MATA32 - TUT0017 S.V.

Week 05

11.1 The Derivative

$$* \; m_{tan} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

derivative of a function provided that the limit exists

→ $f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx} f(x)$

if f is differentiable at a, then f is continous at a

Note: continous does not imply differentiable

11.2 Rules for Differentiation

$* \; \frac{d}{dx}(c) = 0$ given c constant

$\frac{d}{dx}(x^a) = ax^{a-1}$ $a \in \mathbb{R}$

$\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x)$ c constant, f differentiable

$\frac{d}{dx}(f(x) \pm g(x)) = f'(x) \pm g'(x)$ f, g differentiable

11.3 Rate of Change

→ applicable derivatives

→ marginal cost = $\frac{dc}{dq}$ marginal revenue $\frac{dr}{dq}$

→ relative rate of change $f'(x) / f(x)$

11.4 Product and Quotient Rule

$* \; \frac{d}{dx}(f \cdot g) = f'g + fg'$ given f, g differentiable

$(fgh)' = (fg)'h + (fg)h'$ (+h) differentiable

$\frac{d}{dx}(f/g) = \frac{f'g - fg'}{g^2}$ f, g differentiable AND $g \neq 0$

→ economics applications $r = pq$ $\frac{dc}{dI}$

11.5 Chain Rule

$* \; \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ given $y = f(u)$ differentiable, $u = f(x)$ differentiable

$\frac{d}{dx}(u^a) = au^{a-1} \cdot \frac{du}{dx}$

→ economics applications

ex. $f(x) = \dfrac{3}{x-2}$ by definition

$$f'(x) = \lim_{z \to x} \left[\frac{H(z) - H(x)}{z - x} \right]$$

$$= \lim_{z \to x} \frac{\left(\frac{3}{z-2} - \frac{2}{x-2} \right)}{z - x}$$

$$= \lim_{z \to x} \left[\frac{3(x-2) - 3(z-2)}{(z-2)(x-2)} \Big/ z-x \right]$$

$$= \lim_{z \to x} 3x - 3z \Big/ (z-2)(x-2)(z-x)$$

$$= \lim_{z \to x} -3(z-x) \Big/ (z-2)(x-2)(z-x)$$

$$= \lim_{z \to x} -3 \Big/ (z-2)(x-2)$$

$$= -3 \Big/ (x-2)(x-2) = -3 \Big/ (x-2)^2$$

ex. $f(x) = 4x^2$ $\quad f'(x) = (4) \frac{d}{dx}(x^2) = 4(2)(x^{2-1}) = 8x$

$f(x) = 2x^{-14/5}$ $\quad f'(x) = 2 \frac{d}{dx}\left(x^{-14/5}\right) =$

$$= 2\left(-\frac{14}{5}\right)\left(x^{-\frac{14}{5} - 1}\right) = \left(-\frac{28}{5}\right)\left(x^{-\frac{14}{5} - \frac{5}{5}}\right)$$

$$= -\frac{28}{5} \cdot x^{-19/5}$$

$f(x) = \sqrt[8]{x^2}$ $\quad f'(x) = \frac{d}{dx}\left(x^{1/4}\right)$

$= (x^2)^{1/8} = x^{1/4}$ $\quad = \frac{1}{4} x^{\frac{1}{4} - 1} = \frac{1}{4} x^{-3/4}$

ex. $f(x) = \dfrac{7x^3 + x}{6\sqrt{x}}$ a) quotient rule b) simplify!

$$= 7x^3 \Big/ 6x^{1/2} + x \Big/ 6x^{1/2}$$

$$= \frac{7}{6} x^{3 - 1/2} + \frac{1}{6} x^{1 - 1/2} = \frac{7}{6}(x^{5/2}) + \frac{1}{6}(x^{1/2})$$

$$f'(x) = \left(\frac{7}{6}\right)\left(\frac{5}{2}\right)\left(x^{5/2 - 2/2}\right) + \left(\frac{1}{6}\right)\left(\frac{1}{2}\right)\left(x^{1/2 - 2/2}\right)$$

$$= \frac{35}{12}(x^{3/2}) + \frac{1}{12} x^{-1/2}$$

$$= \frac{35\sqrt{x^3}}{12} + \frac{1}{12\sqrt{x}}$$

ex. $f(x) = (x^2 + 3x)(7x^2 - 5)$ a) expand b) product rule!

$$f'(x) = (2x + 3)(7x^2 - 5) + (x^2 + 3x)(14x - 0)$$

$$= 14x^3 - 10x + 21x^2 - 15 + 14x^3 + 42x^2$$

$$= 28x^3 + 63x^2 - 10x - 15$$

ex. $y = \frac{3x-5}{7x+11}$

$y' = \frac{(3)(7x+11) - (3x-5)(7)}{(7x+11)^2}$

$= 14x + 33 - 14x + 35 \,/\, (7x+11)^2$

$= 68 \,/\, (7x+11)^2$

ex. $y = \frac{x-5}{8\sqrt{x}}$ a) quotient rule b) simplify!

$= \frac{x}{8\sqrt{x}} - \frac{5}{8\sqrt{x}} = \frac{1}{8}x^{1-1/2} - \frac{5}{8}x^{-1/2}$

$y' = \left(\frac{1}{8}\right)\left(\frac{1}{2}\right)\left(x^{1/2-1}\right) - \left(\frac{5}{8}\right)\left(-\frac{1}{2}\right)\left(x^{-1/2-1}\right)$

$= \frac{1}{16}x^{-1/2} + \frac{5}{16}x^{-3/2}$

$= \frac{1}{16\sqrt{x}} + \frac{5}{16\sqrt{x^3}}$

ex. $f(x) = \sqrt{9+x^2/4-x} = \left(9+x^2/4-x\right)^{1/2}$

$f'(x) = \left(\frac{1}{2}\right)\left(9+x^2/4-x\right)^{1/2-1} \cdot \frac{d}{dx}\left(9+x^2/4-x\right)$

$= \left(\frac{1}{2}\right)\left(9+x^2/4-x\right)^{-1/2}\left(\frac{(2x)(4-x) - (9+x^2)(-1)}{(4-x)^2}\right)$

* $\left.\frac{dy}{dx}\right|_{x=0}$

$f'(0) = \left(\frac{1}{2}\right)\left(\frac{9+0^2}{4-0}\right)^{-1/2}\left(\frac{(2\cdot 0)(4-0) + (9+0^2)}{(4-0)^2}\right)$

$= \left(\frac{1}{2}\right)\left(\frac{9}{4}\right)^{-1/2}\left(\frac{0+9}{4^2}\right)$

$= \left(\frac{1}{2}\right)\left(\frac{4}{9}\right)^{1/2}\left(\frac{9}{16}\right) = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{9}{16}\right) = \left(\frac{1}{3}\right)\left(\frac{9}{16}\right) = \frac{3}{16}$

ex. given $y = 2u^3 - 8u$ $u = 7x - x^3$ a) plug in + expand b) chain rule!

$y' = \frac{dy}{du} \cdot \frac{du}{dx}$

$= (6u^2 - 8)(7 - 3x^2)$

$= \left[6(7x-x^3)^2 - 8\right] \cdot \left[7 - 3x^2\right]$

$= 6(7x-x^3)^3 - 8(7-3x^2)$

ex. given $y = -\sqrt[3]{x}$, find equation of the tangent line to the curve at the point $(8, -2)$.

(i) Note: $-2 = -\sqrt[3]{8}$ does lie on the curve.

(ii) slope is given by the derivative

$$y = -x^{1/3} \qquad y' = (-1)\left(\tfrac{1}{3}\right)\left(x^{1/3 - 3/3}\right)$$
$$= -\tfrac{1}{3}x^{-2/3}$$

(iii) m= slope when $x = 8$

$$= \left(-\tfrac{1}{3}\right)(8)^{-2/3} = \left(-\tfrac{1}{3}\right)\left(\frac{1}{\sqrt[3]{8^2}}\right)$$
$$= \left(-\tfrac{1}{3}\right)\left(\frac{1}{\sqrt[3]{64}}\right) = \left(-\tfrac{1}{3}\right)\left(\tfrac{1}{4}\right) = -\tfrac{1}{12}$$

(iv) equation of tangent line $\quad y - y_1 = m(x - x_1)$ $\qquad \frac{\text{rise}}{\text{run}}$

$$y - (-2) = \left(-\tfrac{1}{12}\right)(x - 8)$$
$$y + 2 = -\tfrac{1}{12}x + \tfrac{8}{12}$$
$$y = -\tfrac{1}{12}x + \tfrac{2}{3} - \tfrac{6}{3} \quad (=2)$$
$$y = \left(-\tfrac{1}{12}\right)x - \tfrac{4}{3} \qquad y = mx + b$$