Multivariable Calculus
Day 10
Partial derivatives

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Last time

Let $f : \mathbb{R}^n \to \mathbb{R}$.

$$\partial_{x_i} f(x_1, \ldots, x_n) = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{h}$$

$$\partial^m_{x_{k_1}, \ldots, x_{k_m}} f = \partial_{x_{k_1}}(\ldots(\partial_{x_{k_m}} f)\ldots)$$

where $k_i \in \{1, \ldots, n\}$.

Theorem (Clairaut's Theorem)
Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)\,.$$

Important notations

Let $f : D \to \mathbb{R}$ be a function. We write the following, if exist,

$$\nabla f = \begin{bmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{bmatrix}$$

$$\Delta f = \partial_{x_1}^2 f + \ldots \partial_{x_n}^2 f \,.$$

Worksheet

Compute directional derivative of the functions

1. $f(x) = |x|, x \in \mathbb{R}^n$
2. $$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & (x,y) \neq 0 \\ 0 & (x,y) = 0\,. \end{cases}$$
3. $$f(x) = \begin{cases} |x|^2 \sin(1/|x|) & x \neq 0 \\ 0 & x = 0\,. \end{cases}$$

Differentiability

Definition
Let $f : D \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Let $z = f(x)$ and $\Delta z = f(a + \Delta x) - f(a)$. Then f is **differentiable at** a if Δz can be expressed in the form

$$\Delta z = \sum_{i=1}^{n} \partial_i f(a)\Delta x_i + \epsilon_i \Delta x_i \, ,$$

where $\epsilon_i \to 0$ as $\Delta x_i \to (0, 0)$.

f is said to be **differentiable** if it is differentiable at every point on the domain.

Chain rule

Theorem
Let $f(x_1, \ldots, x_n), g_i(y_1, \ldots, y_m)$ *(*$i = 1, \ldots, n$*) be differentiable functions. Then,*

$$z(y_1, \ldots, y_m) = f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m))$$

is differentiable and

$$\frac{\partial z}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial g_j}{\partial y_i}.$$

Directional derivative

Definition
Let $\mathbf{u} \in \mathbb{R}^n$. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathbb{R}^n$ in the direction of $\mathbf{u}$ is the following limit (if exists)

$$D_{\mathbf{u}} f(a) = \lim_{h \to 0} \frac{f(a + h\mathbf{u}) - f(a)}{h}.$$

Worksheet

Compute $\partial_{xy} f$ and $\partial_{yx} f$ of the function

$$f(x, y) = \begin{cases} \frac{xy(y^2 - x^2)}{x^2 + y^2} & (x, y) \neq 0\,, \\ 0 & (x, y) = 0\,. \end{cases}$$