Combinatorics

Permutations

Musical Chairs, Bryant Park, NYC, Summer 2013

Permutations

A permutation is an ordering of a set of objects

permutations of n objects = ?

Objects can be anything

For most excitement

letters!

# letters	permu-tations	# permu-tations
1	a	1
2	a b b a	2
3	a b c a c b b a c b c a c a b c b a	6

General n?

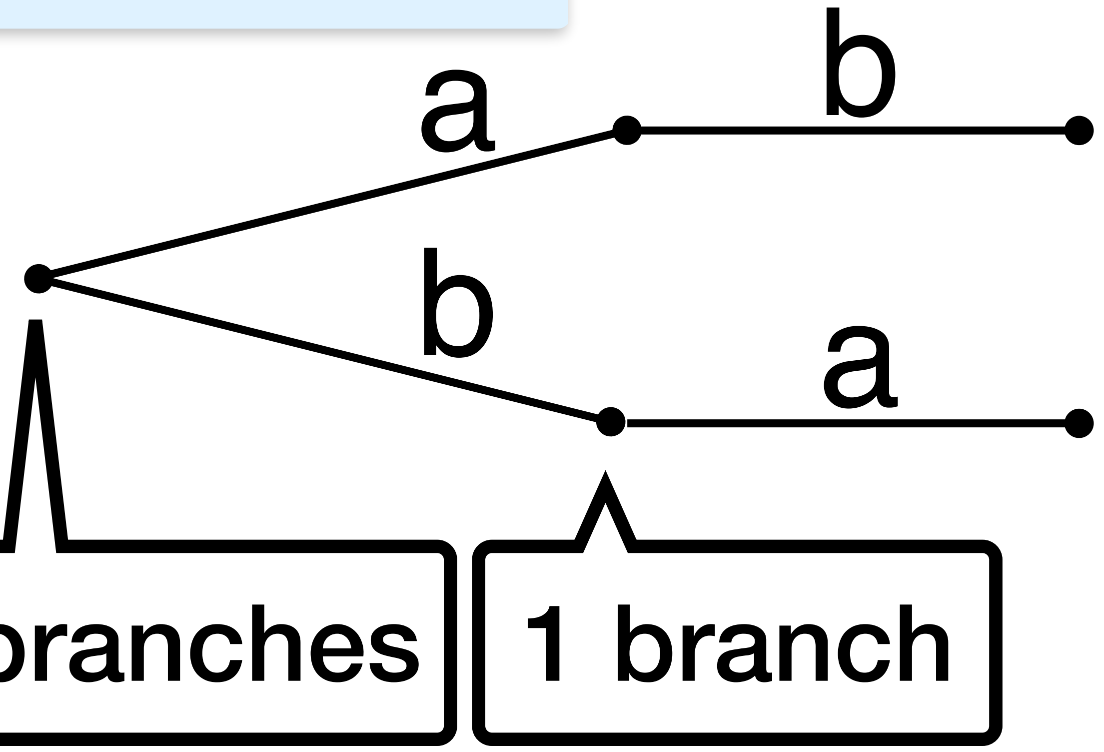
Counting Permutations

2 objects

2 choices

1 choice

a b
b a



$2 \times 1 = 2$

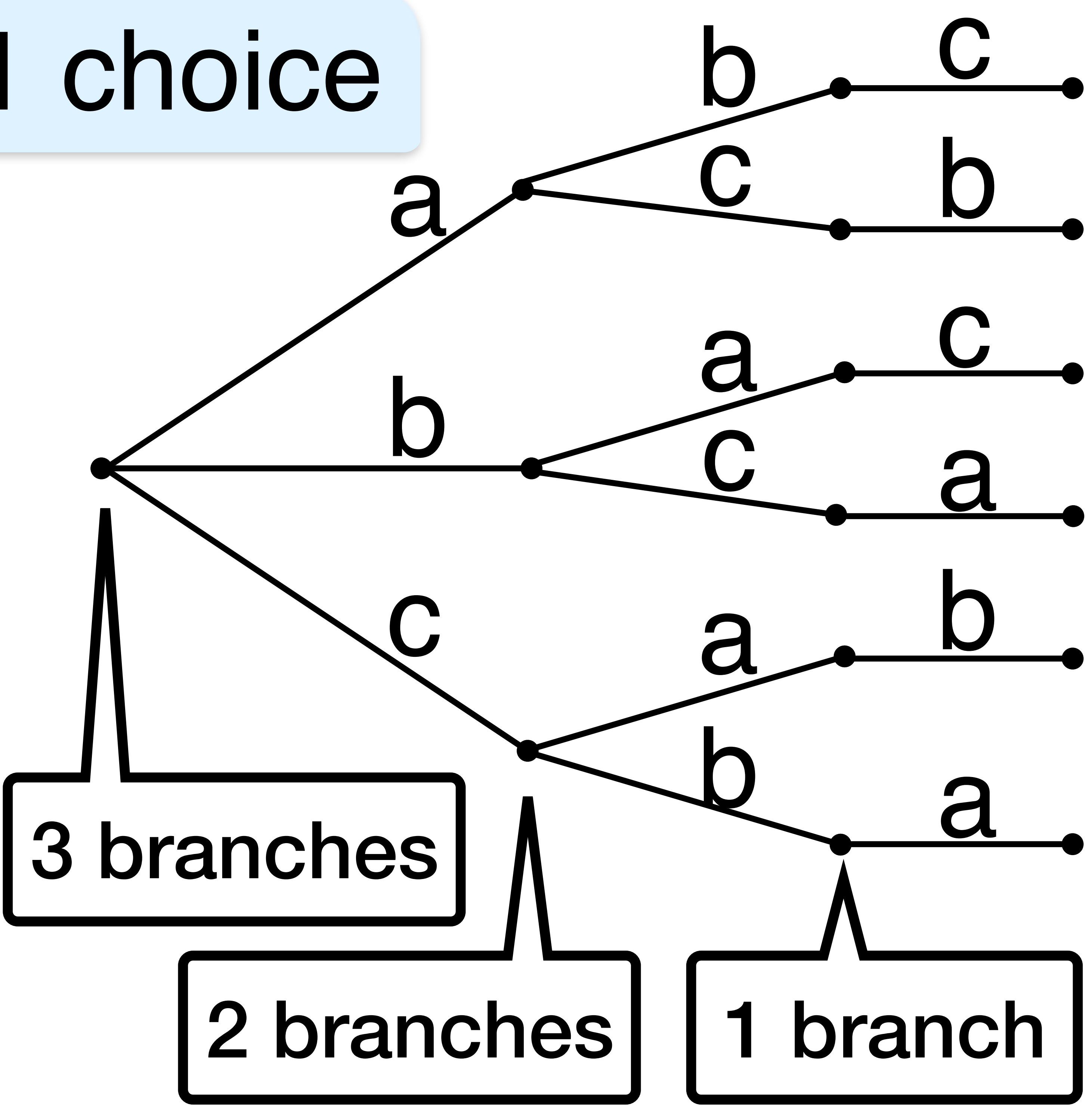
3 objects

2 choices

3 choices

1 choice

a b c
a c b
b a c
b c a
c a b
c b a



“X”

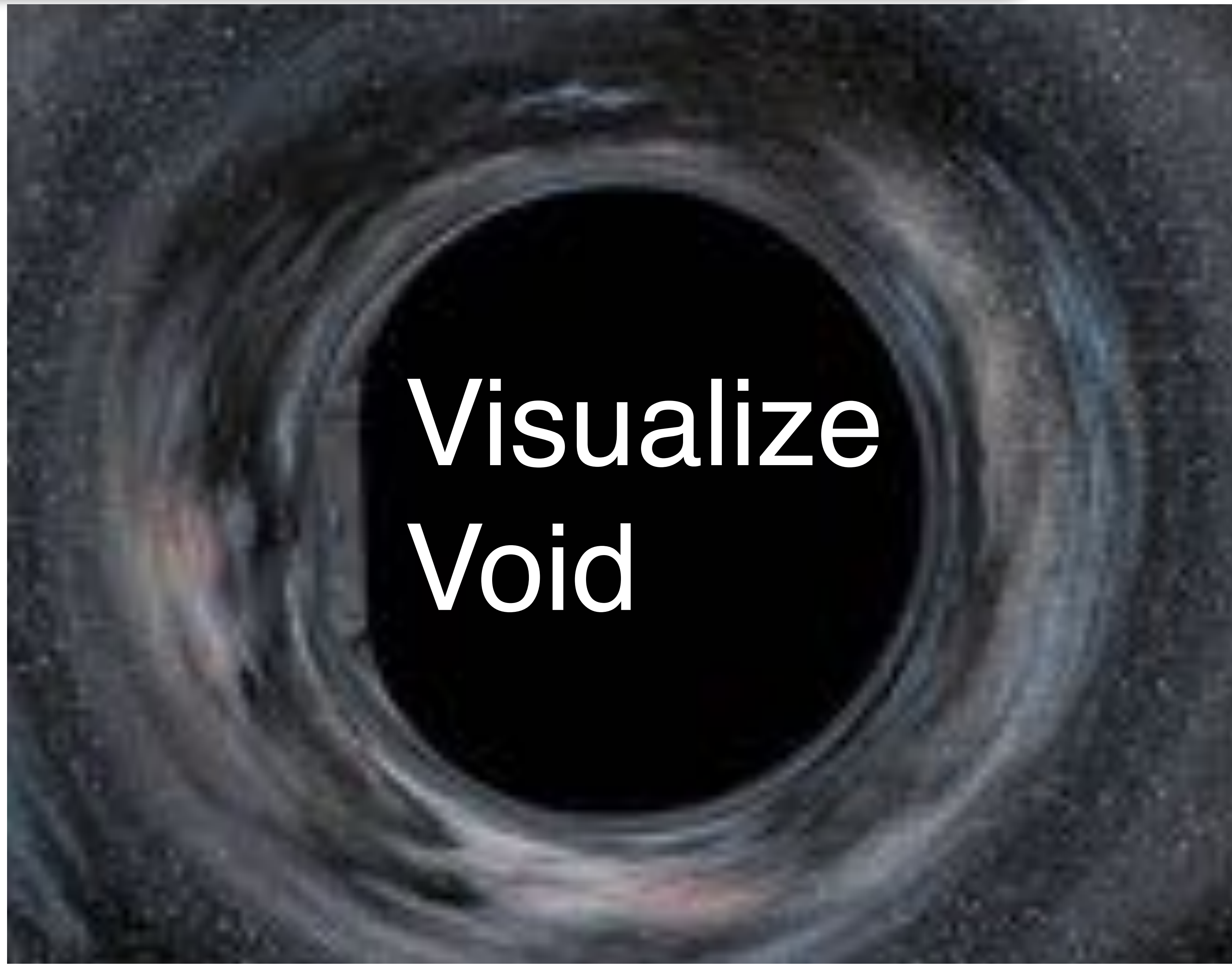
$3 \times 2 \times 1 = 6$

\# permutations of n objects = $n \times (n-1) \times \dots \times 2 \times 1 \triangleq n!$

n factorial

0 Factorial

For $n \geq 1$ $\quad$ $n! = \#$ permutations of n objects $= n \times (n-1) \times \ldots \times 2 \times 1$



What about 0! ?

How many ways can you permute 0 objects?

a,b: ab, ba $\quad$ a: a $\quad$: ?

a,b: (ab), (ba) $\quad$ a: (a) $\quad$: ( )

$0! = 1$

Exact same reason as $2^0=1$

Alternative Factorial View

Write Left to right → Smallest to largest

One position for the 1st object

Two for 2nd • 1 •

Three for 3rd • 2 • 1 •

Four for 4th • 2 • 1 • 3 •

$\vdots$

1 x 2 x 3 x… x n = n!

Recursive Definition

n! can be defined recursively

$$n! = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1$$
$$= n \cdot [(n-1) \cdot \ldots \cdot 2 \cdot 1]$$
$$= n \cdot (n-1)! \qquad \forall n \geq 1$$

1!=1x0!
Extends to negatives

n	Product	n!	
0	1	1	
			x 1
1	1	1	
			x 2
2	2 × 1	2	
			x 3
3	3 × 2 × 1	6	
			x 4
4	4 × 3 × 2 × 1	24	
			x 5
5	5 × 4 × 3 × 2 × 1	120	
			x 6
6	6 × 5 × 4 × 3 × 2 × 1	720	

Examples and applications

Basic Permutations

orders to visit 3 cities

LA, SD, SF

$$\left.\begin{array}{l}\text{LA SD SF}\\ \cdots\cdot\\ \text{SF SD LA}\end{array}\right\}\ 3! = 3 \times 2 \times 1 = 6$$

anagrams of 5 distinct letters

PEARS

$$\left.\begin{array}{l}\text{SPEAR}\\ \cdots\cdot\\ \text{EAPRS}\end{array}\right\}\ 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Constrained Anagrams of PEARS

A,R stay adjacent in order

PARSE
.....
SEPAR

Permutations of P E AR S

$4! = 4 \times 3 \times 2 \times 1 = 24$

A,R are adjacent in any order

SPARE
.....
RAESP

2 orders

24 anagrams each

"$\times$"

$2 \times 24 = 48$

A,R are not adjacent

AESPR
.....
SRPAE

$-$

$5! - 48 = 120 - 48 = 72$

More Constrained Permutations

ways 3 distinct boys and 2 distinct girls can stand in a row

Unconstrained

$(3+2)! = 5! = 120$

Alternating boys and girls

$3! \times 2! = 6 \times 2 = 12$

Boys together and girls together

$2 \times 3! \times 2! = 24$

Unconstrained, but orientation (left to right) doesn't matter $5! / 2 = 60$

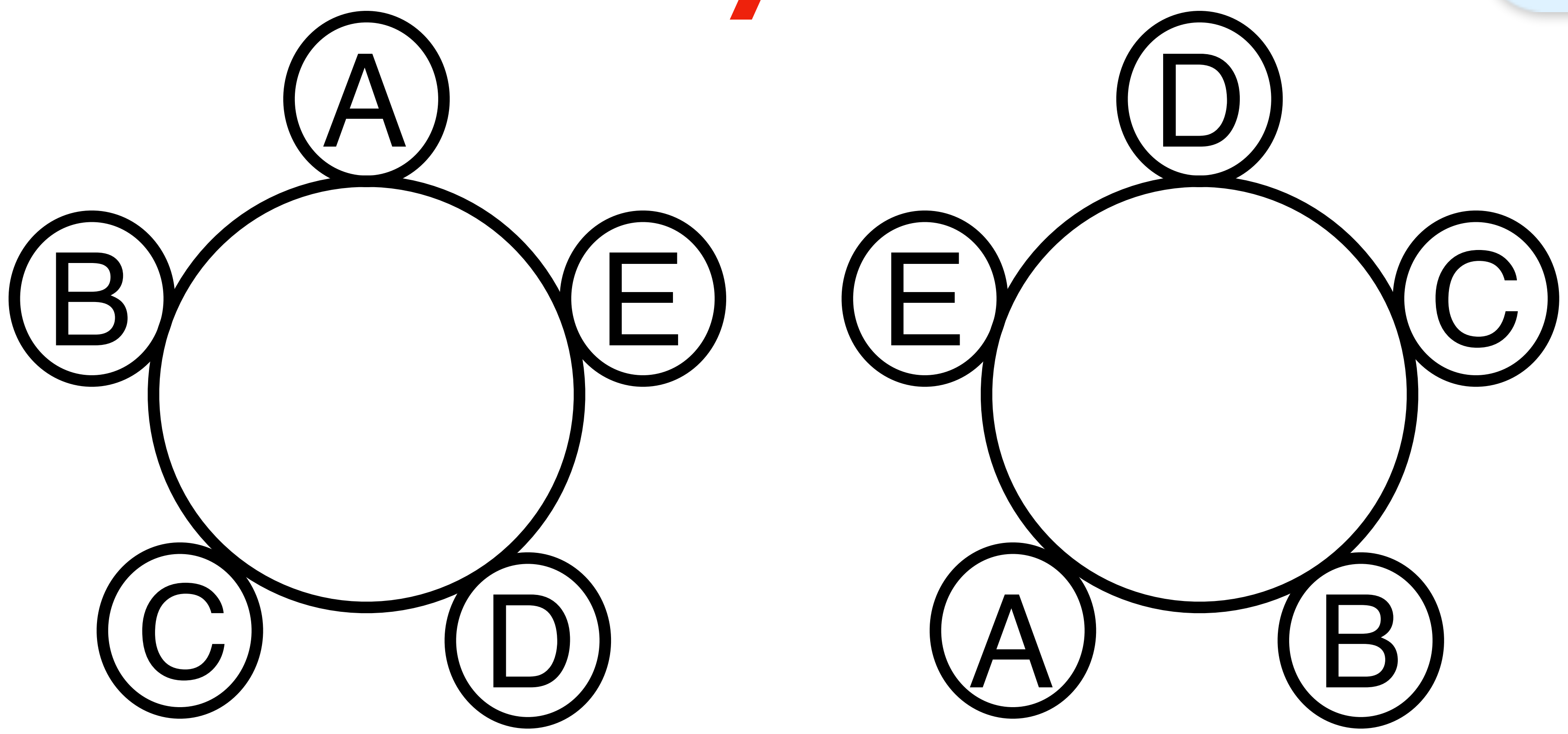
Circular Arrangements

ways 5 people can sit at a round table = ?

Rotations matter

$\neq$

5! = 120



Rotations don't matter

=

5! / 5 = 4! = 24

Partial Permutations

Partial Permutations

orders of n objects = n!

orders of some of the n objects = ?

2-digit PINs

any digits

11 45

10 x 10

distinct digits

05 32 ~~33~~

10 x 9

1st digit 2nd digit

3-letter words

any letters

mom xyz

26 x 26 x 26

distinct letters

abc ~~dad~~

26 x 25 x 24

1st 2nd 3rd

Sequences without Repetition

k-permutation of [n]

Length-k sequences over [n]

with repetition n^k

without repetition $n^{\underline{k}}$

k-Permutations

Set of size n: n-set

ation

An ordering of k elements in a set S is a k-permutation of S

2-permutations of {a,b,c} ab, ac, ba, bc, ca, cb

n-permutation of an n-set is just a permutation of the set

k-permutations of an n-set $n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \stackrel{\text{def}}{=} n^{\underline{k}}$

k	$n^{\underline{k}}$
1	n
2	n(n-1)
3	n(n-1)(n-2)
...	...
k	n(n-1)...(n-k+1)

k^{th} falling power of n

Also denoted P(n,k) or $(n)_k$

Falling Powers and Factorials

Falling powers are simply related to factorials

$$n^{\underline{k}} \qquad n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \quad = \tfrac{n!}{(n-k)!}$$

Factorials and Permutations

4 programming, 5 probability, 6 machine-learning books

ordered lists with 2 books from each subject
where same subject books are listed consecutively

= ?

Prob 3, Prob1, ML 5, ML 2, Prog 1, Prog 4

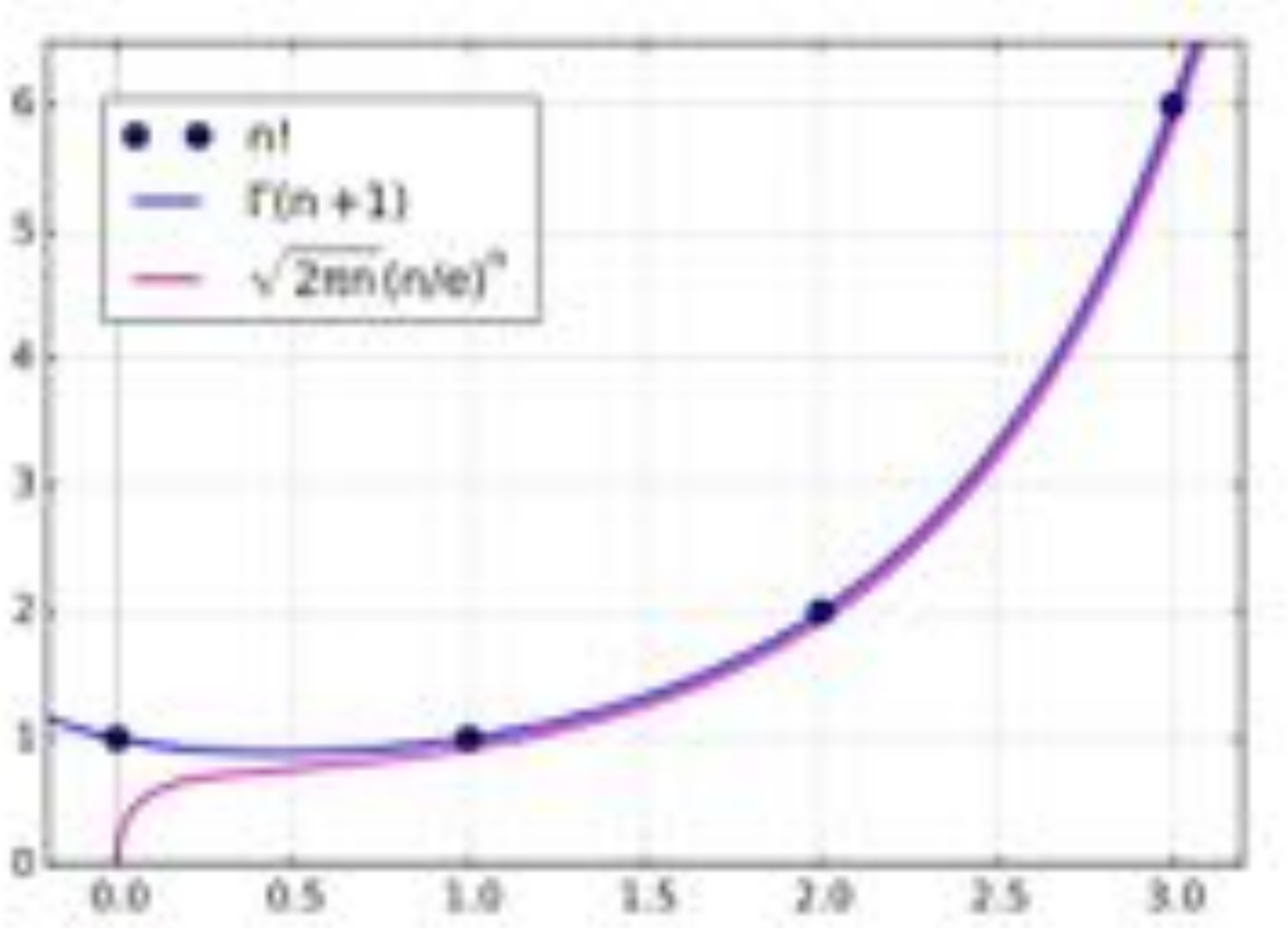
ML 2, ML 6, Prog 1, Prog 2, Prob 5, Prob 1

$$\underbrace{3!}_{\text{Subject Order}} \cdot \underbrace{4^{\underline{2}}}_{\text{Programming}} \cdot \underbrace{5^{\underline{2}}}_{\text{Probability}} \cdot \underbrace{6^{\underline{2}}}_{\text{Machine Learning}}$$

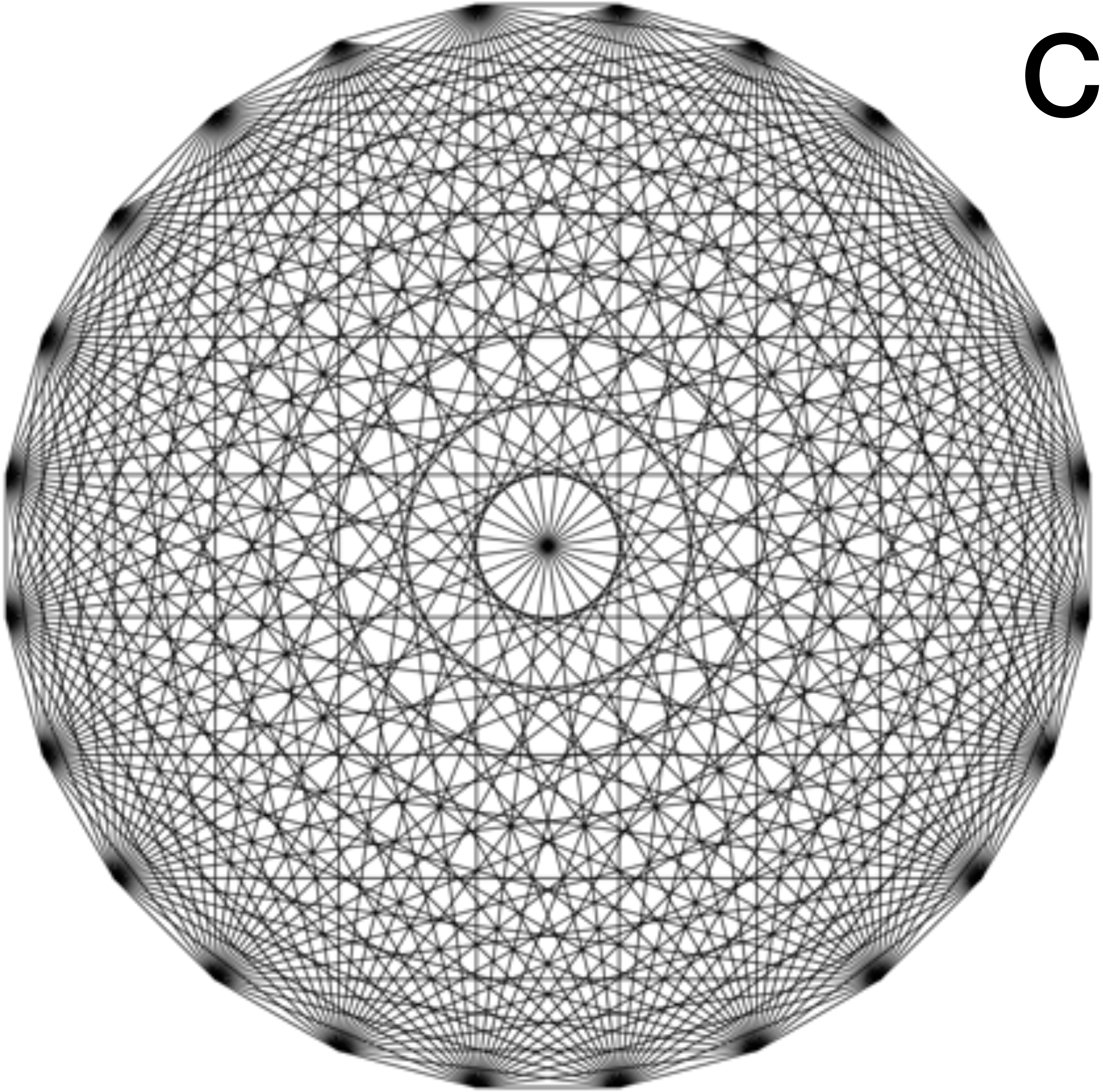
= 6 x (4x3) x (5x4) x (6x5) = 43,200

Stirling's Approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$



N	N!	Stirling approximation	Error %
1	1	1.00	0.227445%
2	2	2.00	0.032602%
3	6	6.00	0.009986%
4	24	24.00	0.004266%
5	120	120.00	0.002198%
6	720	720.01	0.001276%
7	5040	5040.04	0.000805%
8	40320	40320.22	0.000540%
9	362880	362881.38	0.000380%
10	3628800	3628810.05	0.000277%
11	39916800	39916883.11	0.000208%
12	479001600	479002368.48	0.000160%
13	6227020800	6227028659.89	0.000126%
14	87178291200	87178379323.32	0.000101%
15	1307674368000	1307675442913.47	0.000082%
16	20922789888000	20922804061389.80	0.000068%
17	355687428096000	355687629001078.00	0.000056%
18	6402373705728000	6402376752492220.00	0.000048%
19	121645100408832000	121645149634119000.00	0.000040%
20	2432902008176640000	2432902852332160000.00	0.000035%



Combinations

k-subsets

A k-element set is called a k-set, a k-element subset is a k-subset

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

$\binom{[3]}{1}$ { {1}, {2}, {3} }

$\binom{[3]}{2}$ { {1,2}, {1,3}, {2,3} }

$\binom{[4]}{2}$ { {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {3,4} }

Sequences with k 1’s

$\binom{[n]}{k}$ collection of k-subsets of [n] = {1,2,…,n}

1-1 correspondence to n-bit sequences with k 1’s

	Subsets	Binary Sequences
$\binom{[3]}{1}$	{1}, {2}, {3}	100, 010, 001
$\binom{[3]}{2}$	{1, 2}, {1, 3}, {2, 3}	110, 101, 011
$\binom{[4]}{2}$	{1, 2}, {1, 3}, …, {3, 4}	1100, 1010, …, 0011

Two Interpretations

$\binom{[n]}{k}$

k-subsets of an n-set

n-bit sequences with k 1's

$\binom{[3]}{2}$

$\{\{1,2\}, \{1,3\}, \{2,3\}\}$

110, 101, 011

1 - 1 correspondence

Same number of elements

Mostly count sequences

Same applies to subsets

Number of n-Bit Sequences with k 1's

$$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right|$$ = # n-bit sequences with k 1's

binomial coefficient

$$\binom{3}{2} = \left|\binom{[3]}{2}\right| = |\{110, 101, 011\}| = 3$$

Locations of 1's: Ordered Pairs from {1,2,3}

$$\# = 3^2 = 6$$

12	110
13	101
21	110
23	011
31	101
32	011

110

101

011

$$\binom{3}{2} = \frac{3^2}{2} = \frac{6}{2} = 3$$

Calculating the Binomial Coefficients

$$\binom{n}{k} \triangleq \left|\binom{[n]}{k}\right| = ?$$

number set

Each location $\in$ [n]

Specify locations of the k 1's in order

123, 531, 213, …

\# ordered locations $n^{\underline{k}}$

Every binary sequence with k 1's corresponds to k! ordered locations

10101 $\leftrightarrow$ 1,3,5 1,5,3 3,1,5 3,5,1 5,1,3 5,3,1

$$k!\binom{n}{k} = n^{\underline{k}} \qquad \binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}$$

$$\left|\binom{[n]}{k}\right| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$ **for small n,k**

$$\binom{[3]}{1} = \left\{\begin{matrix} 001 \\ 010 \\ 100 \end{matrix}\right\} \qquad \binom{3}{1} = \frac{3!}{1!\,2!} = 3$$

Choose location of 1

$$\binom{[3]}{2} = \left\{\begin{matrix} 011 \\ 101 \\ 110 \end{matrix}\right\} \qquad \binom{3}{2} = \frac{3!}{2!\,1!} = 3$$

Choose location of 0

$$\binom{[4]}{2} = \left\{\begin{matrix} 0011 \\ 0101 \\ 0110 \\ 1001 \\ 1010 \\ 1100 \end{matrix}\right\} \qquad \binom{4}{2} = \frac{4!}{2!\,2!} = 6$$

Choose location of 1's
1st location: 4 choices
2nd location: 3 choices
Each chosen twice

Simple $\binom{n}{k}$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

All-zero sequence

$$\binom{n}{n} = \frac{n!}{n!\,0!} = 1$$

All-one sequence

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = n$$

Choose location of single 1

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

1st location: n ways
2nd location: n-1 ways
Each sequence chosen twice

Alternative Explanation of $\binom{n}{2} = \frac{n(n-1)}{2}$

$\binom{[n]}{2} = \{\text{n-bit strings with two 1's}\}$

$A_i = \{x^n\text{: first 1 at location i}\} \quad (1 \le i \le n-1)$

$|A_i| = n - i \qquad A_i\text{'s disjoint}$

$$\underset{\uparrow \atop 1}{0}\ \ldots\ 0\ \underset{\uparrow \atop i}{1}\ \overbrace{x\ \ y\ \ldots\ \underset{\uparrow \atop n}{z}}^{\text{n-i locations w/ exactly one 1}}$$

$\binom{[n]}{2} = A_1 \uplus A_2 \uplus \ldots \uplus A_{n-1}$

$\binom{n}{2} = |\binom{[n]}{2}| = |A_1| + \ldots + |A_{n-1}| = (n-1) + \ldots + 1 = \frac{n(n-1)}{2}$

Binomial Coefficients by Hand

Simply calculate binomial coefficients

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!}$$

$$\binom{7}{2} = \frac{7\cdot\cancel{6}^{\,3}}{\cancel{2}\cdot 1} = 7 \cdot 3 = 21$$

$$\binom{n}{k} = \binom{n}{n-k}, \text{ if k} > \frac{n}{2} \text{ calculate } \binom{n}{n-k}$$

$$\binom{7}{5} = \binom{7}{2} = 21$$

Cancel terms

$$\binom{12}{9} = \binom{12}{3} = \frac{{}^{2}\cancel{12}\cdot 11\cdot 10}{\cancel{3}\cdot\cancel{2}\cdot 1} = 220$$

Can calculate fairly large binomial coefficients by hand

Permutations vs. Combinations

Permutations

Order matters

Combinations

Order doesn't matter

Type of lock you used to secure your bike?

Combinations

Applications of Binomial Coefficients

Subsets

k-element subsets of an n-set

1-1 correspondence between such sets and n-bit sequence with k 1's

$$\# = \binom{n}{k}$$

Committees with Constraints

4 boys 3 girls

choose 2 boys and 2 girls

$$\binom{4}{2}\binom{3}{2} = 6 \cdot 3 = 18$$

Conflict Resolution

4 boys and 3 girls

John Mary cannot serve together

How many committees of 4

with John $\binom{5}{3} = 10$

with Mary $\binom{5}{3} = 10$

neither $\binom{5}{4} = 5$

$$\overline{25}$$

25

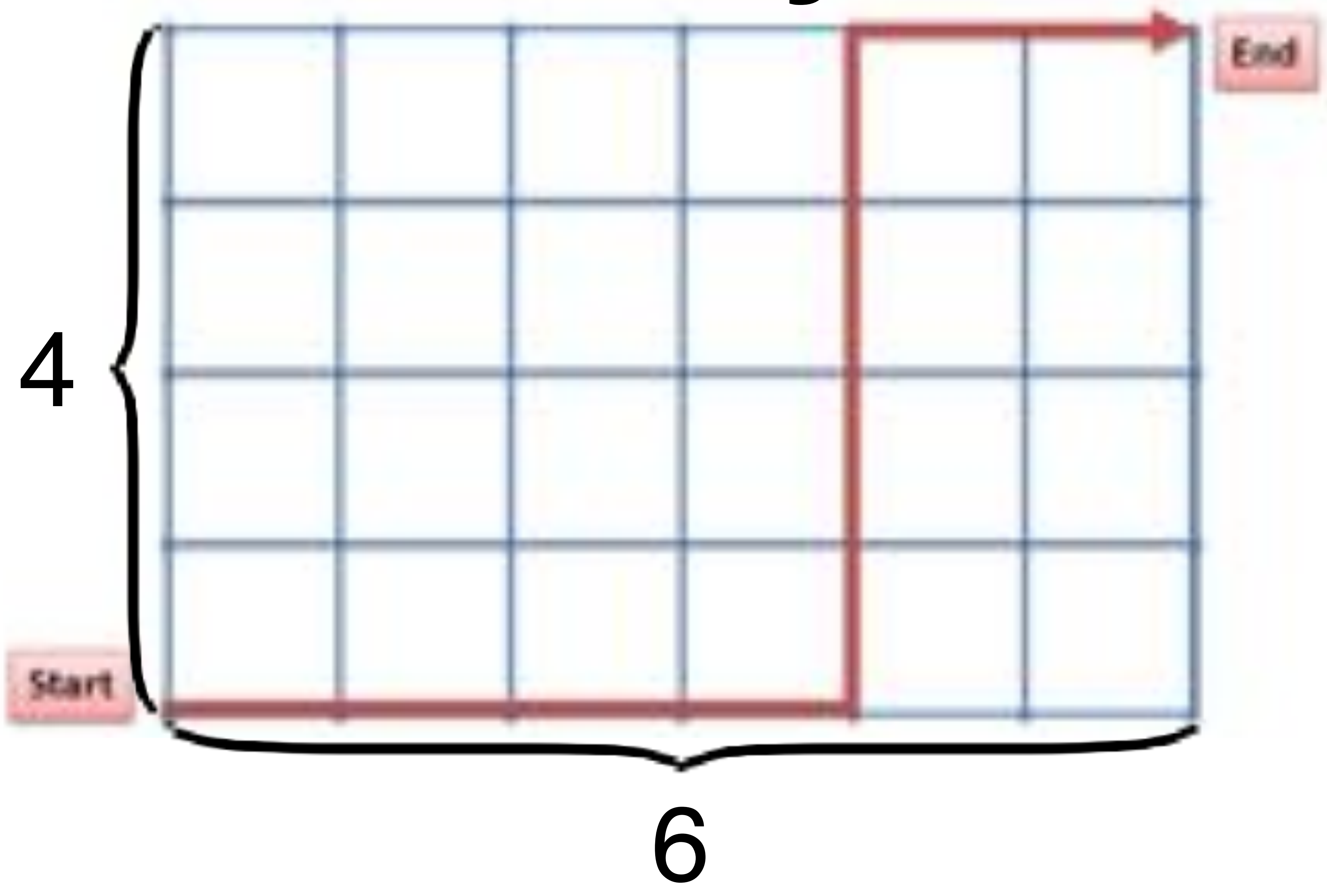
Triangles

n points in general position in plane

How many triangles can be formed?

$$\binom{n}{3}$$

How Many Paths?

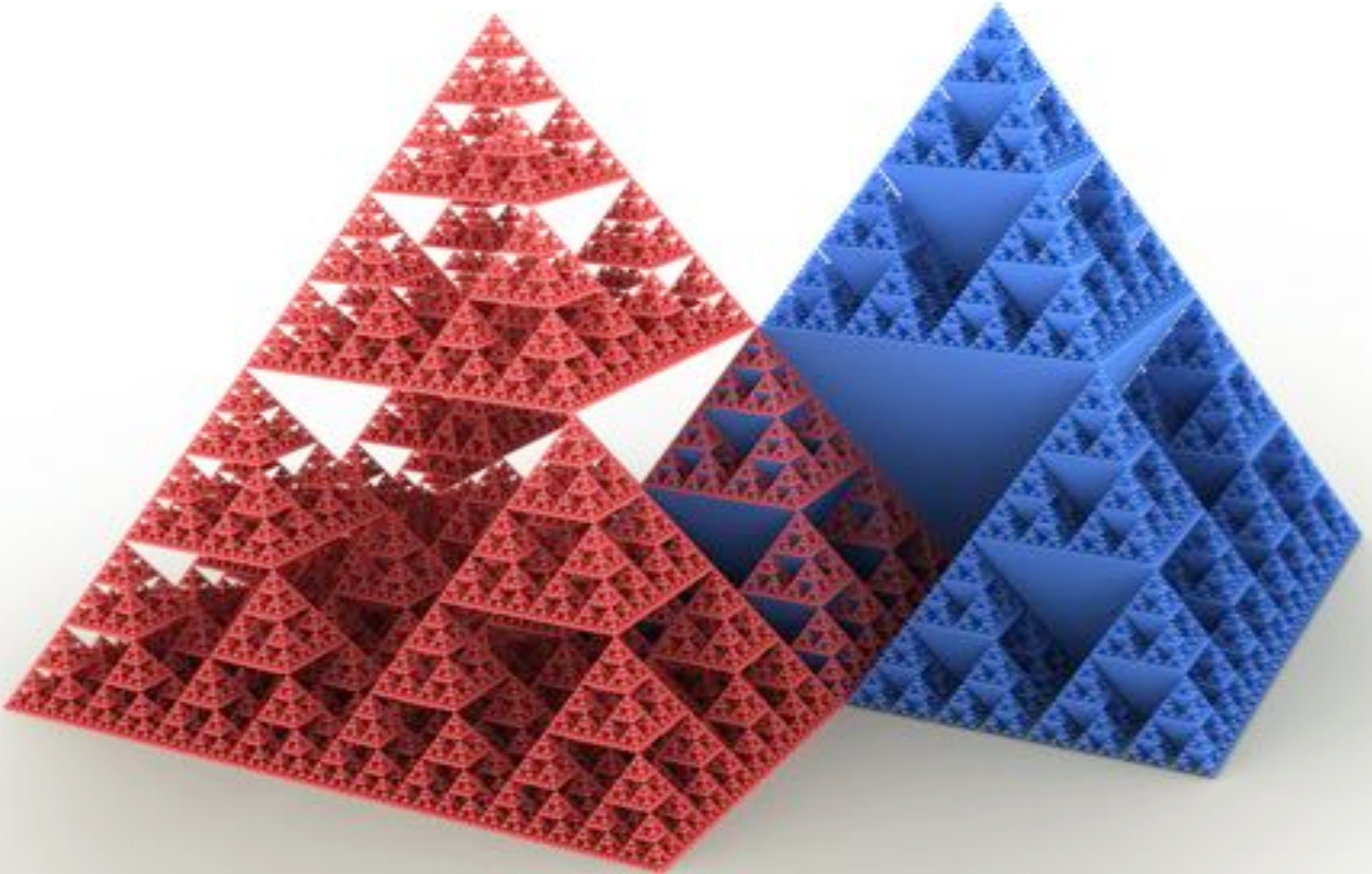


Every path from Start to End is a length-10 sequence of U,R with 6 R's

$$\#\text{ of paths } = \binom{10}{6} = 210$$

Next: Properties

Properties
of Binomial Coefficients



$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5!}{2!\,3!} = \binom{5}{2}$$

Two proofs

Algebraic

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial

$$\binom{[n]}{k} \longleftrightarrow \binom{[n]}{n-k}$$

Complement bits to create 1-1 correspondence

$$\binom{[4]}{3} \longleftrightarrow \binom{[4]}{4-3}$$

$$\left\{\begin{matrix} 1110 \\ 1101 \\ 1011 \\ 0111 \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} 0001 \\ 0010 \\ 0100 \\ 1000 \end{matrix}\right\}$$

$$\binom{n}{k} = \left|\binom{[n]}{k}\right| = \left|\binom{[n]}{n-k}\right| = \binom{n}{n-k}$$

$$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = 10 = \frac{5}{3} \cdot 6 = \frac{5}{3}\binom{4}{2}$$

Recursive definition

choose one of these locations to be 2

choose location of 2

choose k locations of k non-zeros (1's & 2)

$$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$$

from remaining n-1 locations choose locations of the k-1 1's

Number of length-n ternary strings with k-1 1's and one 2

n=4   k=3

{ 0112, 1012, 2110,… }

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\tbinom{3}{0} + \tbinom{3}{1} + \tbinom{3}{2} + \tbinom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3$$

%2^{[n]} = |2^{[n]}| =
\sum^n_{i=0} \left | \binom{[n]}i
\right | = \sum^n_{i=0} \binom ni
%2^{[n]} = \uplus^n_{i=0} \binom

Combinatorial proof

$$2^{[n]} = \biguplus_{i=0}^{n} \binom{[n]}{i}$$

$$2^n = |2^{[n]}| = \sum_{i=0}^{n} \left| \binom{[n]}{i} \right| = \sum_{i=0}^{n} \binom{n}{i}$$

Algebraic proof: next video

Outside the circle

subsets of [n] of size ≤ n-1

n-bit sequences with ≤ n-1 1's

n = 3

$$\underbrace{000}_{1},\ \underbrace{001,\ 010,\ 100}_{3},\ \underbrace{011,\ 101,\ 110}_{3}$$

1 + 3 + 3 = 7

Two ways

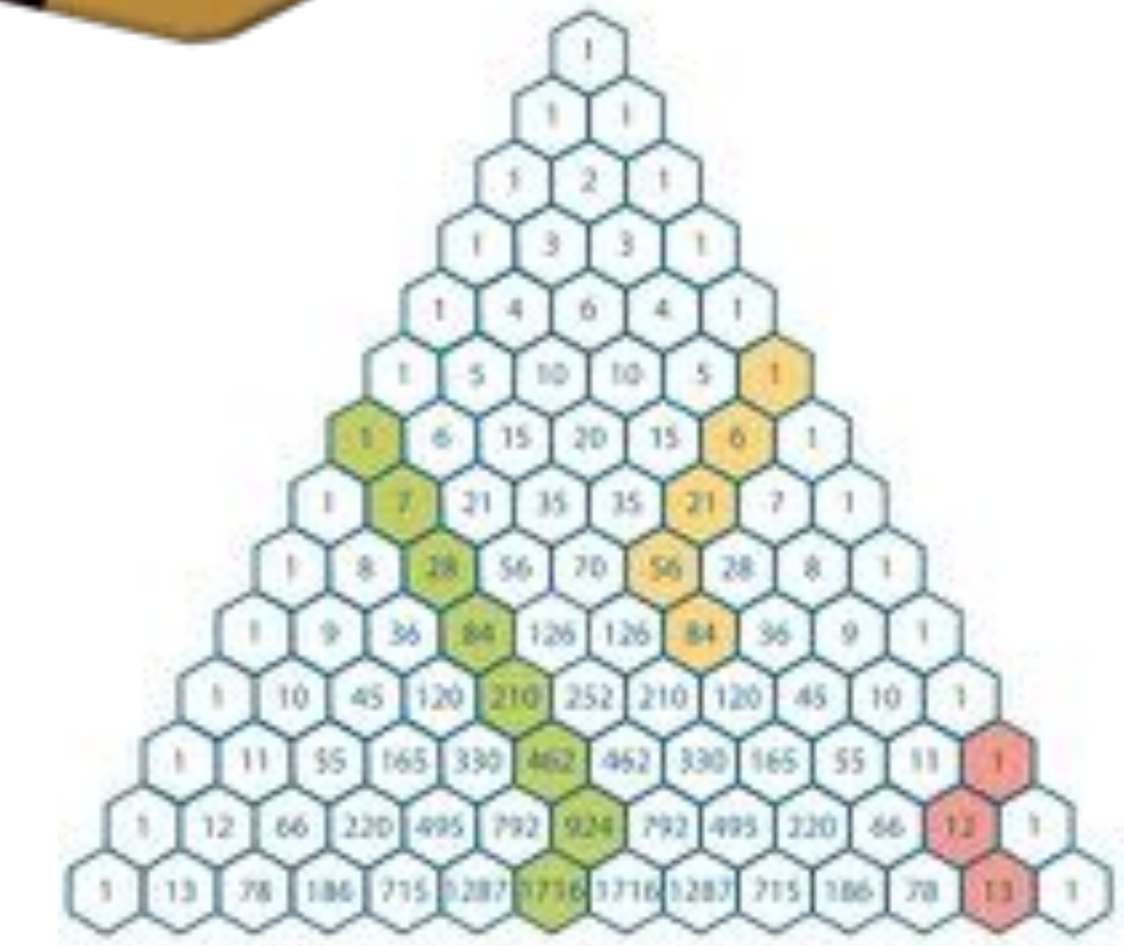
$$\sum_{i=0}^{n-1} \binom{n}{i}$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n \qquad \sum_{i=0}^{n-1} \binom{n}{i} = 2^n - \binom{n}{n} = 2^n - 1$$

$2^3 - 1 = 7$

Hockey Stick Identity

$$\sum_{i=0}^{n} \binom{i+k-1}{k-1} = \binom{n+k}{k}$$



$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

k=3, n=2 $\quad \binom{2}{2} + \binom{3}{2} + \binom{4}{2} = 1 + 3 + 6 = 10 = \binom{5}{3}$

Combinatorial Proof

$$\binom{k-1}{k-1} + \binom{k}{k-1} + \ldots + \binom{n+k-1}{k-1} = \binom{n+k}{k}$$

A = { binary sequences of length n+k with k 1's }

$$|A| = \binom{n+k}{k}$$

A_i = { (n+k)-bit sequences with k 1's where the last 1 is at location i}

$$|A_i| = \binom{i-1}{k-1}$$

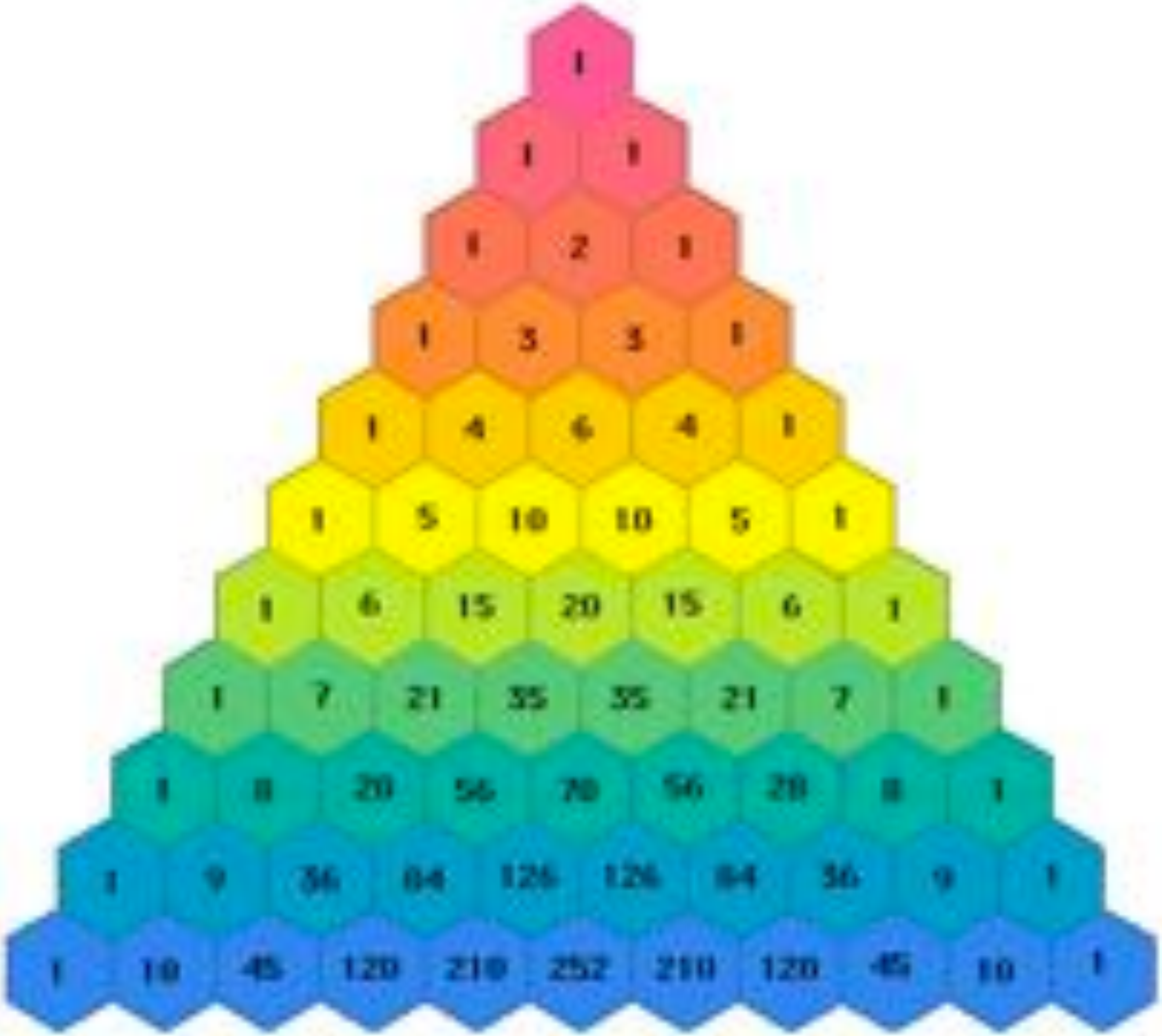
Example

$$A = A_k \uplus A_{k+1} \uplus \ldots \uplus A_{n+k}$$

Next: Pascal Triangle and Binomial Theorem

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

Pascal Triangle and Binomial Theorem


1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{4}{3} = \binom{3}{3} + \binom{3}{2}$$

$$\underbrace{\begin{matrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{matrix}}_{\text{three } 1\text{'s}} \qquad \underbrace{1\ 1\ 1}_{\text{three } 1\text{'s}}\ \boxed{0} \qquad \begin{matrix}\underbrace{\begin{matrix}0&1&1\\1&0&1\\1&1&0\end{matrix}}_{\text{two } 1\text{'s}}\end{matrix}\ \boxed{\begin{matrix}1\\1\\1\end{matrix}}$$

Pascal's Triangle

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{2}{1} = \binom{1}{1} + \binom{1}{0}$$

$$\binom{3}{1} = \binom{2}{1} + \binom{2}{0}$$

$$\binom{3}{2} = \binom{2}{2} + \binom{2}{1}$$

k

n	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$\binom{n}{0} = 1 \qquad \binom{n}{n} = 1$$

Binomial Theorem

$(a + b)^1 = a + b$

$(a + b)^2 = a^2 + 2ab + b^2$

$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

$$1 = \binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4} = 1$$

n \ k	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i \quad \forall\, a, b \quad \forall\, n \geq 0$$

So important it gives the binomial coefficients their name

Explanation

$$(a+b)^3 = (a+b)(a+b)(a+b)$$

Three (a+b) factors

$$= \underbrace{aaa + aab + aba + abb + baa + bab + bba + bbb}$$

Each term: product of a's and b's
one selected from each factor

Sum of terms

a's + # b's = 3

$$= a^3 + 3a^2b + 3ab^2 + b^3$$

# of terms with i b's = # ways to select i factors out of the 3 = $\binom{3}{i}$

$$= \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3$$

Generally

$$(a+b)^3 = \binom{3}{0}a^3 + \binom{3}{1}a^2b + \binom{3}{2}ab^2 + \binom{3}{3}b^3 = \sum_{i=0}^{3}\binom{3}{i}a^{3-i}b^i$$

Similarly

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}b^i$$

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Gave combinatorial proof

Algebraic proof

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}$$

Polynomial Coefficients

$$(1+x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i$$

Coefficient of x^2 in $(1+x)^7$

$$(1+x)^7 = \sum_{i=0}^{7} \binom{7}{i} x^i$$

$$\binom{7}{2} = 21$$

Coefficient of x^3 in $(3+2x)^5$

$$(3+2x)^5 = \sum_{i=0}^{5} \binom{5}{i} 3^{5-i} (2x)^i$$

$$\binom{5}{3} 3^2 \cdot 2^3 = 720$$

Binomial → Taylor

Taylor expansion

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{n^{\underline{i}}}{i!} \left(\frac{x}{n}\right)^i$$

$$= \sum_{i=0}^{n} \frac{x^i}{i!} \cdot \frac{n^{\underline{i}}}{n^i}$$

$$n \to \infty$$

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Binomial Distribution

$$\sum_{i=0}^{n} \binom{n}{i} p^{n-i}(1-p)^i = (p + (1-p))^n = 1^n = 1$$

Example

$$\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)^2 = 1$$

More in binomial distribution

So far: Binary Sequences

Next: Larger Alphabets

Multinomials

Multinomial Coefficients

Beyond binary - ternary alphabets

$$k_1 + k_2 + k_3 = n$$

$$\# \ \{1,\ 2,\ 3\} \text{ sequences with } \begin{cases} k_1 \ \ 1\text{'s} \\ k_2 \ \ 2\text{'s} \\ k_3 \ \ 3\text{'s} \end{cases}$$

$$\underbrace{\binom{n}{k,n-k}}_{\text{sum to n}} = \binom{n}{k}$$

$$\binom{n}{k_1}\binom{n-k_1}{k_2} = \frac{n!}{k_1!\cdot\cancel{(n-k_1)!}} \cdot \frac{\cancel{(n-k_1)!}}{k_2!\cdot\underbrace{(n-k_1-k_2)!}_{k_3}} = \frac{n!}{k_1!\cdot k_2!\cdot k_3!} \triangleq \binom{n}{k_1,k_2,k_3}$$

k_1 location of 1's

k_2 location of 2's out of n-k_1 locations left

(location of k_3 3's is determined)

symmetric in k_1, k_2 ,k_3

Simple Example

sequences over {1,2,3,4}

digit	1	2	3	4
# times	1	4	4	2

length 11

31222334243

$$\binom{11}{1,4,4,2} = \frac{11!}{1!\cdot 4!\cdot 4!\cdot 2!}$$

$$= \frac{11\cdot 10\cdot 9\cdot \cancel{8}\cdot 7\cdot \cancel{6}\cdot 5\cdot \cancel{4!}}{\cancel{4!}(\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot 1)(\cancel{2}\cdot 1)}$$

$$= 11\cdot 10\cdot 9\cdot 7\cdot 5 = 34,650$$

MISSISSIPPI

anagrams = ?

sequences over {M,I,S,P}

letter	M	I	S	P
# times	1	4	4	2

length 11 (SISSISIPPIM)

Same as sequences over {1,2,3,4} in previous slide

$$\binom{11}{\underset{\substack{\uparrow\\ \text{M}}}{1}, \underset{\substack{\uparrow\\ \text{I}}}{4}, \underset{\substack{\uparrow\\ \text{S}}}{4}, \underset{\substack{\uparrow\\ \text{P}}}{2}} = 34,650$$

Students in Class

10 students

3 classes: morning, afternoon, evening

Any number of students in each class 3^{10}

6 morning

3 afternoon

1 evening

$$\underbrace{\binom{10}{6,3,1}}_{\text{morning students, afternoon students, evening students}} = \frac{10!}{6!\cdot 3!\cdot 1!} = \frac{10\cdot \overset{3}{\cancel{9}}\cdot \overset{4}{\cancel{8}}\cdot 7}{\cancel{3}\cdot \cancel{2}} = 840$$

morning students

afternoon students

evening students

Multinomial Theorem

$$(a_1 + a_2 + ... + a_m)^n = \sum_{\substack{k_1 + k_2 + ... + k_m = n \\ k_1, k_2, ..., k_m \geq 0}} \binom{n}{k_1, k_2, ..., k_m} \prod_{t=1}^{m} a_t^{k_t}$$

$$(a+b+c)^2$$

$$(a+b+c)^2 = \sum_{\substack{i+j+k=2 \\ i,j,k \geq 0}} \binom{2}{i,j,k} a^i b^j c^k$$

$$= \tbinom{2}{2,0,0} a^2 + \tbinom{2}{0,2,0} b^2 + \tbinom{2}{0,0,2} c^2 + \tbinom{2}{1,1,0} ab + \tbinom{2}{1,0,1} ac + \tbinom{2}{0,1,1} bc$$

$$= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$$

Sum of Multinomials

Recall binomial identity $$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Similarly for multinomials

$$m^n = (1 + 1 + \ldots + 1)^n = \sum_{\substack{k_1 + \ldots + k_m = n \\ k_1, \ldots, k_m \geq 0}} \binom{n}{k_1, k_2, \ldots, k_m}$$

$$3^2 = 9 = \underbrace{\tbinom{2}{2,0,0}}_{1} + \underbrace{\tbinom{2}{0,2,0}}_{1} + \underbrace{\tbinom{2}{0,0,2}}_{1} + \underbrace{\tbinom{2}{1,1,0}}_{2} + \underbrace{\tbinom{2}{1,0,1}}_{2} + \underbrace{\tbinom{2}{0,1,1}}_{2}$$

Students in Class

2 students

3 classes: morning, afternoon, evening $\quad 3^2 = 9$

Broken by class

$$\underbrace{\binom{2}{2,0,0}}_{1} + \underbrace{\binom{2}{0,2,0}}_{1} + \underbrace{\binom{2}{0,0,2}}_{1} + \underbrace{\binom{2}{1,1,0}}_{2} + \underbrace{\binom{2}{1,0,1}}_{2} + \underbrace{\binom{2}{0,1,1}}_{2} = 9$$

same as last slide

Next: Application

Combinatorics Stars and Bars

- Final application
- Beyond combinatorics - algebra
- Simple derivation of unintuitive result

Counting Sums

\# ways to

write 5 as a sum of 3 positive integers, when order matters

partition 5 items into 3 groups, when order matters

3 + 1 + 1 ☆ ☆ ☆ | ☆ | ☆

2 + 2 + 1 ☆ ☆ | ☆ ☆ | ☆

2 + 1 + 2 ☆ ☆ | ☆ | ☆ ☆

1 + 3 + 1 ☆ | ☆ ☆ ☆ | ☆

1 + 2 + 2 ☆ | ☆ ☆ | ☆ ☆

1 + 1 + 3 ☆ | ☆ | ☆ ☆ ☆

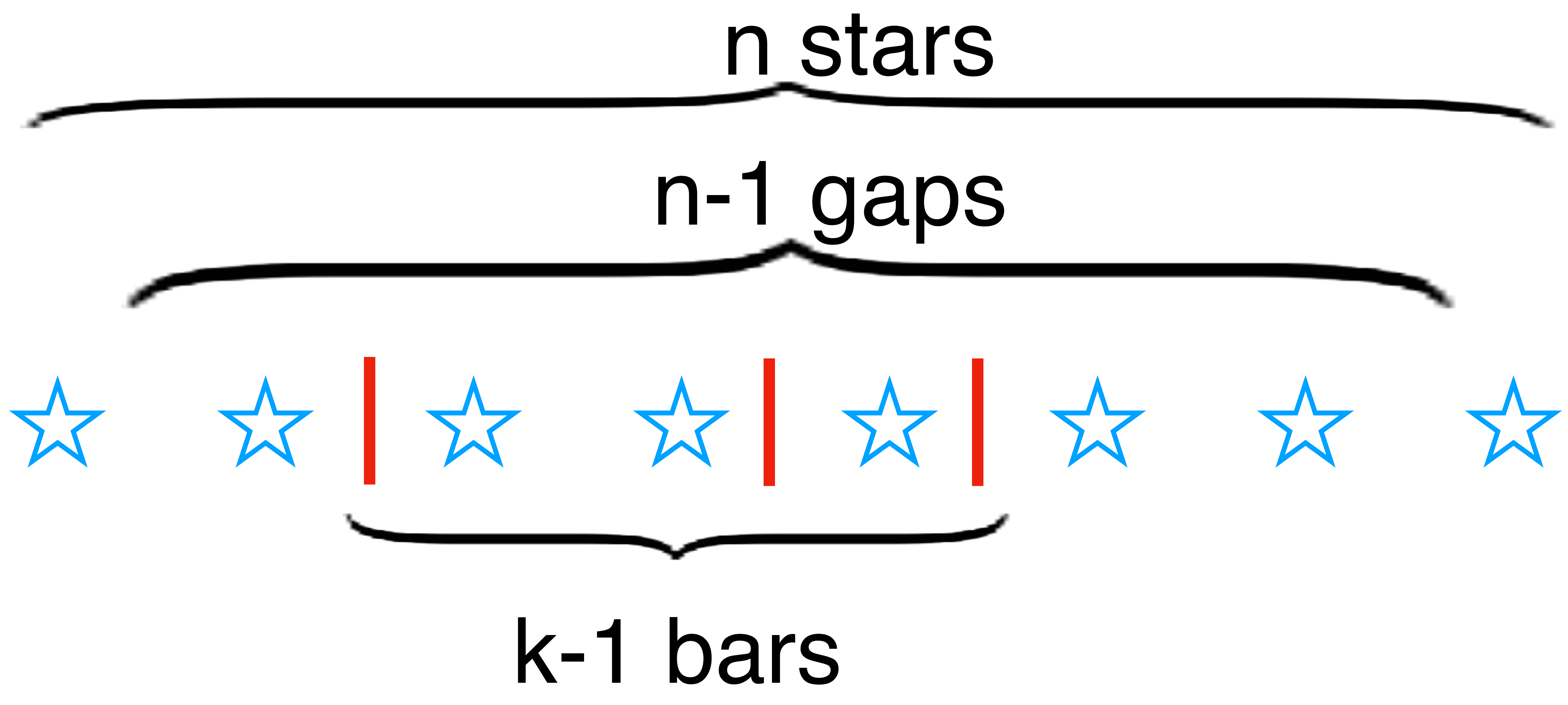
\# = 6

Addition	Partition
sum to 5	5 stars (items)
3 positive terms	3 consecutive star intervals
2 +'s separating the numbers	2 bars separating the intervals
	4 inter-star gaps
	Choose 2 of 4 gaps

$\binom{4}{2} = 6$

k Terms Adding to n

ways to write n as a sum of k positive integers, when order matters



$$\boxed{\#\text{ sums} = \binom{n-1}{k-1}}$$

Simple Examples

ways to write n as sum of k positive integers, when order matters $= \binom{n-1}{k-1}$

k	sums	$\binom{n-1}{k-1}$
1	n	$\binom{n-1}{1-1} = 1$
n	1+1+…+1	$\binom{n-1}{n-1} = 1$
2	1+(n-1), 2+(n-2), …, (n-1)+1	$\binom{n-1}{2-1} = n-1$
n-1	2+1+..+1, …, 1+..+1+2	$\binom{n-1}{n-2} = n-1$

8 as sum of 4 positive integers $\binom{8-1}{4-1} = \binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35$

Any Sum to n

ways to write n as a sum of (any # of) positive integers

n=3

3	☆ ☆ ☆
1+2	☆ | ☆ ☆
2+1	☆ ☆ | ☆
1+1+1	☆ | ☆ | ☆

= 4 = 2^2

general n

$\overbrace{☆ \quad ☆ \quad ☆ \quad ☆}^{n}$

bar or no bar

= 2^{n-1}

n as sum of k∈[n]: $\binom{n-1}{k-1}$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = 2^{n-1}$$

Nonnegative Terms

ways to write n as sum of k nonnegative integers

2 as sum of 3 nonnegatives

2+0+0 ☆ ☆ | |

0+2+0 | ☆ ☆ |

0+0+2 | | ☆ ☆

1+1+0 ☆ | ☆ |

1+0+1 ☆ | | ☆

0+1+1 | ☆ | ☆

= 6

As before n ☆ k-1 |

Now ☆ and | can appearing any order

sum ↔ ☆ and |

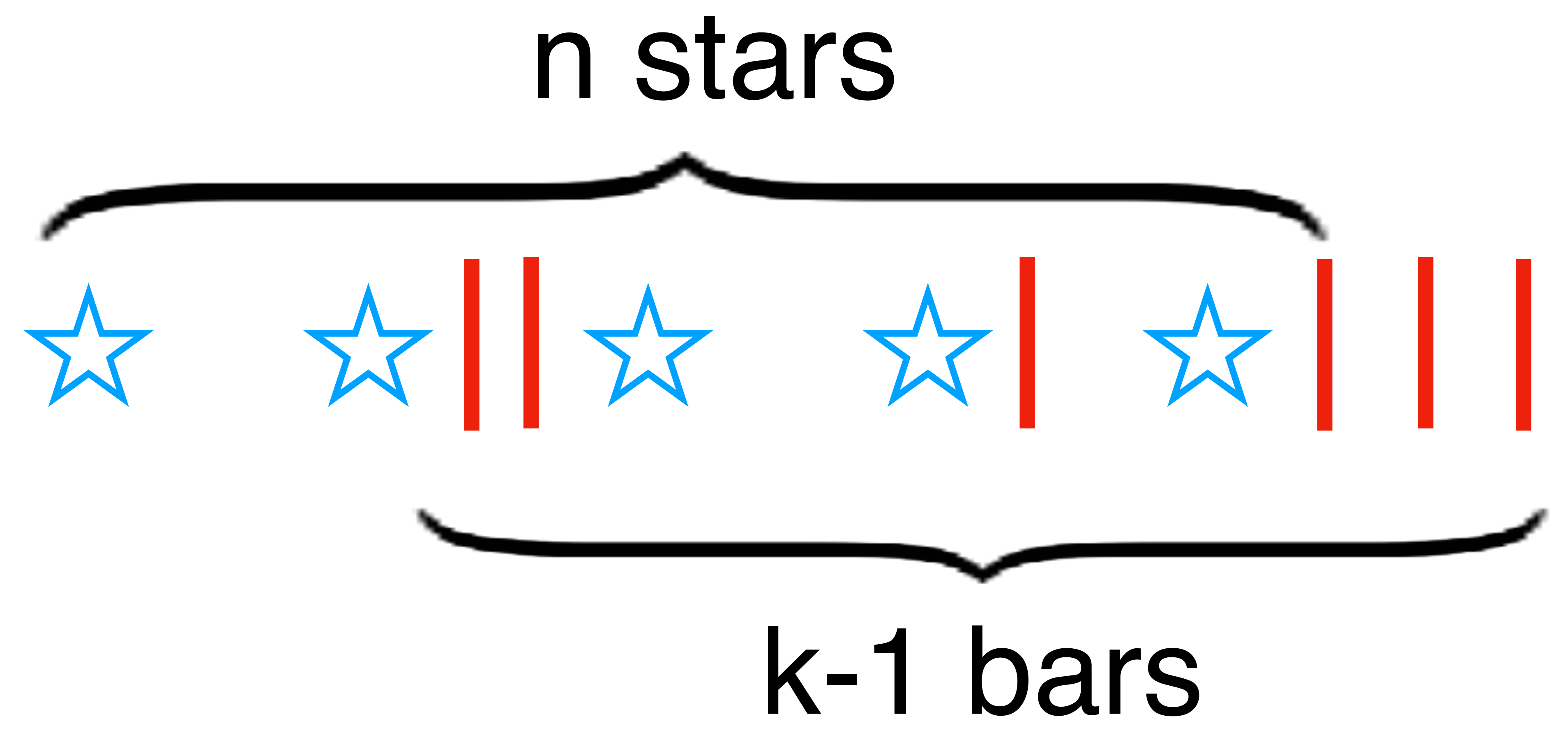
sums = # ☆ and |

ways to order 2 ☆ and 2 |

$\binom{4}{2}$ = 6

k Non-negatives Adding to n

ways to write n as a sum of k non-negative integers, when order matters



sequences with n stars and k-1 bars

length n+(k-1) sequences with k-1 bars

$$\boxed{\binom{n+k-1}{k-1}}$$

4-Letter Words

4-letter words when order matters

$26^4 = 456{,}976$

4-letter words when order does not matter

evil = vile = live = veil = eilv = liev = …

doom = mood = odom = …

Determined by composition:

#a, #b, #c, …, #z

#a + #b + #c + … + #z = 4

26 nonnegative terms

k=26

sum to 4

n=4

$$\binom{4+26-1}{26-1} = \binom{29}{25} = \binom{29}{4}$$

= 23,751

a little ≥ 26^4 / 4!

More Applications

Can derive # positive adding to n

k positive adding to n = # k non-negative adding to n-k

1+2+1+3 = 7 0+1+0+2 = 7-4 = 3

$$\binom{(n-k)+k-1}{k-1} = \binom{n-1}{k-1}$$

Can derive # non-negative adding to ≤ n

k non-negative adding to ≤ n = # k+1 non-negative adding to n

2+0+3 ≤ 7 2+0+3+2 =7

$$\binom{n+(k+1)-1}{(k+1)-1} = \binom{n+k}{k}$$