2018 **Term Test Answer**

MATB61 Linear Programming and Optimization

1. A supermarket requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Staff Needed	16	20	17	21	23	25	18

Let x_i = no. of employees starting at day i

LP problem:

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

Subject to

$$
\begin{aligned}
x_1 \qquad\qquad\qquad + x_4 + x_5 + x_6 + x_7 &\geq 16 \\
x_1 + x_2 \qquad\qquad\qquad + x_5 + x_6 + x_7 &\geq 20 \\
x_1 + x_2 + x_3 \qquad\qquad\qquad + x_6 + x_7 &\geq 17 \\
x_1 + x_2 + x_3 + x_4 \qquad\qquad\qquad + x_7 &\geq 21 \\
x_1 + x_2 + x_3 + x_4 + x_5 &\geq 23 \\
x_2 + x_3 + x_4 + x_5 + x_6 &\geq 25 \\
x_3 + x_4 + x_5 + x_6 + x_7 &\geq 18 \\
x_i \geq 0, \quad i = 1, 2, 3, 4, 5, 6, 7.
\end{aligned}
$$

2. Suppose that we apply the Simplex method to a given LP problem and obtain the following tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	0	a	0	1	1	0	e
x_1	1	f	0	2	0	-4	5
x_3	0	-3	1	-4	0	1	1
	0	c -1	0	5	0	b	12

Specify the ranges of values for the parameters *a*, *b*, *c*, *d*, *e*, *f* ($\geq$, $>$, $\leq$, $<$, $=$) that make each of the following statements true. Assume that the original problem was a maximization problem.

a) The tableau describes an infeasible basic solution.

$e < 0$

b) The tableau describes an optimal basic feasible solution.

$c - 1 \geq 0$, $b \geq 0$, $e \geq 0$

c) The tableau describes a basic feasible solution, but it is not the optimal solution.

$c - 1 < 0$ or $b < 0$, $e \geq 0$

d) The tableau describes a basic feasible solution, but the problem is unbounded and the simplex algorithm cannot proceed any further.

$c - 1 < 0$, $b \geq 0$, $e \geq 0$, $a \leq 0$, $f \leq 0$

e) The tableau describes an optimal basic feasible solution which is not unique.

$c - 1 \geq 0$, $b \geq 0$, $e \geq 0$, $c - 1 = 0$ or $b = 0$

f) The current basic solution is feasible. At the next iteration, x_2 is the only candidate for entering the basis.

$b \geq 0$, $e \geq 0$, $c - 1 < 0$

g) The current basic solution is feasible. At the next iteration, a degenerate BFS will occur in the next tableau.

$a \geq 0$, $e \geq 0$, $b \geq 0$, $f \geq 0$, $c - 1 < 0$, $c - 1 < b$, $e/a = 5/f$

3. [22 points]

1) [10 points]Convert the following LP problem into standard form:

$$\begin{aligned} \text{Minimize} \quad & 2x_1 - x_2 + x_3 \\ \text{Subject to} \quad & |x_1 - 2x_2| \le 2 \\ & 3x_1 - 5x_2 - x_3 = -1 \\ & x_1 \ge 0,\ 2 \le x_2 \le x_3. \end{aligned}$$

Set $x_2 = x_2' + 2$ and $x_3 = x_3' + 2$.

$$\begin{aligned} \text{Maximize} \; & -2x_1 + x_2' - x_3' \\ \text{Subject to} \; & x_1 - 2x_2' \le 6 \\ & -x_1 + 2x_2' \le -2 \\ & 3x_1 - 5x_2' - x_3' \le 11 \\ & -3x_1 + 5x_2' + x_3' \le -11 \\ & x_2' - x_3' \le 0 \\ & x_1 \ge 0,\ x_2' \ge 0,\ x_3' \ge 0. \end{aligned}$$

2) **[8 points]** Assume a LP problem: Max $z = c^T x$, subject to $Ax \le b$, $x \ge 0$ has an unique optimal solution. Prove that the optimal solution may not be reached at x_0 if x_0 is not an extreme point of the feasible region.

Proof: Assume that the unique optimal solution was $z_0 = c^T x_0$.

Since x_0 is not an extreme point of the feasible region, x_0 must be an interior point of some two distinct points x_1 and x_2 in the feasible region. Then we have

$$x_0 = rx_1 + (1 - r)x_2 \text{ for } 0 < r < 1.$$

Multiply c^T to both sides to obtain

$$z_0 = c^T x_0 = rc^T x_1 + (1 - r)c^T x_2.$$

Let $z_1 = c^T x_1$ and $z_2 = c^T x_2$. We have $z_0 = rz_1 + (1 - r)z_2$.

If $z_1 = z_2$, then $z_0 = rz_1 + (1 - r)z_1 = z_1 = z_2$.

It contradicts that the LP problem has unique solution.

If $z_1 \le z_2$, $z_0 = rz_1 + (1 - r)z_2 \le rz_2 + (1 - r)z_2 = z_2$.

Then $z_0 = c^T x_0$ can't be the optimal solution.

Similarly for $z_1 \ge z_2$.

Therefore, the optimal solution may not be reached at x_0 if x_0 is not an extreme point of the feasible region.

3) **[4 points]** Multiple choices:

a) Given an LP problem

$$\begin{aligned} \text{Maximize } & p = 4x + 2y + 7z \\ \text{subject to } & 2x - y + 4z \le 18 \\ & 4x + 2y + 5z \le 10 \\ & x \ge 0,\ y \ge 0,\ z \ge 0 \end{aligned}$$

Which of the following is(are) basic feasible solution(s) of the LP problem?

i) [5/2, 0, 1], ii) [9, 0, 0], iii) [0, 0, 9/2], iv) [0, 5, 0]

b) Which of the following statements is true?

i) The set of solutions to an inequality is a hyperplane.
ii) An extreme point is a basic feasible solution.
iii) [1, 0] is a convex combination of [-1, 0], [0, 5] and [4, 1].
iv) any point in S is an interior point of any two distinct points in S.

4. [18 points] Solve the following LP problems graphically.

a) Maximize $z = -x + y$
Subject to $x - 3y \le 4$
$x + 2y \ge 4$
$x, y \ge 0.$

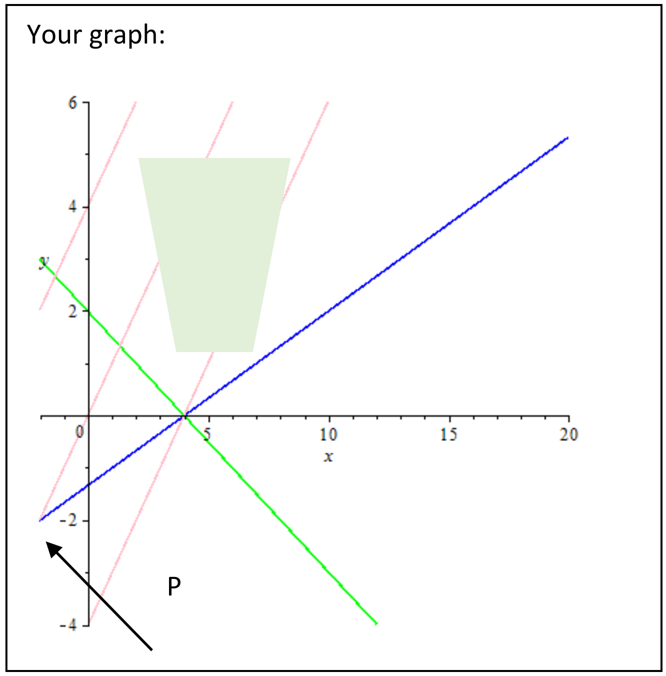
Set $y = x + P$.

$P = 0, y = x$

$P = 4, y = x + 4$

There is no optimal solution.

The objective function is unbounded,



b) Minimize $z = 4x + 2y$
Subject to $2x - 3y \le 12$
$4x - y \ge 1$
$2x + y \ge 8$
$x, y \ge 0.$

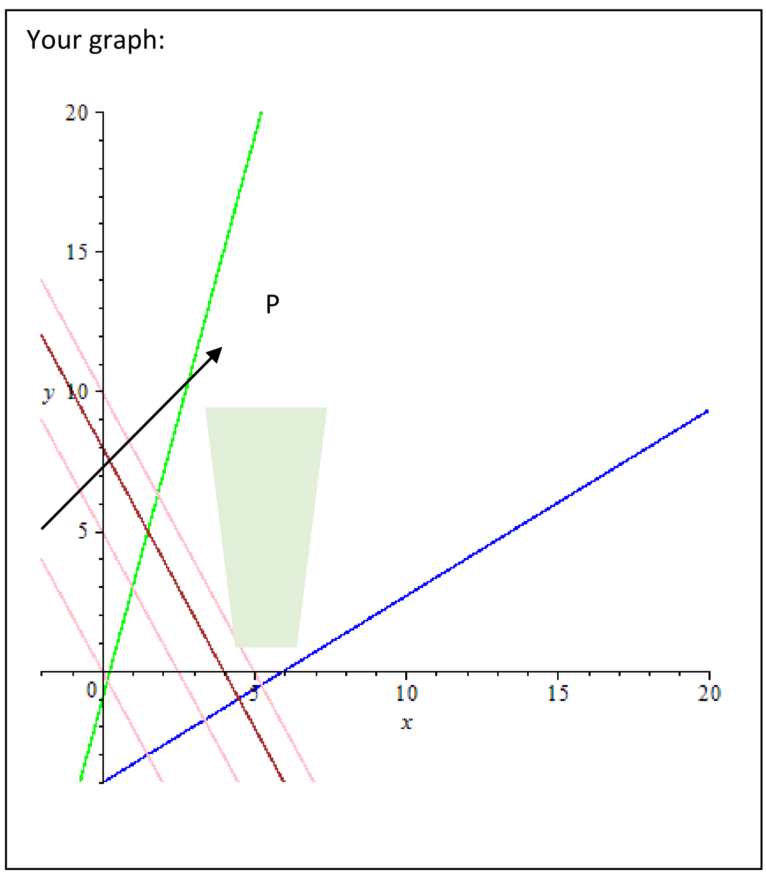
Set $y = -2x + P/2$

$P = 0, y = -2x.$

$P = 10, y = -2x + 5,$

The optimal solution is

$Z = 16$ at the line segment $2x + y = 8$ between $(3/2, 5)$ and $(4, 0)$.



5. [15 points] Solve the following LP Problem by the Big M method:

$$\text{Maximize } z = 4x_1 + 5x_2 - 3x_3$$

$$\begin{aligned} \text{Subject to} \quad & x_1 + 2x_2 + x_3 = 10 \\ & x_1 - x_2 \geq 6 \\ & x_1 + 3x_2 + x_3 \leq 14 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

T1	x_1	x_2	x_3	s_1	s_2	y_1	y_2	
y_1	1	2	1	0	0	1	0	10
y_2	1	-1	0	-1	0	0	1	6
s_2	1	3	1	0	1	0	0	4
	$-(2M+4)$	$-(M+5)$	$-(M-3)$	M	0	0	0	$-16M$

T_3	x_1	x_2	x_3	s_1	s_2	y_1	y_2	
x_2	0	1	1/3	1/3	0	1/3	-1/3	4/3
x_1	1	0	1/3	-2/3	0	1/3	2/3	22/3
s_2	0	0	-1/3	-1/3	1	-4/3	1/3	8/3
	0	0	0	-1	0	M+3	M+1	36

T_f	x_1	x_2	x_3	s_1	s_2	
s_1	0	3	1	1	0	4
x_1	1	2	1	0	0	10
s_2	0	1	0	0	1	4
	0	3	1	0	0	40

6. [15 points] Use the two phase method to solve the following LP problem.

Maximize $z = 5x_1 + 3x_2 + x_3$

Subject to:

$2x_1 + x_2 + x_3 \leq 4$

$x_1 + 2x_2 - x_3 \geq 3$

$x_2 + 2x_3 \leq 2$

$x_i \geq 0$, $i = 1, 2, 3$

Auxiliary LP

Maximize $z' - x_1 - 2x_2 + x_3 + s_2 = -3$

Subject to:

$2x_1 + x_2 + x_3 + s_1 = 4$

$x_1 + 2x_2 - x_3 - s_2 + y_1 = 3$

$x_2 + 2x_3 + s_3 = 2$

$x_i \geq 0$, $i = 1, 2, 3$

Phase I:

T1	x_1	x_2	x_3	s_1	s_2	s_3	y	
s_1	2	1	1	1	0	0	0	4
y	1	2	-1	0	-1	0	1	3
s_3	0	1	2	0	0	1	0	2
	-1	-2	1	0	1	0	0	-3

Tf	x_1	x_2	x_3	s_1	s_2	s_3	y	
s_1	3/2	0	3/2	1	1/2	0	-1/2	5/2
x_2	1/2	1	-1/2	0	-1/2	0	1/2	3/2
s_3	-1/2	0	5/2	0	1/2	1	1/2	1/2
	0	0	0	0	0	0	1	0

Phase II:

Objective row: $z - 5x_1 - 3x_2 - x_3 = 0$

$$\begin{array}{rrrrrrrr} & -5 & -3 & -1 & 0 & 0 & 0 & \mid 0 \\ + & 3(\ 1/2 & 1 & -1/2 & 0 & -1/2 & 0 & \mid 3/2\ ) \\ \hline & -7/2 & 0 & -5/2 & 0 & -3/2 & 0 & \mid 9/2 \end{array}$$

T1	x_1	x_2	x_3	s_1	s_2	s_3	
s_1	3/2	0	3/2	1	1/2	0	5/2
x_2	1/2	1	-1/2	0	-1/2	0	3/2
s_3	-1/2	0	5/2	0	1/2	1	1/2
	-7/2	0	-5/2	0	-3/2	0	9/2

Tf	x_1	x_2	x_3	s_1	s_2	s_3	
x_1	1	0	-1/2	1/2	0	1/2	1
x_2	0	1	2	0	0	1	2
s_2	0	0	9/2	1/2	1	3/2	2
	0	0	5/2	5/2	0	1/2	11