Convergence rate to equilibrium to a Hamilton-Jacobi equation

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1 Introduction

We consider the following Hamilton-Jacobi equation

eq:main (1.1)
$$\begin{cases} \partial_t u(x,t) + \dfrac{u_x(x,t)^2}{2} + c(x)u(x,t) = 0 \\ u(x,0) = u_0(x)\,. \end{cases} \tag{1.1}$$

The questions at hand are that suppose $c(x) \to 0$ as $x \to \infty$, e.g., $c(x) = \frac{1}{x}$, what are the

- Wellposedness of stationary solution.
- Rate of convergence to stationary solution.

There are papers that show exponential convergence of (1.1) when $c(x) \geqslant \gamma > 0$ but nothing if $c(x)$ vanishes. In particular, suppose $c(x) \stackrel{\text{def}}{=} 1$, then the problem

$$\bar{u} + \frac{\bar{u}_x^2}{2} = 0 \tag{1.2}$$

has a unique solution because we can use comparison prinple here. Then, it is known that

$$\lim_{t\to\infty} |u(x,t) - \bar{u}(x)| \leqslant C_1 e^{-C_2 t}\,.$$

See [FL09]. The proof of this boils down to constructing supersolution, which is not hard.

2 Stationary problem

We consider the ergodic problem

eq:ergodic

$$\text{(E)} \qquad \frac{\bar{u}}{x} + H(\bar{u}_x) = 0 \qquad \text{for } x \in \mathbb{R}\,.$$

This equation when $H(p) = p^2$ has two smooth solutions, $\bar{u} = x$ and $\bar{u} = 0$. Thus, to study wellposedness, we ought to impose more condition, e.g. sublinear property. To this end, we can use the cut-off trick in the paper [TV20].

References

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