MATH 312: HOMEWORK 1

In all the questions below, show your work.

Question 1. *Let X be a set. Let $\{\tau_\alpha\}$ be a family of topologies on X.*

(1) Is $\cap_{\alpha \in I} \tau_\alpha$ a topology on X?
(2) Is $\cup_{\alpha \in I} \tau_\alpha$ a topology on X?

Question 2. *Compare the following topologies on the real line $\mathbb{R}$. Are they comparable? Which one is finer/coarser than the others?*

(1) τ_1 is the topology is generated by the base $\mathcal{B}_1 = \{(a,b) : a, b \in \mathbb{R}\}$. This is the standard topology on $\mathbb{R}$.
(2) τ_2 is the topology is generated by the base $\mathcal{B}_2 = \{[a,b) : a, b \in \mathbb{R}\}$. This is called the lower limit *topology on $\mathbb{R}$.*
(3) Let $K = \{1/n : n \in \mathbb{N}\}$. τ_3 is the topology generated by the base $\mathcal{B}_3 = \{(a,b) \backslash K : a, b \in \mathbb{R}\} \cup \mathcal{B}_1$. This is called the K topology on $\mathbb{R}$.

Question 3. *Compare the box topology and the product topology on $\mathbb{R}^{\mathbb{N}}$.*

Question 4. *Given topological space (X, τ). Let A be a subset of X. Show that $\bar{A}$ (the closure of A) is the smallest closed set in X that contains A.*

Question 5. *Consider the metric spaces $(\mathbb{R}^N, d_p)$, where $p \in [1, \infty)$*

$$d_p(x,y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p \right)^{1/p} .$$

(1) Show that for $p > q \geqslant 1$,

$$d_p(x,y) \leqslant d_q(x,y) .$$

(2) Explain why we can now define

$$d_\infty(x,y) = \lim_{p \to \infty} d_p(x,y) ?$$

(3) Show that d_∞ is a metric on $\mathbb{R}^N$.
(4) Show that $d_\infty(x,y) = \max_i |x_i - y_i|$.
(5) Let τ_p be the topology associated with the metric d_p. Is it true that $\tau_p = \tau_q$ for all $p, q \in [1, \infty)$?
(6) Is it true that $\tau_p = \tau_\infty$ for all $p \in [1, \infty)$?

Question 6. *Exercise 20.2 in Bass.*

Question 7. *Exercise 20.3 in Bass.*

Question 8. *Exercise 20.4 in Bass.*

Question 9. *Exercise 20.5 in Bass.*

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Question 10. *Exercise 20.6 in Bass.*

Question 11. *Exercise 20.11 in Bass.*

Question 12. *Exercise 20.12 in Bass.*