MATH 102: IDEAS OF MATH

WORKSHEET 6

Problem 1. (1) Prove that $\mathbb{Q} \subseteq \mathbb{R}$.
(2) Prove that $\mathbb{R} \not\subseteq \mathbb{Q}$.

Problem 2. Prove that,

$$\left\{x \in \mathbb{R} \,\middle|\, x^2 \leqslant 1\right\} = [-1, 1] \,.$$

Problem 3. Prove that for any two sets X and Y,

$$X \cap Y \subseteq X \cup Y \,.$$

Problem 4. Prove that

$$\bigcap_{n \geqslant 1} \left[0, 1 + \frac{1}{n}\right) = [0, 1] \,.$$

Problem 5. Prove that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$.

Here's a theorem that we will discuss more in depth later.

Theorem 0.1. *For all natural numbers n and m, if $m > 0$, then there are natural numbers q and r such that $n = mq + r$ and $r < m$. (The numbers q and r are called the quotient and remainder when n is divided by m.)*

Problem 6. Prove that for every integer, the remainder when x^2 is divided by 4 is either 0 or 1.

Definition 1. Let X be a set. The *power set* of X, written $\mathcal{P}(X)$, is the set of all subsets of X.

For example,

$$\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} \,.$$

Problem 7. Write out elements of

(1) $\mathcal{P}(\{1, 2, 3\})$,
(2) $\mathcal{P}(\emptyset)$,
(3) $\mathcal{P}(\mathcal{P}(\emptyset))$,
(4) $\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))$.

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The following problem is to help you clear some of the common confusion between the use of $\in$ and $\subseteq$ in power set.

Problem 8. True or false and prove your claim.

(1) $\mathcal{P}(\emptyset) \in \mathcal{P}(\mathcal{P}(\emptyset))$
(2) $\emptyset \in \{\{\emptyset\}\}$
(3) $\{\emptyset\} \in \{\{\emptyset\}\}$
(4) $\mathcal{P}(\mathcal{P}(\emptyset)) \in \{\emptyset, \{\emptyset, \{\emptyset\}\}\}$

Problem 9. True or False and prove your claim.

(1) $\mathcal{P}(X \cup Y) = \mathcal{P}(X) \cup \mathcal{P}(Y)$,
(2) $\mathcal{P}(X \cap Y) = \mathcal{P}(X) \cap \mathcal{P}(Y)$.