MATH 170: MIDTERM 2

GOOD LUCK!

There are three questions. Make sure you justify all your work for complete credit.

Rules

- You have 50 minutes to complete your work and 10 minutes to upload your work.
- Open notes (you can use your notes freely).
- No use of internet, textbooks, computer algebra systems, calculators.
- No collaboration.

Questions

1. Let A, B be sets. Recall that the symmetric difference of two sets A and B is

$$A \triangle B = \{x \mid x \in A \text{ or } x \in B \text{ but not both}\}.$$

Determine whether the following are true or false via the Venn diagram.

$$A = A \cup ((A \cup B) \setminus (A \triangle B)).$$

Solution. True. □

2. (a) Give an example of a function f from $X = \mathbb{N}$ to $Y = \mathbb{N}$ that is injective but not surjective.
   (b) Let $Z = \{y \in Y \mid f(x) = y\}$. Now consider the function f above but a new codomain Z,i.e., $f : \mathbb{N} \to Z$. Is this new function a bijection? Explain your reasoning.

Solution. a. There are numerous functions. Two common ones are

$$f(n) = 2n$$

and

$$f(n) = n^2.$$

b. It is a bijection. In words, the set Z is the set of natural numbers so that every element in Z gets mapped to by an element in X, i.e, pick a $y \in Z$, then $y = f(x)$ for some $x \in X$. So this function $f : \mathbb{N} \to Z$ is surjective. It was injective by design. So it is bijective. □

3. Prove by induction that $\forall n \in \mathbb{N}$,

$$1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution. 1. Determine $P(n)$, which says

$$1 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

2. Base case: $P(0)$ says

$$0 = 0,$$

which is true.

3. Suppose $P(0) \wedge \cdots \wedge P(k)$ is true. In particular, it is true that

$$1 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}\,.$$

We want to show that $P(k+1)$ is true. In particular, we want to show the following is true

$$1 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}\,.$$

To do this, consider

$$\begin{aligned}
1 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k+1)(k(2k+1) + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\
&= \frac{(k+1)(2k^2 + 3k + 4k + 6)}{6} \\
&= \frac{(k+1)(k+2)(2k+3)}{6}\,.
\end{aligned}$$

This shows that $P(k+1)$ is true.

4. By induction theorem, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$