2. The Simplex method

2.1 The Simplex method

The standard LP problem

$$\text{maximize } z = c^T x \quad \text{for } x \in R^n$$
$$\text{subject to}$$
$$Ax \le b$$
$$x \ge 0 \qquad (1)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Corresponding LP in canonical form

$$\text{maximize } z = c^T x \quad \text{for } x \in R^n$$
$$\text{subject to}$$
$$Ax = b$$
$$x \ge 0 \qquad (2)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \\ \vdots \\ x_{n+m} \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Consider the LP problem (2).

1) To find an extreme point (the initial basic feasible solution)

Let **b** $\ge$ **0**.

Set all the nonslack variables in the system $Ax = b$ equal to zero. One of the basic solutions is

$$x = [0, 0, \ldots, 0, x_{n+1}, \ldots, x_{n+m}]. \qquad (3)$$

2) To seek adjacent extreme point

Def. Two distinct extreme points are said to be ______________ if as basic solutions they have all but one basic variable in common.

Let $x^{(0)} = [0,0,...,0, x_{n+1}^0, ..., x_{n+m}^0]$ be the initial basic feasible solution of a LP problem with

$$x_{n+1}^0 A_{n+1} + x_{n+2}^0 A_{n+2} + ... + x_{n+m}^0 A_{n+m} = b \qquad (4)$$

Where A_i are column vectors of

$$\begin{array}{cc} & \begin{array}{cccccccc} A_1 & A_2 & \dots & A_n & A_{n+1} & A_{n+2} & \dots & A_{n+m} \end{array} \\ A = & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 & 0 & \cdots & 1 \end{bmatrix} \end{array}$$

Since A_{n+1} A_{n+2} … A_{n+m} form a basis for the column space of A, then

3) To compare basic feasible solutions

Substituting the basic feasible solutions ______and ______ into the objective function

The Simplex algorithm:

a) find out whether a given basic feasible solution is an optimal solution,
b) search an adjacent basic feasible solution with the same or larger value for the objective function.

4) To build an initial tableau

The objective function $z = c^T x$ may be rewritten as

$$- c_1 x_1 - c_2 x_2 - \dots - c_n x_n + z = 0.$$

The initial tableau for the LP problem (1) is

	x_1	x_2	$\dots$	x_n	x_{n+1}	x_{n+2}	$\dots$	x_{n+m}	z	
x_{n+1} x_{n+2} • • • x_{n+m}										

In the tableau, a basic variable has the following properties:

a) It appears in exactly one equation and in that equation it has a coefficient of + 1.
b) The column that it labels has all zeros (including the objective row entry) except for the +1 in the row that is labelled by the basic variable.
c) The value of a basic variable is the entry in the same row in the rightmost column.

Example: Maximize $z = c^T x$ for $x \in R^2$

subject to

$$Ax \le b$$

$$x \ge 0 \qquad (8)$$

where

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \qquad c = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

5) To check an optimal solution

Note that the objective function

Optimality Criterion

If the objective row of a tableau has zero entries in the columns labelled by basic variables and no negative entries in the column labelled by nonbasic variables, then the solution represented by the tableau is optimal.

6) To select the entering variable

The largest increase in z per unit increase in a variable occurs for the______ negative entry in the objective row.

If the feasible set is bounded,

In this text, we choose to increase the variable with the _____negative entry in the objective row.

The variable to be increased is called __________ *variable* .

7) To choose the departing variable

The variable has departed from the set of the basic variables is called ___________ *variable* .The column of the entering variable is called the ______ *column* and the row that is labelled with the departing variable is called the _______ *row*.

To determine the departing variable by how much we could increase the entering variable is decided by the ______________________ θ ratio.

8) To form a new tableau – the process is called ___________:

a) Locate and circle the entry at the intersection of the pivotal row and pivotal column. This entry is called the ________. Mark the pivotal column by placing an arrow above the entering variable, and mark the pivotal row by placing an arrow to the left of the departing variable.

b) If the pivot is k, multiply the pivot row by 1/k, making the entry in the pivot position in the new tableau equal to 1.

c) Adding suitable multiples of the new pivotal row to all other rows (including the objective row), so that all other elements in the pivotal column become zero.

d) In the new tableau, replace the label on the pivotal row by the entering variable.

Example 2.1: Maximize $p = 4x + 2y + 6z$
subject to

$$
\begin{aligned}
3x + 2y \quad\quad &\leq 10 \\
y + 2z &\leq 8 \\
2x + y + z &\leq 8 \\
x \geq 0, y \geq 0, z &\geq 0.
\end{aligned}
$$

Recall: Example 1.2:

A furniture maker has a line of four types of desks. They vary in the manufacturing process and their profitability. The furniture maker has available 6000 hours of time in the carpentry shop each six months, and 4000 hours of time in the finishing shop. Each desk of type 1 requires 4 hours of carpentry and 1 hour of finishing. Each desk of type 2 requires 9 hours of carpentry and 1 hour of finishing. Each desk of type 3 requires 7 hours of carpentry and 3 hours of finishing. Each desk of type 4 requires 10 hours of carpentry and 40 hours of finishing. The profit is \$12 for each desk of type 1, \$20 for each desk of type 2, \$28 for each desk of type 3, \$40 for each desk of type 4. How should the production be scheduled to maximize the profit?

Example 2.2: Maximize $p = 2x + 7y$

Subject to
$$\begin{aligned} 4x - 3y &\leq 4 \\ 3x - y &\leq 6 \\ 5x &\leq 8 \\ x \geq 0,\ y &\geq 0. \end{aligned}$$

Example 2.3: Maximize $p = 3x + 3y - 3z - w$

Subject to $x + y - w \leq 3$

$x - y + w \leq 6$

$x + y - z + w \leq 5$

$x \geq 0, y \geq 0, z \geq 0, w \geq 0.$

2.2 Degeneracy

Example 2.4: Maximize $p = 3x + 2y$

Subject to $2x + y \leq 4$

$x - y \leq 2$

$3x + 2y \leq 10$

$x \geq 0, y \geq 0$

Example2.5: Maximize p = 12x + 9y + z - 6w + 4s

Subject to $x + 10y + z - 9w - (5/3)s \leq 2$

$x + 2y + (1/3)z - 2w \leq 2$

$x + y + (1/3)s \leq 2$

$x \geq 0, y \geq 0, z \geq 0, w \geq 0, s \geq 0.$

Tableau #1

x	y	z	w	s	s1	s2	s3	
1	2	1/3	-2	0	1	0	0	2
1	10	1	-9	-5/3	0	1	0	2
1	1	0	0	1/3	0	0	1	2
-12	-9	-1	6	-4	0	0	0	0

Tableau #2

x	y	z	w	s	s1	s2	s3	
0	1	1/3	-2	-1/3	1	0	-1	0
0	9	1	-9	-2	0	1	-1	0
1	1	0	0	1/3	0	0	1	2
0	3	-1	6	0	0	0	12	24

Tableau #3

x	y	z	w	s	s1	s2	s3	
0	-2	0	1	1/3	1	-1/3	-2/3	0
0	9	1	-9	-2	0	1	-1	0
1	1	0	0	1/3	0	0	1	2
0	12	0	-3	-2	0	1	11	24

Tableau #4

x	y	z	w	s	s1	s2	s3	
0	-2	0	1	1/3	1	-1/3	-2/3	0
0	-9	1	0	1	9	-2	-7	0
1	1	0	0	1/3	0	0	1	2
0	6	0	0	-1	3	0	9	24

Tableau #5

x	y	z	w	s	s1	s2	s3	
0	1	-1/3	1	0	-2	1/3	5/3	0
0	-9	1	0	1	9	-2	-7	0
1	4	-1/3	0	0	-3	2/3	10/3	2
0	-3	1	0	0	12	-2	2	24

Tableau #6

x	y	z	w	s	s1	s2	s3	
0	1	-1/3	1	0	-2	1/3	5/3	0
0	0	-2	9	1	-9	1	8	0
1	0	1	-4	0	5	-2/3	-10/3	2
0	0	0	3	0	6	-1	7	24

Tableau #7

x	y	z	w	s	s1	s2	s3	
0	1	1/3	-2	-1/3	1	0	-1	0
0	0	-2	9	1	-9	1	8	0
1	0	-1/3	2	2/3	-1	0	2	2
0	0	-2	12	1	-3	0	15	24

Tableau #8

x	y	z	w	s	s1	s2	s3	
0	1	1/3	-2	-1/3	1	0	-1	0
0	9	1	-9	-2	0	1	-1	0
1	1	0	0	1/3	0	0	1	2
0	3	-1	6	0	0	0	12	24

Bland's rule:

- Selecting the pivotal column: choose the column with the smallest subscript among those with negative entries.
- Selecting the pivotal row: If two rows have equal θ ratios, choose the row corresponding to the basic variable with lowest subscript.

2.3 Two phases method

2.3.1 Introduction of artificial variables

General LP problem

Maximize $z = c^T x$

Subject to the restrictions

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge)(=) b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le (\ge)(=) b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge)(=) b_m$$
$$x_j \ge 0, \quad j = 1, 2, \ldots, n.$$

To **rewrite the constraints so that its right-hand side is nonnegative**, i.e._________.

$$a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1n}x_n \le (\ge)(=) b'_1$$
$$a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2n}x_n \le (\ge)(=) b'_2$$
$$\vdots$$
$$a'_{m1}x_1 + a'_{m2}x_2 + \cdots + a'_{mn}x_n \le (\ge)(=) b'_m$$

Example 2.6: Maximize $z = 5x_1 + 3x_2$

Subject to:

$$2x_1 + x_2 \le 1$$
$$x_1 + 4x_2 \ge 6$$
$$x_i \ge 0, \quad i = 1, 2.$$

The general LP problem may be rewritten as

Maximize $z = c^T x$

Subject to the restrictions $\sum_{j=1}^{s} a'_{ij} x_j = b'_i$ i = 1, 2, …, m.

$x_j \geq 0$, j = 1, 2, …, s

where $b'_i \geq 0$ i = 1, 2, …, m.

Introduce artificial variables y_i to each of the equation and assuming the profit associated with each y_i to be ____________, to obtain

Maximize $z = c^T x$

Subject to the restrictions $\sum_{j=1}^{s} a'_{ij} x_j + y_i = b'_i$

$x_j \geq 0$, j = 1, 2, …, s, $y_i \geq 0$, i = 1, 2, …, m.

2.3.2 Phase I

Minimize $z^* = \sum_{i=1}^{m} y_i$

Subject to the restrictions $\sum_{j=1}^{s} a'_{ij} x_j + y_i = b'_i$

$x_j \geq 0$, j = 1, 2, …, s, $y_i \geq 0$, i = 1, 2, …, m

with $b'_i \geq 0$, i = 1, 2, …, m.

Step 1: assign –1 to convert a maximization problem.

Step 2: The auxiliary LP problem is

Step 3: Solve the auxiliary LP problem by simplex method until either of the following three cases arise:

Max z < 0 and at least one artificial variable appears in the BFS at positive level.

Max z = 0 and at least one artificial variable appears in BFS at positive level.

Max z = 0 and no artificial variable appears in the BFS.

2.3.3 Phase II

Example 2.7: Maximize $z = 5x_1 - 4x_2 + 3x_3$

Subject to:

$$2x_1 + x_2 - 6x_3 = 20$$

$$6x_1 + 5x_2 + 10x_3 \leq 76$$

$$8x_1 - 3x_2 + 6x_3 \leq 50$$

$$x_i \geq 0 \text{ , } i = 1, 2, 3.$$

Phase I

Phase II

Example 2.8: Maximize $z = -3x_1 + x_3$

Subject to $x_1 + x_2 + x_3 \leqslant 4$

$$-2x_1 + x_2 - x_3 \geqslant 1$$

$$3x_2 + x_3 = 9$$

$$x_i \geqslant 0, \quad i = 1, 2, 3$$

2.4 Big M method

After introducing slack variables and artificial variables to a LP problem, we have

Maximize $z = \sum_{j=1}^{s} c_j x_j$

Subject to the restrictions $\sum_{j=1}^{s} a'_{ij} x_j + y_i = b'_i$

$x_j \geq 0$, j = 1, 2, …, s, $y_i \geq 0$, i = 1, 2, …, m.

The objective function is rewritten as

$$z = \sum_{j=1}^{s} c_j x_j - \sum_{i=1}^{m} M y_i \quad \text{with} \quad y_i = b'_i - \sum_{j=1}^{s} a'_{ij} x_j .$$

Example 2.9: Maximize $z = x_1 + 2x_2$

Subject to $x_1 + x_2 \leq 9$

$x_1 - x_2 \geq 1$

$x_i \geq 0$, i = 1, 2.

Example 2.10: Maximize $z = 2x_1 + x_2$

Subject to $-x_1 + x_2 \geq 2$

$x_1 + x_2 \leq 1$

$x_i \geq 0, \quad i = 1, 2.$

Example 2.11: Maximize $z = x_1 + 3x_2 - 2x_3$

Subject to $x_1 + 2x_2 + 2x_3 = 6$

$x_1 + x_2 - x_3 \geq 2$

$x_i \geq 0, \quad i = 1, 2, 3.$