Multivariable Calculus
Day 8
Limits and Continuity

Truong-Son Van

Spring 2023

Fulbright University Vietnam

Limits

Definition
Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x, y) - L| < \epsilon$ if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$.

Sadly, we will not go deeply about this concept because it requires real analysis and we have bigger fishes to fry.

What we will learn:

1. Simple cases when limits exist
2. Typical cases when limits don't exist

Simple cases when limits exist

Theorem
Let L, M and k be real numbers and that

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\,, \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M\,.$$

We then have

1. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)+g(x,y)) = L + M,$
2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (kf(x,y)) = kL,$
3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)g(x,y)) = LM,$
4. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$ *if* $M \neq 0,$
5. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y)^p = L^p$ *for* $p > 0,$

Worksheet

Determine if the limit exists and if it is, find it.

1. $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3},$$
2. $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2},$$
3. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}},$$
4. $$\lim_{(x,y)\to(0,0)} \frac{x}{y}.$$