WORKSHEET 3

MATH 101

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Please make sure you have a graphical example for each of the definitions below.

Definition 1. Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a **vertical** asymptote of $f(x)$.

$$\lim_{x\to a^-} f(x) = +\infty \text{ or } -\infty$$
$$\lim_{x\to a^+} f(x) = +\infty \text{ or } -\infty$$
$$\text{or}$$
$$\lim_{x\to a} f(x) = +\infty \text{ or } -\infty$$

Question 1. *Find all the vertical asymptotes of the following function*

$$f(x) = \frac{x}{(x-1)^2(x-4)}.$$

What are the behaviors of the function on the left and right of each asymptote.

Theorem 1 (Squeeze Theorem)**.** *Let $f(x)$, $g(x)$, and $h(x)$ be functions defined for all $x \neq a$ over an open interval containing a. Suppose:*

$$f(x) \leqslant g(x) \leqslant h(x) \quad \textit{for all } x \neq a \textit{ in an open interval containing } a$$

and

$$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$$

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where L is a real number. Then,

$$\lim_{x \to a} g(x) = L.$$

Theorem 2. *The following limits are true.*

$$\lim_{\theta \to a} \sin(\theta) = \sin(a)\,,$$

and

$$\lim_{\theta \to a} \cos(\theta) = \cos(a)\,,$$

Question 2. *Evaluate the following limits*

(1) $\displaystyle \lim_{x \to 1} \frac{x^3 + 3x^2 + 5}{4 - 7x}$

(2) $\displaystyle \lim_{x \to 3} \ln e^{3x}$

(3) $\displaystyle \lim_{x \to 6} \frac{3x - 18}{2x - 12}$

(4) $\lim_{\theta \to \pi} \frac{\sin \theta}{\tan \theta}$

(5) $\lim_{x \to \pi} \frac{\sqrt{x+4} - 1}{x+3}$

(6) $\lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h}$

Question 3. *Evaluate the following one-sided limits*

(1) $\lim_{x \to 1^+} \frac{2x^2 + 7x - 4}{x^2 + x - 2}$

(2) $\displaystyle\lim_{x\to 1^-} \frac{2x^2+7x-4}{x^2+x-2}$

(3) $\displaystyle\lim_{x\to 2^+} \frac{2x^2+7x-4}{x^2+x-2}$

(4) $\displaystyle\lim_{x\to 2^-} \frac{2x^2+7x-4}{x^2+x-2}$

Question 4. *Suppose*

$$2x-1 \leqslant g(x) \leqslant x^2-2x+3\,.$$

True or False.

$$\lim_{x\to 2} g(x) = 0\,.$$

Definition 2. A function $f(x)$ is continuous at a point a if and only if the following conditions are satisfied:

(1) $f(a)$ is defined,
(2) $\lim_{x\to a} f(x) = f(a)$.

A function is discontinuous at a point a if it fails to be continuous at a.

Definition 3. If $f(x)$ is discontinuous at a, then

(1) f has a **removable discontinuity** at a if $\lim_{x\to a} f(x)$ exists.
(2) f has a **jump discontinuity** at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist, but $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$.
(3) f has an **infinite discontinuity** at a if $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$.

Question 5. *Find all the discontinuous points in Questions 3 and 4. Which function do you think can be mofified slightly at a few points to become continuous?*

Question 6. *Determine whether the function is continuous at a given point.*

(1) $h(\theta) = \frac{\sin\theta - \cos\theta}{\tan\theta}$, $\theta = \pi$.

(2) $f(x) = \begin{cases} x^2 - e^x, & x < 0 \\ x - 1, & x \geqslant 0\,, \end{cases}$ $x = 0$.

Question 7. *Find the value of k so that the following function is continuous*

$$f(x) = \begin{cases} 3x + 2, & x < k\,, \\ 2x - 3, & k \leqslant x \leqslant 8\,. \end{cases}$$

Theorem 3 (Intermediate Value Theorem)**.** *Let f be continuous over a closed, bounded interval $[a, b]$. If z is any real number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ so that $f(c) = z$.*

Question 8. *(1) Show that $f(x) = x^3 - x^2 - 3x + 1$ has at least one zero over $[0, 1]$.*

(2) Show that $f(x) = x - \cos x$ has at least one zero.