# MATH 102: IDEAS OF MATH

WORKSHEET 8

## 1. Concepts

**Definition 1.** We call a relation from $A$ to $A$ a relation on $A$.

**Definition 2.** Let $R$ be a relation from $A$ to $B$ and $S$ be a relation from $B$ to $C$. We define the *composition* of relations $S \circ R$ as follows

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B, (a, b) \in R \wedge (b, c) \in S\}.$$

**Definition 3.** Suppose $R$ is a relation from $A$ to $B$. The *inverse* of $R$ is the relation $R^{-1}$ from $B$ to $A$, defined as follows:

$$R^{-1} = \{(b, a) \in B \times A \mid (a, b) \in R\}$$

**Definition 4.** Suppose $R$ is a relation on $A$.

(1) $R$ is said to be reflexive on $A$ (or just reflexive, if $A$ is clear from context) if $\forall x \in A(xRx)$, or in other words, $\forall x \in A((x, x) \in R)$.
(2) $R$ is symmetric if $\forall x \in A \forall y \in A(xRy \to yRx)$.
(3) $R$ is transitive if $\forall x \in A \forall y \in A \forall z \in A((xRy \wedge yRz) \to xRz)$.

## 2. Problems

*Problem* 1. Let $A = \{1, 2\}$.

(1) Write down a relation $R$ that describe the following thing from subsets of $A$: $XRY$ if $X \subseteq Y$.
(2) Represent this relation in the form of arrows and dots.

*Problem* 2. Suppose that $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6, 7, 8\}$, $R = \{(1, 7), (3, 6), (3, 7)\}$, and $S = \{(4, 7), (4, 8), (5, 6)\}$. Note that $R$ is a relation from $A$ to $C$, and $S$ is a relation from $B$ to $C$. Find the following relations:

(1) $S^{-1} \circ R$.
(2) $R^{-1} \circ S$.

*Problem* 3. Suppose $R$ is a relation from $A$ to $B$, $S$ is a relation from $B$ to $C$, and $T$ a relation from $C$ to $D$. Prove the following

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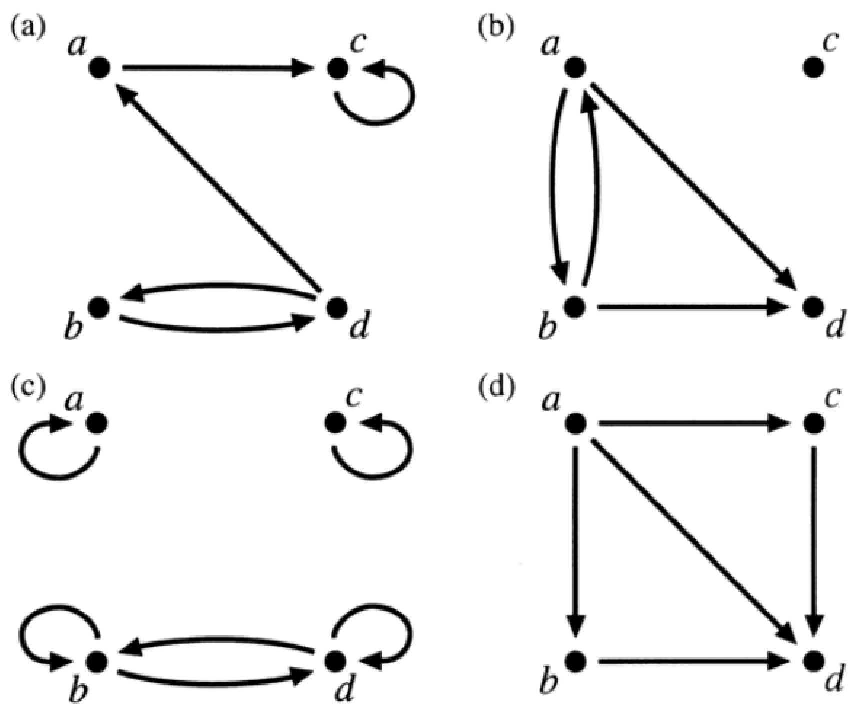
(1) $(R^{-1})^{-1} = R$
(2) $\mathrm{Dom}(R^{-1}) = \mathrm{Ran}(R)$
(3) $\mathrm{Ran}(R^{-1}) = \mathrm{Dom}(R)$
(4) $T \circ (S \circ R) = (T \circ S) \circ R$
(5) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

*Problem* 4. Determine whether each relation represented by the graph is reflexive, symmetric, or transitive.




*Problem* 5. Suppose $R$ is a relation on a set $A$. Prove

(1) $R$ is reflexive if and only if $i_A \subseteq R$, where $i_A$ is the identity relation on $A$.
(2) $R$ is symmetric if and only if $R = R^{-1}$.
(3) $R$ is transitive if and only if $R \circ R \subseteq R$.