MATHEMATICS

1. Suppose a, b denote the distinct real roots of the quadratic polynomial $x^2 + 20x - 2020$ and suppose c, d denote the distinct complex roots of the quadratic polynomial $x^2 + 20x + 2020$. Then the value of

$$\mathrm{ac(a - c) + ad(a - d) + bc(b - c) + bd(b - d)}$$

is:

   (a) 0
   (b) 8000
   (c) 80800
   (d) 16000

2. If the function $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = |x|(x - \sin x)$, then which of the following statements is TRUE?

   (a) f is one-one, but NOT onto
   (b) f is onto, but NOT one-one
   (c) f is BOTH one-one and onto
   (d) f is NEITHER one-one NOR onto

3. Let the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{(x-1)} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}\left(e^{(x-1)} + e^{(1-x)}\right)$
Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is:

   (a) $(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
   (b) $(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$
   (c) $(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$
   (d) $(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$

4. Let a, b and λ be positive real numbers. Suppose P is end point of the latus rectum of the parabola $y^2 = 4\lambda x$, and the suppose ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the point P, If the tangents to the parabola and the ellipse at the point P are perpendicular to each other, then the eccentricity of the ellipse is:

(a) $\frac{1}{\sqrt{2}}$

(b) $\frac{1}{2}$

(c) $\frac{1}{3}$

(d) $\frac{2}{5}$

5. Let C_1 and C_2 be two biased coins such that the probabilities of getting heads in a single toss are $\frac{2}{3}$ and $\frac{1}{3}$, respectively. Suppose α is the number of heads that appear when C_1 is tossed twice, independently, and suppose β is the number of heads that appear when C_2 is tossed twice, independently. Then the probability that the roots of the quadratic polynomial $x^2-\alpha x+\beta$ are real and equal, is:

   (a) $\frac{40}{81}$

   (b) $\frac{20}{81}$

   (c) $\frac{1}{2}$

   (d) $\frac{1}{4}$

6. Consider all rectangles lying in the region:

$$(x, y) \in \mathbb{R} \times \mathbb{R} : 0 \leq x \leq \frac{2\pi}{1}, 0 \leq y \leq \frac{2\sin(2x)}{2}$$

   and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is:

   (a) $\frac{3\pi}{2}$

   (b) π

   (c) $\frac{\pi}{2\sqrt{3}}$

   (d) $\frac{\pi\sqrt{3}}{2}$

7. Let the function $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 - x^2 + (x-1)\sin x$ and let $g : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Let $fg : \mathbb{R} \to \mathbb{R}$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

   (a) If g is continuous at $x = 1$ then fg is differentiable at $x = 1$

   (b) If fg is differentiable at $x = 1$ then g is continuous at $x = 1$

   (c) If g is differentiable at $x = 1$ then fg is differentiable at $x = 1$

   (d) If fg is differentiable at $x = 1$ then g is differentiable at $x = 1$

8. Let M be a 3×3 invertible matrix with real entries and let I denote the 3×3 identity matrix. If $M^{-1} = \text{adj}(\text{adj } M)$ then which of the following statements is/are ALWAYS TRUE?

(a) $M = I$

(b) $\det M = 1$

(c) $M^2 = I$

(d) $(\text{adj } M)^2 = I$

9. Let S be the set of all complex numbers z satisfying $|z^2+z+1| = 1$. Then which of the following statements is/are TRUE?

(a) $|z + \frac{1}{2}| \leq \frac{1}{2}$ for all $z \in S$

(b) $|z| \leq 2$ for all $z \in S$

(c) $|z + \frac{1}{2}| \geq \frac{1}{2}$ for all $z \in S$

(d) The set S has exactly four elements

10. Let x, y and z be positive real numbers. Suppose x, y and z are the lengths of the sides of a triangle opposite to its angles X, Y and Z, respectively. If

$$\tan\left(\frac{X}{2}\right) + \tan\left(\frac{Z}{2}\right) = \frac{2y}{x+y+z}$$

then which of the following statements is/are TRUE?

(a) $2Y = X + Z$

(b) $Y = X + 2$

(c) $\tan\left(\frac{X}{2}\right) = \frac{x}{y+x}$

(d) $x^2 + z^2 - y^2 = xz$

11. Let L_1 and L_2 be the following straight lines:

$$L_1 : \frac{x-1}{1} = \frac{y}{-1} = \frac{z-1}{3}, \quad L_2 : \frac{x-1}{-3} = \frac{y}{-1} = \frac{z-1}{1}$$

Suppose the straight line

$$L : \frac{x-a}{l} = \frac{y-1}{m} = \frac{z-y}{-2}$$

lies in the plane containing L_1 and L_2 and passes through the point of intersection of L_1 and L_2. If the line L bisects the acute angle between the lines L_1 and L_2, then which of the following statements is/are TRUE?

(a) $\alpha - \gamma = 3$

(b) $l + m = 2$

(c) $\alpha - \gamma = 1$

(d) $l + m = 0$

12. Which of the following inequalities is/are TRUE?

(a) $\int_0^1 x \cos x \, dx \geq \frac{3}{8}$

(b) $\int_0^1 x \sin x \, dx \geq \frac{3}{10}$

(c) $\int_0^1 x^2 \cos x \, dx \geq \frac{1}{2}$

(d) $\int_0^1 x^2 \sin x \, dx \geq \frac{2}{9}$

13. Let m be the minimum possible value of $\log_3(3^{y_1} + 3^{y_2} + 3^{y_3})$ y_1, y_2, y_3 are real numbers for which $y_1 + y_2 + y_3 = 9$. Let M be the maximum possible value of $\log_3(x_1) + \log_3(x_2) + \log_3(x_3)$ where x_1, x_2, x_3 are positive real numbers for which $x_1+x_2+x_3 = 9$. Then the value of $\log_2(m^3)+\log_3(M^2)$ is ____________.

14. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers in arithmetic progression with common difference 2. Also, let $b_1, b_2, b_3, \ldots$ be a sequence of positive integers in geometric progression with common ratio 2. If $a_1 = b_1 = c$, then the number of all possible values of c for which the equality

$$2(a_1 + a_2 + \cdots + a_n) = b_1 + b_2 + \cdots + b_n$$

holds for some positive integer n, is ____________.

15. Let $f : [0, 2] \to \mathbb{R}$ be the function defined by

$$f(x) = (3 - \sin(2\pi x)) \sin\left(\pi x - \frac{\pi}{4}\right) - \sin(3\pi x + \frac{\pi}{4}).$$

If $\alpha, \beta \in [0, 2]$ are such that$\{x \in [0, 2] : f(x) \geq 0\} = [\alpha, \beta]$,then the value of $\beta - \alpha$ is ____________.

16. In a triangle PQR, let

$$\vec{a} = \overrightarrow{QR}, \quad \vec{b} = \overrightarrow{RP}, \quad \vec{c} = \overrightarrow{PQ}.$$

$\det \vec{a} = 3, \quad \det \vec{b} = 4$, and $\frac{\vec{a}\cdot(\vec{c}-\vec{b})}{\vec{c}\cdot(\vec{a}-\vec{b})} = \frac{|\vec{a}|}{|\vec{a}|+|\vec{b}|}$

then the value of $|\vec{a} \times \vec{b}|^2$ is ____________.

17. For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$$S = \{(x^2 - 1)^2(a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}\}.$$

For a polynomial f', let f' and f'' denote itsfirst and second order derivatives, respectively. Then the minimum possible value of $(m_{f'}+m_{f''})$ where $f \in S$, is ____________.

18. Let e denote the base of the natural logarithm. The value of the real number a for which the right-hand limit

$$\lim_{x\to 0^+} \frac{(1-x)^{\frac{1}{x}} - e^{-1}}{x^a}$$

is equal to a nonzero real number, is ___________.