MATH 102: HOMEWORK 3

SOLUTION

Grade: 3.1, 3.8, 3.9, 3.12, 3.14, 3.15, 3.21, 3.22

Problem 1 (Exercise 3.1). a. $\{\{\emptyset\}\}$
b. $\{\{\{Porco\}\}\}$
c. $\{Dragon, \{\{Porco\}\}\}$
d. $\{Tofu, \{Dragon, \{Porco\}\}\}$
e. $\{\{Dragon, \emptyset\}\}$
f. $\{Porco, \{\emptyset\}\}$

Problem 2 (Exercise 3.8). (a)T, (b)F, (c)F, (d)T, (e)T, (f)T, (g)F, (h)T, (i)T, (j)F, (k)T, (l)T, (m)T, (n)F, (o)T

Problem 3 (Exercise 3.9).
(1) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$
(2) $\{\emptyset, \{\mathbb{N}\}, \{\{\mathbb{Q}, \mathbb{R}\}\}, \{\mathbb{N}, \{\mathbb{Q}, \mathbb{R}\}\}\}$
(3) $\{\emptyset, \{\mathbb{N}\}, \{\mathbb{Q}\}, \{\mathbb{R}\}, \{\mathbb{N}, \mathbb{Q}\}, \{\mathbb{N}, \mathbb{R}\}, \{\mathbb{Q}, \mathbb{R}\}, \{\mathbb{N}, \mathbb{Q}, \mathbb{R}\}\}$
(4) $\{\emptyset\}$

Problem 4 (Exercise 3.12). Prove

$$(A \cap B)^c = A^c \cup B^c\,.$$

Proof. We need to prove two ways.

1. First we need to show $(A \cap B)^c \subseteq A^c \cup B^c$.
   Let $x \in (A \cap B)^c$. This means $x \notin A \cap B$. There are 3 cases.
   Case 1. $x \in A$ and $x \notin B$. Then, $x \in B^c \subseteq A^c \cup B^c$.
   Case 2. $x \in B$ and $x \notin A$. Then $x \in A^c \subseteq A^c \cup B^c$.
   Case 3. x does not belong to either A or B, i.e., $x \notin A \cup B$. Then, $x \in (A \cup B)^c = A^c \cap B^c \subseteq A^c \cup B^c$ (by the first De Morgan's law in the book, Theorem 3.16).
   In all cases, $x \in A^c \cap B^c$. So, $(A \cap B)^c \subseteq A^c \cup B^c$.

2. Then we need to show $(A \cap B)^c \supseteq A^c \cup B^c$.
   Let $x \in A^c \cup B^c$. Therefore, $x \in A^c$ or $x \in B^c$. There are two cases.
   Case 1. If $x \in A^c$, then $x \notin A$. Therefore, $x \notin A \cap B$. Thus, $x \in (A \cap B)^c$.

Date: October 3, 2024.

Case 2. If $x \in B^c$, then $x \notin B$. Therefore, $x \notin A \cap B$. Thus, $x \in (A \cap B)^c$.

From both cases, we have $x \in (A \cap B)^c$. Therefore, $(A \cap B)^c \supseteq A^c \cup B^c$

□

Problem 5 (Exercise 3.14). (1) $(A \backslash B) \cup (B \cap C)$
(2) $(A \cap C) \backslash B$
(3) $(A \cup B \cup C) \backslash (A \cap B \cap C)$

Problem 6 (Exercise 3.15). Let A and B be sets. Prove

$$\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B) .$$

For the example, come up with your own and check.

Proof. Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. This means $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. There are two cases.

Case 1. $X \in \mathcal{P}(A)$. Then, $X \subset A$. Therefore, $X \subset A \cup B$. By definition, $X \in (A \cup B)$.

Case 2. $X \in \mathcal{P}(B)$. Then, $X \subset B$. Therefore, $X \subset A \cup B$. By definition, $X \in (A \cup B)$.

In both cases, we have $X \in (A \cup B)$. Therefore, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. □

Problem 7 (Exercise 3.21). Prove

$$\{n \in \mathbb{Z} : 2 \mid n\} \cap \{n \in \mathbb{Z} : 9 \mid n\} \subseteq \{n \in \mathbb{Z} : 6 \mid n\} .$$

Proof. Let $x \in \{n \in \mathbb{Z} : 2 \mid n\} \cap \{n \in \mathbb{Z} : 9 \mid n\}$. Then, it must be true that $2 \mid x$ and $9 \mid x$. By definition, $x = 2n$ for some $n \in \mathbb{Z}$ and $x = 9m$ for some $m \in \mathbb{Z}$. So, $2n = 9m$ and $2 \mid 9m$. By Question 4 in Worksheet 4, either $2 \mid 9$ or $2 \mid m$. Since it is for sure that $2 \nmid 9$, it must be the case that $2 \mid m$.

Therefore, $m = 2k$. This means that

$$x = 3m = 3 \cdot 2k = 6k .$$

So, $6 \mid x$ and $x \in \{n \in \mathbb{Z} : 6 \mid n\}$. □

Problem 8 (Exercise 3.22). Prove

$$\{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod} 6)\} \subseteq \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod} 2)\} .$$

Proof. Let $(x, y) \in \{(m, n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod} 6)\}$. Then $x \equiv y \,(\text{mod} 6)$. By definition, $6 \mid (x - y)$ and so $x - y = 6k$ for some $k \in \mathbb{Z}$. But then,

$$x - y = 2 \cdot (3k) .$$

Therefore, $2 \mid (x-y)$ and $x \equiv y \,(\text{mod}2)$. Thus, $(x,y) \in \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod}2)\}$. So,

$$\{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod}6)\} \subseteq \{(m,n) \in \mathbb{Z} \times \mathbb{Z} : m \equiv n \,(\text{mod}2)\}.$$

$\square$