# MATH 170: HOMEWORK 1 SOLUTIONS

TEACHING TEAM

1. Are these numbers divisible by 2, 3, 5? To get points, use the criteria of divisibility that we covered in class. Optional: check your answers with a calculator.
   a. 12345;
   b. 10101010.

   *Solution.* a. Divisible by 3 and 5 as the sums of the digits is 15 which is divisible by 3 and the last digit is 5.
   b. divisible by 2 and 5 as the last digit is 0. □

2. a. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a, b) = \gcd(a, -b)$.
   b. Compute $\gcd(12, -30)$ using Euclid's algorithm.
   c. Compute $\gcd(12\,345, 11\,100)$ using Euclid's algorithm.
   d. Optional: check your answers on https://www.wolframalpha.com/.


*Solution.* a. We have that $\gcd(a, b) \mid b$. So there exists $q \in \mathbb{Z}$ such that

$$b = q \gcd(a, b)\,.$$

This means,

$$-b = (-q) \gcd(a, b)\,.$$

So $\gcd(a, b) \mid -b$ and, therefore, $\gcd(a, b) \leqslant \gcd(a, -b)$. Argue similarly, $\gcd(a, -b) \leqslant \gcd(a, b)$. Thus, the equality holds.

b. By part (a), $\gcd(12, -30) = \gcd(12, 30)$. Therefore, we can apply Euclid's algorithm

$$30 = 2 \cdot 12 + 6$$
$$12 = 2 \cdot 6 + 0\,.$$

*Date*: September 14, 2021.

So $\gcd(12, -30) = 6$.

c.

$$\begin{aligned}
12345 &= 1 \cdot 11100 + 1245 \\
11100 &= 8 \cdot 1245 + 1140 \\
1245 &= 1 \cdot 1140 + 105 \\
1140 &= 10 \cdot 105 + 90 \\
105 &= 1 \cdot 90 + 15 \\
90 &= 60 \cdot 15 + 0\,.
\end{aligned}$$

Thus, $\gcd(12345, 11100) = 15$. □

3. a. Divide the following numbers by 10 with remainder: 1234, 2021. That is, write them in the form $10 \cdot q + r$, where $0 \leqslant r < 10$. Compute $1234 + 2021$ and divide the result by 10 with remainder. Observe that the remainder you get is the sum of the first two remainders.

   b. Compute the remainder of 1236 and 2027 when dividing by 10, then compute the remainder of their sum. Why is it no longer just the sum of remainders?

   c. Rephrase the following rule in your own words: If you divide $a_1, a_2 \in \mathbb{Z}$ by $b > 0$ with remainder, write the remainder of $a_1$ as $r_1$ and the remainder of $a_2$ as $r_2$. Then the remainder of $a_1 + a_2$ is $r_1 + r_2$ or $r_1 + r_2 - b$.

   * Bonus: prove c.

*Solution.* a.

$$\begin{aligned}
1234 &= 10 \cdot 123 + 4\,, \\
2021 &= 10 \cdot 202 + 1\,, \\
1234 + 2021 = 3255 &= 10 \cdot 325 + 5\,.
\end{aligned}$$

b.

$$\begin{aligned}
1236 &= 10 \cdot 123 + 6\,, \\
2027 &= 10 \cdot 202 + 7\,, \\
1236 + 2027 = 3263 &= 10 \cdot 326 + 3\,.
\end{aligned}$$

c. I will give a proof of this problem, not rephras it.

Let's rewriting $a_1$ and $a_2$ so we have a visual effect of what is going on.

$$a_1 = q_1 b + r_1$$

$$a_2 = q_2 b + r_2 \,,$$

for some $q_1, q_2 \in \mathbb{Z}$. We know that because $b > 0$ and by the definition of remainders,

$$0 \leqslant r_1, r_2 < b \,.$$

Note that there is no restriction for $r_1 + r_2$ as they are just another natural number. So, there are 2 cases

(a) When $r_1 + r_2 < b$. Then,

$$a_1 + a_2 = q_1 b + q_2 b + r_1 + r_2 = (q_1 + q_2)b + (r_1 + r_2) \,.$$

This is the unique representation of $a_1 + a_2$ with its remainder (by the division theorem (look it up again)) as $0 \leqslant r_1 + r_2 < b$. In other words, we have found the unique $q$ and $r$ in the division theorem so that $a_1 + a_2$ can be written in terms of $b$. So, the remainder of $a_1 + a_2$ divided by $b$ in this case is $r_1 + r_2$.

(b) When $r_1 + r_2 > b$. Note that $r_1 + r_2 < 2b$ as each of them is less than $b$. Therefore, $r_1 + r_2 - b < b$. We want to find the unique $q$ and $r$ in the division theorem in this case as well. To do that, we write

$$\begin{aligned} a_1 + a_2 &= q_1 b + q_2 b + r_1 + r_2 \\ &= (q_1 + q_2)b + (r_1 + r_2 - b) + b \\ &= (q_1 + q_2 + 1)b + (r_1 + r_2 - b) \,. \end{aligned}$$

As $r_1 + r_2 - b < b$, we just show that the remainder of $a_1 + a_2$ divided by $b$ in this case is $r_1 + r_2 - b$.

$\square$