

MATH 104: Multivariable Calculus

Name: _____

June 19, 2023

Rules

- 5 questions, 90 minutes
- Closed books
- Show all your work. Mere numbers for solutions will not count for grades.
- No sharing of calculators

Scores

Problem 1. ____/20

Problem 2. ____/20

Problem 3. ____/20

Problem 4. ____/20

Problem 5. ____/20

Total _____/100

Questions

Problem 1. (4 points each subproblem) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$.

- (a) What does it mean for f to be differentiable at (a, b) ?
- (b) What does it mean for f to have a directional derivative in the direction of \mathbf{u} ? What's a notation for this notion?
- (c) Write directional derivative of function f in the direction \mathbf{u} in terms of partial derivative/gradient of f .
- (d) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function and C be a smooth curve in \mathbb{R}^2 parametrized by $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$. Write down the formula to compute the line integral of f along C .
- (e) Let $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a vector field and C be a smooth curve in \mathbb{R}^2 parametrized by $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$. Write down a formula to compute the line integral of \mathbf{F} along C .

Problem 2. Compute the following

(a) (10 points)

$$\oint_C xy \, dy - y^2 \, dx ,$$

where C is the boundary of the square $[0, 1] \times [0, 1]$.

(b) (10 points)

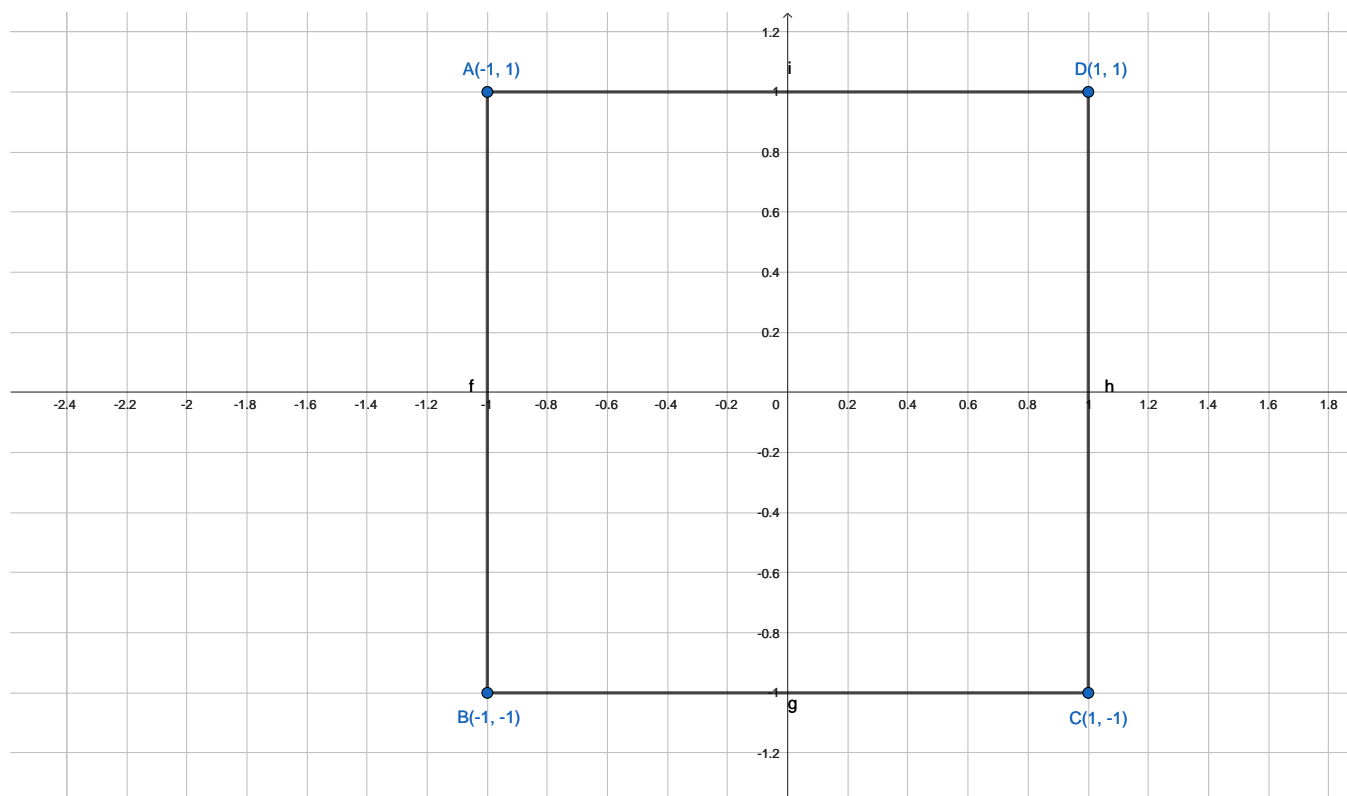
$$\int_0^1 \int_{3y}^3 e^{x^2} \, dx dy .$$

Problem 3. (a) (5 points) Consider the vector field

$$\mathbf{F}(x, y) = x^2 \mathbf{j}.$$

Is the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path?

(b) (10 points) Compare the path integrals of \mathbf{F} on two paths $A \rightarrow B \rightarrow C$ and $A \rightarrow D \rightarrow C$, where the paths are from the figure below.



(c) (5 points) Are parts (a) and (b) consistent with each other? Why or why not?

Problem 4. (a) (10 points) State the change of variable theorem. That is, for a change of coordinate $\varphi : D \rightarrow S$ such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(u, v),$$

what is the formula for $\iint_S f \, dA$?

(b) (10 points) Evaluate

$$\iint_S xy \, dA$$

where S is the disk with radius 2 and has center at $(0, 0)$.

Problem 5. A rectangular box with a lid is made from $12m^2$ of cardboard. Find the maximum volume of such a box.

(Hint: set up a constraint optimization problem for the function $V(x, y, z) = xyz$ and then solve it.)