MATH 104: Multivariable Calculus (brief notes)

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Contents

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1 Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n-dimensional vectors doesn't require much more effort, we will talk about n-dimensional vectors instead.

Definition 1.1. An n-dimensional Euclidean space $\mathbb{R}^n$ is the Cartesian product of n Euclidean spaces $\mathbb{R}$.

Definition 1.2. An n-dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \ldots, v_n \rangle , \tag{1}$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in $\mathbb{R}^n$ is also represented by an n-tuple but with round brackets. A vector connecting two points $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \ldots, b_n - a_n \rangle .$$

We denote the above vector as $\vec{AB}$ where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \ldots, 0 \rangle .$$

Definition 1.3. The length of a vector $\mathbf{v}$ (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2} . \tag{2}$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \ldots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b} ,$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a} .$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1\mathbf{e_1} + \cdots + u_n\mathbf{e_n}\,,$$

where $e_i = \langle 0, \ldots, 1, \ldots, 0\rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e_1} = \mathbf{i}\,, \qquad \mathbf{e_2} = \mathbf{j}\,, \qquad \mathbf{e_3} = \mathbf{k}\,.$$

Properties of vector operations

Read the book

(Make sure you understand the geometric intepretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in $\mathbb{R}^n$ are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \ldots, u_n\rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n\rangle$ in $\mathbb{R}^n$ is the scalar

$$\mathbf{u}\cdot\mathbf{v} = u_1v_1 + \cdots + u_nv_n\,.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$,
2. $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = (\mathbf{u}\cdot\mathbf{w}) + (\mathbf{v}\cdot\mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u})\cdot\mathbf{w} = c(\mathbf{u}\cdot\mathbf{w})$.

Theorem 1.1 (Law of cosine)**.** *If θ is the angle between the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, then*

$$\boldsymbol{u}\cdot\boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}|\cos\theta\,.$$

Corollary 1.1. *Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal to each other if $\boldsymbol{u}\cdot\boldsymbol{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of $\mathbf{u}$ in the direction of $\mathbf{v}$ is the scalar

$$\mathrm{comp}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{v}|}\,,$$

and the projection of $\mathbf{u}$ onto $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|}\right)\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v}\,.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to $\mathbb{R}^3$. It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle \,.$$

Theorem 1.2. *Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Then,*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta \,.$$

Theorem 1.3. *The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.*

2 Some basic equations in $\mathbb{R}^3$

Just to build some toy examples for the future, we will play with some basic equations in three dimensions.

2.1 Equations for lines

A line is a collection of points that is parallel to a vector and goes through a specific point. To capture this idea, we have the following representation for a line

$$L = \{\mathbf{r}(t)\,|\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, t \in \mathbb{R}\}\,,$$

where r_0 is the initial position and $\mathbf{v}$ is the direction. The equation for $\mathbf{r}(t)$ is called a **vector equation for a line** L.

Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$. The **parametric equations** of L is the following system of equations

$$\begin{aligned} x &= x_0 + v_1 t\,, \\ y &= y_0 + v_2 t\,, \\ z &= z_0 + v_3 t\,. \end{aligned}$$

This leads to the **symmetric equations** of L

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}\,.$$

Definition 2.1. Two lines are parallel if their directional vectors are parallel (scalar multiple of each other).

Two lines that are not parallel and don't intersect each other are said to be skew.

2.2 Equations for planes

A plane is a collection of points that is perpendicular to one specific direction represented by a some vector called a **normal vector**. Note that due to scaling, there are more than one normal vector. To capture this idea, we have the following representation of a plane

$$P = \{\mathbf{r} \,|\, \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}\,.$$

This is called a **vector equation for the plane** P.

Multiplying things out, we have the **scalar equation of the plane** P with normal vector $\mathbf{n} = \langle n_1, n_2, n_3\rangle$ through a point $P_0(x_0, y_0, z_0)$

$$n_1(r_1 - x_0) + n_2(r_2 - y_0) + n_3(r_3 - z_0) = 0\,.$$

The equation of the form

$$ax + by + cz + d = 0$$

is called a **linear equation**.

Definition 2.2. Two planes are said to be parallel if their normal vectors are parallel. If two planes are not parallel, they intersect in a straight line and the angle between the two planes is defined to be the angle between the two normal vectors.

2.3 Cylinders

Definition 2.3. A cylinder is a surface that consists of all lines (called **rulings**) that are parallel to a given line.

Example 2.1.

1. $z = x^2$
2. $x^2 + y^2 = 1$

2.4 Quadric surfaces

Definition 2.4. A quadric surface is the graph of a second-degree equation in three variables x, y and z. The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E\,.$$

Example 2.2.

1. Ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\,.$$
2. Hyperbolic paraboloid
$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}\,.$$
3. Elliptical cone
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}\,.$$

Read the books for more surfaces and pictures.

3 Functions in higher dimensions

Reading: Stewart Chapter 12, 13, Thomas Calculus Chapter 12, 13, Active Calculus Chapter 9

3.1 Functions of several variables

Definition 3.1. A function of several variables is a function $f : D \to C$ where $D \subseteq \mathbb{R}^m$ and $C \subseteq \mathbb{R}^n$, where $m \geq 2$ and $n \geq 1$.

$$f(x) = (f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)) .$$

D is called the domain of f and C is called the codomain of f.

The domain of f is where each of the component f_i of f is defined.

Example 3.1. The following are some examples of multivariable functions

1. $f(x, y) = x^2 - 2xy + y^2$
2. $f(x, y, z) = \frac{1}{1-xy^2}$

3.2 Vector functions

3.2.1 Limit, continuity and differentiation

The expression in the vector equation for a line is an example of a function that maps from $\mathbb{R}$ to $\mathbb{R}^n$. There's no one who would stop us from considering more general kinds of function.

Definition 3.2. A **vector function** (**vector-valued function**) is a function that has the codomain that belongs to $\mathbb{R}^n$ where $n \geq 2$. In other words, $f : D \to \mathbb{R}^n$.

Example 3.2. The following are some examples of vector functions.

1. $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$
2. $\mathbf{f}(t) = \langle \cos(t), \sin(t), t \rangle$

Note that my definition is more general than that in the book. However, **In this course, whenever we talk about vector valued function, we will only refer to that which has one dimensional domain ($D \subseteq \mathbb{R}$).**

By and large, there's nothing different between a vector function and a one-variable scalar function. All the concepts such as limit, continuity and differentiability are applied to each coordinate the same way as in one dimensional case.

Theorem 3.1. *Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^n$, given by $\mathbf{r}(t) = \langle r_1(t), \ldots, r_n(t) \rangle$. Then, $\mathbf{r}$ is said to be continuous at t_0 if*

$$\mathbf{r}(t_0) = \lim_{t \to t_0} \mathbf{r}(t) ,$$

where

$$\lim_{t \to t_0} \mathbf{r}(t) = \langle \lim_{t \to t_0} r_1(t), \ldots, \lim_{t \to t_0} r_n(t) \rangle .$$

Furthermore, we can define the derivative of $\mathbf{r}$

$$\frac{d}{dt}\mathbf{r}(t) = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists.

When $\mathbf{r} : I \to \mathbb{R}^n$ (I is an interval in $\mathbb{R}$) is continuous, we call it a **space curve** (to describe the intuitive picture of what a curve should look like in our mind).

Geometrically, if $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$, it represents the **tangent vector** of the curve $\mathbf{r}$ at t.

Definition 3.3. A **parametric equation** for a curve is an equation of the form

$$x = x(t)\,, \quad y = y(t)\,, \quad z = z(t)\,.$$

Typical differentiation rules apply.

Theorem 3.2 (Differentiation rules)**.**

1. $(\mathbf{u}(t) + \mathbf{v}(t))' = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $(c\mathbf{u}(t))' = c\mathbf{u}'(t)$
3. $(f(t)\mathbf{u}(t))' = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $(\mathbf{u}(t) \cdot \mathbf{v}(t))' = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $(\mathbf{u}(t) \times \mathbf{v}(t))' = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $(\mathbf{u}(f(t)))' = \mathbf{u}'(f(t))f'(t)$

3.2.2 Integrals

There are different ways to play with integrals for vector functions, each has its own interpretation and physical applications.

3.2.2.1 Indefinite integral

$$\int_a^b \mathbf{r}(t)\,dt = \left\langle \int_a^b r_1(t)\,dt, \int_a^b r_2(t)\,dt, \int_a^b r_3(t)\,dt \right\rangle$$

3.2.2.2 Arc Length and curvature

Definition 3.4. The length the curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ is defined to be

$$L = \int_a^b |\mathbf{r}'(t)|\; dt\,.$$

If one wants to keep track the length of the curve $\mathbf{r} : [a, b] \to \mathbb{R}^n$ made by an airplane at any time t, one uses the **arc length function**

$$\ell(t) = \int_a^t |\mathbf{r}'(u)|\; du\,.$$

For those who are interested in the geometrical meaning of the curvature without having to accept from the book that the curvature is the inverse of the radius of the osculating circle, please take a look at this expository note.

Re-parametrize with respect to arc length The nice thing about $s(t)$ is that it is a strictly increasing function with respect to t, given that $\mathbf{r}'$ is non-zero for all t. Therefore, we can talk about the inverse of ℓ, $\ell^{-1} : [0, L] \to [a, b]$

$$t = \ell^{-1}(s)\,.$$

Therefore, we can re-write

$$\mathbf{r}(t) = \mathbf{r}(\ell^{-1}(s))\,.$$

Theorem 3.3.

$$\left|\frac{dr(t)}{ds}\right| = 1\,.$$

Thus,

$$l(s) = \int_0^s \left|\frac{d}{ds}\mathbf{r}(t)\right| \, dt = s\,.$$