Multivariable Calculus
Day 7
Paramtrization by arc length and introduction multivariable scalar functions

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Parametrize by arc length

Review

Arc length $=?$

Arc length function

If one wants to keep track the length of the curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ made by an airplane at any time t, one uses the *arc length function*

$$\ell(t) = \int_a^t \left|\mathbf{r}'(u)\right| \, du \, .$$

$\ell(t)$ is that it is a strictly increasing function with respect to t, given that $\mathbf{r}'$ is non-zero for all t.

Worksheet

Let $\mathbf{r} : [-\ln 4, 0] \to \mathbb{R}^2$ be a space curve such that

$$\mathbf{r}(t) = e^t \cos(t)\mathbf{i} + e^t \sin(t)\mathbf{j} + e^t\mathbf{k}\,.$$

1. Compute the arc length function of this curve.
2. Is there an inverse function of this?

Re-parametrize with respect to arc length

Letting $s = \ell(t)$, we can talk about the inverse of ℓ, $\ell^{-1} : [0, L] \to [a, b]$

$$t = \ell^{-1}(s)\,.$$

Therefore, we can re-write

$$\mathbf{r}(t) = \mathbf{r}(\ell^{-1}(s))\,.$$

Theorem

$$\left| \frac{d\mathbf{r}(t)}{ds} \right| = 1\,.$$

Thus,

$$\int_0^s \left| \frac{d}{ds}\mathbf{r}(t) \right| \, dt = s\,.$$

Worksheet

Verify the theorem with the space curve in the previous problem. In particular, show that

$$\left| \frac{d\mathbf{r}(t)}{ds} \right| = 1$$

and

$$\int_0^s \left| \frac{d}{ds}\mathbf{r}(t) \right| \, dt = s \,,$$

where

$$\mathbf{r}(t) = e^t \cos(t)\mathbf{i} + e^t \sin(t)\mathbf{j} + e^t \mathbf{k} \,.$$

Definition
Let $\mathbf{T}(t)$ be the unit tangent vector of the curve $\mathbf{r} : [a, b] \to \mathbb{R}^3$. The curvature of $\mathbf{r}(t(s))$ is defined to be

$$\kappa(s) = \left| \frac{d\mathbf{T}(t(s))}{ds} \right| .$$

Theorem

$$\kappa(s(t)) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} .$$

Remember the osculating circle?

Multivariable scalar functions

Definition
Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A real-valued/scalar function f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's domain. The set of w-values taken on by f is the function's range. The symbol w is the dependent variable of f, and f is said to be a function of the n independent variables x_1 to x_n. We also call the x_j's the function's input variables and call w the function's output variable.

FOCUS: two-variable functions. Higher dimensions will be the same.

Definition
Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.