Multivariable Calculus

Truong-Son Van

03/04/2024

Fulbright University Vietnam

Level sets

We will restrict our attention to functions with one output for the next few weeks.

$$f : \mathbb{R}^n \to \mathbb{R}.$$

A level c-level set of a function is defined to be

$$f^{-1}(c) = \{x : f(x) = c\}.$$

If $n = 2$, the level set is called the level curve.

If $n = 3$, the level set is called the level surface.

c-level curve

Suppose one can parametrize the c-level curve by $\mathbf{r}(t)$.

That means

$$\frac{d}{dt}F(\mathbf{r}(t)) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0\,.$$

Suppose at t_0, $\mathbf{r}(t_0) = (a, b)$. We then have that, the tangent line of the c-level curve of F at (a, b) must satisfy the relation

$$\nabla F(a, b) \cdot \langle x_1 - a, x_2 - b\rangle = 0\,.$$

Another way to write this:

$$\partial_{x_1} F(a, b)(x_1 - a) + \partial_{x_2} F(a, b)(x_2 - b) = 0\,.$$

`https://www.youtube.com/watch?v=ZTbTYEMvo10`

c-level surface

Similar to c-level curves, a c-level surface is a surface that satisfies

$$F(x, y, z) = c\,.$$

Reasoning similarly to the case of the c-level curve, we have that for ANY curve $\mathbf{r}(t)$ on the c-level surface,

$$\frac{d}{dt}F(\mathbf{r}(t)) = \nabla F(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0\,.$$

That means for any curve that goes through the point (a, b, c) at time t_0, it must be the case that

$$\nabla F(a, b, c) \cdot \mathbf{r}'(t_0) = 0\,.$$

$\implies \nabla F(a, b, c)$ is perpendicular to ALL curves on the c-level surface that goes through (a, b, c).

$\Longrightarrow$ $\nabla F(a, b, c)$ is perpendicular to ALL curves on the c-level surface that goes through (a, b, c).

$\Longrightarrow$ There is ONE vector that perpendicular to ANY 2 curves on the c-level surface at the point (a, b, c).

$\implies$ $\nabla F(a, b, c)$ is perpendicular to ALL curves on the c-level surface that goes through (a, b, c).

$\implies$ There is ONE vector that perpendicular to ANY 2 curves on the c-level surface at the point (a, b, c).

$\implies$ The plane made by this very one vector will be tangent to ALL the curves.

$\implies$ $\nabla F(a, b, c)$ is perpendicular to ALL curves on the c-level surface that goes through (a, b, c).

$\implies$ There is ONE vector that perpendicular to ANY 2 curves on the c-level surface at the point (a, b, c).

$\implies$ The plane made by this very one vector will be tangent to ALL the curves.

$\implies$ The tangent plane is unique and satisfies the formula (analogous to the curve case)

$$\nabla F(a, b, c) \cdot \langle x - a, y - b, z - c \rangle = 0\,.$$

Steepest ascent

Steepest ascent

Directional derivative tells you the change of the function F in certain direction $\mathbf{u}$.

Question: what is the direction that gives me the largest change PER UNIT?

Steepest ascent

Directional derivative tells you the change of the function F in certain direction $\mathbf{u}$.

Question: what is the direction that gives me the largest change PER UNIT?

We know that for a unit vector $\mathbf{u}$,

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \nabla f(\mathbf{x_0}) \cdot \mathbf{u} = |\nabla f(\mathbf{x_0})| \cos\theta \,.$$

Steepest ascent

Directional derivative tells you the change of the function F in certain direction $\mathbf{u}$.

Question: what is the direction that gives me the largest change PER UNIT?

We know that for a unit vector $\mathbf{u}$,

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \nabla f(\mathbf{x_0}) \cdot \mathbf{u} = |\nabla f(\mathbf{x_0})| \cos\theta \,.$$

This is maximized at $\theta = 0$.

`https://www.youtube.com/watch?v=TEB2z7ZlRAw`

Let $f : \mathbb{R} \to \mathbb{R}$ and $T(x)$ be a linear function such that $T(a) = f(a)$. So,

$$T(x) = f(a) + m(x - a)\,.$$

This function is any arbitrary linear function that may give us an "approximation" for f near a.

Let $f : \mathbb{R} \to \mathbb{R}$ and $T(x)$ be a linear function such that $T(a) = f(a)$. So,

$$T(x) = f(a) + m(x - a)\,.$$

This function is any arbitrary linear function that may give us an "approximation" for f near a.

To determine the "best" approximation, we want to see what happens to $f(x) - T(x)$ as $x \to a$.

Let $f:\mathbb{R}\to\mathbb{R}$ and $T(x)$ be a linear function such that $T(a)=f(a)$. So,

$$T(x)=f(a)+m(x-a)\,.$$

This function is any arbitrary linear function that may give us an "approximation" for f near a.

To determine the "best" approximation, we want to see what happens to $f(x)-T(x)$ as $x\to a$.

Naturally, $\lim_{x\to a} f(x)-T(x)=0$, by the first requirement.

Let $f : \mathbb{R} \to \mathbb{R}$ and $T(x)$ be a linear function such that $T(a) = f(a)$. So,

$$T(x) = f(a) + m(x - a)\,.$$

This function is any arbitrary linear function that may give us an "approximation" for f near a.

To determine the "best" approximation, we want to see what happens to $f(x) - T(x)$ as $x \to a$.

Naturally, $\lim_{x \to a} f(x) - T(x) = 0$, by the first requirement.

Can we do better?

From Taylor's theorem, if

$$T(x) = f(a) + f'(a)(x - a) \tag{1}$$

then

From Taylor's theorem, if

$$T(x) = f(a) + f'(a)(x - a) \tag{1}$$

then

$$\lim_{x \to a} \frac{f(x) - T(x)}{|x - a|} = 0\,, \tag{2}$$

which is a more significant statement than just

$$\lim_{x \to a} f(x) - T(x) = 0\,.$$

From Taylor's theorem, if

$$T(x) = f(a) + f'(a)(x - a) \tag{1}$$

then

$$\lim_{x \to a} \frac{f(x) - T(x)}{|x - a|} = 0\,, \tag{2}$$

which is a more significant statement than just

$$\lim_{x \to a} f(x) - T(x) = 0\,.$$

From analysis class (take it when you have a chance!), we will know that (2) is the necessary and sufficient condition to determine the linear approximation you learned before (e.g. (1)).

From Taylor's theorem, if

$$T(x) = f(a) + f'(a)(x - a) \tag{1}$$

then

$$\lim_{x \to a} \frac{f(x) - T(x)}{|x - a|} = 0\,, \tag{2}$$

which is a more significant statement than just

$$\lim_{x \to a} f(x) - T(x) = 0\,.$$

From analysis class (take it when you have a chance!), we will know that (2) is the necessary and sufficient condition to determine the linear approximation you learned before (e.g. (1)).

Furthermore, condition (2) is equivalent to the definition of differentiability. This justifies the complicated definition we learned in higher dimensions.