MATH 102: IDEAS OF MATH

WORKSHEET 9

1. Concepts

Definition 1. We call a relation from A to A a relation on A.

Definition 2. Let R be a relation from A to B and S be a relation from B to C. We define the *composition* of relations $S \circ R$ as follows

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}.$$

Definition 3. Suppose R is a relation from A to B. The *inverse* of R is the relation R^{-1} from B to A, defined as follows:

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}$$

Definition 4. Suppose R is a relation on A.

(1) R is said to be reflexive on A (or just reflexive, if A is clear from context) if $\forall x \in A(xRx)$, or in other words, $\forall x \in A((x, x) \in R)$.
(2) R is symmetric if $\forall x \in A \forall y \in A(xRy \to yRx)$.
(3) R is transitive if $\forall x \in A \forall y \in A \forall z \in A((xRy \wedge yRz) \to xRz)$.

2. Problems

Problem 1. Let $A = \{1, 2\}$.

(1) Write down a relation R that describe the following thing from subsets of A: XRY if $X \subseteq Y$.
(2) Represent this relation in the form of arrows and dots.

Problem 2. Suppose that $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6, 7, 8\}$, $R = \{(1, 7), (3, 6), (3, 7)\}$, and $S = \{(4, 7), (4, 8), (5, 6)\}$. Note that R is a relation from A to C, and S is a relation from B to C. Find the following relations:

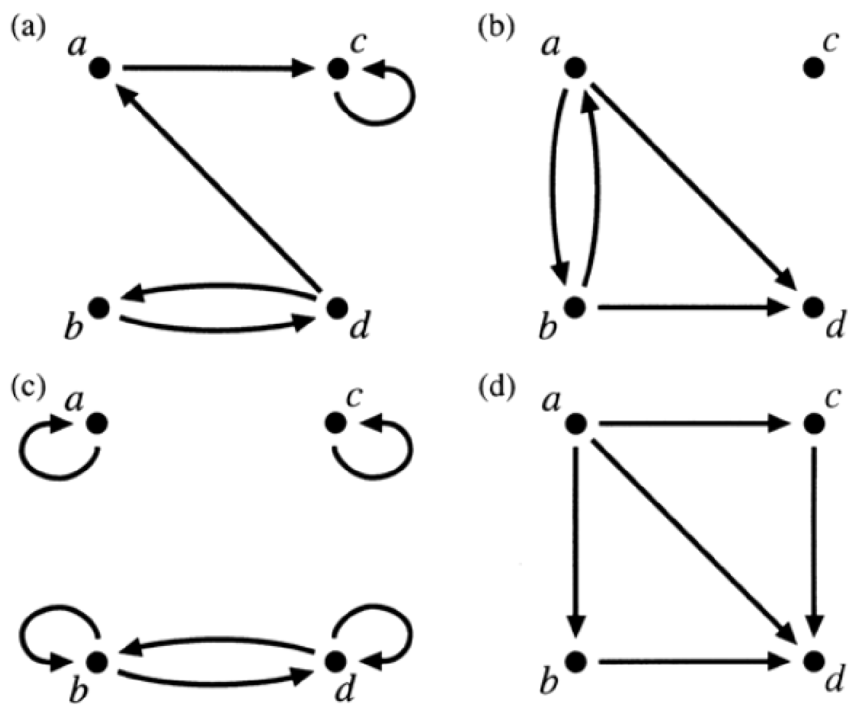
(1) $S^{-1} \circ R$.
(2) $R^{-1} \circ S$.

Problem 3. Suppose R is a relation from A to B, S is a relation from B to C, and T a relation from C to D. Prove the following

Date: Nov 6, 2023.

(1) $(R^{-1})^{-1} = R$
(2) $\mathrm{Dom}(R^{-1}) = \mathrm{Ran}(R)$
(3) $\mathrm{Ran}(R^{-1}) = \mathrm{Dom}(R)$
(4) $T \circ (S \circ R) = (T \circ S) \circ R$
(5) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Problem 4. Determine whether each relation represented by the graph is reflexive, symmetric, or transitive.



Problem 5. Suppose R is a relation on a set A. Prove

(1) R is reflexive if and only if $i_A \subseteq R$, where i_A is the identity relation on A.
(2) R is symmetric if and only if $R = R^{-1}$.
(3) R is transitive if and only if $R \circ R \subseteq R$.