Multivariable Calculus
Day 14
Optimization (cont.)

Spring 2023

Basics

Definition
A function $f : D \to \mathbb{R}$ has a **local maximum** at $\mathbf{x_0}$ if $f(\mathbf{x_0}) \geq f(\mathbf{x})$ for $\mathbf{x} \in B_\delta(\mathbf{x_0})$ for small enough δ. f has a **global maximum** at $\mathbf{x_0}$ if $f(\mathbf{x_0}) \geq f(\mathbf{x})$ for $\mathbf{x} \in D$. f has a **local (global) minimum** at $\mathbf{x_0}$ if $-f$ has a local (global) maximum at $\mathbf{x_0}$

A necessary condition

Theorem (First derivative test)
Let $f : D \to \mathbb{R}$ be a function. If $\mathbf{x_0}$ is a local minimum and f has partial derivatives at $\mathbf{x_0}$. Then

$$\partial_{x_i} f(\mathbf{x}_0) = 0\,.$$

Definition
$\mathbf{x}_0$ is said to be a **critical point** of $f : D \to \mathbb{R}$ if

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or one of the partial derivatives $\partial_{x_i} f(\mathbf{x}_0)$ fails to exist.

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Worksheet: Find critical points of the following functions

1. $f(x, y) = |x| + |y|$
2. $f(x, y) = 2xy - 4x + 2y - 3$
3. $f(x, y) = x^2 + y^2 - qxy$ where $q \in \mathbb{R}$ is a given constant

Second derivative test

Suppose the second partial derivatives of f are continuous near (a, b) and suppose that (a, b) is a critical point of f. Let

$$D = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2 .$$

1. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
2. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
3. If $D < 0$, then $f(a, b)$ is neither a local maximum nor local minimum.
4. If $D = 0$, then we cannot conclude.

D is called the discriminant of the function f at (a, b).

Examples

Find the critical points of the following functions and use the second derivative test to classify them

1. $f(x, y) = 3x^2 + y^2 - 9x + 4y$
2. $f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$

Theorem (Extreme value theorem)
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Algorithm to find absolute maxima and minima on closed bounded regions

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Worksheet

Suppose the temperature T at each point on a disk of radius 1 is given by

$$T(x, y) = 2x^2 + y^2 - y\,.$$

What would be the hottest and coldest point on this disk?

Constrained optimization

Constrained optimization takes various forms, depending on the assumptions. We will deal with the most straight forward form. The problem we will study is the following:

Maximize/minimize a function $f : D \to \mathbb{R}$, subject to a constraint (side condition) of the form $g(\mathbf{x}) = k$, for some constant $k \in \mathbb{R}$.

Typically, people will write as follows

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) = k\,. \end{aligned}$$

Constrained optimization

Theorem (Method of Lagrange Multiplier)
Suppose the maximum/minimum values of f *exist and* $\nabla g(\mathbf{x}) \neq 0$ *where* $g(\mathbf{x}) = k$. *To find the maximum and minimum values of* f *subject to constraint* $g(\mathbf{x}) = k$, *we do the following:*

1. *Find all values of* $\mathbf{x}$ *and* $\lambda \in \mathbb{R}$ *such that*

$$\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}),$$

   and

$$g(\mathbf{x}) = k.$$

2. *Evaluate* f *at all the points* $\mathbf{x}$ *that result from step 1. The largest of these values is the maximum of* f; *the smallest is the minimum value of* f.

Example

https://youtu.be/hQ4UNu1P2kw