Homework 3

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```r
if (!require(pacman)) install.packages("pacman")
```

```
Loading required package: pacman
```

```r
pacman::p_load(ggplot2, tidyverse)
```

Exercise 1

We define the probability mass function of X_i, a poisson random variable as

$$P(X_i = x_i) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

We will determine the likelihood by considering the product of the the poisson PMF in order to account for all observations which occur for i = 1, ..., n.

This is done under the assumption that each occurrence of X_i is iid.

$$X_i \overset{\text{i.i.d.}}{\sim} Poisson(\lambda)$$

$$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

This can be simplified to below.

$$L(\lambda) = \frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!}$$

Exercise 2

In order to find the maximum likelihood estimate, we are able to find the λ for which the log likelihood function is maximized.

This works because log is a monotonic function.

$$log(L(\lambda)) = log(\frac{\lambda^{\sum_{i=1}^{n} x_i} e^{-n\lambda}}{\prod_{i=1}^{n} x_i!})$$

$$log(L(\lambda)) = \sum_{i=1}^{n} x_i \; log(\lambda) - n\lambda - log(\prod_{i=1}^{n} x_i!)$$

$$\frac{\partial}{\partial\lambda} log(L(\lambda)) = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

$$\frac{\sum_{i=1}^{n} x_i}{\lambda} - n = 0$$

$$\hat{\lambda}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n}$$

To ensure this is a maximum, we can check the second-derivative of the log-likelihood function.

$$\frac{\partial^2}{\partial\lambda^2} log(\lambda) = \frac{-\sum_{i=1}^{n} x_i}{\lambda^2}$$

$$\frac{\partial^2}{\partial\lambda^2} log(\lambda) < 0$$

so we know our MLE for lambda is a maximum.

Exercise 3

```r
sumX = 500
n = 100

logLikelihood <- function(lambda){
  sumX*log(lambda) -n*lambda
}
```

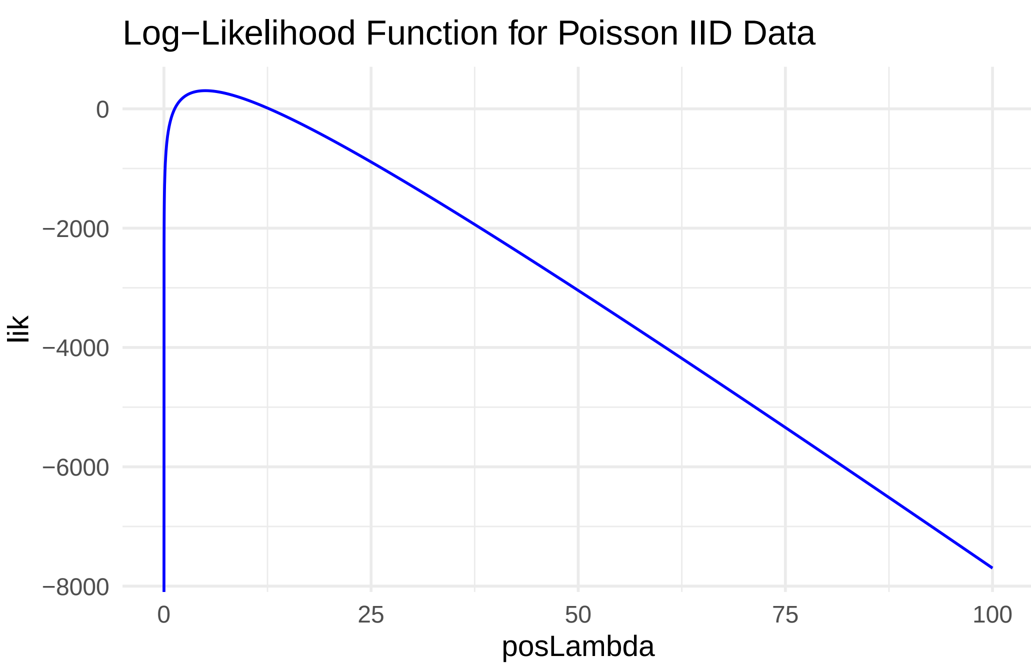
```r
nGrid = 10000

posLambda <- seq(0, 100, length = nGrid)

lik <- logLikelihood(posLambda)

df <- data.frame(posLambda, lik)

ggplot(df, aes(x = posLambda, y = lik)) +
  geom_line(color = "blue") +
  labs(
    title = "Log-Likelihood Function for Poisson IID Data"
  ) +
  theme_minimal()
```




```r
df |>
  filter(lik == max(lik))
```

```
  posLambda     lik
1    5.0005 304.719
```

Exercise 4

Game	First Five Shots	Likelihood (No Hot Hand)	Likelihood (Hot Hand)
1	BMMBB	$(p_b)^3(1-p_b)^2$	$p_b(p_{b\|b})(1-p_{b\|b})(1-p_b)$
2	MBMBM	$(p_b)^2(1-p_b)^3$	$(1-p_b)(p_b)^2(1-p_{b\|b})^2$
3	MMBBB	$p_b^3(1-p_b)^2$	$(1-p_b)^2(p_b)(p_{b\|b})^2$
4	BMMMB	$p_b^2(1-p_b)^3$	$p_b^2(1-p_{b\|b})(1-p_b)^2$
5	MMMMM	$(1-p_b)^5$	$(1-p_b)^5$

b)

0.4, the MLE, is a better estimate than some arbitrary value like 0.3 because by maximizing the likelihood, we are providing an estimate for which the data we observed is relatively probable.

c)

We specify the likelihood of model 1, the no hot hand model.

$$Lik(p_b) = p_b^{10}(1-p_b)^{15}$$

We can find the value of p_b for which the log likelihood is maximized which will be the same value for which log is maximized due to monotonicity of the log function.

$$log(Lik(p_b))) = 10log(p_b) + 15log(1-p_b)$$

Differentiating will serve as our means of finding extrema under the assumption that the rate of change is 0 at extrema.

$$\frac{\partial}{\partial p_b} log(Lik(p_b)) = \frac{10}{p_b} - \frac{15}{1-p_b} = 0$$

$$\frac{10}{p_b} = \frac{15}{1-p_b}$$

$$\hat{p_b}_{MLE} = 2/5$$

We check the second derivative with respect to p_b to ensure the estimate is, in fact, a maximum.

$$\frac{\partial^2}{\partial p_b^2} log(Lik(p_b)) = \frac{-10}{p_b^2} - \frac{15}{(1-p_b)^2} < 0$$

so we know our $\hat{p_b}$ value is a maximum.

Model 2: Hot Hand Model

$$Lik(p_b, p_{b|b}) = p_b^7(1-p_b)^{11}(p_{b|b})^3(1-p_{b|b})^4$$

$$log\ Lik(p_b, p_{b|b}) = 7 * log(p_b) + 11log(1-p_b) + 3log(p_{b|b}) + 4log(1-p_{b|b})$$

We check the second derivative with respect to each parameter to ensure the estimates are, in fact, maxima.

$$\frac{\partial}{\partial p_b} log(Lik(p_b, p_{b|b})) = \frac{7}{p_b} - \frac{11}{1-p_b} = 0$$

$$\frac{7}{p_b} = \frac{11}{1-p_b}$$

$$\hat{p_b}_{MLE} = \frac{7}{18}$$

$$\frac{\partial^2}{\partial p_b^2} = \frac{-7}{p_b^2} - \frac{11}{(1-p_b)^2} < 0$$

so our $\hat{p_b}$ is a maximum.

$$\frac{\partial}{\partial p_{b|b}} log(Lik(p_b, p_{b|b})) = \frac{3}{p_{b|b}} - \frac{4}{1-p_{b|b}}$$

$$\frac{3}{p_{b|b}} = \frac{4}{1-p_{b|b}}$$

$$\hat{p_{b|b}}_{MLE} = \frac{3}{7}$$

$$\frac{\partial^2}{\partial p_{b|b}^2} = \frac{-3}{p_{b|b}} - \frac{4}{(1-p_{b|b})^2} < 0$$

so this $\hat{p_{b|b}}$ is a maximum.

```r
logLik1<- function(pb){
  10*log(pb) + 15*log(1-pb)
}

logLik2 <- function(pb, pbtwo){
  7*log(pb)+11*log(1-pb)+3*log(pbtwo)+4*log(1-pbtwo)
}
```

```r
#the mle for likelihood of model 1

mle = 2/5

# mles for likelihood of model 2
mle1 = 7/18
mle2 = 3/7

#LRT
LRT <- 2 *(logLik2(mle1, mle2)-logLik1(mle))

#chose to utilize 1 degree of freedom since there's only one new parameter
pchisq(LRT, 1, lower.tail = FALSE)
```

```
[1] 0.8560131
```

The chi-squared test p value is large so we fail to reject evidence that the not hot hand is not sufficient. For this reason, we choose the first model without the additional parameter.