MATH 104: WORKSHEET 7

1. Concepts

(1) Arclength

$$L = \int_a^b |\mathbf{r}'(t)|\, dt\,.$$

(2) Curvature

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

where $\mathbf{T}(t)$ is the unit tangent vector.

2. Discussions

Question 1. Consider the path $\mathbf{r}(t) = \langle 14t, 7t^2, 7\ln t\rangle$.
Find the length of the curve between $(14, 7, 0)$ and $(70, 175, 7\ln 5)$.

Question 2. Find the perimeter of the ellipse:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1\,.$$

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Question 3. Find the curvature $\kappa(t)$ of the curve

$$\mathbf{r}(t) = (3\sin t)\mathbf{i} + (3\sin t)\mathbf{j} - \cos t\mathbf{k}\,.$$

Question 4. From the arclength formula, define

$$\ell(t) = \int_a^t |\mathbf{r}'(q)|\,dq\,.$$

Observe, $L = \ell(b)$ and for a given $s \in [0, \ell(b)]$ we have a unique t so that $s = \ell(t)$. That means, $t = \ell^{-1}(s)$. Therefore, we can write

$$\mathbf{r}(t) = \mathbf{r}(\ell^{-1}(s))\,.$$

The RHS is called the parametrization by arclength because the RHS is a function by the arclength.

Show that

(1)

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

(2)

$$\left|\frac{d}{ds}\mathbf{r}(t)\right| = 1$$

(3)

$$\ell(s) = \int_0^s \left|\frac{d}{ds}\mathbf{r}(t)\right| \, dt = s\,.$$

Question 5. Show that in the parametrization by arclength,

$$\kappa(s) = \left|\frac{d\mathbf{T}(t)}{ds}\right|$$

Question 6 (Hard). Show that

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Solution is on page 865 in the book.