Multivariable Calculus
Day 20
Vector Calculus: Line integrals

Spring 2023

Warning

Don't be confused with arc length!!!

Definition
Let D be a domain on $\mathbb{R}^n$. A vector field on $\mathbb{R}^n$ is a function $\mathbf{F} : D \to \mathbb{R}^n$ that assign each point $\mathbf{x} \in D$ to a vector $\mathbf{F}(\mathbf{x}) \in \mathbb{R}^n$.

We only think about vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$.

Line integrals

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In $\mathbb{R}^2$, one typically write the vector fields in terms of **component functions** P, Q

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}\,.$$

In $\mathbb{R}^3$, one typically write the vector fields in terms of **component functions** P, Q, R

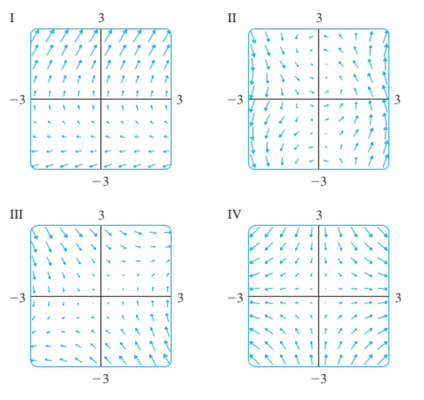
$$\mathbf{F}(x, y, z) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + R(x, y, z)\mathbf{k}\,.$$

Example

$$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}\,.$$

Matching the vector fields with the pictures

$\mathbf{F}(x,y) = \langle x, -y\rangle$, $\mathbf{F}(x,y) = \langle y, x-y\rangle$, $\mathbf{F}(x,y) = \langle y, y+2\rangle$, $\mathbf{F}(x,y) = \langle \cos(x+y), x\rangle$,



Line integrals

This is similar to integration of parametric curves but there are differences.

We now perform a Riemann-sum-like action.

Definition
Let C be a smooth curve. The **line integral of f along C** is defined as

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \,,$$

where Δs_i is the length of a subarc of C.

Worksheet

Suppose the curve C is given by the parametric equation

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle, \qquad t \in [a, b].$$

Find a formula for

$$\int_C f(x, y)\, ds$$

that can relate to the parametric equation. (Hint: use the arclength formula)

Example

1. Evalutate
$$\int_C (2 + x^2 y)\, ds$$
where C is the upper half of the unit circle.
2. Evaluate
$$\int_C 2x\, ds$$
where C consists of the arc C_1 of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ followed by the vertical line segment C_2 from $(1, 1)$ to $(1, 2)$.