Multivariable Calculus
Day 22
Vector Calculus: Line integrals (cont.)

Spring 2023

Recap: Worksheet

Work done by a force field along a path is

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} \,.$$

A force field is given $\mathbf{F}(x, y) = x^2\mathbf{i} - xy\mathbf{j}$. Suppose we want to move a particle along the quarter circle $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, 0 \leq t \leq \pi/2$. Compute the work done.

Fundamental Theorem for line integrals

Theorem (Fundamental Theorem for line integrals)
Let C be a smooth curve given by the parametrization $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then,

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Worksheet

Prove the Fundamental Theorem of line integrals.

1. Find the work done by the gravitational field in $\mathbb{R}^3$

$$\mathbf{F}(\mathbf{x}) = -\frac{-mMG}{|\mathbf{x}|^3}\mathbf{x}$$

in moving a particle with mass m from $(3, 4, 12) \to (2, 2, 0)$ along a straight line.

Closed curves

Definition
A **closed curve** is a curve that starts and ends at the same point.

A **simple closed curve** is a closed curve that never crosses itself.

Corollary
If C is a closed curve and $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then

$$\oint_C \nabla f \cdot d\mathbf{r} = 0\,.$$

Conservative vector fields

Definition
A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is there exists a function f such that

$$\nabla f = \mathbf{F}\,.$$

True or False? If $\mathbf{F}$ is a conservative vector field, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0\,.$$

Independence of path

Worksheet

Evaluate

$$\int_C y^2\,dx + x\,dy\,, \qquad i = 1,2$$

1. where C is the line segment from $(-5,-3) \to (0,2)$
2. where C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3) \to (0,2)$.
3. repeat the above two steps with

$$P = x\,, \qquad Q = y\,.$$

Definition
Let $\mathbf{F}$ be a continuous vector field with domain D, we say that the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

Independence of path and conservative vector fields

Theorem
$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Worksheet

Prove the above theorem.