Multivariable Calculus
Day 4
Multivariable and vector functions

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From last time

1. Find the angle between the planes $x + y + z = 1$ and $x - 2y + 3z = 1$. Find an equation for the line of intersection.
2. Find a formula for the distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Quadric surfaces

A quadric surface is the graph of a second-degree equation in three variables x, y and z. The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E\,.$$

Functions of several variables

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Definition
A function of several variables is a function $f : D \to C$ where $D \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$.

$$f(x) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) .$$

D is called the domain of f and C is called the codomain of f.

Examples

1. $f(x, y) = x^2 - 2xy + y^2$
2. $f(x, y, z) = \frac{1}{1-xy^2}$

Vector functions

Vector functions

Definition
A vector function (vector-valued function) is a function that has the codomain that belongs to $\mathbb{R}^n$ where $n \geq 2$. In other words, $f : D \to \mathbb{R}^n$.

Example
The following are some examples of vector functions.

- Line: $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
- Helix: $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$

Theorem
Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$*, given by* $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$*. Then,* $\mathbf{r}$ *is said to be continuous at* t_0 *if*

$$\mathbf{r}(t_0) = \lim_{t \to t_0} \mathbf{r}(t)\,,$$

where

$$\lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} r_1(t), \ldots, \lim_{t \to t_0} r_n(t) \rangle\,.$$

Furthermore, we can define the derivative of $\mathbf{r}$

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

Worksheet

- Draw the helix $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$
- Find $\lim_{t \to \pi} \mathbf{f}(t)$
- Find $\mathbf{f}'(t)$
- Fill in the right-hand side
  1. $(\mathbf{u}(t) + \mathbf{v}(t))' =$
  2. $(c\mathbf{u}(t))' =$
  3. $(f(t)\mathbf{u}(t))' =$
  4. $(\mathbf{u}(t) \cdot \mathbf{v}(t))' =$
  5. $(\mathbf{u}(t) \times \mathbf{v}(t))' =$
  6. $(\mathbf{u}(f(t)))' =$