3. Duality

3.1 Example :

Suppose that a company produces two types of garden shears, manual and electric, and each requires the use of machines A and B in its production as following:

	Machine A	Machine B	Profit/unit
Manual	1 hr	1 hr	$10
Electric	2 hr	4 hr	$24
Hours available	120	180	

Assuming that the company can sell all the shears it can produce, we will determine the maximum monthly profit.

Let x_1 and x_2 be the numbers of manual and electric shears produced per month. Then

Now let us look at the situation from a different point of view.

Suppose that the company wishes to rent out machines A and B. What is the minimum monthly rental fee they should charge?

.

3.2 LP duality

The pair LP problems are called *dual problems*.

$$\begin{aligned} \text{Maximize} \quad & z = c^T x \\ \text{Subject to} \quad & Ax \leqslant b \qquad (1) \\ & x \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} \text{Minimize} \quad & w = b^T y \\ \text{Subject to} \quad & A^T y \geqslant c \qquad (2) \\ & y \geqslant 0 \end{aligned}$$

Where (1) is the *primal problem* and (2) is the *dual problem*.

Each linear program has an associated dual linear program.
The dual problem of the dual problem (2) is the primal problem (1).

The dual problem of a LP problem in canonical form

$$\begin{aligned} \text{Maximize} \quad & z = c^T x \\ \text{Subject to} \quad & Ax = b \\ & x \geqslant 0 \end{aligned}$$

The dual problem of a LP problem

$$\text{Maximize } z = c^T x$$
$$\text{Subject to } Ax \leqslant b$$
$$x \text{ unrestricted}$$

The relationships between the primal and dual problems:

Primal problem	Dual problem
Maximization	
Coefficients of objective function	
Coefficients of ith constraint	
Right-hand sides of constraints	
ith constraint is "$\leqslant$"	
ith constraint is " = "	
jth variable is unrestricted	
jth variable $\geqslant 0$	
Number of variables	

Example3.1: Let the primal problem be

$$\text{Minimize } z = -4x_1 - 5x_2$$
$$\text{Subject to } 2x_1 + x_2 \leqslant 3$$
$$x_1 + 2x_2 \leqslant 3$$
$$x_1 \geq 0,\ x_2 \geq 0.$$

Example3.2: Maximize $z = 3x_1 + x_2 - x_3$

$$
\begin{aligned}
\text{Subject to } \quad x_1 + x_2 + x_3 &= 10 \\
2x_1 - x_2 \quad &\geqslant 2 \\
x_1 - 2x_2 + x_3 &\leqslant 6 \\
x_1, x_2, x_3 &\geqslant 0
\end{aligned}
$$

3.3 Weak duality

Example3.3:

Maximize $z = 5x_1 + 3x_2 + 2x_3$

$$
\begin{aligned}
\text{Subject to:} \quad x_1 &\leqslant 15 && (1) \\
x_3 &\geqslant 3 && (2) \\
x_1 + x_2 + x_3 &\leqslant 25 && (3) \\
4x_1 + 2x_2 + 2x_3 &\leqslant 75 && (4) \\
2x_1 - x_3 &\leqslant 20 && (5) \\
x_1, x_2, x_3 &\geqslant 0 && (6),(7),(8)
\end{aligned}
$$

The lower the right hand side of the final inequality is, the better the bound is.

The dual LP provides a general method to obtain the best possible lower bound. The basic idea is to obtain multipliers y_i so that they give a bound with a right hand side that is as small as possible. A valid set of values of the y_i has to satisfy the following constraints:

(a) All multipliers should be non-negative, and

(b) When the addition of the multiplied inequalities is performed,
the result should be the objective function (expression for z).

LP problem:

Maximize $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$

Subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m$$
$$x_j \ge 0,\ j = 1, 2, \ldots, n.$$

Dual LP problem:

Minimize $z = b_1y_1 + b_2y_2 + \ldots + b_my_m$

Subject to the restrictions

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \ge c_1$$
$$a_{12}x_1 + a_{22}y_2 + \cdots + a_{m2}y_m \ge c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{nm}y_m \ge c_n$$
$$y_i \ge 0,\ i = 1, 2, \ldots, m.$$

Or, in matrix form:

	LP	Dual LP
Maximize	$z = c^T x$	
Subject to	$Ax \le b$	
	$x \ge 0$	

Every feasible solution of the dual LP gives an _____________ on the maximum value of the original LP. The best ____________________ is achieved by the _______________of the dual LP.

<u>Weak Duality Theorem</u>

Let x and y be two feasible solutions of an LP and its dual, respectively, and let $z = c^Tx$ and $w = b^Ty$ be the values of the objective functions at x and y, respectively. Then $z \le w$.

Corollary: If z^* and w^* are the optimal values of an LP and its dual, respectively, then $z^* \le w^*$.

		Primal		
		Finite	Unbounded	Infeasible
	Finite			
Dual	Unbounded			
	Infeasible			

3.4 Strong duality

Consider LP problem in standard form with slack variables:

Maximize $z = c^T x + 0x_s$

Subject to $Ax + x_sI = b$

$x \geq 0,\ x_s \geq 0$

	Nonbasic variables		Basic variables	
	x_B	x_N	x_s	
x_s	B	N	I	b
	$-c_B$	$-c_N$	0	0

	Basic variables	Nonbasic variables		
	x_B	x_N	x_s	
x_B	I	$B^{-1}N$	B^{-1}	$B^{-1}b$
	0	$c_BB^{-1}N - c_N$	c_BB^{-1}	$c_BB^{-1}b$

Example 3.4: LP problem

$$\begin{aligned} &\text{Max z} = 6x_1 - 2x_2 + 3x_3 \\ &\text{Subject to} \\ &\quad 2x_1 - x_2 + 2x_3 \le 2 \\ &\quad x_1 \qquad + 4x_3 \le 4 \\ &\quad x_i \ge 0,\ i = 1,\ 2,\ 3. \end{aligned}$$

Initial Tableau

	x_1	x_2	x_3	s_1	s_2	
s_1	2	– 1	2	1	0	2
s_2	1	0	4	0	1	4
	–6	2	– 3	0	0	0

Final Tableau

	x_1	x_2	x_3	s_1	s_2	
x_1	1	0	4	0	1	4
x_2	0	1	6	– 1	2	6
	0	0	9	2	2	12

It turns out that
the simplex method always provides a solution to the original and a solution to the dual LP with matching z- and w-values.

In other words, we have
if the LP is not unbounded or infeasible, the maximum z-value of the LP is always equal to the minimum w-value of the dual.

Strong Duality Theorem
If an LP has an optimal solution, then so does its dual, and furthermore, the values of their objective functions (optimal) are the same.

3.5 Complementary slackness

Consider the LP problem and its dual

LP	Dual LP
P: Maximize $z = c^T x$	D: Minimize $w = b^T y$
Subject to $Ax \le b$	Subject to $A^T y \ge c$
$x \ge 0$	$y \ge 0$

Example 3.5:

The LP problem	The dual
Maximize $P = 10x_1 + 24x_2$	Minimize $F = 120y_1 + 180y_2$
S. t. $x_1 + 2x_2 \le 120$	S. t. $y_1 + y_2 \ge 10$
$x_1 + 4x_2 \le 180$	$2y_1 + 4y_2 \ge 24$
$x_i \ge 0$, i = 1, 2.	$y_i \ge 0$, i = 1, 2.

the final tableau of the primal

	x_1	x_2	s_1	s_2	
x_1	1	0	2	-1	60
x_2	0	1	-1/2	1/2	30
	0	0	8	2	1320

and the final tableau of the daul

	y_1	y_2	t_1	t_2	
y_1	1	0	-2	1/2	8
y_2	0	1	1	-1/2	2
	0	0	60	30	-1320

Suppose that $x = (x_1, x_2, \ldots, x_n)$ is primal feasible and that $y = (y_1, y_2, \ldots, y_m)$ is dual feasible. Let $(s_1, s_2, \ldots, s_m)$ denote the corresponding primal slack variables, and let $(t_1, t_2, \ldots, t_n)$ denote the corresponding dual slack variables. Then x and y are optimal for their respective problems if and only if

Suppose x and y are feasible for P and D respectively. Then x and y are optimal for their respective problems if and only if

$$(A^T y - c)_i x_i = 0 \text{ for all } i$$

$$(Ax - b)_j y_j = 0 \text{ for all } j.$$

These equations are called the ______________________________________.

Example 3.5: The LP problem

Maximize $z = 2x_1 + 8x_2 - x_3 - 2x_4$

subject to

$$2x_1 + 3x_2 + 6x_4 \le 6$$
$$-2x_1 + 4x_2 + 3x_3 \le 1.5$$
$$3x_1 + 2x_2 - 2x_3 - x_4 \le 4$$
$$x_i \ge 0,\ i = 1, \ldots, 4$$

Final Tableau

	x_1	x_2	x_3	x_4	s_1	s_2	s_3	
s_1	0	0	0.8125	6.875	1	-0.3125	-0.875	
x_2	0	1	0.3125	-0.125	0	0.1875	0.125	
x_1	1	0	-0.875	-0.25	0	-0.125	0.25	

Dual Problem:

Minimize $w = 6y_1 + 1.5y_2 + 4y_3$

subject to

$$2y_1 - 2y_2 + 3y_3 \ge 2$$
$$3y_1 + 4y_2 + 2y_3 \ge 8$$
$$3y_2 - 2y_3 \ge -1$$
$$6y_1 - y_3 \ge -2$$
$$y_i \ge 0,\ i = 1, 2,$$

Final Tableau

	y_1	y_2	y_3	t_1	t_2	t_3	t_4	
y_2	0.3125	1	0	0.125	-0.1875	0	0	
t_3	-0.8125	0	0	0.875	-0.3125	1	0	
y_3	0.875	0	1	-0.25	-0.125	0	0	
t_4	-6.875	0	0	0.25	0.125	0	1	

"Why do we care?".

1. It's an easy way to check whether a pair of primal/dual feasible solutions are optimal.
2. Given one optimal solution (obtained e.g. from the simplex algorithm) complementary slackness makes it easy to find the optimal solution of the dual problem.
3. Things can be even easier than that: see the following example.

Example 3.6: Maximize $p = 10x_1 + 10x_2 + 20x_3 + 20x_4$

Subject to

$$12x_1 + 8x_2 + 6x_3 + 4x_4 \le 210$$
$$3x_1 + 6x_2 + 12x_3 + 24x_4 \le 210$$
$$x_i \ge 0,\ i = 1, \ldots, 4$$

3.6 The dual simplex method:

Example 3.7: Using the simplex method to solve the LP problem

Maximize $z = 2x_1 + x_2$

Subject to

$$5x_2 \le 15$$
$$6x_1 + 2x_2 \le 24$$
$$x_1 + x_2 \le 5$$
$$x_i \ge 0, i = 1, 2$$

and its dual, Also, verifying Complementary slackness and dual theorem.

The goal for the <u>primal problem</u> when using simplex method is to achieve <u>optimality</u>. The goal for a corresponding method for the <u>dual problem</u> is to achieve <u>feasibility</u>.

The dual simplex method deals with LP problems which is easy to obtain

The dual simplex method includes:
a feasibility criterion.
a procedure for getting a new solution that removes some of the infeasibilities of current solution and consequently drives the current solution toward a feasible solution.

Proceeding:

1. Find an initial basic solution such that all entries in the objective row are nonnegative and at least one basic variable has a negative value.
2. Select a departing variable by examining the basic variables and choosing the most negative one.
3. Select an entering variable. This selection depends on the ratios of the objective row entries to the corresponding pivotal row entries. The ratios are formed only for those entries of the pivotal row that are negative. If all entries in the pivotal row are nonnegative, the problem has no feasible solution. Among all the ratios (nonpositive), select the maximum ratio.
4. Perform pivoting to obtain a new tableau.

The process stops when a basic solution that is feasible.

Example 3.8: Minimize $z = 2x_1 + 3x_2 + 4x_3 + 5x_4$

Subject to
$$x_1 - x_2 + x_3 - x_4 \geqslant 10$$
$$x_1 - 2x_2 + 3x_3 - 4x_4 \geqslant 6$$
$$3x_1 - 4x_2 + 5x_3 - 6x_4 \geqslant 15$$
$$x_i \geqslant 0,\ i = 1, 2, 3, 4.$$

Example 3.9: TV circuit has 30 large-screen televisions in a warehouse in Erie and 60 large-screen televisions in a warehouse in Pittsburgh. Thirty-five are needed in a store in Blairsville, and 40 are needed in a store in Youngstown. It costs \$18 to ship from Pittsburgh to Blairsville and \$22 to ship from Pittsburgh to Youngstown, whereas it costs \$20 to ship from Erie to Blairsville and \$25 to ship from Erie to Youngstown. How many televisions should be shipped from each warehouse to each store to minimize the shipping cost?

Example 3.10: A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Staff needed	13	15	19	14	16	11	17

3.7 Economic interpretation

P: Maximize $z = c^T x$
Subject to $Ax \leqslant b$
$x \geqslant 0$

D: Minimize $w = b^T y$
Subject to $A^T y \geqslant c$
$y \geqslant 0$

a_{ij}: the number of units of resource i required to produce one unit of commodity j;
b_i: the maximum number of units of resource i available;
c_j: the value (profit) per unit of commodity j produced.

Interpreting the primal and dual problems in terms of physical units in order to determine the meaning of the dual variables.

P: Maximize $\sum_{j=1}^{n} \left(\frac{value}{output \quad j} \right) (output \quad j) = (value)$

Subject to $\sum_{j=1}^{n} \left(\frac{input \quad i}{output \quad j} \right) (output \quad j) \leq (input \quad i)$, i = 1, 2, …, m

$(output \quad j) \geq 0$ j = 1, 2, …, n

D: minimize: $\sum_{i=1}^{m} (input \quad i) y_i = ?$

Subject to $\sum_{i=1}^{m} \left(\frac{input \quad i}{output \quad j} \right) y_i \geq \left(\frac{value}{output \quad j} \right)$ j = 1, 2, …, n

$y_i \geq 0$ i = 1, 2, …, m

Verbal descriptions of the primal and dual problems can then be stated as follows:

P: With a given unit of value of each output (c_j) and a given upper limit for the available of each input (b_i), how much of each output (x_j) should be produced in order to maximize the value of the total output.

D: With a given available of each input (b_i) and a given lower limit of unit value for each output (c_j), what unit values should be assigned to each input (y_i) in order to minimize the value of the total input.

The variables y_i are referred to by various names, e.g. shadow prices, accounting or marginal values.

Note: If z* at x* and w* at y* are the optimal values of an LP and its dual, then

$$z^* = c^T x^* = b^T y^* = b_1 y^*_1 + b_2 y^*_2 + \dots b_m y^*_m.$$

Therefore,

If other conditions remain the same, the change of the ith input (resource) yields the change of the optimal value.

It estimates the contribution of the value of a unit of the corresponding resource (ith input) to the optimal value.

A manufacturer may compare the market price of the ith input with the shadow price of the ith input. If the market price is lower than his shadow price, he may purchase more to produce more products. Otherwise, he might have to sell some of his resource.

Recall:
A company produces two types of garden shears, manual and electric, and each requires the use of machines A and B in its production as following:

	Machine A	Machine B	Profit/unit
Manual	1 hr	1 hr	\$10
Electric	2 hr	4 hr	\$24
Hours available	120	180	

Assuming that the company can sell all the shears it can produce, we will determine the maximum monthly profit.

Let x_1 and x_2 be the numbers of manual and electric shears produced per month. Then

$$\begin{aligned} &\text{Maximize } P = 10x_1 + 24x_2 \\ &\text{Subject to} \quad x_1 + 2x_2 \leqslant 120 \\ &\qquad\qquad\quad x_1 + 4x_2 \leqslant 180 \\ &\qquad\qquad\quad x_i \geqslant 0,\ i = 1, 2. \end{aligned}$$

The maximum profit per month is \$1320, which occurs when $x_1 = 60$ and $x_2 = 30$.

Let F be the total monthly rental fee. Suppose the company assigns y_1 and y_2 dollars to each hour of capacity on machines A and respectively.

$$\begin{aligned} &\text{Minimize } F = 120y_1 + 180y_2 \\ &\text{Subject to} \quad y_1 + y_2 \geqslant 10 \\ &\qquad\qquad\quad 2y_1 + 4y_2 \geqslant 24 \\ &\qquad\qquad\quad y_i \geqslant 0,\ i = 1, 2. \end{aligned}$$

the maximum value of monthly rental fee is also1320, the minimum value of F is \$1320. It occurs when $y_1 = 8$ and $y_2 = 2$.