Multivariable Calculus
Day 8
Limits and Continuity

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Limits

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$.

Sadly, we will not go deeply about this concept because it requires real analysis and we have bigger fishes to fry.

What we will learn:

1. Simple cases when limits exist
2. Typical cases when limits don't exist

Simple cases when limits exist

Theorem
Let L, M and k be real numbers and that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\,, \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M\,.$$

We then have

1. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$,
2. $\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL$,
3. $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)g(x,y)) = LM$,
4. $\lim_{(x,y)\to(x_0,y_0)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ *if* $M \neq 0$,
5. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)^p = L^p$ *for* $p > 0$,

Worksheet

Determine if the limit exists and if it is, find it.

1. $\displaystyle \lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$,
2. $\displaystyle \lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2}$,
3. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$,
4. $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x}{y}$.

Two-path test

If $\lim_{(x,y)\to(a,b)} f(x,y) = L_1$ as $(x,y)\to(a,b)$ along a path C_1, and $\lim_{(x,y)\to(a,b)} f(x,y) = L_2$ as $(x,y)\to(a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Worksheet

Determine if the limit exists and if it is, find it.

1. $$\lim_{(x,y)\to(0,0)} \frac{x^2}{3x^2+4y^2}$$
2. $$\lim_{(x,y,z)\to(0,0,0)} \frac{4xy+2yz+2xz}{16x^2+4y^2+4z^2}$$
3. $$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$$

Continuity

Continuity

A function f is continuous at $\mathbf{a} \in \mathbb{R}^n$ if

$$\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})\,.$$

We say that f is continuous on D if f is continuous at every point $\mathbf{x} \in D$.