Previous year unitwise questions with the solution

Unit-IV.

Ques 1) Show that the following function is harmonic and find its harmonic conjugate function.

$$u = \frac{1}{2}\log(x^2+y^2)$$

(RGPV May 2019)

Solution :- Given function

$$\boxed{u = \frac{1}{2}\log(x^2+y^2)} \quad -①$$

Partially differentiate u wrt x & y,

$$\boxed{\frac{\partial u}{\partial x} = \frac{1}{\cancel{2}} \times \frac{1}{x^2+y^2}(\cancel{2}x) = \frac{x}{x^2+y^2}} \quad -②$$

Similarly $\boxed{\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}}$ $-③$

Again partially differentiate eqn ② wrt x & eqn ③ wrt y.

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)\cdot 1 - x(2x)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{y^2-x^2}{(x^2+y^2)^2}} \quad -④$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)\cdot 1 - y\,(2y)}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}} \quad -5$$

Hence

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Hence u is harmonic function.

Now to find harmonic conjugate of u.

Given that

$$u = \frac{1}{2}\log(x^2+y^2) \quad \text{and}$$

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2+y^2)} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{y}{(x^2+y^2)}$$

Now by formula

$$\boxed{dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy} \quad -6.$$

By C-R Eq^n

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \;\&\; \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

Hence eq^n 6 becomes.

$$dV = \left(-\frac{\partial u}{\partial y}\right) dx + \left(\frac{\partial u}{\partial x}\right) dy$$

ie $$\boxed{dV = \left(\frac{-y}{x^2+y^2}\right) dx + \left(\frac{x}{x^2+y^2}\right) dy} \quad \text{--- (7)}$$

Let

$$dV = M\,dx + N\,dy$$

clearly eq^n (7) is exact. Hence Solution of eq^n (7) is

$$\boxed{\int dV = \int_{y\ \text{Constant}} M\,dx + \int \left(\begin{array}{c}\text{Those terms of N free}\\ \text{from } x\end{array}\right) dy + C}$$

ie $$V = \int_{y\ \text{constant}} \frac{-y}{x^2+y^2}\, dx + \int 0\, dy + C$$

$$\Rightarrow V = -y \int_{y\ \text{constant}} \frac{1}{x^2+y^2}\, dx + C.$$

$$\Rightarrow V = -\cancel{y} \times \frac{1}{\cancel{y}} \tan^{-1}\left(\frac{x}{y}\right) + C$$

$$\left[\text{Since } \int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\Rightarrow \boxed{V = -\tan^{-1}\left(\frac{x}{y}\right) + C} \quad \text{Ans}$$

This is the required harmonic conjugate of u.

b) Determine the analytic function, whose real part is

$$e^{2x}(x\cos 2y - y\sin 2y).$$

(RGPV May 2019)

Solution :- Given that

$$u = e^{2x}(x\cos 2y - y\sin 2y) \quad -①$$

$$\Rightarrow \frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + (x\cos 2y - y\sin 2y)2e^{2x}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = e^{2x}[\cos 2y + 2x\cos 2y - 2y\sin 2y]} \quad -②$$

Again

$$\frac{\partial u}{\partial y} = e^{2x}[x\{-2\sin 2y\} - \{2y\cos 2y + \sin 2y\}]$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial y} = e^{2x}[-2x\sin 2y - 2y\cos 2y - \sin 2y]} \quad -③$$

Now put $x = z$ & $y = 0$ in eq^n ②, to get

$$\phi_1(z,0) = \left(\frac{\partial u}{\partial x}\right)_{(x=z,\,y=0)}$$

$$\boxed{\phi_1(z,0) = e^{2z}[1+2z]}$$

Again put $x = z$ & $y = 0$ in eq^n ③, to get

$$\phi_2(z,0) = \left(\frac{\partial u}{\partial y}\right)_{(x=z,\, y=0)}$$

$$\Rightarrow \boxed{\phi_2(z,0) = e^{2z}[0] = 0}$$

Hence by Milne Thomson Method

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + C$$

$$\Rightarrow f(z) = \int e^{2z}(1+2z)\,dz - i\int 0\,dz + C$$

$$\Rightarrow f(z) = \int \underset{\mathrm{II}}{e^{2z}}\,\underset{\mathrm{I}}{(1+2z)}\,dz + C$$

$+(1+2z)$	e^{2z}
-2	$\frac{e^{2z}}{2}$
$+0$	$\frac{e^{2z}}{4}$

$$\Rightarrow f(z) = \frac{1}{2}(1+2z)e^{2z} - \frac{1}{2}e^{2z} + C$$

$$\Rightarrow f(z) = \cancel{\frac{e^{2z}}{2}} + ze^{2z} - \cancel{\frac{1}{2}e^{2z}} + C$$

$$\Rightarrow \boxed{f(z) = ze^{2z} + C}$$ Ans

This is the required analytic function $f(z)$.

Ques 2) (a) Evaluate the following using Cauchy Integral Formula

$$\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz$$ where c is the circle $|z| = \frac{3}{2}$

(RGPV May 2019).

Solution : Here the poles are given by

$$z(z-1)(z-2) = 0.$$

$$\Rightarrow z = 0, 1, 2$$

For $z = 0$, $|0| = 0 < \frac{3}{2}$ (Hence $z=0$ lies inside c).

For $z = 1$, $|1| = 1 < \frac{3}{2}$ (Hence $z=1$ lies inside c).

For $z = 2$, $|2| = 2 > \frac{3}{2}$ (Hence $z=2$ lies outside c).

Hence given integral can be written as.

$$\boxed{\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_c \frac{\frac{4-3z}{z-2}}{z(z-1)}\,dz.} \quad \text{--- (1)}$$

Let $f(z) = \frac{4-3z}{z-2}$ --- (2)

clearly $f(z)$ is analytic inside c

Also $$\boxed{\frac{1}{z(z-1)} = \frac{1}{z-1} - \frac{1}{z}} \quad \text{--- (3)}$$

Hence put the values from eqⁿ (2) & (3) in eqⁿ (1), we get

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_C \frac{f(z)}{z(z-1)}\,dz$$

$$= \int_C f(z)\left[\frac{1}{z-1} - \frac{1}{z}\right]dz$$

$$= \int_C \frac{f(z)}{z-1}\,dz - \int_C \frac{f(z)}{z}\,dz$$

$$= 2\pi i f(1) - 2\pi i f(0) \quad (4)$$

$$\left[\text{Since} \int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)\right]$$

Here $f(z) = \dfrac{4-3z}{z-2}$

$\Rightarrow f(1) = \dfrac{4-3}{1-2} = -1$

$f(0) = \dfrac{4}{-2} = -2$

Hence eqⁿ (4) becomes

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz = 2\pi i(-1) - 2\pi i(-2)$$

$$= -2\pi i + 4\pi i$$

$$= 2\pi i$$

Hence $\boxed{\int_C \frac{4-3z}{z(z-1)(z+2)}\,dz = 2\pi i}$ Ans

Que (b) Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ for the circle $|z|=1$

Solution: let $\boxed{I = \int_0^{2\pi} \frac{1}{2+\cos\theta} d\theta}$ — (1)

Let $z = e^{i\theta}$

$\Rightarrow dz = i e^{i\theta} d\theta$

$\Rightarrow dz = iz\, d\theta$

$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$ — (2)

Also given that closed curve is $|z|=1$.

Hence $I = \int_{|z|=1} \frac{1}{(2+\cos\theta)} \frac{dz}{iz}$ — (3)

Here $z = e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow \frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$\Rightarrow \boxed{\cos\theta = \frac{z+\frac{1}{z}}{2}}$ — (4)

Hence eq^n (3) becomes

$$I = \int_{|z|=1} \frac{1}{2+\left(\frac{z+\frac{1}{z}}{2}\right)} \frac{dz}{iz}$$

$$\Rightarrow I = \int_{|z|=1} \frac{2}{4+z+\frac{1}{z}} \frac{dz}{iz}$$

$$\Rightarrow I = \frac{2}{i} \int_{|z|=1} \frac{1}{\cancel{z}} \times \frac{\cancel{z}}{4z+z^2+1} dz$$

$$\Rightarrow I = \frac{2}{i} \int_{|z|=1} \frac{1}{z^2+4z+1} dz \quad \text{— (5)}$$

Now here the poles are given by

$$z^2+4z+1 = 0.$$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16-4\times1\times1}}{2\times1}$$

$$\Rightarrow z = \frac{-4 \pm \sqrt{16-4}}{2}$$

$$\Rightarrow z = \frac{-4 \pm \sqrt{12}}{2}$$

$$\Rightarrow z = -2 \pm \sqrt{3}$$

Let $z = -2+\sqrt{3} = \alpha$ and
$z = -2-\sqrt{3} = \beta$

Hence eq^n (5) becomes.

$$\boxed{I = \frac{2}{i}\int_{|z|=1} \frac{1}{(z-\alpha)(z-\beta)}\,dz} \quad -(6)$$

For $z=\alpha$, $|\alpha| = |-2+1.732| = 0.268 < 1$

Hence $z=\alpha$ lies inside C

For $z=\beta$, $|\beta| = |-2-\sqrt{3}| = 2+\sqrt{3} > 1$

Hence $z=\beta$ lies outside C.

Hence by eq^n (6)

$$I = \frac{2}{i}\int_{|z|=1} \frac{1/z-\beta}{(z-\alpha)}\,dz$$

Let $f(z) = \dfrac{1}{z-\beta}$.

clearly $\frac{1}{z-\beta}$ ~~this~~ is analytic inside $|z| = 1$.

Hence $I = \frac{2}{i}\left[2\pi i \times f(\alpha)\right]$.

$$\left[\because \int \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)\right]$$

$$\Rightarrow I = \frac{2}{\cancel{i}} \times 2\pi \cancel{i} \times \frac{1}{\alpha - \beta}$$

$$\Rightarrow I = 4\pi \times \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})}$$

$$\Rightarrow I = \frac{4\pi}{-\cancel{2}+\sqrt{3}+\cancel{2}+\sqrt{3}} = \frac{4\pi}{2\sqrt{3}}$$

$$\Rightarrow \boxed{I = \frac{2\pi}{\sqrt{3}}} \quad \underline{\text{Ans}}.$$

Ques3) (a) Determine whether $\frac{1}{2}$ is analytic or not.

(RGPV NOV 2019)
(RGPV June 2020)

Solution :- Let $\boxed{f(z) = \frac{1}{2}}$

or $f(z) = \frac{1}{2} + i \times 0$.

on compairing $f(z) = U + iV$, we get

$$\boxed{U = \frac{1}{2}, \quad V = 0}$$

partially diff. U wrt x & y respectivly and also partially diff. V wrt x & y respec., we get

$U_x = 0$ | $V_x = 0$

$U_y = 0$ | $V_y = 0$

Hence

$$\boxed{U_x = V_y \text{ and } U_y = -V_x}$$

Hence $f(z)$ satisfies C-R Eqn.

This implies $f(z)$ is analytic.

(b) Show that the function

$u = e^{-2xy}\sin(x^2-y^2)$ is harmonic

(RGPV NOV 2019)
(RGPV June 2020)

Solution:- Given function

$$\boxed{u = e^{-2xy}\sin(x^2-y^2)} \quad \text{—} ①$$

Now partially differentiate u wrt x & y

$$\Rightarrow \frac{\partial u}{\partial x} = e^{-2xy}\cos(x^2-y^2)(2x) + \sin(x^2-y^2)e^{-2xy}(-2y)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = 2e^{-2xy}\left(x\cos(x^2-y^2) - y\sin(x^2-y^2)\right)} \quad \text{—} ②$$

Similarly

$$\frac{\partial u}{\partial y} = e^{-2xy}\cos(x^2-y^2)(-2y) + \sin(x^2-y^2)e^{-2xy}(-2x)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial y} = -2e^{-2xy}\left[y\cos(x^2-y^2) + x\sin(x^2-y^2)\right]} \quad ③$$

Again partially differentiating eqⁿ (2) wrt x & eqⁿ (3) wrt y.

$$\frac{\partial^2 u}{\partial x^2} = 2\Big[e^{-2xy}\Big\{ -x\sin(x^2-y^2)(2x) + \cos(x^2-y^2) - y\cos(x^2-y^2)(2x) \Big\} + \Big\{ x\cos(x^2-y^2) - y\sin(x^2-y^2)\Big\} e^{-2xy}(-2y) \Big]$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = 2e^{-2xy}\Big[-2x^2\sin(x^2-y^2) + \cos(x^2-y^2) - 2xy\cos(x^2-y^2) - 2xy\cos(x^2-y^2) + 2y^2\sin(x^2-y^2) \Big]} \quad -(4)$$

Similarly

$$\frac{\partial^2 u}{\partial y^2} = -2\Big[e^{-2xy}\Big\{ -y\sin(x^2-y^2)(-2y) + \cos(x^2-y^2) + x\cos(x^2-y^2)(-2y) \Big\} + \Big\{ y\cos(x^2-y^2) + x\sin(x^2-y^2)\Big\} e^{-2xy}(-2x) \Big]$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = -2e^{-2xy}\Big[2y^2\sin(x^2-y^2) + \cos(x^2-y^2) - 2xy\cos(x^2-y^2) - 2xy\cos(x^2-y^2) - 2x^2\sin(x^2-y^2) \Big]} \quad (5)$$

Adding eqⁿ (4) & (5), we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2e^{-2xy}\Big[-2x^2\sin(x^2-y^2)+\cos(x^2-y^2)$$

$$-4xy\cos(x^2-y^2)+2y^2\sin(x^2-y^2)\Big]$$

$$-2e^{-2xy}\Big[2y^2\sin(x^2-y^2)+\cos(x^2-y^2)$$

$$-4xy\cos(x^2-y^2)-2x^2\sin(x^2-y^2)\Big]$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 2e^{-2xy}\Big[\cancel{-2x^2\sin(x^2-y^2)}+\cancel{\cos(x^2-y^2)}$$

$$\cancel{-4xy\cos(x^2-y^2)}+\cancel{2y^2\sin(x^2-y^2)}\cancel{-2y^2\sin(x^2-y^2)}$$

$$\cancel{-\cos(x^2-y^2)}+\cancel{4xy\cos(x^2-y^2)}+\cancel{2x^2\sin(x^2-y^2)}\Big]$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0}$$ Ans

Hence proved that u is a harmonic function

Ques 4 (a) Show that the function $f(z)=e^z$ is analytic everywhere.

Solution :- Given $f(z)=e^z$

$$\Rightarrow f(z) = e^{x+iy} \quad (\text{put } z=x+iy)$$

$$\Rightarrow f(z) = e^x \times e^{iy}$$

$$\Rightarrow f(z) = e^{x}(\cos y + i\sin y).$$

$$\Rightarrow \boxed{f(z) = e^{x}\cos y + i\,e^{x}\sin y.} \quad -(1).$$

Compairing eq^n (1) with $f(z) = U + iV$, we get

$$U = e^{x}\cos y, \quad V = e^{x}\sin y \quad -(2)$$

Now partially diff. U wrt x & y. & V wrt x & y,

$$\frac{\partial U}{\partial x} = e^{x}\cos y. \qquad \frac{\partial V}{\partial x} = e^{x}\sin y.$$

$$\frac{\partial U}{\partial y} = -e^{x}\sin y \qquad \frac{\partial V}{\partial y} = e^{x}\cos y.$$

clearly $\boxed{\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = e^{x}\cos y.}$

and $\frac{\partial U}{\partial y} = -e^{x}\sin y = -\frac{\partial V}{\partial x}.$

ie $\boxed{\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.}$

Hence $f(z)$ satisfies C-R Eq^n everywhere

Hence $f(z)$ is analytic everywhere. Ans

(b) Evaluate $\oint_C \frac{z}{z^2+9}\,dz$, where C is the circle

$$|z-2i| = 4.$$

Solution:- Given integral

$$\boxed{I = \int_C \frac{z}{z^2+9}\,dz.} \quad -(1)$$

Here poles are given by $z^2+9=0$

i.e. $z^2 = -9$

$\Rightarrow z = \pm\sqrt{-9}$

$\Rightarrow z = \pm 3i$

For $z = 3i$ $\quad |3i-2i| = |i| = 1 < 4$

(Hence $z=3i$ lies inside c).

For $z=-3i$ $\quad |-3i-2i| = |-5i| = 5 > 4$

(Hence $z=-3i$ lies outside c).

Hence by eq^n ①

$$I = \int_{|z-3i|=4} \frac{z}{(z+3i)(z-3i)} dz$$

let $$f(z) = \frac{z}{(z+3i)(z-3i)}$$

$z=3i$ is a simple pole of $f(z)$.

So

$$\text{Res}[f(z), z=3i] = \lim_{z\to 3i} (z-3i) f(z)$$

$$= \lim_{z\to 3i} \cancel{(z-3i)} \frac{z}{(z+3i)\cancel{(z-3i)}}$$

$$= \frac{3i}{(3i+3i)}$$

$$= \frac{3\cancel{i}}{6\cancel{i}}$$

$$= \frac{1}{2}$$

Hence by Residue Theorem

$$\boxed{\begin{aligned} I &= \int_{|z-3i|=4} \frac{z}{(z+3i)(z-3i)}\,dz \\ &= 2\pi i\left[\operatorname{Res}[f(z), z=3i]\right] \\ &= 2\pi i\left(\frac{1}{2}\right) \\ &= \pi i \end{aligned}}$$

Ans.

Ques 5) (a) Show that $f(z) = z\bar{z}$ is differentiable but not analytic at origin

(RGPV Nov 2022)

Solution :- Given Function

$$f(z) = z\bar{z}$$

$$\Rightarrow f(z) = (x+iy)(x-iy)$$

$$\Rightarrow f(z) = (x^2+y^2) + 0i$$

Hence $U = x^2+y^2,\ V = 0$

Now

$\frac{\partial U}{\partial x} = 2x$	$\frac{\partial V}{\partial x} = 0$
$\frac{\partial U}{\partial y} = 2y$	$\frac{\partial V}{\partial y} = 0$

Hence at $x=0, y=0$

$$\boxed{\begin{aligned} \frac{\partial U}{\partial x} &= \frac{\partial V}{\partial y} = 0 \\ \frac{\partial U}{\partial y} &= -\frac{\partial V}{\partial x} = 0 \end{aligned}}$$

And at $x \neq 0, y \neq 0$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

ie $f(z)$ satisfies C-R Eq^n only at origin

$\Rightarrow$ $f(z)$ is not differentiable at any point except origin.

Also at $z=0$

$$= \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$$

$$= \lim_{z \to 0} \frac{z\bar{z} - 0}{z}$$

$$= \lim_{z \to 0} \bar{z}$$

$$= 0.$$

Hence $f(z)$ is differentiable at origin only & by C-R eq^n $f(z)$ is nowhere differentiable for $x \neq 0$ & $y \neq 0$.

Hence we can say that $f(z)$ is differentiable only at origin and not on any other non zero point in a domain D.

Hence $f(z)$ is not analytic at origin but it is differentiable at origin

(b) Show that $u(x,y) = e^{-2x} \sin 2y$ is harmonic and determine its Harmonic Conjugate.

(RGPV Nov 2022)

Solution : Given function

$$\boxed{u(x,y) = e^{-2x} \sin 2y} \quad -(1)$$

Partially differentiate eqn (1) wrt x & y.

$$\boxed{\frac{\partial u}{\partial x} = e^{-2x}(-2)\sin 2y = -2e^{-2x}\sin 2y} \quad -(2)$$

$$\boxed{\frac{\partial u}{\partial y} = 2e^{-2x}\cos 2y} \quad -(3)$$

Again diffn partially eqn (2) wrt x & eqn (3) wrt y.

$$\boxed{\frac{\partial^2 u}{\partial x^2} = -2e^{-2x}(-2)\sin 2y}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} = 4e^{-2x}\sin 2y} \quad -(4)$$

Similarly $\frac{\partial^2 u}{\partial y^2} = 2e^{-2x}(-\sin 2y)\,2.$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial y^2} = -4e^{-2x}\sin 2y} \quad -(5)$$

Adding eqn (4) & (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{4e^{-2x}\sin 2y} - \cancel{4e^{-2x}\sin 2y}$$

$$\Rightarrow \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.}$$

Hence $f(z)$ is harmonic function.

Now to find harmonic conjugate of $u(x,y)$

ie to find V.

By Formula.

$$\boxed{dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy} \quad -(6)$$

By C-R Eqⁿ $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ & $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

Hence eqⁿ (6) becomes

$$dV = \left(-\frac{\partial U}{\partial y}\right)dx + \left(\frac{\partial U}{\partial x}\right)dy$$

$$\Rightarrow \boxed{dV = \left(-2e^{-2x}\cos 2y\right)dx + \left(-2e^{-2x}\sin 2y\right)dy} \quad -(7)$$

let

$$\boxed{dV = M\,dx + N\,dy} \quad -(8)$$

clearly eqⁿ (7) is exact Hence the solution is

$$\boxed{\int dV = \int_{y\text{ constant}} M\,dx + \int \left(\begin{matrix}\text{Those terms of } N \\ \text{free from } x\end{matrix}\right)dy + C}$$

$$\Rightarrow V = \int_{y\text{ Constant}} \left(-2e^{-2x}\cos 2y\right)dx + \int 0 \cdot dy + C$$

$$\Rightarrow V = -\cancel{2}e^{-2x}\frac{\sin 2y}{\cancel{2}} + C$$

$$\Rightarrow \boxed{V = -e^{-2x}\sin 2y + C}$$

This is the required harmonic Conjugate of U.

Ques 6)(a) By Residue theorem, Evaluate

$$\oint_C \frac{\tan z}{z^2-1}\,dz \quad , \quad \text{where } c : |z| = 2$$

(RGPV NOV' 2022).

Solution ⇒ Let $I = \int_{|z|=2} \frac{\tan z}{z^2-1}\,dz$

$$\Rightarrow \boxed{I = \int_{|z|=2} \frac{\sin z}{\cos z(z^2-1)}\,dz} \quad \text{—— ①}$$

Hence the poles are given by $\cos z(z^2-1) = 0$.

$$\Rightarrow \cos z = 0 \quad , \quad z^2-1 = 0$$

i) when $\cos z = 0$

$$\Rightarrow z = (2n+1)\frac{\pi}{2}$$

For $n=0$, $z = \frac{\pi}{2} \Rightarrow \left|\frac{\pi}{2}\right| = 1.56 < 2$.

For $n=1$, $z = \frac{3\pi}{2} \Rightarrow \left|\frac{3\pi}{2}\right| = 4.5 > 2$.

For $n=2$, $z = \frac{5\pi}{2} \Rightarrow \left|\frac{5\pi}{2}\right| > 2$

⋮

Similarly For $n=-1$, $z = -\frac{\pi}{2} \Rightarrow \left|-\frac{\pi}{2}\right| = 1.56 < 2$.

and For $n < -1$, $|z| = \left|(2n+1)\frac{\pi}{2}\right| > 2$.

Hence we can say that when

$\cos z = 0$, then the poles which lie inside c are $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$

and all other poles lie outside c.

ii) when $z^2 - 1 = 0$

$\Rightarrow z^2 = 1$

$\Rightarrow z = \pm 1$

For $z = 1$, $|1| = 1 < 2$

For $z = -1$, $|-1| = 1 < 2$

Hence when $z^2 - 1 = 0$, then $z = 1$ & $z = -1$ are the poles which lie inside C

Hence the poles of eqⁿ ① which lie inside C

are $z = 1, -1, \frac{\pi}{2}, -\frac{\pi}{2}$

Now all these poles are simple poles

So

(a) $\text{Res}[f(z), z=1] = \lim_{z \to 1} (z-1) f(z)$

$$= \lim_{z \to 1} (z-1) \left(\frac{\sin z}{\cos z (z^2 - 1)} \right)$$

$$= \lim_{z \to 1} \cancel{(z-1)} \frac{\sin z}{\cos z (z+1) \cancel{(z-1)}}$$

$$= \frac{\sin 1}{\cos 1 \cdot (2)}$$

$\Rightarrow \text{Res}[f(z), z=1] = \frac{1}{2} \tan 1$ — ②

(b) $\text{Res}[f(z), z=-1] = \lim_{z \to -1} (z+1) f(z)$

$$= \lim_{z \to -1} (z+1) \frac{\sin z}{\cos z (z^2 - 1)}$$

$$= \lim_{z \to -1} \cancel{(z+1)} \frac{\sin z}{\cos z (z-1) \cancel{(z+1)}}$$

$$= \frac{\sin(-1)}{\cos(-1)(-1-1)}$$

$$= -\frac{1}{2}\tan(-1)$$

$$\Rightarrow \text{Res}[f(z), z=-1] = \frac{1}{2}\tan(1) \quad \text{— (3)} \qquad \left[\text{Since } \tan(-\theta) = -\tan\theta\right]$$

(c) $$\text{Res}\left[f(z), \frac{\pi}{2}\right] = \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z \to \frac{\pi}{2}} \left(z - \frac{\pi}{2}\right) \frac{\sin z}{\cos z (z^2 - 1)}$$

$$= \lim_{z \to \frac{\pi}{2}} \left[\left(\frac{z - \pi/2}{\cos z}\right) \cdot \left(\frac{\sin z}{z^2 - 1}\right)\right]$$

$$= \lim_{z \to \frac{\pi}{2}} \left(\frac{\sin z}{z^2 - 1}\right) \cdot \lim_{z \to \frac{\pi}{2}} \left(\frac{z - \pi/2}{\cos z}\right)$$

$$= \frac{\sin(\pi/2)}{\left(\frac{\pi^2}{4} - 1\right)} \times \left(\frac{1}{-\sin z}\right)_{z \to \frac{\pi}{2}}$$

$$= \left(\frac{4}{\pi^2 - 4}\right) \times (-1)$$

$$= \frac{-4}{\pi^2 - 4}. \quad \text{— (4)}$$

(d) $$\text{Res}\left[f(z), -\frac{\pi}{2}\right] = \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) f(z)$$

$$= \lim_{z \to -\frac{\pi}{2}} \left(z + \frac{\pi}{2}\right) \frac{\sin z}{\cos z (z^2 - 1)}$$

$$= \lim_{z \to -\frac{\pi}{2}} \left(\frac{\sin z}{(z^2 - 1)}\right) \cdot \lim_{z \to -\frac{\pi}{2}} \left(\frac{z + \pi/2}{\cos z}\right)$$

$$= \frac{\sin(-\pi/2)}{\left(\frac{\pi^2}{4}-1\right)} \times \left(\frac{1}{-\sin z}\right)_{z \to -\frac{\pi}{2}}$$

$$= \frac{-\sin(\pi/2)}{\left(\frac{\pi^2}{4}-1\right)} \times \frac{-1}{\sin\left(-\frac{\pi}{2}\right)}$$

$$= \frac{-1}{\left(\frac{\pi^2}{4}-1\right)} \times \frac{\cancel{+1}}{\cancel{+1}}$$

$$= \frac{-4}{\pi^2-4} \quad \text{--- (5)}$$

Now by applying Residue Theorem

$$\int_{|z|=2} f(z)\,dz = 2\pi i\Big[\text{Res}[f(z), z=1] + \text{Res}[f(z), z=-1] + \text{Res}[f(z), z=\tfrac{\pi}{2}] + \text{Res}[f(z), z=-\tfrac{\pi}{2}]\Big]$$

$$= 2\pi i\left[\frac{1}{2}\tan 1 + \frac{1}{2}\tan 1 - \frac{4}{\pi^2-4} - \frac{4}{\pi^2-4}\right]$$

$$\Rightarrow \boxed{\int_{|z|=2} f(z)\,dz = 2\pi i\left[\tan 1 - \frac{8}{\pi^2-4}\right]}$$

Ans

This is the required solution.

(b) using Cauchy integral theorem, to evaluate the integral

$$\int_C \frac{e^{2z}}{(z-1)^2(z-3)}\,dz \quad \text{where } c \text{ is the circle } |z|=2.$$

(RGPV NOV 2022)

Solution :- Given integral

$$I = \int_C \frac{e^{2z}}{(z-1)^2(z-3)} dz \quad -(1)$$

Here poles are given by $(z-1)^2(z-3) = 0$.

$$\Rightarrow z = 1, 1, 3.$$

For $z = 1$, $|1| = 1 < 2$.

Hence $z = 1$ lies inside C.

For $z = 3$, $|3| = 3 < 2$.

Hence $z = 3$ lies outside C.

$$\text{Hence } I = \int_C \frac{e^{2z}/(z-3)}{(z-1)^2} dz \quad -(2)$$

$$\text{Let } f(z) = \frac{e^{2z}}{(z-3)}$$

Hence eq^n (2) becomes.

$$I = \int_{|z|=2} \frac{f(z)}{(z-1)^2} dz \quad -(3)$$

Hence by Cauchy Integral Formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$

$$\text{Hence } I = \int_{|z|=2} \frac{f(z)}{(z-1)^{1+1}} dz = \frac{2\pi i}{1!} f'(1)$$

$$\Rightarrow I = \frac{2\pi i}{1} f'(1) \quad \text{---(4)}$$

Here $f(z) = \dfrac{e^{2z}}{(z-3)}$

$$\Rightarrow f'(z) = \frac{2(z-3)e^{2z} - e^{2z}(1-0)}{(z-3)^2}$$

$$\Rightarrow f'(z) = \frac{e^{2z}(2z-6-1)}{(z-3)^2}$$

$$\Rightarrow f'(z) = \frac{e^{2z}(2z-7)}{(z-3)^2}$$

put $z=1$ in $f'(z)$, we get

$$\Rightarrow f'(1) = \frac{e^2(2-7)}{(1-3)^2}$$

$$\Rightarrow f'(1) = \frac{e^2(-5)}{4}$$

$$\Rightarrow f'(1) = -\frac{5}{4}e^2$$

Hence $I = \dfrac{\cancel{2}\pi i}{1} \times \left(-\dfrac{5}{\cancel{4}_2}e^2\right)$

$$\Rightarrow I = -\frac{5}{2}\pi i e^2$$

Ques 7) (a) Prove that $\left[\dfrac{d^2}{dx^2} + \dfrac{d^2}{dy^2}\right] |f(z)|^2 = 4|f'(z)|^2$

(RGPV June 2022).

Solution :- First we will prove that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

We have $z = x + iy$ and $\bar{z} = x - iy$.

$$\Rightarrow x = \frac{z+\bar{z}}{2}, \quad y = \frac{z-\bar{z}}{2i}$$

$$\Rightarrow \boxed{x = \frac{1}{2}(z+\bar{z}), \quad y = \frac{-i}{2}(z-\bar{z})} \quad (1)$$

Partially differentiating x & y wrt z & $\bar{z}$

$$\left.\begin{aligned} \frac{\partial x}{\partial z} &= \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{-i}{2} \\ \frac{\partial x}{\partial \bar{z}} &= \frac{1}{2}, \quad \frac{\partial y}{\partial \bar{z}} = \frac{i}{2} \end{aligned}\right\} (2)$$

Let $f = f(x, y)$.

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial z} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial z}$$

$$\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(-\frac{i}{2}\right)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)} \quad (3)$$

Again

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} \times \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} \times \frac{\partial y}{\partial \bar{z}}$$

$$\Rightarrow \frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial y}\left(\frac{i}{2}\right)$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)} \quad -④$$

Now

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \bar{z}}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \times \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$= \frac{1}{4}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \cdot \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f$$

$$= \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} - i^2\frac{\partial^2}{\partial y^2}\right) f$$

$$\Rightarrow \frac{\partial^2 f}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f$$

$$\Rightarrow 4\frac{\partial^2}{\partial z \partial \bar{z}}(f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(f)$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}} \quad -⑤$$

Now Apply $|f(z)|^2$ from Right Hand Side

$$\Rightarrow \boxed{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^2} \quad ⑥$$

$$\Rightarrow \text{Take RHS} = 4\frac{\partial^2}{\partial z \partial \bar{z}}|f(z)|^2$$

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left[f(z)\overline{f(z)}\right].$$

[Since $|z|^2 = z\bar{z}$]

$$= 4\frac{\partial^2}{\partial z \partial \bar{z}}\left[f(z) f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z)\frac{\partial}{\partial \bar{z}} f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}\left[f(z) f'(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}(f(z)) f'(\bar{z})$$

$$= 4 f'(z) f'(\bar{z})$$

$$= 4 f'(z) \overline{f'(z)}$$

$$= 4|f'(z)|^2$$

Hence eqⁿ (6) becomes

$$\boxed{\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2}$$

(Hence Proved)

(b) Construct the analytic function $f(z)$, whose real part is $e^x \cos y$.

(RGPV June 2022)

Solution :- $U = e^x \cos y$

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y = \phi_1(x,y) \\ \frac{\partial u}{\partial y} &= -e^x \sin y = \phi_2(x,y) \end{aligned}}$$

Put $x = z$, $y = 0$ in $\phi_1(x,y)$ & $\phi_2(x,y)$ to get

$\phi_1(z,0) = e^z \cos 0 = e^z$

$\phi_2(z,0) = -e^z \sin 0 = 0.$

Hence by Milne Thomson Method.

$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz + c$

$\Rightarrow f(z) = \int e^z\,dz - i\int 0\,dz + c$

$\Rightarrow \boxed{f(z) = e^z + c}$ Ans

This is the required analytic function f(z).

Ques 8) (a) Using Cauchy's integral formula, find $\int_C \frac{e^{2z}}{(z+1)^3}\,dz$ where c is the curve $|z| = 2$

(RGPV JUNE 2022)

Solution :- let $\boxed{I = \int_C \frac{e^{2z}}{(z+1)^3}\,dz}$ —(1)

Here the poles are given by

$(z+1)^3 = 0$

$\Rightarrow z = -1, -1, -1$

For $z = -1$, $|-1| = 1 < 2$

Hence $z = -1$ is a pole of order 3 which lies inside C.

Hence let $f(z) = e^{2z}$

then $I = \int_C \frac{f(z)}{(z+1)^3} dz$

$$\Rightarrow \boxed{I = \int_C \frac{f(z)}{(z+1)^{2+1}} dz} \quad -(2)$$

By Cauchy's Integral Formula

$$\boxed{I = \int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a).}$$

Hence eq^n (2) becomes

$$I = \int_C \frac{f(z)}{(z+1)^{2+1}} dz = \frac{2\pi i}{2!} f^2(-1). \quad -(3)$$

Here $f(z) = e^{2z}$

$\Rightarrow f'(z) = 2e^{2z}$

$\Rightarrow f^2(z) = 4e^{2z}$

$$\Rightarrow \boxed{f^2(-1) = 4e^{-2}} \quad -(4)$$

Put above value from eq^n (4) in eq^n (3), we get

$$I = \int_C \frac{e^{2z}}{(z+1)^3} dz = \frac{\cancel{2}\pi i}{\cancel{2}} \times 4e^{-2}$$

$$\Rightarrow \boxed{I = 4\pi e^{-2} i} \quad \text{Ans.}$$

This is the required solution.

(b) Evaluate $\int_C \frac{1}{(z+4)z^8} dz$ where C is the circle $|z| = 2$

(RGPV June 2022)

Solution :- let $I = \int_C \frac{1}{(z+4)z^8} dz$ —(1)

Here poles are given by $(z+4)z^8 = 0$

$\Rightarrow z = 0, -4$

For $z=0$, $|0| = 0 < 2$

Hence $z=0$ lies inside C.

For $z=-4$, $|-4| = 4 > 2$

Hence $z=-4$ lies outside C.

Hence

$$\text{Res}[f(z), z=0] = \frac{1}{(m-1)!} \lim_{z \to 0} \frac{d^{m-1}}{dz^{m-1}} (z-0)^m f(z)$$ —(2)

Here m is a order of a pole $z=0$.

So here $m = 8$.

Hence

$$\text{Res}[f(z), z=0] = \frac{1}{7!} \lim_{z \to 0} \frac{d^7}{dz^7} (z)^8 f(z)$$

$$= \frac{1}{7!} \lim_{z \to 0} \frac{d^7}{dz^7} \cancel{z^8} \times \frac{1}{(z+4)\cancel{z^8}}$$

$\left[\text{Here } f(z) = \frac{1}{(z+4)z^8} \right]$

$$\Rightarrow \quad Res[f(z), z=0] = \frac{1}{7!} \lim_{z \to 0} \frac{d^7}{dz^7}\left(\frac{1}{(z+4)}\right) \quad (3)$$

$$i) \quad \frac{d}{dz}\left(\frac{1}{(z+4)}\right) = \frac{-1}{(z+4)^2}$$

$$ii) \quad \frac{d^2}{dz^2}\left(\frac{1}{(z+4)}\right) = \frac{d}{dz}\left(\frac{-1}{(z+4)^2}\right) = -1 \times -2 (z+4)^{-3}$$

$$iii) \quad \frac{d^3}{dz^3}\left(\frac{1}{z+4}\right) = \frac{d}{dz}\left(-1 \times -2 (z+4)^{-3}\right) = -1 \times -2 \times -3 (z+4)^{-4}$$

$\vdots \qquad \vdots$

$$vii) \quad \frac{d^7}{dz^7}\left(\frac{1}{z+4}\right) = -1 \times -2 \times -3 \times -4 \times -5 \times -6 \times -7 (z+4)^{-8}$$

$$= -7! (z+4)^{-8}$$

Hence eqⁿ (3) becomes

$$Res[f(z), z=0] = \frac{1}{7!} \lim_{z \to 0} \times \left(\frac{-7!}{(z+4)^8}\right)$$

$$= -\frac{1}{\cancel{7!}} \times \cancel{7!} \lim_{z \to 0} \left(\frac{1}{(z+4)^8}\right)$$

$$= -\frac{1}{4^8}$$

Hence by Residue theorem

$$\int_C f(z)\,dz = 2\pi i \left[\text{Sum of Residue of } f(z) \text{ inside } C\right]$$

$$\Rightarrow \int_C \frac{1}{(z+4)z^8}\,dz = 2\pi i \left[Res[f(z), z=0]\right]$$

$$= 2\pi i\left[-\frac{1}{4^8}\right]$$

$$= -\pi i\left(\frac{2}{2^{16}}\right) \quad \left[\because 4^8 = (2^2)^8 = 2^{16}\right]$$

$$= \frac{-\pi i}{2^{15}}$$

Ques 9) (a) Prove that an analytic function with Constant modulus is constant.

(RGPV June 2023)

Solution :- Let $f(z) = U + iV$ be an analytic function.

$\Rightarrow$ $f(z)$ satisfies C-R Eq^n

ie
$$\left.\begin{array}{l} \boxed{U_x = V_y} \;\& \\ \boxed{U_y = -V_x} \end{array}\right\} \text{①}$$

Also given that $|f(z)|$ is Constant.

ie $|f(z)| = \sqrt{U^2+V^2} = C_1$

$$\Rightarrow \boxed{U^2 + V^2 = C_1^2} \quad \text{—②}$$

Partially differentiating eq^n ② wrt x & y respectively.

$$2U U_x + 2V V_x = 0.$$

$$\Rightarrow \boxed{U U_x + V V_x = 0} \quad \text{—③}$$

Similarly $2UU_y + 2VV_y = 0$

$$\Rightarrow \boxed{UU_y + VV_y = 0} \quad -(4).$$

From C-R eq^n (1), put values of U_x & U_y in eq^n (3):

$$\boxed{UV_y - VU_y = 0.} \quad -(5).$$

Multiply eq^n (4) by U & eq^n (5) by V.

$$U^2 U_y + \cancel{UVV_y} = 0.$$

$$\cancel{UVV_y} - V^2 U_y = 0.$$

$$- \quad +$$

$$U^2 U_y + V^2 U_y = 0.$$

$$\Rightarrow (U^2 + V^2)\, U_y = 0.$$

$$\Rightarrow C_1^2 U_y = 0. \quad \left\{ \because U^2 + V^2 = C_1^2 \right\}$$

$$\Rightarrow \boxed{U_y = 0.}$$

Again Multiply eq^n (4) by V & eq^n (5) by U.

$$\cancel{UVU_y} + V^2 V_y = 0.$$

$$U^2 V_y - \cancel{UVU_y} = 0$$

$$\Rightarrow (U^2 + V^2)\, V_y = 0.$$

$$\Rightarrow C_1^2 V_y = 0.$$

$$\Rightarrow \boxed{V_y = 0}$$

Hence $U_y = -V_x = 0$
and $V_y = U_x = 0$ $\Bigg\}$ (5)

From eqⁿ (5) $U_x = 0$ and $U_y = 0$

$U_x = 0 \Rightarrow U = F(y)$ alone.

$U_y = 0 \Rightarrow U = F(x)$ alone.

$\Rightarrow U =$ Constant
let $\boxed{U = C_2}$

Similarly from eqⁿ (5) $V_x = 0$ and $V_y = 0$.

$V_x = 0 \Rightarrow V = F(y)$ alone
$V_y = 0 \Rightarrow V = F(x)$ alone.

$\Rightarrow V =$ Constant.
$\Rightarrow \boxed{V = C_3}$

Hence if $U = C_2$ & $V = C_3$

then $f(z) = U + iV$

$\Rightarrow \boxed{f(z) = C_2 + iC_3}$

$\Rightarrow \boxed{f(z) = C.}$ Ans.

ie $f(z)$ is Constant.

Que 9) (b) use Cauchy Integral formula to solve

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$$ where c is the circle $|z| = 3$.

(RGPV June 2023)

Solution : Let $I = \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$. — ①

Here the poles are given by $(z-1)(z-2) = 0$.

$\Rightarrow z = 1, 2$.

For $z = 1$, $|1| = 1 < 3 \Rightarrow z = 1$ lies inside c.

For $z = 2$, $|2| = 2 < 3 \Rightarrow z = 2$ lies inside c.

Also $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

Hence eqn ① becomes.

$$I = \oint_{|z|=3} [\sin \pi z^2 + \cos \pi z^2]\left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$$

$$\Rightarrow I = \int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$- \int_{|z|=3} \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz$$

— ②

Now by Cauchy Integral Formula

$$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i f(a),$$ where $f(z)$ is analytic

Hence eqn ② becomes.

$$I = 2\pi i f(2) - 2\pi i f(1). \quad — ③$$

Here $f(z) = \sin \pi z^2 + \cos \pi z^2$

Hence $f(2) = \sin \pi 4 + \cos \pi 4$.

$$\Rightarrow f(2) = \sin 4\pi + \cos 4\pi$$

$$\Rightarrow f(2) = 1 \quad \left\{ \because \sin 4\pi = 0 \atop \cos 4\pi = 1 \right\}$$

Also $f(1) = \sin \pi \cdot 1 + \cos \pi \cdot 1$

$$\Rightarrow f(1) = \sin \pi + \cos \pi$$

$$\Rightarrow f(1) = -1 \quad \left[\because \sin\pi = 0 \text{ \& } \atop \cos\pi = -1 \right]$$

Hence by eqⁿ (3).

$$I = 2\pi i \times 1 - 2\pi i(-1)$$
$$I = 2\pi i + 2\pi i$$
$$I = 4\pi i$$

Ques 9) (C) Using complex integration method, solve : $\int_0^{2\pi} \frac{\cos 4\theta}{5 + 4\cos\theta} d\theta$

(RGPV June 2023

Solution :- Let

$$I = \int_0^{2\pi} \frac{\cos 4\theta}{5+4\cos\theta} d\theta \text{ —— (i)}$$

Also. Let

$$I_1 = \int_0^{2\pi} \frac{e^{i4\theta}}{5+4\cos\theta} d\theta \text{ — (ii)}$$

then clearly I = Real part of I_1.

So we will evaluate I_1 and its real part will give required integral I

So. $$\boxed{I_1 = \int_0^{2\pi} \frac{e^{i4\theta}}{5+4\cos\theta}\,d\theta.}$$

put $z = e^{i\theta}$

$\Rightarrow dz = i e^{i\theta} d\theta$

$\Rightarrow dz = i z\, d\theta$

Hence $d\theta = \dfrac{dz}{iz}$ — (iii)

As $z = e^{i\theta} \Rightarrow \dfrac{1}{z} = e^{-i\theta}$

$\Rightarrow \quad z + \dfrac{1}{z} = 2\cos\theta$

ie $\boxed{\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)}$

$\left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$

Hence (ii) becomes

$$I_1 = \int_{|z|=1} \frac{z^{\cancel{4}3}}{5+\frac{4}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{i\cancel{z}}$$

$$\Rightarrow I_1 = \frac{1}{i}\int_{|z|=1} \frac{z^3}{5+2\left(z+\frac{1}{z}\right)}\,dz$$

$$\Rightarrow I_1 = \frac{1}{i}\int_{|z|=1} \frac{z^4}{5z+2z^2+2}\,dz.$$

$$\Rightarrow I_1 = -i\int_{|z|=1} \frac{z^4}{2z^2+5z+2}\,dz. \quad -\text{(iv)}$$

Hence the poles are given by

$$2z^2+5z+2 = 0.$$

$$2z^2 + 4z + z + 2 = 0$$
$$\Rightarrow 2z(z+2) + 1(z+2) = 0$$
$$\Rightarrow (z+2)(2z+1) = 0$$
$$\Rightarrow z = -2,\ z = -\frac{1}{2}$$

At $z = -2$, $|-2| = 2 > 1$

(Hence $z = -2$ lies outside c)

At $z = -\frac{1}{2}$, $\left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

(Hence $z = -\frac{1}{2}$ lies inside c)

Hence
$$\boxed{I_1 = -i \int_{|z|=1} \frac{z^4}{(z+2)(2z+1)}\, dz} \quad -(v)$$

~~Hence Res~~ let $f(z) = \frac{z^4}{(z+2)(2z+1)}$

then $\boxed{\text{Res}\left[f(z), z = -\frac{1}{2}\right] = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) f(z)}$

(since $z = -\frac{1}{2}$ is a simple pole)

$$\Rightarrow \text{Res}\left[f(z), z = -\frac{1}{2}\right] = \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \frac{z^4}{(z+2)(2z+1)}$$

$$= \frac{1}{2} \lim_{z \to -\frac{1}{2}} \cancel{\left(z + \frac{1}{2}\right)} \frac{z^4}{(z+2)\cancel{\left(z + \frac{1}{2}\right)}}$$

$$= \frac{1}{2} \left[\frac{\left(-\frac{1}{2}\right)^4}{\left(-\frac{1}{2} + 2\right)} \right]$$

$$= \frac{1}{2}\left[\frac{1/16}{3/2}\right].$$

$$= \frac{1}{2} \times \frac{1}{\cancel{16}_{8}} \times \frac{\cancel{2}}{3}$$

$$= \frac{1}{48}. \quad \text{---(vi)}$$

Hence by Residue Theorem

$$I_1 = -i \int_{|z|=1} \frac{z^4}{(z+2)(2z+1)} dz$$

$$= -i\left[2\pi i \left\{ Res\left[f(z), z = -\frac{1}{2}\right] \right\}\right]$$

$$= -2\pi i^2 \left(\frac{1}{48}\right)$$

$$= \frac{2\pi}{48} \qquad (i^2 = -1)$$

$$= \frac{\pi}{24}$$

Hence $\boxed{I_1 = \frac{\pi}{24}.}$

So I = Real part of I_1

I = Real part of $\left(\frac{\pi}{24}\right)$.

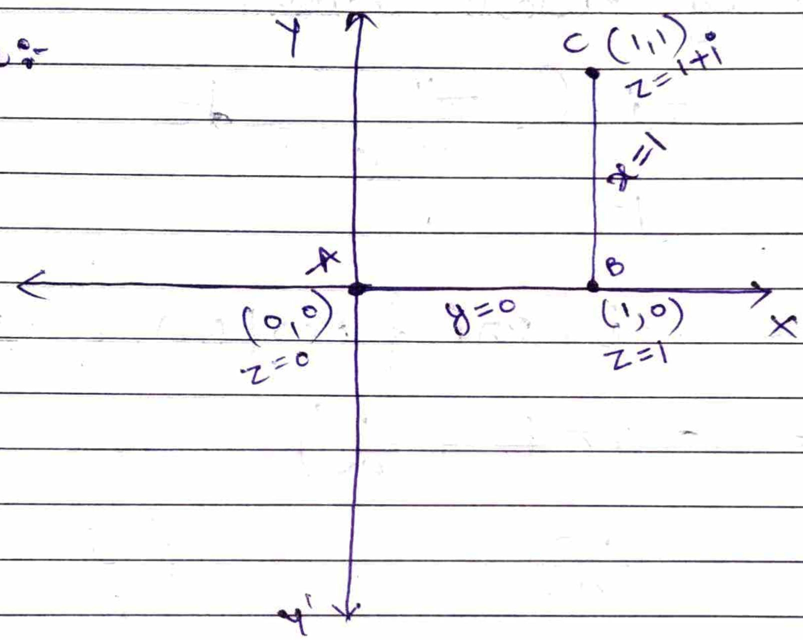
$\boxed{I = \frac{\pi}{24}.}$ Ans.

This is the required solution.

Ques 10) Solve : $\int_0^{1+i} (x - y + ix^2)\, dz$ along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1+i$

(RGPV June 2023).

Solution :-



Given integral

$$I = \int_0^{1+i} (x - y + ix^2)\, dz \quad \text{---(1)}$$

So we have to integrate along A to B & then along B to C.

Hence eqn (1) can be written as

$$\boxed{I = \int_A^B (x - y + ix^2)\, dz + \int_B^C (x - y + ix^2)\, dz. \quad \text{---(2)}}$$

Along the line AB :- x varies from 0 to 1
and $y = 0$
$\Rightarrow dy = 0$.

Hence

$$\int_A^B (x - y + ix^2)\,dz = \int_A^B (x + ix^2)(dx + idy)$$

$$\left[\because \begin{array}{l} z = x + iy \\ dz = dx + idy \end{array}\right].$$

$$\Rightarrow \int_A^B (x - y + ix^2)\,dz = \int_A^B (x + ix^2)(dx)$$

$$(\because dy = 0).$$

$$= \int_0^1 (x\,dx + ix^2\,dx)$$

$$= \left(\frac{x^2}{2} + i\frac{x^3}{3}\right)_0^1$$

$$\left[= \left(\frac{1}{2} + \frac{i}{3}\right) - 0 \right] \quad \text{—(3)}$$

Along the line BC : y varies from 0 to 1
and $x = 1$
$\Rightarrow dx = 0$.

$$\text{Hence} \int_B^C (x - y + ix^2)\,dz = \int_B^C (1 - y + i)(dx + idy)$$

$$\left[\because \begin{array}{l} z = x + iy \\ \Rightarrow dz = dx + idy \end{array}\right].$$

$$= \int_B^C (1 - y + i)(0 + idy)$$

$$= \int_0^1 i(1-y+i)\,dy$$

$$= i\left[y - \frac{y^2}{2} + iy\right]_0^1$$

$$= i\left[1 - \frac{1}{2} + i\right]$$

$$= i\left[\frac{1}{2} + i\right]$$

$$\left[= \frac{i}{2} - 1 \right] \quad \text{--- (4)}$$

Put values from eq^n (3) & eq^n (4) in eq^n (2)

Hence

$$I = \left(\frac{1}{2} + \frac{i}{3}\right) + \left(\frac{i}{2} - 1\right)$$

$$\Rightarrow I = \left(\frac{1}{2} - 1\right) + \frac{i}{3} + \frac{i}{2}$$

$$\Rightarrow I = -\frac{1}{2} + \frac{2i + 3i}{6}$$

$$\Rightarrow \boxed{I = \frac{5}{6}i - \frac{1}{2}.}$$ Ans

This is the required integral.