We define a probability model to demostrate.

let C be the prediction.

Y be the ground truth.

A be the protected value.

for all pairs (a, y), let $P(C) = \frac{1}{2}$.

let A and Y be dependent variables.

st $P(A) = \frac{1}{2}$, $P(Y) = \frac{1}{2}$. let $P(A|Y) = \frac{1}{3}$. $P(\neg A|Y) = \frac{2}{3}$

$P(A|\neg Y) = \frac{2}{3}$. $P(\neg A|\neg Y) = \frac{1}{3}$.

WTS. this model is not sufficient but seperateble.

we have:

$P(A \wedge Y) = P(A|Y) \cdot P(Y) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$

$P(A \wedge \neg Y) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

$P(\neg A \wedge Y) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

$P(\neg A \wedge \neg Y) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

A joint probability table:

C	Y	A	P(C),P(A),P(Y)	P(A|Y)	P(AY)	P(AC)	P(C|YA)	P(Y|AC)	P(YCA)
F	F	F	1/2	2/3	1/3	1/4	1/2	2/3	1/6
F	F	T	1/2	1/3	1/6	1/4	1/2	1/3	1/12
F	T	F	1/2	1/3	1/6	1/4	1/2	1/3	1/12
F	T	T	1/2	2/3	1/3	1/4	1/2	2/3	1/6
T	F	F	1/2	2/3	1/3	1/4	1/2	2/3	1/6
T	F	T	1/2	1/3	1/6	1/4	1/2	1/3	1/12
T	T	F	1/2	1/3	1/6	1/4	1/2	1/3	1/12
T	T	T	1/2	2/3	1/3	1/4	1/2	2/3	1/6

Since $P(C|YA) = P(C|Y\neg A)$, this is seperateble.

But since $P(Y|CA) \neq P(Y|C\neg A)$. this is not sufficient.

Similarly, for sufficient but not seperateble,

make C the protected value and A the prediction.

Then we have $P(C|YA) \neq P(C|Y\neg A)$ which means not seperable,

and $P(Y|CA) = P(Y|C\neg A)$ which means sufficient.