# MULTIVARIABLE CALCULUS
# DAY 11 WORKSHEET
# DIFFERENTIABILITY AND DIRECTIONAL DERIVATIVE

**Question 1.** *We will deal with implicit differentiation. Re-call in calculus, there are situations like the following*

$$y^2 + y = e^x \,, \tag{1}$$

*and you were asked to compute* $\frac{dy}{dx}$*. Now, you will learn how to do similar things with more variables.*

- *Compute* $\frac{dy}{dx}$ *for* (1)*.*
- *Suppose* $F(x, y, z) = 0$*. Use the chain rule to find a formula for* $\frac{\partial y}{\partial x}$ *(or* $\frac{\partial x}{\partial z}$*).*
- *Let* $z = \sin(x) + e^{xy}$*. Compute* $\frac{\partial x}{\partial y}$ *and* $\frac{\partial z}{\partial x}$*.*

**Question 2.** *Read section 10.5.2 of the book Active Calculus about the tree diagrams to represent the chain rule.* `https://activecalculus.org/multi/S-10-5-Chain-Rule.html`

*Complete Activity 10.5.3.*

**Question 3.** *Let's think about the following theorem.*

**Theorem 1.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function.

(1) If $f$ is differentiable at $\mathbf{x}_0$, then $f$ is continuous at $\mathbf{x}_0$.
(2) If $f$ is differentiable at $\mathbf{x}_0$, then $\partial_{x_i} f$ exists for every $i = 1, \ldots, n$.
(3) If $\partial_{x_i} f$ exists AND is continuous for all $i = 1, \ldots, n$, then $f$ is continuously differentiable (differentiable and the derivative is continuous (whatever that means)).

*To have some feel for this theorem, let's try the following. Compute directional derivative in the direction* $\mathbf{u} \neq \mathbf{0}$ *of the functions*

- $f(\mathbf{x}) = |\mathbf{x}|^2$.
- $$f(\mathbf{x}) = \begin{cases} \frac{x_1^2 x_2}{x_1^4 + x_2^2} & \mathbf{x} \neq 0 \\ 0 & \mathbf{x} = 0 \,. \end{cases}$$
- *Does having directional derivative in ALL direction* $\mathbf{u}$ *imply that the function* $f$ *is differentiable?*
- *Read the following article and make some sort of flow-chart / mind map to cover all of the connections between differentiability and partial derivative.* `https://mathinsight.org/differentiable_function_discontinuous_partial_derivatives`

*Date*: Spring 2023.

**Question 4.** *Re-call that Taylor's theorem for one dimensional a smooth function $f$ is*

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(b)(x-a)^2\,,$$

*where $b$ is a number in between $a$ and $x$.*

- *The first two terms of the Taylor's theorem gives us the linear approximation of $f$. This is a little bit puzzling. Without any restriction, a linear approximation should be a function $T(x) = mx + b$ where $T(a) = f(a)$. That means, any line that would coincide with $f$ at $a$ should be okay.*

  *However, you were taught that there's only one linear approximation. Surely, there must be other condition(s) that help determine that one line you were taught. Discuss among friends to find out what's missing.*
- *Find the linear approximation for $f(x) = e^{x^2}$.*
- *Generalize the previous two questions to find appropriate conditions to have a good linear approximation for the function $f(\mathbf{x}) = e^{|\mathbf{x}|^2}$, where $\mathbf{x} \in \mathbb{R}^2$.*
- *Find the linear approximation $L(\mathbf{x})$ for $f(\mathbf{x})$.*

**Question 5.** *Relate what you've just done in the previous question to the definition of differentiability. Explain the meaning of the definition in human language.*

**Question 6.** *Read more about differentials in Stewart Section 14.4 to get to know the terminology. This terminology is used a lot but it is just a convention with no clear definition. Most elementary calculus books (including the books we use) go as far as letting $dx = \Delta x$, which is absolutely horrendous.*