University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Linear Programming and Optimazation

MATB61 Winter 2020

Selected answers to the assignment # 5

Section 2.2

#2 The optimal solution: $z = 22$; $x_1 = 2$, $x_2 = 3$.
The simplex algorithm examines the following extreme points:
O(0, 0), A(4, 0), A(4, 0), B(2, 3).
Note that in the initial tableau, last slack variable $x_5$ could also have been chosen as the departing variable.

#4 The optimal solution: $z = 32$; $x_1 = 0$, $x_2 = 4$, $x_3 = 0$.
The simplex algorithm examines the following extreme points:
O(0, 0, 0), A(0, 4, 0), A(0, 4, 0).
Note that in the second tableau, the slack variable $x_5$ could also have been chosen as the departing variable.

#6 The optimal solution: $z = 33/2$; $x_1 = 3/2$, $x_2 = 3$.
The simplex algorithm examines the following extreme points:
O(0, 0), O(0, 0), A(2, 2), B(3/2, 3).

8. (a)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| ← $x_6$ | .5 | −5.5 | −2.5 | 9 | 0 | 1 | 0 | 0 |
| $x_7$ | .5 | −1.5 | −.5 | 1 | 0 | 0 | 1 | 0 |
| | −1 | 7 | 1 | 2 | 0 | 0 | 0 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 0 | 12 | 6 | −17 | 1 | −2 | 0 | 1 |
| $x_1$ | 1 | −11 | −5 | 18 | 0 | 2 | 0 | 0 |
| ← $x_7$ | 0 | 4 | 2 | −8 | 0 | −1 | 1 | 0 |
| | 0 | −4 | −4 | 20 | 0 | 2 | 0 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 0 | 0 | 0 | 7 | 1 | 1 | −3 | 1 |
| ← $x_1$ | 1 | 0 | .5 | −4 | 0 | −.75 | 2.75 | 0 |
| $x_2$ | 0 | 1 | .5 | −2 | 0 | −.25 | .25 | 0 |
| | 0 | 0 | −2 | 12 | 0 | 1 | 1 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 0 | 0 | 0 | 7 | 1 | 1 | −3 | 1 |
| $x_3$ | 2 | 0 | 1 | −8 | 0 | −1.5 | 5.5 | 0 |
| ← $x_2$ | −1 | 1 | 0 | 2 | 0 | .5 | −2.5 | 0 |
| | 4 | 0 | 0 | −4 | 0 | −2 | 12 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 3.5 | −3.5 | 0 | 0 | 1 | −.75 | 5.75 | 1 |
| ← $x_3$ | −2 | 4 | 1 | 0 | 0 | .5 | −4.5 | 0 |
| $x_4$ | −.5 | .5 | 0 | 1 | 0 | .25 | −1.25 | 0 |
| | 2 | 2 | 0 | 0 | 0 | −1 | 7 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | .5 | 2.5 | 1.5 | 0 | 1 | 0 | −1 | 1 |
| $x_6$ | −4 | 8 | 2 | 0 | 0 | 1 | −9 | 0 |
| $x_4$ | .5 | −1.5 | −.5 | 1 | 0 | 0 | 1 | 0 |
| | −2 | 10 | 2 | 0 | 0 | 0 | −2 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_5$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 1 |
| $x_6$ | .5 | −5.5 | −2.5 | 9 | 0 | 1 | 0 | 0 |
| $x_7$ | .5 | −1.5 | −.5 | 1 | 0 | 0 | 1 | 0 |
| | −1 | 7 | 1 | 2 | 0 | 0 | 0 | 0 |

(b) In the sequence of tableaux leading to an optimal solution, the variables marked with a B, are chosen by Bland's rule:

| Tableau | Entering variable | Departing variable |
|---|---|---|
| 1 | $x_1$ | $x_6$ (B) |
| 2 | $x_2$ (B) | $x_7$ |
| 3 | $x_3$ | $x_1$ (B) |
| 4 | $x_4$ (B) | $x_2$ |
| 5 | $x_6$ | $x_3$ (B) |
| 6 | $x_1$ (B) | $x_4$ |

Optimal solution: $[0\ 0\ 0\ 0\ 1\ 0\ 0]^T$; $z = 0$

9. (a)

| | | ↓ $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|---|
| ← | $x_4$ | .6 | −6.4 | 4.8 | 1 | 0 | 0 | 0 | 0 |
| | $x_5$ | .2 | −1.8 | .6 | 0 | 1 | 0 | 0 | 0 |
| | $x_6$ | .4 | −1.6 | .2 | 0 | 0 | 1 | 0 | 0 |
| | $x_7$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | | −.4 | −.4 | 1.8 | 0 | 0 | 0 | 0 | 0 |

| | | $x_1$ | ↓ $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 1 | −10.67 | 8 | 1.67 | 0 | 0 | 0 | 0 |
| ← | $x_5$ | 0 | .33 | −1 | −.33 | 1 | 0 | 0 | 0 |
| | $x_6$ | 0 | 2.67 | −3 | −.67 | 0 | 1 | 0 | 0 |
| | $x_7$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | | 0 | −4.67 | 5 | .67 | 0 | 0 | 0 | 0 |

| | | $x_1$ | $x_2$ | ↓ $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | 1 | 0 | −24 | −9 | 32 | 0 | 0 | 0 |
| | $x_2$ | 0 | 1 | −3 | −1 | 3 | 0 | 0 | 0 |
| ← | $x_6$ | 0 | 0 | 5 | 2 | −8 | 1 | 0 | 0 |
| | $x_7$ | 0 | 0 | 3 | 1 | −3 | 0 | 1 | 1 |
| | | 0 | 0 | −9 | −4 | 14 | 0 | 0 | 0 |

| | | $x_1$ | $x_2$ | $x_3$ | ↓ $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|---|
| ← | $x_1$ | 1 | 0 | 0 | .6 | −6.4 | 4.8 | 0 | 0 |
| | $x_2$ | 0 | 1 | 0 | .2 | −1.8 | .6 | 0 | 0 |
| | $x_3$ | 0 | 0 | 1 | .4 | −1.6 | .2 | 0 | 0 |
| | $x_7$ | 0 | 0 | 0 | −.2 | 1.8 | −.6 | 1 | 1 |
| | | 0 | 0 | 0 | −.4 | −.4 | 1.8 | 0 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | 1.67 | 0 | 0 | 1 | −10.67 | 8 | 0 | 0 |
| ← $x_2$ | −.33 | 1 | 0 | 0 | .33 | −1 | 0 | 0 |
| $x_3$ | −.67 | 0 | 1 | 0 | 2.67 | −3 | 0 | 0 |
| $x_7$ | .33 | 0 | 0 | 0 | −.33 | 1 | 1 | 1 |
| | .67 | 0 | 0 | 0 | −4.67 | 5 | 0 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | −9 | 32 | 0 | 1 | 0 | −24 | 0 | 0 |
| $x_5$ | −1 | 3 | 0 | 0 | 1 | −3 | 0 | 0 |
| ← $x_3$ | 2 | −8 | 1 | 0 | 0 | 5 | 0 | 0 |
| $x_7$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | −4 | 0 | 0 | 0 | 0 | −9 | 0 | 0 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_4$ | .6 | −6.4 | 4.8 | 1 | 0 | 0 | 0 | 0 |
| $x_5$ | .2 | −1.8 | .6 | 0 | 1 | 0 | 0 | 0 |
| $x_6$ | .4 | −1.6 | .2 | 0 | 0 | 1 | 0 | 0 |
| $x_7$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 |
| | −.4 | −.4 | 1.8 | 0 | 0 | 0 | 0 | 0 |

(b) In the sequence of tableaux leading to an optimal solution, the variables marked with a B, are chosen by Bland's rule:

| Tableau | Entering variable | Departing variable |
|---|---|---|
| 1 | $x_1$ (B) | $x_4$ (B) |
| 2 | $x_2$ | $x_5$ (B) |
| 3 | $x_3$ (B) | $x_6$ |
| 4 | $x_4$ (B) | $x_1$ (B) |
| 5 | $x_5$ | $x_2$ (B) |
| 6 | $x_1$ (B) | $x_3$ |
| 7 | $x_2$ | $x_7$ |

Section 2.3

2. (a)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_5$ | 1 | 3 | −1 | 1 | 1 | 0 | 0 | 0 | 5 |
| $y_1$ | 1 | 7 | 1 | 0 | 0 | −1 | 1 | 0 | 4 |
| $y_2$ | 4 | 2 | 0 | 1 | 0 | 0 | 0 | 1 | 3 |
| | −5 | −9 | −1 | −1 | 0 | 1 | 0 | 0 | −7 |

(b)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_5$ | 1 | 3 | $-1$ | 1 | 1 | 0 | 0 | 0 | 5 |
| $y_1$ | 1 | 7 | 1 | 0 | 0 | $-1$ | 1 | 0 | 4 |
| $y_2$ | 4 | 2 | 0 | 1 | 0 | 0 | 0 | 1 | 3 |
| | $-1-5M$ | $-2-9M$ | $-M$ | $-1-M$ | 0 | $M$ | 0 | 0 | $-7M$ |

4. (a)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|---|---|
| $y_1$ | 3 | 1 | $-2$ | $-1$ | 0 | 1 | 0 | 2 |
| $y_2$ | 2 | 4 | 7 | 0 | $-1$ | 0 | 1 | 3 |
| | $-5$ | $-5$ | $-5$ | 1 | 1 | 0 | 0 | $-5$ |

(b)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|---|---|
| $y_1$ | 3 | 1 | $-2$ | $-1$ | 0 | 1 | 0 | 2 |
| $y_2$ | 2 | 4 | 7 | 0 | $-1$ | 0 | 1 | 3 |
| | $-1-5M$ | $2-5M$ | $7-5M$ | $-1+M$ | $M$ | 0 | 0 | $-5M$ |

6.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $y_1$ | $y_2$ | |
|---|---|---|---|---|---|---|---|---|---|
| $x_1$ | 1 | $-1/2$ | 0 | $1/2$ | 0 | $1/2$ | $1/2$ | $-1/2$ | 3 |
| $x_5$ | 0 | $-25/6$ | 0 | $-5/6$ | 1 | $1/2$ | $-1/6$ | $-1/2$ | $10/3$ |
| $x_3$ | 0 | $5/6$ | 1 | $1/6$ | 0 | $-1/2$ | $-1/6$ | $1/2$ | $1/3$ |
| | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |

8. (a)

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | |
|---|---|---|---|---|---|---|---|---|
| $x_7$ | $-3$ | 0 | $-2$ | $-1$ | 0 | $-3/4$ | 1 | 0 |
| $x_2$ | 0 | 1 | 1 | 3 | 0 | $1/2$ | 0 | 2 |
| $x_5$ | 1 | 0 | 3 | 0 | 1 | $-1/2$ | 0 | 4 |
| | 0 | 0 | $-2$ | 2 | 0 | $1/2$ | 0 | 2 |

(b) $\begin{bmatrix} 0 & \frac{2}{3} & \frac{4}{3} & 0 & 0 & \frac{8}{3} \end{bmatrix}^T; \quad z = \frac{14}{3}$

10. Use 3 oz of food A and 4 oz of food B; $z = \$1.60$

12. $[0 \quad 7 \quad 0 \quad 3]^T; \quad z = 2$

14. $\begin{bmatrix} \frac{19}{13} & 0 & \frac{5}{13} & 0 & 0 \end{bmatrix}^T; \quad z = \frac{43}{13}$

16. Chewy should consist of 53 1/3 kg of sunflower seeds and 80 kg of raisins. Crunchy should consist of 46 2/3 kg of sunflower seeds and 31 1/9 kg of peanuts. Nutty should consist of 28 8/9 kg of of peanuts only. Profit is $157 7/9.

18. Make 1000 glazed doughnuts and 400 powdered sugar doughnuts. Profit is $90.

20. No feasible solutions.

22. No finite optimal solution.

24. Assume that $\mathbf{x}$ is a feasible solution to (10), (11), and (12). Then for the $i$th constraint in (14) we have

$$\sum_{j=1}^{s} a_{ij}x_j + 0 = \sum_{j=1}^{s} a_{ij}x_j = b_i$$

Also, $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} \geq \begin{bmatrix} \mathbf{0} \\ 0 \end{bmatrix}$ so that $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ is a feasible solution to (13), (14), and (15). Conversely, if $\begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}$ is a feasible solution to (13), (14), and (15), then $\mathbf{x} \geq \mathbf{0}$. Furthermore, for the $i$th constraint in (14) we have

$$\sum_{j=1}^{s} a_{ij}x_j + 0 = b_i = \sum_{j=1}^{s} a_{ij}x_j$$

**In addition:**

a) $d \leqslant 0$, $e \leqslant 0$ and $c < 0$.
b) $d \geqslant 0$, $e \geqslant 0$ and $c \geqslant 0$.
c) $e \geqslant 0$, $c \geqslant 0$, $d < 0$ and $a \leqslant 0$.
d) $d \geqslant 0$, $e \geqslant 0$ and $c \geqslant 0$, one of d or e is zero.
e) $d \geqslant 0$, $e < 0$, $b > 0$ and $c \geqslant 0$, $1/b < c$.
f) $e < 0$, $e < d$, $b > 0$ and $c \geqslant 0$, $1/b = c$.