CSC384 Assignment#4

Question 3

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Claim.
We cannot enforce **Sufficiency** and **Separation** at the same time.

Proof. We define below abbreviation here:

$$\begin{aligned}\text{Hyperlipidemia} = YES &\rightarrow H = h\\ \text{Prediction} = YES &\rightarrow P = p\\ \text{Gender} = \textit{Female} &\rightarrow G = g\end{aligned}$$

Hence, by the handout,

$$\begin{aligned}\textbf{Separation} &:= \\ P(P=p|H=h) &= P(P=p|H=h, G=g)\end{aligned}$$

$$\begin{aligned}\textbf{Sufficiency} &:= \\ P(H=h|P=p) &= P(H=h|P=p, G=g)\end{aligned}$$

$$\begin{aligned}P(P=p, H=h, G=g) &= P(P=p|H=h, G=g)P(H=h, G=g)\\ &= P(P=p|H=h, G=g)P(H=h|G=g)P(G=g) &(1)\\ P(P=p, H=h, G=g) &= P(P=p|H=h)P(H=h)P(G=g|P=p, H=h) &(2)\end{aligned}$$

$$\begin{aligned}P(P=p, H=h, G=g) &= P(H=h|P=p, G=g)P(P=p, G=g)\\ &= P(H=h|P=p, G=g)P(P=p|G=g)P(G=g) &(3)\\ P(P=p, H=h, G=g) &= P(H=h|P=p)P(P=p)P(G=g|P=p, H=h) &(4)\end{aligned}$$

So, by **Separation** and equations $(1), (2)$, we can cancel the terms and get

$$\begin{aligned}P(H=h|G=g)P(G=g) &= P(H=h)P(G=g|P=p, H=h)\\ \frac{P(H=h|G=g)}{P(H=h)} &= \frac{P(G=g|P=p, H=h)}{P(G=g)} &(5)\\ &= \frac{P(G=g|H=h)}{P(G=g)} &(6)\end{aligned}$$

By (5) and (6),

$$P(G=g|P=p, H=h) = P(G=g|H=h). \qquad (7)$$

Pluging (7) into (4) and (3) we have

$$\begin{aligned}P(H=h|P=p)P(P=p)P(G=g|H=h) &= P(H=h|P=p, G=g)P(P=p|G=g)P(G=g)\\ \frac{P(H=h|P=p)}{P(H=h|P=p, G=g)} &= \frac{P(P=p|G=g)}{P(P=p)} \times \frac{P(G=g)}{P(G=g|H=h)}\\ &= \frac{P(G=g|P=p)}{P(G=g)} \times \frac{P(G=g)}{P(G=g|H=h)}\\ &= \frac{P(G=g|P=p)}{P(G=g|H=h)} &(8)\end{aligned}$$

Thus we can conclude from equation (8), konwn **Separation**, whether **Sufficiency** also holds, i.e.

$$P(H=h|P=p) \quad = \quad P(H=h|P=p, G=g),$$

depends on whether

$$P(G=g|P=p) \quad = \quad P(G=g|H=h). \tag{9}$$

Here the claim "We cannot enforce **Sufficiency** and **Separation** at the same time." is proved - it holds unless in the particular and lucky case described in (9).

□

Note. According to handout, we are supposed to provide the examples as the form of joint probability distribution of (A, C, Y) where A is the "protected attribute", C is a classification, and Y is a label representing "ground truth" and

$$P(A=a, C=c, Y=y) > 0 \text{ for all } a \in A.\text{domain}, c \in C.\text{domain}, y \in Y.\text{domain}.$$

And **Sufficiency** means $P(Y|C) = P(Y|C, A)$ while **Separation** means $P(C|Y) = P(C|Y, A)$.

Also in accordance with piazza question @564, Professor Allin said "You do not need to directly relate these two joints to the medicalDiagnosis network; it's fine to make up your own joints in order to illustrate the point."

Suppose A, C, Y respectively represents *Gender, Prediction, Hyperlipidemia*. Therefore we can fabricate two data examples to demonstrate the claim and our proof.

Example 1. *where **suficiency** holds but not separation*

That is $P(Y|C) = P(Y|C, A)$ and $P(C|Y) \neq P(C|Y, A)$

Suppose we have the tables

#	**C**	**P(c)**
1	YES	0.3
2	NO	0.7

#	**Y**	**P(y)**
1	YES	0.375
2	NO	0.625

#	**A**	**C**	**P(a|c)**
1	Female	YES	0.3
2	Female	NO	0.5
3	Male	YES	0.7
4	Male	NO	0.5

#	**A**	**Y**	**P(a|y)**
1	Female	YES	0.356
2	Female	NO	0.4904
3	Male	YES	0.644
4	Male	NO	0.5096

#	**A**	**C**	**Y**	**P(y|c)=P(y|c,a)**
1	Female	YES	YES	0.9
2	Female	YES	NO	0.1
3	Female	NO	YES	0.15
4	Female	NO	NO	0.85
5	Male	YES	YES	0.9
6	Male	YES	NO	0.1
7	Male	NO	YES	0.15
8	Male	NO	NO	0.85

Hence, the joint probability distribution $P(a,c,y)$ is illustrated in the table below:

#	A	C	Y	P(a,c,y)=P(y|c)×P(a|c)×P(c)
1	Female	YES	YES	$0.9\times0.3\times0.3=0.081$
2	Female	YES	NO	$0.1\times0.3\times0.3=0.009$
3	Female	NO	YES	$0.15\times0.5\times0.7=0.0525$
4	Female	NO	NO	$0.85\times0.5\times0.7=0.2975$
5	Male	YES	YES	$0.9\times0.7\times0.3=0.189$
6	Male	YES	NO	$0.1\times0.7\times0.3=0.021$
7	Male	NO	YES	$0.15\times0.5\times0.7=0.0525$
8	Male	NO	NO	$0.85\times0.5\times0.7=0.2975$

Now to check if **Separation** holds, there are many ways to compute $P(c|y)$:

$$\begin{aligned}P(c|y) &= \frac{P(c,y)}{P(y)}=\frac{\sum_{a\in A}P(c,y|a)P(a)}{P(y)}=\frac{P(a,c,y)+P(\neg a,c,y)}{P(y)}\\ &= \frac{P(a,c,y)}{P(a|c)P(y)}\\ &= \frac{P(y|c)P(c)}{P(y)}\end{aligned}$$

Take the first one of this, and compute $P(c|y,a)$ by:

$$P(c|y,a) = \frac{P(a,c,y)}{P(y)P(a|y)}$$

#	A	C	Y	P(c|y)	P(c|y,a)
1	Female	YES	YES	$\frac{0.081+0.189}{0.375}=0.72$	$\frac{0.081}{0.375\times0.356}=0.606741573$
2	Female	YES	NO	$\frac{0.009+0.021}{0.625}=0.048$	$\frac{0.009}{0.625\times0.4904}=0.029363785$
3	Female	NO	YES	$\frac{0.0525+0.0525}{0.375}=0.28$	$\frac{0.0525}{0.375\times0.356}=0.393258427$
4	Female	NO	NO	$\frac{0.2975+0.2975}{0.625}=0.952$	$\frac{0.2975}{0.625\times0.4904}=0.970636215$
5	Male	YES	YES	$\frac{0.081+0.189}{0.375}=0.72$	$\frac{0.189}{0.375\times0.644}=0.782608696$
6	Male	YES	NO	$\frac{0.009+0.021}{0.625}=0.048$	$\frac{0.021}{0.625\times0.5096}=0.065934066$
7	Male	NO	YES	$\frac{0.0525+0.0525}{0.375}=0.28$	$\frac{0.0525}{0.375\times0.644}=0.217391304$
8	Male	NO	NO	$\frac{0.2975+0.2975}{0.625}=0.952$	$\frac{0.2975}{0.625\times0.5096}=0.934065934$

And we can see from the table that $P(C|Y)\neq P(C|Y,A)$ which means **Separation** doesn't hold.

Example 2. *where **separation** holds but not sufficiency*

That is $P(C|Y)=P(C|Y,A)$ and $P(Y|C)\neq P(Y|C,A)$

Suppose we have the tables

#	C	P(c)
1	YES	0.375
2	NO	0.625

#	Y	P(y)
1	YES	0.3
2	NO	0.7

#	A	C	P(a|c)
1	Female	YES	0.356
2	Female	NO	0.4904
3	Male	YES	0.644
4	Male	NO	0.5096

#	A	Y	P(a|y)
1	Female	YES	0.3
2	Female	NO	0.5
3	Male	YES	0.7
4	Male	NO	0.5

#	A	C	Y	P(c|y)=P(c|y,a)
1	Female	YES	YES	0.9
2	Female	YES	NO	0.15
3	Female	NO	YES	0.1
4	Female	NO	NO	0.85
5	Male	YES	YES	0.9
6	Male	YES	NO	0.15
7	Male	NO	YES	0.1
8	Male	NO	NO	0.85

Hence, the joint probability distribution $P(a,c,y)$ is illustrated in the table below:

#	A	C	Y	P(a,c,y)=P(c|y)×P(a|y)×P(y)
1	Female	YES	YES	$0.9 \times 0.3 \times 0.3 = 0.081$
2	Female	YES	NO	$0.15 \times 0.5 \times 0.7 = 0.0525$
3	Female	NO	YES	0.1×0.3×0.3=0.009
4	Female	NO	NO	$0.85 \times 0.5 \times 0.7 = 0.2975$
5	Male	YES	YES	$0.9 \times 0.7 \times 0.3 = 0.189$
6	Male	YES	NO	$0.15 \times 0.5 \times 0.7 = 0.0525$
7	Male	NO	YES	$0.1 \times 0.7 \times 0.3 = 0.021$
8	Male	NO	NO	$0.85 \times 0.5 \times 0.7 = 0.2975$

Now to check if **Sufficiency** holds, there are many ways to compute $P(y|c)$:

$$\begin{aligned} P(y|c) &= \frac{P(c,y)}{P(c)} = \frac{\sum_{a\in A} P(c,y|a)P(a)}{P(c)} = \frac{P(a,c,y)+P(\neg a,c,y)}{P(c)} \\ &= \frac{P(a,c,y)}{P(a|y)P(c)} \\ &= \frac{P(c|y)P(y)}{P(c)} \end{aligned}$$

Take the first one of this, and compute $P(y|c,a)$ by:

$$P(y|c,a) = \frac{P(a,c,y)}{P(c)P(a|c)}$$

#	A	C	Y	P(y|c)	P(y|c,a)
1	Female	YES	YES	$\frac{0.081+0.189}{0.375} = 0.72$	$\frac{0.081}{0.375 \times 0.356} = 0.606741573$
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8	Male	NO	NO	$\frac{0.2975+0.2975}{0.625} = 0.952$	$\frac{0.2975}{0.625 \times 0.5096} = 0.934065934$

And we can see from the table that $P(Y|C) \neq P(Y|C,A)$ which means **Sufficiency** doesn't hold.