MATA32 – TUT0017 S.V.

Week 10

14.2 The Indefinite Integral

* F is an antiderivative of f given $F'(x) = f(x)$

$$dF = f(x)\,dx$$

$$\int f(x)\,dx = F(x) + c \qquad c \text{ constant}$$

* $\int k\,dx = kx + c$

$$\int x^a\,dx = \left(\frac{1}{a+1}\right)x^{a+1} + c \qquad a \neq -1$$

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(x) + c \qquad x > 0$$

$$\int e^x\,dx = e^x + c$$

$$\int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx \qquad k \text{ constant}$$

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Note: test corrections V2PAQ11 V4PAQ3

ex. $\int 7x\,dx = 7 \cdot \int x\,dx = 7\left(\frac{1}{2}x^2 + C_1\right) = \frac{7}{2}x^2 + c$

ex. $\int -\frac{3}{5}e^x\,dx = -\frac{3}{5}\int e^x\,dx = -\frac{3}{5}\left(e^x + C_1\right) = -\frac{3}{5}e^x + c$

ex. $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right)dx$

$= 2 \cdot \int x^{4/5}\,dx - 7 \cdot \int x^3\,dx + 10\int e^x\,dx - \int dx$

$= 2\left(\frac{1}{\frac{4}{5}+1}x^{\frac{4}{5}+1} + C_1\right) - 7\left(\frac{1}{4}x^4 + C_2\right) + 10\left(e^x + C_3\right) - \left(x + C_4\right)$

$= 2\left(\frac{1}{9/5}x^{9/5}\right) - 7\left(\frac{1}{4}x^4\right) + 10e^x - x + \underbrace{\left(2C_1 - 7C_2 + 10C_3 - C_4\right)}_{\text{is just a constant}}$

$= \frac{10}{9}x^{9/5} - \frac{7}{4}x^4 + 10e^x - x + c$

ex. $\int \frac{x^3 - 1}{x^2}\,dx$

$= \int\left(\frac{x^3}{x^2} - \frac{1}{x^2}\right)dx$

$= \int\left(x - x^{-2}\right)dx$

$= \frac{1}{2}x^2 - \left(\frac{1}{-2+1}x^{-2+1}\right) + c$

$= \frac{x^2}{2} - \left(\frac{x^{-1}}{-1}\right) + c$

$= \frac{x^2}{2} + \frac{1}{x} + c$

ex. $\int \frac{x^4 - 5x^2 + 2x}{5x^2}\,dx$

$= \int\left(\frac{x^4}{5x^2} - \frac{5x^2}{5x^2} + \frac{2x}{5x^2}\right)dx$

$= \int\left(\frac{1}{5}x^2 - 1 + \frac{2}{5}x^{-1}\right)dx$

$= \left(\frac{1}{5}\right)\left(\frac{1}{3}x^3\right) - (x) + \left(\frac{2}{5}\right)(\ln x) +$

$= \frac{1}{15}x^3 - x + \frac{2}{5}\ln(x) + c$

Note: $x > 0$

$= \frac{1}{15}x^3 - x + \frac{2}{5}\ln(|x|) + c$

ex. $\int e^x + x^e\,dx = e^x + \frac{x^{e+1}}{e+1} + c$

ex. $\int \sqrt[3]{6x}\,dx = \int 6^{1/3}x^{1/3}\,dx = 6^{1/3}\int x^{1/3}\,dx = 6^{1/3}\left(\frac{3}{4}x^{4/3} + C\right) = \frac{3 \cdot \sqrt[3]{6}}{4}x^{4/3} + c$

14.3 Integration with Initial Conditions

- initial condition: a condition which gives a function value of f for a specific value of x

ex. given $y'' = x^2 - 6$, $y'(0) = 2$, and $y(1) = -1$, find y.

(i) $y' = \int x^2 - 6\,dx$

$= \frac{1}{3}x^3 - 6x + C \longrightarrow y'(0) = \frac{1}{3}0^2 - 6(0) + C$

$2 = 0 - 0 + C$

$2 = C$

$y' = \frac{1}{3}x^3 - 6x + 2$

(ii) $y = \int \left(\frac{1}{3}x^3 - 6x + 2\right)dx$

$= \left(\frac{1}{3}\right)\left(\frac{1}{4}x^4\right) - (6)\left(\frac{1}{2}x^2\right) + 2x + C$

$= \frac{1}{12}x^4 - 3x^2 + 2x + C \longrightarrow y(1) = \frac{1}{12}(1)^4 - 3(1)^2 + 2(1) + C$

$-1 = \frac{1}{12} - 3 + 2 + C$

$-\frac{1}{12} = C$

(iii) $y = \frac{1}{12}x^4 - 3x^2 + 2x - \frac{1}{12}$

ex. find the demand function given the marginal revenue

$\frac{dr}{dq} = 5000 - 3(2q + 2q^3)$

* assume $r(0) = 0$ (given initial condition)

revenue $= r = \int \frac{dr}{dq}\,dq$

$= \int (5000 - 3(2q + 2q^3))\,dq$

$= \int 5000 - 6q - 6q^3\,dq$

$= 5000q - (6)\left(\frac{1}{2}q^2\right) - (6)\left(\frac{1}{4}q^4\right) + C$

$= 5000q - 3q^2 - \frac{3}{2}q^4 + C$

$r|_0 = 0 = 5000(0) - 3(0)^2 - \frac{3}{2}(0)^4 + C$

$0 = C$

$r = 5000q - 3q^2 - \frac{3}{2}q^4$

$r = pq$, $p = \frac{r}{q}$

$p = 5000 - 3q - \frac{3}{2}q^3$ demand function.

14.4 Integration Formula's

$\star \quad \int u^a \, du = \left(\frac{1}{a+1}\right) u^{a+1} + C \quad , \quad a \neq -1$

used for u-sub, where $u(x)$ is differentiable

ex. $\int 3x^2(x^3+7)^3 \, dx = A$

consider $u = x^3 + 7 \qquad du = 3x^2 \, dx$

$$
\begin{aligned}
A &= \int (x^3+7)^3 \, 3x^2 dx \\
&= \int u^3 \, du \\
&= \frac{1}{4} u^4 + C = \frac{1}{4}(x^3+7)^4 + C
\end{aligned}
$$

ex. $\int x\sqrt{x^2+5} \, dx$

$= \int (x)(x^2+5)^{1/2} \, dx = B$

consider $u = x^2 + 5 \qquad du = 2x dx$

$x dx = \frac{1}{2} du$

$$
\begin{aligned}
B &= \int (u)^{1/2} \left(\frac{1}{2} du\right) \\
&= \frac{1}{2} \cdot \int u^{1/2} \, du = \frac{1}{2}\left(\frac{1}{\frac{1}{2}+1} u^{1/2+1}\right) + C \\
&= \frac{1}{2}\left(\frac{2}{3} u^{3/2}\right) + C \\
&= \frac{1}{3}(x^2+5)^{3/2} + C
\end{aligned}
$$

ex. $\int \frac{4x}{\sqrt{x^2+1}} \, dx$, how to choose u-sub?

$u = \{ 4x, \; x^2+1, \; \sqrt{x^2+1} \}$

$du = \{ \underbrace{4dx, \; 2x dx}, \;$ makes more complicated $\}$

one removes the 'x' extra term, try to sub all x out

$$
\begin{aligned}
&= \int (4)(x^2+1)^{-1/2} (x dx) \\
&= 4 \cdot \int u^{-1/2} \left(\frac{1}{2} du\right) = \frac{4}{2} \cdot \int u^{-1/2} du \\
&= 2\left(2u^{1/2}\right) + C \\
&= 4u^{1/2} + C \\
&= 4(x^2+1)^{1/2} + C
\end{aligned}
$$

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$\ast$ $\int e^u du = e^u + c$

* use u-sub $\quad \int \frac{1}{u} du = \ln|u| + c \qquad u \neq 0$

ex. $\int 2xe^{x^2}\,dx$

let $u = x^2 \qquad du = 2x\,dx$

$= \int e^u du$

$= e^u + C = e^{x^2} + C$

$\frac{d}{dx}(e^{x^2}+C) = (e^{x^2})(2x) + 0 = 2xe^{x^2}$

ex. $\int \frac{2x}{x^2+5}\,dx$

let $u = x^2+5 \qquad du = 2x\,dx$

$= \int \left(\frac{1}{x^2+5}\right)(2x\,dx)$

$= \int \frac{1}{u}$

$= \ln|u| + C$

$= \ln|x^2+5| + C \qquad$ Note: $x^2+5 > 0$

$= \ln(x^2+5) + C$

ex. $\int \frac{2x^3+3x}{x^4+3x^2+7}\,dx$

let $u = x^4+3x^2+7 \qquad du = (4x^3+6x)\,dx$

$\frac{1}{2} du = (2x^3+3x)\,dx$

$= \int \frac{1}{2u} du = \frac{1}{2} \cdot \ln|u| + C$

$= \frac{1}{2} \cdot \ln|x^4+3x^2+7| + C$

ex. find y given $y' = \frac{x}{x^2+6}$, $y(1) = 0$

$y = \int y'\,dx = \int \frac{x}{x^2+6}\,dx$

let $u = x^2+6 \qquad du = 2x\,dx \qquad \frac{1}{2} du = x\,dx$

$= \int \left(\frac{1}{u}\right)\left(\frac{1}{2} du\right)$

$= \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln|u| + C$

$= \frac{1}{2} \ln|x^2+6| + C$

$y'(1) = \frac{1}{2} \ln(1^2+6) + C$

$0 = \frac{1}{2} \ln(7) + C \longrightarrow C = -\frac{1}{2}\ln(7)$

then, $y = \frac{1}{2} \ln(x^2+6) - \frac{1}{2}\ln(7) = \frac{1}{2} \ln\left(\frac{x^2+6}{7}\right) = \ln\left(\frac{x^2+6}{7}\right)^{1/2}$ by ln() properties