Multivariable Calculus
Day 21
Vector Calculus: Line integrals (cont.)

Spring 2023

Line integrals

We now perform a Riemann-sum-like action.

Definition
Let C be a smooth curve. The **line integral of f along C** is defined as

$$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i \,,$$

where Δs_i is the length of a subarc of C.

Formula

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt\,.$$

Note: everything you do for 2 dimension will generalize directly to n dimension. So, we will not cover the details for 3 or higher dimensions. However, questions about those will be fair game.

Worksheet

Let's relate a few things with each other.

1. Look up Wikipedia or recall from your memory the way to compute the expectation of a random variable.
2. Watch the videos about center of mass
    1. `https://youtu.be/ol1COj0LACs`,
    2. `https://youtu.be/e548hRYcXlg`
3. What is the relationship between the center of mass and the expectation?
4. What is the analog of the density function in the calculation of the center of mass?

Worksheet

A wire takes the shape of the semicircle $x^2 + y^2 = 1, y > 1$ and is thicker near its base than near the top. Find the center of mass of the wire if the linear density at any point is proportional to its distance from the line $y = 1$.

Worksheet

Evaluate

$$\int_C y^2\,dx + x\,dy\,, \qquad i = 1, 2$$

1. where C is the line segment from $(-5,-3) \to (0,2)$
2. where C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3) \to (0,2)$.
3. repeat the above two steps with

$$P = x\,, \qquad Q = y\,.$$

Line integrals for vector fields

Definition
Let $\mathbf{F}$ be a continuous vector field defined on a curve C. Then the **line integral of F along** C is defined as

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds\,,$$

where $\mathbf{T}$ is the unit tangent vector.

Proposition

Suppose C is smooth and parametrized by $\mathbf{r}(t), a \leq t \leq b$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt\,.$$

Notations

$$\int_C f(x,y)dx := \int_a^b f(x(t),y(t))\,x'(t)\,dt\,,$$
$$\int_C f(x,y)dy := \int_a^b f(x(t),y(t))\,y'(t)\,dt\,,$$

We can abbreviate the above by

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)\,dx + Q(x,y)\,dy\,.$$

Note: when integrating with respect to arc length like this, reverse the direction of traversing the curve C will NOT result in a change of sign of the final solution.

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds\,.$$

On the other hand,

$$\int_{-C} P(x,y)\,dx + Q(x,y)\,dy = -\int_C P(x,y)\,dx + Q(x,y)\,dy\,.$$

Worksheet

Work is: `https://www.youtube.com/watch?v=oQqskrRWGco`

A force field is given $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$. Suppose we want to move a particle along the quarter circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, 0 \leq t \leq \pi/2$. Compute the work done.

Theorem (Fundamental Theorem for line integrals)
Let C be a smooth curve given by the parametrization $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$