

Multivariable Calculus

Day 4

Multivariable and vector functions

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From last time

- ① Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Find an equation for the line of intersection.
- ② Find a formula for the distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Quadric surfaces

A quadric surface is the graph of a second-degree equation in three variables x, y and z . The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E .$$

Functions of several variables

Functions of several variables

Definition

A function of several variables is a function $f : D \rightarrow C$ where $D \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$.

$$f(x) = (f_1(x_1, \dots, x_m), \dots, f_n(x_1, \dots, x_m)).$$

D is called the domain of f and C is called the codomain of f .

Examples

❶ $f(x, y) = x^2 - 2xy + y^2$

❷ $f(x, y, z) = \frac{1}{1-xy^2}$

Vector functions

Definition

A vector function (vector-valued function) is a function that has the codomain that belongs to \mathbb{R}^n where $n \geq 2$. In other words, $f : D \rightarrow \mathbb{R}^n$.

Example

The following are some examples of vector functions.

- Line: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- Helix: $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$

Theorem

Let $\mathbf{r} : \mathbb{R} \rightarrow \mathbb{R}^n$, given by $\mathbf{r}(t) = \langle r_1(t), \dots, r_n(t) \rangle$. Then, \mathbf{r} is said to be continuous at t_0 if

$$\mathbf{r}(t_0) = \lim_{t \rightarrow t_0} \mathbf{r}(t),$$

where

$$\lim_{t \rightarrow t_0} \mathbf{r}(t) = \langle \lim_{t \rightarrow t_0} r_1(t), \dots, \lim_{t \rightarrow t_0} r_n(t) \rangle.$$

Furthermore, we can define the derivative of \mathbf{r}

$$\frac{d}{dt} \mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

- Draw the helix $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$
- Find $\lim_{t \rightarrow \pi} \mathbf{f}(t)$
- Find $\mathbf{f}'(t)$
- Fill in the right-hand side
 - ① $(\mathbf{u}(t) + \mathbf{v}(t))' =$
 - ② $(c\mathbf{u}(t))' =$
 - ③ $(f(t)\mathbf{u}(t))' =$
 - ④ $(\mathbf{u}(t) \cdot \mathbf{v}(t))' =$
 - ⑤ $(\mathbf{u}(t) \times \mathbf{v}(t))' =$
 - ⑥ $(\mathbf{u}(f(t)))' =$

Integrals

Indefinite integral

$$\int_a^b \mathbf{r}(t) dt = \left\langle \int_a^b r_1(t) dt, \int_a^b r_2(t) dt, \int_a^b r_3(t) dt \right\rangle$$

How would one measure the length of a curve?