Queens College

Caribbean Advanced Proficiency Examination

Pure Mathematics – Unit 2

Paper 02

Time : 2 hours 30 minutes

End of Term Test Easter 2014

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This examination paper consists of THREE sections: Complex Numbers & Calculus II, Sequences, Series 7 Approximations, Counting, Matrices & Differential Equations.

Each section consists 2 questions.

The maximum mark for each question is 50.

The maximum mark for this examination is 150.

INSTRUCTIONS TO CANDIDATES

1. Answer **ALL** questions from the **THREE** sections.
2. Unless otherwise stated in the question, any numerical answers **MUST** given exactly **OR** to three significant figures as appropriate.

Section A (Module 1)
Answer Both Questions

1. (a) Given that $xy + 2x^2y^2 = 3x$, show that $\dfrac{dy}{dx} = -\dfrac{2}{5}$ when $x = 1$ and $y = 1$. (5)

(b) Show that the equation $z = x^2 - y^2$ satisfies the equation $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$. (5)

(c) On a single Argand diagram sketch the loci given by

(i) $|z - 2 - 3i| = 2$, (2)

(ii) $\arg(z - 2 - 3i) = \dfrac{\pi}{4}$. (2)

(iii) Hence shade the region that satisfies both

$|z - 2 - 3i| \le 2$ and $\arg(z - 2 - 3i) \le \dfrac{\pi}{4}$. (1)

(d) (i) Find the complex number $v = x + iy$ such that $x, y \in \mathbb{R}$ and $v^2 = 3 + 4i$. (5)

(ii) Hence, or otherwise, solve for z the equation

$$z^2 - (4 + 3i)z + 1 + 5i = 0. \qquad (5)$$

Total 25 marks

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2. (a) (i) Use the substitution $u = 1 - x^2$ to evaluate $\int \frac{x}{\sqrt{(1 - x^2)}}\,dx$. (3)

(ii) Hence, find $\int_0^1 \cos^{-1}(x)\,dx$. (5)

(b) If $I_n = \int_0^{\pi/2} x^n \cos(x)\,dx$, prove that $I_n = \left(\frac{\pi}{2}\right)^n - n(n-1)I_{n-2}$, $n \geq 2$. (7)

(c) (i) Show that $\frac{x^3 + x^2 + 2x + 1}{x(x^2 + 1)}$ can be written in the form $A + \frac{B}{x} + \frac{Cx + D}{x^2 + 1}$, where A, B, C and D are constants to be determined. (5)

(ii) Hence evaluate $\int_1^2 \frac{x^3 + x^2 + 2x + 1}{x(x^2 + 1)}\,dx$. (5)

Total 25 marks

Section B (Module 2)
Answer Both Questions

3. (a) A sequence $\{u_n\}$ is defined by the recurrence relation

$$u_{n+1} = u_n + 2n + \frac{1}{2},\ n \in \mathbb{N},\ u_1 = 2.$$

(i) State the first four terms of the series. (2)

(ii) Prove by mathematical induction that $u_n = \frac{2n^2 - n + 3}{2}$. (6)

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3. (b) Given that $f(k) = \frac{1}{k^3}$, $k > 0$, show that

(i) $f(k) - f(k+1) = \frac{3k^2 + 3k + 1}{k^3 (k+1)^3}$. (2)

(ii) $\sum_{r=1}^{n} \frac{3r^2 + 3r + 1}{r^3 (r+1)^3} = 1 - \frac{1}{(n+1)^3}$. (5)

(iii) Find $\sum_{r=1}^{\infty} \frac{3r^2 + 3r + 1}{r^3 (r+1)^3}$. (2)

(c) (i) Obtain the first 4 non-zero terms of the Taylor Series expansion of $\sin x$ in ascending powers of $\left(x - \frac{\pi}{4}\right)$. (5)

(ii) Hence, determine $\sin 46^0$, correct to 5 decimal places. (3)

Total 25 marks

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4. (a) (i) Expand $\sqrt{\frac{1+x}{1-x}}$ in ascending powers of x up to and including the term in x^3, stating the values of x for which the expansion is valid. (3)

(ii) Use your expansion to show that $\sqrt{11} \approx \frac{663}{200}$. (3)

(b) Show that the coefficient of x^3 in the expansion of $\left(1 - \frac{x}{4}\right)^6 \left(\frac{1}{2+x}\right)^6$ is $-\frac{9}{32}$. (8)

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4. (c) (i) Using the Intermediate Value Theorem, show that the equation $x^2 - 4\cos x = 0$ has a positive root in the interval [1, 2]. (3)

(ii) Using linear interpolation once find a first approximation to this root, x, correct to 3 decimal places. (3)

(iii) Use the Newton-Raphson method to show that

$$x_{n+1} = \frac{x_n^{\ 2} + 4x_n \sin x_n + 4\cos x_n}{2x_n + 4\sin x_n}. \qquad (3)$$

(iv) Hence, using your approximation for x found in (c) (ii), and the Newton-Raphson method in (c) (iii) twice, find an approximation to this root correct to 3 decimal places.

(2)

Total 25 marks

Section C (Module 3)
Answer Both Questions

5. (a) (i) How many numbers can be formed using the digits 1, 3, 3, 4, 5, 7, 8? (2)

(ii) How many of these numbers are odd? (3)

(iii) How many of these numbers are even? (2)

(b) Ian travels to school either by bus or by taxi or by private car. The probability that he travels by bus is 0.2 and the corresponding probabilities that he travels by taxi or by private car are 0.3 and 0.5 respectively. The probability that he arrives on time if he travels by bus is 0.6. The probabilities that he arrives on time if he travels by taxi or by private car are 0.9 and 0.8 respectively.

(i) Draw a carefully labeled tree diagram to illustrate this information. (3)

(ii) What is the probability that Ian arrives on time? (2)

(iii) Given that Ian does not arrive on time, what is the probability that he travelled by private car?

(3)

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5. (c) A system of linear equations is given by

$$\begin{aligned} x + y + z &= 4 \\ 2x + y - 3z &= 2 \\ 4x + 3y - z &= 10. \end{aligned}$$

(i) Show that the system has no unique solution. (4)

(ii) Row reduce the augmented matrix of the system of equations to echelon form and deduce that the system has infinitely many solutions. (3)

(iii) Hence, find the general solution of the system of linear equations. (3)

Total 25 marks

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6. (a) In a certain chemical reaction the amount, y grams, of a substance present is decreasing. The rate of decrease of y is proportional to the product of y and t where t is the time in seconds after the start of the reaction.

(i) Show that the differential equation that satisfies this situation is given by $\frac{\mathrm{d}y}{\mathrm{d}t} = -kyt$, where k is a positive constant. (2)

(ii) At the start of the reaction $y = 50$. Show that $y = 50\mathrm{e}^{-\frac{1}{2}kt^2}$. (4)

Ten seconds after the start of reaction the amount of substance present is 40 grams.

(iii) Find the time after the start of the reaction at which the amount of substance present is 25 grams. (6)

(b) (i) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 8\frac{\mathrm{d}y}{\mathrm{d}x} + 12y = \cos x + \sin x. \quad (8)$$

(ii) Show that when x is large and positive, $y \approx \frac{5\sqrt{2}}{69}\cos\left(x - 81.9^0\right)$. (5)

Total 25 marks

End of Test

Formulae

Binomial Expansion: $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + b^n$ where n is a positive integer.

$$\binom{n}{r} = \frac{n!}{(n-r)!\, r!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

for n real and $|x| < 1$

Newton-Raphson iteration: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Maclaurin's Series:

$$\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \mathrm{f}''(0)\frac{x^2}{2!} + \mathrm{f}'''(0)\frac{x^3}{3!} + \ldots + \mathrm{f}^r(0)\frac{x^r}{r!} + \ldots$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad (-1 < x \le x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

Differentiation:

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\tan kx$	$k\sec^2 kx$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$	$\dfrac{\mathrm{f}^1(x)\mathrm{g}(x)-\mathrm{g}'(x)\mathrm{f}(x)}{[\mathrm{g}(x)]^2}$
$\arcsin x$	$\dfrac{1}{\sqrt{(1-x^2)}}$
$\arccos x$	$-\dfrac{1}{\sqrt{(1-x^2)}}$
$\arctan x$	$\dfrac{1}{1+x^2}$

Integration: (+ constant: $a > 0$ where relevant)

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\,\mathrm{d}x$
$\sec^2 kx$	$\dfrac{1}{k}\tan kx$
$\tan x$	$\ln\lvert\sec x\rvert$
$\cot x$	$\ln\lvert\sin x\rvert$
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x+\cot x\rvert = \ln\lvert\tan \tfrac{1}{2}x\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$

$$\int u \frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

$\dfrac{1}{\sqrt{\left(a^2 - x^2\right)}}$	$\arcsin\left(\dfrac{x}{a}\right)$	$\lvert x \rvert < a$
$\dfrac{1}{a^2 + x^2}$	$\dfrac{1}{a}\arctan\left(\dfrac{x}{a}\right)$	

Complex Numbers:

$\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ $\qquad$ $\left\{r\left(\cos\theta + \mathrm{i}\sin\theta\right)^n\right\} = r^n \cos n\theta + \mathrm{i}\sin n\theta$