4. Game theory

Examples 4.1: Paper-Scissors-Rock -- A two person game.
Rules: At the count of three declare one of: Paper, Scissors, Rock;
Winner Selection. Identical selection is a draw.
Otherwise:

- Rock beats Scissors
- Paper beats Rock
- Scissors beats Paper

Payoff Matrix --Payoffs are from row player to column player:

The column player wishes to maximize the payoff and the row player wishes to minimize the payoff.

Given: m× n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Row player selects a strategy $i \in \{1, ..., m\}$.
- Column player selects a strategy $j \in \{1, ..., n\}$.
- Row player pays column player a_{ij} dollars.

Note: The rows of A represent deterministic strategies for Row player, while columns of A represent deterministic strategies for column player.

Randomized Strategies.

- Suppose row player picks i with probability y_i.
- Suppose column player picks j with probability x_j .

Throughout, $x = [x_1, x_2, \cdot\cdot\cdot, x_n]^T$ and $y = [y_1, y_2, \cdot\cdot\cdot, y_m]^T$ will denote stochastic vectors:

$$x_j \geq 0, j = 1, 2, \ldots, n, \qquad y_i \geq 0, i = 1, 2, \ldots, m$$

If row player uses random strategy y and column player uses x, then expected payoff from row player to column player is

Suppose that column player were to adopt strategy x.
Then, row player's best defense is to use y that minimizes $y^T Ax$:

And so column player should choose that x which maximizes these possibilities:

Let $\min_i \sum_{j=1}^{n} a_{ij} x_j = t$. Then

On the other hand,

Since each y with one component equal to one and the remaining components equal to zero is a candidate for minimizing $y^T Ax$, we have

Therefore.

The problem of finding the column player's optimal strategy is

Maximize
Subject to

$x_j \geq 0$, $j = 1, 2, \ldots, n$.

This is equivalent to the LP problem

Maximize z
Subject to

Similarly, suppose that row player were to adopt strategy y.Then, column player's best defense is to use x that maximizes $y^T Ax$:

And so row player should choose that y which minimizes these possibilities:

The problem of finding the row player's optimal strategy is

Minimize
Subject to

$y_i \geq 0$, i = 1, 2, . . . , m.

This is equivalent to the LP problem

Minimize z
Subject to

$y_i \geq 0$, i = 1, 2, . . . , m.

(Von Neumann) Min-Max Theorem:
Let x* denote column player's solution to her max–min problem.
Let y* denote row player's solution to his min–max problem. Then

The problem of finding the column player's optimal strategy is

Example 4.2: Given a payoff matrix $\begin{bmatrix} 1 & -6 \\ -2 & 7 \end{bmatrix}$.

Example 4.3: The payoff matrix of the game Paper-Scissors-Rock is given by

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Example 4.4: Two players, say an even player and an odd player, each secretly think of an integer between 1 and 3 inclusive. Both players reveal their numbers simultaneously. If the sum of the numbers is even, the even player wins a number of dollars from the odd player equal to the difference of the numbers if they are distinct or the sum of the numbers if they are the same. If the sum of the numbers is odd, the odd player wins \$3 from the even player.

Always use domination to reduce a given matrix game:

1) ~~One row~~ (term-by-term ≥) another row
2) ~~One column~~ (term-by-term ≤) another column

Example 4.5: Two players simultaneously choose a positive integer. Both players may choose an integer between 1 and 100. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1. The player that chooses a number two or more larger than that chosen by his opponent loses 2.

Example 4.6: Using domination for simplifying the matrix

$$A = \begin{bmatrix} -6 & 2 & -4 & -7 & -5 \\ 4 & 4 & -2 & -9 & -1 \\ -7 & 3 & -3 & -8 & -2 \\ 2 & -3 & 6 & 0 & 3 \end{bmatrix}.$$

Example 4.7: Using domination for simplifying the game matrix.

$$\begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 13 & 18 & 18 & 11 & 18 & 23 \\ 29 & 22 & 22 & 22 & 16 & 22 \\ 8 & 21 & 31 & 12 & 19 & 19 \\ 12 & 22 & 31 & 21 & 14 & 23 \\ 16 & 29 & 27 & 21 & 19 & 30 \\ 23 & 31 & 37 & 26 & 28 & 34 \end{bmatrix} \end{array}$$