MATH 102: IDEAS OF MATH

WORKSHEET 5

1. Set

Definition 1. A *set* is a collection of objects.

Set of objects x satisfying some property $P(x)$ is denoted by

$$\left\{ x \;\middle|\; P(x) \right\}. \tag{1.1}$$

Denote

(1) The set of all integers to be $\mathbb{Z}$
(2) The set of all natural numbers to be $\mathbb{N}$
(3) the set of all rational numbers to be $\mathbb{Q}$

Problem 1. From high school knowledge, try to describe $\mathbb{Q}$ in terms of $\mathbb{Z}$ using set notation (1.1).

Problem 2. What's the difference between a logical formula and a propositional formula?

Definition 2 (Notations)**.** The following are standard notations

(1) $x \in X$ represents "x belongs to set X".
(2) $x \notin X$ represents "x does not belong to set X".

Typically, a set is written by a capitalized letter, unless it is a special set such as $\mathbb{Z}, \mathbb{N}, \mathbb{Q}$.

2. Logical formula

Last time, we talked briefly about universal quantifier and existential quantifier.

Definition 3 (**The universal quantifier** $\forall$)**.** If $p(x)$ is a logical formula with free variable with free variable x with domain X, then $\forall x \in X, p(x)$ is the logical formula defined according to the following rules:

Date: September 21, 2023.

- If $p(x)$ can be derrived from the assumption that x is an arbitrary element of X, then $\forall x \in X, p(x)$ is true;
- If $a \in X$ and $\forall x \in X, p(x)$ is true, then $p(a)$ is true.

Definition 4 (The existential quantifier $\exists$). If $p(x)$ is a logical formula with free variable x with domain X, then $\exists x \in X, p(x)$ is the logical formula defined according to the following rules:

- If $a \in X$ and $p(a)$ is true, then $\exists x \in X, p(x)$ is true;
- If $\exists x \in X, p(x)$ is true, and q can be derived from the assumption that $p(a)$ is true for some fixed $a \in X$, then q is true.

Problem 3. In your own words, re-interpret the definitions of the quantifiers.

Problem 4. Represent the following sentences in logical formula form. Some sentences need to be rephrased so that things are clear to identify variables and predicates.

(1) $x - y$ is rational.
(2) Every even natural number $n \geqslant 2$ is divisible by k.
(3) There is an integer that is divisible by every integer.
(4) There is no greatest odd integer.
(5) Between any two distinct rational numbers is a third distinct rational number.
(6) If any integer has a rational square root, then that root is an integer.

Problem 5. Translate the following into English.

$$\forall a \in \mathbb{R}, (a \geqslant 0 \implies \exists b \in \mathbb{R}, a = b^2)\,.$$

Problem 6. Let P be the set of all prime numbers.

(1) Translate the following logical formulas into English.
    (a) $\forall n \in P, (n > 2 \implies (\exists k \in \mathbb{Z}, n = 2k + 1))$.
    (b) $\neg \exists n \in P, (n > 2 \wedge (\exists k \in \mathbb{Z}, n = 2k))$.
(2) Are they true? How would you go on to prove them?
(3) Are both statements talk about the same thing?

Problem 7. From the previous problem, find equivalent statements to the following

$$\neg(\exists x \in X, P(x))\,,$$

and

$$\neg(\forall x \in X, P(x))\,.$$

Hint: re-call the following from Worksheet 4

Theorem 2.1 (De Morgan's laws for logical operators)**.** *Let P, Q be propositional variables. Then,*

(1) $\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$,
(2) $\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$.