UNIT-4

Electrostatics in Vacuum

Gradient of scalar field

It represents maximum rate of change in the scalar function. It is a vector quantity.

let $\phi(x,y,z)$ is a scalar function then its gradient is represented by

$$\text{grad}\,\phi = \vec{\nabla}\phi$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Q Find out the gradient of $\phi = 3x^2 + 2y + z^2$ at the points $(1,3,4)$

$$\vec{\nabla}\phi = \hat{i}\frac{\partial(3x^2+2y+z^2)}{\partial x} + \hat{j}\frac{\partial(3x^2+2y+z^2)}{\partial y} + \hat{k}\frac{\partial(3x^2+2y+z^2)}{\partial z}$$

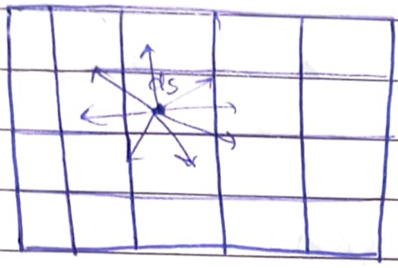
$$\vec{\nabla}\phi = \hat{i}\,6x + 2\hat{j} + 2z\hat{k}$$

$$6(1)\hat{i} + 2\hat{j} + 2(4)\hat{k}$$

$$= 6\hat{i} + 2\hat{j} + 8\hat{k}$$

Electric flux

Electric lines of forces passing normally to the surface ds is called electric flux.



$$\phi = E\,ds\cos\theta$$

Divergence of a vector field

It represents the amount of flux per unit volume through a point in the vector field. It is a scalar quantity.

Let vector A is a vector function and it is given by –

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Then divergence of vector A is represented by,

$$\text{Divergence of } \vec{A} = \vec{\nabla}\cdot\vec{A}$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{\nabla}\vec{A} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\left[\hat{i}\cdot\hat{i} = 1\right]$$

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Line integral $\rightarrow \int \vec{F} \cdot \vec{dl}$

Surface integral $\rightarrow \int \vec{F} \cdot \vec{ds}$ $\quad \left[\vec{ds} = dx\,dy \right.$

Volume integral $\rightarrow \int \vec{F} \cdot \vec{dv}$ $\quad \left. dv = dx\,dy\,dz \right.$

Curl of a vector field

The curl of a vector field is defined as maximum line integral of the vector per unit area. It is a vector quantity.

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

$$\text{Curl } \vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Guass Divergence Theoram

This theoram states that the flux of a vector field over any closed surface S is equal to the volume integral of divergence of the vector over the volume V enclosed by the surface S

$$\iint \vec{F} \cdot \vec{ds} = \iiint \operatorname{div} \vec{F}\, dV$$

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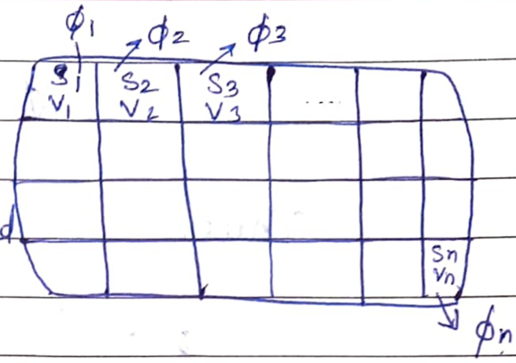
Proof -

$$\phi = \iint \vec{F} \cdot \vec{ds}.$$



let a closed surface inclosed volume V such that

$$S = S_1 + S_2 + S_3 - - - S_n$$
$$V = V_1 + V_2 + V_3 - - - V_n$$

Then, total flux ϕ passing through the surface S

$$\phi = \iint_S \vec{F} \cdot ds$$

flux associated with surface S_1

$$\phi_1 = \iint_S \vec{F} \cdot \vec{ds_1}$$

Similarly,

$$\phi_2 = \iint_S \vec{F} \cdot \vec{ds_2}$$
$$\phi_3 = \iint_S \vec{F} \cdot \vec{ds_3}$$
$$\vdots$$
$$\phi_n = \iint_S \vec{F} \cdot \vec{ds_n}$$

Total flux -

$$\phi = \phi_1 + \phi_2 + \phi_3 + - - - \phi_n$$

$$\phi = \iint_S \vec{F} \cdot \vec{ds_1} + \iint_S \vec{F} \cdot \vec{ds_2} + - - - \iint_S \vec{F} \cdot \vec{ds_n}$$

$$\phi = \sum_{i=1}^{n} \iint_S \vec{F} \cdot \vec{ds_i}$$

$$\phi = \sum_{i=1}^{n} V_i \left[\iint \frac{\vec{F} \cdot \vec{ds_i}}{V_i} \right]$$

$$\phi = \sum_{i=1}^{n} V_i \, \text{div} \vec{F}$$

Thus,

$$\left\{ \sum V_i = V = \iiint_V dv \right\}$$

$$\phi = \sum_{i=1}^{n} V_i \, \text{div} \vec{F}$$

$$\phi = \iiint_V \text{div} \vec{F} \, dv$$

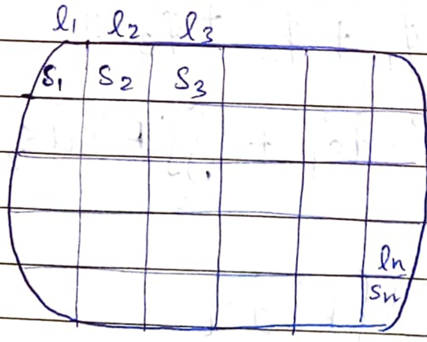
Hence,

$$\iint_S \vec{F} \cdot \vec{ds} = \iiint_V \text{div} \vec{F} \, dv$$

Stokes Theoram

The surface integral of the curl of a vector field $\vec{F}$ over a surface is equal to the closed line integral of vector field $\vec{F}$ over the boundary of that surface.

$$\iint_S \text{curl} \vec{F} \cdot \vec{ds} = \oint_C \vec{F} \cdot \vec{dl}$$



let a boundary L of a vector field F is divided into smaller boundaries

l_1, l_2 ---- l_n and these boundaries enclosed by the surface $S(S_1, S_2 ---- S_n)$ respectively, then, line integral of vector $\vec{F}$ over the boundary L ~~is~~.

Proof → Line integral of $\vec{F}$ over ---- L

$$= \oint_c \vec{F} \cdot \vec{dl} \quad \text{——} \ (1)$$

Line Integral of $\vec{F}$ over boundary l_1

$$= \oint_c \vec{F} \cdot \vec{dl_1}$$

Similarly for l_2

$$= \oint_c \vec{F} \cdot \vec{dl_2}$$

For n^{th} boundary

$$= \oint_c \vec{F} \cdot \vec{dl_n}$$

Then,

$$\oint_c \vec{F} \cdot \vec{dl} = \oint_c \vec{F} \cdot \vec{dl_1} + \oint_c \vec{F} \cdot \vec{dl_2} + ---- \oint_c \vec{F} \cdot \vec{dl_n}$$

$$\oint_c \vec{F} \cdot \vec{dl} = \sum_{i=1}^{n} \oint \vec{F} \cdot \vec{dl_i}$$

$$= \sum_{i=1}^{n} S_i \oint_c \frac{\vec{F} \cdot \vec{dl_i}}{S_i}$$

$\left[\Sigma S_i = \iint_S \vec{ds} \right]$

$$= \sum_{i=1}^{n} S_i \ \text{Curl} \vec{F}$$

$$\cancel{= \oint_C \text{curl}}$$

$$= \iint_S \text{curl}\,\vec{F} \cdot ds$$

Charge Distribution

Linear charge distribution

When the charge is distributed, one dimensionally over the length of linear object, it is called linear charge distribution

Linear charge density

Charge distributed on the unit length of the object is called linear charge density.

$$\text{Linear charge density} = \frac{Q}{\int dl} = \lambda$$

Surface charge Distribution and surface charge density

When the charge is distributed along the surface of the object then this is called surface charge distribution

$$\text{Surface charge density} = \frac{Q}{\iint ds} = \sigma$$

Volume charge Distribution
Charge distributed on the volume of an object is called volume charge distribution

$$\text{Volume charge density} = \frac{Q}{\iiint dV} = \rho$$

Equation of Continunity

We know that

$$I = -\frac{dQ}{dt} \quad \text{——} ①$$

Current density

$$I = \iint_S J \cdot ds \quad \text{——} ②$$

Charge can be represented as

$$Q = \iiint_V \rho \, dv \quad \text{——} ③$$

By placing the value of Q in eq ①

$$I = -\frac{d}{dt}\left[\iiint_V \rho \, dv\right] \quad \text{——} ④$$

from eq ② and ④

$$\iint_S J \cdot ds = \iiint_V \frac{dP}{dt} dV \quad \text{——} ⑤$$

$$\iiint_V \operatorname{div} J \, dv = -\iiint_V \frac{d\rho}{dt}$$

$$\iiint_V \left[\operatorname{div} J + \frac{d\rho}{dt}\right] = 0$$

[for finite volume] $\operatorname{div} \vec{J} + \frac{d\rho}{dt} = 0$

Maxwell's Equation

Equation	Differential form	Integral form	Remark
I	$\vec{\nabla} \cdot \vec{D} = \rho$	$\iint_S \vec{D} \cdot \vec{ds} = \iiint \rho \, dv$	Guass law of electric field
II	$\vec{\nabla} \cdot \vec{B} = 0$	$\iint \vec{B} \cdot \vec{ds} = 0$	Guass law of magnetic f[illegible]
III	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint \vec{E} \cdot \vec{dl} = -\iint \frac{\partial B}{\partial t} ds$	Faraday's law
IV	$\vec{\nabla} \times \vec{H} = \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right]$	$\oint \vec{H} \cdot \vec{dl} = \iint_S \left[\vec{J} + \frac{\partial D}{\partial t}\right] ds$	Ampere's law

Maxwell's equation in Vacuum or in free space

$\rho = 0$, $J = 0$, $\epsilon = \epsilon_0$, $\mu = \mu_0$

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Maxwell's Ist equation $\vec{\nabla} \cdot \vec{D} = 0$

Maxwell's IInd equation $\vec{\nabla} \cdot \vec{B} = 0$

Maxwell's IIIrd equation $\vec{\nabla} \times \vec{E} - \frac{\partial B}{\partial t} = -\mu_0 \frac{\partial \vec{H}}{\partial t}$

$\{ B = \mu H \}$

Maxwell's IVth equation $\vec{\nabla} \times \vec{H} = \frac{\partial D}{\partial t} = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\{ D = \epsilon E \}$

Maxwell's 1st equation $\Rightarrow \vec{\nabla} \cdot \vec{D} = \rho$

Proof - According to Guass law of Electric field.

$$\iint \vec{E} \cdot \vec{ds} = \frac{1}{\varepsilon} [Q]$$

$$\iint \epsilon \vec{E} \vec{ds} = Q$$

$$\iint \vec{D} \cdot \vec{ds} = Q. = \iiint_V \rho \, dv.$$

$$\iint_S \vec{D} \cdot \vec{ds} = \iiint_V \rho \, dv$$

$$\iiint_V \operatorname{div} \vec{D} \, dv = \iiint_V \rho \, dv$$

$$\iiint_V \vec{\nabla} \cdot \vec{D} \, dv = \iiint \rho \, dv$$

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$$\vec{\nabla}\cdot\vec{D} = \rho$$

For vacuum

$$\vec{\nabla}\cdot\vec{D} = 0$$

$$\phi = \frac{1}{\varepsilon}[Q]$$

$$\phi = \iint \vec{E}\cdot\vec{ds}$$

$$\vec{D} = \epsilon\vec{E}$$

$$Q = \iiint_V \rho\, dv$$

Maxwell's IInd equation $\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

Proof : According to Guass of magnetic field

$$\iint \vec{B}\cdot\vec{ds} = 0$$

$$\iint B\cdot ds = 0$$

$$\iiint_V \operatorname{div}\vec{B}\cdot dv = 0 \quad \text{[From Guass Divergence theorem]}$$

$$\iiint_V \vec{\nabla}\cdot\vec{B}\cdot dv = 0$$

For finite volume $\vec{\nabla}\cdot\vec{B} = 0$

Maxwell's IIIrd equation $\Rightarrow \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

Proof - According to Faraday's law emf

$$e = -\frac{d\phi}{dt} \quad \text{——— (1)}$$

$$\phi = \iint_S B\cdot ds \quad \text{——— (2)}$$

$$e = \oint \vec{E}\cdot\vec{dl} \quad \text{——— (3)}$$

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By eq ① and ③

$$\oint \vec{E} \cdot dl = -\frac{d\phi}{dt} \quad \text{— (4)}$$

$$\oint \vec{E} \cdot dl = -\frac{d}{dt} \iint_S \vec{B} \cdot \vec{ds}$$

$$\oint \vec{E} \cdot \vec{dl} = -\iint_S \frac{d\vec{B}}{dt} \cdot \vec{ds}$$

$$\iint_S \text{curl } \vec{E} \cdot \vec{ds} = -\iint_S \frac{d\vec{B}}{dt} \cdot \vec{ds}$$

$$\iint_S [\vec{\nabla} \times \vec{E}] \, \vec{ds} = -\iint_S \frac{dB}{dt} \, ds$$

$$\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt} \qquad [B = \mu H]$$

For Vaccum

$$\vec{\nabla} \times \vec{E} = \mu_0 \frac{\partial \vec{H}}{\partial t}$$

Maxwell's 4^{th} Equation

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Proof -

According to Ampere's law

$$\oint_C \vec{B} \cdot \vec{dl} = \mu I = \mu [I_c + I_d]$$

$$\oint_C B \cdot dl = \mu \left[\iint_S J_c \cdot ds + \iint_S \vec{J_d} \cdot \vec{ds} \right]$$

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$$\oint_c \vec{B}\cdot\vec{dl} = \mu\left[\iint_s \vec{J}\cdot\vec{ds} + \iint_s \frac{\partial \vec{D}}{\partial t}\,\vec{ds}\right]$$

$$\oint_c \vec{B}\cdot\vec{dl} = \mu \iint_s \left[J_c + \frac{\partial \vec{D}}{\partial t}\right] ds$$

$$\cancel{\mu} \oint_c \vec{H}\cdot\vec{dl} = \cancel{\mu} \iint_s \left[\vec{J_c} + \frac{\partial \vec{D}}{\partial t}\right] ds$$

By applying Stoke's theoram

$$\iint_s \text{curl}\,\vec{H}\cdot ds = \iint_s \left[\bar{J_c} + \frac{\partial \vec{D}}{\partial t}\right] ds$$

$$\text{curl}\,\vec{H} = J_c + \frac{\partial \vec{D}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = J_c + \frac{\partial \vec{D}}{\partial t}$$