
Counting
~~Sheep~~
Sets
Goal: Avoid counting!

Set Size

The number of elements in a set S is called its **size**, or **cardinality**, and denoted |S| or # S

Set of size n: n-set

ation

Bits	$\|\{0,1\}\| = 2$		
Coin	$\|\{\text{heads, tails}\}\| = 2$	Die	$\|\{1,2,3,4,5,6\}\| = 6$
Digits	$\|\{0,1,\ldots,9\}\| = 10$	Letters	$\|\{a,\ldots,z\}\| = 26$

Empty set $|\varnothing| = 0$

Integers $|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{P}| = \infty$ Countably infinite $\aleph_0$

Reals $|\mathbb{R}| = \infty$ Uncountably infinite $\aleph$

Integer Intervals

$m \le n$

$\{m,\ldots,n\}$ = {integers from m to n, inclusive}

$\{3,\ldots,5\} = \{3,4,5\}$

$\boxed{|\{m,\ldots,n\}| = n-m+1}$

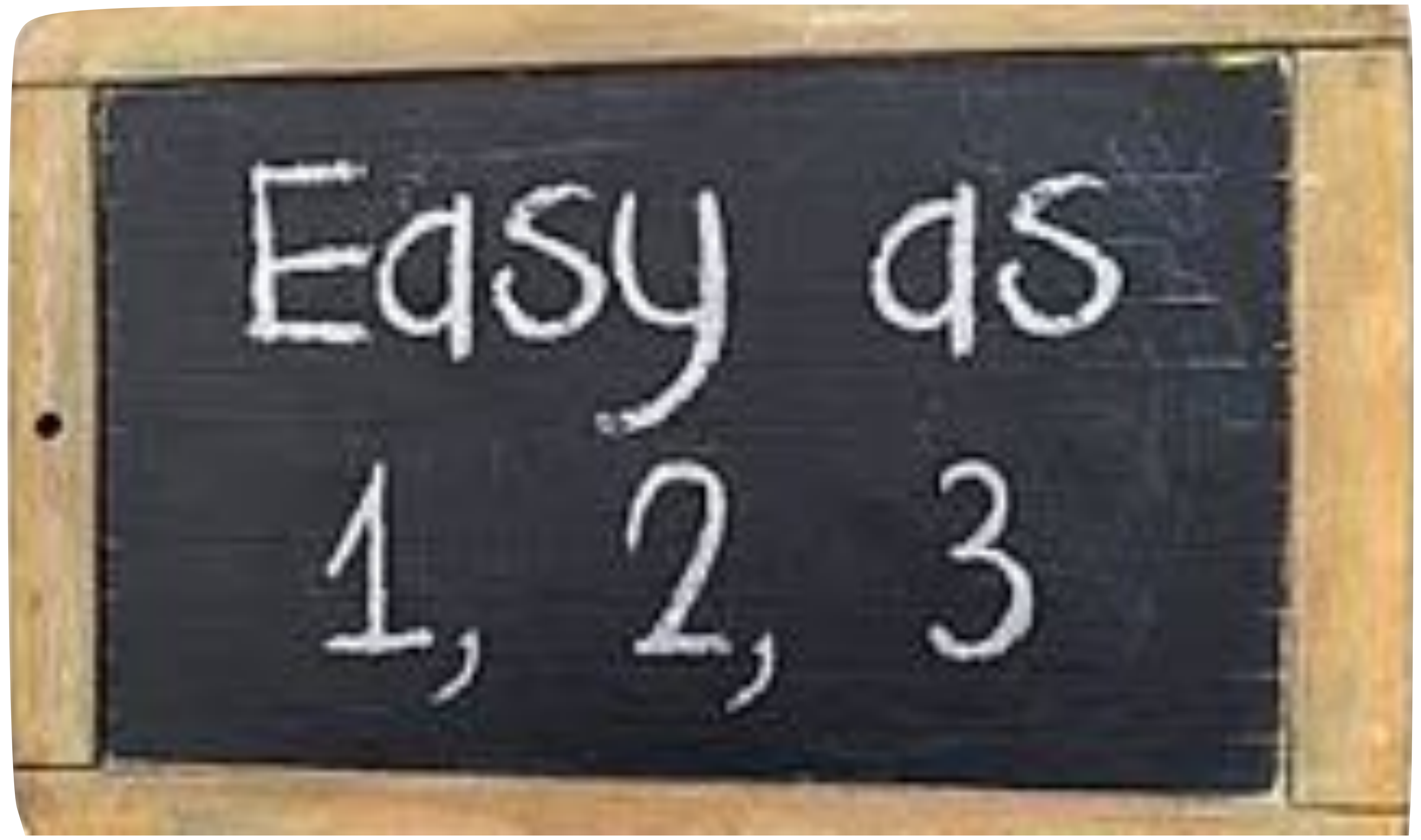
+1?

First

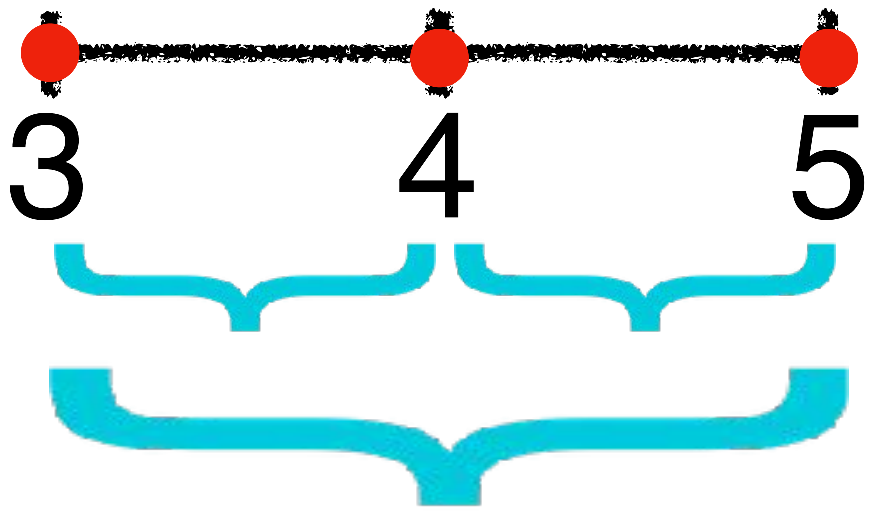
Try small #'s

$|\{5,\ldots,5\}| = |\{5\}| = 1 = 5 - 5 + 1$

$|\{1,\ldots,3\}| = |\{1,2,3\}| = 3 = 3 - 1 + 1$



$\{3,4,5\}$



points = #intervals+1= (5-3)+1 = 3

5-3 = 2 = length = #intervals

Integer Multiples

${}_d(n] = \{\ 1 \le i \le n : d \mid i\ \}$

$(n] = [n] = \{\ 1, \ldots, n\ \}$

${}_3(8] = \{\ 3, 6\ \} = \{1 \cdot 3,\ 2 \cdot 3\}$

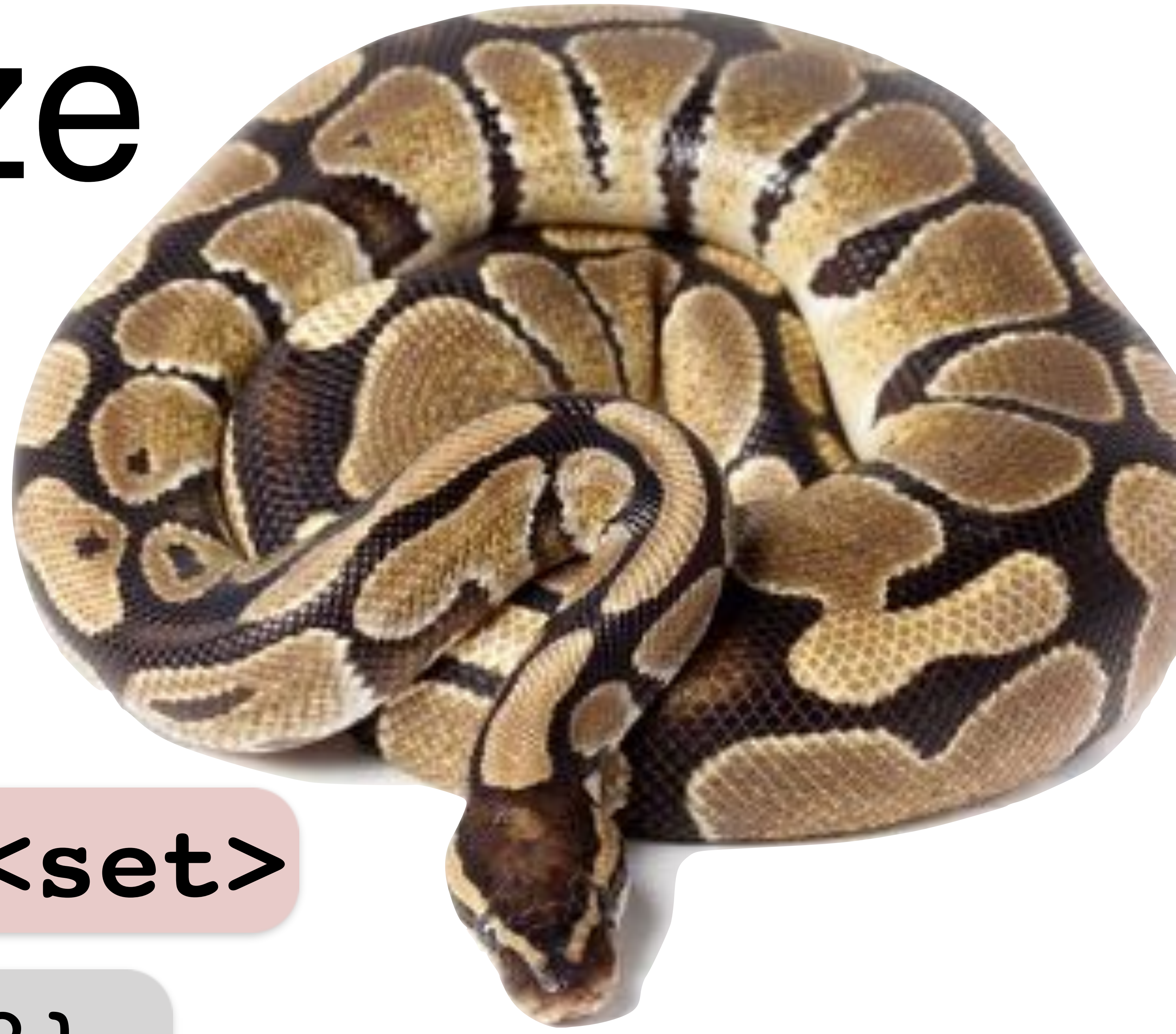
${}_3(9] = \{\ 3, 6, 9\ \} = \{1 \cdot 3,\ 2 \cdot 3,\ 3 \cdot 3\}$

$|\ {}_d(n]\ | = \lfloor\ n/d\ \rfloor$

$|\ {}_3(8]\ | = \lfloor\ 8/3\ \rfloor = 2$

$|\ {}_3(9]\ | = \lfloor\ 9/3\ \rfloor = 3$

Set Size



Size `len`

```
print(len({-1, 1}))
2
```

Sum `sum`

```
print(sum({-1, 1}))
0
```

Minimum `min`

```
print(min({-1, 1}))
-1
```

Maximum `max`

```
print(max({-1, 1}))
1
```

Loops

`for <var> in <set>`

```
A = {1, 2, 3}
print(len(A))
3
num=0
for i in A:
    num += 1
print(num)
3
```

Disjoint Unions

Addition Rule

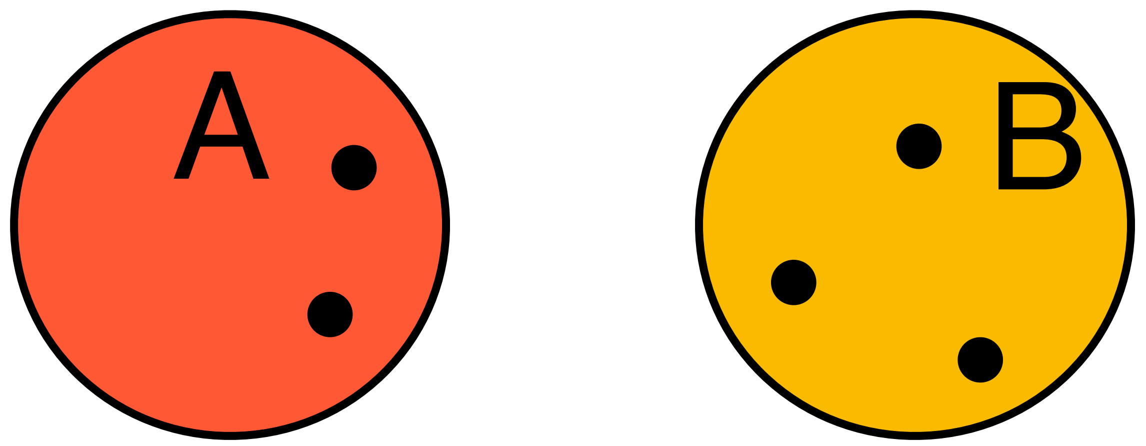
Subtraction Rule

Pets

A Hairy Problem

Disjoint Unions

$|A| = 2$ $|B| = 3$



$|A \cup B| = 2 + 3 = 5$

A union of disjoint sets is called a disjoint union

Use ●● to emphasize a disjoint union

$\{0\} \cup \{1\} = \{0,1\}$

For disjoint sets,
the size of the union
is the sum of the sizes

$$\boxed{|A \cup B| = |A| + |B|}$$

Addition Rule

Numerous applications & implications

Reason $\cup \approx +$

Kids Play

Class has 2 boys and 3 girls

students = ?

∪ ⊕ # students = 2 + 3 = 5

Jar with Marbles

1 blue, 2 green, 3 red

marbles = ?

∪ of 3 sets

⊕ twice

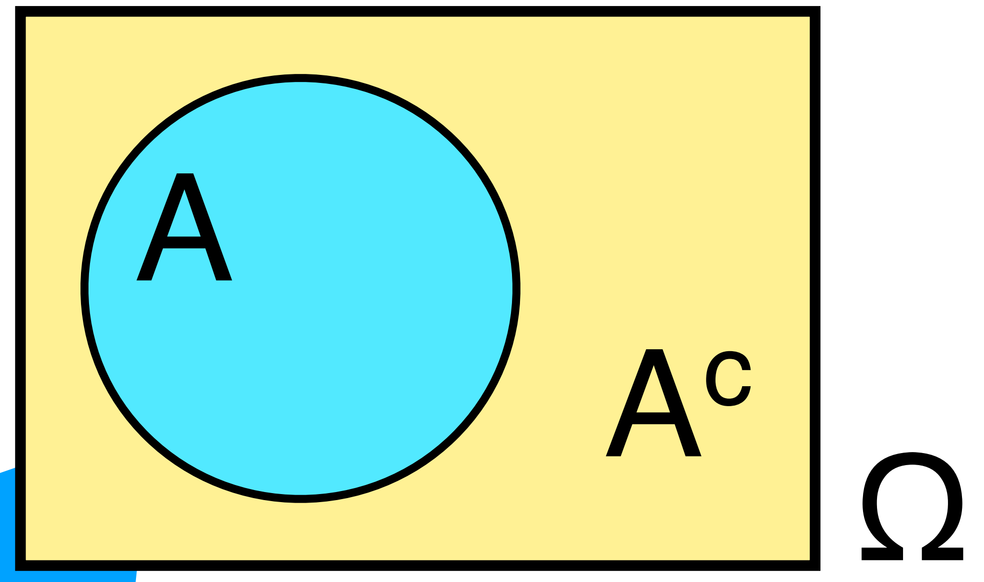
=1+2+3 = 6

Complements

Quintessential disjoint sets

A and A^c

$A \cup A^c = \Omega$



$| \Omega | = | A \cup A^c | = |A| + |A^c|$

Subtraction (or complement) Rule

$$\boxed{|A^c| = | \Omega | - |A|}$$

$D = \{ i \in [6] : 3 \mid i \} = \{ 3, 6 \}$

$|D|=2$

Application of + rule

$D^c = \{ i \in [6] : 3 \nmid i \} = \{1, 2, 4, 5\}$

$|D^c|=4$

$\Omega = [6] = \{1,..,6\}$

$|D^c| = 4 = 6 - 2 = |\Omega| - |D|$ ✅

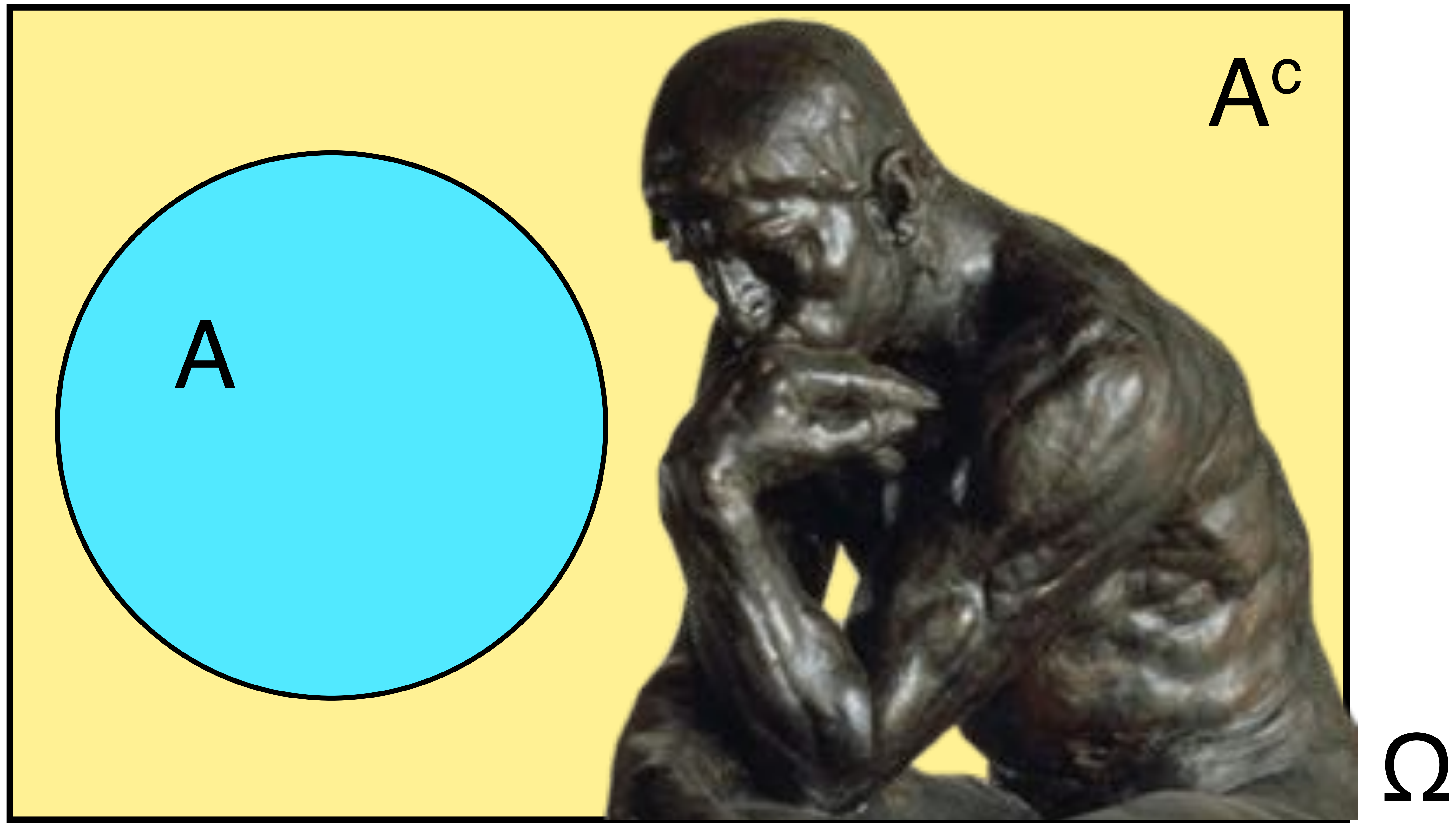
Reason set difference ≈ -

Think Outside the Circle

$|A^c| = |\Omega| - |A|$

$|A| = |\Omega| - |A^c|$

Handy for large or complex sets!



$A = \{ i \in [100] : 3 \nmid i \} = \{1,2,4,5,7,\ldots,100\}$

integers between 1 and 100 not divisible by 3

$\Omega = \{1,\ldots,100\}$

$A^c = \{i \in [100] : 3 \mid i\} = \{3, 6, 9, \ldots, 99\}$

$|A^c| = 33$

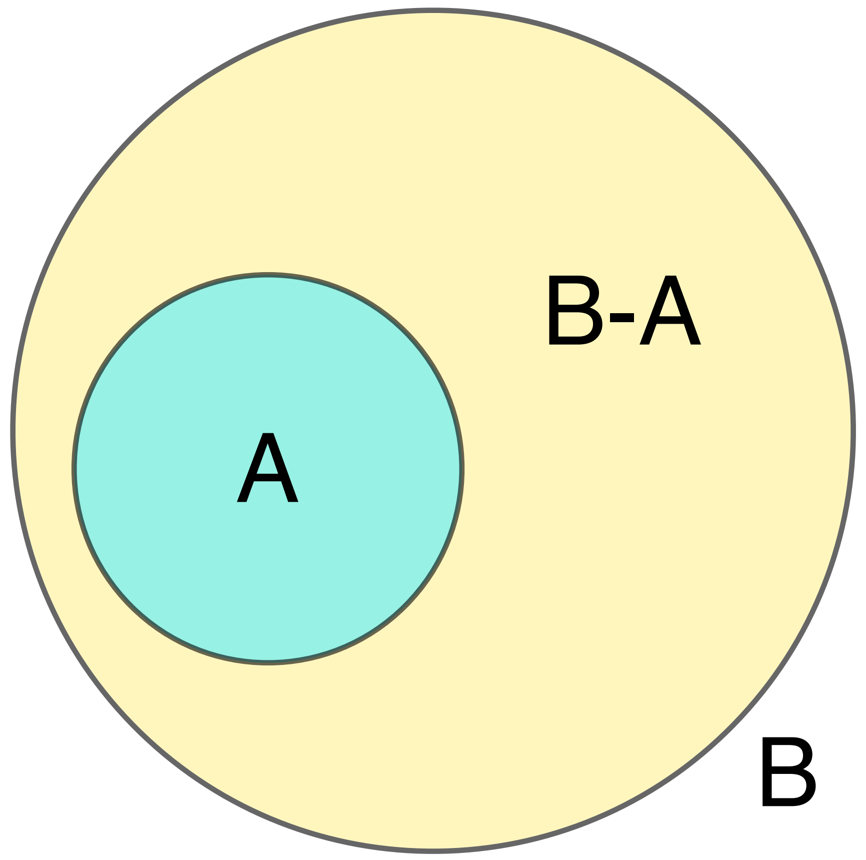
$|A| = |\Omega| - |A^c| = 100 - 33 = 67$

General Subtraction Rule

$A \subseteq B \rightarrow B = A \cup (B-A)$

$|B| = |A| + |B-A|$

$|B-A| = |B| - |A|$



Subtraction Rule

Disjoint Unions

- Addition Rule
- Subtraction Rule
- Pets
- A Hairy Problem

Another Hairy Problem

A friend claims she can determine the size of any set instantly and exactly

She can determine the exact # of hairs on your head

Can you ask her some questions to be fairly certain if she tells the truth?

Hint

It doesn't just work for humans

See you next video

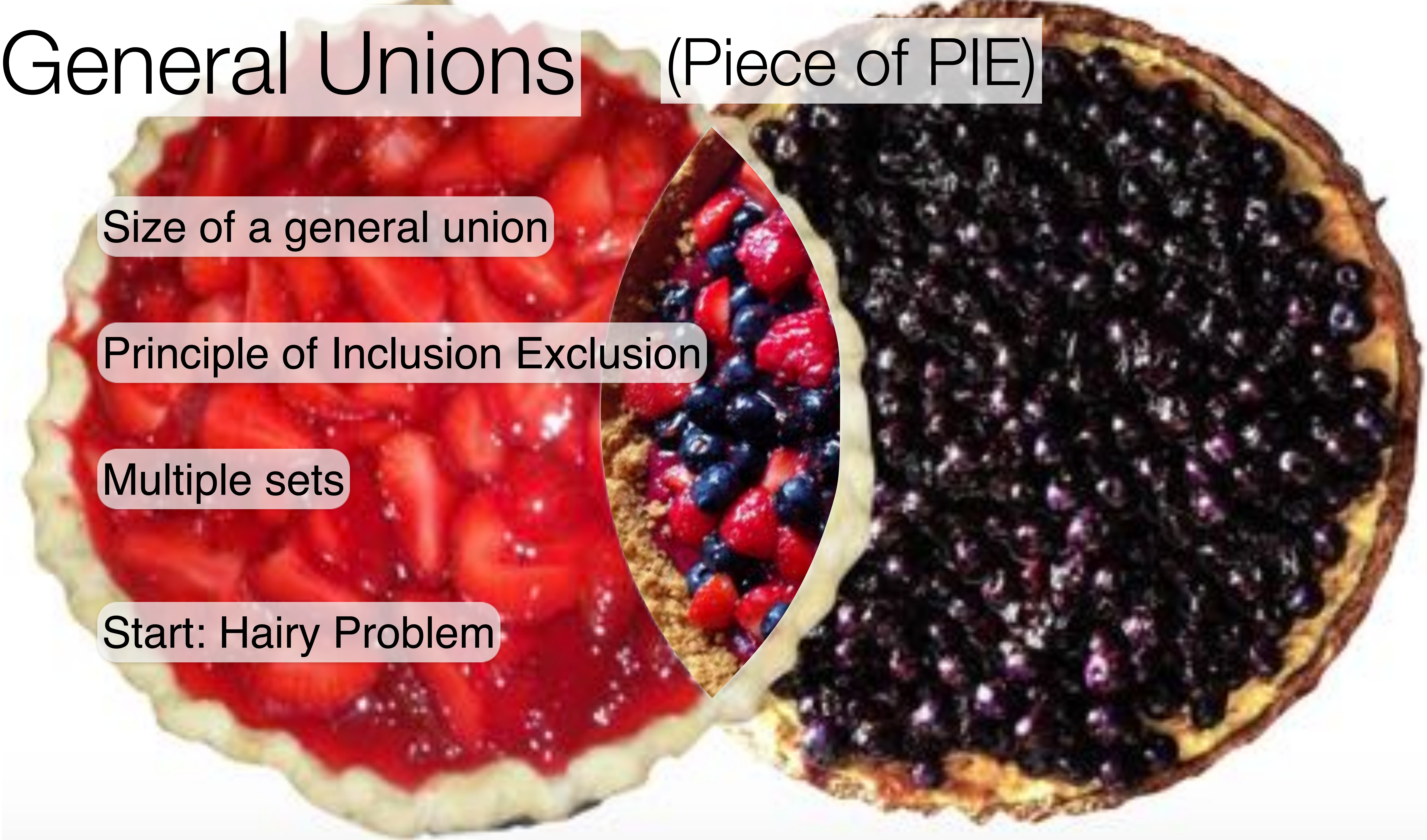
General Unions (Piece of PIE)

Size of a general union

Principle of Inclusion Exclusion

Multiple sets

Start: Hairy Problem



Hairy Problem

A friend claims she can determine the size of any set instantly and exactly

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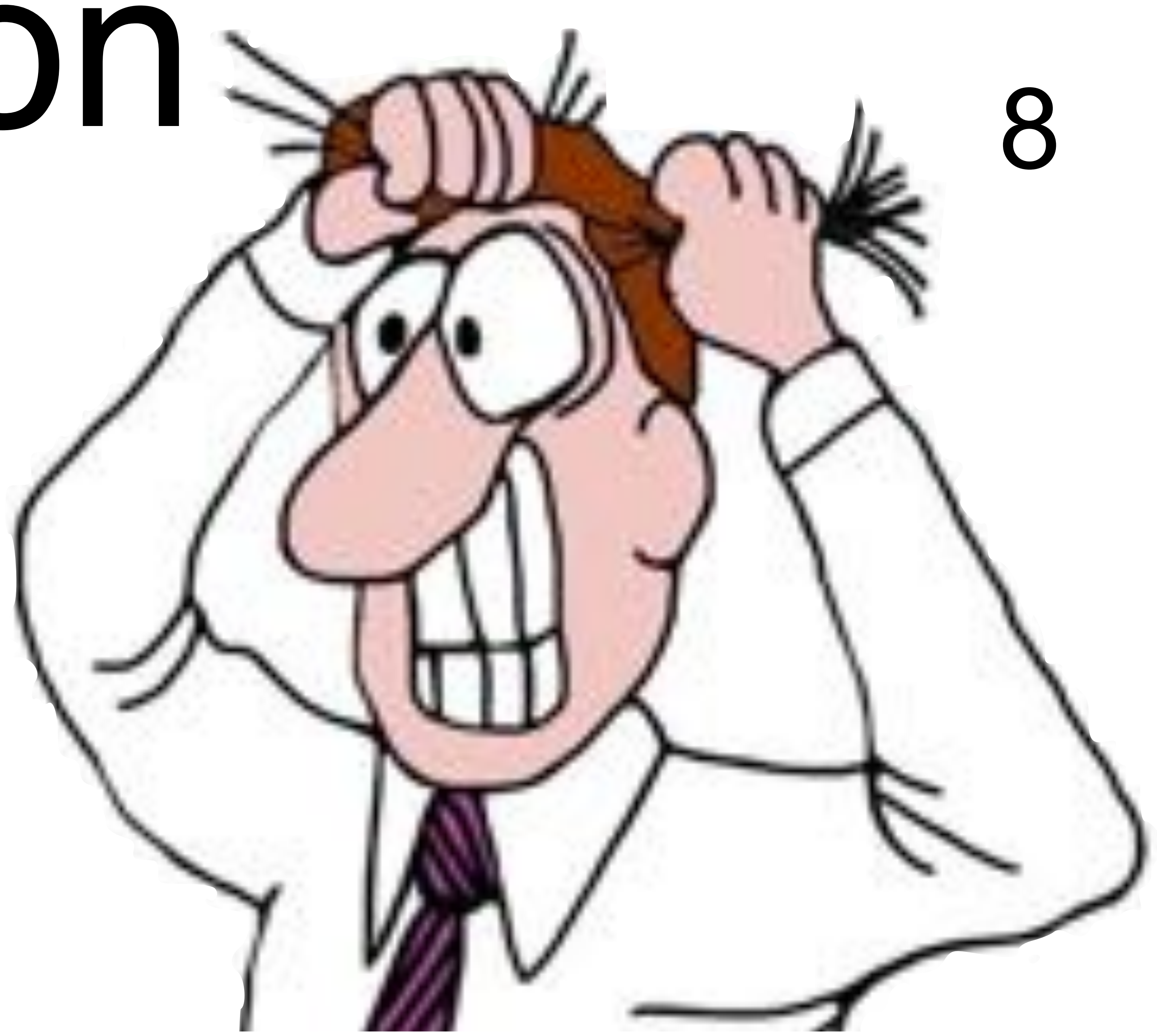
Simple Solution



Subtraction Rule

Ask her how many hairs you have

Remove a small number of hairs, say 8

Ask her how many hairs you have now

Difference between her answers should be # hairs removed (8)

Can you ask a single question?

Ask just how many hairs you removed

8 hairs

Zero-knowledge proofs

Prove identity without revealing password

General Unions

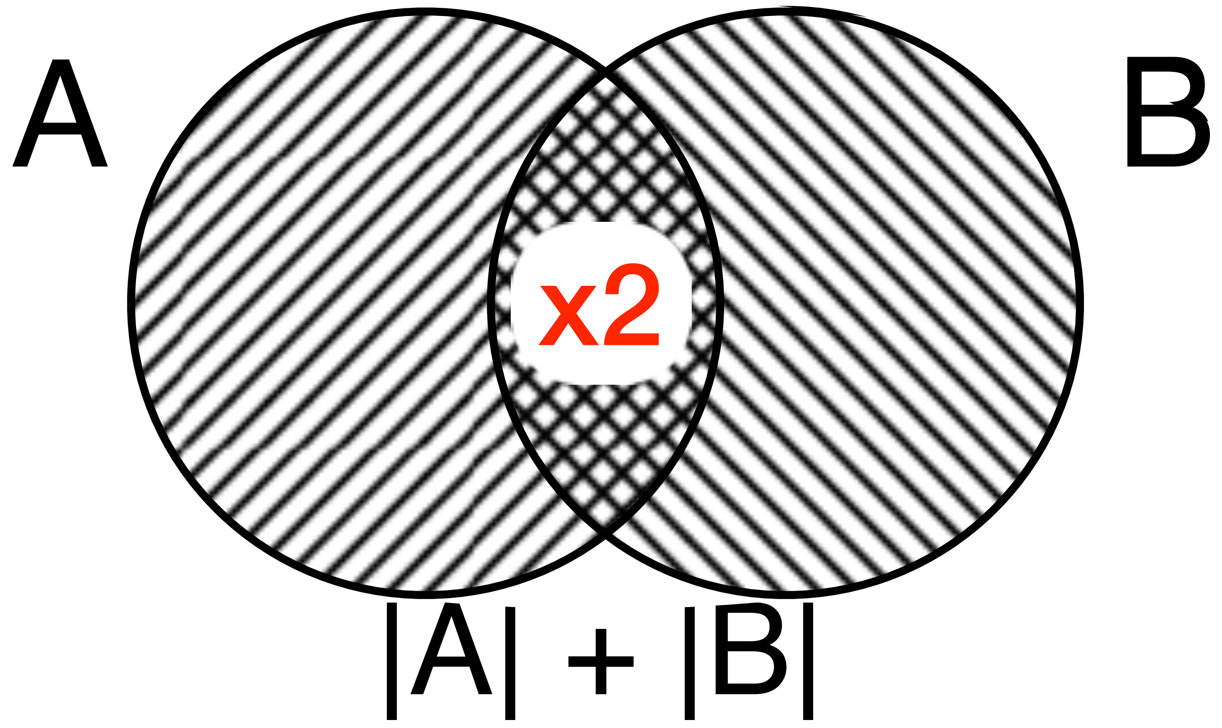
Disjoint A and B: $|A \cup B| = |A| + |B|$

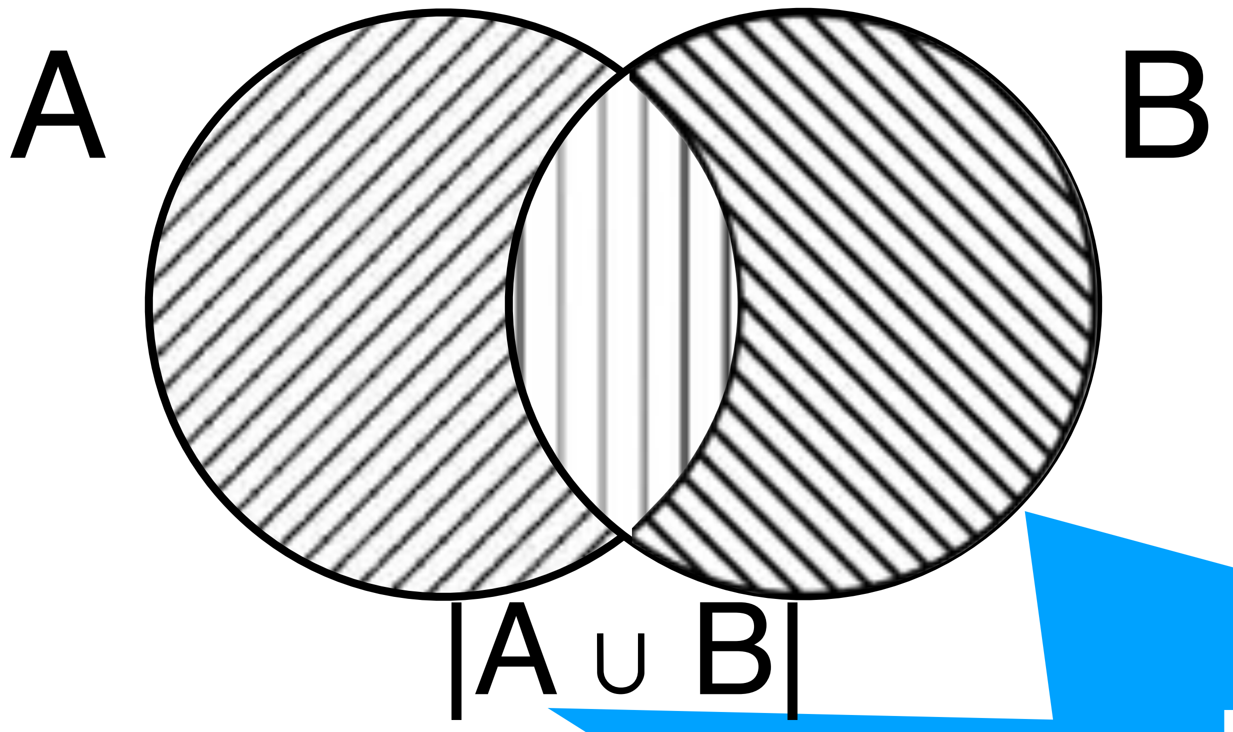
Size of union = sum of sizes

In general: $|A \cup B| \neq |A| + |B|$

$|\{a\} \cup \{a\}| = |\{a\}| = 1 \neq 2 = |\{a\}| + |\{a\}|$

Can we determine $|A \cup B|$ in general?

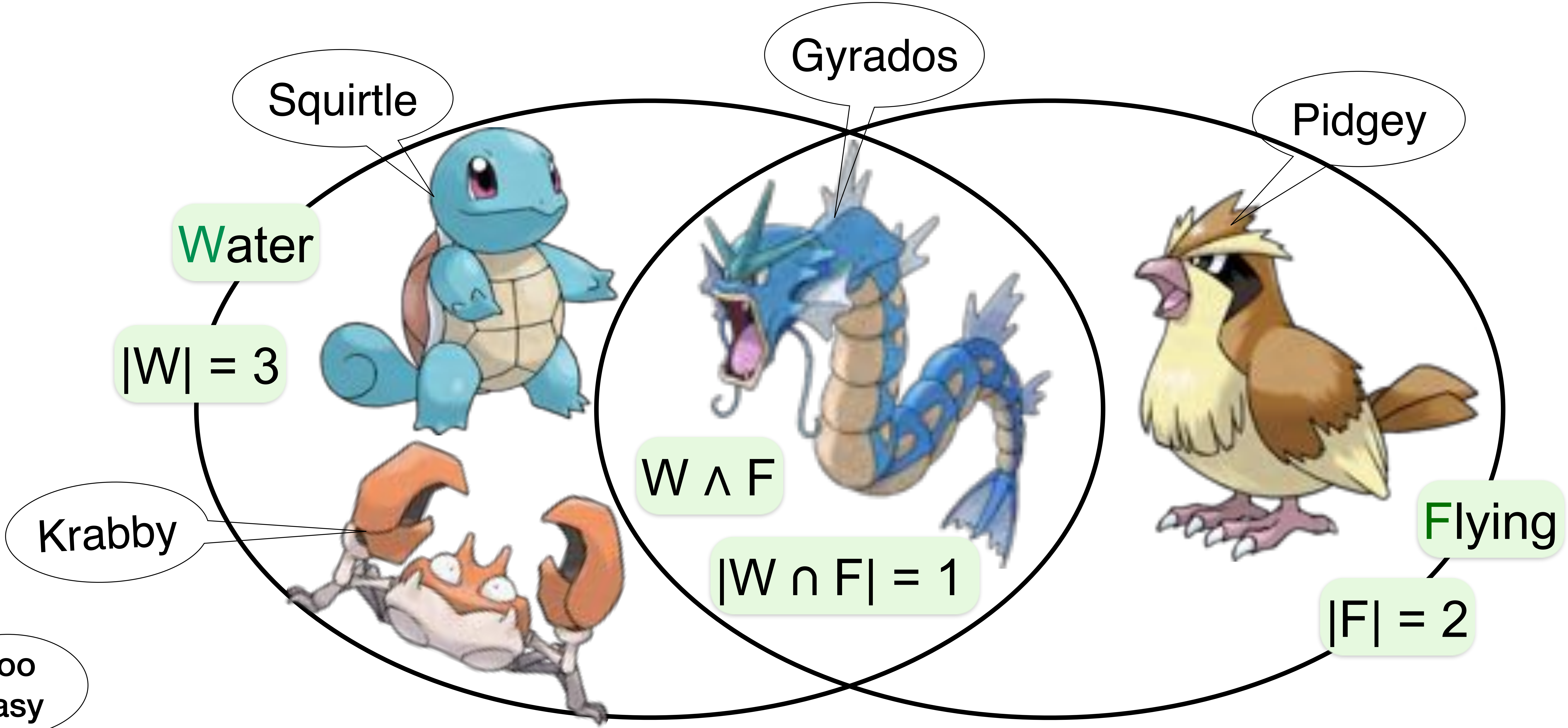




$|A \cup B| = |A| + |B| - |A \cap B|$

Principle of Inclusion-Exclusion (PIE)

Pokémons



Too easy

$\# = 4$

$|W \cup F| = |W| + |F| - |W \cap F| = 3+2-1 = 4$ ✅

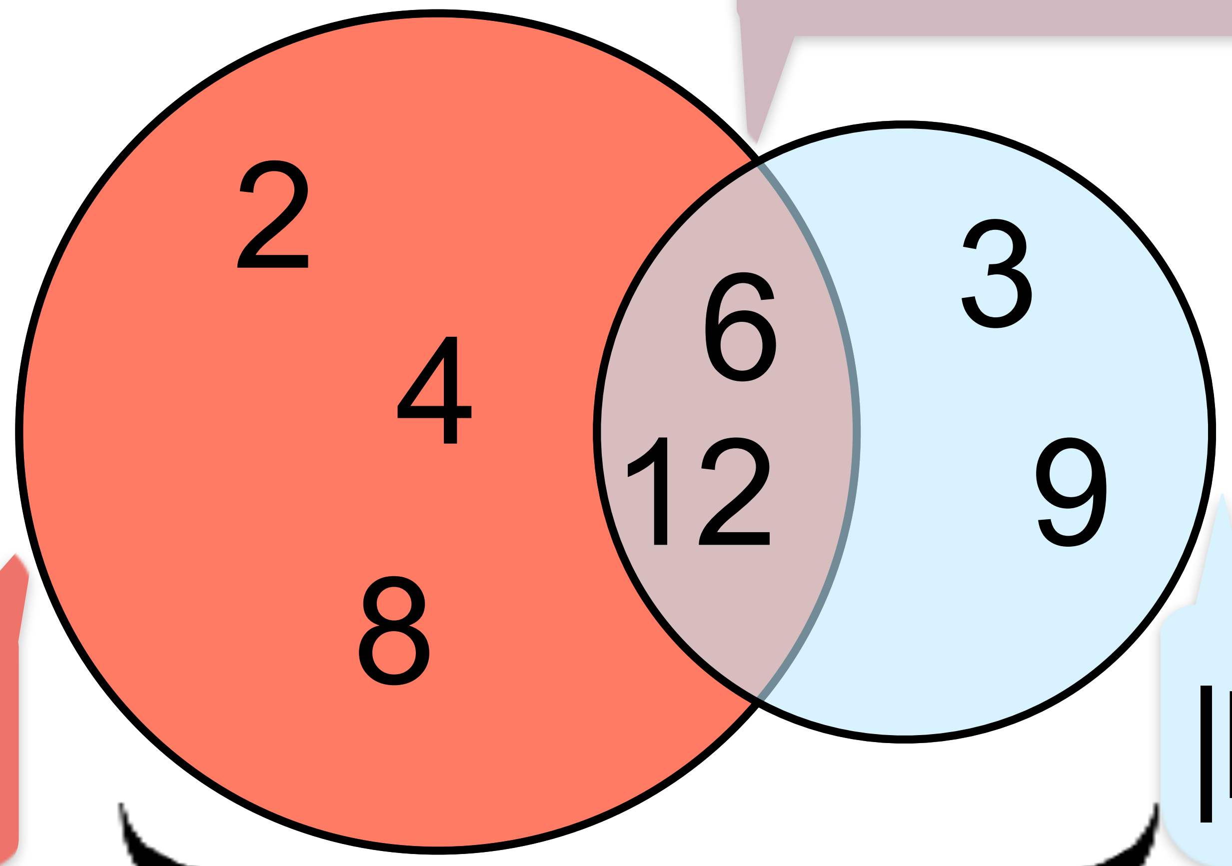
Divisibility by 2 Numbers

$D_{2\vee 3} = \{ i \in [100] : 2|i \vee 3|i \} = \{2,3,4,6,8,\ldots,100\}$

$| D_{2\vee 3} | = ?$

$|D_2 \cap D_3|=16$

$D_2 = \{ i \in [100] : 2|i \} = \{2, 4, 6,\ldots, 100\}$



$D_3 = \{ i \in [100] : 3|i \} = \{3, 6, 9,\ldots, 99\}$

$|D_2|=50$

$|D_3|=33$

$| D_{2\vee 3} | = 67$

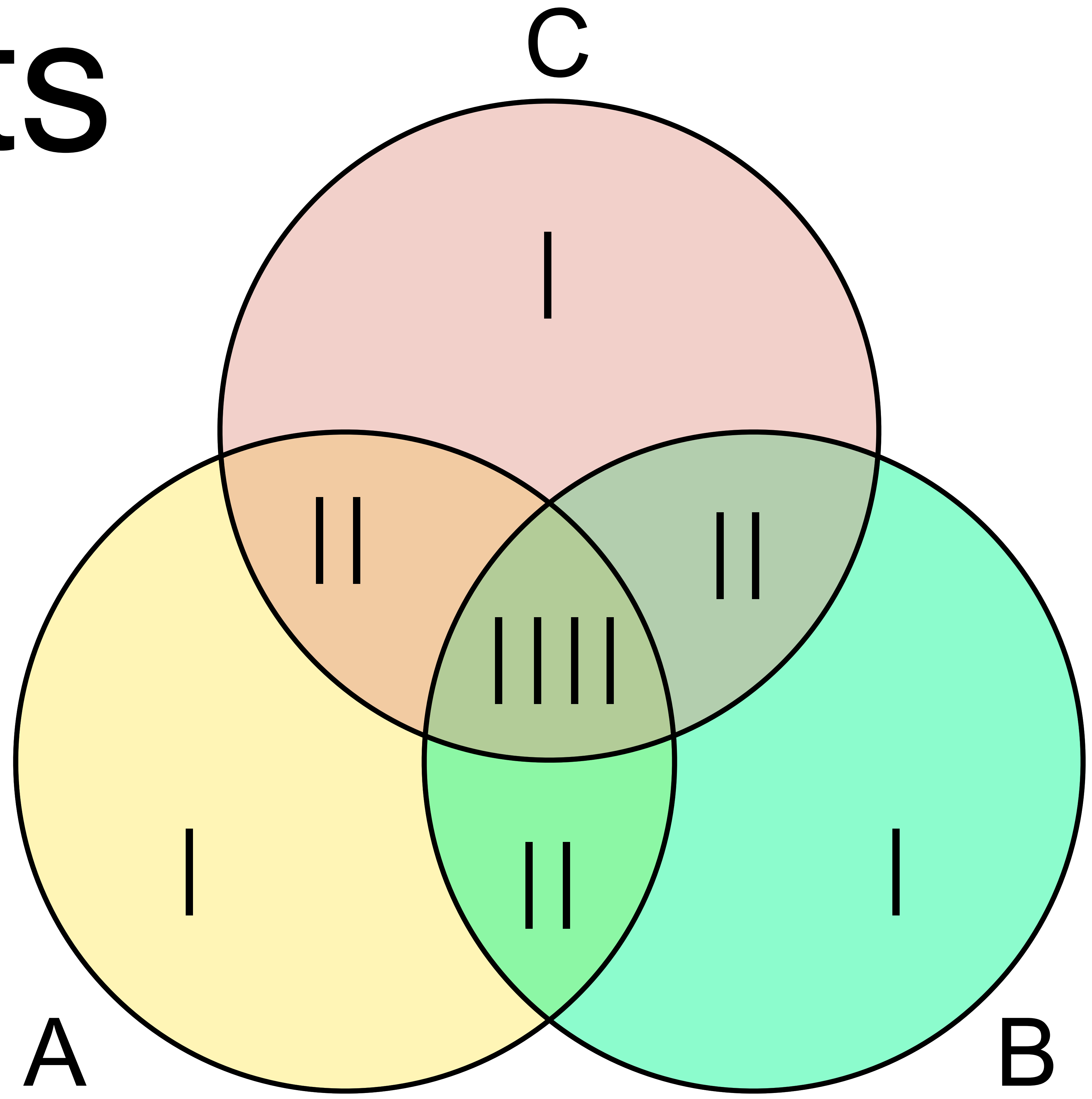
$D_{2\vee 3} = D_2 \cup D_2$

$|D_2 \cup D_3| = |D_2| + |D_3| - |D_2 \cap D_3|$

$D_2 \cap D_3 = \{ i \in [100] : 2|i \wedge 3|i \} = \{ i \in [100] : 6|i \}$

$| D_{2\vee 3} | = |D_2| + |D_3| - |D_2 \cap D_3| = 50 + 33 - 16 = 67$

Multiple Sets



Two sets 

$|A \cup B| = |A| + |B| - |A \cap B|$

Three sets

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

n sets

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

Polyglots

8 students in class, each “speaks” **C**, **R**, or **Π**thon

$|C \cup R \cup \Pi| = 8$

Each language spoken by 5 students

$|C| = |R| = |\Pi| = 5$

Every language *pair* is spoken by 3 students

$|C \cap R| = |C \cap \Pi| = |R \cap \Pi| = 3$

How many students speak all 3 languages?

$|C \cap R \cap \Pi| = ?$

$|C \cup R \cup \Pi| = |C|+|R|+|F|-|C \cap R|-|C \cap \Pi|-|R \cap \Pi|+|C \cap R \cap \Pi|$

$$8 = 5+5+5-3-3-3+?$$

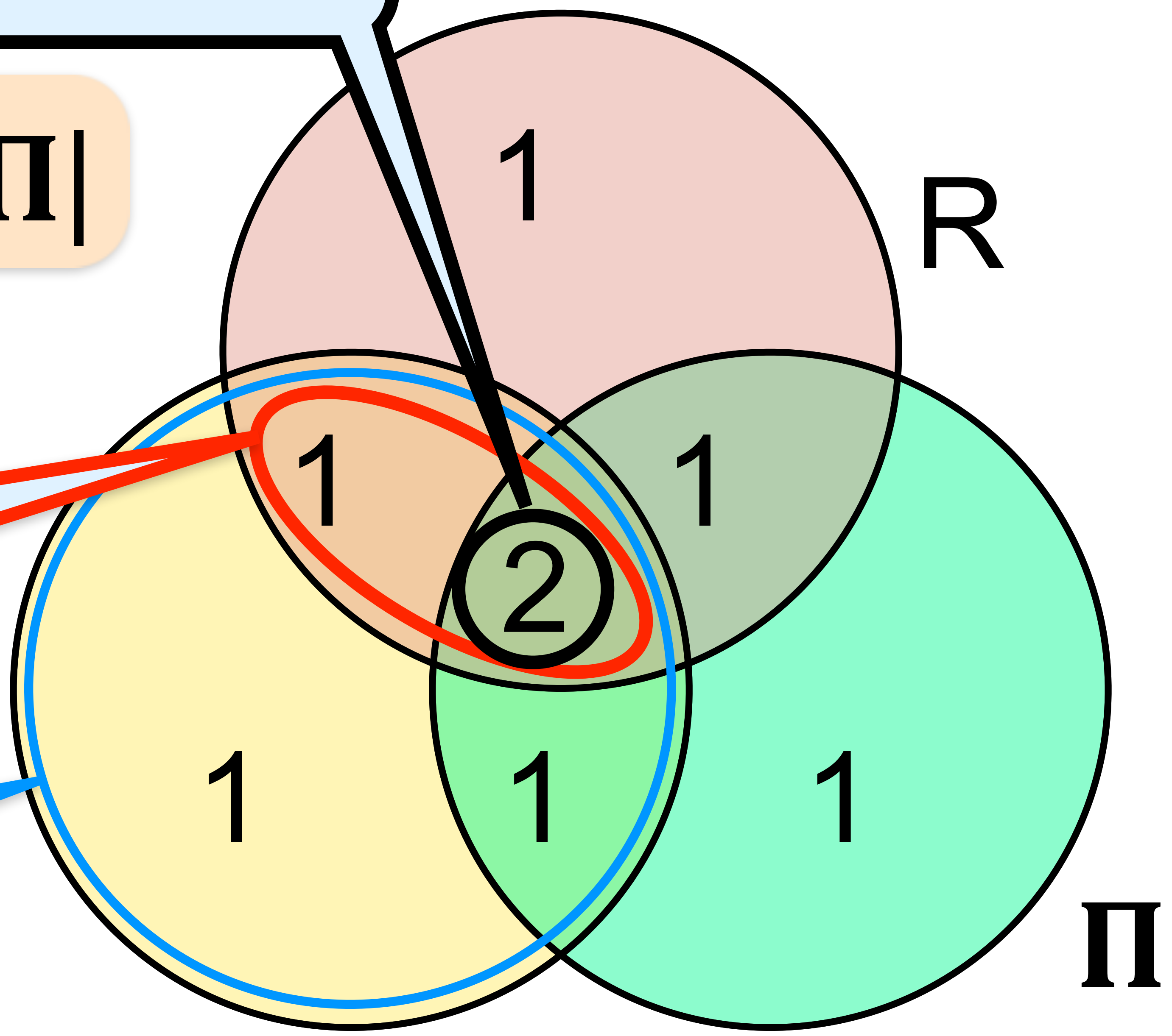
$$? = 8-5-5-5+3+3+3$$

$$= 2$$

With that, can say more

$|C \cap R| = 3$

$|C| = 5$



Sanity Checks

Compare  to some expected outcomes

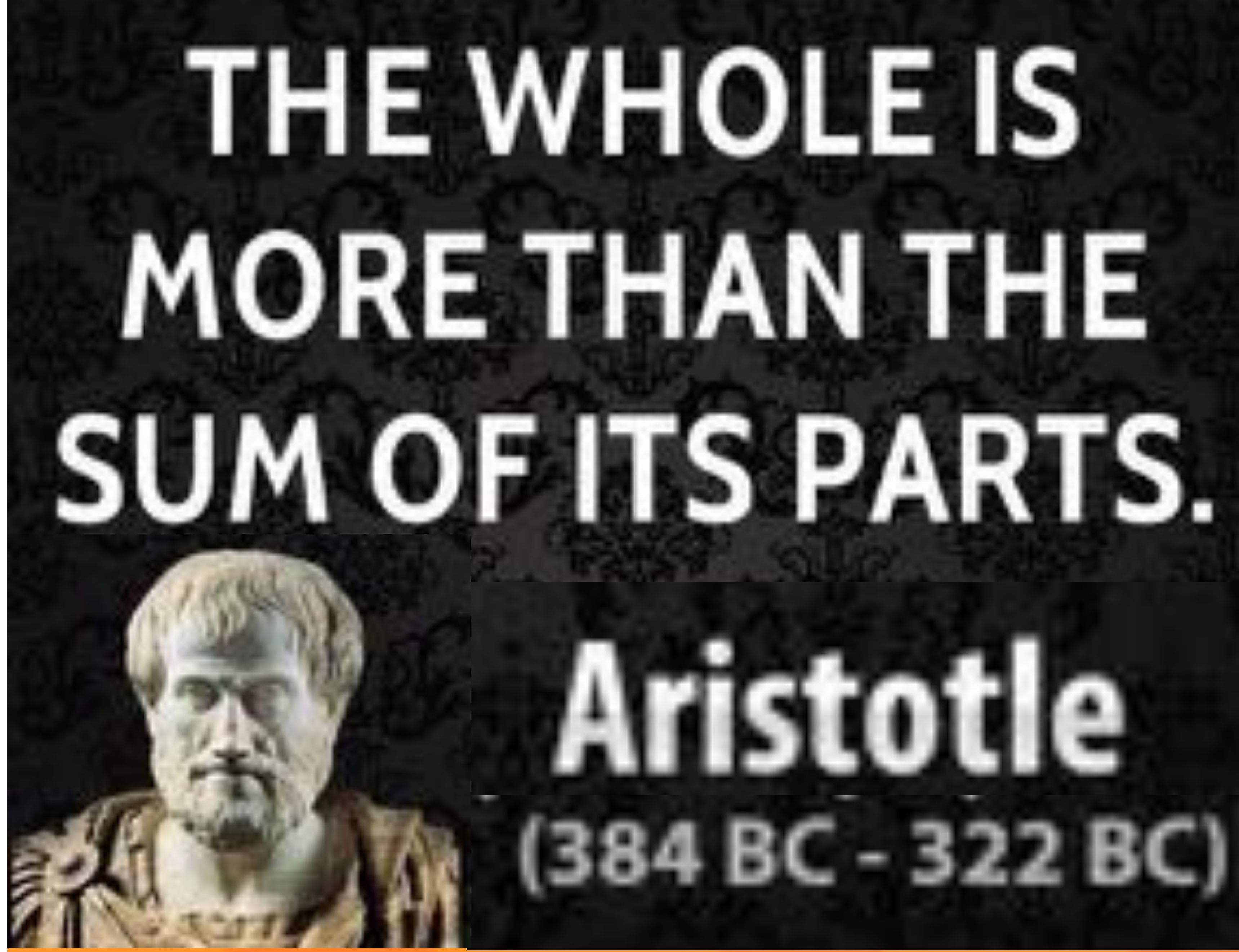
A, B disjoint $|A \cup B| = |A| + |B| - \underbrace{|A \cap B|}_{0} = |A| + |B|$ ✅

Equal sets $|A \cup A| = |A| + |A| - \underbrace{|A \cap A|}_{|A|} = |A|$ ✅

$\max\{|A|, |B|\} \leq |A \cup B| \leq |A| + |B|$ ✅

= iff nested

= iff disjoint



Union ≤ sum of its parts


Counting
Cartesian
Products

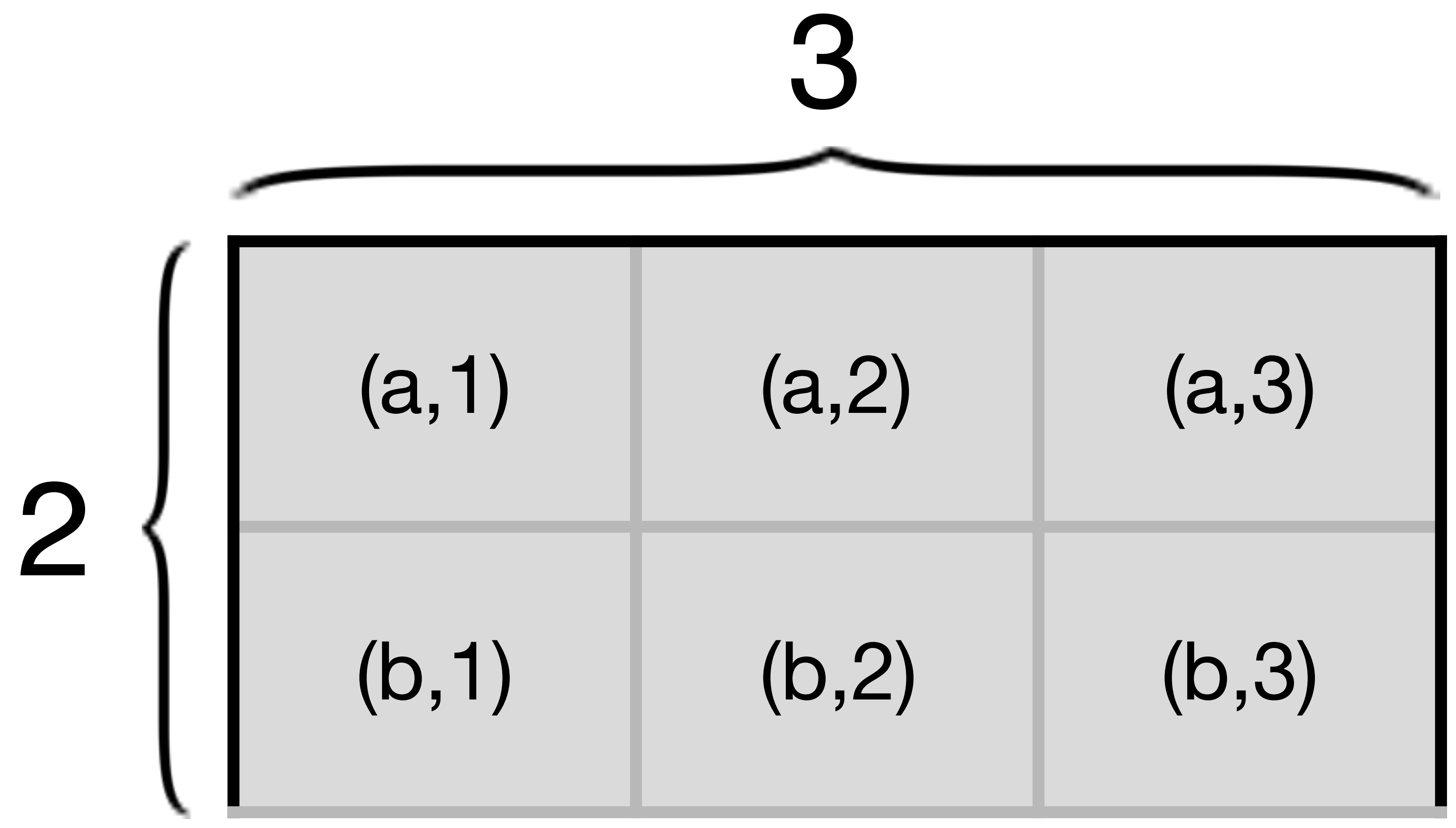
Cartesian Products

$|\{a,b\}| = 2$　　$|\{1,2,3\}| = 3$

$$\{a,b\} \times \{1,2,3\} = \left\{\begin{matrix}(a,1) & (a,2) & (a,3) \\ (b,1) & (b,2) & (b,3)\end{matrix}\right\} \begin{matrix}\leftarrow 3 \\ \leftarrow 3\end{matrix}$$

+

$$|\{a,b\} \times \{1,2,3\}| = 3+3 = 2 \times 3 = 6$$



area 2x3 = 6

The size of a Cartesian Product is the product of the set sizes

$$|A \times B| = |A| \times |B|$$

Product Rule X

Another application of + rule

Tables

Cartesian product

3 attributes

5 records

Adam	M	3.5
Eve	F	3.7
John	M	3.4
Lisa	F	3.2
Mary	F	3.9

X

5x3=15 cells

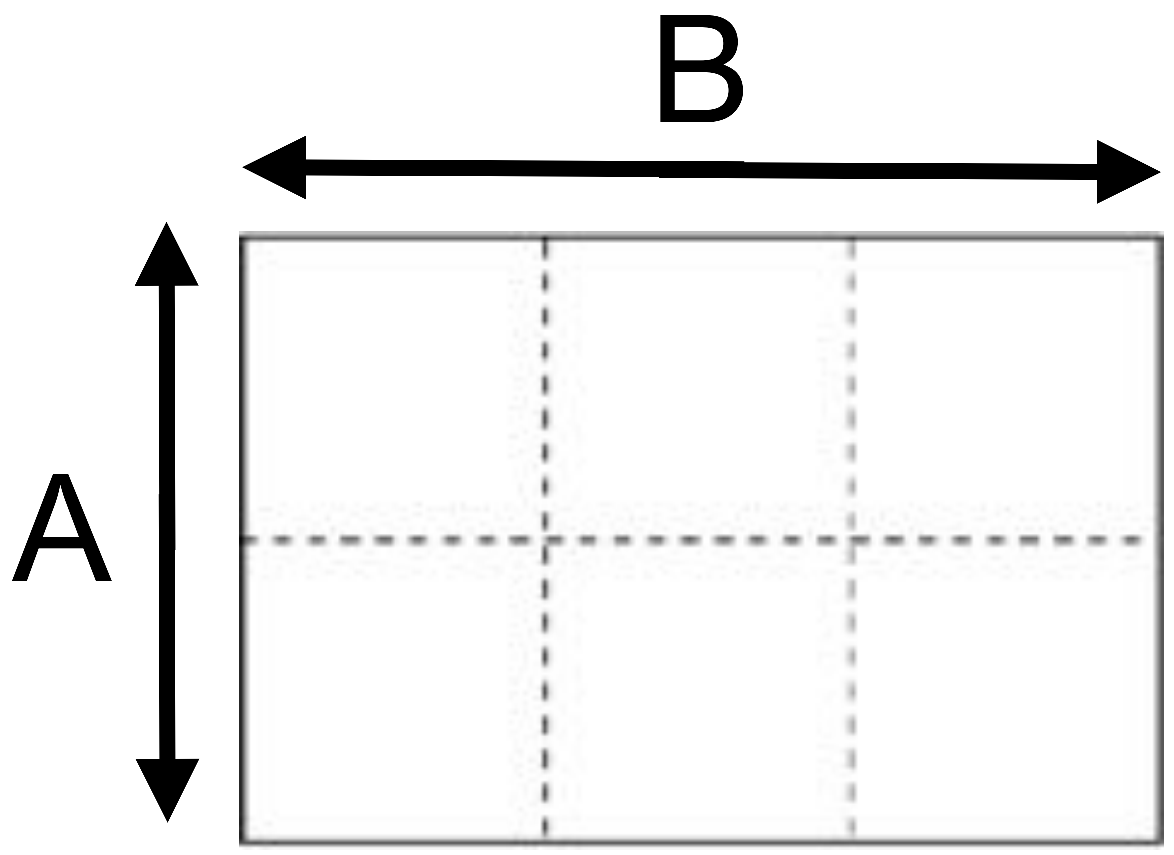
Three Sets

A x B

{(a,b): a ∈ A, b ∈ B}

rectangle

|A x B| = |A|x|B|

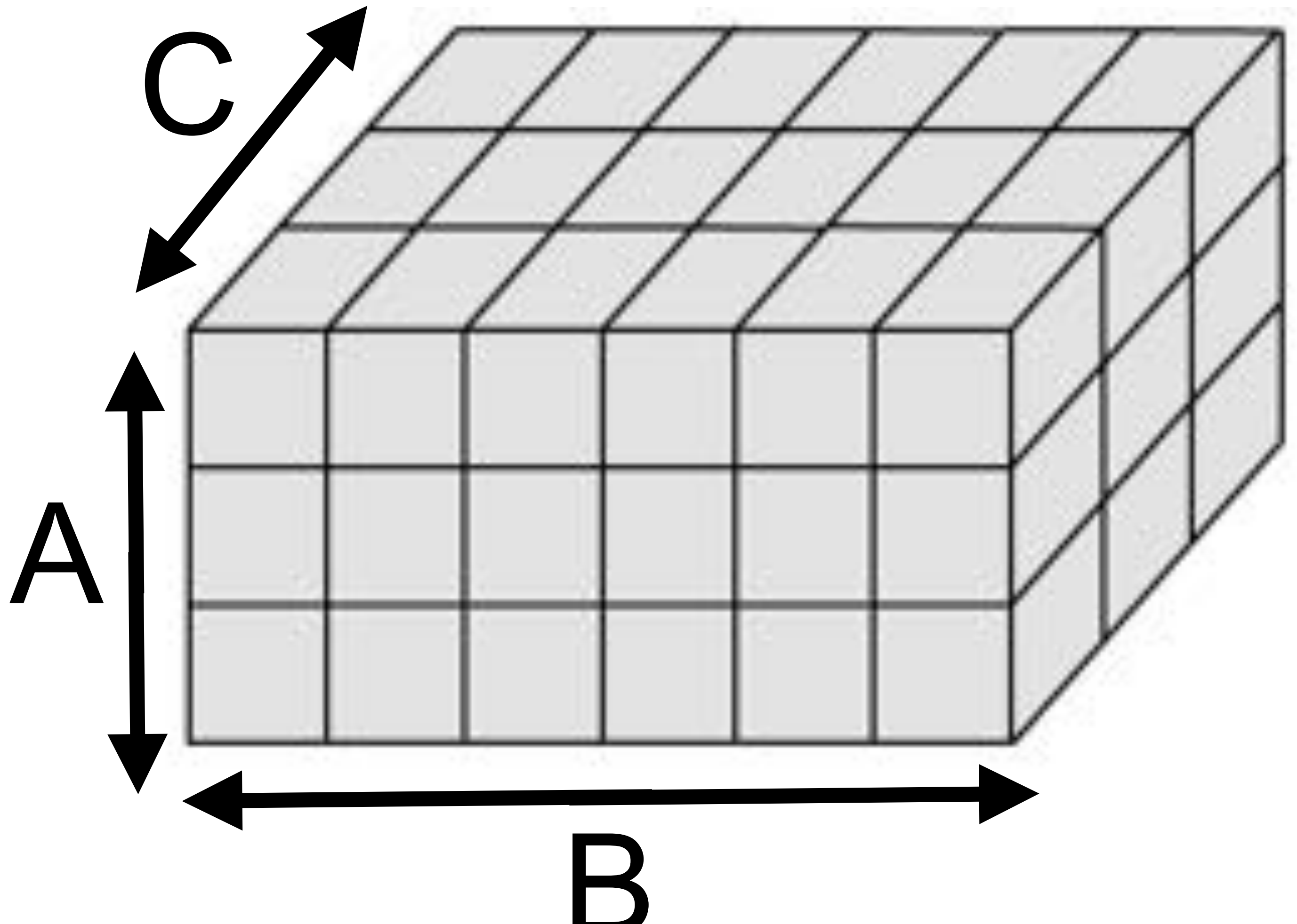


A x B x C

{(a,b,c): a ∈ A, b ∈ B, c ∈ C}

“cuboid”

|A x B x C| = |A|x|B|x|C|



Dandy Dresser

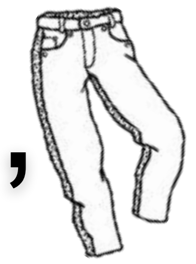
3 shirts

2 pants

5 pairs of shoes

How many outfits can he have?

Outfit = (shirt, pants, shoes)

( ,  , )

3-tuple

{Outfits} = {(shirt, pants, shoes)} = {shirts} x {pants} x {shoes}

Cartesian Product

| {Outfits} | = | {shirts} | x | {pants} | x | {shoes} | = 3 x 2 x 5 = 30

X

Useful?

COSTCO WHOLESALE

X 3 x 3 x 4 36

n Sets

For n sets, by X and induction

$$|A_1 \times \ldots \times A_n| = |A_1| \times \ldots \times |A_n|$$

Subway

How many sandwiches can Subway make?

Bread = {Wheat, Italian}

Meat = {Turkey, Ham, Chicken, Bacon, Beef}

Cheese = {American, Monterey, Cheddar}

Veggie = {Cucumbers, Lettuce, Spinach, Onions}

Sauce = {Ranch, Mustard, Mayonnaise}

Sandwiches = Bread x Meat x Cheese x Veggie x Sauce

|Sand's| = |Bread| x |Meat| x |Cheese| x |Veggie| x |Sauce|

X = 2 x 5 x 3 x 4 x 3 = 360

Cartesian Products

Product rule $|A \times B| = |A| \times |B|$ ⓧ

Multiple sets $|A_1 \times \ldots \times A_n| = |A_1| \times \ldots \times |A_n|$

Cartesian Powers

sequences

subsets

functions

exponential growth

+, -, X, ...

Analogies between number and set operations

Numbers	Sets
Addition	Disjoint union
Subtraction	Complement
Multiplication	Cartesian product
Exponents	?

Cartesian Powers of a Set

Cartesian product of a set with itself is a Cartesian power

$A^2 = A \times A$ Cartesian square

$A^n \stackrel{\text{def}}{=} \underbrace{A \times A \times \ldots \times A}_{n}$ n'th Cartesian power

$|A^n| = | A \times A \times \ldots \times A | = |A| \times |A| \times \ldots \times |A| = |A|^n$

Practical and theoretical applications

California License Plates

Till 1904 — no registration

1905-1912 — various registration formats — one-time \$2 fee

1913

≤6 digits

10^6 = 1 million

If all OK

Sam?

1956

$26^3 \times 10^3 \approx 17.6$ m

1969

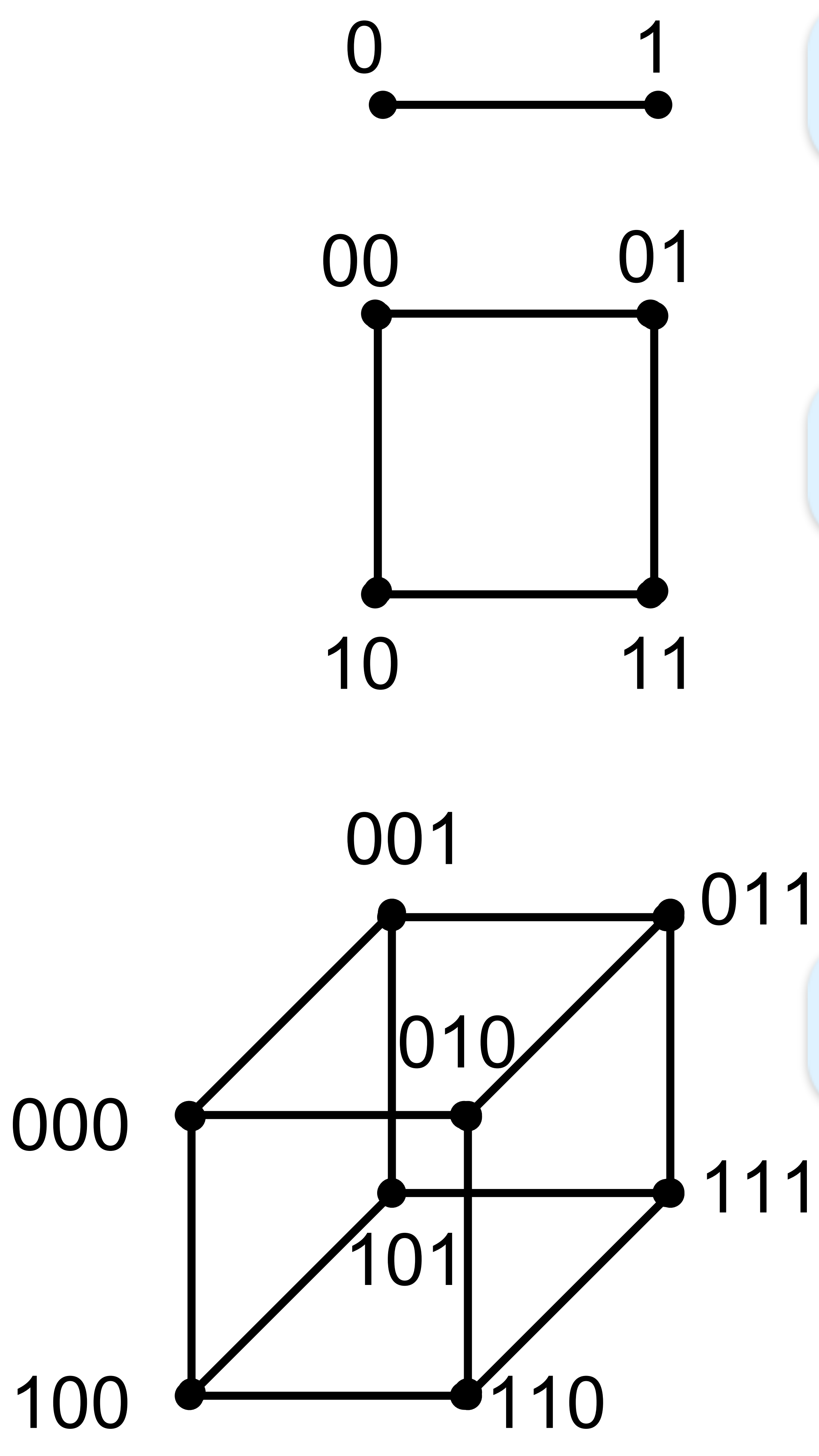
$26^3 \times 10^4 \approx 176$ m

Binary Strings

$\{0,1\}^n = \{$ length-n binary strings $\}$

n-bit strings

n	Set	Strings	Size
0	$\{0,1\}^0$	Λ	1
1	$\{0,1\}^1$	0, 1	2
2	$\{0,1\}^2$	00, 01, 10, 11	4
3	$\{0,1\}^3$	000, 001, 011, 010, 100, 110, 101, 111	8
…	…	…	…
n	$\{0,1\}^n$	0…0, …, 1…1	2^n



$\{0,1\}^1$

$\{0,1\}^2$

$\{0,1\}^3$

$| \{0,1\}^n | = |\{0,1\}|^n = 2^n$

Subsets

The power set of S, denoted $\mathbb{P}(S)$, is the collection of all subsets of S

$\mathbb{P}(\{a,b\}) = \{\ \{\}, \{a\}, \{b\}, \{a,b\}\ \}$

$|\mathbb{P}(S)| = ?$

Subsets of S

Binary strings of length |S|

1-1 correspondence between $\mathbb{P}(S)$ and $\{0,1\}^{|S|}$

$\mathbb{P}(\{a,b\})$ and $\{0,1\}^2$

$\mathbb{P}(\{a,b\})$	a	b	$\{0,1\}^2$
{}	❌	❌	00
{b}	❌	✅	01
{a}	✅	❌	10
{a,b}	✅	✅	11

$$|\mathbb{P}(S)| = |\ \{0,1\}^{|S|}\ | = 2^{|S|}$$

The size of the power set is the power of the set size

Functions

A function from A to B maps every element $a \in A$ to an element $f(a) \in B$

Define a function f: specify $f(a)$ for every $a \in A$

f from {1,2,3} to {p, u} specify f(1), f(2), f(3) f(1)=p, f(2)=u, f(3)=p

f: 3-tuple (f(1), f(2), f(3)) (p, u, p)

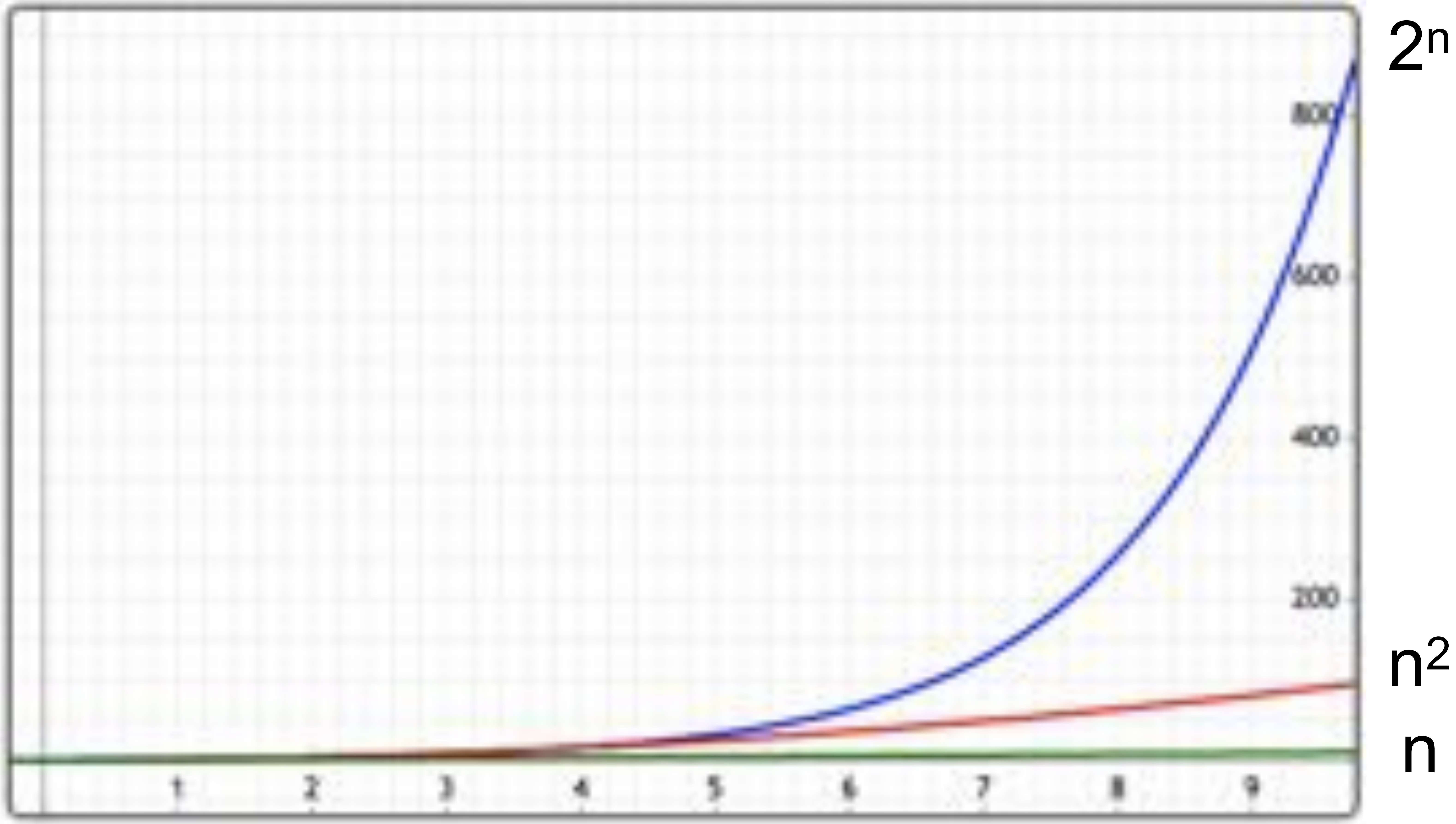
{ functions from {1,2,3} to {p,u} } {p,u} x {p,u} x {p,u}

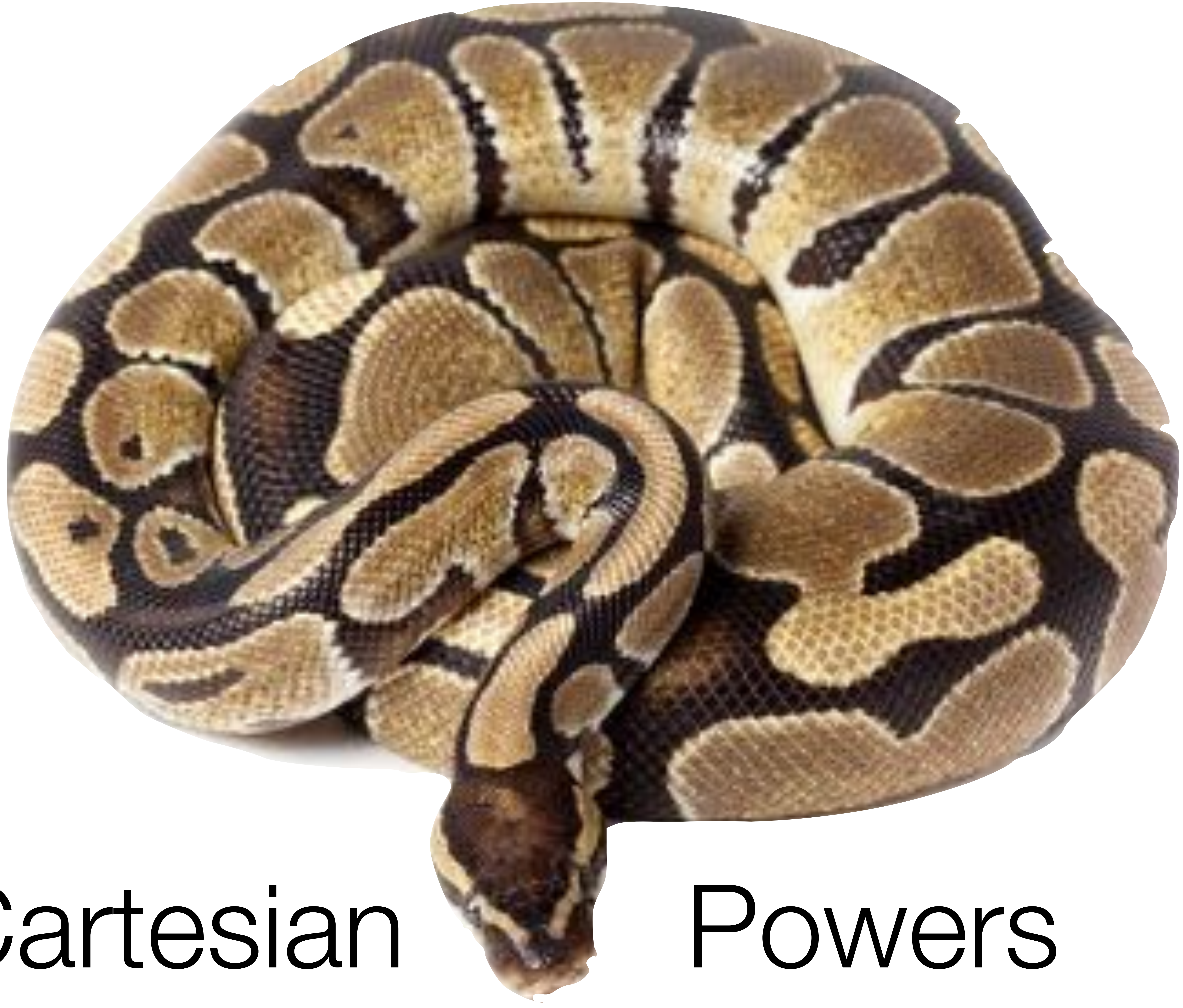
functions from {1,2,3} to {p,u} $= 2 \times 2 \times 2 = 2^3 = |\{p,u\}|^{|\{1,2,3\}|}$

{ functions from A to B } $\underbrace{B \times B \times \ldots \times B}_{|A|} = B^{|A|}$

functions from A to B $= |B^{|A|}| = |B|^{|A|}$ X

Exponential Growth





Cartesian Powers

Cartesian Powers & Exponentials

Cartesian power

Again use **product** function in **itertools** library

```
import itertools
print(set(itertools.product({2, 5, 9}, repeat = 2)))
{(5,9),(5,5),(2,9),(9,2),(9,9),(2,2),(9,5),(2,5),(5,2)}
```

Exponent

**

```
print(3**2)
9
```

Notebook

Compute exponentials

Compare to other growth rates

Chess-Rice Legend (Indian & Persian versions)

Chess Invented by poor yet clever peasant | Became very popular

King liked it | Offered peasant any reward he wished

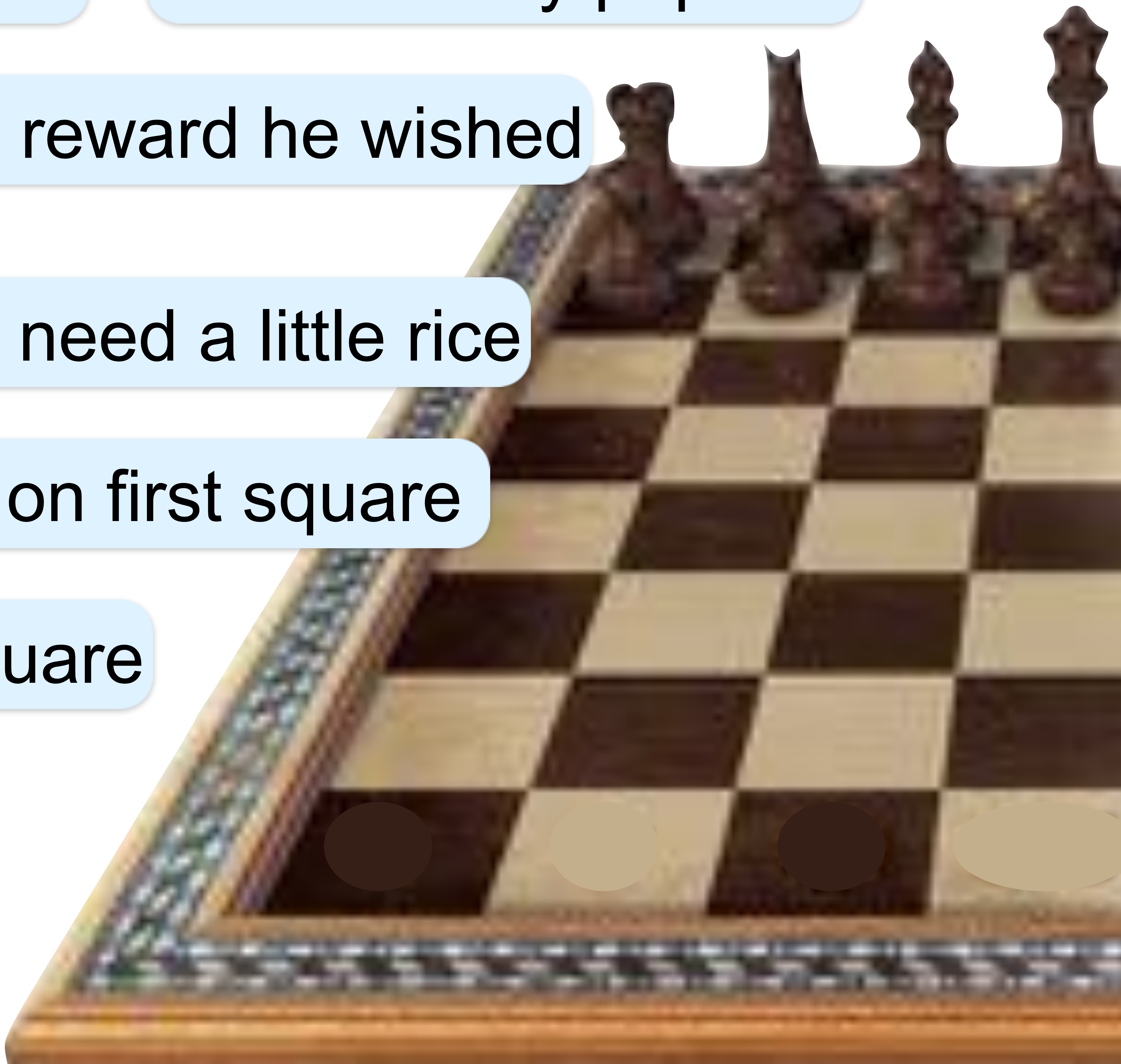
Peasant Poor and humble farmer | Just need a little rice

Kindly place a single rice grain on first square

Double on each subsequent square

King Such a modest request

Granted!



Chess-Rice Legend (ctd.)

King Placed one (2^0) grain on first square

Two (2^1) on second

Four (2^2) on third..

World richest person worth $\$90 \cdot 10^9$

All Americans' worth $\$90 \cdot 10^{12}$

All humans' worth $\$600 \cdot 10^{12}$

64th square: $2^{63} \approx$ 1,000,000,000,000,000,000,000,000,000,000

Two endings Peasant became king

Peasant beheaded

Moral Be peasant or be King: beware of exponentials!

Jeopardy

Counting questions $\longrightarrow$ Answer

$$\# \left\{ \begin{array}{l} \text{n-bit sequences} \\ \text{Subsets of } \{1,\ldots\ldots, n\} \\ \text{Functions: } \{1,\ldots\ldots, n\} \text{ to } \{0,1\} \end{array} \right\} = 2^n$$

? $\longleftarrow$ 2^{2^n}

Find a natural counting question
whose answer is a double exponential

Find a natural counting question whose answer is a double exponential

2 solutions | Subsets | Functions | ∃ more

Power set of S - set of subsets of S | $\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$| \mathbb{P}(S) | = 2^{|S|}$

$| \mathbb{P}(\{a,b\}) | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\ \mathbb{P}(S)\ )$ - set of subsets of $\mathbb{P}(S)$

$|\ \mathbb{P}(S)\ | = 2^{|S|}$

$|\ \mathbb{P}(\ \mathbb{P}(S)\ )\ | = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \}$

$\mathbb{P}(\ \mathbb{P}(\{a, b\})\ ) = \mathbb{P}(\{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \})$

$= \{\ \{\}, \{\ \{\}\ \}, \{\ \{a\}\ \}, \ldots, \{\ \{\}, \{a\}\ \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\}\ \}$

$|\ \mathbb{P}(\ \mathbb{P}(\{a,b\})\ )\ | = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\ \mathbb{P}(\ \mathbb{P}([n])\ )| = 2^{2^{n}}$

Double exponential

Solution 2: Binary Functions

Functions from A to B B^A $\# = |B|^{|A|}$

 # Binary functions of n binary variables

Functions from $\{0,1\}^n$ to $\{0,1\}$ $\{0,1\}^{\{0,1\}^n}$

$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$

Double exponential

x	f(x)	
000	0	2
001	0	2
⋮		⋮
111	1	2

2^n rows; n bits; 2^{2^n}

Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions $2^{63} = ½ \cdot 2^{2^6}$

Which Came First

Cartesian Powers

Cartesian Powers

Cartesian powers $A^n \stackrel{\text{def}}{=} A \times A \times \ldots \times A$ $|A^n| = |A|^n$

Applications binary strings subsets functions

Python A^k `itertools.product(A, repeat = k)`

n^k `n**k`

Variations

Counting Variations

Find a natural counting question whose answer is a double exponential

2 solutions

Subsets

Functions

∃ more

Power set of S - set of subsets of S

$\mathbb{P}(S)$

$\mathbb{P}(\{a, b\}) = \{ \{\}, \{a\}, \{b\}, \{a,b\} \}$

$| \mathbb{P}(S) | = 2^{|S|}$

$| \mathbb{P}(\{a,b\}) | = 4 = 2^2 = 2^{|\{a,b\}|}$ ✅

$\mathbb{P}(S)$ is a set

What about power set of $\mathbb{P}(S)$?

Find a natural counting question whose answer is a double exponential

$\mathbb{P}(\ \mathbb{P}(S)\ )$ - set of subsets of $\mathbb{P}(S)$

$|\ \mathbb{P}(S)\ | = 2^{|S|}$

$|\ \mathbb{P}(\ \mathbb{P}(S)\ )\ | = 2^{|\mathbb{P}(S)|} = 2^{2^{|S|}}$

$\mathbb{P}(\{a, b\}) = \{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \}$

$\mathbb{P}(\ \mathbb{P}(\{a, b\})\ ) = \mathbb{P}(\{\ \{\},\ \{a\}, \{b\}, \{a,b\}\ \})$

$= \{\ \{\}, \{\ \{\}\ \}, \{\ \{a\}\ \}, \ldots, \{\ \{\}, \{a\}\ \}, \ldots, \{\{\},\{a\},\{b\},\{a,b\}\}\ \}$

$|\ \mathbb{P}(\ \mathbb{P}(\{a,b\})\ )\ | = 2^{|\mathbb{P}(\{a,b\})|} = 2^{2^{|\{a,b\}|}}$

$|\ \mathbb{P}(\ \mathbb{P}([n])\ )| = 2^{2^{n}}$

Double exponential

Solution 2: Binary Functions

Functions from A to B

B^A

$\# = |B|^{|A|}$

Binary functions of n binary variables

Functions from $\{0,1\}^n$ to $\{0,1\}$

$\{0,1\}^{\{0,1\}^n}$

$\# = |\{0,1\}|^{|\{0,1\}^n|} = 2^{2^n}$

Double exponential

x	f(x)	
000	0	2
001	0	2
⋮		⋮
111	1	2

2^n (rows), n (width of x), 2^{2^n}

Circuit with n binary inputs, one binary output

Can implement 2^{2^n} functions

$2^{63} = \frac{1}{2} \cdot 2^{2^6}$

Combine addition subtraction multiplication

Determine size of more complex sets

PINs with various constraints

Counting Variations

PIN

Personal Identification Number   2174

\# 4-digit PIN's = ?

$D = \{0, \ldots, 9\}$ < Set of digits

$\{\text{4-digit PIN's}\} = D^4$ < Cartesian Power

$|D^4| \overset{\times}{=} |D|^4 = 10^4 = 10{,}000$

Variable Length

3-5 digit PINs

314 2246 79380

$D = \{0,\ldots,9\}$

$\{\text{PINs}\} = D^3 \cup D^4 \cup D^5$

Disjoint Union

$\#\text{ PINs} = |D^3 \cup D^4 \cup D^5| \overset{+}{=} |D^3| + |D^4| + |D^5|$

$= 10^3 + 10^4 + 10^5$

$= 1{,}000 + 10{,}000 + 100{,}000$

$= 111{,}000$

Forbidden Patterns

4-digit PIN's

Forbidden All same 3333 0000, 1111, ..., 9999 # = 10

Consecutive 3456 0123, 1234, ..., 6789 # = 7

Forbidden All same ∪ Consecutive ⊕ 10 + 7 = 17

Allowed D^4 - Forbidden ⊖ 10,000 - 17 = 9,983

4-digit PINs

PINs Containing Zero

PIN's containing 0

8093

~~2534~~

Use latex for formulas
→ slides less colorful

D = {0,1, …, 9}

$Z = \{0\}$ $\overline{Z} = \{1, 2, ..., 9\}$

$\overline{Z} = Z^c$ set of non-zero digits

$x^n \triangleq x_1, \ldots, x_n$

n-digit sequence

$\exists Z = \{x^n \in D^n : \exists i \ x_i \in Z\}$

{n-digit PINs containing 0}

$|\exists Z| = \ ?$

Start with 2 digits

2 ways

Inclusion exclusion

Subtraction rule

2-Digits: Inclusion-Exclusion

$$\exists Z = \{x_1 x_2 \colon \exists i\ x_i = 0\}$$ 00 03 50 ~~73~~

$$Z_1 = \{x_1 x_2 \colon x_1 = 0\}$$ 00 03 ~~50~~ ~~73~~ $|Z_1| = 10$

$$Z_2 = \{x_1 x_2 \colon x_2 = 0\}$$ 00 ~~03~~ 50 ~~73~~ $|Z_2| = 10$

$$\exists Z = Z_1 \cup Z_2$$

$$|\exists Z| = |Z_1| + |Z_2| - |Z_1 \cap Z_2|$$

$$= 10 + 10 - 1 = 19$$

$$Z_1 \cap Z_2 = \{00\}$$

$$|Z_1 \cap Z_2| = 1$$

2-Digits: Complement Rule

$\exists Z = \{x_1x_2\colon \exists i\ x_i = 0\}$ 00 03 50 ~~73~~

$\Omega = D^2$ All 2-digit PINs

$\overline{\exists Z} = \overline{\{x_1x_2 : \exists i\ x_i = 0\}} = \{x_1x_2 : \forall i\ x_i \neq 0\} = \overline{Z} \times \overline{Z}$ 73 44 19 ~~50~~

both digits nonzero

{1,..,9} x {1,..,9}

$$|\overline{\exists Z}| = |\overline{Z} \times \overline{Z}| = |\overline{Z}|^2 = 9^2 = 81$$

$$\exists Z = D^2 - \overline{\exists Z}$$

$$|\exists Z| = |D^2| - |\overline{\exists Z}| = 100 - 81 = 19$$

 or ⊖ Better?

$\exists Z$

00	10	20	30	…	…	…	…	80	90
01	11								
02									
03									
…									
…									
…									
…									
08									
09									99

Z_2

Z_1

D^2

$\overline{\exists Z}$

Inclusion-Exclusion

$|Z_1| + |Z_2| - |Z_1 \cap Z_2|$

$= 10 + 10 - 1 = 19$

Complement $|\overline{\exists Z}| = 9^2 = 81$ $100 - 81 = 19$

n Digit: Inclusion Exclusion

$$\exists Z = \{x^n \colon \exists i \; x_i = 0\} \qquad x^n \triangleq x_1, \ldots, x_n$$

$$Z_i = \{x^n \colon x_i = 0\}$$

n=4 $\quad Z_2 = \{x0yz\} \quad Z_4 = \{xyz0\}$

$$\exists Z = Z_1 \cup \ldots \cup Z_n$$

$$\begin{aligned}
|\exists Z| \;\; = \;\; & |Z_1| + |Z_2| + \ldots + |Z_n| \\
& - |Z_1 \cap Z_2| - |Z_1 \cap Z_3| - \ldots - |Z_{n-1} \cap Z_n| \\
& + |Z_1 \cap Z_2 \cap Z_3| + \ldots + |Z_{n-2} \cap Z_{n-1} \cap Z_n| \\
& \ldots \\
& + (-1)^{n-1} |Z_1 \cap Z_2 \cap \ldots \cap Z_n|
\end{aligned}$$

n Digits: Complement

$$\overline{\exists Z} = \overline{\{x^n | \exists i \ x_i \in Z\}} = \{x^n | \forall i \ x_i \notin Z\} = (\overline{Z})^n \triangleq \forall \overline{Z}$$

all digits nonzero

X

$$|\forall \overline{Z}| = |\overline{Z}|^n = 9^n$$

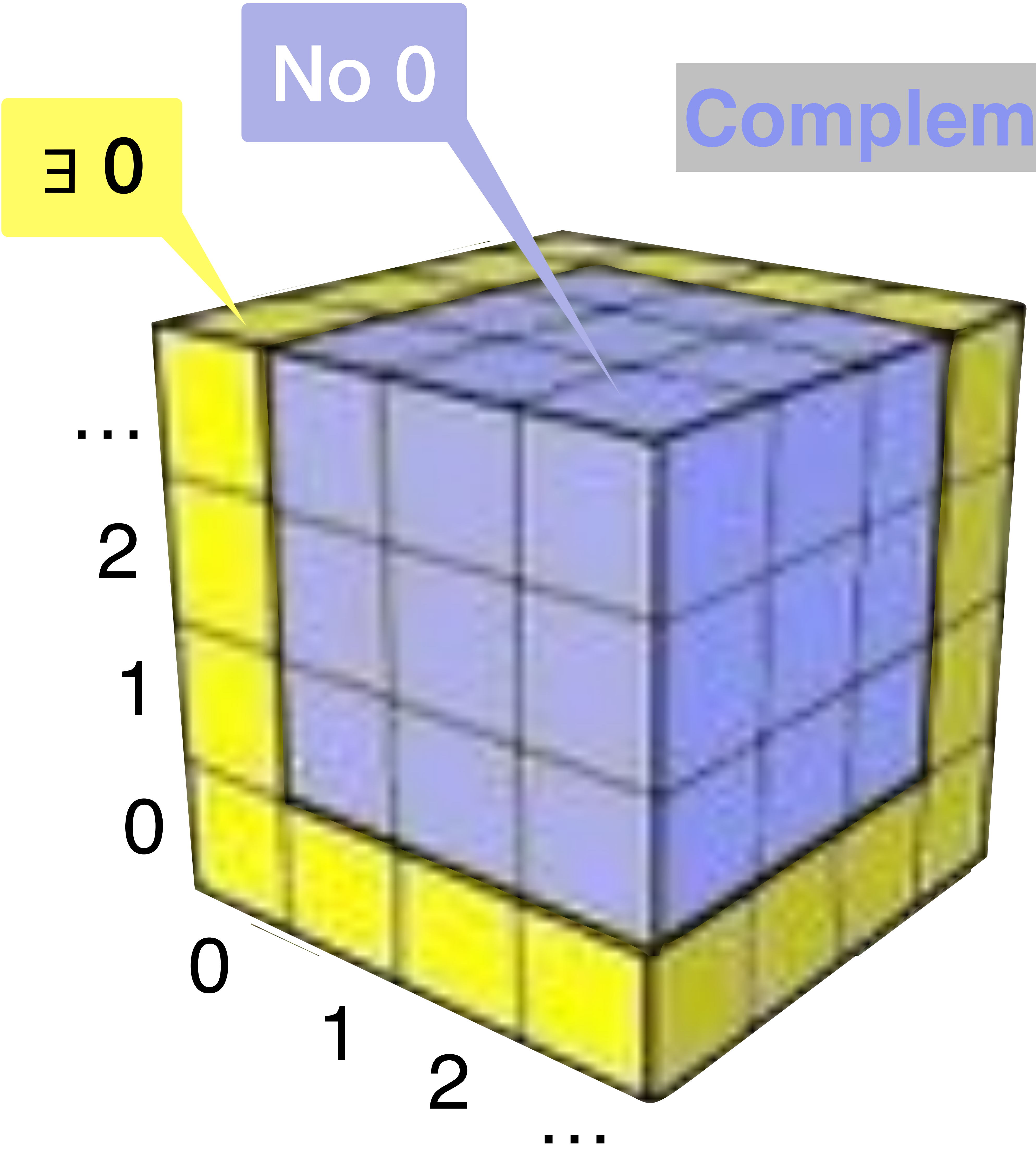
$$\exists Z = D^n - \forall \overline{Z}$$

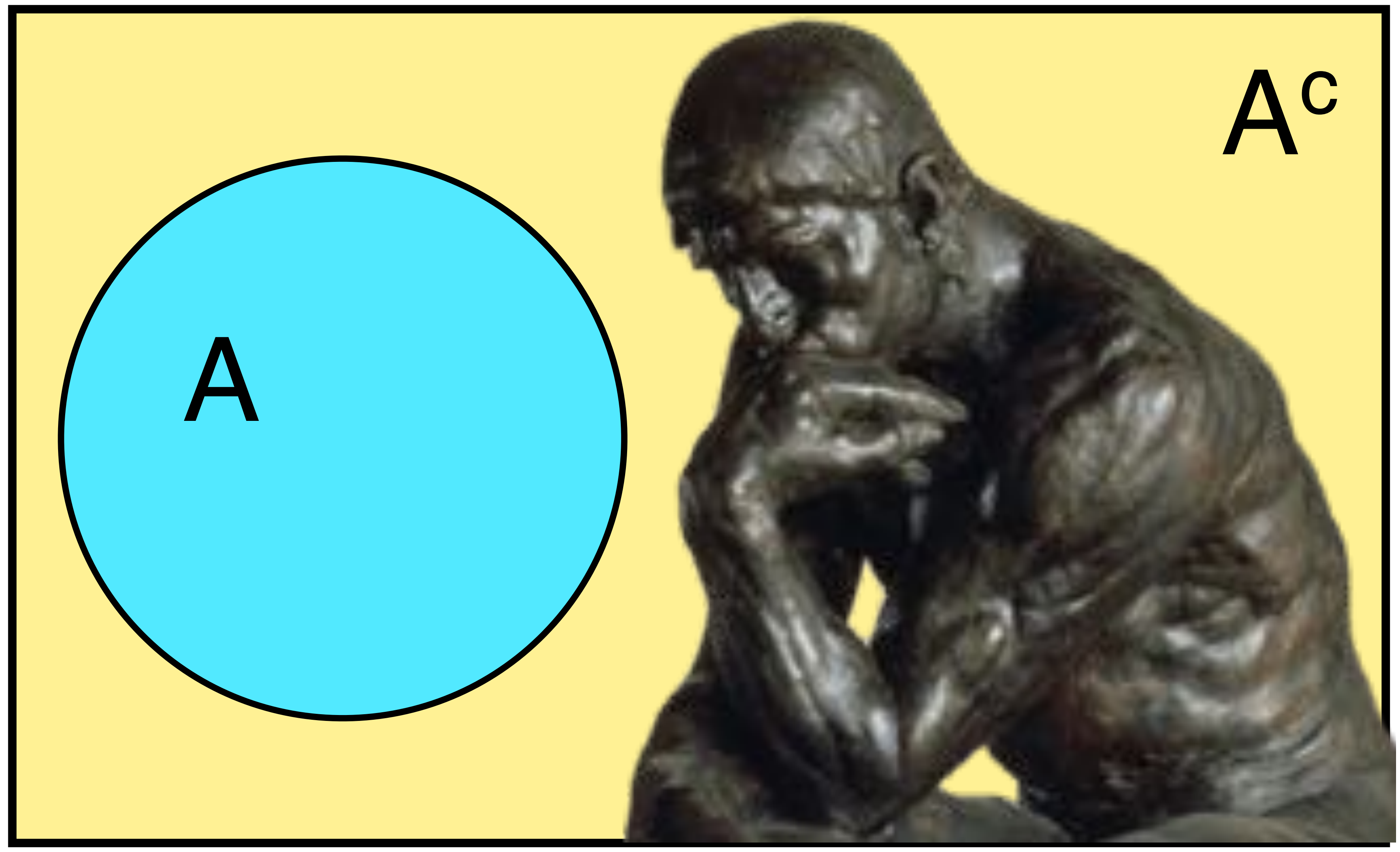
$$|\exists Z| = |D^n| - |\forall \overline{Z}| = 10^n - 9^n$$

Visualize

Complement

Inclusion exclusion





Counting **Variations**

Combined

addition

subtraction

multiplication

Determine size of more complex sets

PINs with various constraints

Trees

Cartesian products as trees

Trees generalize products

Systematic tree counts

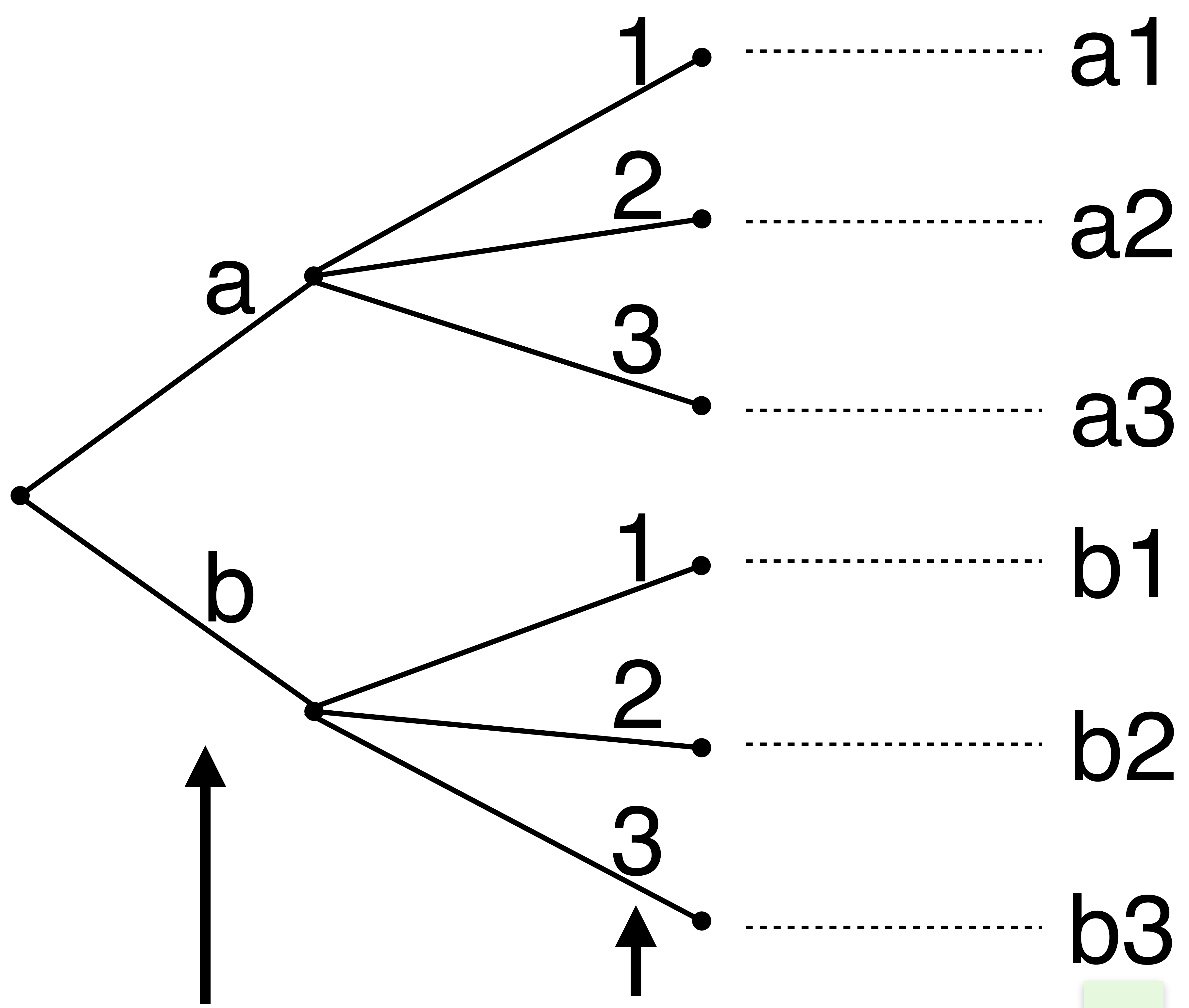
Paths in graphs

Useful for modeling randomness

Counting Trees

Cartesian Products as Trees

Tree ← Sequences ← Cartesian product



$\{a,b\} \times \{1,2,3\}$

2 x 3 = 6

6

$|\{a,b\} \times \{1,2,3\}| = 2\times3 = 6$

Used only

At any level, all nodes have same degree

Trees are More General

San-Diego University of Data Science (SUDS)

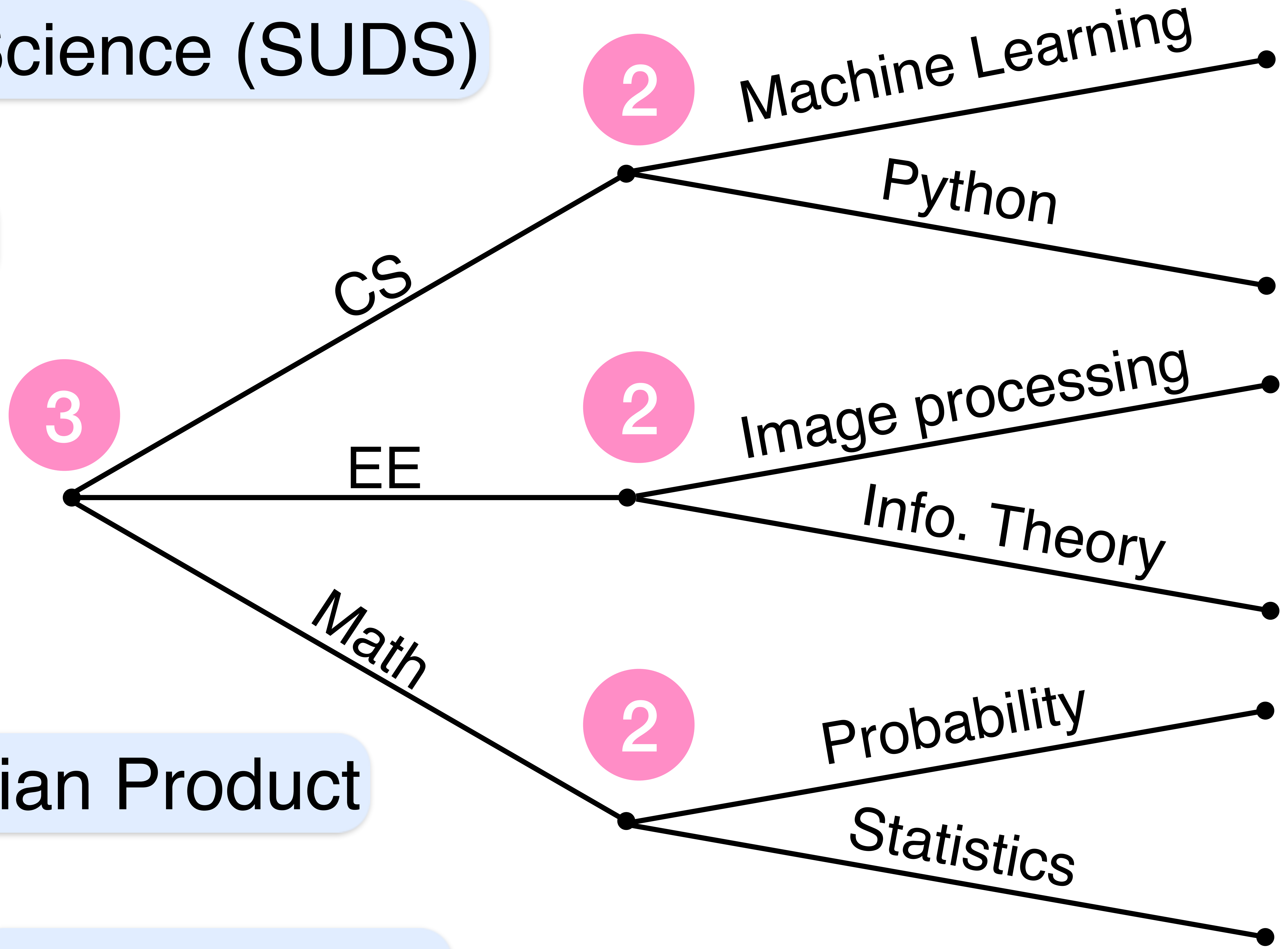
3 departments: CS, EE, Math

Each offers two courses

courses = ?

Departments offer *different* courses

Not Cartesian Product



Still

2 courses / department

Each level, all degrees equal

courses = 3 x 2 = 6

“X”

Why Trees

A tree can represent any set of sequences, not just Cartesian Products

Enable systematic counting technique

Useful in modeling random phenomena

Best of n

Many sports

Two teams or players compete to determine stronger

Single competition too random

Play odd # games

n

NBA Playoffs

n = 7 games

Tennis matches

n = 3 or 5 sets

Goal

Win majority of n games

Once someone wins > n/2

Stop

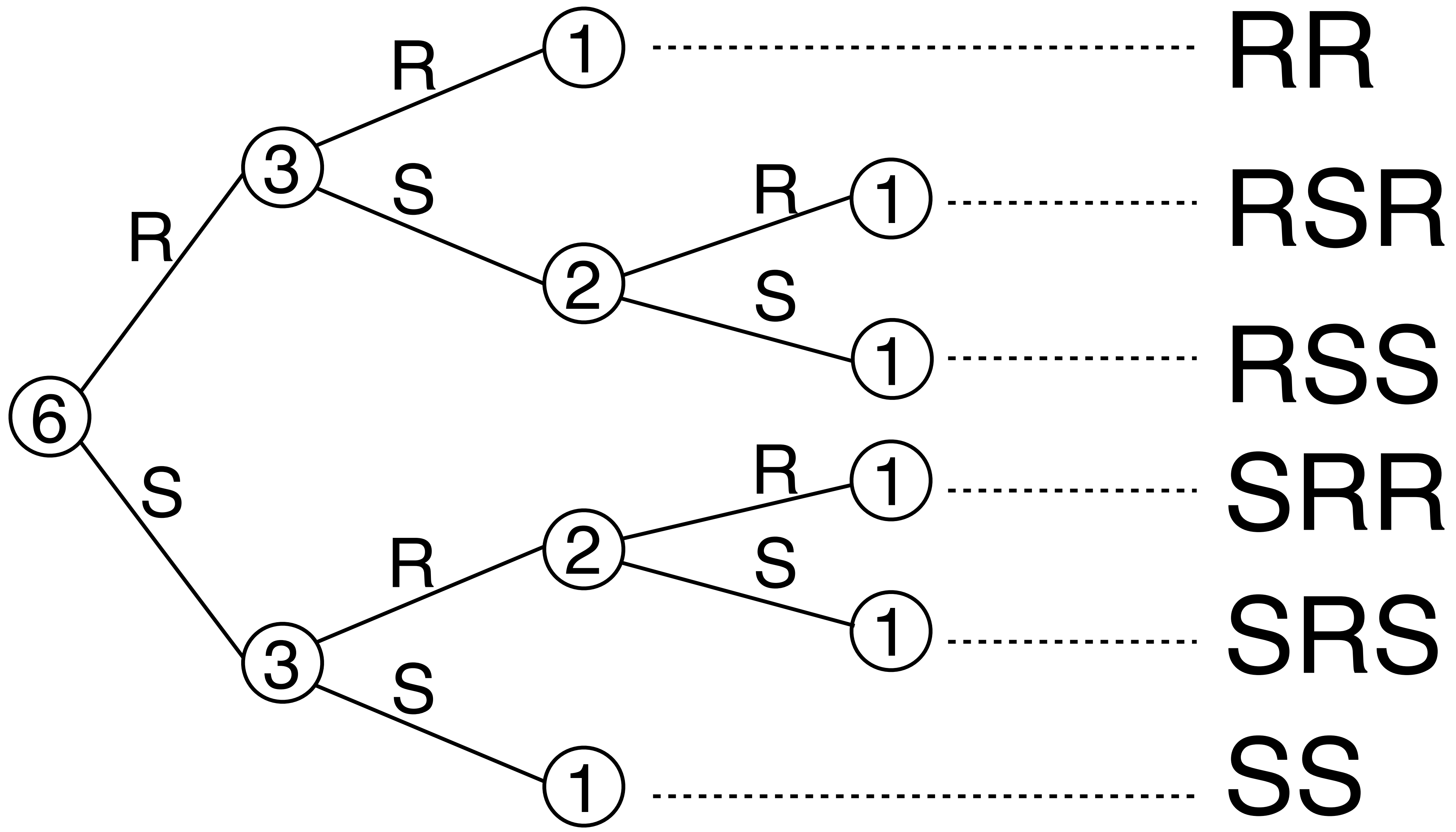
Paths to Victory

Roger and Serena

3-set match

Stop when one wins two sets

win sequences = ?



More later

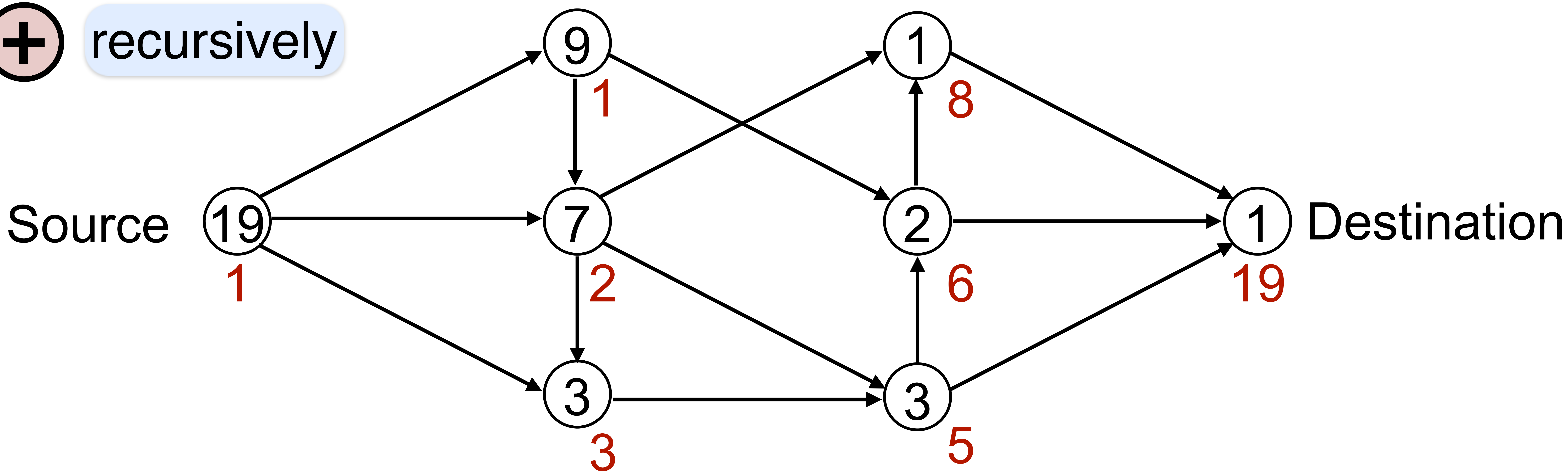
Paths from Source to Destination

Generalize to directed acyclic graph

paths from source to destination

Recursively determine # paths from a node to destination

+ recursively



Look for Structure

Don’t lose sight of the forest

Cartesian products as trees

Trees generalize products

Systematic tree counts

Paths in graphs

Useful for modeling randomness

Counting Trees

Counting Sets

Disjoint unions +

Complements −

General unions 

Cartesian products X

Cartesian powers 

Variations 

Sequences

Trees

Graphs

Combinatorics

“Advanced counting”

Useful for determining probabilities