

Math 170: Brief notes

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Introduction

You may still be thinking about your major and trying this class as a part of exploration. Or this may be the only math class that you take in your four years of undergraduate studies, and you may be wondering why the university imposed such a requirement on you.

Instead of trying to find my own words for a motivational speech, let me cite Abraham Lincoln when he answered in 1864 how he had acquired his persuasive rhetorical skill:

“In the course of my law-reading I constantly came upon the word demonstrate. I thought, at first, that I understood its meaning, but soon became satisfied that I did not. . . . I consulted Webster’s dictionary. That told of “certain proof”, “proof beyond the possibility of doubt”; but I could form no idea what sort of proof that was. I thought a great many things were proved beyond a possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood “demonstration” to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined blue to a blind man. At last I said, “Lincoln, you can never make a lawyer if you do not understand what demonstrate means”; and I left my situation in Springfield, went home to my father’s house, and staid there till I could give any propositions in the six books of Euclid at sight. I then found out what “demostrate” means, and went back to my law studies.”

We see that the 16th president of the US highly regarded Euclid’s “Elements” for its teaching of rigor and reasoning, and not as much for its content. The textbook itself is 2300 years old and you may think that it may have outdated material (arguably, this is a correct guess), but it has survived more than a thousand of editions, and mathematicians only came

up with other logically consistent geometries in the nineteenth century, thus rendering Euclid's work as one of the many possibilities. For more than two thousand years, this book was considered something all educated people had read, and it only came down this pedestal in the 20th century, by which time its content was universally taught through other school textbooks.

But as much as this book was central to the western European civilization (second to Bible), not all people are fond of planar geometry. In this course, we will offer an alternative invitation to the land of reason by means of showing many possible facets of mathematics. If you wish, you may consider it a collection of trailers for higher-level math courses.

CHAPTER 1

Warm-up: pigeonhole and numbers

1. Pigeonhole principle

Let me start with a question to you.

EXERCISE 1. *Last time I checked, there were 79 students enrolled in this course. Can anyone tell me if there is at least one pair of students whose birthday happen on the same week? Any guesses?*

In fact, I guarantee you, without knowing any of your birthdays, that yes. Moreover, I can guarantee that for some seven people in our class, their birthdays fall on the same month. And I am so sure of it because I can prove it using the “pigeonhole principle”.

THEOREM 1.1 (Pigeonhole principle). *If $k > 0$ is a number of pigeonholes, and n pigeons try to occupy them, with $n > k$, then there will necessarily be a pigeonhole with at least two pigeons in it.*

The statement sure seems obvious, but let us prove it as a warm-up, and then we’ll use it to prove our first claim about birthdays.

PROOF. We use a proof by contradiction. Assume that none of the boxes has more than one item. Then we would only have at most k items on our hands. But we assumed that the number of items is greater than k , so we arrived at a contradiction, and our assumption was false. So there must be a box with at least two items. \square

EXAMPLE 1.2. In a class of 79 students, there are at least two students whose birthdays fall on the same week.

PROOF. Here the weeks in a year – total of 52 – play the role of pigeonholes, and students – total of 79 – play the role of pigeons. \square

EXAMPLE 1.3. There are two people in Pennsylvania with the same number of hairs on their body.

PROOF. Possible numbers of hairs on a human body play the role of pigeonholes and people play the role of pigeons. By googling, we can find that people usually have 5 million hairs on their body and the population of Pennsylvania is 13 million. Therefore, by pigeonhole principle, there should exist at least two people with the exact same number of hairs. \square

Now you may think that that was easy. Let us take it up a notch and see what happens when you have many more pigeons than pigeonholes.

THEOREM 1.4 (Generalized pigeonhole principle). *If $k > 0$ is a number of pigeonholes, and n pigeons try to occupy them, with $n > mk$ for some positive integer m , then there will necessarily be a pigeonhole with at least $m + 1$ pigeons in it.*

In other words, we can say that if n pigeons occupy k pigeonholes, then there is at least one pigeonhole containing at least $\lceil \frac{n}{k} \rceil$ items. The number $\lceil \frac{n}{k} \rceil$ is defined as the smallest integer that is larger than $\frac{n}{k}$.

EXERCISE 2. Prove Theorem 1.4.

EXAMPLE 1.5. In a class of 79 students, there are at least seven students whose birthdays fall on the same month.

PROOF. Here the months in a year – total of $k = 12$ – play the role of pigeonholes, and students – total of $n = 79$ – play the role of pigeons. Then we can calculate that $n = 6 \cdot k + 7$, so by the generalized pigeonhole principle (Theorem 1.4) there are at least $6 + 1 = 7$ students whose birthdays occur in the same month. \square

EXAMPLE 1.6. Pennsylvania needs to have at least two area codes for the phone numbers, while the US needs at least 42.

PROOF. Phone numbers in the US have the format $+1(AAA)N*****$, where AAA is the area code. The six stars are digits from 0 to 9, while N can only be from 2 to 9. There are more subtle rules that you can find in Wikipedia on the page “[North American numbering plan](#)”, but further restrictions do not affect the order of magnitude, so let us ignore them for now. With this, the total possible number of variants to fill the stars is $8 \cdot 10^6 = 8\,000\,000$, so $k = 8$ million. As we have found out before, there are $n = 13$ million people in Pennsylvania. So minimal number of area codes is $\lceil \frac{13}{8} \rceil = 2$.

The population of the US is $n = 328.2$ million, so you will need at least $\lceil \frac{328.2}{8} \rceil = 42$ area codes. \square

In fact, if you google, you will find out that major cities in any state have their own area codes, sometimes a couple, and the total number of area codes is now around 320.

EXERCISE 3. *Can you think of reasons to have so many area codes? (Logistical, social, etc.) Do you think there is a restriction on applying mathematical ideas in real life?*

2. Proving things with the pigeonhole principle

Material for this section is take from <https://www.math.uvic.ca/faculty/gmacgill/guide/pigeonhole.pdf>.

As we saw in the above examples, there are four steps involved:

- (1) Decide what pigeons are. They will be the things among which we want to find several of that have the same property.
- (2) Set up pigeonholes. In order for the pigeonhole principle to work, it is necessary to have fewer pigeonholes than pigeons. Sometimes you need an astute observation to do this.
- (3) Give a rule for assigning the pigeons to the pigeonholes. The pigeonhole principle works for any rule – you just need to choose the rule that works best for your situation.
- (4) Apply the pigeonhole principle to your setup and get the desired conclusion.

EXERCISE 4. *Prove that if seven distinct numbers are selected from $\{1, 2, \dots, 11\}$ (braces are used to denote a set of objects, e.g. numbers), then some two of these numbers sum to 12. Show that you can find six numbers so that no pair among those sums up to 12.*

- (1) Let the pigeons be the numbers selected.
- (2) Let the pigeonholes be labeled by the following sets of numbers: $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6\}$.
- (3) The rule: when a number is selected, it is placed in the pigeonhole with the corresponding label.
- (4) There are seven numbers and six pigeonholes, so two of the selected numbers will end up in the same pigeonhole. They cannot both end up in the pigeonhole labeled $\{6\}$ because we are choosing distinct numbers, so it's one of the first five. But then they sum up to 12.

EXERCISE 5. *A party is attended by $n \geq 2$ people. Prove that there will always be two people in attendance who have the same number of friends at*

the party. (Assume that the relation “is a friend of” is symmetric, that is, if X is a friend of Y then Y is a friend of X .)

Each person either is, or is not, a friend of each of the the other $n - 1$ people in attendance. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \dots, n - 1$. However, it can not be the case that there is someone at the party with 0 friends and someone else with $n - 1$ friends simultaneously: if a person is friends with everyone, then everyone at the party has at least one friend there. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \dots, n - 2$ or $1, 2, \dots, n - 1$. In either case, there are n numbers (of friends among the people in attendance) that can take on at most $n - 1$ different values. By the Pigeonhole Principle, two of the numbers are equal. Thus, some two people in attendance who have the same number of friends at the party.

3. Divisibility

Let us now make things a little more abstract with numbers. We will not concern ourselves with where numbers come from (although this is a worthy subject in itself) but will learn how to do things with them. In particular, we will spend some time thinking about something that is very closely related to the pigeonhole principle: divisibility.

Let us first equip ourselves with a vocabulary.

- Natural numbers are numbers that are used for counting, starting from 0. We denote \mathbb{N} to be the set of all natural numbers. Thus, using set notation¹,

$$\mathbb{N} = \{0, 1, 2, \dots\}.$$

- Integer numbers are numbers that are used to measure the difference between two instances of counting. We denote \mathbb{Z} to be the set of all integers. Thus, using set notation,

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}.$$

The shorthand of saying “ a belongs to a set S ” is by using the notation

$$a \in S.$$

For example, the shorthand of “ a is an integer” is “ $a \in \mathbb{Z}$ ”.

¹We will discuss about set later.

For what to come, we need the notion of the absolute value of a number.

DEFINITION 3.1. The absolute value of a number a is a non-negative quantity that represents the size of that number. In mathematical terms,

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

DEFINITION 3.2. A number $a \in \mathbb{Z}$ is said to be divisible by $b \in \mathbb{Z}$ if there exists a number $q \in \mathbb{Z}$ such that

$$a = bq.$$

b is called to be a divisor (or a factor) of a and we can also say that b divides a (notation: $b \mid a$).

a is not divisible by b if a can be written in the form

$$a = bq + r,$$

where $q, r \in \mathbb{Z}$ and $0 < |r| < |a|$.

Just because I can define something, it doesn't mean that my definition is something that makes sense. For example, I can define a cat to be a mammal that lays eggs (what's wrong?). A good principle in life: question yourself often. Just because there are things you can imagine/define, it doesn't mean that those things exist or even make sense.

QUESTION. Does Definition 3.2 make sense? Can there be a number that is both divisible and not divisible by another number?

There is a theorem that guarantees that the situation described in the previous question cannot happen. In other words, Definition 3.2 does make sense.

THEOREM 3.3 (Division theorem). Let $a, b \in \mathbb{Z}$ with $b \neq 0$. There exist unique $q, r \in \mathbb{Z}$ such that

$$a = qb + r \quad \text{and} \quad 0 \leq r < |b|.$$

We will not prove this theorem for now as it is quite involved.

EXERCISE 6. Read the proof of this theorem in Newstead's book [New] (Theorem 6.1.1).

One of the most fundamental concepts about numbers is the greatest common divisor.

DEFINITION 3.4. Let $a, b \in \mathbb{Z}$. An integer d is a greatest common divisor of a and b if:

- (1) $d \mid a$ and $d \mid b$,
- (2) if q is another integer such that $q \mid a$ and $q \mid b$ then $q \mid d$.

We denote the (unique) non-negative greatest common divisor of a and b as $\gcd(a, b)$.

EXERCISE 7. *Why is it that in our definition, we have “a” greatest common divisor, but not “the” greatest common divisor?*

Let’s have some fun with greatest common divisors.

THEOREM 3.5. *Let $a, b, q, r \in \mathbb{Z}$ and suppose that $a = qb + r$. Then*

$$\gcd(a, b) = \gcd(b, r).$$

PROOF. □

One question arise, how can we find the greatest common divisor of any two numbers? For example, how on earth are we supposed to know $\gcd(199888, 4987774)$? Luckily, ancient people were pretty smart and there is a method that was invented at least 2300 years ago as it first appeared in Euclid’s Elements (300 BC). We, as modern people, still benefit from this method as it has a lot of practical applications ².

THEOREM 3.6 (Euclid’s algorithm). *Let $a, b \in \mathbb{Z}$. The $\gcd(a, b)$ is computed as follows.*

- Set $r_0 = |a|$ and $r_1 = |b|$.
- Given r_{n-2} and r_{n-1} , define r_n to be the remainder of r_{n-2} divided by r_{n-1} .
- Stop when $r_n = 0$; then $r_{n-1} = \gcd(a, b)$.

²For a curious mind, you can read more about it here: https://en.wikipedia.org/wiki/Euclidean_algorithm

Bibliography

- [New] Clive Newstead. Infinite Descent. URL: <https://infinitedescent.xyz/dl/infdesc.pdf>.