MATH 312: MIDTERM

NAME: ________________________________

Instructions

(1) Pick 5 out of 6 questions to do. Specify your choices. If you don't I'll take the minimum of all possible points.
(2) Show your work, be as thorough as possible.
(3) You have 80 minutes

Date: March 13, 2025.

Question 1 (Quick questions, 5 points each). *Complete the following:*

(1) *Give the definition of a topological space.*
(2) *Give the definition of a measure space (include definitions of sigma algebra and measure).*
(3) *Give an example of a topological space where every set is a clopen set.*
(4) *What does it mean for a set to be compact in a topology?*
(5) *What is a continuous function from* (X, τ_X) *to* (Y, τ_Y)*?*

Question 2 (True or False? Justify. 20 points.). *A set* K *in a topological space* (X, τ) *is closed if it is compact.*

Question 3 (True or False? Justify. 20 points.). *A closed subset of a compact set is compact.*

Question 4 (True or False? Justify. 20 points.). *Let* $f : (X, \tau_X) \to (Y, \tau_Y)$ *be continuous. If* K *is compact, then* $f(K)$ *is compact.*

Question 5 (20 points). *Let* (X, τ_X) *and* (Y, τ_Y) *be topological spaces where* Y *is compact. If* N *is an open set of* $X \times Y$ *containing the slice* $\{x_0\} \times Y$ *of* $X \times Y$*. Prove that there exists a neighborhood* W *of* x_0 *such that* $W \times Y \subseteq N$*.*

Question 6 (20 points). *Complete the following:*

(1) *Prove that if* c_{nj} *are non-negative real numbers that increase in* n *to* c_j *for each* j*, then*
$$\lim_{n\to\infty} \sum_{j=1}^{\infty} c_{nj} = \sum_{j=1}^{\infty} c_j \,.$$
(2) *(True or False? Justify) Suppose* $\mu_1, \mu_2, \ldots$ *are measures on a measurable space* $(X, \mathcal{A})$ *and* $\mu_n(A)$ *is increasing for each* $A \in \mathcal{A}$*. Define*
$$\mu(A) = \lim_{n\to\infty} \mu_n(A)\,.$$
Then, μ *is a measure on* $(X, \mathcal{A})$*.*