Multivariable Calculus
Day 24
Vector Calculus: Surface integrals

Spring 2023

Curl and divergence

$$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\,.$$

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\,.$$

`https://www.youtube.com/watch?v=rB83DpBJQsE`

Orientation of a surface

Given a surface S, we define the orientation of it as following

1. If S has a boundary, then the **positive orientation** of the surface is that when one walks along the boundary of the surface with the head points in that direction, the surface is on the left.
2. If S does not have a boundary, then the **positive orientation** is the direction of the outward normal vector.

Unless specified otherwise, the normal vector of a surface is conventionally be thought of as pointing in the positive direction.

Parametrization

$$\mathbf{r} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3 \,.$$

We often write

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \,.$$

Surface integral

Definition

Let S be a surface with parametrization. The surface integral of f over the surface S is

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}\,.$$

Similarly to the line integral, one can show that

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r_u} \times \mathbf{r_v}|\,dA\,.$$

Definition
If $\mathbf{F}$ is a continuous vector field on an oriented surface S (parametrized by $\mathbf{r}(u, v)$) with unit normal vector $\mathbf{n}$, then the **surface integral of F over** S is

$$\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA \, .$$

The integral is called the **flux of F** across S.

Stokes' Theorem

Theorem
Let S be an oriented smooth surface that is bounded by a simple closed smooth boundary curve ∂S with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}\,.$$

https://www.youtube.com/watch?v=LqNqqidw2mg

Divergence Theorem

Theorem
Let E be a simple solid region and let surface ∂E be the boundary of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives. Then,

$$\iint_{\partial E} \mathbf{F} \cdot d\mathbf{S} = \iiint_{E} \operatorname{div} \mathbf{F} \, dV \, .$$

`https://www.youtube.com/watch?v=TORt2O_HjMY`