MATA32 - TUT0017 S.V.

Week 08

12.6 Newton's Method

$$* \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 1, 2, 3, \ldots$$

Note: exam schedule out!

ex. given $x^3 = 3x - 1$, approx the root between $x \in (-1, -2)$

Note: $f(x) = x^3 - 3x + 1$ where $f(-1) = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3$

$f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$

$f(-1)$ and $f(-2)$ have opposite signs, so $f(x) = 0$ between ↳ x = root

() preface notes

(i) $f(x) = x^3 - 3x + 1$, $f'(x) = 3x^2 - 3$

(ii) $f(-2) = -1$ closer to 0, then:

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

$$= x_n - \frac{x_n^3 - 3x_n + 1}{3x_n^2 - 3} = \frac{3x_n^3 - 3x_n - x_n^3 + 3x_n - 1}{3x_n^2 - 3}$$

$$x_{n+1} = 2x_n^3 - 1 / 3x_n^2 - 3$$

(iii) $n = 0$ $\quad x_{n+1} = x_1 = -2$

$n = 1$ $\quad x_2 = \frac{2(-2)^3 - 1}{3(-2)^2 - 3} = \frac{2(-8) - 1}{3(4) - 3} = \frac{-16 - 1}{12 - 3} = \frac{-17}{9} \approx -1.889$

(iv) $f(-\frac{17}{9}) = (-\frac{17}{9})^3 - 3(\frac{17}{9}) + 1 = \frac{-4913}{729} + \frac{51}{9} + 1 \approx -0.07$

can continue approximation.

ex. given $x^2 = 26$, find constants b, c such that $x_{n+1} = bx_n + cx_n^{-1}$

$f(x) = x^2 - 26$

$f'(x) = 2x$

$$x_{n+1} = x_n - f(x_n) / f'(x_n)$$

$$= x_n - x_n^2 - 26 / 2x_n$$

$$= 2x_n^2 - x_n^2 + 26 / 2x_n$$

$$= x_n^2 + 26 / 2x_n$$

$$= \frac{x_n^2}{2x_n} + \frac{26}{2x_n}$$

$$= \frac{1}{2}x_n + 13x_n^{-1}$$

(will solve $x = \pm\sqrt{26}$)

then, $b = \frac{1}{2}$ $\quad c = 13$

12.7 Higher Order Derivatives

* first derivative	y'	$f'(x)$	$\frac{dy}{dx}$
second derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$

ex. $f(x) = 6x^3 - 12x^2 + 6x - 2$

$f'(x) = 18x^2 - 24x + 6$

$f''(x) = 36x - 24$

$f'''(x) = 36$

$f^{(4)}(x) = 0$

ex. $y = e^{x^2}$

$\frac{dy}{dx} = e^{x^2} \cdot 2x$

$= 2xe^{x^2}$

$\frac{d^2y}{dx^2} = 2\left[e^{x^2} + xe^{x^2}(2x)\right]$

$= 2e^{x^2}(2x^2+1)$

ex. $f(x) = x \cdot \ln x$

$f'(x) = \ln x + x\left(\frac{1}{x}\right)$

$= \ln x + 1$

$f''(x) = \frac{1}{x}$

$f'''(x) = -\frac{1}{x^2}$

ex. $y = e^{ax^2}$

$y' = e^{ax^2} \cdot (2ax)$

$= 2axe^{ax^2}$

$y'' = (2a)(e^{ax^2} + xe^{ax^2}(2ax))$

$= 2ae^{ax^2} + 4x^2a^2e^{ax^2}$

Note: higher order with implicit differentiation

ex. $4 = x^2 + 4y^2$

$0 = 2x + 4(2y)\frac{dy}{dx}$

$-2x = 8y \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = -\frac{x}{4y}$

$\frac{d^2y}{dx^2} = \frac{(-1)(4y) - (-x)(4)(\frac{dy}{dx})}{(4y)^2}$

$= \frac{-4y + 4x(\frac{dy}{dx})}{16y^2}$

$= \frac{-y + x(-\frac{x}{4y})}{4y^2}$

$= \left(\frac{1}{4y^2}\right)(-4y^2 - x^2)\left(\frac{1}{4y}\right)$

$= -\frac{4y^2 + x^2}{16y^3}$ Note: $x^2 + 4y^2 = 4$

$= -\frac{4}{16y^3} = -\frac{1}{4y^3}$

ex. given $1 = x^2 + 2xy + y^2$, verify $y'' = 0$.

$$1 = (x+y)^2$$
$$\frac{d}{dx}(1) = \frac{d}{dx}(x+y)^2$$
$$0 = 2(x+y)\left(1 + \frac{dy}{dx}\right)$$
$$0 = 1 + \frac{dy}{dx}$$
$$\frac{dy}{dx} = -1$$
$$\frac{d^2y}{dx^2} = 0$$

[OR]

$$\frac{d}{dx}(1) = \frac{d}{dx}(x^2 + 2xy + y^2)$$
$$0 = 2x + 2y + 2x\frac{dy}{dx} + 2y\frac{dy}{dx}$$
$$-x - y = \frac{dy}{dx}(x+y)$$
$$\frac{dy}{dx} = -\frac{(x+y)}{x+y}$$
$$= -1$$
$$\frac{d^2y}{dx^2} = 0$$

13.1 Relative Extrema

* f differentiable on (a,b), if $f'(x) > 0$ for all $x \in (a,b)$ then f increasing on (a,b)
  $f'(x) < 0$ " f decreasing "
* relative maximum at a if in open interval $f(a) \geq f(x)$ for all $x \in$ interval
  " minimum at a " $f(a) \leq f(x)$ "
  absolute maximum at a " $f(a) \geq f(x)$ for all $x \in f$ domain
  minimum at a $f(a) \leq f(x)$ "
* critical point $a \in f(x)$ if $f'(a) = 0$ or DNE
* First Derivative Test: (1) find $f'(x)$
  (2) determine critical values, evaluate sign chart
  (3) for each a continous, determine $f'(x)$ sign changes to define relative extremas
  (4) for each a not continous, evaluate extrema definition

ex. given $f(x) = x + \dfrac{4}{x+1}$ for $x \neq -1$

(1) $f(x) = x + 4(x+1)^{-1}$

$$f'(x) = 1 - 4(x+1)^{-2} = 1 - \frac{4}{(x+1)^2}$$
$$= \frac{(x+3)(x-1)}{(x+1)^2}$$

(2) $$0 = \frac{(x+3)(x-1)}{(x+1)^2}$$
$$0 = (x+3)(x-1)$$

$0 = x+3$ $\quad$ $0 = x-1$

$-3 = x$ $\quad$ $1 = x$

→ ...continued

... continuing

(...2) sign chart:

	$(-\infty,-3)$	-3	$(-3,-1)$	-1	$(-1,1)$	1	$(1,\infty)$
$(x+3)$	$-$		$+$		$+$		$+$
$(x+1)^{-2}$	$+$		$+$		$+$		$+$
$(x-1)$	$-$		$-$		$-$		$+$
$f'(x)$	$+$		$-$		$-$		$+$
$f(x)$	↗	0	↘	*	↘		↗

↳ $x \neq -1$

(3) -3 = relative maximum

(since $f'(x)$ switches signs $+ \to -$) → $(-3,-5)$

1 = relative minimum

(since $f'(x)$ switches signs $- \to +$) → $(1,3)$

(4) no critical values for which $f(x)$ is not continous

then, $f(x)$ is increasing on $(-\infty,-3)$, $(1,\infty)$

decreasing on $(-3,-1)$, $(-1,1)$

ex. given $y = x^{-1}e^{x} = \dfrac{e^{x}}{x}$

$y' = \dfrac{e^{x}x - e^{x}}{x^{2}} = \dfrac{e^{x}(x-1)}{x^{2}}$

$y' = 0 = \dfrac{e^{x}(x-1)}{x}$

$0 = e^{x}(x-1)$

↳ critical numbers $x = 1, 0$ $\left(0 \text{ DNE} \to \frac{1}{x}\right)$

following,

	$(-\infty,0)$	0	$(0,1)$	1	$(1,\infty)$
y':	$-$	X	$-$	0	$+$
y:	↓		↓		↑

so, y is decreasing on $(-\infty,0)$, $(0,1)$

increasing on $(1,\infty)$

y has a local minimum at $x=1$

13.2 Absolute Extrema on a Closed Interval

* extreme value theorem
* find absolute extrema: (0) f is continous on [a,b]
  (1) find critical values
  (2) evaluate endpoints
  (3) maximum = greatest , minimum = lowest

ex. given $y = x^2 - 4x + 5$ over $x \in [1,4]$

(0) no discontinuities

(1) $y' = 2x - 4 = 0$

$2x = 4$, $x = 2$

(2) critical: $f(2) = 2^2 - 4(2) + 5$
$= 1$

endpoints: $f(1) = 1^2 - 4 \cdot 1 + 5 = 2$

$f(4) = 4^2 - 4 \cdot 4 + 5 = 5$

(3) absolute maximum (4,5) and minimum (2,1)

ex. $f(x) = x^{2/3}$, $x \in [-8,8]$

$f'(x) = \frac{2}{3} x^{-1/3}$

where $f'(x)$ DNE at $x = 0$, only critical point

following, $f(-8) = (-8)^{2/3}$
$= 4$

$f(0) = 0$

$f(8) = (8)^{2/3}$
$= 4$

then, absolute maximum $(\pm 8, 4)$

minimum $(0,0)$

* Chapter 13 builds!

learn basics well