MATA32 - TUT0017 S.U.

Week 09

13.3 concavity

* concave up: f' increasing on interval (a,b), $f''(x) > 0$ for all x in interval

concave down: f' decreasing " $f''(x) < 0$ "

inflection point: at a if f continuous on a and $f''(a) = 0$ or DNE

ex. $y = 6x^4 - 8x^3 + 1$

$y' = 24x^3 - 24x^2$

$y'' = 72x^2 - 48x$

$= 24x(3x-2)$

$24x = 0$ $\quad$ $3x - 2 = 0$

$x = 0$ $\quad$ $x = \frac{2}{3}$

sign chart:

		0		2/3	
$(24x)$	$-$		$+$		$+$
$(3x-2)$	$-$		$-$		$+$
y''	$+$		$-$		$+$
y	$\cup$		$\cap$		$\cup$

ex. curve sketching

(1) $y = 2x^3 - 9x^2 + 12x$

$y' = 6x^2 - 18x + 12$

$= 6(x-1)(x-2)$

$y'' = 12x - 18$

$= 6(2x-3)$

(2) intercepts: $(x,0)$ $\quad$ $(0,y)$

$(0,0)$

no real roots

(3) symmetry: $y(x) \neq y(-x)$

$y(x) \neq -y(x)$

(4) critical points: $0 = y'$

$(1,5)$ $\quad$ $(2,4)$

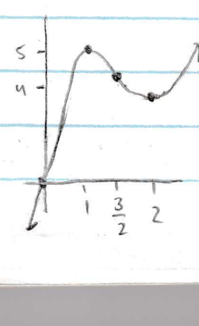
		1		2	
$(x-1)$	$-$		$+$		$+$
$(x-2)$	$-$		$-$		$+$
y'	$+$		$-$		$+$
y	↗		↘		↗

(5) concavity: $0 = y''$

$(\frac{3}{2}, \frac{9}{2})$

		3/2	
$2x-3$	$-$		$+$
y''	$-$		$+$
y	$\cap$		$\cup$

(6) sketch:



ex. [1] $f(x) = 2x^2 - x^4$ $\quad$ $f'(x) = 4x - 4x^3$ $\quad$ $f''(x) = 4 - 12x^2$

$= x^2(\sqrt{2}+x)(\sqrt{2}-x)$ $\quad$ $= 4x(1+x)(1-x)$ $\quad$ $= 4(1-3x^2)$

[2] intercepts: $x = 0$ $\quad$ $y = 0$

$f(0) = 0$ $\quad$ $x = 0, \pm\sqrt{2}$

[3] symetry: $f(x) = f(-x)$ $\quad$ $f(x) \neq -f(x)$

$2x^2 - x^4 = 2(-x)^2 - (-x)^4$ $\quad$ $2x^2 - x^4 \neq x^4 - 2x^2$

[4] critical points: $(0,0)$ $\quad$ $(-1,1)$ $\quad$ $(1,1)$ $\quad$ Note: $f(1) = f(-1)$

[5] inflection points: $(\sqrt{1/3}, 1/3)$, $(-\sqrt{1/3}, 1/3)$

[6] sign charts:

(y)	$-\sqrt{2}$	0	$\sqrt{2}$	
(x^2) +	+	+	+	
$(\sqrt{2}+x)$ −	+	+	+	
$(\sqrt{2}-x)$ +	+	+	−	
y −	+	+	−	

(above, below x-axis)

(y')	−1	0	1	
$(4x)$	−	−	+	+
$(1+x)$	−	+	+	+
$(1-x)$	+	+	+	−
y'	+	−	+	−
y	↗	↘	↗	↘

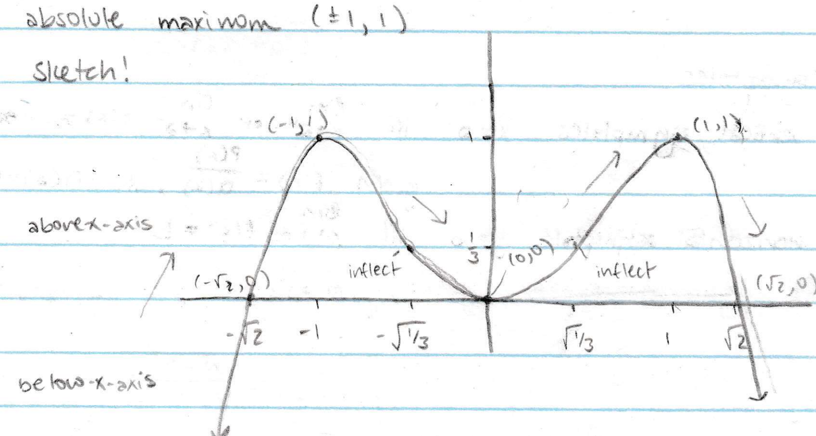
(y'')	$-\sqrt{1/3}$	$\sqrt{1/3}$	
$(3x^2)$	+	+	+
$(1-3x^2)$	−	+	−
y''	−	+	−
y	∩	∪	∩

[7] relative minimum: $(0,0)$

relative maximum: $(\pm 1, 1)$

absolute maximum $(\pm 1, 1)$

[8] sketch!



$f(-x) = f(-x)$

13.4 The Second Derivative Test

* find relative extrema: (0) f differentiable
  (1) find $f'(a) = 0$
  (2) relative minimum at a if $f''(a) > 0$
  relative maximum at a if $f''(a) < 0$

ex. from (13.3), $f'(x) = 0 \to x = \pm 1, 0$

where $f''(1) = 4 - 12(1)^2 < 0$ relative maximum

$f''(-1) = 4 - 12(-1)^2 < 0$ maximum

$f''(0) = 4 - 12(0)^2 > 0$ minimum

ex. $y = x^4 - 2x^2 + 4$

$y' = 4x^3 - 4x = 4x(x^2 - 1)$

$= 4x(x-1)(x+1)$ critical points: $x = 0, \pm 1$

$y'' = 12x^2 - 4$

where $y''(0) = -4 < 0$ local maximum

$y''(\pm 1) = 8 > 0$ minimum

13.5 Asymptotes

* vertical asymptotes $x = a$ iff $\lim_{x \to a^+}$ or $\lim_{x \to a^-} f(x) = \pm\infty$

given $f(x) = \frac{P(x)}{Q(x)}$, iff $Q(a) = 0$ and $P(a) \neq 0$

* horizontal asymptote $y = b$ iff $\lim_{x \to \pm\infty} f(x) = b$

- a polynomial function with degree ≥ 1 has no asymptotes

ex. $f(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$

$= \frac{x(x-4)}{(x-3)(x-1)}$

denominator $= 0$

if $x = 1, 3$

numerator $\neq 0$

then, vertical asymptotes

ex. $f(x) = \frac{x^2 - 4x}{x^2 - 4x + 3}$

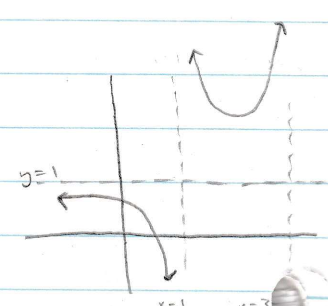
$\lim_{x \to \infty} f(x)$

$= \lim_{x \to \infty} \frac{x^2(1 - 4/x)}{x^2(1 - 4/x + 3/x^2)}$

$= \lim_{x \to \infty} \frac{1 - 4/x}{1 - 4/x + 3/x^2}$

$= \frac{1}{1}$

then, horizontal asymptote



ex. Curve sketching

[1] $y = \frac{1}{4-x^2}$

$= \frac{1}{(2+x)(2-x)}$

	-2		2	
$(2+x)$	$-$	$+$	$+$	
$(2-x)$	$+$	$+$	$-$	
y	$-$	$+$	$-$	

intercepts: $(0, \frac{1}{4})$, $y \neq 0$

symmetry: $y(x) = y(-x)$ $\quad y(x) \neq -y(x)$

[2] $y' = \frac{2x}{(4-x^2)^2}$

$y'(0) = 0$

undefined: ± 2

	$x<-2$	$-2<x<0$	$0<x<2$	$x>2$
$(2x)$	$-$	$-$	$+$	$+$
$(4-x^2)^2$	$+$	$+$	$+$	$+$
y'	$-$	$-$	$+$	$+$
y	↘	↘	↗	↗

[3] $y'' = \frac{2(4+3x^2)}{(4-x^2)^3}$

no real roots

undefined: ± 2

	$x<-2$	$-2<x<2$	$x>2$
$3x^2$	$+$	$+$	$+$
$(4-x^2)^3$	$-$	$+$	$-$
y''	$-$	$+$	$-$
y	$\frown$	$\smile$	$\frown$

[4] asymptotes: vertical − $P(x) = 1$ $\quad Q(x) = (2+x)(2-x)$

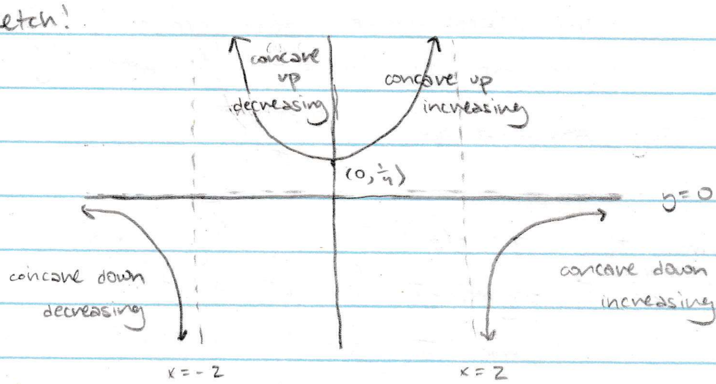
$= 0$ when $x = \pm 2$

horizontal − $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \left(\frac{1}{-x^2}\right) = 0$

[5] $(0,0)$ critical point

$f''(0) = \frac{2(4+3 \cdot 0^2)}{(4-0^2)^3} > 0$ local minimum

[6] sketch!



13.6 Applied Maxima and Minima

ex. what are two non-negative numbers such that their sum is 20 and we maximize the product of twice one number and square the other

(i) $x+y=20$

(ii) $\max = 2xy^2$ then, $y = 20-x$

$$\max = 2x(20-x)^2$$
$$= 2x^3 - 80x^2 + 800x$$
$$(\max)' = 6x^2 - 160x + 800$$
$$0 = 2(3x-20)(x-20)$$
$$x = \frac{20}{3}, 20 \qquad (20 = \text{endpoint})$$
$$(\max)'' = 12x - 160$$
$$(\max)''\Big|_{\frac{20}{3}} = -80 < 0$$

by the Second Derivative Test, there is a max at $\left(\frac{20}{3}, \frac{40}{3}\right)$

ex. maximize number of people in a program after t number of years

$$n = \frac{t^3}{3} - 6t^2 + 32t, \qquad 0 \le t \le 12$$
$$\frac{dn}{dt} = t^2 - 12t + 32 = 0$$
$$(t-4)(t-8) = 0$$

test critical values, and endpoints on closed interval

$n(0) = 0$ $\quad n(4) = \frac{160}{3}$ $\quad n(8) = \frac{128}{3}$ $\quad n(12) = 96$

then maximum at $(12, 96)$

ex. minimize [average cost]

$$c = c(q) = \frac{1}{4}q^2 + 3q + 400 \qquad q \ge 0$$
$$\text{avg cost} = \bar{c} = \frac{c}{q} = \frac{1}{4}q + 3 + \frac{1}{q}\cdot 400$$
$$\text{min: } \frac{d\bar{c}}{dq} = \frac{1}{4} - \frac{400}{q^2} = (4q^2)^{-1}(q^2 - 1600)$$
$$\text{critical: } \frac{d\bar{c}}{dq} = 0 \text{ when } q = 40 \qquad (\text{given } q \ge 0)$$
$$\text{relative extrema: } \frac{d^2\bar{c}}{dq^2} = \frac{1}{q^3}\cdot 800$$

positive for $q = 40$

then, $\bar{c}$ has a relative minimim at $q = 40$

$\bar{c}$ is continuous for $q > 0$

following, $\bar{c}$ has an absolute minimum at $(40, 23)$