

# MATH 104: Multivariable Calculus (brief notes)

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# 1 Vectors

## 1.1 Basics

**Reading:** Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about  $n$ -dimensional vectors doesn't require much more effort, we will talk about  $n$ -dimensional vectors instead.

**Definition 1.1.** An  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is the Cartesian product of  $n$  Euclidean spaces  $\mathbb{R}$ .

**Definition 1.2.** An  $n$ -dimensional vector  $\mathbf{v} \in \mathbb{R}^n$  is a tuple

$$\mathbf{v} = \langle v_1, \dots, v_n \rangle, \quad (1)$$

where  $v_i \in \mathbb{R}$ .

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

*Remark.* A point in  $\mathbb{R}^n$  is also represented by an  $n$ -tuple but with round brackets. A vector connecting two points  $A = (a_1, \dots, a_n)$  and  $B = (b_1, \dots, b_n)$  can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle.$$

We denote the above vector as  $\vec{AB}$  where  $A$  is the tail (initial point) and  $B$  is the tip/head (terminal point). We denote  $\mathbf{0}$  to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle.$$

**Definition 1.3.** The length of a vector  $\mathbf{v}$  (denoted by  $|\mathbf{v}|$ ) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \dots + v_n^2}. \quad (2)$$

**Definition 1.4.** A unit vector is a vector that has magnitude 1.

**Exercise 1.1.** Turn a vector  $\mathbf{v} \in \mathbb{R}^n$  into a unit vector with the same direction.

### Rules to manipulate vectors

Let  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$  and  $c, d \in \mathbb{R}$ . Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b},$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}.$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1 \mathbf{e}_1 + \cdots + u_n \mathbf{e}_n,$$

where  $e_i = \langle 0, \dots, 1, \dots, 0 \rangle$  is the vector which has zero at all entries except that the  $i^{th}$  entry is 1.

In 3D,

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}.$$

## Properties of vector operations

Read the book

(Make sure you understand the geometric interpretation)

## 1.2 Products

### 1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in  $\mathbb{R}^n$  are perpendicular?
- How do we find the projection of one vector onto another?

**Definition 1.5.** The dot product of vectors  $\mathbf{u} = \langle u_1, \dots, u_n \rangle$  and  $\mathbf{v} = \langle v_1, \dots, v_n \rangle$  in  $\mathbb{R}^n$  is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

### Properties of dot product

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ . Then,

1.  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ ,
2.  $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$ ,
3. If  $c$  is a scalar, then  $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$ .

**Theorem 1.1** (Law of cosine). *If  $\theta$  is the angle between the vectors  $\mathbf{u}$  and  $\mathbf{v}$ , then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta.$$

**Corollary 1.1.** *Two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal to each other if  $\mathbf{u} \cdot \mathbf{v} = 0$ .*

### Projection

Let  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ . The component of  $\mathbf{u}$  in the direction of  $\mathbf{v}$  is the scalar

$$\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|},$$

and the projection of  $\mathbf{u}$  onto  $\mathbf{v}$  is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Read the book for more details. Make sure you understand the geometric meaning.

### 1.2.2 3D special: Cross product

This concept is very specific to  $\mathbb{R}^3$ . It will not make sense in other dimensions.

**Definition 1.6.** Let  $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ . The cross product of  $\mathbf{a}$  and  $\mathbf{b}$  is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

**Theorem 1.2.** Let  $\theta$  be the angle between  $\mathbf{a}$  and  $\mathbf{b}$ . Then,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta.$$

**Theorem 1.3.** The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ .