University of Toronto at Scarborough
Department of Computer and Mathematical Sciences

Linear Programming and Optimazation

MATB61 Winter 2020

Answers for some problems may be found at the end of the textbook.

Selected answers to the assignment # 2

Section 1.2

#2

$$\text{Maximize } z = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 3 & 2 & -3 \\ 2 & 3 & 2 \\ -3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

#4

$$\text{Maximize } z = \begin{bmatrix} 40 & 30 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 6 & 6 & 2 \\ 36 & 24 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \leq \begin{bmatrix} 12 \\ 48 \\ 360 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

subject to

#6

$$\text{Maximize } z = \begin{bmatrix} 2 & 5 & -5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y^+ \\ y^- \\ u \\ v \end{bmatrix}$$

subject to

$$\begin{bmatrix} 3 & 2 & -2 & 1 & 0 \\ 2 & 9 & -9 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y^+ \\ y^- \\ u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} x \\ y^+ \\ y^- \\ u \\ v \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

#8

$$\text{Maximize } z = [-3 \quad -2.5 \quad 0 \quad 0]\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix}$$

subject to

$$\begin{bmatrix} -30 & -40 & 1 & 0 \\ 40 & 20 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} \le \begin{bmatrix} -120 \\ 80 \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

#10

$$\text{Maximize } z = [300 \quad 500 \quad 400 \quad 0 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 4 & 4 & 2 & 1 & 0 \\ 2 & 3 & 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 80 \\ 50 \end{bmatrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

#11

(a) We have

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 13 \end{bmatrix} \le \begin{bmatrix} 8 \\ 15 \end{bmatrix}; \quad z = 340$$

and

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 14 \end{bmatrix} \le \begin{bmatrix} 8 \\ 15 \end{bmatrix}; \quad z = 420$$

Also,

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) We have

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix} \ngeq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

#12

$$\text{Max } z = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Subject to

$$\begin{bmatrix} 3 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & 5 & 4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 10 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \geq 0.$$

#14

(a) We have

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 11 \end{bmatrix} \leq \begin{bmatrix} 10 \\ 12 \\ 15 \end{bmatrix}$$

and $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(b) $x_4 = 2, \quad x_5 = 5, \quad x_6 = 4$

#15

If $\mathbf{x}_1$ and $\mathbf{x}_2$ are feasible solutions, then $\mathbf{x}_1 \geq 0$ and $\mathbf{x}_2 \geq 0$, so $\mathbf{x} = \frac{1}{3}\mathbf{x}_1 + \frac{2}{3}\mathbf{x}_2 \geq \mathbf{0}$.

Also, if $\mathbf{A}\mathbf{x}_1 \leq \mathbf{b}$ and $\mathbf{A}\mathbf{x}_2 \leq \mathbf{b}$, then

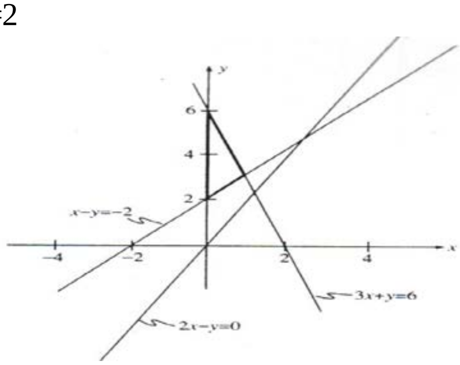
$$\begin{aligned} \mathbf{A}\mathbf{x} &= \mathbf{A}\left(\frac{1}{3}\mathbf{x}_1 + \frac{2}{3}\mathbf{x}_2\right) \\ &= \frac{1}{3}\mathbf{A}\mathbf{x}_1 + \frac{2}{3}\mathbf{A}\mathbf{x}_2 \\ &\leq \frac{1}{3}\mathbf{b} + \frac{2}{3}\mathbf{b} = \mathbf{b} \end{aligned}$$

#16

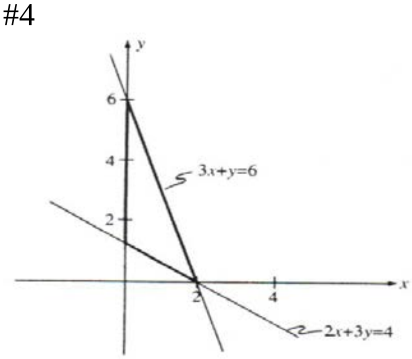
Show that if x_1 and x_2 satisfy $Ax \geq b$ and $x \geq 0$, then $x = rx_1 + sx_2$ should also satisfies $Ax \geq b$ and $x \geq 0$ as we did in class.

Section 1.3

#2



#4



18 The set is not convex.

#20 The set is convex.

#22 The set is not convex.

#24 The set is convex.

26. Let V be a subspace of R^n and let $\mathbf{v}$ and $\mathbf{w} \in V$. Then since V is a subspace of R^n,

$$\mathbf{x} = \lambda\mathbf{v} + (1-\lambda)\,\mathbf{w} \in V \text{ for } 0 < \lambda < 1.$$

Hence, V is a convex set.

27. Let $R = \{\mathbf{x} \in R^n \mid a_i \le x_i \le b_i\}$ be a rectangle in R^n. Let $\mathbf{v}$ and $\mathbf{w} \in R$. Then

$$a_i \le v_i \le b_i \text{ and } a_i \le w_i \le b_i.$$

If

$$\mathbf{x} = \lambda\mathbf{v} + (1-\lambda)\mathbf{w}, \quad 0 < \lambda < 1,$$

then

$$x_i = \lambda v_i + (1-\lambda)\,w_i$$

We now have

$$a_i = \lambda a_i + (1-\lambda)\,a_i \le \lambda v_i + (1-\lambda)\,w_i \le \lambda b_i + (1-\lambda)\,b_i = b_i$$

or

$$a_i \le x_i \le b_i$$

Hence, $\mathbf{x} \in R$.

28. Let $\mathbf{v}$ and $\mathbf{w} \in H_2$ so that $\mathbf{c}^T\mathbf{v} \ge k$ and $\mathbf{c}^T\mathbf{w} \ge k$. If

$$\mathbf{x} = \lambda\mathbf{v} + (1-\lambda)\mathbf{w}, \quad 0 < \lambda < 1,$$

then

$$\begin{aligned}\mathbf{c}^T\mathbf{x} = \mathbf{c}^T(\lambda\mathbf{v} + (1-\lambda)\,\mathbf{w}) &= \mathbf{c}^T(\lambda\mathbf{v}) + (1-\lambda)\,\mathbf{c}^T\mathbf{w} \\ &\ge \lambda k + (1-\lambda)\,k = k\end{aligned}$$

Hence, $\mathbf{x} \in H_2$.

29. We have $H = \left\{\mathbf{x} \in R^n \mid \mathbf{a}^T\mathbf{x} = \mathbf{b}\right\}$. Let $\mathbf{x}_1$ and $\mathbf{x}_2 \in H$ so that $\mathbf{a}^T\mathbf{x}_1 = \mathbf{b}$ and $\mathbf{a}^T\mathbf{x}_2 = \mathbf{b}$. If $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$, $0 < \lambda < 1$, then

$$\begin{aligned}\mathbf{a}^T\mathbf{x} &= \mathbf{a}^T(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\ &= \lambda\mathbf{a}^T\mathbf{x}_1 + (1-\lambda)\mathbf{a}^T\mathbf{x}_2 \\ &= \lambda\mathbf{b} + (1-\lambda)\mathbf{b} = \mathbf{b}.\end{aligned}$$

Hence, $\mathbf{x} \in H$.

32. We have $\mathbf{c}^T\mathbf{x}_1 = k$ and $\mathbf{c}^T\mathbf{x}_2 = k$. If $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$, $0 < \lambda < 1$, then

$$\begin{aligned}\mathbf{c}^T\mathbf{x} &= \mathbf{c}^T(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)\\ &= \lambda\mathbf{c}^T\mathbf{x}_1 + (1-\lambda)\mathbf{c}^T\mathbf{x}_2\\ &= \lambda k + (1-\lambda)k = k.\end{aligned}$$

33. Let S be the set of all solutions to $\mathbf{Ax} \leq \mathbf{b}$ and let $\mathbf{x}_1$ and $\mathbf{x}_2 \in S$. If $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$, $0 < \lambda < 1$, then

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}\,(\lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2)\\ &= \lambda\mathbf{Ax}_1 + (1-\lambda)\,\mathbf{Ax}_2\\ &\leq \lambda\mathbf{b} + (1-\lambda)\,\mathbf{b} = \mathbf{b}.\end{aligned}$$

Hence, $\mathbf{x} \in S$.

34. Let S be the set of all solutions to $\mathbf{Ax} > \mathbf{b}$ and let $\mathbf{x}_1$ and $\mathbf{x}_2 \in S$. If $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$, $0 < \lambda < 1$, then

$$\begin{aligned}\mathbf{Ax} &= \mathbf{A}\,(\lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2)\\ &= \lambda\mathbf{Ax}_1 + (1-\lambda)\,\mathbf{Ax}_2\\ &> \lambda\mathbf{b} + (1-\lambda)\,\mathbf{b} = \mathbf{b}.\end{aligned}$$

Hence, $\mathbf{x} \in S$.

35. Suppose that S is a convex set and let $\mathbf{v}'$ and $\mathbf{w}' \in f\,(S)$. Then $\mathbf{v}' = f\,(\mathbf{v})$ and $\mathbf{w}' = f\,(\mathbf{w})$ for some $\mathbf{v}$ and $\mathbf{w}$ in S. Now let

$$\mathbf{y} = \lambda\mathbf{v}' + (1-\lambda)\,\mathbf{w}'$$

Then

$$\begin{aligned}\mathbf{y} &= \lambda f\,(\mathbf{v}) + (1-\lambda)\,f\,(\mathbf{w})\\ &= f\,(\lambda\mathbf{v} + (1-\lambda)\,\mathbf{w})\end{aligned}$$

Since S is a convex set, $\mathbf{x} = \lambda\mathbf{v} + (1-\lambda)\,\mathbf{w} \in S$, so $\mathbf{y} = f\,(\mathbf{x}) \in f\,(S)$. Hence, $f\,(S)$.is a convex set.

36. We defined f to be convex if

$$f(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \leq \lambda f(\mathbf{x}_1) + (1-\lambda)f(\mathbf{x}_2).$$

The left side of this inequality is the second coordinate of the point $(\mathbf{x}, f(\mathbf{x}))$ on the graph of f, where $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\,\mathbf{x}_2$. The right side gives the points on the line segment joining the points $(\mathbf{x}_1, f(\mathbf{x}_1))$ and $(\mathbf{x}_2, f(\mathbf{x}_2))$ on the graph of f.

37. Let $\mathbf{x}_1$ and $\mathbf{x}_2 \in R^n$. Then

$$
\begin{aligned}
f(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) &= \mathbf{c}^T\mathbf{x} \\
&= \mathbf{c}^T(\lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) \\
&= \lambda \mathbf{c}^T\mathbf{x}_1 + (1-\lambda)\mathbf{c}^T\mathbf{x}_2 \\
&= \lambda f(\mathbf{x}_1) + (1-\lambda) f(\mathbf{x}_2)
\end{aligned}
$$

In addition:

1. $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ -0.5 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$

2. Let $c^T x_1 = 3$ and $c^T x_2 = 3$.
Let $x = \lambda x_1 + (1-\lambda)x_2$. Then $c^T x = \lambda c^T x_1 + (1-\lambda)c^T x_2 = \lambda 3 + (1-\lambda)3 = 3$.
So any point on the line between x_1 and x_2 will have $c^T x = 3$.

3. 1) $\{(0, 0, a, 0), a \geq 11\}$
2) $\{(5, 0, 6, 0)\}$
3) empty

4. The two feasible region are different. The region of Set B is the intersection of two planes, which is a line. The region of set A is an area bounded by the two inequalities. For example, x = 2 and y = 3 satisfy the inequalities of set A. You may not find a z such that (2, 3, z) satisfies the inequalities of set B.
This shows that when introducing slack variables, the same variable can not be used for different inequalities.