Book of Problems

SUMS

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1　Preface

This is the SUMS Maths Space Book of Problems. It will be a repository for problems donated by SUMS members.

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2.1 Density of Natural numbers mod π

Consider the function $f : \mathbb{N} \to [0, \pi]$ defined $\forall n$ by $f(n) = n - \lfloor \frac{n}{\pi} \rfloor n$.
Determine - with proof - if the following is true.

$$\forall x \in [0, \pi] \epsilon > 0 \quad \exists n \in \mathbb{N} \text{ s.t } |f(n) - x| < \epsilon$$

You may use without proof the fact that π is irrational

2.2 A block containing primes

Prove that there exists a block of 1000 consecutive natural numbers of which exactly 10 are prime

2.3 Primes in the Fibonacci sequence

The first few terms of the Fibonacci sequence are as follows
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$
$F_5 = 5$
$F_6 = 8$
$F_7 = 13$

Note that $F_4 = 3$ is the 4th number in the sequence (4 not a prime) but the number is 3 (a prime number).

Show that this is the only case in the Fibonacci sequence where a prime number has a non-prime index

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2.4   Cyclogon conundrum

Consider a n sided convex polygon resting with two vertices on a line (as exemplified below).

IMAGE TO BE ADDED

Pick a vertex to be denoted X.
Consider the 'leftmost' of the two points lying on the line and imagine rotating the polygon anticlockwise around that vertex until the next vertex lies on the line.
Repeat this process with the new leftmost vertex on the line until the processes has been done n times.

You should find that our original two vertices should be the two now lying on the line (again). What is the length of the path traced by X?

2.5   Isomorphisms with some subset of the Natural numbers under binary XOR

Let $A \subseteq \mathbb{N}$ and $S \subseteq \{\{a, b, c\} : a, b, c \in A\}$ where $\forall a, b \in A$ there exists a unique $c \in A$ s.t $\{a, b, c\} \in S$.

We define the binary operation $\bullet : S \times S \rightarrow S$ s.t $\forall \{a, b, c\} \in S$
$a \bullet b = c$
Show that $(S, \bullet)$ is isomorphic to some subset of the natural number under binary XOR

2.6   Intermediate value property of derivatives

Given a differentiable function $f : [a, b] \rightarrow \mathbb{R}$ $(a < b)$

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Is it true that given d s.t $f'(a) < d < f'(b)$ there exists a $c \in [a, b]$ s.t $f'(c) = d$?

2.7   Impossible position on a Rubik's cube

Given a 3x3x3 Rubik's cube. Is there a position that can be reached by disassembling and reassembling the cube that isn't reachable by simply making moves from the starting position?