Multivariable Calculus
Constrained Optimization

Spring 2024

Constrained optimization

Constrained optimization takes various forms, depending on the assumptions.

Three kinds of constraints:

1. Domain
2. Side condition of the form $g(x) = k$
3. A mix of those

Typically, people will write as follows

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = k\,. \end{aligned}$$

Absolute Max / Min

Theorem
If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 1/2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Constrained optimization

Theorem (Method of Lagrange Multiplier)
Suppose the maximum/minimum values of f exist and $\nabla g(\mathbf{x}) \neq 0$ where $g(\mathbf{x}) = k$. To find the maximum and minimum values of f subject to constraint $g(\mathbf{x}) = k$, we do the following:

1. *Find all values of $\mathbf{x}$ and $\lambda \in \mathbb{R}$ such that*

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}),$$

   and

$$g(\mathbf{x}) = k.$$

2. *Evaluate f at all the points $\mathbf{x}$ that result from step 1. The largest of these values is the maximum of f; the smallest is the minimum value of f.*

Example

https://youtu.be/hQ4UNu1P2kw

Worksheet

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(1, 1, 1)$.

Two constraints

Problem 15 from

`https://activecalculus.org/multi/S-10-8-Lagrange-Multipliers.html`