

# MATH 104: Multivariable Calculus

Name: \_\_\_\_\_

May 23, 2023

## Rules

- 5 questions, 90 minutes
- Closed books
- Show all your work. Mere numbers for solutions will not count for grades.
- No sharing of calculators

## Scores

Problem 1. \_\_\_\_/20

Problem 2. \_\_\_\_/20

Problem 3. \_\_\_\_/20

Problem 4. \_\_\_\_/20

Problem 5. \_\_\_\_/20

Total \_\_\_\_\_/100

## Questions

*Problem 1.* (4 points each subproblem) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ .

- (a) What does it mean for  $f$  to be differentiable at  $(a, b)$ ?
- (b) What does it mean for  $f$  to have a directional derivative in the direction of  $\mathbf{u}$ ? What's a notation for this notion?
- (c) Write directional derivative of function  $f$  in the direction  $\mathbf{u}$  in terms of partial derivative/gradient of  $f$ .
- (d) Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function and  $C$  be a smooth curve in  $\mathbb{R}^2$  parametrized by  $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$ . Write down the formula to compute the line integral of  $f$  along  $C$ .
- (e) Let  $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a vector field and  $C$  be a smooth curve in  $\mathbb{R}^2$  parametrized by  $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$ . Write down a formula to compute the line integral of  $\mathbf{F}$  along  $C$ .

*Problem 2.* Compute the following

(a) (10 points)

$$\oint_C xy \, dy - y^2 \, dx ,$$

where  $C$  is the boundary of the square  $[0, 1] \times [0, 1]$ .

(b) (10 points)

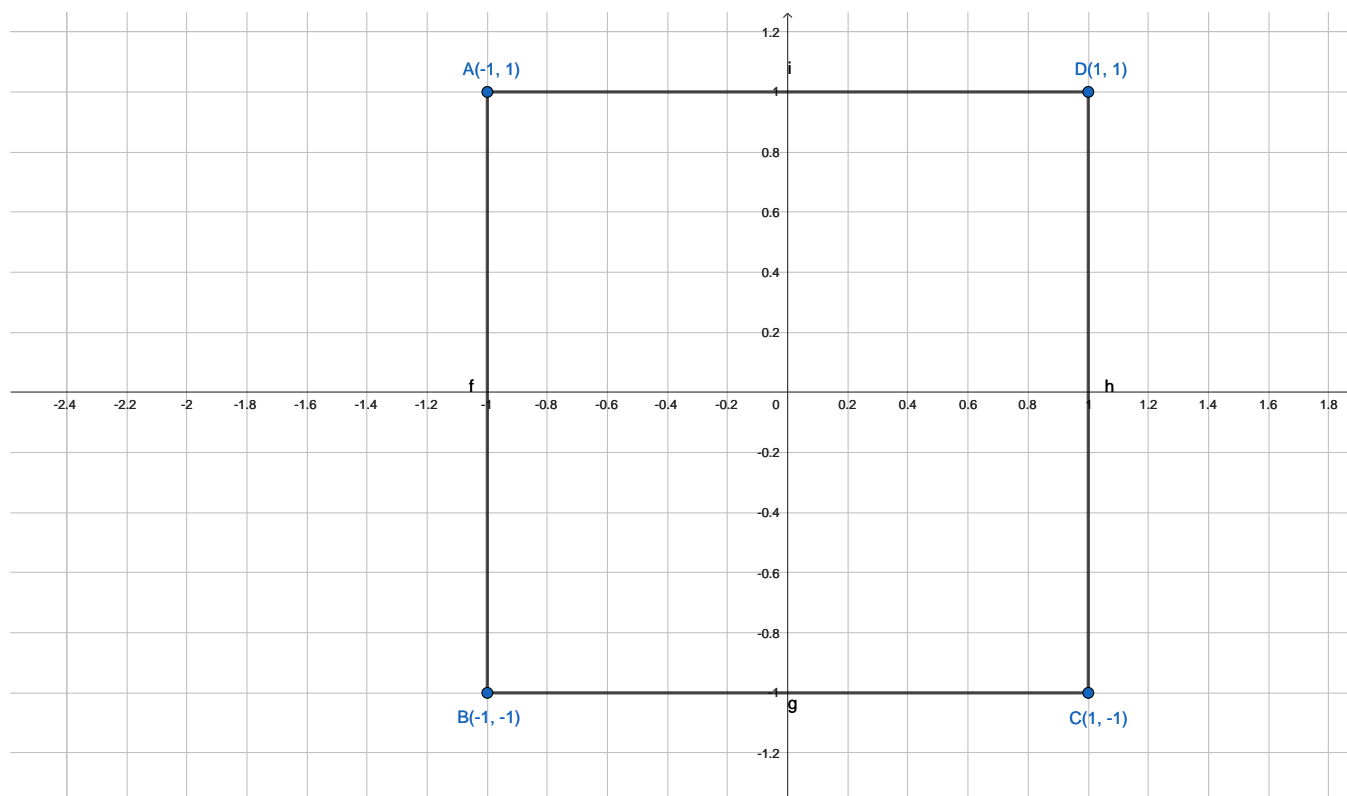
$$\int_0^1 \int_{3y}^3 e^{x^2} \, dx dy .$$

Problem 3. (a) (5 points) Consider the vector field

$$\mathbf{F}(x, y) = x^2 \mathbf{j}.$$

Is the integral  $\int_C \mathbf{F} \cdot d\mathbf{r}$  independent of path?

(b) (10 points) Compare the path integrals of  $\mathbf{F}$  on two paths  $A \rightarrow B \rightarrow C$  and  $A \rightarrow D \rightarrow C$ , where the paths are from the figure below.



(c) (5 points) Are parts (a) and (b) consistent with each other? Why or why not?

*Problem 4.* (a) (10 points) State the change of variable theorem. That is, for a change of coordinate  $\varphi : D \rightarrow S$  such that

$$\begin{pmatrix} x \\ y \end{pmatrix} = \varphi(u, v),$$

what is the formula for  $\iint_S f \, dA$ ?

(b) (10 points) Evaluate

$$\iint_S xy \, dA$$

where  $S$  is the disk with radius 2 and has center at  $(0, 0)$ .

*Problem 5.* A rectangular box with a lid is made from  $12m^2$  of cardboard. Find the maximum volume of such a box.

(Hint: set up a constraint optimization problem for the function  $V(x, y, z) = xyz$  and then solve it.)