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Module-03

UNIT-03

Partial Diffrential Equations

Definations:-

1) Partial Diffrential Equation:-

A diffrential equation which involves partial derivative of one or more dependent variables with regard to two or more than two independent variables is called Partial diffrential Equation.

In case of two independent variables, we shall usually take them x and y and

z to be dependent variable

We shall use the following standard notations

$$\frac{\delta z}{\delta x} = p, \quad \frac{dz}{\delta y} = q, \quad \frac{\delta^2 z}{\delta x^2} = r$$

$$\frac{\delta^2 z}{\delta y^2} = t, \quad \frac{\delta^2 z}{\delta x \cdot \delta y} = s$$

Examples :-

1) The partial diffrential equation

$$yzp + zxq = xy$$

or $$yz\frac{\delta z}{\delta x} + zx\frac{\delta z}{\delta y} = xy$$

is of order one and degree one

2) The Partial Diffrential Equation

$$p + r + s = 1 \text{ or}$$

$$\frac{\delta z}{\delta x} + \frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta x \cdot \delta y} = 1 \text{ is}$$

of order two and degree one.

Linear and Non-Linear Partial Diffrential Equations

Defination :- A diffrential equation is called a linear partial diffrential equation if z(dependent variable) and all it's derivatives appear in it first degree.

A Partial diffrential equation which is not linear is called a non-linear partial Diffrential equation.

Topic - 01 Formulation (or Derivation) of

Partial Diffrential Equation.

Suppose we are given a functial relation.

$$f(x, y, z, a, b) = 0 \quad ---- (1)$$

The partial diffrential equations may be drived eq. (1) by

Mathod-01 :- by the elimination of arbistrary constaints from a given functional relation b/w the variables a and b.

Mathod-02 :- by the elimination of arbistrary functions of the variables.

Mathod-01 :- by the elimination of arbitrary constaints

Let $f(x, y, z, a, b) = 0 \quad ---- (1)$

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Where a, b are arbitrary constants x and y are independent variables and z is dependent variable.

Differentiating eq. (1) partially w.r.t. to x we get

$$\frac{\delta f}{\delta x} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta x} = 0$$

$$\Rightarrow \quad \frac{\delta f}{\delta x} + p \cdot \frac{\delta f}{\delta z} = 0 \quad \text{----- (2)}$$

Again Differentiating (1) partially w.r.t. to y we get

$$\frac{\delta f}{\delta y} + \frac{\delta f}{\delta z} \cdot \frac{\delta z}{\delta y} = 0$$

$$\Rightarrow \frac{\delta f}{\delta y} + q \frac{\delta f}{\delta z} = 0 \quad \text{----- (3)}$$

Eliminating a, b from equations (1), (2) and (3)

we get $\quad \phi(x, y, z, p, q) = 0$

Which is partial diffrential Equation of first order.

Remark :- If the number of arbitrary constaints is greater then the number of independent variables, then the order of the partial diffrential equation obtained will be more than 1.

Mathod-02 :- By the Elimination of arbitrary function.

Let u, v be two given functions of x, y, z connected by the given relation

$$f(u, v) = 0 \quad \text{---} (1)$$

where f is an arbitrary function.

Then linear P.D.E. is

$$Pp + Qq = R$$

~~where~~ it is the required partial diffrential equation

where

$$P = \frac{\delta(u,v)}{\delta(y,z)}, \quad Q = \frac{\delta(u,v)}{\delta(z,x)}.$$

$$R = \frac{\delta(u,v)}{\delta(x,y)}$$

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and $P = \frac{\delta(u,v)}{\delta(y,z)} = \frac{\delta u}{\delta y}\cdot\frac{\delta v}{\delta z} - \frac{\delta u}{\delta z}\cdot\frac{\delta v}{\delta y}$

$$= \begin{vmatrix} \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \end{vmatrix}$$

$$Q = \frac{\delta(u,v)}{\delta(z,x)} = \frac{\delta u}{\delta z}\cdot\frac{\delta v}{\delta x} - \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta z} = \begin{vmatrix} \frac{\delta u}{\delta z} & \frac{\delta u}{\delta x} \\ \frac{\delta v}{\delta z} & \frac{\delta v}{\delta x} \end{vmatrix}$$

$$R = \frac{\delta(u,v)}{\delta(x,y)} = \frac{\delta u}{\delta x}\cdot\frac{\delta v}{\delta y} - \frac{\delta u}{\delta y}\cdot\frac{\delta v}{\delta x} = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{vmatrix}$$

Q.1 Form Partial diffrential equations from the following equations by elimination the arbitrary constaints.

Solution:- (i) $z = ax + by + ab$

(ii) $z = ax + a^2y^2 + b$

(iii) $z = (x^2+a)(y^2+b)$

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

(10)

olutions:- (i) $z = ax + by + ab$

Solve:- Now Given eq.

$$z = ax + by + ab \quad \text{---} (1)$$

Diffrentiating eq. (1) partially w.r.to x.

$$\frac{\delta z}{\delta x} = a \quad \text{---} (2)$$

Diffren- eq. (1) partially w.r.t. to y

$$\frac{\delta z}{\delta y} = b \quad \text{---} (3)$$

Now Putting a and b from eq. (2) and eq. (3) in eq. (1)

eliminating a and b from eq. (1) (2) and (3)

Then

$$z = \left(\frac{\delta z}{\delta x}\right) \cdot x + \left(\frac{\delta z}{\delta y}\right) y + \left(\frac{\delta z}{\delta x}\right) \cdot \left(\frac{\delta z}{\delta y}\right)$$

Hence

$$z = px + qy + p \cdot q.$$

This is the required solution of eq. (1).

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(ii) $z = (x^2+a)(y^2+b)$

Solut:- Given eq. $z = (x^2+a)(y^2+b)$ --- (1)

D. w.r. to x Partially eq (1)

$$\frac{\delta z}{\delta x} = 2x(y^2+b)$$

$$\Rightarrow (y^2+b) = \frac{\delta z}{\delta x} \cdot \frac{1}{2x} \quad \text{---} (2)$$

Again D. w. r. to y Partially eq. (1)

$$\frac{\delta z}{\delta y} = (x^2+a)\,2y$$

$$\Rightarrow (x^2+a) = \frac{\delta z}{\delta y} \cdot \frac{1}{2y} \quad \text{---} (3)$$

Now Putting (x^2+a) and (y^2+b) from eq. (2) and (3) in eq. (1)

we get

$$z = \left(\frac{\delta z}{\delta y} \cdot \frac{1}{2y}\right)\left(\frac{\delta z}{\delta x} \cdot \frac{1}{2x}\right)$$

Hence $z = \frac{1}{4xy} p \cdot q \Rightarrow \boxed{pq = 4xyz}$

This is the required Solution of eq. (1).

(iii) $z = ax + a^2y^2 + b$

Solve:- Given $z = ax + a^2y^2 + b$ ---(1)

D. w. r. to x Partially eq. (1)

$$\frac{\delta z}{\delta x} = a \quad \text{---(2)}$$

D. w. r. to y Partially eq. (2).

$$\frac{\delta z}{\delta y} = 2a^2y$$

Eliminating a from eq. (2)

$$\frac{\delta z}{\delta y} = 2\left(\frac{\delta z}{\delta x}\right)^2 y$$

Then $\boxed{q = 2p^2y}$

Hence this is the required Solution of eq. (1).

(iv) $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Solution:- Given $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ---(1)

Here the number of arbitrary constaints a, b, c is greater that the number of independent variables (x, y).

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There fore, order of P.D.E. will be greater than one.

Now Diffrentiating Partially w.r.r. to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\delta z}{\delta x} = 0$$

$$\Rightarrow \quad \frac{\cancel{2}x}{a^2} = -\frac{\cancel{2}z}{c^2} \frac{\delta z}{\delta x}$$

$$c^2 x = -a^2 z \frac{\delta z}{\delta x}$$

$$\boxed{c^2 x + a^2 z \frac{\delta z}{\delta x} = 0} \quad \text{----} (2)$$

and Similarly

$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\delta z}{\delta y} = 0$$

Hence $\boxed{c^2 y + b^2 z \frac{\delta z}{\delta y} = 0}$ ---- (3)

Again Diffrentiating eq. (2) Partially w.r.t. x we get

$$c^2 + a^2 \frac{\delta z}{\delta x} \cdot \frac{\delta z}{\delta x} + a^2 z \frac{\delta^2 z}{\delta x^2} = 0$$

$$= c^2 + a^2\left(\frac{\delta z}{\delta x}\right)^2 + a^2 z \frac{\delta^2 z}{\delta x^2} = 0$$

Divided by a^2 both side

$$\frac{c^2}{a^2} + \left(\frac{\delta z}{\delta x}\right)^2 + z\frac{\delta^2 z}{\delta x^2} = 0 \quad \text{---- (4)}$$

Now

eq. (2) $c^2 x + a^2 z \frac{\delta z}{\delta x} = 0$

Divided by a^2 both side

$$\frac{c^2}{a^2} x + z\frac{\delta z}{\delta x} = 0$$

$$\frac{c^2}{a^2} x = -z\frac{\delta z}{\delta x}$$

Finally $\boxed{\frac{c^2}{a^2} = -\frac{z}{x}\frac{\delta z}{\delta x}}$

Put value $\frac{c^2}{a^2}$ in eq. (4)

$$-\frac{z}{x}\frac{\delta z}{\delta x} + \left(\frac{\delta z}{\delta x}\right)^2 + z\frac{\delta^2 z}{\delta x^2} = 0$$

$$\boxed{z\frac{\delta^2 z}{\delta x^2} - \frac{z}{x}\frac{\delta z}{\delta x} + \left(\frac{\delta z}{\delta x}\right)^2 = 0}$$

Hence this is the required Solution of eq. (1).

Q.2 From the partial differential equation by eliminating the arbitrary functions from the following.

(i) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

(ii) $z = f(x+iy) + g(x-iy)$

Now (1) $\Rightarrow$ $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

Solution:- Now Given eq. of the form

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{---} (1)$$

Differentiating z Partially w. r. to x

$$\frac{\delta z}{\delta x} = 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(-\frac{1}{x^2}\right)$$

$$-px^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{---} (2)$$

$$\boxed{\because p = \frac{\delta z}{\delta x}}$$

and

Differentiating z Partially w. r. t. to y

$$\frac{\delta z}{\delta y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\cdot\left(\frac{1}{y}\right)$$

$$q - 2y = 2f'\left(\frac{1}{x} + \log\frac{1}{y}\right)\cdot\frac{1}{y}$$

Then
$$qy - 2y^2 = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{---} (3)$$

Now compair eq. (2) and (3)

$$-px^2 = qy - 2y^2$$

or $$\boxed{2y^2 = x^2p + yq}$$

Hence this is the required solution of eq. (1).

(ii) $z = f(x+iy) + g(x-iy)$

Solution: Now given

$$z = f(x+iy) + g(x-iy) \quad \text{---} (1)$$

Here z involving two functions.

∴ Diffrentiating two times w.r.to x and y respectivly we get.

$$\frac{\delta z}{\delta x} = f'(x+iy).1 + g(x-iy).1$$

$$\frac{\delta z}{\delta y} = f'(x+iy).i + g'(x-iy)(-i)$$

again.

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$$\frac{\delta^2 z}{\delta x^2} = f''(x+iy) + g''(x-iy) \quad \text{---} (2)$$

and

$$\frac{\delta^2 z}{\delta y^2} = f''(x+iy)(i^2) + g''(x-iy)(-i)(-i)$$

$$\frac{\delta^2 z}{\delta y^2} = -f''(x+iy) - g''(x-iy)$$

$$\frac{\delta^2 z}{\delta y^2} = -[f''(x+iy) + g''(x-iy)]$$

Put value eq. (2)

$$\frac{\delta^2 z}{\delta y^2} = -\frac{\delta^2 z}{\delta x^2}$$

$$\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta y^2} = 0$$

$$r + s = 0$$

Hence this is required solution of eq. (1).

Q.3 from Partial diffrential equations fr.

(i) $f(x^2+y^2, z-xy)=0$

Solution:- Given eq.

$$f(x^2+y^2, z-xy)=0 \quad \text{---(1)}$$

we know that

$$f(u,v)=0 \quad \text{---(2)}$$

Compair eq. (1) and (2)

$$u = x^2+y^2 \quad \text{and} \quad v = z-xy$$

$$\therefore \frac{\delta u}{\delta x} = 2x, \quad \frac{\delta u}{\delta y} = 2y, \quad \frac{\delta u}{\delta z} = 0$$

and $$\frac{\delta v}{\delta x} = -y, \quad \frac{\delta v}{\delta y} = -x, \quad \frac{\delta v}{\delta z} = 1$$

we know that P.D.E of eq. (1) is

$$Pp + Qq = R \quad \text{-----(3)}$$

Now find value of P, Q and R.

where $$P = \begin{vmatrix} \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \end{vmatrix} = \begin{vmatrix} 2y & 0 \\ -x & 1 \end{vmatrix}$$

$$P = 2y \cdot 1 - (-x) \cdot 0 = 2y$$

$$Q = \begin{vmatrix} \frac{\delta u}{\delta z} & \frac{\delta u}{\delta x} \\ \frac{\delta v}{\delta z} & \frac{\delta v}{\delta x} \end{vmatrix} = \begin{vmatrix} 0 & 2x \\ 1 & -y \end{vmatrix} = 0 \cdot (-y) - 1 \cdot 2x$$

$$Q = -2x$$

$$R = \begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ -y & -x \end{vmatrix} = 2x(-x) - (2y)(-y)$$

$$R = -2x^2 + 2y^2 = 2(y^2 - x^2)$$

Now put values eq. (3)

$$Pp + Qq = R$$

$$2yp + (-2x)q = 2(y^2 - x^2)$$

$$\cancel{2}(yp - xq) = \cancel{2}(y^2 - x^2)$$

$$\boxed{yp - xq = y^2 - x}$$

this is required solution.

(ii) $f(x+y+z, x^2+y^2+z^2)$

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Topic-02 :- Linear Partial Diffrential equations of First Order.

(Legrange's Linear equations)

Given Lagrange's equation:

$$Pp + Qq = R \quad \text{------} \; (1)$$

find the value of P, Q, and R.

Then Complete Solution of eq. (1) is

$$f(u, v) = 0 \quad \text{or} \quad f(a, b) = 0 \quad \text{------} \; (2)$$

where f is an arbitrary function and

$$u = u(x, y, z) = a \quad \text{and}$$
$$v = v(x, y, z) = b \quad \text{------} \; (3)$$

The Lagrange's auxiliary equations (A.E.) of eq. (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{------} \; (4)$$

A.E. eq. (4) can be solved by 3 ways.

(i) **Method of Groping :-**

In this case, we take two numbers from the auxillary eq. (4)

Say $\frac{dx}{P} = \frac{dy}{Q}$ and find diffrential equation in x and y only.

This equation can be easily solved and we gett one solution.

Similarly, we take other two members, i.e.

$$\frac{dx}{P} = \frac{dz}{R} \quad \text{or} \quad \frac{dy}{Q} = \frac{dz}{R}$$

and find the second solution.

(ii) **Method of multipliers :-**

In this case, we use the multipliers l, m, n (Which are not always constant) and find

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

These multipliers can be so chosen that

$$lP + mQ + nR = 0$$

Then $l\,dx + n\,dy + n\,dz = 0$

∴ If $\frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} = k,$

then $l\,dx + m\,dy + n\,dz + k(lP + mQ + nR)$

Now If $lP + mQ + nR = 0$ then
$$l\,dx + m\,dy + n\,dz = 0$$

After integration, we get solution.

Again by using another set of multipliers l_1, m_1, n_1 we get another solution.

(iii) Combination of method (i) and (ii)

The complete solution or general Integral of ① is one of the following form

$$f(u, v) = 0 \quad \text{or} \quad u = f(v)$$

$$\text{or} \quad v = f(u).$$

Q.1 Solve $y^2zp + x^2zq = xy^2$.

Solutions: Given P.D.E. is

$$y^2zp + x^2zq = xy^2 \quad \text{---} (1)$$

we know that
$$Pp + Qq = R \quad \text{---} (2)$$

Comparing eq. 1 and 2

$P = y^2 z$, $Q = x^2 z$, $R = xy^2$

$\therefore$ Auxilary equartion are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\Rightarrow \frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

Now Taking first two members

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}} \Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2}$$

$$\Rightarrow x^2 dx = y^2 dy$$

$$= \frac{x^3}{3} = \frac{y^3}{3} + c_1 \Rightarrow x^3 - y^3 = 3c_1$$

$$\Rightarrow x^3 - y^3 = 3c_1 = a$$

again Taking first and last member

$$\frac{dx}{\cancel{y^2} z} = \frac{dz}{x\cancel{y^2}} \Rightarrow \frac{dx}{z} = \frac{dz}{x}$$

$$\Rightarrow x\,dx = z\,dz \Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow x^2 - z^2 = 2c_2 = b$$

Hence Complete solution of given equation is

$$f(a, b) = 0$$

$$\Rightarrow f(x^3 - y^3, x^2 - z^2) = 0$$

Q.2 Solve $y^2 p - xyq = x(z - 2y)$

Solution:- Hence $P = y^2$, $Q = -xy$

and $R = x(z - 2y)$

$\therefore$ Auxilary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

Taking first two members

$$\frac{dx}{y^2} = \frac{dy}{-xy} \Rightarrow \frac{dx}{y} = \frac{dy}{-x}$$

$$\Rightarrow -x\,dx = y\,dy$$

$$\Rightarrow x\,dx + y\,dy = 0$$

$$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c_1$$

$$\Rightarrow x^2 + y^2 = 2c_1 = a$$

Now taking last two members

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)} \Rightarrow \frac{dy}{-y} = \frac{dz}{z-2y}$$

$$\Rightarrow (z-2y)\,dy = -y\,dz$$

$$\Rightarrow \frac{dz}{dy} = -\frac{(z-2y)}{y}$$

$$= \frac{dz}{dy} = -\frac{z}{y} + \frac{2\cancel{y}}{\cancel{y}}$$

$$= \frac{dz}{dy} + \frac{1}{y}z = 2$$

which is linear diffrential equation in z

$$I.F = e^{\int \frac{1}{y}dy} = \cancel{e}^{\log y} = y$$

$$\boxed{I.F. = y}$$

$$z.(I.f) = \int Q\,.(If)\,dy + c_2$$

$$zy = \int 2y\,dy + c_2$$

$$\Rightarrow zy = \frac{\cancel{2}y^2}{\cancel{2}} + c_2$$

$$\Rightarrow zy - y^2 = c_2 = b$$

Hence Solution is $f(a, b) = 0$

$$f(x^2+y^2,\ zy - y^2) = 0$$

Q.3 Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

Solution:- Here $P = x^2(y-z)$

$Q = y^2(z-x)$ and $R = z^2(x-y)$

$\therefore$ A.E. $- \dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$ ---- (1)

(1) Using multipliers as

$l = 1/x^2, \quad m = \dfrac{1}{y^2}, \quad n = \dfrac{1}{z^2}$

we get $\dfrac{l\,dx + m\,dy + n\,dz}{l.P + mQ + nR}$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\frac{\cancel{x^2}(y-z)}{\cancel{x^2}} + \frac{\cancel{y^2}(z-x)}{\cancel{y^2}} + \frac{\cancel{z^2}(x-y)}{\cancel{z^2}}}$$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{\cancel{y} - \cancel{z} + \cancel{z} - \cancel{x} + \cancel{x} - \cancel{y}}$$

$$\Rightarrow \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

Hence A.E. are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$ ---- (2)

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Now taking last two members

$$\frac{dz}{z^2(x-y)} = \frac{\frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz}{0}$$

$$\Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0$$

Integration both side.

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c_1$$

$$\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = -c_1 = a$$

(ii) Again using multipliers as

$$l = \frac{1}{x}, \quad m = \frac{1}{y}, \quad n = \frac{1}{z}$$

we get

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{x^2(y-z)}{x} + \frac{y^2(z-x)}{y} + \frac{z^2(x-y)}{z}}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{zy}}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

Hence A.E. are

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}.$$

Now Taking Second and fourth mem

$$\frac{dy}{y^2(z-x)} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz = 0$$

Now Integration

$$\log x + \log y + \log z = \log c_2$$

$$\cancel{\log}(x.y.z) = \cancel{\log} c_2$$

$$\Rightarrow x.y.z = c_2 = b$$

Hence Complete Solution is

$$f(a, b) = 0$$

$$\Rightarrow f\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ x.y.z\right) = 0$$

Q.4 Solve $x^2p + y^2q = (x+y).z$

Solution:- Here $P = x^2$, $Q = y^2$

$$R = (x+y).z$$

Hence Auxilary equat Are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \quad --- (1)$$

Now taking first two members:

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Now integration

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$$\Rightarrow \quad \frac{1}{y} - \frac{1}{x} = c_1 = a$$

$$\Rightarrow \quad \frac{x-y}{xy} = c_1 = a$$

(II) Now using multipliers as

$$l = \frac{1}{x},\ m = \frac{1}{y},\ \text{and}\ n = -\frac{1}{z}$$

Hence $$\frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{\frac{x^2}{\cancel{x}} + \frac{y^2}{\cancel{y}} + \left\{\frac{-(x+y)\cancel{z}}{\cancel{z}}\right\}}$$

$$\Rightarrow \quad \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{x + y - (x+y)}$$

$$\Rightarrow \quad \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{0}$$

Hence Auxilary eq. are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{\frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz}{0} \quad ---(2)$$

Taking first and last members

$$\frac{dx}{x^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy \cancel{+} \frac{1}{z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0$$

Now Integration.

$$\log x + \log y - \log z = \log c_2$$

$$\cancel{\log}\left(\frac{x \cdot y}{z}\right) = \cancel{\log} c_2$$

$$\Rightarrow \frac{xy}{z} = c_2 = b$$

Hence complete solution is

$$f(a,b)=0 \Rightarrow f\left(\frac{x-y}{xy}, \frac{xy}{z}\right)=0$$

Q.5 $(y^2+z^2-x^2)p - 2xyq = -2zx$

Q.6 $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$

Hint divid $x \cdot y \cdot z$ both side.

Q.7 $(z^2-2yz-y^2)p + (xy+zx)q$

$= xy - zx$

Q.8 Solve $yzp + zxq = xy$

Topic - 03 :-

Non-Linear Partial Diffrential Equations of first order

[A] Standard Form I :- $f(p,q)=0$

Equation of the form $f(p,q)=0$ (i.e. equations containing p and q only)

Working Rule :-

The given equation is $f(p,q)=0$ ---(1)

The complete integral of eq.(1) is

$$z = ax + by + c \quad \text{----} (2)$$

where a and b connected by the relation

$$f(a,b)=0 \quad \text{----} (3)$$

Now from (2) $\frac{\delta z}{\delta x} = a \Rightarrow p = a$

and

$$\frac{\delta z}{\delta y} = b \Rightarrow q = b$$

Now putting the values of a and b in eq (3),

we get $f(p,q)=0$. Which is the same as eq (1) This implies that eq.(2) satisfied the given eq.(1) from eq.(3), we can find b in term of a.

Let $b = \phi(a)$, say

$\therefore$ Complete integral eq. (2) beomes

$$z = ax + \phi(a)y + c$$

Remarks:- Sometimes change of variables transforms the equation to standard form.

Q.1:- Solve $p^2 - q^2 = 25$

Solution:- The given equation is of the form I.

$$f(p,q) = 0 \quad \text{------ (1)}$$
$$p^2 - q^2 = 0$$

$\therefore$ It's complete integral is

$$z = ax + by + c \quad \text{---- (2)}$$

D.w.r.to x $\quad \frac{\delta z}{\delta x} = p = a$

D.w.r.to y $\quad \frac{\delta z}{\delta y} = q = b$

Now Put values eq. (1)

$$p^2 - q^2 = 25$$

$$a^2 - b^2 = 25$$

$$\boxed{b = \sqrt{a^2 - 25}}$$

Hence eq. (2)

$\therefore$ Complete integral of eq. (1)

is $z = ax + by + c$

$$\boxed{z = ax + (\sqrt{a^2 - 25})\, y + c}$$

Hence proved.

<u>Q.2:-</u> $x^2 p^2 + y^2 q^2 = z^2$

<u>Solution:-</u> Given P.D.E is

$$x^2 p^2 + y^2 q^2 = z^2 \quad \text{----(1)}$$

Given eq. (1) is not a standard form I but can be transformed to Standard for I by change variables as follows.

Putting $x = e^X$, $y = e^Y$, $z = e^Z$

So that $X = \log x$, $Y = \log y$, $Z = \log z$

Now $p = \frac{\delta z}{\delta x} = \frac{\delta z}{\delta X} \cdot \frac{\delta X}{\delta x} + \frac{\delta z}{\delta Y} \cdot \frac{\delta Y}{\delta x}$

$$= \frac{\delta z}{\delta X} \cdot \frac{1}{x} + \frac{\delta z}{\delta Y} \cdot 0$$

$$\Rightarrow \boxed{px = \frac{\delta z}{\delta X}}$$

Again $\frac{\delta Z}{\delta X} = \frac{\delta Z}{\delta z} \cdot \frac{\delta z}{\delta X}$

$$= \frac{\delta Z}{\delta z} \cdot (px)$$

$$= \frac{1}{z} \cdot px$$

$\Rightarrow$ $\boxed{px = z \cdot \frac{\delta Z}{\delta X}}$

Similarly $\boxed{q \cdot y = z \frac{\delta Z}{\delta Y}}$

Substituting above values px and qy in the eq. 1

$$x^2 p^2 + y^2 q^2 = z^2$$

$$z^2 \left(\frac{\delta Z}{\delta X}\right)^2 + z^2 \left(\frac{\delta Z}{\delta Y}\right)^2 = z^2$$

$$P^2 + Q^2 = 1 \quad ---- (2)$$

where $P = \frac{\delta Z}{\delta X}$ and $Q = \frac{\delta Z}{\delta Y}$

Now eq (2) is of standard form I therefore, it's complete integral is given by

$$Z = aX + bY + c_1 \quad --- (3)$$

Now D.w.r. to X and Y Repectively

$\frac{\delta Z}{\delta X} = a$ and $\frac{\delta Z}{\delta Y} = b$

so eq. 2 $a^2 + b^2 = 1$

$$\boxed{b = \sqrt{1-a^2}}$$

Putting value of b in eq. 3 the complete integral is given by

$$Z = aX + (\sqrt{1-a^2})Y + c_1$$

Put values X, Y and Z

$$\log z = a \cdot \log x + (\sqrt{1-a^2}) \log y + c_1$$

Now Putting $a = \cos\alpha$ and
$\sqrt{1-\cos^2\alpha} = \sin\alpha$
$c_1 = \log c$

we get

$$\Rightarrow \log z = \cos\alpha \cdot \log x + \sin\alpha \cdot \log y + \log c$$

$$\Rightarrow \log z = \log x^{\cos\alpha} + \log y^{\sin\alpha} + \log c$$

$$\Rightarrow \boxed{z = c \cdot x^{\cos\alpha} \cdot y^{\sin\alpha}}$$

Hence this is complete integral of eq. 1.

Standard form-II :- Equation involving only p, q and z i.e. equations of the form

$$f(p, q, z) = 0$$

Working Rule :-

Step-01 :- Given equation is of the form $f(p, q, z) = 0$ --- (1)

Step-02 :- Putting $p = \frac{\delta z}{\delta X}$ and $q = a\frac{\delta z}{\delta X}$ is eq. (1)

where $z = f(X)$ and $X = x + ay$.

Step-03 :- Solve the resulting ordinary diffrential equation b/w x and z.

Then result so obtained is the complete integral of eq. (1).

Q.1 :- Solve $z = p^2 + q^2$

Solution :- Given P.D.E $z = p^2 + q^2$ --- (1)

eq. (1) is of the form II i.e. $f(z, p, q) = 0$.

Putting $z = f(X)$ and $X = x + ay$

Then $\frac{\delta z}{\delta x} = p = \frac{\delta z}{\delta X} \cdot \frac{\delta X}{\delta x} = \frac{\delta z}{\delta X} \cdot 1$

So $p = \frac{\delta z}{\delta X}$

and $\frac{\delta z}{\delta y} = q = \frac{\delta z}{\delta X} \cdot \frac{\delta X}{\delta y} = \frac{\delta z}{\delta X} \cdot a$

So $q = a \cdot \frac{\delta z}{\delta X}$

Now put values eq. (1)

$$z = p^2 + q^2$$

$$z = \left(\frac{\delta z}{\delta X}\right)^2 + a^2\left(\frac{\delta z}{\delta X}\right)^2$$

$$z = (1+a^2)\left(\frac{\delta z}{\delta X}\right)^2$$

Now square root both side

$$\sqrt{z} = \sqrt{1+a^2} \cdot \frac{\delta z}{\delta X}$$

$$\delta X = \sqrt{1+a^2} \frac{\delta z}{\sqrt{z}}$$

Integration both side

$$X + b = \sqrt{1+a^2} \cdot 2\sqrt{z}$$

$$2\sqrt{z} = \frac{X+b}{\sqrt{1+a^2}}$$

squaring both side.

$$4z = \frac{(X+b)^2}{\sqrt{1+a^2}} \times$$

$$\boxed{4z(1+a^2) = (\cancel{x} + ay + b)^2}$$

this is required complete Integral of eq. 1.

Q.2 Find the complete integral of

$$z^2(p^2z^2+q^2) = 1$$

Solution:- Given

$$z^2(p^2z^2+q^2) = 1$$

This is of the form

$f(z, p, q) = 0$ i.e. standard form II

Putting $z = f(X)$ and $X = \cancel{x} + ay$

So that $p = \frac{\delta z}{\delta X}$ and $q = a \cdot \frac{\delta z}{\delta X}$

in eq. 1 we get

$$z^2\left[\left(\frac{\delta z}{\delta X}\right)^2 z^2 + a^2\left(\frac{\delta z}{\delta X}\right)^2\right] = 1$$

$$z^2(z^2+a^2)\left(\frac{\delta z}{\delta X}\right)^2 = 1$$

Now square root both side

$$\Rightarrow \quad z\sqrt{z^2+a^2}\,\frac{dz}{\delta X} = 1$$

$$z\sqrt{z^2+a^2}\,\delta z = \delta X$$

Now integrating both side.

$$\frac{1}{3}(z^2+a^2)^{3/2} = X+b$$

squaring both side

$$\frac{1}{9}(z^2+a^2)^3 = (X+b)^2$$

$$\boxed{(x+ay+b)^2 \cdot 9 = (z^2+a^2)^3}$$

this is required solution of eq. (1).

Q.3 $p(1+q) = qz$

Q.4 $p^3+q^3 = 27(z)$

[C] Standard form-III :-

$$f_1(x,p) = f_2(y,q)$$

Working Rule :-

Step-I :- Given $f_1(x,p) = f_2(y,q)$ ---(1)

Putting $f_1(x,p) = a$ and $f_2(y,q) = a$ ----(2)

where (a is a arbitrary constant

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Step-II :- Solving eq. (2) for p and q we get

$$p = \phi_1(x, a) \text{ and } q = \phi_2(y, a)$$

Step-III :- Complete integral is

$$dz = p\,dx + q\,dy$$

or

$$dz = \phi_1(x, a)\,dx + \phi_2(y, a)\,dy$$

$$\therefore z = \int \phi_1(x, a)\,dx + \int \phi_2(y, a)\,dy$$

Q.1 Solve $p^2 - q^2 = x - y$

Solution :- Given P.D.E is

$$p^2 - q^2 = x - y$$

$$p^2 - x = q^2 - y \quad \text{----} (1)$$

which is of the standard form III

i.e.

$$f_1(x, p) = f_2(y, q)$$

Here.

$$f_1(x, p) = p^2 - x$$

and $f_2(y, q) = q^2 - y$

$\Rightarrow$ $p^2 - x = a$ so $p^2 = a + x$

square root both side

$$p = \sqrt{x + a}$$

and

$\Rightarrow$ $q^2 - y = a$ so $q^2 = a + y$

square both side

$$q = \sqrt{y + a}$$

we know that complete integral

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

Now Integrating both side

$$z = \frac{(x+a)^{3/2}}{3/2} + \frac{(y+a)^{3/2}}{3/2} + b$$

$$\boxed{z = \frac{2}{3}\left[(x+a)^{3/2} + (y+b)^{3/2}\right] + b}$$

This is required solution of eq. (1).

Q.2 Solve $z^2(p^2+q^2) = x^2+y^2$

Solution:- Given P.D.E. is

$$z^2(p^2+q^2) = x^2+y^2$$

$$\left(z.\frac{\delta z}{\delta x}\right)^2 + \left(z.\frac{\delta z}{\delta y}\right)^2 = x^2+y^2 \quad \text{---- (1)}$$

Putting $z\,dz = dZ \Rightarrow$

$$\Rightarrow \boxed{\frac{z^2}{2} = Z}$$

Hence eq. (1). becomes

$$\left(\frac{\delta Z}{\delta x}\right)^2 + \left(\frac{\delta Z}{\delta y}\right)^2 = x^2+y^2$$

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① $\int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\log\{x+\sqrt{a^2+x^2}\}$

② $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\{x+\sqrt{x^2-a^2}\}$

$\Rightarrow P^2+Q^2 = x^2+y^2$ where $P = \frac{\delta Z}{\delta x}$

and $Q = \frac{\delta Z}{\delta y}$.

$\Rightarrow P^2 - x^2 = y^2 - Q^2$ ------ ②

Now eq.(2) is in standard form III

$f_1(P,x) = f_2(Q,y)$

Let $P^2 - x^2 = a \Rightarrow P^2 = a + x^2$.

so $\Rightarrow P = \sqrt{a+x^2}$

and $-Q^2+y^2 = a \Rightarrow Q^2 = y^2 - a$

so $\Rightarrow Q = \sqrt{y^2-a}$

$\therefore$ Complete integral is

$$dZ = \sqrt{a+x^2}\,dx + \sqrt{y^2-a}\,dy$$

Now integration both side we have.

$$Z = \frac{x}{2}\sqrt{a+x^2} + \frac{a}{2}\log\{x+\sqrt{a+x^2}\} + \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\log\{y+\sqrt{y^2-a}\} + b.$$

$$\Rightarrow \boxed{\frac{Z^2}{2} = \frac{x}{2}\sqrt{a+x^2} + \frac{a}{2}\log\{x+\sqrt{a+x^2}\} + \frac{y}{2}\sqrt{y^2-a} - \frac{a}{2}\log\{y+\sqrt{y^2-a}\} + b}$$

[D] Standard form IV :-

$$z = px + qy + f(p, q)$$

Working Rule :-

Step-I :- $z = px + qy + f(p, q)$ ---- (1)

Step-II :- Put $p = a$ and $q = b$ in eq. (1) we get complete solution or complete integral.

i.e. $z = ax + by + f(a, b)$ ---- (1)

Step-III :- Differentiating eq. (2) w.r.t. 'a' and 'b'

we get

$$\frac{\delta z}{\delta a} = 0 = x + \frac{\delta f}{\delta a} \quad \text{---- (3)}$$

and

$$\frac{\delta z}{\delta b} = 0 = y + \frac{\delta f}{\delta b} \quad \text{--- (4)}$$

The singular solution is obtained by eliminating a and b from eq. (2), (3) and (4).

Q.1 :- Find the complete and Singular solution of

$$z = px + qy + \log pq.$$

<u>Solution:-</u> Given $z = px + qy + \log p.q$ --- (1)

which is standard form IV from i.e.

$$z = px + qy + f(p, q).$$

Putting $p = a$ and $q = b$

$\therefore$ Complete integral is

$$z = ax + by + \log \cancel{a.b}\; ab \quad \text{--- (2)}$$

Now we find <u>Singular Solution:-</u>

Then Diffrentiating partially w.r.to a and b, eq.(2)

we get. $\frac{\delta z}{\delta a} = 0 = x + \frac{1}{a}$,

and $\frac{\delta z}{\delta b} = 0 = y + \frac{1}{b}$

$\Rightarrow \quad x + \frac{1}{a} = 0 \quad \Rightarrow \quad \boxed{a = -\frac{1}{x}}$

$\Rightarrow \quad y + \frac{1}{b} = 0 \quad \Rightarrow \quad \boxed{b = -\frac{1}{y}}$

Putting in (2), then Singular Solution

$$z = ax + by + \log a.b$$

$$z = -1 - 1 + \log\left(\frac{1}{xy}\right) \qquad \left[\because \log(xy)^{-1} = -\log(xy) \right.$$

$$\Rightarrow \boxed{z = -2 - \log(xy)}$$

Q.2 Find the complete and singular Solution of $z = px + qy + p^2 + q^2$

Solution:- Given $z = px + qy + p^2 + q^2$ --- ①

eq.① is in standard form IV.

then Put $p = a$ and $q = b$ in eq.①

∴ Complete integral is

$$z = ax + by + a^2 + b^2 \quad \text{---} ②$$

Now we find Singular Solution.

Then D.E.② partially D.w.r. to a and b

we have

$$\frac{\delta z}{\delta a} = 0 = x + 2a \Rightarrow \boxed{a = -\frac{x}{2}}$$

$$\frac{\delta z}{\delta b} = 0 = y + 2b \Rightarrow \boxed{b = -\frac{y}{2}}$$

Putting values eq.②

~~z=~~

$$z = ax + by + a^2 + b^2$$

$$z = \left(-\frac{x}{2}\right)x + \left(-\frac{y}{2}\right)y + \left(-\frac{x}{2}\right)^2 + \left(-\frac{y}{2}\right)^2$$

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = \frac{-2x^2 - 2y^2 + x^2 + y^2}{4}$$

$$4z = -x^2 - y^2$$

$$\boxed{x^2 + y^2 + 4z = 0}$$

Hence this is singular Solution.

Topic 04:- CHARPIT'S METHOD

This is general method for finding the complete solution of a non-linear partial diffrential equation of first Order.

Let $f(x, y, z, p, q) = 0$ ---- (1)

Working Rule :-

Step-01:- Transfer all the term of given P.D.E. (1) to L.H.S. and denote entire expression in L.H.S. by f.

and find the value of $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}$ $\frac{\delta f}{\delta p}$ and $\frac{\delta f}{\delta q}$ using eq. 1

Step-02:- Write down the charpit's axiliary equations.

$$\frac{dp}{\frac{\delta f}{\delta x} + p.\frac{\delta f}{\delta z}} = \frac{dq}{\frac{\delta f}{\delta y} + q.\frac{\delta f}{\delta z}} = \frac{dz}{-p\frac{\delta f}{\delta p} - q\frac{\delta f}{\delta q}}$$

$$= \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}} = \frac{dF}{0} \quad \text{---} (2)$$

Step-03:- Solve charpit auxilary eq. 2 with 1 and find the Simplest relation ~~and value of p and q.~~ involving at least one of p and q.

Step-04:- The simplest relation of step-03 is solved along with the given equation to find p and q. Put these values of p and q in the complete integral.

Step-05:- Find Complete integral by $\boxed{dz = p\,dx + q\,dy}$ ---(3)

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Step-06:- Now integrating eq. (3) and the find ~~value~~ ~~of~~ ~~z~~. complete solution.

Q.1:- Solve $(p^2+q^2)y = qz$ by using charpit's mathod.

Solution:- Now given $(p^2+q^2)y = qz$

Let $f = (x, y, z, p, q) = (p^2+q^2)y - qz = 0$ ---- (1)

Now D.w.r. to x, y, z, p, q respectively

$$\frac{\delta f}{\delta x} = 0, \quad \frac{\delta f}{\delta y} = p^2 + q^2, \quad \frac{\delta f}{\delta z} = -q$$

and $$\frac{\delta f}{\delta p} = 2yp, \quad \frac{\delta f}{\delta q} = 2yq - z$$

∴ then we know that charpit's auxilary equations are.

$$\frac{dp}{\frac{\delta f}{\delta x} + p\frac{\delta f}{\delta z}} = \frac{dq}{\frac{\delta f}{\delta y} + q\frac{\delta f}{\delta z}} = \frac{dz}{-p\frac{\delta f}{\delta p} - q\frac{\delta f}{\delta q}}$$

$$= \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}}$$

Put values. in auxilary equations.

we get

$$\frac{dp}{-pq} = \frac{dq}{p^2+\cancel{q^2}-\cancel{q^2}} = \frac{dz}{-2yp^2-q(2yq-z)}$$

$$= \frac{dx}{-2yp} = \frac{dy}{-(2yq-z)}$$

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2yp^2-2yq^2+zq}$$

$$= \frac{dx}{-2yp} = \frac{dy}{z-2yq} \quad \text{----} (2)$$

Now taking the first two member of eq. (2)

we get $\frac{dp}{-pq} = \frac{dq}{p^2}$

$$\frac{dp}{-q} = \frac{dq}{p}$$

$$p.dp = -q\,dq \Rightarrow p.dp+q\,dq=0$$

Now Integrating both side.

$$\Rightarrow \frac{p^2}{2}+\frac{q^2}{2} = \frac{a^2}{2}$$

where a is arbitrary constaints

$$\boxed{p^2+q^2=a^2} \quad \text{----} (3)$$

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Now Put value eq. 1

$$(p^2+q^2)y - qz = 0$$

$$\boxed{a^2 y = qz} \quad \text{---} (4)$$

So

$$\boxed{q = \frac{a^2 y}{z}}$$

then palue q put in eq. 3

$$p^2 + q^2 = a^2$$

$$p^2 + \left(\frac{a^2 y}{z}\right)^2 = a^2$$

$$p^2 = a^2 - \frac{a^4 y^2}{z^2}$$

$$p^2 = \frac{z^2 a^2 - a^4 . y^2}{z^2}$$

$$p^2 = \frac{a^2}{z^2}(z^2 - a^2 y^2)$$

Square root both sige.

$$\boxed{p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}}$$

∴ Complete integral.

$$dz = p\,dx + q\,dy$$

Put values P and q

$$dz = \frac{a}{z}\left(\sqrt{z^2-a^2y^2}\right)dx + \frac{a^2y}{z}dy.$$

$$z.dz = a\left(\sqrt{z^2-a^2y^2}\right)dx + a^2y\,dy$$

$$z\,dz - a^2y\,dy = a\left(\sqrt{z^2-a^2y^2}\right)dx$$

$$\Rightarrow \frac{z.dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx.$$

by Sepration of variables.

Now Integrating both side.

Now $z^2 - a^2y^2 = t$

$$2z\,dz - 2a^2y\,dy = dt$$

$$2\left(z\,dz - a^2y\,dy\right) = dt.$$

$$\Rightarrow \int \frac{dt}{\sqrt{t}\cdot 2} = \int a\,dx$$

$$\Rightarrow \frac{\cancel{2}\sqrt{t}}{\cancel{2}} = ax + b$$

b is arbitrary constaints.

$$\Rightarrow \boxed{\sqrt{z^2-a^2y^2} = ax+b}$$

both side squring.

$$\boxed{z^2 - a^2y^2 = (ax+b)^2}$$

Ans

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Q.2 Solve $z = px + qy + p^2 + q^2$ by Charpit's method.

Solution:- Let $f = px + qy + p^2 + q^2 - z = 0$ ---(1)

Now D.w.r.to x, y, z, p, q respectively

$$\frac{\delta f}{\delta x} = p, \quad \frac{\delta f}{\delta y} = q, \quad \frac{\delta f}{\delta z} = -1$$

$$\frac{\delta f}{\delta p} = x + 2p, \quad \frac{\delta f}{\delta q} = y + 2q$$

$\therefore$ Charpit's auxiliary equations are.

$$\frac{dp}{\frac{\delta f}{\delta x} + p\frac{\delta f}{\delta z}} = \frac{dq}{\frac{\delta f}{\delta y} + q\cdot\frac{\delta f}{\delta z}} = \frac{dz}{-p\frac{\delta f}{\delta p} - q\frac{\delta f}{\delta q}}$$

$$= \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}}$$

Put. values:.

$$\frac{dp}{p-p} = \frac{dq}{q-q} = \frac{dz}{-p(x+2p) - q(y+2q)}$$

$$= \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)}$$

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$$\Rightarrow \frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px-2p^2-qy-2q^2}$$

$$= \frac{dx}{-x-2p} = \frac{dy}{-y-2q} \quad \text{---} (2)$$

Now taking first and fourth mem in eq. (2)

$$\frac{dp}{0} = \frac{dx}{-x-2p}$$

$\Rightarrow \quad dp = 0$

Now Integrating

$$\boxed{p = a} \quad \text{---} (2) \quad \{a \text{ is a A. C.}\}$$

Now taking second and fifth meml in eq. (2)

$$\frac{dq}{0} = \frac{dy}{-y-2q}$$

$\Rightarrow \quad dq = 0$

Now Integrating.

$$\boxed{q = b} \quad \text{---} (3)$$

∴ Complete integral.

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + b\,dy$$

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Now Integrating.

$$z = ax + by + c$$

$$\Rightarrow \boxed{z = ax + by + a^2 + b^2}$$

where $c = a^2 + b^2$ are arbitrary constants.

Hence this is required complete integral.

Q.3 Find the complete integral of the equation.

$$2(z + xp + yq) = yp^2$$

Solution:- $\Rightarrow$ Let $f = 2(z + xp + yq) - yp^2 = 0$ ---- (1)

Now D.w.r. to x, y, z, p, q Partially eq. (1)

$$\frac{\delta f}{\delta x} = 2p, \quad \frac{\delta f}{\delta y} = 2q - p^2, \quad \frac{\delta f}{\delta z} = 2$$

$$\frac{\delta f}{\delta p} = 2x - 2yp, \quad \frac{\delta f}{\delta q} = 2y$$

Now Charpit's auxilliary equation is

$$\frac{dp}{\frac{\delta f}{\delta x} + p\frac{\delta f}{\delta z}} = \frac{dq}{\frac{\delta f}{\delta y} + q\frac{\delta f}{\delta z}} = \frac{dz}{-p\frac{\delta f}{\delta p} - q\frac{\delta f}{\delta q}} = \frac{dx}{-\frac{\delta f}{\delta p}} = \frac{dy}{-\frac{\delta f}{\delta q}}$$

Put values

$$\frac{dp}{2p+p.2} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p(2x-2yp)-2qy}$$

$$\frac{dx}{-(2x-2yp)} = \frac{dy}{-2y} \quad ----(2)$$

Now taking first and fifth mem. of eq. (2)

$$\frac{dp}{2\cancel{+}p} = \frac{dy}{-\cancel{2}y}$$

$$\Rightarrow \frac{dp}{p} = 2\frac{dy}{-y}$$

$$\Rightarrow \frac{dp}{p} + 2\frac{dy}{y} = 0$$

Now integrating both side.

$$\log p + 2\log y = \log a$$

$$\log p + \log y^2 = \log a$$

$$\cancel{\log}\, p.y^2 = \cancel{\log}\, a$$

$$py^2 = a$$

$$\boxed{p = \frac{a}{y^2}} \quad ----(3)$$

Now Put $p = \frac{a}{y^2}$ in eq. (1)

$$2(z + xp + yq) - yp^2 = 0$$

$$2z + 2x\frac{a}{y^2} + 2yq - y\cdot\left(\frac{a}{y^2}\right)^2 = 0$$

$$2z + \frac{2ax}{y^2} + 2yq - \frac{a^2}{y^3} = 0$$

$$2yq = \frac{a^2}{y^3} - 2z - \frac{2ax}{y^2}$$

then

$$\Rightarrow \boxed{q = \frac{a^2}{2y^4} - \frac{z}{y} - \frac{ax}{y^3}}$$

∴ Complete integral is

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{y^2}dx + \left(\frac{a^2}{2y^4} - \frac{z}{y} - \frac{ax}{y^3}\right)dy$$

$$dz = \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy - \frac{z}{y}dy$$

$$dz + \frac{z}{y}dy = \frac{a}{y^2}dx - \frac{ax}{y^3}dy + \frac{a^2}{2y^4}dy$$

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multiple y both Side.

$$y\,dz + z\,dy = \frac{a}{y}dx - \frac{ax}{y^2}dy + \frac{a^2}{2y^3}dy$$

$$y\,dz + z\,dy = a\left(\frac{dx}{y} - \frac{x}{y^2}dy\right) + \frac{a^2}{2y^3}dy$$

$$y\,dz + z\,dy = a\left(\frac{y\,dx - x\,dy}{y^2}\right) + \frac{a^2}{2y^3}dy$$

$$\Rightarrow d(yz) = a\,d\left(\frac{x}{y}\right) + \frac{a^2}{2y^3}dy$$

Now integration.

$$yz = a\frac{x}{y} + \frac{a^2}{-(2)(2)(y^2)} + b$$

then $$\boxed{z = \frac{ax}{y^2} - \frac{a^2}{4y^3} + \frac{b}{y}}$$

Hence this is required Complete integral.

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Topic-05 :- Homogeneous Linear partial Diffrential Equation with constaint Coeficient.

Consider n^{th} order linear P.D.E. :

$$A_0 \frac{\delta^n z}{\delta x^n} + A_1 \frac{\delta^n z}{\delta x^{n-1} \delta y} + A_2 \frac{\delta^n z}{\delta x^{n-2} \delta y^2} + \cdots + A_n \frac{\delta^n z}{\delta y^n} = f(x,y) \quad \text{----} (1)$$

using $D = \frac{\delta}{\delta x}$ and $D' = \frac{\delta}{\delta y}$ we get

$$(A_0 D^n + A_1 D^{n-1} D' + A_2 D^{n-2} D'^2 + \cdots + A_n D'^n) z = f(x,y)$$

i.e.

$$f(D, D') z = f(x,y) \quad \text{----} (2)$$

This is called Homogenous linear Partial Diffrential Equation.

i.e. The complete solution of eq (1) is

$$\boxed{z = C.F. + P.I.} \quad \text{----} (3)$$

The solution of $f(D, D') = 0$ is called complementry function (C.F.) of eq (2).

A Particular Integral (P.I.)

of eq. (2) which is represet by the symbol.

$$P.I. = \frac{1}{f(D, D')} f(x, y)$$

<u>finding Complementry function (C.F.)</u>

The equation by putting

$D = m$, and $D' = 1$ in $f(D, D') = 0$ is called the auxiliary eq. of eq (2)

i.e. $(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \cdots + A_n) = 0$

let $m_1, m_2 \cdots m_n$ be the roots of eq. (2)

Then, the following cases may arise

<u>Case-I</u> :- Now when roots are real and equal.

Suppose $m_1 = m_2 = m_3 = \cdots = m_r = m$

Then $C.F. = \phi_1(y+mx) + x.\phi_2(y+mx) + x^2\phi_3(y+mx) + \cdots + x^{r-1}\phi_r(y+mx)$

where $\phi_1, \phi_2, \phi_3 \cdots \phi_r$ are arbitrary functions.

Imaginary | irrational $\alpha \pm \sqrt{\beta}$

104 $m_1 = \alpha + \sqrt{\beta}$, $m_2 = \alpha - \sqrt{\beta}$

$\alpha \pm i\beta$ then $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$

Case-II :- When roots are real and distinct.

$m_1, m_2, m_3 \dots, m_r$

Then $C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \dots + \phi_r(y + m_r x)$

Finding Particular Integral (P.I.)

Rule 01 :- $\frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$.

If $f(a,b) \neq 0$.

i.e. Putting $D = a$, and $D' = b$.

Rule-02 :=> $\frac{1}{f(D,D')} e^{ax+by} = \frac{1}{f(a,b)} e^{ax+by}$

If $f(a,b) = 0$, Put $D = a$, and $D = b$

then $\frac{1}{f(D,D')} e^{ax+by} = \frac{x}{\frac{\delta}{\delta D}\{f(D,D')\}} e^{ax+by}$

after case and Put value $D = a$, $D' = b$.

again fail.

so

$$\frac{1}{f(D,D')} e^{ax+by} = \frac{x^2}{\frac{\delta}{\delta D} \cdot \frac{\delta}{\delta D}\{f(D,D')\}} e^{ax+by}$$

Put value $D = a$, and $D' = b$

Rule-03 :-

$$\frac{1}{f(D^2, D\cdot D', D'^2)} \sin(ax+by) = \frac{1}{f\{-(a)^2, -(a\cdot b), -(b)^2\}} \sin(ax+by)$$

Putting $D^2 = -(a)^2$, $D'^2 = -(b)^2$

and $D\cdot D' = -(a\cdot b)$

If $f(-a^2, -a\cdot b, -b^2) \neq 0$

Rule-04 :-

$$\frac{1}{f(D^2, D\cdot D', D'^2)} \cos(ax+by) = \frac{1}{f\{-(a)^2, -(a\cdot b), -(b)^2\}} \cos(ax+by)$$

Putting $D^2 = -(a)^2$, $D'^2 = -(b)^2$, $D\cdot D' = -(ab)$

If $f\{-(a)^2, -(a\cdot b), -(b)^2\} \neq 0$.

Rule-05 :-

$$\frac{1}{F(D^2, D'^2)} \{\sin ax \cdot \sin by\} = \frac{1}{F\{-(a)^2, -(b)^2\}} \sin ax \cdot \sin by$$

If $f\{-(a)^2, -(b)^2\} \neq 0$

Put $D = -(a)^2$, $D' = -(b)^2$

Rule-06 :-

$$\frac{1}{f(D^2, D'^2)} (\cos ax \cdot \cos by) = \frac{1}{f\{-(a)^2, -(b)^2\}} \{\cos ax \cdot \cos by\}$$

If $f\{-(a)^2, -(b)^2\} \neq 0$

Putting $D^2 = -(a)^2$, $D'^2 = -(b)^2$

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Rule-07:-

$$\frac{1}{f(D,D')}x^r y^s = [F(D,D')]^{-1}.x^r.y^s$$

which calculated by expanding $[F(D,D')]^{-1}$ in powers of D and by using Binomial expansion.

Rule-08:-

$$\frac{1}{f(D,D')}e^{ax+by}.V(x,y) = e^{ax+by}.\left\{\frac{1}{f(D+a,D'+b)}V(x,y)\right.$$

Putting $D \to D+a$, $D' \to D'+b$

Rule-09:- General formula.

$$\frac{1}{(D-mD')}f(x,y) = \int f(x, c-mx)dx$$

Putting $y = c-mx$ where c is arbitrary constaint

where after integration, put $c \to y+mx$

Only apply easily linear factors of $f(D,D')$.

Q.1:- Finding the ~~particular integral~~ complete solutions of the P.D.E.

$$\frac{\delta^3 z}{\delta x^3} - 7\frac{\delta^3 z}{\delta x.\delta y^2} - 6\frac{\delta^3 z}{\delta y^3} = \sin(x+2y)$$

Solution:- Now Given P.D.E.

$$\frac{\delta^3 z}{\delta x^3} - 7\frac{\delta^3 z}{\delta x.\delta y^2} - 6\frac{\delta^3 z}{\delta y^3} = \sin(x+2y)$$

Put $D = \frac{\delta}{\delta x}$ and $D' = \frac{\delta}{\delta y}$

$$\{D^3 - 7D.D'^2 - 6D'^3\}z = \sin(x+2y) \quad \text{-----(1)}$$

we know that

$$f(D,D')z = f(x,y) \quad \text{---(2)}$$

Compair eq. (1) and (2)

$$f(D,D') = D^3 - 7D.D'^2 - 6D'^3$$

$$f(x,y) = \sin(x+2y)$$

Hence

Complete solution of eq.(1)

$$\boxed{z = C.F. + P.I} \quad \text{-----(3)}$$

firstly we find

C.F.:- Put $D = m$, $D' = 1$ and $f(x,y) = 0$ in eq.(2)

represent $f(D,D') = 0$

So

$$D^3 - 7D.D'^2 - 6D'^3 = 0$$

$$m^3 - 7m - 6 = 0$$

$$(m+1)(m^2 - m - 6) = 0$$

$$(m+1)(m+2)(m-3) = 0$$

$$m = 1, -2, 3$$

Then roots real and Distinct

$$m_1 = 1, \ m_2 = -2, \ m_3 = 3$$

So.

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \\ + \phi_3(y + m_3 x)$$

$$\boxed{C.F. = \phi_1(y + x) + \phi_2(y - 2x) + \phi_3(y + 3x)}$$

Now we find P.I.

$$P.I. = \frac{1}{f(D, D')} . f(x, y)$$

So

$$P.I. = \frac{1}{\{D^3 - 7D.D'^2 - 6D'^3\}} \sin(x + 2y)$$

Now apply Rule (3).

$$D^2 = -a^2, \ D.D' = -(a.b), \ D'^2 = -(b)^2$$

Then P.I. $= \dfrac{1}{D^2 \cdot D - 7DD'^2 - 6D'^2 \cdot D'} \sin(x+2y)$

$a = 1$, and $b = 2$

P.I. $= \dfrac{1}{-(1)^2 \cdot D - 7D\{-(2)^2\} - 6\{-(2)^2\}D'} \sin(x+2y)$

$= \dfrac{1}{-D + 28D + 24D'} \sin(x+2y)$

$= \dfrac{1}{27D + 24D'} \sin(x+2y)$

$= \dfrac{(27D - 24D')\sin(x+2y)}{(27D + 24D')(27D - 24D')}$

$= \dfrac{27D\sin(x+2y) - 24D'\sin(x+2y)}{729D^2 - 576D'^2}$

again Put $D^2 = -(a)^2 = -(1)^2$, $D'^2 = -(b)^2 = -(2)^2$

$= \dfrac{27\frac{\delta}{\delta x}\sin(x+2y) - 24\frac{\delta}{\delta y}\sin(x+2y)}{729\{-(1)^2\} - 576\{-(2)^2\}}$

$= \dfrac{27\cos(x+2y)\cdot 1 - 24\cos(x+2y)\cdot 2}{-729 + 2304}$

$$\boxed{\text{P.I.} = \frac{1}{1575}\left[27\cos(x+2y) - 48\cos(x+2y)\right]}$$

Put C.F and P.I. values eq. (3)

$\therefore$ $z = C.F. + P.I.$

$$z = \phi_1(y+x) + \phi_2(y-2x) + \phi_3(y+3x) +$$
$$+ \frac{1}{1575}\left[27\cos(x+2y) - 48\cos(x+2y)\right]$$

Hence this is required complete Solutions.

Q.2 Solve $(D^2 - DD')z = \sin x . \cos 2y$

Solution:- ~~Solve~~ Given P.D.E.

$$(D^2 - DD')z = \sin x . \cos 2y \quad \text{----}(1)$$

we know that.

$$f(D, D')z = f(x, y) \quad \text{----}(2)$$

compair eq.(1) and (2)

$f(D, D') = D^2 - DD'$, $\quad f(x, y) = \sin x . \cos 2y$

Then complete solution of eq.(1) is

$$\boxed{z = C.F + P.I.} \quad \text{---}(3)$$

we find C.F. $\Rightarrow$ Put value $D = m$ and $D' = 1$ in $f(D, D') = 0$

$$D^2 - D.D' = 0$$

$$m^2 - m.1 = 0$$

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$m(m-1) = 0$

then $m = 0, 1$

- then roots are real and distinct.

$m_1 = 0, \quad m_2 = 1$

then.

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$

$$= \phi_1(y + 0x) + \phi_2(y + 1x)$$

$$\boxed{C.F. = \phi_1(y) + \phi_2(y + x)}$$

Now we find P.I.

$$P.I = \frac{1}{f(D, D')} \cdot f(x, y)$$

$$= \frac{1}{D^2 - DD'} \sin x \cdot \cos 2y$$

we know that

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$P.I. = \frac{1}{2\{D^2 - DD'\}} \sin(x + 2y) + \sin(x - 2y)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'} \sin(x + 2y) + \frac{1}{D^2 - DD'} \sin(x - 2y)\right]$$

apply Rule - 03

$\sin(x + 2y)$	$\sin(x - 2y)$
$a = 1, \; a = 2$	$a = 1, \; b = -2$
$D^2 = -(a)^2 = -(1)^2 = -1$	$D^2 = -(a)^2 = -(1)^2 = -1$
$D.D' = -(a \cdot b) = -(1 \cdot 2) = -2$	$D.D' = -(a \cdot b) = -\{1 \cdot (-2)\} = 2$

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then

$$P.I. = \frac{1}{2}\left[\frac{1}{-1-(-2)}\sin(x+2y) + \frac{1}{-1-2}\sin(x-2y)\right]$$

$$= \frac{1}{2}\left[\sin(x+2y) + \frac{1}{-3}\sin(x-2y)\right]$$

$$\boxed{P.I. = \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)}$$

Now C.F. and P.I Put values eq. (3)

$$Z = C.F. + P.I.$$

$$\boxed{Z = \phi_1(y) + \phi_2(y+x) + \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)}$$

Q.3:- Solve $\frac{\delta^2 z}{\delta x^2} - 4\frac{\delta^2 z}{\delta x \cdot \delta y} + 4\frac{\delta^2 z}{\delta y^2} = e^{2x+y}$

Solution:- Given P.D.E can be written as

$$(D^2 - 4DD' + 4D'^2)z = e^{2x+y} \quad \text{---(1)}$$

C.F:

it A.E. is $m^2 - 4m + 4 = 0$ $\left\{\begin{matrix}\text{Put} \\ D = m \\ D' = 1\end{matrix}\right\}$

$(m-2)^2 = 2, 2$

when roots are real and equal.

$$C.F. = \phi_1(y+m_1x) + x\,\phi_2(y+m_1x)$$

$$\boxed{C.F. = \phi_1(y+2x) + x\,\phi_2(y+2x)}$$

we find P.I $= \frac{1}{f(D,D')} \cdot f(x,y)$

$= \frac{1}{D^2 - 4DD' + 4D'^2} e^{2x+y}$

$e^{ax+y} = e^{2x+y}$ $a = 2$, and $b = 1$
then $D = 2$, and $D' = 1$
when $= \frac{1}{D^2 - 4DD' + 4D'^2} = \frac{1}{4 - 4.2.1 + 4} = \frac{1}{0}$
failled.

Then apply Rule-02

$$P.I = \frac{x}{\frac{\delta}{\delta D}(D^2 - 4DD' + 4D'^2)} e^{2x+y}$$

$$= \frac{x}{2D - 4D'} e^{2x+y}$$

Put $D = 2$ and $D' = 1$ $\frac{1}{2D - 4D'} = \frac{1}{2.2 - 4} = \frac{1}{0}$
again failled

Now again Apply Rule-02

$$P.I. = \frac{x}{2D - 4D'} e^{2x+y}$$

$$= \frac{x^2}{\frac{\delta}{\delta D}(2D - 4D')} e^{2x+y}$$

$$\boxed{P.I. = \frac{x^2}{2} e^{2x+y}}$$

Hence complete solution is

$z = C.F. + P.I.$

$$\boxed{z = \phi_1(y+2x) + x\,\phi_2(y+2x) + \frac{x^2}{2}e^{2x+y}}$$

Q.4:- Solve $(D^2 + 3DD' + 2D'^2)z = 24xy$

Solution:- Given $(D^2 + 3DD' + 2D'^2)z = 24x.y$ ---- (1)

For C.F. It's A.E is

Put $D = m, D' = 1$

$$m^2 + 3m + 2 = 0$$

$$(m+1)(m+2) = 0 \qquad m = -1, -2$$

when roots are real and distinct.

$$m_1 = -1, \quad m_2 = -2$$

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$

$$\boxed{C.F. = \phi_1(y - x) + \phi_2(y - 2x)}$$

Now we find $P.I. = \frac{1}{f(D,D')} . f(x.y)$

$$P.I. = \frac{1}{(D^2 + 3DD' + 2D'^2)} 24x.y.$$

Now Apply Rule-07.

$$\frac{1}{f(D,D')} . x^r y^s = [f(D,D')]^{-1} . x^r y^s$$

expendend $[f(D,D')]^{-1}$ in Bionomial form.

$$P.I = \frac{1}{D^2\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]}\, 24x.y.$$

$$= \frac{1}{D^2}\left[1+\frac{3D'}{D}+\frac{2D'^2}{D^2}\right]^{-1} \cdot 24x.y.$$

$$\left[(1+x)^{-1} = 1-x+x^2-x^3+\cdots\right]$$

$$= \frac{1}{D^2}\left[1-\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)+\left(\frac{3D'}{D}+\frac{2D'^2}{D^2}\right)^2+\cdots\right] 24x.y$$

$$= \frac{1}{D^2}\left[24xy - \frac{3D'}{D}24x.y + \frac{2D'^2}{D^2}24.xy + \cdots\right]$$

$$= \frac{1}{D^2}\left[24xy - \frac{3\frac{\partial}{\partial y}24xy}{D} + \frac{2\frac{\partial^2}{\partial y^2}24xy}{D^2} + \cdots\right]$$

$$= \frac{1}{D^2}\left[24xy - \frac{3.24x}{D}\right]$$

$$= \frac{1}{D^2}\left[24xy - \int 3.24x\, dx\right]$$

$$= \frac{1}{D^2}\left[24xy - 3.\overset{12}{\cancel{24}}.\frac{x^2}{\cancel{2}}\right]$$

$$= \frac{1}{D^2}\left[24xy - 36x^2\right]$$

$$= \frac{1}{D}\left[\int 24.xy\, dx - \int 36x^2 dx\right]$$

$$P.I. = \frac{1}{D}\left[\cancel{24}^{12} y.\frac{x^2}{\cancel{2}} - \cancel{36}^{12}.\frac{x^3}{\cancel{3}}\right]$$

$$= \frac{1}{D}\left[12yx^2 - 12x^3\right]$$

$$= \int 12yx^2 dx - \int 12x^3 dx$$

$$= \cancel{12}^{4} y.\frac{x^3}{\cancel{3}} - \cancel{12}^{3}.\frac{x^4}{\cancel{4}}$$

$$\boxed{P.I. = 4x^3y - 3x^4}$$

Hence complete solution

$$z = C.F. + P.I.$$

$$\boxed{z = \phi_1(y-x) + \phi_2(y-2x) + 4x^3y - 3x^4}$$

Q.5:- Solve $\frac{\delta^2 z}{\delta x^2} - 2\frac{\delta^2 z}{\delta x . \delta y} + \frac{\delta^2 z}{\delta y^2} = e^{x+2y} + x^3$

Solution: Given P.D.E is

$$\frac{\delta^2 z}{\delta x^2} - 2\frac{\delta^2 z}{\delta x . \delta y} + \frac{\delta^2 z}{\delta y^2} = e^{x+2y} + x^3$$

$$D^2 - 2DD' + D'^2 = e^{x+2y} + x^3 \quad \text{---} (1)$$

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for C.F.: it's A.E. $m^2 - 2m + 1 = 0$
Put $D = m, D' = 1$

$$(m-1)^2 = 0$$

$m = 1, 1$ when roots are equal

$$C.F. = \phi_1(y+mx) + x\phi_2(y+mx)$$

$$\boxed{C.F = \phi_1(y+x) + x\phi_2(y+x)}$$

Now we find $P.I. = \frac{1}{f(D, D')} f(x,y)$

$$P.I. = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} + x^3$$

$$= \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y} + \frac{1}{D^2 - 2DD' + D'^2} x^3$$

Apply Rule-01 and Rule-07

$e^{x+2y} = e^{ax+by}$ | expend Binomial form

$a=1, b=2 \quad D=1, D'=2$

$$P.I. = \frac{1}{(1)^2 - 2(1)(2) + (2)^2} e^{x+2y} + \frac{1}{D^2}\left[1 - \frac{2D'}{D} + \frac{D'^2}{D^2}\right]^{-1} x^3$$

$$= \frac{1}{1-4+4} e^{x+2y} + \frac{1}{D^2}\left[1 + \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right) + \left(\frac{2D'}{D} + \frac{D'^2}{D^2}\right)^2 - - - -\right] x^3$$

$$= e^{x+2y} + \frac{1}{D^2}\left[x^3 + \frac{2D'}{D}x^3 + \frac{D'^2}{D^2}x^3 + \dots\right]$$

$$= e^{x+2y} + \frac{1}{D^2}\left[x^3 + 0 + 0 - - -\right]$$

$$P.I. = e^{x+2y} + \frac{1}{D}\int x^3\,dx$$

$$= e^{x+2y} + \int \frac{x^4}{4}\,dx$$

$$= e^{x+2y} + \frac{x^5}{4\cdot 5}$$

$$\boxed{P.I. = e^{x+2y} + \frac{x^5}{20}}$$

Hence complete solution

$$Z = C.F. + P.I.$$

$$\boxed{Z = \phi_1(y-x) + x\,\phi_2(y-x) + e^{x+2y} + \frac{x^5}{20}}$$

Solve (6):- $\frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta x\cdot\delta y} + 6\frac{\delta^2 z}{\delta y^2} = y\cdot\cos x$

Solution:- Given P.D.E. $D^2 + DD' - 6D'^2 = y\cos x$ ---(1)

For C.F.: it's A.E. is $m^2 + m - 6 = 0$

$D = m,\ D' = 1$ $\quad (m+3)(m-2) = 0$

$$m = -3,\ 2$$

When roots are real and distict

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$$

$$\boxed{C.F. = \phi_1(y - 3x) + \phi_2(y + 2x)}$$

we find P.I. $= \frac{1}{f(D,D')} \cdot f(x,y)$

$$P.I. = \frac{1}{D^2+DD'-6D'^2} y\cdot\cos x$$

$$= \frac{1}{(D+3D')(D-2D')} y\cdot\cos x$$

Now Apply General Rule

$$P.I. = \frac{1}{(D+3D')}\left[\frac{1}{(D-2D')} y\cdot\cos x\right]$$

$$= \frac{1}{(D+3D')}\left[\frac{1}{D-2D'} y\cdot\cos x\right]$$

{Since $D-2D' = D-mD'$, $m=2$
Put $y = c-mx = c-2x$
where c is arbitrary constant
after Int. then $c = y+mx = y+2x$ }

$$= \frac{1}{(D+3D')}\left[\int (c-mx)\cos x\,dx\right]$$

$$= \frac{1}{D+3D'}\left[\int \underset{\mathrm{I}}{\underline{(c-2x)}}\ \underset{\mathrm{II}}{\underline{\cos x}}\,dx\right]$$

$$P.I = \frac{1}{(D+3D')}\left[(c-2x)\sin x - \int \frac{\delta}{\delta x}(c-2x)\cdot\int \cos x \, dx\right]$$

$$= \frac{1}{(D+3D')}\left[(c-2x)\sin x - \int (-2)\sin x \, dx\right]$$

$$= \frac{1}{D+3D'}\left[(c\cancel{\sin x}-2x)\sin x - 2\cos x\right]$$

$$= \frac{1}{D+3D'}\left[y\sin x - 2\cos x\right]$$

Again Put General rule.

$$\left\{ \begin{array}{l} D+3D' = D - mD' \quad \text{so } m = -3 \\ \text{put } y = c - mx = c + 3x \end{array} \right\}$$

$$= \int \underbrace{(c+3x)}_{I}\underbrace{\sin x}_{II} - 2\cos x \, dx$$

$$= (c+3x)(-\cos x) - \int \frac{\delta}{\delta x}(c+3x)\int \sin x \, dx - \int 2\cos x \, dx$$

$$= -(c+3x)\cos x - \int 3(\cancel{\cos x})dx - 2\sin x$$

$$= -y\cos x + 3\sin x - 2\sin x$$

$$\boxed{P.I. = \sin x - y\cos x}$$

$$z = C.F + P.I.$$

$$\boxed{z = \phi_1(y+3x) + \phi_2(y+2x) + \sin x - y\cos x}$$

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Q.7: Solve. $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Solution:- Given $(D^2 - DD' - 2D'^2)z = (y-1)e^x$

for C.F. it's A.E. is

$D = m, D' = 1$

$$m^2 - m - 1 = 0$$
$$(m-2)(m+1) = 0$$
$$m = 2, -1$$

$$C.F. = \phi_1(y + m_1 x) + \phi_2(y + m_2 x))$$

$$\boxed{C.F = \phi_1(y+2x) + \phi_2(y-x)}$$

Now we find $P.I. = \frac{1}{f(D, D')} \cdot f(x, y)$

$$P.I. = \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x$$

$$= \frac{1}{(D+D')(D-2D')}(y-1)e^x$$

$$= \frac{1}{D+D'}\left[\frac{1}{D-2D'}(y-1)e^x\right]$$

Now Apply General Rule.

$D - mD' = D - 2D'$, $m = 2$

$y = c - mx$

Put $y = c - 2x$

$$= \frac{1}{D+D'}\int \underbrace{(c-2x-1)}_{I}\underbrace{e^x}_{II}\,dx$$

$$= \frac{1}{D+D'}\left[(c-2x-1)\int e^x dx - \int \frac{\delta}{\delta x}(c-2x-1)\int e^x dx\right.$$

$$P.I = \frac{1}{D+D'}\left[(c-2x-1)e^x - \int(-2)\,e^x\,dx\right]$$

$$= \frac{1}{D+D'}\left[(y-1)e^x + 2e^x\right]$$

again Apply General Rule

$D - mD' = D + D'$, $m = -1$, $y = c - mx$

Put. $y = c + x$

$$P.I. = \int \underset{I}{(c+x-1)}\,\underset{II}{e^x}\,dx + 2\int e^x\,dx$$

$$= (c+x-1)e^x - \int 1\cdot e^x\,dx + 2e^x$$

$$= (y-1)e^x - e^x + 2e^x$$

$$= (y-1)e^x + e^x$$

$$= e^x(y-\cancel{1}+\cancel{1})$$

$$\boxed{P.I. = e^x \cdot y}$$

Hence complete solution is

$$z = C.F. + P.I.$$

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-x) + e^x \cdot y}$$