Multivariable Calculus
Day 23
Vector Calculus: Line integrals (cont.)

Spring 2023

Recap worksheet

Evaluate

$$\int_C y^2\,dx + x\,dy\,, \qquad i = 1, 2$$

1. where C is the line segment from $(-5, -3) \to (0, 2)$
2. where C is the arc of the parabola $x = 4 - y^2$ from $(-5, -3) \to (0, 2)$.
3. repeat the above two steps with

$$P = x\,, \qquad Q = y\,.$$

Question

When is a vector field $\mathbf{F}$ in $\mathbb{R}^2$ conservative?

Definition
Let $\mathbf{F}$ be a continuous vector field with domain D, we say that the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Independence of path and conservative vector fields

Theorem
$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path Γ in D.

Worksheet

Prove the above theorem.

Definition
A domain D is said to be **open** if around each point, we can draw an open ball around it. A domain D is said to be **connected** if for any two points, there is a path that connect them together. A domain D is said to be **simply connected** if is connected and there's no hole in it.

Theorem
Suppose $\mathbf{F}$ *is a vector field that is continuous on an open connected region* D*. If* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *is independent of path in* D*, then* $\mathbf{F}$ *is a conservative vector field on* D*.*

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Clairaut's theorem If $\mathbf{F}$ is conservative then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

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Clairaut's theorem If **F** is conservative then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}\,.$$

Is the converse true?

What happens when $\mathbf{F}$ is not conservative?

Green's Theorem

Theorem (Green's Theorem)
Let D be an open bounded simply connected domain in $\mathbb{R}^2$, Γ be the boundary of D, and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_\Gamma \mathbf{F} \cdot d\ell = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,.$$

Theorem
Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ *be a vector field on an open simply connected region* D*. Suppose that* P *and* Q *have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

through out D*. Then* $\mathbf{F}$ *is conservative.*

Computing the area of any region bounded by a curve

https://www.youtube.com/watch?v=aLSx1eM27P4