Caribbean Advanced Proficiency Examinations
Internal Moderation Test 2014
Pure Mathematics Unit 2
Module 2

School: Queen's College
Guyana

Time: 1 hour 30 minutes
Tutor: Rudolph Deoraj

Instruction to Candidates

Answer **ALL** questions. Unless otherwise stated in the question, all numerical answers **MUST** be given exactly **OR** to three significant figures as appropriate.
This paper consists 7 questions. The maximum mark on this paper is 70.

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1. (a) A sequence $\{a_n\}$ is given by $a_{n+1} = a_n + (2n+1)$, $a_1 = 1$.

   (i) Write down the values of $u_2,\ u_3,\ u_4$ and hence deduce that $a_n = n^2$.. (2)

   (ii) Prove by mathematical induction that $a_{n+1} = a_n + (2n+1)$ for all $n \in \mathbb{N}$. (6)

   (b) (i) A sequence $\{u_n\}$ is given by $u_{n+1} = \dfrac{u_n^{\ 2} + 2}{2u_n}$, $u_1 = 1$. By considering u_2, u_3, u_4, u_5 show that the sequence is convergent. (2)

   (ii) State the exact value of u_n. (1)

Total 11 marks

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2. (a) Given that $\sum_{r=1}^{m} u_r = m^2(m-1)$, show that $u_r = (r-1)(3r-2)$. (3)

(b) (i) Express $\frac{1}{r^2} - \frac{1}{(r+1)^2}$ as a single fraction in its simplest form. (1)

(ii) Hence find the sum to n terms of the series $\frac{3}{1 \times 4} + \frac{5}{4 \times 9} + \frac{7}{9 \times 16} + \ldots$

(3)

(iii) State the convergent sum of the series in (ii). (1)

Total 8 marks

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3. (a) A company determines that its plant and machinery depreciate at a rate R on the value at the beginning of each succeeding year.

(i) Find and expression, in terms of A and R, for the beginning of the n^{th} year, where A is the value of the plant and machinery at the start of the accounting period.

(3)

(ii) Given that A = \$3 000 000 and R = 5%, determine the value of the plant and machinery at the beginning of the 20^{th} year.

(2)

(b) A series is given as $4^2 + 7^2 + 10^2 + 13^2 + \ldots$ Express, using the summation $\left(\sum_{r=1}^{n} u_r\right)$, stating the value of u_r. (1)

Total 6 marks

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4. (a) Use Maclaurin's series expansion to show that for x small so that x^3 and higher powers of x may be neglected,

$$\ln\left(\frac{1-2x}{1-x}\right) = \mathrm{e}^{x} - \mathrm{e}^{2x}.$$

(6)

(b) (i) Given that y satisfies the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = (x+y)^3$,

and that $y = 1$ at $x = 0$, find expressions for $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ and $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3}$.
(You may leave your answers unsimplified). (7)

(ii) Hence, or otherwise, find y as a series in ascending powers of x up to and including the term in x^3.

(4)

Total 17 marks

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5. (a) Find the binomial expansion of $(8+x)^{1/3}$ up to and including the term in x^3, stating the value of x for which the expansion is valid.

(6)

(b) Using your expansion at (a) find $\sqrt[3]{(8.4)}$ correct to 4 decimal places.

(3)

Total 9 marks

6.

$$\mathrm{f}(x) = \tan x + 1 - 4x^2, \; -\frac{\pi}{2} < x < \frac{\pi}{2}.$$

(a) Show that the equation $\mathrm{f}(x) = 0$ has a root α in the interval [1.42, 1.44]. (3)

(b) Use linear interpolation once on the interval [1.42, 1.44] to find an estimate of α, giving your answer to 3 decimal places.

(2)

Total 5 marks

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7. (a) It is given that $e^{0.5x} + x^2 - 3.5x = 0$ has exactly one root in the interval [0, 1].

Apply the interval bisection method once to find a more accurate determination of the interval containing the root.

(3)

(b) Starting with $x_0 = 0.5$ apply the Newton-Raphson method twice to obtain a better approximation for this root in the interval [0, 1], correct to 4 decimal places.

(4)

(c) (i) Verify that a root of the equation $x + 3 = 2 \tan x$ lies in the interval [1, 1.5]

(3)

A rearrangement of the equation $x + 3 = 2 \tan x$ gives the iterative formula

$$x_{n+1} = \tan^{-1}\left(\frac{x_n + 3}{2}\right).$$

(ii) Taking $x_0 = 1$ find an approximation to this root correct to 3 decimal places.

(4)

Total 14 marks

End of Test

Formulae

Arithmetic Series: $u_n + a + (n-1)d \qquad S_n = \frac{n}{2}\{2a + (n-1)d\}$

Geometric Series: $u_n = ar^{n-1}$

$$\begin{cases} S_n = \dfrac{a(r^n - 1)}{r-1},\ r > 1 \quad S_n = \dfrac{a(1 - r^n)}{1-r},\ r < 1 \\ S_\infty = \dfrac{a}{1-r},\ |r| < 1 \end{cases}$$

Binomial Expansion: $(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + b^n$ where n is a positive integer.

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots$$

for n real and $|x| < 1$

Newton-Raphson iteration: $x_{n+1} = x_n - \dfrac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}$

Maclaurin's Series:

$$\mathrm{f}(x) = \mathrm{f}(0) + \mathrm{f}'(0)x + \mathrm{f}''(0)\frac{x^2}{2!} + \mathrm{f}'''(0)\frac{x^3}{3!} + \ldots + \mathrm{f}^r(0)\frac{x^r}{r!} + \ldots$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^r}{r!} + \ldots \quad \text{for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{r+1}\frac{x^r}{r} + \ldots \quad (-1 < x \leq x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^r\frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^r\frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$