Multivariable Calculus
Day 16
Integration

Spring 2023

Riemann sums

Let f be a function on a rectangle R. An n-fold Riemann sum for f over R is a sum of the following form

$$\sum_{i_1=1}^{m_1} \cdots \sum_{i_n=1}^{m_n} f(\xi_{i_1 \ldots i_n}) \Delta V \, ,$$

where

- $\Delta V = \Delta x_1 \times \cdots \times \Delta x_n$,
- $\Delta x_i = (b_i - a_i)/m_i$,
- $\xi_{i_1 \ldots i_n} \in R_{i_1 \ldots i_n}$,
- $R_{i_1 \ldots i_n} = \prod_{i=1}^{n} [a_i + (i_1 - 1)\Delta x, a_i + i_1 \Delta x_i]$.

In two dimension ΔV is replaced by ΔA.

Definition
The double integral of f over a rectangle $R \subseteq \mathbb{R}^2$ is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_{ij})\Delta A$$

if the limit exists.

The triple integral of f over a rectangle $R \subseteq \mathbb{R}^3$ is

$$\iiint_R f(x,y)\,dV = \lim_{m,n,l\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} f(\xi_{ijk})\Delta V$$

if the limit exists.

Worksheet

How can you define the double/triple integral for a region that is not a rectangle?

Iterated integral

Suppose that f is integrable on $R = [a, b] \times [c, d]$. An iterated integral of f is defined as

$$\int_a^b A(x)\, dx\,,$$

where

$$A(x) = \int_c^d f(x, y)\, dy\,.$$

Typically, we write the above as

$$\int_a^b \int_c^d f(x, y)\, dydx\,.$$

This means that we integrate in y before in x– always integrate the inner part first.

Fubini

Theorem (Speicial case of Fubini)
If f is continuous on the rectangle R, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dydx = \int_c^d \int_a^b f(x, y)\, dxdy\,.$$

Worksheet

Compute

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dxdy$$

and

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dydx$$

Worksheet

Let

$$f(x,y) = \begin{cases} 1\,, & y \in (x, x+1), x \geq 0 \\ -1\,, & y \in (x-1, x), x \geq 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dydx\,.$$

Worksheet

1. Set up a problem with double integral to find the area of triangle with vertices $(0,0),(2,0),(2,3)$.
2. Let $f(x,y) = x^2y$. Find
$$\iint_D f(x,y)\,dA$$
where D is the triangle with vertices $(0,0),(2,0),(2,3)$.
3. Compute
$$\iint_D e^{-y^2}\,dA$$