MATH 104: HOMEWORK 2

DUE DATE: IN CLASS – FRIDAY, FEB 2, 2024

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Problem 1. Consider the following planes in $\mathbb{R}^3$, where C is constant:

$$2Cx - 3y + (C+4)z = 5$$
$$(C+1)x + Cy - z = 1$$

(1) Find the value(s) of C that makes these planes orthogonal.
(2) Explain why if the planes are orthogonal at one intersection point they are orthogonal at all intersection points.

Problem 2. Consider the following hyperplanes in $\mathbb{R}^4$:

$$x_1 - 2x_2 + 5x_3 + x_4 = 8$$
$$3x_1 + 6x_2 + x_3 + 4x_4 = 17$$

Identify vectors orthogonal to each hyperplane and use these to show that the hyperplanes are orthogonal.

Problem 3. Consider the following planes in $\mathbb{R}^3$:

$$2x - 3y + z = 5$$
$$3x + y - 2z = -1$$

Identify vectors orthogonal to each plane and use these to compute a vector that is tangent to both planes.

Problem 4. Consider the following vector in $\mathbb{R}^5$:

$$\vec{v} = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -1 \\ 1 \end{pmatrix}$$

(1) Give an example of a nonzero vector that is orthogonal to $\vec{v}$.
(2) Compute the angle between the vector $\vec{v}$ and the basis vector $\hat{e}_1$.

Date: January 24, 2024.

Problem 5. Consider the following vectors in $\mathbb{R}^4$:

$$\vec{u} = \begin{pmatrix} 2 \\ 1 \\ -2 \\ 0 \end{pmatrix}, \vec{v} = \begin{pmatrix} 2 \\ 1 \\ 2 \\ -4 \end{pmatrix}$$

(1) Compute the angle between these two vectors..
(2) Find a vector that is orthogonal to both u and v.

Problem 6. Consider the following vectors in $\mathbb{R}^4$:

$$\vec{u} = \begin{pmatrix} 4 \\ -5 \\ 2 \\ -2 \end{pmatrix}, \vec{v} = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix}$$

(1) Compute the angle between these two vectors.
(2) Compute the projected length of $\vec{u}$ onto the " v-axis".

Problem 7. Consider the following four vectors in $\mathbb{R}^3$:

$$\vec{a} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \vec{b} = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}, \vec{c} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \vec{d} = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$$

(1) Is there a pair of orthogonal vectors among the above? Explain.
(2) Which three of the vectors above span a parallelopiped with the largest volume?

Problem 8. Consider the following two curves in $\mathbb{R}^3$:

$$\gamma_1(s) = \begin{pmatrix} s^2 - 3s \\ e^s - 1 \\ 1 - \cos 2s \end{pmatrix}, \quad \gamma_2(t) = \begin{pmatrix} \sin(t-1) \\ t^2 + t - 2 \\ 1 - \sqrt{t} \end{pmatrix}$$

(1) Verify that these curves intersect at the origin for some values of s and t.
(2) At what angle do these curves intersect at the origin?
(3) Find a vector that is orthogonal to both curves at the origin.

Problem 9. Consider the parametrized curve in 3-D given by

$$\vec{\gamma}(t) = \begin{pmatrix} t^2 - t + 4 \\ t^3 - 3t^2 + 2t - 1 \\ 2t \end{pmatrix} \begin{array}{l} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{array}$$

(1) Compute the velocity vector of this curve.
(2) Write down the equation of a plane orthogonal to this curve at $\vec{\gamma}(0)$.

(3) At what angle does this curve cross the (x, y) plane $z = 0$? Explain your reasoning and give your answer as best you can without a calculator.

Problem 10. Consider the following matrix and vectors:

$$A = \begin{bmatrix} 1 & -1 & 2 & 2 \\ -1 & 3 & 0 & 1 \\ 2 & 1 & -3 & 1 \\ 4 & 0 & 1 & 0 \end{bmatrix}, \vec{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix}, \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \vec{w} = \begin{pmatrix} -2 \\ -1 \\ 1 \\ -3 \end{pmatrix}$$

(1) Evaluate the dot product $\vec{v} \cdot A\vec{u}$ if possible. If not, explain why not.
(2) Compute $A\vec{u} + A\vec{v} + A\vec{w}$.
(3) Compute the quantity $\left(\vec{v}\vec{w}^T\right)\vec{u}$ if it exists; if not, explain why not.

Problem 11. Consider the following matrices / vectors:

$$A = \begin{bmatrix} 3 & 6 \\ -2 & 5 \\ 7 & -1 \end{bmatrix}, B = \begin{bmatrix} 4 & 1 & 3 \\ 2 & -6 & 0 \end{bmatrix}, \vec{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \vec{y} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$$

Compute the following products, if possible: if not, explain why not.

(1) AB
(2) $B\vec{y}$
(3) $\vec{x}B$
(4) $\vec{x}^T B\vec{y}$
(5) A^T

Problem 12. Consider the following row-reduction:

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 3 & -4 & 8 & 3 \\ 1 & 0 & 7 & 4 \\ 0 & 2 & 5 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = B$$

(1) Write out the steps of the row-reduction, identifying each row operation.
(2) Solve the system of equations given by

$$Bx = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 6 \end{pmatrix}$$

Problem 13. Consider the following system of linear equations

$$\begin{aligned} x - 2y + 3u - v &= -10 \\ 2x - 7y - 4u + v &= 16 \\ 3u - 2v &= -13 \\ 6u + 3v &= -12 \end{aligned}$$

(1) Rewrite this as a linear system of the form $A\vec{x} = \vec{b}$, specifying $A, \vec{x}$, and $\vec{b}$ carefully.
(2) Row-reduce the augmented matrix of this system to lower-triangular form.
(3) Solve the original equations for the unknowns using your answer to part (2) and back-substitution.