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ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF HAMILTON–JACOBI EQUATIONS*

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Abstract. We consider the long time behavior of viscosity solutions of first-order Hamilton–Jacobi equations with periodic space dependence. We prove, under sharp conditions, that as time goes to infinity, solutions converge to solutions of the corresponding stationary equation.

Key words. Hamilton–Jacobi equations, periodicity, ergodic problem, long time behavior, viscosity solutions, hamiltonian systems

AMS subject classifications. 70H20, 58F11, 58F05, 49L25

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1. Introduction.

In this article we are interested in the behavior, as $t \to +\infty$, of the viscosity solutions of first-order Hamilton–Jacobi equations of the form

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty)\ , \tag{1.1}$$

$$u = u_0 \quad \text{on } \mathbb{R}^N \times \{0\}\ , \tag{1.2}$$

where the hamiltonian H, the initial datum u_0, and the solution u are assumed to be real-valued continuous functions and $Du = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N})$ denotes the gradient of u.

Throughout the paper we suppose that both H and u_0 are $\mathbb{Z}^N$-*periodic* in x, i.e., that for all $x, p \in \mathbb{R}^N$ and $z \in \mathbb{Z}^N$,

$$H(x + z, p) = H(x, p) \quad \text{and} \quad u_0(x + z) = u_0(x)\ . \tag{1.3}$$

We also assume that a comparison (uniqueness) result holds for (1.1)–(1.2). The first consequence of this assumption is the $\mathbb{Z}^N$-periodicity in x of the solution for any $t > 0$.

The study of the long time behavior of solutions of (1.1) and (1.2) first leads to an *ergodic* problem. Indeed, the first step is to show the existence of a constant c_0, depending only on H and not u_0, such that the function $u(\cdot, t) + c_0 t$ remains bounded, as $t \to +\infty$. The classical result in this direction is due to Lions, Papanicolaou, and Varadhan [7], who obtained the existence of such a constant c_0, the so-called *ergodic cost*, under the following coercivity assumption on H:

$$H(x, p) \to +\infty \quad \text{when } |p| \to +\infty, \text{ uniformly in } x \in \mathbb{R}^N. \tag{1.4}$$

Another way to define c_0 is by using the stationary Hamilton–Jacobi equation. Indeed, c_0 is the unique constant c for which the equation

$$H(x, Du) = c \quad \text{in } \mathbb{R}^N \tag{1.5}$$

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has a continuous, periodic viscosity solution.

It is also worth pointing out that if $u_0 \in W^{1,\infty}(\mathbb{R}^N)$, then (1.4) yields that the function $(x,t) \mapsto u(x,t) + c_0 t$ is in $W^{1,\infty}(\mathbb{R}^N \times [0,+\infty))$. This is a key fact, since it provides the compactness in $C(\mathbb{R}^N)$ of the functions $u(\cdot,t) + c_0 t$ for $t > 0$, a fact which is essential in the study of the behavior of these functions when $t \to +\infty$. Before coming back to this question, which is the central purpose of our work, we mention that such an ergodic problem in the deterministic control framework was systematically studied by Arisawa [1, 2].

In this article we are interested in the next step, i.e., in the behavior of $u(\cdot,t)+c_0 t$ as $t \to +\infty$. To simplify the exposition, we are going to assume, without any loss of generality, that $c_0 = 0$. With this convention, the question we address here can be formulated in the following way:

Assume that $u \in W^{1,\infty}(\mathbb{R}^N \times (0,+\infty))$ *or* $u \in BUC(\mathbb{R}^N \times (0,+\infty))$. *Is it true that, as* $t \to \infty$,

$$u(\cdot,t) \to u_\infty(\cdot) \quad \textit{in } C(\mathbb{R}^N)\ , \tag{1.6}$$

where u_∞ *is a viscosity solution of the stationary equation*

$$H(x,Du) = 0 \quad \textit{in } \mathbb{R}^N\ ? \tag{1.7}$$

The apparent simplicity of this question is misleading. In fact, this problem has remained open for a long time. The first results on the asymptotic behavior of viscosity solutions of Hamilton–Jacobi equations were obtained in Lions [6] and in Barles [3] essentially for either x-independent cases or equations involving a suitable dependence on u. It was only very recently that Namah and Roquejoffre [8] and Fathi [5] succeeded in proving rather general results related to the above questions, which we now briefly describe and compare.

The results of [5] and [8] are obtained for equations set on compact manifolds and for hamiltonians which are convex in the p-variable and satisfy (1.4). The result of [5] was proved under the additional assumption that H is smooth and strictly convex; i.e., there exists a constant $\alpha > 0$ such that

$$D^2_{pp}H(x,p) \geq \alpha I \quad \text{in } \mathbb{R}^N \times \mathbb{R}^N\ . \tag{1.8}$$

The proof relies on the representation of the solution u by the so-called Oleinik–Lax formula and is based on dynamical systems methods. In particular, [5] emphasizes the central role played by the Aubry–Mather set, an attractor set for the geodesics associated with the Lax–Oleinik formula. This result was revisited recently by Roquejoffre [10], who uses a combination of partial differential equations and dynamical systems methods. (See also Roquejoffre [9] for results in dimension 1.)

The approach of [8] is based on partial differential equations methods and requires a condition, which in the $\mathbb{R}^N$-framework can be stated as follows:

$$\begin{cases} \text{There exists a } C^1\text{-function } \phi \in BUC(\mathbb{R}^N) \text{ such that} \\ H(x,D\phi(x)) \leq 0 \quad \text{in } \mathbb{R}^N \text{ and} \\ \quad H(x,p+D\phi(x)) > H(x,D\phi(x)) \text{ for all } x \in \mathbb{R}^N \text{ and } p \in \mathbb{R}^N\backslash\{0\}. \end{cases} \tag{1.9}$$

The two key arguments of [8] are that $u(\cdot,t)$ is decreasing (and therefore uniformly convergent) on the set $K = \{x \in \mathbb{R}^N : H(x,D\phi(x)) = 0\}$—note that this set is necessarily a nonempty subset of $\mathbb{R}^N$ as a consequence of the fact that $c_0 = 0$—and that there is a strong comparison principle for the Dirichlet problem

$$H(x, Dw) = 0 \ \text{ in } \mathbb{R}^N \backslash K, \qquad w = \varphi \ \text{ on } \partial K,$$

where φ is a continuous function. This last property holds because ϕ is a (local) strict subsolution of the equation in $\mathbb{R}^N \backslash K$. The strong comparison principle for viscosity solutions then allows the use of the half-relaxed limits methods.

Here we provide a generalization of these two types of results. In particular, we are able to treat hamiltonians which are not necessarily convex and to remove the assumptions on the regularity of H and ϕ. Moreover the $\mathbb{Z}^N$-periodic setting we chose here for the sake of simplicity can be replaced without any additional difficulty by a general compact manifold one. It is, however, worth mentioning that some kind of compactness assumption on the domain is necessary at least to apply our strategy of proof.

Our main argument, which is completely different from those given in [5], [8], [9], and [10] can be described roughly in the following way: We first show that

$$||(u_t)^-(\cdot, t)||_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{ as } t \to +\infty \ . \tag{1.10}$$

For the reader's convenience we provide in section 3, under simplified assumptions, a formal argument which shows why such property should be true. In fact, the formulation of the precise results (Theorems 3.1 and 3.2) is a bit more general but unfortunately rather technical. The main consequence of (1.10) is that the ω-limit set of the function $u(\cdot, t)$ contains only subsolutions of (1.7). In turn, this property is enough to prove (1.6). It is in this last step that the compactness property of the domain seems to play a key role.

This paper is organized as follows: In section 2 we state the assumptions and the main results of the paper. Section 3 is devoted to the statement of the weak versions of (1.10) which are proved in the appendix. In section 4 we prove the main results and in section 5 we discuss the main assumptions on the hamiltonian and some extensions.

2. The main results and their applications. To formulate the main results we recall that we are interested in the asymptotic behavior of solutions $u \in BUC(\mathbb{R}^N \times [0, \infty))$ of the initial value problem

$$\begin{cases} u_t + H(x, Du) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ u = u_0 & \text{on } \mathbb{R}^N \times \{0\}, \end{cases} \tag{2.1}$$

under the assumptions that

(H1) $\left\{\begin{array}{l} H \text{ is continuous in } \mathbb{R}^N \times \mathbb{R}^N \text{ and } \mathbb{Z}^N\textit{-periodic} \text{ with respect to } x, \\ \text{i.e., for all } x, p \in \mathbb{R}^N \text{ and } z \in \mathbb{Z}^N, \\ \qquad\qquad H(x + z, p) = H(x, p). \end{array}\right.$

We also assume the following.

(H2) There exists a viscosity subsolution $\phi \in BUC(\mathbb{R}^N)$ of $H(x, D\phi) \le 0 \quad \text{in } \mathbb{R}^N$.

(H3) $\left\{\begin{array}{l} \text{Either } u \text{ and } \phi \text{ are in } W^{1,\infty}(\mathbb{R}^N \times (0, \infty)) \text{ or there exists a} \\ \text{continuous function } m : [0, +\infty) \to \mathbb{R}^+ \text{ such that } m(0^+) = 0 \\ \text{and, for all } x, y \in \mathbb{R}^N \text{ and } p \in \mathbb{R}^N, \\ \qquad\qquad |H(x, p) - H(y, p)| \le m(|x - y|(1 + |p|)), \end{array}\right.$

and

$$\text{(H4)}\quad \begin{cases} \text{there exist } \eta > 0 \text{ and } \psi(\eta) > 0 \text{ such that, if } H(x, p+q) \geq \eta \text{ and} \\ H(x,q) \leq 0 \text{ for some } x \in A \subset \mathbb{R}^N,\ p, q \in \mathbb{R}^N, \text{ then, for all } \mu \in (0,1], \\ \\ \qquad \mu H\Big(x, \mu^{-1}p + q\Big) \geq H(x, p+q) + \psi(\eta)(1-\mu). \end{cases}$$

Note that if H is C^1 in p, then (H4) reduces to

$$\text{(H4)}'\quad \begin{cases} \qquad H_p(x, p+q) \cdot p - H(x, p+q) \geq \psi(\eta), \\ \\ \text{for any } x \in A,\ p, q \in \mathbb{R}^N \text{ such that } H(x, p+q) \geq \eta \text{ and } H(x,q) \leq 0. \end{cases}$$

The final assumption is as follows.

$$\text{(H5)}\quad \begin{cases} \text{There exists a, possibly empty, compact subset } K \text{ of } \mathbb{R}^N \text{ such that} \\ \\ \text{(i)} \quad H(x,p) \geq 0 \text{ on } K \times \mathbb{R}^N, \text{ and} \\ \\ \text{(ii)} \quad \text{for all } \delta > 0, \text{ (H4) holds with } A = (K_\delta)^c \text{ for all } \eta > 0, \\ \qquad \text{where } (K_\delta)^c = \{x \in \mathbb{R}^N : d(x, K) > \delta\} \text{ with } \psi \text{ depending on } \delta. \end{cases}$$

The result about the asymptotic behavior of the solution u of (1.1) and (1.2) is the following.

THEOREM 2.1. *Assume that* (H1)–(H3) *and* (H5) *hold. If* $u \in BUC(\mathbb{R}^N \times (0,\infty))$ *is a* $\mathbb{Z}^N$*-periodic in* x *solution of* (1.1), *then there exists a* $\mathbb{Z}^N$*-periodic* $\overline{u} \in BUC(\mathbb{R}^N)$ *such that*

(i) $H(x, D\bar{u}) = 0$ *in* $\mathbb{R}^N$, *and*

(ii) $u(x,t) \to \bar{u}(x)$, *uniformly in* $\mathbb{R}^N$, *as* $t \to \infty$.

Before stating a variant of this result, which holds under simpler hypotheses on H, we want to point out that it is generally rather difficult to show that the solution u of (1.1)–(1.2) is actually in $BUC(\mathbb{R}^N \times (0,\infty))$. The classical existence results provide a solution which is only in $BUC(\mathbb{R}^N \times (0,T))$ for all $T > 0$ (see Barles [4]). To the best of our knowledge, as we already mentioned in the introduction, the only general result which gives the compactness in $C(\mathbb{R}^N)$ of the functions $u(\cdot,t)$—the important information—is the one obtained by Lions, Papanicolaou, and Varadhan [7] (see also Namah and Roquejoffre [8]) under the following assumption:

(H6) $\qquad H(x,p) \to \infty$ as $|p| \to \infty$, uniformly in $x \in \mathbb{R}^N$.

As a consequence of this, the reader can replace in any of our results the assumption "$u \in W^{1,\infty}(\mathbb{R}^N \times (0,\infty))$" with "(H6) and $u_0 \in W^{1,\infty}(\mathbb{R}^N)$" and in the same way "$u \in BUC(\mathbb{R}^N \times (0,\infty))$" with "(H6) and $u_0 \in BUC(\mathbb{R}^N)$," which implies in both cases the existence of such a solution.

It is also worth mentioning that if we assume that the sets $\{p \in \mathbb{R}^N : H(x,p) \leq 0\}$ are bounded uniformly for $x \in \mathbb{R}^N$ and that (H4) holds with $A = \mathbb{R}^N$, then (H6) are a direct consequence of (H4). Hence, in these cases we do not lose any generality by assuming (H6).

Finally we write about the existence of a BUC-subsolution ϕ as an assumption. In fact, this property together with the global boundedness of u is a direct consequence of the definition of c_0 (recall that we assume $c_0 = 0$).

To state a variant of Theorem 2.1 we introduce the following simpler hypotheses.

$$\text{(H7)}\quad \begin{cases} \text{There exists a family } (\phi_\varepsilon)_{\varepsilon>0} \text{ of } C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)\text{-functions, which} \\ \text{are uniformly bounded in } \varepsilon > 0 \text{ and satisfy} \\ \\ \qquad H(x, D\phi_\varepsilon) \leq \varepsilon \quad \text{in } \mathbb{R}^N. \end{cases}$$

(H8) For every $x \in \mathbb{R}^N$, the function $p \mapsto H(x,p)$ is locally Lipschitz.

$$
\text{(H9)} \quad \begin{cases} \text{There exists a, possibly empty, compact subset } K \text{ of } \mathbb{R}^N \text{ such that} \\ \text{(i)} \;\; H(x,p) \geq 0 \text{ on } K \times \mathbb{R}^N \text{, and} \\ \text{(ii)} \;\; \text{if } H(x,p) \geq \eta > 0 \text{ and } d(x,K) \geq \eta \text{, then for all sufficiently} \\ \qquad \text{small, compared to } \eta,\ \varepsilon > 0, \\ H_p(x,p) \cdot (p - D\phi_\varepsilon(x)) - H(x,p) \geq \psi(\eta) > 0 \;\text{ for all } x \text{ and a.e. in } p. \end{cases}
$$

We have the following theorem.

THEOREM 2.2. *Assume* (H1) *and* (H7)–(H9). *Then any $\mathbb{Z}^N$-periodic in x solution $u \in W^{1,\infty}(\mathbb{R}^N \times (0,\infty))$ of* (2.1) *converges, uniformly in x, as $t \to \infty$, to a $\mathbb{Z}^N$-periodic in x solution $\bar{u}$ of* (1.7).

The differences between Theorem 2.1 and Theorem 2.2 (which are not so obvious at first glance) will become clear in their proofs. Indeed, to prove Theorem 2.2 we will use Theorem 3.2, the proof of which is far simpler than the one of Theorem 3.1, which provides the key argument in the proof of Theorem 2.1.

On the other hand, for hamiltonians H, which are convex in p, and for Lipschitz continuous solutions, Theorem 2.2 is as general as Theorem 2.1 since, in particular, the existence of the ϕ_ε can be obtained from (H2) by a standard regularization argument.

Here we discuss in detail the two classes of examples presented in the introduction and show how they follow from the above theorems. We also present an example not covered by [5], [8], [9], and [10].

(i) *The Namah–Roquejoffre case.* The hamiltonian H is assumed to be of the form

$$
H(x,p) = F(x,p) - f(x), \tag{2.2}
$$

where

$$
\begin{cases} F \in C(\mathbb{R}^N \times \mathbb{R}^N) \text{ is } \mathbb{Z}^N\text{-periodic in } x, \text{ convex in } p, \text{ and} \\ F(x,p) > F(x,0) \equiv 0 \text{ for all } x \in \mathbb{R}^N \text{ and } p \in \mathbb{R}^N \backslash \{0\}, \end{cases} \tag{2.3}
$$

and

$$
\begin{cases} f \text{ is continuous, } \mathbb{Z}^N\text{-periodic in } x,\ f \geq 0 \text{ on } \mathbb{R}^N, \text{ and} \\ K_f = \{x \in \mathbb{R}^N : f(x) = 0\} \text{ is a nonempty compact subset of } \mathbb{R}^N. \end{cases} \tag{2.4}
$$

In [8] the authors considered Lipschitz continuous solutions. Hence (H1), (H3) are clearly satisfied, while (H7) is satisfied with $\phi_\varepsilon \equiv 0$. It only remains to check (H8) and (H9). To this end, we observe that the convexity of H in p yields (H8). Moreover, for all x and for almost all $p \in \mathbb{R}^N$,

$$
H(x,0) \geq H(x,p) + H_p(x,p)(0-p),
$$

and, hence,

$$
H_p(x,p) \cdot p - H(x,p) \geq H(x,0) = f(x).
$$

It follows that (H9) is satisfied with $K = K_f$. Indeed, it is clear that if $d(x,K_f) \geq \delta$, then $f(x) \geq \psi(\delta) > 0$, with ψ independent of η.

Finally it is worth pointing out that the result is also true when (H7) holds with $\phi_\varepsilon \equiv \phi \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. This is a consequence of the above analysis after changing u to $u - \phi$ in the equation.

(ii) *The Fathi case and extensions.* The main assumption on H in this case is that

$$\begin{cases} H \text{ is } C^1(\mathbb{R}^N \times \mathbb{R}^N) \text{ and there exists } \alpha > 0 \text{ such that} \\ \qquad D^2_{pp}H(x,p) \geq \alpha \text{ Id} \quad \text{in } \mathcal{D}'(\mathbb{R}^N \times \mathbb{R}^N). \end{cases} \tag{2.5}$$

It is immediate from (2.5), at least when $H \in C^2(\mathbb{R}^N \times \mathbb{R}^N)$, that for all $x, q \in \mathbb{R}^N$ and for almost all $p \in \mathbb{R}^N$,

$$H(x,q) \geq H(x,p) + H_p(x,p) \cdot (q-p) + \frac{\alpha}{2}|p-q|^2,$$

and, hence,

$$H_p(x,p+q) \cdot p - H(x,p+q) \geq -H(x,q) + \frac{\alpha}{2}|p|^2.$$

If $H(x,q) \leq 0$ and $H(x,p+q) \geq \eta$, it is clear that there exists $\psi(\eta) > 0$ such that $|p| \geq \psi(\eta)$. In this case we may take $K = \emptyset$ in (H9). When H is not smooth, we argue by approximations. Note that in [5] H is assumed to be smooth.

(iii) *Another example.* Consider a hamiltonian H of the form

$$H(x,p) = \psi(x,p)F\Big(x, \frac{p}{|p|}\Big) - f(x), \tag{2.6}$$

where $f \in C(\mathbb{R}^N)$ is nonnegative and $\mathbb{Z}^N$-periodic in x; $F \in C(\mathbb{R}^N \times \mathbb{R}^N \backslash \{0\})$ is continuous, strictly positive, bounded, and $\mathbb{Z}^N$-periodic in x; and $\psi(x,p) = |p + q(x)|^2 - |q(x)|^2$, where $q \in C(\mathbb{R}^N)$ is $\mathbb{Z}^N$-periodic in x. In addition we assume that for some $x_0 \in \mathbb{R}^N$, $q(x_0) = 0$ and $f(x_0) = 0$. It turns out then that $c_0 = 0$ and ϕ can be chosen to be any constant.

It is a bit tedious but straightforward to check that H satisfies the assumptions of Theorem 2.1 but neither (2.3)–(2.4) nor (2.5).

3. Some preliminary results. Here we present two results about the behavior in time, and for large times, of solutions of (1.1). These results are of independent interest themselves. Their proofs, however, are rather technical. In order not to confuse the issue here and for the reader's convenience, we present them in the appendix.

Both results hold for hamiltonians which are not necessarily periodic in x. Instead of restating the assumption of the previous section here, without the (H1) and for any domain, we first introduce the assumption that

(H1)′ H is uniformly continuous on $\mathbb{R}^N \times \overline{B}_R$, for all $R > 0$ where $\overline{B}_R = \{p \in \mathbb{R}^N : |p| \leq R\}$

and summarize the other hypotheses as follows:

(H10) $\begin{cases} \text{(H1)}', \text{(H2), (H3), and (H4) hold for } (x,p) \in \Omega \times \mathbb{R}^N, \text{ with } \Omega \text{ a given open} \\ \text{subset of } \mathbb{R}^N, \text{ and } w \in BUC(\bar{\Omega} \times [0,\infty)) \text{ a solution of (1.1) in } \Omega \times (0,\infty). \end{cases}$

Before we state the main result, we remark that we may assume, without any loss of generality, that

$$w - \phi \geq 1 \quad \text{in } \bar{\Omega} \times [0, +\infty). \tag{3.1}$$

Indeed, the form of H allows the change ϕ to $\phi - K$ for any constant K. Since $w \in BUC(\bar{\Omega} \times [0,\infty))$, to achieve (3.1) it suffices to choose K sufficiently large.

We also need to introduce, for $\eta > 0$, the functions

$$\mu_\eta(t) = \min_{x\in\bar\Omega, s\geq t} \Big[\frac{w(x,s)-\phi(x)+2\eta(s-t)}{w(x,t)-\phi(x)}\Big], \tag{3.2}$$

and

$$\chi_\eta(t) = \min_{x\in\partial\Omega, s\geq t} \Big[\frac{w(x,s)-\phi(x)+2\eta(s-t)}{w(x,t)-\phi(x)}\Big]. \tag{3.3}$$

It follows easily that $\mu_\eta,\ \chi_\eta : [0,\infty) \to \mathbb{R}$ are uniformly continuous and that $0 \leq \mu_\eta \leq \chi_\eta \leq 1$. Finally, if $\Omega = \mathbb{R}^N$, we define $\chi_\eta \equiv -\infty$.

The first result is the following.

THEOREM 3.1. *Assume* (H10). *Then there exists a constant C depending only on w and ϕ such that*

$$\mu_\eta(t) \geq 1 + \inf_{\theta\leq t}[(\chi_\eta(\theta)-1)e^{-C\psi(\eta)(t-\theta)}] \wedge (\mu_\eta(0)-1)e^{-C\psi(\eta)t}. \tag{3.4}$$

Moreover, if $w\big|_{\partial\Omega}$ converges, uniformly in $x \in \partial\Omega$, as $t \to \infty$, then, for all $s \geq t$ and $x \in \bar\Omega$,

$$w(x,t) - w(x,s) - 2\eta(s-t) \leq \delta_\eta(t),$$

where $\delta_\eta : [0,\infty) \to [0,\infty)$ is such that $\delta_\eta(t) \to 0$ as $t \to \infty$.

We continue with some preliminaries for the second result, which is also proved in the appendix. To this end, we consider the solution $w \in BUC(\overline{\Omega} \times (0,\infty))$ of the equation

$$\frac{\partial w}{\partial t} + F(x,w,Dw) = 0 \quad \text{in } \Omega \times (0,\infty), \tag{3.5}$$

where

(H11) F is uniformly continuous on $\bar\Omega \times [-R,R] \times \bar B_R$, for all $R > 0$,

(H12) $\begin{cases} \text{either } w \in W^{1,\infty}(\mathbb{R}^N \times (0,\infty)) \text{ or for each } R > 0, \text{ there exists a} \\ \text{continuous function } m_R : [0,\infty) \to [0,\infty) \text{ such that } m_R(0^+) = 0 \\ \text{and, for all } x, y \in \bar\Omega,\ p \in \mathbb{R}^N, \text{ and } w \in [-R,R], \\ \qquad |F(x,w,p) - F(y,w,p)| \leq m_R(|x-y|(1+|p|)), \end{cases}$

and

(H13) $\dfrac{\partial F}{\partial w}(x,w,p) \geq \psi(\eta) > 0$ a.e., if $F(x,w,p) \geq \eta > 0$.

To state the result we need to introduce, for $\eta > 0$, the functions

$$M_\eta(t) = \sup_{x\in\bar\Omega, s\geq t} [w(x,t) - w(x,s) - 2\eta(s-t)] \tag{3.6}$$

and

$$X_\eta(t) = \sup_{x\in\partial\Omega, s\geq t} [w(x,t) - w(x,s) - 2\eta(s-t)], \tag{3.7}$$

with the convention that $X_\eta = -\infty$ if $\partial\Omega = \phi$.

We have the following theorem.

THEOREM 3.2. *Assume* (H11), (H12), *and* (H13). *Then for all* $\eta > 0$ *and* $t \geq 0$,

$$M_\eta(t) = \sup_{\theta \leq t}[X_\eta(\theta)e^{-\psi(\eta)(t-\theta)}] \vee M_\eta(0)e^{-\psi(\eta)t}. \tag{3.8}$$

Moreover, if $w\big|_{\partial\Omega}$ *converges, uniformly in* $x \in \partial\Omega$, *as* $t \to \infty$, *then, for all* $s \geq t$ *and* $x \in \bar{\Omega}$, *we have*

$$w(x,t) - w(x,s) - 2\eta(s-t) \leq \delta_\eta(t),$$

where $\delta_\eta : [0,\infty) \to [0,\infty)$ *is such that* $\delta_\eta(t) \to 0$ *as* $t \to \infty$.

The conclusions of Theorems 3.1 and 3.2 are in some sense weak versions of (1.10). For the reader's convenience we present below a formal argument, which explains why (1.10) should hold.

To this end let us assume that w is a smooth solution of (3.5) in $\mathbb{R}^N \times (0,\infty)$. A straightforward application of the maximum principle yields that the function $t \mapsto \|(w_t)^-\|_\infty$ is decreasing in time. If (1.10) were not true, then there must exist some $\eta > 0$ and t_0 such that for all $t \geq t_0$,

$$\|(w_t)^-\|_\infty \geq \eta. \tag{3.9}$$

Let $z = w_t$ and $m(t) = \|z^-\|_\infty$. Differentiating (3.5) with respect to t, we find that

$$z_t + F_w(x,w,t,Dw)z + D_pF \cdot Dz = 0.$$

It then follows that

$$m' + F_w(x,w,t,Dw)m = 0.$$

Using (3.9) and (H13) we find

$$m' + \psi(\eta)m = 0,$$

which yields

$$m(t) = m(t_0)e^{-\psi(\eta)(t-t_0)}.$$

Letting $t \to \infty$ contradicts (3.9).

4. The proofs of Theorems 2.1 and 2.2. We begin with the following proof.

Proof of Theorem 2.1. 1. Assumption (H5) yields that u is decreasing on K and, hence, $u\big|_K$ converges, uniformly in x, as $t \to \infty$.

2. Consider the function χ_η defined by (3.3), with $\Omega = (K_\eta)^c$, where, for $\eta > 0$, $K_\eta = \{x \in \mathbb{R}^N : d(x,K) \geq \eta\}$. Step 1 and the uniform of continuity of u then imply that

$$\varliminf_{t\to\infty} \chi_\eta(t) \geq 1 - \nu(\eta), \quad \text{where} \quad \nu(\eta) \to 0 \quad \text{as} \quad \eta \to 0.$$

Using this last observation and applying Theorem 3.1 we obtain that

$$\varliminf_{t\to\infty} \mu_\eta(t) \geq 1 - \tilde{\nu}(\eta), \quad \text{where} \quad \tilde{\nu}(\eta) \to 0 \quad \text{as} \quad \eta \to 0.$$

Therefore, for all $s > 0$ and for all $(x,t) \in (K_\eta)^c \times [0,s]$, we have

$$u(x,t) - u(x,s) - 2\eta(s-t) \le \tilde{\delta}_\eta(t), \tag{4.1}$$

where

$$\overline{\lim_{t\to\infty}}\, \tilde{\delta}_\eta(t) \le \tilde{\nu}(\eta).$$

3. Since the family $(u(\cdot,t))_{t\ge 0}$ is compact in $BUC(\mathbb{R}^N)$ and the functions $u(\cdot,t)$ are periodic in x for all t, we may consider a subsequence $u(\cdot,T_n)$, with $T_n \to +\infty$, converging uniformly in $\mathbb{R}^N$.

The maximum principle for viscosity solutions implies that for any $n,p \in \mathbb{N}$, we have

$$\|u(\cdot,T_n+\cdot) - u(\cdot,T_p+\cdot)\|_{L^\infty(\mathbb{R}^N\times(0,\infty))} \le \|u(\cdot,T_n) - u(\cdot,T_p)\|_{L^\infty(\mathbb{R}^N)}\ . \tag{4.2}$$

It follows from this inequality that $(u(\cdot,T_n+\cdot))_n$ is a Cauchy sequence in $BUC(\mathbb{R}^N\times(0,\infty))$ and therefore it converges uniformly to a function $u_\infty \in BUC(\mathbb{R}^N\times(0,\infty))$.

4. Using (4.1) we find, for all $0 \le t \le s$ and for all $x \in (K_\eta)^c$,

$$u(x,t+T_n) - u(x,s+T_n) - 2\eta(s-t) \le \tilde{\delta}_\eta(t+T_n).$$

Letting $n \to \infty$ and then $\eta \to 0$ yields, for all $0 \le t \le s$ and for all $x \in (K)^c$,

$$u_\infty(x,t) - u_\infty(x,s) \le 0, \tag{4.3}$$

i.e., that u_∞ is increasing in t for $x \in (K)^c$.

On an other hand, step 1 yields that $u\big|_K$ converges, uniformly in x, as $t \to \infty$. Hence u_∞ is constant in time on K.

5. The stability property of viscosity solutions applied to the sequence $(u(\cdot,T_n+\cdot))_n$ then implies that u_∞ is a solution of

$$(u_\infty)_t + H(x,Du_\infty) = 0 \quad \text{in} \quad \mathbb{R}^N\times(0,\infty),$$

and, since u_∞ is increasing in t for all $x \in \mathbb{R}^N$,

$$H(x,Du_\infty(\cdot,t)) \le 0 \quad \text{in } \mathbb{R}^N\times\{t\} \text{ and for all } t>0.$$

Again, the stability implies

$$H(x,Du_\infty(\cdot,0)) \le 0 \quad \text{in } \mathbb{R}^N.$$

This last assertion shows that *any function in the ω-limit set of u is a subsolution of the stationary equation in $\mathbb{R}^N$.*

6. The uniform convergence of $u(\cdot,T_n+\cdot)$ to u_∞ on $\mathbb{R}^N\times(0,\infty)$ yields

$$-o_n(1) + u_\infty(x,t) \le u(x,T_n+t) \le u_\infty(x,t) + o_n(1) \quad \text{in } \mathbb{R}^N. \tag{4.4}$$

Since $u_\infty \in BUC(\mathbb{R}^N\times(0,\infty))$ is increasing with respect to t, it follows that $u_\infty(\cdot,t) \to \overline{u}(\cdot)$, uniformly in x, as $t \to \infty$.

Finally, taking the relaxed half-limits $\limsup^*$ and $\liminf_*$ in t[1] in (4.4) yields

$$-o_n(1) + \overline{u}(x) \le \underline{\lim}_* u(x) \le \overline{\lim}^* u(x) \le \overline{u}(x) + o_n(1) \quad \text{in } \mathbb{R}^N.$$

[1]For $z \in BUC(\mathbb{R}^N\times(0,\infty))$, $\limsup^* z(x) = \limsup_{\substack{y\to x\\ t\to\infty}} z(y,t)$ and $\liminf_* z(x) = \liminf_{\substack{y\to x\\ t\to\infty}} z(y,t)$.

Letting $n \to +\infty$, we obtain

$$\underline{\lim}_* u = \overline{\lim}^* u = \overline{u} \quad \text{in } \mathbb{R}^N,$$

which yields the *uniform convergence of* $u(\cdot,t)$ *to* $\overline{u}(\cdot)$ *as* $t \to \infty$.

7. Finally, by the stability result, the limit of u, as $t \to \infty$, which we still denote by $\overline{u}$, is a (viscosity) solution of $H(x, D\overline{u}) = 0$ in $\mathbb{R}^N$.

We continue with the following proof.

Proof of Theorem 2.2. 1. For each $\epsilon > 0$, we define

$$w^\varepsilon = -\exp[-(u - \phi_\varepsilon)].$$

It is then immediate that

$$w_t^\epsilon + F(x, w^\epsilon, Dw^\epsilon) = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where, for $x, p \in \mathbb{R}^N$ and $w \in \mathbb{R}$,

$$F(x,w,p) = -wH\left(x, -\frac{p}{w} + D\phi_\epsilon(x)\right). \tag{4.5}$$

After this change of variable, the proof consists essentially in following readily the proof of Theorem 2.1, replacing only the use of Theorem 3.1 by the use of Theorem 3.2.

2. It is straightforward to verify that F satisfies assumptions (H11), (H12), and (H13) of Theorem 3.2. Therefore, for all $\eta > 0$, for all $x \in (K_\eta)^c$, and for all $t \geq 0$, we have

$$w_\epsilon(x,t) - w_\epsilon(x,s) - \eta(s-t) \leq \tilde{\delta}_\eta(t), \tag{4.6}$$

where $\tilde{\delta}_\eta(t) \to 0$ as $t \to +\infty$.

3. The functions w_ε are uniformly bounded, since u is bounded and the ϕ_ε's are uniformly bounded in ε. Since

$$u(x,t) - u(x,s) = -\log(-w_\epsilon(x,t)) + \log(-w_\epsilon(x,s)),$$

there exists a constant $\tilde{C} > 0$ such that, for all $x \in (K_\eta)^c$ and for all $s \geq t$, we have

$$u(x,t) - u(x,s) \leq \tilde{C}\left[\eta(s-t) + \tilde{\delta}_\eta(t)\right].$$

4. Using this last inequality, the conclusion follows by applying readily the arguments of the proof of Theorem 2.1.

5. Remarks and extensions. A natural question is whether one needs an assumption like, for example, (H4)$'$, on H. The example of the eikonal equation

$$u_t + |Du| = 0 \quad \text{in } \mathbb{R}^N \times (0,+\infty) \tag{5.1}$$

shows that, if we restrict our attention to convex hamiltonians, (H5)(ii) is not necessary to obtain the convergence. Indeed, in this example, we can apply either the result of [8] or Theorem 2.1, since assumption (H5)(i) holds with $K = \mathbb{R}^N$.

The following one-dimensional example shows, however, that, except for equations like (5.1), such an extension does not seem to be possible. Indeed, consider the problem

$$\begin{cases} u_t + |u_x + \alpha| - |\alpha| = 0 & \text{in } \mathbb{R} \times (0,+\infty), \\ u(x,0) = \sin(x) & \text{in } \mathbb{R}. \end{cases} \tag{5.2}$$

If $\alpha > 1$ it is easily checked that the unique viscosity solution of (5.2) is

$$u(x,t) = \sin(x-t) \ ,$$

which is clearly in $W^{1,\infty}(\mathbb{R}^N \times (0,+\infty))$ but does not converge as $t \to +\infty$. For the hamiltonian $H(p) = |p+\alpha| - |\alpha|$, the quantity $H_p \cdot p - H$ vanishes for p such that $\alpha(p+\alpha) \geq 0$ and, therefore, it does not satisfy any of the (H4)-type assumptions.

On the contrary, we remark that Theorem 2.2 applies to the equation

$$u_t + |u_x + \alpha|^2 - |\alpha|^2 = 0 \quad \text{ in } \mathbb{R} \times (0,+\infty) \ ,$$

which essentially has the same limiting equation as (5.2), in the sense that both limiting equations have the same viscosity solutions. This example shows that some kind of strict convexity-type property is really playing a role in the asymptotic behavior of the solutions of Hamilton–Jacobi equations.

Typically assumption (H4)$'$ implies that the set $\{p \in \mathbb{R}^N : H(x,p+q) \leq 0\}$ is starshaped. This geometric condition alone does not seem to be sufficient as is shown by (5.2) above. On the other hand, if H is strictly convex and $c_0 = 0$, then any function F which equals H on the set $\{H > 0\}$ and is strictly negative on the set $\{H < 0\}$ satisfies the assumptions of Theorem 2.1.

Appendix. To prove Theorem 3.1 we need the following.

LEMMA A.1. *Under the assumptions of Theorem* 3.1, *the function* μ_η *defined by* (3.2) *is a viscosity solution of the variational inequality*

$$\max(\mu_\eta'(t) + C\psi(\eta)(\mu_\eta(t) - 1), \mu_\eta(t) - \chi_\eta(t)) \geq 0 \quad \textit{in } (0,+\infty). \tag{A.1}$$

Assuming for the moment this lemma we proceed with the following.

Proof of Theorem 3.1. 1. The first part of the claim follows from the facts that (A.1) admits a comparison principle and the right-hand side of (3.4) is a solution of the variational inequality (A.1) with initial datum $\mu_\eta(0)$.

2. The uniform convergence of $w\big|_{\partial\Omega}$ implies that $\chi_\eta(t) \to 1$ as $t \to \infty$. Then (3.4) yields that $\mu_\eta(t) \to 1$ as $t \to \infty$.

It then follows for all $x \in \overline{\Omega}$ and all $s \geq t$ that

$$\mu_\eta(t)(w(x,t) - \phi(x)) - w(x,s) + \phi(x) - 2\eta(s-t) \leq 0,$$

and, hence,

$$w(x,t) - w(x,s) - 2\eta(s-t) \leq \max_{x\in\bar{\Omega}}((1-\mu_\eta(t))(w(x,t) - \phi(x)).$$

It is now clear that the right-hand side of this last inequality tends to 0 as $t \to \infty$. ∎

Proof of Lemma A.1. 1. Let $\widetilde{\psi} \in C^1((0,+\infty))$ and t be a strict local minimum point of $\mu - \widetilde{\psi}$. Since there is nothing to check if $\mu_\eta(t) \geq \chi_\eta(t)$, we may assume that $\mu_\eta(t) < \chi_\eta(t)$, and, in particular, $\mu_\eta(t) < 1$.

2. For $\varepsilon > 0$ and $\alpha > 0$ we introduce the function

$$\Psi^{\varepsilon,\alpha}(x,y,z,t,s) = \frac{w(x,s) - \phi(z) + 2\eta(s-t)}{w(y,t) - \phi(z)} + \frac{|x-y|^2}{2\varepsilon} + \frac{|x-z|^2}{2\varepsilon} - \widetilde{\psi}(t) + \alpha|x|^2.$$

Classical arguments from the theory of viscosity solutions (see, for example, Barles [3]) yield that the function $\Psi^{\varepsilon,\alpha}$ achieves its minimum over $\overline{\Omega}\times\overline{\Omega}\times\overline{\Omega}\times\{(\tau,s)\backslash s\geq\tau,\tau\in[t-\delta,t+\delta]\}$ at some point $(\bar{x},\bar{y},\bar{z},\bar{t},\bar{s})$ (as usual we drop the dependence of $\bar{x},\bar{y},\bar{z},\bar{t}$, and $\bar{s}$ in ε and α for the sake of simplicity of notations). Moreover, as $(\varepsilon,\alpha)\to(0,0)$, we have

$$\text{(A.2)}\qquad \begin{cases} \text{(i)}\quad \bar{\mu}=\dfrac{w(\bar{x},\bar{s})-\phi(\bar{z})+2\eta(\bar{s}-\bar{t})}{w(\bar{y},\bar{t})-\phi(\bar{z})}\to\mu_\eta(t),\\[2ex] \text{(ii)}\quad \dfrac{|\bar{x}-\bar{y}|^2}{2\varepsilon},\quad \dfrac{|\bar{x}-\bar{z}|^2}{2\varepsilon}\to 0,\;\; \alpha|\bar{x}|^2\to 0,\\[2ex] \text{and}\\[2ex] \text{(iii)}\quad \bar{s}>\bar{t}\text{ and }\bar{x},\bar{y},\bar{z}\in\Omega\text{ for }\varepsilon\text{ and }\alpha\text{ small enough,}\end{cases}$$

with the last point being a consequence of the inequality $\mu_\eta(t)<\chi_\eta(t)$.

3. Set

$$\text{(A.3)}\qquad P=\frac{1}{\bar{\mu}}\frac{(\bar{y}-\bar{x})}{\varepsilon}(w(\bar{y},\bar{t})-\phi(\bar{z}))\;\text{ and }\;Q=\frac{1}{1-\bar{\mu}}\frac{(\bar{z}-\bar{x})}{\varepsilon}(w(\bar{y},\bar{t})-\phi(\bar{z})).$$

The viscosity inequalities for $w(x,s)$, $w(y,t)$, and ϕ are

$$\text{(A.4)}\qquad \begin{cases} \text{(i)}\quad -2\eta+H(\bar{x},\bar{\mu}P+(1-\bar{\mu})Q+2\alpha\bar{x}(w(\bar{y},\bar{t})-\phi(\bar{z})))\geq 0,\\[1ex] \text{(ii)}\quad -\widetilde{\psi}'(\bar{t})(w(\bar{y},\bar{t})-\phi(\bar{z}))-2\eta\bar{\mu}^{-1}+H(\bar{y},P)\leq 0,\\[1ex] \text{and}\\[1ex] \text{(iii)}\quad H(\bar{z},Q)\leq 0.\end{cases}$$

Using (A.2(ii)) and (A.2(iii)), we may rewrite (A.4) as

$$\text{(A.5)}\qquad \begin{cases} \text{(i)}\quad -2\eta+H(\bar{z},\bar{\mu}P+(1-\bar{\mu})Q)+\tilde{n}_\varepsilon(\alpha)+\xi(\varepsilon,\alpha)\geq 0,\\[1ex] \text{(ii)}\quad -\widetilde{\psi}'(\bar{t})(w(\bar{y},\bar{t})-\phi(\bar{z}))-2\eta\bar{\mu}^{-1}+H(\bar{z},P)-\xi(\varepsilon,\alpha)\leq 0,\\[1ex] \text{and}\\[1ex] \text{(iii)}\quad H(\bar{z},Q)\leq 0,\end{cases}$$

where $\tilde{n}_\varepsilon(\alpha)\to 0$ when $\alpha\to 0$ if ε is fixed and $\xi(\varepsilon,\alpha)\to 0$ when $(\varepsilon,\alpha)\to(0,0)$.

4. Set

$$\widetilde{P}=\bar{\mu}(P-Q).$$

If ε and α are chosen sufficiently small and α is small compared to ε, then (A.5(i)) yields

$$H(\bar{z},\widetilde{P}+Q)\geq\eta\,,$$

while (A.5(iii)) reads

$$H(\bar{z}, Q) \leq 0.$$

Moreover, again if ε and α are chosen small enough, (A.2(i)) implies that $0 < \bar{\mu} < 1$. Assumption (H4) with $\mu = \bar{\mu}$ then yields

$$\bar{\mu} H(\bar{z}, P) \geq H(\bar{z}, \widetilde{P} + Q) + \psi(\eta)(1 - \bar{\mu}). \tag{A.6}$$

Dividing (A.5(i)) by $\bar{\mu}$ and subtracting (A.5(ii)) we obtain

$$\widetilde{\psi}'(\bar{t})(w(\bar{y}, \bar{t}) - \phi(\bar{z})) + \frac{1}{\bar{\mu}} H(\bar{z}, \widetilde{P} + Q) - H(\bar{z}, P) \geq -\tilde{n}_\varepsilon(\alpha) - \xi(\varepsilon, \alpha)\Big(\frac{1}{\bar{\mu}} + 1\Big), \tag{A.7}$$

and, in view of (A.6),

$$\widetilde{\psi}'(\bar{t})(w(\bar{y}, \bar{t}) - \phi(\bar{z})) + \frac{\psi(\eta)(\bar{\mu} - 1)}{\bar{\mu}} \geq -\tilde{n}_\varepsilon(\alpha) - \xi(\varepsilon, \alpha)\Big(\frac{1}{\bar{\mu}} + 1\Big). \tag{A.8}$$

Dividing by $w(\bar{y}, \bar{t}) - \phi(\bar{z}) \geq 1$, and letting $\alpha \to 0$ and then $\varepsilon \to 0$ we obtain

$$\widetilde{\psi}'(t) + C\psi(\eta) \frac{(\mu_\eta(t) - 1)}{\mu_\eta(t)} \geq 0.$$

Since $0 \leq \mu_\eta(t) \leq 1$, this reduces to

$$\widetilde{\psi}'(t) + C\psi(\eta)(\mu_\eta(t) - 1) \geq 0. \qquad \square$$

We continue with the following proof.

Proof of Theorem 3.2. Since the variational inequality (A.9) below admits a comparison principle, the conclusion follows immediately from the lemma which is stated and proved below. $\square$

LEMMA A.2. *Under the assumptions of Theorem* 2.2, *the function* M_η *defined by* (3.6) *is a viscosity subsolution of the variational inequality*

$$\min[M' + \psi(\eta) M, M - X_\eta] \leq 0 \quad \textit{in } (0, \infty). \tag{A.9}$$

Proof. 1. M_η is clearly positive, as it can be seen by letting $s = t$, uniformly continuous, and bounded, since $w \in BUC(\mathbb{R}^N \times (0, \infty))$.

2. Let $\Phi \in C^1((0, \infty))$ and assume that τ is a local maximum point of $M_\eta - \Phi$ in $[\tau - \delta, \tau + \delta]$ for some $\delta > 0$. Since there is nothing to show if $M_\eta(\tau) \leq X_\eta(\tau)$, we may assume that $M_\eta(\tau) > X_\eta(\tau) \geq 0$.

3. Consider, for $x, y \in \mathbb{R}^N$, $t \in [\tau - \delta, \tau + \delta]$, and $s \geq t$, the function

$$\Psi^{\varepsilon, \alpha}(x, y, t, s) = w(x, t) - w(y, s) - \frac{|x - y|^2}{2\varepsilon^2} - 2\eta(s - t) - \alpha(|x|^2 + |y|^2) - \Phi(t).$$

Classical arguments from the theory of viscosity solutions yield (see [3]) that the function $\Psi^{\varepsilon,\alpha}$ achieves its maximum at some point $(\bar{x},\bar{y},\bar{t},\bar{s})$ and that, when $(\varepsilon,\alpha)\to(0,0)$,

$$\text{(A.10)}\qquad \begin{cases} \text{(i)}\ \ \Psi^{\varepsilon,\alpha}(\bar{x},\bar{y},\bar{t},\bar{s})\to M_\eta(\tau), \\[2mm] \text{(ii)}\ \ \alpha(|\bar{x}|^2+|\bar{y}|^2)\to 0,\ \dfrac{|\bar{x}-\bar{y}|^2}{\varepsilon^2}\to 0, \\[2mm] \text{(iii)}\ \ w(\bar{x},\bar{t})-w(\bar{y},\bar{s})>M_\eta(\bar{t}), \\[2mm] \text{and} \\[2mm] \text{(iv)}\ \ \bar{x},\bar{y}\in\Omega \text{ and } |\bar{t}-\bar{s}|>0, \text{ for } (\varepsilon,\alpha) \text{ small, since} \\ \qquad M_\eta(\tau)>X_\eta(\tau)\geq 0. \end{cases}$$

Using (H12) and (H13) we may rewrite the viscosity inequalities

$$\Phi'(\bar{t})-2\eta+F(\bar{x},w(\bar{x},\bar{t}),\ p+2\alpha\bar{x})\leq 0 \ \text{ and } \ -2\eta+F(\bar{y},w(\bar{y},\bar{s}),\ p-2\alpha\bar{y})\geq 0,$$

where $p=\frac{(\bar{x}-\bar{y})}{\varepsilon^2}$ as

$$\text{(A.11)}\ \begin{cases} \text{(i)}\ \ \Phi'(\bar{t})-2\eta+F(\bar{x},w(\bar{x},\bar{t}),\bar{p})+\tilde{n}_\varepsilon(2\alpha|\bar{x}|)\leq 0 \\[2mm] \text{and} \\[2mm] \text{(ii)}\ \ -2\eta+F(\bar{x},w(\bar{y},\bar{s}),\bar{p})+m_R(|\bar{x}-\bar{y}|(1+|p|))+\tilde{n}_\varepsilon(2\alpha|\bar{y}|)\geq 0. \end{cases}$$

Using (A.10(ii)) we obtain that $F(\bar{x},w(\bar{y},\bar{s}),\bar{p})>\eta$ for α and ε small enough. Since $w(\bar{x},\bar{t})\geq w(\bar{y},s)$, using sufficiently small ε and α in (A.10(iii)) and (H10), yields

$$F(\bar{x},w(\bar{x},\bar{t}),\bar{p})-F(\bar{x},w(\bar{y},\bar{s}),\bar{p})\geq\psi(\eta)(w(\bar{x},\bar{t})-w(\bar{y},\bar{s}))\geq\psi(\eta)M_\eta(\bar{t}).$$

Finally, subtracting (A.11(ii)) for (A.11(i)) we obtain

$$\Phi'(\bar{t})+\psi(\eta)M_\eta(\bar{t})+\tilde{n}_\varepsilon(\alpha)+\widetilde{m}(\varepsilon)\leq 0,$$

and we conclude, letting first $\alpha\to 0$ and then $\varepsilon\to 0$. ∎

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