MATH 104: Multivariable Calculus (brief notes)

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Contents

Syllabus

Key information

- Instructor: Truong-Son Van
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- Class time: T & Th: 9:45a - 11:15a
- Class Location: CR 502
- Office hours: M & W, 10a-11a (or by appointment)
- Prerequisites: Calculus (MATH 101)

Textbooks and references

It is highly recommended that students read the textbooks.

1. In-class worksheets.
2. Active Calculus: Multivariable by Schlicker et al. 2018 edition. (https://activecalculus.org/multi/preface-2.html)
3. Thomas' Calculus: Early Transcendentals by Hass, Heil, et al. 14^{th} edition.
4. Calculus Early Transcendental by Stewart. 8^{th} edition.
5. Anything you can find on Google would work. Calculus is a subject that people have written about so much. So, there's no excuse for not having access to the knowledge.
6. 3-D grapher: https://www.math3d.org/

Course description

How do we describe the trajectory of a space shuttle? How is the human body affected by scuba diving to different depths for different lengths of time? The mathematics required to describe most real life systems involves functions of more than one variable. The concepts of the derivative and integral from a first course in calculus must therefore be extended to higher dimensional settings. In this course students will be guided through the essential ideas of multivariable calculus, including partial derivatives, multiple integrals and vector calculus, and their applications. These mathematical tools are used extensively in the physical sciences and engineering, and in other areas including economics and computer graphics.

Learning objectives

After the course, students are expected to:

- Be confident in handling functions of two or more variables and familiar with how they can be represented graphically
- Understand the key concepts of multivariable calculus, including partial derivatives, the gradient vector, multiple integrals, line and surface integrals, the divergence and curl of a vector function
- Know how such derivatives and integrals are calculated and some of their uses
- Be able to apply these ideas to real world problems
- Have improved analytic, computational and problem solving skills

Assessment

During the course, students are expected to compute their own percentage points based on the following scheme. The instructor is not responsible for providing the running percentage.

Form of assessment	**Weight**
Weekly homeworks	25%
Class worksheets	15%
Mini-project	15%
Midterm	15%
Final	30%

The following is the non-negotiable letter grade breakdown. It is based on common practice in the United States for standard courses such as Calculus.

Letter Grade	**Percentage**
A	[93,100]
A-	[90,93)
B+	[87,90)
B	[83,87)
B-	[80, 83)

Letter Grade	Percentage
C+	[77,80)
C	[73,77)
C-	[70,73)
D+	[67,70)
D	[60, 66)
F	[0,60)

Core content

1. Introduction

- Functions of two variables
- Graphs in three dimensions, surfaces and level curves
- Functions of three or more variables
- Limits and continuity
- Vectors (review)

2. Partial Derivatives

- Partial derivatives
- Tangent planes, linear approximations and differentials
- Chain rule
- Directional derivatives and gradient vectors
- Extrema and optimization
- Lagrange Multipliers

3. Multiple Integrals

- Double integrals
- Double integrals in polar coordinates
- Triple integrals
- Triple integrals in cylindrical and spherical coordinates
- Applications of multiple integrals

4. Vector Calculus

- Vector functions and their derivatives
- Vector fields
- Line integrals
- The fundamental theorem of line integrals
- Green's Theorem
- Parametric surfaces and surface integrals
- Curl and divergence
- Divergence Theorem
- Stokes Theorem

Late assignments

- 15% of the possible total mark will be deducted for every 24 hrs (or part of 24 hrs) after the deadline. Work more than 2 days late will not be accepted.
- Except for exceptional circumstances (see definition), I will not extend the deadlines.

Time expectations

Some materials require time to be accustomed to. Some students are quicker than others. However, on average, you should expect 10-15 hours per week (including class time) on the materials in order to know the subject relatively well.

Collaboration & Plagiarism

Plagiarism is the act of submitting the intellectual property of another person as your own. It is one of the most serious of academic offenses. Acts of plagiarism include, but are not limited to:

- Copying, or allowing someone to copy, all or a part of another person's work and presenting it as your own, or not giving proper credit.
- Purchasing a paper from someone (or a website) and presenting it as your own work.
- Re-submitting your work from another course to fulfill a requirement in another course.

Further details can be found in the Code of Academic Integrity [link].

Learning Support

In addition to your course instructors, there are other resources available to support your academic work at Fulbright, including one-on-one consultations with learning support staff, supplementary workshops, and both individual and group tutoring and mentoring in course content, language learning, and academic skills. If you would like to request learning support, please contact Fulbright Learning Support (https://learning-support.notion.site).

Wellbeing

Mental health and wellbeing are essential for the success of your academic journey. The Fulbright Wellness Center provides various services including counseling, safer community, and accessibility services. If you are experiencing undue personal or academic stress, are feeling unsafe, or would like to know more about issues related to wellbeing, please contact the Wellness Center via wellness@fulbright.edu.vn or visit the Wellness Center office on Level 5 of the Crescent campus.

For more information, pleaes check https://onestop.fulbright.edu.vn/s/article/Health-and-Wellness-Introduction

Tentative Course Schedule

The following schedule will be updated as we go so that students will know what to read before/after class.

Date	Session	Content	Deadlines
Feb 7 & 9	1&2	Linear algebra crash course & Vector-valued functions	
Feb 14 & 16	3 & 4	Integration: arc length	
Feb 21 & 23	5 & 6	Introduction and functions of several variables	
Feb 28 & Mar 2	7 & 8	Limits & continuity	
Mar 7 & 9	9 & 10	Partial derivatives & Tangent planes, chain rules & higher derivatives	
Mar 14 & 16	11 & 12	Directional derivatives, gradient vectors & Extrema and Optimization	
Mar 21 & 23	13 & 14	Extrema and Optimization (cont.)	
Mar 28	15	Midterm exam	
Mar 30		**Break**	
Apr 4 & 6	16 & 17	Double integrals	
Apr 11	18	Applications of Double Integrals	
Apr 13	19	Surface Area	
Apr 18 & 20	20 & 21	Triple Integrals	
Apr 25 & 27	22 & 23	Change of variables	
May 2		**Break**	
May 4	24		
May 9	25		
May 11	26		
May 16	27		
May 18	28		
May 23	29		
May 25	30		

Chapter 1

Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n-dimensional vectors doesn't require much more effort, we will talk about n-dimensional vectors instead.

Definition 1.1. An n-dimensional Euclidean space $\mathbb{R}^n$ is the Cartesian product of n Euclidean spaces $\mathbb{R}$.

Definition 1.2. An n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \ldots, v_n \rangle, \tag{1.1}$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in $\mathbb{R}^n$ is also represented by an n-tuple but with round brackets. A vector connecting two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle .$$

We denote the above vector as $\vec{AB}$ where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle .$$

Definition 1.3. The length of a vector $\mathbf{v}$ (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} . \tag{1.2}$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b} ,$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} .$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1\mathbf{e_1} + \cdots + u_n\mathbf{e_n} ,$$

where $e_i = \langle 0, \dots, 1, \dots, 0 \rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e_1} = \mathbf{i} , \qquad \mathbf{e_2} = \mathbf{j} , \qquad \mathbf{e_3} = \mathbf{k} .$$

Properties of vector operations

Read the book

(Make sure you understand the geometric intepretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n \,.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$.

Theorem 1.1 (Law of cosine)**.** *If θ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then*

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta \,.$$

Corollary 1.1. *Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|} \,,$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} \,.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \,.$$

Theorem 1.2. *Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \,.$$

Theorem 1.3. *The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*