# MATHEMATICS

1. Q1 Suppose $a, b$ denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose $c, d$ denote the distinct complex roots of the quadratic polynomial $x^2 + 20x + 2020$. Then the value of

$$ac(a-c) + ad(a-d) + bc(b-c) + bd(b-d)$$

is:

   (a) 0
   (b) 8000
   (c) 80800
   (d) 16000

2. Q2 If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is TRUE?

   (a) $f$ is one-one, but NOT onto
   (b) $f$ is onto, but NOT one-one
   (c) $f$ is BOTH one-one and onto
   (d) $f$ is NEITHER one-one NOR onto

3. Q3 Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by
$f(x) = e^{(x-1)} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}\left(e^{(x-1)} + e^{(1-x)}\right)$
Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is:

   (a) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
   (b) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
   (c) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
   (d) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

4. Q4 Let $a, b$ and $\lambda$ be positive real numbers. Suppose $P$ is end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and the suppose ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point $P$, If the tangents to the parabola and the ellipse at the point $P$ are perpendicular to each other, then the eccentricity of the ellipse is:

(a) $\frac{1}{\sqrt{2}}$
(b) $\frac{1}{2}$
(c) $\frac{1}{3}$
(d) $\frac{2}{5}$

5. Q5 Let $C_1$ and $C_2$ be two biased coins such that the probabilities of getting heads in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose $\alpha$ is the number of heads that appear when $C_1$ is tossed twice, independently, and suppose $\beta$ is the number of heads that appear when $C_2$ is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2-\alpha x+\beta$ are real and equal, is:

   (a) $\frac{40}{81}$
   (b) $\frac{20}{81}$
   (c) $\frac{1}{2}$
   (d) $\frac{1}{4}$

6. Q6 Consider all rectangles lying in the region:

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{2\pi}{1}, 0 \leq y \leq \frac{2\sin(2x)}{2}\}$$

   and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is:

   (a) $\frac{3\pi}{2}$
   (b) $\pi$
   (c) $\frac{\pi}{2\sqrt{3}}$
   (d) $\frac{\pi\sqrt{3}}{2}$

7. Q7 Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \to \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

   (a) If $g$ is continuous at $x = 1$ then $fg$ is differentiable at $x = 1$
   (b) If $fg$ is differentiable at $x = 1$ then $g$ is continuous at $x = 1$
   (c) If $g$ is differentiable at $x = 1$ then $fg$ is differentiable at $x = 1$
   (d) If $fg$ is differentiable at $x = 1$ then $g$ is differentiable at $x = 1$

8. Q8 Let $M$ be a $3\times3$ invertible matrix with real entries and let $I$ denote the $3 \times 3$ identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$ then which of the following statements is/are ALWAYS TRUE?

(a) $M = I$

(b) $\det M = 1$

(c) $M^2 = I$

(d) $(\text{adj } M)^2 = I$

9. Q9 Let $S$ be the set of all complex numbers $z$ satisfying $|z^2 + z + 1| = 1$. Then which of the following statements is/are TRUE?

    (a) $|z + \frac{1}{2}| \leq \frac{1}{2}$ for all $z \in S$

    (b) $|z| \leq 2$ for all $z \in S$

    (c) $|z + \frac{1}{2}| \geq \frac{1}{2}$ for all $z \in S$

    (d) The set $S$ has exactly four elements

10. Q10 Let $x, y$ and $z$ be positive real numbers. Suppose $x, y$ and $z$ are the lengths of the sides of a triangle opposite to its angles $X, Y$ and $Z$, respectively. If

$$\tan\left(\frac{X}{2}\right) + \tan\left(\frac{Z}{2}\right) = \frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE?

    (a) $2Y = X + Z$

    (b) $Y = X + 2$

    (c) $\tan\left(\frac{X}{2}\right) = \frac{x}{y+x}$

    (d) $x^2 + z^2 - y^2 = xz$

11. Q11 Let $L_1$ and $L_2$ be the following straight lines:

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}, \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-a}{l} = \frac{y-1}{m} = \frac{z-y}{-2}$$

lies in the plane containing $L_1$ and $L_2$ and passes through the point of intersection of $L_1$ and $L_2$. If the line $L$ bisects the acute angle between the lines $L_1$ and $L_2$, then which of the following statements is/are TRUE?

    (a) $\alpha - \gamma = 3$

    (b) $l + m = 2$

    (c) $\alpha - \gamma = 1$

(d) $l + m = 0$

12. Q12 Which of the following inequalities is/are TRUE?

    (a) $\int_0^1 x \cos x \, dx \geq \frac{3}{8}$
    (b) $\int_0^1 x \sin x \, dx \geq \frac{3}{10}$
    (c) $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$
    (d) $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9}$

13. Q13 Let $m$ be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$ $y_1, y_2, y_3$ are real numbers for which $y_1 + y_2 + y_3 = 9$. Let $M$ be the maximum possible value of $\log_3(x_1) + \log_3(x_2) + \log_3(x_3)$ where $x_1, x_2, x_3$ are positive real numbers for which $x_1 + x_2 + x_3 = 9$. Then the value of $\log_2(m^3) + \log_3(M^2)$ is ___________.

14. Q14 Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of $c$ for which the equality

$$2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$$

    holds for some positive integer $n$, is ___________.

15. Q15 Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin(3\pi x + \frac{\pi}{4}).$$

    If $\alpha, \beta \in [0, 2]$ are such that$\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$,then the value of $\beta - \alpha$ is ___________.

16. Q16 In a triangle $PQR$, let

$$\vec{a} = \overrightarrow{QR}, \quad \vec{b} = \overrightarrow{RP}, \quad \vec{c} = \overrightarrow{PQ}.$$

    If

$$|\vec{a}| = 3, |\vec{b}| = 4, ]and[\frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} =][\frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}]$$

    then the value of $|\vec{a} \times \vec{b}|^2$ is ___________.

17. Q17 For a polynomial $g(x)$ with real coefficients, let $m_g$ denote the number of distinct real roots of $g(x)$. Suppose $S$ is the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1 x + a_2 x^2 + a_3 x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial $f'$, let $f'$ and $f''$ denote itsfirst and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$ where $f \in S$, is ____________.

18. Q18 Let $e$ denote the base of the natural logarithm. The value of the real number $a$ for which the right-hand limit

$$\lim_{x \to 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is ____________.