MATH 102: MIDTERM

GOOD LUCK!

There are five questions. Make sure you justify all your work for complete credit.

Rules

- You have 90 minutes to complete your work..
- Closed books.
- No use of internet, textbooks, computer algebra systems, calculators.
- No collaboration.
- 1 person per bathroom break. When you go to the bathroom, turn in your cellphone and exam until return.

Date: October 11, 2023.

Questions

1. (a) *[10 points.]* Use the truth table to show that

$$P \implies Q \equiv (\neg P) \vee Q\,.$$

   (b) *[10 points.]* Find an equivalent expression of

$$\neg(P \implies Q)$$

   and use the truth table to check it.

2. The symmetric difference of two sets A and B is defined as follows

$$A \triangle B = (A \setminus B) \cup (B \setminus A) .$$

(a) *[10 points.]* Use the Venn diagram to represent $A \triangle B$.

(b) *[10 points.]* Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and $B = \{2, 4, 6\}$. What is $A \triangle B$?

3. *[20 points.]* For any sets A and B, prove that

$$A \triangle B \subseteq A \cup B\,.$$

Hint: Start with "Let x..." and use definitions of the operations.

4. *[20 points.]* Analyze the logical forms of the following statements. The universe of discourse is $\mathbb{R}$. What are the free variables in each statement?

   (a) Every number that is larger than x is larger than y.

   (b) For every number a, the equation $ax^2 + 4x - 2 = 0$ has at least one solution if $a \geq -2$.

   (c) All solutions of the inequality $x^3 - 3x < 3$ are smaller than 10.

   (d) If there is a number x such that $x^2 + 5x = w$ and there is a number y such that $4 - y^2 = w$, then w is between -10 and 10.

5. (a) *[10 points.]* Translate the following statement into a logical formula with predicates and quantifiers.
   "If all humans are heroes all the time then no humans are heroes anytime."
   Hint: Define X to be set of humans and T set of time. Suggested variables: x for humans, t for time. Suggested predicate: $P(x,t) =$"x is a hero in time t".
   (b) *[5 points.] A logical formula is said to be simplest if the negation signs (if there is any) are right in front of the predicate and NOT in front of quantifiers. For example, "$\neg(\forall x \in X, P(x))$" is NOT simplest, but "$\exists x \in X, \neg P(x)$" is simplest.*
   Find the simplest logical formulae to express the negation of the statement in (a).
   Hint: Use Question 1 recall $\neg(\forall x \in X, P(x)) \equiv \exists x \in X, \neg P(x)$ and $\neg(\exists x \in X, P(x)) \equiv \forall x \in X, \neg P(x)$.
   (c) *[5 points.]* Translate the resulting formula in (b) back into English.