Multivariable Calculus
Day 12
Tangent plane and ∇F

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Worksheet Problem 4

Let $f : \mathbb{R} \to \mathbb{R}$ and $T(x)$ be a linear function such that $T(a) = f(a)$. So,

$$T(x) = f(a) + m(x - a)\,.$$

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Can we do better?

From Taylor's theorem, if

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Furthermore, condition (2) is equivalent to the definition of differentiability. This justifies the complicated definition we learned in higher dimensions.

Example

- 1D: $f(x) = e^{-x^2}$, $a = 1$
- 2D: $f(\mathbf{x}) = e^{-|\mathbf{x}|^2}$, $\mathbf{a} = \langle 1, 1 \rangle$

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Are you in crisis yet?

Let's slow down

Let's consider the curve situation first.

Typically, one thinks of a curve as

$$z = f(x)\,.$$

However, there's a more general form for a curve as level curve of a two-variable differentiable function $F(x, y)$. In particular, a c-level curve of $F(x, y)$ is a curve $\mathbf{r}(t)$ such that

$$F(\mathbf{r}(t)) = c\,.$$

c-level curve

That means

$$\frac{d}{dt}F(\mathbf{r}(t)) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0\,.$$

Suppose at t_0, $\mathbf{r}(t_0) = (a, b)$. We then have that, the tangent line of the c-level curve of F at (a, b) must satisfy the relation

$$\nabla F(a, b) \cdot \langle x_1 - a, x_2 - b \rangle = 0\,.$$

Another way to write this:

$$\partial_{x_1} F(a, b)(x_1 - a) + \partial_{x_2} F(a, b)(x_2 - b) = 0\,.$$

`https://www.youtube.com/watch?v=ZTbTYEMvo10`

c-level surface

Similar to c-level curves, a c-level surface is a surface that satisfies

$$F(x, y, z) = c\,.$$

Reasoning similarly to the case of the c-level curve, we have that for ANY curve $\mathbf{r}(t)$ on the c-level surface,

$$\frac{d}{dt}F(\mathbf{r}(t)) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0\,.$$

That means for any curve that goes through the point (a, b, c) at time t_0, it must be the case that

$$\nabla F(a, b, c) \cdot \mathbf{r}'(t_0) = 0\,.$$

$\implies \nabla F(a, b, c)$ is perpendicular to ALL curves on the c-level surface that goes through (a, b, c).

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$\implies$ The tangent plane is unique and satisfies the formula (analogous to the curve case)

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0 \,.$$

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We know that for a unit vector $\mathbf{u}$,

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This is maximized at $\theta = 0$.

`https://www.youtube.com/watch?v=TEB2z7ZlRAw`