MATA32 - TUT0017 S.V.

Week 07

12.3 Elasticity of Demand

* $\eta = \dfrac{p/q}{dp/dq} = \dfrac{p}{q} \cdot \dfrac{dq}{dp} = \dfrac{\text{percentage change in quantity}}{\text{percentage change in price}}$

ex. given demand function $p = 1200 - q^2$

$$\eta = \frac{p/q}{dp/dq} = \frac{1200-q^2/q}{-2q} = \frac{q^2-1200}{2q^2}$$

if $q = 10$, $\eta = -5.5 = $ % change in demand / % change in price

then, if price increase by 1%:

% change in demand = $\eta \cdot$ % change in price

= $(-5.5)(1\%)$

= -5.5% >> demand decreases

Note: $|\eta| = |-5.5| = 5.5 > 1$, demand is elastic

12.4 Implicit Differentiation

* given an equation that defines y implicitly as a differentiable function of x

to find $\frac{dy}{dx}$: (i) differentiate both sides with respect to x

(ii) solve for $\frac{dy}{dx}$, note restrictions

ex. $y^4 = x^3 + 4xy^2 - 27$

$$\frac{d}{dx}(y^4) = \frac{d}{dx}(x^3 + 4xy^2 - 27)$$

$$4y^3 \cdot \frac{dy}{dx} = 3x^2 + 4x\left(\frac{d}{dx}y^2\right) + 4y^2 - 0$$

$$= 3x^2 + 8xy \cdot \frac{dy}{dx} + 4y^2$$

$$\frac{dy}{dx}(4y^3 - 8xy) = 3x^2 + 4y^2$$

$$\frac{dy}{dx} = \frac{3x^2 + 4y^2}{4y^3 - 8xy}$$

where $4y^3 - 8xy \neq 0$

$y^3 - 2xy \neq 0$

$y(y^2 - 2x) \neq 0$

ex. $0 = x^2 + xy - 2y^2$ $\qquad (\frac{dy}{dx} = y')$

$\frac{d}{dx}(0) = \frac{d}{dx}(x^2 + xy - 2y^2)$

$0 = 2x + y + xy' - 4y'$

$xy' - 4y' = -2x - y$

$y'(x-4) = -2x - y$

$y' = \frac{-2x-y}{x-4} = \frac{2x+y}{4-x}$, $\quad 4-x \neq 0$, $x \neq 4$

ex. $10 = x^3 + 3y^2 - 6xy$

$0 = 3x^2 + 6y \cdot \frac{dy}{dx} - 6x\frac{dy}{dx} - 6y$

$\frac{dy}{dx}(6y - 6x) = 6y - 3x^2$

$\frac{dy}{dx} = 6y - 3x^2 / 6y - 6x = \frac{2y - x^2}{2y - 2x}$

; $2y \neq 2x$, $y \neq x$

ex. $x + y - 1 = (x - y)^2$

$1 + \frac{dy}{dx} - 0 = 2(x-y)(1 - \frac{dy}{dx})$

$1 + \frac{dy}{dx} = (2x - 2y)(1 - \frac{dy}{dx})$

$= 2x - 2x\frac{dy}{dx} - 2y + 2y\frac{dy}{dx}$

$1 - 2x + 2y = -2x\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx}$

$= \frac{dy}{dx}(-2x + 2y - 1)$

$\frac{dy}{dx} = \frac{1 - 2x + 2y}{2y - 2x - 1}$; $\quad 2y - 2x - 1 \neq 0$

ex. $y^2 + y = \ln(x)$

$2y \cdot \frac{dy}{dx} + \frac{dy}{dx} = \frac{1}{x}$

$\frac{dy}{dx}(2y+1) = \frac{1}{x}$

$\frac{dy}{dx} = \frac{1}{x(2y+1)} = [x(2y+1)]^{-1}$; $\quad x(2y+1) \neq 0$

ex. $4 = x^2 + y^3$ $\qquad (y^3 = 4 - x^2 > y = (4-x^2)^{1/3})$

$0 = 2x + 3y^2 \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-2x}{3y^2} = \frac{-2x}{3(4-x^2)^{2/3}} = \frac{1}{3}(-2x)(4-x^2)^{-2/3}$

[OR] $y = (4-x^2)^{1/3}$

$y' = (\frac{1}{3})(4-x^2)^{-2/3}(-2x)$

↳ explicit only works if one 'y' term

ex. find the slope and equation of the tangent line at a given point

$y^2 e^{2x} = 3y + x^2$, $p = (0,3)$

(i) $m = \text{slope} = \frac{dy}{dx}$

$$e^{2x} \cdot 2y \cdot \frac{dy}{dx} + y^2 e^{2x} \cdot 2 = 3\frac{dy}{dx} + 2x$$

$$\frac{dy}{dx}(2ye^{2x} - 3) = 2x - 2y^2e^{2x}$$

$$\frac{dy}{dx} = \frac{2x - 2y^2e^{2x}}{2ye^{2x} - 3}$$

Note restrictions!

$$\left.\frac{dy}{dx}\right|_{(0,3)} = \frac{2(0) - 2(3)^2 e^{2\cdot 0}}{2(3)e^{2\cdot 0} - 3} = \frac{-18}{3} = -6$$

(ii) equation

$$y - y_1 = m(x - x_1)$$

$$y - 3 = -6(x - 0)$$

$$y = 3 - 6x$$

Note: $p = (0,3)$ on both lines exists

12.5 Logarithmic Differentiation

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$$y = f(x)$$

$$\ln y = \ln(f(x))$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\ln(f(x)))$$

simplify using ln properties

express answer in terms of x only

ex. $y = (3x^3 - 1)^2(x+3)^4$

$$\ln(y) = \ln[(3x^3-1)^2(x+3)^4]$$

$$= \ln[(3x^3-1)^2] + \ln[(x+3)^4]$$

$$= 2 \cdot \ln(3x^3-1) + 4 \cdot \ln(x+3)$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}[2 \cdot \ln(3x^3-1) + 4 \cdot \ln(x+3)]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2\left(\frac{1}{3x^3-1}\right)(9x^2) + 4\left(\frac{1}{x+3}\right)(1)$$

$$\frac{dy}{dx} = y\left[2\left(\frac{9x^2}{3x^3-1}\right) + 4\left(\frac{1}{x+3}\right)\right]$$

$$= (3x^3-1)^2(x+3)^4\left(\frac{18x^2}{3x^3-1} + \frac{4}{x+3}\right)$$

$$= (18x^2)(3x^3-1)(x+3)^4 + (4)(3x^3-1)^2(x+3)^3$$

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ex. $y = \sqrt{x^2+5/x+9} = \left(\frac{x^2+5}{x+9}\right)^{1/2}$

$\ln(y) = \ln\left(\frac{x^2+5}{x+9}\right)^{1/2} = \frac{1}{2}\cdot\ln\left(\frac{x^2+5}{x+9}\right)$

$\ln(y) = \frac{1}{2}\left[\ln(x^2+5) - \ln(x+9)\right]$

$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2}\left(\left(\frac{1}{x^2+5}\right)(2x) - \left(\frac{1}{x+9}\right)\right)$

$\frac{dy}{dx} = \left(\frac{x^2+5}{x+9}\right)^{1/2}\left(\frac{1}{2}\right)\left(\frac{2x}{x^2+5} - \frac{1}{x+9}\right)$

ex. $y = x^x$

$\ln(y) = \ln(x^x) = x\cdot\ln(x)$

Note: need implicit differentiation since $\ln(y) = \ln(f(x))$
not $y = f(x)$

$\frac{1}{y}\cdot\frac{dy}{dx} = (1)(\ln x) + (x)\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = y(\ln x + 1)$

$= (x^x)(\ln x + 1)$

Note: $x^x = (e^{\ln x})^x$, $y = (e^{x\cdot\ln x})$

$\rightarrow y' = (e^{x\cdot\ln x})(1\cdot\ln x + x\cdot\frac{1}{x})$

$= x^x(\ln x + 1)$

ex. $y = (\sqrt{x})^x$

$\ln(y) = \ln\left[(\sqrt{x})^x\right] = x\cdot\ln(\sqrt{x}) = x\cdot\ln(x^{1/2})$

$\ln(y) = \frac{x}{2}\ln(x)$

$\frac{1}{y}\cdot\frac{dy}{dx} = \left(\frac{1}{2}\right)(\ln x) + \left(\frac{x}{2}\right)\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = y\left(\frac{1}{2}\ln x + \frac{1}{2}\right)$

$= (\sqrt{x})^x\left(\frac{\ln x + 1}{2}\right)$

ex. $y = (1+e^x)^{\ln x}$

$\ln(y) = \ln\left[(1+e^x)^{\ln x}\right] = \ln x\cdot\left[\ln(1+e^x)\right]$

$\frac{1}{y}\cdot\frac{dy}{dx} = \left(\frac{1}{x}\right)(\ln(1+e^x)) + (\ln x)\left(\frac{1}{1+e^x}\right)(e^x)$

$\frac{dy}{dx} = y\left[\frac{\ln(1+e^x)}{x} + \frac{e^x\ln x}{1+e^x}\right]$

$= (1+e^x)^{\ln x}\cdot\left(\frac{\ln(1+e^x)}{x} + \frac{e^x\ln x}{1+e^x}\right)$