

# MATHEMATICS

1. 1 Suppose  $a, b$  denote the distinct real roots of the quadratic polynomial  $x^2 + 20x - 2020$  and suppose  $c, d$  denote the distinct complex roots of the quadratic polynomial  $x^2 + 20x + 2020$ . Then the value of

$$ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)$$

is:

- (a) 0
  - (b) 8000
  - (c) 80800
  - (d) 16000
2. 2 If the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = |x|(x - \sin x)$ , then which of the following statements is TRUE?
- (a)  $f$  is one-one, but NOT onto
  - (b)  $f$  is onto, but NOT one-one
  - (c)  $f$  is BOTH one-one and onto
  - (d)  $f$  is NEITHER one-one NOR onto
3. 3 Let the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = e^{(x-1)} - e^{-|x-1|}$  and  $g(x) = \frac{1}{2}(e^{(x-1)} + e^{(1-x)})$ . Then the area of the region in the first quadrant bounded by the curves  $y = f(x)$ ,  $y = g(x)$  and  $x = 0$  is:
- (a)  $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
  - (b)  $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
  - (c)  $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
  - (d)  $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
4. 4 Let  $a, b$  and  $\lambda$  be positive real numbers. Suppose  $P$  is end point of the latus rectum of the parabola  $y^2 = 4\lambda x$ , and the suppose ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  passes through the point  $P$ . If the tangents to the parabola and the ellipse at the point  $P$  are perpendicular to each other, then the eccentricity of the ellipse is:

- (a)  $\frac{1}{\sqrt{2}}$   
 (b)  $\frac{1}{2}$   
 (c)  $\frac{1}{3}$   
 (d)  $\frac{2}{5}$
5. 5 Let  $C_1$  and  $C_2$  be two biased coins such that the probabilities of getting heads in a single toss are  $\frac{2}{3}$  and  $\frac{1}{3}$ , respectively. Suppose  $\alpha$  is the number of heads that appear when  $C_1$  is tossed twice, independently, and suppose  $\beta$  is the number of heads that appear when  $C_2$  is tossed twice, independently. Then the probability that the roots of the quadratic polynomial  $x^2 - \alpha x + \beta$  are real and equal, is:
- (a)  $\frac{40}{81}$   
 (b)  $\frac{20}{81}$   
 (c)  $\frac{1}{2}$   
 (d)  $\frac{1}{4}$
6. 6 Consider all rectangles lying in the region:

$$(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{2\pi}{1}, 0 \leq y \leq \frac{2\sin(2x)}{2}$$

and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is:

- (a)  $\frac{3\pi}{2}$   
 (b)  $\pi$   
 (c)  $\frac{\pi}{2\sqrt{3}}$   
 (d)  $\frac{\pi\sqrt{3}}{2}$
7. 7 Let the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $f(x) = x^3 - x^2 + (x-1)\sin x$  and let  $g : \mathbb{R} \rightarrow \mathbb{R}$  be an arbitrary function. Let  $fg : \mathbb{R} \rightarrow \mathbb{R}$  be the product function defined by  $(fg)(x) = f(x)g(x)$ . Then which of the following statements is/are TRUE?
- (a) If  $g$  is continuous at  $x = 1$  then  $fg$  is differentiable at  $x = 1$   
 (b) If  $fg$  is differentiable at  $x = 1$  then  $g$  is continuous at  $x = 1$   
 (c) If  $g$  is differentiable at  $x = 1$  then  $fg$  is differentiable at  $x = 1$   
 (d) If  $fg$  is differentiable at  $x = 1$  then  $g$  is differentiable at  $x = 1$
8. 8 Let  $M$  be a  $3 \times 3$  invertible matrix with real entries and let  $I$  denote the  $3 \times 3$  identity matrix. If  $M^{-1} = \text{adj}(\text{adj } M)$  then which of the following statements is/are ALWAYS TRUE?

- (a)  $M = I$
- (b)  $\det M = 1$
- (c)  $M^2 = I$
- (d)  $(\operatorname{adj} M)^2 = I$

9. 9 Let  $S$  be the set of all complex numbers  $z$  satisfying  $|z^2 + z + 1| = 1$ . Then which of the following statements is/are TRUE?

- (a)  $|z + \frac{1}{2}| \leq \frac{1}{2}$  for all  $z \in S$
- (b)  $|z| \leq 2$  for all  $z \in S$
- (c)  $|z + \frac{1}{2}| \geq \frac{1}{2}$  for all  $z \in S$
- (d) The set  $S$  has exactly four elements

10. 10 Let  $x, y$  and  $z$  be positive real numbers. Suppose  $x, y$  and  $z$  are the lengths of the sides of a triangle opposite to its angles  $X, Y$  and  $Z$ , respectively. If

$$\tan\left(\frac{X}{2}\right) + \tan\left(\frac{Z}{2}\right) = \frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE?

- (a)  $2Y = X + Z$
- (b)  $Y = X + 2$
- (c)  $\tan\left(\frac{X}{2}\right) = \frac{x}{y+x}$
- (d)  $x^2 + z^2 - y^2 = xz$

11. 11 Let  $L_1$  and  $L_2$  be the following straight lines:

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}, \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-a}{l} = \frac{y-1}{m} = \frac{z-y}{-2}$$

lies in the plane containing  $L_1$  and  $L_2$  and passes through the point of intersection of  $L_1$  and  $L_2$ . If the line  $L$  bisects the acute angle between the lines  $L_1$  and  $L_2$ , then which of the following statements is/are TRUE?

- (a)  $\alpha - \gamma = 3$
- (b)  $l + m = 2$
- (c)  $\alpha - \gamma = 1$
- (d)  $l + m = 0$

12. 12 Which of the following inequalities is/are TRUE?

- (a)  $\int_0^1 x \cos x \, dx \geq \frac{3}{8}$
- (b)  $\int_0^1 x \sin x \, dx \geq \frac{3}{10}$
- (c)  $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$
- (d)  $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9}$

13. 13 Let  $m$  be the minimum possible value of  $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$   $y_1, y_2, y_3$  are real numbers for which  $y_1 + y_2 + y_3 = 9$ . Let  $M$  be the maximum possible value of  $\log_3(x_1) + \log_3(x_2) + \log_3(x_3)$  where  $x_1, x_2, x_3$  are positive real numbers for which  $x_1 + x_2 + x_3 = 9$ . Then the value of  $\log_2(m^3) + \log_3(M^2)$  is \_\_\_\_\_.

14. 14 Let  $a_1, a_2, a_3, \dots$  be a sequence of positive integers in arithmetic progression with common difference 2. Also, let  $b_1, b_2, b_3, \dots$  be a sequence of positive integers in geometric progression with common ratio 2. If  $a_1 = b_1 = c$ , then the number of all possible values of  $c$  for which the equality

$$2(a_1 + a_2 + \dots + a_n) = b_1 + b_2 + \dots + b_n$$

holds for some positive integer  $n$ , is \_\_\_\_\_.

15. 15 Let  $f : [0, 2] \rightarrow \mathbb{R}$  be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin\left(3\pi x + \frac{\pi}{4}\right).$$

If  $\alpha, \beta \in [0, 2]$  are such that  $\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$ , then the value of  $\beta - \alpha$  is \_\_\_\_\_.

16. 16 In a triangle  $PQR$ , let

$$\vec{a} = \overrightarrow{QR}, \quad \vec{b} = \overrightarrow{RP}, \quad \vec{c} = \overrightarrow{PQ}.$$

$$\det \vec{a} = 3, \quad \det \vec{b} = 4, \quad \text{and} \quad \frac{\vec{a} \cdot (\vec{c} - \vec{b})}{\vec{c} \cdot (\vec{a} - \vec{b})} = \frac{|\vec{a}|}{|\vec{a}| + |\vec{b}|}$$

then the value of  $|\vec{a} \times \vec{b}|^2$  is \_\_\_\_\_.

17. 17 For a polynomial  $g(x)$  with real coefficients, let  $m_g$  denote the number of distinct real roots of  $g(x)$ . Suppose  $S$  is the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial  $f'$ , let  $f'$  and  $f''$  denote its first and second order derivatives, respectively. Then the minimum possible value of  $(m_{f'} + m_{f''})$  where  $f \in S$ , is \_\_\_\_\_.

18. 18 Let  $e$  denote the base of the natural logarithm. The value of the real number  $a$  for which the right-hand limit

$$\lim_{x \rightarrow 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is \_\_\_\_\_.