WORKSHEET 12

MATH 101

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Approximations

Optimization

Definition 1. Let f be a function defined over an interval I and let $c \in I$. We say f has an absolute maximum on I at c if $f(c) \geqslant f(x)$ for all $x \in I$. We say f has an absolute minimum on I at c if $f(c) \leqslant f(x)$ for all $x \in I$. If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.

Definition 2. A function f has a **local maximum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geqslant f(x)$ for all $x \in I$. A function f has a **local minimum** at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leqslant f(x)$ for all $x \in I$. A function f has a **local extremum** at c if f has a local maximum at c or f has a local minimum at c.

Question 1. *What is the difference betwee local and absolute extrema?*

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Theorem 1. *If f is a continuous function over the closed, bounded interval $[a, b]$, then there is a point in $[a, b]$ at which f has an absolute maximum over $[a, b]$, and there is a point in $[a, b]$ at which f has an absolute minimum over $[a, b]$.*

Definition 3. Given an interval, an interior point is a point that is not the endpoint. Let c be an interior point in the domain of f. We say that c is a critical number of f if $f'(c) = 0$ or $f'(c)$ is undefined. We call the point $(c, f(c))$ a critical point of f.

Theorem 2 (Fermat's theorem)**.** *If f has a local extrememum at c and f is differentiable at c, then $f'(c) = 0$.*

Question 2. *Find the critical points of the following:*

(1) $y = 4x^3 - 3x$

(2) $y = 4\sqrt{x} - x^2$

(3) $y = \sin^2(x)$

Theorem 3 (First derivative test). *Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possibly at point c, then $f(c)$ satisfies one of the following descriptions:*

(1) *If f' changes sign from positive when $x < c$ to negative when $x > c$, then $f(c)$ is a local maximum of f.*
(2) *If f' changes sign from negative when $x < c$ to positive when $x > c$, then $f(c)$ is a local minimum of f.*
(3) *If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.*

Theorem 4 (Closed interval method). *To find the absolute extrema of a continuous function f on a closed interval $[a, b]$ we follow the following steps:*

(1) *Find the critical points and the values of f at those points*
(2) *Find the values of f at the end points*
(3) *Compare all the values from the above steps to find absolute max/min*

Question 3. *Find local and/or absolute extrema.*

(1) $y = x^2 + 2/x$ *over* $[1, 4]$

(2) $y = \sqrt{9 - x^2}$ *over* $[1, 9]$

(3) $y = xe^{x/2}$ *on* $[-3, 1]$

(4) $y = \ln(x^2 + x + 1)$ *on* $[-1, 1]$

The meaning of second derivative is that it tells us about the concavity of the graph of a function.

Definition 4. Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.