MATA32 - TUT0017 S.V.

week 11

14.5 Techniques of Integration

* preliminary division $\int \frac{f}{g} = \int (q + \frac{r}{g}) = \int q + \int \frac{r}{g}$

given $\frac{f}{g} = q + \frac{r}{g}$

ex. $\int \frac{2x^3 + 3x^2 + x + 1}{2x+1} dx = \int \left(\frac{2x^3 + 3x^2 + x}{2x+1} + \frac{1}{2x+1} \right) dx$

$= \int \left(\frac{x(2x^2 + 3x + 1)}{2x+1} + \frac{1}{2x+1} \right) dx$

$= \int \left(\frac{x(2x+1)(x+1)}{2x+1} + \frac{1}{2x+1} \right) dx$

$= \int \left(x(x+1) + \frac{1}{2x+1} \right) dx$

$= \int x^2 dx + \int x dx + \int \frac{1}{2x+1} dx$

$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{2}\ln|2x+1| + C$

ex. $\int \frac{\sqrt{2+\sqrt{x}}}{\sqrt{x}} dx$

let $u = 2 + \sqrt{x}$ $du = \frac{1}{2\sqrt{x}} dx$, $2du = \frac{1}{\sqrt{x}} dx$

$= \int \sqrt{u}\,(2du)$

$= 2 \cdot \int u^{1/2} du$

$= 2\left(\frac{2}{3}u^{3/2}\right) + C = \frac{4}{3}(2+\sqrt{x})^{3/2} + C$

* $\int b^u du = \frac{1}{\ln b} b^u + C$ (Note: $\int e^u du = e^u + C$, $b^u = e^{(\ln b)u}$

ex. $\int 2^{3-x} dx$

where $b = 2$ $u = 3-x$ $du = -dx$, $-du = dx$

$= \int b^u (-du)$

$= -\int e^{\ln(b)u} du$

$= -\frac{1}{\ln(b)}(b^u) + C = -\frac{2^{3-x}}{\ln(2)} + C$

ex. $\int e^{f(x) + \ln(f'(x))} dx$, assume $f'(x) > 0$

$= \int e^{f(x)} \cdot \underline{e^{\ln(f'(x))}} dx$

$e^{\ln(b)u} = b^u = (f'(x))^1$ (one)

$= \int e^{f(x)} \cdot f'(x) dx$

substitute $u = f(x)$, $du = f'(x)dx$

$= \int e^u du = e^u + C = e^{f(x)} + C$

* Integration by Parts

given u, v functions of x, apply product rule for differentiation

$$\frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$u \frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

by integration: $\int u\,dv = uv - \int v\,du$

ex. $\int x^2 \ln(4x)\,dx$

let $u = \ln(4x)$ $\quad dv = x^2\,dx$

$du = \frac{4}{4x}\,dx = \frac{1}{x}\,dx$ $\quad v = \frac{1}{3}x^3$

then, $\int u\,dv = uv - \int v\,du$

$$= \ln(4x) \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{3}x^3 \ln(4x) - \frac{1}{3}\int x^2\,dx$$

$$= \frac{1}{3}x^3 \ln(4x) - \frac{1}{9}x^3\,dx + c$$

ex. $\int \frac{1}{x^2} \ln(x)\,dx$

let $u = \ln(x)$ $\quad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx$ $\quad v = -\frac{1}{x}$

$$\int u\,dv = (\ln(x))\left(-\frac{1}{x}\right) - \int\left(-\frac{1}{x}\right)\left(\frac{1}{x}\right)dx$$

$$= -\frac{\ln(x)}{x} + \int \frac{1}{x^2}\,dx$$

$$= -\frac{\ln(x)}{x} - \frac{1}{x} + c$$

ex. $\int x e^{ax}\,dx$

let $u = x$ $\quad dv = e^{ax}\,dx$

$du = dx$ $\quad v = \frac{1}{a}e^{ax}$

$$\int u\,dv = x \cdot \frac{1}{a}e^{ax} - \int \frac{1}{a}e^{ax}\,dx$$

$$= \frac{x}{a}e^{ax} - \frac{1}{a^2}e^{ax} + c$$

ex. $\int \frac{2x+7}{e^{3x}}\,dx$

$u = 2x+7$ $\quad du = 2dx$; $\quad dv = e^{-3x}\,dx$ $\quad v = -\frac{1}{3}e^{-3x}$

$$= -\frac{1}{3}(2x+7)(e^{-3x}) + \frac{2}{3}\int e^{-3x}\,dx$$

$$= -\frac{(2x+7)}{3}e^{-3x} - \frac{2}{9}e^{-3x} + c$$

14.6 The Definite Integral

- read textbook!; measuring the area underneath a graph
- the common limit of $\underline{S_n}$ and $\overline{S_n}$ as $n \to \infty$, if it exists, is called the definite integral of f over $[a,b]$ written a

$\int_a^b f(x)\,dx$ where a, b upper/lower bounds of integration

x variable of integration

$f(x)$ integrand

ex. integrate $f(x) = x-5$ from $x=0$ to $x=3$ $\iff \int_0^3 (x-5)\,dx$

(i) divide $[0,3]$ into n intervals of equal length $\Delta x = \frac{3}{n}$

gives endpoints: $0, \frac{3}{n}, 2\cdot\frac{3}{n}, \ldots (n-1)\frac{3}{n}, n\cdot\frac{3}{n} = 3$

(ii) form sum and simplify

$$S_n = \sum_{k=1}^{n} f\cdot\left(k\cdot\frac{3}{n}\right)\cdot\frac{3}{n}$$

$$= \sum_{k=1}^{n}\left(\left(k\cdot\frac{3}{n}-5\right)\frac{3}{n}\right) = \sum_{k=1}^{n}\left(\frac{9}{n^2}k - \frac{15}{n}\right)$$

$$= \frac{9}{n^2}\sum_{k=1}^{n} k - \frac{15}{n}\sum_{k=1}^{n} 1$$

$$= \frac{9}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{15}{n}(n)$$

$$S_n = \frac{9}{2}\left(\frac{n+1}{n}\right) - 15$$

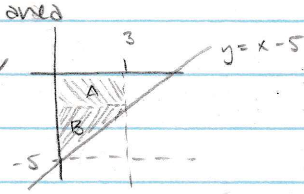
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{9}{2}\left(1+\frac{1}{n}\right)-15\right)$$

$$= \frac{9}{2}(1+0) - 15$$

$$\int_0^3 (x-5)\,dx = -\frac{21}{2}$$

(iii) measure as area

consider



$\text{area}(A) = bh = (-2)(3) = -6$

$\text{area}(B) = \frac{1}{2}bh = \frac{1}{2}(3)(-3) = -\frac{9}{2}$

then area of $y = x-5$ between $x=0, x=3$

$= \text{area}(A) + \text{area}(B)$

$= -6 + \left(-\frac{9}{2}\right)$

$= -\frac{12}{2} - \frac{9}{2}$

$= -\frac{21}{2}$

Note: understand this, will be very useful for chapter 15

14.7 The Fundamental Theorem of Calculus

* $\int_a^b f(x)\,dx = F(b) - F(a)$ given f is continous and F is the antiderivative of f on the interval $[a,b]$

- properties of the definite integral
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ given f continous, $a \le b \le c \in$ interval

ex. $\int_0^1 \frac{x^3}{\sqrt{1+x^4}}\,dx = \int_0^1 x^3(1+x^4)^{-1/2}\,dx$

$u = 1+x^4$ $\quad du = 4x^3\,dx$, $\frac{1}{4}\,du = x^3\,dx$

$= \int_0^1 \frac{1}{4} u^{-1/2}\,du$

$= \frac{1}{4} \cdot \int_0^1 u^{-1/2}\,du$

$= \frac{1}{4}\left(2u^{1/2}\right)\Big|_{0=x}^{1=x}$ Note: no $+ C$

$= \frac{1}{2}(1+x^4)^{1/2}\Big|_0^1$

$= \frac{1}{2}\left((1+1^4)^{1/2} - (1+0^4)^{1/2}\right)$

$= \frac{1}{2}\left(2^{1/2} - 1^{1/2}\right) = \frac{1}{2}(\sqrt{2}-1)$

ex. $\int_1^3 \frac{4x}{\sqrt{x^2+1}}\,dx$

$u = x^2+1$ $\quad du = 2x\,dx$, $2du = 4x\,dx$

$= \int_{1=x}^{3=x} 2u^{-1/2}\,du$

$u(1) = 1^2+1 = 2$ $\quad u(3) = 3^2+1 = 10$

$= 2\int_2^{10} u^{-1/2}\,du$

$= 2\left(2u^{1/2}\right)\Big|_2^{10}$

$= 2\left(2(10)^{1/2} - 2(2)^{1/2}\right)$

$= 4(\sqrt{10} - \sqrt{2})$

ex. $\int_1^{95} \frac{x}{\ln(e^x)}\,dx$

$= \int_1^{95} \frac{x}{x}\,dx = \int_1^{95} 1\,dx$

$= x\Big|_1^{95} = 95 - 1 = 94$

ex. $\int_1^e \frac{(\ln(x))^{1/n}}{x}\,dx$

let $u = \ln(x)$ $\quad u(1) = \ln(1) = 0$

$du = \frac{1}{x}dx$ $\quad u(e) = \ln(e) = 1$

$= \int_0^1 u^{1/n}\,du$

$= \left(\frac{1}{\frac{1}{n}+1}\right) u^{\frac{1}{n}+1} \Big|_0^1$

$= \left(\frac{1}{\frac{n+1}{n}}\right) 1^{\frac{n+1}{n}} - (\ldots)0^{\cdots} = \left(\frac{n}{n+1}\right)(1) - 0 = \frac{n}{n+1}$

ex. $\int_0^{\sqrt{3}} 4x^3 (x^2+1)^{1/2}\,dx$

consider $u = x^2+1$ $\quad du = 2x\,dx$, $\frac{1}{2}du = x\,dx$

$= 4 \cdot \int_0^{\sqrt{3}} (x^2)(x^2+1)^{1/2}(x\,dx)$

note: if $u = x^2+1$ then $x^2 = u-1$

$u(0) = 1$. $u(\sqrt{3}) = 4$

$= 4 \cdot \int_1^4 (u-1)(u)^{1/2}\,du$

$= 4 \cdot \int_1^4 u^{3/2} - u^{1/2}\,du$

$= 4 \cdot \left[\int_1^4 u^{3/2}du - \int_1^4 u^{1/2}\,du\right]$

$A = \frac{2}{5}u^{5/2}\Big|_1^4$ $\quad B = \frac{2}{3}u^{3/2}\Big|_1^4$

$= \frac{2}{5}(4^{5/2} - 1^{5/2})$ $\quad = \frac{2}{3}(4^{3/2} - 1^{3/2})$

$= \frac{2}{5}(2^5 - 1)$ $\quad = \frac{2}{3}(2^3 - 1)$

$= \frac{2}{5}(32-1)$ $\quad = \frac{2}{3}(8-1)$

$= 4\left(\frac{2}{5}(31) - \frac{2}{3}(7)\right)$

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