(2n+1/m)$-module of volume forms in F_{2n+2-m}. In particular, $t(2n+1/2n+2) \subset sve(2n+1/2n+2)$.

2) As $t(2n+1/m)$-module $t(2n+1/m)$ is $\mathcal{F}_{-1}$ for $n = m = 0$ and $\mathcal{F}_{-2}$ otherwise. In particular, $t(1/4) \simeq vol$.

Define the residue on $S^{1/n}$ setting

$$Res : Vol \to C, \ f v_{t,\xi} \mapsto \text{coeff. of } \xi_1 \ldots \xi_n / t$$

It follows from example 2 that functions with zero residue on $S^{1/4}$ form a subspace or, rather, ideal in $t^l(4)$.

References

[B] Berezin F.A., Introduction to analysis with anticommuting variables. Kluwer, Dodrecht, 1987.

[F] Fuks(Fuchs) D., Cohomology of infinite dimensional Lie algebras. Consultants Bureau, NY, 1987.

[Ka1] Kac V.G., Lie superalgebras . Adv. Math. v. 26, 1977. 8-96.
[Ka2] Kac V.G., Classification of simple Z-graded Lie superalgebras and simple Jordan superalgebras. Commun. Alg. 5(13), 1977, 1375-1400.
[L1] Leites D., Lie superalgebras . J. Sov. Math. (JOSMAR), 30, 1984.
[L2] Leites D. (ed.), Seminar on supermanifolds, nn1-34. Reports of Dept. of Math. of Stockholm Univ. 1986-89.
[M] Manin Yu., Gauge fields and complex geometry. Springer, NY. 1987

Status of the Algebraic Approach to Super Riemann Surfaces

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Abstract

I review the description of super Riemann surfaces as algebraic curves in projective superspace, as opposed to the usual descriptions as 2d supergravity geometries or (1,1) supermanifolds with superconformal structure. Some applications and several open problems in this approach are described.

1 Introduction

Just as Riemann surfaces are the basic geometric objects which appear in conformal field theory and perturbative string theory, super Riemann surfaces (SRSs) assume this role for superconformal theories and superstrings. The mathematical literature contains three complementary but equivalent descriptions of Riemann surfaces: a geometric description, as conformal equivalence classes of Riemannian metrics on smooth 2-manifolds; an analytic description, as complex analytic 1-manifolds; and an algebraic one, as algebraic curves in some complex projective space P^n. Although the geometric and analytic approaches have been most exploited in string theory thus far, the algebraic approach offers numerous potential advantages. First, it makes available all the powerful machinery of algebraic geometry (sheaf cohomology, schemes, divisors, intersection theory, analysis of singularities, and so forth) for the study of questions in string theory. Closely related is the prospect of recruiting algebraic geometers themselves to attack these questions, in keeping with the trend of the current mutual brain drain from mathematics to physics and vice versa. Second is the possibility of simplifying the actual calculations of perturbative string theory. Such calculations routinely require the construction of functions, differentials, and sections of various bundles having prescribed zeros and poles on a Riemann surface (or on moduli space). In view of the well-developed technology for expressing such objects in terms of theta functions and prime forms, it may be outdated to describe these as complicated and unfamiliar transcendental functions. Nevertheless, Serre's so-called GAGA principle [1] guarantees that all such functions and sections on a projective curve are actually simple rational functions in terms of the coordinates of the ambient projective space. The simplicity of this description has been exploited in some loop calculations, particularly on hyperelliptic curves [2,3]. Next, the algebraic description makes manifest the connection between Riemann surfaces and number theory, reflected in the interest in strings over the p-adics and other number fields [4,5,6,7,8]. Once a Riemann surface has been described as the zero set of polynomial equations, it is clear that the variables in these equations need not be restricted to complex values but may range over any number field containing the coefficients. There are other algebraic structures of interest in string theory whose connection to Riemann surfaces is conveniently described in an algebraic viewpoint. For example, the KP system of equations, closely related to the operator formalism and the Krichever-Novikov algebras, is usefully viewed in this way [9]. The KP system can be viewed as a description of isospectral deformations of a commutative ring of differential operators in one variable. It can be shown that a pair of operators in such a ring is always connected by a polynomial relation, which becomes the equation of the associated Riemann surface considered as a (possibly singular) variety. Finally, there is the vision of Friedan and Shenker of a purely algebraic or even arithmetic nonperturbative formulation of string theory [10]. The algebraic representation of Riemann surfaces is but a small step toward this goal, which requires an algebraic description of moduli space, and indeed universal moduli space, as well. Clearly this list of motivations argues equally strongly for an algebraic description of SRSs as for one of Riemann surfaces.

The geometric [11] and analytic [12,13,14,15] viewpoints on SRSs are well established in the literature. The newer algebraic viewpoint was introduced only last year [16,17,18,19,20] with the proof that any SRS, and indeed any supermanifold of dimension (1,N), can be embedded in some projective superspace as the zero set of some (even and odd) polynomial equations. In [16] this was shown by explicit construction for supertori, using the theory of superelliptic functions. In [17] the proof was extended to all SRSs, although without explicit formulas for the embeddings or the polynomial equations. Then LeBrun, Poon, and Wells [18,19] independently gave a super extension of the Kodaira embedding theorem which gives the conditions for a supermanifold to embed in projective superspace, and of Chow's theorem (or lemma, according to authors less easily impressed) which guarantees that any embedded supermanifold is the zero set of polynomials. This work is described in Wells' contribution to these Proceedings. [20] gives a more abstract treatment of these theorems a la Hartshorne [21] and includes a super GAGA principle. Although the methods, language, and generality of these papers differ slightly, the ideas and results are essentially the same.

In this paper I will sketch the proofs of the Kodaira and Chow theorems and then discuss the most important outstanding problems in the algebraic theory of SRSs. In contrast to Wells' discussion of arbitrary supermanifolds, I will focus specifically on SRSs and emphasize the language and concerns of superconformal field theory. In Section 2 I review the basic facts about SRSs, explain how to embed them in projective superspaces, and indicate why an embedded SRS is algebraic. Section 3 discusses the important open problems.

2 Projective Embeddings of Super Riemann Surfaces

Recall that in the usual analytic description, a SRS is a complex supermanifold M of dimension (1,1) with an additional "superconformal" structure. In the overlap region of two charts with coordinates (z,θ) and $(\tilde{z},\tilde{\theta})$ the transition functions give a superanalytic relation,

$$\tilde{z} = f(z) + \theta\zeta(z), \quad \tilde{\theta} = \psi(z) + \theta g(z). \tag{1}$$

The coordinates are superconformal if all transition functions obey $D\tilde{z} = \tilde{\theta}D\tilde{\theta}$, where $D = \partial_\theta + \theta\partial_z$. This condition implies that

$$g = \sqrt{f' + \psi\psi'}, \quad \zeta = \psi\sqrt{f' + \psi\psi'} = \psi\sqrt{f'}, \tag{2}$$

leaving only two independent functions in (1).

The superconformal condition implies that D transforms homogeneously, $D = (D\tilde{\theta})\tilde{D}$, so that there is a well-defined global notion of a field belonging to the kernel of D. Another way to say this is that the vector field D generates a line subbundle, often denoted $\hat{\omega}^{-1}$, of the tangent bundle $T(M)$. An equivalent formulation is that the 1-form $dz + \theta d\theta$ generates the subbundle $\hat{\omega}^2$ of the cotangent bundle. $\hat{\omega}$ itself is nothing but Ber, the canonical bundle of M, generated by the holomorphic volume element $[dz\, d\theta]$ or dz

appearing in super contour integrals. Sections of the bundle $\hat{\omega}^p$ are called superconformal tensors of weight $p/2$ [or $(p/2, 0)$]. That is, *the superconformal condition on the transition functions picks out those supermanifolds on which the physically crucial notion of superconformal tensor makes sense.*

The transition functions of a SRS may depend on a set of anticommuting parameters which I will call $\beta_1, \beta_2, \ldots, \beta_L$. What you think these are depends on what you think a supermanifold is. If you adopt the description of a supermanifold as a ringed space, as in Wells' article in this volume, you will say that I am not discussing a single supermanifold at all, but rather a family of supermanifolds, and the β_i are parameters in the parameter space of this family. If you prefer the less abstract supermanifold theory of DeWitt [22] and Rogers [23], the idea is that odd constants β_i may appear in the transition functions of a single supermanifold in the same way that even constants such as $\sqrt{2}$ or π may. Alternatively, one may think of the β_i as the supermoduli of the SRS: if odd parameters appear in the transition functions, one may try to absorb them into a redefinition of the coordinates in the various charts. One finds that there is normally an irreducible set of parameters left over, which cannot be "gauged away", and these are the supermoduli. All these viewpoints are equivalent in practice; it does not matter how you view these parameters, but it's important that they are there. I will not commit myself to any particular supermanifold theory in this talk, letting the experts translate my remarks into their own preferred formalisms. The few SRSs for which all the β_i can be absorbed in coordinate redefinitions, giving a set of transition functions involving no odd parameters, are called *split*. In the supergravity interpretation, these have vanishing (or pure gauge) gravitino fields.

It is only for split SRSs that much of the folklore of superconformal field theory is actually true globally. In the split case only, we have $\psi = 0$ in all transition functions (since there are no odd parameters for it to depend on), while the $f(z)$ can be viewed as transition functions for an ordinary Riemann surface M_0 (the body, or reduced space, of the SRS). Further, θ transforms just like $dz^{1/2}$ when changing charts, and as a consequence $[A(z) + \theta B(z)]d\mathbf{z}^p$ is a superconformal tensor precisely when $A(z)dz^{p/2}$ and $B(z)dz^{(p+1)/2}$ are conformal tensors on M_0. In the nonsplit case, a superconformal tensor does *not* consist of a pair of conformal tensors; corrections involving the β_i must be added. The conditions under which this is possible will be a major theme of this talk.

The projective superspaces $P^{m,n}$ in which we will embed SRSs consist of the even lines through the origin in the flat superspaces $C^{m+1,n}$. More precisely, $P^{m,n}$ consists of the points (x^i, ϕ^α) in $C^{m+1,n}$ such that at least one x^i is invertible, modulo the equivalence relation $(x^i, \phi^\alpha) \sim (kx^i, k\phi^\alpha)$ for all even invertible constants k.

Maps from a SRS M into $P^{m,n}$, $(z,\theta) \to [x^i(z,\theta), \phi^\alpha(z,\theta)]$ are obtained by taking the x^i and ϕ^α to be even and odd sections of any line bundle L over M. Recall that sections of line bundles do not have well-defined values at a point independent of a choice of trivialization. However, ratios of sections are intrinsically defined. This fits perfectly with the fact that only ratios of the homogeneous coordinates of $P^{m,n}$ are meaningful, making line bundles the natural tools for constructing projective embeddings. A line bundle is called *very ample* if the map defined by some choice of sections is actually an embedding; for this there must exist enough sections to take distinct values at distinct points of M. (A bundle is *ample* if some power of it is very ample.) Since every SRS carries the canonical bundle $\hat{\omega}$, it is natural to try to construct an embedding from sections of $\hat{\omega}^p$; that is, from superconformal tensors of weight $p/2$. I will show in a moment that this bundle is indeed very ample for M of genus > 1 and p large enough ($p > 3$). The result is a super analog of the standard p-canonical embedding of a Riemann surface.

Let us begin with the split case. Then superconformal tensors on M are simply pairs of conformal tensors on M_0. From standard Kodaira embedding theory [24] it is known that conformal tensors $s^i(z)$ of weight $p/2$, and $t^j(z)$ of weight $(p+1)/2$ can be found so that each of the maps $z \to [s^i(z)]$ and $z \to [t^j(z)]$ embeds M_0 in a projective space. Then it is easy to check that $x^i(z,\theta) = s^i(z)$, $\phi^\alpha(z,\theta) = \theta t^\alpha(z)$ are superconformal tensors of weight p which embed M in projective superspace. To claim that M is actually embedded as an algebraic curve is to assert that these tensors satisfy sufficiently many homogeneous polynomial relations that locally only one even and one odd tensor are independent. To see why such polynomial relations should exist, note that a homogeneous polynomial of degree d in tensors of weight q is itself a tensor of weight qd. An easy count using the Riemann-Roch theorem shows that for large d there are many more distinct polynomials than there are independent tensors of weight qd. Therefore, many of the polynomials must equal zero, giving the desired relations. More concretely, the usual Chow theorem shows that the conformal tensors s^i satisfy enough polynomial relations $P(s^i) = 0$ to locally express all of them in terms of one. So the even equations will be $P(x^i) = 0$. The same reasoning gives equations obeyed by the t^j, but they will not do for our purposes since they have degree $d > 1$: to reexpress them as equations for ϕ^α we would have to multiply through by θ^d, which vanishes. To get equations which are linear in the t^j, note that any ratio t^j/t^k is a meromorphic function on M_0. By standard function theory on Riemann surfaces [25] it can be expressed as a rational function of some "primitive pair" of functions which without loss of generality can be taken to be s^1/s^0 and s^2/s^0. Then for any j,k we get rational relations

$$\frac{t^j}{t^k} = \frac{Q(s^1/s^0, s^2/s^0)}{R(s^1/s^0, s^2/s^0)}, \tag{3}$$

with Q and R polynomials. Clearing denominators and multiplying through by θ gives the desired odd homogeneous polynomial equations of the form,

$$\phi^j \tilde{R}(s^0, s^1, s^2) = \phi^k \tilde{Q}(s^0, s^1, s^2), \tag{4}$$

which can be locally solved for all the ϕ^i but one. (There is a technical point to check here: if Q and R should both vanish at some point, they can be replaced by another pair of polynomials which do not.)

Now suppose that M is not split. We treat this case as a "perturbation" of the split case by the "small" parameters β_i. So, first set all the β_i formally to zero (mod out by the ideal they generate) and construct a projective embedding of the resulting split SRS as above. The resulting objects $s^i(z)$ and $\theta t^j(z)$, which were superconformal tensors in the split case, now transform as such tensors only to zeroth order in the β_i. Can they be extended to superconformal tensors by adding correction terms order by order in the β_i? This question, of extending conformal tensors to superconformal ones in the general nonsplit case, is highly nontrivial and will be discussed further below. Fortunately, for tensors of sufficiently high weight ($p \geq 3$) the answer is always yes. Furthermore, the homogeneous polynomial relations among the tensors can also be corrected order by order in the β_i so that they remain valid [19]. One way to see this is by an extension of function theory on Riemann surfaces to SRSs: it can be shown that any meromorphic function is rationally expressible in terms of a primitive triple of functions, two even and one odd [17]. This leads to polynomial relations as in the split case.

3 Open Problems

The results discussed above merely prove the existence of an algebraic viewpoint on super Riemann surfaces. The development and application of this viewpoint still lie ahead. Rather than simply urge the reader to pick up an algebraic geometry text [21,24] and prove super analogs of all the theorems, I have compiled a list of problems which seem particularly pressing for the application of these methods to string theory.

3.1 Can superconformal tensors be constructed via super Poincaré series?

One method for constructing superconformal tensors on arbitrary SRSs, which comes highly recommended in the physics literature, is the use of super Poincaré series [26,27,28,29]. By the uniformization theorem [14,30], any SRS M of genus > 1 has a representation as $M \cong SU/G$, with SU the unique SRS with body the upper half plane and G a "Fuchsian" supergroup of super Möbius transformations,

$$\begin{aligned}
\tilde{z} &= \frac{az+b}{cz+d} + \theta\frac{\gamma z+\delta}{(cz+d)^2}, \\
\tilde{\theta} &= \frac{\gamma z+\delta}{cz+d} + \frac{\theta}{cz+d}, \\
ad-bc &= 1+\gamma\delta.
\end{aligned} \tag{5}$$

Given a holomorphic (or meromorphic) function $f(z,\theta)$ on SU, the weighted sum over all its group transforms,

$$F(z,\theta) = \sum_G (D\tilde{\theta})^p f(\tilde{z},\tilde{\theta}), \tag{6}$$

$$D\tilde{\theta} = [cz+d+\theta(\delta c-\gamma d)]^{-1}, \tag{7}$$

formally transforms as

$$F(\tilde{z},\tilde{\theta}) = (D\tilde{\theta})^{-p}F(z,\theta), \tag{8}$$

and therefore projects down to a tensor of weight $p/2$ on M. This representation of tensors is particularly convenient for discussing the extension from split to nonsplit SRSs required above. The extension simply involves turning on the odd parameters in the group elements while keeping the same function $f(z,\theta)$. This was the method employed in [17].

Unfortunately, no valid proof of the convergence of the super Poincaré series has appeared in the literature. The standard methods which work for ordinary Poincaré series do not seem to generalize. A classical method relates the terms in the series to the areas of regions containing isometric circles of group elements, which can be bounded by geometric arguments [31]. Very roughly, an isometric circle in the z-plane becomes a cylinder in (z,θ) space which is unbounded in the θ direction. There is no notion of area or volume which can be used to control the behavior of group orbits in the unbounded direction. Indeed, there is no norm on a Grassmann algebra with the nice properties of the length in the complex plane, which might be used to bound the terms in the series. For example, we can show that a complex number is small by showing that its inverse is large, but if a is a large number then both $1+a\beta_1\beta_2$ and its inverse $1-a\beta_1\beta_2$ can well be large in any reasonable norm. (In a standard norm [23], both have length $1+|a|$.) In [17] we proposed to prove convergence by using a quasisuperconformal map

W to relate G to the corresponding group G_0 with all odd parameters set to zero, $G = WG_0W^{-1}$. The series for G_0 are ordinary Poincaré series with known convergence properties, and the super Beltrami equations [14] provide considerable control over the map W. The method fails for a rather subtle reason. The conjugation WG_0W^{-1} must produce some group of superconformal automorphisms of SU, with quotient space M. However, this group need not be the original group G, because super Möbius transformations do not exhaust the automorphisms of SU [14,30]. SU has superconformal automorphisms whose bodies are Möbius transformations but whose nilpotent terms are essentially arbitrary analytic functions, with possibly very singular boundary behavior. The method fails without control over this behavior. Alternatively, one can demand that G be the original group, at the cost of losing the control over the conjugation W which is also crucial.

Thus, the convergence behavior of super Poincaré series is an entirely open problem. Essentially, one must try to bound the terms (7) in the series by showing that the odd parameters γ, δ in the group elements grow no faster than the even ones c, d as we run through the group. To explain the possibly unfamiliar notion of bounding the growth rate of odd parameters, choose a set of generators for the group, and view all other elements as words in these generators. The odd parameters in any such word are expressible in terms of the odd (and even) parameters in the generators. It is the coefficients in such a representation, which are ordinary numbers, which must be bounded as the word length grows.

3.2 Algebraic characterization of superconformal structure

Although we have discussed projective embeddings only for SRSs, the methods and conclusions in fact apply to all (1,1) supermanifolds (and more generally). Thus, given the polynomial equations for a (1,1) supermanifold M in some $P^{m,n}$, how can we determine algebraically whether it admits a SRS structure? And if it does, can we characterize this structure algebraically, for example by giving a rational expression on $P^{m,n}$ which restricts to a superconformal tensor on M? It is known [32] that a (1,1) supermanifold can admit at most one superconformal structure, but most of them admit none: the moduli space of SRSs has complex dimension $(3g-3, 2g-2)$ while that of (1,1) supermanifolds has dimension $(4g-3, 4g-4)$, at least near the locus of SRSs. Indeed, although we know which infinitesimal deformations of SRSs preserve the SRS structure, we do not know how to locate the SRSs inside the larger moduli space in terms of any set of coordinates at all. This problem must be solved if algebraic methods are to be useful in superstring theory, where only SRSs are relevant. Recall that SRSs are distinguished by the existence of a subbundle $\hat{\omega}^{-1}$ of the tangent bundle generated by D; a coordinate-free characterization of D is that D and D^2 together span all vector fields. In practice one might look for such a subbundle by the method, familiar from the operator formalism, of covering a (1,1) supermanifold by two open sets: an affine piece omitting a point at infinity, and a neighborhood of the omitted point. This method is adapted to a description in terms of polynomial equations, since the affine piece can be a Zariski open set on which some variable does not vanish. The tangent bundle can be trivialized on each piece, so is completely characterized by a single transition matrix across the overlap, with no cocycle condition to be satisfied. In a specific example it should be straightforward to see whether coordinate redefinitions could bring the 2×2 transition matrix to the triangular form it assumes in the basis $\{D, D^2\}$. It is not clear whether a general criterion can be formulated in terms of the vanishing of some algebraic invariant.

3.3 Give explicit polynomial equations for specific SRSs

At present, the explicit polynomial equations for a SRS are known only for the atypical case of genus 1 — the supertori [16]. For example, a supertorus with odd (periodic-periodic) spin structure, obtained as the quotient of $C^{1,1}$ by the two supertranslations

$$z \to z+1, \quad \theta \to \theta \tag{9}$$

and

$$z \to z+\tau+\theta\delta, \quad \theta \to \theta+\delta, \tag{10}$$

can be explicitly embedded in $P^{3,2}$ using the superelliptic (supertranslation invariant) function

$$R(z,\theta) = \wp(z;\tau+\theta\delta) = \wp(z;\tau) + \theta\delta\partial_\tau\wp(z;\tau), \tag{11}$$

with $\wp$ the Weierstrass $\wp$ function. The embedding is given by

$$\begin{aligned} (z,\theta) \quad &\to \quad (R, D^2R, R^2, 1; DR, D^3R) &(12)\\ &= \quad (x, y, u, 1; \phi, \psi) \in P^{3,2}, &(13)\end{aligned}$$

and characterized by the equations

$$\begin{aligned} y^2 - 4x^3 + g_2x + g_3 + 2\psi\phi &= 0, &(14)\\ 2y\psi - 12x^2\phi + g_2\phi + \delta\partial_\tau g_2 x + \delta\partial_\tau g_3 &= 0, &(15)\\ u &= x^2, &(16)\end{aligned}$$

where $g_2(\tau)$ and $g_3(\tau)$ are the usual modular forms. The equations are not homogeneous because the coordinates of $P^{3,2}$ have been normalized so that one of them is unity. We see here the mixing of even and odd variables in the equations of a nonsplit SRS. In the split limit $\delta = 0$ the equations reduce to the form expected from Section 2: the second equation becomes a proportionality between ϕ and ψ, so that the term $2\psi\phi$ in the first equation vanishes.

The simplest SRSs of higher genus for which explicit equations might be found are the hyperelliptic SRSs, which generalize supertori and have been studied to some extent [33,34]. Equations of Riemann surfaces are easily generated by checking that the codimension is correct, and incorporating possible symmetries, e.g. a Z_2 automorphism in the hyperelliptic case. For SRSs one must also solve the problem of characterizing SRSs among all supercurves. Unless one can exploit special properties of the function theory on hyperelliptic SRSs to circumvent this, obtaining their equations may be no simpler than for an arbitrary SRS.

3.4 Extension of conformal tensors to superconformal tensors

We saw that in the split case, superconformal tensors consist of pairs of conformal tensors whose weights differ by a half unit. In the nonsplit case, the best we can hope for is that pairs of conformal tensors can be extended to superconformal tensors by adding correction terms involving the β_i, and that this construction can produce a basis for the space of all superconformal tensors. In general, this optimistic guess is false: not every pair of conformal tensors will extend, and the set of superconformal tensors is not generally a (super) vector space! The simplest illustration of this phenomenon is provided by the superconformal tensors of weight zero (i.e., functions) on the supertorus described in Eqs. (9,10). One easily sees that a global function must take the form $a + \theta b$, where the constant a represents an ordinary function (conformal tensor of weight zero) and the constant b is the holomorphic spinor (conformal tensor of weight 1/2) which is present in the odd spin structure. If this were the whole story, then the space of even superfunctions would be a vector space of dimension (1,1) with basis $\{1, \theta\}$. However, invariance of $a + \theta b$ under (10) imposes the constraint $b\delta = 0$, so that the coefficient of θ cannot be chosen freely. One says that the superconformal tensors form a module (over the Grassmann algebra that δ lives in), but not a freely generated module (a super vector space). Alternatively, one can view the superconformal tensors as forming a vector space over C rather than over the Grassmann algebra, with basis $\{1, \delta, \theta\delta\}$ (in the simplest case where δ is the only generator of the Grassmann algebra). In this language the pathology is that the dimension of this vector space jumps at the split locus $\delta = 0$, where the two basis vectors involving δ disappear and are replaced by the single vector θ.

This example may suggest that the problem is peculiar to genus 1, but in fact it occurs generically for any genus, odd spin structures, and superconformal tensors of weights $p/2, p = -1, 0, 1, 2$. The most complete treatment of this issue is in [35], which characterizes the first obstruction to extending conformal tensors. It is also discussed in [17,19,36,37,38]. In the Russian literature on superconformal tensors there is often a disclaimer in fine print which restricts consideration to "normal" SRSs, namely those for which the problem is absent. This mathematical issue impacts several areas of concern in superstring theory. First, it accounts for the absence of a general super Riemann-Roch theorem (except in the normal case), since such a theorem should give the dimensions of the spaces of superconformal tensors, and dimension is defined only for free modules. Second, it leads to subtleties in the application of the super Serre duality theorem [39,40], since the cohomology spaces which are dual are not vector spaces (except over $\mathbf{C}$). More physically, it raises unresolved issues concerning the method of conserved charges used to construct vacuum states in the operator formalism for superstrings [41,42,43]. This method is supposed to produce a conserved charge annihilating the vacuum for every element of a basis for harmonic functions on a punctured SRS. If such a free basis does not exist, there may not be enough charges to characterize the vacuum. This question has not been investigated in the literature, which is restricted to the normal case. Finally, it has so far prevented writing simple formulas for loop amplitudes in odd spin structures analogous to existing formulas for even ones, by complicating the construction of basic elements of function theory such as Picard and Jacobian varieties, theta functions and prime forms, and period matrices. All these constructions assume vector spaces of holomorphic forms.

Recently I have been studying the Picard group (it is probably not a variety) of the supertorus in an effort to understand these issues. The Picard group, or group of degree zero line bundles, makes sense on a SRS M and is given by

$$Pic^0(M) \cong H^1(M, \hat{\omega}^0)/H^1(M, Z). \tag{17}$$

This is calculable for the supertorus (9,10) even though the cohomology group in the numerator is not free. An ordinary torus is isomorphic to its Picard group; here one finds that Pic^0 is isomorphic to the supertorus modulo the equivalence $(z, \theta) \sim (z + \alpha\delta, \theta)$ for all odd constants α. The map $M \to Pic^0(M)$, which would be an isomorphism for a torus, can be given as usual in terms of divisors — here, the principal divisors of Rosly, Schwarz, and Voronov [38]. Choose a basepoint P_0 on the supertorus and map any other point P to the line bundle in Pic^0 having principal divisor $P - P_0$. If $P = (w, \phi)$ and $P_0 = (w_0, \phi_0)$, this means the bundle which has sections behaving like $(z - w - \theta\phi)(z - w_0 - \theta\phi_0)^{-1}$, with no zeros or poles elsewhere.

The relation between a supertorus and its Picard group leads to other topics. The group structure on Pic^0 comes from the "addition" of bundles by tensor product. For the ordinary torus this group law can be pulled back to M and implemented geometrically by embedding the torus in projective space and intersecting it with lines; the three points of intersection add to zero. A similar geometric construction is possible for the supertorus embedded as in Eqs. (14–16). The fact that the points with rational coordinates on the torus form a subgroup is basic in the number theory of elliptic curves, and this generalizes to the super context as well. These issues will be discussed in a separate publication [44].

3.5 The super KP hierarchy

Since the relation between the Kadomtsev-Petviashvili hierarchy and Riemann surfaces is conveniently discussed in algebraic terms, a natural application of the algebraic viewpoint on SRSs is to understand the geometry associated with the super KP (SKP) hierarchy. In the form proposed by Manin and Radul [45], this is

$$\begin{aligned} \partial L/\partial t_n &= [L_+^{2n}, L], && (18) \\ \partial L/\partial \tau_n &= \{L_+^{2n-1}, L\} - 2L^{2n} + \sum_{k=1}^{\infty} \tau_k \, \partial L/\partial t_{n+k-1}. && (19) \end{aligned}$$

Here L is a pseudosuperdifferential operator of the form,

$$L = D + u_1 + u_2 D^{-1} + u_3 D^{-2} + \cdots, \tag{20}$$

where the coefficient functions u_i depend on the independent variables z and θ and on the even and odd evolution parameters t_i and τ_i, and L_+^n denotes the differential operator part of the pseudodifferential operator L^n. Although the algebraic structure of these equations and their solutions is understood [46,47], the geometric picture has remained unclear. By analogy with the ordinary KP hierarchy, one would expect a class of solutions for which some power L^k is a pure differential operator belonging to a ring R of supercommuting operators. There should be an associated supercurve, possibly a SRS, given by Spec R, meaning that R is isomorphic to the ring of holomorphic functions on the affine part of the supercurve (the curve punctured at a point at infinity). As L flows via the SKP hierarchy, R does not change as an abstract ring, but the specific operators representing its elements change. These operators define a line bundle on the supercurve, whose fibers are essentially their simultaneous eigenspaces. Thus, SKP is realized as a flow on the Jacobian (actually the Picard group of line bundles) of a supercurve. Recently, Motohico Mulase and I have shown that this picture is indeed correct, with the disappointing feature that the supercurves which appear are definitely not SRSs [48]. The possible connections between the operator formalism on SRSs and SKP, and between the various different versions of SKP itself [49,50], remain as open questions.

I hope this discussion has shown that the algebraic viewpoint on SRSs is a new and fascinating area for investigation, with connections and likely applications to string theory, the operator formalism, algebraic geometry, and number theory.

References

[1] J.P. Serre, "Géométrie algébrique et géométrie analytique", Ann. Inst. Fourier **6**, 1–42 (1956).

[2] D.J. Gross and P.F. Mende, Nucl. Phys. **B303**, 407–454 (1988).

[3] D. Montano, Nucl. Phys. **B297**, 125–140 (1988).

[4] B. Grossman, Phys. Lett. **197B**, 101–104 (1987).

[5] I. Volovich, Class. Quant. Grav. **4**, L83–L87 (1987).

[6] L. Brekke, P.G.O. Freund, and M. Olson, Nucl. Phys. **B302**, 365–402 (1988).

[7] Z. Hlousek and D. Spector, "p-adic string theory", Cornell preprint CLNS 88/832.

[8] P.H. Frampton, also I.Y. Araf'eva, in this volume.

[9] M. Mulase, J. Diff. Geom. **19**, 403–430 (1984), and in this volume.

[10] D. Friedan and S.H. Shenker, Phys. Lett. **175B**, 287–296 (1986).

[11] E. D'Hoker and D.H. Phong, Rev. Mod. Phys. **60**, 917–1065 (1988).

[12] D. Friedan, in *Unified String Theories*, ed. D. Gross and M. Green, World Press, Singapore, 1986.

[13] M.A. Baranov and A.S. Schwarz, JETP Lett. **42**, 419–421 (1986).

[14] L. Crane and J.M. Rabin, Commun. Math. Phys. **113**, 601–623 (1988).

[15] C. LeBrun and M. Rothstein, Commun. Math. Phys. **117**, 159–176 (1988).

[16] J.M. Rabin and P.G.O. Freund, Commun. Math. Phys. **114**, 131–145 (1988).

[17] J.M. Rabin and P. Topiwala, "Super Riemann surfaces are algebraic curves", U.C.S.D. preprint, June 1988.

[18] C. LeBrun, "The Kodaira embedding theorem for complex supermanifolds", S.U.N.Y. Stonybrook preprint, August 1988.

[19] C. LeBrun, Y.-S. Poon, and R.O. Wells, Jr., "Projective embeddings of complex supermanifolds", to appear in Commun. Math. Phys.

[20] P. Topiwala and J.M. Rabin, "Super GAGA and moduli of super Riemann surfaces", submitted to Commun. Math. Phys.

[21] R. Hartshorne, *Algebraic Geometry*, Springer-Verlag, New York, 1977.

[22] B.S. DeWitt, *Supermanifolds*, Cambridge University Press, Cambridge, 1984.

[23] A. Rogers, J. Math. Phys. **21**, 1352–1365 (1980).

[24] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.

[25] H.M. Farkas and I. Kra, *Riemann Surfaces*, Springer-Verlag, New York, 1980.

[26] E. Martinec, Nucl. Phys. **B281**, 157–210 (1987).

[27] P. Teofilatto, J. Math. Phys. **29**, 2389–2396 (1988).

[28] K. Aoki, Commun. Math. Phys. **117**, 405–429 (1988).

[29] M.A. Baranov, Yu. I. Manin, I.V. Frolov, and A.S. Schwarz, Commun. Math. Phys. **111**, 373–392 (1987).

[30] L. Hodgkin, Lett. Math. Phys. **14**, 177–184 (1987).

[31] L. Ford, *Automorphic Functions*, second edition, Chelsea, New York 1951.

[32] S.B. Giddings and P. Nelson, Commun. Math. Phys. **116**, 607–634 (1988).

[33] M. Bershadsky, Mod. Phys. Lett. **A3**, 91–108 (1988).

[34] C. Črnković and G.M. Sotkov, "Strings on hyperelliptic supersurfaces", ICTP preprint IC/88/356.

[35] L. Hodgkin, "Problems of fields on super Riemann surfaces", King's College (London) preprint 1988.

[36] E. D'Hoker and D.H. Phong, "Conformal scalar fields and chiral splitting on super Riemann surfaces", preprint Columbia-UCLA/88/TEP/38.

[37] S.B. Giddings and P. Nelson, Commun. Math. Phys. **118**, 289–302 (1988).

[38] A.A. Rosly, A.S. Schwarz, and A.A. Voronov, Commun. Math. Phys. **119**, 129–152 (1988).

[39] C. Haske and R.O. Wells, Jr., Duke Math. J. **54**, 493–500 (1987).

[40] I.B. Penkov, Invent. Math. **71**, 501–512 (1983).

[41] L. Alvarez-Gaumé, C. Gomez, G. Moore, and C. Vafa, Nucl. Phys. **B303**, 455–521 (1988).

[42] L. Alvarez-Gaumé, C. Gomez, P. Nelson, G. Sierra, and C. Vafa, Nucl. Phys. **B311**, 333–400 (1988).

[43] A.S. Schwarz, Nucl. Phys. **B317**, 323–343 (1989).

[44] J.M. Rabin, "Super elliptic curves", in preparation.

[45] Yu. I. Manin and A.O. Radul, Commun. Math. Phys. **98**, 65–77 (1985).

[46] M. Mulase, Invent. Math. **92**, 1–46 (1988).

[47] K. Ueno and H. Yamada, Adv. Stud. in Pure Math. **16**, 373 (1988).

[48] M. Mulase and J.M. Rabin, "Super Grassmannians, algebraic supercurves, and a characterization of super Jacobians", in preparation.

[49] A. LeClair, "Supersymmetric KP hierarchy: free field construction", Princeton preprint PUPT-1107, September 1988.

[50] V.G. Kac and J.W. van de Leur, Ann. Inst. Fourier **37**, 99–137 (1987); "Super boson-fermion correspondence of type B", preprint 1989.

PROJECTIVE EMBEDDINGS OF COMPLEX SUPERMANIFOLDS*

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This lecture represents joint work with Claude LeBrun of State University of New York at Stony Brook and Yat-Sun Poon of Rice University, [**LPW**]. There is some overlap with similar work by Jeffrey Rabin and P.T. Topiwala in two recent reprints [**RT1**], [**RT2**], and this work is reported in [**R**].

We begin by recalling some classical embedding theorems from complex manifold theory (as a general reference for this topic see [**GH**], [**W**]). There are classical explicit embeddings of Plücker:

$$G(k, \mathbf{C}^n) \hookrightarrow \mathbf{P}_N(\mathbf{C})$$

and Segré:

$$\mathbf{P}_n \times \mathbf{P}_m \hookrightarrow \mathbf{P}_N(\mathbf{C}),$$

where $\mathbf{P}_n(\mathbf{C})$ denotes n-dimensional complex projective space, and $G(k, \mathbf{C}^n)$ denotes the Grassmannian manifold of k-dimensional subspaces of n-dimensional complex Euclidean space $\mathbf{C}^n$. Once one has an embedding of a complex manifold X into complex projective space, what can you say about the embedded submanifold (or subvariety, if we allow singularities)? The answer is the celebrated results of Chow from 1949:

- Any analytic subvariety of projective space is defined by polynomial equations (i.e. is "projective algebraic").

A related result due to Serre in 1956 is the GAGA theorem.

- What's true analytically in projective space is also true algebraically.

The acronym GAGA refers to the title of the original paper in French, "Geometrie analytique et geometrie algebrique."

Shortly before Serre established his results came the powerful theorem of Kodaira, the *Kodaira embedding theorem,* in 1954, for which Kodaira was awarded the Fields Medal somewhat later. This result asserts:

- Let X be a compact complex manifold, and if there exists a holomorphic line bundle $L \to X$, such that L is *positive,* i.e. such that L admits an Hermitian metric h so that the *curvature* of h, $F_h := \frac{i}{2\pi}\partial\bar{\partial}h$, is a positive-definite $(1,1)$-form on X, then there exists an embedding

$$f : X \to \mathbf{P}_N(\mathbf{C})$$

for sufficiently large N.

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We remark that the mapping f is given by $(N+1)$ sections of the line bundle L $(S_0, \ldots, S_N)$ whose values $[S_0(n), \ldots, S_N(x)]$ at any point $x \in X$ give well-defined homogeneous coordinates for a point (the image point) in $\mathbf{P}_N(\mathbf{C})$.

Kodaira showed that to obtain such a positive line bundle L, it is sufficient to have a *Hodge form* ω on X, that is a $(1,1)$-form ω defined on X with the following properties:

1) Locally, $\omega = \sum \omega_{ij} dz^i \wedge d\bar{z}^j$ where ω_{ij} is positive-definite Hermitian form,
2) ω is d-closed
3) $\int_\gamma \omega \in \mathbf{Z}$, for each cycle integral cycle γ on X, i.e. the homology class $\gamma \in H_2(X, \mathbf{Z})$.

Kodaira shows that any such ω will be the Chern form of a positive line bundle L on X.

Let's give some further examples of projective embeddings in light of Kodaira's theorem. First, consider a Riemann surface X, that is a compact complex manifold of complex dimension one.

Example 1 (Riemann Surface): Let X be a compact complex manifold of complex dimension one. Then

$$\dim_{\mathbf{C}} H^2(X, \mathbf{Z}) = 1$$

Let ω be an *Hermitian metric* on X, i.e., a form of the type

$$\omega = i f(x)\, dz \wedge d\bar{z},$$

where $f > 0$. Since $\dim_{\mathbf{C}} X = 1$, it follows that $d\omega = 0$, and if we let $\int\limits_X \omega = c$, then $c^{-1}\omega$ is a Hodge form and X admits a projective embedding by Kodaira's theorem.

Example 2 (Complex Torus): Let X be a complex torus of the form $X = \mathbf{C}^n/\Lambda$, where Λ is a lattice of rank $2n$. The complex vector space of 1-forms $H^o(X, \Omega^1)$ has dimension n, and if we let $\{\omega_1, \ldots, \omega_n\}$ be a basis for $H^o(X, \Omega^1)$, and let $\{\gamma_1, \ldots, \gamma_{2n}\}$ be a basis for $H_2(X, \mathbf{Z})$, then the period matrix

$$\Omega := (\Omega_{ij}) = \left(\int\limits_{\gamma_j} \omega_i \right)$$

is an $n \times 2n$ matrix which, by a suitable choice of bases, can be put in the form

$$\Omega = (I_n, Z)$$

where Z is an $n \times n$ complex matrix.

If Z satisfies:

$$Z = Z^t, \qquad \operatorname{Im} Z \text{ positive definite},$$

(the Riemann-Lefschetz period matrix conditions), the two form

$$\omega = \sum \omega_i \wedge \bar{\omega}_i$$

will be a Hodge form and such a torus will be embedded. These conditions of Riemann-Lefschetz are also necessary, as we can see readily.

Example 3 (Hopf): Let

$$X = \mathbf{C}^2 - \{0\}/\mathbf{Z},$$

where $\mathbf{Z}$ acts on $\mathbf{C}^2$ by $z \to e^m z$, $m \in \mathbf{Z}$, $Z \in \mathbf{C}^2$. Then X is diffeomorphic to $S^1 \times S^3$, and we see that

$$\dim H^1(X, \mathbf{C}) = 1,$$

but every Kähler manifold and hence every projective algebraic manifold has

$$\dim H^1(X, \mathbf{C}) \in 2\mathbf{Z}.$$

Hence, for topological reasons the complex manifold X cannot be embedded.

Thus we have examples of complex manifolds which can always be projectively embedded (Riemann surfaces), sometimes can be projectively embedded (complex tori), and can never be projectively embedded (Hopf manifolds).

Three dimensional Calabi-Yau manifolds which arise in the work of Candelas *et al* [C] are all projective algebraic as one can see by an application of the Kodaira theorem. This is useful in the classification of possible manifolds which arise in this theory (see [**GH1**], [**GH2**]).

We now turn to the question of embedding complex supermanifolds. For references on complex supermanifolds we refer you to [**M**], [**SW**], but we recall the fundamental definitions here. We'll use the terminology of ringed spaces (see [**W**], [**SW**], [**W2**].

Definition: A *complex supermanifold* is a ringed space $(M, \mathcal{A})$ where M is a topological space and $\mathcal{A}$ is a sheaf of super commutative rings on M such that, if $\mathcal{N}$ is the ideal of nilpotents in $\mathcal{A}$, then:

(a) $(M, \mathcal{O})$ is a complex manifold where $\mathcal{O} := \mathcal{A}/\mathcal{N}$.
(b) $\mathcal{E} := \mathcal{N}/\mathcal{N}^2$ is a locally free $\mathcal{O}$-module
(c) A is locally isomorphic to the $\mathbf{Z}_2$-graded exterior algebra $\wedge^*_{\mathcal{O}}\mathcal{E}$.

We say that a complex supermanifold $(X, \mathcal{A})$ is *split* if $\mathcal{A} \cong \wedge^*_{\mathcal{O}} F$ on all of X, where $\mathcal{F}$ is a locally free sheaf on the complex manifold $(X, \mathcal{O})$.

We now give some examples of complex supermanifolds.

Example 4 (Superprojective space): Let $\mathbf{P}_n = \mathbf{P}_n(\mathbf{C})$ be complex projective space, and define the ringed space

$$\mathbf{P}_{n|m} := \left(\mathbf{P}_n, \wedge^*_{\mathcal{O}} \left(\mathbf{C}^m \otimes_{\mathbf{C}} \mathcal{O}(-1)\right)\right).$$

where $\mathcal{O}(-1)$ denotes the sheaf of sections of H^{-1}, where H is the hyperplane section bundle on $\mathbf{P}_n$ (i.e. the holomorphic line bundles whose first Chern number is 1). The motivation for this definition comes from the fact that holomorphic functions on $\mathbf{P}_{n|m}$ should be of total homogenity 0 in $n+1$ even coordinates $(Z_0, \ldots, Z_n)$ and m odd coordinates $(\theta^1, \ldots, \theta^m)$, i.e.

$$f = \sum_{k=0}^{m} \sum_{0 \le i_1 < \cdots < i_k \le m} f_{i_1 \ldots i_k}(Z_0, \ldots, Z_n)\theta^{i_1} \ldots \theta^{i_k}$$

where $f_{i_1 \ldots i_k}$ has homogeneity $-k$. This is a local representation for a section of the Grassmann bundle

$$\wedge^* \left(\mathbf{C}^m \otimes \mathcal{O}(-1)\right)$$

Example 5 (Supergrassmannian manifolds): Let $\mathbf{C}^{n|m}$ be super Euclidean space, i.e. $\mathbf{C}^{n|m} = (\mathbf{C}^n, \wedge^*(\mathcal{O}_{\mathbf{C}^n} \otimes \mathbf{C}^m))$, and let $G(k|n, \ell|m)$ be the set of all free submodules of $\mathbf{C}^{n|m}$ of rank $k|\ell$, equipped with its natural structure sheaf (see [M], [**SW**]). This is defined analogously to the classical definition of a Grassmannian manifolds, and its reduced space is of the form

$$G(k, n) \times G(\ell, m)$$

a product of classical Grassmannians, i.e.

$$G(k|n,\ell|m) = (G(k,n) \times G(\ell,m),\mathcal{A}),$$

for a suitable sheaf $\mathcal{A}$. For $G(|n+1,o|m)$ we obtain $\mathbf{P}_{n|m}$ and $\mathcal{A}$ is given as in the previous example. But in general $\mathcal{A}$ is *not* the exterior algebra of a locally free sheaf, and the generic supergrassmannian manifold is *not* split, although superprojective space is.

We now introduce the notion of restrictions of sheaves on supermanifolds to the *reduced space*. If $(X,\mathcal{A})$ is a complex supermanifold, then $(X,\mathcal{A}/\mathcal{N})$ is a complex manifold called the *reduction* of $(X,\mathcal{A})$, on the *base space* for the complex supermanifold. In other words, it is the complex manifold underlying the given complex supermanifold. We have a natural injection

$$(X,\mathcal{O}) \to (X,\mathcal{A})$$

given by the surjection $\mathcal{A} \to \mathcal{A}/\mathcal{O}$, recalling the notion of ringed space mapping from [**W2**]. In fact, there is a sequence of such mappings if we consider

$$(X,\mathcal{O}) \to (X,\mathcal{A}/\mathcal{N}^2) \to (X,\mathcal{A}/\mathcal{N}^3) \to \cdots \to (X,\mathcal{A})$$

we describe this as

$$X^{(0)} \hookrightarrow X^{(1)} \hookrightarrow \cdots \hookrightarrow X^{(k)} = (X,\mathcal{A})$$

for a given supermanifold $(X,\mathcal{A})$, and think of $X^{(j)}$ as a sequence of supermanifolds which are embedded one in other, starting out with the initial classical manifolds. If $\mathcal{F}$ is an $\mathcal{A}$-module on $(X,\mathcal{A})$, then

$$\mathcal{F}_{rd} := \mathcal{F}/\mathcal{N}\mathcal{F}$$

is the *restriction* of $\mathcal{F}$ to $(X,\mathcal{O})$. Similarly one could restrict to the other submanifolds $X^{(j)}$.

Definition: A locally free rank one sheaf of $\mathcal{A}$-modules $\mathcal{L}$ on (X,A) is *positive* if $\mathcal{L}_{rd}$ is a positive rank one sheaf on $(X,\mathcal{O})$.

Theorem [LPW]: A complex supermanifold M with underlying complex supermanifold M_{rd} compact can be embedded in some superprojective space $\mathbf{P}_{n|m}$ if and only if it admits a positive rank one sheaf of $\mathcal{A}_M$-modules. Here $\mathcal{A}_M$ is the structure sheaf of the complex supermanifold $M = (M_{rd},\mathcal{A}_M)$. The proof in one direction is easy. If $f: M \to P_{n|m}$ is a given embedding, then

$$\mathcal{L} = f^*(\mathcal{O}(1) \otimes_{\mathcal{O}} \wedge^*(\mathbf{C}^m \otimes \mathcal{O}(-1)))$$

will be a positive rank one sheaf on $(X,\mathcal{O})$. In the other direction, if $\mathcal{L}_0 \to M_{rd}$ is a given positive rank one sheaf on M_{rd}, then we need to *extend* $\mathcal{L}_0$ to $\mathcal{L}$ on all of M. This is the hypothesis, that such an extension exists, then we have to use the extension to generate the embedding. The proof is a variant of the original Kodaira proof, and uses specific versions of a cohomology vanishing theorem due to Grauert, inductively, obtaining extensions of the embedding from $X^{(i)}$ to $X^{(i+1)}$ in a step-wise fashion.

A corollary of the above theorem is that any split complex supermanifold admits a projective embedding.

A second question is: how can we measure whether a complex supermanifold is embeddable in terms of classical invariants? It's not easy to see, off hand, if a given positive line bundle on the underlying manifold extends to the ambient supermanifold.

As we indicated above, the proof depends on extending the embedding

$$\begin{tikzcd} M^{(k)} \arrow[r, "\varphi^{(k)}"] & \mathbf{P}_{n|m} \\ M_{rd} \arrow[u] \arrow[r, "\varphi"] & \mathbf{P}_n \arrow[u] \end{tikzcd}$$

where we let $\varphi^{(0)} = \varphi$ be the original embedding, and let $\varphi^{(k)}$ be the k-th extension. Then we have the following theorem, assuming the above context.

THEOREM. *(a) If K is odd there is no obstruction to extending $\varphi^{(k-1)}$ to $\varphi^{(k)}$*
(b) If k is even, the only obstruction to extension lies in $H^2(M_{rd}, \wedge^k \mathcal{E})$, where $\mathcal{E} = \eta/\eta^2$ is a locally free characteristic sheaf of the supermanifolds $M = (M_{rd}, \mathcal{A})$.

Remark: In the C^∞ category all obstructions vanish since the sheaves are fine sheaves.

Corollary: Any supermanifold of dimension $1|m$ is superprojective.

This follows from the fact that on any Riemann surface (the underlying complex manifold is a Riemann surface in'this case) the cohomology groups $H^2(M_{rd}, \mathcal{F}) = 0$, for any locally free sheaf, by dimensional reasons (i.e. $2 > 1$).

We now give examples of complex supermanifolds which are not superprojective.

Example 6: The complex supergrassmannian manifold $G(1|1, 2|2)$ is not superprojective, as is pointed out in Manin's book [**M**]. This was also a basic example of a nonsplit manifold. We also can give an example of a complex supermanifold which is not split but which is superprojective, and which is a consequence of the theorem above. Namely there is a two-dimensional family of complex supermanifolds $M(s,t)$ such that

(a) $M(s,t)_{rd} = \mathbf{P}_1 \times \mathbf{P}_1$
(b) $M(s,t)$ has characteristic sheaf $\mathcal{E} = \mathcal{O}(-1,-1) \oplus \mathcal{O}(-1,-1)$
(c) $M(0,0)$ is split, i.e. $\mathcal{A}_{M(0,0)} = \wedge^* \mathcal{E}$.
(d) $M(1,1)$ is $\mathbf{G}(1|1, 2|2)$
(e) $M(1,-1)$ is not split, but is embeddable.

The details of this example are in [**LPW**]. Finally we mention a generalized Segre embedding

$$\mathbf{P}_{m|n} \times \mathbf{P}_{k|\ell} \to \mathbf{P}_{mk+m+k+1|n(k+1)+\ell(m+1)}$$

which in the special case of $\mathbf{P}_{1|1}$ yields

$$\mathbf{P}_{1|1} \times \mathbf{P}_{1|1} \to \mathbf{P}_{4|4}.$$

This has the underlying projection embedding

$$\mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_4$$

which contrasts with the classical Segre embedding

$$\mathbf{P}_1 \times \mathbf{P}_1 \to \mathbf{P}_3$$

Finally we mention that there is a superprojective family embedding theorem which, in particular applies to families of super Riemann surfaces see [**LPW**], and this topic is discussed in more detail by Rabin [**R**].

REFERENCES

[C] Philip Candelas, Gary Horowitz, Andrew Strominger, and Edward Witten, *Vacuum configurations for superstrings*, Nucl. Phys. **B258** (1985), 46-.

[GH1] Paul Green and Tristan Hübsh, *Polynomial deformations and cohomology of Calabi-Yau manifolds*, Commun. Math. Phys. **113** (1987), 505–528.

[GH2] Paul Green and Tristan Hübsh, *Connecting moduli space of Calabi-Yau threefolds*, Commun. Math. Phys. **119** (1988), 431–441.

[GH] P.A. Griffith and J. Harris, *Principles of Algebraic Geometry*, J. Wiley, New York, 1978.

[LPW] Claude LeBrun, Yat-Sun Poon, and R.O. Wells, *Projective embeddings of complex supermanifolds*, Commun. Math. Phys. (to appear), 1989.

[M] Yu I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, NY, 1988.

[R] J. Rabin, *Status of the algebraic approach to super Riemann surfaces, these proceedings.*

[RT1] J. Rabin and P.T. Topiwala, *Super Riemann surfaces and algebraic curves*, (preprint, U. of California at San Diego), 1989.

[RT2] J. Rabin and P.T. Topiwala, *Super GAGA and moduli of super Riemann surfaces*, (preprint, U. of California at San Diego), 1989.

[SW] S. Shnider and R.O. Wells, Jr., *Supermanifolds, Super Twistor Spaces, and Super Yang-Mills Fields*, Univ. of Montreal Press, 1989.

[W] R.O. Wells, Jr., *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980.

[W2] R.O. Wells, Jr., *Supermanifolds, twistor geometry, and wavelets*, these proceedings.

SOME RESULTS ON LINE BUNDLES OVER SUSY-CURVES†

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1. INTRODUCTION

It is usually thought that any approach to superstrings à la Polyakov should involve a suitable generalization of the notion of Riemann surface. Physical arguments[1] require that the data needed to specify a "super Riemann surface" consist of
(i) an ordinary Riemann surface X;
(ii) a spin structure over X, say $\kappa_X^{\frac{1}{2}}$;
(iii) a gravitino field over X, i.e. a section of $\kappa_X^{\frac{3}{2}}$.

Working in the framework of graded manifolds à la Berezin-Leĭtes-Kostant, the simplest thing to do is to consider a *graded Riemann surface*,[2] i.e. a $(1,1)$ dimensional complex analytic graded manifold $(X, \mathcal{B})$, such that $\mathcal{B}_1$ is a spin structure over X. However, in this way only the data (i) and (ii) are encoded. To include the last point, one can consider *families of graded Riemann surfaces*, also called *super Riemann surfaces*[3] or *SUSY-curves*[4]. The definition of a SUSY-curve is recalled in Section 3.

This Communication is devoted to the generalization of two classical results in the theory of Riemann surfaces to the setting of SUSY-curves.
1. Any holomorphic line bundle over a compact Riemann surface with vanishing Chern class is isomorphic to a flat bundle, The transposition of this result to SUSY-curves requires the introduction of the notions of relative bundle, relative Chern class, and relatively flat bundle. If $\pi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a SUSY-curve, and λ is an arbitrary line bundle over $(X, \mathcal{A})$, all these relative concepts classify, in a sense, the behaviour of all the bundles obtained by restricting λ to the fibres of π.
2. The Chern-Weil theorem for SUSY-curves: by working with the relative Berezinian sheaf, it is possible to define a superconformal connection over λ. Then, by means of a relative integration of the curvature of the superconformal connection, one gets the relative Chern class.

† Talk given by the second author.

Due to the lack of space, these two results are here stated without proofs. In Section 2 a proof of result 1 in the simpler, but enlightening case of families of ordinary Riemann surfaces is skechted.

2. RELATIVE BUNDLES OVER FAMILIES OF RIEMANN SURFACES

Let X be a compact Riemann surface, and denote as usual by κ_X its canonical bundle, by $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and, finally, by $\mathcal{O}_X^*$ the invertible subsheaf of $\mathcal{O}_X$. It is a classical result[5] that any holomorphic line bundle over X, having a vanishing Chern class, is isomorphic with a flat bundle (we recall that a bundle is flat if it can be given locally constant transition functions). This is shown first by considering the commutative diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & & \kappa_X \arrow[u] & & \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[u] & \mathcal{O}_X^* \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[u] & \mathbb{C} \arrow[r] \arrow[u] & \mathbb{C}^* \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and writing the associated commutative diagram of cohomology; then, since $H^1(X, \kappa_X) \simeq H^2(X, \mathbb{C}) \simeq \mathbb{C}$, we obtain the diagram

$$\begin{tikzcd}
0 & & & \\
H^1(X, \mathcal{O}_X) \arrow[r] \arrow[u] & H^1(X, \mathcal{O}_X^*) \arrow[r, "c_1"] & H^2(X, \mathbb{Z}) \arrow[r] & 0 \\
H^1(X, \mathbb{C}) \arrow[r] \arrow[u] & H^1(X, \mathbb{C}^*) \arrow[u] & &
\end{tikzcd}$$

which proves the claim.

Let us consider now the generalization of this result to the case of *families of Riemann surfaces*. We recall that a family of Riemann surfaces is a fibration $\pi : X \to Y$, where X and Y are complex manifolds, and
(i) π is proper and flat;[6]
(ii) $\pi^{-1}(y)$ is a (compact) Riemann surface for all $y \in Y$.

The relative tangent sheaf is the $\mathcal{O}_X$-module $T(X/Y)$ defined by the exact sequence

$$0 \to T(X/Y) \to Der\mathcal{O}_X \to \pi^* \, Der\mathcal{O}_Y \to 0$$

and its dual (relative cotangent sheaf) is denoted by $\kappa_{X/Y}$.

The relative Picard group[7] is defined as $\mathrm{Pic}(X/Y) = H^0(Y, R^1\pi_*\mathcal{O}_X^*)$. The elements in $\mathrm{Pic}(X/Y)$ are isomorphism classes of holomorphic line bundles on X

which differ multiplicatively by the pull-back of a bundle on Y, i.e. $[\lambda_1] = [\lambda_2]$ if $\lambda_1 \simeq \lambda_2 \otimes \pi^*\eta$. Notice that, if $Y = \{pt\}$, the family reduces to a single Riemann surface X, and we obtain $\mathrm{Pic}(X/Y) = \mathrm{Pic}(X)$.

The relative Chern class of an element $\xi \in \mathrm{Pic}(X/Y)$ is the element $c_1(\xi) \in H^0(Y, R^2\pi_*\mathbf{Z})$ determined by the natural map $H^0(Y, R^1\pi_*O_X^*) \to H^0(Y, R^2\pi_*\mathbf{Z})$. Let us notice that, if λ is a line bundle over X, and $\xi = [\lambda]$, then $c_1(\lambda_{|\pi^{-1}(y)}) = (c_1(\xi))_y$ for all $y \in Y$; that is, the germ at y of the relative Chern class of $[\lambda]$ coincides with the ordinary Chern class of the restriction of λ to the Riemann surface sitting on y. Thus, the relative Chern class of $[\lambda]$ "classifies" simultaneously all the restricted bundles $\lambda_{|\pi^{-1}(y)}$, $y \in Y$.

A relatively flat bundle is an element in $\mathrm{Pic}(X/Y)$ which lies in the image of the natural morphism

$$H^0(Y, R^1\pi_*\pi^{-1}O_Y^*) \to H^0(Y, R^1\pi_*O_X^*) \equiv \mathrm{Pic}(X/Y).$$

It is easily shown that the elements in $H^0(Y, R^1\pi_*\pi^{-1}O_Y^*)$ are represented by bundles on X which are flat along the fibres of $X \to Y$.

Theorem 1. *Any relative bundle with vanishing Chern class is relatively flat locally on the parameter space Y.*

Here "locally" means that any y has a neighbourhood U such that our claim holds for the family restricted to U. This result is proved by considering the diagram

$$\begin{tikzcd}
 & & 0 & & \\
 & & \kappa_{X/Y} \arrow[u] & & \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & \mathcal{O}_x \arrow[r] \arrow[u] & \mathcal{O}_X^* \arrow[r] & 0 \\
0 \arrow[r] & \mathbf{Z} \arrow[r] \arrow[u] & \pi^{-1}\mathcal{O}_Y \arrow[r] \arrow[u] & \pi^{-1}\mathcal{O}_Y^* \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

and then applying the derived functor R, thus getting

$$\begin{tikzcd}
0 & & & \\
R^2\pi_*\pi^{-1}\mathcal{O}_Y \arrow[u] & & & \\
R^1\pi_*\kappa_{X/Y} \arrow[u, "\alpha"] & & & \\
R^1\pi_*\mathcal{O}_X \arrow[r] \arrow[u] & R^1\pi_*\mathcal{O}_X^* \arrow[r, "c"] & R^2\pi_*\mathbf{Z} \arrow[r] & 0 \\
R^1\pi_*\pi^{-1}\mathcal{O}_Y \arrow[r] \arrow[u, "\beta"] & R^1\pi_*\pi^{-1}\mathcal{O}_Y^* \arrow[u] & &
\end{tikzcd} \tag{1}$$

The morphism α is bijective, as one can check by tracing through the isomorphisms

$$R^1\pi_*\kappa_{X/Y}\otimes_{\mathcal{O}_y} k(y)\simeq H^1(\pi^{-1}(y),\kappa_{X/Y}\otimes_{\mathcal{O}_y} k(y))$$
$$\text{(Grauert semicontinuity theorem)}$$
$$\simeq H^1(\pi^{-1}(y),\kappa_{\pi^{-1}(y)})\simeq \mathrm{C}$$

$$R^2\pi_*\pi^{-1}\mathcal{O}_Y\otimes_{\mathcal{O}_y} k(y)\simeq H^2(\pi^{-1}(y),\pi^{-1}\mathcal{O}_y)\otimes_{\mathcal{O}_y} k(y)$$
$$\simeq H^2(\pi^{-1}(y),\mathrm{C})\otimes_{\mathrm{C}}\mathcal{O}_y\otimes_{\mathcal{O}_y} k(y)\simeq \mathrm{C}.$$

and using Nakayama's Lemma. Therefore, β is surjective, so that any $\tau\in\left(R^1\pi_*\mathcal{O}_X^*\right)_y$ such that $c(\tau)=0$ comes from an element in $\left(R^1\pi_*\pi^{-1}\mathcal{O}_Y^*\right)_y$. In order to prove Theorem 1, we have to apply the functor H^0 to the diagram (1); then, the morphisms induced in cohomology by β and c get cokernels. However, if for any $y\in Y$ we restrict to a suitable neighbourhood U of y, these cokernels vanish, which implies our claim.

We wish also to state a relative Chern-Weil theorem, whose proof is quite straightforward: let λ be a line bundle over X, and let ω be any smooth connection on λ, whose curvature is Ω.

Theorem 2. *If $[\lambda]$ is the class in $\mathrm{Pic}(X/Y)$ determined by λ, then*

$$\frac{i}{2\pi}\int_{X/Y}\Omega=c_1[\lambda]\,, \tag{2}$$

where the $\int_{X/Y}$ denotes integration along the fibres of $X\to Y$.

3. SUSY-CURVES

We briefly recall some basic definitions and results concerning the theory of graded manifolds.[8] A graded analytic manifolds of dimension (m,n) is a pair $(X,\mathcal{B})$, where X is a topological space, and $\mathcal{B}$ is a sheaf of graded-commutative complex algebras, such that:
(i) $(X,\mathcal{B}/\mathcal{N})$ is an m-dimensional complex analytic variety, $\mathcal{N}$ being the ideal of nilpotents of $\mathcal{B}$;
(ii) $\mathcal{N}/\mathcal{N}^2$ is a locally free $\mathcal{B}/\mathcal{N}$-module of rank n, and $\mathcal{B}$ is locally isomorphic with the exterior bundle $\bigwedge_{\mathcal{B}/\mathcal{N}}(\mathcal{N}/\mathcal{N}^2)$.

If $\dim(X,\mathcal{B})=(m,1)$, the natural epimorphism $\mathcal{B}\to\mathcal{O}_X=\mathcal{B}/\mathcal{N}$ induces an isomorphism $\mathcal{B}_0\xrightarrow{\sim}\mathcal{O}_X$, so that:

$$\mathcal{B}=\mathcal{O}_X\oplus\mathcal{B}_1\xrightarrow{\sim}\bigwedge\nolimits_{\mathcal{O}_X}(\mathcal{B}_1).$$

A holomorphic line bundle λ over $(X,\mathcal{B})$ is a $\mathcal{B}$-module either of rank (1,0) or (0,1). Let $\tilde\lambda=\lambda\otimes_{\mathcal{B}}\mathcal{O}_X$ be the induced line bundle over X. Then one has natural isomorphisms of graded $\mathcal{B}$-modules[9]

$$\lambda\xrightarrow{\sim}\tilde\lambda\oplus\tilde\lambda\cdot\mathcal{B}_1\xrightarrow{\sim}\tilde\lambda\otimes_{\mathcal{B}_0}\mathcal{B}\qquad\text{if rank }\lambda=(1,0)$$
$$\lambda\xrightarrow{\sim}\tilde\lambda\cdot\mathcal{B}_1\oplus\tilde\lambda\xrightarrow{\sim}\tilde\lambda\otimes_{\mathcal{B}_0}\hat{\mathcal{B}}\qquad\text{if rank }\lambda=(0,1),$$

where $\hat{\mathcal{B}} = \mathcal{B}_1 \oplus \mathcal{B}_0 = \Pi\mathcal{B}$. Thus, denoting by $\mathrm{Pic}(X, \mathcal{B})$ the group of isomorphism classes of line bundles over $(X, \mathcal{B})$, we have

$$\mathrm{Pic}(X, \mathcal{B}) \simeq \mathrm{Pic}(X).$$

A fundamental example of rank $(0,1)$ bundle is the analytic Berezinian sheaf[10] $\mathrm{Ber}(\mathcal{B})$ of $(X, \mathcal{B})$, which plays the rôle of the dualizing sheaf[11] of $(X, \mathcal{B})$.

A *family of graded manifolds* is a fibration $\pi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ of graded manifolds, where π is a proper and flat morphism of graded ringed spaces. We assume that the relative dimension of the family, i.e. $\dim \pi^{-1}(y)$, $y \in Y$, is $(1,1)$. A *SUSY-curve structure* is a a rank $(0,1)$ submodule $\mathcal{D}$ of the relative tangent sheaf $T(\mathcal{A}/\mathcal{B})$, such that the Fröbenius map

$$\mathcal{D} \otimes_{\mathcal{A}} \mathcal{D} \xrightarrow{[\,,\,]} T(\mathcal{A}/\mathcal{B}) \to T(\mathcal{A}/\mathcal{B})/\mathcal{D}$$

is an isomorphism of $\mathcal{A}$-modules. A SUSY-curve (i.e. a family of graded manifolds with a SUSY-curve structure) is also called a family of graded Riemann surfaces, or a super Riemann surface.

We denote by $\mathrm{Ber}(\mathcal{A}/\mathcal{B})$ the *relative Berezinian sheaf* of a SUSY-curve, which can be shown[9] to be canonically isomorphic with $\mathcal{D}^*$. Let $\tilde{d}$ be the composition of the exterior differential on $(X, \mathcal{A})$ with the natural projection $\kappa_{\mathcal{A}} \to \kappa_{\mathcal{A}/\mathcal{B}}$, where $\kappa_{\mathcal{A}/\mathcal{B}} = (T(\mathcal{A}/\mathcal{B})^*$ is the relative cotangent sheaf. If $p : \kappa_{\mathcal{A}/\mathcal{B}} \to \mathcal{D}^* \simeq \mathrm{Ber}(\mathcal{A}/\mathcal{B})$ is the natural projection, its composition with $\tilde{d}$ yields a C-module morphism $\delta : \mathcal{A} \to \mathrm{Ber}(\mathcal{A}/\mathcal{B})$. Letting

$$\mathcal{A}^{p,q} \equiv \mathrm{Ber}(\mathcal{A}/\mathcal{B}) \otimes \overline{\mathrm{Ber}(\mathcal{A}/\mathcal{B})} \otimes \mathcal{S} \qquad p, q = 0, 1\,,$$

where $\mathcal{S}$ is the smooth structure sheaf of $(X, \mathcal{A})$, we may introduce the notion of *superconformal connection*[12] on a line bundle λ as a pair of morphisms

$$\hat{\nabla} : \lambda \to \lambda \otimes \mathcal{A}^{1,0}\,; \qquad \hat{\bar{\nabla}} : \lambda \to \lambda \otimes \mathcal{A}^{0,1}$$

fulfilling suitable Leibniz rules involving δ and $\bar{\delta}$.

We can introduce a relative Picard group, a relative Chern class, etc., by mimicking the non-graded case. Finally, we state the fundamental results of this communication, whose proofs can be found in Ref. 9.

Theorem 3. *1. Any relative line bundle on a SUSY-curve having a vanishing Chern class is relatively flat.*

2. For any superconformal connection on a line bundle λ on a SUSY-curve, whose curvature is $K^{1,1}$, one has

$$\frac{i}{2\pi} \int_{\mathcal{A}/\mathcal{B}} K^{1,1} = c_1[\lambda]\,,$$

where $\int_{\mathcal{A}/\mathcal{B}}$ denotes Berezin integration along the fibres of the SUSY-curve.

REFERENCES

1. Friedan, D., Notes on string theory and two conformal field theory, in: Green, M., and Gross, D., "Unified string theories," World-Scientific, Singapore (1986).
2. Batchelor, M., and Bryant, P., Graded Riemann surfaces, *Commun. Math. Phys.,* **114**:243 (1988).
3. LeBrun, C., and Rothstein, M., Moduli of super Riemann surfaces, *Commun. Math. Phys.,* **117**:159 (1988).
4. Manin, Yu.I., Critical dimensions of the string theories and the dualizing sheaf of the moduli space of (super) curves, *Funct. Anal. Appl.,* **20**:244 (1987).
5. Gunning, R.C., "Lectures on Riemann surfaces," Princeton Univ. Press, Princeton (1966).
6. Bănică, C., and Stănăşilă, O., "Méthodes algébriques dans la théorie globale des espaces complexes. II," Gauthier-Villars, Paris (1977).
7. Grothendieck, A., 1962, "Fondements de la géometrie algébrique," Séminaire Bourbaki 1957–62, Paris (1962).
8. Manin, Yu.I., "Gauge field theory and complex geometry," Springer-Verlag, Berlin (1988).
9. Bartocci, C., Bruzzo, U., and Hernández Ruipérez, D., forthcoming paper.
10. Hernández Ruipérez, D., and Muñoz Masqué, J., Construction intrinsèque du faisceau de Berezin, *C. R. Acad. Sc. Paris,* **301**:915 (1985).
11. Penkov, I.B., $\mathcal{D}$-modules on supermanifolds, *Invent. Math.,* **71**:501 (1983).
12. Giddings, S.B., and Nelson, P., Line bundles on super Riemann surfaces, *Commun. Math. Phys.,* **118**:289 (1988).

INSTANTONS FROM SUPERSYMMETRIC CONFORMAL CHIRAL SCALAR SUPERFIELD THEORIES

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ABSTRACT

It is shown that generalized supersymmetric field equations depending on a chiral scalar supermultiplet admit instanton solutions when transition is made to Euclidean space-time. Implications and possible applications are discussed.

INTRODUCTION

Four dimensional second order supersymmetric equations involving an arbitrary function of an unconstrained chiral superfields were proposed in an earlier paper[1]. In three dimensions an invariant operator provides the generalization of analogous two dimensional equations that are first order in the fermion field. Recently partial results were obtained[2] using methods developed previously[1,3] in connection with two and three dimensional supersymmetric equations considered by Witten[4], Fronsdal[5], and Affleck, et al.[6] in the conformally invariant case.

Let S be a chiral scalar supermultiplet and if Θ_α (α = 1,2) are two component complex Grassmann numbers that transform under Lorentz as $\Theta' = L\Theta$ (Det L = 1) with L being a 2x2 complex matrix representing the Lorentz group, then $\hat{\Theta}$ which denotes the CP conjugate of Θ , i.e., $\hat{\Theta} = -i\sigma_2\Theta^*$ and $\hat{\Theta}^{+} = (-\Theta_2 \;,\; \Theta_1)$ transform under the inverse Lorentz transformation as $\hat{\Theta}' = L^{+-1}\,\Theta$, $\hat{\Theta}^{+\prime} = \hat{\Theta}^{+} L^{-1}$, respectively. It follows that if the fermionic field Ψ transforms as a left handed spinor, then $\hat{\Theta}^{+}\Psi = \Theta^\alpha \Psi_\alpha$ is Lorentz invariant, which is fashionably abbreviated as $\Theta\Psi$. Since Ψ_α and Θ_β commute, the supermultiplet S can be written in the form

$$S = \Phi(y) + \sqrt{2}\,\Theta\,\psi(y) + \Theta\,\Theta\,A(y) \; . \tag{1}$$

Writing

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$$y^\mu = x^\mu + i\,\Theta^+ \sigma^\mu \Theta \tag{2}$$

where the connection between SL(2,C) and the Lorentz group is established through the σ-matrices

$$\sigma^0 = -\sigma_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3}$$

and σ^k are the usual Pauli matrices, whose conjugates are defined by $\bar{\sigma}^0 = \sigma^0$ and $\bar{\sigma}^k = -\sigma^k$. Hence

$$\begin{aligned} S &= \Phi(x) + i\,\Theta^+ \sigma^\mu \Theta\, \partial_\mu \Phi(x) + \sqrt{2}\,\Theta\, \Psi(x) \\ &\quad + i\sqrt{2}\,\Theta^+ \sigma^\mu \Theta\, \Theta\, \partial_\mu \Psi(x) - (1/4)\Theta\,\Theta\,\bar{\Theta}\,\bar{\Theta}\,\Box\Phi(x) \\ &\quad + \Theta\Theta\, A(x) \end{aligned} \tag{4}$$

If f(S) is a power series in S, then

$$\begin{aligned} f(S) &= f(\Phi) + \sqrt{2}\, f'(\Phi)\, \Theta\Psi + \Theta\,\Theta\, f'(\Phi)\, A \\ &\quad - (1/2)\, f''(\Phi)\Psi\Psi \end{aligned} \tag{5}$$

which has chiral multiplet structure. Free part of the Lagrangian is given by the integration over the $\Theta\,\Theta\,\bar{\Theta}\,\bar{\Theta}$ component of S^+S, which up to a total divergence, is given by

$$\begin{aligned} L_0 &= -1/2\,(\Phi\,\Box\Phi^* + \Phi^*\Box\,\Phi) + A\,A^* - i/2\,\Psi^+\,\bar{\sigma}^\mu\,\partial_\mu\Psi \\ &\quad + i\,\partial_\mu\,\Psi^+\,\bar{\sigma}^\mu\Psi \end{aligned} \tag{6}$$

Part that includes the interaction is

$$\begin{aligned} L_I &= \int d^2\Theta\; f(s) + \int d^2\Theta\; f(S)^+ \\ &= A\, f'(\Phi) - 1/2\, f''(\Phi)\Psi\Psi + A^*\, f'(\Phi)^* \\ &\quad - 1/2\, f''(\Phi)^*\,\bar{\Psi}\Psi \end{aligned} \tag{7}$$

with the total Lagrangian given by $L = L_0 + L_I$. The fields A can be eliminated in a usual manner by varying A and A^* so that

$$f'(\Phi) + A^* = 0 \tag{8a}$$

$$-\Box\Phi + A^*\, f''(\Phi)^* - 1/2\, f'''(\Phi)^*\bar{\Psi}\bar{\Psi} = 0 \tag{8b}$$

$$i\bar{\sigma}^\mu\,\partial_\mu\Psi + f''(\Phi)\hat{\Psi} = 0 \tag{8c}$$

Equations (8a) and (8b) then yield

$$\Box\Phi = f'(\Phi)\, f''(\Phi)^* - 1/2\, f''(\Phi)^*\,\Psi^+\hat{\Psi} \tag{9}$$

The choice of $f(S) = -(g/3)S^3$ leads to conformally invariant Lagrangian, where g is dimensionless. Then the equations of motion for the propagating field Φ and ψ are

$$i\,\sigma^\mu\,\partial_\mu\Psi = 2\,g\,\Phi^*\hat{\Psi}\,, \tag{10a}$$

$$\Box\Phi = 2\ g^2\ \Phi^2\ \Phi^* + g\ \Psi^+\Psi\ . \tag{10b}$$

To write these in Dirac form we introduce the Majorana field represented by 4x1 column π with

$$\pi = \begin{pmatrix} \Psi \\ -\hat{\Psi} \end{pmatrix} \quad , \qquad \gamma^\mu = \begin{pmatrix} 0 & -i\sigma^\mu \\ -i\bar{\sigma}^\mu & 0 \end{pmatrix} \tag{11}$$

Taking complex conjugate of Eq.(10a) and multiplying it from the left by $-i\sigma_2$ yields

$$i\ \sigma^\mu\ \partial_\mu \hat{\Psi} = 2\ g\ \Phi\ \Psi\ . \tag{12}$$

If we let $\Phi^\mu = \Omega + i\ \Sigma$, where Ω is a real scalar and Σ a real pseudoscalar field, then Eqs.(8c) and (12) can be represented by an equivalent Majorana equation

$$\gamma^\mu\ \partial_\mu\ \pi = 2\ g\ (\Omega + i\ \gamma_5\ \Sigma)\ \pi \tag{13}$$

Here we note that all the four components of π, $\gamma^\mu\ \partial_\mu$ and $i\ \gamma_5$ are all real. Now Eq.(10b) takes the form

$$\Box\Phi = 2\ g^2\ (\Phi^*\ \Phi)\ \Phi + g\ \bar{\pi}\ ((1 + \gamma_5)/2)\cdot\pi. \tag{14}$$

where $\bar{\Psi} = \Psi^+\ \gamma_4 = i\ \Psi^+\ \ \gamma^0$. Taking the complex conjugate of Eq.(14), adding it to itself, and subtracting from itself results in

$$\Box\Omega = 2\ g^2\ (\Omega^2 + \Sigma^2)\ \Omega + 1/2\ g\ \bar{\pi}\ \pi, \tag{15}$$

and

$$\Box\Sigma = 2\ g^2\ (\Omega^2 + \Sigma^2)\ \Sigma - i\ g/2\ \bar{\pi}\ \gamma_5\ \pi. \tag{16}$$

Supersymmetry is formulated in Minkowski space. When we make a transition to Euclidean space on replacing time by imaginary time, the four-spinor still splits into two component right and left handed spinors Ψ_R and Ψ_L. But the transformation properties are drastically altered since the Lorentz group SL(2,C) is now replaced by SU(2)xSU(2)$\sim$O(4). We have

$$\Psi_L' = U\ \Psi_L \qquad , \qquad \Psi_R' = V\ \Psi_R \tag{17}$$

under four dimensional rotation, where U and V are 2x2 unitary matrices. The O(4) invariant quantities are

$$\Psi_L^+\ \Psi_L = \bar{\Psi}((1+\gamma_5)/2)\Psi \quad , \quad (\ \bar{\Psi} = \Psi^+)\ , \tag{18a}$$

$$\Psi_R^+\ \ \Psi_R = \bar{\Psi}((1-\gamma_5)/2)\Psi = \Psi^+\ \ ((1-\gamma_5)/2)\Psi\ \ . \tag{18b}$$

It is easy to show that $\Psi_L^+\ \hat{\Psi}_L$ and $\Psi_R^+\ \hat{\Psi}_R$ are also O(4) invariant. Note that although the first set of invariants Eq.(18) are real, last set are complex. In the Euclidean case if we define D by

$$D = \partial_4 + i\ \vec{\sigma}\cdot\vec{\nabla} \qquad , \quad \bar{D} = \partial_4 - i\ \vec{\sigma}\cdot\vec{\nabla} \tag{19}$$

then they transform as

$$D \to UDV^{+}, \qquad D^{+} \to VD^{+}U \tag{20}$$

so that the Lorentz invariants are $\Psi_R^{+} D^{+} \Psi_L$ and $\Psi_L^{+} D \Psi_R$. Looking at the $\Theta_L \Theta_L \bar{\Theta}_R \bar{\Theta}_R$ component of the product $S_R^{+} S_L$, where

$$S_{L,R} = \Phi_{L,R} + \sqrt{2}\, \Theta_{L,R}^{+} \Psi_{L,R} + \hat{\Theta}_{L,R} \Theta_{L,R} A_{L,R} , \tag{21}$$

we have

$$(S_R^{+} S_L) = \Psi_R^{+} D^{+} \Psi_L + A_R^{*} A_L + \Phi_R^{*} \Box \Phi_L + \Phi_L \Box \Phi_R^{*} , \tag{22}$$

so that the kinetic part of the Lagrangian, up to a total divergence, may now be written as

$$L_0 = \int S_R^{+} d^2\Theta_L\, d^2\bar{\Theta}_R + \int S_L^{+} S_R\, d^2\Theta_R\, d^2\bar{\Theta}_L . \tag{23}$$

The part that includes the interaction is

$$L_1' = g/3 \int (S_L^3\, d^2\Theta_L + S_L^3\, d^2\bar{\Theta}_L) \tag{24}$$

or

$$L_2' = g'/3 \int (S_R^3\, d^2\Theta_R + S_R^{+3}\, d^2\bar{\Theta}_R) \tag{25}$$

so that the general conformal invariant Lagrangian is given by $L = L_0 + L_1' + L_2'$. If $g = g'$ we have parity invariance as well. If $g' = 0$ we recover the results of an earlier work[1].

In the Euclidean case the vector x^μ is associated with a matrix (also called a quaternion) that is proportional to a unitary matrix

$$x = \sum i\, \sigma_\mu\, x^\mu = x^4 + i\, \vec{\sigma}\cdot\vec{x} \tag{26}$$

where the conjugate matrix $\bar{x} = x^{+}$ does satisfy

$$\bar{x}\, x = x^{+} x = \mathrm{Det}\, x = (x^4)^2 + \vec{x}\cdot\vec{x} = x^2 \tag{27}$$

and is O(4) invariant. Thus transformation law for x reads

$$x' = U\, x\, V^{+} = U\, x\, V^{-1}. \tag{28}$$

Since also $\hat{x} = \sigma_2\, x^{+}\, \sigma_2 = \bar{x}$, then x transforms like

$$x = \Psi_L \Psi_R^{+} + \hat{\Psi}_L \Psi_R^{+}. \tag{29}$$

If we write $x\,\bar{y} = s + i\,\vec{\sigma}\cdot\vec{f}$, and $\bar{x}\,y = s + i\,\vec{\sigma}\cdot\vec{\tau}$, where s is the invariant $x^\mu\, y_\mu$, while f and τ are real with the transformation laws

$$i\vec{\sigma}\cdot\vec{f}' = U\, i\vec{\sigma}\cdot\vec{f}\, U^{-1} \tag{30a}$$

and

$$i\vec{\sigma}\cdot\vec{\tau}' = V\, i\vec{\sigma}\cdot\vec{\tau}\, V^{-1} \tag{30b}$$

we have the invariants

$$(i\vec{\sigma}\cdot\vec{f})^2 = a + ib = \alpha \tag{31a}$$

and

$$(i\vec{\sigma}\cdot\vec{\tau})^2 = a - ib = \beta \tag{31b}$$

Thus α corresponds to the Minkowski invariant $(\vec{\sigma}\cdot\vec{f})^2$ with $f^{\mu\nu} = x^\mu y^\nu - x^\nu y^\mu$, and β to its complex conjugate. Further we have the correspondences

$$\pi \rightarrow \Psi \quad , \text{ or } \quad \pi \rightarrow \Psi^c = \gamma_5 \hat{\Psi} = -i\sigma_2 \gamma_5 \Psi^* \; . \tag{32}$$

Also

$$\bar{\pi}\gamma_5 = \bar{\pi}^c \rightarrow \hat{\Psi}^+ \gamma_5 \tag{33a}$$

and

$$i\bar{\pi}\, \gamma_5\, \pi \rightarrow i\, \Psi^+ \gamma_5 \hat{\Psi} \tag{33b}$$

To find the Euclidean version of the free Dirac equation, with our metric in Minkowski space we have the relation

$$i\, \partial_\mu \leftrightarrow p_\mu \tag{34a}$$

and

$$-\, p_\mu\, p^\mu = -p_0^2 + \vec{p}^2 = -m^2 \; . \tag{34b}$$

When $p_0 = ip_4$, this becomes $p_4^2 + \vec{p}^2 = -m^2 = \mu^2 > 0$, so that m, the rest value of the energy, becomes imaginary, since the Euclidean energy is $p_4 = ip_0 = -iE$, E being the Minkowski energy. The Euclidean Dirac equation is then

$$i\, \gamma^\mu\, \partial_\mu\, \Psi = \mu \Psi \quad , \quad (\mu = im) \tag{35}$$

which has a plane wave solution $\Psi(x) = e^{-ip\,x}\, \Psi(0)$. In Euclidean space-time there is no Majorana equation for which μ should be imaginary. The Euclidean Lagrangian leading to Eq.(35) is

$$L = -i\, \Psi^+\, \gamma^\mu\, \partial_\mu \Psi + \mu\, \Psi^+ \Psi + \text{ h.c.} \tag{36}$$

If μ is replaced by a scalar field Ω in interaction with the Dirac field Ψ (also adding a pseudoscalar field Σ) we obtain the extended Lagrangian

$$\begin{aligned} L = &-i\, \Psi^+ \gamma^\mu\, \partial_\mu \Psi + 2g\, \Psi^+ (\, \Omega + i\gamma_5\, \Sigma) \\ &- 1/2\, \partial_\lambda\, \Omega\, \partial^\lambda \Omega - 1/2\, \partial_\lambda\, \Sigma\, \partial^\lambda \Sigma + \text{h.c.} \end{aligned} \tag{37}$$

leading to the Euclidean equations

$$i\, \gamma^\mu\, \partial_\mu \Psi = 2g\, (\, \Omega + i\gamma_5\, \Sigma\,)\Psi \; , \tag{38a}$$

$$\Box\Omega = 2g^2\, (\, \Omega^2 + \Sigma^2\,)\, \Omega + 1/2\, g\, \Psi^+ \Psi \quad , \tag{38b}$$

$$\Box\Sigma = 2g^2 (\Omega^2 + \Sigma^2) \Sigma - 1/2 \quad g \Psi^+ \gamma_5 \Psi . \tag{38c}$$

INSTANTON SOLUTIONS AND CONCLUSIONS

The form of the fermionic solution is

$$\Psi = 8/g \; e^{-i\gamma_5 \alpha/2} \; (k/1+ik\gamma x) \; (k/1+k^2x^2)\eta \tag{39}$$

where k has dimension of inverse length, and the fermionic spinor η obeys

$$\eta^+\eta = -\cos\alpha \quad , \tag{40a}$$

and

$$\eta^+\gamma_5\eta = - \sin\alpha \quad . \tag{40b}$$

The bosonic fields are given by

$$\Omega = 2/g \; w \cos\alpha \quad , \tag{41}$$

$$\Sigma = 2/g \; w \sin\alpha \quad , \tag{42}$$

where

$$w = k / 1 + k^2x^2 \quad . \tag{43}$$

We can now write

$$\Phi = \Omega + i\Sigma = we^{i\alpha} \; , \tag{44a}$$

and

$$\Omega + i\gamma_5\Sigma = we^{i\gamma_5\alpha} \; , \tag{44b}$$

so that Eqs.(38b,c) can now be expressed as

$$\Box\Omega = -16/g \; w^3 \cos\alpha \quad , \tag{45a}$$

and

$$\Box\Sigma = -16/g \; w^3 \sin\alpha \; , \tag{45b}$$

where we used $\Box w = -8w^3$. Using

$$\sigma_2\gamma_5^*\sigma_2 = \gamma_5 \tag{46a}$$

and

$$\sigma_2 (\gamma\cdot x)^* \sigma_2 = \gamma\cdot x \tag{46b}$$

we arrive at

$$\Psi^+\Psi = -64/g^2 \quad w^3 \cos\alpha \; , \tag{47}$$

and

$$\Psi^{+} \gamma_5 \Psi = -64i/g\ w^3 \sin\alpha\ , \tag{48}$$

where we utilized Eq.(40). We further note that

$$(\Omega^2 + \Sigma^2)\ \Omega = 8/g^3\ w^3 \cos\alpha\ , \tag{49}$$

and

$$(\Omega^2 + \Sigma^2)\ \Sigma = 8/g^3\ w^3 \sin\alpha\ . \tag{50}$$

Substituting Eqs.(47-50) into Eq.(38) we see that they are consistent with Eq.(44). In order to show that Eq.(38a) is also satisfied we calculate its left hand side:

$$\begin{aligned}\partial_\lambda \Psi = (8/g)\ e^{-i\gamma_5\ \alpha/2}\ [&(-\sqrt{k}/1+ik\gamma\cdot x)(ik\ \gamma_\lambda\ ) \\ &* (1/1+ik\gamma\cdot x)(k/1+k^2x^2) \\ &-(\sqrt{k}/1+ik\gamma\cdot x)(2k^3\ x_\lambda\ /(1+k^2x^2)^2)\ ]\ \eta\end{aligned} \tag{51}$$

and after some algebra we arrive at

$$\begin{aligned}i\gamma^\mu\ \partial_\mu \Psi = 8/g\ e^{i\gamma_5\ \alpha/2}\ [&\sqrt{k}\ [(4+2ik\gamma\cdot x)k^2\ /\ (1+k^2x^2)^2] \\ &* (1/1+ik\gamma\cdot x) - 2ik\gamma\cdot x\ (\sqrt{k}\ /1+ik\gamma\cdot x) \\ &* (k^2/(1+k^2x^2)^2]\ \eta\end{aligned} \tag{52}$$

where we made use of

$$\gamma_5\ \gamma_\mu\ = -\ \gamma_\mu\ \gamma_5 \tag{53}$$

and

$$\partial_\mu(M^{-1}) = -M^{-1}\ (\ \partial_\mu\ M)\ M^{-1} \tag{54}$$

with M being

$$M = 1/1+ik\gamma\cdot x \tag{55}$$

in our case. Hence

$$i\ \gamma^\mu\ \partial_\mu \Psi = [\ 4k\ e^{i\gamma_5\ \alpha}\ /(1+k^2x^2)\ ]\ \Psi \tag{56}$$

and using Eqs. (41), (43) and (44) we see that Eq.(38a) is exactly satisfied.

The four-dimensional equations we proposed could arise as chiral fermion theories (involving left handed spinors only) which might be derived from effective Lagrangian approximating supersymmetric grand unified theories or weak-interaction theories. Applications to the high temperature limits of four-dimensional supersymmetry are also possible. We shall explore these aspects in subsequent publications.

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REFERENCES

1. S. Catto and Y. Gürsey, Nucl. Phys. B (Proc. Suppl.) 6, 148(1989).

2. F. Gürsey and H. C. Tze, Phys. Lett. B129, 205(1983).

3. S. Catto and F. Gürsey, Phys. Rev. D29, 653(1984).

4. E. Witten, Nucl. Phys. B185, 513(1981).

5. C. Fronsdal, Phys. Rev. D26, 1988(1982).

6. I. Affleck, J. Harvey and E. Witten, Nucl. Phys. B206, 413(1982).

EINSTEIN-HERMITIAN BUNDLES OVER COMPLEX SURFACES

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INTRODUCTION

The notion of an Einstein-Hermitian bundle over a complex manifold has proved to be extremely useful both in algebraic geometry and in superstring theory. In algebraic geometry it has provided a means by which the set of all holomorphic vector bundles on a complex manifold can be studied. In superstring theory it has yielded a way to generalize the Yang-Mills fields to high dimensional complex manifolds.

Einstein-Hermitian bundles were first introduced independently by Hitchin and Kobayashi. They are immediate generalizations of Kähler-Einstein manifolds. Terminology is not fixed and you may see references to Hermitian-Einstein or Hermitian-Yang-Mills connections.

Again I want to thank the organizers of this conference, Ling-Lie Chau and Werner Nahm, for inviting me to this conference and I also want to dedicate this note to Professor Konrad Blueler, whose comments were always to the point and who has worked unstintingly to bring mathematicians and physicists together.

§1. BASIC DEFINITIONS

Suppose that M is a complex holomorphic manifold and that E is a smooth complex vector bundle over M. The fiber dimension or rank of E will be denoted by r. It is easy to put a smooth hermitian structure h, on the fibers of E and so you obtain an Hermitian bundle. To simplify matters let us assume that M is a Kähler manifold.

The bundle E carries a unique connection (covariant derivative) D such that

$$Dh = 0 \quad \text{and} \quad D'' = d''$$

in which the covariant derivative D and the exterior derivative d have been decomposed into $(1,0)$ and $(0,1)$ parts:

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Edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990

$D = D' + D''$
$d = d' + d''$

This connection is called the Hermitian connection on E.

The curvature F of D is defined as D^2 and as usual can be represented by the matrix of 2-forms $d\omega + \omega \wedge \omega$, in which D is locally represented by ω.

In this case the hermitian connection satisfies $\omega = h^{-1} d'h$, while $F = d''\omega$.

The mean curvature of D is defined to be:

$$K^i_j := g^{\alpha\overline{\beta}} R^i{}_{j\alpha\overline{\beta}}$$

in which $g_{\alpha\overline{\beta}}$ is the Kähler metric on M and $F = R^i_{j\alpha\overline{\beta}} \; dz^{\alpha} \wedge d\overline{z}^{\beta}$ locally.

Definition. E satisfies the weak Einstein condition iff $K^i_j = \phi\, \delta^i{}_j$, ϕ real valued on M. If ϕ is constant, then E is called an Einstein-Hermitian bundle.

Note that when the complex dimension of M is two, an Einstein-Hermitian bundle with $\phi = 0$ qualifies as an anti-instanton bundle. Various properties of the basic idea can be found in the book of Kobayashi in the references.

The usefulness of the idea can best be demonstrated by the next theorem due to Donaldson.

Theorem. If M is a projective algebraic manifold, then every Einstein-Hermitian bundle is a direct sum of stable bundles.

Remember that projective algebraic manifolds are those that can holomorphically embedded in complex projective space. I do not want to digress about stable bundles. Suffice it to say that stable bundle represent some type of non-degenerate bundle. For simplicity I am thinking of Mumford-Takemoto stability.

§2. EXAMPLES

In this section I want to give some examples of Einstein-Hermitian bundles and indicate a method of generating many such bundles.

The first example is the complex 2-dimensional torus, denoted by T_2. Thus $T_2 = C^2 / \Lambda$ in which Λ is a lattice in C^2. All of the bundles on T_2 can be realized as 'equivariant bundles' on C^2. Suppose that the multipliers $a: C^2 \times \Lambda \to GL(V,C)$ satisfy the (co-cycle) condition:

$$a(x,\lambda'\lambda) = a(\lambda(x),\lambda')\; a(x,\lambda) .$$

The bundle map $A: C^2 \times C^r \to C^2 \times C^r$ given by:

$$A(x,\omega) = (\lambda(x),\; a(x,\lambda)\omega)$$

yields bundles on T_2 by quotienting out the trivial bundle by the group generated by the bundle maps. Further if you choose

$$a(x,\lambda) = U(\lambda)\ \exp(\frac{1}{r}\ F(x,\lambda) + \frac{1}{2r}\ F(\lambda,\lambda))$$

in which F is hermitian, Im F/π is an integer on Λ and U is a 'semi-representation', all of the projectively flat Einstein-Hermitian bundles of rank can be produced in this manner. This fact is proved in Kobayashi.

On the other hand, if T_2 is an abelian variety (that is, T_2 is projective algebraic) and $\theta : T_2 \to P_N(C)$ is the embedding via theta-functions, then the pull back of an Einstein-Hermitian bundle on $P_N(C)$ is an Einstein-Hermitian bundle with respect to the pulled back metric on T_2. This is not difficult to verify. Moreover, the pull back will satisfy the weak Einstein condition with respect to the flat metric on T_2.

The last example I want to discuss involves the twistor space of T_2. Recall that the twistor space of T_2 is topologically $P_1(C) \times T_2 =: Z$, but that the complex structure on Z is not the product complex structure. To be precise $P_1(C)$ is the Riemann sphere so $(\alpha,q) \to q\alpha$ yields a complex structure. Some explanation is needed here. $P_1(C)$ represents the unit sphere of imaginary quaternions while the points of T_2 are thought of as elements of the one-dimensional (additive) quaternonic torus. It is not difficult to show that Z is a holomorphic 3-fold, but it is much more delicate to prove that Z is <u>not</u> a Kähler manifold. In fact, a theorem of André Blanchard settles the matter, in that all of the fibers of $Z \to P_1(C)$ would have to have the same complex structure in this case.

As in the case of embedding an abelian variety you can pull back Einstein-Hermitian bundles over T_2.

Theorem. Given the twistor fibering $Z \to T_2$ any Einstein-Hermitian bundle over T_2 pulls back to an Einstein-Hermitian bundle over Z. To prove the theorem, it is necessary to verify that the projection map $Z \to T_2$ pulls (1,1) forms back to (1,1)-forms even though it is <u>not</u> a holomorphic map.

This theorem can be viewed as a type of Penrose-Ward correspondence.

Discussion. The emphasis on stable bundles as noted earlier is solely based on the fact that they are non-degenerate in some sense and therefore you have control over them, just as continuous functions are easier to study than discontinuous functions. In fact, on the projective spaces of low dimensions you can 'describe' stable bundles and hence solve the classical instanton problem.

The K-theory of vector bundles yields a type of extraordinary cohomology ring and is more remote than studying a moduli space of bundles, which is what is proposed here.

The usefulness of projective algebraic holomorphic manifolds lies in their simplicity. They are in some sense easier to handle than general holomorphic manifolds and allow the use of Serre's theorems on analytic geometry. The theorems of Serre were also discussed by R.O. Wells, Jr. in his earlier lecture.

Finally, the equations which arise from the Einstein-Hermitian condition can be treated in the basic manner that Yang used to look at instantons many years ago. Interesting as this is, it would take us too far afield to go into details here.

REFERENCES

S. Donaldson, 1985, Anti Self-dual Yang-Mills...., Proc. London Math. Soc., 50:1-26.

S. Kobayashi, 1987, "Differential Geometry of Complex Vector Bundles," Plenum Press, NY.

A. Blanchard, 1956, Sur les varietes analytiques complexes, Ann. Sci. E.N.S., t73:1517-202.

K. Uhlenbeck and S.-T. Yau, 1986, On the Existence of Hermitian-Yang-Mills Connections in Stable Vector Bundles, Comm. Pure and Applied Math., 39:257-293.

Symplectic Reduction of the Minimally Coupled Massless Superparticle in D=10

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Abstract

In this study of the Brink-Schwarz superparticle in the presence of background non-abelian gauge field, we derive the D=10 super Yang-Mills curvature constraints by demanding an analog of the Siegel supersymmetry. We then show that, assuming $F_{\alpha\beta} = 0$, the twistor transform proposed by Witten can be described using symplectic reduction.

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I *Curvature Constraints Implied by Siegel Symmetry*

In this section, we study the Brink-Schwarz superparticle in a $D = 10$ Majorana-Weyl superspace in the presence of a background Yang-Mills superfield. We show that a necessary condition for the existence of an analog of the Siegel supersymmetry of the Brink-Schwarz particle in the absence of background gauge fields is that the spinor-spinor component of the non-abelian field strength vanish. This condition is shown to be equivalent to the full super Yang-Mills equations in [2].

We begin with a brief review of superparticle dynamics in the absence of a background gauge field. Our superspace is parametrized by supercoordinates $Z^M = (x^m, \theta^\mu)$. In addition, we introduce a set of N Grassman variables, ψ^I representing the internal degrees of freedom of our superparticle which describe the coupling to the background Yang-Mills gauge field.

We begin by studying the system defined by the Lagrangian

$$L = \frac{1}{2} e^{-1} \eta_{ab} \dot{Z}^M E_M{}^a \dot{Z}^N E_N{}^b + i \dot{\psi}^I \psi_I \tag{1}$$

In writing this, we have introduced an einbein, e, enforcing reparametrization invariance. We have also introduced the vielbein super 1-forms $E^A = (E^a, E^\alpha)$, which are related to the coordinate differentials by $E^A = dZ^M E_M{}^A$. We shall assume in the following discussion that the E^A correspond to a "flat" superspace. The indices on the internal space are raised and lowered with an N-dimensional Euclidean metric.

Passage to the Hamiltonian formalism is effected by defining

$$P_M = \frac{\partial L}{\partial \dot{Z}^M} = e^{-1}(\tau) \eta_{ab} E_M{}^a \dot{Z}^N E_N{}^b, \tag{2}$$

$$P_I = \frac{\partial L}{\partial \dot{\psi}^I} = i\psi_I, \tag{3}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0. \tag{4}$$

It follows that P_e is a primary constraint in Dirac's language, as is

$$\phi_I = P_I - i\psi_I \approx 0, \tag{5}$$

and

$$u_\alpha = V_\alpha{}^M P_M \approx 0, \tag{6}$$

where $V_A{}^M$ is the inverse super-vielbein, satisfying $V_A{}^M E_M{}^B = \delta_A^B$.

The canonical Hamiltonian is $H_C = \dot{Z}^M P_M - \dot{\psi}^I P_I - L \approx \frac{1}{2} e u_a u^a$, where $u_a = V_a{}^M P_M$. The primary Hamiltonian is given by $H_P = H_C + \lambda^\alpha u_\alpha + \lambda^I \phi_I + \rho P_e$, where λ^α, λ_I and ρ are multipliers which are arbitrary at this stage of the analysis.

Dynamics in the super phase space can be formulated by introducing the super Poisson brackets

$$\{A, B\} = \sum_i (-)^{A\, q^i} \frac{\partial A}{\partial q^i} \frac{\partial B}{\partial p_i} - (-)^{AB} (-)^{B\, q^i} \frac{\partial B}{\partial q^i} \frac{\partial A}{\partial p_i}, \tag{7}$$

where q^i is a complete set of canonical coordinates and p_i their conjugate momenta.

Demanding that the primary constraint $P_e = 0$ be preserved by time translations, $\{H_P, P_e\} \approx 0$, yields the secondary constraint

$$\chi = \frac{1}{2}u^a u_a \approx 0. \tag{8}$$

Before proceeding further, it is convenient to eliminate the second class constraints ϕ_I. This can be done either via Dirac brackets, or, equivalently, by noting that on the constraint surface, one can add an arbitrary linear combination of the constraints. Our Poisson brackets with respect to the ψ_I are

$$\{A(\psi), B(\psi)\} = (-)^A \frac{\partial A}{\partial \psi^I}\frac{\partial B}{\partial P_I} - (-)^{AB}(-)^B \frac{\partial B}{\partial \psi^I}\frac{\partial A}{\partial P_I} \tag{9}$$

Thus, we consider replacing the ψ_I everywhere with with an arbitrary linear combination

$$\psi'^I = \psi^I + C^{IJ}\phi_J. \tag{10}$$

Requiring that

$$\{\psi'^I, \phi_J\} = 0 \tag{11}$$

determines that

$$C^{IJ} = -\frac{i}{2}\delta^{IJ} \tag{12}$$

The ψ'^I satisfy

$$\{\psi'^I, \psi'^J\} = \frac{i}{2}\delta^{IJ} \tag{13}$$

which is just the Dirac bracket of the ψ^I.

Since the other constraints do not depend on the ψ^I in the absence of a gauge field, the ψ^I will play no further role in the discussion until we introduce a background gauge field.

The constraints χ and u_α satisfy the algebra

$$\begin{aligned}
\{\chi, \chi\} &= 0, &(14)\\
\{u_\alpha, u_\beta\} &= \Omega^A_{\alpha\beta} u_A, &(15)\\
\{u_\alpha, \chi\} &= \eta^{ab} u_b \Omega^B_{\alpha a} u_B, &(16)
\end{aligned}$$

where we have introduced the object of anholonomicity,

$$\Omega^A = dE^A = \frac{1}{2}E^C \wedge E^B\, \Omega^A_{BC}. \tag{17}$$

In flat superspace, $\Omega^a_{\alpha\beta} = \Gamma^a_{\alpha\beta}$, with all other components of the anholonomicity 2-form vanishing. Then χ is first class in Dirac's language, generating reparameterizations, while the u_α are not first class since $\Gamma^a_{\alpha\beta}u_a$ does not vanish. Nevertheless, $(\Gamma^a u_a)^2 = 2\chi \approx 0$, and hence there are first class constraints. Explicitly, these can be taken to be $\pi^\alpha = \Gamma^{a\alpha\beta}u_a u_\beta$ in flat superspace. π^α is the generator of the local fermionic symmetry discovered by Siegel. The fact that the u_α cannot be covariantly decomposed into 8 first class and 8 second class constraints is the origin of the difficulties in quantizing this formulation of the superparticle.

We now turn to the study of the $D = 10$ Brink-Schwarz superparticle in a background Yang-Mills superfield, $A_{M\,IJ}(Z) = A^a_M \lambda^a_{IJ}$. The λ matrices are the usual adjoint representation of the algebra, which we take to be $SO(N)$. Thus, we consider the Lagrangian

$$L = \frac{1}{2}e^{-1}\eta_{ab}\dot{Z}^M E^a_M \dot{Z}^N E^b_N + i\dot{\psi}^I\psi_I + \dot{Z}^M A_{M\,IJ}\psi^I\psi^J, \tag{18}$$

We define the momenta

$$P_M = \frac{\partial L}{\partial(\dot{Z}^M)} = e^{-1}\eta_{ab}E^a_M\dot{Z}^N E^b_N + A_{M\,IJ}\psi^I\psi^J \tag{19}$$

It is useful to introduce the velocity-like variables

$$u_A = V_A{}^M(P_M - A_{M\,IJ}\psi^I\psi^J). \tag{20}$$

As before, the spinorial velocity variables are constrained to vanish:

$$u_\alpha \approx 0 \tag{21}$$

as is the momentum conjugate to the einbein:

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 \tag{22}$$

The momenta conjugate to the the ψ^I:

$$P_I = \frac{\partial L}{\partial \dot{\psi}^I} = i\psi_I \tag{23}$$

also define primary constraints:

$$\phi_I = P_I - i\psi_I \approx 0 \tag{24}$$

Performing the Legendre transformation, we find the canonical Hamiltonian,

$$\begin{aligned} H_c &= \dot{Z}^M P_M + i\dot{\psi}^I\psi_I - L \\ &= \frac{1}{2}eu^a u_a \end{aligned} \tag{25}$$

Requiring that the constraint (22) be preserved by time translation tells us that we have a secondary constraint,

$$\chi = \frac{1}{2}u^a u_a \approx 0 \tag{26}$$

Before proceeding with our analysis, we eliminate the second-class constraints, ϕ_I as before. Now, in our constraint algebra, the only place that the ψ^I enter is quadratically, in the velocity variables u_A. Noting that the Dirac bracket

$$\{\lambda^a{}_{IJ}\psi^I\psi^J, \lambda^b{}_{KL}\psi^K\psi^L\}|_D = if^{ab}{}_c\lambda^c{}_{IL}\psi^I\psi^L \tag{27}$$

it follows that brackets involving the gauge fields will behave like commutators in the adjoint representation of the group as far as the ψ^I are concerned after we eliminate the constraints ϕ_I. All future Poisson brackets will be understood to be Dirac brackets with respect to the ψ^I. With this provision, we suppress the explicit dependence on ψ^I, and write our equations in terms of

$$\tilde{A}_M = A_{M\,IJ}\psi^I\psi^J \tag{28}$$

In the absence of a gauge field, there are, as we have seen no further constraints. Therefore, we look for an analog of the Siegel symmetry in the presence of the gauge field. Specifically, we look for an even constraint

$$\hat{\chi} = \chi + \lambda^\alpha u_\alpha \tag{29}$$

and a set of spinorial first class constraints

$$\pi^\alpha = H^{\alpha\beta} u_\beta \tag{30}$$

such that

$$\{\hat{\chi}, u_\gamma\} \approx 0 \tag{31}$$

and

$$\{\pi^\alpha, u_\gamma\} \approx 0, \tag{32}$$

that is, $\hat{\chi}$ and π^α are first class. In these equation, λ^α and $H^{\alpha\beta}$ are thought of as smooth functions of the dynamical variables. Requiring this set to be first class constraints does not prevent mixing, so any putative term in π^α of the form $\rho^\alpha\chi$ could be eliminated by subtracting a multiple of $\hat{\chi}$, which just involves a different choice of basis for the first class constraints.

We shall assume that to zeroth order term in powers of ψ^I, everything reduces to the gauge-free case. This should be understood as follows. The combination $\psi^I\psi^J$ determines the coupling of the background gauge field to the particle; our conditions govern the limit in which the coupling is "turned off". Then $H^{\alpha\beta} \rightarrow \hat{H}^{\alpha\beta} = \Gamma^{a\,\alpha\beta} u_a$ and $\{\pi^\alpha, u_\gamma\} \rightarrow 2\hat{\chi}\delta^\alpha_\gamma \approx 0$.

It is straightforward to check, in flat superspace with a general gauge background, that

$$\{u_\alpha, u_\beta\} = \Gamma^c_{\alpha\beta} u_c + F_{\alpha\beta\,IJ}\psi^I\psi^J. \tag{33}$$

Here F is the usual Yang-Mills field strength, defined as a matrix (IJ) valued 2-form

$$\begin{aligned} F &= \frac{1}{2} dZ^N dZ^M (\partial_M A_N - (-)^{MN}\partial_N A_M - i[A_M, A_N]) \\ &= \frac{1}{2} dZ^N dZ^M F_{MN} = \frac{1}{2} E^B E^A F_{AB} \end{aligned} \tag{34}$$

in terms of the matrix A_M whose (IJ)'th element is $A_{M\,IJ}$. Note we have written F both in terms of its manifold coordinates F_{MN} and in its "inertial" coordinates F_{AB}, which are different. It is the latter which enters (33).

Our requirement that the π^α be first class can be given explicitly as

$$H^{\alpha\beta}(\Gamma^a{}_{\beta\gamma} u_a + F_{\beta\gamma\,IJ}\psi^I\psi^J) \approx 0. \tag{35}$$

We now expand $H^{\alpha\beta}$ in powers of ψ^I

$$H^{\alpha\beta} = \Gamma^{a\,\alpha\beta} u_a + \sum_{k=1}^{\infty} \hat{H}^{\alpha\beta}_{I_1\ldots I_k}\psi^{I_1}\cdots\psi^{I_k} \tag{36}$$

and find that the first and second order terms of (35) give

$$\psi^I \hat{H}^{\alpha\beta}_I \Gamma^a_{\beta\gamma} u_a \approx 0 \tag{37}$$

$$\psi^I\psi^J u_a\left(\hat{H}^{\alpha\beta}_{IJ}\Gamma^a_{\beta\gamma} + \Gamma^{a\,\alpha\beta} F_{\beta\gamma\,IJ}\right) \approx 0 \tag{38}$$

The field strength is not a function of u_b, so the vanishing of (38) means that, dropping possible terms[1] in $\hat{H}^{\alpha\beta}_{IJ}$ which vanish for $u_b \to 0$, we have

$$\hat{H}^{\alpha\beta}_{IJ}\Gamma^{a}_{\beta\gamma} + \Gamma^{a\,\alpha\beta}F_{\beta\gamma\,IJ} = 0 \tag{39}$$

In solving this, it is useful to decompose the symmetric matrix F as

$$F_{IJ\,\beta\gamma} = F^{a}_{IJ}\Gamma_{a\,\beta\gamma} + F^{(5)}_{IJ}\Gamma_{(5)\,\beta\gamma} \tag{40}$$

where $\Gamma_{(5)\,\beta\gamma}$ denotes the antisymmetric product of 5 Γ-matrices. On post-multiplying (39) by $\Gamma^{\gamma\delta}_{a}$, and using the 10-D identity

$$\Gamma^{\alpha\beta}_{a}\Gamma^{(n)}_{\beta\gamma}\Gamma^{a\,\gamma\delta} = (-1)^n(10-2n)\tilde{\Gamma}^{(n)\,\alpha\delta}, \tag{41}$$

we find,

$$\hat{H}^{\alpha\beta}_{IJ} = \frac{8}{10}F^{a}_{IJ}\Gamma^{\alpha\beta}_{a} \tag{42}$$

One can now plug this back into (39), then pre-multiply by $\Gamma_{a\,\delta\alpha}$, from which it follows that

$$-\frac{64}{10}F^{b}_{IJ}\Gamma_{b\,\delta\gamma} + 10F_{IJ\,\delta\gamma} = 0 \tag{43}$$

From this, it follows that both F^a and $F^{(5)}$ must vanish. Hence, we conclude that, in order for there to be a first class spinorial constraint which reduces to the Siegel symmetry as the coupling to the gauge field is turned off,

$$F_{IJ\,\alpha\beta} = 0. \tag{44}$$

To continue the analysis of the first class constraints, we observe that, when the spinor-spinor curvature vanishes, the Bianchi identity implies that the vector-spinor curvature has the form

$$F_{a\alpha} = \phi^{\beta}\Gamma_{a\alpha\beta}. \tag{45}$$

One can evaluate λ^α using Eq. (31).

$$0 \approx \{\chi, u_\gamma\} + \lambda^\alpha\{u_\alpha, u_\gamma\} = u^a F_{a\gamma} + \lambda^\alpha\Gamma^{a}_{\alpha\gamma}u_a = u^a(\phi^\alpha + \lambda^\alpha)\Gamma_{a\alpha\gamma}, \tag{46}$$

As ϕ^α is not a function of u_b, and there are no constraints linear in u_b, the coefficient must vanish, which determines [2] λ^α.

The result just proved can be generalized to the case of a vielbein which does not necessarily represent flat superspace. The argument given in [3] applies here also if we consider terms order by order in ψ^I. Looking at 0'th order terms we conclude that in the proper basis for the vielbein, Ω^{A}_{BC} satisfies the Witten constraints (eq. 27 in [5]):

$$\Omega^{a}_{\alpha\beta} = \Gamma^{a}_{\alpha\beta} \quad , \quad \Omega^{(ab)}_{\alpha} = 0 \tag{47}$$

These conditions are sufficient to apply the proof presented above.

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[1] We have not ruled out such terms in $\hat{H}^{\alpha\beta}_{IJ}$, but any such terms will have to satisfy (38) by themselves, and are irrelevant to our discussion.

[2] λ^α is not compeletely determined, because a term of the form $\zeta_\beta\Gamma^{b\,\beta\alpha}u_b$ would add only a piece with a factor of χ to the bracket, still preserving the constraints. But this corresponds only to adding a multiple of π^α, a first class constraint, into the definition of the first class constraint $\hat{\chi}$, so this indeterminancy is of no significance.

II *Symplectic Reduction*

Assuming the curvature constraints $F_{\alpha\beta} = 0$ on the super Yang Mills field strength, the twistor transform proposed by Witten [5] can be described using symplectic reduction. We shall show that the algebra of superfunctions on the constraint set (eq. 20, 23, 25) invariant under the first class constraints (eq. 28,30) equals the algebra of superfunctions on the supermanifold corresponding to a super vector bundle over super null line space. It will be clear from the construction that this super vector bundle is precisely the one given by the supersymmetric extension of the Penrose Ward transform.

Since Poisson bracket of any constraint with a first class constraint vanishes on the constraint set, the Poisson bracket with first class constraints is defined for functions on the constraint set. Let $f(Z^M, u_a, \psi^I)$ be such a function.

$$\begin{aligned}
\{\hat{\chi}, f\} &\approx \{\hat{\chi}, Z^M\}\frac{\partial f}{\partial Z^M} + \{\hat{\chi}, u_a\}\frac{\partial f}{\partial u_a} + \{\hat{\chi}, \psi_I\}\frac{\partial f}{\partial \psi_I} \\
&= u^a\{u_a, Z^M\}\frac{\partial f}{\partial Z^M} + \lambda^\alpha\{u_\alpha, Z^M\}\frac{\partial f}{\partial Z^M} \\
&\quad + u^b\{u_b, u_a\}\frac{\partial f}{\partial u_a} + \lambda^\alpha\{u_\alpha, u_a\}\frac{\partial f}{\partial u_a} \\
&\quad + u^a\{u_a, \psi_I,\}\frac{\partial f}{\partial \psi_I} + \lambda^\alpha\{u_\alpha, \psi_I\}\frac{\partial f}{\partial \psi_I} \\
&= -u^a V_a^M\frac{\partial f}{\partial Z^M} - \lambda^\alpha V_\alpha^M\frac{\partial f}{\partial Z^M} \\
&\quad + u^b F_{ba}\frac{\partial f}{\partial u_a} + \lambda^\alpha F_{\alpha a}\frac{\partial f}{\partial u_a} \\
&\quad + iu^a A_{aIJ}\psi^J\frac{\partial f}{\partial \psi_I} + i\lambda^\alpha A_{\alpha IJ}\psi^J\frac{\partial f}{\partial \psi_I} \\
&= \left[-V_a^M u^a\frac{\partial f}{\partial Z^M}\right] + \left[iu^a A_{aIJ}\psi^J\frac{\partial f}{\partial \psi_I}\right] \\
&\quad + \left[F_{ba}u^b\frac{\partial f}{\partial u_a} - \lambda^\alpha(V_\alpha^M\frac{\partial f}{\partial Z^M} + F_{\alpha a}\frac{\partial f}{\partial u_a} - iA_{\alpha IJ}\psi^J\frac{\partial f}{\partial \psi_I})\right]
\end{aligned} \tag{48}$$

$$\begin{aligned}
\{\pi^\alpha, f\} &\approx u^a\Gamma_a^{\alpha\beta}\{u_\beta, f\} \\
&= u^a\Gamma_a^{\alpha\beta}\Big(\{u_\beta, Z^M\}\frac{\partial f}{\partial Z^M} + \{u_\beta, \psi^I\}\frac{\partial f}{\partial \psi_I} + \{u_\beta, u_a\}\frac{\partial f}{\partial u_a}\Big) \\
&= \Big(\left[-u^a\Gamma_a^{\alpha\beta}V_\beta^M\frac{\partial f}{\partial Z^M}\right] + \left[u^a\Gamma_a^{\alpha\beta}iA_{\beta IJ}\psi^J\frac{\partial f}{\partial \psi_I}\right] + \left[u^b\Gamma_b^{\alpha\beta}F_{\beta a}\frac{\partial f}{\partial u_a}\right]\Big)
\end{aligned} \tag{49}$$

where $A_{a\,IJ} = V_a^M A_{M\,IJ}$ and $A_{\alpha\,IJ} = V_\alpha^M A_{M\,IJ}$. The three summands enclosed in [] at the end of equation (48) define three vector fields, denoted X_1, X_2, X_3, operating on the superfunction f. Similarly the three summands in the last line of equation (49) define vector fields $X_1^\alpha, X_2^\alpha, X_3^\alpha$. The vector fields X_1, X_2, X_1^α,and X_2^α preserve the order in ψ^I. From the definition of λ it follows that $\lambda^\alpha F_{\alpha a} = 0$. Thus the vectorfields X_3 and X_3^α raise the order in ψ^I by two. The vector fields X_1 and X_1^α define the tangents to super null lines (cf. [2]), and so the algebra of invariant superfunctions on the submanifold of (Z^M, u_a) space defined by the constraint $u^a u_a = 0$ is isomorphic to the algebra of superfunctions on (non-projective) super null line space.

We have

$$\{\hat{\chi}, f\} = X_1 f + X_2 f + X_3 f, \quad \{\pi^\alpha, f\} = X_1^\alpha f + X_2^\alpha f + X_3^\alpha f. \tag{50}$$

Expand the function $f(Z^M, u_a, \psi^I)$ in powers of ψ^I

$$\begin{aligned} f(Z^M, u_a, \psi^I) &= f_0(Z^M, u_a) + f_I(Z^M, u_a)\psi^I \\ &\quad + \sum_{k\geq 2, I_1\dots I_k} f_{I_1\dots I_k}(Z^M, u_a)\psi^{I_1}\cdots\psi^{I_k}. \end{aligned} \tag{51}$$

It is clear that invariance under the first class constraints implies that f_0 is invariant along the super null lines. If we consider the ψ^I as sections of a super vector bundle over super Minkowski space which has been pulled back to the phase space supermanifold with coordinates (Z^M, P_M) and restricted to the constraint submanifold with coordinates (Z^M, u_a), subject to the further constraint $u^a u_a = 0$, then the expression,

$$\nabla_{\frac{\partial}{\partial Z^M}} = \frac{\partial}{\partial Z^M} - iA_{M\,IJ}\psi^I \frac{\partial}{\partial \psi_J}$$

defines a covariant derivative. The invariance conditions imply that $f_I(Z^M, u_a)\psi^I$ represents a section covariant constant along the super null lines.

Let $f_k = \sum f_{I_1\dots I_k}(Z^M, u_a)\psi^{I_1}\cdots\psi^{I_k}$ be a homogeneous superfunction of order k in ψ^I. Suppose that $(X_1 + X_2)f_k = (X_1^\alpha + X_2^\alpha)f_k = 0$. Then there exists a super function $f = \sum_{j=0}^\infty f_{k+2j}$ whose expansion in powers of ψ^I begins with f_k and is invariant under the first class constraints.

The solution is generated by solving the recursive system

$$(X_1 + X_2)f_{k+2j+2} = -X_3 f_{k+2j} \ , \ (X_1^\alpha + X_2^\alpha)f_{2k+2j+2} = -X_3^\alpha f_{k+2j}. \tag{52}$$

The compatibility conditions are equivalent to closure of the algebra of first class constraints. The corresponding homogeneous system without the terms involving X_3, X_3^α describes covariant constant sections of the exterior algebra bundle and the method of solution is described in [1] and [4].

The superfunctions in (Z^M, u_a) which are invariant under X_1 and X_1^α define coordinates ω^σ on (non-projective) super null line space. The theorem given above guarantees the existence of a superfunction invariant under the first class constraints whose leading term in the ψ^I expansion is ω^σ. This will form one part of the basis for invariant functions on the constraint manifold, that is, coordinates on super null line space. We denote this superfunction by $\tilde{\omega^\sigma}$. To complete the argument we first assume that ψ^I form a local basis for the sections of the given vector bundle. When the bundle is "pulled-back" to super phase space the pull-backs of the same sections, which we denote by the same symbol, form a local basis. In order to solve for covariant constant sections of the form $\hat{\psi}^I = f_I(Z^M, u_a)\psi^I$ we use a method described in [1]. Let $\tilde{\psi}^I$ be the superfunction invariant under the first class constraints (a full expansion in powers of ψ^I) leading term $\hat{\psi}^I$.

Local coordinates for the quotient of the constraint supermanifold by the first class constraints are given by the superfunctions $\tilde{\omega^\sigma}$ and $\tilde{\psi}^I$.

To demonstrate the result we first observe that any superfunction on the constraint manifold can be written in the form

$$f = \sum f_{I_1\dots I_k}(Z^M, u_a)\tilde{\psi}^{I_1}\cdots\tilde{\psi}^{I_k}.$$

Invariance implies that $X_1 f_0 = 0$ and thus f_0 is a function of ω^σ. Replacing ω^σ with $\tilde{\omega^\sigma}$ only changes higher order terms in the ψ^I expansion, so we can write

$$f = f_0(\tilde{\omega^\sigma}) + \sum_{k\geq 1, I_1\dots I_k} f^*_{I_1\dots I_k}(Z^M, u_a)\tilde{\psi}^{I_1}\cdots\tilde{\psi}^{I_k}.$$

Suppose that

$$f = \sum_{j<k, I_1 \ldots I_j} f_{I_1 \ldots I_j}(\tilde{\omega^\sigma})\tilde{\psi^{I_1}} \cdots \tilde{\psi^{I_j}} + \sum_{j \geq k, I_1 \ldots I_j} f_{I_1 \ldots I_j}(Z^M, u_a)\tilde{\psi^{I_1}} \cdots \tilde{\psi^{I_j}}.$$

The next step in the induction is to show that the kth order term can be expressed in terms of $\tilde{\omega^\sigma}$. Invariance implies that each $f_{I_1 \ldots I_k}$ is invariant under X_1 and X_1^α and thus may be expressed in terms of ω^σ. Once again we replace ω^σ by $\tilde{\omega^\sigma}$ and affect only the higher order terms in ψ^I or equivalently in $\tilde{\psi}^I$ so we have proved the induction.

We conclude that the structure sheaf of the quotient supermanifold is equivalent to the extension of the superfunctions on super null line space by the exterior algebra of covariant constant sections.

REFERENCES

[1] Harnad, J., Hurtubise, J., and Shnider, S., Supersymmetric Yang Mills Equations and Supertwistors, *Annals of Physics* (to appear)

[2] Harnad, J. and Shnider, S., *Comm. Math. Phys.*, **106** (1986) 183

[3] Shapiro, J. A. and Taylor, C. C., *Phys. Lett. B* **181** (1986) 67 **186** (1986) 69

[4] Shnider, S. and Wells, R. O. Jr., Supermanifolds, Supertwistor Spaces, and Yang Mills Fields, *Seminaires de Mathematiques Superieures* **105** Universite de Montreal, 1989

[5] Witten, E. *Nuc. Phys. B* **266** (1986) 245.

QUANTUM GRAVITY AND THE BERRY PHASE

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ABSTRACT

The equations describing quantum gravity in the presence of matter are decomposed into coupled equations by writing the total wave function as the product of a gravitational and a matter-gravity part. Although the total wave function is gauge-invariant ("neutral") the gravitation and matter parts acquire opposite and, in general, non-integrable phases which are the anholonomy associated with the parallel transport of the state vector with respect to a connection. Further in the coupled equations describing the system one also has a scalar potential which is associated with a metric tensor on a manifold of quantum states and is related to fluctuations. The (Berry) phases, like the fluctuations, can lead to observable effects in suitably constructed matter-gravity systems.

I. INTRODUCTION

The invariance of the matter-gravity action under arbitrary space-time transformations has as a consequence that time does not appear in the corresponding Hamiltonian formulation. Such a feature is maintained in the canonical quantization of gravity within the superspace approach[1)]. In such an approach the space-time dynamical variables are the three geometries of space-like surfaces with their conjugate momenta while matter is described by the corresponding fields and their conjugate momenta. As a consequence Einstein's equations become geodesic equations in the manifold of three geometries [1,2)] (superspace) modified by the presence of a "force term". Canonical quantization then leads to a Schrödinger-like equation (Wheeler-DeWitt) and a corresponding wave function satisfying it.

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On examining the Wheeler-DeWitt equation for the matter-gravity system one observes that the natural mass scale for the gravitational part is the Planck mass whereas the matter part involves more conventional (usual particle masses) mass scales. Further, the matter-related part generally only contains the three metric and not its conjugate momenta.

This immediately suggests an analogy with other composite systems, such as molecules, which involve two mass, or time,scales and which, as a consequence, are studied within the Born-Oppenheimer or adiabatic approximation [3]. The case of the present system, however, is particularly interesting since, as mentioned above, time does not appear explicitly in the composite system [4] and one is effectively dealing with the quantum mechanical version of constraints associated with the invariance properties of the action. Thus beginning from the resulting quantum equations one may examine the consequences and implications of the separation of the total wave function into a matter and a gravitation part and see under what conditions matter will follow gravitation adiabatically and be described by the usual quantum dynamics. This is done in the next Section.

In Section III we illustrate the geometrical and topological structures induced in superspace, through the presence of matter, within the above approach and in the last Section our results are summarized and discussed.

II. THE MATTER-GRAVITY QUANTUM SYSTEM

If we write our four dimensional line element in the following form [5]:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (-\beta^2 + \beta_i\beta^i)\,dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j \qquad (2-1)$$

corresponding to the standard (3+1) decomposition of space-time, one has that the total Hamiltonian (gravity plus matter) is given by:

$$H = H^G + H^M = \int d^3x(\beta\mathcal{H}_0^T + \beta^i\mathcal{H}_i^T) = \sum_x \varepsilon^3(\beta\mathcal{H}_0^T + \beta^i\mathcal{H}_i^T) \equiv \sum_x H_x \quad (2-2)$$

where in the second line of the above we have replaced the integral over three-space by a sum over all points of a lattice of volume ε^3, thus obtaining a sum of Hamiltonians at each point. The introduction of such a lattice, besides being necessary for justifying the formal manipulations involving functional derivatives, is the most natural approach to superspace which is the topological product of the 6-dimensional space of "points" $\{\gamma^{ij}\}$ with itself over all points of 3-space x. We note that the lapse and shift functions, β and β^i, are not true degrees of freedom but play the role of Lagrange multipliers and we have chosen the so-called "proper time" gauge conditions [6]:

$$\frac{\partial\beta}{\partial t} = \frac{\partial\beta^i}{\partial t} = 0 \qquad (2-3)$$

From the above one obtains Hamiltonian constraints for the classical system and on quantizing these become conditions on a wave function $\Psi(\gamma, \varphi)$ which is a functional of the three metric γ_{ij} and the matter field φ [7]. In particular these latter are:

$$\begin{aligned}
\mathcal{H}_0^T\Psi &= (\mathcal{H}_0^G + \mathcal{H}_0^M)\Psi \\
&= \left\{\frac{1}{8\pi G}\left[\frac{(8\pi G)^2}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl})\pi^{ij}\pi^{kl} - \gamma^{1/2}(^3R - 2\Lambda)\right] + \mathcal{H}_0^M\right\}\Psi \\
&= \left\{\frac{1}{8\pi G}\left[(8\pi G)^2 G^{RS}\pi_R\pi_S - \gamma^{1/2}(^3R - 2\Lambda)\right] + \mathcal{H}_0^M\right\}\Psi \\
&= \left\{\frac{1}{8\pi G}\left[-(8\pi G\hbar)^2 G^{LM}\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M} - \gamma^{1/2}(^3R - 2\Lambda)\right] + \mathcal{H}_0^M\right\}\Psi \\
&\equiv \left(-\frac{l^2}{2}G^{LM}\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M} + V_G + \mathcal{H}_0^M\right)\Psi(\gamma,\varphi) = 0
\end{aligned} \tag{2-4}$$

where Λ is a cosmological constraint [8], G is Newton's constant, l is the Planck length and we have not exhibited a term associated with factor ordering which, in any case, can be included in the gravitational "potential" V_G.

In the above we have identified pairs of indices into single indices according to [2]:

$$\begin{aligned}
&\gamma_{11} = \gamma^1 \quad , \quad \gamma_{22} = \gamma^2 , \quad \gamma_{33} = \gamma^3 \\
&\gamma_{23} = 2^{-1/2}\gamma^4, \ \gamma_{31} = 2^{-1/2}\gamma^5, \gamma_{12} = 2^{-1/2}\gamma^6
\end{aligned} \tag{2-5}$$

and we note that $\gamma = |\gamma_{ij}|, ^3R$ is the curvature scalar associated with the 3-metric γ_{ij}, π^{ij} is the momentum conjugate to γ_{ij} and is given by:

$$\pi^{ij} = -\frac{1}{8\pi G}(K^{ij} - \gamma^{ij}K) \tag{2-6}$$

with:

$$K^{ij} = \gamma^{im}\gamma^{jl}\frac{1}{2}\beta^{-1}(\beta_{m;l} + \beta_{l;m} - \gamma_{ml,0}) \tag{2-7}$$

where K^{ij} is the extrinsic curvature and $K = \gamma_{ij}K^{ij}$. Lastly $\mathcal{H}_0^M$ and $\mathcal{H}_i^M$ are the matter field stress energy tensor projected in a direction normal to the three-dimensional space-like surface and with one component normal and one tangential respectively. Further we have assumed that $\mathcal{H}_0^M$ depends only on the matter field variables and the spatial metric tensor (γ^L) and not on its conjugate momenta.

As a consequence of reparametrization invariance, in the presence of matter, on the space-like three-surface, one has the further condition on the wave function:

$$\frac{i\hbar}{4\pi G}\gamma^{jp}\left(\frac{\delta}{\delta\gamma^{ij}}\Psi\right)_{;p} + \mathcal{H}_i^M\Psi = 0 \tag{2-8}$$

In order to further study the matter-gravity system let us write its wave function as:

$$\Psi(\gamma,\varphi)=\psi(\gamma)\chi(\varphi,\gamma) \tag{2-9}$$

and determine the coupled equations of motion for χ and ψ by first substituting into Eq. (2-4) and contracting over χ. One then obtains

$$\begin{aligned}\Bigg\{\frac{-l^2}{2}\Bigg[&G^{LM}\left(\frac{\delta}{\delta\gamma^L}+iA_L\right)\left(\frac{\delta}{\delta\gamma^M}+iA_M\right)+\frac{G^{LM}}{\langle\chi|\chi\rangle}\langle\chi|\left(\frac{\delta}{\delta\gamma^L}-iA_L\right)\cdot\\ &\cdot\left(\frac{\delta}{\delta\gamma^M}-iA_M\right)|\chi\rangle\Bigg]+V_G+\frac{1}{\langle\chi|\chi\rangle}\langle\chi|\mathcal{H}_0^M|\chi\rangle\Bigg\}\psi(\gamma)=0\end{aligned} \tag{2-10}$$

where:

$$A_L=\frac{-i}{\langle\chi|\chi\rangle}\langle\chi|\frac{\delta}{\delta\gamma^L}|\chi\rangle \tag{2-11}$$

and:

$$\langle\chi|\chi\rangle\equiv\int d\varphi\chi^{\star}(\gamma,\varphi)\chi(\gamma,\varphi) \tag{2-12}$$

where the integral is over the different matter modes. The above can be obtained through formal manipulations involving functional derivatives which, however, can be justified by considering, as in Eq.(2-2), a spatial lattice in which case the functional derivatives are related to normal derivatives.

At this point it is convenient to introduce:

$$\psi=e^{-i\int d^3x\int^{\gamma}A_L\delta\gamma'^L}\,\tilde{\psi} \tag{2-13}$$

$$\chi=e^{i\int d^3x\int^{\gamma}A_L\delta\gamma'^L}\,\tilde{\chi} \tag{2-14}$$

where, because our states χ are normalizable, the phase is real and we observe that Ψ is neutral. On substituting the above in Eq.(2-10) we obtain:

$$\begin{aligned}&\left[\frac{-l^2}{2}\,G^{LM}\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M}+V_G+\frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle\right]\tilde{\psi}\\ &=\frac{l^2}{2}\frac{G^{LM}}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\frac{\delta}{\delta\gamma^L}\,\frac{\delta}{\delta\gamma^M}|\tilde{\rangle}\tilde{\psi}\end{aligned} \tag{2-15}$$

If we now multiply the above by $\tilde{\chi}$ and subtract it from Eq.(2-4) we have:

$$\begin{aligned}&\tilde{\psi}\left(\mathcal{H}_0^M-\frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle\right)\tilde{\chi}-l^2G^{LM}\frac{\delta\tilde{\psi}}{\delta\gamma^L}\frac{\delta\tilde{\chi}}{\delta\gamma^M}\\ &=\frac{l^2}{2}\tilde{\psi}G^{LM}\left(\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M}-\frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M}|\tilde{\chi}\rangle\right)\tilde{\chi}\end{aligned} \tag{2-16}$$

which of course may be rewritten in terms of ψ and χ by using Eqs.(2-13) and (2-14). Let us further observe that Eqs.(2-15) and (2-16) do not, as yet, involve any approximation.

In order to interpret Eq.(2-16) we observe that γ^L is the induced metric on a certain space-like surface and its degrees of freedom can be changed by changing for fixed three-geometry satisfies a Tomonaga-Schwinger equation describing the change in the wave function of the matter field when one makes an infinitesimal local change on the space-like surface on which it is defined.

Let us examine when the neglect of the R.H.S. of Eq.(2-19) is justified. The first term is associated with the shift function β^i which (like the lapse function β) is arbitrary, at least in sufficiently small finite regions, and may be chosen to be zero. The last term is associated with a "metric" on the manifold of matter quantum states and is related to quantum fluctuations. We shall return to this in the next Section. We further note that the same term also occurs in the R.H.S. surfaces ($t \to t + dt$) and coordinates within a fixed classical space-time or by changing the classical solution (geometry). One may then write:

$$\begin{aligned}\tilde{\psi}^* \delta\gamma_{ij} \tilde{\psi} &= \tilde{\psi}^* \left(16\pi G \beta \delta t G_{ijkl} \pi^{kl} + \beta_{i;j} \delta t + \beta_{j;i} \delta t \right) \tilde{\psi} \\ &= \tilde{\psi}^* \left(-16\pi G \delta\tau G_{ijkl} i\hbar \frac{\delta}{\delta\gamma_{kl}} + \delta\tau_{i;j} + \delta\tau_{j;i} \right) \tilde{\psi}\end{aligned} \qquad (2-17)$$

where we have set $\pi^L = -i\hbar \frac{\delta}{\delta\gamma^L}$ and we have introduced local proper times τ and τ_i .

On substituting the above into Eq.(2-16) and multiplying by $\tilde{\psi}^*$ one has:

$$\begin{aligned}&\tilde{\psi}^*\tilde{\psi} \left\{ \left(\mathcal{H}_0^M - \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle} \langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle \right) \tilde{\chi} - i\hbar \left(\frac{\delta\gamma_{ij}}{\delta\tau} - \frac{\delta\tau_{i;j}}{\delta\tau} - \frac{\delta\tau_{j;i}}{\delta\tau} \right) \frac{\delta\tilde{\chi}}{\delta\gamma_{ij}} \right\} \\ &= \tilde{\psi}^*\tilde{\psi} \left\{ \left(\mathcal{H}_0^M - \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle} \langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle \right) \tilde{\chi} - i\hbar \left[\frac{\delta\gamma_{ij}}{\delta\tau} - \frac{\delta\tau_{(i;j)}}{\delta\tau} \right] \frac{\delta\tilde{\chi}}{\delta\gamma_{ij}} \right\} \\ &= \tilde{\psi}^*\tilde{\psi} \left\{ \frac{l^2}{2} G^{LM} \left(\frac{\delta}{\delta\gamma^L} \frac{\delta}{\delta\gamma^M} - \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle} \langle\tilde{\chi}| \frac{\delta}{\delta\gamma^L} \frac{\delta}{\delta\gamma^M} |\tilde{\chi}\rangle \right\} \tilde{\chi}\end{aligned} \qquad (2-18)$$

which we may further write as:

$$\begin{aligned}&\tilde{\psi}^*\tilde{\psi} \left(\mathcal{H}_0^M - i\hbar \frac{\delta}{\delta\tau} \right) e^{-\frac{i}{\hbar} \int d^3x \int^\tau \langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle \frac{\delta\tau'}{\langle\tilde{\chi}|\tilde{\chi}\rangle}} \tilde{\chi} \\ &= e^{-\frac{i}{\hbar} \int d^3x \int^\tau \langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle \frac{\delta\tau'}{\langle\tilde{\chi}|\tilde{\chi}\rangle}} \tilde{\psi}^*\tilde{\psi} \left[-i\hbar \frac{\delta\tau_{(i;j)}}{\delta\tau} \frac{\delta\tilde{\chi}}{\delta\gamma_{ij}} + \right. \\ &\left. + i\hbar \frac{\delta\gamma_{ij}}{\delta\tau} \frac{\delta\tilde{\chi}}{\delta\gamma_{ij}} + \frac{l^2}{2} G^{LM} \left(\frac{\delta}{\delta\gamma^L} \frac{\delta}{\delta\gamma^M} - \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle} \langle\tilde{\chi}| \frac{\delta}{\delta\gamma^L} \frac{\delta}{\delta\gamma^M} |\tilde{\chi}\rangle \right) \right] \tilde{\chi}\end{aligned} \qquad (2-19)$$

We note that we have kept the factor $\tilde{\psi}^*\tilde{\psi}$ since there may be values of γ_L for which the wave function ψ does not have any support. In particular if one considers a semiclassical limit for the gravitational wave function, if it exists, one finds that Eq.(2-19) will only be valid for values of the three metric lying on the corresponding classical trajectory.

In such a case we see that if the R.H.S. of Eq.(2-19) is negligible, $[\chi]_s$ defined by:

$$[\chi]_s \equiv e^{-\frac{i}{\hbar} \int d^3x \int^\tau \langle\tilde{\chi}|\mathcal{H}_0^M|\tilde{\chi}\rangle \frac{\delta\tau'}{\langle\tilde{\chi}|\tilde{\chi}\rangle}} \tilde{\chi} \qquad (2-20)$$

of Eq.(2-15) and that in Eq.(2-19) the last two terms together correspond to an operator minus its average value with respect to a state acting on that state. This can be estimated as being proportional to the square root of the fluctuation of that operator about its mean. Certainly we expect such a difference to be smaller than the individual terms. The third last term in Eq.(2-19) deserves a special mention. It consists of the rate of change of the three metric multiplying an operator minus its average value with respect to a state acting on that state (remembering Eq.(2-14)). Unless there are compensations and remembering that in particular Eq.(2-19) contains a factor $\tilde{\psi}^*\tilde{\psi}$, we expect such a term to be small.

Thus the R.H.S. of Eqs.(2-15) and (2-19) are associated with fluctuations and their neglect, which, together with the semiclassical approximation to $\tilde{\psi}$ are sufficient to ensure adiabaticity and matter satisfying a Tomonaga-Schwinger equation. Hence in the semiclassical and adiabatic approximations $\chi(\varphi,\gamma)$ describes a matter wave function in which φ follows γ adiabatically.

On using Eqs.(2-9), (2-11) and (2-14) and proceeding as before we obtain from the remaining constraint Eq.(2-8):

$$\frac{i\hbar}{4\pi G}\gamma^{jp}\left(\frac{\delta}{\delta\gamma^{ij}}\tilde{\psi}\right)_{;p} + \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\mathcal{H}_i^M|\tilde{\chi}\rangle\tilde{\psi} = 0 \qquad (2-21)$$

and:

$$\tilde{\psi}^*\tilde{\psi}\left\{\frac{i\hbar}{4\pi G}\gamma^{jp}\left(\frac{\delta}{\delta\gamma_{ij}}\tilde{\chi}\right)_{;p} + \left(\mathcal{H}_i^M - \frac{1}{\langle\tilde{\chi}|\tilde{\chi}\rangle}\langle\tilde{\chi}|\mathcal{H}_i^M|\tilde{\chi}\rangle\right)\tilde{\chi}\right\} = 0 \qquad (2-22)$$

We observe that in the presence of the gauge connection A_L reparametrization invariance on the space-like surface can be realized in a nontrivial way if one also allows for a gauge transformation of the connection. In such a case the constants of motion associated with the symmetries of a system may be modified with interesting physical consequences [9]. Let us end this Section by remarking that the matter-gravity system in the adiabatic approximation has already been examined, with analogous results, using a path integral approach [10]. Further similar equations to the above have been obtained in the context of mini superspace and the consequences examined for an inflationary universe [11]. It is worth noting that in the latter case, since mini-superspace is unidimensional, the phase in Eqs.(2-13) and (2-14) can be "gauged" away.

III. FLUCTUATIONS, PHASES, GEOMETRY AND TOPOLOGY

In the previous Section we obtained equations of motion both for the gravitational wave function ψ and the matter wave function χ. In particular the latter was identified on observing that, in the semiclassical (for the gravitational wave function) and adiabatic limits, for fixed three geometry it satisfies the Tomonaga-Schwinger equation. Of remarkable interest is the emergence of the induced gauge connection A_L and the associated, in general, non-integrable phase factor (Berry phase) [3,12]. Such a gauge connection can of course lead to interesting topological structures in the manifold of all three geometries (superspace) and the associated phase cannot be eliminated in the presence of nontrivial homotopic mappings of the gauge connection on the parameter manifold. Some of these structures are analogous to the θ-vacua structure in Yang-Mills theories [13] and the possibility of such structures in superspace has been previously suggested [14].

In order to examine the diverse factors occuring in our expressions in more detail let us, for simplicity, consider the case for which matter is in a normalized eigenstate $|n\rangle$ of $\mathcal{H}_0^M$:

$$\mathcal{H}_0^M|n\rangle = \mathcal{E}_n(\gamma)|n\rangle \qquad (3-1)$$

then in the adiabatic approximation the system will remain in an eigenstate of $\mathcal{H}_0^M$ as the parameters γ_L are slowly varied. Let us for $|\chi\rangle = |n\rangle$ consider the matrix element B^n_{LM} occurring in both Eqs.(2-15) and (2-16):

$$\begin{aligned} B^n_{LM} &= \langle\tilde{n}|\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M}|\tilde{n}\rangle = \langle n|\left(\frac{\delta}{\delta\gamma^L} - iA^n_L\right)\left(\frac{\delta}{\delta\gamma^M} - iA^n_M\right)|n\rangle \\ &= \sum_{l\neq n}\langle n|\frac{\delta}{\delta\gamma^L}|l\rangle\langle l|\frac{\delta}{\delta\gamma^M}|n\rangle \\ &= -\sum_{l\neq n}\frac{1}{(\mathcal{E}_n - \mathcal{E}_l)^2}\langle n|\frac{\delta\mathcal{H}_0^M}{\delta\gamma^L}|l\rangle\langle l|\frac{\delta\mathcal{H}_0^M}{\delta\gamma^M}|n\rangle \end{aligned} \qquad (3-2)$$

where of course $A^n_L \equiv -i\langle n|\frac{\delta}{\delta\gamma^L}|n\rangle, |\tilde{n}\rangle$ is defined analogously to $|\tilde{\chi}\rangle$ and B^n_{LM} is Hermitean [15]. It can be decomposed into real and imaginary parts:

$$ReB^n_{LM} = \frac{1}{2}\left(B^n_{LM} + B^n_{ML}\right) \qquad (3-3)$$

and

$$ImB^n_{LM} = \frac{1}{2}\left(B^n_{LM} - B^n_{ML}\right) \qquad (3-4)$$

The first term, Eq.(3-3), provides a possible means of measuring distances along paths in parameter space: it corresponds to a metric tensor on a manifold of quantum states [16]. Further we see that the quantum metric tensor for a given eigenstate of the energy is related to the energy fluctuations about that state [17]. Since the term occurs in Eqs.(2-15) and (2-16) contracted with G^{LM} one essentially obtains an additional scalar potential which is the trace of the quantum metric tensor.

The second term, Eq.(3-4), can be related to the connection A^n_L . Indeed it is straightforward to see that:

$$\frac{\delta}{\delta\gamma^L}A^n_M - \frac{\delta}{\delta\gamma^M}A^n_L = -2iImB^n_{LM}$$
$$= i\sum_{l\neq n}\frac{1}{(\mathcal{E}_n-\mathcal{E}_l)^2}\left[\langle n|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^L}|l\rangle\langle l|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^M}|n\rangle - \langle n|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^M}|l\rangle\langle l|\frac{\delta\mathcal{H}^M_o}{\delta\gamma^L}|n\rangle\right] \quad (3-5)$$

which is a phase 2-form in parameter space. It is worth noting that if for some values of the parameter γ some states $|l\rangle$ become almost degenerate with $|n\rangle$ then they will dominate the sum in Eq.(3-5) and in the absence of such singularities, if the matter Hamiltonian is real, Eq.(2-5) is zero (the eigenfunctions can then be made real). Further if we consider a closed cycle C in parameter space (superspace), we have using Eq.(3-5) that the phase factor in Eqs.(2-13) and (2-14) becomes:

$$\int d^3x \oint_C \langle n|i\frac{\delta}{\delta\gamma^L}|n\rangle\delta\gamma^L$$
$$= \int d^3x \int\!\!\!\!/\int_C \delta\sigma^{LM} Im\sum_{l\neq n}\frac{1}{(\mathcal{E}_l-\mathcal{E}_n)^2}\langle n|\frac{\delta\mathcal{H}^M_o}{\delta\gamma^L}|l\rangle\langle l|\frac{\delta\mathcal{H}^M_0}{\delta\gamma^M}|n\rangle \quad (3-6)$$

where the slash on the integration symbol on the R.H.S. means that it is only over the three geometries and $\delta\sigma^{LM}$ denotes an element of a two dimensional surface in superspace bounded by the circuit C.

In the last Section we have seen, within the context of a functional approach, the emergence of a covariant functional derivative acting on our functional state vectors. This naturally leads to the emergence of induced phase factors of opposite signs for the two wave functions in the product decomposition of the total wave functions of the system. This phase factor can be related either to the parallel transport of states either adiabatically in parameter space[18)] or in the space of all physical states of the system[19)] the latter coinciding with the former in the adiabatic approximation. We emphasize that in our case these considerations are for functional state vectors; thus one can imagine that one has such a structure for each spatial point and therefore the total phase is the sum of the phases at each point.

We have seen the emergence of a quantum geometric tensor whose antisymmetric part is a two form whose flux is the phase. Particularly interesting is the symmetric part of this tensor which is a possible metric on parameter space. The presence of the phase and the quantum metric have dynamical consequences: the former leads to the presence of a gauge field and the latter to a scalar field in the equation of motion in parameter space (Eq.(2-10)).

IV. CONCLUSIONS

We have examined the matter-gravity quantum system by writing its wave function as the product of two: one involving only gravitational degrees of freedom and the other involving both the gravitational and the matter degrees of freedom. This has allowed us to separate the quantum constraint equations into coupled equations involving the two factor wave functions.

In particular one of the equations is the Wheeler-DeWitt equation for gravitation in the presence of matter modified by the additional presence of an induced gauge connection and a scalar potential. This scalar potential is related to the trace of a metric tensor on a manifold of quantum states and is associated with fluctuations. Analogously the corresponding coupled equation also involves a gauge connection (with the opposite sign) and terms associated with fluctuations.

On considering a semiclassical limit for the factor wave function satisfying the modified Wheeler-DeWitt equation, which implies that we are introducing a classical time and trajectories for the three metric, and neglecting fluctuations, one finds that the coupled equation becomes the Tomonaga-Schwinger equation. Thus in such a limit one can interpret its wave function as describing matter which follows gravitation adiabatically.

In order to better understand the approximations involved we have examined the case for which matter is in an energy eigenfunction (one could also consider it to be in a superposition of eigenstates of the matter Hamiltonian or other operators). In general the adiabatic approximation consists of the neglect of the off-diagonal terms (or fluctuations) since it is expected that the system will remain (in the absence of degeneracies) in the same eigenstate during its time evolution. We observe that the expressions obtained in Section II are exact and therefore we may use them to obtain corrections to the adiabatic approximation from them and in particular study the consequence of degeneracies and the resulting large fluctuations on, for example, the time evolution of a system.

Similar considerations may also be applied to the introduction of a classical time: in particular one may consider a minimal wave packet for the gravitational wave function, obtained as a solution to the modified Wheeler-DeWitt equation after expanding about the equilibrium position of the gravitational potential, and allow for its oscillations. This again is analogous to what is done for other composite systems in the context of the Born-Oppenheimer approximation.

Besides the quantum metric tensor we have seen that our equations also involve a gauge connection and an associated phase which occurs with opposite signs in the matter and gravity wave functions so that their product is "neutral". The presence of such a connection is of particular interest since one can then obtain nontrivial topological structures in the manifold of all three metrics (superspace) [20] , some of them analogous to the θ-vacua structure in Yang-Mills theories. Such structures can lead to a modification of the constants of motion associated with a symmetry of a system; this of course is of particular interest for physically observable quantities. Further reparametrization invariance on the space-like three-surface can be realized in a nontrivial way if one also allows for a gauge transformation of the connection.

Finally let us comment on the induced phase. A time evolution in which the state of a system returns to its original state is of particular interest in physics.

An example of this for a quantum system in adiabatic evolution is when a Hamiltonian returns to its original value and the state evolves as an eigenstate of the Hamiltonian and returns to the original state. Other examples are periodic particle motions or the splitting and recombination of a beam so that the system may be regarded as going backwards in time along one beam and returning along the other beam to its original state at the same time. In all the above cases of cyclic evolution the initial and final states can differ by a phase factor which is the holonomy transformation due to the parallel transport of the state vector with respect to a connection. Such a geometrical phase factor can lead to observable consequences such as the modification of the energy levels of particles executing periodic motions in slowly varying three geometries.

REFERENCES

1) J.A. Wheeler, in Relativity, Groups and Topology, 1963 Les Houches Lectures. Edited by DeWitt and DeWitt (Gordon and Breach Science Publishers Inc., New York, 1964).
2) B.S. DeWitt, Phys.Rev. 160 , 113 (1967).
3) C.A. Mead and D.G. Truhlar, J. Chem.Phys. 70 , 2284 (1979)
C.A. Mead, Chem.Phys. 49 , 23,33 (1984).
4) By a matter-gravity system we mean any composite system, involving matter and gravitational fields, which could even be microscopic (such as an electron in orbit around a charged black hole).
5) Greek indices run from 0 to 3, small Latin indices from 1 to 3 and capital Latin indices from 1 to 6. Differentiation is denoted by a comma, covariant differentiation by a semicolon and we use units for which $c = 1$.
6) See C. Teitelboim, in Quantum Structure of Space and Time, edited by M.J. Duff and C.J. Isham (Cambridge University Press, 1982).
7) Naturally one may also consider the case of more matter fields.
8) Or a generic term contributing to the gravitational potential and which one could envisage changing slowly from one space-like surface to another.
9) See, e.g., R. Jackiw, Int. Journal of Mod.Phys. A, 3 , 285 (1988).
10) R. Balbinot, A. Barletta and G. Venturi, 1988 preprint.
11) R. Brout and G. Venturi, Phys.Rev. *D15* , 2436 (1989).
12) M.V. Berry, Proc. Roy. Soc. London, *A392* , 45 (1984).
13) θ-vacua for a gauge field are associated with vacuum solutions of the form $-ig^{-1}\partial_i g$ where g is related to a transformation with respect to an internal symmetry group. Clearly our gauge connection $A_L = -i\chi^{-1}\frac{\delta}{\delta\gamma^L}\chi$ is highly reminiscent of such a structure.
14) C.J. Isham, Phys.Lett. 106*B*, 188 (1981).
15) We observe that in the general case $B_{LM} = Tr\left(\rho\frac{\delta}{\delta\gamma^L}\frac{\delta}{\delta\gamma^M}\right)$
$-Tr\left(\rho\frac{\delta}{\delta\gamma^L}\right)Tr\left(\rho\frac{\delta}{\gamma^L}\right)$, where $\rho = \frac{|\chi\rangle\langle\chi|}{\langle\chi|\chi\rangle}$.
16) J.P. Provost and C. Vallee, Comm. Math. Phys. 76 , 286 (1982).

17) One may also consider eigenstates of other operators.

18) B. Simon, Phys.Rev. Lett., 51 , 2167 (1983).

19) Y. Aharanov and J. Anandan, Phys.Rev. Lett. 58 , 1593 (1987).

20) This could occur for example if during the circuit C (see Eq.(3-6)) one passes near a point where the state n becomes degenerate. For the case of three dimensions and a double degeneracy this leads to a monopole-like singularity (see ref. 12).

GRAVITY AND LORENTZ BREAKDOWN IN HIGHER-DIMENSIONAL THEORIES AND STRINGS

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The gravitational phenomenology of theories compactified from higher dimensions is investigated. Emphasis is placed on the consequences in string theory of tensor-induced spontaneous breaking of the higher-dimensional Lorentz symmetry. The role played by this mechanism in causing a gravitational version of the Higgs effect and in compactification is studied. The phenomenology of compactified theories with massless modes is compared with experiment via an examination of non-leading but observable gravitational effects arising in the presence of a localized matter distribution. Further constraints from known cosmological features of the universe are presented. The results significantly constrain many theories involving extra dimensions in their perturbative regime. A mechanism is needed that leaves massless the physical spacetime components of the higher-dimensional metric while generating masses for other components. Some suggestions for overcoming this metric-mass problem are made.

1. Introduction

The imaginative notion that the universe may have more than four dimensions lies at the core of many modern attempts to unify gravity with the other fundamental forces, including Kaluza-Klein theories [1-3], supergravities [4], and superstring theories [5]. The latter have been under intensive investigation in recent years because they are prime candidates for a consistent theory of quantum gravity.

The observation of precisely four macroscopic dimensions suggests that any higher dimensions are of microscopic size. This means a symmetry-breaking mechanism is required to separate the D dimensions of a higher-dimensional theory into the four macroscopic spacetime dimensions and $n = D - 4$ microscopic dimensions. In particular, the D-dimensional Lorentz group must break to the four-dimensional one.

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Edited by L.-L. Chau and W. Nahm, Plenum Press, New York, 1990

A natural mechanism for spontaneous breaking of higher-dimensional Lorentz symmetry may occur in string theories [6]. This mechanism is described in Section 2. The role of Lorentz breaking in compactification and in causing a gravitational version of the Higgs effect, together with topological constraints induced on the compactified manifold, are explored in Section 3 in the context of a string-inspired model. The phenomenology of small fluctuations about a flat background metric induced by a localized matter distribution, both with and without higher-dimensional Lorentz breaking, are considered in Section 4. Cosmological constraints on higher-dimensional theories are presented in Section 5. Section 6 provides a discussion of the results.

2. Tensor–Induced Lorentz–Symmetry Breakdown

The basic idea is that tensor-induced spontaneous breaking of the higher-dimensional Lorentz symmetry may be a natural mechanism in many string theories [6]. Any string theory with on-shell cubic interaction terms between Lorentz scalars S and Lorentz tensors T of the form $ST \cdot T$ will have such terms present in the off-shell extension, i.e., in the string field theory. Then, if any scalars S have finite vacuum expectation values of the appropriate sign and magnitude, some Lorentz tensors T acquire mass-squared terms of the wrong sign. This means these tensors must also acquire expectation values, which results in spontaneous breaking of the Lorentz group.

As it is relatively well understood, the bosonic string is an ideal choice for the study of tensor-induced higher-dimensional Lorentz breaking. Furthermore, the perturbative string vacuum is unstable because it contains a tachyon. This Section examines a consequence of the assumption that the string vacuum is stabilized by a finite tachyon expectation value $< \phi >$.

In the covariant string field theory [7], the static potential is

$$V(\{S^i\},\{T^i_M\}) = \frac{1}{2}\sum_{i,j} m_{ij}{}^2 S^i S^j + \frac{1}{2}\sum_{i,j} M_{ij}{}^2 T^i_M T^{jM} + \frac{1}{3!}\sum_{i,j,k} g^{SSS}_{ijk} S^i S^j S^k + \frac{1}{2}\sum_{i,j,k} g^{STT}_{ijk} S^i T^j_M T^{kM} + (TTT\text{--}term) \ ,$$

where S^i denotes a generic scalar field, $m_{ij}{}^2$ and $M_{ij}{}^2$ are the scalar and tensor mass-squared matrices, and g^{SSS}_{ijk}, g^{STT}_{ijk} are coupling constants. The TTT-term involves three tensor fields with indices appropriately contracted. In Siegel-Feynman gauge [8], the static potential begins as follows:

$$V(\phi) = -\frac{1}{2\alpha'}\phi^2 + \frac{\bar{g}}{6}\phi^3 + \frac{8}{27}\bar{g} A_\mu A^\mu \phi + \dots \quad ,$$

where $\bar{g}$ is the three-tachyon coupling at zero momentum. Disregarding tree-level contributions from other scalars and loop effects, the squared mass of A^μ is $M_A^2 = \frac{16}{27}\bar{g} < \phi >$. If $< \phi >$ is negative, this squared mass is negative. Then, spontaneous breaking of the higher-dimensional Lorentz symmetry will occur.

The presence of static S–T–T couplings is a stringy effect. In fact, tensor-induced breaking is unique to string theory because renormalizable particle gauge theories lack appropriate cubic couplings. For instance, four-dimensional renormalizable theories of scalars and vectors must be gauge theories because gauge invariance is needed to remove unphysical degrees of freedom. The trilinear terms of the form $A^{i\mu}\partial_\mu\phi^j\phi^k$ do not contribute to the static potential because of the derivative. Although gauge theories can also contain scalar–scalar–vector–vector couplings, such terms will lead only to positive quadratic coefficients for the vector fields when the scalars acquire vacuum expectation values. Remarkably, the field theory of the open bosonic string avoids these constraints despite having a large gauge invariance and being well-behaved at short distances. The infinite number of trilinear interactions and particle fields in the string field theory make this possible.

If Lorentz breaking occurs, it is nonperturbative in $\bar{g}$. Set $\bar{S}^i = \bar{g}\alpha' S^i$, $\bar{T}^i_M = \bar{g}\alpha' T^i_M$, $\alpha' {m_{ij}}^2 = c^{SS}_{ij}$, $\alpha' {M_{ij}}^2 = C^{TT}_{ij}$, $g^{SSS}_{ijk} = c^{SSS}_{ijk}\bar{g}$, and $g^{STT}_{ijk} = c^{STT}_{ijk}\bar{g}$. As g^{SSS}_{ijk} and g^{STT}_{ijk} are proportional to $\bar{g}$, c^{SSS}_{ijk} and c^{STT}_{ijk} are pure numbers. The static potential then becomes a polynomial independent of α' and $\bar{g}$, with an overall factor of $(\bar{g}^2{\alpha'}^3)^{-1}$. This means that any finite $< \phi >$ or $< S^i >$ is of order $1/(\bar{g}\alpha')$ and is nonperturbative in $\bar{g}$.

Solutions to the equations of motion obtained by varying the static potential may also be sought. A solution in terms of constants corresponds to a set of vacuum expectation values. If $< \phi > \neq 0$ then $< S^i > \neq 0$ for other scalars because whenever $g^{\phi\phi}{}^S_i \neq 0$ a term proportional to $< \phi >^2 S^i$ is generated. These linear terms in S^i drive further instabilities, which makes the theory radically different from the first-quantized approach. Most scalars will acquire a nonzero vacuum expectation value. Any negative eigenvalues in the resulting mass matrix for the tensor fields signal tensor-induced spontaneous breaking of Lorentz symmetry.

3. A String–Inspired Model

Higher-dimensional theories must reproduce phenomenological observations other than the existence of four macroscopic dimensions. For example, they must contain Newton gravity with Einstein corrections. The analysis of gravitational predictions of Kaluza–Klein and supergravity theories is possible, but an explicit field theory with calculable couplings describing closed string theories is not yet established. A direct field-theoretic analysis of gravitational effects in closed strings is therefore presently impractical. Instead, insight can be acquired by exploring via standard methods a D-dimensional effective field theory containing the essential features of the effective action arising from a string theory with tensor-induced breaking [9].

Consider a simple model for which the only tensor field is a vector A_μ. This might be viewed as a model of the effective action for the massless vector A_μ and the metric $g_{\mu\nu}$ in the open bosonic string. Neglecting higher-derivative effects and setting the cosmological constant to zero, the model is given by the D-dimensional Einstein-Maxwell lagrangian with a potential V for A_μ inducing spontaneous breaking of Lorentz symmetry:

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{16\pi G_D}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V\right) \quad .$$

Here, G_D is the D-dimensional Newton coupling constant, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and indices are raised using $g^{\mu\nu}$. Also, $V = V(A_\mu A^\mu - a^2)$, where the quantity a is a constant and $V(x)$ is assumed positive except at $x = 0$, where it vanishes. Making the choice $V = \frac{1}{2}\lambda(A_\mu A^\mu - a^2)$, with λ a lagrange-multiplier field, freezes motion about the minimum and hence simplifies the analysis of gravity. In fact, the results presented are independent of the choice of V [10].

A solution of the equations of motion is $g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = a_\mu$, $\lambda = 0$, where a_μ is a constant vector satisfying $a_\mu a^\mu = a^2$. The linearized equations for small fluctuations are found by perturbing about this vacuum solution. They are invariant under the infinitesimal local transformations $\delta h_{\mu\nu} = \partial_\mu \chi_\nu + \partial_\nu \chi_\mu$, $\delta\epsilon_\mu = \partial_\mu(\chi_\nu a^\nu)$, $\delta\lambda_1 = 0$, where $\partial_\mu a^\nu = 0$. These transformations follow from the invariance of the full theory under general coordinate transformations and are distinct from a $U(1)$ gauge transformation, which is excluded by the presence of the potential V for A_μ. The functional constraint $f[h_{\mu\nu}] = 0$ provides an acceptable gauge choice for $h_{\mu\nu}$, whenever there exists a solution for χ_μ of the equation $f[h_{\mu\nu} + \delta h_{\mu\nu}] = f[h_{\mu\nu}] + f[\delta h_{\mu\nu}] = 0$ with $\delta h_{\mu\nu}$ as above. An especially useful gauge choice is the restriction to harmonic coordinates, $\partial^\mu \bar{h}_{\mu\nu} = 0$, where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$.

A natural question is whether tensor-induced Lorentz breaking destabilizes a given background metric. An instability might be a signal of compactification of the extra dimensions. For a generic background solution this is a difficult question to address. Consider instead the the second-order fluctuations of the action about a flat background. Denote the background fields by $g_{\mu\nu} = \eta_{\mu\nu}$ and $A_\mu = a_\mu = a\delta_{\mu d}$ and the associated fluctuations by $h_{\mu\nu}$ and ϵ_μ. A calculation of the second-order variation $\delta^2 I$ subject to the equations of motion yields

$$\delta^2 I \approx \int d^D x \left[\frac{1}{32\pi G_D} (h^{\mu\nu}\partial_\rho\partial^\rho h_{\mu\nu} - h\partial_\rho\partial^\rho h + 2h\partial^\mu\partial^\nu h_{\mu\nu} + 2\partial_\lambda h^{\lambda\mu}\partial^\nu h_{\mu\nu}) \right.$$
$$\left. - \frac{1}{2} f^{\mu\nu} f_{\mu\nu} - (a^\mu h_{\mu\nu} a^\nu - 2\epsilon_\mu a^\mu)^2 V''(0) \right] \quad .$$

The first part of this expression represents the flat-space background fluctuations for standard gravity, which are stable after a gravitational gauge is fixed. The remainder is a sum of perfect squares. Therefore, flat D-dimensional spacetime is stable to perturbations even in the presence of Lorentz breaking.

Another natural question arises from the analogy between gravity and non-abelian gauge theory. The question is whether a gravitational version of the Higgs mechanism can occur. In the present model, A_μ must play the role of the Higgs field. When A_μ acquires a constant expectation value $a\delta_{\mu D-1}$, the potential V is minimized. A longitudinal component of the metric arising from the nonzero expectation value of the vector field in the kinetic term would then provide the analogue of the Higgs effect. However, a nonzero expectation value for A_μ cannot give a new contribution to $h_{\mu\nu}$ in $F^{\mu\nu}F_{\mu\nu}$ because the affine connection coupling gravity to tensor fields cancels in the antisymmetric combination $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This argument seems to depend on the use of $F^{\mu\nu}F_{\mu\nu}$ as a kinetic energy term. In fact, the absence of a Higgs effect holds quite generally for spontaneous Lorentz-symmetry breaking induced by any tensor field [10]. This suggests that the generation of mass for metric components from tensor-induced breaking is unlikely even in a full string theory. Another possible source of mass terms for the metric might be the potential $V = V(A_\mu g^{\mu\nu} A_\nu - a^2)$. As this depends on $g_{\mu\nu}$ as well as on A_μ, a quadratic derivative-free term for the metric might arise. In a nonabelian gauge theory this cannot happen because the scalar potential does not contain the gauge field. If it occurs, this alternative mechanism would be qualitatively different from the usual particle-theory Higgs mechanism. In fact, no component of $h_{\mu\nu}$ gains mass; instead, the combination $2\epsilon_\mu g^{\mu\nu} A_\nu - A_\mu h^{\mu\nu} A_\nu$ is massive.

Some features of the compactified dimensions in the effective theory may be found from topological considerations. Assume there exists a solution for the gravitational sector of the form $M^4 \times M^n$, where M^4 is a four-dimensional flat space and M^n is a compact n-dimensional manifold. One solution for A_μ minimizing the terms $\mathcal{L}_{gA} \equiv \frac{1}{4} F^{\mu\nu}F_{\mu\nu} + \frac{1}{2} V$ is $A_\mu = a\delta_{\mu d}$, where d points in one of the three physical dimensions. This implies Lorentz-symmetry breaking in these dimensions. To avoid this undesirable result, it is necessary to have a competing solution $A_\mu = a_\mu$ lying in M^n and rendering zero each term in $\mathcal{L}_{gA}$. Then, treating a_μ as the components of a one–form $\hat{a} = a_\mu dx^\mu$, the vanishing of the first term of $\mathcal{L}_{gA}$ implies that $\hat{a}$ is closed, i.e., $d\hat{a} = 0$. Treating a^μ as the components of a vector field $\vec{a}$, the vanishing of the second term in $\mathcal{L}_{gA}$ implies that M^n admits a nowhere–vanishing vector field of constant magnitude.

These constraints imply that the compactified manifold must have vanishing Euler characteristic and nonvanishing first Betti number [10]. In fact, there exists a theorem [11] stating that any smooth arcwise–connected compact manifold M^n satisfying the above conditions is a fiber bundle over S^1. This therefore is the case for physically acceptable solutions of the effective action being considered. Analogous topological features are likely to be present in a full string theory.

4. Effects from Localized Matter Distributions

Consider the gravitational field about a localized static distribution of total mass M. This might represent a planetary body or the sun, for example. The question of interest is the behavior of the four–dimensional gravitational potential at large radial distances r, where perturbative methods are applicable. For the leading large–r behavior, it can be shown that the localized mass distribution can be approximated by a delta–function source [10]. Take the stress–energy tensor to have the form $T_{\mu\nu} = \hat{\rho}\eta_{\mu 0}\eta_{\nu 0}$. A static solution is sought for small fluctuations $h_{\mu\nu}$ of the metric and no fluctuations of the vector field A_μ about the flat background solution $g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = a_\mu$, $\lambda = 0$, where a_μ is a constant vector satisfying $a_\mu a^\mu = a^2$. Without loss of generality, assume a_μ is aligned so that $a_\mu = a\delta_{\mu D-1}$. For simplicity, choose the harmonic gauge $\partial^\alpha \bar{h}_{\alpha\beta} = 0$. Then, the Einstein equations become

$$\nabla^2 \bar{h}_{\mu\nu} = 0 \quad ,$$
$$\nabla^2 \bar{h}_{00} = -16\pi G_D \hat{\rho} \quad ,$$
$$(1+k)\nabla^2 \bar{h}_{D-1\ D-1} - k\partial^2_{D-1}\bar{h}_{D-1\ D-1} + \frac{k}{D-2}(\partial^2_{D-1} - \nabla^2)\bar{h} = 0 \quad .$$

Here, we define $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu}$. Also, in the first of these equations, $\mu = \nu = 0$ and $\mu = \nu = D-1$ are excluded. A general solution to these equations is

$$h_{00} = \frac{\mu(l-1)}{lr} \quad , \qquad h_{jj} = \frac{\mu}{lr} \quad , \qquad h_{D-1\ D-1} = \frac{\mu}{(k+1)lr} \quad ,$$

where $\mu = 4MG_D/V_n$ with V_n the volume of the internal dimensions, $k = 8\pi G_D a^2$, and $l = (1+k)^{-1}\big((n+2) + (n+1)k\big)$.

To compare with experiment, expand the metric generated by a static, spherically symmetric body of mass M in r^{-1}:

$$ds^2 = -(1 - 2MG_N r^{-1} + \ldots)dt^2 + (1 + 2\gamma MG_N r^{-1} + \ldots)d\mathbf{x}^2 \quad .$$

The Eddington–Robertson parameters [12,13] β and γ have experimental values [14,15] $\beta = 1.003 \pm 0.005$, $\gamma = 1.000 \pm 0.002$. Then, the first r^{-1} term in h_{00} fixes the definition of MG_N as $G_N \equiv 2(l-1)G_D/lV_n$. This ensures standard four-dimensional Newton gravity, independent of the details of the compactification.

Leading corrections arising from Einstein gravity are represented in lowest order by the term containing γ. The value for γ is found to be

$$\gamma = \frac{1+k}{1+n+nk} \quad .$$

This result is independent of the choice of gravitational gauge, i.e., of harmonic coordinates. It is incompatible with the experimental one unless either $D = 4$ and $k = 0$ or $D = 5$ and $k \to \infty$. The latter value is unnatural for strings since the Planck scale determines the scale of a, which implies that k is of order unity.

The analysis presented above is based on perturbative calculations in higher-dimensional theories with purely an R term, with or without tensor-induced breaking. However, the results should be independent of lagrangian terms of higher power in R, since on dimensional grounds these leave unaffected the leading large-r behavior.

The assumptions made have thus led to the result that the perturbative sector of any higher-dimensional theory is excluded by experiment. The analysis is classical and is independent of other difficulties with higher dimensions such as natural chirality generation [16] and renormalizability or finiteness. A discussion of this phenomenological difficulty and of possibilities for avoiding it is given in Section 6.

Throughout this Section, the background vector a_μ has been been taken as aligned in one of the internal dimensions. One can instead consider the consequences of aligning it instead in one of the physical spacetime dimensions. In fact, in leading order there are solutions that are quite different from the one given above but that incorporate standard Newton gravity [10]. However, tensor-induced spontaneous Lorentz-symmetry breaking in the physical space probably causes phenomenological difficulties at higher order.

5. Cosmological Constraints

Given the results of Section 4, it is natural to ask whether additional constraints arise from cosmological considerations. The standard procedure for investigating higher-dimensional cosmologies, followed in this Section, is to allow both the physical spatial dimensions and the compactified dimensions to evolve separately as homogeneous maximally symmetric spaces. This assumption is incompatible with non-zero expectation values of tensor fields. Therefore, tensor-induced spontaneous Lorentz-symmetry breaking is disregarded here.

Consider cosmologies in a pure D-dimensional Einstein theory involving two homogeneous spaces: a four-dimensional one and an n-dimensional one, governed by scale parameters $a(t)$ and $b(t)$, respectively. The appropriate metric is

$$ds^2 = -dt^2 + a^2(t) g_{ab}(x) dx^a dx^b + b^2(t) g_{jk} dx^j dx^k \quad ,$$

where $a = 0, \ldots, 3$ labels the physical spacetime dimensions and $j = 1, \ldots, n$ labels the internal dimensions. Approximate the cosmological matter distribution as a perfect fluid with D-dimensional density $\hat{\rho}$, pressure $\hat{p}$ in the physical dimensions, and pressure $\hat{\tau}$ in the internal dimensions. The stress-energy tensor $T_{\mu\nu}$ is then given by the diagonal $D \times D$ matrix with entries $(\hat{\rho}, \hat{p}, \hat{p}, \hat{p}, \hat{\tau}, \ldots, \hat{\tau})$. Note that the four-dimensional density and pressure are given by $\rho = \hat{\rho} V_n$ and $p = \hat{p} V_n$. It can be shown that for the physical situation, where $b \ll a$, the approximation $\hat{\tau} = 0$ is good [10].

The Einstein equations for the case $n > 0$ reduce to three coupled second-order differential equations for a and b, with coefficients dependent on D and on two parameters k and k_n. As usual, $k = -1$, 0 or $+1$ determines whether the geometry of the four-dimensional space is open, flat or closed; likewise, $k_n = 0$ or $+1$ determines the geometry of the n-dimensional space. Independently of the detailed nature of the internal manifold, it is possible to argue that $k_n = 1$ can be excluded for realistic evolutions provided no special choice of the density $\hat{\rho}$ is made: a contradiction can be obtained from the assumption that the present-day value of b is microscopic [10]. For the rest of this Section, $k_n = 0$.

An important aspect of the D-dimensional theory for $D > 4$ is that the cosmology is governed by equations that are qualitatively different from the usual four-dimensional case. For $n = 0$, the Einstein equations describe Friedmann cosmologies and the evolution of the spacetime scale parameter a is governed entirely by a first-order differential equation controlled by the mass density $\hat{\rho}$ and by the cosmological constant Λ, if present. However, for $n = 1$, i.e., $D = 5$, $a(t)$ is described by a second-order equation with the value of ρ being entirely irrelevant.

Furthermore, for the higher–dimensional theories the evolution of a is tied to that of b via two second–order equations and is also dependent on $\hat{\rho}$, $\hat{\tau}$, and Λ.

For arbitrary D, one set of constraints arises from the dependence of the Newton gravitational constant G_N on b^{-n} derived in Section 4. The point is that the dynamical evolution of b means that G_N varies with time. Denoting present–day values by a subscript 0, the variation of G_N in the current epoch is [10]

$$\frac{1}{H}\frac{\dot{G}_N}{G_N}\bigg|_0 = \begin{cases} \frac{3n}{n-1} \pm \left[\frac{3n(n+2)}{(n-1)^2} + \frac{3n(n+2)\Omega_0}{(n-1)(n+1)} - \frac{6n}{(n-1)}\frac{k}{a_0^2 H_0^2}\right]^{\frac{1}{2}} & \text{, when } n > 1, \\ 1 - \frac{3\Omega_0}{4} + \frac{k}{a_0^2 H_0^2} & \text{, when } n = 1, \end{cases}$$

where $H_0 = \dot{a}_0/a_0$ and $\Omega_0 = 8\pi G_N \rho_0 / 3H_0^2$. In many models, the variation of G_N with time exceeds the current experimental upper limit [17], given by $|\dot{G}_N/G_N| < 6 \times 10^{-12}$ yr^{-1}. This bound implies that $|\frac{1}{H}\frac{\dot{G}_N}{G_N}|_0$ must be less than 0.12 using a conservatively low value of $H_0 = 50$ km s^{-1} Mpc^{-1}. However, $|\frac{1}{H}\frac{\dot{G}_N}{G_N}|_0$ is of order one unless Ω_0 and a_0 are finely adjusted [10]. Thus, all admissible models have a fine–tuning problem.

The simplest model, which has $n = 1$, can be studied analytically and provides more insight. The exact solutions yield four classes of models: class (a) has $k = 0$; class (b) has $k = 1$; class (c) has $k = -1$ and $\dot{a}_0 \leq 1$; and class (d) has $k = -1$ and $\dot{a}_0 \geq 1$. Classes (c) and (d) coincide when $\dot{a}_0 = 1$.

Two experimental criteria can be used to exclude many of these models. First, measurements of the deceleration parameter q_0 favor a positive value [18], whereas models of type (c) all have a negative value [10]. Second, the physical age of the universe t_u is known to be $1.5 - 1.9 \times 10^{10}$ yr from radioactive dating [19], the age of globular clusters [20-22], the age of the chemical elements [23], and the age of the earth [24]. The exact solution provides [10] the bounds $t_u = \frac{1}{2H_0}$ for class a, $0 \leq t_u \leq \frac{1}{2H_0}$ for class b, $0 \leq t_u \leq \frac{1}{H_0}$ for class c, and $\frac{1}{2H_0} \leq t_u \leq \frac{1}{H_0}$ for class d. Since $(2H_0)^{-1} = 0.9 \times 10^{10}$ yr for $H_0 = 55$ km s^{-1} Mpc^{-1}, classes (a) and (b) are incompatible with experiment. A higher value of H_0 yields even smaller values of t_u for these models, thereby increasing the disagreement with experiment. It appears that the only consistent class of five–dimensional models is (d); even for this case $\dot{a}_0$ must be fine–tuned to be slightly greater than one. Given the freedom to adjust parameters in this class of models, present–day features of standard Friedmann cosmologies can be numerically reproduced [10]. Typically, differences occur only during the early universe.

It is also possible to find exact solutions for the internal scale parameter b in the five–dimensional model [10], with an energy density comprised of both matter and radiation. An interesting feature of the solutions is that $b(t)$ contracts as $a(t)$ expands with t, which might explain why the scale of the internal dimensions is small.

6. Discussion

The previous Sections explored some consequences of tensor-induced spontaneous Lorentz-symmetry breaking in higher-dimensional theories, via a string-inspired model. Phenomenological gravitational effects were investigated, both with and without the presence of tensor-induced breaking. The results are applicable not only to strings but also to Kaluza-Klein and $D > 4$ supergravity theories.

The strongest constraint comes from the requirement that a localized mass distribution reproduce a four-dimensional metric of the Eddington-Robertson form with parameters in agreement with experimental. Theories in higher dimensions can provide parameters disagreeing with experimental data. Other phenomenological constraints are provided by cosmological considerations. Many theories can be excluded by current experimental knowledge of the variation of the Newton gravitational constant with time, the deceleration parameter, and the age of the universe. Models with compactified dimensions providing a satisfactory cosmology satisfying observational constraints are possible only with fine-tuning of parameters.

The assumptions made therefore imply that the perturbative sector of many higher-dimensional theories is excluded on the basis of current experimental observations. The source of the phenomenological difficulties may be viewed as the long-range propagation of massless modes of Lorentz-scalar components h_{jk}, $j,k \geq 4$, of the components of the metric $h_{\mu\nu}$. These modes are generic in the perturbative analysis of compactification schemes. Like the graviton, they couple weakly and cannot be directly detected. Nonetheless, without changing Newton gravity they cause measurable deviations from Einstein gravity and changes in cosmological parameters.

As some of the most attractive theoretical proposals for the quantum unification of the fundamental forces involve higher dimensions and can have massless modes of the type discussed, circumventing this result is desirable. In the remainder of this Section, we mention some possibilities that may be relevant to this issue. Note that certain compactification schemes might avoid such massless modes altogether. However, a compactified manifold must satisfy the equations of motion and be stable under small fluctuations. Frequently, compactifications have zero modes; for example, it is common in Kaluza-Klein, supergravity and superstring theories to consider Ricci–flat manifolds, which typically have zero modes. Zero modes occur in toroidal compactifications, including the original five–dimensional Kaluza–Klein model [1,2], and generically occur in models based on Calabi–Yau manifolds [25].

Given the presence of such modes in a theory, the phenomenological difficulties may nonetheless be circumvented by a mechanism suppressing their long–range propagation. The simplest solution is to give them masses. However, introducing explicit mass terms violates general coordinate invariance. An analogous situation exists in nonabelian gauge theories, for which gauge invariance precludes an explicit mass term for any gauge field. Spontaneous symmetry breaking and the Higgs effect bypass this problem. However, as demonstrated in Section 3, the gravitational version of the Higgs effect effect neither provides masses for components of the gravition nor destabilizes flat higher–dimensional spacetime.

What mechanisms exist for mass generation in nonabelian gauge theories other than the Higgs effect? Perturbatively, pure SU(3) nonabelian gauge theory describes eight massless gluons. However, the physical spectrum has massive glueballs and confinement prevents the isolation of individual gluons: the effective gluon mass increases away from a glueball. A similar mechanism may exist in strings, for which interactions involve an infinite number of fields and are highly nonlinear. In fact, strong coupling in strings has been advocated previously [26]. Strong coupling at large distances would cause massive bound states to form. If the higher–dimensional components of the metric are involved then they effectively decouple, as desired.

Assuming the existence of some mechanism for effective metric–mass generation, another question must be addressed. The perturbative analysis of strings yields four–dimensional gravitational and nonabelian gauge interactions. These parts of the perturbative sector should therefore remain unaffected by any mechanism for mass generation. Then, one must understand why the implementation of the damping mechanism occurs asymmetrically, in only one part of the theory. It may be that the origin of the asymmetry is related to tensor–induced spontaneous Lorentz–symmetry breaking, even though no damping mechanism directly results. From this viewpoint, strings are favored over other higher–dimensional theories.

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References

1. T. Kaluza, Sitzungsber. Preuss Akad. Wiss, Phys. Math. **K1**, 966 (1921).
2. O. Klein, Z. Phys. **37**, 875 (1926); Nature (London) **118**, 516 (1926).

3. See, for example, T. Appelquist, A. Chodos and P.G.O. Freund, eds., *Modern Kaluza–Klein Theories* (Addison Wesley, Menlo Park, 1987).
4. See, for example, M.J. Duff, B.E.W. Nilsson and C.N. Pope, Phys. Rep. **130**, 1 (1986) and references therein.
5. See, for example, M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory,* Vols. I and II (Cambridge University Press, Cambridge, 1987).
6. V.A. Kostelecký and S. Samuel, Phys. Rev. D **39**, 683 (1989).
7. E. Witten, Nucl. Phys. **B268**, 253 (1986).
8. W. Siegel, Phys. Lett. B **142**, 276 (1984); **151**, 391, 396 (1985).
9. V.A. Kostelecký and S. Samuel, Phys. Rev. Lett. **63**, 224 (1989).
10. V.A. Kostelecký and S. Samuel, Phys. Rev. D, in press.
11. D. Tischler, Topology **9**, 153 (1970).
12. A.S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1922).
13. H.P. Robertson, in A.J. Deutsch and W. B. Klemperer, eds., *Space Age Astronomy* (Academic Press, New York, 1962).
14. I.I. Shapiro, C.C. Counselman and R.W. King, Phys. Rev. Lett. **36**, 555 (1976).
15. R.D. Reasenberg, I.I. Shapiro, P.E. MacNeil, R.B. Goldstein, J.C. Breidenthal, J.P. Brenkle, D.L. Cain, T.M. Kaufman, T.A. Komarek and A.I. Zygielbaum, Ap. J. **234**, L219 (1979).
16. E. Witten, in R. Jackiw, N.N. Khuri, S. Weinberg and E. Witten, eds., *Proceedings of the 1983 Shelter Island Conference on Quantum Field Theory and the Fundamental Problems of Physics* (MIT, Cambridge, 1985).
17. R.W. Hellings, P.J. Adams, J.D. Andersen, M.S. Keesey, E.L. Lau, E.M. Standish, V.M. Canuto and I. Goldman, Phys. Rev. Lett. **51**, 1609 (1983).
18. See, for example, L. Z. Fang, T. Kiang, F. H. Cheng and F. X. Hu, Q. Jl. R. Astr. Soc. **23**, 363 (1982) and references therein.
19. H. R. Butcher, Nature **328**, 127 (1987); **330**, 704 (1987).
20. A. Sandage, Ap. J. **252**, 553 (1982); Ap. J. **331**, 583 (1988).
21. K. Janes and P. Demarque, Ap. J. **264**, 206 (1983).
22. D. A. VandenBerg, Ap. J. Suppl. **51**, 29 (1983).
23. F.-K. Thielemann, J. Metzinger and H. V. Klapdor, Z. Phys. A **309**, 301 (1983); Astron. Astrophys. **123**, 162 (1983).
24. R. G. Ostic, R. D. Russell and D. H. Reynolds, Nature **199**, 1150 (1963).
25. P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, Nucl. Phys. **B258**, 46 (1985).
26. M. Dine and N. Seiberg, Phys. Lett. **162B**, 299 (1985).

CURRENT ALGEBRA AND EXTENDED 2D GRAVITY WITH HIGHER SPIN GAUGE FIELD

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Abstract: We discuss quantum field theory of extended two dimensional gravity associated with Zamolodchikov's W-algebra. We introduce new type of gauge symmetry whose gauge potentials are higher rank symmetric tensor fields. In the limit that those gauge potentials are infinitesimaly small, we write down the invariant action for matter fields and extended gravity. By investigating the gauge symmetry in the light cone gauge, we find that induced quantum gravity associated with this symmetry is described by simply laced affine Lie algebras. Our result is a natural generalization of the result of Polyakov's quantum gravity.

1. Introduction: It has been recognized [1] that the quantization of two dimensional gravity will play a fundamental role in understanding the off-critical string theory. For example, if the dimensions of space-time are different from critical dimension, the Weyl mode of the metric no longer can be regarded as gauge degree of freedom and becomes propagating. Furthermore, if one wants to consider off-shell amplitude of the string theory, one cannot neglect the contribution of Weyl mode to the vertex operators.

Recently, Polyakov [2] made an important observation that if we use light-cone gauge fixing for the metric,

$$ds^2 = dx^+ dx^- + h_{++}(dx^+)^2, \tag{1}$$

the symmetry of induced gravity is enlarged to $sl(2, \mathbf{R})$ current algebra. Later, this Polyakov's observation is more explicitly studied by Knizhnik, Polyakov and Zamolodchikov [3] that critical exponents of matter fields on random lattice [4] can be systematically obtained in terms of representation theory of this affine Lie algebra.

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In this article, we would like to indicate that this Polyakov's discussion can be rather straightforwardly generalized to the quantum field theory of "W-gravity" [5], the extended gravity which corresponds to Zamolodchikov's W-algebra and its generalization [6]. Although no explicit algebras are not known, there are many evidences that for each simply laced affine Lie algebra $\hat{\mathbf{g}}$ based on Lie algebra g, there exists a extended conformal algebra whose generators have dimensions given by orders of Casimir operators of g. Interestingly, the minimal models of such chiral algebras are described by coset space conformal field theory of corresponding Lie algebra. The simplest correspondence is that of Virasoro algebra and $sl(2)$ affine algebra. Judging from these correspondences, it seems very natural to expect that the W-gravity associated with $\hat{\mathbf{g}}$ is described by noncompact realization of $\hat{\mathbf{g}}$.

Recently, this problem was investigated from totally different directions. The one approach [7] is to use the correspondence between generalized KdV equations and affine Lie algebra.[8] It was indicated by Drinfeld and Sokolov that the phase space of classical KdV type equations can be obtained by "Hamiltonian Reduction" of the phase space of the affine Lie algebra. On the other hand, it was also known that the W-algebra can be obtained by quantizing the poisson bracket of the potentials of KdV equations (Gelfand Dikii algebra).[9] From these correspondences, it is natural to expect that we can get the Hilbert space of the W-algebra from that of affine Lie algebra by Quantum version of Hamilton reduction. In this approach, the correspondence between representations of W-algebra and affine algebras are manifest. The drawback of this approach is that we can not understand the gauge symmetry underlying the system. Due to that, the physical meaning of this approach is not clear.

On the other hand, in the other approach employed by us,[10] we first constructed the geometry of W-gravity and quantize it by generalized light cone fixing. Hence the gauge symmetry behind the W-algebra is manifest. The difficulty in this approach is that one can construct the gauge symmetry only when the gauge potential associated with higher spin field is infinitesimally small. Furthermore, the appearance of affine algebra seems rather accidental.

Although these approaches looks totally different, the result seems to be similar. Hence the another formulation which interpolates between these formulations is highly desirable.

2. Linearized Geometry of W-gravity: In this section, we construct the geometry of the W-algebra in some restriced sense, i.e. the gauge potential associated with higher spin chiral current is infinitesimally small. Our restricted description is enough for our later discussion on the relation between W-gravity and affine Lie algebra. However, it is needless to say that the investigation of the gauge symmetry to the next order is highly desirable.

Since the chiral generator $W^{(n)}$ is a chiral component of symmetric tensor with rank n, it is natural to introduce the rank n symmetric gauge potential $A^{(n)}_{\mu_1\cdots\mu_n}$ which couples to W,

$$\frac{\delta S}{\delta A^{(n)}_{\mu_1\cdots\mu_n}} = W^{(n)}_{\mu_1\cdots\mu_n}. \tag{2}$$

In (2), we have to introduce additional higher spin currents which vanish when the system is conformal invariant. We can recover the original chiral current if we impose conformal invariance and conservation laws of currents.

$$\begin{aligned} g^{\mu_1\mu_2} W_{\mu_1\mu_2\cdots\mu_n} &= 0 & (3)\\ \nabla^{\mu_1} W_{\mu_1\cdots\mu_n} &= 0. & (4) \end{aligned}$$

The covariant derivative in this equation is that of two dimensional gravity in the ordinary sense. Conservation laws for W currents (4) are consequences of following gauge transformation laws for A,

$$\delta A^{(n)}_{\mu_1\cdots\mu_n} = \nabla_{(\mu_1}\kappa^{(n-1)}_{\mu_2\cdots\mu_n)} + \cdots, \tag{5}$$

where $\kappa^{(n-1)}$ are parameters of gauge transformations and are rank $(n-1)$ symmetric tensor. These equations are the zeroth approximitations with respect to the higher gauge potentials. $\cdots$ means the nonabelian contributions.

At least in this linearized limit, it is obvious that we can choose generalized light-cone gauge for A,

$$A^{(n)}_{+\cdots+} \neq 0, \qquad \text{other components} = 0, \tag{6}$$

since the degree of freedom for $A^{(n)}$ is $n+1$ and that of $\kappa^{(n-1)}$ is n.

Let us investigate the analogue of "curvature" for this generalized gravity. It is well known that for Yang-Mills and reparametrization symmetries, the independent degree of freedom for curvature is one,

$$\begin{aligned} F_{+-} &\sim \partial_- A_+ - \partial_+ A_-, & (7)\\ R &\sim \partial_-^2 h_{++} - 2\partial_-\partial_+ h_{+-} + \partial_+^2 h_{--}. & (8) \end{aligned}$$

This fact is still true for our generalized gravity. By simple investigation, it is clear there is only one covariant object (actually invariant) for our generalized symmetry,

$$S^{(n)} = \sum_{p=0}^{n}(-1)^p \begin{pmatrix} n \\ p \end{pmatrix} \partial_-^p \partial_+^{n-p} A^{(n)}_{\underbrace{+\cdots+}_{p}\underbrace{-\cdots-}_{n-p}}. \tag{9}$$

We defined this generalized curvature by investigating the gauge transformation property. It is a very important problem to investigate that it has the meaning of the ordinary "curvature".

3. Invariant Actions for Linearized W-Gravity: It is very important to discuss the invariant actions for linearlized W-gravity in order to understand the nature of W-gravity. The most typical example is the action for free bosons. We should remember that the explicit realization of W-algebras are given in terms of free bosons.[6] Let us illustrate it in the case of W_3-algebra. In this case, the symmetry is realized by two boson fields. The gauge transformations of scalor bosons are given by,

$$\begin{aligned} \delta_\epsilon \phi^i &= \epsilon_+ \partial_+ \phi^i + \epsilon_- \partial_- \phi^i, & (10) \\ \delta_\kappa \phi^1 &= \kappa_{++} \left(\left(\partial_- \phi^1\right)^2 - \left(\partial_- \phi^2\right)^2 \right) + (+ \leftrightarrow -) + \cdots \\ \delta_\kappa \phi^2 &= -2\kappa_{++} \partial_- \phi^1 \partial_- \phi^2 + (+ \leftrightarrow -). & (11) \end{aligned}$$

The action which is invariant under these transformations is,

$$S = S_0 + S_1 + \cdots, \tag{12}$$

where,

$$S_0 = \int d^2x \partial_+ \phi^i \partial_- \phi^i, \tag{13}$$

$$\begin{aligned} S_1 = & - \int d^2x h_{++} \left(\partial_- \phi^i\right)^2 \\ & + \frac{3}{2} A_{+++} \left(\left(\partial_- \phi^1\right)^3 - 3 \partial_- \phi^1 \left(\partial_- \phi^2\right)^2 \right) \\ & + (+ \leftrightarrow -). \end{aligned} \tag{14}$$

It is easy to observe that the variation of S_0 can be absorbed by the variation of gauge potentials in S_1 defined in (5) without using equation of motion. Some of the components of gauge fields do not appear explicitly in our lagrangian because our free boson system has conformal invariance. We remark that the new gauge symmetry is defined between spin zero boson and spin 2 composite primary fields. Due to this nonlinearlity, it seems very nontrivial to write down the lagrangian which is invariant under gauge transformations up to all orders.

Since we got the example of matter lagrangian for W-gravity, we can make some computations to get the action of induced W-gravity. The standard computation methods are illustrated in Polyakov's book.[11] In our case, we have to calculate following Feynmann diagrams.

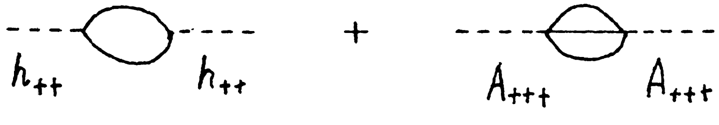

Fig.1

Since these computations produce nonlocal Lagrangian which is not invariant under gauge transformations, we have to add to it local counter terms such that the Lagrangian is gauge invariant.[11] The resulting action has following form,

$$S^{grav} \sim \int d^2x \left(R(x)\Box^{-1}R(x) + \frac{1}{30}S^{(3)}(x)\Box^{-1}S^{(3)}(x) \right), \tag{15}$$

where $R(x)$ and $S^{(3)}$ are Ricci scalors of W-gravity (8,9). This action seems to be the natural generalization of action for induced gravity.

From this invariant action for induced W-gravity, it is easy to discuss generalization of Liouvile action. In the case of ordinary gravity, the choice of Liouvile field is almost unique, i.e. $\phi_1 = h_{+-}$. However, in our case, there are some ambiguity to identify which component of gauge potential to be Liouvile field. Let us examine the gauge fixing different from our previous choice (6),

$$A^{(3)}_{+--} \neq 0, \qquad A^{(3)}_{+++} = A^{(3)}_{++-} = A^{(3)}_{---} = 0. \tag{16}$$

Of course, to set $A_{++-} \neq 0$ is an alternative choice. If we put, $h_{+-} = 1 + \phi_1$ and $\partial_+ A_{+--} = \phi_2$, the action for induced W-gravity reduces to the generalized Liouvile action,

$$S^{Liouvile} \sim \int d^2x \, (\partial_-\phi_1\partial_+\phi_1 + \partial_-\phi_2\partial_+\phi_2), \tag{17}$$

which coincides with the previous conjectures made from different contexts by some people.[5,12] Our discussion gives geometrical background to those works.

Let us discuss the hidden symmetry of our generalized conformal gauge. In the ordinary gravity case, the residual gauge transformation which fixes the conformal metric is so-called conformal transformation,

$$\epsilon_+ = \epsilon_+(x^-), \qquad \epsilon_- = \epsilon_-(x^+). \tag{18}$$

This residual transformation redefines Liouvile field as follows,

$$\delta\phi_1 = \partial_-\epsilon_+ + \partial_+\epsilon_-. \tag{19}$$

This redefinition coincides with on-shell transformation of ϕ_1. We can observe similar phenomena for our case. The gauge transformation which fix our generalized conformal gauge (16) is given by,

$$\begin{aligned} \kappa_{++} &= \kappa_{++}(x^-), \\ \kappa_{--} &= \kappa_{--}(x^+), \\ \kappa_{+-} &= -\frac{1}{2}x^+\partial_-\kappa_{++}(x^-) + \eta(x^-). \end{aligned} \tag{20}$$

We should remark that we have to introduce three (anti-) holomorphic parameter in this case. However, since we introduce Liouvile field ϕ_2 by derivative of A_{+--}, the transformation of ϕ_2 is completely analogous to ϕ_1,

$$\delta\phi_2 = \partial_+^2\kappa_{--} - \partial_-^2\kappa_{++}. \tag{21}$$

The paramater κ can be regarded as analogue of reparametrization parameter ϵ associated with W. The appearance of third parameter $\eta(x^-)$ in (20) seems inevitable even if we can pick another gauge. In the conformal field theory, the generater associated with $\eta(x^-)$ vanished identically. This fact seems to imply that this parameter is unphysical.

4. Semi-classical Analysis in the Light Cone Gauge: Since the geometry of W-gravity has been clarified to some extent now, it is time to proceed to quantize the system. In this paper, we generalize the strategy of Polyakov [2] which is also powerful for our case. The key point of his approach is an essential use of light cone gauge to analyze the system semi-classically. Perhaps, it is appropriate for us to start from following action,

$$S = S_0 + \frac{1}{\pi}\int d^2x\,(h_{++}T_{--} + A_{+++}W_{---})\,, \tag{22}$$

which is an abstract form of free boson action (12). We neglected the h_{--} and A_{---} part because we picked light cone gauge here. Since we will use only OPE of T_{--} and W_{---}, our argument can be applicable to any matter system which has W-symmetry. In the following, we neglect the subscripts if it is obvious to understand. In order to get essential feature of quantized system, we need to introduce the gauge symmetry of next order. In the light cone gauge, the necessary information can be obtained from OPE of W-algebra itself. Let us illustrate it in the case of W_3-algebra, whose OPE is known explicitly.

$$\begin{aligned} T(x)T(0) &\sim \frac{c}{2}x^{-4} + 2x^{-2}T(0) + x^{-1}\partial T(0), && (23)\\ T(x)W(0) &\sim 3x^{-2}W(0) + x^{-1}\partial W(0) && (24)\\ W(x)W(0) &\sim \frac{c}{3}x^{-6} + 2x^{-4}T(0) + x^{-3}\partial T(0) + \frac{3}{10}x^{-2}\partial^2 T(0) \\ &\quad + \frac{1}{15}x^{-1}\partial^3 T(0) + \alpha\left(2x^{-2}\Lambda(0) + x^{-1}\partial\Lambda(0)\right), && (25)\end{aligned}$$

where Λ is a normalized square of T,[6] and $\alpha = 16/(22+5c)$.

We can translate these OPEs to the geometry in the light cone gauge. The procedure is to use the anomalous conservation laws of chiral field under the perturbation of light cone gauge potential. For example, let us evaluate the derivative of energy momentum tensor under such perturbation,

$$\partial_+ < T_{--}(x^+, x^-)\cdots >, \tag{26}$$

where $< \cdots >$ means the vacuum expectiation value. If there is no perturbation, this quantity vanishes identically if we use the equation of motion. However, after we switch on the perturbation, there are singularities as in (23,24,25) between T and W fields in the action and T_{--} in the vacuum expectation value.

These singularities give rise to non-analyticity if we use the well-known equations, $\partial_+ \frac{1}{x^-} \sim \delta^{(2)}(x)$. The modification of the conservation laws of currents is given as follows,

$$\begin{aligned}
\nabla_+ T &= \partial_+ T - 2\partial_- h T - h\partial_- T - 3\partial_- A W - 2A\partial_- W \\
&= \frac{c}{12}\partial_-^3 h,
\end{aligned} \tag{27}$$

$$\begin{aligned}
\nabla_+ W &= \partial_+ W - (3\partial_- h + h\partial_-)W \\
&\quad - \left(\frac{1}{3}\partial_-^3 A + \frac{1}{2}\partial_-^2 A\partial_- + \frac{3}{10}\partial_- A\partial_-^2 + \frac{1}{15}A\partial_-^3\right) T \\
&\quad -\alpha(2\partial_- A + A\partial_-)\Lambda \\
&= \frac{c}{360}\partial_-^5 A.
\end{aligned} \tag{28}$$

In the following, we shall neglect the contribution of the composite operator in (28) in our semi-classical ($c \longrightarrow \infty$) argument. Once we found the definition of covariant derivative, the gauge transformations can be deduced from that.

$$\begin{aligned}
\delta_{\epsilon,\kappa} h &= \partial_+ \epsilon - h\partial_- \epsilon + \partial_- h \epsilon \\
&\quad + \frac{1}{30}\left(-2A\partial_-^3 + 3\partial_- A\partial_-^2 - 3\partial_-^2 A\partial_- + 2\partial_-^3 A\right)\kappa,
\end{aligned} \tag{29}$$

$$\delta_{\epsilon,\kappa} A = \partial_+ \kappa - h\partial_- \kappa + 2\kappa\partial_- h + \epsilon\partial_- A - 2\partial_- \epsilon A. \tag{30}$$

This formula can be regarded as next order gauge transformation for the light cone gauge potentials and we find quite naturally h and A are mixed in this order. Owing to the anomaly term (27,28), the action is not invariant under these transformation.

$$\begin{aligned}
\delta_{\epsilon,\kappa} S &= \int d^2x\,(\delta h T + \delta A W) \\
&= -\int d^2x\,(h\nabla_+ T + A\nabla_+ W) \\
&= -\frac{c}{12\pi}\int d^2x \left(\epsilon\partial_-^3 h + \frac{1}{30}\kappa\partial_-^5 A\right).
\end{aligned} \tag{31}$$

Since we are considering the anomalous gauge theory, the action is not invariant under gauge transformation. In such theory, one should interprete that the gauge degree of freedom becomes dynamical. In this viewpoint, the above equation should be regarded as variation of the action with respect to this dynamical field. In this way, one gets the following equation of motion for light cone gauge potentials,

$$\partial_-^3 h = 0, \qquad \partial_-^5 A = 0. \tag{32}$$

This equation is a natural generalization of Polyakov's equation, $\partial_-^3 h = 0$. It can also be derived from our action for induced W-gravity (15).

By investigation of Ward identities,

$$\delta_{\epsilon,\kappa}\langle h(x_1)\cdots h(x_n)A(y_1)\cdots A(y_m)\rangle = 0, \tag{33}$$

and by using gauge transformation laws,(29,30,31) one can readily observe that h and A should have following OPEs.

$$h^c(x)h^c(0) \sim -\frac{c}{24}\frac{(x^-)^2}{(x^+)^2} - \frac{x^-}{x^+}h^c - \frac{1}{2}\frac{(x^-)^2}{x^+}\partial_- h^c, \tag{34}$$

$$h^c(x)A^c(0) \sim -2\frac{x^-}{x^+}A^c - \frac{1}{2}\frac{(x^-)^2}{x^+}\partial_- A^c, \tag{35}$$

$$A^c(x)A^c(0) \sim -\frac{c}{96}\frac{(x^-)^4}{(x^+)^2} - \left(\frac{1}{2}\frac{(x^-)^3}{x^+} + \frac{1}{4}\frac{(x^-)^4}{x^+}\partial_-\right)h^c, \tag{36}$$

where we have rescaled the gauge fields, $h^c = (c/12)h$, $A^c = (c/12\sqrt{10})A$. Due to the generalization of Polyakov's equation in (32), we can expand h^c and A^c as follows,

$$h^c(x) = \sum_{p=-1}^{1}(x^-)^{p+1}I^{(p)}(x^+), \qquad A^c(x) = \sum_{p=-2}^{2}(x^-)^{p+2}J^{(p)}(x^+). \tag{37}$$

One can easily check that the OPEs (34,35,36) are equivalent to OPE of non-compact current algebra $sl(3,\mathbf{R})$ in terms of currents Is and Js defined above. The central charge of the algebra is given by $k = c/24$. Division of currents into two groups is nothing but the decomposition of adjoint representation of $sl(3)$ into representations of $sl(2)$ subalgebra, $(8) \to (3) + (5)$.

The extension of above argument for general W-algebra seems difficult since there is no explicit OPEs for general W-algebra. However, we can make following argument which strongly suggest that our argument can be extended to those cases. Let us remember that the W-algebra associated with simply-laced Lie algebra g consists of generators whose spins are given by orders of Casimir operators. Since OPE of spin n fields yields the singularity $(x^-)^{-2n}$, our discussion in this section suggests that light cone component $A^{(n)}$ satisfies the equation of motion, $\partial_-^{2n-1}A^{(n)} = 0$. This equation implies that $A^{(n)}$ is described by $2n-1$ chiral gauge fields as,

$$A^{(n)} = \sum_{p=-n+1}^{n-1}(x^-)^{p+n-1}J^{(n,p)}(x^+). \tag{38}$$

Now, let us compute how many chiral currents appear in the theory. It is given by the sum, $\sum_{n\in C}(2n-1)$. (C is the set of dimensions of Casimir operators.) Interestingly, this summation can be carried out explcitly for any simple Lie algebra and the answer is the dimensions of g itself. It is not accidental. Kostant [13] proved that every simple Lie algebra has "fundamental $sl(2)$" algebra and that adjoint representation of $\mathbf{g}$ can be decomposed as that. The explicit formula for the fundamental $sl(2)$ is given as follows. Let $\{X^\alpha, Y^\alpha, H^\alpha\}_{\alpha=1\cdots n}$ be Weyl Cartan basis of $\mathbf{g}$. Then the fundamental $sl(2)$ can be represented as follows,

$$I^- = \sum_\alpha X^\alpha, \qquad I^0 = \sum_{\alpha,\beta} A^{-1}_{\alpha,\beta} H^\beta, \qquad I^+ = \sum_{\alpha,\beta} A^{-1}_{\alpha,\beta} Y^\beta, \tag{39}$$

where $A^{-1}_{\alpha,\beta}$ is the inverse of the Cartan matrix. By using the OPE between energy momentum tensor and $W^{(n)}$, one can show that,

$$h^c(x)A^{(n)}(0) \sim -(n-1)\frac{x^-}{x^+}A^{(n)} - \frac{1}{2}\frac{(x^-)^2}{x^+}\partial_- A^{(n)}. \tag{40}$$

This equation shows that each currents which are contained in $A^{(n)}$ form the $(2n-1)$ dimensional representation of fundamental $sl(2)$ if we use the representation for h^c,

$$h^c(x) = I^-(x^+) - 2I^0(x^+)x^- + I^+(x^+)(x^-)^2. \tag{41}$$

If our argument can be generalized in this way, the relation between central charges of current algebra and W-algebra is obtained from (41). In this computation, we have to use,

$$A^{-1}_{\alpha,\beta} = (\omega_\alpha, \omega_\beta), \tag{42}$$

where ω_α is the fundamental weight vectors, and Freudenthal de Vries identity,

$$\left(\sum_\alpha \omega_\alpha\right)^2 = \frac{gD}{12}, \tag{43}$$

where g and D are respectively dual Coxeter number and dimension of the Lie algebra $\mathbf{g}$. After some simple computations, we get following relation between central charges,

$$k = \frac{c}{gD}. \tag{44}$$

As expected, this result generalizes the special cases when $\mathbf{g}$ is $sl(2)$ or $sl(3)$.

For later discussions, let us write down component forms of OPE (41).

$$I^-(x^+)A^{(n)}(0) \sim \frac{1}{x^+}\partial_- A^{(n)}, \tag{45}$$

$$I^0(x^+)J^{(n,p)}(0) \sim \frac{p}{x^+}J^{(n,p)}. \tag{46}$$

5. Quantum Corrections: In this section, we will discuss quantum corrections to our semi-classical analysis we made in the previous section. Our strategy is a straightforward extension of the analysis by KPZ.[3]

In order to define the path-integral with respect to gauge potentials, we have to devide the functional integral by gauge volume as in the ordinary string theory. If we define the gauge slice defined by our light cone gauge (6), we have to insert delta-functions $\prod_{I\neq\{+\cdots+\}}\delta\left(A_I^{(n)}\right)$ in the integrand. As usual, the contributions of these terms can be represented by introducing ghost fields. In our case, we have to introduce a set of ghost fields $\left(b^{(n,p)},c^{(n,p)}\right)(\ p=0,\cdots n-1)$ corresponding to each gauge transformation parameters $\kappa_{+\cdots+-\cdots-}$. These ghost fields have dimensions $2p-n$ and $1-2p+n$.

After the introduction of these fields, we have to investigate whether we can consistently impose the constraint coming from light cone gauge fixing. For example, we have to check whether we can put the currents corresponding to $A^{(n)}_{+\cdots+-\cdots-}$ to be zero. Investigation of general consistency condition seems difficult because of the appearance of nonlinear terms. However, the consistency of gravity part,

$$\frac{\delta S}{\delta h_{--}}=T^{tot}_{++}\quad\approx\quad 0, \tag{47}$$

$$\frac{\delta S}{\delta h_{+-}}=T^{tot}_{+-}\quad\approx\quad 0, \tag{48}$$

can be done in completely similar fashion as KPZ [3]. Their method to obtain the explicit formula for T^{tot}_{+-} and T^{tot}_{++} is to introduce the transformations which these fields generate. Let us consider the following reparametrization,

$$\delta x^+=\epsilon_-(x^+),\qquad \delta x^-=-x^-\partial_+\epsilon_-(x^+)+\eta_+(x^+). \tag{49}$$

These transformations preserves the light cone metric (1) and changes the h_{++} as follows.

$$\delta h_{++}=(2h_{++}-x^-\partial_-h_{++})\partial_+\epsilon_-+\partial_+h_{++}\epsilon_-+\partial_-h_{++}\eta_+-\partial_+^2\epsilon_-x^-+\partial_+\eta_+. \tag{50}$$

The last two terms come from diagonal part of the metric. We remark that ϵ (resp. η) transformation is generated by T_{++} (resp. T_{+-}). Since there is an expression for h_{++} in terms of currents, KPZ can find the correct form for energy momentum tensor.

This KPZ's strategy can also be applicable to our case, because transformation (49) preserves our generalized light cone gauge. Light cone component $A^{(n)}_{+\cdots+}$ tramsforms,

$$\delta A^{(n)}=\epsilon_-\partial_+A^{(n)}-x^-\partial_+\epsilon_-\partial_-A^{(n)}+n\partial_+\epsilon_-A^{(n)}+\eta_+\partial_-A^{(n)}. \tag{51}$$

We should note that η_+ transformation is given by x^- derivative. By comparing it with (45), we find T_{+-} is given by current I^-. On the other hand, if we use Taylor expansion for $A^{(n)}$ in terms of chiral currents (38), these currents transform by ϵ transformation as follows,

$$\delta_\epsilon J^{(n,p)}=\epsilon_-\partial_+J^{(n,p)}+(1-p)\partial_+\epsilon_-J^{(n,p)}. \tag{52}$$

This equation means that current $J^{(n,p)}$ transforms spin $(1-p)$ field with respect to stress tensor T^{tot}_{++}. It is well known that if we use the Sugawara type energy momentum tensor, currents have dimension one. Hence we need to introduce some correction term to it. Desired expression can be obtained from equation (46). We get,

$$T^{grav}_{++} = T^{Sugawara}_{++} + \partial_+ I^0. \tag{53}$$

The evaluation of ghost fields' contribution to energy momentum tensor is straightforward since we have already known their dimensions. Explicitly, it is given by following sum,

$$T^{gh} = \sum_{n \in C} \sum_{p=0}^{n-1} \left((2p-n)\partial_+ b^{(n,p)} c^{(n,p)} + (2p-n-1) b^{(n,p)} \partial_+ c^{(n,p)} \right). \tag{54}$$

The total energy momentum tensor is now obtained by adding them the matter energy momentum tensor.

$$T^{tot}_{++} = T^{mat}_{++} + T^{grav}_{++} + T^{gh}_{++}. \tag{55}$$

The constraint that this operator vanish gives nontrival constraint for the realization of current algebra. In partiacular, we have now the condition,

$$c_{matter} + c_{gravity} + c_{ghost} = 0. \tag{56}$$

We put c_{matter} to be d. $c_{gravity}$ is given by contributions of Sugawara energy momentoum tensor and additional terms. By using Freudental de-Vries identity (43), we get

$$c_{gravity} = \frac{kD}{k+g} - kgD. \tag{57}$$

The contribution from ghost fields is given by the sum,

$$\begin{aligned} c_{ghost} &= \sum_{n \in C} \sum_{p=0}^{n-1} (-12p^2 + 12p - 2) \\ &= -r - (1+h)^2 D. \end{aligned} \tag{58}$$

The formula in the second line can be proved for simply-laced Lie algebras by explicit computations. By using these formulae, we have now obtained the central charge of current algebra in terms of that of W-algebra.

$$-(k+g) = \frac{2gD + r - d \pm \sqrt{(r-d)(r+4gD-d)}}{2gD} \tag{59}$$

If we pick the + sign in (59), the semiclasical limit ($c \to -\infty$) coincides with our formula in the last section. As one can readily observe, the formula is consistent only when $d \leq r, d \geq r + 4gD$. This result shows that by introducing another gauge symmetries, the weak gravity region is enlarged from original gravitational theory ($c \leq 1$).

The interesting point of our formula is that it implies interesting connection between the representations of current algebra and W-algebra. It is known that irreducible representations of W-algebra associated with current algebra X is given by coset space construction

$$\frac{X^k \otimes X^1}{X^{diag}}. \tag{60}$$

In particular, if we set $k + g = \frac{q}{p-q}$, we get central charge of mimimal model of W-algebra,

$$d = r - gD\frac{(p-q)^2}{pq}. \tag{61}$$

If we put this value to our formula (59), we find that k takes rational value in this case,

$$k + g = -\frac{p}{q}. \tag{62}$$

We remark here that in order to compare this result to that of compact group, we have to convert to mirror quantity for k, $k' + g = -(k + g)$. These values for k' are exactly the values of central charges when there are modular invariant representations.[14] Furthermore, if we use the modified Sugawara type energy momentum tensor,(53) the primary fields of current algebra have the dimensions of those of W-algebra. This fact suggests that there is one to one correspondence between representation of each theory.

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References

[1] Polyakov,A.M., *Physics Letters* **103B** (1981) 207.

[2] Polyakov,A.M., *Modern Physics Letters***A2** (1987) 893.

[3] Knizhnik,V.M., Polyakov,A.M. and Zamolodchiov,A.B., *Modern Physics Letters* **A3** (1988) 819.

[4] Kazakov,V.A., *Physics Letter* **150B** (1985) 282; David,F., *Nuclear Physics* **B257** (1985) 543; Kazakov,V.A., *Physics Letter* **A119** (1986) 140.

[5] Bilal,A. and Gervais,J.-L., preprint LPTENS 88/36 (1988).

[6] Zamolodchikov,A.B., *Theor.Math.Phys.* **99** (1985) 108; Zamolodchikov,A.B. and Fateev,V.A., *Nuclear Physics* **B280** (1987) 644; Fateev,V.A. and Lykyanov,S.L., *Int.J.Mod.Phys.* **A3** (1988)

507; Lukyanov,S.L. and Fateev,V.A., Kiev preprints ITP-88-74P, 75P,76P; Bilal,A. and Gervais, J.-L., *Physics Letters* **206B** (1988) 412.

[7] Bershadsky,M. and Ooguri,H., Princeton preprint IASSN-HEP-89/09 (1989); Alekseev,A. and Shatashivili,S., *Nuclear Physics* (1989); Polyakov,A.M., MIT preprint (1989).

[8] Drinfeld,V.G. and Sokolov,V.V., *J.Sov.Math.* **30** (1985) 1975; see also Jimbo,M. and Miwa,T., *Publ.RIMS. Kyoto Univ.* **19** (1983) 943.

[9] Gervais,J.-L., *Physics Letters* **160B** (1985) 277; Yamagishi,K., *Physics Letters* **205B** (1988) 406; Mathiew,P., *Physics Letters* **208B** (1988) 101; Bakas,I., *Physics Letters* **213B** (1988) 313.

[10] Matsuo, Y., *Physics Letters* **227B** (1989) 209.

[11] Polyakov,A.M., *"Gauge Field and Strings"* (1988)

[12] Park,H., preprint (1989).

[13] Kostant,V., *Am.J.Math.* **131** (1959) 973.

[14] Kac,V. and Wakimoto,M., *Proc.Natl.Acad.Sci.USA* **85**

THE PARAMETRIC MANIFOLD PICTURE OF SPACE-TIME

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ABSTRACT

Parametric manifolds are reparametrization-invariant geometric structures describing space-time and internal degrees of freedom in a unified framework. Using the theory of parametric spinors, a decomposition of the space-time in General Relativity is developed with respect to the 3-space of trajectories of a time-like or space-like vector field. The parametric 3+1 decomposition surpasses the ADM formalism in generality since it is possible even in space-times which do not admit a space-like foliation.

1.INTRODUCTION

The theory of gravitational fields is a prolific source of geometrical insight. Concepts like Weyl's gauge invariance or the dimensional reduction of Kaluza, distilled from the relativistic geometry of the gravitational field, contribute significantly to the present edifice of theoretical physics. In this paper we shall consider another facet of the gravitational field which presents us with the notion of a parametric manifold.

A parametric manifold may be taken as an abstract geometrical manifestation of the properties of non-null congruences. Time-like congruences, in particular, represent a space-time filling array of material particles or observers, each of which having the property that it slices out a preferential time and 3-space aspect from the 4-geometry of the space-time. Indeed, such an attitude is required by many canonical quantization schemes. Following mainly the work of Arnowitt, Deser and Misner[1] (ADM), it has widely been accepted that the requisite initial-value formulation can be secured for the canonical scheme by foliating the space-time with a family of space-like hypersurfaces. Such a foliation always exists in a stably causal space-time. In the quantized picture, however, acausal contributions must be taken into account even at the tree level. The ADM approach appears to be unable to describe space-times like Gödel's universe[2] which have no global space-like foliation.

A further handicap of the ADM formalism is that fields of rotating sources do not lend themselves easily to a decomposition with respect to a family of hypersurfaces. Ominously, in the case of the Kerr black hole this is due to the fact that the Killing vectors are not hypersurface-forming. (The 3+1 decomposition of Kerr space-time can, however, easily be done with respect to the Riemannian 3-manifold[3] of the Killing trajectories.)

The notion of a parametric space arises in General Relativity in connection with distance measurements yielding a 3+1 decomposition of the space-time with respect to a generic non-null congruence of curves. Similar decompositions have been considered in the literature[4,5,6]. The closest to our parametric picture appears to be Zelmanov's 'chronometrically invariant' decomposition[7,8]. The reason that he failed to pin down the parametric manifold structure as a sovereign entity is that in his chronometrically invariant formalism quantities are mixed in (notably those involving the redshift factor f) that are foreign to the parametric structure. The 3-manifold defined by distance measurements is not, in the generic case, a Riemannian one.

In the first part of this paper we shall outline a theory of parametric manifolds. In Sec. 2, we describe some properties of parametric manifolds in D dimensions. We restrict the subsequent discussion, however, to $D = 3$. In Sec. 3, we consider the behaviour of the curvature. Parametric spinors will be introduced in Sec. 4.

The second part presents the special applications of parametric manifolds in General Relativity. We shall show in Sec. 5 that the space-time is a 3-dimensional parametric manifold, endowed with a redshift potential f. In Sec. 6 we discuss the adjungation of spinors. In the final sections we shall achieve a complete 3+1 decomposition of curvature relations and the gravitational equations of Einstein using parametric spinor techniques. The seducing elegance of spinorial manipulations should not go without mention. The 3+1 disentangling goes about by the algebra of soldering forms. Following this, spinors can be ripped off the final expressions with ease. The ADM formulation of the gravitational equations and the theory of stationary space-times will emerge as limiting cases of the parametric theory.

2.PARAMETRIC MANIFOLDS

The archetype of a parametric manifold is the pair $(\phi, \vec{\omega})$ consisting of the one-parameter group of diffeomorphisms

$$\phi : R \rightarrow \{\mathcal{M}\} \tag{2.1}$$

in the category $\{\mathcal{M}\}$ of (D-dimensional) differentiable manifolds $\mathcal{M}$ and the covectorfield $\vec{\omega}(x,t)$, where $t \in R$, defined on the image bundle of ϕ. A point x of the parametric manifold is defined as an equivalence class of points $\{x_t \epsilon \mathcal{M}_t \epsilon \{\mathcal{M}\}\}$ under the diffeomorphism group ϕ. Transiting to a new system of horizontal sections by re-parametrizations of the form

$$t' = t + F(x) \qquad t' \epsilon R \tag{2.2}$$

where $F(x)$ is some smooth function, the covectorfield $\vec{\omega}$ transforms

$$\vec{\omega}' = \vec{\omega} + \nabla F. \tag{2.3}$$

Functions on a parametric manifold, $\varphi(x,t)$ have the re-parametrization-invariant 'starry derivative'

$$\partial_{*i}\varphi \stackrel{\text{def}}{=} \partial_i \varphi - \omega_i \dot{\varphi} \qquad (i = 1, 2, ..., D) \tag{2.4}$$

where a dot denotes $\frac{\partial}{\partial t}$. The starry derivative* of a function is a generalization of what is meant by the invariant derivative in gauge theories. Indeed, in the case of a 'stationary' fields, i.e., when

$$\varphi(x,t) = \varphi_\circ(x) exp(iet), \tag{2.5}$$

the starry derivative reduces to the familiar local gauge invariant form

$$\partial_{*i}\varphi_\circ = \partial_i \varphi_\circ - ie\omega_i \varphi_\circ. \tag{2.6}$$

* The operator ∂_{*i} is a horizontal connection in the image bundle of ϕ. A related notion 'chronometrically invariant derivative' has been introduced by Zelmanov[7,8] in the context of relativistic cosmology

Of particular interest are parametric manifolds on sections $\mathcal{M}_t$ which are Riemannian manifolds with metric g_t. One can also take the Riemannian limit $\vec{\omega} \to 0$ of a *parametric manifold.* It is easy to construct parametric manifolds of more general nature.

The notion of covariant derivative ∇ can be transferred to a parametric manifold. For a scalar field φ we have $\nabla_i \varphi = \partial_{*i}\varphi$. In a local section the covariant derivative of a covectorfield is

$$\nabla_i v_k = v_{k*i} - \Gamma_{ik}{}^l v_l. \tag{2.7}$$

Requiring that the parametric metric g on $(\phi, \vec{\omega})$has a vanishing covariant derivative, we get the connection $\Gamma_{ik}{}^l = \frac{1}{2} g^{lr}(g_{ir*k} + g_{kr*i} - g_{ik*r})$. The parameter derivative ∇_o of a (p,q) tensor,

$$\nabla_o T^{ij\ldots}{}_{kl\ldots} = \dot{T}^{ij\ldots}{}_{kl\ldots} \tag{2.8}$$

again is a (p,q) tensor.

The commutators of the derivatives acting on a scalar φ are

$$\begin{aligned} (\nabla_i \nabla_k - \nabla_k \nabla_i)\varphi &= (\omega_{i*k} - \omega_{k*i})\dot{\varphi} \\ (\nabla_i \nabla_o - \nabla_o \nabla_i)\varphi &= \dot{\omega}_i \dot{\varphi}. \end{aligned} \tag{2.9}$$

The *Zelmanov curvature* Z_{rijk}, defined by the identity

$$[\nabla_k \nabla_j - \nabla_j \nabla_k + (\omega_{j*k} - \omega_{k*j})\nabla_o] v_i = Z_{rijk} v^r, \tag{2.10}$$

possesses the familiar algebraic properties of curvature

$$Z_{ijkl} = Z_{ij[kl]}, \qquad Z_{i[jkl]} = 0 \tag{2.11}$$

(But note the absence of the pairwise interchange of indices). The action of the commutator on the metric yields the decomposition

$$Z_{ijkl} = K_{ijkl} + \dot{g}_{ij}\omega_{[k*l]} \tag{2.12}$$

with

$$K_{ijkl} = K_{[ij][kl]}. \tag{2.13}$$

It is possible to further isolate a part R_{ijkl} of the curvature Z_{ijkl} with the algebraic symmetry properties of the Riemann tensor,

$$R_{ijkl} = R_{[ij][kl]} = R_{i[jkl]}. \tag{2.14}$$

Action of the commutator (2.10) on the gradient of a scalar yields

$$Z_{ijkl} = R_{ijkl} + \dot{g}_{ij}\omega_{[k*l]} + \frac{1}{2}(\dot{g}_{ik}\omega_{[j*l]} + \dot{g}_{il}\omega_{[k*j]} + \dot{g}_{jk}\omega_{[l*i]} + \dot{g}_{jl}\omega_{[i*k]}). \tag{2.15}$$

The *parametric curvature* H_{ikl} is defined by the commutator

$$[\nabla_i \nabla_o - \nabla_o \nabla_i - \dot{\omega}_i \nabla_o] v_k = [H_{ikl} + \frac{1}{2}(\nabla_i - \dot{\omega}_i)\dot{g}_{kl}] v^l. \tag{2.16}$$

The parametric curvature has the symmetry properties $H_{ijk} = H_{i[jk]}$ and $H_{[ijk]} = 0$. The Bianchi identities take the form

$$\nabla_{[i} K_{|jk|lm]} = 2\omega_{[i*l} H_{m]kj}. \tag{2.17}$$

3.THE CURVATURE IN 3 DIMENSIONS

In three dimensions, a skew tensor $F_{ik} = -F_{ki}$ has the equivalent representation

$$F_i \stackrel{\text{def}}{=} \sqrt{1/2}\epsilon_{ijk}F^{jk}, \qquad F_{jk} = \sqrt{1/2}\epsilon_{ijk}F^i \tag{3.1}$$

where ϵ_{ijk} is the unit skew tensor. The curvature $K_{ijkl} = K_{[ij][kl]}$ can be written equivalently

$$K^{mn} \stackrel{\text{def}}{=} \frac{1}{2}\epsilon^{ijm}\epsilon^{kln}K_{ijkl} = 1/2(g^{mn}g^{ik}g^{jl} + g^{mk}g^{il}g^{jn} + g^{ml}g^{in}g^{jk}) \\ \times(2R_{ijkl} + \dot{g}_{ik}\omega_{[j*l]} + \dot{g}_{il}\omega_{[k*j]} + \dot{g}_{jk}\omega_{[l*i]} + \dot{g}_{jl}\omega_{[i*k]}) \tag{3.2}$$

where the decomposition (2.15) has been employed (Note that $K_{ij} = 2K_{ir}{}^{r}{}_{j} + g_{ij}K_{rs}{}^{rs}$). Conversely,

$$K_{ijkl} = \frac{1}{2}\epsilon_{ijm}\epsilon_{kln}K^{mn}. \tag{3.3}$$

Introducing the tensor

$$P_{ik} \stackrel{\text{def}}{=} R_{ik} - \frac{1}{3}g_{ik}R + \frac{3}{2}g^{mn}\dot{g}_{m[n}\omega_{i*k]} \tag{3.4}$$

where

$$R_{ik} \stackrel{\text{def}}{=} R_{ir}{}^{r}{}_{k} \tag{3.5}$$

is the Ricci curvature and $R = R^i_i$, we have

$$K_{ijki} = g_{il}P_{kj} - g_{ik}P_{lj} + g_{jk}P_{li} - g_{jl}P_{ki} - \frac{1}{6}(g_{ik}g_{jl} - g_{il}g_{jk})R. \tag{3.6}$$

The tensor P_{ik} is trace-free, $P^k_k = 0$ and in the Riemannian limit, P_{ik} is symmetric. A classification of stationary space-times based on the algebraic properties of the tensor P_{ik} has been given in Ref. 9.

The parametric curvature has the equivalent representation*

$$\Xi_k{}^i = \frac{1}{2\sqrt{2}}i\epsilon^{irs}H_{krs} = \frac{1}{2\sqrt{2}}i\epsilon^{irs}(\nabla_s - \dot{\omega}_s)\dot{g}_{rk}.$$

The tensor $\Xi_k{}^i$ again, as is $P_k{}^i$, is trace-free, $\Xi_k{}^k = 0$.

4.PARAMETRIC SPINORS

The algebraic relations for spinors in three dimensions (Cf. Ref. 10) remain valid in parametric 3-manifolds. We shall work in local sections of the spin bundle, though one can argue[11] that this restriction is unnecessary. We introduce the soldering forms σ in the 3-space, symmetric in their spinor indices

$$\sigma^i_{AB} = \sigma^i_{BA}. \tag{4.1}$$

The soldering forms satisfy the defining relation

$$\sigma^i{}_C{}^A\sigma^j{}_B{}^C + \sigma^j{}_C{}^A\sigma^i{}_B{}^C = g^{ij}\delta^A_B \tag{4.2}$$

* The curvature $P_k{}^i$ is real and $\Xi_k{}^i$ complex when the metric is positive-definite. Due to the presence of a factor $det^{-\frac{1}{2}}[g_{ik}]$ in the tensor ϵ^{ijk}, the converse holds for the metric signature $(+,+,-)$

and their product can be expressed

$$\sigma^{i}_{\ A B}\sigma_{i C D} = \epsilon_{A(C}\epsilon_{D)B} \tag{4.3}$$

where the index parentheses denote symmetrization and

$$(\epsilon_{A B}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.4}$$

is the skew unit spinor.

From (4.2) we get the useful identity

$$\sigma^{i}_{\ A B}\sigma_{k}^{A B} = -\delta^{i}_{k}. \tag{4.5}$$

We have the Lie product rule

$$\sigma_{i A C}\sigma_{j B}^{\ \ C} - \sigma_{j A C}\sigma_{i B}^{\ \ C} = \sqrt{2}i\epsilon_{i j k}\sigma^{k}_{\ A B}. \tag{4.6}$$

This defines an $SU(2)$ or $SU(1,1)$ algebra, depending on the signature of the metric. With the signature $(+,+,-)$, the tensor ϵ_{ijk} is imaginary, and the right-hand side of Eq. (4.6) as well as all spinorial quantities can be made real.

It is still possible to have parameter-dependent soldering forms even when the metric does not depend on any parameter. However, one can introduce the Ashtekar-type gauge condition[18] $M^{[ik]} = \dot{\sigma}^{[i}_{\ A B}\sigma^{k]A B} = 0$ in which there exist relations of proportionality between the parameter derivatives of these quantities. Typically,

$$\dot{\sigma}^{i}_{\ A B} = -\frac{1}{2}\sigma^{k}_{\ A B}g^{ij}\dot{g}_{jk}. \tag{4.7}$$

The covariant derivative of a spinor with respect to the space-time metric g is defined such that the soldering forms and the fundamental spinor are covariantly constant:

$$\nabla_{i}\sigma^{k\ \ B}_{\ A} = 0, \qquad \nabla_{i}\epsilon_{A B} = 0. \tag{4.8}$$

Hence the derivative

$$\nabla_{i}\xi_{A} = \partial_{*i}\xi_{A} - \Gamma_{iA}^{\ \ B}\xi_{B} \tag{4.9}$$

contains a spinor affine connection

$$\Gamma_{iA}^{\ \ B} = -\frac{1}{2}\sigma_{j}^{\ B C}(\sigma^{j}_{\ A C*i} + \sigma^{k}_{\ A C}\Gamma^{j}_{ik}). \tag{4.10}$$

The *spinor Ricci identity* can be obtained by application of the commutator (2.10) on the null vector $v^{i} = \sigma^{i}_{\ A B}\xi^{A}\xi^{B}$:

$$\nabla^{P}_{(A}\nabla_{B)P}\xi_{C} - \frac{1}{\sqrt{2}}\epsilon^{ijk}\sigma_{i A B}\omega_{j*k}\dot{\xi}_{C} = \phi_{C D A B}\xi^{D} - 2\xi_{(A}\epsilon_{B)C}\Lambda \tag{4.11}$$

where the curvature spinor is defined

$$\phi_{A B C D} = \frac{1}{2}P_{A(C D)B} \tag{4.12}$$

with $P_{A B C D} = \sigma^{i}_{\ A B}\sigma^{k}_{\ C D}P_{ik}$ and

$$\Lambda = \frac{1}{24}R. \tag{4.13}$$

The curvature spinor has the algebraic properties

$$\phi_{ABCD} = \phi_{(AB)(CD)}, \qquad \phi_{AB}{}^{AB} = 0. \tag{4.14}$$

The parameter derivative of spinors,

$$\nabla_o \xi_A = \dot{\xi}_A \tag{4.15}$$

has the commutation property

$$\sigma^i_{AB}(\nabla_i \nabla_o - \nabla_o \nabla_i - \dot{\omega}_i \nabla_o)\xi_C = \Xi_{ABCD}\,\xi^D \tag{4.16}$$

where the *spinor parametric curvature*,

$$\Xi_{ABCD} \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2}} i\epsilon^{ijk}\sigma_{kCD}\sigma^l_{AB}H_{lij} \tag{4.17}$$

is symmetric in two pairs of indices: $\Xi_{ABCD} = \Xi_{(AB)(CD)}$.

5.SPACE-TIME AS A PARAMETRIC MANIFOLD

The parametric manifold structure of a 4-dimensional space-time with a Lorentzian metric $\hat{g}$ is induced by a non-null vectorfield a. We shall have to require that the global properties of the set S of integral curves of a are such that it can be given the structure of a parametric manifold. Following Carter and Quintana[12], we define the mapping $\psi : \ \mathcal{M} \to S$ such that for each point $p \epsilon \mathcal{M}$, $\psi(p)$ is the trajectory of a passing through p.

Let φ be a smooth scalar field on S containing also a parameter t. At each a-trajectory of the space-time, φ can still take real values determined by the parameter t. Define the composition $\hat{\varphi} = \varphi \circ \psi$ which is a scalar field on the space-time. Consider the gradient $\nabla_\mu \hat{\varphi}$. This has the orthogonal decomposition with respect to the non-null vectorfield a: $\nabla_\mu \hat{\varphi} = [\delta^\nu_\mu - (a^\rho a_\rho)^{-1} a_\mu a^\nu]\nabla_\nu \hat{\varphi} + (a^\rho a_\rho)^{-1} a_\mu a^\nu \nabla_\nu \hat{\varphi} = \partial_{*\mu}\hat{\varphi} + (a^\rho a_\rho)^{-1} a_\mu \dot{\varphi}$ where $\dot{\phi} = a^\mu \nabla_\mu \hat{\phi}$. Thus we have $\partial_{*\mu}\varphi = (\partial_i \varphi - \omega_i \dot{\varphi},\ 0)$ with $\omega_i = a_i / a^\rho a_\rho$.

Tensor fields on S represent the space-time tensor fields satisfying $a^\mu T^{\lambda\ldots\kappa}_{\mu\ldots\nu} = 0, \ \ldots, \ a_\lambda T^{\lambda\ldots\kappa}_{\mu\ldots\nu} = 0$. The decomposition of tensors will be exemplified by the decomposition of the 4-vector

$$\hat{u} = \{u^i, u_o\}. \tag{5.1}$$

A hat will signify that the entity refers to the 4-space-time whenever such an emphasis is necessary. The invariant parts of $\hat{u}$ are*

$$u_o \stackrel{\text{def}}{=} \hat{u}^\mu a_\mu, \qquad (u^i, 0) \stackrel{\text{def}}{=} (\delta^\mu_\nu - f^{-1} a^\mu a_\nu)\hat{u}^\nu \tag{5.2}$$

where Greek indices $\mu, \nu, \ldots$ range through the values 0, 1, 2 and 3 and we denote the norm of the vector a by

$$f = a^\mu a_\mu = \hat{g}_{oo}. \tag{5.3}$$

Then the 4-metric has the decomposition

$$(\hat{g}_{\mu\nu}) = \begin{pmatrix} -f^{-1}g_{ij} + f\omega_i\omega_j & f\omega_i \\ f\omega_j & f \end{pmatrix} \tag{5.4}$$

* Note that in a local coordinate system adapted to the vectorfield a such that $a = \partial/\partial x^0$ we have $u_o = u_0$

with the inverse

$$(\hat{g}^{\mu\nu}) = \begin{pmatrix} -fg^{ij} & f\omega^i \\ f\omega^j & f^{-1} - f\omega^2 \end{pmatrix} \tag{5.5}$$

where g is the metric of the parametric 3-manifold of trajectories*. The signature of the 3-metric g is positive-definite when the vectorfield a is time-like and is taken to be $(+,-,-)$ in the space-like case. It follows that the components of the vector $\hat{u}$ are

$$\hat{u}^\mu = (u^i, f^{-1}u_o - u^r\omega_r), \qquad \hat{u}_\mu = (-f^{-1}u_i + \omega_i, u_o). \tag{5.6}$$

The scalar product of vectors $\hat{u}$ and $\hat{v}$ has the parametric decomposition

$$\hat{u}^\mu \hat{v}_\mu = f^{-1}(u_o v_o - u^i v^k g_{ik}). \tag{5.7}$$

Let us introduce the vector

$$\mathcal{E}_i = f_{*i} + i\varphi_i \tag{5.8}$$

where

$$\varphi_i \stackrel{\text{def}}{=} \epsilon_{ijk}\omega^{j;k} f^2, \tag{5.9}$$

and $\vec{\varphi}$ satisfies the identity $f\nabla \cdot \vec{\varphi} = (2\nabla f - f\dot{\vec{\omega}}) \cdot \vec{\varphi}$. Note that $\vec{\mathcal{E}}$ is a complex vector, and in a stationary vacuum space-time, the vector $\vec{\mathcal{E}}$ is the gradient of the Ernst potential[13] $\mathcal{E}$. We shall employ the further vector

$$\vec{G} \stackrel{\text{def}}{=} \frac{\vec{\mathcal{E}}}{2f} - \frac{1}{2}\dot{\vec{\omega}}. \tag{5.10}$$

The soldering forms σ satisfy the defining relation[11]

$$\sigma_{\mu AC'}\sigma_{\nu B}{}^{C'} + \sigma_{\nu AC'}\sigma_{\mu B}{}^{C'} = \hat{g}_{\mu\nu}\epsilon_{AB}. \tag{5.11}$$

We now decompose Eq. (5.11). Contraction with $a^\mu a^\nu$ yields the product rule for the $a_{AB'}$ spinors:

$$a_{AC'}a_B{}^{C'} = \frac{1}{2}f\epsilon_{AB} \tag{5.12}$$

where $a_{AB'} = \sigma_{oAB'}$. From the mixed components with $(\mu,\nu) = (i,0)$ we get

$$\sigma^i{}_{AC'}a_B{}^{C'} + a_{AC'}\sigma^{iC'}_B = 0. \tag{5.13}$$

We now introduce the soldering forms in the 3-space by

$$\sigma^i{}_{AB} = -\frac{\sqrt{2}}{f}\sigma^i{}_{AC'}a_B{}^{C'}. \tag{5.14}$$

Eq. (5.13) then expresses the symmetry of the 3-dimensional soldering forms in their spinor indices.

The components of Eq. (5.11) with $(\mu,\nu) = (i,j)$ yield Eq. (4.2). The product of soldering forms in the 4-space-time[11] $\sigma_{\mu AB'}\sigma^\mu{}_{CD'} = \epsilon_{AC}\epsilon_{B'D'}$ when decomposed according to the scheme (5.7) of scalar products yields the product relation (4.3) in the 3-space.

---

* The 3-metric in Zelmanov's paper[7] is related to g by a conformal map

6.ADJUNCTION

The spinor $a_{AB'}$ establishes, in a natural way, a map between the primed and unprimed spin spaces. We introduce the *adjoint spinor* $\xi^{\dagger A}$ by[10]

$$\xi^{\dagger A} \stackrel{\text{def}}{=} \sqrt{2/|f|}a^{AB'}\bar{\xi}_{B'}. \tag{6.1}$$

In effect, the spinor $a_A{}^{B'}$ annihilates a primed index and creates an unprimed index. Application of Eq. (5.12) yields for the double adjoint

$$(\xi^\dagger)^{\dagger A} = \begin{cases} -\xi^A & \text{if } f > 0 \ \ \textit{(time-like a)}, \\ \xi^A & \text{if } f < 0 \ \ \textit{(space-like a)}. \end{cases} \tag{6.2}$$

The covariant derivation commutes with adjunction.

Spinors may completely be freed from their primed indices by use of the spinor $a_{AB'}$. The 'unpriming' of spinors consists in an indexwise application of Eq. (6.1). The 3+1 decomposition of world tensors turns into *symmetrization* in the spinor picture. For example, the vector $\hat{v}^\mu$ can be rewritten, using Eq. (5.7),

$$\hat{v}_{AB'} = \hat{v}^\mu \sigma_{\mu AB'} = f^{-1}(v_o\sigma_{oAB'} - v^i\sigma^k_{AB'}g_{ik}). \tag{6.3}$$

The unprimed version, $\hat{v}_{AB} = \sqrt{\frac{2}{|f|}}a_B{}^{B'}\hat{v}_{AB'}$ has the irreducible parts

$$\hat{v}_{(AB)} = v^i\sigma_{iAB}, \qquad \hat{v}_{[AB]} = \epsilon_{AB}v_o. \tag{6.4}$$

The complex conjugation of scalar products proceeds as follows,

$$\begin{aligned}\overline{(\xi_A\eta^A)} &= \bar{\xi}_{A'}\bar{\eta}^{A'} = \frac{2}{f}\bar{\xi}_{A'}a_C{}^{A'}a^{CB'}\bar{\eta}_{B'} \\ &= \pm\, \xi^\dagger_A\eta^{\dagger A}\end{aligned} \tag{6.5}$$

where the upper and lower signs hold for $f > 0$ and $f < 0$, respectively. In particular, $\overline{\xi^\dagger_A\eta^A} = \pm\xi^{\dagger\dagger}_A\eta^{\dagger A} = \xi^A\eta^\dagger_A$. Hence it follows that the norm of the spinor ξ defined by $||\xi||^2 = \xi^{\dagger A}\xi_A$ is real. The Hermiticity of the soldering forms reads $\sigma^{i\dagger}_{A\ B} = \mp\sigma^i_{AB}$. Although the timelike case carries here a negative sign, this is exactly the kind of behavior a Hermitian product $\sigma_{AB} = \xi_{(A}\xi^\dagger_{B)}$ exhibits.

7.SPINOR DERIVATIVES

The covariant derivative of a spinor with respect to the space-time metric $\hat{g}$ is defined in the contemporary literature such that the soldering forms and the fundamental spinor are covariantly constant:

$$\hat{\nabla}_\nu\sigma^\mu_{AB'} = 0, \qquad \hat{\nabla}_\mu\epsilon_{AB} = 0. \tag{7.1}$$

Hence the spinor affine connection $\hat{\Gamma}^{\ C}_{\mu A}$ in the covariant derivative

$$\hat{\nabla}_\mu\xi_A = \xi_{A,\mu} - \hat{\Gamma}^{\ C}_{\mu A} \tag{7.2}$$

can be expressed in the form[13]

$$\hat{\Gamma}^{\ B}_{\mu A} = \frac{1}{2}\sigma_\alpha{}^{BF'}(\sigma^\beta_{AF'}\hat{\Gamma}^\alpha_{\mu\beta} + \sigma^\alpha_{AF',\mu}). \tag{7.3}$$

One can decompose the connection $\hat{\Gamma}$ in terms of Γ. Introduce the spinor $\mathcal{E}_A^B = \mathcal{E}_i \sigma^i{}_A{}^B$. Straightforward computation yields[15]

$$
\begin{aligned}
\hat{\Gamma}_{oA}^{\ \ B} &= -\frac{1}{2\sqrt{2}}(\mathcal{E}_A^B + \Omega_A^B) \\
\hat{\Gamma}_{iA}^{\ \ B} &= \Gamma_{iA}^{\ \ B} - \frac{1}{2\sqrt{2}}\omega_i(\mathcal{E}_A^B + \Omega_A^B) \\
&\quad - i\frac{1}{2\sqrt{2}}\epsilon_{ijk}\frac{\mathcal{E}^j}{f}\sigma^k{}_A{}^B - \frac{1}{\sqrt{2}}H_{ij}\sigma^j{}_A{}^B
\end{aligned}
\tag{7.4}
$$

where*

$$
\Omega_A^B = -2\sigma^j{}_A{}^B f\dot{\omega}_j \qquad H_{ij} = \frac{1}{2}(\dot{g}_{ij} - g_{ij}\frac{\dot{f}}{f})f^{-1}
\tag{7.5}
$$

The spinor derivative operator can be decomposed by use of (6.3)

$$
\hat{\nabla}_{AB'} = \sigma^i_{AB'}\hat{\nabla}_{*i} + f^{-1}a_{AB'}\hat{\nabla}_o
\tag{7.6}
$$

where

$$
\hat{\nabla}_{*i} \stackrel{\text{def}}{=} \hat{\nabla}_i - \omega_i\hat{\nabla}_o .
\tag{7.7}
$$

The Lorentzian and parametric covariant derivatives of spinors are related by

$$
\hat{\nabla}_{*i}\xi_A = \nabla_i\xi_A - \Pi_{iA}{}^B\xi_B
\tag{7.8}
$$

$$
\hat{\nabla}_o\xi_A = \dot{\xi}_A - \Theta_A{}^B\xi_B
\tag{7.9}
$$

Introducing the complex vectors

$$
G_i^{(\pm)} \stackrel{\text{def}}{=} G_i \pm \frac{1}{2}\dot{\omega}_i ,
\tag{7.10}
$$

the connections $\Pi_{iAB} = \Pi_{ik}\sigma^k_{AB}$ and $\Theta_{AB} = \Theta_i\sigma^i_{AB}$ have the form

$$
\Pi_{ik} = -\frac{1}{\sqrt{2}}(i\epsilon_{ijk}G^{(+)j} + H_{ik}), \qquad \Theta_i = -\frac{1}{\sqrt{2}}fG_i^{(-)} .
\tag{7.11}
$$

The Riemannian limit ($\vec{\omega} \to 0$) of the connection (7.11) has been considered by Sen[16].

The covariant derivative of the spinor $a_{BP'}$ can be computed as follows,

$$
a_{(A}^{P'}\hat{\nabla}_o a_{B)P'} = \frac{1}{2\sqrt{2}}f^2\sigma^i_{AB}(\frac{f_{*i}}{f} - 2\dot{\omega}_i)
\tag{7.12}
$$

$$
\sigma^{iP'}_{(A}\hat{\nabla}_{*i}a_{B)P'} = \frac{1}{2\sqrt{2}}\sigma^i_{AB}(f_{*i} + 2i\varphi_i).
\tag{7.13}
$$

Using this information, we can rewrite the spinor operator $\hat{\nabla}_{(A}^{P'}\hat{\nabla}_{B)P'}$ in the form

$$
\begin{aligned}
\hat{\nabla}_{(A}^{P'}\hat{\nabla}_{B)P'} = -\frac{1}{\sqrt{2}}\sigma^k_{AB}\big[&(\hat{\nabla}_o\hat{\nabla}_{*k} - \hat{\nabla}_{*k}\hat{\nabla}_o + \dot{\omega}_k\hat{\nabla}_o) \\
&+ \frac{1}{2}i\epsilon^{ij}{}_k f(\hat{\nabla}_{*i}\hat{\nabla}_{*j} - \hat{\nabla}_{*j}\hat{\nabla}_{*i} + 2\omega_{j*i}\hat{\nabla}_o)\big]
\end{aligned}
\tag{7.14}
$$

* The symmetric tensor H_{ik} should not be confused with the parametric curvature H_{ijk}

which will be utilized in the next section when decomposing the Ricci identity.

8. DECOMPOSITION OF CURVATURE

In this section we obtain the parametric decomposition of the spinor Ricci and Bianchi identities. First we produce the parametric form of the space-time curvature by use of the Ricci identities[14]

$$\hat{\nabla}_{(A}^{\ \ P'} \hat{\nabla}_{B)P'} \xi^B = 3\hat{\Lambda}\xi_A \tag{8.1}$$

$$\hat{\nabla}_{C(P'} \hat{\nabla}_{Q')}^{C} \xi_A = \hat{\Phi}_{ABP'Q'} \xi^B \tag{8.2}$$

$$\hat{\nabla}_{P'(A} \hat{\nabla}_{B}^{\ \ P'} \xi_{C)} = \Psi_{ABCD}\, \xi^D \tag{8.3}$$

where $\hat{\Lambda} = \hat{R}/24$ and

$$\hat{\Phi}_{ABP'Q'} = -\frac{1}{2}(\hat{R}_{\mu\nu} - \frac{1}{4}\hat{g}_{\mu\nu}\hat{R})\sigma^{\mu}_{AP'}\sigma^{\nu}_{BQ'} \tag{8.4}$$

and Ψ_{ABCD} are the Ricci and Weyl spinors, respectively. Substitution of the connections (7.8) and (7.9) and use of the commutators [Eqs.(4.11) and (4.16)] with

$$\nabla_{P(A} \nabla_{B)}^{P} \xi_C = \frac{1}{2\sqrt{2}} i\epsilon^{ikl} \sigma_{lAB} (\nabla_i \nabla_k - \nabla_k \nabla_i)\xi_C \tag{8.5}$$

yields

$$(\hat{\nabla}_o \hat{\nabla}_{*k} - \hat{\nabla}_{*k}\hat{\nabla}_o + \dot{\omega}_k \hat{\nabla}_o)\xi_A = -(\Xi_{kAB} + E_{kAB})\xi^B \tag{8.6}$$

$$\begin{aligned} &\frac{1}{2} i\epsilon^{ijk}\sigma_{kAB}(\hat{\nabla}_{*i}\hat{\nabla}_{*j} - \hat{\nabla}_{*j}\hat{\nabla}_{*i} + 2\omega_{j*i}\hat{\nabla}_o)\xi_C \\ &\qquad = \sqrt{2}(-\phi_{CDAB}\,\xi^D + 2\xi_{(A}\epsilon_{B)C}\Lambda) + iF_{ABCD}\,\xi^D \end{aligned} \tag{8.7}$$

where

$$E_{kAB} = \sigma^{i}_{\ AB} E_{ki}, \qquad F_{ABCD} = \sigma^{i}_{\ AB}\sigma^{k}_{CD} F_{ik}, \qquad \Xi_{kAB} = \sigma^{i}_{\ AB}\Xi_{ki}$$

and

$$E_{ik} = \nabla_i \Theta_k - \dot{\Pi}_{ik} + \frac{1}{2}\Pi_i^{\ j}\dot{g}_{jk} - \dot{\omega}_i\Theta_k + \sqrt{2}i\Pi_i^{\ j}\epsilon_{kjl}\Theta^l \tag{8.8}$$

$$\begin{aligned} F_{ik} &= \epsilon_{ijl}(\nabla^j \Pi^l_{\ k} + \omega^{l*j}\Theta_k) \\ &\quad + \sqrt{2}i(\Pi_{ri}\Pi_k^{\ r} - \Pi_r^{\ r}\Pi_{ki}) - \frac{1}{\sqrt{2}} i g_{ik}(\Pi_s^{\ r}\Pi_r^{\ s} - \Pi_r^{\ r}\Pi_s^{\ s}). \end{aligned} \tag{8.9}$$

The unprimed version of the Ricci spinor,

$$\hat{\Phi}_{ABCD} \stackrel{\text{def}}{=} \frac{2}{f} a_C^{\ C'} a_D^{\ D'} \hat{\Phi}_{ABC'D'} \tag{8.10}$$

is easily written in a 3+1 fashion by repeated use of Eq. (6.3):

$$\begin{aligned} \hat{\Phi}_{ABCD} &= -f^{-1}\Big[\frac{1}{4}\epsilon_{AC}\epsilon_{BD}\hat{R}_{oo} + \frac{1}{2\sqrt{2}}(\epsilon_{AC}\sigma^{i}_{\ BD} + \epsilon_{BD}\sigma^{i}_{\ AC})\hat{R}^k_o g_{ik} \\ &\qquad + \frac{1}{2}\sigma^{i}_{\ AC}\sigma^{k}_{BD}\hat{R}^{jl}g_{ij}g_{kl}\Big] + 3\epsilon_{AB}\epsilon_{CD}\hat{\Lambda}. \end{aligned} \tag{8.11}$$

The tensor part is projected out by contraction with $\sigma^{AB}_{(a}\sigma^{CD}_{b)}$ and using the relations

$$\sigma^{AB}_{(a}\sigma^{CD}_{b)}\sigma^{(i}_{AC}\sigma^{k)}_{BD} = \delta^{(i}_a\delta^{k)}_b - \frac{1}{2}g^{ik}g_{ab} \tag{8.12}$$

and

$$24\hat{\Lambda} = f^{-1}(\hat{R}_{oo} - \hat{R}^{ab}g_{ab}). \tag{8.13}$$

We get

$$\sigma^{AB}_{(a}\sigma^{CD}_{b)}(\hat{\Phi}_{(ABCD)} - 6\epsilon_{AC}\epsilon_{BD}\hat{\Lambda}) = -\frac{1}{2f}(g_{aj}g_{bl}\hat{R}^{jl} - g_{ab}\hat{R}_{oo}). \tag{8.14}$$

Further irreducible parts are

$$\hat{\Phi}_{B(A}{}^{B}{}_{D)} = -\frac{1}{\sqrt{2}f}\sigma_{iAD}\hat{R}^{i}_{o} \tag{8.15}$$

$$\hat{\Phi}_{AB}{}^{AB} - 6\hat{\Lambda} = -f^{-1}\hat{R}_{oo}. \tag{8.16}$$

Employing Eq. (7.14) for the operator $\hat{\nabla}^{P'}_{(A}\hat{\nabla}_{B)P'}$ in the Ricci identity (8.1), and finally removing the arbitrary spinor ξ_A, we obtain

$$3\hat{\Lambda}\epsilon_{CA} = \frac{1}{\sqrt{2}}\sigma^{i}_{AB}\sigma^{kB}_{C}(\Xi_{ik} + E_{ik} - ifF_{ik}) + f(\phi_{CB}{}^{B}{}_{A} - 3\Lambda\epsilon_{CA}). \tag{8.17}$$

Symmetrization in the spinor indices gives the identity involving the skew part of the parametric curvature

$$\Xi_{[ik]} + E_{[ik]} - ifF_{[ik]} + \frac{1}{\sqrt{2}}fP_{[ik]} = 0. \tag{8.18}$$

From (8.17) we get the curvature scalar

$$3\hat{\Lambda} = -3f\Lambda + \frac{1}{2\sqrt{2}}g^{ik}(E_{ik} - ifF_{ik}). \tag{8.19}$$

Eq. (8.2) yields the decomposition of the Ricci tensor:

$$f^{-1}\hat{R}_{oo} = \sqrt{2}g^{ik}E_{ik} \tag{8.20}$$

$$\frac{1}{\sqrt{2}f}\sigma_{iAB}\hat{R}^{i}_{o} = \frac{1}{2}i\sigma_{kAB}\epsilon^{ijk}(\Xi_{ij} + E_{ij} + ifF_{ij}) - f\phi_{P(AB)}{}^{P} \tag{8.21}$$

$$\begin{aligned} f^{-1}(g_{ij}g_{kl}\hat{R}^{jl} - g_{ik}\hat{R}_{oo}) = {} & fR_{ik} - \sqrt{2}(\Xi_{(ik)} + E_{(ik)} + ifF_{(ik)}) \\ & - \sqrt{2}g_{ik}g^{ab}(E_{ab} - ifF_{ab}). \end{aligned} \tag{8.22}$$

Similarly, Eq. (8.3) provides us with the decomposition of the Weyl curvature:

$$\Psi_{ABCD} = -\frac{1}{\sqrt{2}}\sigma^{i}_{(AB}\sigma^{k}_{CD)}Q_{ik} \tag{8.23}$$

where the symmetric and traceless tensor Q_{ik} is defined

$$\begin{aligned} Q_{ik} = {} & \Xi_{(ik)} + E_{(ik)} - ifF_{(ik)} - \frac{1}{\sqrt{2}}fR_{ik} \\ & - \frac{1}{3}g_{ik}g^{ab}(E_{ab} - ifF_{ab} - \frac{1}{\sqrt{2}}fR_{ab}). \end{aligned} \tag{8.24}$$

The Petrov classification of gravitational fields is nothing but the eigenvalue problem of the tensor Q_{ab}.

The spinor form of the Bianchi identities has the irreducible parts[14]

$$\hat{\nabla}_{B'}^{A}\Psi_{ABCD} = \hat{\nabla}_{(B}^{A'}\hat{\Phi}_{CD)A'B'} \tag{8.25}$$

$$\hat{\nabla}^{CA'}\hat{\Phi}_{CDA'B'} + 3\hat{\nabla}_{DB'}\hat{\Lambda} = 0. \tag{8.26}$$

Consider first Eq. (8.25) describing the propagation of the gravitational degrees of freedom. The left hand side contains the derivative of the gravitational spinor Ψ_{ABCD}:

$$X_{P'BCD} = \hat{\nabla}_{P'}^{A}\Psi_{ABCD}. \tag{8.27}$$

Application of Eq. (7.6) and unpriming yields

$$\begin{aligned} -\frac{\sqrt{2}}{f}a_P^{P'}X_{P'BCD} &= \sigma_P^{i\ A}(\nabla_i\Psi_{ABCD} - 4\Pi_{i(A}{}^{R}\Psi_{|R|BCD)}) \\ &+ \frac{1}{\sqrt{2}f}(\nabla_o\Psi_{PBCD} - 4\Theta_{(P}^{\ R}\Psi_{|R|BCD)}). \end{aligned} \tag{8.28}$$

Upon transvecting with ϵ^{PB} we get the divergence equation

$$\nabla_a Q_b^a - \sqrt{2}i\epsilon^{akl}\Pi_{lk}Q_{ab} - \sqrt{2}i\epsilon^{ak}{}_b\Pi^l{}_kQ_{al} = \frac{\sqrt{2}}{f}a_P^{P'}X_{P'}{}^{P}{}_{b}. \tag{8.29}$$

Symmetrization of Eq. (8.28) in the spinor indices yields the evolution equation for the tensor Q_{ab}:

$$\begin{aligned} &\dot{Q}_{ab} - \dot{g}_{ar}Q_b^r + if\epsilon_{ik(a}(\nabla^i - 2\sqrt{2}f^{-1}\Theta^i)Q_{b)}^k + \sqrt{2}f(2\Pi^r{}_{(a}Q_{b)r} \\ &\quad +\Pi_{r(a}Q_{b)}^r - 2\Pi_r^rQ_{ab} - g_{ab}\Pi_{rs}Q^{rs}) = 2a_{P'(P}X^{P'}{}_{BCD)}\sigma_a^{PB}\sigma_b^{CD}. \end{aligned} \tag{8.30}$$

The second set (8.26) of the Bianchi identities expresses the condition that the Einstein tensor is divergence-free. The curvature spinors $\hat{\Phi}_{ABC'D'}$ and $\hat{\Lambda}$ are locally determined, via the Einstein equations, by the energy-momentum distribution of matter. Upon decomposition, Eq. (8.26) gives rise to the identity (2.17) involving the parametric curvature.

9.PARAMETRIC FORM OF GRAVITATIONAL EQUATIONS

Insertion of the detailed form (7.11) of the connections into the tensors E_{ik} and F_{ik} [Eqs. (8.8) and (8.9)] yields

$$\begin{aligned} E_{ik} = -\dot{\Pi}_{ik} + \frac{1}{2}\Pi_i{}^j\dot{g}_{jk} + \frac{1}{\sqrt{2}}f[-(\nabla_i + G_i^{(+)} + \bar{G}_i^{(-)})G_k^{(-)} \\ - G_i^{(-)}G_k^{(+)} + g_{ik}G_r^{(-)}G^{(+)r} - i\epsilon_k{}^{rs}G_r^{(-)}H_{is}] \end{aligned} \tag{9.1}$$

where

$$\begin{aligned} -\dot{\Pi}_{ik} + \frac{1}{2}\Pi_i{}^j\dot{g}_{jk} = \frac{1}{\sqrt{2}}\Big[\dot{H}_{ik} - \frac{1}{2}\frac{\dot{f}}{f}H_{ik} \\ + i\epsilon_{ijk}(\dot{G}^{(+)j} + fG^{(+)j}H) - f(H_{ir} + i\epsilon_{ijr}G^{(+)j})H_k^r\Big], \end{aligned} \tag{9.2}$$

$H = H^r_r$, and

$$\sqrt{2}F_{ik} = -\epsilon_{ijl}\nabla^j H^l{}_k + i\Big[g_{ik}\nabla^j G^{(+)}_j - \nabla_k G^{(+)}_i + (\bar{G}^{(-)}_i - G^{(-)}_i)G^{(-)}_k$$
$$- G^{(+)}_i G^{(+)}_k + \frac{1}{2}g_{ik}(H^2 - H^r_s H^s_r) + H_{ir}H^r_k - H_{ik}H \tag{9.3}$$
$$- i\epsilon_{irs}G^{(+)r}H^s_k + i\epsilon_{krs}G^{(+)r}H^s_i + i\epsilon_{ijk}G^{(+)j}H\Big].$$

We now use these expressions and the parametric curvature

$$\Xi_{ik} = \frac{1}{\sqrt{2}}i\epsilon_i{}^{rs}(\nabla_r - G^{(+)}_r + G^{(-)}_r)(2fH_{sk} + g_{sk}\frac{\dot{f}}{f}) \tag{9.4}$$

to obtain the decomposition of the Ricci tensor

$$f^{-1}\hat{R}_{oo} = \dot{H} + \frac{1}{2}\frac{\dot{f}}{f}H + f[H^{ik}H_{ik} - (\nabla^i + \bar{G}^{(-)i})G^{(-)}_i] \tag{9.5}$$

$$\begin{aligned} f^{-2}\hat{R}^i_o &= -\sqrt{2}\epsilon^{ijk}F_{jk} \\ &= \nabla^k H^i{}_k - \nabla^i H + i\epsilon^{ijk}(\nabla_k G^{(+)}_j + \bar{G}^{(-)}_k G^{(-)}_j) - 2G^{(+)k}H_k{}^i \end{aligned} \tag{9.6}$$

$$\begin{aligned} f^{-1}(g_{ij}g_{kl}\hat{R}^{jl} - g_{ik}\hat{R}_{oo}) &= fR_{ik} - \dot{H}_{ik} - g_{ik}\dot{H} + \frac{1}{2}\frac{\dot{f}}{f}(H_{ik} - g_{ik}H) \\ &- f\Big[(\nabla_{(i} - G^{(-)}_{(i} + G^{(+)}_{(i})(G^{(+)}_{k)} - G^{(-)}_{k)}) - 2G^{(-)}_{(i}\bar{G}^{(-)}_{k)} \\ &+ i\epsilon_{(i}{}^{rs}H_{k)s}(\bar{G}^{(-)}_r - G^{(-)}_r) - 2H_{ir}H^r_k + H_{ik}H\Big] \\ &- g_{ik}f\Big[(\nabla + \vec{G}^{(-)} - \vec{G}^{(+)})\cdot(\vec{G}^{(+)} - \vec{G}^{(-)}) + H_{rs}H^{rs}\Big] \end{aligned} \tag{9.7}$$

and the curvature scalar

$$\begin{aligned} \hat{R} &= -fR + 2\dot{H} + \frac{\dot{f}}{f}H + f(H^2 - H_{rs}H^{rs}) \\ &+ 2f\left[2\nabla\cdot\vec{G}^{(+)} - \nabla\cdot\vec{G}^{(-)} + \vec{G}^{(+)}\cdot\vec{G}^{(-)} - G^{(-)2} - G^{(+)2}\right]. \end{aligned} \tag{9.8}$$

It is sometimes advantageous to introduce the quantities[1]

$$\pi_{ik} \stackrel{\text{def}}{=} H_{ik} - g_{ik}H; \qquad \pi = \pi^r_r. \tag{9.9}$$

In this notation, the Ricci tensor can be written

$$\begin{aligned} f^{-1}\hat{R}_{oo} &= -\frac{1}{2}\dot{\pi} - \frac{1}{4}\frac{\dot{f}}{f}\pi \\ &+ f[\pi^{ik}\pi_{ik} - \frac{1}{4}\pi^2 - (\nabla - \vec{G}^{(+)} + \vec{\bar{G}}^{(-)})\cdot\vec{G}^{(-)}] \end{aligned} \tag{9.10}$$

$$\begin{aligned} -f^{-2}\hat{R}^i_o &= -\nabla^k\pi^i{}_k \\ &+ i\epsilon^{ijk}(\nabla_j G^{(+)}_k + \bar{G}^{(-)}_j G^{(-)}_k) + 2G^{(+)k}\pi_k{}^i - G^{(+)i}\pi \end{aligned} \tag{9.11}$$

$$
\begin{aligned}
f^{-2}(g_{ij}g_{kl}\hat{R}^{jl} - g_{ik}\hat{R}_{oo}) &= R_{ik} - f^{-1}(\dot{\pi}_{ik} - \frac{\dot{f}}{2f}\pi_{ik}) + 2\pi_{ir}\pi^r_k \\
&- (\nabla_{(i} - G^{(-)}_{(i} + G^{(+)}_{(i})(G^{(+)}_{k)} - G^{(-)}_{k)}) - i\epsilon_{(i}{}^{rs}\pi_{k)s}(\bar{G}^{(-)}_r - G^{(-)}_r) \\
&+ 2G^{(-)}_{(i}\bar{G}^{(-)}_{k)} - g_{ik}\big[(\nabla + \vec{G}^{(-)} - \vec{G}^{(+)})\cdot(\vec{G}^{(+)} - \vec{G}^{(-)}) + \frac{1}{2}\pi^2\big],
\end{aligned} \tag{9.12}
$$

and the Ricci scalar takes the form

$$
\begin{aligned}
\hat{R} &= -fR - \dot{\pi} - \frac{\dot{f}}{2f}\pi + f\pi_{rs}\pi^{rs} \\
&+ 2f\big[2\nabla\cdot\vec{G}^{(+)} - \nabla\cdot\vec{G}^{(-)} + \vec{G}^{(+)}\cdot\vec{G}^{(-)} - G^{(-)^2} - G^{(+)^2}\big].
\end{aligned} \tag{9.13}
$$

In *vacuo*, $\hat{R}_{\mu\nu} = 0$, the stationary limit is[10]

$$(\nabla - \vec{G} + \bar{\vec{G}})\cdot\vec{G} = 0 \tag{9.14}$$

$$(\nabla + \bar{\vec{G}})\times\vec{G} = 0 \tag{9.15}$$

$$R_{ik} + G_i\bar{G}_k + \bar{G}_iG_k = 0. \tag{9.16}$$

In the limit $\vec{G}^{(\pm)} \to 0$, we obtain the *Hamiltonian constraint*[1]

$$R = -\pi_{ik}\pi^{ik} + \frac{1}{2}\pi^2, \tag{9.17}$$

the *momentum constraint*

$$\nabla_k\pi^k_i = 0, \tag{9.18}$$

and the 'evolution equations'

$$\dot{\pi} = 2\pi_{ik}\pi^{ik} - \frac{1}{2}\pi^2 \tag{9.19}$$

$$R_{ik} = \dot{\pi}_{ik} - 2\pi_{ir}\pi^r_k + \frac{1}{2}g_{ik}\pi^2. \tag{9.20}$$

10. CONCLUDING REMARKS

Clearly much remains to be done to bear out the full flexibility of the parametric formulation of gravity in applications. The structure of the vacuum Einstein field equations $\hat{R}_{\mu\nu} = 0$, Eqs. (9.10)-(9.12), and Bianchi identities (8.29)-(8.30) suggests that a semi-quantized theory of gravitation is possible where the Weyl curvature alone is operator-valued. According to the the approach known as quantum field theory on a fixed background[17], the Einstein field equations contain the vacuum expectation value of the energy-momentum tensor: $\hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} = -k\,\langle T_{\mu\nu}\rangle$. In a semi-quantized theory of gravitation, the separation of fields into quantized versus 'external' components would be modified such that the curvature tensor Q_{ik} is treated as quantum field. (Note that the Bianchi identities are linear in Q_{ik}.)

While it is often stressed that the true dynamical degrees of freedom of the gravitational field are associated with the Weyl curvature, still in many attempts at the canonical quantization, the dynamical variables are chosen to be the induced metric of hypersurfaces or the 'SU(2) soldering forms'[18]. The parametric decomposition of the field equations might help clarify such issues.

The modalities offered by the parametric formalism for gauge fixing may involve the non-null congruence whose tangent is a, as well as the re-parametrization and coordinate transformations. It is possible, for instance, to adopt the *eigenray condition* first introduced for stationary space-times[19]. The eigenrays are integral curves to the principal null directions directions of the spinor $G_{AB} = \hat{\nabla}^{B'}_{(A} a_{B)B'}$. The spinor G_{AB} determines the vectorfield a up to an arbitrary gradient term, and further conditions should be imposed to fix the vectorfield a completely.

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REFERENCES

[1] Arnowitt, R., Deser, S. and Misner, C. W., in *Gravitation*, Ed. Witten, L. (Wiley, 1962)

[2] Gödel, K., Rev. Mod. Phys. **21**, 447 (1949)

[3] Ehlers, J., in: *Les théories relativistes de la gravitation* (CNRS, Paris, 1959)

[4] Schouten, J. A.: Ricci-calculus (Springer, Berlin, 1954)

[5] Stachel, J.: Congruences of subspaces, in: *Gravitation and Geometry*, Eds. Rindler, W. and Trautman, A. (Bibliopolis, 1987)

[6] Lottermoser, M.: Über den Newtonschen Grenzwert der Allgemeinen Relativitätstheorie und die relativistische Erweiterung Newtonscher Anfangsdaten (Dissertation, Munich, 1988)

[7] Zelmanov, A. L.: K relativistskoj teorii anizotropnoj nieodnorodnoj vselennoj, in: *Trudi shestovo soveshanija po voprosam kosmologii*, p. 144, Acad. Publ. USSR, 1959,

[8] Zelmanov, A. L., Dokl. Akad. USSR **107**, 815 (1956)

[9] Perjés, Z., Int. J. Theor. Phys. **10**, 217 (1974), Perjés, Z., Commun. Math. Phys. **12**, 275 (1969)

[10] Perjés, Z., J. Math. Phys. **11**, 3383 (1970)

[11] Penrose, R.: Structure of space-time, in: *Battelle Rencontres*, Eds. DeWitt, C. M. and Wheeler, J. A. (Benjamin, 1968)

[12] Carter, B. and Quintana, H., Proc. Roy. Soc. **331**, 57 (1972)

[13] Ernst, F. J., Phys. Rev. **167**, 1175 (1968)

[14] Pirani, F. A. E., in: *Lectures in General Relativity* Brandeis Summer Institute in Theoretical Physics (Prentice-Hall, Englewood Cliffs, N. J. 1964)

[15] Perjés, Z., unpublished (1971)

[16] Sen, A., J. Math. Phys. **22**, 1718 (1981)

[17] Isham, C., in: *Quantum Gravity*, Eds. Isham, C., Penrose, R. and Sciama, D., Oxford University Press, 1984

[18] Ashtekar, A., Phys. Rev. **D36**, 1587 (1987)

[19] Perjés, Z., Dissertation, Magyar Fizikai Folyóirat **XXIV**, 173 (1976)

GRAVITY AS AN SO(3,2) GAUGE THEORY

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Abstract

EINSTEIN–CARTAN-gravity can be viewed as a gauge theory with the gauge group SO(3,2), where a suitable quadratic lagrangian leads to the EINSTEIN-equations with cosmological term. Using the soldering of KOBAYASHI we show how the theory is realized in a CLIFFORD-algebrabundle.

1. Introduction

It is a paradigma of modern physics that the fundamental interactions can be described by renormalizable quantum gauge theories. This is supported by the success of QED and the WEINBERG–SALAM-theory and by the fact that QCD is a good candidate to deal with the strong interaction. EINSTEIN's general relativity is also a gauge theory, which gives a classical description of gravity. This was not clear from the beginning as EINSTEIN used to work only in a fixed gauge. (For a description of the EINSTEIN-gauge see [7].) The gauge group in general relativity is SO(3,1), where due to É. CARTAN, the dynamical variable is the vierbein, or Gl(4) with a symmetry breaking down to SO(3,1). Also larger gauge groups have been considered ([9], [21], [22], [19]).

Beside these there are atempts to describe gravity as a gauge theory of the DESITTER-group SO(4,1) or anti-DESITTER-group SO(3,2) ([9], [28], [26], [14]), see also ([20]), where Sp(4) was used. Here some quadratic lagrangians for gravity are proposed ([11], [23]), which nevertheless yield the usual EINSTEIN–CARTAN-theory. In this paper we consider an SO(3,2) gauge theory with a subgroup SO(3,1).

We will consider a CLIFFORD-bundle, $\mathrm{Cliff}^{(3,1)}M$, as the fundamental geometric arena of the theory, since on a suitable, which means parallelizable ([6]), fourmanifold the SO(3,2)- and SO(3,1)-bundle, as well as the tangent bundle TM, can be imbedded in such a CLIFFORD-bundle ([1], [3]). Here spin-$^1/_2$-matter fields can be added to the theory as sections of an associated vectorbundle, and it is possible to build the CARTAN-connection and its curvature ([15]) as CLIFFORD-bundle valued quantities. We write here the langrangian $\mathfrak{L}=\mathrm{str}\Omega_c^2$, where Ω_c is the curvature of the CARTAN-connection (yielding the torsion and the SO(3,1) curvature) and str is a supertrace on the CLIFFORD-algebra viewed as a graded algebra.

This paper is organized as follows: In section 2 we will give, just to fix notation, a short résumé of the mathematics used. The reader should be familiar with the calculus of vectorbundle valued differential forms. The notion of soldering is given in section 3 where we define the soldering form and soldering curvature. In section 4 the lagrangian is postulated and the EINSTEIN-CARTAN-theory recovered.

2. Some Mathematical Preliminaries

A principal bundle with group G and projection π: P $\longrightarrow$ M will be denoted by ξ = (P, π, M; G) and a bundle associated to ξ with typical fibre Y, carrying a free representation ρ of G, will be denoted by ξ_Y = (E, π', M; G; Y), with total space

$$E = {}^{(P \times Y)}/_G \; . \qquad (1)$$

A section σ in a bundle is a mapping σ: M $\longrightarrow$ P with $\pi \circ \sigma = \mathrm{Id}_M$; a map f: P $\longrightarrow$ Y is called equivariant, or ρ-equivariant, if

$$f \circ R_g = \rho(g^{-1}) \circ f \; .$$

Proposition 2.1 [23] There is a bijection between the equivariant maps f: P $\longrightarrow$ Y and the sections of ξ_Y.

The space of k-forms on M with values in the vectorbundle ξ_V is denoted by $\Omega^k(\xi_V)$. The space of k-forms on P which are ρ-equivariant and vectorspace-valued is $\Omega^k_\rho(P,V)$. A k-form on P is called horizontal if it vanishes on vertical vectors. These forms lie in $\overline{\Omega^k_\rho(P,V)}$.

The following proposition shows that there are two equivalent ways to do the calculations in a gauge theory: by means of the vectorbundle-valued forms on M or of the horizontal equivariant forms on P.

Proposition 2.2 [24] There is an isomorphism $\Omega^k(\xi_V) \simeq \overline{\Omega^k_\rho(P,V)}$.

We denote a principal connection form $\omega \in \Omega^1_{Ad}(P,\mathfrak{g})$ ($\mathfrak{g}$ is the LIE-algebra of G). The subspace of horizontal vectors of TP is defined as the kernel of ω.

With the connection there is the covariant derivative in a vectorbundle, see

[24], and this leads to the notion of the curvature

$$\Omega = D\omega = d\omega + {}^1/_2[\omega,\omega] \qquad (2)$$

and to the BIANCHI-identity

$$D\Omega = 0. \qquad (3)$$

The YANG-MILLS-equations can be found here by varying the YANG-MILLS-action:

$$S_{Y-M} = {}^1/_2\int tr(\Omega \wedge *\Omega)\ {*1} \qquad (4)$$

(For the HODGE-star * see [7].)

Experiments show up to now that fundamental matter fields are always spin-$^1/_2$-fields and lie therefore, stated mathematically, in a representation space of a CLIFFORD-algebra. In the bundle language we say that these fields are sections in a vectorbundle associated with a CLIFFORD-bundle. This CLIFFORD-bundle is an algebrabundle with a CLIFFORD-algebra as typical fibre.

Though in this paper the CLIFFORD-algebrabundle is to some extent a tool to simplify calculations the theoretical idea behind its introduction is to simplify the introduction of spin-$^1/_2$-matterfields in the theory, with the spin-density of the matterfields coupling to the torsion of space-time as in the POINCARÉ-gauge theory [9].

Given a vectorspace V with metric g we will view a CLIFFORD-algebra Cl^g as generated by dim V = n generators $\gamma_i = \gamma(e_i)$ (e_i being a basis of V and γ: $V \longrightarrow Cl^g$ an injective homomorphism) which satisfy the anticommutation relations

$$\{\gamma_i,\gamma_j\} = 2\ g(e_i,e_j). \qquad (5)$$

The CLIFFORD-algebra is then the space of all linear combinations of these n^2 elements of Cl^g:

$$1,\ \gamma_i,\ \gamma_i\gamma_j\ (i<j),\ \gamma_i\gamma_j\gamma_k\ (i<j<k),\ \dots,\ \gamma_1\gamma_2\cdots\gamma_n\ .$$

The last element is called volume element η [3] or, if dim V = 4, γ_5. Note that in contrary to most physics textbooks we have no i in the definition of γ_5!

The canonical imbedding homomorphism γ will no longer be written explicitly and for g = (p,q) we will write $Cl^g =: Cl(p,q)$.

The automorphism α which carries γ_i into $-\gamma_i$ is called the main automorphism of the CLIFFORD-algebra and by this the CLIFFORD-algebra is a $\mathbb{Z}_2$-graded algebra: $Cl^g = Cl^g_0 \oplus Cl^g_1$, where the elements of Cl^g_0 and Cl^g_1 are even, resp. odd under α. In this way we will view Cl^g as a superalgebra and define a supertrace. As the trace vanishes on commutators the supertrace is defined by the property to vanish on graded commutators $[x_i,x_j]_g = x_ix_j - (-1)^{ij}x_jx_i$, $x_i \in Cl^g_i$. We restrict ourselves here to Cl(3,1) which has a representation in $\mathfrak{M}(4,4;\mathbb{C})$, the space of complex 4x4-matrices (e. g. DIRAC-representation). Identifying the CLIFFORD-algebra elements with their representation we have

Lemma 2.3 str: $Cl(3,1) \longrightarrow \mathbb{C}$

$X \longrightarrow tr(\gamma_5 X)$ is a supertrace.

It is easy to realize that this str vanishes on graded commutators.

We mention here that the group Pin(g) has a representation in Cl^g (see [1]) and $Pin(g) \cap Cl_0^g =: Spin(g)$ is the twofold covering of SO(g).

Proposition 2.4 [1], [5] The LIE-algebra $\mathfrak{so}(g)$ of SO(g) (and therefore of its covering groups) lies in Cl_0^g and is generated by the $\frac{1}{2}n(n-1)$ elements $\gamma_i\gamma_j$, $i<j$.

Theorem 2.5 (PAULI) The CLIFFORD-algebra Cl^g has for even n=dim V exactly one equivalence class of irreducible representations, for odd n exanctly two and the dimension of the representation space is dim S = $2^{n/2}$ for even n and dim S = $2^{(n-1)/2}$ for odd n.

There is an interesting interlocking of these representations of CLIFFORD-algebras of ascending dimension, see e. g. [3]. In particular we will need the following proposition which is also proved in [3].

Proposition 2.6 Let p+q=2m+1. There is an algebra-isomorphism

$$i : Cl_0(p,q) \longrightarrow Cl(p,q-1)$$

and there exists a faithful representation $\gamma: Cl_0(p,q) \longrightarrow End\ \mathbb{C}^m$ so that

$$\gamma \circ i^{-1} : Cl(p,q-1) \longrightarrow End\ \mathbb{C}^m$$

is the irreducible representation of the PAULI-theorem.

We now proceed to the notion of CLIFFORD-bundles which is closely related to that of spinstructure. A spinstructure is a principalbundle over the base space M with group Spin(g) and is denoted by $SpinM^g$ (this is the total space of ($SpinM^g$, pr, M; Spin(g))). A CLIFFORD-bundle is an algebrabundle over M where the fibre over $x \in M$ is Cl^g, the CLIFFORD-algebra belonging to the vectorspace T_xM and the metric g of M. As Spin(g) lies in Cl^g a spinstructure clearly exists if a CLIFFORD-bundle over M exists.

Obviously a CLIFFORD-bundle and a spinstructure exists if the frame bundle LM over M is trivial. In this case there is a basis of T_xM in every x : $e_i(x)$ (i=1,...,n), which is continuous in x and we can build the CLIFFORD-algebra via homomorphisms $\gamma(x)(e_i(x)) = \gamma_i(x) \in Cl^g$. These homomorphisms satisfy $\{\gamma_i(x),\gamma_j(x)\} = 2g(e_i(x),e_j(x))$ also continuously in x, so in every $x \in M$ the algebra is defined by (5).

So we get the implications:

$$\text{LM trivial} \Rightarrow \text{CliffM}^g \text{ exists} \Rightarrow \text{SpinM}^g \text{ exists.}$$

For $g = (3,1)$ and M noncompact, GEROCH proved also the converse to be true:

Theorem 2.7 [6] (GEROCH) Let $g = (3,1)$ and M noncompact. The following statements are equivalent:

(i) M is parallelizable

(ii) LM is trivial

(iii) M admits a spinstructure $\text{SpinM}^{(3,1)}$.

3. Soldering

Throughout the rest of the paper we will conventionally use the following notation:

G, H are LIE-groups with H being a subgroup of G, and M is a manifold with $\dim M = \dim {}^G/_H = n$. We build principal bundles over M with structure groups G and H, denoted $\xi = (P, \pi, M; G)$ and $\zeta = (Q, \pi', M; H)$, so that ζ is a a subbundle of ξ in the sense that $Q \subset P$ and $H \subset G$ in every fibre.

We mention:

Proposition 3.1 [12], [24] Let $\xi_{G/H} = (E, pr, M; G; {}^G/_H)$ (E as in (1)) associated to ξ with typical fibre ${}^G/_H$. Then there is a bijection between the principal-H-subbundles of ξ over M and the sections of $\xi_{G/H}$.

Let $\mathfrak{g}$ be the LIE-algebra of G and $\mathfrak{h}$ the one of H than is

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}, \tag{6}$$

definig the vectorspace $\mathfrak{r}$. In general $\mathfrak{r}$ is not a LIE-algebra.

With [16] we call ${}^G/_H$ reductiv if

$$Ad(H)\mathfrak{r} \subseteq \mathfrak{r}. \tag{7}$$

Let ξ, ζ as described above with ${}^G/_H$ reductiv. We call the Ad-equivariant horizontal 1-form θ on Q with values in $\mathfrak{r}$ ($\theta \in \overline{\Omega^1_{Ad}(Q,\mathfrak{r})}$) soldering form if it has everywhere maximal rank.

Now we can also call the $\mathfrak{r}$-bundle valued 1-form on TM which corresponds to θ via proposition 2.2 soldering form. It is clear that $\dim M = \dim \mathfrak{r} = \text{rank}(\theta)$, so these spaces are "soldered" by the form in the sense that forms defined on one of them (e. g. the metric on M) can be transported to the other by θ_* (or ${\theta^{-1}}_*$ which always exists due to the maximal rank).

We will give now two examples of reductive structures where soldering occurs. The first one is the standard example (see [24]), the second will be used in the rest of the paper.

First example: Let $\dim M = n$ and G be the n-dimensional affine group: $G = A(n) = Gl(n) \otimes \mathbb{R}^n$ with the subgroup $H = Gl(n)$, the LIE-algebras are

$\mathfrak{g} = \mathfrak{gl}(n) \quad \mathbb{R}^n$ and $\mathfrak{h} = \mathfrak{gl}(n)$, so $\mathfrak{r} = \mathbb{R}^n$.

$^G/_H$ has reductiv structure.

So we can define a soldering form $\theta \in \Omega^1_{Ad}(Q,\mathfrak{r})$ which is essentially the soldering form $\theta \in \overline{\Omega^1_{Id}(Q,\mathbb{R}^n)}$ (or $\theta : TM \longrightarrow \mathbb{R}^n$ by proposition 2.3) considered in [24], [21], [7] (in [16] this form is called canonical form) – the vierbein in the POINCARÉ-gauge theory.

Second example: Let dim M = 4, G = SO(3,2), H = SO(3,1). If we define again $\mathfrak{r}$ by $\mathfrak{so}(3,2) = \mathfrak{so}(3,1) \oplus \mathfrak{r}$, it is obvious that

$$[\mathfrak{so}(3,1), \mathfrak{r}] \subseteq \mathfrak{r}$$

as can be seen from the generators. Since SO(3,1) has two connection components this is not sufficient to show the reductivity of $^{SO(3,2)}/_{SO(3,1)}$. But these components are connected with +1 and −1 respectively so in the equation

$$exp\ (\mathrm{Ad}(h^{-1})r) = h^{-1}\ (exp\ r)\ h$$

we can write $h = \pm\ exp\ H\ (H \in \mathfrak{h})$, where the sign shows if h lies in the connection component of +1 or −1 and with this it is easy to see that $h^{-1}(exp\ r)h \in \mathfrak{r}$.

So in this case a soldering form θ: TQ $\longrightarrow \mathfrak{r}$ can be defined.

Now we will introduce the notion of a CARTAN connection [15] or soldering connection. Although the contribution of ÉLIE CARTAN to the theory should not be underestimated the second name seems more appropriate since it reminds the reader that this soldering connection form splits into a soldering form and a connection form (proposition 3.3 below). We will put an index C to the soldering connection form in honour of É. CARTAN.

Let now ξ, ζ as defined above, then $\omega_C \in \Omega^1_{Ad}(Q,\mathfrak{g})$ is called a soldering connection if

(i) ω_C carries fundamental vector fields A into their generators $a \in \mathfrak{h}$,

(ii) $\omega_C(v) = 0 \Leftrightarrow v = 0$.

Proposition 3.2 [15] ξ, ζ as above. Let ω be a connection form in P and $\theta : TQ \longrightarrow \mathfrak{r}$ a soldering form ($TQ \subseteq TP$). Then exists a soldering connection form $\omega_C : TQ \longrightarrow \mathfrak{g}$ defined by

$$\omega_C = \omega|_{TQ} + \theta.$$

Proposition 3.3 [15] Let $\omega_C : TQ \longrightarrow \mathfrak{g}$ be a soldering connection form. Then there exists a connection form ω and a soldering form θ with

$$\omega_C = \omega + \theta. \qquad (8)$$

In the same way as a connection form in a principal bundle induces a curvature by (2) the soldering connection form induces a soldering curvature defined by

$$\Omega_C := d\omega_C + \tfrac{1}{2}[\omega_C, \omega_C]. \qquad (9)$$

Decomposing ω_C according to proposition 3.3 we get immediately the identity

$$\Omega_C = \Omega + \Theta + \tfrac{1}{2}[\theta, \theta] \qquad (10)$$

where Ω is the curvature of ω and Θ is defined by

$$\Theta := d\theta + [\omega, \theta],$$

the covariant derivative of the soldering form. Θ is called torsion of θ and ω.

Now we decompose (10) into the parts falling into the LIE-algebra $\mathfrak{h}$ and the space $\mathfrak{r}$. We denote by $\Omega_{C\mathfrak{r}}$ the projection of Ω_C onto $\mathfrak{r}$, by $\Omega_{C\mathfrak{h}}$ the one onto $\mathfrak{h}$. The decomposition can be written in different forms depending on the structure of G and H:

Theorem 3.4 [15]

(i) Let $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{h}$ then $\Theta = \Omega_{C\mathfrak{r}}$, $\Omega = \Omega_{C\mathfrak{h}} - \tfrac{1}{2}[\theta,\theta]$

(ii) Let $[\mathfrak{r}, \mathfrak{r}] = 0$ then $\Theta = \Omega_{C\mathfrak{r}}$, $\Omega = \Omega_{C\mathfrak{h}}$

(iii) Let $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r}$ then $\Theta = \Omega_{C\mathfrak{r}} - \tfrac{1}{2}[\theta,\theta]$, $\Omega = \Omega_{C\mathfrak{h}}$.

(i) is the case if G/H is a symmetric space (see [10], [16]), $SO(3,2)/SO(3,1)$ falls into this class. (ii) is the case if G/H is an euclidic or affine space as in the first example we gave above, and in (iii) H is normal in G, that is G/H is a LIE-group.

4. SO(3,2) Gauge Theory

Since we view the CLIFFORD-algebrabundle as the fundamental entity of the theory on grounds given above, we imbed now the groups SO(3,2), SO(3,1) and their LIE-algebras into the CLIFFORD-algebra Cl(3,1) and with that the appropriate principal bundles into the CLIFFORD-algebrabundle $\mathrm{CliffM}^{(3,1)}$.

It is clear how SO(3,1) lies in Cl(3,1) – every element of the LIE-algebra $\mathfrak{so}(3,1)$ has by proposition 2.4 a representation $\omega^{ab}\gamma_a\gamma_b$ $(0 \le a < b \le 3)$ in Cl(3,1), where γ_a are the generators of Cl(3,1). (We use latin indices in the algebra, greek indices are reserved for space-time indices.) So the elements of SO(3,1) are given by $\pm\, exp(\omega^{ab}\gamma_a\gamma_b)$ where the sign determines the connection component as described in 3. .

In the same way SO(3,2) is imbedded in Cl(3,2) by $\pm\, exp(\omega^{ab}\gamma_a\gamma_b)$ $(0 \le a < b \le 4)$. But since the generators always appear in pairs the groups and algebras lie in $Cl_0(3,1)$ resp. $Cl_0(3,2)$. This allows to use the isomorphism from proposition 2.6 to imbed $Cl_0(3,2)$ into $Cl(3,1)$ by

$$i : Cl_0(3,2) \longrightarrow Cl(3,1)$$

with $i(\gamma_a) = \gamma_a$ (a=0,...,3) and $i(\gamma_4) = \gamma_5 = \eta$.

So for an element of SO(3,2) we have the representation $\pm\, exp(\omega^{ab}\gamma_a\gamma_b)$ (a<b, a,b∈{0,1,2,3,5}) in Cl(3,1).

With these imbeddings we can view all the quantities defined in 3., that is ω, Ω, ω_C, Ω_C, θ, Θ (in the case G=SO(3,2), H=SO(3,1)) as CLIFFORD-algebra valued forms and by this we may write the langrangian:

$$\mathfrak{L} = \mathrm{str}\ \Omega_C{}^2 \qquad (11)$$

To show now that this lagrangian yields – by variation of the soldering form – the EINSTEIN-HILBERT-theory with a cosmological term, we decompose the soldering curvature as in eqn (10) and get with lemma 2.3:

$$\mathrm{str}\ \Omega_C{}^2 = \mathrm{tr}\ (\gamma_5\ \Omega_C{}^{ab}\gamma_a\gamma_b \wedge \Omega_C{}^{cd}\gamma_c\gamma_d) \quad (a,b,c,d \in \{0,1,2,3,5\})$$

$$= -4\epsilon_{abcd}\ \Omega^{ab}\wedge\Omega^{cd} + 8\epsilon_{abcd}\ \Omega^{ab}\wedge\theta^c\wedge\theta^d - 4\epsilon_{abcd}\ \theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d$$

$$(a,b,c,d \in \{0,1,2,3\}) \qquad (12)$$

Here we used the familiar traces of γ-matrices.

Note that the torsion Θ does not occur in the explicit form of the langrangian. Now we calculate the action as an integral of (11) over the base manifold M:

$$S = \int \mathfrak{L} *1 =: S_o + S_{EH} + S_C \qquad (13)$$

Here we separated the terms as it was done in [11], where an analogous action was derived. The individual terms are well known:

$$S_{EH} = const \int \epsilon_{abcd}\ \Omega^{ab}\wedge\theta^c\wedge\theta^d *1$$

is the EINSTEIN-HILBERT-action ([11], [7]) which gives by variation of the soldering form or vierbein θ the homogeneous EINSTEIN equations and by variation of the connection ω the vanishing torsion (in absence of matter fields, otherwise the energy-momentum tensor and the spin-density of the matter occur in the equations).

The independent variation of ω and θ is equivalent to the variation of ω_C in (11). So here we recover the EINSTEIN-HILBERT-theory.

The term

$$S_C = const \int \epsilon_{abcd}\ \theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d *1$$

is just a cosmological constant of the theory which cannot be avoided in an SO(3,2) theory ([11], [28], [14]).

It is essential to realize that the quadratic part of the action

$$S_o = const \int \epsilon_{abcd}\ \Omega^{ab}\wedge\Omega^{cd} *1$$

is a topological invariant, as

$$\mathfrak{E} := \frac{1}{32\,\pi^2}\,\epsilon_{abcd}\,\Omega^{ab}\wedge\Omega^{cd} \tag{14}$$

is the EULER characteristic class (see [25]).

We want to emphasize that the main difference between this EULER class and the YANG-MILLS-lagrangian is the HODGE star occuring in the latter (4). The HODGE star contains the metric $g^{\mu\nu}$ which is by soldering connected to the fibre metric g^{ab}:

$$g^{\mu\nu} = \theta^{\mu}_{a}\,\theta^{\nu}_{b}\,g^{ab}$$

so a variation of θ changes the YANG-MILLS-lagrangian due to the HODGE star but leaves the EULER class invariant.

We add the remark that the proposed lagrangian (11) is not invariant under the complete SO(3,2) group, since Ω_c is defined on an SO(3,1) principal-subbundle (see (9) and the definition of ω_c). But ω_c can be viewed as a restriction of a connection ω defined on a SO(3,2) principal bundle (see proposition 3.2). By proposition 3.1 there may exist a large number of SO(3,1) subbundles and choosing one of these breaks the SO(3,2) symmetry of the theory down to SO(3,1). In [28] a nonpropagating auxiliary field was proposed to do this. Whether the symmetry breaking is done by a gravitational analogue to the HIGGS-mechanism or if the GOLDSTONE theorem is appliable ([13]) we will not discuss here.

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References

[1] M. Atiyah, R. Bott, A. Shapiro: Topology 3 Suppl. 1, 3 (1964)

[2] D. Bleecker: *Gauge Theory and Variational Principles* Addison-Wesley 1981

[3] P. Budinich, A. Trautman: *The Spinorial Chessboard* Springer 1988

[4] K. Bugajska: Int. Journ. of Theor. Phys. 18.2 , 77 (1979)

[5] Y. Choquet-Bruhat, C. DeWitt-Morette: *Analysis, Manifolds and Physics* North Holland 1982

[6] R. Geroch: Journ. Math. Phys. 9 , 1739 (1958); 11 , 343 (1970)

[7] M. Göckeler, T Schücker: *Differential Geometry, Gauge Theory and Gravity* Cambridge University Press 1987

[8] W. Greub: *Multilinear Algebra* Springer 1978

[9] F. Hehl: Gen. Rel. Grav. 4.4 , 333 (1973); 5.5 , 491 (1974)
[10] S. Helgason in Battelle Rencontres (eds deWitt, Wheeler) 1 (1967)
[11] A. Hirshfeld in The Fundamental Interaction (eds Debrus, Hirshfeld) 165 (1988)
[12] D. Husemoller: *Fibre Bundles* Springer 1966
[13] D. Ivanenko, S. Sardanashvili: Phys. Rep. 94.1 (1983)
[14] K. Kaminura, T. Fukuyama: Mod. Phys. Lett. A 4.7 , 621 (1989)
[15] S. Kobayashi: Annali di Mat. 43 , 119 (1957)
[16] S. Kobayashi, K. Nomizu: *Foundations of Differential Geometry I, II* J. Wiley & Sons 1969
[17] B. Kostant, S. Sternberg: Ann. Phys, 176 , 45 (1987)
[18] A. Lichnerowicz in Battelle Rencontres (eds DeWitt, Wheeler) 107 (1968)
[19] E. Lord: Phys. Lett. 65A.1 , 1 (1978)
[20] S. MacDowell, F. Mansouri: Phys. Rev. Lett. 38.14 , 739 (1977)
[21] E. Mielke: *Geometrodynamics of Gauge Fields* Akademie-Verlag 1987
[22] E. Mielke: *Progress in Metric-Affine Gauge Theory* Talk given at the Physikertagung Bonn 1989
[23] F. Müller-Hoissen: Dissertation, Göttingen 1983
[24] R. Percacci: *Geometry of Nonlinear Field Theories* World Scientific 1986
[25] R. Sexl, H. Urbantke: *Gravitation und Kosmologie* Wissenschaftsverlag 1983
[26] T. Shirafuji, M. Suzuki to be published in Prog. Theor. Phys.
[27] N. Steenrod: *The Topology of Fibre Bundles* Princeton Univ. Press 1951
[28] K. Stelle, P. West: Phys. Rev. D 21.6 , 1466 (1980)

HEURISTICS OF SOLITARY WAVES IN NON-INTEGRABLE FIELD THEORIES

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ABSTRACT

We discuss two aspects of the behavior of solitary waves in non-integrable classical field theories in one spatial dimension and contrast this behavior with that observed in closely related integrable theories. First, we examine the nature of kink/anti-kink interactions in the "ϕ^4" theory, described by the classical field equation

$$\phi_{tt} - \phi_{xx} - \phi + \phi^3 = 0.$$

We illustrate a remarkable "resonance" structure in the velocity dependence of the kink/anti-kink interactions in ϕ^4 and develop a simple but accurate heuristic explanation for this structure. Second, we discuss the possible existence of spatially-localized, time-periodic "breather" solutions to this equation. We review numerical evidence that, for discrete approximations to this equation, there may exist (the lattice analogs of) breathers, but we then indicate how recent analytic studies have shown that effects "beyond all orders" in perturbation theory destroy (at least the) small amplitude breathers in the continuum limit. We compare and contrast these results with the related solutions in the integrable sine-Gordon theory.

LATTICE APPROACH OF THE ANTIFERROMAGNETIC HEISENBERG MODEL IN 2+1 DIMENSIONS AND THE HOPF CHERN-SIMONS TERMS

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1. INTRODUCTION

The 2+1 dimensional antiferromagnetic (A.F.) Heisenberg model, which is equivalent to the Hubbard model at half filling, had been intensively studied recently, since P.W. Anderson proposed the pioneering ideas[1] about the high Tc superconductivity. Two space dimensions has the novel feature that the quantum statistics may be the fractional statistics. The fractional quantum Hall effects provide the first novel example of the application of fractional statistics to condensed matter physics[2]. In high Tc superconductor, as the cupper oxide layers play a crucial role, there has been much speculation recently[3] that the novel feature of high Tc superconductivity should come from the novel feature of fractional statistics.

In 2+1 quantum field theories, the fractional statistics can occur in σ model with Hopf terms[4], or in gauge theories with Chern-Simons terms[5], and the Hopf Chern-Simons terms can be derived in one flavor fermion gauge theories resulting from integrating out the two-component fermions to get the effective field theories[6].

For A.F. Heisenberg model, there are two approaches to get the continuum limit formulation in the long wave length approximation to reduce it to field theories. One of the approaches is with the large s approximation to reduce it to a nonlinear O(3) σ-model or CP(1) model, which in 1+1 dimensions has topological terms, but in 2+1 dimensions the microscopic derivation of the conjectured Hopf terms[7] get the negative results[8].

The other approach starts from the recently discovered local U(1)[9] and SU(2)[10] gauge symmetries in the 2+1 dimensional A.F. Heisenbrg model which in the continuum limit can be reduced to the field theories of fermion interacting with gauge fields.

But to get the correct continuum limit, we should formulate the lattice gauge theories. In even space-time dimensions, a lot of work has been done, but in the literature it seems little attention has been paid

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to lattice fermions interacting with gauge fields on lattices in three space-time dimensions.

In this talk I will recapitulate the derivation of the naive SU(2) lattice gauge action of the A.F. Heisenberg model in 3 space time dimensions, then I will formulate the SU(2) lattice gauge theories to get the field theories in continuum limit, with two copies of two-flavoured fermions with opposite signs of the mass interacting with gauge fields. Finally, I will give some discussions of the results with regard to the Chern-Simons terms.

This talk is based on two works. One is with the Hamiltonian formulation approach of the theories, which was collaborated with Carleton Detar and W.S. Wu, and the other with the Lagrangian formulation approach, which was collaborated with Chung-I Tan.

2. THE SU(2) LOCAL GAUGE SYMMETRY OF THE A.F. HEISENBERG MODEL AND THE NAIVE LATTICE ACTION

The 1/2 spin operator $\vec{S}_{\vec{x}}$ at lattice site $\vec{x}$ can be expressed by the fermion creation annihilation operators $C^{+}_{\vec{x}\alpha}$, $C_{\vec{x},\beta}$ and the Pauli matrices $\vec{\sigma}$,

$$\vec{S}_{\vec{x}} = \frac{1}{2} C^{+}_{\vec{x}\alpha} \vec{\sigma}_{\alpha\beta} C_{\vec{x},\beta} \tag{1}$$

or explicitly,

$$S^{+}_{\vec{x}} = C^{+}_{\vec{x}\uparrow} C_{\vec{x}\downarrow}, S^{-}_{\vec{x}} = C^{+}_{\vec{x}\downarrow} C_{\vec{x}\uparrow}, \quad S^{z} = \frac{1}{2} (C^{+}_{\vec{x}\uparrow} C_{\vec{x}\uparrow} - C^{+}_{\vec{x}\downarrow} C_{\vec{x}\downarrow})$$

According to E. Dagotto *et al's* treatment, by using a particle-hole transformation in the spin down operator with the following change of notations,

$$C_{x,\uparrow}, C^{+}_{\vec{x}\uparrow} \to \psi_{\vec{x}1}, \psi^{+}_{\vec{x}1}; \; C_{\vec{x}\downarrow}, C^{+}_{x\downarrow} \to \psi^{+}_{\vec{x},2}, \psi_{\vec{x},2} \tag{2}$$

and by introducing the following notations,

$$M_{\vec{x}} = \sum_{a=1,2} \psi^{+}_{\vec{x},a} \psi_{\vec{x},a} = (C^{+}_{1}C_{1})_{\vec{x}} + (C_{2}C^{+}_{2})_{\vec{x}} \tag{3}$$

$$B_{\vec{x}} = \sum_{a,b=1,2} \frac{1}{2} \varepsilon_{ab} \psi_{\vec{x},a} \psi_{\vec{x},b} = \psi_{\vec{x},1} \psi_{\vec{x},2} = (C_{1}C^{+}_{2})_{\vec{x}} \tag{4}$$

it is easy to check that the A.F. Heisenberg Hamiltonian

$$H_{AF} = \frac{J}{2} \sum_{\vec{x},\vec{\ell}} \vec{S}_{\vec{x}} \cdot \vec{S}_{\vec{x}+\vec{\ell}} \tag{5}$$

with $\vec{\ell}$ to denote the two space direction on the square lattice, can be expressed as

$$H_{AF} = \frac{J}{8} \sum_{\vec{x},\vec{\ell}} \left[M_{\vec{x}} M_{\vec{x}+\vec{\ell}} + 2(B^{+}_{\vec{x}} B_{\vec{x}+\vec{\ell}} + B^{+}_{\vec{x}+\vec{\ell}} B_{\vec{x}}) \right] - \frac{J}{2} \sum_{\vec{x}} (M_{\vec{x}} - \frac{1}{2}) \tag{6}$$

The derivation is following,

$$H = \frac{J}{2}\sum_{\vec{x},\hat{\ell}} \vec{S}_{\vec{x}}\cdot\vec{S}_{\vec{x}+\hat{\ell}} = \frac{J}{2}\sum_{\vec{x},\hat{\ell}} \left\{\frac{1}{2}\left(S^+_{\vec{x}}S^-_{\vec{x}+\hat{\ell}} + S^-_{\vec{x}}S^+_{\vec{x}+\hat{\ell}}\right)\right.$$

$$\left. + \left(S^z_{\vec{x}}+\frac{1}{2}\right)\left(S^z_{\vec{\lambda}+\hat{\ell}}+\frac{1}{2}\right)\right\} - \frac{J}{4}\sum_{\vec{x}}\left(2S^z_{\vec{x}}+\frac{1}{2}\right)$$

$$= \frac{J}{8}\sum_{\vec{x},\hat{l}} \left\{2\left[(C^+_1C_2)_{\vec{x}}(C_1C^+_2)_{\vec{x}+\hat{\ell}} + (C^+_2C_1)_{\vec{x}}(C^+_1C_2)_{\vec{x}+\hat{\ell}}\right]\right.$$

$$\left. + (C^+_1C_1-C^+_2C_2+1)_{\vec{x}}(C^+_1C_1-C^+_2C_2+1)_{\vec{x}+\hat{\ell}}\right\} - \frac{J}{2}\sum_{\vec{x}}\left\{(C^+_1C_1-C^+_2C_2)_{\vec{x}} + \frac{1}{2}\right\}$$

$$= \frac{J}{8}\sum_{\vec{x},\hat{\ell}} \left\{2\left[B^+_{\vec{x}}B_{\vec{x}+\hat{\ell}}+B^+_{\vec{x}+\hat{\ell}}B_{\vec{x}}\right] + M_xM_{\vec{x}+\hat{\ell}}\right\} - \frac{J}{2}\sum_{\vec{x}}\left(M_{\vec{x}} - \frac{1}{2}\right) .$$

By second order perturbation, the A.F. Heisenberg Hamiltonian (6) can be proved to be equivalent to the SU(2) gauge invariant Kogut Susskind Hamiltonian

$$H_{AF} = \frac{8}{3J}\sum_{\vec{x},\ell,\alpha} E^{\alpha}_{\vec{x},\hat{\ell}}E^{\alpha}_{\vec{x},\hat{\ell}} + \frac{i}{2}\sum_{\substack{\vec{x},\hat{\ell}\\ a,b}} \left[\psi^+_{\vec{x},a}U^{ab}_{\vec{x},\hat{\ell}}\psi_{\vec{x}+\hat{\ell},b} - h.c.\right] \tag{7}$$

where the original coefficient $g^2/2a$ have been changed to 8/3J.

The proof of the equivalence is based on the fact that

$$|ab\rangle = U^{ab}|0\rangle ,$$

where $|0\rangle$ is the vacuum and $E|0\rangle=0$, form complete and orthogonal basis of the Hilbert space, and have the following properties:

$$\langle ab|cd\rangle = \langle 0|U^{+ab}\,U^{cd}|0\rangle = \delta_{ad}\delta_{bc} \tag{8}$$

$$\langle 0|U^{ab}U^{cd}|0\rangle = \varepsilon_{ac}\varepsilon_{bd} . \tag{9}$$

In order to apply perturbation theory, we change H to

$$W = \frac{3J}{8}H_{KS} = \sum_{\vec{x},\hat{\ell}} E^2_{\vec{x},\hat{\ell}} + \frac{3J}{16}W_q = W_o+xW_q \tag{10}$$

where $x = \frac{3J}{16}$ and

$$W_q = i\sum_{\substack{\vec{x},\hat{\ell}\\ a,b}} \left(\psi^+_{\vec{x},a}U^a_{\vec{x},\hat{\ell}}\psi_{\vec{x},\hat{\ell},b} - b.c.\right) \tag{11}$$

For SU(2) gauge invariant theory, it is easy to find that

$$E^{\alpha}E^{\alpha}\,U|0\rangle = \frac{3}{4}\,U|0\rangle .$$

Since $<0|U|0>=0$, the first order perturbation is zero,

$$W^{(1)}_{ij} = <\Psi_{i,0}|x\ W_q|0,\Psi_j> = 0\ . \tag{12}$$

The second order perturbation gives

$$W^{(2)}_{ij} = <0\Psi_i|W_q\ x^2/w_o - W_o\ W_q|0,\Psi_i>$$

$$= x^2 \sum_{ab} <0\Psi_i|W_q|ab> \frac{1}{(-3/4)} <ab|W_q|0\Psi_j>$$

$$= -\frac{4}{3} x^2 <0,\Psi_i|W_q W_q|0,\Psi_j> \tag{13}$$

where the unperturbed energy W_o of the state $|0,\Psi_o)$ has been put to zero.

By substituting the expression (11) of W_q into (13), and using the relations (8) and (9), it is easy to get

$$W^{(2)}_{ij} = \frac{4}{3} x^2 \left\{ \sum_{\vec{x},\hat{\ell}} [M_{\vec{x}} M_{\vec{x}+\hat{\ell}} + 2(B^{+}_{\vec{x}} B_{\vec{x}+\hat{\ell}} + B^{+}_{\vec{x}+\hat{\ell}} B_{\vec{x}}] - 4 \sum_{\vec{x}} (M_{\vec{x}} - \frac{1}{2}) \right\} \tag{14}$$

which gives the effective Hamiltonian of the Kogut Susskind Hamiltonian,

$$H_{KS} = \frac{8}{3J} W^{(2)} = \frac{J}{8} \sum_{\vec{x},\hat{\ell}} \left\{ 2[B^{+}_{\vec{x}} B_{\vec{x}+\hat{\ell}} + B^{+}_{\vec{x}+\hat{\ell}} B_{\vec{x}}] + M_{\vec{x}} M_{\vec{x}+\hat{\ell}} \right\} - \frac{J}{2} \sum_{\vec{x}} (M_{\vec{x}} - \frac{1}{2})$$

$$= H_{AF} \tag{15}$$

The temporal part of the Lagrangian

$$L = \sum_{\vec{x},\vec{\sigma}} C^{+}_{\vec{x}\sigma} (i \frac{d}{dt}) C_{\vec{x}\sigma} - H_{K.S.} \tag{16}$$

can be expressed by the spinor (2)

$$L = \sum_{\vec{x}} \psi^{+}_{\vec{x}}\ i \frac{d}{dt} \psi_{x} - H_{K.S.} + \frac{J}{4} \sum_{\vec{x}} \psi^{+}_{\vec{x}} \psi_{\vec{x}} \tag{17}$$

upon integrating by parts with respect to time and throwing away a constant term. The last term comes from the anticommutator

$$\left\{ \frac{\partial}{\partial t} C_{\vec{x},2}(t), C_{\vec{x},2}(a) \right\} = -\frac{J}{4}\ i \sum_{\vec{x}} M_{\vec{x}}\ . \tag{18}$$

The half-filled constraints imposed on the question are

$\psi^{+}\ \sigma_3 \psi = 0$, per electron per site;

$$\left.\begin{array}{l} \psi^{+}\ \sigma_2 \psi = 0 \\ \\ \psi^{+}\ \sigma_1 \psi = 0 \end{array}\right\} \text{ no double occupancy} \tag{19}$$

The Lagrangian multipliers of the above constraint are just the gauge potential to make the Lagrangian gauge invariant under time dependent gauge transformation,

$$L = \sum_{\vec{x}} \psi^{+}_{\vec{x}}(i \frac{d}{dt} + A_{o\vec{x}})\psi_{\vec{x}} - H_{K.S.} + \frac{J}{4}\sum_{\vec{x}} \psi^{+}_{\vec{x}}\psi_{\vec{x}} \ . \tag{20}$$

where $A_{o\vec{x}} = \sum_{i=1}^{3} A^{i}_{o\vec{x}} \sigma_i$.

By discreting the time and putting the spinor velocity to 1, after transforming to Eucleadean metric, we get the naive action in Euclidean space-time:

$$S = S_F + S_m = - a^2 \sum_{\gamma,\mu} [\bar{\psi}(\gamma)\gamma_\mu U_\mu(r)\psi(r+m^2) - h.c.] + i\ ma^3 \Sigma\bar{\psi}(r)\psi(r)$$

$$\mu = 1,2,3. \tag{21}$$

$$r:\ r_1, r_2, r_3 \text{ coordinates of } r,\ m = \frac{J}{4} ,$$

where we have written the Dirac matrix γ_n explicitly, which satisfy the anticommutation relations:

$$\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \tag{22}$$

and have the antihermitian property,

$$\gamma^{+}_{\mu} = - \gamma_\mu \ . \tag{23}$$

3. THE LATTICE GAUGE THEORIES OF THE 3 DIMENSIONAL A.F. HEISENBERG MODEL

Before taking continuum limit, we should solve the fermion doubling problem by considering the Susskind scheme. On even dimension, Kluberg Stern et al.[12] had developed Susskind scheme to construct an explicit representation of the lattice theory in terms of flavored fermion fields. C. Burden et al.[13] had extended the above treatment in odd dimension, but only with free fermions. We need to extend the above works to fermions interacting with gauge fields.

We start from the naive SU(2) lattice action (21). With a Kawamo Smit transformation, this action may be diagonalized in the spinor indices with the transformation,

$$\psi(r) = \gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{r_3}\ \chi(r),\ \bar{\psi}(r) = \bar{\chi}(r)\left(\gamma_1^{r_1}\gamma_2^{r_2}\gamma_3^{r_3}\right) \tag{24}$$

$$(r_1, r_2, r_3\text{: coordinates of } r)$$

$$S = S_F + S_m = a^2 \sum_{r,\mu} (-1)^{r_1 + \dots r_{\mu-1}} (\bar{\chi}(r)U_\mu(r)\chi(r+\hat{\mu}) - \bar{\chi}(r+\hat{\mu})U^{+}_{\mu}(r)\chi(r))$$

$$+ ima^3 \sum_{r} \bar{\chi}(r)\chi(r) \tag{25}$$

The important feature of 3-dimensional is that the $\gamma_4 = \gamma_1\gamma_2\gamma_3$ is a diagonal matrix, and $\{\gamma_\mu\}$, $\{-\gamma_\mu\}$ are two inequivalent representations of the Clifford algebra. In the following we will use these two inequivalent representations, denoted by $\{\gamma_\mu\}$, $\{\beta_\mu\}$ respectively.

Consider the 3-dimensional hypercube H(y) with its origin at y (y_μ = integer), and corners

$$r_\mu = 2y_\mu + \eta_\mu, \qquad \eta_\mu = 0, \text{ or } 1, \qquad \mu = 1,2,3, \tag{26}$$

and relabel the fields $\chi(r)$, $\bar{\chi}(r)$

$$\chi(2y+q) \equiv (-)^y \chi_\eta(y) \tag{27}$$

$$\bar{\chi}(2y+q) \equiv (-)^y \bar{\chi}_\eta(y) \tag{28}$$

with

$$(-1)^y = (-1)^{y_1+y_2+y_3} \tag{29}$$

The link in H(y) between 2y and $2y+\eta$ is expressed by the original link $U_\mu(r) = U_\mu(2y+\eta_\mu)$ with the notation,

$$U_\eta(y) = [U_1(2y)]^{\eta_1} \, [U_2(2y+\eta_1)]^{\eta_2} \, [U_3(2y+\eta_1+\eta_2)] \ . \tag{30}$$

The links from $2y+\eta$ to $2y+\eta'$, $2y+\eta$ to $2(y+\hat{\mu})+\eta'$ and $2y+\eta$ to $2(\eta-\mu)+\eta'$ can be found to be the following forms, respectively,

$$2y+\eta \rightarrow 2y+\eta', \qquad U^{+}_\eta(y)U_{\eta'}(y) \tag{31}$$

$$2y+\eta \rightarrow 2(y+\hat{\mu})+\eta', \qquad U_\eta(y)U^2_\mu(y)U_{\eta'}(y+\hat{\mu}) \tag{32}$$

$$2y+\eta \rightarrow 2(y-\hat{\mu})+\eta', \qquad U^{-}_\eta(y)\, U^{+2}_\mu(y)U_{\eta'}(y+\hat{\mu}) \tag{33}$$

The $U^2_\eta(2y)$ factor in the above expressions, at the small a approximation can be expressed as

$$U^2_\mu(y) = (e^{igaA_\mu})^2 = 1+2aigA_\mu(y). \tag{34}$$

We further introduce first and second central derivative on the lattice with spacing 2a, formed by the sites r=2y and the covariant derivative,

$$\nabla_{\hat{\mu}} U_\eta(y)\chi_\eta(y) = \frac{U_\eta(y+\hat{\mu})\chi_\eta(y+\hat{\mu}) - U_\eta(y-\hat{\mu})\chi_\eta(y-\hat{\mu})}{4a} \tag{35}$$

$$\nabla^2_\mu U_\eta(y)\chi_\eta(y) = \frac{U_\eta(y+\hat{\mu})\chi_\eta(y+\hat{\mu}) - 2U_\eta(y)\chi_\eta(y) + U_\eta(y-\hat{\mu})\chi_\eta(y-\hat{\mu})}{4a^2} \tag{36}$$

$$D_\mu = (\nabla_\mu + igA_\mu) \ . \tag{37}$$

With small a approximation, S_F can be written in the following form:

$$S_F = a^3 \sum_{\eta,\mu} \sum_{\eta,\eta} \left\{ \bar{\chi}_\eta(y) U^+_\eta(y) \left(\frac{\Gamma^\mu_{\eta\,\eta'} + B^\mu_{\eta\,\eta'}}{2} \right) D_\mu U_{\eta'}(y) \chi_{\eta'}(y) \right.$$

$$\left. + \, a \, \bar{\chi}_\eta(y) U^+_\eta(y) \left(\frac{\tilde{\Gamma}^\mu_{\eta\,\eta'} + \tilde{\Gamma}^\mu_{\eta\,\eta'}}{2} \right) (D^2_\mu(y) - g^2 A^2_\mu) U_{\eta'}(y) \chi_{\eta'}(y) \right\} \tag{38}$$

where

$$\begin{aligned}
\Gamma^\mu_{\eta\eta'} &= \frac{1}{2} \, tr(\Gamma^+_\eta \, \gamma_\mu \, \Gamma_{\eta'}) \\
B^\mu_{\eta\eta'} &= \frac{1}{2} \, tr(B^+_\eta \, \beta_\mu B_{\eta'}) \\
\tilde{\Gamma}^\mu_{\eta\eta'} &= \frac{1}{2} \, tr(\Gamma^+_\eta \, \gamma_\mu \Gamma_{\eta'})(\delta_{\eta',\eta-\hat{\mu}} - \delta_{\eta',\eta+\hat{\mu}}) \\
\tilde{B}^\mu_{\eta\eta'} &= \frac{1}{2} \, tr(B^+_\eta B_\mu B_{\eta'})(\delta_{\eta',\eta-\hat{\mu}} - \delta_{\eta',\eta+\hat{\mu}})
\end{aligned} \tag{39}$$

and

$$\Gamma_\eta = \gamma_1^{\eta_1} \gamma_2^{\eta_2} \gamma_3^{\eta_3}, \quad B_\eta = \beta_1^{\eta_1} \beta_2^{\eta_2} \beta_3^{\eta_3} \tag{40}$$

What we need is to find a linear transformation which takes the eight fields $U_\eta(y)\chi_\eta(y)$ to four flavors of two-component spinors. The suitable transformation of $U_\eta \chi_\eta$ and $\bar{\chi}_\eta U^+_\eta$ is

$$\begin{aligned}
u^{\alpha a}(y) &= \frac{1}{4\sqrt{2}} \sum_\eta \Gamma^{\alpha a}_\eta \; U_\eta(y) \chi_\eta(y) \; , \\
d^{\alpha a}(y) &= \frac{1}{4\sqrt{2}} \sum_\eta B^{\alpha a}_\eta \; U_\eta(y) \chi_\eta(y) \; , \\
\bar{u}^{\alpha a}(y) &= \frac{1}{4\sqrt{2}} \sum_\eta \bar{\chi}_\eta(y) \; U^+_\eta(y) \Gamma^{*\alpha a}_\eta \\
\bar{d}^{\alpha a}(y) &= \frac{1}{4\sqrt{2}} \sum_\eta \bar{\chi}_\eta(y) \; U^+_\eta(y) B^{*\alpha a}_\eta
\end{aligned} \tag{41}$$

These transformations can be inverted,

$$U_\eta(y) \chi_\eta(y) = \sqrt{2} \sum_{\alpha,a} (\Gamma^{*\alpha a}_\eta u^{\alpha a}_\eta(y) + B^{*\alpha a}_\eta d^{\alpha a}(y)) \tag{42}$$

$$\bar{\chi}_\eta(y) U^+_\eta(y) = \sqrt{2} \sum_{\alpha,a} (\bar{u}^{\alpha\eta}(y)^{\alpha a} \Gamma^{\alpha a}_\eta + \bar{d}^{\alpha a} B^{\alpha a}_\eta) \tag{43}$$

by using the identities,

$$tr(\Gamma^+_\eta \Gamma_{\eta'} + B^+_\eta B_{\eta'}) = 4\delta_{\eta\eta'} \tag{44}$$

Once the transformations are inserted in the action S_F+S_m, and make use of the identities,

$$\sum_{\eta} \Gamma_{\eta}^{\alpha a}\Gamma_{\eta}^{*\beta a} = \sum_{\eta} B_{\eta}^{\alpha a}B_{\eta}^{*\beta a} = 4\delta_{\alpha\beta}\delta_{ab}, \qquad \sum_{\eta} \Gamma_{\eta}^{\alpha a}B_{\eta}^{*\beta b} = 0 \ , \tag{45}$$

some algebra leads to

$$S_F+S_m = (2a)^3 \sum_{\eta,\mu} \Big\{\bar{u}(\gamma_{\mu}\otimes 1)D_{\mu}\mu + \bar{d}(\beta_{\mu}\otimes 1)D_{\mu}d$$
$$+ im[\bar{u}(1\otimes 1)u + \bar{d}(1\otimes 1)d] + O(a)\Big\} \ , \tag{46}$$

in which in the fermion bilinear, the first matrix acts on Greek spinor indices, the second matrix acts on Latin flavour indices.

In the continuum limit, by replacing $-\gamma_{\mu}$ with β_{μ}, we get the Lagrangian density

$$\mathcal{L} = \sum_{f=1}^{2} \Big\{\bar{u}_f\gamma_{\mu}D_{\mu}u_f - \bar{d}_f\gamma_{\mu}D_{\mu}d_f + im(\bar{u}_fu_f + \bar{d}_fd_f)\Big\} \ . \tag{47}$$

Therefore, we get two copies of 2-flavoured fermion interacting with gauge fields, with opposite signs of the mass.

4. DISCUSSION

It has been found that in 3-dimensional space time,[6] for each species of fermions coupling with gauge fields induce a topologically nontrivial vacuum current, with the relations,

$$\frac{\delta S_{eff}}{\delta A_{a\mu}} = \langle j_a^{\mu}\rangle \tag{48}$$

where S_{eff} is the effective action obtained by integration over the two components fermion fields $\psi(x)$ in the functional integral for one flavour fermions with the Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(\gamma^{\mu}D_{\mu} + im)\psi \ , \tag{49}$$

$$S_{eff}[A] = -\ell n \det i(\gamma^{\mu}D_{\mu} + m) \ . \tag{50}$$

Integrating (49) shows that the effective action contains the topological Chern-Simons terms,

$$W[A] = \frac{1}{8\pi^2} \int d^3x \ \mathrm{tr}({}^*F_{\mu}A^{\mu} - \frac{1}{3} A^{\mu}A^{\nu}A^{\alpha}\varepsilon_{\mu\nu\alpha}), \qquad {}^*F^{\mu} = \frac{1}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} \ , \tag{51}$$

Now the 2+1 dimensional A.F. Heisenberg model (47) has two copies of two flavours fermions with opposite sign of mass, therefore the induced vacuum current vanishes and no Chern-Simons terms can be derived.

Redlich[6] had demonstrated that the detD and, by (51), the effective action $S_{eff}[A]$ in the massless limit are not gauge invariant under a large homotopically nontrivial gauge transformations, with winding number n,

$$\det(D) \to (-)^{|n|}\det(D) \ . \tag{52}$$

But with Lagrangian (47), we have two copies of even number of species of fermions, therefore the gauge invariance of effective action has no problem.

In odd dimension, the mass term is a pseudo scalar which violates parity. If we redefine parity such that, in addition to flipping one space dimension, it also interchanges the u and d copies, then we can keep the Lagrangian (47) invariant. Polychronakos[13] had studies the flavour and parity symmetry breaking patterns in 3 dimensional Abelian gauge theories.

In 3-dimensions, there is no chirality, because there is no analog "γ^5" anticommuting all γ_μ. In order to introduce chirality, Pisarski[14] uses four component spinors. We define 4x4 Dirac matrices by

$$\tilde{\gamma}_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix}, \qquad \mu = 1,2,3 \tag{53}$$

which anticommuting the following to "γ^5" matrices:

$$\tilde{\gamma}_4 = \begin{pmatrix} 0 & i1 \\ i1 & 0 \end{pmatrix}, \qquad \tilde{\gamma}_5 = \begin{pmatrix} 0 & i1 \\ -1 & 0 \end{pmatrix} \tag{54}$$

If we define the 4 components fermion field Ψ as

$$\Psi = \binom{u}{d} \ , \tag{55}$$

then the A.F. Heisenberg model reduced to

$$\mathcal{L} = \sum_{f=1}^{2} \left\{ \bar{\Psi}_f \, \tilde{\gamma}_\mu \, D^\mu \, \Psi_f + im \, \bar{\Psi}_f \Psi_f \right\} \tag{56}$$

Chiral symmetry is broken in four dimensional lattice QCD have already established. It is interesting to find if it is broken in three dimensions. Pisanski showed with large N approximation, that for N large, massless QED spontaneously breaks chiral symmetry.

The result that we can not derive Chern-Simons terms is not conclusive, as we are confined to the 1/2 filling case. The question is, how about doping? Only after doping the Hubbard model is relevant to high Tc superconductivity. Another question is, how about the ground state? X.G. Wen, F. Wilczek and A. Zee[15] had proposed the idea of chiral spin states and had elegant discussions about these problems. D. Kheshchenko and P.B. Wiegman had interesting approaches to these problems.[16] Our approach to these problems will be discussed elsewhere.

REFERENCES

1] P.W. Anderson, Science 235, 1196 (1987)

2] The Quantum Hall effect, edited by R.E. Prange and S.M. Girvin, (Springer, New York 1987)
3] R.B. Langhlin, Science 242, 525 (1988); Phys. Rev. Lett. 60, 2677; V. Kalmeyes and R.B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987)
4] F. Wilczek and A. Zee, Phys. Rev. Lett. 51, 2250 (1980)
5] D.P. Arovas, R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. B251, 117 (1985)
6] N. Redlich, Phys. Rev. Lett. 52, 18 (1984); Phys. Rev. D29, 2366 (1984);
7] A.M. Polyakov, Mod. Phys. Lett. A3, 325 (1988); Z.E. Dzyakoshinskii, A.M. Polyakov, P.W. Wiegmann, Phys. Lett. A127, 112 (1988)
8] F.D.M. Haldane, Phys. Rev. Lettt. 61, 1029 (1988); X.G. Wen, A. Zee, Phys. Rev. Lett. 61, 1025 (1988); E. Frandkin, M. Stone, Phys. Rev. B38, 7215 (1988); L.B. Zoffe, A.Z. Larkin, Int. J. Mod. Phys. B2, 203 (1988)
9] G. Baskaran and P.W. Anderson, Phys. Rev. B37, 580 (R.C.) (1988); P.W. Anderson, G. Baskaran, Z. Zou, T. Hsu, Phys. Rev. Lett. 58, 2790 (1987); I. Affleck and J.B. Marston, Phys. Rev. B37, 3774 (1988); G. Kotliar, Phys. Rev. B37, 3644 (1988)
10] P.W. Anderson in Proc. of the International School of Physics "Enrico Fermi", July 1987: *Frontiers and Borderlines in Many Body Particle Physics* (North Holland, Amsterdam, 1988); I. Affleck, Z. Zou, T. Hsu and P.W. Anderson, Phys. Rev. B38, 2926 (1988); E. Dagotto, E. Fradkin and A. Moreo, Phys. Rev. B38, 2926 (1988)
11] H. Kluberg-Stern, A. Morel, and O. Nopoly, Nucl. Phys. B220, 447 (1983)
12] C. Burden and A.N. Burkitt, Europhys. Lett. 3, 545 (1987)
13] A.P. Polychronakos, Phys. Rev. Lett. 60, 1920 (1988)
14] R.D. Pisarski, Phys. Rev. D29, 2423 (1984)
15] X.G. Wen, Frank Wilczek, and A. Zee, Phys. Rev. B39, 11413 (1989)
16] D. Kheshchenko, P.B. Wiegmann, Phys. Lett. B225, 279 (1989), and Landau Institute preprint (1989)

IS IT POSSIBLE TO DO CANONICAL QUANTUM FIELD THEORY RIGOROUSLY?

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INTRODUCTION

The origins of this work lie in the mystical belief that quantum electrodynamics (QED), as developed by Tomonaga-Schwinger -Feynman-Dyson and others, cannot possibly be a "wrong" theory, despite its mathematical contradictions. More precisely, a theory which agrees so well with experiment cannot be fundamentally unsound. Its formalism requires not a massive change but rather a large-scale reinterpretation. Nature has been sending us a signal which we have not yet been able to interpret. Assuming this, the present work is an attempt to decipher this signal.

QED, as is well known, is heavily dependent on the interaction picture.[1,2] In the second part of his Cargèse lectures of 1964, which may be said to have ushered in the age of constructive quantum field theory, Wightman [3] discusses "What is wrong with the interaction picture", and isolates Haag's theorem [4,5] and the ultraviolet catastrophe as the main problems. A whimsical, but nevertheless perfectly correct, formulation of Haag's theorem would be that the interaction picture exists only if there is no interaction. The argument, roughly speaking, is based on symmetry principles (for fields obeying the equal-time commutation or anticommutation relations, ETCR). More recently, Baumann [6] has obtained results, based on rate-of-growth estimates in Wightman field theory, for scalar Bose and spin 1/2 fields, proving that for space dimensions $d > 3$ such fields must necessarily be free, while for $d \leq 3$ the possible forms of interaction are strongly restricted. In brief, the prospects for a quantum field theory (QFT) obeying the ETCR *on a single Hilbert space* appear to be dim. This is a fact of life which QED enthusiasts, who ought really to constitute the entire community of physicists, have to contend with.

What, then, is to be done? There have been two broad responses. (1) First, try to understand QFT on a single Hilbert space, and construct nontrivial models: Wightman field theory, constructive field theory, Euclidean field theory, field theories on lattices. (2) Try to understand the general stucture of quantum field theories in an algebraic setting - the approaches of Araki-Haag, Segal-Borchers-Haag-Kastler, Doplicher-Haag-Roberts. Somewhere in between lies the algebraic formulation, due to Borchers, of Wightman field theory. None of these approaches show any visible signs of perplexity at the success of QED.

We have decided, therefore, to reverse the priorities, and attempt a formulation of quantum field theory which (a) uses the ETCR and is not a free-field theory, and (b) admits a perturbation theory which coincides with the standard covariant perturbation theory in the case of QED. The first requirement means that, owing to the theorems of Haag and Baumann, we cannot formulate the theory on a single Hilbert space. The second requirement means that we set aside, at least temporarily, any idea of formulating a general, *structural* field theory.

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Our attempt will be to develop a QFT not on a Hilbert space but on a *Hilbert bundle.* The original observation that Hilbert bundles might somehow be linked to physics came from an analysis of Landau's theory of superfluidity, [7,8] the essential feature of which was that the excitations of the superfluid ground state had *zero* nonrelativistic mass. But these, according to a celebrated paper by Inönü and Wigner, [9] did not make sense in a single-Hilbert-space framework, because there existed no physically acceptable unitary representation of the inhomogeneous Galilei group corresponding to mass zero.

It turned out that the introduction of Hilbert bundles as state spaces solved this problem satisfactorily, and indeed several other problems of relativity group action on physical states of infinitely large systems, and yielded new insights. [10,11,12] It became clear, somewhat later, that the required representation theory of relativity groups on Hilbert bundles was provided by the bundle structure theorem for a Lie group G and the left-action of G on G itself viewed as a principal bundle over G/H, H being a closed subgroup of G, and on Hilbert bundles associated with this bundle. [13] It transpired simultaneously, that "classical" unitarily non-implementable transformations in physics such as the Bogoliubov transformations could also be understood more clearly within a Hilbert bundle framework. [14] It was therefore natural, in this climate of mind, to ask whether or not a Hilbert bundle framework could accomodate a QFT of the type which we are looking for.

QUANTUM FIELD THEORY ON HILBERT BUNDLES

The account given below is both sketchy and informal. It is also lacking in generality. These defects wil be remedied elsewhere.

We shall assume that our bundle B is already given in product form: $B = M \times \mathbf{H}$, where the base space M may be practically anything; a finite set, a real line, a real 4-manifold, an automorphism group of the ETCR algebra, generally infinite-dimensional. The latter case is useful when we want to to work with many inequivalent representations of the ETCR. We shall write $b = (x, \psi)$, where $b \in B, x \in M$ and $\psi \in \mathbf{H}$. $\mathbf{H}$ is a separable, infinite-dimensional Hilbert space over the complex numbers C.

Let G be the relativity group of the theory, Poincaré or SL (2, C). G will act on B as follows:

$$g(x, \psi) = (gx, u(g, x)\psi)$$

where gx is an action of G on the base space M and $u(g, x)$ a unitary operator on $\mathbf{H}$ which satisfies the cocycle identity

$$u(g', gx)\; u(g, x) = u(g'g, x).$$

The general procedure for obtaining u is to construct a map

$$k: G \times M \to H$$

(where H is a closed subgroup of G) which satisfies the cocycle identity

$$k(g', gx)\; k(g, x) = k(g'g, x),$$

and then set

$$u(g, x) = D(k(g, x))$$

where D is a continuous unitary representation of H on $\mathbf{H}$. Now we make the assumption that *we don't want to change the theory of free fields*; if the field is free, it, and the bundle on which it is defined, should split into a bunch of standard free fields, each defined on a single Hilbert space. In this case the entire relativity group G should act on each fibre, i.e. the subgroup H must equal G itself, and the map

$$k\colon G \times M \to G$$

must be surjective. This is possible only if k, and therefore u, is independent of x, i.e. if

$$g(x, \psi) = (gx, u(g)\psi)$$

where u is now a unitary representation of G upon H. Next, if $gx \neq x$, then even the (free) motion of the free field takes us from one fibre to another, that is, we are still departing from the assumption made above. Hence we must have $gx = x$, and therefore, finally,

$$g(x, \psi) = (x, u(g)\psi).$$

Next, we come to the definition of the *field*. Here we have only to mimic Wightman and Gårding for our more general case, and define fields as fibre-preserving-map-valued distributions on B. This means that if $\phi\colon B \to B$ is a field (making allowances for unbounded cases) and $f \in S^3(R)$ is a test function, then

$$\phi(f, t) = \int \phi(\mathbf{x}, t)\, f(\mathbf{x})\, d^3x$$

is a fibre-preserving map, i.e

$$\pi b_1 = \pi b_2 \Rightarrow \pi(\phi(f, t)\, b_1) = \pi(\phi(f, t)\, b_2).$$

Here

$$\pi\colon B \to M$$

is the projection in the bundle, and $b_1, b_2 \in B$. Note that ϕ has to be defined at sharp times, because we want the ETCR.

Finally, we demand that there should exist a perturbation theory. Then, if ϕ is a field and $b \in B$, the points ϕb, $\phi \phi b$, $\phi \phi \phi b$, etc. should all lie on the same fibre (the short-hand here is easily understood). It is worth emphasizing at this stage that our bundle is not a fibration of a large linear space but a Hilbert bundle in the sense of Steenrod, in which the fibres, although isomorphic, are distinct; one *cannot* superpose a vector from one fibre onto a vector from a different fibre *without introducing a notion of parallel transport*, which we do not have. Therefore terms like ϕb, $\phi \phi b$, $\phi \phi \phi b$, etc will generally belong to different fibres, and perturbation theory will be a non-starter! The only generic way to save the situation is to demand that *the field ϕ should not map one fibre into another*, which we now proceed to do. Then ϕb, $\phi \phi b$, etc. all belong to the fibre π^{-1} (b).

Now we are in a position to develop a Wightman-like theory on our enlarged framework. At first sight it appears that the enlargement has been illusory; the twin restrictions of (a) not changing the theory of free fields, and (b) admitting perturbation theory, have led us to a situation in which *the field cannot induce a motion on the base space*. Consequently, the required extension of the Wightman framework becomes trivial if we are working with a base space like R or R^4, somewhat more involved if the base space is an automorphism group of the ETCR which brings with it many inequivalent representations of the latter. However, in all cases the theorems of Haag and Baumann return in full force on each fibre. What we have constructed, after considerable effort, is a theory, not of a single free field but of a parametrized family of free fields, the parameters being those of a point on the base space. But this, if we cast our minds back, *is very similar to the interaction picture*. The difference is that in the interaction picture one tries to define all of these free fields on the same Hilbert space, whereas in our case they are defined on different points on the base space, and a vector from one fibre cannot be superposed upon a vector from another. Is this departure sufficient to develop a nontrivial theory?

DYNAMICS ON HILBERT BUNDLES

The partial mental picture of dynamics which we have formed so far is the following: In the absence of interaction, all motion takes place *within* the fibres; there is no induced motion on the base space. We would now like to take "affirmative action" and force motion on the base space when there is an interaction. But how can we achieve this, when the fields themselves are unable to induce such motion? The answer is obvious: *This role will have to be assumed by the coupling constant* !

By *dynamics* on a Hilbert bundle B we shall mean a one-parameter family of bundle maps

$$\Omega(t): B \to B, \quad t \in \mathrm{R}$$

such that $\Omega(t)\,\Omega(t') = \Omega(t+t')$ and $\Omega(t)^{-1} = \Omega(-t)$, which satisfy some continuity and differentiability properties, to be specified later. Such a family of bundle maps induces a family of base maps according to the following commutative diagram:

$$\begin{tikzcd} B \arrow[r, "\Omega(t)"] \arrow[d, "\pi"'] & B \arrow[d, "\pi"] \\ M \arrow[r, "\overline{\Omega(t)}"'] & M \end{tikzcd}$$

Each base map is a "motion" of the base manifold, and therefore, given certain smoothness properties, the induced one-parameter group of base maps is generated by a vector field X on the base space M:

$$\overline{\Omega(t)} = \exp(tX). \tag{1}$$

We wish to "lift" the above formula to the bundle, that is, to write a formula like

$$\Omega(t) = \exp(tQ) \tag{2}$$

which combines and generalizes Stone's theorem and formula (1) for motions of a manifold. In other words, we would like to find an infinitesimal generator for the group of bundle maps $\{\Omega(t);\, t \in \mathrm{R}\}$. It cannot be assumed, *a priori*, that every one-parameter group of bundle maps of B, with enough continuity and differentiability properties, has an infinitesimal generator; *that* would be a far-reaching generalization of Stone's theorem. We are not attempting such a generalization at this stage, but are interested in isolating a few special cases which might allow the programme outlined earlier to get off the ground.

As a starting point, let us recall that a vector field X on a differentiable manifold M can be defined either as a cross-section of TM, the tangent bundle of M, or as a derivation of the algebra $C^\infty(M, \mathrm{R})$ of smooth real-valued function on M. Now let $\mathbf{A}$ be the algebra of instantaneous observables of the field ϕ. For simplicity, we assume the following: (i) $\mathbf{A}$ is a C^* - algebra with identity; (ii) $\mathbf{A}$ is irreducible; (iii) the centre of $\mathbf{A}$ consists of scalar multiples of the identity (the reader will notice that we use nowhere near the full force of these assumptions in the sequel). We consider the algebra $C^0(M, \mathbf{A})$ of continuous functions on M with values in $\mathbf{A}$, which may alternatively be regarded as the algebra of sections of the bundle $M \times \mathbf{A}$ with base M. Both addition and multiplication are defined pointwse, i.e. if $\sigma, \mu \in C^0(M, \mathbf{A})$ then

$$(\sigma + \mu)x = \sigma(x) + \mu(x),$$

$$(\sigma\,\mu)x = \sigma(x)\,\mu(x) \qquad \forall\, x \in M.$$

Then $C^0(M, \mathbf{A})$ is an algebra over C, and if we introduce the norm

$$\|\sigma\| = \sup\, \|\sigma(x)\|,\ x \in M$$

then it becomes a C^* - algebra. The norm on the right is the C^* - norm of $\mathbf{A}$.

Now we shall look at some of the derivations of this algebra. The aim is not to classify, or even to list, all derivations, but rather to obtain enough information to enable us to identify those in which we are interested.

i) Let Δ be a derivation of the algebra A. Then Δ extends straightforwardly to a derivation D_Δ of $C^0(M, \mathrm{A})$:

$$(D_\Delta\ \sigma)x = \Delta\,\sigma(x) \qquad \forall\ \sigma \in C^0(M, \mathrm{A}),\ x \in M.$$

Since $\Delta\,\mathrm{c} = 0\ \forall\ \mathrm{c} \in \mathrm{C}$, $D_\Delta f = 0 \quad \forall f \in C^0(M, \mathrm{C})$.

(ii) Let $\sigma \in C^0(M, \mathrm{A})$. Then σ defines an inner derivation of $C^0(M, \mathrm{A})$ by the Lie bracket

$$(D_\sigma\,\mu)x = \sigma(x)\,\mu(x) - \mu(x)\,\sigma(x).$$

Clearly, $D_\sigma\,\mu = 0$ if either σ or $\mu \in C^0(M, \mathrm{C})$.

(iii) Let D be a derivation of $C^0(M, \mathrm{A})$. Then if $\sigma \in C^0(M, \mathrm{A})$, σD is a derivation of $C^0(M, \mathrm{A})$ if $D = 0$ or if $\sigma \in C^0(M, C)$. If $\sigma \notin C^0(M, C)$ then σD is a derivation only in the non-generic case $(\sigma\tau - \tau\sigma)\,D\mu = 0\ \forall\ \tau \in C^0(M, \mathrm{A})$.

We shall say that a nonzero derivation of $C^\infty(M, \mathrm{A})$ is *vertical* if $Df = 0\ \forall f \in C^0(M, \mathrm{C})$. all inner derivations are vertical.

Next, we shall say that a nonzero derivation of $C^\infty(M, \mathrm{R}) \subset C^0(M, \mathrm{A})$ is a *horizontal* derivation. A horizontal derivation is clearly a vector field on M. To find an interpretation for vertical derivations — rather, a subset of the vertical derivations in which we are particularly interested, consider the following situation:

The base M of the bundle B is the real line R, and $\Omega(t)$ acts as follows on B:

$$\Omega(t)(x, \psi) = (x + t, U(t)\,\psi)$$

Here $U(t)$ is a unitary operator on **H** which satisfies

$$U(t)\ U(t') = U(t + t'). \qquad U(t)^{-1} = U(-t).$$

Therefore, by Stone's theorem, there exists a self-adjoint operator H on **H** such that

$$U(t) = \exp\ (iHt).$$

The constant section

$$H(x) = H\ \forall\ x \in M$$

is a vertical derivation of the algebra $C^0(M, \mathrm{A})$. Being constant, it commutes with every horizontal derivation, and therefore the formula

$$\Omega(t) = \exp\ (iHt)\ \exp(Xt) = \exp(iH + X)\,t \tag{3}$$

gives a well-defined one-parameter group of bundle maps, generated by $iH + X$, where H generates the motion on the fibres, and X that on the base space.

To recover the physics, we have to introduce the coupling constant explicitly, and write, instead of (3), the formula

$$\Omega(t) = \exp\ g\,(iH + X)\,t. \tag{4}$$

Thus, as $g \to 0$, all motion comes to a stop. The final formula is

$$\Omega(t)(x, \psi) = (x + t, \exp(igHt)\psi). \quad (5)$$

If we assume that we are using the Fock representation on each fibre, and interpret gH as the interaction Hamiltonian of the interaction picture (written in terms of the fields defined on B as earlier), then formula (5) reproduces the Feynman-Dyson perturbation theory exactly, and bypasses the problems due to the theorems of Haag and Baumann, because motion on the fibres here is accompanied by motion on the base space!

PHYSICAL INTERPRETATION

In the simple example we have just considered, the base space was R, a timelike line of the "instant" parameter" in covariant (interaction picture) perturbation theory. On each point of R we had a Hilbert space, the Fock representation of the ETCR algebra, and a free field, or a set of free fields. Thus we had only trivial generalizations of the free-field Wightman functions. To introduce the interaction, we had essentially to fall back on the Lagrangian formalism and the interaction picture. However, in our scheme the coupling constant played a dual role: that of inducing the motion on the fibres which led to the Feynman-Dyson perturbation theory, and that of inducing a motion on the base space which allowed us to escape the theorems of Haag and Baumann. Fine! But does this make any physical sense?

Let us agree, temporarily, to use the word "time" instead of the longer "instant parameter" of covariant perturbation theory. Then, in our scheme, states of an interacting system at different times *cannot be superposed*, for they belong to disjoint Hilbert spaces. We have, with great deliberation, avoided changing the theory of free fields, in which states at different "times" can indeed be superposed. Does this dichotomy make sense? Yes, because for free fields there is neither observer nor observation. Hence the theory of free fields can, indeed should, be chosen for mathematical convenience alone, if there is a choice. But the moment we introduce an interaction, we enter into a different world. It is there that we have to ask the question: Given a fixed observer, does it make physical sense to superpose states at different times – irrespective of the existence of a "parallel transport" or "connection" along the time axis, natural or otherwise? Is there any experiment in which we can prepare a state of time t_1, another at time t_2 (with $|t_1 - t_2|$ sufficiently large to overcome the inherent imprecision of observation), superpose them, and observe the result at time t_3? If the answer to the above question is in the negative, then our formalism makes sense; if it is in the affirmative, then it does not.

Suppose that the answer is in the negative, and that our formalism (to that extent) makes sense. We then note that it makes a clear distinction between "dynamical" time in which proceses take place, and non-dynamical time, which says that, under certain circumstances, it does not matter where one chooses the origin of the time axis. Perhaps the time has come to gve a felicitous new interpretation to Wigner's terminology of "active" and "passive" transformations. An active transformation is one in which the passage of time changes the Hilbert space from one moment to the next; a passive transformation is one which can be implemented within the same Hilbert space, and is associated with free, or inertial motion. But then: Why has this not been noticed in quantum mechanics? Why is T-violation, if that is what it is, so small?

Let us now look at the clearly positive aspects of the above formalism. What we have done is to avoid some of the really difficult questions of "field quantization" on a single Hilbert space, and learnt instead to work with families of free quantized fields. If this world-view is correct, it surely implies that *field quantization will not introduce new problems which are not present at the semiclassical level.* It makes the indefinite metric method of quantization (Gupter-Bleuler) much more acceptable, both for massless and massive gauge fields.

Finally, let us consider the case in which the base space is a 4-manifold, equipped with a Lorentz structure, and consider the three particle interactions, strong, electromagnetic and weak. With each of these interactions there is now associated a timelike vector field on the base manifold. There appears to be no *a priori* reason to believe that these three timelike vector fields would be the same. They would, if the three interactions were parts of a single, unified interaction, that is, unification is not purely an aesthetic demand. On the other hand, it might be interesting to explore the possibility that the vector fields associated with the strong and the electroweak interactions are *not* the same. Why shouldn't quarks be confined if time has not the same meaning for the strong interaction as it has for the electroweak interaction?

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REFERENCES

1. J. M. Jauch and F. Rohrlich, "The Theory of Photons and Electrons," Addison-Wesley, Cambridge, Mass. (1955).
2. S. S. Schweber, "An Introduction to Relativistic Quantum Field Theory," Harper and Row, New York (1961).
3. A.S. Wightman, "Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields," Cargèse Lectures (1964).
4. R. Haag, "On Quantum Field Theories," Det Kgl. Danske Vidensk. Selsk. Mat.-fys. Medd. 29: No. 12 (1955).
5. R.F. Streater and A.S. Wightman, "PCT, Spin and Statistics, and All That," Benjamin, New York (1964).
6. K. Baumann, On Relativistic Irreducible Quantum Fields Fulfilling CCr, *in*: "IXth International Congress on Mathematical Physics, " B. Simon, A. Truman and I.M. Davies, eds., Adam Hilger, Bristol (1989), and references cited therein.
7. R.N. Sen, "Galilei Invariance and Superfluidity II," Int. J. Theoret. Phys., 8: 277 (1973).
8. R.N. Sen, The Galilei Group and Landau Excitations, *in*: "Statistical Mechanics and Field Theory", R.N. Sen and C. Weil, eds., Israel Universities Press, Jerusalem (1972).
9. E. Inönü and E.P. Wigner, "Representations of the Galilei Group," Nuovo Cimento, IX: 705 (1952).
10. R.N. Sen, "Nonrelativistic Zero-Mass Systems", Göttingen lectures, 1973-74 (unpublished).
11. H.J. Borchers and R.N. Sen, "Relativity Groups in the Presence of Matter," Commun. Math. Phys. 42:101 (1975).
12. R.N. Sen, "Theory of Symmetry in the Quantum Mechanics of Infinite Systems, I, II," Physica 94A: 39, 55 (1978).
13. R.N. Sen, "The Representation of Lie Groups by Bundle Maps," J. Math. Phys., 27: 2002 (1986).
14. S.K. Bose and R.N. Sen, "Implementing the Bogolubov-Valatin Transformations on a Hilbert Bundle", Rep. Math. Phys. 23: 139 (1986).

A GLOBAL THEORY OF PARAMETRIZED QUANTUM MECHANICS

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ABSTRACT

We study the dependence of energy eigenvalues and eigenfunctions on parameters in the Hamiltonian from a global perspective. This dependence can be realized by topological quantum numbers in general. A general conservation law is found for these additive topological charges in the metamorphoses of energy bands. Many topological properties of "accidental" degeneracies are revealed with these results. In particular, global constraints can make degeneracies more generic(smaller codimension). Non-trivial topological charges can also tie energy bands together and make them inseparable. Generally, non-abelian structures arise due to metamorphoses of energy bands. These conclusions are very useful in understanding the meaning of higher cocycles in gauge theories.

In complex quantum systems, we often have to deal with families of Hamiltonian operators. These families can be parametrized by some space. As we move in the parameter space, the eigenvalues and eigenfunctions change accordingly. For example, as we learned in solid state physics, the quasi-momentum in a periodic potential can be regarded as parameter on a momentum torus(the Brillouin zone) for Hamiltonians on a spatial torus(the unit cell in the lattice). As the quasi-momentum changes, energy levels become bands. In chiral gauge theories, we can regard families of gauge connections as parametrizing the chiral Dirac Hamiltonians. As we shall see later, the global behavior of these bands that follows is dictated by gauge theory cocycles of even(2,4,6,...) order, while the odd cocycles have an correponding impact on the Euclidean formulation of the theory. In quantum theories, one is particularly interested in degeneracies among energy levels. How does one characterize the global behavior of the eigenvalues, the degeneracies among them and the eigenfunctions? In what follows, we describe a general method for achieving this end. The issue is basically a homotopy question, but we point out that the most useful mathematical constucts in this context are the fiber bundles and their classification theory rather than homotopy groups(homotopy theory is not limited to homotopy groups). The context of previous discussions on the topology of bands were limited by the use of homotopy groups or some variants thereof. Bundle theory on the other hand gives very general conclusions almost effortlessly. Applying the theory, we describe the possible patterns of knotting together of bands by toplogical charges and the metamorphoses of these "accidental" degeneracies under deformations. We also discuss how topological constraints give new counting rules for codimensions of degeneracies, which tell of how generic(typical) the specific degeneracies are.

Let us then set up the general framework for this theory. We consider a family of Hamiltonians $\{H(y)|y \in Y\}$, where Y is the parameter space and y denotes all the background parameters. Each $H(y)$ is assumed to have a discrete spectrum for each y. As y is varied, eigenvalues change continuously. The collection of the eigenvalues for all y then form bands. Some bands can touch each other at various places in the parameter space Y. If a set of k bands are connected in this fashion without any one of these bands touching other bands, we call this a connected k-band(Fig.1). A 1-band(non-degenerate band) is a band that does not touch any other band. At any point in Y then, every non-degenerate eigenvalue has its one-dimensional eigenspace; a k-fold degenerate eigenvalue has its k-dimensional eigenspace. Our proposition[1] is that *for a connected k-band, the direct sums of the eigenspaces of the eigenvalues form a continuous family of vector spaces on Y; i.e., the direct sums form a vector bundle of rank k on Y.* This proposition will allow us to describe topological properties of bands very easily. It also gives a very graphic description of bundle theory in quantum systems.

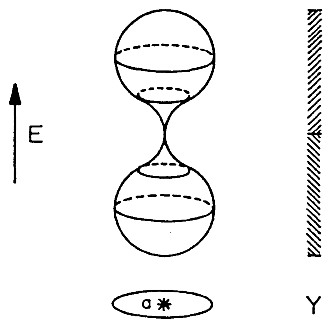

Fig. 1 Connected 2-band

The first general conclusion we can draw is that the most general topological(*i.e.*, homotopy) invariant for an k-band takes value in $Vect_k(Y)$(set of isomorphism classes of k-dimensional vector bundles on Y). This conclusion holds for arbitrary k and Y. According to the theory of bundle classification, this can always be expressed in terms of homotopy(not homotopy groups). There is a natural identification $Vect_k(Y)$ to the set of homotopy classes of maps from Y to G_k, $[Y, G_k]$, where G_k is the infinite Grassmannian. The correspondence is given by the map whose pull-back of the universal $U(k)(O(k)$ for the real case) bundle on G_k gives the associated principal bundle for the vector bundle in $Vect(Y)$. In some simple cases, $Vect_k(Y)$ can be expressed in terms of homotopy groups or cohomology groups. For example, for a non-degenerate band, $Vect_1(Y) = H^2(Y, Z)$ for the complex case and $= H^1(Y, Z_2)$ for the real case. For the torus T^n, $H^2(T^n, Z) = Z^{(n+1)n/2}$, $H^1(T^n, Z_2) = (Z_2)^n$. Notive On the other hand, if the parameter is a sphere $Y = S^n$, $Vect_k(Y) = \pi_{n-1}(U(k))$ for the complex case and $= \pi_{n-1}(O(k))$.

Secondly, the topological invariants describe structurally stable degeneracies. For example, suppose that the parameter space is a 6-sphere and there is a 3-band with non-vanishing 3rd Chern character. Obviously, it is logically possible for such a band that three eigenvalues are always degenerate on the whole 6-sphere. But this is not stable. Under generic small deformations of this six parameter family, the degeneracy will only remain at isolated points on the sphere. These points, however, cannot be deformed away because the topological index(3rd Chern character in this case) is a homotopy invariant. Similarly, for $Y = S^6 \times S^{20}$ for instance, a non-vanishing 3rd Chern character implies degeneracies of codimension 6. Generally, whether a connected k-band can split into smaller bands without interacting with other bands is determined by whether the isomorphism class in $Vect_k(Y)$ correspond to the direct sum of lower dimensional vector bundles. A codimension counting rule with topological constraints can also be stated as follows. If a cohomological characteristic class in dimension d obstructs the splitting, then there are degeneracies at codimension d. This is because the vector bundle can be pulled back to d-dimensional skeletons of Y without losing the characteristic class. This is to be compared to the codimension counting without topological constraints: for degeneracies of multiplicities m_1, m_2, m_3, ..., to occur at the same time the codimension is $\sum_i (m_i^2 - 1)$ for complex Hamiltonians; $\sum_i (m_i - 1)(m_i + 2)/2$ for the real case; and $\sum_i (m_i^2 - 1)$ for skew imaginary case counting only the positive eigenvalues.

The third point is about merging and re-splitting of bands. The question is what relationship exists between the bands before and those after this happens. We call this situation metamorphosis of energy bands. This can be studied with a deformation parameter τ. As τ changes, bands merge and split. Taking the direct sum of eigenspaces for all participating bands, globally, this results in the total bundle. The summation operation $\oplus$ gives $Vect(Y)$(set of isomorphism classes of vector bundles regardless of their rank) a abelian semi-group structure. Because of the homotopy invariance of bundle class, the isomorphism class of the total bundle is a constant. In other words, *the general topological indices for bands, which takes values in $Vect(Y)$ are conserved in metamorphises of bands.* One corollary is that the Chern characters of the bands are conserved. Likewise, any other topological index which is additive with respect to $\oplus$ is conserved. What happens during the meatamorphsis, therefore, can also be described as exchange of topological charges among the bands.

As illustration of these concepts, we now describe the spectral meaning of higher cocycles in gauge theories. Recall that gauge theory cocycles can be constructed through descent from the Chern-Simon form in higher dimensions to the correct space(or space-time) dimension. The first cocycle in, say, four dimensions and the second cocycle three dimensions manifest themselves as anomalies of gauge symmetries in theories with chiral fermions. Take for example the spatial manifold as S^3 and a gauge group $SU(4)$ and chiral Dirac Hamiltonian in the fundamental representation. The 4-cocycle is a non-trivial cocycle. The group $\mathcal{G}$ of local gauge transformations has the homotopy type of $\Omega^3 SU(4)$. In particular, $\pi_4(\mathcal{G}) = Z$ which comes from $\pi_7(SU(4)) = Z$. We take a 4-sphere in $\mathcal{G}$ which represents the generator of the homotopy group. We then study the spectra of the family of chiral Dirac Hamiltonians parametrized by the 4-sphere and a τ parameter which runs from 0 to 1, $\{i\sigma \cdot D(A(\tau, g))\}$ where $A(\tau, g) = (1-\tau)A_0 + g(A_1 + d)g^{-1}$ and g is in the S^4 we chose. Here, we can identify the 4-sphere to the parameter space Y and τ the τ above. As τ goes from 0 to 1, the bands of positive energy and the bands of negative energy can interact. At $\tau = 0$, all topological charges are trivial because A_0 does not depend on g. But as a result of band collisions, the bands can acquire non-trivial topological charges at $\tau = 1$ through exchanges at zero energy. The 4-th cocycle is the total exchange of the second Chern character of bands at zero energy. The simplest senario for this to happen is for the two 2-bands nearest to zero, one above, one below, to collide at zero energy. In Fig. 2, we schematically show how this happens. The bands are unknotted before the collision and knotted after the collision. The knotted band carries a second Chern character equal to $+1, -1$ respectively for the upper and lower band. Similarly, the 6rd, 8th cocycles, *etc.*, give the 3rd, 4th ... Chern characters of the positive energy(or negative energy) bands.

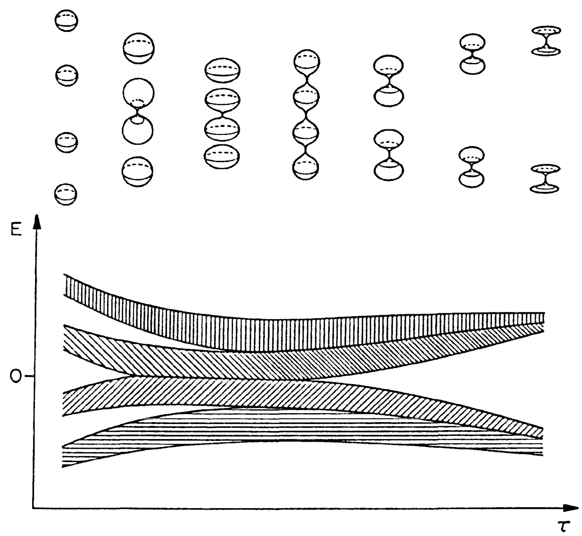

Fig. 2

Just as in the anomaly case, this Hamiltonian view has a counterpart in the Eulcidean version for the odd cocycles. These give the $SU(2)$, $SU(3)$,... windings. They can be related to multi-point functions of currents with some "flavor" indices. The 3-cocycle, for instance, gives the three point function of $D_i^j \cdot (\bar{\psi}\gamma(\gamma_5 - 1)\lambda_j \tau^a \psi)$. Here, λ has gauge group indices and τ carries indices relating to different species(generations) of fermions. This is very similar to the anomaly equation. But it only appears for currents with extra indices. The higher cocycles therefore can be relevant for studying models that are sensitive to generation structures.

To conclude, we have presented a theory of degeneracies in quantum mechanics with parameters with emphasis of the role of global topology. [1] and [2] contain more details and an explicitly worked out two dimensional version of the example discussed above.

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REFERENCES

1. L.N. Chang, Y. Liang, Mod. Phys. Lett. **A3**,1839(1988)
2. L.N. Chang, Y. Liang, "A Topological Theory of Parametrized Quantum Mechanics", preprint, April, 1989.

PRINCIPAL BUNDLES VERSUS LIE GROUPOIDS IN GAUGE THEORY

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ABSTRACT. This talk discusses the relative merits of using connections in Lie groupoids instead of connections in principal bundles for the description of gauge theories. The basic definitions of groupoids as well as Lie groupoids and Lie algebroids are recalled. After recalling the Atiyah definition of a connection in terms of a splitting of an exact sequence of vector bundles, I review the definitions of connections and curvatures for Lie groupoids. I briefly mention the differences between the group of automorphisms of a Lie groupoid associated to a principal bundle, from the group of gauge transformations of the latter. I discuss the subtle differences arising from this when applied to gauge theories. The final section contains some conjectures and speculations relative to possible use of groupoids in quantizing gauge theories as well as formulating quantum theories on (pseudo-)Riemannian manifolds.

1 Introduction

It usually takes physicists between 10 and 25 years to accept and adopt new mathematical concepts for the description of known physical facts. One has only to think of the term "Gruppenpest" used in the late 1920-s and early 1930-s to designate the use of group-theoretical methods in quantum mechanics, which turned into a common tool for physics only in the 1940's, or Wigner's construction of irreducible unitary representations of the Poincaré group in 1939, which was not fully appreciated by theoretical physicists before, say, the middle or late 1950's. Similarly, although some adventurous spirits advocated the use of connections in principal bundles as early as 1963-1965 (E. Lubkin, R. Hermann, M.E. Mayer, among others) physicists resisted their use till the middle or late 1970's.[2]

In another area, much discussed at this conference, the *Doplicher-Haag-Roberts theory of superselection sectors* has been known since 1968, and its relation to Hopf-algebraic notions was noticed as early as 1970; however only recent work (some of it presented here by Fredenhagen and Schroer) is beginning to catch the interest of a wider audience.

One should, therefore, not be too surprised if the use of Lie-groupoid techniques will resurface in a few years, and become the accepted tool for the description of what will then take the place of

[1]The author would like to apologize to all those who have advocated the use of groupoids in gauge theories, and whom he does not mention explicitly in the references. Time and space limitations have made it impossible to carry out a thorough literature search; this will be corrected in an extended version, to be published elsewhere.

[2]There is an, apocryphal story that, at about that time, a physicist asked a famous a mathematician at the Institute for Advanced Study about the possible use of fiber bundles in gauge theory . The reply was: "... but your bundles are always trivial...," which convinced the physicist not to consider the subject for another decade.

gauge theories. I want to use this occasion to amplify some of the remarks on groupoids I made at the preceding Conference at Lake Como [May87], stressing in particular some recent results which came to my attention during the past two years.

In spite of the fact that the notions of *Lie groupoid* and *Lie algebroid*, their connection theory, and their relation to principal bundles, have formed the subject of a beautiful book by Kirill Mackenzie [Mac87] (which in turn was reviewed by Kumpera [Kum88], few differential geometers (let alone mathematical physicists) are aware of their existence, or of their potential uses in physics. I will therefore devote a good part of this contribution to definitions and examples, and will only outline the potential physical uses of the concepts. A more detailed article (and a chapter in my forthcoming book [May90]) will provide the needed details.

A good survey of the history of the groupoid concept can be found in [Bro88],[Mac87], and [Kum88]. Their use in gauge theories (explicitly, or implicitly, as Wilson parallel transport) has surfaced from time to time [work by Mandelstam, Bialynicki-Birula, Loos, Mayer, Aref'eva, Menskiĭ , Gross]. Although the appearance of groupoids of parallel transport along reparametrization classes of paths is probably the easiest to explain (see, e.g., [May87]), I will not do so here, but rather will try to emphasize the equivalence of connection theory in Lie groupoids to that in principal bundles (based on a *splitting of* the *Atiyah sequence of vector bundles.* This could lead to a natural reformulation of classical Yang-Mills theory in terms of so-called *PGB*-groupoids and their Lie algebroids. I will not have time to explain the subtle difference between the automorphism group of such a *PGB*-groupoid and the group of gauge transformations of the underlying (if this is the correct term) principal bundle. one could argue that a study of this automorphism group may shed additional light on BRS- symmetry. Finally, since measured groupoids play an important role in the theory of C*-algebras and in noncommutative differential geometry ([Con82], [Ren80], [Ren82], [Kas82]) it is conceivable that a representation theory of Lie groupoids by measured groupoids (or more general "noncommutative" objects) may lead to a new view of quantum gauge theory, different from both the Kugo-Ojima canonical quantization, and the quantization in terms of path integrals used by most physicists.

Space restrictions and time pressures do not allow me to discuss these topics fully. A more detailed account will be published as an article, and will form a chapter in my forthcoming book on applications of differential geometry to physics.

2 What are groupoids

Definition 2.1 *A* groupoid $\Omega = (\Gamma, B)$ *consists of two sets, the* groupoid Γ *and the* base B, *together with two maps* $s, t : \Gamma \to B$ *(the* source *and* target *projections), and a map* $\varepsilon : B \to \Gamma$ *(the* object inclusion map*).* Γ *can be thought of as a collection of transformations ("*arrows*")* ν, η, ξ, *of the set* B *of* objects *forming the base of the groupoid, with a partially defined composition (multiplication)* $(\nu, \xi) \mapsto \nu\xi$. *The elements of* B *will be denoted by* $x, y, z \in B$, *and their images produced by the* "object inclusion map" ε *in* Γ *are denoted by* $\tilde{x}, \tilde{y}, \tilde{z} \in \Gamma$, *respectively. The* " object inclusion map" ε *identifies elements of the* base space B *with elements of* Γ *called* units *in (the set of units is often denoted by* Γ^0*). If* γ *is an arrow from* x *to* y *then* $x := s(\gamma)$ *is called its* source *and* $y := t(\gamma)$ *is called its* target *(or source and target* projection, *respectively. The reversed arrow is called the inverse of* γ *and is denoted by* γ^{-1}. *The composition of arrows* $\xi\eta$ *is* defined only *if the target of* η *is the source of* ξ : $t(\eta) = s(\xi)$ *(mappings compose from right to left!). Composition is* associative *iff both terms* $\xi(\eta\nu)$ *and* $(\xi\eta)\nu$ *are defined.*

Obviously, for any $\gamma \in \Gamma$*:*

(1) $$s(\gamma^{-1}) = t(\gamma),$$

(2) $$t(\gamma^{-1}) = s(\gamma),$$

(3) $$\gamma^{-1}\gamma = \widetilde{s(\gamma)},$$

(4) $$\gamma\gamma^{-1} = \widetilde{t(\gamma)}.$$

This set of equations is the reason why the arrows $\tilde{x}$ corresponding to the objects x are called *units* or *identities*. It is easy to see that they actually behave like units, i.e., if $s(\xi) = x$, $t(\xi) = y$, then if $\eta \in \Gamma$ is such that $s(\eta) = y$ and $\xi\eta = \eta$, then $\eta = \tilde{x}$, etc.

The set of *composable pairs* of arrows is denoted by Γ^2, or by $\Gamma * \Gamma$. If $\xi \in \Gamma$ we denote the set of its possible sources by $d(\xi) = \xi^{-1}\xi$ and call this set the *domain of* ξ and the set of its possible targets by $r(\xi) = \xi\xi^{-1}$ and call it the *range of* ξ.

For $x, y \in B$ we will denote the set of arrows which have x as a source by $\Gamma_x = s^{-1}(x)$, and the set of arrows which have y as target by $\Gamma^y = t^{-1}(y)$. The set of arrows which have x as a source and y as target is denoted by $\Gamma_x^y = \Gamma_x \cap \Gamma^y$. In the language of fibrations, Γ_x is the s-fiber (*source-fiber*) over x, and Γ^y is the t-fiber (*target-fiber*) over y. For two subsets A and C of B one uses the similar notations: $\Gamma_A = s^{-1}(A)$, $\Gamma^C = t^{-1}(C)$ and $\Gamma_A^C = t^{-1}(C) \cap s^{-1}(A)$. The restriction of the groupoid Γ to a subset $A \subset B$ is $\Gamma\mid_A = \Gamma_A^A$. If A consists of a single element $A = \{x\}$ then $\Gamma_x^x = G(x)$ is the *isotropy group* (or *vertex group*) of Γ at $x \in B$.

There are two trivial extreme cases:

a. When a groupoid consists of units only (no arrows), it is just a space without any structure; it can be identified with its base B; following [Mac87] we call such a groupoid a *base groupoid* (see Example Sec 2.1**a** below).

b. when the groupoid has only one object (unit), i.e., the arrows are always composable, and the groupoid reduces to a *group*.

It is sometimes convenient to identify the objects with the units or identities, i.e., omit the tilde and write, when defined: $x\gamma = \gamma$ and/or $\gamma x = x$.

The definition of a *subgroupoid* is the obvious one. The set of units $\Gamma^0 = \tilde{B}$ is sometimes called the *base subgroupoid*.

A groupoid is called *transitive* if for any two units x, y there exists an arrow α such that $x = s(\alpha), y = t(\alpha)$ (algebraists call such groupoids connected). This is true if the map (s, t) from Γ to $B \times B = \epsilon^{-1}(\Gamma^0 \times \Gamma^0)$ (called the *anchor* by Mackenzie [Mac87]), is surjective (onto) and it is called *principal*[3] if it is injective (one-to-one).

It is sometimes convenient to think of a groupoid as "paths" (more precisely, as *homotopy classes or reparametrization classes of paths*,): the inverse path is defined, two paths can be composed if the end of one is the beginning of the next, associativity holds only if both compositions are defined, and left and right identities are defined as the addition of a zero path at the left or right end.

2.1 Examples of groupoids

This subsection lists a number of examples of groupoids; some have been introduced earlier, others will only be mentioned here

a. *Base groupoids.* As already mentioned, a set B may be regarded as a groupoid over itself with the source and target maps equal to the identity $s = t = \mathrm{Id}_B$, and every element is a unit. Such groupoids have been called by Mackenzie *base groupoids.*

b. *Action groupoid of a transformation group.* Much of the terminology introduced above can be understood in terms of the special case when the groupoid is a transformation group (G, M) acting on a manifold M on the right. The groupoid can be identified with $\Gamma = M \times G$ with the following definition of composition and inverse: the pairs (x, g) and $(y, h) \in M \times G$ are

[3]This is the terminology of [Ren80]; Mackenzie, in his book [Mac87] uses this term for what he later decided to call 'Cartan groupoids' (e-mail communication from Kirill Mackenzie). I will adopt this terminology below. The reader should be careful to avoid confusion.

composable if $y = x \cdot g$, with the result of the composition $(x, g)(x \cdot g, h) = (x, gh)$, and the inverse $(x, g)^{-1} = (x \cdot g, g^{-1})$. The unit space $\Gamma^0 = \{(x, e) | e \in G\}$ can be identified with M by the map $(x, e) \mapsto x$. It is clear that the groupoid is transitive if the group action is transitive and it is principal (in the sense defined above) if the action is free (cf. Appendix G).

c. *Equivalence relations on a set.* Consider a set E and an equivalence relation $\sim$ on E. We consider the *graph* Γ of this equivalence relation, i.e, the set of pairs $(x, y) \in E$ such that $x \sim y$ as a groupoid with the obvious composition law: the pairs $(u, v), (w, x)$ are composable iff $v = w$. The inverse is defined as the reversed pair: $(u, v)^{-1} = (v, u)$. The source and target projections are given respectively by $s(u, v) = \tilde{u} = (u, u)$, $t(u, v) = \tilde{v} = (v, v)$, thus we can identify E with the base of this groupoid. The unit space is the diagonal of the graph Γ. This also shows us that if Γ is a principal groupoid, the pair of projections (s, t) identifies this groupoid with the graph of the equivalence relation defined by the arrows.

d. *Group bundles.* A group bundle is a groupoid for which the domains and ranges coincide, i. e., $d(\alpha) = r(\alpha)$. It may be considered as the union of the isotropy groups $G(x)$. It is clear that two elements can only be composed if they lie in the same fiber.

e. *The groupoid* Γ^{-2}. The subset of composable elements of a groupoid can be given a different groupoid structure: we define the pairs (α, β) and $(\gamma, \delta) \in \Gamma^2$ to be composable iff $\gamma = \alpha\beta$ (in the original groupoid). The product and inverse are then defined by: $(\alpha, \beta)(\alpha\beta, \delta) = (\alpha, \beta\delta)$ and $(\alpha, \beta)^{-1} = (\alpha\beta, \beta^{-1})$. Denoting the target and source maps in the new groupoid by t^2, s^2, respectively, it is easy to see that $t^2(\alpha, \beta) = (\alpha, t(\beta)) = (\alpha, s(\alpha))$ and $s^2(\alpha, \beta) = (\alpha\beta, s(\alpha\beta))$. Finally, the object inclusion map $\varepsilon : \gamma \mapsto \tilde{\gamma} = (\gamma, s(\gamma))$ shows us that the unit space of Γ^2 is the original groupoid Γ. The new groupoid Γ^2 with these operations is a principal groupoid. It will be transitive iff the composition is everywhere defined, i.e., iff Γ is a group.

f. *Groupoid associated to a principal bundle.* Let $\xi = (P, G, M, \pi)$ be a principal bundle and consider the right action of G on $P \times P : (p_2, p_1) \cdot g \mapsto (p_2 \cdot g, p_1 \cdot g)$. Let us denote the orbit of the pair (p_2, p_1) by $\langle p_2, p_1 \rangle$ and the orbit space by $\Pi = (P \times P)/G$. The manifold Π can be turned into a groupoid over the base M by the following definitions: the *source and target projections* are, respectively: $s(\langle p_2, p_1 \rangle) = \pi(p_1)$, $t\langle p_2, p_1 \rangle = \pi(p_2)$; the *object inclusion map* is $x \mapsto \tilde{x} = \langle p, p \rangle$, with $p \in \pi^{-1}(x)$ any element of the fiber over x; the groupoid multiplication, when defined, is given by the following composition of orbits: $\langle p_3, p_2' \rangle \langle p_2, p_1 \rangle = \langle p_3, p_1 \cdot g_{p_2 p_2'} \rangle)$, where $g_{p_2 p_2'} \in G$ is the group element which takes p_2 into p_2' in the same fiber,i.e., $p_2' = p_2 \cdot g_{p_2 p_2'}$. In other words, the groupoid elements are composed in such a way that the "path" in the bundle space is shifted vertically to make "ends meet." This description is equivalent to the use of the *division map* [Mac87, Appendix A,1] $\delta : P \times_\pi P \to G : (p \cdot g, p) \mapsto g$ which maps the fibered product of P with itself, i. e., pairs of points in the same fiber, into the group elements which map the second point onto the first. Then the condition that the source of the second arrow be equal to the target of the first: $s\langle p_3, p_2' \rangle = t\langle p_2, p_1 \rangle$ guarantees that p_2' and p_2 are on the same fiber, i. e., belong to the fibered product $P \times_\pi P$. By means of a "gauge transformation" one can always choose representatives on the orbits so that $p_2' = p_2$, i.e. $g_{p_2 p_2'} = e \in G$ so that the groupoid multiplication becomes simply "cancellation of the middle": $\langle p_3, p_2 \rangle \langle p_2, p_1 \rangle = \langle p_3, p_1 \rangle$, i.e., the division map becomes the identity $e \in G$. The *inverse* of the orbit $\langle p_1, p_2 \rangle$ is the orbit of the transposed pair $\langle p_2, p_1 \rangle$. We will elaborate on some aspects of this example in the next section. The association of groupoids to principal bundles, and the fact that the Atiyah sequence is naturally defined in Lie groupoids, suggest the use of groupoids in gauge theory,

g. *The fundamental groupoid.* Let B be a topological space The set $\Pi(B)$ of homotopy classes of paths $c : [0, 1] \to B$ with prescribed endpoints is a groupoid with the following elements: if $[c]$ denotes a homotopy class of paths from $x \in B$ to $y \in B$ the source and target projections are $s([c]) = c(0) = x, t([c]) = c(1) = y$, the object inclusion map is $x \mapsto \tilde{x} = [\kappa_x]$, with κ_x the constant path at x ($\kappa_x(s) = x, 0 \leq s \leq 1$); partial multiplication is juxtaposition from (left to right) of paths with reparametrization (see Sec. 3.2), and the inverse of $[c]$ is $[c^{-1}], c^{-1}(s) = c(1 - s)]$. Elements $[c_1], [c_2]$ are composable only if $t([c_1] = c_1(1) = c_2(0) = s([c_2])$. Note that some authors use the reversed convention (concatenation of paths right to left).

3 Topological groupoids and Lie groupoids

A *topological groupoid* is a groupoid which is a topological space (needless to say, Hausdorff, locally compact, etc., as discussed in Appendix D) and where composition and taking of inverses (when they exist) are continuous. (Alternatively, one may require the range map $r : \Gamma \longrightarrow \Gamma^0$ to be an open map.) The properties of topological groupoids can be established locally in terms of germs. Since we will not have occasion to use general topological groupoids (which may have some rather counterintuitive roperties, we will turn directly to a more useful subclass, namely "Cartan groupoids". It turns out that an iportant subclass of topological groupoids are equivalent to Cartan principal bundles (principal bundles which are not necessarily locally trivial [Mac87]). Similarly one can define *smooth groupoids* and (see below, Sec. ??) *Lie groupoids* and their associated *Lie algebroids*, structures which are closely related to principal bundles.

4 Lie Groupoids and Algebroids

This section introduces the important notions of Lie groupoid and Lie algebroid and explores their relation to principal bundles. The terminology and definitions follow [Mac87], to which the reader is referred for full proofs of many of the asserted facts.

4.1 Differentiable and Lie groupoids

Definition 4.1 *A* differentiable groupoid *(DG)* (Γ, B) *is a groupoid where both* Γ *and* B *are differential manifolds, such that the source and target projections are surjective submersions and the object inclusion map* $\varepsilon : B \ni x \mapsto \tilde{x} \in \Gamma$, *and the partial multiplication* $\Gamma * \Gamma \longrightarrow \Gamma$ *are* smooth *maps. A morphism of DG-s is a* smooth morphism, *i.e., one where the pair of maps is smooth.*

The tangent bundle to $\Gamma * \Gamma$ is

$$(5) \qquad T\Gamma * T\Gamma = \{Y \oplus X \in T(\Gamma \times \Gamma) | T(s)(Y) = T(t)(X)\}.$$

This implies that the tangent to the partially defined multiplication follows a Leibniz rule and from this it can be shown ([Mac87, p.85]) that inversion is automatically smooth. Furthermore, it can be shown that ε is an immersion, and therefore a homeomorphism onto $\tilde{B}$, which is therefore a closed embedded submanifold of Γ. The source and target fibers are also submanifold. The definition of differentiable groupoid is due to Pradines (1966). It goes back to Ehresmann (1959). In fact, we will only be concerned with *locally trivial* DG in the sequel. However, there exists an interesting mathematical literature on non-locally-trivial DG (*microdifferentiable groupoids*), which was the motivation for discussing this more general case. The terminology is not well established, and some authors (e.g., A. Weinstein) call Mackenzie's differentiable groupoids Lie groupoids (without the requirement of local triviality).

Definition 4.2 *A* Lie groupoid *(LG) is a* locally trivial *differentiable groupoid.*

Here *locally trivial* means almost the same thing as in the fiber bundle context: A topological groupoid Γ, B is locally trivial if it is *transitive* and there exists a covering $\bigcup_i U_i = B$ of the base by open sets such that each restriction $\Gamma|_{U_i}$ is isomorphic to a trivial groupoid $U_i \times G \times U_i$ (with s, t respectively the projections on the third and first factor, G a Lie group $\varepsilon : B \ni x \mapsto \tilde{x} = (x, e, x)$, the partial multiplication $(z, g_2, w)(y, g_1 x) = (z, g_2 g_1, x)$ iff $w = y$, and the inverse element is $(y, g, x)^{-1} = (x, g^{-1}, y)$). Just as in the case of principal bundles, local triviality is easiest to describe in terms of *local sections*:

Definition 4.3 *Let* (Γ, B) *be a DG. It is locally trivial if there exists a point* $b \in B$ *an open cover* $\bigcup_i U_i = B$ *and smooth* sections $\sigma_i : U_i \longrightarrow \Gamma_b$ *such that* $t_b \circ \sigma_i = Id_{U_i}$.

It is simple to show that the differentiable subgroupoids $\Gamma_{U_i}^{U_i}$ are isomorphic to the trivial groupoids $U_i \times \Gamma_b^b \times U_i$ under the mapping

$$(6) \qquad \Sigma : (y, \gamma, x) \mapsto \sigma_i(x) \gamma \sigma_i(y)^{-1}$$

The collection of trivializing sections $\{\sigma_i\}$ is called a *section-atlas* of the groupoid.

In [Mac87, ChapterII] one can find various equivalent criteria for local triviality of topological groupoids. In Chapter III the reader can find a proof that any transitive differentiable groupoid is locally trivial, and hence a Lie groupoid.

4.2 Lie Algebroids and Lie-Algebra Bundles

The concept of *Lie algebroid* was originally intruduced by Pradines in [Pra67] in an attempt to generalize the notion of Lie algebra. In effect, a Lie algebroid is a *Lie-algebra bundle (LAB)* with some additional structure. Lie-algebra bundles made their appearance (implicitly) in Atiyah's definition of a connection, see section 5 (the original references are [Ati57],[Nic61]). Most of this section is adapted from [Mac87, Section 3.2].

Definition 4.4 *A* Lie-algebra bundle *(LAB), is a vector bundle $\mathcal{L} = (L, \pi, B)$ with a* field of Lie brackets *$[\phi, \varphi]$ defined for the vector space of smooth sections,* sections *$\phi, \varphi \in \Gamma L$, such that each $[\ ,\ ]_x : L_x \times L_x \to L_x$ is a Lie algebra bracket (i.e., bilinear, antisymmetric and satisfies the Jacobi identity). In addition, there exists a Lie algebra $\mathcal{G}$ and $\mathcal{L}$ admits an* atlas *$\{\psi_i : U_i \times \mathcal{G} \to \mathcal{L}|_{\mathcal{U}_i}\}$ where $\psi_{i,x}$ is a Lie algebra isomorphism at each $x \in B$.*

In other words, a LAB is a smooth bundle of Lie algebras, and we have already used such bundles in our discussion of connections.

Lie-algebra bundles are a special case of Lie algebroids, namely they are totally transitive algebroids. In order to define Lie algebroids in general Pradines introduced a vector bundle map from the algebroid to the tangent bundle of the base manifold, which Mackenzie has renamed the *anchor* of the algebroid, and which turned out to lead to the following useful definition:

Definition 4.5 *A* Lie algebroid *over a smooth manifold B is a vector bundle A, p, B, together with a Lie bracket $[\ ,\ ] : \Gamma A \times \Gamma A \to \Gamma A$ (biliniear, antisymmetric and Jacobi), and vector bundle map (linear on each fiber and depending smoothly on the base-point) $q : A \to TB$ called the* anchor *(French: flèche) of A with the following properties:*

$$([X,Y]) = [q(X), q(Y)], \qquad X, Y \in \Gamma A, \tag{7}$$

$$[X, fY] = f[X,Y] + q(X)(f)Y, \qquad X, Y \in \Gamma A, f \in C^\infty(B). \tag{8}$$

A Lie algebroid A is *transitive* if q is a submersion, *regular* if q is of locally constant rank, and *totally intransititve* if $q = 0$. The latter is obviously the case for a Lia algebra bundle (LAB).

The anchor contains the basic information about the Lie algebroid: It measures the relation between the bracket structure on ΓA and the ordinary Lie or Poisson bracket structure onγTB. If A is transitive, it will turn out that the right inverse of q defines a connection which is identical to the Atiyah definition introduced in Section 5. If A is regular the image of q foliates the base manifold into leaves where A is transitive.

The simplest example of a Lie algebroid is the *trivial* Lie algebroid.

5 Connections Defined as Split Exact Sequences

This section starts with a brief review of the definition of a connection, due to Michael Atiyah, which has the advantage both of containing all the definitions commonly known in the principal bundle context (i.e., in terms of a connection form, or the horizontal distribution in the tangent bundle, or the horizontal lift of tangent vectors from the tangent bundle of the base to the tangent bundle of the bundle space), and being directly applicable to the Lie algebroid of the Lie groupoid associated to the principal bundle. The reader is reminded that a split exact sequence of vector spaces or vector bundles is essentially a direct sum decomposition of the vector spaces (respectively of the fibers of the vector bundle), such that the splitting maps are the "local inverses" of the injection and surjection of the short exact sequence.

We introduce some notations: $\xi = (P, M, G, \pi)$ denotes a principal bundle with bundle space P, base space M, structure group G, and projection π. The Lie algebra of G is denoted by $\mathcal{G}$ and its bracket is $[\ \ ,\ \ ]$.

The action of an element $a \in G$ on an element $b \in G$ or a point $z \in P$ is denoted by $\mathrm{R}_a b = b \cdot a$, $\mathrm{R}_a z = z \cdot g$ (the dot may be omitted). The adjoint action of a on a group element b and a Lie algebra element $A \in \mathcal{G}$ is defined by $\mathrm{Ad}_G(a)b = a^{-1}ba$, $\mathrm{ad}_{\mathcal{G}}(a)A = (\mathrm{Ad}(a)_*)_e A$, respectively, where the subscript $*$ denotes, as usual the derivative of the map. Recall that $\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}X)$, $X \in \mathcal{G}$, and $\mathrm{ad}\, XY = [X, Y]$, $X, Y \in \mathcal{G}$.

In addition, we may consider the right action of g on P given by $z \cdot g$ as a left action of the point $z \in P$ on $G : \mathrm{L}_z g = z \cdot g$. Its derivative establishes a bijection between the Lie algebra $\mathcal{G}$ and the tangent space to the fiber of P at $p(z)$, which is in fact a Lie algebra homomorphism. Then both $T(P)$ and $T(M)$ can be considered as vector bundles over M and the quotient of $T(P)$ under the adjoint action of G on the fibers is also a vector bundle, which will be denoted by $\mathrm{Ad}(P)$.

Recall that in a principal bundle $\xi = (P, M, G, \pi)$ the vertical action of the structure group on the bundle space induces a homomorphism σ of the Lie algebra $\mathcal{G}$ of G and the Lie algebra $X(P) = \Gamma(T(P))$ of vector fields (differentiations) of P. This homomorphism assigns to each element $A \in \mathcal{G}$ a vector field $A^* \in X(P)$, which can be thought of either as the image in $T(P)$ of the action of the one-parameter subgroup $\exp(At)$, which acts on the right in P, (i.e., $[A^*, Y] = \lim_{t\to 0} \frac{1}{t}[Y - \exp(ta)_* Y]$, where Y is a vector field in $T(P)$ i.e., the bracket is the bracket of vector fields on P). Using the same reasoning, i.e., exponentiating the Lie algebra element to a group element, letting it act on P and then lifting to the tangent bundle $T(P)$, the reader will see that σ commutes with the bracket:

$$\sigma[A, B] = [A^*, B^*]. \tag{9}$$

Moreover, since the 0 element of the Lie algebra is taken into the zero vector field (the lifting of the action of the group identity e to $T(P)$), we have established that σ is a homomorphism. Since that action of G on P is along the fibers, i. e., *vertical*, the *fundamental vector* field A^* is tangent to a fiber of P, and thus, a vertical vector field. The reasoning which led to the introduction of this concept shows us that the right action of G on P induces an action on $X(P)$ given by the bracket of vector fields, i. e., the adjoint action from the right:

$$(\mathrm{R}_b)_* A^* = \{\mathrm{Ad}\ (b^{-1})A\}^*. \tag{10}$$

Another way of seeing this is to note that $(\mathrm{R}_b)_* A^*$ is induced by the one-parameter group $\mathrm{R}_{b^{-1}\exp(At)b}$ $= \mathrm{R}_b \mathrm{R}_{\exp(At)} \mathrm{R}_{b^{-1}}$, and Eq. 10 is obtained by differentiating this relation, or by noting that $b^{-1}\exp(At)b$ is the one-parameter subgroup of G with generator $\mathrm{Ad}\ (b^{-1})A = [A, B] \in \mathcal{G}$.

5.1 The Atiyah sequence

Now, consider the exact sequence of vector spaces over a given point $\pi(z) \in M$ of the base manifold induced by this mapping

$$0 \longrightarrow \mathcal{G} \xrightarrow{\sigma} T_z P \xrightarrow{\pi_*} T_{\pi(z)} M \longrightarrow 0 \tag{11}$$

In order for the sequence of linear mappings of vector spaces to be exact the image of each map must be the kernel of the next, i.e., im $\sigma =$ ker π^*. The image of the Lie algebra $\mathcal{G}$ in $T_z(p)$, consisting of all the fundamental vectors A^* at that point, forms a subspace of $T_z P$ which is projected by π_* into the null vector of $T_{p(z)}M$, i.e., the *vertical tangent vectors*. We can now patch the vector spaces together into a vector bundle over M, with the obvious group actions on $T(P)/G$ and $T(M)$ and the adjoint action induced on $\mathcal{G}$ yielding the bundle denoted above by Ad (P).

More precisely, the bundle Ad (P) is the fiber bundle with fiber $\mathcal{G}$ associated to P by the adjoint action of G on the Lie algebra: Ad $(P) = P \otimes_{\mathrm{Ad}} \mathcal{G}$, and since G is a subgroup of $T(G)$, the projection $\pi : P \to M$ makes the tangent map $\pi_* : T(P)/G \to T(M)$ into a vector bundle projection. (It helps to consider the trivial bundle case and visualize these correspondences in a local chart.) We are then led to the following *general* definition of connection on a principal bundle.

Definition 5.1 *A connection Γ on P is defined as a splitting of the exact sequence of vector bundles*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ad}(P) & \xrightarrow{i} & T(P)/G & \xrightarrow{\pi_*} & T(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M & & M & & M & & \end{array} \tag{12}$$

Here all bundles are fibered over the base space and the projections are not indicated. The vector bundle map (linear on each fiber) i is the *injection* of the bundle $\mathrm{Ad}(P) \hookrightarrow T(P)/G$ (coming from the injection of the Lie algebra $\mathcal{G} \hookrightarrow T_z(P)$) and π_* is the restriction of the tangent projection to the quotient, denoted by the same symbol.

A *splitting* of the sequence 12 is defined by a pair of vector bundle maps θ and λ which are inverses of the injection i and the projection π_*, respectively:

$$0 \longleftarrow \mathrm{Ad}(P) \xleftarrow{\theta} T(P)/G \xleftarrow{\lambda} T(M) \longleftarrow 0 \tag{13}$$

i.e.,

$$i \circ \theta = \mathrm{Id}_{T(P)}, \quad \pi_* \circ \lambda = \mathrm{Id}_{T(M)}. \tag{14}$$

This is equivalent to saying that the sequence of vector spaces (11) at each point splits into the kernel of the map $\theta, \mathrm{Ker}\,\theta = \mathrm{Hor}\ T(P)$ — the horizontal tangent vectors — and the image of λ in $T(P)$ (since G acts vertically, the horizontal spaces are the same at each point of a fiber), $\mathrm{im}\,\lambda = \mathrm{Hor}\ T(P)$, which are the *horizontal lifts* of vectors tangent to M. Thus, at each point $p \in P$ we have a splitting of the tangent space into a horizontal and vertical part

$$T_pP = \mathrm{Hor}_pP \oplus \mathrm{Ver}_pP, \tag{15}$$

the sum being a direct sum of vector spaces. It is clear from our construction that the splitting 13 depends smoothly on the point p, and is *equivariant* under the right action of G on P, in the sense that

$$\mathrm{Hor}_{p.g}P = \mathrm{R}_{g*}\mathrm{Hor}_p, \tag{16}$$

where R_g denotes the right action of G on P and R_{g*} its derivative, acting on the right on the tangent space T_pP. Each smooth vector field X on $T(P)$ thus splits in a smooth fashion into a horizontal vector field $\mathrm{Hor}\,X$ and a vertical vector field $\mathrm{Ver}\,X$, the latter being isomorphic to an element of the Lie algebra $\mathcal{G}$ of G.

Let us now relate the definition given here to the ones you are familiar with. and analyze in more detail the properties of the two maps θ and λ. We are looking for a Lie-algebra valued one-form, which in a trivialization can be identified with the "gauge potential" A For this we recall that since locally P can be considered as a product of $U \in M$ with G, the tangent space at a point $z \in P$ will locally be isomorphic to a product of $T_{\pi_* z}M$ with T_gG, the latter being in turn isomorphic to the Lie algebra $\mathcal{G} = T_eG$ of G, with $\mathrm{Ver}Z$ identified with a fundamental vector A^* associated to an element A of $\mathcal{G}$. This, allows us to identify T_aG with the vertical tangent space Ver_zP.

The mapping θ has the following properties:

i. θ is a *linear map* (as a homomorphism of vector spaces);

ii. it depends smoothly on p;

iii. in order to preserve verticality, θ must be right-equivariant under the action of G, i.e., $R_a*\theta(z) = \lambda(z \cdot a)$, where $\lambda(z)$ denotes the value of the map at the projection of z on M.

Since the fiber of Ad (P) is the Lie-algebra $\mathcal{G}$ and θ is a linear mapping, it is in fact a *Lie-algebra valued one-form with values in* $\mathcal{G}$, which can be thought of as mapping each tangent vector $Z \in T_zP$ into the Lie-algebra element $A \in g$ whose fundamental vector A^* equals the vertical part of Z and yielding zero when applied to any horizontal vector B:

$$\theta(A^*) = A, \qquad \theta(B) = 0 \tag{17}$$

This form is called the connection form associated to the connection 12

This definition of a connection Γ in a principal bundle is equivalent to the usual ones (compare with the discussion in [KN69], for a treatment which has become classical).

In place of the 1-form θ which goes from $T(P)/G$ to Ad $(P) = P\times_G\mathcal{G}$ we can consider the globally defined $\mathcal{G}$-valued 1-form ω, i.e., a section of the product bundle $P\times\mathcal{G}$, which reduces to θ when the action of G is "quotiented out,", i.e., when we consider the Ad-associated bundle Ad (P) leading to the exact sequence 12. The equivariance property of ω is expressed as $(R_a)* = \mathrm{Ad}\ (a^{-1})\omega$, i. e., for a vector field Z on P we will have

$$\langle (R_a)^*\omega, Z\rangle = \langle \mathrm{Ad}(a^{-1})\omega, Z\rangle. \tag{18}$$

6 Lie Groupoids and Gauge Theory

This section contains a few remarks about how groupoids could be made to play an important role in gauge theories. Because of space limitations these remarks are of necessity brief, and somewhat speculative. For a more detailed discussion the reader is referred to a forthcoming paper and the already mentioned book [May90] for more details.

The most obvious groupoid aspect of gauge theory is that of parallel transport, which leads to the concept of *holonomy groupoid* of a connection. In view of the fact that connections in principal bundles and connections in Lie groupoids are almost equivalent (there is the mentioned difference in their automorphism groups, which may lead to a better understanding of BRS-cohomology in the new context), is will beinteresting to discuss the holonomy groupoid in this framework.

A more interesting topic which requires careful discussion is the role representation theory of groupoids may play in a quantized gauge theory. Since measured groupoids and Haar systems [Con82], [Ren82], play an important role in the application of groupoids to C*-algebras, one would be naturally tempted to "quantize" the Lie groupoid with connection which describes the classical gauge theory, by representing it in an appropriate measured (C*-algebraic?) groupoid.

Another potential application of the concepts outlined in this talk is to quantum field theory in a gravitational background [Haa84], [Fre86]. The most natural approach to a quantum field theory on a curved gravitational background is to define the algebraic framework in tangent spaces, and to use the holonomy groupoid of the gravitational background as a means to "parallel-transport" the quantum-algebras from one tangent space to another. Curvature in the underlying manifold is associated with the gravitaional action, whereas the morphisms induced by reparametrization classes of paths will form a representation of the holonomy groupoid by morphisms of the local algebras. It will be interesting to analyze potential connections to the theory of superselection sectors.

References

[Ati57] M. F. Atiyah, Complex-analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, **85**,181–207 (1957).

[Bro88] R. Brown, Symmetry, groupoids, and higher-dimensional analogues. *Comput. Math. Applic.*, **00**, 181–207 (1988).

[Con82] A. Connes, A survey of foliations and operator algebras. In Richard Kadison, editor, *Operator Algebras and Applications, Proc. Symp. Pure Math.*, Part 1, pp. 521–628, American Mathematical Society, Providence, RI, 1982.

[Fre86] K. Fredenhagen and R. Haag, Generally Covariant Quantum Field Theory and Scaling Limits, *Desy Preprint*, **86-066** (1986).

[Haa84] R. Haag, H. Narnhofer, and U. Stein *Commun. Math. Phys.*, **94**, 219 (1984).

[KN69] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry.* Volume 1 and 2 of *Interscience Tracts, No.15*, Wiley-Interscience, New York, 1963-69.

[Kum88] A. Kumpera, Book Review: *Lie Groupoids and Lie Algebroids in Differential Geometry* by Kirill Mackenzie. *Bull. Amer. Math. Soc*, **19**, 358–365, (July 1988).

[Kas82] D. Kastler, On A. Connes Noncommutative Integration Theory. *Commun. Math. Phys.*, **85**, 99–120 (1982).

[Mac87] Kirill Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry.* Volume 124 of *London Math. Soc. Lecture Note Series*, Cambridge University Press, Cambridge, 1987. Contains a detailed discussion of the topic; see review [Kum88].

[May87] M .E. Mayer, Groupoids and Lie Bigebras in Gauge and String Theories in *P roccedings of the International Confernece on Differential-Geometric Methods in Physics, Como, August 1987*, K. Bleuler and M. Werner, Eds., Kluwer, Dordrecht, 1988

[May90] M .E. Mayer, Fiber Bundles and Gauge Theory, *To be published*, Springer Verlag 1990.

[Nic61] Helen K. Nickerson, On differential operators and connections. *Trans. Amer. Math. Soc.*, **99**, 509–539, (1961). To my knowledge the most detailed discussion of connections and curvatures in terms of the Atiyah-sequence and Frölicher-Nijenhuis brackets.

[Pra67] J. Pradines, Théorie de Lie pour les groupoides différentiables, *C.R. Acad. Sci. Paris.*, **263**, 907–910, (1966). **264**, 245–248 (1967).

[Ren80] J. Renault, A Groupoid Approach to C*-Algebras Lecture Notes in Mathematics, 793, Springer Verlag, Berlin, Heidelberg, New York, 1980.

[Ren82] J. Renault, C*-algebras of groupoids and foliations. In Richard Kadison, editor, *Operator Algebras and Applications, Proc. Symp. Pure Math.*, Part 1, P. 339, American Mathematical Society, Providence, RI, 1982.

STATIC AND AXIALLY SYMMETRIC SOLITON SOLUTIONS TO THE SELF-DUAL SU(3) AND SU(2) GAUGE FIELDS IN A EUCLIDEAN SPACE

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1. Introduction

The Euler-Lagrange equations for a non Abelian gauge theory can be derived from a Lagrangian of the form[1]

$$L=-(1/2)I_{ab}F^a_{\mu\nu}F^{b\mu\nu} \tag{1.1}$$

where I_{ab} is the metric tensor of the bilinear form;a,b=1,2,3....,d,with d being the dimension of the metric

$$F^i_{\mu\nu}=b^i_{\mu.\nu}-b^i_{\nu.\mu}-C^i{}_{jk}b^j_\mu b^k_\nu \tag{1.2}$$

$C^i{}_{jk}$ are the structure coefficients and b^i_μ are the self-dual gauge potentials.

In a 4-dimensional manifold M,we may use the $\varepsilon_{\mu\nu ab}$ symbols to define the *F,dual of the form F. Then the action of the Lagrangian L is

$$S=-\int_M tr(F\hat{}\,{}^*F) \tag{1.3}$$

where "^" is the Wedge product.

The Euler-Lagrange equations for the extrema of S are

$$D^*F=0 \tag{1.4}$$

where D,in the components language,is defined by the relation

$$D_\mu=\partial_\mu-A^a_\mu(\lambda^a/2i)g\partial_g \tag{1.5}$$

where A is a connection and ∂ means partial derivative.

If we can find a connection A such that

$$F=\mu^*F,\ \mu=\text{constant} \tag{1.6}$$

then the Euler-Lagrange equations are automatically satisfied and

$$*F=\mu**F=>F=\mu^2**F \tag{1.7}$$

or

$$F=\begin{cases}\mu^2F \text{ for Euclidean metric}\\ -\mu^2F \text{ for Lorentz metric}\end{cases} \tag{1.8}$$

Hence, if we solve $F=\mu*F$ in Euclidean space $\mu=\pm1$, but if we solve in Minkowski spacee $\mu=\pm i$ and these are the values of μ that we may have.

This result has important consequences for the choice of the gauge group. Because, if M has a Lorentz metric, then we are interested to solve $*F=+iF$ with $ig=g$(recall that F is g-valued) and this condition is not satisfied by the Lie algebras of any compact group, because we must choose non-compact Lie group like SL(n,L)or Gl(n,C) say. But this is a serious restriction since in physics the gauge groups choosen are actually compact. However, there is no restriction, if we work with $*F=\pm F$ and especially when $*F=F$(e.g self-dual fields).

Thus, in section 2. we write down the self-dual SU(3) gauge field equations and present the Belinski-Zakharov technique to obtain soliton solutions in SU(3) from a diagonal seed solution g_0. In section 3. we find explicitly the diagonal matrix function Ψ_0 and in section 4. we derive the 1-soliton solution to the self-dual SU(3) gauge field equations. In section 5. we apply the Belinski-Zakharov technique to self-dual SU(2) gauge field equations, for simplicity, to derive 1-soliton solutions from a non-diagonal seed solution g_0.

2. The Yang-Mills equations in the R-gauge

Yang[2], has written the self-dual SU(2) gauge field equations in a complexified Euclidean space in the R-gauge in terms of a real variable φ and two complex variables ρ and ρ. Also, Witten[3] studied those equations and verified that some solutions to the Yang-Mills equations are known solutions to the Einstein field equations.

Prasad[4], has shown that in the R-gauge, in a complexified Euclidean space, the self-dual equations for the SU(3) Lie group may obtained by appropriate variation of the Lagrangian density

$$L(SU(3))=\int d^4x\{\varphi_1^{-2}[(\varphi_{1\mu}\varphi_{1\nu}+\varphi_2\rho_{1\mu}\bar\rho_{1\nu})g^{\mu\nu}+\varepsilon^{\mu\nu}\varphi_2\rho_{1\mu}\bar\rho_{1\nu}]$$

$$+\varphi_2^{-2}[(\varphi_{2\mu}\varphi_{2\nu}+\varphi_1\rho_{2\mu}\bar\rho_{2\nu})g^{\mu\nu}+\varepsilon^{\mu\nu}\varphi_1\rho_{2\mu}\bar\rho_{2\nu}]+\varphi_1^{-1}\varphi_2^{-1}$$

$$[(\rho_{3\mu}-\rho_2\rho_{1\mu})(\bar\rho_{3\nu}-\bar\rho_2\bar\rho_{1\nu})(g^{\mu\nu}+\varepsilon^{\mu\nu})-\varphi_{1\mu}\varphi_{2\nu}g^{\mu\nu}]\} \tag{2.1}$$

where φ_1,φ_2 are real functions of x^μ (μ=0,1,2,3), Greek subscripts mean partial derivative in terms of x^μ; $\rho_1,\bar\rho_1,\rho_2,\bar\rho_2,\rho_3,\bar\rho_3$ are complex functions of x^μ; $d^4x=dyd\bar y dzd\bar z$

$$g_{\bar y y}=g_{y\bar y}=g_{\bar z z}=g_{z\bar z}=1 \tag{2.2}$$

$$\varepsilon^{\mu\nu}=\varepsilon^{\mu\nu\lambda\sigma}(\hat y_\lambda \hat{\bar y}_\sigma+\hat z_\lambda \hat{\bar z}_\sigma) \tag{2.3a}$$

$$\varepsilon^{\mu\nu\lambda\sigma}=1 \text{ when } \mu\nu\lambda\sigma=y\bar y z\bar z \tag{2.3b}$$

and $\hat y_\sigma, \hat{\bar y}_\beta, \hat z_\gamma, \hat{\bar z}_\sigma$ are unit vectors.

Furthermore, the self-dual field equations for the SU(3) Lie group, in the R-gauge are

$$(\vartheta_y\vartheta_{\bar y}+\vartheta_z\vartheta_{\bar z})\ln\varphi_1+(\varphi_1/\varphi_2)(\rho_{1y}\bar\rho_{1\bar y}+\rho_{1z}\bar\rho_{1\bar z})+(1/\varphi_1\varphi_2)$$

$$[(\rho_{3y}-\rho_2\rho_{1y})(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y})+(\rho_{3z}-\rho_2\rho_{1z})(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})]=0 \qquad (2.4a)$$

$$(\vartheta_y\vartheta_{\bar y}+\vartheta_z\vartheta_{\bar z})\ln(\varphi_1/\varphi_2)+(\varphi_2/\varphi_1)(\rho_{1y}\bar\rho_{1\bar y}+\rho_{1z}\bar\rho_{1\bar z})$$

$$-(\varphi_1/\varphi_2{}^2)(\rho_{2y}\bar\rho_{2\bar y}+\rho_{2z}\bar\rho_{2\bar z})=0 \qquad (2.4b)$$

$$[(\rho_{3y}-\rho_2\rho_{1y})/\varphi_1\varphi_2]_{\bar y}+[(\rho_{3z}-\rho_2\rho_{3z})/\varphi_1\varphi_2]_{\bar z}=0 \qquad (2.4c)$$

$$[(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{3\bar y})/\varphi_1\varphi_2]_y+[(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})/\varphi_1\varphi_2]_z=0 \qquad (2.4d)$$

$$(\varphi_2\rho_{1y}/\varphi_1{}^2)_{\bar y}+(\varphi_2\rho_{1z}/\varphi_1{}^2)_{\bar z}-(1/\varphi_1\varphi_2)[\bar\rho_{2\bar y}(\rho_{3y}-\rho_2\rho_{1y})+\bar\rho_{2\bar z}$$

$$(\rho_{3z}-\rho_2\rho_{1z})]=0 \qquad (2.4e)$$

$$(\varphi_2\bar\rho_{1\bar y}/\varphi_1{}^2)_y+(\varphi_2\bar\rho_{1\bar z}/\varphi_1{}^2)_z-(1/\varphi_1\varphi_2)[\rho_{2y}(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y})+\rho_{2z}$$

$$(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})]=0 \qquad (2.4f)$$

$$(\varphi_1\rho_{2y}/\varphi_2{}^2)_{\bar y}+(\varphi_1\rho_{2z}/\varphi_2{}^2)_{\bar z}+(1/\varphi_1\varphi_2)[\bar\rho_{1\bar y}(\rho_{3y}-\rho_2\rho_{1y})+\bar\rho_{1\bar z}$$

$$(\rho_{3z}-\rho_2\rho_{1z})]=0 \qquad (2.4g)$$

$$(\varphi_1\bar\rho_{2\bar y}/\varphi_2{}^2)_y+(\varphi_1\bar\rho_{2\bar z}/\varphi_2{}^2)_z+(1/\varphi_1\varphi_2)[\rho_{1y}(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y})+\rho_{1z}$$

$$(\bar\rho_{3z}-\bar\rho_2\bar\rho_{1\bar z})]=0 \qquad (2.4h)$$

where

$$y\sqrt{2}=x_1+ix_2,\ z\sqrt{2}=x_3-ix_4 \qquad (2.5)$$

We are looking to find soliton solutions to the Eqs.(2.4) when $\varphi_A,\rho_i,\bar\rho_i$ $(A=1,2;i=1,2,3)$ are functions of

$$r=\sqrt{(2y\bar y)} \text{ and } w=(1/\sqrt{2})(z+\bar z) \qquad (2.6)$$

After tedious but straightforward calculations we end up with an equation

$$\vartheta_r U+\vartheta_w V=0 \qquad (2.7)$$

where U and V are 3X3 matrices defined by the relations

$$U=rg_r g^{-1},V=rg_w g^{-1} \qquad (2.8)$$

$$g=(r^{n/6)}/\varphi_1)\begin{vmatrix} 1 & \bar\rho_1 & \bar\rho_3 \\ \rho_1 & \rho_1\bar\rho_1+\varphi_1{}^2/\varphi_2 & \rho_1\bar\rho_3+\bar\rho_2(\varphi_1{}^2/\varphi_2) \\ \rho_3 & \bar\rho_1\rho_3+\rho_2(\varphi_1{}^2/\varphi_2 & \rho_2\bar\rho_2(\varphi_1{}^2/\varphi_2)+\varphi_1\varphi_2 \end{vmatrix} \qquad (2.9)$$

with

$$g=g^+ \qquad \text{and } \det g=r^{6/n} \qquad (n=3) \qquad (2.10)$$

The Belinski-Zakharov technique[5-7] for finding soliton solutions to the Eqs.(2.4) is associated to the linear eingevalue problem

$$D_r\Psi=A\Psi \qquad \text{with} \qquad A=(rU+\lambda V)/(\lambda^2+r^2)$$

$$D_w\Psi=B\Psi \qquad \text{with} \qquad B=(rV-\lambda U)/(\lambda^2+r^2) \qquad (2.11)$$

where

$$D_r=\partial_r+2\lambda r/(\lambda^2+r^2)\partial_\lambda\,,D_w=\partial_w-2\lambda^2/(\lambda^2+r^2)\partial_\lambda \tag{2.12}$$

λ is a complex spectra parameter,$\Psi=\Psi(\lambda,r,w)$is a 3x3 matrix which satifies the condition

$$\Psi(\lambda=0,r,w)=g_0 \tag{2.13}$$

The solitonic character of the solutions of the Eqs.(2.4) is associated with solutions of the form

$$\Psi=X\Psi_0 \tag{2.14}$$

where

$$X=I+\Sigma R_k/(\lambda-\mu_k) \tag{2.15}$$

$R_k=R_k(r,w)$ are complex 3X3 matrix functions of r and w and $\mu_k=\mu_k(r,w)$ are scalar complex functions of r and w. For $\lambda=0$ the Eqs.(2.14) gives

$$g=X(\lambda=0,r,w)g_0 \tag{2.16}$$

The condition $g^+=g$ is insured regarding the expression

$$g=X(-r^2/\bar{\lambda},r,w)g_0\,[X(\lambda,r,w)]^+ \tag{2.17}$$

where $g_0{}^+=g_0$.

Knowing the Ψ_0,a solution Ψ can be generated by purely algebraic operations.Thus,if Ψ_0 is known,then from Eqs.(2.11-2.13) we have

$$D_r\Psi_0=A_0\,,D_w=B_0\Psi_0 \tag{2.18}$$

and for Ψ

$$D_rX=AX\Psi_0-XA_0\Psi_0\,,D_wX=BX\Psi_0-XB_0\Psi_0 \tag{2.19}$$

From Eq.(2.19) we obtain

$$\mu_{k,r}=2\mu_k r/(r^2+\mu_k{}^2)\,,\mu_{k,w}=-2r^2/(r^2+\mu_k{}^2) \tag{2.20}$$

The solution to the Eq.(2.20) is

$$\mu_k=a_k-w\pm\sqrt{[(a_k-w)^2+r^2]}\quad k=1,2...n \tag{2.21}$$

where a_k are arbitrary complex constants.

The new solution generated by this method is completely determined by the function Ψ_0 as following:

$$g_{ab}=(g_0)_{ab}-\Sigma\Sigma(\bar{N}_a D_{kl} N_b)/(\mu_k\bar{\mu}_l) \tag{2.22}$$

where

$$(R_k)_{ab}=n_a{}^{(k)}m_b{}^{(k)} \tag{2.23a}$$

$$n_a{}^{(k)}=D_{lk}(N_a{}^{(l)}/\bar{\mu}_l) \tag{2.23b}$$

$$N_a{}^{(l)}=m_c{}^{(l)}(g_0)_{ca} \tag{2.23c}$$

$$G_{kl}=[m_a{}^{(k)}(g_0)_{ab}\bar{m}_b{}^{(k)}]/)r^2+\mu_k\bar{\mu}_l)=G_{lk} \quad (2.23d)$$

$$m_a{}^{(k)}=m_{0c}{}^{(k)}[\Psi_0{}^{-1}(\mu_k,r,w)]_{ca} \quad (2.23e)$$

$$D_{lk}=(G_{lk})^{-1} \quad (2.23d)$$

with $m_{0c}{}^{(k)}$ arbitrary complex constants.

The Eq.(2.22) satisfies the condition $g=g^+$ but not the condition $\det g=r^2$.The problem can be overcomed defining a new matrix solution to the Eq.(2.4)

$$g^{ph}=rg(\det g)^{-1/2} \quad (2.24)$$

which satisfies the conditions (2.10)

3.The function Ψ_0.

The function Ψ_0 satisfies the equations

$$D_r\Psi_0=A_0\Psi_0,D_w\Psi_0=B_0\Psi_0 \quad \text{with } A_0=A(\lambda=0,r,w);B_0=B(\lambda=0,r,w) \quad (3.1a)$$

and $\Psi_0(\lambda=0,r,w)=g_0$. (3.1b)

To find out the explicit form of the matrix Ψ_0 we consider that g_0 and Ψ_0 are diagonal matrices and that the matrix g_0 is

$$(g_0)_{11}=r^{v1}e^{X_1} \quad (3.2a)$$

$$(g_0)_{22}=r^{v2}e^{X_2} \quad (3.2b)$$

$$(g_0)_{33}=r^{v3}e^{X_3} \quad (3.2c)$$

where

$$X_i=-[a_iw+c_i(r^2/2-w^2)] \text{ with } i=1,2,3 \quad (3.3)$$

$$v1=2(1-b)/n,v2=2(1+b)/n,\text{and } v3=k \quad (3.4)$$

and a_i,c_i are arbitrary real constants.

The fact that $\det g_0=r^2$,implies the condition

$$\Sigma X_i=0\Rightarrow\Sigma a_i=\Sigma c_i=0 \quad (3.5)$$

Besides,g_0 has to satisfy the Eq.(2.7)

$$\partial_rU_0+\partial_wV_0=0 \text{ where } U_0=U(\lambda=0,r,w);V_0=V(\lambda=0,r,w) \quad (3.6)$$

which implies that the X_i have to satisfy the equation

$$(rX_{i,r})_{,r}+(rX_{i,r})_{,w}=0 \quad (3.7)$$

e.g the usual Laplace equation in cylindrical coordinates.

From the Eq.(3.1) we find the solution Ψ_0 which is

$$(\Psi_0)_{11}=(r^2-2\lambda w-\lambda^2)^{v1/2}e^{Y_1} \quad (3.8a)$$

$$(\Psi_0)_{22}=(r^2-2\lambda w-\lambda^2)^{v2/2}e^{Y_2} \quad (3.8b)$$

$$(\Psi_0)_{33}=(r^2-2\lambda w-\lambda^2)^{v3/2}e^{Y_3} \quad (3.8c)$$

$$(\Psi_0)_{ij}=0,\ i\neq j \quad (3.8d)$$

where

$$Y_i=-[a_i(w+\lambda/2)+c_i(r^2/2-(w+\lambda/2)^2)],i=1,2,3 \quad (3.9)$$

4.The 1-soliton solution

In the previous sections we have established the theory to obtain soliton solutions to the Eq.(2.4). Thus,if the matrix X has only one pole e.g $X=I+R_1/(\lambda-\mu_1)$,then the 1-soliton solution to Eq.(2.4) is

$$g_{11}=-(r^{2/n}/\varphi_1|\mu|^2\Delta)\{|\mu|^2[|m_{02}{}^{(1)}\Psi_1\Psi_3|^2\varphi_1\varphi_2-|m_{03}{}^{(1)}\Psi_1\Psi_2|^2$$

$$r^{k-2/n}(\varphi_1{}^2\varphi_2)]+r^2|m_{01}{}^{(1)}\Psi_1\Psi_3|^2 \quad (4.1a)$$

$$g_{12}=\bar{g}_{21}=r^{2/n}m_{01}{}^{(1)}\bar{m}_{02}{}^{(1)}((r^2+|\mu|^2)/(|\mu|^2\Delta))(2\bar{\alpha}\bar{\mu}\varphi_2/\Psi_3) \quad (4.1b)$$

$$g_{13}=\bar{g}_{31}=-r^k m_{01}{}^{(1)}\bar{m}_{03}{}^{(1)}((r^2+|\mu|^2)/(|\mu|^2\Delta))(2\bar{\alpha}\bar{\mu}\varphi_1/\Psi_2\varphi_2) \quad (4.1c)$$

$$g_{22}=(r^{2/n}\varphi_2/|\mu|^2\Delta)\{|m_{01}{}^{(1)}\Psi_2\Psi_3|^2$$

$$+r^{k-2/n}|m_{03}{}^{(1)}\Psi_1\Psi_2|^2(\varphi_1{}^2/\varphi_2)-r^2\varphi_1\varphi_2|m_{02}{}^{(1)}\Psi_1\Psi_2|^2\} \quad (4.1d)$$

$$g_{23}=\bar{g}_{32}=r^k m_{02}{}^{(1)}m_{03}{}^{(1)}((r^2+|\mu|^2)/(|\mu|^2\Delta))(2\bar{\alpha}\bar{\mu}\varphi_1{}^2/\Psi_1) \quad (4.1e)$$

$$g_{33}=(r^k\varphi_1/|\mu|^2\Delta\varphi_2)\{|\mu|^2[|m_{01}{}^{(1)}\Psi_2\Psi_3|^2-|m_{02}{}^{(1)}\Psi_1\Psi_3|^2]$$

$$-r^{2(1+k-1/n)}|m_{03}{}^{(1)}\Psi_1\Psi_2|^2(\varphi_1{}^3/\varphi_2)\} \quad (4.1f)$$

where

$$\Delta=|m_{01}{}^{(1)}\Psi_2\Psi_3|^2-|m_{02}{}^{(1)}\Psi_1\Psi_3|^2\varphi_1\varphi_2+|m_{03}{}^{(1)}\Psi_2\Psi_1|^2(\varphi_1{}^2/\varphi_2) \quad (4.2)$$

$$\mu=\alpha-w+\sqrt{(r^2+(\alpha-w)^2)} \quad (4.3)$$

$$\Psi_1\Psi_2\Psi_3=-(-2\mu\alpha)^{2/n+k/2} \quad (4.4)$$

with

$$\varphi_1=r^{2b/n}e^{-X_1},\ \varphi_2=r^{2/n}e^{X_1} \quad (4.5)$$

The physical g is given by the expression

$$g^{ph}=(|\mu|/r)g \quad (4.6)$$

The procedure can repeated n times to give the n-soliton solution to the Eq.(2.4).

Note that for $n=2,\varphi_1=\varphi_2=\varphi,\rho_1=\rho_2=0$ and $\rho_3=\rho$ Eq.(2.19) gives the g of the SU(2)[3,6,8] and the subject presented on the above sections is valid for the SU(2) Lie group.

5.Soliton solutions to the Yang-Mills equations obtained from an non-diagonal seed solution

To obtain soliton solutions to the Yang-Mills equations from an non-diagonal seed solution,we start with the Yang-Mills equations in the R-gauge for the SU(2) Lie group and repeate everything we presented in sections 2. and 3.,but instead of having g_0 diagonal, as in Eq.(3.2),we specialize it as

$$g_0=r\begin{vmatrix} 0 & 1 \\ 1 & \varphi \end{vmatrix} \tag{5.1}$$

where φ satisfies the equation

$$\varphi_{rr}+(1/r)\varphi_r+\varphi_{ww}=0 \tag{5.2}$$

and write the 2X2 matrix Ψ as following

$$\Psi=(r^2-2\lambda w-\lambda^2)^{-1/2}Q(\lambda,r,w) \tag{5.3}$$

Then Eqs(2.11) read

$$D_rQ=[r(U-I)\lambda V]Q/(\lambda^2+r^2), \tag{5.4a}$$

$$D_wQ=[rV-\lambda(U-I)]Q/(\lambda^2+r^2) \tag{5.4b}$$

with

$$\Psi_0=Q(\lambda=0,r,w)=g_0/r \tag{5.4c}$$

Subsequently,we re-write Eq.(5.4) as following

$$D_rQ=[r^2\varphi_r+\lambda r\varphi_w]\sigma Q/(\lambda^2+r^2) \tag{5.5a}$$

$$D_wQ=[r^2\varphi_w-\lambda r\varphi_r]\sigma Q/(\lambda^2+r^2) \tag{5.5b}$$

with

$$\Psi_0=g_0/r \quad \text{and} \quad \sigma=\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

Specifying the matrix Q as

$$Q=\begin{vmatrix} 0 & 1 \\ 1 & F \end{vmatrix} \tag{5.6}$$

Eqs.(5.5) become

$$D_rF=[r^2\varphi_r+\lambda r\varphi_w]/(\lambda^2+r^2) \tag{5.7a}$$

$$D_wF=[r^2\varphi_w-\lambda\varphi_r]/(\lambda^2+r^2) \tag{5.7b}$$

and

$$F(\lambda=0,r,w)=\varphi \tag{5.7c}$$

To constract the soliton solution we need

$$F^{(k)}=F(\lambda=\mu_k,r,w) \tag{5.8}$$

e.g the function F along the pole's tranjectories which obey the Eqs.(2.19). From Eqs.(5.7) and (2.19) we obtain

$$r\vartheta_r F^{(k)} - \mu_k \vartheta_w F^{(k)} = r\varphi_r \tag{5.9a}$$

$$\mu_k \vartheta_r F^{(k)} + r\vartheta_w F^{(k)} = r\varphi_w \tag{5.9b}$$

and

$$F^{(k)} = (1/2)\int (r/\mu_k)\{[\mu_{k,r}\varphi_r - \mu_{k,w}\varphi_w]dr$$

$$+[\mu_{k,r}\varphi_w + \mu_{k,w}\varphi_w]dw\} \tag{5.10}$$

Since

$$(\mu_{k,r}/\mu_k) \to 2/r, (\mu_{k,w}/\mu_k) \to 0 \text{ as } \mu_k \to 0 \tag{5.11}$$

the integral (5.10)exists and Eq.(5.10) is compatible to the initial condition (5.8);e.g the system of Eqs.(5.7) is completely determined along the poles' tranjectories and its solution reduces to a single quadrature. Thus,we can find F;furthermore from Eq.(5.6) we find Q and then the $\Psi_0 = Q|_{\lambda=0}$ from Eq(5.3).Following the procedure presented in sections 2. and 3., we find the 1-soliton solution to the Eqs.(2.4),written for the SU(2), which is

$$g_{11} = -(r/\Delta)|m_{01}^{(1)}|^2(r^2+|\mu|^2) \tag{5.12a}$$

$$g_{12} = \bar{g}_{21} = r\{1-(r^2+|\mu|^2)/(|\mu|^2\Delta)[\bar{m}_{01}^{(1)}m_{02}^{(1)}+|m_{01}^{(1)}|^2(\varphi-F)]\} \tag{5.12b}$$

$$g_{22} = r\varphi\{1-(r^2+|\mu|^2)/(\varphi\Delta|\mu|^2)|[m_{02}^{(1)}+m_{01}^{(1)}(\varphi-F)]|^2\} \tag{5.12c}$$

where

$$\Delta = \bar{m}_{01}^{(1)}m_{02}^{(1)} + m_{01}^{(1)}\bar{m}_{02}^{(1)} + m_{01}^{(1)}\bar{m}_{01}^{(1)}(\varphi-F-\bar{F}) \tag{5.13}$$

Again,if we will repeat the procedure with a matrix X with n poles,we can obtain an n-soliton solution to the field equations.

6.Conclusion and Discussion

In the R-gauge,dealing with the self-dual gauge fields in the SU(3) and SU(2) lie groups,we found soliton solutions to the Eq.(2.12) using diagonal and non-diagonal seed solutions in the SU(3) and SU(2) Lie groups, respectively.

In case of using diagonal seed solutions, experts have tried[6,7], unsuccessfully,to verify that the obtained solitonic solutions are solutions to the Einstein field equations.However,in case where we work in the SU(2) with a non-diagonal seed solution we verify that the so obtained solitonic solutions are solitonic solutions to Einstein field equations[8]

References

1.Charles Nash and Siddhartha Sen,"Topology and Geometry for Physicists", Academic Press, New York (1983).
2.C.N.Yang, "Condition of self-duality for SU(2) gauge fields on Euclidean four-dimensional space", Phys.Rev.Lett., **38**:1377 (1977).
3.L.Witten, "Static axially symmetric solutions of self-dual SU(2) gauge fields in Euclidean four-dimensional space", Pys.Rev.D., **18**:3877 (1979).
4.M.K.Prasad, "Yang's R gauge for self-dual SU(3) gauge fields", Phys.Rev.D., **17**:3243 (1978).
5.V.A.Belinski and V.E.Zakharov, Zh.Eksp.Teor.Fiz., **75**:1955 (1978) [Sov.Phys.JETP, **48**:985 (1978)]

6.P.S.Letelier, "Soliton solutions for self-dual Su(2) gauge fields on Euclidean space", J.Math.Phys., **23**:1175 (1982).
7.D.Papadopoulos, "Soliton solutions for self-dual SU(3) gauge field theory", J.Math.Phys., **26**:2990 (1985).
8.D.Papadopoulos, "Soliton solutions for self-dual SU(2) gauge fields on Euclidean space obtained from a nondiagonal seed solution", Phys.Lett.A., **127**:34 (1988).
9.P.S.Letelier, "Soliton solutions to the vacuum Einstein equations obtained from a nondiagonal seed solution", J.Math.Phys., **27**:564 (1986).

SUPERALGEBRA AND SUPERSPACE OF VECTOR SPINOR GENERATORS

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Abstract

The super Poincare algebra of generators belonging to the (1, 1/2) + (1/2, 1) representation of the Lorentz group is constructed. The closure of the algebra is demonstrated. The supertranslations and the chiral superfield in the new superspace with vector-spinor coordinates are also given.

This work is done in collaboration with Dr. Christopher Pilot and entertains the possibility of constructing new superalgebras and their corresponding superspaces with higher spin fermionic generators and coordinates.

To date, all supersymmetries(1) employed in field theory are based on spin-1/2 fermionic generators which belong to the (1/2, 0) + (0, 1/2) representation of the Lorentz group. The use of fermionic generators belonging to spins greater than half is forbidden by the theorem of Haag, Lopuszanski and Sohnius(2). In this note(3), the assumptions underlying the Haag, Lopuszanski and Sohnius theorem are relaxed and a consistent supersymmetry algebra with generators belonging to the (1, 1/2) + (1/2, 1) representation of the Lorentz group is constructed. Our starting point is the theorem of Coleman and Mandula(4).

The Coleman-Mandula theorem states that for a symmetry to be a symmetry of the S-matrix, the generators (bosonic) of the symmetry must close on the four vector P_μ, the tensor of rotations and boosts $J_{\mu\nu}$, and a set of scalar charges B_I of an internal symmetry group. This theorem-supplemented by the theorem of Haag, Lopuszanski and Sohnius which states that in a theory consisting of massive particles, the bosonic and fermionic generators must close on P_μ, $J\mu\nu$, B_I and a further set of charges called central charges[5] for the fermionic generators. The theorem restricts the fermionic generators to the (1/2, 0) + (0, 1/2) representations of the Lorentz group. The arguments leading to this restriction run as follows. A fermionic generator Q_s and its hermitian conjugate $\bar{Q}_s$ belong to the same algebra and satisfy the anticommutation relation

$$\{Q_s, \bar{Q}_s\} = \sum_x a_x T^x_{2s} \tag{1}$$

where T^x are all possible covariant terms and a_x are coefficients restricted by the closure of the algebra. The principal assumption in the theorem is that all T^x in eq. (1) belong to spin 2S since one is dealing with an anticommutator of Q_s and $\bar{Q}_s$. Since the Coleman-Mandula theorem requires that the anticommutator close on the generators of the Poincare algebra and a set of scalar charges, it follows that 2S is either zero or one and hence the maximum value of S equals a half.

It is the last assumption that we wish to relax. Our criterion of an admissible fermionic generator for it to be the generator of the symmetry of the S-matrix is that its anti-commutor must close on terms that respect covariance and the symmetry properties of the anti-commutator. The T^x are all proportional to P_μ and the rest of the symmetry resides in the a_x. We demonstrate this in the case of vector spinor generators θ^μ_α belonging to the (1, 1/2) + (1/2, 1) representation of the Lorentz group.

The most general anticommutation relation between $\theta_{\mu\alpha}$ and its hermitian conjugate $\bar{\theta}_{\mu\alpha}$ is

$$\begin{aligned}\{\theta_{\mu\alpha}, \bar{\theta}_{\nu\beta}\} = f_1\, \eta_{\mu\nu}\, \not{P}_{\alpha\beta} + f_2\, (\gamma_\mu P_\nu + \gamma_\nu P_\mu)_{\alpha\beta} \\ + f_3(\gamma_\mu P_\nu \gamma_5 - \gamma_\nu P_\mu \gamma_5)_{\alpha\beta} + f_4\, \Sigma_{\mu\nu\lambda\sigma}\, P^\lambda \gamma^\sigma \gamma_5\end{aligned} \tag{2}$$

Note that all terms in eq. (2) are proportional to the four vector P_μ and respect the symmetry properties of the anticommutator.

The generators $\theta_{\mu\alpha}$ are taken to be Majorana spinors,

$$\theta_\mu = C\, \bar{\theta}_\mu^{\,T} \tag{3}$$

where C is the charge conjugation matrix and T denotes transpose.

In general, the vector spinor generator $\theta_{\mu\alpha}$ is reducible. The spinor part $Q_{\mu\alpha}$ free from the spin (1/2, 0) + (0, 1/2) components is

$$Q_{\mu\alpha} = \theta_{\mu\alpha} - \frac{1}{4}\, (\gamma_\mu \gamma^\sigma \theta_\sigma)_\alpha = \theta_{\mu\alpha} - \frac{1}{4}\, (\gamma_\mu \theta)_\alpha \tag{4}$$

and satisfies

$$\gamma^{\mu} Q_{\mu\alpha} = 0 \tag{5}$$

The $Q_{\mu\alpha}$ are Majorana spinors also. The anticommutator of eq. (2) now consist of four terms; $\{Q_{\mu\alpha}, \bar{Q}_{\nu\beta}\}$, $\{Q_{\mu\alpha}, (\bar{\Theta}\gamma_{\nu})_{\beta}\}$, $\{(\gamma_{\mu}\Theta)_{\alpha}, \bar{Q}_{\nu\beta}\}$ and $\{(\gamma_{\mu}\Theta)_{\alpha}, (\bar{\Theta}\gamma_{\nu})_{\beta}\}$. These anticommutators can be disentangled from one another by requiring that

$$(\gamma^{\mu})_{\alpha'\alpha} \{Q_{\mu\alpha}, \bar{Q}_{\nu\beta}\} = 0$$

$$\{Q_{\mu\alpha}, \bar{Q}_{\nu\beta}\} (\gamma^{\nu})_{\beta\beta'} = 0$$

$$(\gamma^{\mu})_{\alpha'\alpha} \{Q_{\mu\alpha}, \bar{Q}_{\nu\beta}\} (\gamma^{\nu})_{\beta\beta'} = 0$$

$$(\gamma^{\mu})_{\alpha'\alpha} \{Q_{\mu\alpha}, (\Theta\gamma_{\nu})_{\beta}\} = 0$$

$$\{(\gamma_{\mu}\Theta)_{\alpha}, \bar{Q}_{\nu\beta}\} (\gamma^{\nu})_{\beta\beta'} = 0$$

which are consequences of eq. (5). We find

$$\{\Theta_{\alpha}, \bar{\Theta}_{\beta}\} = (-2f_1 + 8f_2 + 6if_4)\, \not{P}_{\alpha\beta} \tag{6}$$

$$\{Q_{\mu\alpha}, \bar{\Theta}_{\beta}\} = \frac{1}{2}(f_1 + 2f_2 - if_4)\, [3P_{\mu}\, \delta_{\alpha\beta} - \frac{1}{2} [\gamma_{\mu}, \gamma_{\nu}]_{\alpha\beta}\, P^{\nu}]$$

$$+ 3f_3\, P_{\mu}\, (\gamma_5)_{\alpha\beta} - \frac{1}{2} f_3\, ([\gamma_{\mu}, \gamma_{\nu}]\, \gamma_5)_{\alpha\beta}\, P^{\nu} \tag{7}$$

$$\{\Theta_{\alpha}, \bar{Q}_{\nu\beta}\} = \frac{1}{2}(f_1 + 2f_2 - if_4)\, [3P_{\nu}\, \delta_{\alpha\beta} + \frac{1}{2} [\gamma_{\nu}, \gamma_{\mu}]\, P^{\mu}]$$

$$+ \quad 3f_3\, P_{\nu}\, (\gamma_5)_{\alpha\beta} + \frac{1}{2} f_3\, ([\gamma_{\nu}, \gamma_{\mu}]\, \gamma_5)_{\alpha\beta}\, P^{\mu} \tag{8}$$

$$\{Q_{\mu\alpha}, \bar{Q}_{\nu\beta}\} = \frac{5}{8}(f_1 + if_4)\, \{\eta_{\mu\nu}\, \not{P} - \frac{1}{5}(P_{\mu}\gamma_{\nu} + P_{\nu}\, \gamma_{\mu})$$

$$+ \frac{3}{5}\, i\epsilon_{\mu\nu\lambda\sigma}\, \gamma^{\lambda}\gamma_5 P^{\sigma}\}_{\alpha\beta} \tag{9}$$

where we have made use of the relations

$$\epsilon_{\lambda\mu\sigma\nu}\, \gamma^{\mu}\gamma^{\sigma}\gamma^{\nu} = -\, 6i\, \gamma_{\lambda}\gamma_5, \quad \epsilon_{\lambda\mu\sigma\nu}\, \gamma^{\sigma}\gamma^{\nu}\gamma_5 = -i\, [\gamma_{\lambda}\gamma_{\mu} - \gamma_{\mu}\gamma_{\lambda}] \tag{10}$$

for simplification. Since we are only interested in the anti-commutator of two <u>irreducible vector spinor</u> generators we impose the following constraints,

$$\{\Theta_{\alpha}, \bar{\Theta}_{\beta}\} = \{Q_{\mu\alpha}, \bar{\Theta}_{\beta}\} = \{\Theta_{\alpha}, \bar{Q}_{\nu\beta}\} = 0 \tag{11}$$

which are satisfied if

$$f_1 = +\frac{5}{3}\, i f_4$$

$$f_2 = -\frac{1}{3}\, i f_4$$

$$f_3 = 0 \tag{12}$$

We take $f_4 = -i$ without loss of generality. With these values for f_1, f_2, f_3 and f_4, the super Poincare algebra involving irreducible vector-spinor generator $Q_{\mu\alpha}$ is

$$[J_{\mu\nu},\ J_{\sigma\lambda}] = i\ (\eta_{\nu\sigma}\ J_{\mu\lambda} - \eta_{\nu\lambda}\ J_{\mu\sigma} + \eta_{\mu\lambda}\ J_{\nu\sigma} - \eta_{\mu\sigma}\ J_{\nu\lambda}) \tag{13}$$

$$[P_\mu,\ J_{\nu\lambda}] = i\ (\eta_{\nu\mu}\ P_\lambda - \eta_{\lambda\mu}\ P_\nu) \tag{14}$$

$$[P_\mu,\ P_\nu] = 0 \tag{15}$$

$$[Q_{\mu\alpha},\ P_\nu] = [\bar{Q}_{\mu\alpha},\ P_\mu] = 0 \tag{16}$$

$$[Q_{\mu\alpha},\ J_{\nu\lambda}] = i\ (\eta_{\nu\mu}\ Q_{\lambda\mu} - \eta_{\lambda\mu}\ Q_{\nu\alpha}) + 1/2\ (\sigma_{\nu\lambda}\ Q_\mu)_\alpha \tag{17}$$

$$[\bar{Q}_{\mu\alpha},\ J_{\nu\lambda}] = i\ (\eta_{\nu\mu}\ \bar{Q}_{\lambda\alpha} - \eta_{\lambda\mu}\ \bar{Q}_{\nu\alpha}) - 1/2\ (\sigma_{\mu\lambda}\ \bar{Q}_\mu)_\alpha \tag{18}$$

$$[Q_{\mu\alpha},\ \bar{Q}_{\nu\beta}\} = 5/3\ \eta_{\mu\nu}\ \not{P}_{\alpha\beta} - 1/3\ (\gamma_\mu P_\nu + \gamma_\nu P_\mu)_{\alpha\beta} + i\ \epsilon_{\mu\nu\lambda\sigma}\ P^\lambda \gamma^\sigma \gamma_5 \tag{19}$$

where $\sigma_{\mu\nu} = i/2\ [\gamma_\mu, \gamma_\nu]$.

It needs to be demonstrated that the above algebra closes under the Jacobi identities. The Jacobi identities involving the bosonic generators $J_{\mu\nu}$ and P_λ are satisfied since the Poincare algebra is consistent. The {P, P, Q} Jacobi identity is satisfied because $[P_\mu,\ P_\nu] = 0$ and $[P_\mu,\ Q_{\nu\alpha}] = 0$. The Jacobi identities for $[Q, Q, \bar{Q}]$ and $[Q, \bar{Q}, P]$ are satisfied due to $[P_\mu,\ P_\nu] = 0$, $[P_\mu,\ Q_{\nu\alpha}] = 0$, $[P_\mu,\ \bar{Q}_{\mu\alpha}] = 0$ and the fact that all terms in $\{Q_{\mu\alpha},\ \bar{Q}_{\nu\beta}\}$, $\{Q_{\mu\alpha},\ Q_{\nu\beta}\}$ are proportional to P_μ. The Jacobi identity for [P, Q, J] is satisfied due to the fact that terms in [Q, J] are proportional to Q and terms in [P, J] are proportional to P which in turn give terms like [P, P] and [P, Q] which are zero due to the algebra. The non-trivial Jacobi identities are:

A.
$$[J_{\mu\nu},\ [J_{\lambda p},\ Q_{\sigma\alpha}]] + [Q_{\sigma\alpha},\ [J_{\mu\nu},\ J_{\lambda\rho}]] + [J_{\lambda\rho},\ [Q_{\sigma\alpha},\ J_{\mu\nu}]] = 0 \tag{20}$$

This relation is satisfied due to the algebra of equations (13) and (17).

B.
$$\{Q_{\mu\alpha},\ [\bar{Q}_{\nu\beta},\ J_{\lambda\rho}]\} + [J_{\lambda\rho},\ \{Q_{\mu\alpha},\ \bar{Q}_{\nu\beta}\}] + \{\bar{Q}_{\nu\beta},\ [Q_{\mu\alpha},\ J_{\lambda\rho}] \tag{21}$$

This requires that the following relation be satisfied.

$$\epsilon_{\mu\nu\delta\rho}\ \eta_{\lambda\sigma} - \epsilon_{\mu\nu\delta\lambda}\ \eta_{\rho\sigma} + \epsilon_{\mu\rho\delta\sigma}\ \eta_{\lambda\nu} - \epsilon_{\mu\lambda\delta\sigma}\ \eta_{\rho\nu} - \epsilon_{\mu\nu\lambda\sigma}\ \eta_{\delta\rho} + \epsilon_{\mu\nu\rho\sigma}\ \eta_{\delta\lambda} - \epsilon_{\nu\rho\delta\sigma}\ \eta_{\lambda\mu} + \epsilon_{\nu\lambda\delta\sigma}\ \eta_{\rho\mu} = 0 \tag{22}$$

This relation is satisfied due to the Schouten identity

$$\epsilon_{\mu\nu\lambda\rho}\,\eta_{\delta\sigma} + \epsilon_{\mu\lambda\rho\delta}\,\eta_{\mu\sigma} + \epsilon_{\lambda\rho\delta\mu}\,\eta_{\nu\sigma} + \epsilon_{\rho\delta\mu\nu}\,\eta_{\lambda\sigma} + \epsilon_{\delta\mu\nu\lambda}\,\eta_{\rho\sigma} = 0 \qquad (23)$$

Thus the closure of the algebra under all Jacobi identities has been demonstrated. The super Poincane algebra of eq. (13) to (19) is a consistent algebra.

It is to be noted that for

$$f_1 = -f_2 = -if_4$$
$$f_3 = 0 \qquad (24)$$

the anticommutators of eqs. (7), (8) and (9) vanish. This situation corresponds to the case in which the vector spinor generator is constructed out of pure spin-1/2 generator of conventional supersymmetry, i.e.,

$$Q_{\mu\alpha} = (\gamma_\mu\, Q)_\alpha$$

In this case, the algebra of eq. (2) is trivially constructed out of the algebra of N = 1 supersymmetry and the Dirac algebra. This demonstrates that for $f_1 \neq -\, if_4$ $f_2 \neq -\, f_1$ and $f_3 \neq 0$ the super poincare algebra of vector spinor generators is non-trivial.

The superspace is taken to consist of four bosonic coordinates and twelve fermionic coordinates $\theta_{\mu\alpha}$. It can be shown that under the new supersymmetry, the coordinates of the superspace transform as

$$x^\mu \rightarrow x^\mu + \frac{1}{2}\,\bar{\epsilon}^\nu \gamma^\mu \theta_\nu$$

$$\theta^\mu \rightarrow \theta^\mu + \epsilon^\mu \qquad (25)$$

where ϵ^μ is an anticommuting majorana spinor parameter. The superfield expansion is given in terms of the chiral coordinates $\theta_{L\mu\alpha} = 1/2\ (1+\gamma_5)\ \theta_{\mu\alpha}$. Also, it is convenient to work with spinor coordinates. Thus the superfield expansion parameter is θ_{ABC} where all indeces are symmetric. There are six independent coordinates in θ_{ABC} and the superfield is

$$\Phi_L = \phi_1 + \theta^{(AB\dot{C})}\varphi_{1(AB\dot{C})} + \theta^{(AB\dot{C})}\ \theta_{(AB\dot{C})}\phi_2$$

$$+\ \theta^{(AB\dot{E})}\ \theta^{(CD}{}_{\dot{E})}\ \phi_{2(ABCD)} + \theta^{(AE\dot{C})}\ \theta^{(B}{}_{E}{}^{\dot{D})}\ \phi_{2}{}^{(AB\dot{C}\dot{D})}$$

$$+\ \theta^{(DF\dot{A})}\ \theta_{(D}{}^{E\dot{B})}\ \theta_{(EF}{}^{\dot{C})}\ \varphi_{2(\dot{A}\dot{B}\dot{C})}$$

$$+\ \theta^{(AF\dot{D})}\ \theta^{(BE}{}_{\dot{D})}\ \theta_{(EF}{}^{\dot{C})}\ \varphi_{2(AB\dot{C})}$$

$$+\ \theta^{(AB\dot{G})}\ \theta^{(CF\dot{E})}\ \theta^{(D}{}_{F\dot{G})}\ \varphi_{2(ABCD\dot{E})}$$

$$+\ \theta^{(AB\dot{E})}\ \theta_{(B}{}^{C\dot{F})}\ \theta_{(C}{}^{D}{}_{\dot{E})}\ \theta_{(DA\dot{F})}\ \phi_3$$

$$+ \theta^{(AH\dot{E})} \theta^{(BF}{}_{\dot{E})} \theta^{(C}{}_{F}{}^{\dot{G})} \theta^{(D}{}_{H\dot{G})} \phi_{3(ABCD)}$$

$$+ \theta^{(AH\dot{E})} \theta^{(BF}{}_{\dot{E})} \theta_{(F}{}^{G\dot{C})} \theta_{(GH}{}^{\dot{D})} \phi_{3(AB\dot{C}\dot{D})}$$

$$+ \theta^{(AD\dot{H})} \theta^{(BE\dot{I})} \theta_{(D}{}^{F}{}_{\dot{I})} \theta_{(F}{}^{G\dot{C})} \theta_{(GE\dot{H})} \varphi_{3(AB\dot{C})}$$

$$+ \theta^{(A\dot{A})}{}_{F} \theta^{(B\dot{B})}{}_{A} \theta^{(C\dot{C})}{}_{B} \theta^{(D}{}_{C\dot{A})} \theta^{(E}{}_{D\dot{B})} \theta^{(F}{}_{B\dot{C})} \phi_4 \qquad (26)$$

In the above ϕ_1, ϕ_2, ϕ_3, ϕ_4 are spin (0,0) scalars; $\varphi_{1(AB\dot{C})}$, $\varphi_{2(AB\dot{C})}$ and $\varphi_{3(AB\dot{C})}$ are spin (1,1/2) spinors; $\phi_{2(ABCD)}$, $\phi_{3(ABCD)}$ are spin (2,0) conformal tensors, $\phi_{2(AB\dot{C}\dot{D})}$, $\phi_{3(AB\dot{C}\dot{D})}$ are symmetric spin (1,1) tensors: $\varphi_{2(\dot{A}\dot{B}\dot{C})}$ is a spin (0,3/2) spinor and, finally, $\varphi_{2(ABCD\dot{E})}$ is a spin (2,1/2) spinor. Even powers of $\theta_{(AB\dot{C})}$ always have bosonic components associated with them; an odd number will lead to fermionic components. As in any supersymmetric theory the total number of fermionic components equals the total number of bosonic. In this case we have 32 + 32 giving a total of 64 components.

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References

1. Y. A. Gol Fand and E. P. Likhtam, JETP 6H. 13 (1971) 323. J. Wess and B. Zumino, Nucl. Phys. 070 (1974) 39.
2. R. Haag, J. T. Lopuszanski and M. Sohnius, Nucl. Phys. B88 (1975) 257.
3. C. Pilot and S. Rajpoot, Mod. Phys. Lett. A, Vol. 4, No. 9, 831 (1989).
4. S. Coleman and J. Mandula, Phys. Rev. 159 (1987) 1251.
5. R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275.

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