MATA32 - TUT0017 S.V.

week 04

10.1 Limits

* $\lim_{x \to a} f(x) = L$ the limit of $f(x)$ as x approaches a is the number L

  if there is no such number, the limit DNE

  Note: $f(a) \neq \lim_{x \to a} f(x)$ in all cases

* Properties:
  - $\lim_{x \to a} f(x) = \lim_{x \to a} c = c$ if $f(x) = c$, a constant function
  - $\lim_{x \to a} x^n = a^n$ given n positive integer
  - $\lim_{x \to a} [f(x) \pm_{\times} g(x)] = \lim_{x \to a} f(x) \pm_{\times} \lim_{x \to a} g(x)$ add, subtract, multiply
  - $\lim_{x \to a} c \cdot f(x) = c \cdot \lim_{x \to a} f(x)$ for a constant c
  - $\lim_{x \to a} f(x) = f(a)$ given f a polynomial function
  - $\lim_{x \to a} f(x)/g(x) = \lim_{x \to a} f(x) / \lim_{x \to a} g(x)$ given $\lim_{x \to a} f(x), g(x)$ exist and $\lim_{x \to a} g(x) \neq 0$
  - $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ given n positive integer
  - $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ given f, g functions for which $f(x) = g(x)$ for all $x \neq a$ (if either limit exists, they are equal)

given form $0/0$: $\lim_{h \to 0} f(x+h) - f(x) / h$

$\lim_{x \to 0} (1+x)^{1/x} = e$

ex. $\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{(x-1)} = \lim_{x \to 1} (x^2+x+1) = (1)^2 + (1) + 1 = 3$

↳ $x \to 1$ but $x \neq 1$ so avoid $\frac{f(x)}{0}$

ex. $\lim_{x \to -2} \frac{8x^2 + 16x}{x^3 - 4x} = \lim_{x \to -2} \frac{8x + 16}{x^2 - 4} = \lim_{x \to -2} \frac{8(x+2)}{(x-2)(x+2)} = \lim_{x \to -2} \frac{8}{x-2} = \frac{8}{-2-2} = \frac{8}{-4} = -2$

ex. $\lim_{x \to 4} \sqrt{x^2 + 1} = \sqrt{\lim_{x \to 4} x^2 + 1} = \sqrt{(4)^2 + 1} = \sqrt{17}$

ex. given $f(x) = x^2 + x + 1$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{(x+h)^2 + (x+h) + 1 - (x^2 + x + 1)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + x + h + 1 - x^2 - x - 1}{h}$$
$$= \lim_{h \to 0} \frac{2xh + h^2 + h}{h}$$
$$= \lim_{h \to 0} 2x + h + 1$$
$$= 2x + 0 + 1 = 2x + 1$$

ex. * conjugates!

$\lim_{x \to 0} x / \sqrt{x+16} - 4$ (gives $0/0$ form)

given $(\sqrt{x+16} - 4)$, conjugate: $(\sqrt{x+16} + 4)$

a - b   a + b

$= \lim_{x \to 0} \left(x / \sqrt{x+16} - 4 \right)\left(\sqrt{x+16} + 4 / \sqrt{x+16} + 4 \right)$ * multiply by $\frac{a}{a} = 1$

$= \lim_{x \to 0} \dfrac{x(\sqrt{x+16} + 4)}{(\sqrt{x+16})^2 + 4\sqrt{x+16} - 4\sqrt{x+16} + (4)(-4)}$

$= \lim_{x \to 0} x\sqrt{x+16} + 4x / (x+16) - 16$

$= \lim_{x \to 0} x(\sqrt{x+16} + 4) / x$

$= \lim_{x \to 0} \sqrt{x+16} + 4 = \sqrt{0+16} + 4 = \sqrt{16} + 4 = 4 + 4 = 8$

ex. $\lim_{h \to 0} \dfrac{\sqrt{4+2h} - 2}{h} = \lim_{h \to 0} \left(\dfrac{\sqrt{4+2h} - 2}{h} \right)\left(\sqrt{4+2h} + 2 / \sqrt{4+2h} + 2 \right)$

$= \lim_{h \to 0} \left(4 + 2h - 4 / h(\sqrt{4+2h} + 2) \right)$

$= \lim_{h \to 0} 2 / \sqrt{4+2h} + 2$

$= 2 / \sqrt{4+2 \cdot 0} + 2 = 2 / \sqrt{4} + 2 = 2 / 2 + 2 = \frac{2}{4} = \frac{1}{2}$

10.2 Limits continued

* One-Sided Limits $\lim_{x \to a^-} f(x)$ $\lim_{x \to a^+} f(x)$

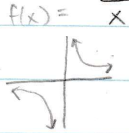
approach from left   right

where $\lim_{x \to a} f(x)$ exists if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

* Infinite Limits $\lim_{x \to a} f(x) = \pm\infty$ increasing/decreasing without bound

Limits at Infinity $\lim_{x \to \pm\infty} f(x)$

ex. consider $f(x) = \dfrac{1}{x}$



$\lim_{x \to 0^-} \frac{1}{x} = -\infty$ since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\lim_{x \to 0^+} \frac{1}{x} = \infty$ then $\lim_{x \to 0} f(x)$ DNE

ex. $\lim_{x \to 3^+} \dfrac{x^2 + x - 12}{\sqrt{x-3}} = \lim_{x \to 3^+} \dfrac{(x-3)(x+4)}{\sqrt{x-3}}$

$= \lim_{x \to 3^+} (\sqrt{x-3})(\sqrt{x-3})(x+4) / \sqrt{x-3}$

$= \lim_{x \to 3^+} (\sqrt{x-3})(x+4)$

$= (\sqrt{3-3})(3+4)$

$= \sqrt{0}(7) = 0 \cdot 7 = 0$

ex. $\lim_{x \to 5^-} \left(\frac{10}{x-5} \right) = \left(-\frac{1}{0} \right) = -\infty$

for $x < 5$, $x - 5 < 0$, neg approaching 0 from the left

ex. $\lim_{x \to 1^-} \frac{|x^2-4|}{x+2}$ given $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ $(|-2| = -(-2) = 2)$

for $x < 1$, $x^2 - 4 < 0$

$= \lim_{x \to 1^-} \frac{-(x^2-4)}{x+2} = \lim_{x \to 1^-} \frac{-(x+2)(x-2)}{x+2}$

$= \lim_{x \to 1^-} -(x-2) = -(1-2) = -(-1) = 1$

ex. $\lim_{x \to \infty} \frac{x^2+9}{5x^2+6}$

$= \lim_{x \to \infty} \frac{x^2(1 + 9/x^2)}{x^2(5 + 6/x^2)}$ as $x \to \infty$, $\frac{9}{x^2}, \frac{6}{x^2} \to 0$

$= \frac{1}{5}$

Note: as $x \to \pm\infty$, $\frac{1}{x} \to 0$

ex. consider dominant terms

$\lim_{x \to -\infty} \frac{x^4 - 3x}{5 - 2x} = \lim_{x \to -\infty} \frac{x^4}{-2x} = -\frac{1}{2} \cdot \lim_{x \to -\infty} x^3 = \left(-\frac{1}{2}\right)(-\infty) = \infty$

ex. $\lim_{x \to -\infty} \frac{x}{(3x-1)^2} = \lim_{x \to -\infty} \frac{x}{9x^2 - 6x + 1} = \lim_{x \to -\infty} \frac{x}{9x^2} = \frac{1}{9} \cdot \lim_{x \to -\infty} \frac{1}{x} = \frac{1}{9}(0) = 0$

ex. $\lim_{x \to \infty} (8x^2 - 2x) = \lim_{x \to \infty} x^2\left(8 - \frac{2}{x}\right) = \lim_{x \to \infty} x^2(8-0) = \lim_{x \to \infty} 8x^2 = \infty$

ex. $\lim_{x \to -\infty} \left(x^2 + \frac{1}{x}\right) = \lim_{x \to -\infty} (x^2) + \lim_{x \to -\infty} \left(\frac{1}{x}\right) = \infty + 0 = \infty$

10.3 Continuity

* f is continous at a if (a) $f(a)$ exists [AND]

(b) $\lim_{x \to a} f(x)$ exists [AND]

(c) $f(a) = \lim_{x \to a} f(x)$

discontinous if $f(a)$ DNE [or] $\lim_{x \to a} f(x)$ DNE

[or] $f(a) \neq \lim_{x \to a} f(x)$

ex. $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 1 \\ 3 & x < 1 \end{cases}$, is f continous at $a=1$?

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 3 = 3$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2+1) = (1)^2+1 = 1+1 = 2$$

since $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, f is discontinous at $a=1$

ex. $f(x) = \begin{cases} x-2 & \text{if } x > 0 \\ 0 & x = 0 \\ -2 & x < 0 \end{cases}$, continous at $a=0$?

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-2) = -2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x-2 = 0-2 = -2$$

but! $f(a) = f(0) = 0$

since $f(a) = 0 \neq -2 = \lim_{x \to a} f(x)$, f discontinous at $a=0$

ex. given $f(x) = \begin{cases} \frac{x^3-8}{x-2} & \text{if } x < 2 \\ c^2x^2 & x \geq 2 \end{cases}$

for what c is f(x) continous at $a=2$?

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^3-8}{x-2} = \lim_{x \to 2^-} \frac{(x-2)(x^2+2x+4)}{x-2}$$

$$= \lim_{x \to 2^-} x^2+2x+4$$

$$= (2)^2+2(2)+4 = 4+4+4 = 12$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} c^2x^2$$

$$= c^2(2)^2 = 4c^2$$

need $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$

$$12 = 4c^2$$

$$3 = c^2$$

$$\pm\sqrt{3} = c$$

double check: $f(a) = f(2) = (\pm\sqrt{3})^2(2)^2$

$$= (3)(4) = 12$$