Quantum Computation
Meeting 1: Introduction

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September 26, 2022

Logistics

Info (dates, speakers, references) is hosted on:
https://www.tsvan.xyz/reading.html
Follow John Preskill's notes
http://theory.caltech.edu/~preskill/ph219/ph219_2021-22.html
Helpful videos:

1. Preskill's class: https://www.youtube.com/playlist?list=PL0ojjrEqIyPy-1RRD8cTD_lF1hflo89Iu
2. UC Berkeley Vazirani's class: https://www.youtube.com/playlist?list=PLXEJgM3ycgQW5ysL69uaEdPoof4it6seB

Basics: Complex Vector Space

A *complex vector space* is a non-empty set $\mathbb{V}$, whose elements we call vectors, with three operations

1. Addition $+ : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$
2. Negation $- : \mathbb{V} \to \mathbb{V}$
3. Scalar multiplication: $\cdot : \mathbb{C} \times \mathbb{V} \to \mathbb{V}$

and a distinguished element called *zero vector* $0 \in \mathbb{V}$. The operations above obey usual rules with the scalar multiplication obeys rules for complex numbers.

Example 1

$\mathbb{C}^n$, space of complex polynomials $\mathbb{C}[x]$, space of complex-valued square integrable functions $L^2(\mathbb{R}; \mathbb{C})$.

(Un)fortunately, we do have to work with complex numbers...

Basics: Inner Product

Definition 2

Given a vector space $\mathbb{V}$, an *inner product* (dot product or scalar product) is a function

$$\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$$

that satisfies

1. $\langle V, V \rangle \geq 0$ and $\langle V, V \rangle = 0$ iff $V = 0$.
2. $\langle V_1 + V_2, V_3 + V_4 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_4 \rangle$.
3. $\langle cV_1, V_2 \rangle = c\langle V_1, V_2 \rangle$ and $\langle V_1, cV_2 \rangle = \overline{c}\langle V_1, V_2 \rangle$.
4. $\langle V_1, V_2 \rangle = \overline{\langle V_2, V_1 \rangle}$.

Basics: Hilbert Space

Definition 3

A Hilbert space is a complete vector space with an inner product.

Example 4

1. $\mathbb{C}^n$ with the usual inner product
2. $L^2(\mathbb{R};\mathbb{C})$ where

$$\langle f, g\rangle = \int \overline{f} g$$

Fact: finite dimensional Hilbert spaces are isomorphic to $\mathbb{R}^n$. This concept is more useful in infinite dimensional spaces.

Basics: Hilbert Space

Definition 5

Given a Hilbert space $\mathcal{H}$, a hermitian operator is a mapping $A : \mathcal{H} \to \mathcal{H}$ such that $A = A^\dagger$, where $A^\dagger : \mathrm{Dom}(\mathrm{A}^\dagger) \subset \mathcal{H} \to \mathcal{H}$ is defined by the following relation

$$\langle Ax, y\rangle = \langle x, A^\dagger y\rangle .$$

- For a bounded operator or in finite dimensions, hermitian is the same with self-adjoint. Issues come up when we deal with unbounded operators in infinite dimensions.
- We mostly deal with finite dimensions, however. So this is not an issue. Just get used to the terminology of physicists.

Basics: Quantum Mechanics

Basics: Quantum Mechanics (đểu)

We approach this from the axiomatic point of view.

Axiom 1 (States)

A state is a complete description of a physical system. In quantum mechanics, a state is a **ray** *in a Hilbert space.*

Basics: Quantum Mechanics (đểu)

Question

What is a ray?

Let $\sim$ be an equivalence relation on $\mathcal{H}$ defined by

$$x \sim y \iff x = \alpha y \,, \alpha \in \mathbb{C} \,.$$

In QM, a ray, denoted by $|\varphi\rangle$, is an element in the equivalence class $\mathcal{H}/\sim$.
When we write $|\varphi\rangle$, we typically think of representation φ of this equivalence class.

Basics: Quantum Mechanics (đểu)

Notations (abuse):

$$\alpha\left|\varphi\right\rangle := \left|\alpha\varphi\right\rangle \equiv \left|\varphi\right\rangle \ , \tag{1}$$

$$\left|\varphi\right\rangle + \left|\theta\right\rangle := \left|\varphi+\theta\right\rangle \ . \tag{2}$$

Be careful! Despite (1),

$$\alpha\left|\varphi\right\rangle + \left|\theta\right\rangle \neq \left|\varphi+\theta\right\rangle \ .$$

Basics: Quantum Mechanics (đểu)

Given $\varphi \in \mathcal{H}$, we denote $\langle\varphi|$ to be a linear functional (dual vector) mapping from $\mathcal{H} \to \mathbb{C}$, such that

$$\langle\varphi|\,(\theta) = \langle\varphi, \theta\rangle\,.$$

By abuse of notation again, because we think of $|\theta\rangle$ to be the representation θ of the equivalence class, we write

$$\langle\varphi|\theta\rangle := \langle\varphi|\,(\theta) = \langle\varphi, \theta\rangle\,.$$

This is called Dirac notation.

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This is called Dirac notation.
Incredibly annoying to mathematicians!

Basics: Quantum Mechanics (đểu)

Axiom 2 (Observables)

An observable is a property of a physical system that in principle can be measured. In QM, an observable is a hermitian operator.

Facts:

1. Hermitian operators have real eigenvalues
2. A "compact" hermitian operator A can be decomposed into sums orthogonal projectors to eigenvectors.

$$A = \sum_n a_n E_n$$

where a_n's are eigenvalues and E_n are the orthogonal projectors to eigenvectors.

Axiom 3 (Measurement)

A measurement is a process in which information about the state of a physical system is acquired by an observer. In QM, the measurement of an observable A prepares an eigenstate of A, and the observer learns the value of the corresponding eigenvalue As QM is inherently probabilistic, the probabilities for the outcomes are determined by "Born" rule:

$$\mathbb{P}(a_n) = ||E_n |\varphi\rangle ||^2 = \langle\varphi| E_n |\varphi\rangle \ .$$

The expectation value of the measurement outcome:

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The expectation value of the measurement outcome:

$$\langle A \rangle = \sum a_n \mathbb{P}(a_n) = \langle\varphi| A |\varphi\rangle \ .$$

Basics: Quantum Mechanics (để̉u)

Axiom 4 (Time evolution)

Time evolution is determined by the Schrödinger equation:

$$\frac{d}{dt}|\Psi(t)\rangle = -iH(t)|\Psi(t)\rangle$$

Axiom 5 (Composite systems)

The Hilbert space of composite system AB is the tensor product of A and B:

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

of dimension $d_A d_B$ and orthonormal basis $\{|e_i\rangle_A \otimes |k_j\rangle_B\}$.

Basics: Quantum Mechanics (đểu)

Rule: if $|a\rangle = \sum a_i |e_i\rangle \in \mathcal{H}_A$ and $b = \sum b_j |k_j\rangle \in \mathcal{H}_B$, then

$$|a\rangle \otimes |b\rangle := \sum_{i,j} a_i b_j |e_i\rangle \otimes |k_j\rangle \ .$$

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The rest of QM just flows from here, up to our imagination.

Qubits

The building block of classical information is the bit, which takes values in $\{0, 1\}$. The building block of quantum information is the qubit.

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Question: why qubits? other than they live in the simplest non-trivial quantum system?

Qubits

$\mathcal{H} = \mathrm{span}\{|0\rangle, |1\rangle\}$, $\dim(\mathcal{H}) = 2$.
Think

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

All the possible *unitary* transformations are the Pauli operators:

$$I, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, a qubit looks like

$$|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle .$$

As a convention, we would like to normalize the coefficients α and β so that $|| \, |\varphi\rangle \, ||^2 = 1$.

Qubits

Measure the effect of σ_3. Recall Axiom 3.
Two different eigenstates:

$$|0\rangle\ , \quad |1\rangle\ .$$

$$\sigma_3 = 1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - 1\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = E_1 - E_2\,.$$

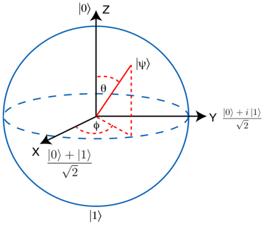
Probability for each eigenstate is the square of the projection:

$$\mathbb{P}(1) = ||E_1|\varphi\rangle||^2 = |\alpha|^2 \qquad \mathbb{P}(-1) = ||E_2|\varphi\rangle||^2 = |\beta|^2\,.$$

Bloch sphere representation of Qubits

Theorem 6

$$|\varphi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\ .$$



Theorem 7

The expectation values of the Pauli operators are

$$\begin{aligned}
\langle \sigma_1 \rangle &= \sin\theta \cos\phi \,, \\
\langle \sigma_2 \rangle &= \sin\theta \sin\phi \,, \\
\langle \sigma_3 \rangle &= \cos\theta \,.
\end{aligned}$$

Open system: two qubits

Let A, B be two quantum systems. We denote AB to be composite system. The Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.
Consider the following state in this system

$$|\Psi\rangle_{AB} = a\,|0\rangle_A \otimes |0\rangle_B + b\,|1\rangle_A \otimes |1\rangle_B = a\,|00\rangle + b\,|11\rangle\ .$$

Open system: two qubits

Consider the following observable $A = M_A \otimes I_B$.

$$
\begin{aligned}
{}_{AB}\langle\Psi| M_A \otimes I_B |\Psi\rangle_{AB} &= (a^* \langle 00| + b^* \langle 11|) M_A \otimes I_B (a|00\rangle + b|11\rangle) \\
&= |a|^2 \langle 0| M_A |1\rangle + |b|^2 \langle 1| M_A |1\rangle \ .
\end{aligned}
$$

Rewrite

$$
{}_{AB}\langle\Psi| M_A \otimes I_B |\Psi\rangle_{AB} = \mathrm{tr}(M_A \rho_A)
$$

where

$$
\rho_A = |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| = \mathrm{tr}_B(|\Psi\rangle\langle\Psi|) = \sum_{i,j,\mu} \psi^*_{j,\mu}\psi_{i,\mu} |i\rangle\langle j| \ .
$$

(density operator)

General case

$$
\begin{aligned}
\langle M_A \rangle &= {}_{AB}\langle \Psi | \, M_A \otimes I_B \, | \Psi \rangle_{AB} \\
&= \sum_{j,\nu} a^*_{j\nu} ({}_A\langle j| \otimes_B \langle \nu|)(M_A \otimes I_B) \sum_{i,\nu} a_{j\nu} ({}_A|i\rangle \otimes_B \langle \nu|) \\
&= \sum_{i,j,\mu} a^*_{j\mu} a_{i,\mu} \langle j| \, M_A \, |i\rangle = \mathrm{tr}(M_A \rho_A) \, .
\end{aligned}
$$

Properties of density operator

$$\rho_A = \sum_{i,j,\mu} \psi^*_{j,\mu}\psi_{i,\mu} \left|i\right\rangle \left\langle j\right| .$$

1. Hermitian
2. Non-negative
3. unit trace

Thus, there is an orthonormal basis in which the density operator is diagonal. The eigenvalues are non-negative real numbers that sum to one.

$$\rho_A = \sum_i p_i \left|i\right\rangle_A \left\langle i\right|_A ,$$

$$\sum_i p_i = 1 .$$

If there is just one nonzero eigenvalue, we say the state of A is “pure”. Otherwise it is “mixed”.

Open system: two qubits

Why the fuss?
For $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we have

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I\,.$$

If we are in a closed one qubit system $|0\rangle = |\uparrow_z\rangle$ and $|1\rangle = |\downarrow_z\rangle$.

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$$

$$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - |\downarrow_z\rangle)$$

Open system: two qubits

Therefore, when projecting to $|\uparrow_z\rangle$,

$$\frac{1}{\sqrt{2}}(|\uparrow_x\rangle + |\downarrow_x\rangle)$$

becomes $|z\rangle$ with probability 1.
On the other hand, in an open system

$$\frac{1}{2}(|\uparrow_z\rangle\langle\uparrow_z| + |\downarrow_z\rangle\langle\downarrow_z|) = \frac{1}{2}I.$$

Then the projection onto $|\uparrow_z\rangle$ has the expectation value

$$\mathrm{tr}(|\uparrow_z\rangle\langle\uparrow_z|\rho) = \langle\uparrow_z|\rho|\uparrow_z\rangle = \frac{1}{2}.$$

Schmidt decomposition

Consider a state in $\mathcal{H}_A \otimes \mathcal{H}_B$

$$|\Psi\rangle_{AB} = \sum_{i,\mu} \psi_{i\mu} |i\rangle_A \otimes |\mu\rangle_B \equiv \sum_i |i\rangle_A \otimes |\tilde{i}\rangle_B ,$$

where

$$|\tilde{i}\rangle_B = \sum_\mu \psi_{i\mu} |\mu\rangle_B .$$

Schmidt decomposition

Choose a basis of so that

$$\rho_A = \sum_i p_i |i\rangle \langle i| \ .$$

Then,

$$\rho_A = \mathrm{tr}_B(|\Psi\rangle \langle\Psi|) = \sum_{i,j} \langle \tilde{j} | \tilde{i} \rangle (|i\rangle \langle j|)$$

Schmidt decomposition

Matching coefficients,

$$\langle \tilde{j} | \tilde{i} \rangle_B = p_i \delta_{ij} \, .$$

Letting

$$|i'\rangle_B = p_i^{-1/2} \, |\tilde{i}\rangle_B \, ,$$

we get

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i} \, |i\rangle_A \otimes |i'\rangle_B \, .$$