(2-4)

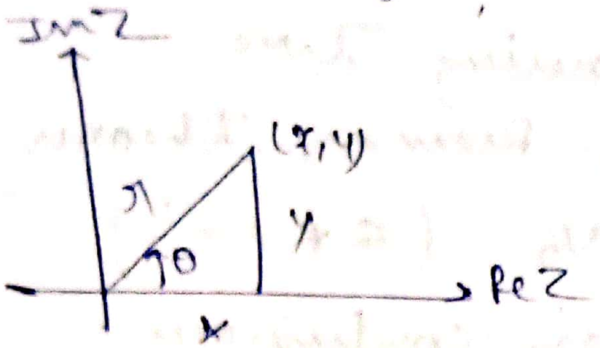
Complex Variable

$z = x + iy$

$z = r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$



$r = |z| = \sqrt{x^2+y^2}$

$\theta = \arg z = \tan^{-1}\left(\frac{y}{x}\right)$

$\cos\theta = \frac{x}{r}$ | $\sin\theta = \frac{y}{r}$

$x = r\cos\theta$ | $y = r\sin\theta$

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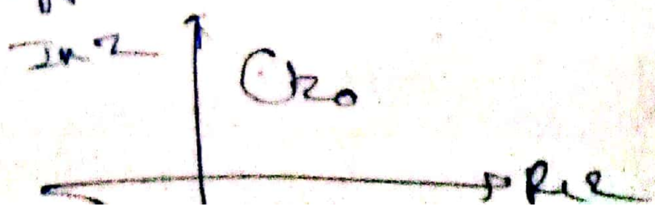
(P) If two complex z_1 and z_2 lies on the right half plain or both lies on imaginary axis then

(1) $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

(2) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

Analytic :→ A function $f(z)$ is said to be analytic at a point z_0 if it is ~~called~~ not only differentiable at z_0 but also differentiable in the neighbourhood of z_0



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(D) The complex function $f(z) = U(x,y) + i\,V(x,y)$ is said to be differentiable at a point z_0 if it satisfies the following two conditions of Cauchy-Riemann theorem

(1) $U_x = V_y$, $U_y = -V_x$ (C-R eq^n)

(2) U_x, U_y, V_x and V_y are continuous function of x and y (function which is analytic everywhere called entire fun^n)

Ex $f(z) = z^2$:-

$U + iV = (x+iy)^2$

$U + iV = x^2 - y^2 + i(2xy)$

$U = x^2 - y^2$, $V = 2xy$

$U_x = 2x$, $U_y = -2y$, $V_x = 2y$, $V_y = 2x$

clearly U_x, U_y, V_x and V_y are continuous function of x and y

$U_x = V_y$, $U_y = -V_x$ (C R eq^n) satisfied for all values of z $\therefore$ $f(z) = z^2$ is analytic every where

Eg - $f(z) = \bar{z}$

$U + iV = x - iy$

$U = x$, $V = -y$

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$\quad , \quad V_x = 0, \quad V_y = -1$

$(U_x \neq V_y)$ CR eqⁿ is not satisfied at any value of z.

$\therefore f(z)$ is nowhere analytic.

* If $f(z) = U(x,y) + Vi(x,y)$ is analytic then $U(x,y)$ and $V(x,y)$ are orthogonal to each other and U and V are harmonic functions of x and y

i.e $\quad U_{xx} + U_{yy} = 0$

$\quad V_{xx} + V_{yy} = 0$

Ex. $f(z) = z^2$

$U + iV = (x+iy)^2 = x^2 - y^2 + i(2xy)$

$U = x^2 - y^2, \quad V = 2xy$

$$m_1 = \frac{dy}{dx} = \frac{-\partial U/\partial x}{\partial U/\partial y} = \frac{-2x}{-2y} = \frac{x}{y}$$

$$m_2 = \frac{dy}{dx} = \frac{-V_x}{V_y} = \frac{-2y}{2x} = \frac{-y}{x}$$

$$m_1 m_2 = \left(\frac{x}{y}\right) \times \left(\frac{-y}{x}\right) = -1$$

$U_x = 2x, \quad U_y = -2y,$ | $V_x = 2y, \quad V_y = 2x$

$U_{xx} = 2, \quad U_{yy} = -2$ | $V_{xx} = 0 \quad V_{yy} = 0$

$U_{xx} + U_{yy} = 0$ | $V_{xx} + V_{yy} = 0$

U is harmonic | (V is harmonic)

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6 If $f(z) = u + iv$ is analytic and $u = \sin x \cosh y$ then $f(z) =$

$\Rightarrow$ $u_x = \cos x \cdot \cosh y$ $\quad u_y = \sin x \sinh y$

we know that

$$f'(z) = u_x + i v_x$$
$$= u_x - i u_y$$

$f'(z) = \cos x \cdot \cosh y - i \sin x \sinh y$

Put z ~~and~~ $y = 0$ in $f'(z)$

$f'(z) = \cos z$

$\int f'(z)\, dz = \int \cos z\, dz$

$\boxed{f(z) = \sin z}$

extension find v. $\Rightarrow$

$f(z) = \sin z$

$u + iv = \sin(x + iy)$

$u + iv = \sin(x) \cos iy + \cos x \sin iy$

$$\boxed{\begin{aligned} \cos i\theta &= \cosh \theta \\ \sin i\theta &= i \sinh \theta \end{aligned}}$$

$u + iv = \sin x \cosh y + i \cos x \sinh y$

$u = \sin x \cosh y$

$v = \cos x \sinh y$

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④ When imaginary part of analytic function $v(x,y)$ is given

① find V_x, V_y $\quad (U_x = V_y)$

② we know that $f'(z) = U_x + iV_x$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad \downarrow$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad V_y + iV_x$

③ put $x = z$, $y = 0$ in $f'(z)$ and then integrate w.r.t. z to get $f(z)$

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Q If $f(z) = U + iV$ is analytic and $V = 3x^2y - y^3$ then $f(z) =$

$V_x = 6xy$, $\quad V_y = 3x^2 - 3y^2$

we know that $f'(z) = U_x + iV_x$
$= V_y + iV_x$

$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = 3x^2 - 3y^2$

$U = x^3 - 3y^2x + c$ (function of y)

$f'(z) = (3x^2 - 3y^2) + i\,6xy$

put $x = z$, $y = 0$ in $f'(z)$

$f'(z) = 3z^2$

$\int f'(z)\,dz = \int 3z^2\,dz$

$\boxed{f(z) = z^3}$

$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

$\Rightarrow \frac{\partial U}{\partial y} = -6yx + \frac{dc(y)}{dy}$

$-6yx = -6xy$

$\frac{dc(y)}{dy} - 6yx = -6xy$

$\frac{dc(y)}{dy} = 0$

$dc(y) = 0\,dy$

$c = \text{const}$

$U = x^3 - 3y^2x + c$

$U + iV = (x+iy)^3$
$= x^3 + i^3y^3 + 3x(iy)^2 + 3x^2 iy$
$= x^3 - iy^3 - 3xy^2 + 3x^2 iy$
$\boxed{x^3 - 3xy^2 + i(3x^2y - y^3)}$

WB-1

$\ln\sqrt{i} = \ln(i)^{1/2}$

$\ln(0 + i)^{1/2}$

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$= \ln(\cos\frac{\pi}{2} + i\sin\pi/2)^{1/2}$

$\ln[(e^{i\pi/2})^{1/2}]$

$= \ln e^{i\pi/4} = \frac{i\pi}{4}$

Gate

$i^i = (0+i)^i$

$= (\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})^i$

$= (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2}$

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WB2 $U = x^2 - y^2 + xy \to U_x = 2x + y$

$(U_x = V_y)$

(a) $x^2 - y^2 - xy \to V_y = -2y - 1$

(b) $x^2 + y^2 + xy \to V_y = 2y + x$

(c) $2xy + \frac{1}{2}(y^2 - x^2) \to V_y = 2x + y$ ✓

(d) $\frac{xy}{2} + 2(y^2 - x^2) \to V_y = \frac{x}{2} + 4y$

---

wb3 $f(z) = z^2$ maps first quadrant onto ——

$u + iV = (x + iy)^2 = x^2 - y^2 + 2xyi$

$-+$	$++$
$--$	$+-$

If z lies in first quadrant. $(x>0, y>0$

imaginary part of $f(z) = z^2$

[illegible]

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$\therefore$ ($2xy > 0$) lie on the upper half plane.

$$((\omega^2)^3) = 1$$

Q.5 $\quad y^3 = 1 \qquad \omega^3 = 1$

$$(x-1)^3 = -8 \qquad (x-1)^3 = -8 \qquad (x-1)^3 = (-2\omega^2)^3$$

$$(x-1)^3 = (-2)^3 \qquad (x-1)^3 = (-2\omega)^3 \qquad \boxed{x = 1 - 2\omega^2}$$

$$x - 1 = -2 \qquad x - 1 = -2\omega$$

$$\boxed{x = -1} \qquad x = 1 - 2\omega$$

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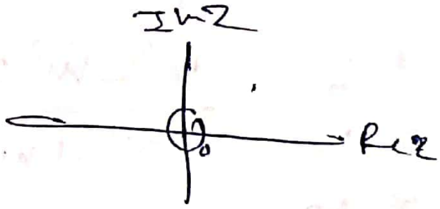
(WB-6) $f(z) = x^2 + iy^2$ at $z = 0$

$$u + iv = x^2 + iy^2$$



$$u = x^2, \quad v = y^2$$

$$u_x = 2x, \quad v_y = 2y$$

$$u_x = 2x, \quad u_y = 0, \quad v_x = 0, \quad v_y = 2y$$

Clearly u_x, u_y, v_x, v_y are continuous.

CR eq^n are satisfied at $z = 0$ but not in its neighbourhood

$\therefore$ $f(z)$ is ($x^2 + y^2$) not analytic at $z = 0$

$\Rightarrow$ similarly $g(z)$ is also not analytic at $z = 0$

(7) $\displaystyle\lim_{z \to 0} \frac{\bar{z}}{z}$

$$\lim_{x+iy \to 0} \frac{x - iy}{x + iy}$$

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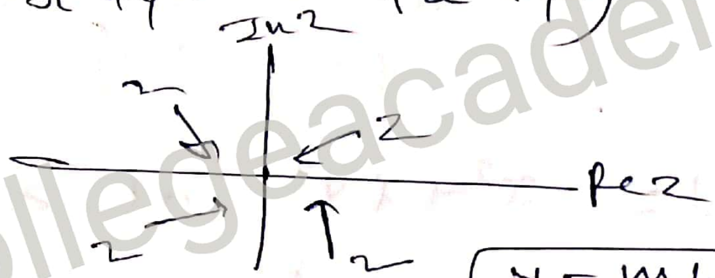
$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{(x-iy)(x-iy)}{(x+iy)(x-iy)}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2-2xyi}{x^2+y^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2-y^2}{x^2+y^2} - i\left(\frac{2xy}{x^2+y^2}\right)$$



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$$\boxed{y = mx}$$

$$\lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2} - i\left(\frac{2x \cdot mx}{x^2+m^2x^2}\right)$$

$$\lim_{x\to 0} \left(\frac{1-m^2}{1+m^2}\right) - i\left(\frac{2m}{1+m^2}\right)$$

slope	path	limit
$m=1$	$y=x$	$l=-1$
$m=2$	$y=2x$	$l=-\frac{3}{5}-i\frac{4}{5}$

} different

$\therefore \lim_{z\to 0} \frac{\bar{z}}{z}$ does not exist

(8) (c) $u = x^2+y^2$

$u_x = 2x, \quad u_y = 2y$

$u_{xx} = 2, \quad u_{yy} = 2$

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$U_{xx} + U_{yy} = 4 \neq 0$

$\therefore$ U is not harmonic

(9) $\int \bar{z}\, dz$ where c is $z = t^2 + it$

from $z = 0$, to $z = 4+2i$

$(0,0)$ $(4,2)$

$x + iy = t^2 + it$ — y varies

$x = t^2,\ y = t$ — t varie from (0 to 2)

$$= \int_0^2 (t^2 - it)(2t + i)\, dt$$

$$= \int_0^2 (2t^3 + it^2 - i2t^2 + t)\, dt$$

sin iθ = i sinh θ

$$= \int_0^2 (2t^3 + t)\, dt - i\int_0^2 t^2\, dt$$

$\ln(x+iy) = \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$

$$\left(2\frac{t^4}{4} + \frac{t^2}{2}\right)\Bigg|_0^2 - i\left(\frac{t^3}{3}\right)\Bigg|_0^2$$

$$(8+2) - i\left(8/3\right) = 10 - \frac{8i}{3}$$

Taylor series :- If $f(z)$ is analytic at a point z_0

then $f(z)$ can be expressed as

$$f(z) = f(z_0) + (z-z_0)f'(z_0) + \frac{(z-z_0)^2}{2!}f''(z_0) + \frac{(z-z_0)^3}{3!}f'''(z_0) + \cdots$$

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The T.S.E of $f(z) = \sin z$ at $z = \frac{\pi}{4}$ is

$f(z) = \sin z$, $f'(z) = \cos z$, $f''(z) = -\sin z$

$f(\pi/4) = \frac{1}{\sqrt{2}}$, $f'(\pi/4) = \frac{1}{\sqrt{2}}$, $f''(\pi/4) = -\frac{1}{\sqrt{2}}$

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0)$$

$$= \frac{1}{\sqrt{2}} + \left(z - \frac{\pi}{4}\right)\frac{1}{\sqrt{2}} + \frac{\left(z - \frac{\pi}{4}\right)^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{\left(z - \frac{\pi}{4}\right)^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \cdots$$

The T.S.E of $f(z) = \frac{1}{z+4}$, at $z = 2$

⊛ IF algebraic function given apply binomial expansion and for other function apply formula.

let $z - 2 = t$, $(z - 2 = 0)$

$z = t + 2$,

$$f = \frac{1}{t+4+2} = \frac{1}{t+6} = \frac{1}{6}\left(1 + \frac{t}{6}\right)^{-1}$$

$$= \frac{1}{6}\left[1 - \frac{t}{6} + \left(\frac{t}{6}\right)^2 + \left(\frac{t}{6}\right)^3 + \cdots\right]$$

$$= \frac{1}{6}\left[1 - \left(\frac{z-2}{6}\right) + \left(\frac{z-2}{6}\right)^2 - \left(\frac{z-2}{6}\right)^3 + \cdots\right]$$

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The laurent series :- of $f(z) = \frac{1}{(z+1)(z+3)}$

in the valid region $(1 < |z| < 3)$

$\Rightarrow \quad (1 < |z| < 3)$

$1 < |z|$ and $|z| < 3$ — maintain always less than one

$\frac{1}{|z|} < 1, \quad \frac{|z|}{3} < 1$

$$f(z) = \frac{1/2}{z+1} - \frac{1/2}{z+3}$$

$$= \frac{1}{2} \times \frac{1}{(z+1)} - \frac{1}{2(3+z)}$$

$$= \frac{1}{2 \cdot z\left(1+\frac{1}{z}\right)} - \frac{1}{2 \times 3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 \cdots\right)$$

$\Rightarrow$ laurent series expansion of $f(z) = \frac{1}{(z+1)(z+3)}$ in the valid region $|z| > 3$

$\Rightarrow \quad |z| > 3$

$3 < |z|$

$\frac{3}{|z|} < 1, \quad \frac{1}{|z|} < 1$

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$f(z) = \frac{1}{(z+1)(z+3)}$ in the valid region $|z| > 3$

$\Rightarrow f(z) = \frac{1/2}{z+1} - \frac{1/2}{z+3}$

$= \frac{1}{2z(1+\frac{1}{z})} - \frac{1}{2z(1+\frac{3}{z})}$

$= \frac{1}{2z}(1+\frac{1}{z})^{-1} - \frac{1}{2z}(1+\frac{3}{z})^{-1}$

$= \frac{1}{2z}(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots) - \frac{1}{2z}(1 - \frac{3}{z} + (\frac{3}{z})^2 + \cdots)$

In laurent series $\Rightarrow$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}}_{\text{principal part}}$$

If there are no terms in the principal part of $f(z)$ then z_0 is called removal singular point.

Eg. $f(z) = \frac{z - \sin z}{z^3} = \frac{z - (z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots)}{z^3}$

$= \frac{z - z + \frac{z^3}{3!} - \frac{z^5}{5!} - \cdots}{z^3}$

$\frac{1}{3!} - \frac{z^2}{5!} + \cdots = \quad z = 0$ is removal

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* IF there are infinite numbers of terms in the principal part of $f(z)$ then z_0 is called essential singular.

Eg. $f(z) = e^{1/z} = 1 + \underbrace{\frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!} + \cdots}_{\text{infinite no. of negative power of } z}$

$z = 0$ is essential singular

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* IF there are finite no. of terms in the principal part of $f(z)$ then z_0 is called pole.

eg. $f(z) = \frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z^4}$

$= \frac{1}{z^3} - \frac{1/z}{3!} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$

finite no. of negative power of z

$z = 0$ is a pole.

Residue :→ In laurent series expansion of $f(z)$ the coefficient of $\frac{1}{z - z_0}$ is called residue of $f(z)$ at $z = z_0$

Ex $f(z) = \frac{z+1}{(z-2)(z-4)}$

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$$\frac{-3/2}{z-2} + \frac{5/2}{z-4}$$

$Res\, f(z)$ at $z=2$ $Res\, f(z)$ at $z=4$

Ex $f(z) = e^{1/z} = 1 + \frac{1/z}{1!} + \frac{1/z^2}{2!} + \frac{1/z^3}{3!}$

$$\operatorname*{Res}_{z=0} e^{1/z} = \text{coe of } \frac{1}{z-0} = 1$$

Ex $f(z) = \frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!\,z} + \frac{z}{5!} - \frac{z^3}{7!}$

$$\operatorname*{Res}_{z=0} f(z) \quad \text{coef of } \frac{1}{z} = -\frac{1}{3!} = -\frac{1}{6}$$

(*) If $z=a$ is a simple pole (Power 1) of $f(z)$ then $\operatorname*{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$

e.g. $f(z) = \frac{z+1}{(z-2)^1(z-4)^1}$

$$\operatorname*{Res}_{z=2} f(z) = \lim_{z\to 2} \cancel{(z-2)} \frac{(z+1)}{\cancel{(z-2)}(z-4)} = -3/2$$

$$\operatorname*{Res}_{z=4} f(z) = \lim_{z\to 4} \cancel{(z-4)} \frac{(z+1)}{(z-2)\cancel{(z-4)}} = 5/2$$

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* If $z = a$ is a pole of order m then

$$\operatorname*{Res}_{z=a} f(z) = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-a)^m f(z) \right)$$

eg. = $f(z) = \dfrac{e^{3z}}{(z-1)^5}$

$$\operatorname*{Res}_{z=1} f(z) = \lim_{z \to 1} \frac{1}{4!} \frac{d^4}{dz^4} \left(\cancel{(z-1)^5} \frac{e^{3z}}{\cancel{(z-1)^5}} \right)$$

11/03/17 $= \dfrac{1}{24} \times 81 e^{3z} = \dfrac{81 e^{3}}{24}$

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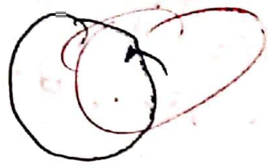
Cauchys integral theorem :->

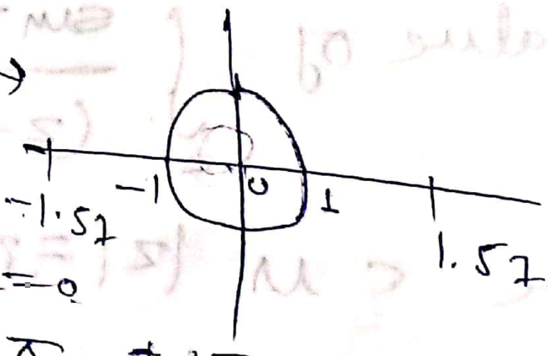
If $f(z)$ is analytic and $f'(z)$ is continuous with in and on the boundary of a simple closed curve c then

$$\oint_c f(z)\,dz = 0$$



$\Rightarrow$ The value of $\oint_c \sec z\,dz$ where c is $|z| = 1$

$= \displaystyle\int \frac{1}{\cos z}\,dz$



singular points $\cos z = 0$

are $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2} \ldots$

$2n + (-1)^n$

1.57

so here all the singular points are outside to the region c so it is analytic and continuous at all points inside the curve

so. $\oint_C \sec z \, dz = 0$

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Cauchy's integral formula :-

If $f(z)$ is analytic with in and on the boundary of a simple closed curve c and a is any point inside c

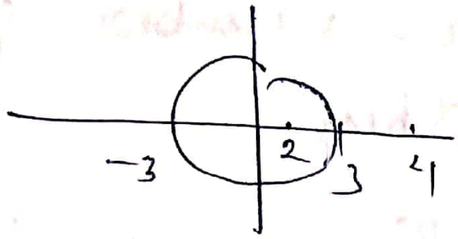
then $\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$ (anticlock wise)

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{n}(a)$$

where $f^{n}(a) = \frac{d^n}{dz^n}\left[f(z)\right]_{z=a}$

Q The value of $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-4)} dz$

where c is $|z| = 3$



$$= \int \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{z-4}}{z-2}\, dz$$

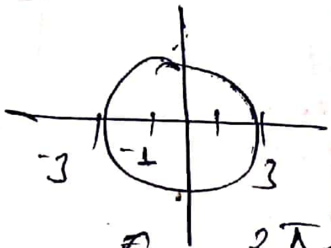
↓

$$2\pi i\, f(2) = 2\pi i \left(\frac{\sin 4\pi + \cos 4\pi}{2-4}\right)$$

$$= \left(\frac{0+1}{-2}\right) 2\pi i = -\pi i$$

(c) The value of $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$ where

C is $|z| = 3$



$$\oint \frac{f(z)}{(z-a)^n}\, dz = \oint \frac{2\pi i}{n!} f^n(a)$$

$$= \frac{2\pi i}{3!} f'''(-1) = \frac{8\pi i}{3!} e^{-2}$$

$$\boxed{f''' = 8e^{2z}}$$

Cauchy's Residue theorem :->

If $f(z)$ is analytic at all points inside and on the boundry of a simple closed curve C.

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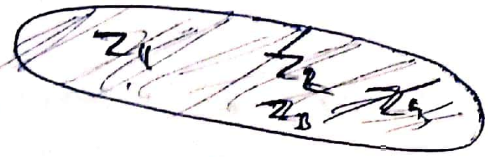
except at finite number of poles lies inside c then

$$\oint_C f(z)\,dz = 2\pi i\,[\text{sum of residues of } f(z) \text{ at all its poles lies inside } C]$$



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6. The value of $\int_C \frac{1}{z^2+4}\,dz$ where

C is $|z-i|=2$

$|x+iy-i|=2$

$\sqrt{[x^2+i(y-1)]^2}=2$

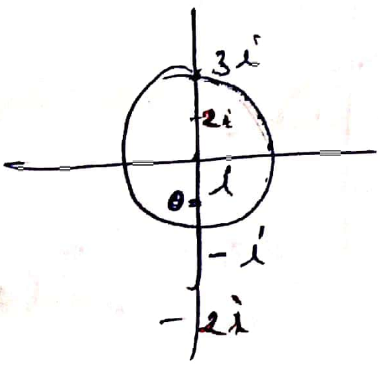
$x^2+(y-1)^2=2^2$

$x^2+(y-1)^2=4$

$h=0,\ k=1$

$r=2$

$\Rightarrow$



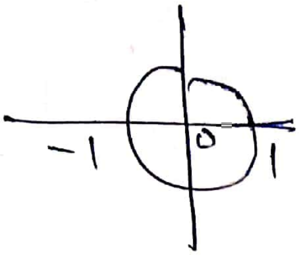
singular points

$z=\pm 2i$

$$\operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i}(z-2i)\frac{1}{(z-2i)(z+2i)} = \frac{1}{4i}$$

$$\int \frac{1}{z^2+4}\,dz = 2\pi i\left(\frac{1}{4i}\right) = \pi/2$$

Q The value of $\int_C e^{1/z}\,dz$ where C is $|z|=1$



singular point = 0 which is inside the curve

$\text{Res } e^{1/z}$ at $z=0$ = coeff of $\frac{1}{z}$ in $\left\{1+\frac{1/z}{1!}+\frac{1/z^2}{2!}\right\}$

$= \frac{1}{1!}$

by C-R-T

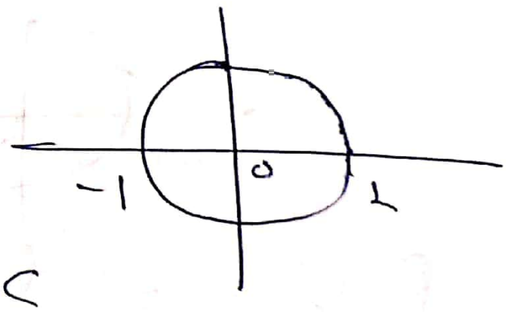
$$\int_C e^{1/z}\,dz = 2\pi i(1)$$

IES-2017 The value of $\int_C \frac{1}{z\sin z}\,dz$ where C is $|z|=1$

Singular points are $z\sin z = 0$

$z = 0$, $\pm n\pi$

$z=0$ inside C, $\pm n\pi$ outside C



$$\frac{1}{z\sin z} = \frac{1}{z\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right)} = \frac{1}{z^2\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}\right)}$$

$$= \frac{1}{z^2\left[1-\left(\frac{z^2}{3!}-\frac{z^4}{5!}+\dots\right)\right]} = \frac{1}{z^2}\left[1-\left\{\frac{z^2}{3!}-\frac{z^4}{5!}-\dots\right\}\right]^{-1}$$

$(1-t)^{-1} = 1+t+t^2+\dots$

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$$= \frac{1}{z^2}\left\{1+\left(\frac{z^2}{3!}-\frac{z^4}{5!}\cdots\right)+\left(\frac{z^2}{3!}-\frac{z^4}{5!}\cdots\right)\right\}$$

$$= \frac{1}{z^2}\left\{1+\frac{z^2}{3!}-\frac{z^4}{5!}\cdots\right\} = \frac{1}{z^2}+\frac{1}{3!}-\frac{z^2}{5!}$$

$$\operatorname{Res}_{z=0} f(z) = \text{coeff of } \frac{1}{z}$$
$$= 0$$

by CRT

$$\int_C \frac{1}{z\sin z}\,dz = 2\pi i(0) = 0$$

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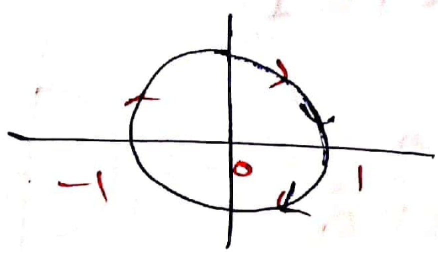
(13) $\int_C \frac{1}{z^2}\,dz$ where C is $|z|=1$

$z=0$ is the pole of order 2 which lies inside C



$$\operatorname{Res}_{z=0} f(z) = \lim_{z\to 0} \frac{1}{1!}\frac{d}{dz}\, z^2 \times \frac{1}{z^2} = 0$$

By CRT

clockwise

$$\int \frac{1}{z^2}\,dz = 2\pi i(0) = 0$$

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P-32

(T-5)

$$f(z) = \frac{1}{1+z^{10}} \qquad \beta = e^{i\pi/10}$$

$$\underset{z=e^{i\pi/10}}{\mathrm{Res}} = \lim_{z \to e^{i\pi/10}} \left(z - e^{i\pi/10}\right) \times \frac{1}{1+z^{10}}$$

$$= \left(\frac{0}{0}\right)$$

$$= \lim_{z \to e^{\pi i/10}} = \frac{1}{10 z^9}$$

$$= \frac{1}{10 e^{\frac{19\pi}{10}}} = \frac{1}{10} e^{-\frac{9\pi}{10} i}$$

$$= \frac{1}{10}\left(\cos\frac{9\pi}{10} - i \sin\frac{9\pi}{10}\right)$$

$$= \frac{1}{10}\left[\cos\left(\pi - \frac{\pi}{10}\right) - i \sin\left(\pi - \frac{\pi}{10}\right)\right]$$

$$= \frac{1}{10}\left(-\cos\frac{\pi}{10} - i \sin\frac{\pi}{10}\right)$$

$$= -\frac{1}{10}\left[\cos\frac{\pi}{10} + i \sin\frac{\pi}{10}\right)$$

$$= -\frac{1}{10} e^{\frac{\pi}{10} i} = -\frac{1}{10} \beta.$$

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