

Multivariable Calculus

Day 2

Linear Algebra (cont.)

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Dot product

Definition

The dot product of vectors $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ in \mathbb{R}^n is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \dots + u_n v_n .$$

Dot product: Geometric meaning

Theorem

If θ is the angle between the vectors \mathbf{u} and \mathbf{v} , then

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta .$$

Dot product: Geometric meaning

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Proof in \mathbb{R}^2 .
Homework



Dot product: Geometric meaning

Theorem

If θ is the angle between the vectors \mathbf{u} and \mathbf{v} , then

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta.$$

Proof in \mathbb{R}^2 .

Homework



Corollary

Two vectors \mathbf{u} and \mathbf{v} are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.

Dot product: Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$\text{comp}_{\mathbf{v}} \mathbf{u} =$$

and the projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} =$$

Example

- ① Let $\mathbf{a} = \langle 3, 0, -1 \rangle$. Find a vector \mathbf{b} such that $\text{comp}_{\mathbf{a}}\mathbf{b} = 2$.
- ② Suppose \mathbf{a} and \mathbf{b} are nonzero vectors. When would it be true that $\text{comp}_{\mathbf{a}}\mathbf{b} = \text{comp}_{\mathbf{b}}\mathbf{a}$?
- ③ When would it be true that $\text{proj}_{\mathbf{a}}\mathbf{b} = \text{proj}_{\mathbf{b}}\mathbf{a}$?

3D special: Cross product

3D special: Cross product

WARNING

This concept is very specific to \mathbb{R}^3 . It will not make sense in other dimensions.

3D special: Cross product

Definition

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The cross product of \mathbf{a} and \mathbf{b} is defined to be

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle.$$

Worksheet

Given a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

the determinant of the matrix A is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

Question

Relate the cross product of vectors \mathbf{u} and \mathbf{v} with the determinant of a 3×3 matrix.

Suppose $\mathbf{u} = \langle 0, 1, 3 \rangle$, $\mathbf{v} = \langle 2, -1, 0 \rangle$.

- ① Find the cross products $\mathbf{u} \times \mathbf{v}$
- ② Find the cross products $\mathbf{v} \times \mathbf{u}$
- ③ Compute $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{v} \cdot \mathbf{u} \times \mathbf{v}$.
- ④ Compute $\mathbf{u} \times \mathbf{u}$.

3D special: Cross product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be vectors in \mathbb{R}^3 and $c \in \mathbb{R}$. Then

- ① (anti-symmetry) $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- ② $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$
- ③ $(c\mathbf{u}) \times \mathbf{w} = c(\mathbf{u} \times \mathbf{w}) = \mathbf{u} \times (c\mathbf{w})$
- ④ $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if \mathbf{u} and \mathbf{v} are parallel
- ⑤ **WARNING:** in general $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} \neq \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$

3D special: Cross product

Theorem

Let θ be the angle between \mathbf{a} and \mathbf{b} . Then,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta .$$

Theorem

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

3D special: Cross product

Theorem

The length, $|\mathbf{u} \times \mathbf{v}|$, of the cross product of vectors \mathbf{u} and \mathbf{v} is the area of the parallelogram determined by \mathbf{u} and \mathbf{v} .

Example

Find the area of the parallelogram created by two vectors $\langle 1, 3, -2 \rangle$ and $\langle 3, 0, 1 \rangle$.

- ① Prove the above theorem.
- ② Find the area of the parallelogram in \mathbb{R}^3 whose vertices are $(1, 0, 1)$, $(0, 0, 1)$, $(2, 1, 0)$, $(1, 1, 0)$. Hint: maybe draw it out.