Queen's College
CAPE Pure Mathematics – Unit 2
Evaluation Test Christmas Term 2013

Tutor: Rudolph Deoraj
Time: 1 hour 30 minutes

You are required to answer all questions. All working must be shown. Answers given without relevant working will gain no reward. This paper consists 9 questions. The maximum mark is 75.

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1. A sequence $a_1,\ a_2,\ a_3,\ldots$ is defined by

$$a_1 = k,$$
$$a_{n+1} = 3a_n + 5,\quad n \geq 1,$$

where *k* is a positive integer.

(a) Write down an expression for a_2 in terms of *k*. (1)

(b) Show that $a_3 = 9k + 20$. (2)

(c) (i) Find $\sum_{r=1}^{4} a_r$ in terms of *k*. (3)

(ii) Show that $\sum_{r=1}^{4} a_r$ is divisible by 10. (1)

Total [7 marks]

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2. A boy saves some money each week over a period of *m* weeks. He saves $10 in week 1, $20 in week 2, $30 in week 3 and so on so that his weekly savings form an arithmetic sequence. He saves a total of $1350 in the *m* weeks.

(a) Show that $m\,(m + 3) = 15 \times 18$. (4)

(b) Hence write down the value of *m*. (1)

Total [5 marks]

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3. The adult population of a town is 25 000 at the end of Year 1. A model predicts that the adult population of the town will increase by 3% each year, forming a geometric sequence.

The model predicts that Year N will be the first year in which the adult population of the town exceeds 40 000.

(a) Show that $(N-1)\ln 1.03 > \ln 1.6$ (3)

(b) Find the value of N. (2)

Total [5 marks]

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4. (a) Prove by mathematical induction that

$$\sum_{r=1}^{n}(r+2)(r+3)=\frac{1}{3}n(n^2+9n+26)$$

for all positive integers n. (7)

(b) Hence show that

$$\sum_{r=n+1}^{3n}(r+2)(r+3)=\frac{2}{3}n(an^2+bn+c)$$

where a, b and c are integers to be found. (4)

Total [11 marks]

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5. (*a*) Find the binomial expansion of $\sqrt[3]{(8-9x)}$, in ascending powers of x, up to and including the term in x^3. Stating the value of x for which the expansion is valid.

(6)

(*b*) Use your expansion to estimate an approximate value for $\sqrt[3]{7100}$, giving your answer to 4 decimal places. State the value of x, which you use in your expansion, and show all your working.

(3)

Total [9 marks]

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6.

$$f(x) = \frac{1}{2}x^4 - x^3 + x - 3$$

(a) Show that the equation $f(x) = 0$ has a root α between $x = 2$ and $x = 2.5$. (3)

(b) Starting with the interval [2, 2.5] use interval bisection twice to find an interval of width 0.125 which contains α. (4)

The equation $f(x) = 0$ has a root β in the interval [–2, –1].

(c) Use linear interpolation once to find a first approximation for β, correct to 3 decimal place. (2)

(d) Using your result from (c), apply the Newton-Raphson process once to $f(x)$ to obtain a second approximation to β. Give your answer to 2 decimal places. (4)

Total [12 marks]

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7. (a) Seven friends together with their respective partners all meet up for a social get together. They arrange for a photograph to be taken of all 14 of them standing in a line.
How many different arrangements are there if

(i) each friend is standing next to his or her partner (3)

(ii) the seven friends all stand together and the seven partners all stand together? (2)

(Answers may be given in factorial form)

(b) A group of nine people consists of 2 boys, 3 girls and 4 adults. In how many ways can a team of 4 be chosen if

(i) both boys are in the team (2)

(ii) the adults are either all in the team or all not in the team (3)

(iii) at least 2 girls are in the team? (2)

Total [12 marks]

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8. Maria has 3 pre-set stations on her radio. When she switches on her radio, there is a probability of 0.3 that it will be set to station 1, with probabilities that it will be set to stations 2 and 3 being 0.45 and 0.25 respectively. On station 1 the probability that the presenter is a male is 0.1, and the probabilities that a male presenter will be on stations 2 and 3 are 0.85 and p respectively.

When Maria switches on her radio, the probability that it is set to station 3 and the presenter is male is 0.075.

(a) Show that the value of p is 0.3. (1)

(b) Given that Maria switches on and hears a male presenter, find the probability that the radio was set to station 2.
(4)

Total [5 marks]

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9. The matrix $\mathbf{P}$ is given by $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & d \\ -1 & 0 & 1 \end{pmatrix}$, where d is constant.

(a) Find the value of d for which $\mathbf{P}$ is non-invertible. (2)

A system of linear equations is given by

$$x + y = 1, \quad 2x + y + dz = -4, \quad -x + z = 1.$$

(b) By reducing $\mathbf{P}$ to echelon form determine the value of d for which the system of linear equations is consistent.
(3)

(c) Hence, or otherwise, solve system of linear equations. (4)

Total [9 marks]

End of Test

Formulae

Arithmetic Progression: $T_n = a + (n-1)d$ $\quad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Geometric Progression: $T_n = ar^{n-1}$ $\quad S_n = \frac{a\left(1 - r^n\right)}{1 - r}$ $\quad S_\infty = \frac{a}{1 - r}$ $\quad |r| < 1$

Binomial Series: $(a + b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + b^n \quad (n \in N)$

where $\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \dots \quad |x| < 1 \quad (n \in R)$$

Newton-Raphson Iteration: $x_{n+1} = x_n - \left(\frac{\mathrm{f}(x_n)}{\mathrm{f}^1(x_n)}\right)$

Statistics: $P(A \cap B) = P(A) \times P(A \mid B)$