ON OSCULATING CIRCLE AND CURVATURE IN 2D

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1. Introduction

In this expository note, we derive the relationship between curvature and the radius of the osculating circle. In particular, let p be a point on a twice differentiable curve, R be the radius of the osculating circle at that point and κ is the curvature of the curve at that point. Then,

$$R = \frac{1}{\kappa}\,.$$

There is an article by Fuchs [Fuc13] about this for those who are interested in the geometry of curves.

From Wikipedia, the definition of an osculating circle is the following.

Definition 1.1. The osculating circle of a sufficiently smooth plane curve at a given point p on the curve has been traditionally defined as the circle passing through p and a pair of additional points on the curve infinitesimally close to p.

2. Derivation

Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be a space curve that is twice differentiable. For a point $A = \mathbf{r}(t)$ on the curve, where $t \in (a, b)$, we have two other points, $B = \mathbf{r}(t - \varepsilon)$ and $C = \mathbf{r}(t + \varepsilon)$ given that ε is small enough.

Furthermore, given any 3 points in $\mathbb{R}^2$, there exists a unique circle that go through all 3 of them. To determine this circle, we build two center lines[1] that divide the segments AB and AC. The center of the circle that passes through A, B, C would be the intersection of these two lines.

Mathematically, denote $c^\varepsilon(t)$ to be the center of the circle passing through A, B, C. We have the following pair of equations

$$\left(c^\varepsilon(t) - \frac{\mathbf{r}(t) + \mathbf{r}(t-\varepsilon)}{2}\right) \cdot (\mathbf{r}(t) - \mathbf{r}(t-\varepsilon)) = 0$$

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[1] A center line of a segment is the line that is perpendicular at the center of that segment

$$\left(c^{\varepsilon}(t) - \frac{\mathbf{r}(t) + \mathbf{r}(t+\varepsilon)}{2}\right) \cdot (\mathbf{r}(t) - \mathbf{r}(t+\varepsilon)) = 0$$

Assume that there is a limit $c^{\varepsilon}(t) \to c(t)$ as $\varepsilon \to 0$ ($c(t)$ will be the center of the osculating circle). Dividing everything by ε and letting $\varepsilon \to 0$, we have

$$(\mathbf{c}(t) - \mathbf{r}(t))\, \mathbf{r}'(t) = 0\,.$$

This implies that the center of the osculating circle lies on the normal direction of the curve. The nice thing about 2D is that, we can produce a formula for the normal direction of the curve at $\mathbf{r}(t)$ explicitly. Let $\mathbf{n}(t) = \langle -r_2'(t), r_1'(t)\rangle$. Then, $\mathbf{n}(t) \cdot \mathbf{r}'(t) = 0$.

For $s \neq t$, the two normal lines at $\mathbf{r}(t)$ and $\mathbf{r}(s)$ will intersect at a point $P(s)$. Furthermore,

$$\lim_{s \to t} P(s) = \mathbf{c}(t)\,.$$

Let's now find the formula for $P(s)$. In particular, let

$$P(s) = \mathbf{r}(t) + \eta \mathbf{n}(t) = \mathbf{r}(t) + \lambda \mathbf{n}(s)\,.$$

This means,

$$\mathbf{r}(t) + \eta\langle -r_2'(t), r_1'(t)\rangle = \mathbf{r}(s) + \lambda\langle -r_2'(s), r_1'(s)\rangle\,.$$

Writing this in the form of system of equations, we have

$$\begin{cases} r_1(t) - \eta r_2'(t) & = r_1(s) - \lambda r_2'(s) \\ r_2(t) + \eta r_1'(t) & = r_2(s) + \lambda r_1'(s) \end{cases}$$

Multiplying the first equation by $r_1'(s)$ and the second equation by $r_2'(s)$, we have

$$\begin{cases} (r_1(t) - \eta r_2'(t)) r_1'(s) & = (r_1(s) - \lambda r_2'(s)) r_1'(s) \\ (r_2(t) + \eta r_1'(t)) r_2'(s) & = (r_2(s) + \lambda r_1'(s)) r_2'(s) \end{cases}$$

Adding both equations, we have

$$(r_1(t) + r_2(t)) + \eta(-r_2'(t) r_1'(s) + r_1'(t) r_2'(s)) = r_1(s) r_1'(s) + r_2(s) r_2'(s)\,.$$

Therefore,

$$\begin{aligned} \eta &= \frac{r_1(s) r_1'(s) + r_2(s) r_2'(s) - r_1(t) r_1'(s) - r_2(t) r_2'(s)}{-r_2'(t) r_1'(s) + r_1'(t) r_2'(s)} \\ &= \frac{r_1'(s) \frac{r_1(s) - r_1(t)}{s-t} + r_2'(s) \frac{r_2(s) - r_2(t)}{s-t}}{r_1'(s) r_2'(s) - r_2'(t) r_1'(s) + r_1'(t) r_2'(s) - r_1'(s) r_2'(s)} \\ &= \frac{r_1'(s) \frac{r_1(s) - r_1(t)}{s-t} + r_2'(s) \frac{r_2(s) - r_2(t)}{s-t}}{r_1'(s) \frac{r_2'(s) - r_2'(t)}{s-t} + \frac{r_1'(t) - r_1'(s)}{s-t} r_2'(s)}\,. \end{aligned}$$

Letting $s \to t$, we have

$$\lim_{s\to t} \eta = \frac{(r_1'(t))^2 + (r_2'(t))^2}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)} \,.$$

Therfore,

$$\begin{aligned}\lim_{s\to t} P(s) &= \mathbf{r}(t) + \frac{(r_1'(t))^2 + (r_2'(t))^2}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)} \langle -r_2'(t), r_1'(t)\rangle \\ &= \mathbf{r}(t) + \frac{|\mathbf{r}'(t)|^{3/2}}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)} \mathbf{u}(t) \,,\end{aligned}$$

where $\mathbf{u}(t)$ is the unit normal vector at $\mathbf{r}(t)$. It is now easy to verify that

$$\frac{|\mathbf{r}'(t)|}{|\mathbf{T}'(t)|} = \frac{|\mathbf{r}'(t)|^{3/2}}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)}$$

as it's a routine algebraic manipulation.

Therefore, the radius of the osculating circle is

$$R = \frac{|\mathbf{r}'(t)|}{|\mathbf{T}'(t)|} \,.$$

From Stewart [Ste15], the curvature of the curve is

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{R} \,.$$

References

[Fuc13] Dmitry Fuchs. "Evolutes and Involutes of Spatial Curves". In: *The American Mathematical Monthly* 120.3 (Mar. 2013), pp. 217–231. DOI: 10.4169/amer.math.monthly.120.03.217.

[Ste15] James Stewart. *Calculus*. Cengage Learning, 2015, p. 1392. ISBN: 9781285740621.

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