

Quantum Computation

Meeting 1: Introduction

Truong-Son Van

Fulbright University Vietnam Reading Group

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Logistics

Info (dates, speakers, references) is hosted on:

<https://www.tsvan.xyz/reading.html>

Follow John Preskill's notes

http://theory.caltech.edu/~preskill/ph219/ph219_2021-22.html

Helpful videos:

- 1 Preskill's class: https://www.youtube.com/playlist?list=PL0ojjrEqIyPy-1RRD8cTD_1F1hf1o89Iu
- 2 UC Berkeley Vazirani's class: <https://www.youtube.com/playlist?list=PLXEJgM3ycgQW5ysL69uaEdPoof4it6seB>

Basics: Complex Vector Space

A *complex vector space* is a non-empty set \mathbb{V} , whose elements we call vectors, with three operations

- 1 Addition $+$: $\mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$
- 2 Negation $-$: $\mathbb{V} \rightarrow \mathbb{V}$
- 3 Scalar multiplication: \cdot : $\mathbb{C} \times \mathbb{V} \rightarrow \mathbb{V}$

and a distinguished element called *zero vector* $0 \in \mathbb{V}$. The operations above obey usual rules with the scalar multiplication obeys rules for complex numbers.

Example 1

\mathbb{C}^n , space of complex polynomials $\mathbb{C}[x]$, space of complex-valued square integrable functions $L^2(\mathbb{R}; \mathbb{C})$.

(Un)fortunately, we do have to work with complex numbers...

Basics: Inner Product

Definition 2

Given a vector space \mathbb{V} , an *inner product* (dot product or scalar product) is a function

$$\langle \cdot, \cdot \rangle : \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{C}$$

that satisfies

- 1 $\langle V, V \rangle \geq 0$ and $\langle V, V \rangle = 0$ iff $V = 0$.
- 2 $\langle V_1 + V_2, V_3 + V_4 \rangle = \langle V_1, V_3 \rangle + \langle V_2, V_4 \rangle$.
- 3 $\langle cV_1, V_2 \rangle = c\langle V_1, V_2 \rangle$ and $\langle V_1, cV_2 \rangle = \bar{c}\langle V_1, V_2 \rangle$.
- 4 $\langle V_1, V_2 \rangle = \overline{\langle V_2, V_1 \rangle}$.

Basics: Hilbert Space

Definition 3

A Hilbert space is a complete vector space with an inner product.

Example 4

- 1 \mathbb{C}^n with the usual inner product
- 2 $L^2(\mathbb{R}; \mathbb{C})$ where

$$\langle f, g \rangle = \int \bar{f} g$$

Fact: finite dimensional Hilbert spaces are isomorphic to \mathbb{R}^n . This concept is more useful in infinite dimensional spaces.

Basics: Hilbert Space

Definition 5

Given a Hilbert space \mathcal{H} , a hermitian operator is a mapping $A : \mathcal{H} \rightarrow \mathcal{H}$ such that $A = A^\dagger$, where $A^\dagger : \text{Dom}(A^\dagger) \subset \mathcal{H} \rightarrow \mathcal{H}$ is defined by the following relation

$$\langle Ax, y \rangle = \langle x, A^\dagger y \rangle.$$

- For a bounded operator or in finite dimensions, hermitian is the same with self-adjoint. Issues come up when we deal with unbounded operators in infinite dimensions.
- We mostly deal with finite dimensions, however. So this is not an issue. Just get used to the terminology of physicists.

Basics: Quantum Mechanics

Basics: Quantum Mechanics (đầu)

We approach this from the axiomatic point of view.

Axiom 1 (States)

*A state is a complete description of a physical system. In quantum mechanics, a state is a **ray** in a Hilbert space.*

Basics: Quantum Mechanics (đầu)

Question

What is a ray?

Let \sim be an equivalence relation on \mathcal{H} defined by

$$x \sim y \iff x = \alpha y, \alpha \in \mathbb{C}.$$

In QM, a ray, denoted by $|\varphi\rangle$, is an element in the equivalence class \mathcal{H}/\sim .

When we write $|\varphi\rangle$, we typically think of representation φ of this equivalence class.

Basics: Quantum Mechanics (đầu)

Notations (abuse):

$$\alpha |\varphi\rangle := |\alpha\varphi\rangle \equiv |\varphi\rangle , \quad (1)$$

$$|\varphi\rangle + |\theta\rangle := |\varphi + \theta\rangle . \quad (2)$$

Be careful! Despite (1),

$$\alpha |\varphi\rangle + |\theta\rangle \neq |\varphi + \theta\rangle .$$

Basics: Quantum Mechanics (đầu)

Given $\varphi \in \mathcal{H}$, we denote $\langle \varphi |$ to be a linear functional (dual vector) mapping from $\mathcal{H} \rightarrow \mathbb{C}$, such that

$$\langle \varphi | (\theta) = \langle \varphi, \theta \rangle .$$

By abuse of notation again, because we think of $|\theta\rangle$ to be the representation θ of the equivalence class, we write

$$\langle \varphi | \theta \rangle := \langle \varphi | (\theta) = \langle \varphi, \theta \rangle .$$

This is called Dirac notation.

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This is called Dirac notation.

Incredibly annoying to mathematicians!

Basics: Quantum Mechanics (đều)

Axiom 2 (Observables)

An observable is a property of a physical system that in principle can be measured. In QM, an observable is a hermitian operator.

Facts:

- 1 Hermitian operators have real eigenvalues
- 2 A “compact” hermitian operator A can be decomposed into sums orthogonal projectors to eigenvectors.

$$A = \sum_n a_n E_n$$

where a_n 's are eigenvalues and E_n are the orthogonal projectors to eigenvectors.

Basics: Quantum Mechanics (đầu)

Axiom 3 (Measurement)

A measurement is a process in which information about the state of a physical system is acquired by an observer. In QM, the measurement of an observable A prepares an eigenstate of A , and the observer learns the value of the corresponding eigenvalue. As QM is inherently probabilistic, the probabilities for the outcomes are determined by "Born" rule:

$$\mathbb{P}(a_n) = ||E_n |\varphi\rangle ||^2 = \langle \varphi | E_n | \varphi \rangle .$$

The expectation value of the measurement outcome:

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The expectation value of the measurement outcome:

$$\langle A \rangle = \sum a_n \mathbb{P}(a_n) = \langle \varphi | A | \varphi \rangle .$$

Basics: Quantum Mechanics (đều)

Axiom 4 (Time evolution)

Time evolution is determined by the Schrödinger equation:

$$\frac{d}{dt} |\Psi(t)\rangle = -iH(t) |\Psi(t)\rangle$$

Axiom 5 (Composite systems)

The Hilbert space of composite system AB is the tensor product of A and B :

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

of dimension $d_A d_B$ and orthonormal basis $\{|e_i\rangle_A \otimes |k_j\rangle_B\}$.

Basics: Quantum Mechanics (đều)

Rule: if $|a\rangle = \sum a_i |e_i\rangle \in \mathcal{H}_A$ and $|b\rangle = \sum b_j |k_j\rangle \in \mathcal{H}_B$, then

$$|a\rangle \otimes |b\rangle := \sum_{i,j} a_i b_j |e_i\rangle \otimes |k_j\rangle .$$

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The rest of QM just flows from here, up to our imagination.

Qubits

The building block of classical information is the bit, which takes values in $\{0, 1\}$. The building block of quantum information is the qubit.

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Question: why qubits? other than they live in the simplest non-trivial quantum system?

Qubits

$\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$, $\dim(\mathcal{H}) = 2$.

Think

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

All the possible *unitary* transformations are the Pauli operators:

$$I, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, a qubit looks like

$$|\varphi\rangle = \alpha |0\rangle + \beta |1\rangle.$$

As a convention, we would like to normalize the coefficients α and β so that

$$||\varphi\rangle||^2 = 1.$$

Qubits

Measure the effect of σ_3 . Recall Axiom 3.

Two different eigenstates:

$$|0\rangle, \quad |1\rangle.$$

$$\sigma_3 = 1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - 1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = E_1 - E_2.$$

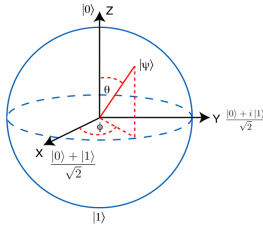
Probability for each eigenstate is the square of the projection:

$$\mathbb{P}(1) = \|E_1 |\varphi\rangle\|^2 = |\alpha|^2 \quad \mathbb{P}(-1) = \|E_2 |\varphi\rangle\|^2 = |\beta|^2.$$

Bloch sphere representation of Qubits

Theorem 6

$$|\varphi\rangle = \cos\left(\frac{\theta}{2}\right) |0\rangle + e^{i\phi} \sin\left(\frac{\theta}{2}\right) |1\rangle .$$



Theorem 7

The expectation values of the Pauli operators are

$$\langle \sigma_1 \rangle = \sin \theta \cos \phi ,$$

$$\langle \sigma_2 \rangle = \sin \theta \sin \phi ,$$

$$\langle \sigma_3 \rangle = \cos \theta .$$

Open system: two qubits

Let A, B be two quantum systems. We denote AB to be composite system. The Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Consider the following state in this system

$$|\psi\rangle_{AB} = a|0\rangle_A \otimes |0\rangle_B + b|1\rangle_A \otimes |1\rangle_B = a|00\rangle + b|11\rangle .$$

Open system: two qubits

Consider the following observable $A = M_A \otimes I_B$.

$$\begin{aligned} {}_{AB} \langle \Psi | M_A \otimes I_B | \Psi \rangle_{AB} &= (a^* \langle 00 | + b^* \langle 11 |) M_A \otimes I_B (a | 00 \rangle + b | 11 \rangle) \\ &= |a|^2 \langle 0 | M_A | 0 \rangle + |b|^2 \langle 1 | M_A | 1 \rangle . \end{aligned}$$

Rewrite

$${}_{AB} \langle \Psi | M_A \otimes I_B | \Psi \rangle_{AB} = \text{tr}(M_A \rho_A)$$

where

$$\rho_A = |a|^2 |0\rangle \langle 0| + |b|^2 |1\rangle \langle 1| .$$

(density operator)

Open system: two qubits

Why the fuss?

For $|\Psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we have

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}I.$$

If we are in a closed one qubit system $|0\rangle = |\uparrow_z\rangle$ and $|1\rangle = |\downarrow_z\rangle$.

$$|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$$

$$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - |\downarrow_z\rangle)$$

Open system: two qubits

Therefore, when projecting to $|\uparrow_z\rangle$,

$$\frac{1}{\sqrt{2}}(|\uparrow_x\rangle + |\downarrow_x\rangle)$$

becomes $|\uparrow_z\rangle$ with probability 1.

On the other hand, in an open system

$$\frac{1}{2}(|\uparrow_z\rangle \langle\uparrow_z| + |\downarrow_z\rangle \langle\downarrow_z|) = \frac{1}{2}I.$$

Then the projection onto $|\uparrow_x\rangle$ has the expectation value

$$\text{tr}(|\uparrow_x\rangle \langle\uparrow_x| \rho) = \langle\uparrow_x| \rho |\uparrow_x\rangle = \frac{1}{2}.$$