MATH 104: Multivariable Calculus (brief notes)

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Contents

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1 Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n-dimensional vectors doesn't require much more effort, we will talk about n-dimensional vectors instead.

Definition 1.1. An n-dimensional Euclidean space $\mathbb{R}^n$ is the Cartesian product of n Euclidean spaces $\mathbb{R}$.

Definition 1.2. An n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \ldots, v_n \rangle \,, \tag{1}$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in $\mathbb{R}^n$ is also represented by an n-tuple but with round brackets. A vector connecting two points $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \ldots, b_n - a_n \rangle \,.$$

We denote the above vector as $\vec{AB}$ where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \ldots, 0 \rangle \,.$$

Definition 1.3. The length of a vector $\mathbf{v}$ (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} \,. \tag{2}$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \ldots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b} \,,$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} \,.$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1\mathbf{e_1} + \cdots + u_n\mathbf{e_n}\,,$$

where $e_i = \langle 0, \ldots, 1, \ldots, 0\rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e_1} = \mathbf{i}\,, \qquad \mathbf{e_2} = \mathbf{j}\,, \qquad \mathbf{e_3} = \mathbf{k}\,.$$

Properties of vector operations

Read the book

(Make sure you understand the geometric intepretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \ldots, u_n\rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n\rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + \cdots + u_nv_n\,.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$,
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u})\cdot\mathbf{w} = c(\mathbf{u}\cdot\mathbf{w})$.

Theorem 1.1 (Law of cosine)**.** *If θ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then*

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta\,.$$

Corollary 1.1. *Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\,,$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\,.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \,.$$

Theorem 1.2. *Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \,.$$

Theorem 1.3. *The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*

2 Some basic equations in $\mathbb{R}^3$

Just to build some toy examples for the future, we will play with some basic equations in three dimensions.

2.1 Equations for lines

A line is a collection of points that is parallel to a vector and goes through a specific point. To capture this idea, we have the following representation for a line

$$L = \{\mathbf{r}(t) \,|\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, t \in \mathbb{R}\}\,,$$

where r_0 is the initial position and $\mathbf{v}$ is the direction. The equation for $\mathbf{r}(t)$ is called a **vector equation for a line** L.

Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$. The **parametric equations** of L is the following system of equations

$$\begin{aligned} x &= x_0 + v_1 t\,, \\ y &= y_0 + v_2 t\,, \\ z &= z_0 + v_3 t\,. \end{aligned}$$

This leads to the **symmetric equations** of L

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}\,.$$

2.2 Equations for planes

A plane is a collection of points that is perpendicular to one specific direction represented by a some vector called a **normal vector**. Note that due to scaling, there are more than one normal vector. To capture this idea, we have the following representation of a plane

$$P = \{\mathbf{r} \,|\, \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}\,.$$

This is called a **vector equation for the plane** P.

Multiplying things out, we have the **scalar equation of the plane** P with normal vector $\mathbf{n} = \langle n_1, n_2, n_3\rangle$ through a point $P_0(x_0, y_0, z_0)$

$$n_1(r_1 - x_0) + n_2(r_2 - y_0) + n_3(r_3 - z_0) = 0\,.$$

2.3 Cylinders

Definition 2.1. A cylinder is a surface that consists of all lines (called **rulings**) that are parallel to a given line.

Example 2.1.

1. $z = x^2$
2. $x^2 + y^2 = 1$

2.4 Quadric surfaces

Definition 2.2. A quadric surface is the graph of a second-degree equation in three variables x, y and z. The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E\,.$$

Example 2.2.

1. Ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$

2. Hyperbolic paraboloid
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}\,.$$

3. Elliptical cone
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\,.$$

Read the books for more surfaces and pictures.

3 Functions of several variables

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

3.1 Functions of several variables

$f : D \to C$ where $D \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$.

$$f(x) = f(x_1, \ldots, x_n) \,.$$

D is called the domain of f and C is called the codomain of f.

Example 3.1. The following are some examples of multivariable functions

1. $f(x, y) = x^2 - 2xy + y^2$
2. $f(x, y, z) = \frac{1}{1-xy^2}$
3. $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$

The expression in the vector equation for a line is an example of a function that maps from $\mathbb{R}$ to $\mathbb{R}^n$. There's no one who would stop us from considering more general kinds of function.

Definition 3.1. A **vector function** (**vector-valued function**) is a function that has the codomain that belongs to $\mathbb{R}^n$ where $n \geq 2$.

In this course, whenever we talk about vector valued function, we will only refer to that which has one dimensional domain ($D \subseteq \mathbb{R}$).