MATH 102: IDEAS OF MATH

WORKSHEET 11

1. Concepts

Definition 1. Let $f : X \to Y$ be a function. A *left inverse* for f is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$.

Definition 2. Let $f : X \to Y$ be a function. A *right inverse* for f is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Definition 3. Let $f : x \to Y$ be a function. An *inverse* (or two-sided inverse) of f is a function $g : Y \to X$ that is both left and right inverse.

Theorem 1.1 (Fundatmental theorem of arithmetic)**.** *Let $a \in \mathbb{N}$ be a nonzero number that is bigger than 1. There exist primes $p_1, \ldots, p_k \in \mathbb{N}$ such that*

$$a = p_1 \cdot \ldots \cdot p_k$$

Moreover, this expression is essentially unique: if $a = q_1 \cdot \ldots \cdot q_l$ is another expression of a as a product of primes, then $k = l$ and, re-ordering the q_i if necessary, for each i, $q_i = p_i$.

2. Problems

Problem 1.
(1) Suppose X is non-empty. Show that function $f : X \to Y$ has a left inverse if and only if it is injective.
(2) Show that a function $f : X \to Y$ has a right inverse if and only if it is surjective.

Problem 2. Let $e : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by

$$e(m, n) = 2^m 3^n \,.$$

(1) Show that e is injective.
(2) Find a left inverse for e.

Problem 3. Let $f : \mathbb{R} \to [0, \infty)$ be a function defined by

$$f(x) = x^2 \,.$$

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(1) Show that f is surjective.
(2) Find a right inverse for f.

Problem 4. Let $D \subseteq \mathbb{Q}$ be the set of *dyadic rational numbers*, that is

$$D = \left\{ x \in \mathbb{Q} | \exists a \in \mathbb{Z}, \exists n \in \mathbb{N} \cup \{0\}, x = \frac{a}{2^n} \right\} .$$

Let $k \in \mathbb{N}$ and define $f : D \to D$ by

$$f(x) = \frac{x}{2^k} .$$

Show that f is bijective.