1. Introduction to Linear Programming

1.1 Introduction of Linear programming (LP)

To understand best how to apply a mathematical theory to solution for some practical problem:

- ***Recognition of the problem***
- ***Formulation of a mathematical model***
- ***Solution of the mathematical problem***
- ***Translation of the results back into the context of the original problem***

Optimization via (Freshman) Calculus

Example: how to enclose the largest area: Suppose we have 100 meters of fencing, and want to enclose a rectangular area up against a long straight wall. How big an area can we enclose?

1.2 Mathematical Models

Example 1.1: A produce grower is purchasing fertilizer containing three nutrients, A, B, and C. The minimum needs are 160 units of A, 200 units of B, and 80 units of C. There are two popular brands of fertilizer on the Market.
Fast Grow: contains 3 units of A, 5 units of B, and 1 units of C, costing $8 a bag.
Easy Grow: contains 2 units each nutrient costing $6 a bag.
If the grower wishes to minimize cost while still maintaining the nutrients required, how many bags of each brand should be bought?

Example 1.2: A furniture maker has a line of four types of desks. They vary in the manufacturing process and their profitability. The furniture maker has available 6000 hours of time in the carpentry shop each six months, and 4000 hours of time in the finishing shop. Each desk of type 1 requires 4 hours of carpentry and 1 hour of finishing. Each desk of type 2 requires 9 hours of carpentry and 1 hour of finishing. Each desk of type 3 requires 7 hours of carpentry and 3 hours of finishing. Each desk of type 4 requires 10 hours of carpentry and 40 hours of finishing. The profit is $12 for each desk of type 1, $20 for each desk of type 2, $28 for each desk of type 3, $40 for each desk of type 4. How should the production be scheduled to maximize the profit?

Example 1.3: A PO requires different numbers of employees on different days of the week. Union rules state each employee must work 5 consecutive days and then receive two days off. Find the minimum number of employees needed.

	Mon	Tue	Wed	Thu	Fri	Sat	Sun
Staff needed	13	15	19	14	16	11	17

Example 1.4: (The diet problem) A nutritionist is planning a menu consisting of two main foods A and B. Each ounce of A contains 2 units of fat, one unit of carbohydrates, and 4 units of protein. Each ounce of B contains 3 units of fat, 3 units of carbohydrates, and 3 units of protein. The nutritionist wants the meal to provide at least 18 units of fat, at least 12 units of carbohydrates, and at least 24 units of protein. If an ounce of A costs $2 and an ounce of B costs $2.5, how many ounces of each food should be served to minimize the cost of the meal yet satisfy the nutritionist's requirements?

Example 1.5: A paper manufacturer having two mills must supply weekly three printing plants with newsprint. Mill 1 produces 350 tons of newsprint a week and Mill 2 550 tons. Plant 1 requires 275 tons/week, plant 2 325 tons, and plant 3 300 tons. The shipping costs, in dollars per ton, are as follows:

	Plant 1	Plant 2	Plant 3
Mill 1	17	22	15
Mill 2	18	16	12

The problem is to determine how many tons each mill should ship to each plant so that the total transportation cost is minimal.

1.3 Linear programming problems

General LP problem

For value of $x_1, x_2, \ldots, x_n$,

maximize or minimize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge)(=) b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le (\ge)(=) b_2 \\
&\vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad\quad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge)(=) b_m
\end{aligned}
$$

A LP problem in <u>standard form</u>

For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
&\qquad\qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
&x_j \ge 0,\ j = 1, 2, \ldots, n.
\end{aligned} \tag{1}
$$

A LP problem in <u>canonical form</u>

For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\qquad\qquad \vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \\
&x_j \ge 0,\ j = 1, 2, \ldots, n.
\end{aligned} \tag{2}
$$

How to convert a GLP to a standard LP:

1) Minimization problem as a maximization problem

2) Reversing an inequality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n \ge b$$

3) Changing an equality to an inequality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n = b$$

4) unconstrained variables

If there is no nonnegative constrain on x_j,

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5) Changing an inequality to an equality

$$k_1x_1 + k_2x_2 + \cdots + k_nx_n \le b$$

To convert a standard LP problem (1) to a canonical form (2) of the problem: For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (1)$$

$x_j \ge 0$, $j = 1, 2, \ldots, n$.

Example 1.6: Let LP be min f(x, y) = 3x+2y subject to the constraints

$$\begin{cases} y \ge x + 2 \\ y \le -x + 3 \\ y \ge \dfrac{2}{5}x + \dfrac{1}{5} \end{cases}$$

convert it to a standard LP problem then a canonical form.

Example 1.7: Convert the following LP problem into standard form:

Min $z = -x_1 + 2x_2 - 3x_3$

s.t. $x_1 - x_2 + x_3 = -2$

$|x_1 - x_3| \le 1$

$x_2 \ge 0$

1.4 Matrix notation

The standard form LP: For value of $x_1, x_2, \ldots, x_n$,

maximize $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$

subject to the restrictions

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (1)$$

$x_j \ge 0$, $j = 1, 2, \ldots, n$.

Letting

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The standard form LP in matrix form

maximize

subject to

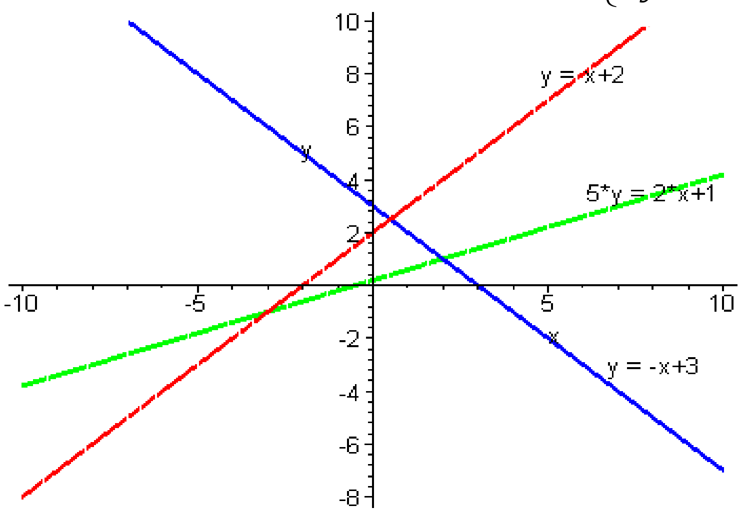
The canonical form LP in matrix form

Maximize

subject to

1.5 Theorem of LP problems

Example1.8: Consider three different non parallel $\begin{cases} y = x+2 \\ y = -x+3 \\ 5y = 2x+1 \end{cases}$ lines



I. $\begin{cases} y \geq x+2 \\ y \geq -x+3 \\ y \geq \frac{2}{5}x+\frac{1}{5} \end{cases}$ II. $\begin{cases} y \leq x+2 \\ y \geq -x+3 \\ y \geq \frac{2}{5}x+\frac{1}{5} \end{cases}$ III. $\begin{cases} y \geq x+2 \\ y \leq -x+3 \\ y \geq \frac{2}{5}x+\frac{1}{5} \end{cases}$

IV. $\begin{cases} y \leq x+2 \\ y \leq -x+3 \\ y \geq \frac{2}{5}x+\frac{1}{5} \end{cases}$ V. $\begin{cases} y \geq x+2 \\ y \leq -x+3 \\ y \leq \frac{2}{5}x+\frac{1}{5} \end{cases}$ VI. $\begin{cases} y \leq x+2 \\ y \geq -x+3 \\ y \leq \frac{2}{5}x+\frac{1}{5} \end{cases}$ VII. $\begin{cases} y \leq x+2 \\ y \leq -x+3 \\ y \leq \frac{2}{5}x+\frac{1}{5} \end{cases}$

A *feasible solution* to a LP problem

$\{x_1, x_2, \ldots, x_n\}$ is a feasible solution of (1)

$\leftrightarrow$

If a feasible region can be contained within a circle then it is ___________. Otherwise it is called _______________. If it contains at least one point then it is ____________, otherwise it is called ____________.

Objective function:

Optimum solution:

Linear Programming Theorem (LP):

Let f be a linear function. Let U be a nonempty region in R^2 such that U is defined by linear inequalities and it includes its boundaries.

a) If U is bounded then f has a maximum and a minimum on U and these values occur at corner points of U.

b) If U is unbounded and if f has a maximum or a minimum then this occurs at a corner point of U.

1.6 Graphic solution of LP problems

Example 1.9: Find the maximum and minimum of f(x, y) = 3x+2y subject to the constraints

$$\begin{cases} y \le x+2 \\ y \le -x+3 \\ y \ge \frac{2}{5}x+\frac{1}{5} \end{cases} \quad \text{with } x \ge 0 \text{ and } y \ge 0.$$

Example 1.1:

Example 1.4:

Example 1.10: Minimize Z = 3x + 9y

Subject to

$$\begin{cases} y \le -\dfrac{3}{2}x + 6 \\ y \ge -\dfrac{1}{3}x + \dfrac{11}{3} \\ y \ge x - 3 \\ x, y \ge 0 \end{cases}$$

1.7 Geometry of LP problems

Geometry of a constraint of a LP problem:

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \le b_i$$

or $a^T x \le b_i$ where $a^T = [a_{i1}, a_{i2}, \cdots, a_{in}]$

The set of points $x = (x_1, x_2, \ldots, x_n)$ in R^n that satisfy this constraint is called a

________________________.

The set of points $x = (x_1, x_2, \ldots, x_n)$ in R^n that satisfy $a^T x = b_i$ is called a _____________.

A hyperplane is a _______________of a closed half-space.

The set of feasible solutions to a LP problem is the intersection of all the closed half-spaces determined by the constraints.

Example 1.11: $\begin{cases} x + y + z \ge 1 \\ x \ge 0 \\ y \ge 0 \\ z \ge 0 \end{cases}$

Geometry of the objective function

An objective function

Let k be a constant.

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Geometrically, the optimal solution is the hyperplane that

Geometry of the set of feasible solutions

Let x_1 and x_2 be feasible solutions.
The ***line segment*** connecting x_1 and x_2

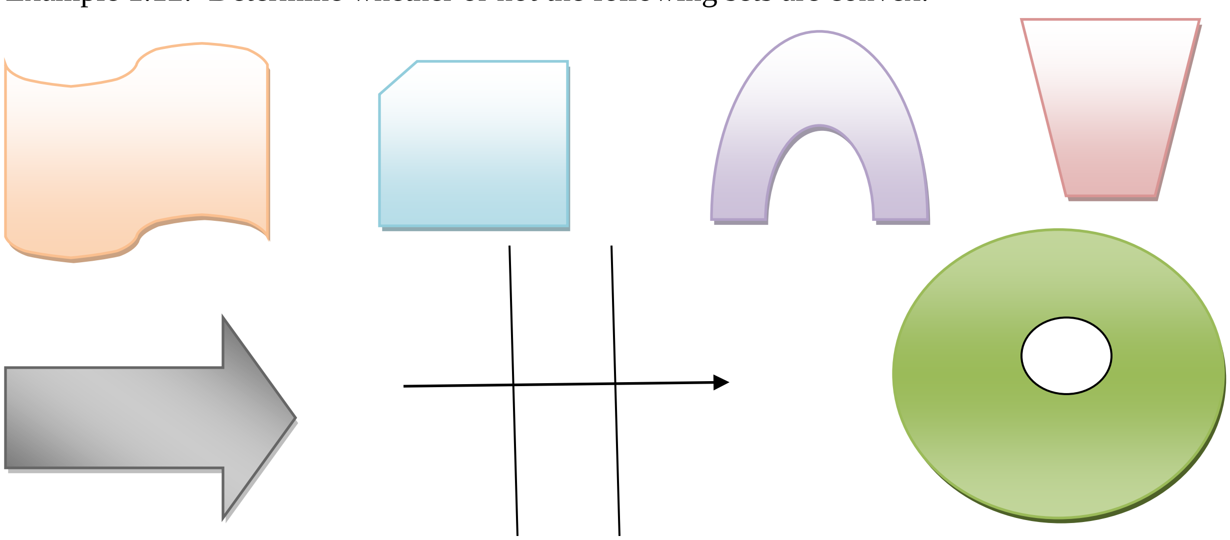
$$= \{ x \in R^n | \; x = \lambda x_1 + (1 - \lambda) x_2 \,, \; 0 \le \lambda \le 1 \}$$

If $a^T x \le b_i$ is a constraint of the problem, and $a^T x_1 \le b_i$ and $a^T x_2 \le b_i$, for any interior point of the line segment,

Interior point is a ___________ solution but not an ____________ solution.
Def: A sub set K of R^n is ________________ if for any x_1, $x_2 \in K$

$$x = \lambda x_1 + (1 - \lambda) x_2 \in K, \qquad 0 \le \lambda \le 1$$

Example 1.12: Determine whether or not the following sets are convex.



Example 1.13. A hyperplane $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

A closed half-space $a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b$

$\{ \|x\| \geq 1 \mid x \in R^n\}$ & $\{ \|x\| = 1 \mid x \in R^n\}$

Thm: The intersection of a finite collection of convex sets is convex.

1.8 The extreme point theorem

Def. A point $x \in R^n$ is a ______________________ of the points $x_1, x_2, \ldots, x_r$ in R^n if for some real numbers $c_1, c_2, \ldots, c_r$ which satisfy

Thm. The set of all convex combinations of a finite set of points in R^n is a convex set.

Def. A point x in a convex set S is called an ____________________ of S if it is not an interior point of any line segment in S.

Thm. Let S be a convex set in R^n. A point x in S is an extreme point of S if and only if x is not a convex combination of other points.

Thm. Let S be the set of feasible solutions to a general LP problem.

1) If S is nonempty and bounded, then an optimal solution to the problem exists and occurs at an extreme point.
2) If S is nonempty and unbounded, and if an optimal solution to the problem exists, then it occurs at an extreme point.
3) If an optimal solution to the problem does not exist, then either S is empty or S is unbounded.

1.9 Basic solutions

Consider a LP problem in canonical form

$$\text{maximize } z = c^T x \quad \text{for } x \in R^s \tag{1}$$

subject to

$$Ax = b \tag{2}$$

$$x \geq 0 \tag{3}$$

where A is an $n \times s$ matrix and b is an $n \times 1$ matrix.

Let S be the convex set of all feasible solutions of (2).

Thm. Suppose that the last m columns of A, which denote by A_1', A_2', ..., A_m' are linearly independent and suppose that

$$x_1' A_1' + x_2' A_1' \cdots + x_m' A_m' = b \tag{5}$$

where $x_i' \geq 0$ for i = 1, 2, …, m. Then the point

x = (0, 0, …, 0, x_1', x_2', …, x_m')

is an extreme point of S.

Thm. If $x = (x_1, x_2, \ldots, x_m)$ is an extreme point of S, then the columns of A that correspond to positive x_i form a linearly independent set of vectors in R^n

Corollary. If x is an extreme point and x_{i_1}, x_{i_2}, …, x_{i_r} are the r positive components of x, then $r \leq m$, and the set of columns A_{i_1}, A_{i_2}, …, A_{i_r} can be extended to a set of m linearly independent vectors in R^n by adjoining a suitably chosen set of m – r columns of A.

Thm. At most m components of any extreme point of S can be positive. The rest must be zero.

A ____________ x to Ax = b is the solution of it obtained from solving this system along with the s – m zeros form x .

Def. A ____________________ to the LP problem given by (1) –(3) is a ______ solution that is also a _______ solution of (2).

Thm. For the LP problem given by (1) – (3), every ____________ solution is an ______ point, and, conversely, every _________ point is a _______________ solution.

Thm. The LP problem given by (1) – (3) has a finite number of ______________ solutions.

Thm. Every extreme point of S yields an extreme point of S' when slack variable are added. Conversely, every extreme point of S', when truncated, yields an extreme point of S.

Example 1.14: Maximize f(x, y) = 3x+2y

Subject to

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$$-x+y \le 2$$
$$x+y \le 3$$
$$\frac{2}{5}x - y \le -\frac{1}{5}$$
$$x, y \ge 0$$