MATH 104: Multivariable Calculus

Name: _

May 22, 2023

Rules

- 5 questions, 90 minutes
- Closed books
- Show all your work. Mere numbers for solutions will not count for grades.
- No sharing of calculators

Scores

Problem 1. _____/20

Problem 2. _____/20

Problem 3. _____/20

Problem 4. _____/20

Problem 5. _____/20

Total ____________/100

Questions

Problem 1. Let $f : \mathbb{R}^2 \to \mathbb{R}$.

a. What does it mean for f to be differentiable at (a, b)?

b. What does it mean for f to have a directional derivative in the direction of $\mathbf{u}$? What's a notation for this notion?

c. Write directional derivative of function f in the direction $\mathbf{u}$ in terms of partial derivative/gradient of f.

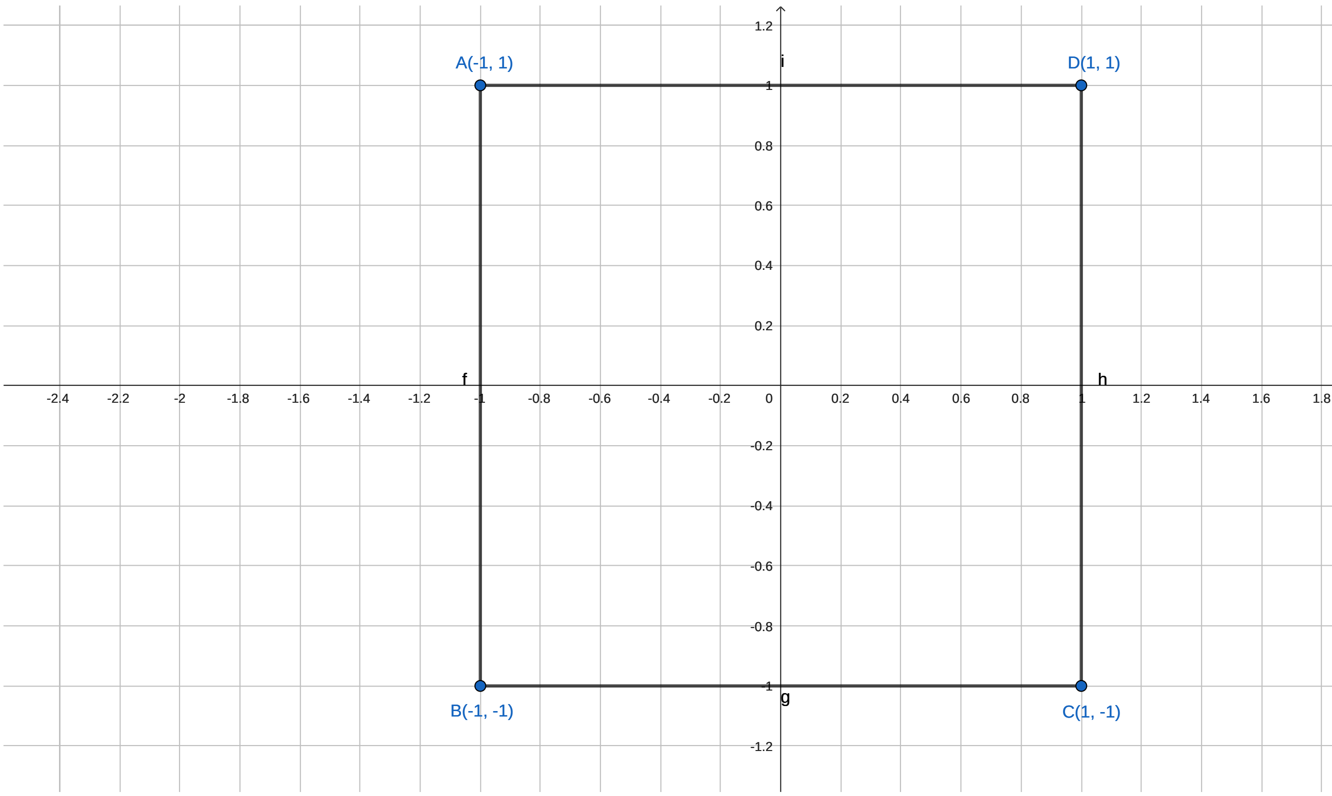
d. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and C be a smooth curve in $\mathbb{R}^2$ parametrized by $\mathbf{r} : [a, b] \to \mathbb{R}^2$. Write down the formula to compute the line integral of f over C.

Problem 2. a. Consider the vector field

$$\mathbf{F}(x, y) = x^2\mathbf{j}\,.$$

Is the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ independent of path?

b. Compare the path integrals of $\mathbf{F}$ on two paths $A \to B \to C$ and $A \to D \to C$, where the paths are from the figure below.



c. Are parts (a) and (b) consistent with each other? Why or why not?

Problem 3. 1. Given a function $f : \mathbb{R}^n \to \mathbb{R}$, where $n \geqslant 2$. What are the properties of ∇f? (List everything that you can think of)

2. Given any surface $F(x, y, z) = C$, where F is differentiable everywhere, how do you know there's only one tangent plane at a given point? (Give the best answer you can)

3. Find the tangent plane to the surface

$$x^2 + 2xy - y^2 + z^2 = 7$$

at point

$$P_0(1, -1, 3)\,.$$

Problem 4. 1. Find the tangent vector, normal vector and curvature of the following curve

$$\mathbf{r}(t) = \langle \cos t + t \sin t, \sin t - t \cos t, 3 \rangle .$$

2. What's the meaning of the curvature of a curve at a point on the curve?

Problem 5.

1. State the second derivative test when you want to optimize a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$.

2. Find all the local maxima, local minima, and saddle points (neither min nor max) of the function

$$f(x, y) = \ln(x + y) + x^2 - y\,.$$