# ON OSCULATING CIRCLE AND CURVATURE IN 2D

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## 1. Introduction

In this expository note, we derive the relationship between curvature and the radius of the osculating circle. In particular, let $p$ be a point on a twice differentiable curve, $R$ be the radius of the osculating circle at that point and $\kappa$ is the curvature of the curve at that point. Then,

$$R = \frac{1}{\kappa}.$$

From Wikipedia, the definition of an osculating circle is the following.

**Definition 1.1.** The osculating circle of a sufficiently smooth plane curve at a given point $p$ on the curve has been traditionally defined as the circle passing through $p$ and a pair of additional points on the curve infinitesimally close to $p$.

## 2. Derivation

Let $\mathbf{r} : [a, b] \to \mathbb{R}^2$ be a space curve that is twice differentiable. For a point $A = \mathbf{r}(t)$ on the curve, where $t \in (a, b)$, we have two other points, $B = \mathbf{r}(t - \varepsilon)$ and $C = \mathbf{r}(t + \varepsilon)$ given that $\varepsilon$ is small enough.

Furthermore, given any 3 points in $\mathbb{R}^2$, there exists a unique circle that go through all 3 of them. To determine this circle, we build two center lines[1] that divide the segments $AB$ and $AC$. The center of the circle that passes through $A, B, C$ would be the intersection of these two lines.

Mathematically, denote $c^\varepsilon(t)$ to be the center of the circle passing through $A, B, C$. We have the following pair of equations

$$\left( c^\varepsilon(t) - \frac{\mathbf{r}(t) + \mathbf{r}(t - \varepsilon)}{2} \right) \cdot (\mathbf{r}(t) - \mathbf{r}(t - \varepsilon)) = 0$$

$$\left( c^\varepsilon(t) - \frac{\mathbf{r}(t) + \mathbf{r}(t + \varepsilon)}{2} \right) \cdot (\mathbf{r}(t) - \mathbf{r}(t + \varepsilon)) = 0$$

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[1]A center line of a segment is the line that is perpendicular at the center of that segment

Assume that there is a limit $c^\varepsilon(t) \to c(t)$ as $\varepsilon \to 0$ ($c(t)$ will be the center of the osculating circle). Dividing everything by $\varepsilon$ and letting $\varepsilon \to 0$, we have

$$\left(\mathbf{c}(t) - \mathbf{r}(t)\right)\mathbf{r}'(t) = 0\,.$$

This implies that the center of the osculating circle lies on the normal direction of the curve. The nice thing about 2D is that, we can produce a formula for the normal direction of the curve at $\mathbf{r}(t)$ explicitly. Let $\mathbf{n}(t) = \langle -r_2'(t), r_1'(t) \rangle$. Then, $\mathbf{n}(t) \cdot \mathbf{r}'(t) = 0$.

For $s \neq t$, the two normal lines at $\mathbf{r}(t)$ and $\mathbf{r}(s)$ will intersect at a point $P(s)$. Furthermore,

$$\lim_{s \to t} P(s) = \mathbf{c}(t)\,.$$

Let's now find the formula for $P(s)$. In particular, let

$$P(s) = \mathbf{r}(t) + \eta\mathbf{n}(t) = \mathbf{r}(t) + \lambda\mathbf{n}(s)\,.$$

This means,

$$\mathbf{r}(t) + \eta\langle -r_2'(t), r_1'(t) \rangle = \mathbf{r}(s) + \lambda\langle -r_2'(s), r_1'(s) \rangle\,.$$

Writing this in the form of system of equations, we have

$$\begin{cases} r_1(t) - \eta r_2'(t) &= r_1(s) - \lambda r_2'(s) \\ r_2(t) + \eta r_1'(t) &= r_2(s) + \lambda r_1'(s) \end{cases}$$

Multiplying the first equation by $r_1'(s)$ and the second equation by $r_2'(s)$, we have

$$\begin{cases} (r_1(t) - \eta r_2'(t))r_1'(s) &= (r_1(s) - \lambda r_2'(s))r_1'(s) \\ (r_2(t) + \eta r_1'(t))r_2'(s) &= (r_2(s) + \lambda r_1'(s))r_2'(s) \end{cases}$$

Adding both equations, we have

$$(r_1(t) + r_2(t)) + \eta(-r_2'(t)r_1'(s) + r_1'(t)r_2'(s)) = r_1(s)r_1'(s) + r_2(s)r_2'(s)\,.$$

Therefore,

$$\begin{aligned} \eta &= \frac{r_1(s)r_1'(s) + r_2(s)r_2'(s) - r_1(t)r_1'(s) - r_2(t)r_2'(s)}{-r_2'(t)r_1'(s) + r_1'(t)r_2'(s)} \\ &= \frac{r_1'(s)\frac{r_1(s)-r_1(t)}{s-t} + r_2'(s)\frac{r_2(s)-r_2(t)}{s-t}}{r_1'(s)r_2'(s) - r_2'(t)r_1'(s) + r_1'(t)r_2'(s) - r_1'(s)r_2'(s)} \\ &= \frac{r_1'(s)\frac{r_1(s)-r_1(t)}{s-t} + r_2'(s)\frac{r_2(s)-r_2(t)}{s-t}}{r_1'(s)\frac{r_2'(s)-r_2'(t)}{s-t} + \frac{r_1'(t)-r_1'(s)}{s-t}r_2'(s)}\,. \end{aligned}$$

Letting $s \to t$, we have

$$\lim_{s \to t} \eta = \frac{(r_1'(t))^2 + (r_2'(t))^2}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)}\,.$$

Therfore,

$$\lim_{s \to t} P(s) = \mathbf{r}(t) + \frac{(r_1'(t))^2 + (r_2'(t))^2}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)} \langle -r_2'(t), r_1'(t) \rangle$$

$$= \mathbf{r}(t) + \frac{|\mathbf{r}'(t)|^{3/2}}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)} \mathbf{u}(t),$$

where $\mathbf{u}(t)$ is the unit normal vector at $\mathbf{r}(t)$. It is now easy to verify that

$$\frac{|\mathbf{r}'(t)|}{|\mathbf{T}'(t)|} = \frac{|\mathbf{r}'(t)|^{3/2}}{r_1'(t)r_2''(t) - r_1''(t)r_2'(t)}$$

as it's a routine algebraic manipulation.

Therefore, the radius of the osculating circle is

$$R = \frac{|\mathbf{r}'(t)|}{|\mathbf{T}'(t)|}.$$

From Stewart, the curvature of the curve is

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{R}.$$

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