MATH 104: Multivariable Calculus

Name: _

May 22, 2023

Rules

- 5 questions, 90 minutes
- Closed books
- Show all your work. Mere numbers for solutions will not count for grades.
- No sharing of calculators

Scores

Problem 1. _____/20

Problem 2. _____/20

Problem 3. _____/20

Problem 4. _____/20

Problem 5. _____/20

Total ____________/100

Questions

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$.

a. What does it mean for f to be differentiable at (a, b)?

b. What does it mean for f to have a directional derivative in the direction of $\mathbf{u}$? What's a notation for this notion?

c. Write directional derivative of function f in the direction $\mathbf{u}$ in terms of partial derivative/gradient of f.

d. What is the dimension of ∇f? What about $D_{\mathbf{u}} f$? What about Δf?

Problem 2. a. State Clairaut's theorem.

b. An example that was introduced in class but we didn't have time to cover in details. Hopefully you've done that at home. If you have, the following will be easy enough. Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$ for the following function

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}\,, & (x,y) \neq (0,0) \\ 0\,, & (x,y) = (0,0)\,. \end{cases}$$

c. Why doesn't the above function follow Clairaut's theorem?

Problem 3. 1. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, where $n \geqslant 2$. What are the properties of ∇f? (List everything that you can think of)

2. Given any surface $F(x, y, z) = C$, where F is differentiable everywhere, how do you know there's only one tangent plane at a given point? (Give the best answer you can)

3. Find the tangent plane to the surface

$$x^2 + 2xy - y^2 + z^2 = 7$$

at point

$$P_0(1, -1, 3)\,.$$

Problem 4. 1. Find the tangent vector, normal vector and curvature of the following curve

$$\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, 3 \rangle .$$

2. What's the meaning of the curvature of a curve at a point on the curve?

Problem 5. 1. State the second derivative test when you want to optimize a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$.

2. Find all the local maxima, local minima, and saddle points (neither min nor max) of the function

$$f(x, y) = \ln(x + y) + x^2 - y\,.$$