

MATH 104: Multivariable Calculus (brief notes)

Truong-Son Van

Contents

1	Vectors	2
1.1	Basics	2
	Rules to manipulate vectors	2
	Properties of vector operations	3
1.2	Products	3
2	Some basic equations in \mathbb{R}^3	5
2.1	Equations for lines	5
2.2	Equations for planes	5
2.3	Cylinders	5
2.4	Quadric surfaces	6
3	Functions of several variables	7
3.1	Functions of several variables	7

Spring 2023

1 Vectors

1.1 Basics

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

You should be able to answer the following questions after reading this section:

- What is a vector?
- What does it mean for two vectors to be equal?
- How do we add two vectors together and multiply a vector by a scalar?
- How do we determine the magnitude of a vector?
- What is a unit vector
- How do we find a unit vector in the direction of a given vector?

Typically, we talk about 3-dimensional vectors (as discussed in Stewart and Thomas). However, since talking about n -dimensional vectors doesn't require much more effort, we will talk about n -dimensional vectors instead.

Definition 1.1. An n -dimensional Euclidean space \mathbb{R}^n is the Cartesian product of n Euclidean spaces \mathbb{R} .

Definition 1.2. An n -dimensional vector $\mathbf{v} \in \mathbb{R}^n$ is a tuple

$$\mathbf{v} = \langle v_1, \dots, v_n \rangle, \quad (1)$$

where $v_i \in \mathbb{R}$.

In dimensions less than or equal to 3, we represent a vector geometrically by an arrow, whose length represents the magnitude.

Remark. A point in \mathbb{R}^n is also represented by an n -tuple but with round brackets. A vector connecting two points $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$ can be constructed as

$$\mathbf{x} = \langle b_1 - a_1, \dots, b_n - a_n \rangle.$$

We denote the above vector as \vec{AB} where A is the tail (initial point) and B is the tip/head (terminal point). We denote $\mathbf{0}$ to be the zero vector, i.e.,

$$\mathbf{0} = \langle 0, \dots, 0 \rangle.$$

Definition 1.3. The length of a vector \mathbf{v} (denoted by $|\mathbf{v}|$) is defined to be

$$|\mathbf{v}| = \sqrt{v_1^2 + \dots + v_n^2}. \quad (2)$$

Definition 1.4. A unit vector is a vector that has magnitude 1.

Exercise 1.1. Turn a vector $\mathbf{v} \in \mathbb{R}^n$ into a unit vector with the same direction.

Rules to manipulate vectors

Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$. Then,

$$c(\mathbf{a} + \mathbf{b}) = \langle ca_1 + cb_1, \dots, ca_n + cb_n \rangle = c\mathbf{a} + c\mathbf{b},$$

and

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}.$$

These formulas are deceptively simple. Make sure you understand all the implications.

Because of this rule, sometimes it is good to write vectors in terms of elementary vectors:

$$\mathbf{u} = u_1 \mathbf{e}_1 + \cdots + u_n \mathbf{e}_n,$$

where $e_i = \langle 0, \dots, 1, \dots, 0 \rangle$ is the vector which has zero at all entries except that the i^{th} entry is 1.

In 3D,

$$\mathbf{e}_1 = \mathbf{i}, \quad \mathbf{e}_2 = \mathbf{j}, \quad \mathbf{e}_3 = \mathbf{k}.$$

Properties of vector operations

Read the book

(Make sure you understand the geometric interpretation)

1.2 Products

1.2.1 Dot product

- How is the dot product of two vectors defined and what geometric information does it tell us?
- How can we tell if two vectors in \mathbb{R}^n are perpendicular?
- How do we find the projection of one vector onto another?

Definition 1.5. The dot product of vectors $\mathbf{u} = \langle u_1, \dots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \dots, v_n \rangle$ in \mathbb{R}^n is the scalar

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n.$$

Properties of dot product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then,

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$,
2. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$,
3. If c is a scalar, then $(c\mathbf{u}) \cdot \mathbf{w} = c(\mathbf{u} \cdot \mathbf{w})$.

Theorem 1.1 (Law of cosine). *If θ is the angle between the vectors \mathbf{u} and \mathbf{v} , then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta.$$

Corollary 1.1. *Two vectors \mathbf{u} and \mathbf{v} are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$.*

Projection

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. The component of \mathbf{u} in the direction of \mathbf{v} is the scalar

$$\text{comp}_{\mathbf{v}} \mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|},$$

and the projection of \mathbf{u} onto \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\mathbf{u} \cdot \frac{\mathbf{v}}{|\mathbf{v}|} \right) \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v}.$$

Read the book for more details. Make sure you understand the geometric meaning.

1.2.2 3D special: Cross product

This concept is very specific to \mathbb{R}^3 . It will not make sense in other dimensions.

Definition 1.6. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$. The cross product of \mathbf{a} and \mathbf{b} is defined to be

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle.$$

Theorem 1.2. Let θ be the angle between \mathbf{a} and \mathbf{b} . Then,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin \theta.$$

Theorem 1.3. The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} .

2 Some basic equations in \mathbb{R}^3

Just to build some toy examples for the future, we will play with some basic equations in three dimensions.

2.1 Equations for lines

A line is a collection of points that is parallel to a vector and goes through a specific point. To capture this idea, we have the following representation for a line

$$L = \{\mathbf{r}(t) \mid \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, t \in \mathbb{R}\},$$

where \mathbf{r}_0 is the initial position and \mathbf{v} is the direction. The equation for $\mathbf{r}(t)$ is called a **vector equation for a line** L .

Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{r}_0 = (x_0, y_0, z_0)$. The **parametric equations** of L is the following system of equations

$$\begin{aligned}x &= x_0 + v_1 t, \\y &= y_0 + v_2 t, \\z &= z_0 + v_3 t.\end{aligned}$$

This leads to the **symmetric equations** of L

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}.$$

2.2 Equations for planes

A plane is a collection of points that is perpendicular to one specific direction represented by a some vector called a **normal vector**. Note that due to scaling, there are more than one normal vector. To capture this idea, we have the following representation of a plane

$$P = \{\mathbf{r} \mid \mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0\}.$$

This is called a **vector equation for the plane** P .

Multiplying things out, we have the **scalar equation of the plane** P with normal vector $\mathbf{n} = \langle n_1, n_2, n_3 \rangle$ through a point $P_0(x_0, y_0, z_0)$

$$n_1(r_1 - x_0) + n_2(r_2 - y_0) + n_3(r_3 - z_0) = 0.$$

2.3 Cylinders

Definition 2.1. A cylinder is a surface that consists of all lines (called **rulings**) that are parallel to a given line.

Example 2.1.

1. $z = x^2$
2. $x^2 + y^2 = 1$

2.4 Quadric surfaces

Definition 2.2. A quadric surface is the graph of a second-degree equation in three variables x, y and z . The equation that represents these surfaces is

$$Ax^2 + By^2 + Cz^2 + Dz = E.$$

Example 2.2.

1. Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

2. Hyperbolic paraboloid

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}.$$

3. Elliptical cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

Read the books for more surfaces and pictures.

3 Functions of several variables

Reading: Stewart Chapter 12, Thomas Calculus Chapter 12, Active Calculus Chapter 9

3.1 Functions of several variables

$f : D \rightarrow C$ where $D \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^m$.

$$f(x) = f(x_1, \dots, x_n).$$

D is called the domain of f and C is called the codomain of f .

Example 3.1. The following are some examples of multivariable functions

1. $f(x, y) = x^2 - 2xy + y^2$
2. $f(x, y, z) = \frac{1}{1-xy^2}$
3. $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$

The expression in the vector equation for a line is an example of a function that maps from \mathbb{R} to \mathbb{R}^n . There's no one who would stop us from considering more general kinds of function.

Definition 3.1. A **vector function** (**vector-valued function**) is a function that has the codomain that belongs to \mathbb{R}^n where $n \geq 2$.

In this course, whenever we talk about vector valued function, we will only refer to that which has one dimensional domain ($D \subseteq \mathbb{R}$).