# MATH 102: IDEAS OF MATH

WORKSHEET 12

## 1. Concepts

**Theorem 1.1** (The Induction Principle). *Suppose that we have a sequence of statements $P(n)$ labeled by the natural numbers $1, 2, \ldots$ such that we know that*

*(1)* $P(1)$ *is true, and*
*(2)* $\Big(P(1) \wedge P(2) \wedge \cdots \wedge P(n)\Big) \Rightarrow P(n+1)$.

*Then all the statements $P(1), P(2), \ldots$ are true.*

We will not discuss the proof of this theorem. However, curious minds can find the proof in Section 4.2 of Newstead.

*Structure of induction proof.* Induction proofs typically have the following structure:

(1) Identify the statements $P(n)$.
(2) Step 2 is also called *base of induction*: prove $P(0)$ or $P(1)$ (sometimes it doesn't make sense to talk about $P(0)$).
(3) Assume that all $P(k)$ with $k \leqslant n$ are true – this is called the *induction hypothesis*. Now perform the *step of induction*: prove that $P(n+1)$ is true.
(4) Finally, conclude by the Principle of Induction that all $P(n)$ are true.

□

## 2. Problems

*Problem* 1. Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}\,.$$

*Proof.* Done. □

*Problem* 2. Prove that for all natural number $n$, $n^3 - n$ is divisible by 3.

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*Proof.* Step 1. $P(n) =$ "$n^3 - n$ is divisible by 3."

Step 2. Base case: For $n = 1$,

$$1^3 - 1 = 0,$$

which is divisible by 3.

Step 3. (Induction hypothesis) Assume now that $P(k)$ is true for some $k \in \mathbb{N}$, i.e.,

$$k^3 - k = 3m$$

for some $m \in \mathbb{N}$.

We want to show that $P(k+1)$ is true. To see this, consider

$$\begin{aligned}(k+1)^3 - (k+1) &= k^3 + 3k^2 + 3k + 1 - k - 1 \\ &= k^3 - k + 3k^2 + 3k \\ &= 3m + 3k^2 + 3k = 3(m + k^2 + k) = 3l\,,\end{aligned}$$

where $l = m + k^2 + k \in \mathbb{N}$. By definition of divisibility,

$$3|(k+1)^3 - (k+1)\,.$$

So, $P(k+1)$ is true.

Step 4. By induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

*Problem* 3. Prove by induction that for all $n \in \mathbb{N}$,

$$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1\,.$$

*Proof.* Step 1. $P(n) =$ "$2^0 + 2^1 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$"

Step 2. Base case. For $n = 1$, we have

$$2^0 + 2^1 = 3 = 2^{1+1} - 1\,.$$

So $P(1)$ is true.

Step 3. Suppose that $P(k)$ is true for some $k \in \mathbb{N}$, i.e.,

$$2^0 + 2^1 + 2^2 + \cdots + 2^k = 2^{k+1} - 1\,.$$

We want to show that $P(k+1)$ is true. To do this, consider

$$2^0 + 2^1 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1\,.$$

So, $P(k+1)$ is true.

Step 4. By induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

*Problem* 4. For all $n \geqslant 4$, we have $3n < 2^n$.

*Proof.* Step 1. $P(n) =$ "$3n < 2^n$".

Step 2. Base case. For $n = 4$, we have

$$3 \cdot 4 = 12 < 16 = 2^4\,.$$

So $P(4)$ is true.

Step 3. Suppose that $P(k)$ is true for some $k \in \mathbb{N}$ and $k \geqslant 4$, i.e.,

$$3k < 2^k \,.$$

We want to show that $P(k+1)$ is true. To do this, consider

$$3(k+1) = 3k + 3 < 2^k + 3 < 2^k + 2^k = 2^{k+1}$$

because $k \geqslant 4$. So, $P(k+1)$ is true.

Step 4. By induction, $P(n)$ is true for all $n \in \mathbb{N}$ and $n \geqslant 4$. $\square$