Multivariable Calculus
Day 11
Differentiability and Directional Derivative

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Spring 2023

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Worksheet Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ and $T(x)$ be a linear function such that $T(a) = f(a)$. So,

$$T(x) = f(a) + m(x - a)\,.$$

This function is any arbitrary linear function that may give us an "approximation" for f near a.

To determine the "best" approximation, we want to see what happens to $f(x) - T(x)$ as $x \to a$.

Naturally, $\lim_{x \to a} f(x) - T(x) = 0$, by the first requirement.

Can we do better?

From Taylor's theorem, if

$$T(x) = f(a) + f'(a)(x-a) \tag{1}$$

then

$$\lim_{x \to a} \frac{f(x) - T(x)}{|x-a|} = 0\,, \tag{2}$$

which is a more significant statement than just

$$\lim_{x \to a} f(x) - T(x) = 0\,.$$

From analysis class (take it when you have a chance!), we will know that (2) is the necessary and sufficient condition to determine the linear approximation you learned before (e.g. (1)).

Furthermore, condition (2) is equivalent to the definition of differentiability. This justifies the complicated definition we learned in higher dimensions.

Example

- 1D: $f(x) = e^{-x^2}$, $a = 1$
- 2D: $f(\mathbf{x}) = e^{-|\mathbf{x}|^2}$, $\mathbf{a} = \langle 1, 1 \rangle$

Tangent plane

A plane is determined by its normal line $\mathbf{n}$.

Question: how do we know that there's only one tangent plane to a graph of a differentiable function?

Are you in crisis yet?

Let's slow down

Let's consider the curve situation first.

Typically, one thinks of a curve as

$$z = f(x)\,.$$

However, there's a more general form for a curve as level curve of a two-variable differentiable function $F(x, y)$. In particular, a c-level curve of $F(x, y)$ is a curve $\mathbf{r}(t)$ such that

$$F(\mathbf{r}(t)) = c\,.$$

c-level curve

That means

$$\frac{d}{dt}F(\mathbf{r}(t)) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0\,.$$

Suppose at t_0, $\mathbf{r}(t_0) = (a, b)$. We then have that, the tangent line of the c-level curve of F at (a, b) must satisfy the relation

$$\nabla F(a, b) \cdot \langle x_1 - a, x_2 - b \rangle = 0\,.$$

Another way to write this:

$$\partial_{x_1} F(a, b)(x_1 - a) + \partial_{x_2} F(a, b)(x_2 - b) = 0\,.$$