

Multivariable Calculus

Day 24

Vector Calculus: Surface integrals

Spring 2023

Curl and divergence

<https://www.youtube.com/watch?v=rB83DpBJQsE>

Orientation of a surface

Given a surface S , we define the orientation of it as following

- ① If S has a boundary, then the **positive orientation** of the surface is that when one walks along the boundary of the surface with the head points in that direction, the surface is on the left.
- ② If S does not have a boundary, then the **positive orientation** is the direction of the outward normal vector.

$$\mathbf{r} : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^3 .$$

We often write

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} .$$

Surface integral

Definition

Let S be a surface with parametrization. The surface integral of f over the surface S is

$$\iint_S f(x, y, z) dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}.$$

Similarly to the line integral, one can show that

$$\iint_S f(x, y, z) dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| dA.$$

Surface integral of vector fields

Definition

If \mathbf{F} is a continuous vector field on an oriented surface S (parametrized by $\mathbf{r}(u, v)$) with unit normal vector \mathbf{n} , then the **surface integral of \mathbf{F} over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA.$$

The integral is called the **flux of \mathbf{F}** across S .

Stokes' Theorem

Theorem

Let S be an oriented smooth surface that is bounded by a simple closed smooth boundary curve ∂S with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

<https://www.youtube.com/watch?v=LqNqqidw2mg>

Divergence Theorem

Theorem

Let E be a simple solid region and let surface ∂E be the boundary of E , given with positive (outward) orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives. Then,

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV .$$

https://www.youtube.com/watch?v=TORt20_HjMY