WORKSHEET 14

MATH 101

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Integration

Definition 1. The sigma notation:

$$\sum_{i=1}^{n} f_i = f_1 + f_2 + \cdots + f_n \,.$$

Problem 1. Show that

(1) $\displaystyle\sum_{i=1}^{n} f_i + \sum_{i=1}^{n} g_i = \sum_{i=1}^{n}(f_i + g_i) \,.$

(2) $\displaystyle\sum_{i=1}^{n} c f_i = c \sum_{i=1}^{n} f_i \,.$

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Definition 2. A set of points $P = \{x_i\}$ for $i = 0, 1, 2, \ldots, n$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, which divides the interval $[a, b]$ into subintervals of the form $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$ is called a *partition* of $[a, b]$. If the subintervals all have the same width, the set of points forms a *regular partition* of the interval $[a, b]$.

Problem 2 (Rules). Read Left-endpoint and Right-endpoint rules from Section 5.1.

Use both left-endpoint and right-endpoint approximations to approximate the area under the curve of $f(x) = x^2$ on the interval $[0, 2]$; use $n = 4$.

Problem 3. Continue Problem 2. Use excel or a programming language to see the trend of

$$|L_n - R_n|$$

for $n = 3, 4, 5, \ldots, 100$.

Problem 4. Compute:

(1) L_4 for $f(x) = 1/(x-1)$ on $[2, 3]$

(2) L_8 for $f(x) = x^2 - 2x + 1$ on $[0, 2]$

Definition 3. Let $f(x)$ be defined on a closed interval $[a, b]$ and let P be a regular partition of $[a, b]$. Let Δx be the width of each subinterval $[x_{i-1}, x_i]$ and for each i, let x_i^* be any point in $[x_{i-1}, x_i]$. A Riemann sum is defined for $f(x)$ as

$$\sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Recall that with the left- and right-endpoint approximations, the estimates seem to get better and better as n get larger and larger. The same thing happens with Riemann sums. Riemann sums give better approximations for larger values of n. We are now ready to define the area under a curve in terms of Riemann sums.

Definition 4. Let $f(x)$ be a continuous, nonnegative function on an interval $[a, b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x$ be a Riemann sum for $f(x)$. Then, the area under the curve $y = f(x)$ on $[a, b]$ is given by

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

If we want an overestimate, for example, we can choose $\{x_i^*\}$ such that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*) \geqslant f(x)$ for all $x \in [x_{i-1}, x_i]$. In other words, we choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the maximum function value on the interval $[x_{i-1}, x_i]$. If we select $\{x_i^*\}$ in this way, then the Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$ is called an **upper sum**. Similarly, if we want an underestimate, we can choose $\{x_i^*\}$ so that for $i = 1, 2, 3, \ldots, n$, $f(x_i^*)$ is the minimum function value on the interval $[x_{i-1}, x_i]$. In this case, the associated Riemann sum is called a **lower sum**. Note that if $f(x)$ is either increasing or decreasing throughout the interval $[a, b]$, then the maximum and minimum values of the function occur at the endpoints of the subintervals, so the upper and lower sums are just the same as the left- and right-endpoint approximations.

Problem 5 (Finding Lower and Upper Sums)*.* Find a lower sum for $f(x) = 10 - x^2$ on $[1, 2]$; let $n = 4$ subintervals.