Math 170: Brief notes

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Introduction

You may still be thinking about your major and trying this class as a part of exploration. Or this may be the only math class that you take in your four years of undergraduate studies, and you may be wondering why the university imposed such a requirement on you.

Instead of trying to find my own words for a motivational speech, let me cite Abraham Lincoln when he answered in 1864 how he had aquired his persuasive rhetorical skill:

> "In the course of my law-reading I constantly came upon the word *demonstrate*. I thought, at first, that I understood its meaning, but soon became satisfied that I did not. ...I consulted Webster's dictionary. That told of "certain proof", "proof beyond the possibility of doubt"; but I could form no idea what sort of proof that was. I thought a great many things were proved beyond a possibility of doubt, without recourse to any such extraordinary process of reasoning as I understood "demonstration" to be. I consulted all the dictionaries and books of reference I could find, but with no better results. You might as well have defined *blue* to a blind man. At last I said, "Lincoln, you can never make a lawyer if you do not understand what *demonstrate* means"; and I left my situation in Springfield, went home to my father's house, and staid there till I could give any propositions in the six books of Euclid at sight. I then found out what "demostrate" means, and went back to my law studies."

We see that the 16th president of the US highly regarded Euclid's "Elements" for its teaching of rigor and reasoning, and not as much for its content. The textbook itself is 2300 years old and you may think that it may have outdated material (arguably, this is a correct guess), but it has survived more than a thousand of editions, and mathematicians only came up with other logically consistent geometries in the nineteenth century, thus rendering Euclid's work as one of the many possibilities. For more than two thousand years, this book was considered something all educated people had read, and it only came down this pedestal in the 20th century, by which time its content was universally taught through other school textbooks.

But as much as this book was central to the western European civilization (second to Bible), not all people are fond of planar geometry. In this course, we will offer an alternative invitation to the land of reason by means of showing many possible facets of mathematics. If you wish, you may consider it a collection of trailers for higher-level math courses.

CHAPTER 1

Warm-up: pigeonhole and numbers

1. Pigeonhole principle

Let me start with a question to you.

EXERCISE 1. *Last time I checked, there were 79 students enrolled in this course. Can anyone tell me if there is at least one pair of students whose birthday happen on the same week? Any guesses?*

In fact, I guarantee you, without knowing any of your birthdays, that yes. Moreover, I can guarantee that for some seven people in our class, their birthdays fall on the same month. And I am so sure of it because I can prove it using the "pigeonhole principle".

THEOREM 1.1 (Pigeonhole principle). *If $k > 0$ is a number of pigeonholes, and n pigeons try to occupy them, with $n > k$, then there will necessarily be a pigeonhole with at least two pigeons in it.*

Here is a brief example with $k = 9$ and $n = 10$:

The statement sure seems obvious, but let us prove it as a warm-up, and then we'll use it to prove our first claim about birthdays.

PROOF. We use a proof by contradiction. Assume that none of the boxes has more than one item. Then we would only have at most k items on our hands. But we assumed that the number of items is greater than k, so we arrived at a contradiction, and our assumption was false. So there must be a box with at least two items. □

EXAMPLE 1.2. In a class of 79 students, there are at least two students whose birthdays fall on the same week.

PROOF. Here the weeks in a year – total of 52 – play the role of pigeonholes, and students – total of 79 – play the role of pigeons. □

EXAMPLE 1.3. There are two people in Pennsylvania with the same number of hairs on their body.

PROOF. Possible numbers of hairs on a human body play the role of pigeonholes and people play the role of pigeons. By googling, we can find that people usually have 5 million hairs on their body and the population of Pennsylvania is 13 million. Therefore, by pigeonhole principle, there should exist at least two people with the exact same number of hairs. □

Now you may think that that was easy. Let us take it up a notch and see what happens when you have many more pigeons than pigeonholes.

THEOREM 1.4 (Generalized pigeonhole principle). *If $k > 0$ is a number of pigeonholes, and n pigeons try to occupy them, with $n > mk$ for some positive integer m, then there will necessarily be a pigeonhole with at least $m + 1$ pigeons in it.*

In other words, we can say that if n pigeons occupy k pigeonholes, then there is at least one pigeonhole containing at least $\lceil \frac{n}{k} \rceil$ items. The number $\lceil \frac{n}{k} \rceil$ is defined as the smallest integer that is larger than $\frac{n}{k}$.

EXERCISE 2. *Prove Theorem 1.4.*

EXAMPLE 1.5. In a class of 79 students, there are at least seven students whose birthdays fall on the same month.

PROOF. Here the months in a year – total of $k = 12$ – play the role of pigeonholes, and students – total of $n = 79$ – play the role of pigeons. Then we can calculate that $n = 6 \cdot k + 7$, so by the generalized pigeonhole principle (Theorem 1.4) there are at least $6 + 1 = 7$ students whose birthdays occur in the same month. □

EXAMPLE 1.6. Pennsylvania needs to have at least two area codes for the phone numbers, while the US needs at least 42.

PROOF. Phone numbers in the US have the format $+1\,(AAA)\,N******$, where AAA is the area code. The six stars are digits from 0 to 9, while N can only be from 2 to 9. There are more subtle rules that you can find in Wikipedia on the page "North American numbering plan", but further restrictions do not affect the order of magnitude, so let us ignore them for now. With this, the total possible number of variants to fill the stars is $8 \cdot 10^6 = 8\,000\,000$, so $k = 8$ million. As we have found out before, there are $n = 13$ million people in Pennsylvania. So minimal number of area codes is $\lceil \frac{13}{8} \rceil = 2$.

The population of the US is $n = 328.2$ million, so you will need at least $\lceil \frac{328.2}{8} \rceil = 42$ area codes. □

In fact, if you google, you will find out that major cities in any state have their own area codes, sometimes a couple, and the total number of area codes is now around 320.

EXERCISE 3. *Can you think of reasons to have so many area codes? (Logistical, social, etc.) Do you think there is a restriction on applying mathematical ideas in real life?*

2. Proving things with the pigeonhole principle

Material for this section is take from https://www.math.uvic.ca/faculty/gmacgill/guide/pigeonhole.pdf.

As we saw in the above examples, there are four steps involved:

(1) Decide what pigeons are. They will be the things among which we want to find several of that have the same property.
(2) Set up pigeonholes. In order for the pigeonhole principle to work, it is necessary to have fewer pigeonholes than pigeons. Sometimes you need an astute observation to do this.
(3) Give a rule for assigning the pigeons to the pigeonholes. The pigeonhole principle works for any rule – you just need to choose the rule that works best for your situation.
(4) Apply the pigeonhole principle to your setup and get the desired conclusion.

EXERCISE 4. *Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$ (braces are used to denote a $\underline{set}$ of objects, e.g. numbers), then some two of these numbers sum to 12. Show that you can find six number so that no pair among those sums up to 12.*

(1) Let the pigeons be the numbers selected.
(2) Let the pigeonholes be labeled by the following sets of numbers: $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, $\{5, 7\}$, $\{6\}$.
(3) The rule: when a number is selected, it is placed in the pigeonhole with the corresponding label.
(4) There are seven numbers and six pigeonholes, so two of the selected numbers will end up in the same pigeonhole. They cannot both end up in the pigeonhole labeled $\{6\}$ because we are choosing distinct numbers, so it's one of the first five. But then they sum up to 12.

EXERCISE 5. *A party is attended by $n \geqslant 2$ people. Prove that there will always be two people in attendance who have the same number of friends at*

the party. (Assume that the relation "is a friend of" is symmetric, that is, if X is a friend of Y then Y is a friend of X.)

Each person either is, or is not, a friend of each of the the other $n-1$ people in attendance. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \cdots, n-1$. However, it can not be the case that there is someone at the party with 0 friends and someone else with $n-1$ friends simultaneously: if a person is friends with everyone, then everyone at the party has at least one friend there. Thus, the possible values for the number of friends a person can have in attendance at the party are $0, 1, \cdots, n-2$ or $1, 2, \cdots, n-1$. In either case, there are n numbers (of friends among the people in attendance) that can take on at most $n-1$ different values. By the Pigeonhole Princple, two of the numbers are equal. Thus, some two people in attendance who have the same number of friends at the party.

3. Divisibility

Let us now make things a little more abstract with numbers. We will not concern ourselves with where numbers come from (although this is a worthy subject in itself) but will learn how to do things with them. In particular, we will spend some time thinking about something that is very closely related to the pigeonhole principle: divisibility.

Let us first equip ourselves with a vocabulary.

- Natural numbers are numbers that are used for counting, starting from 0. We denote $\mathbb{N}$ to be the set of all natural numbers. Thus, using set notation[1],
$$\mathbb{N} = \{0, 1, 2, \ldots\} \,.$$

- Integer numbers are numbers that are used to measure the difference between two instances of counting. We denote $\mathbb{Z}$ to be the set of all integers. Thus, using set notation,
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\} \,.$$

The shorthand of saying "a belongs to a set S " is by using the notation
$$a \in S \,.$$

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[1]We will discuss sets later.

For example, the shorthand of "a is an integer" is "$a \in \mathbb{Z}$".

For what to come, we need the notion of the absolute value of a number.

DEFINITION 3.1. The absolute value of a number a is a non-negative quantity that represents the size of that number. In mathematical terms,
$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \,, \\ -a & \text{if } a < 0 \,. \end{cases}$$

Our strategy for uncovering the structure of the natural numbers is to break down complex objects and ideas into their fundamental components, think about this quote by Desmond Tutu, a South African Anglican cleric and theologian, and a human rights activist[2]:

> There is only one way to eat an elephant: a bite at a time.

For example, when you prepare for an exam, you make a list of topics and learn one topic at a time; when I prepare lectures, I make a list of topics and write about one topic at a time; in both situations we achieve a complex result by taking many small bites. And when we study natural number, we break them down into their simplest building blocks – *prime numbers* – and then study their properties and observe how they interact. One way of breaking the numbers down is to try to divide by a smaller number and observe if there is a remainder. This leads us to the following sequence of definitions.

DEFINITION 3.2. A number $a \in \mathbb{Z}$ is said to be divisible by $b \in \mathbb{Z}$ if there exists a number $q \in \mathbb{Z}$ such that
$$a = bq \,.$$

b is called to be a divisor (or a factor) of a and we can also say that b divides a (notation: $b \mid a$).

The number a is not divisble by b if a can be written in the form
$$a = bq + r \,,$$

where $q, r \in \mathbb{Z}$ and $0 < r < |a|$. The number r is then called the remainder.

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Just because I can define something, it doesn't mean that my definition is something that makes sense. For example, I can define a cat to be a mammal that lays eggs (what's wrong?). A good principle in life: question yourself often. Just because there are things you can imagine/define, it doesn't mean that those things exist or even make sense.

QUESTION. *Does Definition 3.2 make sense? Can there be a number that is both divisible and not divisible by another number?*

There is a theorem that guarantees that the situation described in the previous question cannot happen. In other words, Definition 3.2 does make sense.

THEOREM 3.3 (Division theorem). *Let $a, b \in \mathbb{Z}$ with $b \neq 0$. There exist a pair of integers $q, r \in \mathbb{Z}$ such that*

$$a = qb + r \quad \text{and} \quad 0 \leqslant r < |b|,$$

and moreover this pair is unique. q is called the quotient of a when divided by b; and r is called the remainder of a when divided by b.

EXAMPLE 3.4. Let $a = 101$ and $b = 3$. Then

$$101 = 33 \cdot 3 + 2.$$

Here, $q = 33$ and $r = 2$. As said in the division theorem, $2 < 3$.

EXAMPLE 3.5. A slightly interesting example is when a is positive and b is negative. Say, $a = 23$ and $b = -4$. Then

$$23 = (-5) \cdot (-4) + 3.$$

In this case $q = -6$ and $r = 3 < 4$.

We will not prove this theorem for now as it uses the technique of mathematical induction (CS majors would say "recursion").

EXERCISE 6. *Read the proof of this theorem in Newstead's book [New] (Theorem 6.1.1).*

4. Criteria of divisibility

Numbers that are divisible by two are called even, and they have a special name because in many cultures the distinction between odd and even numbers is quite prominent. For example, ancient Greek and Chinese seem to favor odd numbers like 3 and 5. Russian culture traditionally favors 3 and 7: there is a saying that "God loves groups of three", and seven commonly occurs in folk tales. Conversely, Eastern cultures have negative connotations with even numbers, and number 4 in Chinese is associated with death, because quite ominously, we can write it in two ways as an operation on two twos:

$$4 = 2 \cdot 2 = 2 + 2.$$

Some buildings in China skip the fourth floor, just like the thirteenth floor is skipped in some places in the US.

On the other hand, Western cultures seem to "prefer" even numbers. According to a historian of mathematics (Dr. Nishiyama), ancient preference for odd numbers probably faded in the West with the arrival of modern mathematics as represented by Newton. When counting numbers, odd numbers were incomplete, in-between numbers, whereas even numbers were certainly more "rational". This is even reflected in an English proverb that says that two heads are better than one.

So how do we tell even numbers from odd?

PROPOSITION 4.1. *An integer number n is divisible by 2 if and only if its last digit is even (i.e. 0, 2, 4, 6, 8).*

LEMMA 4.2. *Observe that if a number a is divisible by b, then $a + b$, $a + 2b$, etc. are also divisible by b.*

PROPOSITION 4.3. *An integer number n is divisible by 5 if and only if its last digit is 0 or 5.*

PROPOSITION 4.4. *An integer number n is divisible by 4 if and only if its last two digits comprise a number divisible by 4. An integer number n is divisible by 8 if and only if its last three digits comprise a number divisible by 8.*

PROPOSITION 4.5. *An integer number n is divisible by 3 if and only if the sum of all its digits is divisible by 3.*

PROOF. Prove the criterion of divisibility by 3 by writing the decimal expansion:

$$a = a_k \cdot 10^k + \cdots + a_2 \cdot 100 + a_1 \cdot 10 + a_0 \cdot 1.$$

Notice that $10^k = 9\ldots9 + 1$, and the first summand is divisible by 3. So we can write

$$a - 9\ldots9 = a_k + \cdots + a_2 \cdot 100 + a_1 \cdot 10 + a_0 \cdot 1.$$

Repeat this process and we can see that the right handside would eventually be

$$a_k + \cdots + a_1 + a_0.$$

Note that a and each of the $9\ldots9$ is divisible by 3. Therefore, $a_k + \cdots + a_0$ is divisible by 3.   □

PROPOSITION 4.6. *An integer number n is divisible by* 9 *if and only if the sum of all its digits is divisible by* 9.

THEOREM 4.7. *Let $n > 70$ be a natural number that you want to test for being divisible by* 7.
Step 1: Separate the last digit of the number, call it d.
Step 2: Double the last digit and subtract from the remaining number, call the result n_1:

$$n_1 = \frac{(n - d)}{10} - 2 \cdot d.$$

Then n is divisible by 7 *if and only if n_1 is divisible by* 7.

If after using this test once, you still get a number $n_1 > 70$, you can repeat the test and get some number n_2.

EXERCISE 7. *In 2019, a 12-year old Nigerian boy, Chika Ofili, suggested an alternative test for being divisible by* 7. *Read the news article about Chika's test:* https://www.scilynk.in/divisibility-of-7/. *In this test, instead of subtracting* 2d, *Chika suggests to add* 5d. *What do you think about these two conclusions of the article?*

- *"Multiplying by* 5 *helps to reach a number within* 0–70 *at a faster rate compared to multiplication by* 2."
- *"Adding two numbers is psychologically simpler than subtraction."*

Test these conclusions on some numbers, e.g. 2021, 1234567: *count the number of times you apply each test, and try to observe what is easier for you psychologically.*

5. Prime numbers

When we factor numbers into smaller numbers, at some point we will have to stop, because there are some natural numbers that cannot be factored as the product of two smaller natural numbers. Trivially, zero and one are among them, but also there are 2, 3, 5, 7, 2017 and 2027.

I would like to argue that zero and one are so special that we don't even call them "prime". Most early Greeks did not even consider 1 to be a number, so they could not consider its primality. In modern mathematics, we actually have a formal definition that excludes 0 and 1 as prime numbers.

DEFINITION 5.1. A natural number p is called prime if it has exactly two distinct divisors.

With this definition, we can observe that 0 and 1 are not prime, because 0 is divisible by any number, while 1 is divisible by only one number – 1 itself.

THEOREM 5.2 (Fundamental theorem of arithmetic). *Every natural number $n > 1$ is either a prime or it can be expressed as a product of prime numbers in a unique way.*

SKETCH OF PROOF. We will not prove this theorem because it requires a technique called induction, which we have not yet covered.

However, you should be able to see that the existence of such expression is simply a matter of definition. If a number cannot be factored into products of smaller number, by definition, it is a prime number. Keep factoring the smaller number in the products until you get all primes.

The uniqueness part is the tricky part!   □

REMARK 5.3. An important idea here is that in mathematics, a proof of unique existence of something is often done by two separate steps that are not in any order. Existence step is to establish that there is at least one object that satisfies the definition. Uniqueness step is to establish that if two objects with the same definition exist, they must be the same.

EXAMPLE 5.4. We can first try and factor 2021. Let's see which prime numbers, starting from the smallest, can divide it: not 2, 3, 5, 7... Eventually, we see that

$$2021 = 43 \cdot 47.$$

Now let us do the same with 2020. It is divisible by 2, because its last digit is divisible by 2:

$$2020 = 2 \cdot 1010 = 2^2 \cdot 505 = 2^2 \cdot 5 \cdot 101.$$

While doing prime decomposition and testing which numbers are prime, you can observe that if n is a natural number that is not divisible by any number up to $\sqrt{n}$, then n is prime.

Sieve method for finding prime numbers.

THEOREM 5.5. *There are infinitely many prime numbers.*

PROOF. We prove this statement by proof by contradiction.

Suppose that there are only finitely many primes, listing them as[3]

$$p_1 < p_2 < \cdots < p_n \,.$$

Then we claim that the number

$$a = p_1 p_2 \ldots p_n + 1$$

is a prime number, which is larger than the largest prime number p_n, a contradiction. To see that a is a prime number, we suppose that it is not a prime number, then it must be uniquely factorizable by the primes from $\{p_1, \ldots, p_n\}$. In particular

$$a = p_{l_1}^{m_1} \cdot \cdots \cdot p_{l_k}^{m_k} \,.$$

But then we have

$$1 = p_{l_1} \cdot \cdots \cdot p_{l_k} (p_{l_1}^{m_1-1} \cdot \cdots \cdot p_{l_k-1}^{m_k-1} - q) \,,$$

where q is some natural number. This is a contradiction because that means both

$$p_{l_1} \cdot \cdots \cdot p_{l_k}$$

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[3]Note that it is necessary for a finite list of number to have a largest number and a smallest number. In this case, the largest number in our prime list is p_n.

and

$$(p_{l_1}^{m_1-1} \cdot \cdots \cdot p_{l_k-1}^{m_k-1} - q)$$

must be 1 or (-1). This is impossible as at least $p_{l_1} \cdot \cdots \cdot p_{l_k} > 1$ by definition of primes. $\qquad\square$

REMARK 5.6. The proof above is by Euclid in 300 B.C.. A remarkable feat!

EXERCISE 8. *Infinity is lurking behind us already. You should start asking yourself what infinity really is and try to come up with a definition on your own.*

6. Greatest common divisor

One of the most fundamental concepts about numbers is the greatest common divisor.

DEFINITION 6.1. Let $a, b \in \mathbb{Z}$. An integer d is a greatest common divisor of a and b if:

(1) $d \mid a$ and $d \mid b$,
(2) if q is another integer such that $q \mid a$ and $q \mid b$ then $q \mid d$.

We denote the (unique) non-negative greatest common divisor of a and b as $\gcd(a, b)$.

EXERCISE 9. *Why is it that in our definition, we have "a" greatest commond divisor, but not "the" greatest common divisor?*

THEOREM 6.2. *Let $a, b, q, r \in \mathbb{Z}$ and suppose that $a = qb + r$. Then*

$$\gcd(a, b) = \gcd(b, r) \,.$$

PROOF. First, note that any number c that divides both a and b also divides r. To see this, we can write $a = m_1 c$ and $b = m_2 c$. Therefore,

$$m_1 c = q m_2 c + r \,,$$

which is equivalent to

$$r = c(m_1 - q m_2) \,.$$

By definition, c is a divisor of r.

Second, if d divides both b and r then d divides a. To see this, we write $b = m_3 d$ and $r = m_4 d$. Therefore,

$$a = q m_3 d + m_4 d = (q m_3 + m_4 m) d \, .$$

By those two observations, and definition of greatest common divisor,

$$\gcd(a, b) \leqslant \gcd(b, r) \quad \text{and} \quad \gcd(a, b) \geqslant \gcd(b, r) \, .$$

This means that

$$\gcd(a, b) = \gcd(b, r) \, ,$$

as desired.   □

One question arise, how can we find the greatest common divisor of any two numbers? For example, how on earth are we supposed to know $\gcd(199888, 4987774)$? Luckily, ancient people were pretty smart and there is a method that was invented at least 2300 years ago as it first appeared in Euclid's Elements (300 BC). We, as modern people, still benefit from this method as it has a lot of practical applications [4].

THEOREM 6.3 (Euclid's algorithm). *Let $a, b \in \mathbb{Z}$. The $\gcd(a, b)$ is computed as follows.*

- *Set r_1 to be the remainder of a divided by b .*
- *Set r_2 to be the remainder of b divided by r_2.*
- *Given r_{n-2} and r_{n-1}, define r_n to be the remainder of r_{n-2} divided by r_{n-1}.*
- *Stop when $r_{n+1} = 0$; then $r_n = \gcd(a, b)$.*

Writing everything explicitly, we have

$$\begin{aligned} a &= b q_1 + r_1 && (0 < r_1 < b) \\ b &= r_1 q_2 + r_2 && (0 < r_2 < r_1) \\ r_1 &= r_2 q_3 + r_3 && (0 < r_3 < r_2) \\ &\qquad \cdots \\ r_{n-2} &= r_{n-1} q_n + r_n && (0 < r_n < r_{n-1}) \\ r_{n-1} &= r_n q_{n+1} + 0 \, . \end{aligned}$$

Then $\gcd(a, b) = r_n$.

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[4]For a curious mind, you can read more about it here: `https://en.wikipedia.org/wiki/Euclidean_algorithm`

EXERCISE 10. *Why must there is a 0 remainder at the end of Euclid's algorithm? This is an important question because it answer the following question, "why should the algorithm stop?" Is there a scenario when you have to do this forever?*

EXAMPLE 6.4. Use Euclid's algorithm to find

$$\gcd(242, 66) \, .$$

Let's compute!

$$\begin{aligned} 242 &= 66 \cdot 3 + 44 \\ 66 &= 44 \cdot 1 + 22 \\ 44 &= 22 \cdot 1 + 0 \, . \end{aligned}$$

So $\gcd(242, 66) = 22$.

Let's have some fun with greatest common divisors using Euclid's algorithm.

COROLLARY 6.5. *Let $l = \gcd(a, b)$. Then there exist two integers m, n such that*

$$l = am + bn \, .$$

PROOF. We apply Euclid's algorithm to prove this statement. Writing everything explicitly, we have

$$\begin{aligned} a &= b q_1 + r_1 && (0 < r_1 < b) \\ b &= r_1 q_2 + r_2 && (0 < r_2 < r_1) \\ r_1 &= r_2 q_3 + r_3 && (0 < r_3 < r_2) \\ &\qquad \cdots \\ r_{n-2} &= r_{n-1} q_n + r_n && (0 < r_n < r_{n-1}) \\ r_{n-1} &= r_n q_{n+1} + 0 \, . \end{aligned}$$

We know that, $l = r_n$ because that is the content of Euclid's algorithm.

Now, to show what we want to show, we reverse engineer what's happening. Rewriting

$$r_1 = a - b q_1 \, .$$

Then,

$$r_2 = b - r_1 q_2 = b - (a + b q_1) q_2 = (1 - q_1 q_2) b - q_2 a \, .$$

Keep doing this until r_n and we will arrive at our conclusion. $\square$

CHAPTER 2

Logic

At its core, mathematics is a way of reasoning and is very similar to philosphy. The first part of this chapter wil reflect this basic observation. However, what sets apart mathematics from general philosophy is that the language of mathematics requires precision. There should be no ambiguity in a mathematical statement. The main goal of this chapter is to give the students a taste of what it is like to be mathematically precise.

1. What can logic be about?

We follow [Sai91] for this part.

Most broadly, logic is about reasons and reasoning. There are reasons for *acting*: you may avoid sugary desserts for the reason of wanting to keep thin or lose weight. There are reasons for *believing*: you may think that the potatoes are ready to eat for the reason that they have been boiling for twenty minutes. Historically, logic has concerned itself with reasons for believing. The way it works is that one deduce the reasons for a certain belief by making it a consequence of a "higher", more abstract, belief. Logic as a discipline was first considered as early as the 6th century B.C. and independently in India, Greek and China.

EXAMPLE 1.1. Consider the following chain of sentences.

"I believe that humans breath oxygen to live. I believe that I am a human. Therefore, I believe that I breath oxygen to live."

"All humans breath oxygen" is a higher, more abstract, fact than me breathing oxygen as I am just a particular member of the human race.

WARNING 1.2. *Not every belief has its reasons. Every logical system has its core "believes" (called* axioms*) as taken for granted and there can be no further explaination. These are the most abstract believes that are used to deduce every other belief in the same logical system. To have reasons, one needs to take a leap of faith at some point.*

Logic is a normative discipline. It sets out standard for good and bad arguments. These are technical terms and should not be confused by subjective decisions. However, the these technical terms are somewhat inspired by daily commonsensical distinction between good and bad reasons. In our daily conversation, to make good reasons for something is to create premises so that the something follows.

EXAMPLE 1.3. "James is a banker and all bankers are rich" is a good reason for "James is rich."

EXAMPLE 1.4. "James likes expensive cars" is not a good reason for "James is rich."

It is important to note that one can discern good and bad reasons without having to believe in the reasons themselves. In fact, Einstein himself did not believe in (even reject) quantum physics while being one of the founding fathers of the theory. A lot of modern mathematics revolves around physics or biology but a lot of mathematicians barely know any physics or biology(confession time).

1.1. Inductive and deductive logic. The result of assembling premises and conclusions together is called an argument. An argument is valid, or true, if the conclusions follow from the premises. The two most common forms of logic are inductive logic and deductive logic.

An example of inductive logic is the following.

EXAMPLE 1.5. The sun has risen every morning so far; therefore it (probably) will rise tomorrow.

EXERCISE 11. *Contrast the previous example with the following sentence:* " *The sun has risen every morning so far; therefore it (probably) will NOT rise tomorrow."*
Is one of these sentences more true than the other? How do you know?

An example of deductive logic is the following.

EXAMPLE 1.6. All men are mortal. Socrates is a man. Therefore, Socrates is mortal.

Thus, a way to distinguish between inductive and deductive logic is: for deductive logic, it is impossible for the conclusion to be false if the premises are true. For inductive logic, this is not the case as the conclusion in this case may be false despite the premises being true. We can see that it is only in deductive logic, one can talk about the validity of an argument. In inductive logic, there are degree in strength of an argument but inductive reasoning can <u>never be valid</u> by our definition of validity[1]. However, an inductive argument can be stronger than another inductive argument (just make sure one is talking about the same thing– comparing apples to apples).

Mathematics is all about deductive logic whereas science must involve both inductive and deductive logic. The combination of inductive and deductive logics in science gives birth to the need of probability and statistics (whose theories are all mathematical and deductive).

WARNING 1.7. *Do not confuse inductive logic with the method of induction, which is a method in deductive logic. Although, there are resemblance between the two. The difference is that in the method of induction, one is given the super power in theory to transcend time to "go off to infinity" whereas inductive logic is limited by physical evidence, which time is a major road block...*

EXERCISE 12. *Make a table of comparison between inductive and deductive logic.*

EXERCISE 13. *What's wrong with the following?*
Statement: You have horns.
"Proof": What you haven't lost, you have. You haven't lost your horns. Therefore you have horns.

EXERCISE 14. *What's wrong with the following?*
Statement: You don't know your father.
"Proof": I show you a photo of someone, the photo is covered by a cloth. Do you know who's in the photo? You can't see, so you don't. But it's a photo of your father. Therefore you don't know your father.

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[1]Be careful here that validity is a technical term and should not be confused with the daily use of the word

2. Mathematical logic

We follow [New] for this part.

Mathematical logic is the study of logic restricted to mathematics. Its existence is to address the biggest problem in the foundation of mathematics: are theories of mathematics consistent with each others? That said, most working mathematicians do not pay attention to the question of foundations, which might be a worrisome fact. I can only have my fingers crossed that mathematics will not fall apart one day (which, it did for a period of time, when Georg Cantor discovered different infinities in the 19th century)...

A **mathematical statement** is a statement that at least the statement maker has to be able to assign a **truth value** ('true' or 'false') to it. The truth assignment could be the result of an immediate observation or a long chain of difficult reasoning. To make the truth assignment valid, every single argument in the chain of reasoning must be valid. A **proof** of a mathematical statement is a chain of valid arguments that make the mathematical statement <u>true</u>.

There are many names for a mathematical statement, depending on the use:

- **Proposition:** general term that can be used anytime.
- **Theorem:** a particularly important mathematical statement given the context.
- **Lemma:** a mathematical statement that will be used as a stepping stone to prove a theorem.
- **Corollary:** a mathematical statement whose truth value could be deduced from a theorem without much effort.

A statement that are believed to be verifiable but no human has seen or discovered its proof yet is called a **conjecture**.

2.1. Structure. Every mathematical statement has the following structure:

$$\text{Assumptions} + \text{Goals}$$

EXAMPLE 2.1. Suppose Philadelphia is in Massachusetts and Penn is in Philadelphia, then Penn is in Massachusetts.

WARNING 2.2. *Assumptions themselves need not to be true. We will talk more about this later when we talk about truth table.*

EXERCISE 15. *Find an example of a famous mathematical statement that its assumptions are not yet verified.*

EXAMPLE 2.3. For example, abc conjecture implies Fermat's last theorem. But as of now, the status of the abc conjecture is subject to debate. A Fields medalist Peter Scholze and Jakob Stix found a gap in Mochizuki's proof.

Now a quote from Wikipedia: "Scholze and Stix wrote a report asserting and explaining an error in the logic of the proof and claiming that the resulting gap was "so severe that ... small modifications will not rescue the proof strategy"; Mochizuki claimed that they misunderstood vital aspects of the theory and made invalid simplifications.

On April 3, 2020, two Japanese mathematicians announced that Mochizuki's claimed proof would be published in Publications of the Research Institute for Mathematical Sciences (RIMS), a journal of which Mochizuki is chief editor. In March 2021, Mochizuki's proof was published in RIMS."

3. Symbolic logic

One of the main goals of mathematics is to reduce complicated statements/observations to simple abstract structures that are more tractable to human minds and still keep the essential features of the things one would like to study. This is as much of an art as anything else because too much abstraction would lead to triviality, which may not be very interesting.

Symbolic logic is a system of logic that can be used to reduce a mathematical statement into "agreed" formulas that are easier to determine its truth value.[2]

Let us consider a simple example from [New]:

EXAMPLE 3.1. If c divides b and b divides a, then c divides a.

_____

[2]Gottfried Leibniz (another inventor of Calculus) was among the first people to realize the importance of having a system of logic that is universal and calculatable but couldn't actualize this dream. The goal was to reduce confusions and disputes among philosophers and arguers. (Just turn on the Presidential debates and you will understand why we need such a system...) The first well-known work that successfully made symbolic logic a mathematical field was by George Boole in 1854 [Boo09]. One of the earliest work that started the modern account of logic and foundation of mathematics was by Russel and Whitehead [WR97] (there is a comic book about it [DP09]!).

We see that each of the statements "c divides b", "b divides a", and "c divides a" is a proposition if they stand alone by themselves. Thus, abstractly, each of them could be assigned a symbol

- $P = c$ divides b
- $Q = b$ divides a
- $R = c$ divides a

Then you can write

$$\text{If } P \text{ and } Q, \text{ then } R.$$

The above form of the statement looks easier to follow (at least to the mind of a non-English speaker) since at least we don't need to know what "divides" means. While it is not too useful in terms of conveying knowledge, it is extremely useful when it comes to determining the validity of the statement itself. This leads us to the next question: What makes a statement true?

We will need a few new words.

DEFINITION 3.2. A **propositional variable** is a symbol that represents a proposition.

As we said earlier, propositions are just mathematical statements, which are required to have truth values ('true' or 'false').

DEFINITION 3.3. A **logical operator** is a symbol (or collection of words) that turn one or more propositional variables to a single new proposition.

Basic logical operators are:

- Conjuction ('and', $\wedge$)
- Disjunction ('or', $\vee$)
- Implication ('if...then...', $\implies$)
- Negation ('not', $\neg$)

As simple as they look, these four operators build most of mathematics and anything that require reasoning (philosophy, law, computer science, etc.).

DEFINITION 3.4. A **propositional formula** is an expression that is either a propositional variable, or is built up from simpler propositional formulae using logical operators.

REMARK 3.5. When I ask, "What is the variable for the proposition?", I am more interested in what the symbol you give to the mathematical

statement, not so much the content of it. Similarly, when I ask, "What is the formula for the proposition?" I am more interested in the way the proposition is written up, not so much what the proposition conveys.

The simplest kind of propositions is one that only contains one single propositional variable that already explicitly has the truth value ('true' or 'false') known (whether it is a proven statement, ~~a common knowledge~~ or an assumption).   From these atomic propositions, we could build more complicated kinds of propositions with more complicated propositional formula by obeying certain logical rules of the logical operators[3]. This process is entirely "calculatable".   Here are the rules for the basic logical operators listed above.

Conjunction ('and', $\wedge$). The propositional formula for a proposition made by a conjuction has the following form

$$P \wedge Q.$$

RULE. *The proposition $P \wedge Q$ (we say "P and Q") is true if* **both** *P and Q are true. Otherwise, if either (or both) P or Q is false, $P \wedge Q$ is false.*

Disjunction ('or', $\vee$). The propositional formula for a proposition made by a disjunction has the following form

$$P \vee Q.$$

RULE. *The proposition $P \vee Q$ (we say "P or Q") is true if* **either one** *(or both) of P or Q is true. $P \vee Q$ is flase if* **both** *P and Q are false.*

Implication ('if...then...', $\implies$). The propositional formula for a proposition made by a disjunction has the following form

$$P \implies Q.$$

RULE. *The proposition $P \implies Q$ (we say "P implies Q") is true if one of the following cases holds:*

- *P is true and Q is true.*
- *P is false.*

EXERCISE 16. *This is one of the most confusing rules in logic. Meditate on the rule of implication.*

Negation ('not', $\neg$). The propositional formula for a proposition made by a disjunction has the following form

$$\neg P.$$

RULE. *The proposition $\neg P$ (we say "not P") is true if P is false.*

EXERCISE 17. *Mess with your friends/parents/siblings with 'and', 'or', 'if not … then...'*

4. Variables and quantifiers

Consider the following sentence:

"x is divisible by 7."

QUESTION. *Is this a proposition?*

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[3]Even though we call them rules, they follow an intuitive model of our daily reasoning. The advantage of defining explicit rules is to make the reasoning more consistent by the abstraction.

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