Multivariable Calculus
Day 2
Linear Algebra (cont.)

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Dot product

Definition

The dot product of vectors $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n .$$

Dot product: Geometric meaning

Theorem

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta \,.$$

Dot product: Geometric meaning

Theorem

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta .$$

Proof in $\mathbb{R}^2$.

Homework $\square$

Dot product: Geometric meaning

Theorem

If θ is the angle between the vectors $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \,.$$

Proof in $\mathbb{R}^2$.

Homework $\square$

Corollary

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Dot product: Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} =$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} =$$

Example

Worksheet

1. Let $\mathbf{a} = \langle 3, 0, -1 \rangle$. Find a vector $\mathbf{b}$ such that $\mathrm{comp}_{\mathbf{a}}\mathbf{b} = 2$.
2. Suppose $\mathbf{a}$ and $\mathbf{b}$ are nonzero vectors. When would it be true that $\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \mathrm{comp}_{\mathbf{b}}\mathbf{a}$?
3. When would it be true that $\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \mathrm{comp}_{\mathbf{b}}\mathbf{a}$?

3D special: Cross product

WARNING

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

3D special: Cross product

Definition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The cross product of **a** and **b** is defined to be

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle .$$

Worksheet

Given a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the determinant of the matrix A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

Question

Relate the cross product of vectors $\mathbf{u}$ *and* $\mathbf{v}$ *with the determinant of a* 3×3 *matrix.*

Worksheet

Suppose $\mathbf{u} = \langle 0, 1, 3\rangle, \mathbf{v} = \langle 2, -1, 0\rangle$.

1. Find the cross products $\mathbf{u} \times \mathbf{v}$
2. Find the cross products $\mathbf{v} \times \mathbf{u}$
3. Compute $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot \mathbf{u} \times \mathbf{v}$.
4. Compute $\mathbf{u} \times \mathbf{u}$.

3D special: Cross product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$ and $c \in \mathbb{R}$. Then

1. (anti-symmetry) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
3. $(c\mathbf{u}) \times \mathbf{w} = c(\mathbf{u} \times \mathbf{w}) = \mathbf{u} \times (c\mathbf{w})$
4. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if $\mathbf{u}$ and $\mathbf{v}$ are parallel
5. **WARNING:** in general $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

3D special: Cross product

Theorem

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \,.$$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3D special: Cross product

Theorem

The length, $|\mathbf{u} \times \mathbf{v}|$, of the cross product of vectors $\mathbf{u}$ and $\mathbf{v}$ is the area of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Example

Find the area of the parallelogram created by two vectors $\langle 1, 3, -2\rangle$ and $\langle 3, 0, 1\rangle$.

Worksheet

1. Prove the above theorem.
2. Find the area of the parallelogram in $\mathbb{R}^3$ whose vertices are $(1, 0, 1), (0, 0, 1), (2, 1, 0), (1, 1, 0)$. Hint: maybe draw it out.