University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Linear Programming and Optimazation

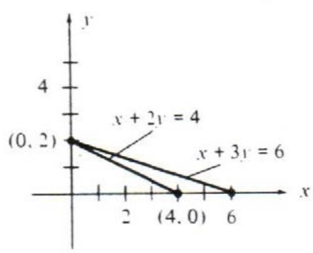
MATB61 Winter 2020

Selected answers to the assignment # 3

Section 1.4

2.

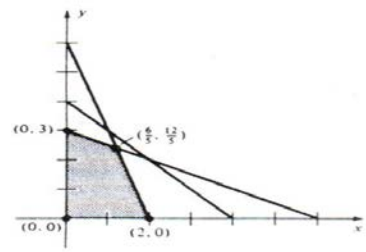
(a)



$\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $z = -6$

4.

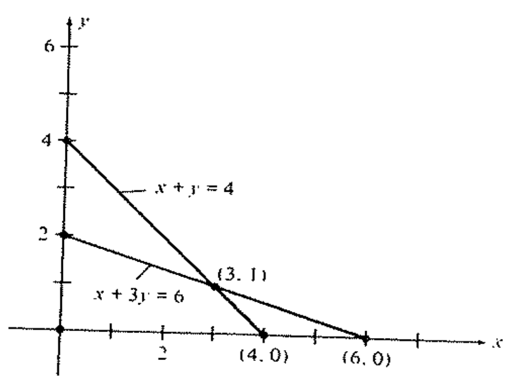
(a)



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$, $\begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}$, $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{6}{5} \\ \frac{12}{5} \end{bmatrix}$; $z = \frac{48}{5}$

6. (a)



$\begin{bmatrix} 6 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 4 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

(b) Any point on the line segment joining $\begin{bmatrix} 6 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$; $z = 3$

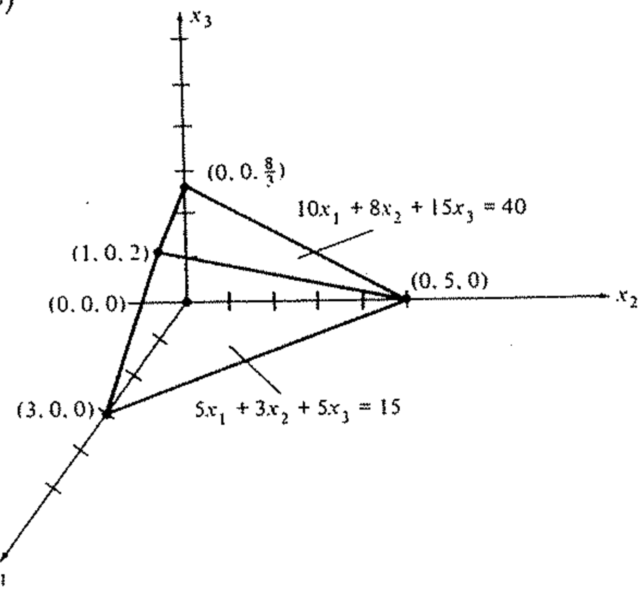
All the basic solutions:

$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, [1, 2], $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$, [1, 3], $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$, [1, 4], $\begin{bmatrix} 4 \\ -6 \end{bmatrix}$, [2, 3], $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$, [2, 4], $\begin{bmatrix} 6 \\ -4 \end{bmatrix}$, [3, 4]

Therefore, all the extreme points:

[3, 1, 0, 0], [4, 0, 2, 0], [6, 0, 0, 2].

8. (a)



$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ \frac{8}{3} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}; \quad z = 20$

All the basic solutions:

$\begin{bmatrix} 0 \\ 5 \end{bmatrix}$, [1, 2], $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, [1, 3], $\begin{bmatrix} 4 \\ -5 \end{bmatrix}$, [1, 4], $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$, [1, 5], $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$, [2, 3], $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$, [2, 4], $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$, [2, 5],

$\begin{bmatrix} \frac{8}{3} \\ \frac{5}{3} \end{bmatrix}$, [3, 4], $\begin{bmatrix} 3 \\ -5 \end{bmatrix}$, [3, 5], $\begin{bmatrix} 15 \\ 40 \end{bmatrix}$, [4, 5]

All the extreme points:

[0, 5, 0, 0, 0], [1, 0, 2, 0, 0], [3, 0, 0, 0, 5], [0, 5, 0, 0, 0], [0, 0, 8/3, 5/3, 0], [0, 0, 0, 15, 40]

15. Suppose that S is convex. Let $\mathbf{x}$ be a convex combination of k points in S. If $k = 2$, then $\mathbf{x} \in S$ because S is convex. For a proof by induction, assume that any convex combination of k points in S belongs to S. Now consider the convex combination of $k+1$ points

$$\mathbf{x} = \sum_{i=1}^{k+1} c_i \mathbf{x}_i, \quad \sum_{i=1}^{k+1} c_i = 1, \quad c_i \geq 0$$

for $i = 1, 2, \ldots k+1$. Assume that $c_{k+1} \neq 1$ and write

$$\mathbf{x} = \left(\sum_{i=1}^{k} c_i\right) \left(\sum_{i=1}^{k} \frac{c_i}{\sum_{i=1}^{k} c_i} \mathbf{x}_i\right) + c_{k+1}\mathbf{x}_{k+1}.$$

Thus, $\mathbf{x}$ is a convex combination of two points, the first of which is in S because it is a convex combination of k points in S. Conversely, if every convex combination of a finite number of points of S is in S, then every convex combination of two points of S is in S.

16. Suppose that the optimal value of z is k and that it is attained at the extreme points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$. Let $\mathbf{x} = \sum_{i=1}^{r} c_i \mathbf{x}_i$, where $\sum_{i=1}^{r} c_i = 1, \quad c_i \geq 0$. Then

$$\begin{aligned} \mathbf{c}^T\mathbf{x} = \mathbf{c}^T\left(\sum_{i=1}^{r} c_i \mathbf{x}_i\right) &= \sum_{i=1}^{r} c_i \left(\mathbf{c}^T \mathbf{x}_i\right) \\ &= \sum_{i=1}^{r} c_i k = k. \end{aligned}$$

Section 1.5

#2 80 Kg pizza-flavored potato chips
0 kg chilli-flavored chip
Net profit = \$9.60

#4

Let

$$\begin{aligned} x_1 &= \text{Amount of pizza-flavored potato chips (in kg)} \\ x_2 &= \text{Amount of chili-flavored potato chips (in kg)} \end{aligned}$$

Maximize $z = 0.12x_1 + 0.10x_2$

subject to

$$\begin{array}{rcrcrcrcrcr} 3x_1 & + & 3x_2 & + & x_3 & & & & & = & 240 \\ 5x_1 & + & 4x_2 & + & & + & x_4 & & & = & 480 \\ 2x_1 & + & 3x_2 & + & & & & + & x_5 & = & 360 \end{array}$$

$$x_j \geq 0, \quad j = 1, 2, ..., 5$$

$$x_3 = 0, \quad x_4 = 80, \quad x_5 = 200$$

$x_3 =$ 0 minutes of unused time on the fryer.

$x_4 =$ 80 minutes of unused time on the flavorer.

$x_5 =$ 200 minutes of unused time on the packer.

The basic variables are x_1, x_4, and x_5.

#6 (b) and (d)

#7 a) no, b) yes, c) yes.

#8 a) Maximize $z = 3x + 2y$

Subject to $2x - y + u = 6$

$2x + y + v = 10$

$x \geq 0, y \geq 0, u \geq 0, v \geq 0.$

b)

x	y	u	v	basic variables
0	0	6	10	u, v
0	10	16	0	y, u
3	0	0	4	x, v
4	2	0	0	x, y

c) Optimal solution: x = 0, y =10; z = 20.

In addition:

We have to prove that if $x, y \in S \setminus \{P\}$ then the line segment connecting x and y is also in $S \setminus \{P\}$. Since S is convex and $x, y \in S$, the line segment will be contained in S, thus the only case when the line segment is not entirely in $S \setminus \{P\}$ if it passes through P. But this is impossible, since P is an extreme point of S. (See the definition of the extreme point.)