MATH 104: Multivariable Calculus (brief notes)

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1 Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n-dimensional vectors doesn't require much more effort, we will talk about n-dimensional vectors instead.

Definition 1.1. An n-dimensional Euclidean space $\mathbb{R}^n$ is the Cartesian product of n Euclidean spaces $\mathbb{R}$.

Definition 1.2. An n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \dots, v_n \rangle , \tag{1}$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in $\mathbb{R}^n$ is also represented by an n-tuple but with round brackets. A vector connecting two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle .$$

We denote the above vector as $\vec{AB}$ where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle .$$

Definition 1.3. The length of a vector $\mathbf{v}$ (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} . \tag{2}$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b} ,$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} .$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1\mathbf{e_1} + \cdots + u_n\mathbf{e_n}\,,$$

where $e_i = \langle 0, \ldots, 1, \ldots, 0\rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e_1} = \mathbf{i}\,, \qquad \mathbf{e_2} = \mathbf{j}\,, \qquad \mathbf{e_3} = \mathbf{k}\,.$$

Properties of vector operations

Read the book

(Make sure you understand the geometric intepretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \ldots, u_n\rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n\rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + \cdots + u_nv_n\,.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$,
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u})\cdot\mathbf{w} = c(\mathbf{u}\cdot\mathbf{w})$.

Theorem 1.1 (Law of cosine)**.** *If θ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then*

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta\,.$$

Corollary 1.1. *Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\,,$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\,.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \,.$$

Theorem 1.2. *Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \,.$$

Theorem 1.3. *The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*

2 Some basic equations in $\mathbb{R}^3$

Just to build some toy examples for the future, we will play with some basic equations in three dimensions.

2.1 Equations for lines

A line is a collection of points that is parallel to a vector and goes through a specific point. To capture this idea, we have the following representation for a line

$$L = \{\mathbf{r}(t) \,|\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, t \in \mathbb{R}\},$$

where r_0 is the initial position and $\mathbf{v}$ is the direction. The equation for $\mathbf{r}(t)$ is called a **vector equation for a line** L.

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$. The **parametric equations** of L is the following system of equations

$$\begin{aligned} x &= x_0 + v_1 t\,, \\ y &= y_0 + v_2 t\,, \\ z &= z_0 + v_3 t\,. \end{aligned}$$

This leads to the **symmetric equations** of L

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}\,.$$

Definition 2.1. Two lines are parallel if their directional vectors are parallel (scalar multiple of each other).

Two lines that are not parallel and don't intersect each other are said to be skew.

2.2 Equations for planes

A plane is a collection of points that is perpendicular to one specific direction represented by a some vector called a **normal vector**. Note that due to scaling, there are more than one normal vector. To capture this idea, we have the following representation of a plane

$$P = \{\mathbf{r} \,|\, \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}\,.$$

This is called a **vector equation for the plane** P.

Multiplying things out, we have the **scalar equation of the plane** P with normal vector $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ through a point $P_0(x_0, y_0, z_0)$

$$n_1(r_1 - x_0) + n_2(r_2 - y_0) + n_3(r_3 - z_0) = 0\,.$$

The equation of the form

$$ax + by + cz + d = 0$$

is called a **linear equation**.

Definition 2.2. Two planes are said to be parallel if their normal vectors are parallel. If two planes are not parallel, they intersect in a straight line and the angle between the two planes is defined to be the angle between the two normal vectors.

2.3 Cylinders

Definition 2.3. A cylinder is a surface that consists of all lines (called **rulings**) that are parallel to a given line.

Example 2.1.

1. $z = x^2$
2. $x^2 + y^2 = 1$

2.4 Quadric surfaces

Definition 2.4. A quadric surface is the graph of a second-degree equation in three variables x, y and z. The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E\,.$$

Example 2.2.

1. Ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$
2. Hyperbolic paraboloid
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}\,.$$
3. Elliptical cone
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\,.$$

Read the books for more surfaces and pictures.

3 Functions in higher dimensions

Reading: Stewart Chapter 12, 13, Thomas Calculus Chapter 12, 13, Active Calculus Chapter 9

3.1 Functions of several variables

Definition 3.1. A function of several variables is a function $f : D \to C$ where $D \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$, where $m \geq 2$ and $n \geq 1$.

$$f(x) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)).$$

D is called the domain of f and C is called the codomain of f.

The domain of f is where each of the component f_i of f is defined.

Example 3.1. The following are some examples of multivariable functions

1. $f(x, y) = x^2 - 2xy + y^2$
2. $f(x, y, z) = \frac{1}{1-xy^2}$

3.2 Vector functions

3.2.1 Limit, continuity and differentiation

The expression in the vector equation for a line is an example of a function that maps from $\mathbb{R}$ to $\mathbb{R}^n$. There's no one who would stop us from considering more general kinds of function.

Definition 3.2. A **vector function** (**vector-valued function**) is a function that has the codomain that belongs to $\mathbb{R}^n$ where $n \geq 2$. In other words, $f : D \to \mathbb{R}^n$.

Example 3.2. The following are some examples of vector functions.

1. $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
2. $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$

Note that my definition is more general than that in the book. However, **In this course, whenever we talk about vector valued function, we will only refer to that which has one dimensional domain ($D \subseteq \mathbb{R}$).**

By and large, there's nothing different between a vector function and a one-variable scalar function. All the concepts such as limit, continuity and differentiability are applied to each coordinate the same way as in one dimensional case.

Theorem 3.1. *Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, given by $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$. Then, $\mathbf{r}$ is said to be continuous at t_0 if*

$$\mathbf{r}(t_0) = \lim_{t \to t_0} \mathbf{r}(t),$$

where

$$\lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} r_1(t), \ldots, \lim_{t \to t_0} r_n(t) \rangle.$$

Furthermore, we can define the derivative of $\mathbf{r}$

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

When $\mathbf{r} : I \to \mathbb{R}^n$ (I is an interval in $\mathbb{R}$) is continuous, we call it a **space curve** (to describe the intuitive picture of what a curve should look like in our mind).

Geometrically, if $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$, it represents the **tangent vector** of the curve $\mathbf{r}$ at t.

Definition 3.3. A **parametric equation** for a curve is an equation of the form

$$x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,.$$

Typical differentiation rules apply.

Theorem 3.2 (Differentiation rules)**.**

1. $(\mathbf{u}(t) + \mathbf{v}(t))' = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $(c\mathbf{u}(t))' = c\mathbf{u}'(t)$
3. $(f(t)\mathbf{u}(t))' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $(\mathbf{u}(f(t)))' = \mathbf{u}'(f(t))f'(t)$

3.2.2 Integrals

There are different ways to play with integrals for vector functions, each has its own interpretation and physical applications.

3.2.2.1 Indefinite integral

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \int_a^b r_3(t)\,dt \right\rangle$$

3.2.2.2 Arc Length and curvature

Definition 3.4. The length the curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is defined to be

$$L = \int_a^b |\mathbf{r}'(t)|\;dt\,.$$

If one wants to keep track the length of the curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ made by an airplane at any time t, one uses the **arc length function**

$$\ell(t) = \int_a^t |\mathbf{r}'(u)|\;du\,.$$

Re-parametrize with respect to arc length The nice thing about $\ell(t)$ is that it is a strictly increasing function with respect to t, given that $\mathbf{r}'$ is non-zero for all t. Therefore, letting $s = \ell(t)$, we can talk about the inverse of ℓ, $\ell^{-1} : [0, L] \to [a, b]$

$$t = \ell^{-1}(s)\,.$$

Therefore, we can re-write

$$\mathbf{r}(t) = \mathbf{r}(\ell^{-1}(s))\,.$$

Theorem 3.3.

$$\left|\frac{dr(t)}{ds}\right| = 1\,.$$

Thus,

$$l(s) = \int_0^s \left|\frac{d}{ds}\mathbf{r}(t)\right|\;dt = s\,.$$

Because of the unchanging nature of the arc-length (with respect to the parametrization), it is used to define a geometric quantity of a space curve called **curvature**.

Definition 3.5 (curvature)**.** Let $\mathbf{T}(t)$ be the unit tangent vector of the curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$. The curvature of $\mathbf{r}(t(s))$ is defined to be

$$\kappa(s) = \left| \frac{d\mathbf{T}(t(s))}{ds} \right| .$$

To convert this into the parameter t, we write $s = s(t)$ and use chain rule to get.

Theorem 3.4. *We have that*

$$\kappa(s(t)) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} .$$

3.2.3 Space curve in $\mathbb{R}^3$ and motion in space

Read the book. This part is not required but it is so beautiful, you may want to read it as an exercise at home (to test how much you understand what we've been discussing so far).

3.3 Activity: on osculating circle and curvature

For those who are interested in the geometrical meaning of the curvature without having to accept from the book that the curvature is the inverse of the radius of the osculating circle, please take a look at https://github.com/sonv/MultiCalc/blob/main/Writing/latexbuild/osculating.pdf.

4 Partial derivatives

Read Stewart Chapter 14, Thomas Chapter 14,

We will study multivariable scalar functions

$$f : D \to \mathbb{R}\,,$$

where $D \subseteq \mathbb{R}^n$, $n \geq 2$.

4.1 Multivariable scalar function

The following definition is from Thomas's book.

Definition 4.1. Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued/scalar function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n. We also call the x_j's the function's input variables and call w the function's output variable.

As prototypes, we only focus on $n = 2$ and $n = 3$.

4.1.1 Graphs

Definition 4.2. The graph of function $f : D \to \mathbb{R}$ is the set of all points $(\mathbf{x}, f(\mathbf{x}))$, where $\mathbf{x} \in D$. Here $D \subseteq \mathbb{R}^n$.

For 2D, the graph of f is also called the **surface** $z = f(x_1, x_2)$.

We cannot visualize the graph of a 3D function since it will be a four dimensional object.

4.1.2 Level

Definition 4.3. In 2D, the ***c*-level curves** of a function f of two variables are curves with equations $f(x, y) = c$, where c is a constant.

In 3D, the ***c*-level surface** of a function f of three variables are surfaces with equations $f(x, y, z) = c$, where c is a constant.

4.2 Limits and continuity

The following definition is from Stewart.

Definition 4.4. Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.

Finding if a function has limit as a point in higher dimension is not as simple as the case for 1 dimension.

Determining whether a multivariable function has a limit sometimes is an art and it requires a lot of experiences and practice. However, there are certain rules that could help us.

Theorem 4.1. *Let L, M and k be real numbers and that*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\,, \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M\,.$$

We then have

1. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)+g(x,y)) = L+M,$
2. $\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL,$
3. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)g(x,y)) = LM,$
4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ *if* $M \neq 0,$
5. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)^p = L^p$ *for* $p > 0,$

Strategy to find out that a two-variable function does NOT have a limit.

If $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ as $(x,y)\to(a,b)$ along a path C_1, and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example 4.1. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^4}$ does not exist.

$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

4.3 Partial derivatives

Given a function $f(x,y)$. The partial derivative of f with respect to x and (a,b), denoted by $f_x(a,b)$, is defined to be

$$f_x(a,b) = \lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}\,.$$

Likewise, the partial derivative of f with respect to y and (a,b), denoted by $f_y(a,b)$, is defined to be

$$f_y(a,b) = \lim_{h\to 0} \frac{f(a,b+h)-f(a,b)}{h}\,.$$

Instead of thinking about derivative with respect to x, y, we could think about derivative with respect to the first and second direction. This way of thinking is a bit better when one thinks about higher dimension.

Notations. If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \partial_x f = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = f_1 = D_1 f = D_x f\,.$$

When you take partial derivatives, just treat other variables as constants and proceed as in the case of one dimension.

More generally, given a function $f(x_1,\ldots,x_n)$, its partial derivative with respect to the ith variable x_i is

$$f_{x_i}(x_1,\ldots,x_n) = \lim_{h\to 0} \frac{f(x_1,\ldots,x_{i-1},x_i+h,x_{i+1},\ldots,x_n) - f(x_1,\ldots,x_{i-1},x_i,x_{i+1},\ldots,x_n)}{h}\,.$$

From here, one can define higher partial derivatives such as the following

$$\partial^3_{x_1x_2x_2} f\,.$$

Note that the power over the symbol ∂ represents the order of derivatives.

Theorem 4.2 (Clairaut's Theorem)**.** *Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then*

$$f_{xy}(a, b) = f_{yx}(a, b)\,.$$

Some important notations Let $f : D \to \mathbb{R}$ be a function. We write the following, if exist,

$$\nabla f = \begin{bmatrix} \partial_{x_1} f \\ \vdots \\ \partial_{x_n} f \end{bmatrix}$$

$$\Delta f = \partial^2_{x_1} f + \ldots \partial^2_{x_n} f\,.$$

4.4 Differentiability

Let S be a surface that has equation $z = f(x, y)$. Let C_1 and C_2 be two different curves on a surface S intersect at a point $P(x_0, y_0, z_0)$.

Heuristically, "the" **tangent plane** to the surface S at point P is defined the be the plane that contains both of the tangent lines of both curves at P.

How do you know if there is only one tangent plane at a point? You actually don't know. That's why the following definition exists.

Definition 4.5 (Differentiability)**.** Let $f : D \to \mathbb{R}$ and $a \in \mathbb{R}^n$. Let $z = f(x)$ and $\Delta z = f(a + \Delta x) - f(a)$. Then f is **differentiable at** a if Δz can be expressed in the form

$$\Delta z = \sum_{i=1}^{n} \partial_i f(a)\Delta x_i + \epsilon_i \Delta x_i\,,$$

where $\epsilon_i \to 0$ as $\Delta x_i \to (0, 0)$ and $\Delta x_i = x_i - a_i$.

f is said to be **differentiable** if it is differentiable at every point on the domain.

The graph of a differentiable function f whose domain is two dimensional is called a **smooth surface**.

Let $\Delta x = (\Delta x_1, \ldots, \Delta x_n)$. Another reformulation of the definition of differentiability is that f is differentiable if

$$\lim_{|\Delta x| \to 0} \frac{\Delta z - \nabla f \cdot \Delta x}{|\Delta x|} = 0\,.$$

It would be a good exercise to see why this is equivalent with the definition above.

For some good intuition, please go to https://mathinsight.org/differentiability_multivariable_definition.

Theorem 4.3. *If the partial derivatives $\partial_i f$ $(i = 1, \ldots, n)$ exist near $a \in \mathbb{R}^n$ and are continuous at a, then f is differentiable at a.*

Theorem 4.4. *If a function $f(x)$ is differentiable at a then f is continuous at a.*

4.5 Chain rule

Theorem 4.5. *Let $f(x_1, \ldots, x_n), g_i(y_1, \ldots, y_m)$ $(i = 1, \ldots, n)$ be differentiable functions. Then,*

$$z(y_1, \ldots, y_m) = f(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m))$$

is differentiable and

$$\frac{\partial z}{\partial y_i} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial g_j}{\partial y_i} \,.$$

4.6 Directional derivative

Definition 4.6. Let $\mathbf{u} \in \mathbb{R}^n$. The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}$ at $a \in \mathbb{R}^n$ in the direction of $\mathbf{u}$ is the following limit (if exists)

$$D_{\mathbf{u}} f(a) = \lim_{h \to 0} \frac{f(a + h\mathbf{u}) - f(a)}{h} \,.$$

How can one compute directional derivative?

Theorem 4.6. *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable then*

$$D_{\mathbf{u}} f(a) = \nabla f(a) \cdot \mathbf{u} \,.$$

4.7 Tangent planes

Let's think about tangent planes in a more systematic way, based on the definition of a plane learned in the first chapter.

Recall the c-level surface of a function $f(x, y, z)$ is the collection

$$\{(x, y, z) | f(x, y, z) = c\} \,.$$

Definition 4.7. The tangent plane at the point $P(x_0, y_0, z_0)$ on the c-level surface of a differentiable f is the plane through P_0, normal to $\nabla f(x_0, y_0, z_0)$.

5 Optimization

5.1 First and second derivative tests

Read Stewart Chapter 14, Thomas Chapter 14,

We will study multivariable scalar functions

$$f : D \to \mathbb{R}\,,$$

where $D \subseteq \mathbb{R}^n$, $n \geq 2$.

Definition 5.1. A function $f : D \to \mathbb{R}$ has a **local maximum** at $\mathbf{x_0}$ if $f(\mathbf{x_0}) \geq f(\mathbf{x})$ for $\mathbf{x} \in B_\delta(\mathbf{x_0})$ for small enough δ. f has a **global maximum** at $\mathbf{x_0}$ if $f(\mathbf{x_0}) \geq f(\mathbf{x})$ for $\mathbf{x} \in D$. f has a **local (global) minimum** at $\mathbf{x_0}$ if $-f$ has a local (global) maximum at $\mathbf{x_0}$

Theorem 5.1 (First derivative test)**.** *Let $f : D \to \mathbb{R}$ be a function. If $\mathbf{x_0}$ is a local minimum and f has partial derivatives at $\mathbf{x_0}$. Then*

$$\partial_{x_i} f(\mathbf{x}_0) = 0\,.$$

The converse is not true, as having $\nabla f(\mathbf{x}_0) = \mathbf{0}$ does not mean that f has a local minimum at $\mathbf{x}_0$.

Exercise 5.1. Think of a function that the converse to the above theorem is not true.

This leads to the following notion.

Definition 5.2. $\mathbf{x}_0$ is said to be a **critical point** of $f : D \to \mathbb{R}$ if

$$\nabla f(\mathbf{x}_0) = 0$$

or one of the partial derivatives $\partial_{x_i} f(\mathbf{x}_0)$ fails to exist.

Please pay attention about the "fail to exist" condition.

Theorem 5.2 (Second derivative test for functions of 2 variables)**.** *Suppose the second partial derivatives of f are continuous near (a, b) and suppose that (a, b) is a critical point of f. Let*

$$D = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2\,.$$

1. *If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.*
2. *If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.*
3. *If $D < 0$, then $f(a, b)$ is neither a local maximum nor local minimum.*
4. *If $D = 0$, then we cannot conclude.*

Theorem 5.3 (Extreme value theorem)**.** *If f is continuous on a* closed *and* bounded *set D. Then, f attains an absolute minimum and an absolute maximum in D.*

5.1.1 Algorithm to find absolute maxima and minima on closed bounded regions

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

5.2 Constrained optimization

Constrained optimization takes various forms, depending on the assumptions. We will deal with the most straight forward form. The problem we will study is the following:

Maximize/minimize a function $f : D \to \mathbb{R}$, subject to a constraint (side condition) of the form $g(\mathbf{x}) = k$, for some constant $k \in \mathbb{R}$.

Theorem 5.4 (Method of Lagrange Multiplier)**.** *Suppose the maximum/minimum values of f exist and $\nabla g(\mathbf{x}) \neq 0$ where $g(\mathbf{x}) = k$. To find the maximum and minimum values of f subject to constraint $g(\mathbf{x}) = k$, we do the following:*

1. *Find all values of $\mathbf{x}$ and $\lambda \in \mathbb{R}$ such that*

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \, ,$$

and

$$g(\mathbf{x}) = k \, .$$

2. *Evaluate f at all the points $\mathbf{x}$ that result from step 1. The largest of these values is the maximum of f; the smallest is the minimum value of f.*

6 Multiple integrals

Read Stewart Chapter 15 and Thomas Chapter 15

Notations:

Rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$.

6.1 Basic definition

Definition 6.1. Let f be a function on a rectangle R. An n-fold Riemann sum for f over R is a sum of the following form

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(\xi_{i_1 \dots i_n}) \Delta A \,,$$

where

- $\Delta A = \Delta x_1 \times \cdots \times \Delta x_n$,
- $\Delta x_i = (b_i - a_i)/m_i$,
- $\xi_{i_1 \dots i_n} \in R_{i_1 \dots i_n}$,
- $R_{i_1 \dots i_n} = \prod [a_i + (i_1 - 1)\Delta x, a_i + i_1 \Delta x_i]$.

Definition 6.2. The double integral of f over a rectangle $R \subseteq \mathbb{R}^2$ is

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_{ij}) \Delta A$$

if the limit exists.

The triple integral of f over a rectangle $R \subseteq \mathbb{R}^3$ is

$$\iiint_R f(x, y)\, dA = \lim_{m,n,l \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(\xi_{ijk}) \Delta A$$

if the limit exists.

6.1.1 Some properties of integrals

1. Let U, V be disjoint domains, then

$$\iint_{U \cup V} f\, dA = \iint_U f\, dA + \iint_V f\, dA \,.$$

2.

$$\iint_U (f + g)\, dA = \iint_U f\, dA + \iint_V f\, dA \,.$$

6.2 Iterated integrals

Suppose that f is integrable on $R = [a, b] \times [c, d]$. An iterated integral of f is defined as

$$\int_a^b A(x)\, dx \,,$$

where

$$A(x) = \int_c^d f(x, y)\, dy \,.$$

Typically, we write the above as

$$\int_a^b \int_c^d f(x, y)\, dydx\,.$$

This means that we integrate in y before in $x-$ always integrate the inner part first.

Similarly, we can define an iterated integral in a different order

$$\int_c^d \int_a^b f(x, y)\, dxdy\,.$$

The biggest question: Is it true that

$$\int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy\,?$$

Theorem 6.1 (Special case of Fubini)**.** *If f is continuous on the rectangle R, then*

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy\,.$$

Example 6.1. Let

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}\,.$$

$$\int_0^1 \int_0^1 f(x, y)\, dydx = \frac{\pi}{4} = -\int_0^1 \int_0^1 f(x, y)\, dxdy\,.$$

Everything we discuss here is true for three-variable functions.

6.3 Change of coordinates

A coordinate transformation is a function φ, which is bijective and differentiable for which $D\varphi$ is invertible at all points in the domain. Here,

$$D\varphi = \begin{pmatrix} \partial_1\varphi_1 & \partial_2\varphi_1 \\ \partial_1\varphi_2 & \partial_2\varphi_2 \end{pmatrix}\,.$$

We will need to re-call the notion of invertible matrix here. For an $n \times n$ matrix A, it is invertible iff $\det A \neq 0$,.

Theorem 6.2. *Let f be a function of (x, y) defined on the domain D. Let*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(u, v)$$

for some coordinate change function $\varphi : D \to S$. If f is continuous and φ is differentiable, then

$$\int_S f\, dA = \int_D f \circ \varphi |\det D\varphi|\, dA$$

6.3.1 Applications of change of coordinates

6.3.1.1 Polar coordinate In $\mathbb{R}^2$, when the region of integration is a section of a disk centered at 0. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(r, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}\,,$$

where $a \leq r \leq b$ and $\alpha \leq \theta \leq \beta$.

6.3.1.2 Cylindrical coordinate

In $\mathbb{R}^3$, when the region of integration is part of a cylinder. Let

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \varphi(r, \phi, \theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix},$$

where $a \le r \le b$, $\alpha \le \theta \le \beta$.

6.3.1.3 Spherical coordinate

In $\mathbb{R}^3$, when the region of integration is a section of a ball centered at 0. Let

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \varphi(\rho, \phi, \theta) = \begin{pmatrix} \rho\sin\phi\cos\theta \\ \rho\sin\phi\sin\theta \\ \rho\cos\phi \end{pmatrix},$$

where $a \le \rho \le b$, $\alpha \le \theta \le \beta$, and $c \le \phi \le d$.

7 Vector Calculus

Read Chapter 16 in Stewart.

7.1 Vector fields

Definition 7.1. Let D be a domain on $\mathbb{R}^n$. A vector field on $\mathbb{R}^n$ is a function $\mathbf{F} : D \to \mathbb{R}^n$ that assign each point $\mathbf{x} \in D$ to a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$.

In $\mathbb{R}^2$, one typically write the vector fields in terms of **component functions** P, Q

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}\,.$$

In $\mathbb{R}^3$, one typically write the vector fields in terms of **component functions** P, Q, R

$$\mathbf{F}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + R(x, y, z)\mathbf{k}\,.$$

Example 7.1. Newton's Law of Gravitation

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x}\,,$$

where $\mathbf{x}$ is the position in $\mathbb{R}^3$.

Example 7.2. Coulomb's Law for the electric force exerted by an electric charge Q at the origin on another charge q at a point $\mathbf{x} \in \mathbb{R}^3$.

$$\mathbf{F}(\mathbf{x}) = \frac{\epsilon q Q}{|\mathbf{x}|^3}\mathbf{x}\,.$$

7.2 Line integrals

Let's focus on $\mathbb{R}^2$. We now perform a Riemann-sum-like action.

Definition 7.2. Let C be a curve. The **line integral of f along C** is defined as

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i\,,$$

where Δs_i is the length of a subarc of C.

Proposition 7.1. *Suppose C is smooth and is parametrized by $\mathbf{r}(t), a \le t \le b$. Then*

$$\int_C f(x, y)\, ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt\,.$$

Note: when integrating with respect to arc length like this, reverse the direction of traversing the curve C will NOT result in a change of sign of the final solution.

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds\,.$$

Now we define line integrals of vector fields.

Definition 7.3. Let $\mathbf{F}$ be a continuous vector field defined on a curve C. Then the **line integral of $\mathbf{F}$ along C** is defined as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}\, ds\,,$$

where $\mathbf{T}$ is the unit tangent vector.

Proposition 7.2. *Suppose C is smooth and parametrized by $\mathbf{r}(t), a \le t \le b$. Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

We also use the following notations

$$\int_C f(x,y)dx := \int_a^b f(x(t),y(t))\, x'(t)\, dt\,,$$
$$\int_C f(x,y)dy := \int_a^b f(x(t),y(t))\, y'(t)\, dt\,,$$

We can abbreviate the above by

$$\int_C P(x,y)dx + \int_C Q(x,y)dy = \int_C P(x,y)\, dx + Q(x,y)\, dy\,.$$

So,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy\,.$$

Note: as oppose to integrating the arc length, reversing the order of the above integrals will change the sign of the integral. This is because the arc length is always positive, while Δx and Δy could be either positive or negative.

$$\int_{-C} P(x,y)\, dx + Q(x,y)\, dy = -\int_C P(x,y)\, dx + Q(x,y)\, dy\,.$$

Theorem 7.1 (Fundamental Theorem for line integrals)**.** *Let C be a smooth curve given by the parametrization $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then,*

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))\,.$$

Definition 7.4. A **closed curve** is a curve that starts and ends at the same point. A **simple closed curve** is a closed curve that never crosses itself.

Sometimes, if C is a closed curve, we signify it by the following notation

$$\oint_C \nabla f \cdot d\mathbf{r}\,.$$

Corollary 7.1. *If C is a closed curve and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then*

$$\oint_C \nabla f \cdot d\mathbf{r} = 0\,.$$

Definition 7.5. A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is there exists a function f such that

$$\nabla f = \mathbf{F}\,.$$

Therefore, if $\mathbf{F}$ is a conservative vector field, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0\,.$$

7.2.1 Independence of path

Suppose C_1 and C_2 are two piecewise smooth curves that have the same initial point A and end point B. Then,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

whenever $\mathbf{F}$ is conservative (Why?). The question is when is the converse true?

The following example is an example when the converse is not always true.

Example 7.3. Evaluate

$$\int_{C_i} x^2 \, dy \,, \qquad i = 1, 2$$

where C_1 is the line segments from $(-1,-1) \to (-1,1) \to (1,1)$ and C_2 is the line segments from $(-1,-1) \to (1,-1) \to (1,1)$.

To further the discussion, we need a few definitions.

Definition 7.6. Let $\mathbf{F}$ be a continuous vector field with domain D, we say that the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for all paths that have the same starting and ending points.

Theorem 7.2. *$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_\Gamma \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path Γ in D.*

Definition 7.7. A domain D is said to be **open** if around each point, we can draw an open ball around it. A domain D is said to be **connected** if for any two points, there is a path that connect them together. A domain D is said to be **simply connected** if is connected and there's no hole in it.

Theorem 7.3. *Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D.*

Proof. todo □

The above theorem gives a way to determine if a vector field is conservative or not, from the point of view of path independence. However, it is often difficult to check the path independence property as one has to integrate over ALL possible curves, and there are a lot of them...

Another way is to take inspiration from Clairaut's theorem. The question is to determine whether $\mathbf{F}$ is conservative, given the mixed partial derivatives of P and Q are the same, i.e.,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \,.$$

(Compare this with Clairaut's)

Theorem 7.4. *Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply connected region D. Suppose that P and Q have continuous first-order partial derivatives and*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

through out D. Then $\mathbf{F}$ is conservative.

Remark. The connectedness of D is crucial (why?).

7.3 Green's Theorem

Theorem 7.5 (Green's Theorem)**.** *Let D be an open bounded simply connected domain in $\mathbb{R}^2$, Γ be the boundary of D, and $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field. If P and Q have continuous partial derivatives on an open region that contains D, then*

$$\int_\Gamma \mathbf{F} \cdot d\ell = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA\,.$$

7.4 Curl and Divergence

Definition 7.8. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$. If all partial derivatives of P, Q, R exist, then we define

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}\,.$$

A different notation for $\operatorname{curl} \mathbf{F}$ is

$$\nabla \times \mathbf{F}\,.$$

Theorem 7.6. *If f is a function of 3 variables that has continuous second partial derivatives, then*

$$\nabla \times (\nabla f) = 0\,.$$

Theorem 7.7. *Suppose $\mathbf{F}$ is a vector field on and simply connected domain D so that P, Q, R all have continuous partial derivatives. F is a conservative vector field if and only if $\nabla \times \mathbf{F} = 0$.*

Definition 7.9. Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$. If all partial derivatives of P, Q, R exist, then we define

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,.$$

A different notation for $\operatorname{div} \mathbf{F}$ is

$$\nabla \cdot \mathbf{F}\,.$$

Theorem 7.8. *Suppose $\mathbf{F}$ is a vector field on a domain D and P, Q, R have continuous second-order partial derivatives. Then,*

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0\,.$$

7.5 Surface integrals

7.5.1 Parametric surfaces

Similar to the way we parametrize a curve by a one-variable vector function $\mathbf{r}(t)$, we can parametrize a surface by a two-variable vector function $\mathbf{r}(u, v)$.

We will only deal with surfaces in $\mathbb{R}^3$ in this section. So, the parametrization of a surface S should be

$$\mathbf{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3\,.$$

We often write

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}\,.$$

From this parametrization, we get to talk about the tangent plane of S at the point $\mathbf{r}(u, v)$, which is the plane that contains two tangent vectors

$$\mathbf{r}_u(u, v) = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\,,$$

and

$$\mathbf{r}_v(u, v) = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\,.$$

7.5.2 Surface integral

Definition 7.10. Let S be a surface with parametrization. The surface integral of f over the surface S is

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}\,.$$

Similarly to the line integral, one can show that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r_u}\times\mathbf{r_v}|\,dA\,.$$

7.5.3 Orientation of the surface

Given a surface S, we define the orientation of it as following

1. If S has a boundary, then the **positive orientation** of the surface is that when one walks along the boundary of the surface with the head points in that direction, the surface is on the left.
2. If S does not have a boundary, then the **positive orientation** is the direction of the outward normal vector.

7.5.4 Surface integral of vector fields

Definition 7.11. If $\mathbf{F}$ is a continuous vector field on an oriented surface S (parametrized by $\mathbf{r}(u,v)$) with unit normal vector $\mathbf{n}$, then the **surface integral of F over** S is

$$\iint_S \mathbf{F}\cdot\,d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_S \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA\,.$$

The integral is called the **flux of F** across S.

7.6 Stokes' and Divergence Theorem

Theorem 7.9 (Stokes' Theorem)**.** *Let S be an oriented smooth surface that is bounded by a simple closed smooth boundary curve ∂S with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\int_{\partial S}\mathbf{F}\cdot\,d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}\,.$$

The boundary of an area is a curve. Similarly, the boundary of a solid is a surface.

Theorem 7.10 (Divergence Theorem)**.** *Let E be a simple solid region and let surface ∂E be the boundary of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives. Then,*

$$\iint_{\partial E}\mathbf{F}\cdot d\mathbf{S} = \iiint_E \mathrm{div}\mathbf{F}\,dV\,.$$