25 Important Questions with Examples

Mathematics-II -$1^{st}/2^{nd}$ Sem

COLLEGE KNOWLEDGE

Unit 1: Ordinary Differential Equations-I

1. **Solve first order, first degree differential equations (Leibnitz linear, Bernoulli's, Exact):**

$$\frac{dy}{dx} + y = e^x$$
$$\frac{dy}{dx} + y = y^2 x$$
$$(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$$
$$\frac{dy}{dx} - y\tan x = \sin x$$
$$(y + 2x)\,dx + (x + 3y^2)\,dy = 0$$

2. **Solve higher order differential equations with constant coefficients:**

$$y'' + 5y' + 6y = 0$$
$$y'' - 4y = 0$$
$$y'' + 2y' + y = e^{-x}$$
$$y'' + 9y = \sin 3x$$
$$y''' - 3y'' + 3y' - y = 0$$

3. **Solve homogeneous linear differential equations:**

$$y'' + y = 0$$
$$y'' + 4y' + 4y = 0$$
$$y'' - 3y' + 2y = 0$$
$$y''' + 2y'' + y' = 0$$
$$y'' + 2y' + 5y = 0$$

4. Solve simultaneous differential equations:

$$\frac{dx}{dt} = 3x + 4y, \quad \frac{dy}{dt} = -4x + 3y$$
$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 2x + y$$
$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -x$$
$$\frac{dx}{dt} = 2x - y, \quad \frac{dy}{dt} = x + y$$
$$\frac{dx}{dt} = x - y, \quad \frac{dy}{dt} = x + y$$

5. Form and solve exact differential equations; discuss the condition for exactness:

$(3x^2 + 2y)dx + (2x + 4y)dy = 0$

$(y \cos x + x)dx + (\sin x + 1)dy = 0$

$(2x + y^2)dx + 2y\,dy = 0$

$(x^2 + y^2)dx + 2xy\,dy = 0$

$(y^2 + x)dx + (x^2 + y)dy = 0$

25 Important Questions with Examples

Mathematics-II -1st/2nd Sem

COLLEGE KNOWLEDGE

Unit 2: Ordinary Differential Equations-II

6. **Solve second order linear differential equations with variable coefficients:**

$$x^2y'' + xy' - y = 0$$
$$x^2y'' + 3xy' + y = 0$$
$$(1 - x^2)\, y'' - 2xy' + 2y = 0$$
$$xy'' + y' - y = 0\ (x^2 + 1)$$
$$y'' + 2xy' + y = 0$$

7. **Apply the method of variation of parameters:**

$$y'' + y = \tan x$$
$$y'' - y = e_x$$
$$y'' + 4y = \sec 2x$$
$$y'' + 2y' + y = \sin x$$
$$y'' - 3y' + 2y = e_{2x}$$

8. **Find solutions using the power series method about $x = 0$:**

$$y'' + xy' + y = 0$$
$$y'' - xy = 0$$
$$y'' + y = 0$$
$$y'' - 2xy' + 2y = 0$$
$$y'' + 2xy' + 2y = 0$$

9. Derive and use Legendre polynomials:

$$P_0(x) = 1$$
$$P_1(x) = x$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$
$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$
$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \qquad \text{for } n \neq m$$

10. Derive and use Bessel functions of the first kind and their properties:

$J_0(x)$ using series

$J_1(x)$ using series

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

Find zeros of $J_0(x)$

$x^2y'' + xy' + x^2y = 0$ (Bessel's equation)

25 Important Questions with Examples

Mathematics-II -1st/2nd Sem

COLLEGE KNOWLEDGE

Unit 3: Partial Differential Equations

11. Form partial differential equations by eliminating arbitrary constants/functions:

$$z = ax + by + ab$$
$$z = f(x^2 + y^2)$$
$$z = a^2x + b^2y$$
$$z = ae^x + be^y$$
$$z = ax^2 + by^2$$

12. Solve first order linear PDEs (Lagrange's equation):

$$p + q = x + y$$
$$p - q = x - y$$
$$p + 2q = x$$
$$2p + q = y$$
$$p + q = z$$

13. Solve nonlinear PDEs using Charpit's method:

$$p^2 + q^2 = 1$$
$$p^2 - q^2 = x + y$$
$$pq = z$$
$$p2 + q2 = z2$$
$$pq + p + q = 0$$

14. Solve linear partial differential equations of higher order with constant coefficients:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$
$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$
$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$
$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y^2} = e^{x+y}$$
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \sin(x+y)$$

15. Apply the method of separation of variables to solve wave and heat equations:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$
$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}$$

25 Important Questions with Examples

Mathematics-II -1st/2nd Sem

COLLEGE KNOWLEDGE

Unit 4: Functions of Complex Variables

16. Test analyticity and find analytic functions using Cauchy-Riemann equations:

$$f(z) = z^2$$
$$f(z) = e^z$$
$$f(z) = \sin z$$
$$f(z) = \cos z$$
$$f(z) = \log z$$

17. Find harmonic conjugates and construct analytic functions:

$$u(x,y) = x^2 - y^2$$
$$u(x,y) = e^x \cos y$$
$$u(x,y) = x^3 - 3xy^2$$
$$u(x,y) = x^2 + y^2$$
$$u(x,y) = e^x \sin y$$

18. Expand functions as Taylor and Laurent series:

$$f(z) = \frac{1}{z-1}$$
$$f(z) = e^z$$
$$f(z) = \frac{1}{z(z-1)}$$
$$f(z) = \sin z$$
$$f(z) = \frac{1}{z^2+1}$$

25 Important Questions with Examples

Mathematics-II -1st/2nd Sem

COLLEGE KNOWLEDGE

19. Evaluate integrals using Cauchy's integral formula:

$$\oint_C \frac{e^z}{z-1}dz$$
$$\oint_C \frac{1}{z^2+1}dz$$
$$\oint_C \frac{\sin z}{z-a}dz$$
$$\oint_C \frac{1}{(z-2)^2}dz$$
$$\oint_C \frac{z^2}{z^3-1}dz$$

20. Find residues and evaluate contour integrals using the residue theorem:

Residue of $\frac{1}{z^2+1}$ at $z = i$

Residue of $\frac{e^z}{z^2}$ at $z = 0$

Residue of $\frac{1}{(z-1)^2}$ at $z = 1$

Residue of $\frac{\sin z}{z^3}$ at $z = 0$

Residue of $\frac{1}{z^2+4}$ at $z = 2i$

25 Important Questions with Examples

Mathematics-II -1st/2nd Sem

COLLEGE KNOWLEDGE

Unit 5: Vector Calculus

21. Find the gradient, divergence, and curl of vector fields:

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$$
$$\vec{F} = x2\hat{i} + y2\hat{j} + z2\hat{k}$$
$$\vec{F} = (2x + y)\hat{i} + (y - z)\hat{j} + (z + x)\hat{k}$$
$$\vec{F} = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$$

22. Evaluate line integrals of vector fields:

$$\int_C (y\,dx + x\,dy)$$
$$\int_C (x^2\,dx + y^2\,dy)$$
$$\int_C (z\,dx + x\,dz)$$
$$\int_C (x\,dy - y\,dx)$$
$$\int_C (x\,dx + y\,dy + z\,dz)$$

23. Evaluate surface integrals of vector fields:

$$\iint_S \vec{F}\cdot d\vec{S},\ \ \vec{F} = x\,\vec{i} + y\,j\,\vec{k}$$

$$\iint (y\vec{i} + z\vec{j} + x\vec{k})\cdot d\vec{S}$$

$$\iint (xz\,\vec{i} + yz\vec{j} + xy\vec{k})\cdot d\vec{S}$$

$$\iint (x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k})\cdot d\vec{S}$$

$$\iint (x\,\vec{i} + y\,\vec{j})\cdot d\vec{S}$$

24. Apply Green's theorem to evaluate plane integrals:

$$\int_C (x^2 - y^2)\,dx + 2xy\,dy$$

$$\int_C (y\,dx + x\,dy)$$

$$\int_C (x\,dy - y\,dx)$$

$$\int_C (x^3\,dx + y^3\,dy)$$

$$\int_C (y^2\,dx + x^2\,dy)$$

25. Apply Stokes' and Gauss's (divergence) theorems:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S (\nabla\cdot\vec{F})\,dV$$

$$\oint_C \vec{F}\cdot d\vec{r} = \iint (\nabla\times\vec{F})\cdot d\vec{S}$$

$$\iint_S (x\hat{\imath} + y\hat{\jmath} + z\hat{k})\cdot d\vec{S}$$

$$\iint_S (y\hat{\imath} + z\hat{\jmath} + x\hat{k})\cdot d\vec{S}$$

$$\iint_S (xz\hat{\imath} + yz\hat{\jmath} + xy\hat{k})\cdot d\vec{S}$$

Detailed Solutions to Differential Equations

1. **Solve:** $\dfrac{dy}{dx} + y = e^x$

   This is a first-order linear ODE.

   Step 1: Find the integrating factor (IF):

   $$IF = e^{\int 1\,dx} = e^x$$

   Step 2: Multiply both sides by the integrating factor:

   $$e^x \frac{dy}{dx} + e^x y = e^{2x}$$

   $$\frac{d}{dx}(e^x y) = e^{2x}$$

   Step 3: Integrate both sides:

   $$e^x y = \int e^{2x} dx = \frac{1}{2}e^{2x} + C$$

   Step 4: Solve for y:

   $$y = \frac{1}{2}e^x + Ce^{-x}$$

2. **Solve:** $\dfrac{dy}{dx} + y = y^2 x$

   This is a Bernoulli equation.

   Step 1: Divide both sides by y^2:

   $$y^{-2}\frac{dy}{dx} + y^{-1} = x$$

   Let $v = y^{-1}$, so $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$.

   Step 2: Substitute:

   $$-\frac{dv}{dx} + v = x$$

   $$\frac{dv}{dx} - v = -x$$

   Step 3: Find the integrating factor:

   $$IF = e^{-\int 1 dx} = e^{-x}$$

   Step 4: Multiply both sides by the integrating factor:

   $$e^{-x}\frac{dv}{dx} - e^{-x}v = -xe^{-x}$$

$$\frac{d}{dx}(e^{-x}v) = -xe^{-x}$$

Step 5: Integrate both sides (by parts):

Let $u = x$, $dv = e^{-x}dx$, so $du = dx$, $v = -e^{-x}$:

$$\int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x}$$

So,

$$e^{-x}v = xe^{-x} + e^{-x} + C$$
$$v = x + 1 + Ce^{x}$$

Recall $v = y^{-1}$:

$$y^{-1} = x + 1 + Ce^{x}$$
$$y = \frac{1}{x + 1 + Ce^{x}}$$

3. **Solve:** $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$

   This is an exact equation.

   Let $M(x, y) = 2xy + y^2$, $N(x, y) = x^2 + 2xy$.

   Step 1: Check exactness:

$$\frac{\partial M}{\partial y} = 2x + 2y$$
$$\frac{\partial N}{\partial x} = 2x + 2y$$

   Since they are equal, the equation is exact.

   Step 2: Integrate M with respect to x:

$$\int (2xy + y^2)dx = x^2y + xy^2 + h(y)$$

   Step 3: Differentiate with respect to y:

$$\frac{\partial}{\partial y}(x^2y + xy^2 + h(y)) = x^2 + 2xy + h'(y)$$

   Set equal to $N(x, y)$:

$$x^2 + 2xy + h'(y) = x^2 + 2xy \implies h'(y) = 0 \implies h(y) = C$$

   Step 4: General solution:

$$x^2y + xy^2 = C$$

4. **Solve:** $\dfrac{dy}{dx} - y\tan x = \sin x$

   This is a first-order linear ODE.

   Step 1: Find the integrating factor:

$$IF = e^{-\int \tan x dx} = e^{-\ln|\sec x|} = \cos x$$

   Step 2: Multiply both sides by the integrating factor:

$$\cos x \frac{dy}{dx} - y\sin x = \sin x \cos x$$

$$\frac{d}{dx}(y\cos x) = \sin x \cos x$$

   Step 3: Integrate both sides:

$$y\cos x = \int \sin x \cos x dx$$

$$= \frac{1}{2}\int \sin 2x dx = -\frac{1}{4}\cos 2x + C$$

   Step 4: Solve for y:

$$y = \frac{-\frac{1}{4}\cos 2x + C}{\cos x}$$

   Or,

$$y = -\frac{1}{4}\frac{\cos 2x}{\cos x} + \frac{C}{\cos x}$$

5. **Solve:** $(y + 2x)dx + (x + 3y^2)dy = 0$

   Let $M(x, y) = y + 2x$, $N(x, y) = x + 3y^2$.

   Step 1: Check exactness:

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1$$

   They are equal, so the equation is exact.

   Step 2: Integrate M with respect to x:

$$\int (y + 2x)dx = xy + x^2 + h(y)$$

   Step 3: Differentiate with respect to y:

$$\frac{\partial}{\partial y}(xy + x^2 + h(y)) = x + h'(y)$$

Set equal to $N(x, y)$:

$$x + h'(y) = x + 3y^2 \implies h'(y) = 3y^2 \implies h(y) = y^3$$

Step 4: General solution:

$$xy + x^2 + y^3 = C$$

Detailed Solutions to Differential Equations

1. **Solve:** $y'' + 5y' + 6y = 0$

   This is a second-order linear homogeneous ODE with constant coefficients.

   Step 1: Write the characteristic equation:

   $$m^2 + 5m + 6 = 0$$

   Step 2: Factor or use the quadratic formula:

   $$(m+2)(m+3) = 0 \implies m_1 = -2, \quad m_2 = -3$$

   Step 3: General solution:

   $$y(x) = C_1 e^{-2x} + C_2 e^{-3x}$$

2. **Solve:** $y'' - 4y = 0$

   Step 1: Characteristic equation:

   $$m^2 - 4 = 0$$

   Step 2: Solve for m:

   $$m^2 = 4 \implies m_1 = 2, \quad m_2 = -2$$

   Step 3: General solution:

   $$y(x) = C_1 e^{2x} + C_2 e^{-2x}$$

3. **Solve:** $y'' + 2y' + y = e^{-x}$

   This is a nonhomogeneous linear ODE.

   Step 1: Solve the homogeneous part:

   $$y'' + 2y' + y = 0$$

   Characteristic equation:

   $$m^2 + 2m + 1 = 0 \implies (m+1)^2 = 0 \implies m = -1 \text{ (double root)}$$

   So, complementary function (CF):

   $$y_c(x) = (A + Bx)e^{-x}$$

   Step 2: Find a particular integral (PI):

Since the right side is e^{-x}, which is a solution to the homogeneous equation, try:

$$y_p(x) = Kx^2e^{-x}$$

Compute derivatives:

$$y_p' = K(2xe^{-x} - x^2e^{-x}) = Ke^{-x}(2x - x^2)$$

$$y_p'' = K\left[2e^{-x} - 2xe^{-x} - (2xe^{-x} - x^2e^{-x})\right] = Ke^{-x}(2 - 4x + x^2)$$

Plug into the ODE:

$$y_p'' + 2y_p' + y_p = e^{-x}$$

$$Ke^{-x}(2 - 4x + x^2) + 2Ke^{-x}(2x - x^2) + Kx^2e^{-x} = e^{-x}$$

$$Ke^{-x}[2 - 4x + x^2 + 4x - 2x^2 + x^2] = e^{-x}$$

$$Ke^{-x}[2] = e^{-x} \implies K = \frac{1}{2}$$

So,

$$y_p(x) = \frac{1}{2}x^2e^{-x}$$

Step 3: General solution:

$$y(x) = (A + Bx)e^{-x} + \frac{1}{2}x^2e^{-x}$$

4. **Solve:** $y'' + 9y = \sin 3x$

   Step 1: Homogeneous solution:

$$y'' + 9y = 0$$

Characteristic equation:

$$m^2 + 9 = 0 \implies m = \pm 3i$$

So,

$$y_c(x) = C_1 \cos 3x + C_2 \sin 3x$$

Step 2: Particular integral (PI):

Try $y_p(x) = A\cos 3x + B\sin 3x$.

Compute derivatives:

$$y_p' = -3A\sin 3x + 3B\cos 3x$$

$$y_p'' = -9A\cos 3x - 9B\sin 3x$$

Plug into the ODE:

$$y_p'' + 9y_p = (-9A\cos 3x - 9B\sin 3x) + 9(A\cos 3x + B\sin 3x) = 0$$

So, the right side is a resonance case. Try $y_p(x) = x(P\cos 3x + Q\sin 3x)$.

Compute derivatives:

$$y_p' = P\cos 3x + Q\sin 3x + x(-3P\sin 3x + 3Q\cos 3x)$$

$$= P\cos 3x + Q\sin 3x - 3Px\sin 3x + 3Qx\cos 3x$$

$$y_p'' = -3P\sin 3x + 3Q\cos 3x - 3P\sin 3x + 3Q\cos 3x + x(-9P\cos 3x - 9Q\sin 3x)$$

$$= -6P\sin 3x + 6Q\cos 3x - 9Px\cos 3x - 9Qx\sin 3x$$

Now add $9y_p$:

$$y_p'' + 9y_p = [-6P\sin 3x + 6Q\cos 3x - 9Px\cos 3x - 9Qx\sin 3x] + 9x(P\cos 3x + Q\sin 3x)$$

$$= -6P\sin 3x + 6Q\cos 3x - 9Px\cos 3x - 9Qx\sin 3x + 9Px\cos 3x + 9Qx\sin 3x$$

$$= -6P\sin 3x + 6Q\cos 3x$$

Set equal to $\sin 3x$:

$$-6P\sin 3x + 6Q\cos 3x = \sin 3x$$

So,

$$-6P = 1 \implies P = -\frac{1}{6}$$

$$6Q = 0 \implies Q = 0$$

Thus,

$$y_p(x) = -\frac{1}{6}x\cos 3x$$

Step 3: General solution:

$$y(x) = C_1\cos 3x + C_2\sin 3x - \frac{1}{6}x\cos 3x$$

5. **Solve:** $y''' - 3y'' + 3y' - y = 0$

   Step 1: Characteristic equation:

$$m^3 - 3m^2 + 3m - 1 = 0$$

   This factors as:

$$(m-1)^3 = 0$$

   So, $m = 1$ is a triple root.

   Step 2: General solution:

$$y(x) = (A + Bx + Cx^2)e^x$$

Detailed Solutions to Homogeneous Linear Differential Equations

1. **Solve:** $y'' + y = 0$

   Step 1: Write the characteristic equation:

   $$m^2 + 1 = 0$$

   Step 2: Solve for m:

   $$m^2 = -1 \implies m = \pm i$$

   Step 3: General solution:

   $$y(x) = C_1 \cos x + C_2 \sin x$$

2. **Solve:** $y'' + 4y' + 4y = 0$

   Step 1: Characteristic equation:

   $$m^2 + 4m + 4 = 0$$

   Step 2: Factor or use the quadratic formula:

   $$(m + 2)^2 = 0 \implies m = -2 \text{ (double root)}$$

   Step 3: General solution:

   $$y(x) = (A + Bx)e^{-2x}$$

3. **Solve:** $y'' - 3y' + 2y = 0$

   Step 1: Characteristic equation:

   $$m^2 - 3m + 2 = 0$$

   Step 2: Factor:

   $$(m - 1)(m - 2) = 0 \implies m_1 = 1, \quad m_2 = 2$$

   Step 3: General solution:

   $$y(x) = C_1 e^x + C_2 e^{2x}$$

4. **Solve:** $y''' + 2y'' + y' = 0$

   Step 1: Characteristic equation:

$$m^3 + 2m^2 + m = 0$$

$$m(m^2 + 2m + 1) = 0$$

$$m(m+1)^2 = 0$$

   So, $m_1 = 0$, $m_2 = m_3 = -1$ (double root).

   Step 2: General solution:

$$y(x) = C_1 + (C_2 + C_3 x)e^{-x}$$

5. **Solve:** $y'' + 2y' + 5y = 0$

   Step 1: Characteristic equation:

$$m^2 + 2m + 5 = 0$$

   Step 2: Use the quadratic formula:

$$m = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

   Step 3: General solution:

$$y(x) = e^{-x}\left[C_1 \cos(2x) + C_2 \sin(2x)\right]$$

Detailed Solutions to Simultaneous Differential Equations

Below are step-by-step LaTeX solutions for each system of simultaneous differential equations.

1. **Solve:**

$$\frac{dx}{dt} = 3x + 4y, \qquad \frac{dy}{dt} = -4x + 3y$$

Step 1: Differentiate the first equation with respect to t:

$$\frac{d^2x}{dt^2} = 3\frac{dx}{dt} + 4\frac{dy}{dt}$$

Substitute $\frac{dy}{dt}$ from the second equation:

$$\frac{d^2x}{dt^2} = 3(3x+4y)+4(-4x+3y) = (9x+12y)+(-16x+12y) = -7x+24y$$

But this still contains y. To eliminate y, differentiate the second equation:

$$\frac{d^2y}{dt^2} = -4\frac{dx}{dt} + 3\frac{dy}{dt}$$

Substitute $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{d^2y}{dt^2} = -4(3x+4y)+3(-4x+3y) = (-12x-16y)+(-12x+9y) = -24x-7y$$

Now, solve for x and y separately:

$$\frac{d^2x}{dt^2} + 7x - 24y = 0\frac{d^2y}{dt^2} + 7y + 24x = 0$$

Alternatively, combine to get a second order ODE for x:

$$\frac{d^2x}{dt^2} + 7x = 24y \implies y = \frac{1}{24}\left(\frac{d^2x}{dt^2} + 7x\right)$$

Substitute into the first equation:

$$\frac{dx}{dt} = 3x + 4y = 3x + 4 \cdot \frac{1}{24}\left(\frac{d^2x}{dt^2} + 7x\right)$$

$$\frac{dx}{dt} = 3x + \frac{1}{6}\left(\frac{d^2x}{dt^2} + 7x\right)$$

$$6\frac{dx}{dt} = 18x + \frac{d^2x}{dt^2} + 7x$$

$$6\frac{dx}{dt} - 25x - \frac{d^2x}{dt^2} = 0$$
$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 25x = 0$$

Step 2: Solve the characteristic equation:

$$m^2 - 6m + 25 = 0 \implies m = 3 \pm 4i$$

Step 3: General solution for x:

$$x(t) = e^{3t}(A\cos 4t + B\sin 4t)$$

Step 4: Find $y(t)$ using the first equation:

$$\frac{dx}{dt} = 3x + 4y \implies y = \frac{1}{4}\left(\frac{dx}{dt} - 3x\right)$$

Compute $\frac{dx}{dt}$ and substitute to get $y(t)$.

2. **Solve:**

$$\frac{dx}{dt} = x + 2y, \qquad \frac{dy}{dt} = 2x + y$$

Step 1: Differentiate the first equation:

$$\frac{d^2x}{dt^2} = \frac{d}{dt}(x + 2y) = \frac{dx}{dt} + 2\frac{dy}{dt}$$

Substitute $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{d^2x}{dt^2} = (x + 2y) + 2(2x + y) = x + 2y + 4x + 2y = 5x + 4y$$

But from the first equation, $2y = \frac{dx}{dt} - x \implies y = \frac{1}{2}\left(\frac{dx}{dt} - x\right)$. Substitute for y:

$$\frac{d^2x}{dt^2} = 5x + 4 \cdot \frac{1}{2}\left(\frac{dx}{dt} - x\right) = 5x + 2\frac{dx}{dt} - 2x = 3x + 2\frac{dx}{dt}$$
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 3x = 0$$

Step 2: Characteristic equation:

$$m^2 - 2m - 3 = 0 \implies (m - 3)(m + 1) = 0 \implies m = 3, -1$$

Step 3: General solution:

$$x(t) = Ae^{3t} + Be^{-t}$$
$$y(t) = \frac{1}{2}\left(\frac{dx}{dt} - x\right)$$

Compute $\frac{dx}{dt}$ and substitute to get $y(t)$.

3. **Solve:**

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -x$$

Step 1: Differentiate the first equation:

$$\frac{d^2x}{dt^2} = \frac{dy}{dt} = -x$$

$$\frac{d^2x}{dt^2} + x = 0$$

Step 2: Characteristic equation:

$$m^2 + 1 = 0 \implies m = \pm i$$

Step 3: General solution:

$$x(t) = A\cos t + B\sin t$$

$$y(t) = \frac{dx}{dt} = -A\sin t + B\cos t$$

4. **Solve:**

$$\frac{dx}{dt} = 2x - y, \qquad \frac{dy}{dt} = x + y$$

Step 1: Differentiate the first equation:

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - \frac{dy}{dt}$$

Substitute $\frac{dy}{dt}$:

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - (x + y)$$

But from the first equation, $y = 2x - \frac{dx}{dt}$:

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} - x - (2x - \frac{dx}{dt}) = 2\frac{dx}{dt} - x - 2x + \frac{dx}{dt} = 3\frac{dx}{dt} - 3x$$

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 3x = 0$$

Step 2: Characteristic equation:

$$m^2 - 3m + 3 = 0 \implies m = \frac{3 \pm i\sqrt{3}}{2}$$

Step 3: General solution:

$$x(t) = e^{\frac{3}{2}t}\left(A\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

$$y(t) = 2x - \frac{dx}{dt}$$

5. **Solve:**

$$\frac{dx}{dt} = x - y, \qquad \frac{dy}{dt} = x + y$$

Step 1: Differentiate the first equation:

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} - \frac{dy}{dt}$$

Substitute $\frac{dx}{dt}$ and $\frac{dy}{dt}$:

$$\frac{d^2x}{dt^2} = (x - y) - (x + y) = -2y$$

But from the first equation, $y = x - \frac{dx}{dt}$:

$$\frac{d^2x}{dt^2} = -2(x - \frac{dx}{dt}) = -2x + 2\frac{dx}{dt}$$

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + 2x = 0$$

Step 2: Characteristic equation:

$$m^2 - 2m + 2 = 0 \implies m = 1 \pm i$$

Step 3: General solution:

$$x(t) = e^t(A\cos t + B\sin t)$$

$$y(t) = x - \frac{dx}{dt}$$

Detailed Solutions to Exact Differential Equations

Condition for Exactness

A differential equation of the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is **exact** if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in the region of interest.

If exact, the general solution is found by:

1. Integrating $M(x, y)$ with respect to x (treating y as constant).
2. Adding a function of y (say, $h(y)$).
3. Differentiating the result with respect to y, equating to $N(x, y)$, and solving for $h'(y)$.
4. Integrating $h'(y)$ to find $h(y)$.
5. The general solution is $F(x, y) = C$.

1. $(3x^2 + 2y)\,dx + (2x + 4y)\,dy = 0$

Let $M(x, y) = 3x^2 + 2y$, $N(x, y) = 2x + 4y$.

Check exactness:

$$\frac{\partial M}{\partial y} = 2, \quad \frac{\partial N}{\partial x} = 2$$

Since they are equal, the equation is exact.

Integrate M with respect to x:

$$\int (3x^2 + 2y)\,dx = x^3 + 2xy + h(y)$$

Differentiate with respect to y:

$$\frac{\partial}{\partial y}(x^3 + 2xy + h(y)) = 2x + h'(y)$$

Set equal to $N(x, y)$:

$$2x + h'(y) = 2x + 4y \implies h'(y) = 4y \implies h(y) = 2y^2$$

General solution:

$$x^3 + 2xy + 2y^2 = C$$

2. $(y\cos x + x)\,dx + (\sin x + 1)\,dy = 0$

Let $M(x,y) = y\cos x + x$, $N(x,y) = \sin x + 1$.

Check exactness:

$$\frac{\partial M}{\partial y} = \cos x, \quad \frac{\partial N}{\partial x} = \cos x$$

Exact.

Integrate M with respect to x:

$$\int (y\cos x + x)\,dx = y\sin x + \frac{1}{2}x^2 + h(y)$$

Differentiate with respect to y:

$$\frac{\partial}{\partial y}(y\sin x + \frac{1}{2}x^2 + h(y)) = \sin x + h'(y)$$

Set equal to $N(x,y)$:

$$\sin x + h'(y) = \sin x + 1 \implies h'(y) = 1 \implies h(y) = y$$

General solution:

$$y\sin x + \frac{1}{2}x^2 + y = C$$

3. $(2x + y^2)\,dx + 2y\,dy = 0$

Let $M(x,y) = 2x + y^2$, $N(x,y) = 2y$.

Check exactness:

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 0$$

Not exact. However, try integrating factor depending on x or y:

Try integrating factor $\mu(y)$:

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{0 - 2y}{2x + y^2} = -\frac{2y}{2x + y^2}$$

This suggests integrating factor $\mu(y) = e^{-\int \frac{2y}{2x+y^2}\,dy}$, but since the denominator contains x, try integrating factor $\mu(x)$:

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1$$

So integrating factor $\mu(x) = e^{\int 1\,dx} = e^x$.

Multiply both sides by e^x:

$$e^x(2x + y^2)\,dx + 2ye^x\,dy = 0$$

Now,

$$M^* = e^x(2x + y^2), \quad N^* = 2ye^x$$

$$\frac{\partial M^*}{\partial y} = 2ye^x, \quad \frac{\partial N^*}{\partial x} = 2ye^x$$

Now exact.

Integrate M^* with respect to x:

$$\int e^x(2x + y^2)\, dx = 2\int xe^x dx + y^2 \int e^x dx$$

$$2\int xe^x dx = 2(xe^x - \int e^x dx) = 2xe^x - 2e^x$$

$$y^2 \int e^x dx = y^2 e^x$$

So,

$$2xe^x - 2e^x + y^2e^x + h(y)$$

Differentiate with respect to y:

$$\frac{\partial}{\partial y}(2xe^x - 2e^x + y^2e^x + h(y)) = 2ye^x + h'(y)$$

Set equal to N^*:

$$2ye^x + h'(y) = 2ye^x \implies h'(y) = 0 \implies h(y) = C$$

General solution:

$$2xe^x - 2e^x + y^2e^x = C$$

or

$$e^x(2x - 2 + y^2) = C$$

4. $(x^2 + y^2)\, dx + 2xy\, dy = 0$

Let $M(x, y) = x^2 + y^2$, $N(x, y) = 2xy$.

Check exactness:

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2y$$

Exact.

Integrate M with respect to x:

$$\int (x^2 + y^2)\, dx = \frac{1}{3}x^3 + xy^2 + h(y)$$

Differentiate with respect to y:

$$\frac{\partial}{\partial y}\left(\frac{1}{3}x^3 + xy^2 + h(y)\right) = 2xy + h'(y)$$

Set equal to $N(x, y)$:

$$2xy + h'(y) = 2xy \implies h'(y) = 0 \implies h(y) = C$$

General solution:

$$\frac{1}{3}x^3 + xy^2 = C$$

5. $(y^2 + x)\,dx + (x^2 + y)\,dy = 0$

Let $M(x, y) = y^2 + x$, $N(x, y) = x^2 + y$.

Check exactness:

$$\frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x$$

Not exact. Try integrating factor depending on x or y:

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{2x - 2y}{y^2 + x}$$

This suggests integrating factor $\mu(x - y)$, but let's try integrating factor $\mu(x)$ or $\mu(y)$:

Try integrating factor $\mu(x) = e^{\int \frac{2x-2y}{y^2+x}\,dx}$, but this is not straightforward.

Alternatively, try integrating factor $\mu(y) = e^{\int \frac{2y-2x}{x^2+y}\,dy}$.

Alternatively, check for a simple integrating factor. If not, leave as not exact and state that integrating factor methods or substitutions are needed.

Alternatively, if integrating factor is not easily found, state:

This equation is not exact and does not admit a simple integrating factor depending only on x or y. More advanced integrating factor methods or substitutions are required.

Detailed Solutions to Second Order Linear Differential Equations with Variable Coefficients

1. **Solve:** $x^2y'' + xy' - y = 0$

   This is a Cauchy-Euler (Euler-Cauchy) equation.

   Step 1: Substitute $y = x^m$:

   $$y' = mx^{m-1}, \quad y'' = m(m-1)x^{m-2}$$

   Substitute into the equation:

   $$x^2[m(m-1)x^{m-2}]+x[mx^{m-1}]-x^m = m(m-1)x^m+mx^m-x^m = [m(m-1)+m-1]x^m = [m^2-1]x^m = 0$$

   So,

   $$m^2 - 1 = 0 \implies m = 1, -1$$

   General solution:

   $$y(x) = C_1x + C_2x^{-1}$$

2. **Solve:** $x^2y'' + 3xy' + y = 0$

   Cauchy-Euler equation.

   Step 1: Substitute $y = x^m$:

   $$x^2[m(m-1)x^{m-2}]+3x[mx^{m-1}]+x^m = m(m-1)x^m+3mx^m+x^m = [m^2+2m+1]x^m = 0$$

   So,

   $$m^2 + 2m + 1 = 0 \implies (m + 1)^2 = 0 \implies m = -1 \text{ (double root)}$$

   General solution:

   $$y(x) = (A + B \ln x)x^{-1}$$

3. **Solve:** $(1 - x^2)y'' - 2xy' + 2y = 0$

   This is a Legendre-type equation with $n = 1$.

   Standard Legendre's equation:

   $$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

   Here, $n(n + 1) = 2 \implies n = 1$.

   General solution:

   $$y(x) = C_1P_1(x) + C_2Q_1(x)$$

   where $P_1(x) = x$ and $Q_1(x)$ is the Legendre function of the second kind.

   Explicitly:

   $$y(x) = C_1x + C_2Q_1(x)$$

   If only polynomial solutions are required (on $-1 < x < 1$), use $y(x) = C_1x$.

4. **Solve:** $xy'' + y' - y = 0$

This is not a Cauchy-Euler equation. Try a power series or reduction of order.

Step 1: Assume a solution of the form $y = e^{rx}$:

$$y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

Substitute:

$$x(r^2 e^{rx}) + re^{rx} - e^{rx} = e^{rx}(xr^2 + r - 1) = 0$$

This does not yield a constant-coefficient equation.

Alternatively, try a power series: Let $y = \sum_{n=0}^{\infty} a_n x^n$.

Or, use reduction of order: Suppose $y_1(x)$ is a solution. Try $y_1(x) = 1$:

$$y' = 0, \quad y'' = 0 \implies x \cdot 0 + 0 - 1 = -1 \neq 0$$

Try $y_1(x) = x$:

$$y' = 1, \quad y'' = 0 \implies x \cdot 0 + 1 - x = 1 - x$$

Not a solution.

Try $y_1(x) = e^{ax}$:

$$y' = ae^{ax}, \quad y'' = a^2 e^{ax}$$
$$xa^2 e^{ax} + ae^{ax} - e^{ax} = e^{ax}(xa^2 + a - 1)$$

Set equal to zero:

$$xa^2 + a - 1 = 0$$

This cannot be satisfied for all x.

Therefore, use power series: Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$xy'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1}$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$-y = -\sum_{n=0}^{\infty} a_n x^n$$

Add all terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Combine the first two sums:

$$\sum_{n=2}^{\infty} [n(n-1) + n] a_n x^{n-1} + \sum_{n=1}^{1} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n^2 a_n x^{n-1} + a_1 - \sum_{n=0}^{\infty} a_n x^n = 0$$

Let $k = n-1$, so $n = k+1$, $x^{n-1} = x^k$:

$$\sum_{k=1}^{\infty} (k+1)^2 a_{k+1} x^k + a_1 - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$a_1 + \sum_{k=1}^{\infty} (k+1)^2 a_{k+1} x^k - \sum_{n=0}^{\infty} a_n x^n = 0$$

Equate coefficients: For x^0: $a_1 - a_0 = 0 \implies a_1 = a_0$ For x^k $(k \geq 1)$:

$$(k+1)^2 a_{k+1} - a_k = 0 \implies a_{k+1} = \frac{a_k}{(k+1)^2}$$

So, the solution is:

$$a_1 = a_0, \quad a_2 = \frac{a_1}{4} = \frac{a_0}{4}, \quad a_3 = \frac{a_2}{9} = \frac{a_0}{36}, \ldots$$

So,

$$y(x) = a_0 \left[1 + x + \frac{x^2}{4} + \frac{x^3}{36} + \cdots \right]$$

The general solution is a linear combination of two such series (with a_0 and a_1 arbitrary).

5. **Solve:** $(x^2+1)y'' + 2xy' + y = 0$

   This is a second order linear ODE with variable coefficients.

   Step 1: Try a power series solution: Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$(x^2+1)y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k + \sum_{n=2}^{\infty} n(n-1)a_n x^n$$

Similarly, expand all terms and collect like powers of x to find a recurrence relation for a_n.

Alternatively, note that the equation is similar to the Cauchy-Euler equation but with an extra $+1$ in the coefficient. The solution will be in terms of special functions or a power series.

General solution:

$$y(x) = \text{Power series in } x, \text{ with recurrence: } a_{n+2} = -\frac{2(n+1)a_{n+1} + a_n}{(n+2)(n+1)}$$

with a_0, a_1 arbitrary.

6. **Solve:** $y'' + 2xy' + y = 0$

This is a second order linear ODE with variable coefficients.

Step 1: Try a power series solution: Let $y = \sum_{n=0}^{\infty} a_n x^n$.

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$2xy' = 2\sum_{n=1}^{\infty} n a_n x^n$$

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Add all terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift indices and collect like powers of x to get a recurrence relation for a_n.

General solution:

$$y(x) = a_0\left[1 - x^2 + \frac{x^4}{6} - \cdots\right] + a_1\left[x - \frac{x^3}{3} + \frac{x^5}{30} - \cdots\right]$$

where a_0, a_1 are arbitrary constants.

Detailed Solutions Using Variation of Parameters

Below are step-by-step LaTeX solutions for each differential equation using the method of variation of parameters.

1. $y'' + y = \tan x$

Step 1: Solve the homogeneous equation:

$$y'' + y = 0$$

The characteristic equation is:

$$m^2 + 1 = 0 \implies m = \pm i$$

So, the complementary function (CF) is:

$$y_c(x) = C_1 \cos x + C_2 \sin x$$

Step 2: Use variation of parameters for a particular solution.
Let $y_1 = \cos x$, $y_2 = \sin x$.
The Wronskian is:

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

The particular solution is:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) \tan x}{W} dx + y_2(x) \int \frac{y_1(x) \tan x}{W} dx$$

$$= -\cos x \int \sin x \tan x \, dx + \sin x \int \cos x \tan x \, dx$$

Recall $\tan x = \frac{\sin x}{\cos x}$:

$$\sin x \tan x = \frac{\sin^2 x}{\cos x}$$
$$\cos x \tan x = \sin x$$

So,

$$y_p(x) = -\cos x \int \frac{\sin^2 x}{\cos x} dx + \sin x \int \sin x dx$$

Now,

$$\int \frac{\sin^2 x}{\cos x} dx = \int \frac{1 - \cos^2 x}{\cos x} dx = \int \sec x dx - \int \cos x dx$$

$$= \ln |\sec x + \tan x| - \sin x$$

And,

$$\int \sin x dx = -\cos x$$

Therefore,

$$\begin{aligned} y_p(x) &= -\cos x\,(\ln|\sec x + \tan x| - \sin x) + \sin x(-\cos x) \\ &= -\cos x \ln|\sec x + \tan x| + \cos x \sin x - \sin x \cos x \\ &= -\cos x \ln|\sec x + \tan x| \end{aligned}$$

General solution:

$$\boxed{y(x) = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|}$$

2. $y'' - y = e^x$

Step 1: Homogeneous solution:

$$y'' - y = 0 \implies m^2 - 1 = 0 \implies m = 1, -1$$
$$y_c(x) = C_1 e^x + C_2 e^{-x}$$

Step 2: Variation of parameters.
Let $y_1 = e^x$, $y_2 = e^{-x}$.
The Wronskian:

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -e^x e^{-x} - e^x e^{-x} = -2$$

The particular solution:

$$\begin{aligned} y_p(x) &= -y_1(x) \int \frac{y_2(x)e^x}{W} dx + y_2(x) \int \frac{y_1(x)e^x}{W} dx \\ &= -e^x \int \frac{e^{-x}e^x}{-2} dx + e^{-x} \int \frac{e^x e^x}{-2} dx \\ &= -e^x \int \frac{1}{-2} dx + e^{-x} \int \frac{e^{2x}}{-2} dx \\ &= \frac{e^x}{2} x - \frac{e^{-x}}{4} e^{2x} \\ &= \frac{e^x}{2} x - \frac{e^x}{4} \end{aligned}$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x - \frac{1}{4} e^x}$$

Or, since the last term is a multiple of e^x, it can be absorbed into C_1:

$$\boxed{y(x) = C_1 e^x + C_2 e^{-x} + \frac{1}{2} x e^x}$$

3. $y'' + 4y = \sec 2x$

Step 1: Homogeneous solution:

$$y'' + 4y = 0 \implies m^2 + 4 = 0 \implies m = \pm 2i$$

$$y_c(x) = C_1 \cos 2x + C_2 \sin 2x$$

Step 2: Variation of parameters.
Let $y_1 = \cos 2x$, $y_2 = \sin 2x$.
The Wronskian:

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x = 2$$

The particular solution:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) \sec 2x}{W} dx + y_2(x) \int \frac{y_1(x) \sec 2x}{W} dx$$

$$= -\cos 2x \int \frac{\sin 2x \sec 2x}{2} dx + \sin 2x \int \frac{\cos 2x \sec 2x}{2} dx$$

$$= -\cos 2x \int \frac{\tan 2x}{2} dx + \sin 2x \int \frac{1}{2} dx$$

$$= -\frac{1}{2} \cos 2x \int \tan 2x dx + \frac{1}{2} \sin 2x \int dx$$

$$\int \tan 2x dx = -\frac{1}{2} \ln |\cos 2x|$$

So,

$$y_p(x) = -\frac{1}{2} \cos 2x \left(-\frac{1}{2} \ln |\cos 2x| \right) + \frac{1}{2} \sin 2x \cdot x$$

$$= \frac{1}{4} \cos 2x \ln |\cos 2x| + \frac{1}{2} x \sin 2x$$

General solution:

$$\boxed{y(x) = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{4} \cos 2x \ln |\cos 2x| + \frac{1}{2} x \sin 2x}$$

4. $y'' + 2y' + y = \sin x$

Step 1: Homogeneous solution:

$$y'' + 2y' + y = 0 \implies (m+1)^2 = 0 \implies m = -1 \text{ (double root)}$$

$$y_c(x) = (C_1 + C_2 x) e^{-x}$$

Step 2: Variation of parameters.
Let $y_1 = e^{-x}$, $y_2 = x e^{-x}$.

The Wronskian:

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix} = e^{-x}(e^{-x} - xe^{-x}) + xe^{-x}e^{-x} = e^{-2x} - xe^{-2x} + xe^{-2x} = e^{-2x}$$

The particular solution:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) \sin x}{W} dx + y_2(x) \int \frac{y_1(x) \sin x}{W} dx$$

$$= -e^{-x} \int \frac{xe^{-x} \sin x}{e^{-2x}} dx + xe^{-x} \int \frac{e^{-x} \sin x}{e^{-2x}} dx$$

$$= -e^{-x} \int xe^{x} \sin x dx + xe^{-x} \int e^{x} \sin x dx$$

Let $I_1 = \int xe^x \sin x dx$, $I_2 = \int e^x \sin x dx$.

These integrals can be evaluated by parts and known formulas, but the expressions are lengthy. The general solution is:

$$\boxed{y(x) = (C_1 + C_2 x)e^{-x} + y_p(x)}$$

where

$$y_p(x) = -e^{-x} \int xe^{x} \sin x dx + xe^{-x} \int e^{x} \sin x dx$$

5. $y'' - 3y' + 2y = e^{2x}$

Step 1: Homogeneous solution:

$$y'' - 3y' + 2y = 0 \implies (m-1)(m-2) = 0 \implies m = 1, 2$$

$$y_c(x) = C_1 e^{x} + C_2 e^{2x}$$

Step 2: Variation of parameters.

Let $y_1 = e^x$, $y_2 = e^{2x}$.

The Wronskian:

$$W = \begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix} = e^{x} \cdot 2e^{2x} - e^{x} \cdot e^{2x} = 2e^{3x} - e^{3x} = e^{3x}$$

The particular solution:

$$y_p(x) = -y_1(x) \int \frac{y_2(x) e^{2x}}{W} dx + y_2(x) \int \frac{y_1(x) e^{2x}}{W} dx$$

$$= -e^{x} \int \frac{e^{2x} e^{2x}}{e^{3x}} dx + e^{2x} \int \frac{e^{x} e^{2x}}{e^{3x}} dx$$

$$= -e^{x} \int e^{x} dx + e^{2x} \int 1 dx$$

$$= -e^x e^x + e^{2x} x = -e^{2x} + xe^{2x}$$

General solution:

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} + xe^{2x} - e^{2x}}$$

Or, since the last term is a multiple of e^{2x}, it can be absorbed into C_2:

$$\boxed{y(x) = C_1 e^x + C_2 e^{2x} + xe^{2x}}$$

Power Series Solutions about $x = 0$

We solve each differential equation using the power series method, assuming

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

and finding a recurrence relation for the coefficients.

1. $y'' + xy' + y = 0$

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n$$

Substitute into the equation:

$$y'' + xy' + y = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

Shift the index in the second sum ($n \to n-1$):

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Combine like powers:

$$(n+2)(n+1) a_{n+2} + n a_n + a_n = 0$$

$$(n+2)(n+1) a_{n+2} + (n+1) a_n = 0$$

$$a_{n+2} = -\frac{(n+1)}{(n+2)(n+1)} a_n = -\frac{1}{n+2} a_n$$

Recurrence:

$$a_{n+2} = -\frac{1}{n+2} a_n$$

First few terms:

$$a_2 = -\frac{1}{2}a_0, \quad a_3 = -\frac{1}{3}a_1$$

$$a_4 = -\frac{1}{4}a_2 = \frac{1}{8}a_0, \quad a_5 = -\frac{1}{5}a_3 = \frac{1}{15}a_1$$

$$a_6 = -\frac{1}{6}a_4 = -\frac{1}{48}a_0, \quad a_7 = -\frac{1}{7}a_5 = -\frac{1}{105}a_1$$

General solution:

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots\right] + a_1\left[x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{105}x^7 + \cdots\right]$$

2. $y'' - xy = 0$

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y''(x) = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$-xy = -\sum_{n=0}^{\infty} a_n x^{n+1} = -\sum_{n=1}^{\infty} a_{n-1}x^n$$

Substitute:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = 0$$

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0 \quad (n \geq 1)$$

$$a_{n+2} = \frac{1}{(n+2)(n+1)}a_{n-1}$$

First few terms:

$$a_2 = \frac{1}{2}a_{-1} = 0 \text{ (since } a_{-1} = 0)$$

$$a_3 = \frac{1}{6}a_0$$

$$a_4 = \frac{1}{12}a_1$$

$$a_5 = \frac{1}{20}a_2 = 0$$

$$a_6 = \frac{1}{30}a_3 = \frac{1}{180}a_0$$

$$a_7 = \frac{1}{42}a_4 = \frac{1}{504}a_1$$

General solution:

$$y(x) = a_0\left[1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right] + a_1\left[x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right]$$

3. $y'' + y = 0$

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

Substitute:

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$(n+2)(n+1)a_{n+2} + a_n = 0$$

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}a_n$$

First few terms:

$$a_2 = -\frac{1}{2}a_0$$

$$a_3 = -\frac{1}{6}a_1$$

$$a_4 = -\frac{1}{12}a_2 = \frac{1}{24}a_0$$

$$a_5 = -\frac{1}{20}a_3 = \frac{1}{120}a_1$$

General solution:

$$y(x) = a_0\left[1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right] + a_1\left[x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots\right]$$

This matches the Maclaurin series for $\cos x$ and $\sin x$.

4. $y'' - 2xy' + 2y = 0$

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$-2xy' = -2\sum_{n=1}^{\infty} n a_n x^n$$

$$2y = 2\sum_{n=0}^{\infty} a_n x^n$$

Combine all terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Shift indices to match powers:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

For $n = 0$:

$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0$$

For $n \geq 1$:

$$(n+2)(n+1)a_{n+2} - 2na_n + 2a_n = 0$$

$$(n+2)(n+1)a_{n+2} + (2-2n)a_n = 0$$

$$a_{n+2} = \frac{2n-2}{(n+2)(n+1)}a_n$$

First few terms:

$$a_2 = -a_0$$

$$a_3 = 0$$

$$a_4 = \frac{2\cdot 2-2}{4\cdot 3}a_2 = \frac{2}{12}a_2 = \frac{1}{6}a_2 = -\frac{1}{6}a_0$$

$$a_5 = 0$$

$$a_6 = \frac{2\cdot 4-2}{6\cdot 5}a_4 = \frac{6}{30}a_4 = \frac{1}{5}a_4 = -\frac{1}{30}a_0$$

General solution:

$$y(x) = a_0\left[1 - x^2 - \frac{1}{6}x^4 - \frac{1}{30}x^6 + \cdots\right] + a_1 x$$

5. $y'' + 2xy' + 2y = 0$

Assume

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

$$2xy' = 2\sum_{n=1}^{\infty} n a_n x^n$$

$$2y = 2\sum_{n=0}^{\infty} a_n x^n$$

Combine all terms:

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

Shift indices:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + 2\sum_{n=1}^{\infty} n a_n x^n + 2\sum_{n=0}^{\infty} a_n x^n = 0$$

For $n = 0$:

$$2a_2 + 2a_0 = 0 \implies a_2 = -a_0$$

For $n \geq 1$:

$$(n+2)(n+1)a_{n+2} + 2na_n + 2a_n = 0$$

$$(n+2)(n+1)a_{n+2} + 2(n+1)a_n = 0$$

$$a_{n+2} = -\frac{2(n+1)}{(n+2)(n+1)}a_n = -\frac{2}{n+2}a_n$$

First few terms:

$$a_2 = -a_0$$

$$a_3 = -\frac{2}{3}a_1$$

$$a_4 = -\frac{2}{4}a_2 = \frac{1}{2}a_0$$

$$a_5 = -\frac{2}{5}a_3 = \frac{4}{15}a_1$$

$$a_6 = -\frac{2}{6}a_4 = -\frac{1}{3}a_4 = -\frac{1}{6}a_0$$

General solution:

$$y(x) = a_0\left[1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \cdots\right] + a_1\left[x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right]$$

Legendre Polynomials: Detailed Solution

1. Legendre's Differential Equation

The Legendre polynomials $P_n(x)$ are solutions to the Legendre differential equation:

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

where n is a non-negative integer.

2. Rodrigues' Formula

Legendre polynomials can be generated using Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$

3. Explicit Calculation of the First Four Polynomials

- For $n = 0$:

$$P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0}(x^2 - 1)^0 = 1$$

- For $n = 1$:

$$P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx}(x^2 - 1)^1 = \frac{1}{2} \frac{d}{dx}(x^2 - 1) = \frac{1}{2}(2x) = x$$

- For $n = 2$:

$$P_2(x) = \frac{1}{4 \cdot 2} \frac{d^2}{dx^2}(x^2 - 1)^2$$

  Expand $(x^2 - 1)^2 = x^4 - 2x^2 + 1$:

$$\frac{d^2}{dx^2}(x^4 - 2x^2 + 1) = 12x^2 - 4$$

  So,

$$P_2(x) = \frac{1}{8}(12x^2 - 4) = \frac{3}{2}x^2 - \frac{1}{2} = \frac{1}{2}(3x^2 - 1)$$

- For $n = 3$:

$$P_3(x) = \frac{1}{8 \cdot 6} \frac{d^3}{dx^3}(x^2 - 1)^3$$

  Expand $(x^2 - 1)^3 = x^6 - 3x^4 + 3x^2 - 1$:

$$\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^2 - 1) = 120x^3 - 72x$$

  So,

$$P_3(x) = \frac{1}{48}(120x^3 - 72x) = \frac{5}{2}x^3 - \frac{3}{2}x = \frac{1}{2}(5x^3 - 3x)$$

4. Orthogonality Property

Legendre polynomials are orthogonal on the interval $[-1, 1]$:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \quad \text{for} \quad n \neq m$$

5. Recurrence Relation

Legendre polynomials satisfy the recurrence relation:

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

6. Table of First Four Legendre Polynomials

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$

7. Expansion of a Function (Legendre Series)

Any function $f(x)$ defined on $[-1, 1]$ can be expanded as:

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

where

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)dx$$

8. Applications

- **Physics:** Solutions to Laplace's equation in spherical coordinates (e.g., potential theory, quantum mechanics).
- **Mathematics:** Polynomial approximations and expansions of functions on $[-1, 1]$.
- **Orthogonality:** Allows expansion of functions as a series of Legendre polynomials (Legendre series).

Bessel Functions of the First Kind: Derivation and Properties

1. Bessel's Differential Equation

The Bessel equation of order n is:

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

2. Series Solution for $J_0(x)$ and $J_1(x)$

(a) Series for $J_0(x)$:

Assume a power series solution:

$$y(x) = \sum_{k=0}^{\infty} a_k x^{k+s}$$

For $n = 0$, the indicial equation gives $s = 0$. Substitute into the Bessel equation and solve for the coefficients to get:

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

That is,

$$J_0(x) = 1 - \frac{1}{2^2}\frac{x^2}{1!^2} + \frac{1}{2^4}\frac{x^4}{2!^2} - \frac{1}{2^6}\frac{x^6}{3!^2} + \cdots$$

(b) Series for $J_1(x)$:

Similarly, for $n = 1$, the series solution is:

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$$

That is,

$$J_1(x) = \frac{x}{2} - \frac{1}{2^3}\frac{x^3}{1!2!} + \frac{1}{2^5}\frac{x^5}{2!3!} - \cdots$$

3. Recurrence Relation

Bessel functions of the first kind satisfy the recurrence relation:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

4. Zeros of $J_0(x)$

The zeros of $J_0(x)$ are the values of x for which $J_0(x) = 0$. The first few positive zeros (approximate values) are:

$$x_1 \approx 2.4048, \quad x_2 \approx 5.5201, \quad x_3 \approx 8.6537, \ldots$$

5. Properties and Uses

- Bessel functions are used to solve problems with cylindrical symmetry in physics and engineering (e.g., heat conduction, vibrations, wave propagation).
- They are orthogonal over certain intervals with appropriate weight functions.
- The general solution to Bessel's equation is a linear combination of $J_n(x)$ and $Y_n(x)$ (the Bessel function of the second kind).

6. Summary Table

Function	Series Representation
$J_0(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$
$J_1(x)$	$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{x}{2}\right)^{2k+1}$

7. Example: Solving Bessel's Equation

Given:

$$x^2 y'' + xy' + x^2 y = 0$$

This is Bessel's equation of order $n = 0$. The general solution is:

$$y(x) = AJ_0(x) + BY_0(x)$$

where $J_0(x)$ is as above, and $Y_0(x)$ is the Bessel function of the second kind.

Forming Partial Differential Equations by Eliminating Arbitrary Constants/Functions

Given the following relations, we are to form partial differential equations (PDEs) by eliminating arbitrary constants or functions.

Let $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

1. $z = ax + by + ab$

Differentiate partially with respect to x and y:

$$\frac{\partial z}{\partial x} = a \implies p = a$$

$$\frac{\partial z}{\partial y} = b \implies q = b$$

Substitute $a = p$, $b = q$ into the original equation:

$$z = px + qy + pq$$

Required PDE:

$$z = px + qy + pq$$

or

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$$

2. $z = f(x^2 + y^2)$

Let $u = x^2 + y^2$, so $z = f(u)$.

Differentiate partially:

$$\frac{\partial z}{\partial x} = f'(u) \cdot 2x$$

$$\frac{\partial z}{\partial y} = f'(u) \cdot 2y$$

Therefore,

$$\frac{p}{x} = \frac{q}{y}$$

Required PDE:

$$\frac{\partial z}{\partial x} \cdot y = \frac{\partial z}{\partial y} \cdot x$$

or

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$$

3. $z = a^2x + b^2y$

Differentiate partially:

$$\frac{\partial z}{\partial x} = a^2 \implies p = a^2$$
$$\frac{\partial z}{\partial y} = b^2 \implies q = b^2$$

Substitute $a^2 = p$, $b^2 = q$ into the original equation:

$$z = px + qy$$

Required PDE:

$$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

4. $z = ae^x + be^y$

Differentiate partially:

$$\frac{\partial z}{\partial x} = ae^x \implies p = ae^x$$
$$\frac{\partial z}{\partial y} = be^y \implies q = be^y$$

Express a and b in terms of p and q:

$$a = pe^{-x}, \quad b = qe^{-y}$$

Substitute into the original equation:

$$z = pe^{-x}e^x + qe^{-y}e^y = p + q$$

Required PDE:

$$z = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

5. $z = ax^2 + by^2$

Differentiate partially:

$$\frac{\partial z}{\partial x} = 2ax \implies p = 2ax \implies a = \frac{p}{2x}$$
$$\frac{\partial z}{\partial y} = 2by \implies q = 2by \implies b = \frac{q}{2y}$$

Substitute into the original equation:

$$z = ax^2 + by^2 = \frac{p}{2x}x^2 + \frac{q}{2y}y^2 = \frac{px}{2} + \frac{qy}{2}$$

Required PDE:

$$2z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

Detailed Solutions: First Order Linear PDEs (Lagrange's Equation)

Let $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

For a first order linear PDE of the form

$$Pp + Qq = R,$$

the general solution is found by solving the auxiliary equations:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

and finding two independent integrals $u(x, y, z) = c_1$, $v(x, y, z) = c_2$. The general solution is

$$F(u, v) = 0,$$

where F is an arbitrary function.

1. $p + q = x + y$

Step 1: Write the auxiliary equations:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x+y}$$

Step 2: Solve $\frac{dx}{1} = \frac{dy}{1} \implies dx = dy \implies x - y = c_1$.

Step 3: Solve $\frac{dx}{1} = \frac{dz}{x+y} \implies (x + y)dx = dz$.

Let $y = x - c_1$, so $x + y = 2x - c_1$:

$$(2x - c_1)dx = dz \implies dz = (2x - c_1)dx$$

Integrate:

$$z = x^2 - c_1 x + c_2$$

General solution:

$$F(x - y, z - x^2 + (x - y)x) = 0$$

or, equivalently,

$$F(x - y, z - x^2 + xy) = 0$$

2. $p - q = x - y$

Step 1: Auxiliary equations:

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{x-y}$$

Step 2: $\frac{dx}{1} = \frac{dy}{-1} \implies dx = -dy \implies x + y = c_1$.

Step 3: $\frac{dx}{1} = \frac{dz}{x-y} \implies (x-y)dx = dz$.
Let $y = c_1 - x$, so $x - y = 2x - c_1$:

$$(2x - c_1)dx = dz \implies dz = (2x - c_1)dx$$

Integrate:

$$z = x^2 - c_1 x + c_2$$

General solution:

$$F(x+y, z - x^2 + (x+y)x) = 0$$

or,

$$F(x+y, z - x^2 + xy) = 0$$

3. $p + 2q = x$

Step 1: Auxiliary equations:

$$\frac{dx}{1} = \frac{dy}{2} = \frac{dz}{x}$$

Step 2: $\frac{dx}{1} = \frac{dy}{2} \implies 2dx = dy \implies y - 2x = c_1$.
Step 3: $\frac{dx}{1} = \frac{dz}{x} \implies x\,dx = dz \implies dz = x\,dx$.
Integrate:

$$z = \frac{1}{2}x^2 + c_2$$

General solution:

$$F(y - 2x, z - \frac{1}{2}x^2) = 0$$

4. $2p + q = y$

Step 1: Auxiliary equations:

$$\frac{dx}{2} = \frac{dy}{1} = \frac{dz}{y}$$

Step 2: $\frac{dx}{2} = \frac{dy}{1} \implies dx = 2dy \implies x - 2y = c_1$.
Step 3: $\frac{dy}{1} = \frac{dz}{y} \implies y\,dy = dz \implies dz = y\,dy$.
Integrate:

$$z = \frac{1}{2}y^2 + c_2$$

General solution:

$$F(x - 2y, z - \frac{1}{2}y^2) = 0$$

5. $p + q = z$

Step 1: Auxiliary equations:

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{z}$$

Step 2: $\frac{dx}{1} = \frac{dy}{1} \implies dx = dy \implies x - y = c_1$.
Step 3: $\frac{dx}{1} = \frac{dz}{z} \implies z\,dx = dz \implies \frac{dz}{z} = dx$.
Integrate:

$$\ln z = x + c_2 \implies z = Ae^x$$

General solution:

$$F(x - y, ze^{-x}) = 0$$

Detailed Solution Using Charpit's Method

We solve the nonlinear PDE:

$$p^2 + q^2 = 1$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

Step 1: Write the PDE in Standard Form

Let

$$f(x, y, z, p, q) = p^2 + q^2 - 1 = 0$$

Step 2: Compute Partial Derivatives

$$\frac{\partial f}{\partial p} = 2p, \quad \frac{\partial f}{\partial q} = 2q, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

Step 3: Write Charpit's Equations

Charpit's auxiliary equations are:

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

Substitute the derivatives:

$$\frac{dx}{-2p} = \frac{dy}{-2q} = \frac{dz}{-2p^2 - 2q^2} = \frac{dp}{0} = \frac{dq}{0}$$

Step 4: Integrate the Equations

From $\frac{dp}{0} = \frac{dq}{0}$, both p and q are constants along the characteristic curves. Let

$$p = a, \quad q = b$$

where a and b are constants.

From the original PDE:

$$a^2 + b^2 = 1$$

So, $a = \sin\theta$, $b = \cos\theta$ for some parameter θ.

Now, from the first two Charpit equations:

$$\frac{dx}{-2a} = \frac{dy}{-2b}$$

Integrate:

$$\frac{dx}{a} = \frac{dy}{b} \implies b\,dx - a\,dy = 0$$

Integrate:

$$bx - ay = c_1$$

From the third Charpit equation:

$$\frac{dz}{-2a^2 - 2b^2} = \frac{dx}{-2a}$$

But $a^2 + b^2 = 1$, so $-2a^2 - 2b^2 = -2$. Thus,

$$\frac{dz}{-2} = \frac{dx}{-2a} \implies dz = \frac{1}{a} dx$$

Integrate:

$$z = \frac{1}{a} x + c_2$$

But since $p = a = \frac{\partial z}{\partial x}$, integrating with respect to x (with y constant) gives:

$$z = ax + \phi(y)$$

Similarly, since $q = b = \frac{\partial z}{\partial y}$, integrating with respect to y (with x constant) gives:

$$z = by + \psi(x)$$

The general solution is a linear combination:

$$z = ax + by + c$$

with the constraint $a^2 + b^2 = 1$.

Step 5: General Solution

Thus, the complete integral is:

$$\boxed{z = ax + by + c, \quad \text{where} \quad a^2 + b^2 = 1}$$

Alternatively, parameterize as:

$$z = \sin\theta\, x + \cos\theta\, y + c$$

where θ and c are arbitrary constants.

Step 6: Verification

Let us check that this satisfies the original PDE:

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

$$p^2 + q^2 = a^2 + b^2 = 1$$

which is satisfied.

Step 7: Particular Solution

If a particular solution passing through a given curve or point is required, substitute the values into the general solution and solve for the constants.

Charpit's Method: Detailed Solution for $p^2 + q^2 = 1$

Let $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

1. Write the PDE in Standard Form

$$f(x, y, z, p, q) = p^2 + q^2 - 1 = 0$$

2. Compute Partial Derivatives

$$\frac{\partial f}{\partial p} = 2p, \quad \frac{\partial f}{\partial q} = 2q, \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial z} = 0$$

3. Charpit's Auxiliary Equations

$$\frac{dx}{-2p} = \frac{dy}{-2q} = \frac{dz}{-2p^2 - 2q^2} = \frac{dp}{0} = \frac{dq}{0}$$

4. Integrate the Equations

From $\frac{dp}{0} = \frac{dq}{0}$, both p and q are constants along the characteristic curves. Let

$$p = a, \quad q = b$$

where a and b are constants.

From the original PDE:

$$a^2 + b^2 = 1$$

Now, from the first two Charpit equations:

$$\frac{dx}{-2a} = \frac{dy}{-2b}$$

Integrate:

$$\frac{dx}{a} = \frac{dy}{b} \implies b\,dx - a\,dy = 0$$

Integrate:

$$bx - ay = c_1$$

From the third Charpit equation:

$$\frac{dz}{-2a^2 - 2b^2} = \frac{dx}{-2a}$$

But $a^2 + b^2 = 1$, so $-2a^2 - 2b^2 = -2$. Thus,

$$\frac{dz}{-2} = \frac{dx}{-2a} \implies dz = \frac{1}{a}dx$$

Integrate:

$$z = \frac{1}{a}x + c_2$$

But since $p = a = \frac{\partial z}{\partial x}$, integrating with respect to x (with y constant) gives:

$$z = ax + \phi(y)$$

Similarly, since $q = b = \frac{\partial z}{\partial y}$, integrating with respect to y (with x constant) gives:

$$z = by + \psi(x)$$

The general solution is a linear combination:

$$z = ax + by + c$$

with the constraint $a^2 + b^2 = 1$.

5. General Solution

Thus, the complete integral is:

$$\boxed{z = ax + by + c, \quad \text{where} \quad a^2 + b^2 = 1}$$

Alternatively, parameterize as:

$$z = \sin\theta\, x + \cos\theta\, y + c$$

where θ and c are arbitrary constants.

6. Verification

Let us check that this satisfies the original PDE:

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

$$p^2 + q^2 = a^2 + b^2 = 1$$

which is satisfied.

7. Particular Solution

If a particular solution passing through a given curve or point is required, substitute the values into the general solution and solve for the constants.

Separation of Variables: Detailed Solution for the Wave and Heat Equations

1. The Heat Equation

Consider the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = f(x)$.

Step 1: Assume a Separable Solution

Let

$$u(x,t) = X(x)T(t)$$

Step 2: Substitute into the PDE

$$\frac{\partial u}{\partial t} = X(x)T'(t), \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

So,

$$X(x)T'(t) = kX''(x)T(t)$$
$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

where λ is a separation constant.

Step 3: Solve the ODEs

- For $X(x)$:

$$X''(x) + \lambda X(x) = 0$$

with $X(0) = X(L) = 0$.

The general solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \ldots$$

- For $T(t)$:

$$T'(t) + k\lambda_n T(t) = 0 \implies T_n(t) = A_n e^{-k\lambda_n t}$$

Step 4: General Solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Step 5: Apply Initial Condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

The coefficients A_n are found by Fourier sine series:

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

2. The Wave Equation

Consider the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions $u(0,t) = u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$.

Step 1: Assume a Separable Solution

Let

$$u(x,t) = X(x)T(t)$$

Step 2: Substitute into the PDE

$$X(x)T''(t) = c^2 X''(x)T(t)$$
$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

Step 3: Solve the ODEs

- For $X(x)$:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0$$

The solution is

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$$

- For $T(t)$:

$$T''(t) + c^2 \lambda_n T(t) = 0$$

The general solution is

$$T_n(t) = A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)$$

Step 4: General Solution

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

Step 5: Apply Initial Conditions

- At $t = 0$:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

So,

$$A_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- For $u_t(x,0) = g(x)$:

$$u_t(x,0) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

So,

$$B_n = \frac{2}{n\pi c} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

3. Summary Table

Equation	Separable Solution	Fourier Coefficients
$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$	$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$	$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$
$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$	$\sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$	$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$ $B_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$

References

For further reading and more examples, see:

- Solution Using Separation of Variables [2]
- Solution of the Wave Equation by Separation of Variables [3]
- Solution of the Heat Equation by Separation of Variables [11]

Analyticity and Cauchy-Riemann Equations: Detailed Solutions

Let $z = x + iy$, and $f(z) = u(x, y) + iv(x, y)$. A function is analytic at a point if it satisfies the Cauchy-Riemann equations in a neighborhood of that point and the partial derivatives are continuous.

The Cauchy-Riemann equations are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We test analyticity and find the analytic functions for each case.

1. $f(z) = z^2$

Let $z = x + iy$:

$$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy$$

So,

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x$$

Check Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

Both equations are satisfied everywhere, and the partial derivatives are continuous.

Conclusion: $f(z) = z^2$ is analytic everywhere in C.

2. $f(z) = e^z$

Let $z = x + iy$:

$$f(z) = e^{x+iy} = e^x(\cos y + i \sin y)$$

So,

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = e^x \sin y, \quad \frac{\partial v}{\partial y} = e^x \cos y$$

Check Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

Both equations are satisfied everywhere, and the partial derivatives are continuous.

Conclusion: $f(z) = e^z$ is analytic everywhere in C.

3. $f(z) = \sin z$

Let $z = x + iy$:

$$f(z) = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

So,

$$u(x, y) = \sin x \cosh y, \quad v(x, y) = \cos x \sinh y$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial u}{\partial y} = \sin x \sinh y$$

$$\frac{\partial v}{\partial x} = -\sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \cosh y$$

Check Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x}$$

Both equations are satisfied everywhere, and the partial derivatives are continuous.

Conclusion: $f(z) = \sin z$ is analytic everywhere in C.

4. $f(z) = \cos z$

Let $z = x + iy$:

$$f(z) = \cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$$

So,

$$u(x, y) = \cos x \cosh y, \quad v(x, y) = -\sin x \sinh y$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \quad \frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y, \quad \frac{\partial v}{\partial y} = -\sin x \cosh y$$

Check Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = -\sin x \cosh y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \cos x \sinh y = -\frac{\partial v}{\partial x}$$

Both equations are satisfied everywhere, and the partial derivatives are continuous.

Conclusion: $f(z) = \cos z$ is analytic everywhere in C.

5. $f(z) = \log z$

Let $z = x + iy$, and write $z = re^{i\theta}$, where $r = \sqrt{x^2 + y^2}$, $\theta = \arctan\left(\frac{y}{x}\right)$:

$$f(z) = \log z = \ln r + i\theta$$

So,

$$u(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2), \quad v(x, y) = \arctan\left(\frac{y}{x}\right)$$

Compute partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial v}{\partial x} = -\frac{y}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

Check Cauchy-Riemann:

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x}$$

Both equations are satisfied for $z \neq 0$, and the partial derivatives are continuous except at $z = 0$.

Conclusion: $f(z) = \log z$ is analytic everywhere except at $z = 0$ and along the branch cut (usually the negative real axis).

Summary Table

$f(z)$	Analyticity	Region of Analyticity
z^2	Analytic everywhere	C
e^z	Analytic everywhere	C
$\sin z$	Analytic everywhere	C
$\cos z$	Analytic everywhere	C
$\log z$	Analytic except at $z = 0$ and branch cut	$C \setminus \{0, \text{branch cut}\}$

Harmonic Conjugates and Analytic Functions: Detailed Solutions

Given the following harmonic functions $u(x, y)$, find their harmonic conjugates $v(x, y)$ and construct the corresponding analytic functions $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$.

Recall: For $f(z) = u(x, y) + iv(x, y)$ to be analytic, the Cauchy-Riemann equations must be satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

1. $u(x, y) = x^2 - y^2$

Compute partial derivatives:

$$u_x = 2x, \quad u_y = -2y$$

By Cauchy-Riemann:

$$v_y = u_x = 2x \implies v = 2xy + \phi(x)$$

$$v_x = -u_y = 2y$$

But $v_x = 2y + \phi'(x) \implies \phi'(x) = 0 \implies \phi(x) = C$ So,

$$v(x, y) = 2xy$$

The analytic function is:

$$f(z) = x^2 - y^2 + i(2xy)$$

But $f(z) = (x + iy)^2 = z^2$.

2. $u(x, y) = e^x \cos y$

Compute partial derivatives:

$$u_x = e^x \cos y, \quad u_y = -e^x \sin y$$

By Cauchy-Riemann:

$$v_y = u_x = e^x \cos y \implies v = e^x \sin y + \phi(x)$$

$$v_x = -u_y = e^x \sin y$$

But $v_x = e^x \sin y + \phi'(x) \implies \phi'(x) = 0 \implies \phi(x) = C$ So,

$$v(x, y) = e^x \sin y$$

The analytic function is:

$$f(z) = e^x \cos y + ie^x \sin y = e^{x+iy} = e^z$$

3. $u(x, y) = x^3 - 3xy^2$

Compute partial derivatives:

$$u_x = 3x^2 - 3y^2, \quad u_y = -6xy$$

By Cauchy-Riemann:

$$v_y = u_x = 3x^2 - 3y^2 \implies v = 3x^2y - y^3 + \phi(x)$$

$$v_x = -u_y = 6xy$$

But $v_x = 6xy + \phi'(x) \implies \phi'(x) = 0 \implies \phi(x) = C$ So,

$$v(x, y) = 3x^2y - y^3$$

The analytic function is:

$$f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$$

But $f(z) = (x + iy)^3 = z^3$.

4. $u(x, y) = x^2 + y^2$

Compute partial derivatives:

$$u_x = 2x, \quad u_y = 2y$$

By Cauchy-Riemann:

$$v_y = u_x = 2x \implies v = 2xy + \phi(x)$$

$$v_x = -u_y = -2y$$

But $v_x = 2y + \phi'(x) \implies 2y + \phi'(x) = -2y \implies 4y = -\phi'(x)$ This is only possible if $y = 0$, so $u(x, y) = x^2 + y^2$ is not harmonic (since $u_{xx} + u_{yy} = 2 + 2 = 4 \neq 0$), so it does not have a harmonic conjugate.

5. $u(x, y) = e^x \sin y$

Compute partial derivatives:

$$u_x = e^x \sin y, \quad u_y = e^x \cos y$$

By Cauchy-Riemann:

$$v_y = u_x = e^x \sin y \implies v = -e^x \cos y + \phi(x)$$

$$v_x = -u_y = -e^x \cos y$$

But $v_x = -e^x \cos y + \phi'(x) \implies \phi'(x) = 0 \implies \phi(x) = C$ So,

$$v(x, y) = -e^x \cos y$$

The analytic function is:

$$f(z) = e^x \sin y - ie^x \cos y = -ie^{x+iy} = -ie^z$$

Summary Table

$u(x,y)$	$v(x,y)$	$f(z)$
$x^2 - y^2$	$2xy$	z^2
$e^x \cos y$	$e^x \sin y$	e^z
$x^3 - 3xy^2$	$3x^2y - y^3$	z^3
$x^2 + y^2$	No harmonic conjugate	Not analytic
$e^x \sin y$	$-e^x \cos y$	$-ie^z$

Detailed Solutions: Taylor and Laurent Series Expansions

Let us expand the following functions as Taylor and Laurent series about $z = 0$ (unless otherwise specified):

1. $f(z) = \dfrac{1}{z-1}$
2. $f(z) = e^z$
3. $f(z) = \dfrac{1}{z(z-1)}$
4. $f(z) = \sin z$
5. $f(z) = \dfrac{1}{z^2+1}$

1. $f(z) = \dfrac{1}{z-1}$

Taylor Series about $z = 0$ ($|z| < 1$)

Rewrite:

$$f(z) = \frac{1}{z-1} = -\frac{1}{1-z}$$

Expand as a geometric series:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

So,

$$f(z) = -\sum_{n=0}^{\infty} z^n = -1 - z - z^2 - z^3 - \cdots, \quad |z| < 1$$

Laurent Series about $z = 0$ ($|z| > 1$)

$$f(z) = \frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty} z^{-(n+1)} = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots, \quad |z| > 1$$

2. $f(z) = e^z$

Taylor Series about $z = 0$ (entire function)

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

This series converges for all z.

Laurent Series

Since e^z is entire (analytic everywhere), its Laurent series about $z = 0$ is the same as its Taylor series (no negative powers).

3. $f(z) = \dfrac{1}{z(z-1)}$

Partial Fraction Decomposition

$$\frac{1}{z(z-1)} = \frac{1}{z} - \frac{1}{z-1}$$

Laurent Series for $0 < |z| < 1$

Expand $\frac{1}{z-1}$ as above:

$$\frac{1}{z-1} = -\sum_{n=0}^{\infty} z^n, \quad |z| < 1$$

So,

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} z^n = \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots, \quad 0 < |z| < 1$$

Laurent Series for $|z| > 1$

Expand $\frac{1}{z-1}$ for $|z| > 1$:

$$\frac{1}{z-1} = \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} = \sum_{n=0}^{\infty} z^{-(n+1)}, \quad |z| > 1$$

So,

$$f(z) = \frac{1}{z} - \sum_{n=0}^{\infty} z^{-(n+1)} = \frac{1}{z} - \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = -\frac{1}{z^2} - \frac{1}{z^3} - \cdots, \quad |z| > 1$$

4. $f(z) = \sin z$

Taylor Series about $z = 0$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

This series converges for all z.

Laurent Series

Since $\sin z$ is entire, its Laurent series about $z = 0$ is the same as its Taylor series (no negative powers).

5. $f(z) = \dfrac{1}{z^2+1}$

Partial Fraction Decomposition

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)} = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

Taylor Series about $z = 0$ ($|z| < 1$)

Expand each term as a geometric series:

$$\frac{1}{z-i} = -\frac{1}{i} \cdot \frac{1}{1-\frac{z}{i}} = -\frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n = -\frac{1}{i}\sum_{n=0}^{\infty}\frac{z^n}{i^n}$$

$$\frac{1}{z+i} = \frac{1}{i} \cdot \frac{1}{1-\frac{z}{-i}} = \frac{1}{i}\sum_{n=0}^{\infty}\left(\frac{z}{-i}\right)^n = \frac{1}{i}\sum_{n=0}^{\infty}\frac{z^n}{(-i)^n}$$

So,

$$\begin{aligned} f(z) &= \frac{1}{2i}\left(-\frac{1}{i}\sum_{n=0}^{\infty}\frac{z^n}{i^n} - \frac{1}{i}\sum_{n=0}^{\infty}\frac{z^n}{(-i)^n}\right) \\ &= -\frac{1}{2i^2}\sum_{n=0}^{\infty}\frac{z^n}{i^n} - \frac{1}{2i^2}\sum_{n=0}^{\infty}\frac{z^n}{(-i)^n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{i^n} + \frac{1}{(-i)^n}\right)z^n \end{aligned}$$

This can be further simplified, but the above is the general form for $|z| < 1$.

Laurent Series for $|z| > 1$

Expand in negative powers using similar geometric series expansions, or use the partial fractions and expand about infinity.

Summary Table

$f(z)$	Taylor Series (about $z = 0$)	Laurent Series (about $z = 0$)
$\frac{1}{z-1}$	$-\sum_{n=0}^{\infty} z^n$, $\lvert z\rvert < 1$	$\sum_{n=0}^{\infty} z^{-(n+1)}$, $\lvert z\rvert > 1$
e^z	$\sum_{n=0}^{\infty} \frac{z^n}{n!}$, $\forall z$	Same as Taylor (entire)
$\frac{1}{z(z-1)}$	$\frac{1}{z} + \sum_{n=0}^{\infty} z^n$, $0 < \lvert z\rvert < 1$	$-\sum_{n=2}^{\infty} z^{-n}$, $\lvert z\rvert > 1$
$\sin z$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$, $\forall z$	Same as Taylor (entire)
$\frac{1}{z^2+1}$	$\frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$, $\lvert z\rvert < 1$	Expand in negative powers for $\lvert z\rvert > 1$

Detailed Solutions Using Cauchy's Integral Formula

Let C be a positively oriented simple closed contour in the complex plane, and let $f(z)$ be analytic inside and on C. Cauchy's integral formula states:

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a} dz$$

and for the n-th derivative,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz$$

We solve each integral in detail:

1. $\int_C \frac{e^z}{z-1} dz$

Here, $f(z) = e^z$ is analytic everywhere, and $a = 1$.

By Cauchy's integral formula:

$$\int_C \frac{e^z}{z-1} dz = 2\pi i \cdot f(1) = 2\pi i \cdot e^1 = 2\pi i e$$

2. $\int_C \frac{1}{z^2+1} dz$

Factor denominator: $z^2 + 1 = (z-i)(z+i)$. The integrand has simple poles at $z = i$ and $z = -i$.

Assume C encloses both poles. By the residue theorem:

$$\int_C \frac{1}{z^2+1} dz = 2\pi i \left(\mathrm{Res}_{z=i} + \mathrm{Res}_{z=-i}\right)$$

Compute residues:

$$\mathrm{Res}_{z=i} = \lim_{z \to i} (z-i) \frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

$$\mathrm{Res}_{z=-i} = \lim_{z \to -i} (z+i) \frac{1}{(z-i)(z+i)} = \frac{1}{-2i}$$

Sum:

$$\frac{1}{2i} + \left(-\frac{1}{2i}\right) = 0$$

So,

$$\int_C \frac{1}{z^2+1} dz = 0$$

3. $\int_C \frac{\sin z}{z-a}dz$

Here, $f(z) = \sin z$ is analytic everywhere, a is inside C.

By Cauchy's integral formula:

$$\int_C \frac{\sin z}{z-a}dz = 2\pi i \cdot \sin a$$

4. $\int_C \frac{1}{(z-2)^2}dz$

This is the case $f(z) = 1$, $a = 2$, and the integrand is $\frac{f(z)}{(z-a)^2}$.

By Cauchy's integral formula for the first derivative:

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$$

Here, $f(z) = 1 \implies f'(z) = 0$.

So,

$$\int_C \frac{1}{(z-2)^2}dz = 2\pi i \cdot f'(2) = 2\pi i \cdot 0 = 0$$

5. $\int_C \frac{z^2}{z^3-1}dz$

Factor denominator: $z^3 - 1 = (z-1)(z-\omega)(z-\omega^2)$, where $\omega = e^{2\pi i/3}$.

Let C enclose all three simple poles. Compute the sum of residues at each pole.

Let $f(z) = z^2$, $g(z) = z^3 - 1$.

The residue at z_k (where $z_k^3 = 1$) is:

$$\mathrm{Res}_{z=z_k} \frac{z^2}{z^3-1} = \frac{z_k^2}{g'(z_k)}$$

But $g'(z) = 3z^2$, so $g'(z_k) = 3z_k^2$.

Thus,

$$\mathrm{Res}_{z=z_k} = \frac{z_k^2}{3z_k^2} = \frac{1}{3}$$

There are three such residues, so the sum is 1.

Therefore,

$$\int_C \frac{z^2}{z^3-1}dz = 2\pi i \cdot 1 = 2\pi i$$

Summary Table

Integral	Value
$\int_C \frac{e^z}{z-1} dz$	$2\pi i e$
$\int_C \frac{1}{z^2+1} dz$	0
$\int_C \frac{\sin z}{z-a} dz$	$2\pi i \sin a$
$\int_C \frac{1}{(z-2)^2} dz$	0
$\int_C \frac{z^2}{z^3-1} dz$	$2\pi i$

Residues and Contour Integrals Using the Residue Theorem

Let us find the residues at the specified points for each function, and explain the process in detail.

1. Residue of $\frac{1}{z^2+1}$ at $z = i$

Factor the denominator:

$$z^2 + 1 = (z - i)(z + i)$$

The function has simple poles at $z = i$ and $z = -i$.

The residue at $z = i$ is:

$$\text{Res}_{z=i} \frac{1}{z^2+1} = \lim_{z \to i} (z - i) \frac{1}{(z - i)(z + i)} = \lim_{z \to i} \frac{1}{z + i} = \frac{1}{i + i} = \frac{1}{2i}$$

2. Residue of $\frac{e^z}{z^2}$ at $z = 0$

This is a pole of order 2 at $z = 0$.

The residue at a double pole is:

$$\text{Res}_{z=0} \frac{e^z}{z^2} = \lim_{z \to 0} \frac{d}{dz} (e^z) = \lim_{z \to 0} e^z = 1$$

Alternatively, expand e^z in a Taylor series:

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots$$

$$\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \cdots$$

The coefficient of $\frac{1}{z}$ is 1, which is the residue.

3. Residue of $\frac{1}{(z-1)^2}$ at $z = 1$

This is a pole of order 2 at $z = 1$.

The residue at a double pole is:

$$\text{Res}_{z=1} \frac{1}{(z - 1)^2} = \lim_{z \to 1} \frac{d}{dz} (1) = 0$$

Since the numerator is constant, its derivative is zero. Thus, the residue is 0.

4. Residue of $\dfrac{\sin z}{z^3}$ at $z = 0$

This is a pole of order 3 at $z = 0$.

The residue is the coefficient of $\frac{1}{z}$ in the Laurent expansion.

Expand $\sin z$ in a Taylor series:

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\frac{\sin z}{z^3} = \frac{z}{z^3} - \frac{z^3}{6z^3} + \frac{z^5}{120z^3} - \cdots = z^{-2} - \frac{1}{6} + \frac{z^2}{120} - \cdots$$

The coefficient of $\frac{1}{z}$ is 0, so the residue is 0.

Alternatively, for a pole of order 3:

$$\mathrm{Res}_{z=0} \frac{\sin z}{z^3} = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} (\sin z) = \frac{1}{2} \lim_{z \to 0} (-\sin z) = 0$$

5. Residue of $\dfrac{1}{z^2 + 4}$ at $z = 2i$

Factor the denominator:

$$z^2 + 4 = (z - 2i)(z + 2i)$$

The function has simple poles at $z = 2i$ and $z = -2i$.

The residue at $z = 2i$ is:

$$\mathrm{Res}_{z=2i} \frac{1}{z^2 + 4} = \lim_{z \to 2i} (z - 2i) \frac{1}{(z - 2i)(z + 2i)} = \frac{1}{2i + 2i} = \frac{1}{4i}$$

Summary Table

Function	Point	Residue
$\frac{1}{z^2+1}$	$z = i$	$\frac{1}{2i}$
$\frac{e^z}{z^2}$	$z = 0$	1
$\frac{1}{(z-1)^2}$	$z = 1$	0
$\frac{\sin z}{z^3}$	$z = 0$	0
$\frac{1}{z^2+4}$	$z = 2i$	$\frac{1}{4i}$

Residue Theorem and Contour Integrals

If a function $f(z)$ is analytic inside and on a simple closed contour C, except for isolated singularities z_k inside C, then

$$\int_C f(z)dz = 2\pi i \sum_k \mathrm{Res}_{z=z_k} f(z)$$

where the sum is over all residues inside C.

Detailed Solutions: Gradient, Divergence, and Curl of Vector Fields

Given the vector fields:

$$\vec{F}_1 = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\vec{F}_2 = yz\hat{i} + zx\hat{j} + xy\hat{k}$$
$$\vec{F}_3 = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$$
$$\vec{F}_4 = (2x + y)\hat{i} + (y - z)\hat{j} + (z + x)\hat{k}$$
$$\vec{F}_5 = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$$

We are to find the gradient (where applicable), divergence, and curl of each vector field.

1. Vector Field $\vec{F}_1 = x\hat{i} + y\hat{j} + z\hat{k}$

Divergence

$$\nabla \cdot \vec{F}_1 = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$$

Curl

$$\nabla \times \vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{i} - \left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right)\hat{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\hat{k} = \vec{0}$$

Gradient

Gradient applies to scalar fields, so not applicable here.

2. Vector Field $\vec{F}_2 = yz\hat{i} + zx\hat{j} + xy\hat{k}$

Divergence

$$\nabla \cdot \vec{F}_2 = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = 0 + 0 + 0 = 0$$

Curl

$$\nabla \times \vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

Calculate each component:

$$\begin{aligned} \hat{i} &: \frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx) = x - x = 0 \\ \hat{j} &: \frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy) = y - y = 0 \\ \hat{k} &: \frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz) = z - z = 0 \end{aligned}$$

So,

$$\nabla \times \vec{F}_2 = \vec{0}$$

3. Vector Field $\vec{F}_3 = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

Divergence

$$\nabla \cdot \vec{F}_3 = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2) = 2x + 2y + 2z$$

Curl

$$\nabla \times \vec{F}_3 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Calculate each component:

$$\begin{aligned} \hat{i} &: \frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2) = 0 - 0 = 0 \\ \hat{j} &: \frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2) = 0 - 0 = 0 \\ \hat{k} &: \frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2) = 0 - 0 = 0 \end{aligned}$$

So,

$$\nabla \times \vec{F}_3 = \vec{0}$$

4. Vector Field $\vec{F}_4 = (2x + y)\hat{i} + (y - z)\hat{j} + (z + x)\hat{k}$

Divergence

$$\nabla \cdot \vec{F}_4 = \frac{\partial}{\partial x}(2x + y) + \frac{\partial}{\partial y}(y - z) + \frac{\partial}{\partial z}(z + x) = 2 + 1 + 1 = 4$$

Curl

$$\nabla \times \vec{F}_4 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y & y - z & z + x \end{vmatrix}$$

Calculate each component:

$$\hat{i} : \frac{\partial}{\partial y}(z + x) - \frac{\partial}{\partial z}(y - z) = 0 - (-1) = 1$$
$$\hat{j} : \frac{\partial}{\partial z}(2x + y) - \frac{\partial}{\partial x}(z + x) = 0 - 1 = -1$$
$$\hat{k} : \frac{\partial}{\partial x}(y - z) - \frac{\partial}{\partial y}(2x + y) = 0 - 1 = -1$$

So,

$$\nabla \times \vec{F}_4 = \hat{i} - \hat{j} - \hat{k}$$

5. Vector Field $\vec{F}_5 = (y + z)\hat{i} + (z + x)\hat{j} + (x + y)\hat{k}$

Divergence

$$\nabla \cdot \vec{F}_5 = \frac{\partial}{\partial x}(y + z) + \frac{\partial}{\partial y}(z + x) + \frac{\partial}{\partial z}(x + y) = 0 + 0 + 0 = 0$$

Curl

$$\nabla \times \vec{F}_5 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix}$$

Calculate each component:

$$\hat{i} : \frac{\partial}{\partial y}(x + y) - \frac{\partial}{\partial z}(z + x) = 1 - 1 = 0$$
$$\hat{j} : \frac{\partial}{\partial z}(y + z) - \frac{\partial}{\partial x}(x + y) = 1 - 1 = 0$$
$$\hat{k} : \frac{\partial}{\partial x}(z + x) - \frac{\partial}{\partial y}(y + z) = 1 - 1 = 0$$

So,

$$\nabla \times \vec{F}_5 = \vec{0}$$

Detailed Solutions: Line Integrals of Vector Fields

Let C be a smooth curve in the plane or space, and let $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ be scalar functions. The line integral of the vector field $\vec{F} = P\,\hat{\imath} + Q\,\hat{\jmath} + R\,\hat{k}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

Below, each integral is solved in detail.

1. $\displaystyle\int_C (y\,dx + x\,dy)$

Let C be a closed curve (e.g., the unit circle $x^2 + y^2 = 1$ traversed counterclockwise).

Method 1: Direct Parameterization

Parameterize the unit circle:

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

Then,

$$dx = -\sin t\,dt, \quad dy = \cos t\,dt$$

Substitute into the integral:

$$\int_C (y\,dx + x\,dy) = \int_0^{2\pi} [\sin t \cdot (-\sin t) + \cos t \cdot \cos t]\,dt = \int_0^{2\pi} \left(-\sin^2 t + \cos^2 t\right) dt$$

$$= \int_0^{2\pi} \cos(2t)\,dt = \frac{1}{2}\sin(2t)\Big|_0^{2\pi} = 0$$

Method 2: Green's Theorem

Let $P = y$, $Q = x$. Then,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$$

So,

$$\int_C (y\,dx + x\,dy) = \iint_R 0\,dA = 0$$

Final Answer:

$$\boxed{0}$$

2. $\displaystyle\int_C (x^2\,dx + y^2\,dy)$

Suppose C is a curve from (x_0, y_0) to (x_1, y_1).

Let $F(x, y) = \big(x^2, y^2\big)$, which is a conservative field.

Find a potential function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = x^2 \implies f(x, y) = \frac{1}{3}x^3 + g(y)$$

$$\frac{\partial f}{\partial y} = y^2 \implies g'(y) = y^2 \implies g(y) = \frac{1}{3}y^3$$

So,

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$$

By the fundamental theorem for line integrals,

$$\int_C (x^2\,dx + y^2\,dy) = f(x_1, y_1) - f(x_0, y_0)$$

If C is closed, the value is 0.

Example: From $(0, 0)$ to $(1, 1)$,

$$\int_C (x^2\,dx + y^2\,dy) = \left[\frac{1}{3}(1)^3 + \frac{1}{3}(1)^3\right] - [0 + 0] = \frac{2}{3}$$

3. $\displaystyle\int_C (z\,dx + x\,dz)$

Suppose C is a curve in space from (x_0, y_0, z_0) to (x_1, y_1, z_1).

Find a potential function:

$$\frac{\partial f}{\partial x} = z \implies f(x, y, z) = xz + g(y, z)$$

$$\frac{\partial f}{\partial z} = x + g_z(y, z)$$

But $\frac{\partial f}{\partial z} = x + g_z(y, z) = x$ implies $g_z(y, z) = 0 \implies g(y, z) = h(y)$.

So,

$$f(x, y, z) = xz + h(y)$$

Thus, the field is conservative, and

$$\int_C (z\,dx + x\,dz) = f(x_1, y_1, z_1) - f(x_0, y_0, z_0) = x_1 z_1 - x_0 z_0$$

4. $\displaystyle\int_C (x\,dy - y\,dx)$

Let C be a closed curve, e.g., the unit circle $x^2 + y^2 = 1$.

Parameterize:

$$x = \cos t, \quad y = \sin t, \quad 0 \le t \le 2\pi$$

$$dx = -\sin t\,dt, \quad dy = \cos t\,dt$$

$$x\,dy - y\,dx = \cos t \cdot \cos t\,dt - \sin t \cdot (-\sin t)\,dt = \cos^2 t + \sin^2 t = 1$$

$$\int_0^{2\pi} 1\,dt = 2\pi$$

Alternatively, by Green's Theorem:

$$P = 0, \quad Q = x$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 0 = 1$$

Area of unit circle is π:

$$\int_C (x\,dy - y\,dx) = 2 \cdot \text{Area} = 2\pi$$

Final Answer:

$$\boxed{2\pi}$$

5. $\displaystyle\int_C (x\,dx + y\,dy + z\,dz)$

Let C be a curve from (x_0, y_0, z_0) to (x_1, y_1, z_1).

Find a potential function:

$$\frac{\partial f}{\partial x} = x \implies f(x,y,z) = \frac{1}{2}x^2 + g(y,z)$$

$$\frac{\partial f}{\partial y} = y \implies g(y,z) = \frac{1}{2}y^2 + h(z)$$

$$\frac{\partial f}{\partial z} = z \implies h(z) = \frac{1}{2}z^2$$

So,

$$f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$$

Thus,

$$\int_C (x\,dx + y\,dy + z\,dz) = f(x_1, y_1, z_1) - f(x_0, y_0, z_0)$$

If C is closed, the value is 0.

Summary Table

Integral	Value/Method
$\int_C (y\,dx + x\,dy)$	0 (if C is closed)
$\int_C (x^2\,dx + y^2\,dy)$	$f(x_1, y_1) - f(x_0, y_0),\ f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$
$\int_C (z\,dx + x\,dz)$	$x_1 z_1 - x_0 z_0$
$\int_C (x\,dy - y\,dx)$	2π (if C is unit circle)
$\int_C (x\,dx + y\,dy + z\,dz)$	$f(x_1, y_1, z_1) - f(x_0, y_0, z_0),\ f = \frac{1}{2}(x^2 + y^2 + z^2)$

Detailed Solutions: Surface Integrals of Vector Fields

Let S be a closed surface (e.g., the unit sphere $x^2 + y^2 + z^2 = 1$) and let $\vec{F}$ be a vector field. The surface integral (flux) of $\vec{F}$ across S is

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $d\vec{S} = \vec{n}\,dS$ and $\vec{n}$ is the outward unit normal.

We use the Divergence Theorem for closed surfaces:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F})\,dV$$

where V is the volume enclosed by S.

1. $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$

Step 1: Compute the divergence.

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

Step 2: Compute the volume of the unit sphere.

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi$$

Step 3: Apply the Divergence Theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3\,dV = 3 \cdot V = 3 \cdot \frac{4}{3}\pi = 4\pi$$

Final Answer:

$$\boxed{\iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\vec{S} = 4\pi}$$

[8][23]

2. $\vec{F} = (y\hat{i} + z\hat{j} + x\hat{k})$ over the unit sphere $x^2 + y^2 + z^2 = 1$

Step 1: Compute the divergence.

$$\nabla \cdot \vec{F} = \frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z} = 0 + 0 + 0 = 0$$

Step 2: Apply the Divergence Theorem.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 0\,dV = 0$$

Final Answer:

$$\boxed{\iint_S (y\hat{\imath} + z\hat{\jmath} + x\hat{k}) \cdot d\vec{S} = 0}$$

[25]

3. $\vec{F} = (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k})$ over the unit sphere $x^2 + y^2 + z^2 = 1$

Step 1: Compute the divergence.

$$\nabla \cdot \vec{F} = \frac{\partial x^2}{\partial x} + \frac{\partial y^2}{\partial y} + \frac{\partial z^2}{\partial z} = 2x + 2y + 2z$$

Step 2: Compute the volume integral. By symmetry, the integral of x, y, or z over the sphere is zero, so

$$\iiint_V (2x + 2y + 2z)\, dV = 0$$

Final Answer:

$$\boxed{\iint_S (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) \cdot d\vec{S} = 0}$$

[33]

4. $\vec{F} = (x\hat{\imath} + y\hat{\jmath})$ over the surface $z = f(x, y)$

Suppose S is the surface $z = f(x, y)$ over a region D in the xy-plane, with upward orientation.

Step 1: The surface element is

$$d\vec{S} = (-f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k})\, dA$$

Step 2: Compute the dot product.

$$\vec{F} \cdot d\vec{S} = (x\hat{\imath} + y\hat{\jmath}) \cdot (-f_x\hat{\imath} - f_y\hat{\jmath} + \hat{k})\, dA = -xf_x - yf_y$$

Step 3: The surface integral is

$$\iint_S (x\hat{\imath} + y\hat{\jmath}) \cdot d\vec{S} = -\iint_D (xf_x + yf_y)\, dA$$

If $f(x, y) = 0$ (the xy-plane), then $f_x = f_y = 0$, so the integral is 0.
Final Answer:

$$\boxed{\iint_S (x\hat{\imath} + y\hat{\jmath}) \cdot d\vec{S} = -\iint_D (xf_x + yf_y)\, dA}$$

[2][5]

5. $\vec{F} = (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k})$ over the sphere $x^2 + y^2 + z^2 = R^2$

Step 1: Compute the divergence.

$$\nabla \cdot \vec{F} = 2x + 2y + 2z$$

By symmetry, the integral of x, y, or z over the sphere is zero, so the total flux is 0.

Final Answer:

$$\boxed{\iint_S (x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}) \cdot d\vec{S} = 0}$$

[33]

Summary Table

Vector Field	Surface Integral over Sphere $x^2 + y^2 + z^2 = 1$
$x\hat{\imath} + y\hat{\jmath} + z\hat{k}$	4π
$y\hat{\imath} + z\hat{\jmath} + x\hat{k}$	0
$x^2\hat{\imath} + y^2\hat{\jmath} + z^2\hat{k}$	0
$x\hat{\imath} + y\hat{\jmath}$	Depends on surface, see formula above

References

- Divergence Theorem and surface integrals: [8][23]
- Surface integral formulas and parameterizations: [2][5]
- Symmetry arguments for zero flux: [25][33]

Applying Green's Theorem to Evaluate Plane Integrals

Let C be a positively oriented, simple, closed curve in the plane, and let D be the region enclosed by C. Green's theorem states:

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

where $P(x, y)$ and $Q(x, y)$ have continuous partial derivatives on an open region containing D.

Below, each integral is solved in detail.

1. $\oint_C (x^2 - y^2)\,dx + 2xy\,dy$

Let $P(x, y) = x^2 - y^2$, $Q(x, y) = 2xy$.

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 2y, \qquad \frac{\partial P}{\partial y} = -2y$$

So,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2y - (-2y) = 4y$$

By Green's theorem:

$$\oint_C (x^2 - y^2)\,dx + 2xy\,dy = \iint_D 4y\,dx\,dy$$

If C is the unit circle $x^2 + y^2 = 1$, use polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

$$\iint_D 4y\,dx\,dy = 4\iint_D y\,dx\,dy = 4\int_0^{2\pi}\int_0^1 (r\sin\theta)r\,dr\,d\theta$$

$$= 4\int_0^{2\pi} \sin\theta\,d\theta \int_0^1 r^2\,dr = 4\left[\int_0^{2\pi} \sin\theta\,d\theta\right]\left[\frac{1}{3}\right]$$

$$\int_0^{2\pi} \sin\theta\,d\theta = 0$$

So,

$$\boxed{0}$$

2. $\oint_C y\,dx + x\,dy$

Let $P(x,y) = y$, $Q(x,y) = x$.
Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 1, \qquad \frac{\partial P}{\partial y} = 1$$

So,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - 1 = 0$$

Thus,

$$\oint_C y\,dx + x\,dy = \iint_D 0\,dx\,dy = 0$$

3. $\oint_C x\,dy - y\,dx$

Let $P(x,y) = -y$, $Q(x,y) = x$.
Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 1, \qquad \frac{\partial P}{\partial y} = -1$$

So,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 - (-1) = 2$$

If C is the circle $x^2 + y^2 = a^2$, the area of the circle is πa^2:

$$\oint_C x\,dy - y\,dx = \iint_D 2\,dx\,dy = 2 \cdot \text{Area}(D) = 2\pi a^2$$

4. $\oint_C x^3\,dx + y^3\,dy$

Let $P(x,y) = x^3$, $Q(x,y) = y^3$.
Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 0, \qquad \frac{\partial P}{\partial y} = 0$$

So,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - 0 = 0$$

Thus,

$$\oint_C x^3\,dx + y^3\,dy = \iint_D 0\,dx\,dy = 0$$

5. $\oint_C y^2\,dx + x^2\,dy$

Let $P(x,y) = y^2$, $Q(x,y) = x^2$.

Compute the partial derivatives:

$$\frac{\partial Q}{\partial x} = 2x, \qquad \frac{\partial P}{\partial y} = 2y$$

So,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$$

If C is the unit circle $x^2 + y^2 = 1$:

$$\iint_D (2x - 2y)\,dx\,dy = 2\iint_D x\,dx\,dy - 2\iint_D y\,dx\,dy$$

By symmetry, both integrals are zero (since x and y are odd functions over the disk):

$$\boxed{0}$$

Summary Table

Integral	Value (for C the unit circle)
$\oint_C (x^2 - y^2)\,dx + 2xy\,dy$	0
$\oint_C y\,dx + x\,dy$	0
$\oint_C x\,dy - y\,dx$	2π
$\oint_C x^3\,dx + y^3\,dy$	0
$\oint_C y^2\,dx + x^2\,dy$	0

Detailed Solution: Application of Stokes' and Gauss' (Divergence) Theorems

1. Gauss's (Divergence) Theorem

Gauss's theorem states:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V (\nabla \cdot \vec{F})\, dV$$

where S is a closed surface bounding the volume V.

2. Stokes' Theorem

Stokes' theorem states:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

where C is the boundary curve of the surface S.

3. Surface Integrals to Volume Integrals (Gauss's Theorem)

Let us apply Gauss's theorem to the following surface integrals:

(a) $\iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\vec{S}$

Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$.

Compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

By Gauss's theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 3\, dV = 3 \cdot (\text{Volume enclosed by } S)$$

If S is the unit sphere $x^2 + y^2 + z^2 = 1$, then

$$\text{Volume} = \frac{4}{3}\pi$$

So,

$$\iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\vec{S} = 3 \cdot \frac{4}{3}\pi = 4\pi$$

(b) $\iint_S (y\hat{\imath} + z\hat{\jmath} + x\hat{k}) \cdot d\vec{S}$

Let $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$.

Compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial y}{\partial x} + \frac{\partial z}{\partial y} + \frac{\partial x}{\partial z} = 0 + 0 + 0 = 0$$

By Gauss's theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 0 \, dV = 0$$

(c) $\iint_S (xz\hat{\imath} + yz\hat{\jmath} + xy\hat{k}) \cdot d\vec{S}$

Let $\vec{F} = xz\hat{\imath} + yz\hat{\jmath} + xy\hat{k}$.

Compute the divergence:

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xy) = z + z + 0 = 2z$$

By Gauss's theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V 2z \, dV$$

If S is the unit sphere, by symmetry, the integral of z over the sphere is zero (since for every z there is a $-z$), so

$$\iint_S (xz\hat{\imath} + yz\hat{\jmath} + xy\hat{k}) \cdot d\vec{S} = 0$$

4. Surface Integrals to Line Integrals (Stokes' Theorem)

(d) $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$

Given a vector field $\vec{F}$ and a surface S with boundary C, Stokes' theorem relates the line integral around C to the surface integral of the curl of $\vec{F}$ over S.

Example: Let $\vec{F} = y\hat{\imath} + z\hat{\jmath} + x\hat{k}$.

Compute the curl:

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = \left(\frac{\partial x}{\partial y} - \frac{\partial z}{\partial z}\right)\hat{\imath} - \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial z}\right)\hat{\jmath} + \left(\frac{\partial z}{\partial x} - \frac{\partial y}{\partial y}\right)\hat{k}$$

$$= (0 - 1)\hat{\imath} - (1 - 0)\hat{\jmath} + (0 - 1)\hat{k} = -\hat{\imath} - \hat{\jmath} - \hat{k}$$

So,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (-\hat{\imath} - \hat{\jmath} - \hat{k}) \cdot d\vec{S}$$

5. Summary Table

Surface Integral	Divergence	Value (unit sphere)
$\iint_S (x\hat{i} + y\hat{j} + z\hat{k}) \cdot d\vec{S}$	3	4π
$\iint_S (y\hat{i} + z\hat{j} + x\hat{k}) \cdot d\vec{S}$	0	0
$\iint_S (xz\hat{i} + yz\hat{j} + xy\hat{k}) \cdot d\vec{S}$	$2z$	0

Conclusion

- Gauss's theorem converts a surface integral over a closed surface to a volume integral of the divergence. - Stokes' theorem converts a line integral around a closed curve to a surface integral of the curl over the surface bounded by the curve. - For symmetric surfaces like the sphere, many integrals involving odd functions vanish due to symmetry.