MATH 104: WORKSHEET 14

1. Concepts

(1) Maximum and minimum values

A function of two variables has a **local maximum** at (a, b) if

$$f(x, y) \leqslant f(a, b)$$

when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If

$$f(x, y) \geqslant f(a, b)$$

when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

Theorem 1.1. *If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist here, then $f_(a, b) = f_y(a, b) = 0$.*

(1) Critical points: $\nabla f(x, y) = 0$ or if one of the partial derivatives doesn't exist.

Theorem 1.2 (Second derivative test)**.** *Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let*

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

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2. Discussions

Problem 2.1. Find the local maximum/minimum values and saddle points.

(1) $f(x, y) = x^2 + xy + y^2 + y$

(2) $f(x, y) = e^x \cos y$

Problem 2.2. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.