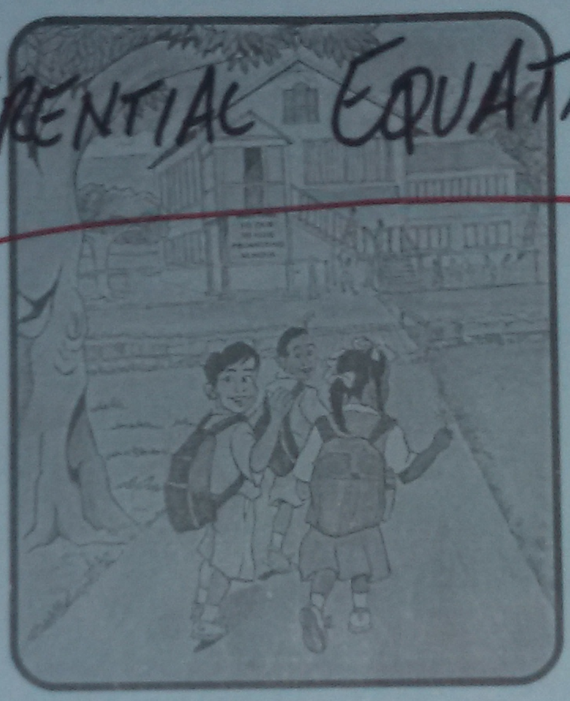
MINISTRY OF EDUCATION
GUYANA

D

EXERCISE BOOK

HEALTHY CHILDREN

DIFFERENTIAL EQUATIONS



HEALTHY ADULTS

campaign for smoke-free Schools

NAME _____

SUBJECT _____

GRADE _____

FOR FREE DISTRIBUTION

LINED

Differential Equations

$y' + y = x+3$

$\frac{dy}{dx} + y = x+3$

$f'(x) + f(x) = x+3$

$x^2 + \cos x = \sqrt{x}$

$y'' + 2y' - 3y = 0$

$y_1(x) = e^{-3x}$

$y_1' = -3e^{-3x}$

$y_1'' = 9e^{-3x}$

$9e^{-3x} + 2(-3e^{-3x}) - 3e^{-3x}$

$9e^{-3x} - 6e^{-3x} - 3e^{-3x} = 0$

$y_2(x) = e^{x}$

$y_2' = e^{x}, \; y_2'' = e^{x}$

$e^{x} + 2e^{x} - 3e^{x} = 0$

Ordinary Differential Equations

Order

Linear/Nonlinear

$$x^2 \underbrace{\frac{d^2y}{dx^2}}_{\text{order 2}} + x\frac{dy}{dx} + 2y = \sin(x)$$

linear

$$x^2\left(\frac{d^2y}{dx^2}\right)^2 = \sin x$$

non linear

$$y \cdot \frac{d^2y}{dx^2} = \sin(x)$$

Separable Equations

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \quad \text{first order} \qquad (1-y^2)\frac{dy}{dx} = x^2$$

$$\Rightarrow \int (1-y^2)\,dy = \int x^2\,dx$$

$$y - \frac{y^3}{3} + C_y = \frac{x^3}{3} + C_x$$

$$y - \frac{y^3}{3} = \frac{x^3}{3} + \cancel{C_x - C_y} \rightarrow C_0$$

$$3y - y^3 = x^3 + C$$

$$\boxed{3y - y^3 - x^3 = C}$$

$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \qquad \underline{y(0) = -1}$$

$$2(y-1)\frac{dy}{dx} = 3x^2+4x+2$$

$$\int (2y-2)\,dy = \int (3x^2+4x+2)\,dx$$

$$y^2 - 2y = x^3 + 2x^2 + 2x + C$$

$$(-1)^2 - 2(-1) = 0^3 + 2\cdot 0^2 + 2\cdot 0 + C$$

$$1 + 2 = C$$

$$C = 3$$

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3}$$

$$y^2 - 2y + 1 = x^3 + 2x^2 + 2x + 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$$y - 1 = \pm\sqrt{x^3 + 2x^2 + 2x + 4}$$

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$$

$$y = 1 \pm \sqrt{4}$$

$$y = 1 \pm 2$$

Since $y(0) = -1$

$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}}$$

$$\frac{dy}{dx} = \frac{y\cos(x)}{1+2y^2} \qquad y(0) = 1$$

$$(1+2y^2)\frac{dy}{dx} = y\cos(x)$$

$$\left(\frac{1}{y} + 2y\right)\frac{dy}{dx} = \cos(x)$$

$$\int\left(\frac{1}{y} + 2y\right)dy = \int\cos(x)\,dx$$

$$\ln|y| + y^2 = \sin(x) + C$$

$$\ln|1| + 1^2 = \sin(0) + C$$

$$0 + 1 = C$$

$$\boxed{\ln|y| + y^2 = \sin(x) + 1}$$

Exact Equations

$$\Psi(x,y) = \Psi_1(x, y(x))$$

$$\frac{d}{dx}\left[\Psi(x,y)\right] = \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

$$\Psi = f_1(x)g_1(y) + \ldots + f_n(x)g_n(y)$$

$$\frac{d\Psi}{dx} = \underline{f_1'(x)g_1(y) + f_1(x)g_1'(y)\frac{dy}{dx}}$$

$$+$$

$$\vdots$$

$$+$$

$$f_n'(x)g_n(y) + f_n(x)g_n'(y)\frac{dy}{dx}$$

$$= \left(f_1'(x)g_1(y) + \ldots + f_n'(x)g_n(y)\right) + \left(f_1(x)g_1'(y) + \ldots + f_n(x)g_n'(y)\right)\frac{dy}{dx}$$

$$= \frac{\partial \Psi}{\partial x} + \frac{\partial \Psi}{\partial y}\frac{dy}{dx}$$

$$\boxed{\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\psi\right) = \frac{\partial^2\psi}{\partial y\partial x} = \psi_{xy} \qquad \boxed{\psi_{xy} = \psi_{yx}}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\psi\right) = \frac{\partial^2\psi}{\partial x\partial y} = \psi_{yx}$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad \psi_x = M$$

$$\underline{\psi_x + \psi_y\frac{dy}{dx} = 0} \qquad \psi_y = N$$

$$\frac{d}{dx}\psi(x,y) = 0 \qquad \psi_{xy} = M_y$$

$$\psi(x,y) = c \qquad \psi_{yx} = N_x$$

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

if $M_y = N_x \Leftrightarrow$ Exact

$\Rightarrow \psi \quad \frac{d}{dx}\psi(x,y) = 0 \qquad \psi_x = M$

$$\psi(x,y) = C \qquad \psi_y = N$$

$$\underbrace{(y\cos(x) + 2xe^y)}_{M(x,y)} + \underbrace{(\sin(x) + x^2e^y - 1)}_{N(x,y)}y' = 0$$

$$M_y = \cos(x) + 2xe^y \qquad N_x = \cos(x) + 2xe^y$$

$$M_y = N_x \text{ Exact}$$

$$\psi_x = M \qquad \psi_y = N \qquad \frac{d}{dx}\psi = 0$$

$$\psi_x = y\cos(x) + 2xe^y$$

$$\int \psi_x\,dx = \int (y\cos(x) + 2xe^y)\,dx + f(y)$$

$$\psi = y\sin(x) + x^2e^y + f(y)$$

$$\frac{\partial\psi}{\partial y} = \sin(x) + x^2e^y + f'(y) = \sin(x) + x^2e^y - 1 = N$$

$$f'(y) = -1$$

$$f(y) = -y + C$$

$$\psi(x,y) = y\sin(x) + x^2e^y - y + C$$

$$\frac{d}{dx}\psi(x,y) = 0$$

$$\psi(x,y) = C$$

$$y\sin(x) + x^2e^y - y = C$$

$$\underbrace{2x+3}_{M} + \underbrace{(2y-2)}_{N}y' = 0$$

$$\frac{d}{dx}\psi = \psi_x + \psi_y y'$$

$M_y = 0 \qquad N_x = 0$

$M_y = N_x \Leftrightarrow$ exact

$$\frac{d}{dx}\psi = 0$$

$$\psi = c$$

$\psi(x,y)$

$$\boxed{\psi_x = 2x+3 \qquad \psi_y = 2y-2}$$

$$\psi = x^2 + 3x + h(y)$$

$$\psi_y = h'(y) = 2y-2$$

$$h(y) = y^2 - 2y$$

$$\psi(x,y) = x^2 + 3x + y^2 - 2y$$

$$\psi = c$$

$$\boxed{x^2 + 3x + y^2 - 2y = c}$$

$$(2x+4y)+(2x-2y)y'=0$$

$M_y = 4 \qquad N_x = 2$ Not EXACT!

$$(3x^2-2xy+2)dx+(6y^2-x^2+3)dy=0$$

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

$$(3x^2-2xy+2)+(6y^2-x^2+3)\frac{dy}{dx}=0$$

$M_y = -2x \qquad N_x = -2x$

$M_y = N_x \Leftrightarrow$ Exact!

$$\psi_x = 3x^2-2xy+2$$

$$\psi = x^3-x^2y+2x+h(y)$$

$$\psi_y = -x^2+h'(y) = 6y^2-x^2+3 = N$$

$$h'(y) = 6y^2+3$$

$$h(y) = 2y^3+3y$$

$$\psi(x,y) = x^3-x^2y+2x+2y^3+3y$$

$$\frac{d}{dx}\psi(x,y) = 3x^2 - (2xy + x^2y') + 2 + 6y^2y' + 3y'$$

$$= 3x^2 - 2xy + 2 + y'(-x^2 + 6y^2 + 3)$$

$$\underline{(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0}$$

$$\frac{d}{dx}\psi(x,y) = 0$$

$$\psi(x,y) = c$$

$$\boxed{x^3 - x^2y + 2x + 2y^3 + 3y = c}$$

Integrating factors

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

$$M_y = 3x + 2y \neq N_x = 2x + y$$

integrating factor $\Rightarrow \underbrace{\mu(x)(3xy + y^2)}_{M} + \underbrace{\mu(x)(x^2 + xy)}_{N}y' = 0$

$$M_y = \mu(x)(3x + 2y) = \mu'(x)(x^2 + xy) + \mu(x)(2x + y) = N_x$$

$$\mu(x)(3x + 2y) - \mu(x)(2x + y) = \mu'(x)(x^2 + xy)$$

$$\mu(x)(3x + 2y - 2x - y) = \mu'(x)(x^2 + xy)$$

$$\mu(x)(x + y) = \mu'(x)\, x(x + y)$$

$$\mu(x) = \mu'(x)\, x \qquad x + y \neq 0$$

$$\mu(x) = \frac{d\mu}{dx} x \qquad x \neq -y$$

$$\frac{\mu(x)}{x} = \frac{d\mu}{dx}$$

$$\frac{1}{x} = \frac{1}{\mu(x)} \frac{d\mu}{dx} \Rightarrow \frac{1}{x} dx = \frac{1}{\mu} d\mu$$

$$\Rightarrow \mu(x) = x$$

$$(3xy^2 + xy^2) + (x^3 + x^2y)y' = 0$$

$$M_y = 3x^2 + 2xy = 3x^2 + 2xy = N_x$$

$$\psi_x = 3xy^2 + xy^2$$

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y)$$

$$\psi_y = x^3 + x^2y + h'(y) = x^3 + x^2y = N$$

$$h'(y) = 0$$

$$h(y) = C_1$$

$$\frac{d}{dx}\psi = 0$$

$$\psi = C_2$$

$$\boxed{x^3y + \frac{1}{2}x^2y^2 = c} \text{ where } x \neq -y$$

$$\mu(x)(3xy + y^2) + \mu(x)(x^3 + xy)y' = 0$$

Homogenous differential equations

$\frac{dy}{dx} = f(x,y)$ $\qquad$ $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

$\frac{dy}{dx} = \frac{x+y}{x}$ $\quad$ $\frac{dy}{dx} = \frac{x}{x} + \frac{y}{x} \Rightarrow \frac{dy}{dx} = 1 + \frac{y}{x}$

$v = \frac{y}{x}$ $\qquad$ $y = xv$ $\qquad$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$v + x\frac{dv}{dx} = 1 + v$ $\qquad\qquad$ $x \neq 0$

$$\frac{dv}{dx} = \frac{1}{x}$$

$$\int dv = \int \frac{1}{x}\,dx$$

$$v = \ln|x| + c$$

$$\frac{y}{x} = \ln|x| + c$$

$\boxed{y = x\ln|x| + cx}$ where $x \neq 0$

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}\,\frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{1+3\frac{y^2}{x^2}}{2\frac{y}{x}} = \frac{1+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)}$$

$$v = \frac{y}{x},\quad y = xv,\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + xv' = \frac{1+3v^2}{2v}$$

$$2v^2 + 2xvv' = 1 + 3v^2$$

$$2xvv' = 1 + v^2$$

$$\frac{2xvv'}{1+v^2} = 1$$

$$\frac{2v}{1+v^2}\frac{dv}{dx} = \frac{1}{x}$$

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln(1+v^2) = \ln|x| + \ln|c|$$

$$\ln(1+v^2) = \ln|cx|$$

$$1+v^2 = cx \qquad cx \geq 0$$

$$1 + \left(\frac{y}{x}\right)^2 = cx \qquad c \geq 0$$

$$x^2 + y^2 = cx^3 \qquad x \geq 0$$

$$\boxed{x^2 + y^2 - cx^3 = 0} \text{ where } cx \geq 0$$

2nd order linear homogenous equations

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

2nd order

$$Ay'' + By' + Cy = 0$$

homogenous

$g(x)$ is a solution

$Ag'' + Bg' + Cg = 0$

$\boxed{C_1 g(x)}$?

$$AC_1g'' + BC_1g' + CC_1g$$

$$C_1\underbrace{(Ag'' + Bg' + Cg)}$$

$h(x)$ is a solution $C_1 \cdot 0 = 0$

is $(g(x) + h(x))$ a solution

$$A(g'' + h'') + B(g' + h') + C(g + h)$$

$$Ag'' + Ah'' + Bg' + Bh' + Cg + Ch$$

$$\underbrace{(Ag'' + Bg' + Cg)}_{=0} + \underbrace{(Ah'' + Bh' + Ch)}_{=0}$$

$$C_1 g(x) + C_2 h(x) = 0$$

— x — x —

$$y'' + 5y' + 6y = 0$$

$y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$r^2 e^{rx} + 5re^{rx} + 6e^{rx} = 0$$

$$e^{rx}(\underline{r^2 + 5r + 6}) = 0$$

$$r^2 + 5r + 6 = 0$$

$$(r+2)(r+3) = 0$$

$$r = -2, \; r = -3$$

$y_1 = e^{-2x}$

$y_1 = c_1 e^{-2x}$

$y_2 = e^{-3x}$

$y_2 = c_2 e^{-3x}$

$$\boxed{y(x) = c_1 e^{-2x} + c_2 e^{-3x}}$$

$$y'' + 5y' + 6y = 0$$

$y(0) = 2, \; y'(0) = 3$

$y(0) = \boxed{2 = c_1 \cdot + c_2}$

$y'(x) = -2c_1 e^{-2x} - 3c_2 e^{-3x}$

$$c_1 + c_2 = 2$$

$$-2c_1 - 3c_2 = 3$$

$y'(0) = \boxed{-2c_1 - 3c_2 = 3}$

$$c_1 = 2 - c_2$$

$$-2(2 - c_2) - 3c_2 = 3$$

$$-4 + 2c_2 - 3c_2 = 3$$

$$-c_2 = 7$$

$$c_2 = -7$$

$c_1 + (-7) = 2$

$c_1 = 9$

$$\boxed{y(x) = 9e^{-2x} - 7e^{-3x}}$$

—— x —— x ——

$4y'' - 8y' + 3y = 0 \qquad y(0) = 2,\ y'(0) = \frac{1}{2}$

$4r^2 - 8r + 3 = 0$

$r = \frac{+8 \pm \sqrt{64 - 4(4)(3)}}{8} = \frac{8 \pm \sqrt{64 - 48}}{8}$

$r = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8} = 1 \pm \frac{1}{2}$

$r = \frac{3}{2},\ r = \frac{1}{2} \qquad y(x) = c_1 e^{\frac{3}{2}x} + c_2 e^{\frac{1}{2}x}$

$y(0) = 2 = c_1 + c_2 = 2 \qquad y'(x) = \frac{3}{2}c_1 e^{\frac{3}{2}x} + \frac{1}{2}c_2 e^{\frac{1}{2}x}$

$y'(0) = \frac{1}{2} = \frac{3}{2}c_1 + \frac{1}{2}c_2 = \frac{1}{2}$

$c_1 = 2 - c_2$

$\frac{3}{2}(2 - c_2) + \frac{1}{2}c_2 = \frac{1}{2}$

$3(2 - c_2) + c_2 = 1$

$6 - 3c_2 + c_2 = 1$

$-2c_2 = -5$

$c_2 = \frac{5}{2}$

$c_1 = 2 - \left(\frac{5}{2}\right)$

$c_1 = \frac{-1}{2}$

$$\boxed{y(x) = \frac{-1}{2}e^{\frac{3}{2}x} + \frac{5}{2}e^{\frac{1}{2}x}}$$

Complex roots of the characteristic equation

$$Ay'' + By' + Cy = 0$$

$$Ar^2 + Br + C = 0$$

Real: $r = r_1, r_2$

$$\boxed{y = c_1 e^{r_1 x} + c_2 e^{r_2 x}}$$

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$r = \frac{-B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A}$$

$$r = \frac{-B}{2A} \pm \frac{\sqrt{4AC - B^2}}{2A} i$$

complex roots:

$$r = \lambda \pm \mu i$$

$$y = c_1 e^{(\lambda + \mu i)x} + c_2 e^{(\lambda - \mu i)x}$$

$$y = c_1 e^{\lambda x + \mu x i} + c_2 e^{\lambda x - \mu x i}$$

$$y = c_1 e^{\lambda x} e^{\mu x i} + c_2 e^{\lambda x} e^{-\mu x i}$$

$$y = e^{\lambda x}(c_1 e^{\mu x i} + c_2 e^{-\mu x i})$$

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$$y = e^{\lambda x}(c_1(\cos \mu x + i \sin \mu x) + c_2(\cos -\mu x + i \sin -\mu x))$$

$$y = e^{\lambda x}(c_1 \cos \mu x + i c_1 \sin \mu x + c_2 \cos \mu x - i c_2 \sin \mu x)$$

$$y = e^{\lambda x}(\underbrace{(c_1 + c_2)}_{c_3} \cos \mu x + \underbrace{(c_1 i - c_2 i)}_{c_4} \sin \mu x)$$

$$y = e^{\lambda x}(c_3 \cos \mu x + c_4 \sin \mu x)$$

$$r = \lambda \pm \mu i$$

$y'' + y' + y = 0 \qquad r^2 + r + 1 = 0$

$\frac{-1 \pm \sqrt{1-4}}{2} \qquad r = \frac{-1 \pm \sqrt{-3}}{2}$

$r = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i = \lambda \pm \mu i$

$y = e^{\lambda x}(c_1 \cos \mu x + c_2 \sin \mu x)$

$y = e^{-\frac{1}{2}x}(c_1 \cos(\frac{\sqrt{3}}{2} x) + c_2 \sin(\frac{\sqrt{3}}{2} x))$

$r = \lambda \pm \mu i$

$y = e^{\lambda x}(c_1 \cos \mu x + c_2 \sin \mu x)$

$y'' + 4y' + 5y = 0 \qquad y(0) = 1, \; y'(0) = 0$

$r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2}$

$r = \underset{\lambda}{-2} \pm \underset{\mu}{1} i \qquad y = e^{-2x}(c_1 \cos x + c_2 \sin x)$

$1 = 1(c_1 \overset{\to 1}{\cos 0} + c_2 \overset{\to 0}{\sin 0}) \qquad y = e^{-2x}(\cos x + c_2 \sin x)$

$1 = c_1$

$y' = -2e^{-2x}(\cos x + c_2 \sin x)$
$+ e^{-2x}(-\sin x + c_2 \cos x)$

$$y'(0) = 0 = -2(1+0) + 1(0 + C_2)$$

$$0 = -2 + C_2$$

$$\underline{\underline{C_2 = 2}}$$

$$\boxed{y(x) = e^{-2x}(\cos x + 2\sin x)}$$

—— x —— x ——

Repeated roots of the characteristic equation

$y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$(r+2)(r+2) = 0$

$r = -2$

$y = ce^{-2x}$

$\cancel{y(0) = A}$

$\cancel{y'(0) = 5A}$

$\cancel{A = ce^0 \Rightarrow c = A}$

$\cancel{y = Ae^{-2x}}$

$\cancel{y' = -2Ae^{-2x}}$

$\cancel{5A = -2Ae^0}$

$\cancel{5A = -2A}$

$g'' = v''e^{-2x} - 2v'e^{-2x}$

$-2v'e^{-2x} + 4ve^{-2x}$

$g'' = v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}$

$g = v(x)e^{-2x}$

$g' = v'e^{-2x} + v(-2e^{-2x})$

$g' = v'e^{-2x} - 2ve^{-2x}$

$e^{-2x}(v'' - 4v' + 4v + 4v' - 8v + 4v) = 0$

$e^{-2x}v''(x) = 0 \Rightarrow v''(x) = 0$

$v'(x) = c_1$

$v(x) = c_1x + c_2$

$g = v(x)e^{-2x} \Rightarrow g = (c_1x + c_2)e^{-2x}$

$= c_1xe^{-2x} + c_2e^{-2x}$

$$y'' - y' + .25y = 0 \qquad y(0) = 2$$
$$r^2 - r + .25 = 0 \qquad y'(0) = \frac{1}{3}$$

$$r = \frac{+1 \pm \sqrt{1 - 4(1)(.25)}}{2} = \frac{1}{2}$$

$$\cancel{y = ce^{-2x}} \qquad \cancel{y = v(x)\, e^{\frac{1}{2}x}}$$

$$\cancel{c_1 + xc_2}$$

$$y = c_1 e^{\frac{1}{2}x} + c_2 x e^{\frac{1}{2}x}$$
$$y' = \frac{1}{2} c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x} + \frac{1}{2} c_2 x e^{\frac{1}{2}x}$$
$$y' = e^{\frac{1}{2}x}\left(\frac{c_1}{2} + c_2\right) + \frac{c_2}{2} x e^{\frac{1}{2}x}$$

$$y(0) = 2 = c_1$$
$$y'(0) = \frac{1}{3} = 1\left(\frac{2}{2} + c_2\right) + \frac{c_2}{2}(0)e^0$$
$$\frac{1}{3} = 1 + c_2 \qquad c_2 = -\frac{2}{3}$$

$$\boxed{y = 2e^{\frac{1}{2}x} - \frac{2}{3} x e^{\frac{1}{2}x}}$$

Undetermined coefficients

$$Ay'' + By' + Cy = g(x)$$

h is ~~homogeneous~~ solution for homogeneous

$$Ah'' + Bh' + Ch = 0$$

$$Aj'' + Bj' + Cj = g(x) \Rightarrow j(x) \text{ is a particular solution}$$

$$A(h'' + j'') + B(h' + j') + C(h + j)$$

$$\underbrace{Ah'' + Bh' + Ch}_{0} + \underbrace{Aj'' + Bj' + Cj}_{g(x)}$$

$$k(x) = h(x) + j(x)$$

$$y'' - 3y' - 4y = 3e^{2x}$$

$$y'' - 3y' - 4y = 0$$

$$r^2 - 3r - 4 = 0$$

$$(r-4)(r+1) = 0$$

$$r = 4, \text{ or } -1$$

$$y_g = C_1 e^{4x} + C_2 e^{-x}$$

$$y = C_1 e^{4x} + C_2 e^{-x} - \frac{1}{2} e^{2x}$$

$$y_p = Ae^{2x}$$

$$y_p' = 2Ae^{2x}$$

$$y_p'' = 4Ae^{2x}$$

$$4Ae^{2x} - 3(2Ae^{2x}) - 4Ae^{2x}$$

$$4Ae^{2x} - 6Ae^{2x} - 4Ae^{2x} = 3e^{2x}$$

$$-6A = 3$$

$$A = \frac{-1}{2}$$

$$y_p = \frac{-1}{2} e^{2x}$$

$$y'' - 3y' - 4y = 2\sin x$$

$$y_h = C_1 e^{4x} + C_2 e^{-x}$$

$$y = A\sin x + B\cos x$$

$$y' = A\cos x - B\sin x$$

$$y'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x - 3(A\cos x - B\sin x) - 4(A\sin x + B\cos x) = 2\sin x$$

$$-A\sin x - B\cos x$$

$$3B\sin x - 3A\cos x$$

$$-4A\sin x - 4B\cos x$$

$$(-5A + 3B)\sin x + (-3A - 5B)\cos x = 2\sin x$$

$$\boxed{\begin{aligned} -5A + 3B &= 2 \\ -3A - 5B &= 0 \end{aligned}}$$

$$-5A+3B=2 \Rightarrow A=\frac{3B-2}{5}$$

$$-3A-5B=0$$

$$-3\left(\frac{3B-2}{5}\right)-5B=0$$

$$\frac{-9B+6}{5}-5B=0$$

$$-9B+6-25B=0$$

$$-34B+6=0$$

$$\boxed{B=\frac{6}{34}}$$

$$\boxed{B=\frac{3}{17}}$$

$$\boxed{A=\frac{-5}{17}}$$

$$A=\frac{3\left(\frac{6}{34}\right)-2}{5}$$

$$A=\frac{\frac{3(6)}{5}-\frac{2(34)}{5}}{34}$$

$$A=\frac{\frac{18}{5}-\frac{68}{5}}{34}$$

$$A=\frac{\frac{-50}{5}}{34}$$

$$\boxed{A=\frac{-10}{34}}$$

$$y_p=\frac{-5}{17}\sin x+\frac{3}{17}\cos x$$

$$y_h=c_1e^{4x}+c_2e^{-x}$$

$$y=y_h+y_p$$

$$y=c_1e^{4x}+c_2e^{-x}-\frac{5}{17}\sin x+\frac{3}{17}\cos x$$

$$y'' - 3y' - 4y = 4x^2$$

$$y_p = Ax^2 + Bx + C$$

$$y_p' = 2Ax + B$$

$$y_p'' = 2A$$

$$\cancel{2A - 6Ax - 3B'}$$

$$2A - 6Ax - 3B - 4Ax^2 - 4Bx - 4C = 4x^2$$

$$-4Ax^2 + (-6A - 4B)x + 2A - 3B - 4C = 4x^2$$

$$-4A = 4 \qquad 6 - 4B = 0 \qquad -2 - \frac{9}{2} - 4C = 0$$

$$A = -1 \qquad B = \frac{3}{2} \qquad \frac{-4-9}{2} = 4C$$

$$y_p = -x^2 + \frac{3}{2}x - \frac{13}{8} \qquad \frac{-13}{2} = 4C$$

$$C = \frac{-13}{8}$$

$$y_g : y'' - 3y' - 4y = 0$$

$$y_g = c_1 e^{4x} + c_2 e^{-x}$$

$$y = y_g + y_p = \boxed{c_1 e^{4x} + c_2 e^{-x} - x^2 + \frac{3}{2}x - \frac{13}{8} = y}$$

$y'' - 3y' - 4y = 3e^{2x} + 2\sin x + 4x^2$

$y'' - 3y' - 4y = 0 \quad y_h = C_1 e^{4x} + C_2 e^{-x}$

$y'' - 3y' - 4y = 3e^{2x} \quad y_p = -\frac{1}{2} e^{2x}$

$y'' - 3y' - 4y = 2\sin x \quad y_p = \frac{-5}{17}\sin x + \frac{3}{17}\cos x$

$y'' - 3y' - 4y = 4x^2 \quad y_p = -x^2 + \frac{3}{2}x - \frac{13}{8}$

$$\boxed{y = C_1 e^{4x} + C_2 e^{-x} - \frac{1}{2}e^{x} - \frac{5}{17}\sin x + \frac{3}{17}\cos x - x^2 + \frac{3}{2}x - \frac{13}{8}}$$

Laplace transform

$$\mathcal{L}\{f(t)\} \longrightarrow F(s)$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \lim_{A\to\infty} \int_0^A e^{-st}\, dt$$

$$= \lim_{A\to\infty} \left[-\frac{1}{s} e^{-st}\right]_0^A = \lim_{A\to\infty} \left(-\frac{1}{s} e^{-sA} - -\frac{1}{s}\right)$$

$s>0$

$$\lim_{A\to\infty} \left(-\frac{1}{s} e^{-sA} + \frac{1}{s}\right) = \frac{1}{s}$$

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt$$

$$= \frac{1}{a-s}\left[e^{(a-s)t}\right]_0^\infty = \frac{1}{a-s}\left[0-1\right]$$

$a-s>0 \Rightarrow$ no limit

$a>s$

$a-s<0$

$\boxed{a<s}$

$$= \frac{-1}{a-s} = \frac{1}{s-a}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \qquad s>a}$$

$$\mathcal{L}\{e^{0t}\} = \frac{1}{s-0} = \frac{1}{s}$$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt = y$$

$$\frac{d}{dt}(uv) = u'v + uv'$$

$$u' = e^{-st} \qquad v' = a\cos at \qquad uv = \int u'v + \int uv'$$

$$u = \frac{-1}{s}e^{-st} \qquad v = \sin at \qquad \int u'v = uv - \int uv'$$

$$y = \frac{-1}{s}e^{-st}\sin at - \int_0^\infty \frac{-1}{s}e^{-st}\,a\cos at\,dt$$

$$y = \frac{-e^{-st}}{s}\sin at + \frac{a}{s}\int_0^\infty e^{-st}\cos at\,dt$$

$$y = \frac{-e^{-st}}{s}\sin at + \frac{a}{s}\left[\frac{-1}{s}e^{-st}\cos at - \int_0^\infty \right.$$

$$\boxed{u' = e^{-st} \qquad v' = -a\sin at \qquad u = \frac{-1}{s}e^{st} \qquad v = \cos at}$$

$$y = \frac{-e^{-st}}{s}\sin at + \frac{a}{s}\left[\frac{-1}{s}e^{-st}\cos at - \int_0^\infty +\frac{1}{s}e^{-st}\,a\sin at\,dt\right]$$

$$y = \frac{-e^{-st}}{s}\sin at - \frac{a}{s^2}e^{-st}\cos at - \frac{a^2}{s^2}y$$

$$y + \frac{a^2}{s^2}y = -e^{-st}\left(\frac{1}{s}\sin at + \frac{a}{s^2}\cos at\right)$$

$$\left(\frac{s^2+a^2}{s^2}\right)y = \left[e^{-st}\left(\frac{1}{s}\sin at + \frac{a}{s^2}\cos at\right)\right]_0^\infty$$

$$\left(\frac{s^2+a^2}{s^2}\right)y = 0 + 1\left(0 + \frac{a}{s^2}\right)$$

$$y = \frac{a}{s^2+a^2} \qquad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}(c_1 f(t) + c_2 g(t))\,dt$$

$$= \int_0^\infty c_1 e^{-st} f(t) + c_2 e^{-st} g(t))\,dt$$

$$= c_1 \int_0^\infty e^{-st} f(t)\,dt + c_2 \int_0^\infty e^{-st} g(t)\,dt$$

$$= c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$\int uv' = uv - \int u'v$$

$u = e^{-st}$ $\quad v' = f'(t)$

$u' = -se^{-st}$ $\quad v = f(t)$

$$\mathcal{L}\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty + \int_0^\infty +se^{-st} f(t)\,dt$$

$$\mathcal{L}\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f'(t)\} = 0 - f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s(s\mathcal{L}\{f(t) - f(0)) - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$f'(t) = \cos at$

$f(t) = \frac{1}{a}\sin at$

$$\mathcal{L}\{\cos at\} = s\mathcal{L}\{\tfrac{1}{a}\sin at\} - \tfrac{1}{a}\sin 0$$

$$= \frac{s}{a}\mathcal{L}\{\sin at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

$$\mathcal{L}\{f'\} + f(0) = s\mathcal{L}\{f\}$$

$$\mathcal{L}\{f\} = \frac{1}{s}(\mathcal{L}\{f'\} + f(0))$$

$$\mathcal{L}\{t\} = \frac{1}{s}\mathcal{L}\{1\} - 0$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^2\} = \frac{1}{s}\mathcal{L}\{2t\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$$

$$\mathcal{L}\{t^3\} = \frac{1}{s}\mathcal{L}\{3t^2\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{6}{s^4}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$y'' + 5y' + 6y = 0 \qquad y(0) = 2$$
$$y'(0) = 3$$

$$\mathcal{L}\{y''\} + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

$$\int_0^\infty (0)e^{-st}\,dt = 0$$

$$s\mathcal{L}\{y'\} - y'(0) + 5\mathcal{L}\{y'\} + 6\mathcal{L}\{y\} = 0$$

$$s(s\mathcal{L}\{y\} - y(0)) - y'(0) + 5(s\mathcal{L}\{y\} - y(0)) + 6\mathcal{L}\{y\} = 0$$

$$s^2\mathcal{L}\{y\} - 2s - 3 + 5s\mathcal{L}\{y\} - 10 + 6\mathcal{L}\{y\} = 0$$

$$\mathcal{L}\{y\}(s^2 + 5s + 6) - 2s - 3 - 10 = 0$$

$$\mathcal{L}\{y\}(s^2+5s+6) = 2s+3+10 = 2s+13$$

$$\mathcal{L}\{y\} = \frac{2s+13}{s^2+5s+6}$$

$$y = \mathcal{L}^{-1}\left\{\frac{2s+13}{s^2+5s+6}\right\}$$

$$\mathcal{L}\{y\} = \frac{2s+13}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$2s+13 = (A+B)s + 3A+2B$$

$$2 = A+B$$

$$3A+2B = 13$$

$$-2A-2B = -4$$

$$A = 9 \qquad B = -7$$

$$\mathcal{L}\{y\} = \frac{9}{s+2} - \frac{7}{s+3} = 9\left(\frac{1}{s+2}\right) - 7\left(\frac{1}{s+3}\right)$$

$$\mathcal{L}\{y\} = 9\mathcal{L}\{e^{-2t}\} - 7\mathcal{L}\{e^{-3t}\} \qquad \boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$

$$\mathcal{L}\{y\} = \mathcal{L}\{9e^{-2t} - 7e^{-3t}\}$$

$$\boxed{y = 9e^{-2t} - 7e^{-3t}}$$

$\mathcal{L}\{y''\}$

$s\mathcal{L}\{y'\} - y'(0)$

$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$

$s(s\mathcal{L}\{y\} - y(0)) - y'(0)$

$s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

$\mathcal{L}\{y^{(4)}\} = s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$

$\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{f(t)\} = F(s)$

$$\boxed{\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\ F(s-a) &= \int_0^\infty e^{-(s-a)t}f(t)\,dt\end{aligned}}$$

$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)\,dt$

$\mathcal{L}\{\cos(2t)\} = F(s) = \dfrac{s}{s^2+4}$

$\mathcal{L}\{e^{3t}\cos 2t\} = F(s-3) = \dfrac{s-3}{(s-3)^2+4}$

$$y'' + y = \sin 2t \qquad y(0) = 2,\ y'(0) = 1$$

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0) = s^2Y(s) - sy(0) - y'(0)$$

$$s^2Y(s) - 2s - 1 + Y(s) = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s}{s^2+1} + \frac{1}{s^2+1}$$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{(As+B)(s^2+1) + (Cs+D)(s^2+4)}{(s^2+4)(s^2+1)}$$

$$As^2 + As + Bs^2 + B$$

$$Cs^3 + 4Cs + Ds^2 + 4D$$

$$2 = (A+C)s^3 + (B+D)s^2 + (A+4C)s + (B+4D)$$

$0 = A+C$	$A = -C$	$0 = A$
$0 = B+D$	$0 = -C+4C$	$B = -D$
$0 = A+4C$	$0 = 3C$	$2 = -D+4D$
$2 = B+4D$	$0 = C$	$2 = 3D$
$0 = B + \frac{2}{3}$	$6 = 3B+2$	$\frac{2}{3} = D$

$$-\frac{2}{3} = B$$

$$3B = -4$$

$$B = -\frac{4}{3}$$

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{-2}{3(s^2+4)} + \frac{2}{3(s^2+1)}$$

$$Y(s) = \frac{-1}{3}\left(\frac{2}{s^2+4}\right) + \frac{2}{3}\left(\frac{1}{s^2+1}\right)$$

$$+ 2\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2+1}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$y = \frac{-1}{3}\sin 2t + \frac{2}{3}\sin t + 2\cos t + \sin t$$

$$\boxed{y(t) = \frac{-1}{3}\sin 2t + \frac{5}{3}\sin t + 2\cos t}$$