MATA32 - TUT0017 S.V.

Week 06

12.1 Derivatives of Logarithmic Functions

* $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ $x \neq 0$ $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$, $x > 0$ $\frac{d}{dx}(\ln(-x)) = \frac{1}{-x} \cdot \frac{d}{dx}(-x)$, $x < 0$

ex. $y = 5 \cdot \ln(x)$

$y' = 5 \cdot \frac{d}{dx}(\ln x) = \frac{5}{x}$ for $x > 0$

ex. $f(x) = \frac{\ln(x)}{x^2}$ [for $x > 0$]

$$f'(x) = \frac{(\frac{1}{x})(x^2) - (\ln x)(2x)}{(x^2)^2} = \frac{\frac{x^2}{x} - 2x \cdot \ln(x)}{x^4} = \frac{x - 2x \cdot \ln(x)}{x^4} = \frac{1 - 2 \cdot \ln(x)}{x^3}$$

* $\frac{d}{dx}(\ln|u|) = \frac{1}{u} \cdot \frac{du}{dx}$, $u \neq 0$

ex. $f(x) = \ln\left(\frac{ax+b}{cx+d}\right)$ txt 12.1 #26

$u = \frac{ax+b}{cx+d}$

$$f'(x) = \frac{1}{u} \cdot \frac{du}{dx} = \left(\frac{1}{\frac{ax+b}{cx+d}}\right)\left(\frac{(a)(cx+d) - (ax+b)(c)}{(cx+d)^2}\right)$$

$$= \left(\frac{cx+d}{ax+b}\right)\left(\frac{acx + ad - acx - bc}{(cx+d)^2}\right)$$

$$= \left(\frac{1}{ax+b}\right)\left(\frac{ad - bc}{cx+d}\right) = \frac{ad - bc}{(ax+b)(cx+d)}$$

ex. $f(x) = \ln\left((p+2)^3 (p+3)^4\right)$

$= \ln(p+2)^3 + \ln(p+3)^4$

$f'(x) = (p+2)^{-3}(3)(p+2)^2(1) + (p+3)^{-4}(4)(p+3)^3(1)$

$= (p+2)^{-1}(3) + (p+3)^{-1}(4) = \frac{3}{p+2} + \frac{4}{p}$

$= \frac{3}{p+2} + \frac{4}{p+3}$

for $p > -2$

Note: $y = \ln(x^a)$

$y' = (x^{-a})(a)(x^{a-1})(1)$

$= \left(\frac{x^{a-1}}{x^a}\right)(a)$

$= \frac{a}{x}$

Note: $\ln a^2 \neq \ln^2 a$

$\ln(a^2) \neq (\ln(a))^2$

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Note: $\ln(x^a) = a \cdot \ln(x)$

$* \ \frac{d}{dx}(\log_b u) = \frac{1}{u \cdot \ln(b)} \cdot \frac{du}{dx}$

$\log_b u = \frac{\ln(u)}{\ln(b)}$

ex. $y = \log(2x+1)$

$y' = \frac{d}{dx}\left(\frac{\ln(2x+1)}{\ln(10)}\right)$

$= \frac{1}{\ln(10)} \cdot \frac{d}{dx}\ln(2x+1) = \frac{2}{\ln 10 \cdot (2x+1)}$

ex. $f(x) = \log(x) + \ln(2x)$

Note: $\ln(x) = \log_e x$

where: $\log(x) = \log_{10}(x) = \frac{\ln(x)}{\ln(10)}$

$f'(x) = \left(\frac{1}{\ln 10}\right)\left(\frac{1}{x}\right) + \left(\frac{1}{2x}\right)(2) = \frac{1}{x \cdot \ln 10} + \frac{1}{x} = \frac{1}{x}\left(\frac{1}{\ln 10} + 1\right)$

12.2 Derivatives of Exponential Functions

$* \ \frac{d}{dx}(e^u) = e^u \cdot \frac{du}{dx}$

Note: $\ln(e) = 1$

$\frac{d}{dx}(a^u) = a^u \cdot \ln(a) \cdot \frac{du}{dx}$

$a = e^{\ln(a)}$

ex. $\frac{d}{dx}(3e^x) = 3 \cdot \frac{d}{dx}(e^x) = 3(e^x)(1) = 3e^x$

ex. $y = \frac{x}{e^x}$

$\frac{dy}{dx} = \frac{d}{dx}(xe^{-x}) = (1)(e^{-x}) + (x)(e^{-x})(-1) = e^{-x} - xe^{-x} = \frac{1-x}{e^x}$

ex. $\frac{d}{dx}(3^x) = \frac{d}{dx}((e^{\ln 3})^x) = \frac{d}{dx}(e^{x \cdot \ln 3}) = (e^{x \cdot \ln 3})(\ln 3) = 3^x \cdot \ln(3)$

ex. $y = e^2 + x^e + 2^{\sqrt{x}}$

$y' = 0 + ex^{e-1} + \frac{d}{dx}(e^{\sqrt{x} \cdot \ln(2)})$

$= ex^{e-1} + (e^{\sqrt{x} \cdot \ln(2)})(\ln 2)(\tfrac{1}{2})(x^{-1/2})$

Note: $(e^{\ln(2)})^{\sqrt{x}}$

$= ex^{e-1} + (\ln 2)(2^{\sqrt{x}})\left(\frac{1}{2\sqrt{x}}\right)$

ex. $y = xe^{ax}$

$y' = (1)(e^{ax}) + (x)(e^{ax})(a)$

$= e^{ax}(1 + ax)$

Midterm Sample Questions

Finance Math

Q. \$1600 rent due at beginning of month, 2.4% APR interest, pay full rent at beginning of two year period, compounding monthly

$$AD_{PV} = R\left(\frac{1-(1+r)^{-n+1}}{r} + 1\right) \qquad n = 2 \times 12 = 24$$

$$r = \frac{0.024}{12} = 0.002$$

$$= 1600\left(\frac{1-(1+0.002)^{-24+1}}{0.002} + 1\right)$$

$$\approx 37,531$$

Q. debt of 5000 due at end of 20 months, 3000 due at end of 3 years (36m) to be repaid as 4000 now + x 14 months from now + $\frac{1}{2}x$ 30 months from now; interest 6% compounded monthly

debt = payments @ now

$$5000(1.005)^{-20} + 3000(1.005)^{-36} = 4000 + x(1.005)^{-14} + \frac{1}{2}x(1.005)^{-30}$$

$$x = \frac{5000(1.005)^{-20} + 3000(1.005)^{-36} - 4000}{(1.005)^{-14} + \frac{1}{2}(1.005)^{-30}}$$

$$\approx 2225$$

Q. maximum amount of compound interest over 10 years at 6.045% APR

$$\left(\frac{S-P}{P}\right) \times 100 = \left(\frac{Pe^{0.06045 \cdot 10} - P}{P}\right)(100) \approx 83.2\%$$

Limits + Continuity

Q. $a \in (1,5)$

$$\lim_{x \to 2}\left(\frac{a^2x^2 - 2a^2x + 3ax - 6a}{x-2}\right)$$

$$\hookrightarrow \frac{a^2x(x-2) + 3a(x-2)}{x-2} = \frac{(x-2)(a^2x+3a)}{(x-2)}$$

$$= \lim_{x \to 2}(a^2x + 3a)$$

$$= 2a^2 + 3a$$

Q. $$\lim_{x\to -\infty}\left(\frac{12x}{\sqrt{36x^2+25}}+1-e^x\right)$$

$$= \lim_{x\to-\infty}\left(\frac{12x}{\sqrt{36x^2+25}}\right) + \lim_{x\to-\infty}(1) - \lim_{x\to-\infty}(e^x)$$

$$= \lim_{x\to-\infty}\left(\frac{x(12)}{\sqrt{x^2(36+25/x^2)}}\right) + 1 - \infty$$

$$= \lim_{x\to-\infty}\left(12x/x\cdot\sqrt{36+25/x^2}\right) + 1 - \infty$$

$$= \lim_{x\to-\infty}\left(12/\sqrt{36+25/x^2}\right) + 1 - \infty$$

$$= \frac{12}{\sqrt{36+0}} + 1 - \infty = \frac{12}{6} + 1 - \infty = 2+1-\infty = -\infty$$

Derivatives

Q. $$f(x) = \frac{12x^4 + 8x^{3/2}}{2\sqrt{x}} + 4x^{5/2}$$

$$= \left(\frac{12}{2}\right)x^{4-1/2} + \left(\frac{8}{2}\right)x^{\frac{3}{2}-\frac{1}{2}} + 4x^{\frac{5}{2}}$$

$$f'(x) = (6)\left(\frac{7}{2}\right)(x^{5/2}) + (4)(1)(x^{1/2}) + (4)\left(\frac{5}{2}\right)(x^{3/2})$$

$$= 21\sqrt{x^5} + 4(\sqrt{x}) + 10\sqrt{x^3}$$

$$f'(1) = 21\sqrt{(1)^5} + 4\sqrt{1} + 10\sqrt{(1)^3}$$

$$= 21\sqrt{1} + 4(1) + 10\sqrt{1} = 21 + 4 + 10 = 35$$

Q. $$y = 4u^4 - 5u^3 + 8 \qquad u = 4x^2 - 2\sqrt{x}$$

$$\frac{dy}{dx} = (16u^3 - 15u^2)(8x - x^{1/2})$$

Note: $x=1 \quad u = 4(1)^2 - 2\sqrt{1} = 4(1) - 2(1) = 2$

$$\left.\frac{dy}{dx}\right|_{x=1} = (16(2)^3 - 15(2)^2)\left(8\cdot 1 - \frac{1}{\sqrt{1}}\right)$$

$$= (16\cdot 8 - 15\cdot 4)(8-1) = (128-60)(7) = (68)(7) = 476$$

Q. $$y = 3^x x^3 + 3e^{x-1}$$

$$y' = (3^x)(\ln 3)(x^3) + (3^x)(3x^2) + (3)(e^{x-1})$$

$$y'(1) = (3^1)(\ln 3)(1^3) + (3^1)(3\cdot 1^2) + (3)(e^{1-1})$$

$$= (3)(\ln 3)(1) + (3)(3\cdot 1) + (3)(e^0)$$

$$= 3\cdot\ln 3 + (3)(3) + (3)(1)$$

$$= 3\cdot\ln 3 + 9 + 3$$

$$= 3\cdot\ln 3 + 12$$