MATH 102: HOMEWORK 4

DUE DATE: THURSDAY, OCT 24

Note that you need to turn in LaTeXversion of this homework.

Do Problem 1 below AND the problems from the book

Problem 1. In class on Oct 8, we discussed about inflation and present value. In short, if someone gives you F amount of dollars every year and the inflation rate is r, then the present value of N year of this fixed income would be

$$PV = F + \frac{F}{1+r} + \cdots + \frac{F}{(1+r)^N}.$$

(1) Use induction to prove that after N year,

$$PV = F\left(\frac{1+r-\frac{1}{(1+r)^N}}{r}\right)$$

(2) Apply this to the situation when $F = \$1$ and $r = 5\%$ (I made a mistake in class about this)

(3) What happen if you were to live forever? How much would you buy a stock if it pays you \$1 per year from now to eternity?

Proof. We shall proceed by induction. Step 1: Base case. For $N = 1$, we have

$$F = F\frac{1+r-1}{r}.$$

Step 2: Induction hypothesis. Suppose that the statement is true for $N = k$, i.e.,

$$F + \cdots + \frac{F}{(1+r)^k} = F\left(\frac{1+r-\frac{1}{(1+r)^k}}{r}\right)$$

Step 3: Inductive step. We need to show that the statement is true for $N = k+1$. To do this, we consider

$$\begin{aligned} & F + \cdots + \frac{F}{(1+r)^k} + \frac{F}{(1+r)^{k+1}} \\ &= F\left(\frac{1+r-\frac{1}{(1+r)^k}}{r}\right) + \frac{F}{(1+r)^{k+1}} \end{aligned}$$

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$$= \frac{F\left(1+r-\frac{1}{(1+r)^k}\right)+\frac{rF}{(1+r)^{k+1}}}{r}$$
$$= F\frac{1+r-\frac{1}{(1+r)^k}+\frac{r}{(1+r)^{k+1}}}{r}$$
$$= F\frac{1+r+\frac{-1-r+r}{(1+r)^{k+1}}}{r}$$
$$= F\frac{1+r-\frac{1}{(1+r)^{k+1}}}{r}$$

as desired.

Step 4: Conclusion. From all of the above steps, we have shown by induction that

$$F+\frac{F}{1+r}+\cdots+\frac{F}{(1+r)^N}$$
$$= F\left(\frac{1+r-\frac{1}{(1+r)^N}}{r}\right)$$

for every $N \in \mathbb{N}$. □

Problems from the file chapter4-hw.pdf

Try: 4.4, 4.5, 4.7, 4.21, 4.23
Grade: 4.4.f, 4.5.a,b, 4.7, 4.21.b, 4.23.a

Problem 2 (4.4.f)*.* We shall proceed by induction.

Step 1: Base Case. For $n = 1$, we check whether the equation holds:

$$1 \cdot 1! = 1$$

Step 2: Inductive Hypothesis. Assume that the statement holds for some $n = k$, i.e.,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$$

Step 3: Inductive Step. We need to show that the statement also holds for $n = k+1$. Starting from the inductive hypothesis, we need to prove:

$$1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! = (k+2)! - 1$$

Using the inductive hypothesis, we can write the left-hand side as:

$$(1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!) + (k+1) \cdot (k+1)!$$

Substitute the inductive hypothesis:

$$(k+1)! - 1 + (k+1) \cdot (k+1)!$$

Factor out $(k+1)!$:

$$(k+1)!\,(1 + (k+1)) - 1$$

$$(k+1)! \cdot (k+2) - 1$$

$$(k+2)! - 1$$

Thus, the statement holds for $n = k+1$.

Step 4: Conclusion. By induction, the formula

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

is true for all $n \in \mathbb{N}$.

Problem 3 (4.5.a). We proceed by induction.

Step 1: Base Case. For $n = 1$, we check if the inequality holds:

$$n + 2 = 1 + 2 = 3$$

$$4n^2 = 4(1)^2 = 4$$

Clearly, $3 < 4$, so the inequality holds for $n = 1$.

Step 2: Induction hypothesis. Now, assume the inequality holds for some $n = k$, i.e.,

$$k + 2 < 4k^2$$

Step 3: Inductive step. We need to show that the inequality holds for $n = k+1$. That is, we want to prove:

$$(k+1) + 2 < 4(k+1)^2\,.$$

We have

$$k + 3 = (k+2) + 1 < 4k^2 + 1 < 4k^2 + 8k + 4 = 4(k+1)^2\,.$$

Thus, the inequality holds for $n = k+1$.

Step 4: Conclusion. By induction, the inequality $n + 2 < 4n^2$ holds for all $n \in \mathbb{N}$.

Problem 4 (4.5.b). We proceed by induction.

Step 1: Base case. For $n = 1$,

$$\frac{1}{\sqrt{1}} \leqslant 2\sqrt{1} - 1\,.$$

Step 2: Induction hypothesis. Suppose the inequality holds for $n = k$, i.e.,

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leqslant 2\sqrt{k} - 1\,.$$

Step 3: Inductive step. We need to show that the inequality holds for $n = k + 1$. To do this, consider

$$\begin{aligned}
&\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&\leqslant 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \\
&= \frac{2\sqrt{k(k+1)} - \sqrt{k+1} + 1}{\sqrt{k+1}} \\
&\leqslant \frac{2(k+1) - \sqrt{k+1}}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - 1\,.
\end{aligned}$$

Step 4: Conclusion. By induction, it is true that

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} \leqslant 2\sqrt{n} - 1\,,$$

for every $n \in \mathbb{N}$.

Problem 5 (4.7). We proceed by induction.

Step 1: Base case. For $n = 0$, $0^2 < 3^0$. For $n = 1$, $1 < 3$.

Step 2: Induction hypothesis. Suppose the inequality is true for $n = k \geqslant 2$, i.e.,

$$k^2 < 3^k\,.$$

Step 3: Inductive step. We need to show the inequality is true for $n = k + 1$. To do this, we consider

$$(k+1)^2 = k^2 + 2k + 1 < 3^k + 2k + 1\,.$$

Note that when $k \geqslant 2$, $2k < k^2$ and $1 < k^2$. Therefore,

$$(k+1)^2 < 3^k + 2k + 1 < 3 * 3^k = 3^{k+1}\,.$$

Step 4: Conclusion. By induction,

$$n^2 < 3^n$$

for $n \geqslant 0$.