MATH 102: IDEAS OF MATH

WORKSHEET 8

1. Concepts

Definition 1 (Newstead, Chapter 4)**.** A function f from a set X to a set Y is a specification of elements $f(x) \in Y$ for $x \in X$ such that

$$\forall x \in X, \exists! y \in Y, y = f(x)\,.$$

Given $x \in X$, the unique element $f(x) \in Y$ is called the value of f at x.

X is called the *domain* of f, and Y is called the *codomain*.

We denote the *range* of f is

$$f(X) = \{f(x) \mid x \in X\}\,.$$

We write $f : X \to Y$ to denote the assertion that f is a function with domain X and codomain Y.

We sometimes write $\mathrm{Dom}(f)$ to mean domain of f and $\mathrm{Ran}(f)$ to mean the range of f.

Definition 2. Let X, Y be sets. The *cartesian product* of X and Y is the set $X \times Y$, defined by

$$X \times Y = \{(a, b) | a \in X \wedge b \in Y\}\ .$$

The elements $(a, b) \in X \times Y$ are called *ordered pairs*, whose defining property is that

$$\forall x \in X, \forall y \in Y, (a, b) = (x, y) \iff a = x \wedge b = y\,.$$

Definition 3. Let $f : X \to Y$ be a function. The *graph* of f is the subset $\mathrm{Gr}(f) \subseteq X \times Y$ defined by

$$\mathrm{Gr}(f) = \{(x, f(x)) | x \in X\} = \{(x, y) \in X \times Y | y = f(x)\}\,.$$

Definition 4. Let A, B be sets. Then the set $R \subseteq A \times B$ is called a relation from A to B.

We also define the domain and range of a relation R.

$$\mathrm{Dom}(R) = \{a \in A \mid \exists b \in B, (a, b) \in R\}$$

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$$\mathrm{Ran}(R) = \{b \in B \mid \exists a \in A, (a,b) \in R\}\,.$$

If $(x,y) \in R$, then we say that x is related to y by R and write xRy.

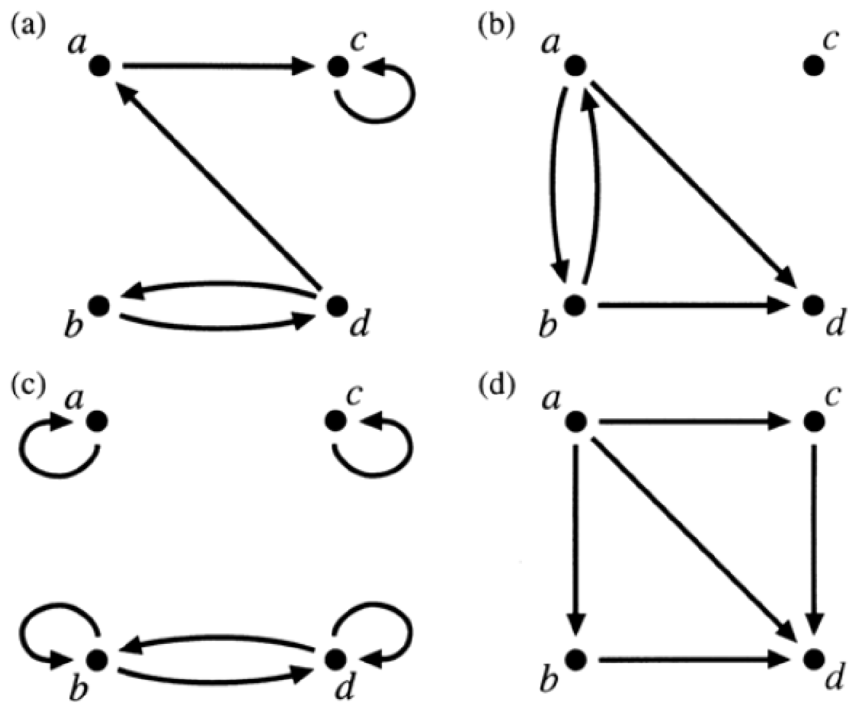
2. Problems

Problem 1. The two concepts of relation and logical predicate are closely related (no pun intended). Can you make this connection?

Problem 2. Let R be a relation so that xRy means "there is an arrow from x to y".

Consider the following pictures and specify in each case

(1) what is R?
(2) what is $\mathrm{Dom}(R)$?
(3) what is $\mathrm{Ran}(R)$?



Problem 3. The two concepts functions and relations are closely related. In fact, one of the standard ways to define a function is as follows.

Definition 5 (Alternative Definition of Functions, Velleman Chapter 5). Suppose F is a relation from A to B. Then F is called a function from A to B if for every $a \in A$ there is exactly one $b \in B$ such that

$(a, b) \in F$. In other words, to say that F is a function from A to B means:

$$\forall a \in A \, \exists! b \in B \, ((a, b) \in F).$$

To indicate that F is a function from A to B, we will write $F : A \to B$.

How are the two definitions related? What is the difference?