* Практика * Интегрирование. Часть 4. 15.06.

8.4.2. $\int \frac{\sqrt[3]{x}\,dx}{\sqrt[3]{x^2}-\sqrt{x}} = \left[\begin{matrix} n=3 \\ q=2 \end{matrix} \right| \Rightarrow k = \text{НОК}(3;2)=6 \Rightarrow$

$\Rightarrow x=t^6 \Rightarrow dx = 6t^5dt] = \int \frac{\sqrt[3]{t^6}\cdot 6t^5 dt}{\sqrt[3]{(t^6)^2}-\sqrt{t^6}} = \int \frac{t^2\cdot 6t^5dt}{t^4-t^3} =$

$= \int \frac{6t^7dt}{t^3(t-1)} = 6\int\frac{t^4dt}{t-1} = 6\int \frac{t^4-1+1}{t-1}dt =$

$= 6\int\frac{t^4-1}{t-1}dt + 6\int\frac{dt}{t-1} = 6\left(\int\frac{(t-1)(t+1)(t^2+1)}{(t-1)}dt +\right.$

$\left.+\int\frac{dt}{t-1}\right) = 6\left(\int(t+1)(t^2+1)dt + \int\frac{dt}{t-1}\right) = \frac{6t^4}{4}+\frac{6t^3}{3}+$

$+\frac{6t^2}{2}+6t+6\ln|t-1|+C = \frac{3(\sqrt[6]{x})^4}{2}+2(\sqrt[6]{x})^3+3(\sqrt[6]{x})^2+$

$+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+C =$

$= \frac{3}{2}\sqrt[3]{x^2}+2\sqrt{x}+3\sqrt[3]{x}+6\sqrt[6]{x}+6\ln|\sqrt[6]{x}-1|+C$

8.4.3. $\int\frac{dx}{\sqrt{x}+\sqrt[4]{x}} = \left[\begin{matrix} n=2 \\ q=4\end{matrix}\right| \Rightarrow k=\text{НОК}(2,4)=4 \Rightarrow x=t^4 \Rightarrow$

$\Rightarrow dx = 4t^3dt] = \int\frac{4t^3dt}{\sqrt{t^4}+\sqrt[4]{t^4}} = 4\int\frac{t^3dt}{t^2+t} = 4\int\frac{t^3dt}{t(t+1)} =$

$= 4\int\frac{t^2dt}{t+1} = 4\int\frac{t^2-1+1}{t+1}dt = 4\int\frac{(t-1)(t+1)}{t+1}dt + 4\int\frac{dt}{t+1} =$

$= 4\int(t-1)dt + 4\int\frac{dt}{t+1} = \frac{4t^2}{2}-4t+4\ln|t+1|+C =$

$= 2t^2-4t+4\ln|t+1|+C = 2(\sqrt[4]{x})^2-4\sqrt[4]{x}+4\ln|\sqrt[4]{x}+1|+C =$

$= 2\sqrt{x}-4\sqrt[4]{x}+4\ln|\sqrt[4]{x}+1|+C$

8.4.5. $\int \frac{dx}{\sqrt[3]{(2x+1)^2} - \sqrt{2x+1}} = \left[\begin{matrix} n=3 \\ q=2 \end{matrix} \right| \Rightarrow K = \text{НОК}(2,3) = 6$

$\Rightarrow 2x+1 = t^6 \Rightarrow dx = 3t^5 dt \Big] = 3\int \frac{t^5 dt}{t^4 - t^3} =$

$= 3\int \frac{t^5 dt}{t^3(t-1)} = 3\int \frac{t^2 dt}{t-1} = 3\int \frac{t^2-1+1}{t-1} dt =$

$= 3\int (t+1)\, dt + 3\int \frac{dt}{t-1} = 3\int dt + 3\int t\, dt + 3\int \frac{dt}{t-1}$

$= \frac{3t^2}{2} + 3t + 3\ln|t-1| + C = \frac{3}{2}(2x+1)^{\frac{1}{3}} + 3(2x+1)^{\frac{1}{6}} +$

$+ 3\ln|(2x+1)^{\frac{1}{6}} - 1| + C$

8.4.6. $\int \frac{dx}{1+\sqrt[3]{x+1}} = \left[x+1 = t^3 \Rightarrow dx = 3t^2 dt \right] =$

$= 3\int \frac{t^2 dt}{1+t} = 3\int \frac{t^2-1+1}{t+1} dt = 3\int (t-1)\, dt + 3\int \frac{dt}{t+1}$

$= 3\int t\, dt - 3\int dt + 3\int \frac{dt}{t+1} = \frac{3t^2}{2} - 3t + 3\ln|t+1| + C =$

$= \frac{3}{2}(x+1)^{\frac{2}{3}} - 3\sqrt[3]{x+1} + 3\ln|\sqrt[3]{x+1} + 1| + C$

8.4.8. $\int \frac{\sqrt{x}}{x^2 \cdot \sqrt{x-1}} dx = \int \frac{1}{x^2} \cdot \frac{\sqrt{x}}{\sqrt{x-1}} dx = \int \frac{1}{x^2} \cdot \sqrt{\frac{x}{x-1}} dx =$

$= \left[\frac{x}{x-1} = t^2 \Rightarrow x = (x-1)t^2 = xt^2 - t^2; \Rightarrow x = \frac{t^2}{t^2-1} \right.;$

$dx = d\left(\frac{t^2}{t^2-1}\right) = \left(\frac{t^2}{t^2-1}\right)' dt = \frac{2t(t^2-1) - t^2 \cdot 2t}{(t^2-1)^2} dt =$

$= \frac{2t^3 - 2t - 2t^3}{(t^2-1)^2} dt = \frac{-2t}{(t^2-1)^2} dt \Big] =$

$= \int \frac{(t^2-1)'}{t^2} \cdot t \left(-\frac{2t}{(t^2-1)^2}\right) dt = -2\int \frac{dt}{t^2} = -2\left(-\frac{1}{t}\right)+C =$

$= \frac{2}{t} + C = 2\cdot \frac{1}{\sqrt{\frac{x}{x-1}}} + C = 2\cdot\frac{\sqrt{x-1}}{\sqrt{x}} + C = 2\sqrt{\frac{x-1}{x}}+C$

8.4.10 $\int \sqrt{x}\,(1+\sqrt[3]{x})^4 dx = \int x^{\frac12}(1+x^{1/3})^4 dx =$

$= [\, m=\frac12,\ n=\frac13,\ p=4 \Rightarrow$ ∗ 1) $p\in Z \Rightarrow x=t^k$;

$K=$ НОК $(2,3)=6 \Rightarrow x=t^6 \Rightarrow dx=6t^5dt\,] =$

$=\int (t^6)^{\frac12}\left(1+(t^6)^{\frac13}\right)^4 6\cdot t^5 dt = 6\int t^3(1+t^2)^4\cdot t^5 dt =$

$=6\int t^8(1+t^2)^4 dt = [(1+t^2)^4 = (1+t^2)^2(1+t^2)^2 =$

$=(1+2t^2+t^4)(1+2t^2+t^4) = 1+2t^2+t^4+2t^2+4t^4+2t^6+$

$+t^4+2t^6+t^8 = 1+4t^2+6t^4+4t^6+t^8\,] = 6\int t^8 dt +$

$+24\int t^{10}dt + 36\int t^{12}dt + 24\int t^{14}dt + 6\int t^{16}dt =$

$= \frac{6t^9}{9} + \frac{24t^{11}}{11} + \frac{36t^{13}}{13} + \frac{24t^{15}}{15} + \frac{6t^{17}}{17} + C = \frac{2}{3}t^9 + \frac{24t^{11}}{11}+$

$+\frac{36}{13}t^{13} + \frac{8}{5}t^{15} + \frac{6}{17}t^{17} + C = \frac23 x\sqrt{x} + \frac{24}{11}x\sqrt[6]{x^5} + \frac{36}{13}x^2\sqrt[6]{x}+$

$+\frac85 x^2\sqrt{x} + \frac{6}{17}x^2\sqrt[6]{x^5}+C$

8.4.11 $\int \frac{dx}{x^4\sqrt{x^2+1}} = \int x^{-4}\cdot(x^2+1)^{-\frac12}dx = [\, m=-4,\ n=2,\ p=-\frac12;$

$\Rightarrow$ ∗ 1) $p\notin Z$ 2) $\frac{m+1}{n} = \frac{-4+1}{2} = -\frac32 \notin Z$ 3) $\frac{m+1}{n}+p = -\frac32-\frac12 = -2\in Z$

$\Rightarrow x^2+1=t^2;\quad x=\frac{1}{\sqrt{t^2-1}} \Rightarrow dx = \frac{-t\,dt}{(t^2-1)^{3/2}}\,] =$

$$= \int \left(\left(t^2-1\right)^{-\frac{1}{2}}\right)^{-4} \left(\left(\left(t^2-1\right)^{-\frac{1}{2}}\right)^2 + 1\right)^{-\frac{1}{2}} \cdot \left(\frac{-t\,dt}{\left(t^2-1\right)^{\frac{3}{2}}}\right) =$$

$$= -\int \left(t^2-1\right)^2 \left(\left(t^2-1\right)^{-1}+1\right)^{-\frac{1}{2}} \cdot \frac{t\,dt}{\left(t^2-1\right)^{3/2}} =$$

$$= -\int \left(t^2-1\right)^2 \sqrt{\frac{t^2-1+1}{t^2-1}} \cdot \frac{t\,dt}{\left(t^2-1\right)^{3/2}} = -\int \frac{\left(t^2-1\right)^2 \cdot 6}{\sqrt{t^2-1}} \cdot$$

$$\cdot \frac{t\,dt}{\left(t^2-1\right)\sqrt{t^2-1}} = -\int t\,dt = -\frac{t^3}{3} + C = -\frac{1}{3}\left(\sqrt{x^{-2}-1}\right)^3 + C$$

$$= -\frac{1}{3}\left(\sqrt{\frac{1-x^2}{x^2}}\right)^3 + C = \frac{\left(x^2-1\right)\sqrt{1-x^2}}{3x3} + C$$