MATH 312: HOMEWORK 2

In all the questions below, show your work.

Question 1. *Show that every metric space is normal.*

Question 2. *Show that every compact Hausdorff space is normal.*

Question 3. *Let* $\{f_n : K \to \mathbb{R}\}_{n\in\mathbb{N}}$ *be a sequence of L-Lipschitz functions where* $K \subset \mathbb{R}$ *is a compact set. Show that* $\{f_n\}$ *is equicontinuous.*

Question 4. *Prove/disprove the following statement: the sequence of functions* $f_n : \mathbb{R} \to \mathbb{R}$ *defined by*

$$f_n(x) = \cos(n+x) + \ln\left(1 + \frac{1}{n+2}\sin^2(n^n x)\right)$$

is equicontinuous.

Question 5. *Let* (M, d) *be a metric space. A family of functions* $\mathcal{F}$ *mapping from* $M \to \mathbb{R}$ *is said to be pointwise equicontinuous if for every* $x \in M$ *and* $\varepsilon > 0$ *there exists* $\delta_x > 0$ *such that for every* $f \in \mathcal{F}$, $|f(x) - f(y)| < \varepsilon$ *whenever* $d(x, y) < \delta_x$.

Suppose that every pointwise equicontinuous sequence of functions $M \to \mathbb{R}$ *is uniformly equicontinuous. Does this mean* (M, d) *is a compact metric space?*

Question 6. *A continuous, strictly increasing function* $\mu : (0, \infty) \to (0, \infty)$ *is called a* modulus of continuity *if* $\mu(s) \to 0$ *as* $s \to 0$. *A function* $f : [a, b] \to \mathbb{R}$ *has modulus of continuity* μ *if* $|f(s) - f(t)| \leqslant \mu(|s - t|)$ *for all* $s, t \in [a, b]$. *Prove*

(1) A function is uniformly continuous if and only if it has a modulus of continuity.

(2) A family of functions is equicontinuous if and only if its members have a common modulus of continuity.

Question 7. *(Challenging) Given* $\lambda \in (0, 1)$, *a function* $u_\lambda : B(0, 1) \subseteq \mathbb{R} \to \mathbb{R}$ *is said to be a solution to a static discount Hamilton-Jacobi equation*[1] *on the ball* $B(0, 1) \subseteq \mathbb{R}$

$$\lambda u_\lambda(x) + H(u'_\lambda(x)) = 1 \qquad \textit{when } x \in B(0,1)\,.$$

Here, u'_λ *denotes the derivative of* u_λ. *The function* $H : \mathbb{R} \to \mathbb{R}$ *is a continuous function called the Hamiltonian of the equation and typically has the following property*

$$\text{(H)} \qquad \lim_{|p|\to\infty} H(p) = \infty\,.$$

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[1]This equation is fundamental in optimal control theory and all kinds of engineering

Suppose magically that you know that there exists a number $M > 0$ *such that for every* $\lambda \in (0,1)$,

$$\|u_\lambda\|_\infty \leqslant M\,.$$

Assuming (H)*, show that the family* $\{u_\lambda\}$ *is uniformly equicontinuous.*

Deduce then there exists a function $\bar{u}$ *such that for some subsequence* $\lambda_j \to 0$ *as* $j \to \infty$,

$$\lim_{j\to\infty} \|u_{\lambda_j} - \bar{u}\|_\infty = 0\,.$$

Guess (NO PROOF[2]) the equation for $\bar{u}$.

[2] A proof for this fact will result in an automatic A+ in this class