MATH 102: HOMEWORK 4

DUE DATE: THURSDAY, OCT 24

Note that you need to turn in LaTeXversion of this homework.

Do Problem 1 below AND the problems from the book

Problem 1. In class on Oct 8, we discussed about inflation and present value. In short, if someone gives you F amount of dollars every year and the inflation rate is r, then the present value of N year of this fixed income would be

$$PV = F + \frac{F}{1+r} + \cdots + \frac{F}{(1+r)^N}\,.$$

(1) Use induction to prove that after N year,
$$PV = F\left(\frac{1 + r - \frac{1}{(1+r)^N}}{r}\right)$$
(2) Apply this to the situation when $F = \$1$ and $r = 5\%$ (I made a mistake in class about this)
(3) What happen if you were to live forever? How much would you buy a stock if it pays you $1 per year from now to eternity?

Proof. We shall proceed by induction. Step 1: Base case. For $N = 1$, we have

$$F = F\frac{1 + r - 1}{r}.$$

Step 2: Induction hypothesis. Suppose that the statement is true for $N = k$, i.e.,

$$F + \cdots + \frac{F}{(1+r)^k} = F\left(\frac{1 + r - \frac{1}{(1+r)^k}}{r}\right)$$

Step 3: Inductive step. We need to show that the statement is true for $N = k + 1$. To do this, we consider

$$\begin{aligned} F + \cdots + \frac{F}{(1+r)^k} + \frac{F}{(1+r)^{k+1}} \\ = F\left(\frac{1 + r - \frac{1}{(1+r)^k}}{r}\right) + \frac{F}{(1+r)^{k+1}} \end{aligned}$$

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$$
\begin{aligned}
&= \frac{F\left(1 + r - \frac{1}{(1+r)^k}\right) + \frac{rF}{(1+r)^{k+1}}}{r} \\
&= F\frac{1 + r - \frac{1}{(1+r)^k} + \frac{r}{(1+r)^{k+1}}}{r} \\
&= F\frac{1 + r + \frac{-1-r+r}{(1+r)^{k+1}}}{r} \\
&= F\frac{1 + r - \frac{1}{(1+r)^{k+1}}}{r}
\end{aligned}
$$

as desired.

Step 4: Conclusion. From all of the above steps, we have shown by induction that

$$
\begin{aligned}
&F + \frac{F}{1+r} + \cdots + \frac{F}{(1+r)^N} \\
&= F\left(\frac{1 + r - \frac{1}{(1+r)^N}}{r}\right)
\end{aligned}
$$

for every $N \in \mathbb{N}$. □

Problems from the file chapter4-hw.pdf

Try: 4.4, 4.5, 4.7, 4.21, 4.23
Grade: 4.4.f, 4.5.a,b, 4.7, 4.21.b, 4.23.a

Problem 2 (4.4.f)*.* We shall proceed by induction.

Step 1: Base Case. For $n = 1$, we check whether the equation holds:

$$1 \cdot 1! = 1$$

Step 2: Inductive Hypothesis. Assume that the statement holds for some $n = k$, i.e.,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$$

Step 3: Inductive Step. We need to show that the statement also holds for $n = k + 1$. Starting from the inductive hypothesis, we need to prove:

$$1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1) \cdot (k+1)! = (k+2)! - 1$$

Using the inductive hypothesis, we can write the left-hand side as:

$$(1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k!) + (k+1) \cdot (k+1)!$$

Substitute the inductive hypothesis:

$$(k+1)! - 1 + (k+1) \cdot (k+1)!$$

Factor out $(k+1)!$:

$$(k+1)!\,(1 + (k+1)) - 1$$

$$(k+1)! \cdot (k+2) - 1$$

$$(k+2)! - 1$$

Thus, the statement holds for $n = k+1$.

Step 4: Conclusion. By induction, the formula

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$$

is true for all $n \in \mathbb{N}$.

Problem 3 (4.5.a). We proceed by induction.

Step 1: Base Case. For $n = 1$, we check if the inequality holds:

$$n + 2 = 1 + 2 = 3$$

$$4n^2 = 4(1)^2 = 4$$

Clearly, $3 < 4$, so the inequality holds for $n = 1$.

Step 2: Induction hypothesis. Now, assume the inequality holds for some $n = k$, i.e.,

$$k + 2 < 4k^2$$

Step 3: Inductive step. We need to show that the inequality holds for $n = k+1$. That is, we want to prove:

$$(k+1) + 2 < 4(k+1)^2 .$$

We have

$$k + 3 = (k+2) + 1 < 4k^2 + 1 < 4k^2 + 8k + 4 = 4(k+1)^2 .$$

Thus, the inequality holds for $n = k+1$.

Step 4: Conclusion. By induction, the inequality $n + 2 < 4n^2$ holds for all $n \in \mathbb{N}$.

Problem 4 (4.5.b). We proceed by induction.

Step 1: Base case. For $n = 1$,

$$\frac{1}{\sqrt{1}} \leqslant 2\sqrt{1} - 1 .$$

Step 2: Induction hypothesis. Suppose the inequality holds for $n = k$, i.e.,

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} \leqslant 2\sqrt{k} - 1 .$$

Step 3: Inductive step. We need to show that the inequality holds for $n = k+1$. To do this, consider

$$\begin{aligned}
&\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \\
&\leqslant 2\sqrt{k} - 1 + \frac{1}{\sqrt{k+1}} \\
&= \frac{2\sqrt{k(k+1)} - \sqrt{k+1} + 1}{\sqrt{k+1}} \\
&\leqslant \frac{2(k+1) - \sqrt{k+1}}{\sqrt{k+1}} \\
&= 2\sqrt{k+1} - 1\,.
\end{aligned}$$

Step 4: Conclusion. By induction, it is true that

$$\frac{1}{\sqrt{1}} + \cdots + \frac{1}{\sqrt{n}} \leqslant 2\sqrt{n} - 1\,,$$

for every $n \in \mathbb{N}$.

Problem 5 (4.7). We proceed by induction.

Step 1: Base case. For $n = 0$, $0^2 < 3^0$. For $n = 1$, $1 < 3$.

Step 2: Induction hypothesis. Suppose the inequality is true for $n = k \geqslant 2$, i.e.,

$$k^2 < 3^k\,.$$

Step 3: Inductive step. We need to show the inequality is true for $n = k+1$. To do this, we consider

$$(k+1)^2 = k^2 + 2k + 1 < 3^k + 2k + 1\,.$$

Note that when $k \geqslant 2$, $2k < k^2$ and $1 < k^2$. Therefore,

$$(k+1)^2 < 3^k + 2k + 1 < 3 * 3^k = 3^{k+1}\,.$$

Step 4: Conclusion. By induction,

$$n^2 < 3^n$$

for $n \geqslant 0$.

Problem 6 (4.21b). We proceed by strong induction.

Step 1: Base case $n = 2$. It is true that $2 = 2$.

Step 2: Induction hypothesis. Suppose the statement is true for every number i such that $2 \leqslant i \leqslant k$, i.e., i has a unique factorization

$$i = p_{i,1}p_{i,2}...p_{i,l}\,.$$

Step 3: Inductive step. We need to show that the statement is true for $k+1$. To do this, we suppose that there are two ways to factorize $k+1$, i.e.,

$$k+1 = q_1 q_2 \dots q_l = r_1 r_2 \dots r_m \,,$$

by Lemma 2.17 (chapter 2), we see that q_1 must divide some r_i. It doesn't matter the order so we may assume $r_i = r_1$. However, because both q_1 and r_1 are primes, it follows that $q_1 = r_1$. Therefore, we may cancel these two numbers and get

$$q_2 q_3 \dots q_l = r_1 r_2 \dots r_m \,.$$

But this is a number less than k (because $q_1 = r_1 \geqslant 2$) and has two differnt prime factorizations. This violates the induction hypothesis. So it must be true that $k+1$ has a unique prime factorization.

Step 4: Conclusion. By induction, it is true that every prime number can be uniquely factorized into primes.

Problem 7 (4.23a). We proceed by strong induction.

Step 1: Base Cases.
For $n = 1$:

$$a_1 = 1 = 2(1) - 1$$

Thus, the formula holds for $n = 1$.
For $n = 2$:

$$a_2 = 3 = 2(2) - 1$$

Thus, the formula holds for $n = 2$.

Step 2: Induction hypothesis. Assume the formula holds for all $k \leqslant n$, that is, assume:

$$a_k = 2k - 1 \quad \text{for all } k \leqslant n$$

Step 3: Inductive step. We need to prove that the formula holds for a_{n+1}, i.e., $a_{n+1} = 2(n+1) - 1$.

Using the recursive relation $a_{n+1} = 2a_n - a_{n-1}$, and substituting the inductive hypotheses $a_n = 2n - 1$ and $a_{n-1} = 2(n-1) - 1$, we get:

$$a_{n+1} = 2(2n-1) - (2(n-1) - 1)$$

Simplifying:

$$a_{n+1} = 2(2n-1) - (2n-3)$$
$$a_{n+1} = (4n-2) - (2n-3)$$
$$a_{n+1} = 4n - 2 - 2n + 3$$
$$a_{n+1} = 2n + 1$$

which is equal to:

$$a_{n+1} = 2(n+1) - 1$$

Thus, the formula holds for a_{n+1}.

Step 4: Conclusion. By the principle of strong induction, we have proven that $a_n = 2n - 1$ for all $n \in \mathbb{N}$.