MATH 102: HOMEWORK 6

DUE DATE: FRIDAY, DEC 8, AT 11:59PM

Problem 1. Do Problem 4 in Worksheet 11.

Problem 2. (1) What's wrong with the following proof that for all $n \in \mathbb{N}$,

$$1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \ldots + (2n+1) \cdot 3^n = n \cdot 3^{n+1}?$$

Proof. We use mathematical induction. Let n be an arbitrary natural number, and suppose that

$$1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \ldots + (2n+1)3^n = n3^{n+1}.$$

Then

$$\begin{aligned} &1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \ldots + (2n+1)3^n + (2n+3)3^{n+1} \\ &= n3^{n+1} + (2n+3)3^{n+1} \\ &= (3n+3)3^{n+1} \\ &= (n+1)3^{n+2}. \end{aligned}$$

(2) Find the correct formula for

$$1 \cdot 3^0 + 3 \cdot 3^1 + 5 \cdot 3^2 + \ldots + (2n+1) \cdot 3^n = n \cdot 3^{n+1}$$

and prove it.

Problem 3. Prove by induction that for all $n \in \mathbb{N}$,

(1)

$$6|(n^3 - n)\,.$$

(2)

$$1 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}\,.$$

Problem 4. Recall factorial from high school:

$$n! = 1 \cdot 2 \cdot \cdots \cdot n\,.$$

We also define the special case $0! = 1$.

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Next, we use this to define the concept of combination:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}\,.$$

(1) Prove the following formula (without induction)

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

(2) Prove the following formula using induction

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n\,.$$

(Hint: use part (1))

(3) (Optional) Prove the general formula using induction

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}\,.$$

Here, we define the sigma notation

$$\sum_{i=0}^{n} f_i = f_0 + f_1 + \cdots + f_n\,.$$