• Практика• Производная. Часть 6. 20.04.

11.5.4. d^2z - ?, $z = \frac{xy}{x-y}$, $z''_{x^2} + 2z''_{xy} + z''_{y^2} = \frac{2}{x-y}$ - док-зать.

① $z'_x = \frac{-y^2}{(x-y)^2}$, $\left[z'_x = \frac{\partial z}{\partial x}, \quad z'_y = \frac{\partial z}{\partial y} \right]$,

$z'_y = \frac{x^2}{(x-y)^2}$ $\left[z''_{xx} = z'_{x^2} = \left(\frac{\partial z}{\partial x}\right)'_x = \frac{\partial^2 z}{\partial x \partial x} = \frac{\partial^2 z}{\partial^2 x} \right]$

$\left[z''_{xy} = \left(\frac{\partial z}{\partial x}\right)'_y = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial z \partial z}{\partial x \partial y} \right]$

$\left[d^2z = d(dz) = \frac{\partial^2 z}{\partial x^2} dx^2 + \frac{2\partial^2 z}{\partial x \partial y} dxdy + \frac{\partial^2 z}{\partial y^2} dy^2 \right]$

① $z'_x = \frac{(xy)'_x(x-y) - (xy)(x-y)'_x}{(x-y)^2} = \frac{-y^2}{(x-y)^2}$

$z'_y = \frac{\partial z}{\partial y} = \frac{(xy)'_y(x-y) - (xy)(x-y)'_y}{(x-y)^2} = \frac{x^2}{(x-y)^2}$

② $z''_{x^2} = \left(\frac{\partial z}{\partial x}\right)'_x = \left(\frac{-y^2}{(x-y)^2}\right)'_x = -y^2\left((x-y)^{-2}\right)'_x = -y^2(-2)(x-y)^{-3} = \frac{-2y^2}{(x-y)^3}$

$z''_{y^2} = \left(\frac{\partial z}{\partial y}\right)'_y = \left(\frac{x^2}{(x-y)^2}\right)'_y = x^2\left((x-y)^{2}\right)'_y = \frac{-2x^2}{(x-y)^{-3}}$

$z''_{xy} = \left(\frac{\partial z}{\partial x}\right)'_y = \left(\frac{-y^2}{(x-y)^2}\right)'_y = -\frac{(y^2)'_y(x-y)^2 - y^2(x-y)'_y}{(x-y)^4} = \frac{-2xy}{(x-y)^3}$

③ $d^2z = \frac{2y^2}{(x-y)^3} dx^2 + \frac{-4xy}{(x-y)^3} dxdy + \frac{2x^2}{(x-y)^3} dy^2$ -

④ $z''_{x^2} + 2z''_{xy} + z''_{y^2} = \frac{2y^2}{(x-y)^3} - \frac{4xy}{(x-y)^3} + \frac{2x^2}{(x-y)^3} = \frac{2(y-x)^2}{(x-y)^3} = \frac{2}{x-y}$ = п.ч.

ч.т.д.

11.5.5. $z = \dfrac{xy}{x+y}$, d^3z

$$d^3z = z'''_{x^3}dx^3 + 3z'''_{x^2y}\,dx^2dy + 3z'''_{xy^2}\,dx\,dy^2 + z'''_{y^3}dy^3$$

① $z'_x = \dfrac{\partial z}{\partial x} = \left(\dfrac{xy}{x+y}\right)'_x = \dfrac{(xy)'_x(x+y) - (xy)(x+y)'_x}{(x+y)^2} = \dfrac{y^2}{(x+y)^2}$

$z'_y = \dfrac{\partial z}{\partial y} = \left(\dfrac{xy}{x+y}\right)'_y = \dfrac{(xy)'_y(x+y) - (xy)(x+y)'_y}{(x+y)^2} = \dfrac{x^2}{(x+y)^2}$

② $z''_x = \left(\dfrac{y^2}{(x+y)^2}\right)'_x = \dfrac{-2y^2}{(x+y)^3}$

$z''_{xy} = \left(\dfrac{\partial z}{\partial x}\right)'_y = \left(\dfrac{y^2}{(x+y)^2}\right)'_y = \dfrac{(y^2)'_y(x+y)^2 - y^2((x+y)^2)'}{(x+y)^4} =$

$= \dfrac{2y(x+y)^2 - y^2\cdot 2(x+y)}{(x+y)^4} = \dfrac{(x+y)(2y(x+y) - 2y^2)}{(x+y)^4} = \dfrac{2y(x+y-y)}{(x+y)^3}$

$= \dfrac{2xy}{(x+y)^3}$

$z''_{yy} = z''_{y^2} = \left(\dfrac{\partial z}{\partial y}\right)'_y = \dots = \dfrac{-2x^2}{(x+y)^3}$

③ $z'''_{x^3} = \left(\dfrac{\partial^2 z}{\partial x^2}\right)'_x = \left(-\dfrac{2y^2}{(x+y)^3}\right)'_x = \dfrac{6y^2}{(x+y)^4}$

$z'''_{x^2y} = \left(\dfrac{\partial^2 z}{\partial x^2}\right)'_y = \left(-\dfrac{2y^2}{(x+y)^3}\right)'_y = \dfrac{(-4y)(x+y)^3 + 2y^2\cdot 3(x+y)^2}{(x+y)^6} =$

$= \dfrac{(x+y)^2(-4y() + 6y^2)}{(x+y)^6} = \left[\dfrac{2y(2x+y) + 6y^2}{(x+y)^4}\right] = \dfrac{2y^2 - 4xy}{(x+y)^4}$

$z'''_{y} = \left(\dfrac{\partial^2 z}{\partial y^2}\right)'_y = \left(-\dfrac{2x^2}{(x+y)^3}\right)'_y = \dfrac{6x^2}{(x+y)^4}$

$$z'''_{xy^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)'_y = \left(\frac{2xy}{(x+y)^3}\right)'_y = \frac{(2xy)'_y (x+y)^3 - (2xy)((x+y)^3)'_y}{(x+y)^6} =$$

$$= \frac{2x(x+y)^3 - 2xy \cdot 3(x+y)^2}{(x+y)^6} = \frac{(x+y)^2(2x(x+y) - 6xy)}{(x+y)^6} =$$

$$= \frac{2x(x+y-3y)}{(x+y)^4} = \frac{2x^2 - 4xy}{(x+y)^4}$$

④ $$d^3 z = \frac{6y^2}{(x+y)^4} dx^3 + \frac{6y^2 - 12xy}{(x+y)^4} dx^2 dy + \frac{6x^2 - 12xy}{(x+y)^4} dx dy^2 +$$

$$+ \frac{6x^2}{(x+y)^4} dy^3 = \frac{6}{(x+y)^4}\left(y^2 dx^3 + (y^2 - 2xy) dx^2 dy + (x^2 - 2xy) dx dy^2 +\right.$$

$$\left. + x^2 dy^3\right) =$$

11.5.6. $d^2 z$, $z = \ln(x^2 + y^2)$

$$d^2 z = d(dz)$$

① dz

$$z'_x = (\ln(x^2+y^2))'_x = \frac{2x}{x^2+y^2}, \quad z'_y = (\ln(x^2+y^2))'_y = \frac{2y}{x^2+y^2}$$

$$dz = \frac{2x}{x^2+y^2} dx + \frac{2y}{x^2+y^2} dy = 2 \cdot \frac{x dx + y dy}{x^2+y^2}$$

② $$d(dz) = (dz)'_x dx + (dz)'_y dy \;\Big|\Rightarrow 2\left(\left(\frac{x dx + y dy}{x^2+y^2}\right)'_x dx +\right.$$

$$\left. + \left(\frac{x dx + y dy}{x^2+y^2}\right)'_y dy\right) = 2\left(\frac{(x dx + y dy)'_x (x^2+y^2) - (x dx + y dy)(x^2+y^2)'_x}{(x^2+y^2)^2} dx +\right.$$

$$\left. + \frac{(x dx + y dy)'_y (x^2+y^2) - (x dx + y dy)(x^2+y^2)'_y}{(x^2+y^2)^2} dy\right) = \left[\begin{array}{l}(x dx + y dy)'_x = \text{примечание} \\ dx \text{ и } dy \text{ за const!}\end{array}\right]$$

$$= \frac{2}{(x^2+y^2)^2} \cdot \left[(dx(x^2+y^2) - (x dx + y dy) 2x)) dx + (dy(x^2+y^2) - 2y(x dx + y dy)) dy\right] =$$

$$= 2/(x^2+y^2)^2 \cdot ((y^2 - x^2) dx^2 - 4xy\, dx dy + (x^2 - y^2) dy^2)$$