

MATH 104: Multivariable Calculus (brief notes)

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1 Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n -dimensional vectors doesn't require much more effort, we will talk about n -dimensional vectors instead.

Definition 1.1. An n -dimensional Euclidean space \mathbb{R}^n is the Cartesian product of n Euclidean spaces \mathbb{R} .

Definition 1.2. An n -dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \dots, v_n \rangle, \quad (1)$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in \mathbb{R}^n is also represented by an n -tuple but with round brackets. A vector connecting two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle.$$

We denote the above vector as \vec{AB} where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle.$$

Definition 1.3. The length of a vector \mathbf{v} (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \dots + v_n^2}. \quad (2)$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b},$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}.$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1 \mathbf{e}_1 + \cdots + u_n \mathbf{e}_n,$$

where $e_i = \langle 0, \dots, 1, \dots, 0 \rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}.$$

Properties of vector operations

Read the book

(Make sure you understand the geometric interpretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in \mathbb{R}^n are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ in \mathbb{R}^n is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$.

Theorem 1.1 (Law of cosine). *If θ is the angle between the vectors \mathbf{u} and \mathbf{v} , then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta.$$

Corollary 1.1. *Two vectors \mathbf{u} and \mathbf{v} are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|},$$

and the projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to \mathbb{R}^3 . It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of \mathbf{a} and \mathbf{b} is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

Theorem 1.2. Let θ be the angle between \mathbf{a} and \mathbf{b} . Then,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta.$$

Theorem 1.3. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .