

# Multivariable Calculus

## Day 10

### Partial derivatives

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Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ .

$$\partial_{x_i} f(x_1, \dots, x_n) = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_{i-1}, x_i + h, x_{i+1}, \dots, x_n) - f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)}{h}$$

$$\partial_{x_{k_1}, \dots, x_{k_m}}^m f = \partial_{x_{k_1}} (\dots (\partial_{x_{k_m}} f) \dots)$$

where  $k_i \in \{1, \dots, n\}$ .

**Theorem (Clairaut's Theorem)**

*Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then*

$$f_{xy}(a, b) = f_{yx}(a, b) .$$

## Important notations

Let  $f : D \rightarrow \mathbb{R}$  be a function. We write the following, if exist,

$$\nabla f = \begin{bmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{bmatrix}$$

$$\Delta f = \partial_{x_1}^2 f + \dots \partial_{x_n}^2 f .$$

Compute directional derivative of the functions

①  $f(x) = |x|, x \in \mathbb{R}^n$

②

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq 0 \\ 0 & (x, y) = 0. \end{cases}$$

③

$$f(x) = \begin{cases} |x|^2 \sin(1/|x|) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

## Definition

Let  $f : D \rightarrow \mathbb{R}$  and  $a \in \mathbb{R}^n$ . Let  $z = f(x)$  and  $\Delta z = f(a + \Delta x) - f(a)$ . Then  $f$  is **differentiable at**  $a$  if  $\Delta z$  can be expressed in the form

$$\Delta z = \sum_{i=1}^n \partial_i f(a) \Delta x_i + \epsilon_i \Delta x_i ,$$

where  $\epsilon_i \rightarrow 0$  as  $\Delta x_i \rightarrow (0, 0)$ .

$f$  is said to be **differentiable** if it is differentiable at every point on the domain.

## Theorem

Let  $f(x_1, \dots, x_n), g_i(y_1, \dots, y_m)$  ( $i = 1, \dots, n$ ) be differentiable functions. Then,

$$z(y_1, \dots, y_m) = f(g_1(y_1, \dots, y_m), \dots, g_n(y_1, \dots, y_m))$$

is differentiable and

$$\frac{\partial z}{\partial y_i} = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \frac{\partial g_j}{\partial y_i}.$$

**Definition**

Let  $\mathbf{u} \in \mathbb{R}^n$ . The directional derivative of  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $a \in \mathbb{R}^n$  in the direction of  $\mathbf{u}$  is the following limit (if exists)

$$D_{\mathbf{u}}f(a) = \lim_{h \rightarrow 0} \frac{f(a + h\mathbf{u}) - f(a)}{h}.$$



Compute  $\partial_{xy}f$  and  $\partial_{yx}f$  of the function

$$f(x, y) = \begin{cases} \frac{xy(y^2 - x^2)}{x^2 + y^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$