# MATH 104: Multivariable Calculus (brief notes)

Truong-Son Van

Spring 2023

# Contents

**1 Vectors** **5**
1.1 Basics . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 5
Rules to manipulate vectors . . . . . . . . . . . . . . . . . . . . . . . . . 6
Properties of vector operations . . . . . . . . . . . . . . . . . . . . . . . 6
1.2 Products . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6

# Chapter 1

# Vectors

## 1.1 Basics

**Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9**

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about $n$-dimensional vectors doesn't require much more effort, we will talk about $n$-dimensional vectors instead.

**Definition 1.1.** An $n$-dimensional Euclidean space $\mathbb{R}^n$ is the Cartesian product of $n$ Euclidean spaces $\mathbb{R}$.

**Definition 1.2.** An $n$-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \ldots, v_n \rangle, \tag{1.1}$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

*Remark.* A point in $\mathbb{R}^n$ is also represented by an $n$-tuple but with round brackets. A vector connecting two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle .$$

We denote the above vector as $\vec{AB}$ where $A$ is the tail (initial point) and $B$ is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle .$$

**Definition 1.3.** The length of a vector $\mathbf{v}$ (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} . \tag{1.2}$$

**Definition 1.4.** A unit vector is a vector that has magnitude 1.

**Exercise 1.1.** Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

## Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b} .$$

This formula is deceptively simple. Make sure you understand all the implications.

## Properties of vector operations

Read the book

(Make sure you understand the geometric intepretation)

## 1.2 Products

### 1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

**Definition 1.5.** The dot product of vectors $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n .$$

## Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$,
3. If $c$ is a scalar, then $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$.

**Theorem 1.1** (Law of cosine)**.** *If $\theta$ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then*

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| \cos\theta\,.$$

**Corollary 1.1.** *Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.*

## Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}\,,$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}\,.$$

Read the book for more details. Make sure you understand the geometric meaning.

### 1.2.2 3D special: Cross product

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

**Definition 1.6.** Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle\,.$$

**Theorem 1.2.** *Let $\theta$ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta\,.$$

**Theorem 1.3.** *The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*