Convergence rate to equilibrium to a Hamilton-Jacobi equation

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September 24, 2020

1 Introduction

We consider the following Hamilton-Jacobi equation

eq:main

$$\begin{cases} \partial_t u(x,t) + \dfrac{u_x(x,t)^2}{2} + c(x)u(x,t) = 0 \\ u(x,0) = u_0(x)\,. \end{cases} \tag{1.1}$$

The questions at hand are that suppose $c(x) \to 0$ as $x \to \infty$, e.g., $c(x) = \frac{1}{x}$, what are the

- Wellposedness of stationary solution.
- Rate of convergence to stationary solution.

There are papers that show exponential convergence of (1.1) when $c(x) \geqslant \gamma > 0$ but nothing if $c(x)$ vanishes. In particular, suppose $c(x) \stackrel{\text{def}}{=} 1$, then the ergodic problem

$$\frac{\bar{u}_x^2}{2} + \bar{u}(x) = c \tag{1.2}$$

has a unique(?) solution. Then, it is known that

$$\lim_{t\to\infty} |u(x,t) - \bar{u}(x)| \leqslant C_1 e^{-C_2 t}\,.$$

See [FL09].

2 Ergodic problem

We consider the ergodic problem

eq:ergodic

$$\frac{u}{x} + H(u_x) = c \qquad \text{for } x \in \mathbb{R}\,. \tag{E}$$

We study the wellposedness of this equation by the method of vanishing viscosity. In particular, by classical theory of elliptic equations, the equation

eq:ergodic-eps

$$\frac{u}{x} + H(u_x) + \varepsilon u_{xx} = 0 \quad \text{for } x \in \mathbb{R} \tag{E$_\varepsilon$}$$

is wellposed. Thus, for each $\varepsilon > 0$, let u^ε be the unique solution of (E$_\varepsilon$).

Lemma 2.1. *There exists a constant c, a function $\bar{u}$ and a subsequence $\{\varepsilon_i\}_{i\in\mathbb{N}}$ such that*

$$\frac{\bar{u}}{x} + H(\bar{u}_x) = c\,, \tag{2.1}$$

and $u_{\varepsilon_i} \to \bar{u}$ locally uniformly.

References

FujitaLoreti2009

[FL09] Yasuhiro Fujita and Paola Loreti. “Long-time behavior of solutions to Hamilton-Jacobi equations with quadratic gradient term”. In: *NoDEA. Nonlinear Differential Equations and Applications* 16.6 (2009), pp. 771–791. ISSN: 1021-9722. DOI: 10.1007/s00030-009-0034-9.