A Martingale Approach to the Predictability of Patterns in Discrete-Event Stochastic Systems

For presentation at the 39th Southern California Control Workshop

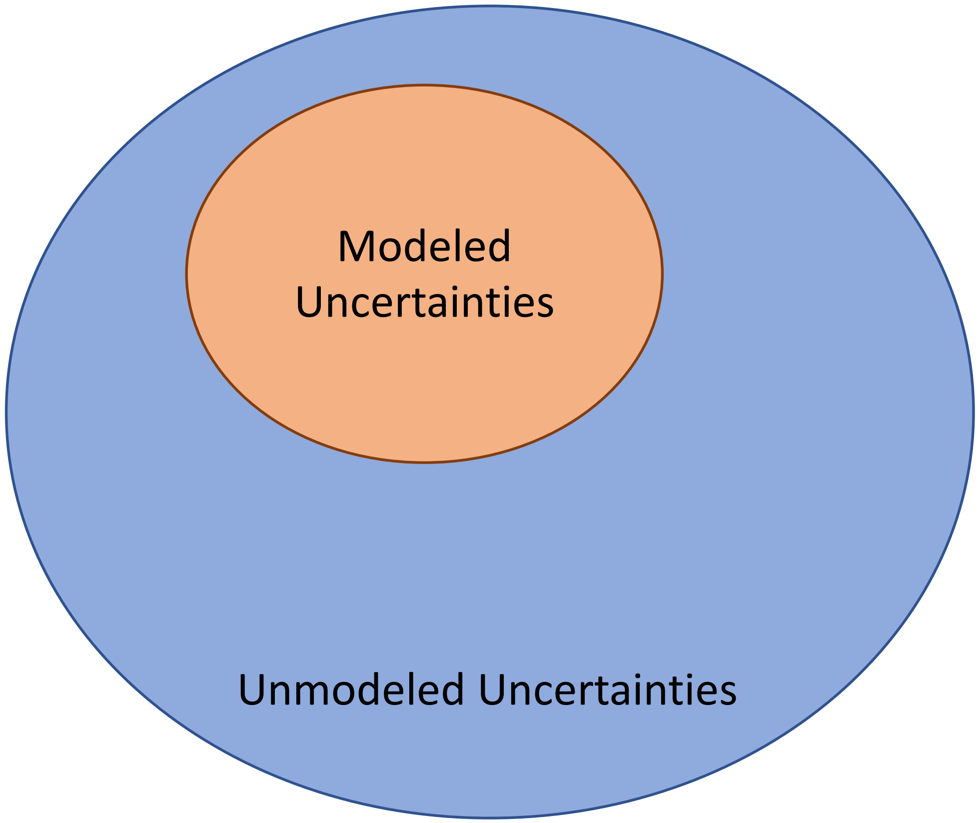
---

SooJean Han

April 8th, 2022

California Institute of Technology

Motivation: Modeled Uncertainties



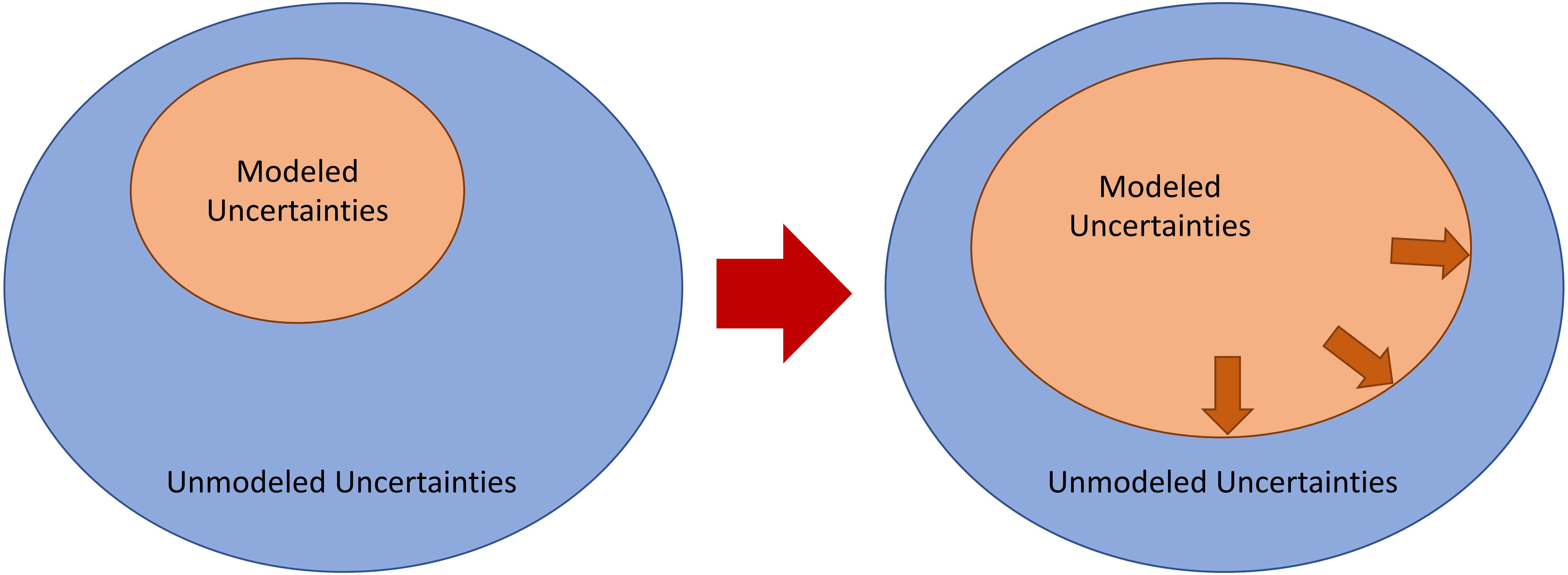
Control of Gaussian white noise systems:

- e.g., Linear Quadratic Gaussian (LQG) [Bernstein et.al. 1988, Doyle 1978], convex optimization-based approaches [Rawlik et.al. 2013, Nakka et.al. 2019], reinforcement learning [Deisenroth et.al. 2015]

Many robotic applications work well in practice with white noise models:

e.g., vision-based localization and mapping [Yang et.al. 2013], spacecraft navigation [Capuano et.al. 2019], motion-planning [Kalakrishnan et.al. 2011], Kalman filtering.

Motivation: Modeled Uncertainties → Learning + Control

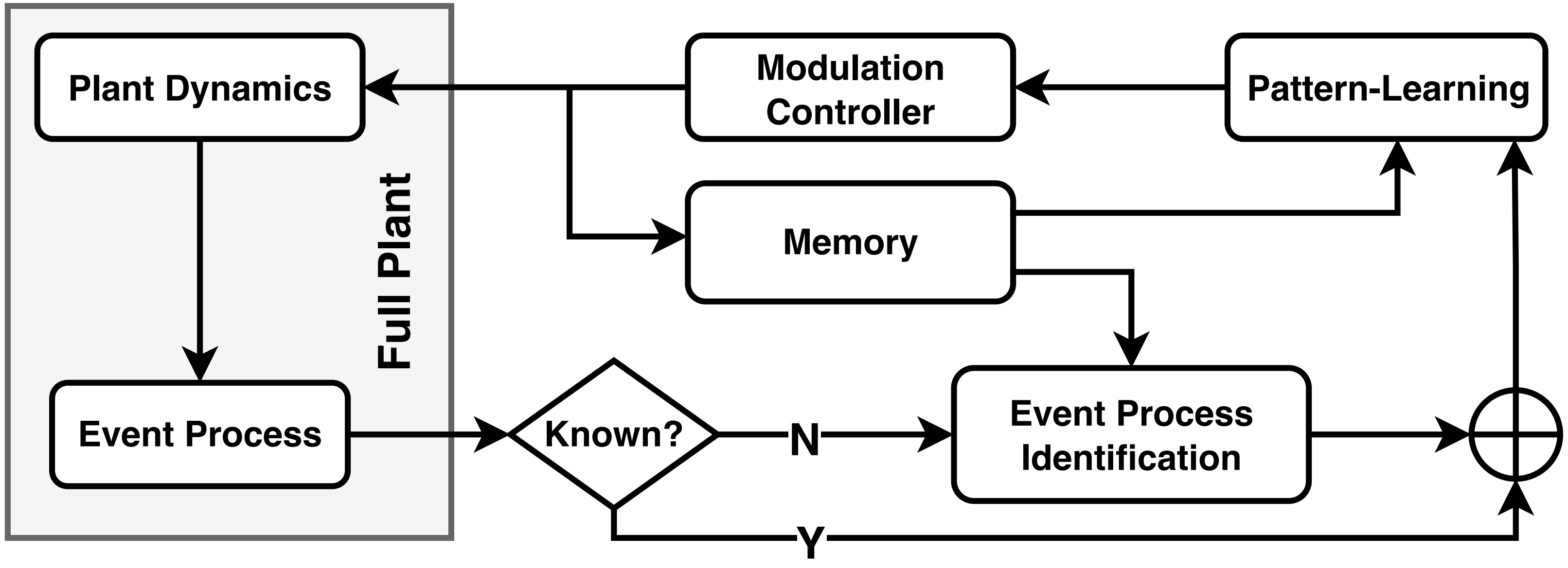


Model-based design before model-free: Expand model assumptions + consider different classes of randomness distributions before enhancing with machine learning.

Proposed Controller Framework

$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$
$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$
$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$
$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

$$\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$$



* SooJean Han and Soon-Jo Chung. *Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*, IEEE Control Systems Letters (L-CSS). Submitted 2022.

Proposed Controller Framework

$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$

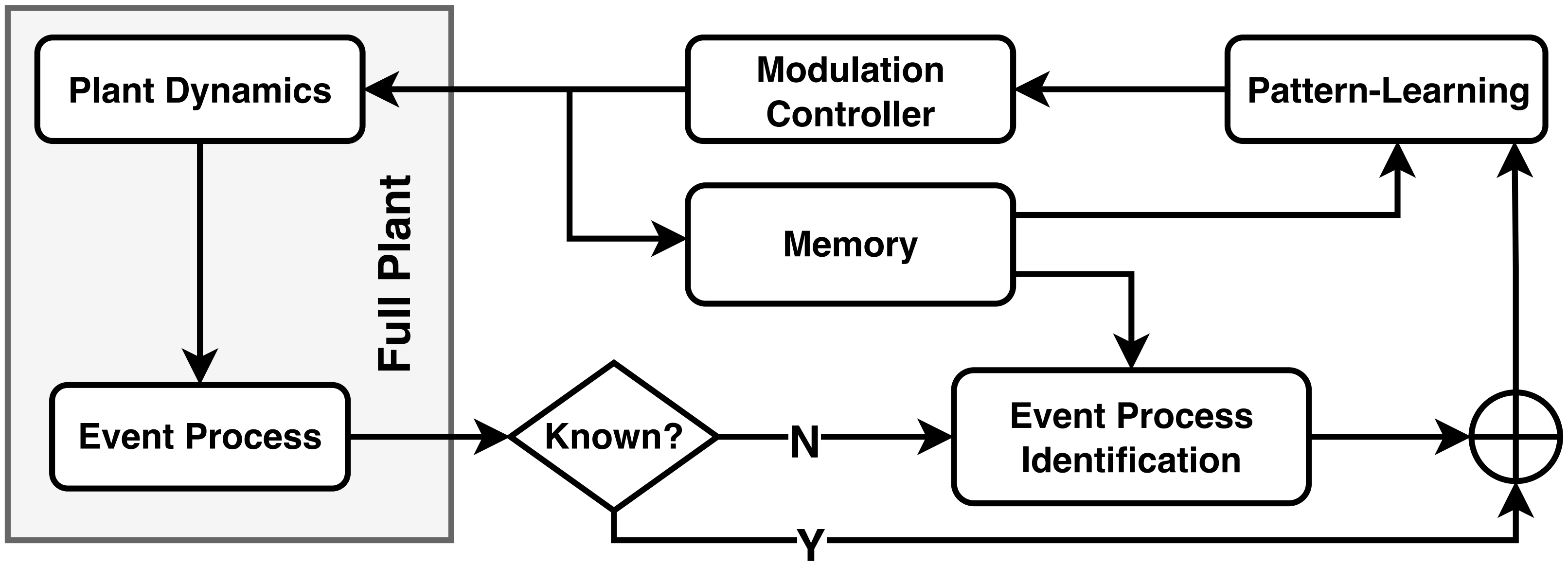
$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$

$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$

$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

"event-index"

$$\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$$



Assumption 1. Event process $\{\xi_n\}$ operates on a timescale $\Delta T \in \mathbb{N}$ times longer than the timescale of the system: if $N[t] = n$, then $N[t + a\Delta T] = n + a$ for any $a \in \mathbb{N}$. $\square$

* **SooJean Han and Soon-Jo Chung.** ***Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*****, IEEE Control Systems Letters (L-CSS). Submitted 2022.**

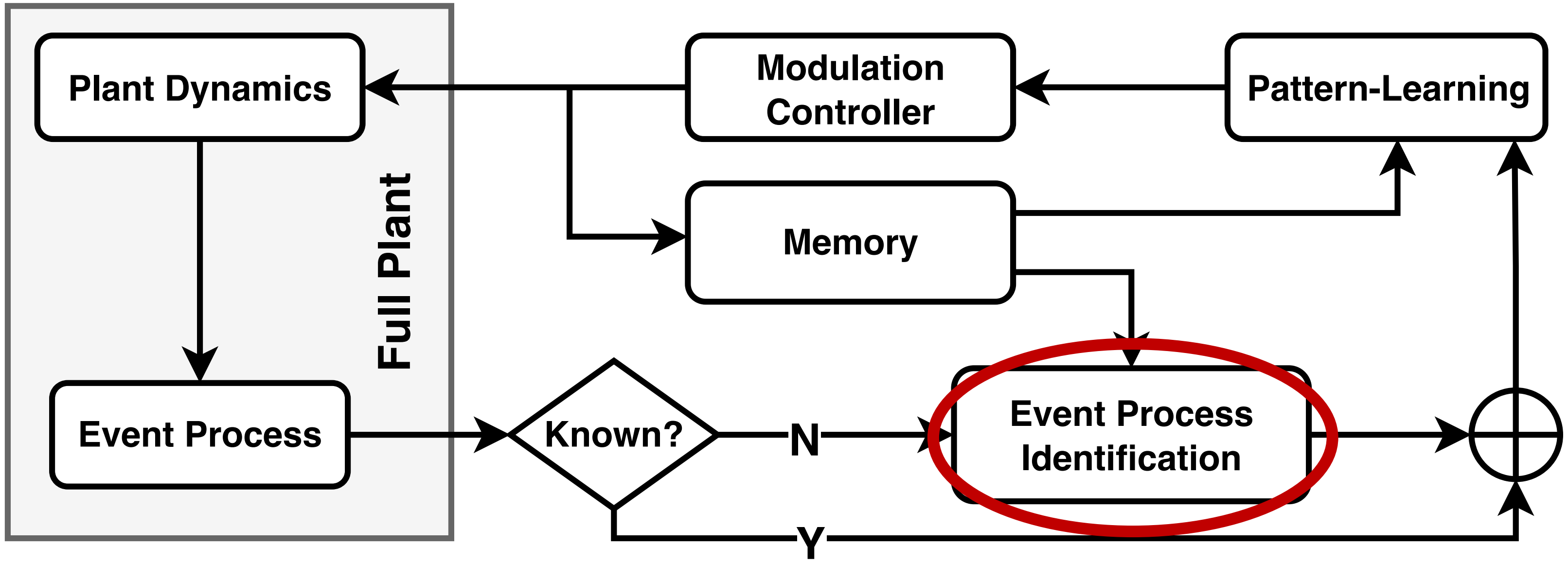
$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$
$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$
$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$
$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

Event Process Identification

$\xi_n^h \triangleq (\xi_n, \cdots, \xi_{n+h})$ $\quad$ P and $\varphi_{1:n}$ are unknown.

$\xi_n^{-h} \triangleq (\xi_{n-h}, \cdots, \xi_n)$ $\quad$ $\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$

$\xi_n^{n-1} = \varphi_{1:n}$

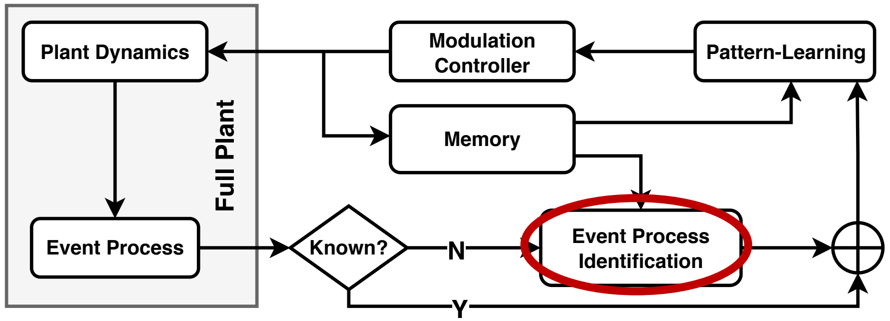


Event process identification for learning the unknown statistics of the event process

* SooJean Han and Soon-Jo Chung. *Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*, IEEE Control Systems Letters (L-CSS). Submitted 2022.

Event Process Identification

$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$
$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$
$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$
$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$



Goal: Use observations $\mathbf{x}[T_{N[t]} : t], \mathbf{u}[T_{N[t]} : t]$ to estimate:

- $\hat{P}^{(t)}$ of the true transition probability matrix P (e.g., Metropolis-Hastings, Gibbs' sampling, or Baum-Welch)
- $\hat{\varphi}_n^{(t)}$ of the true event φ_n (e.g., Viterbi's algorithm)

Pattern Learning

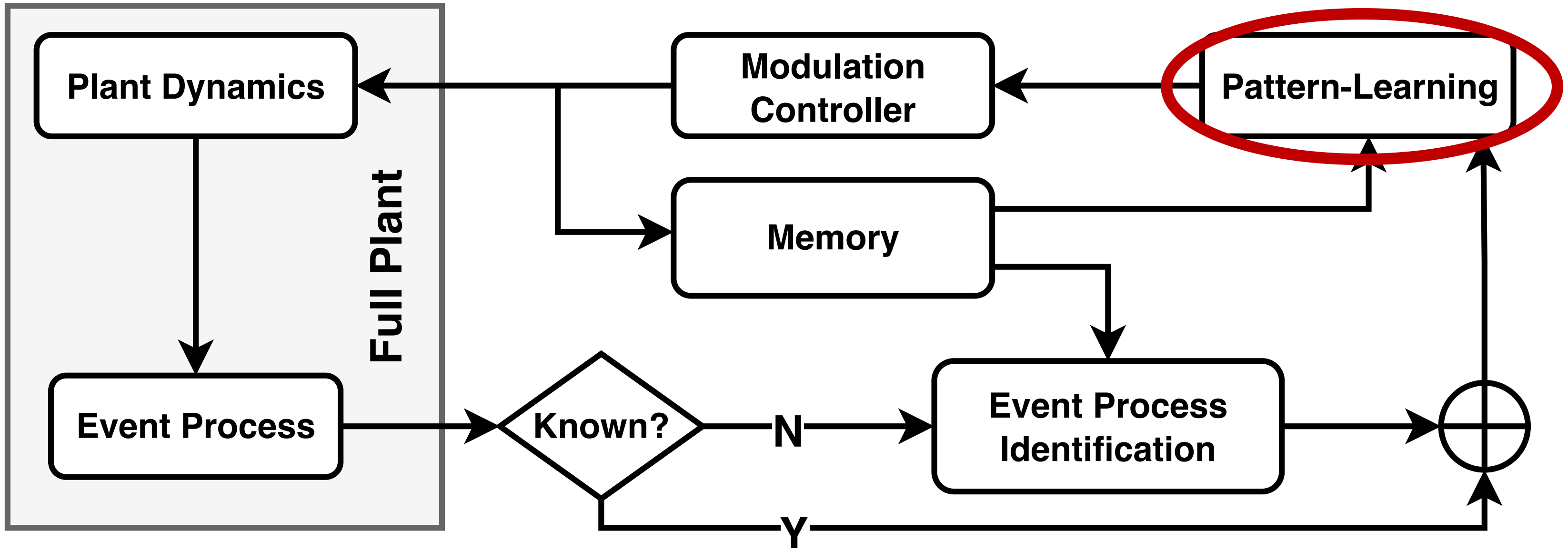
$$T_n \triangleq \min\{s \in \mathbb{N} \mid N[s] = n\}$$

$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$

$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$

$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

$$\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$$



Pattern-learning component to recognize previously-occurred patterns of events.

Pattern: a single event or a sequence of events over time which is "interesting to the user", e.g., faults in fault-tolerance control

*** SooJean Han and Soon-Jo Chung. *Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*, IEEE Control Systems Letters (L-CSS). Submitted 2022.**

Modulation Controller

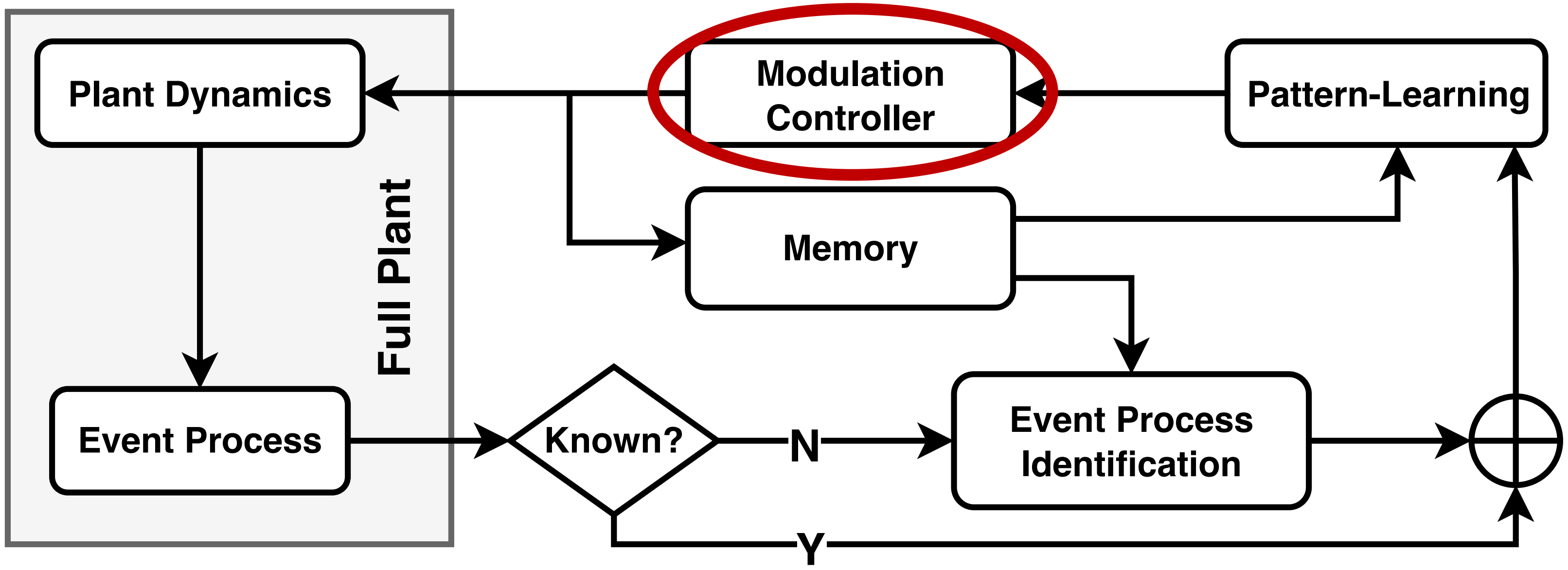
$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$

$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$

$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$

$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

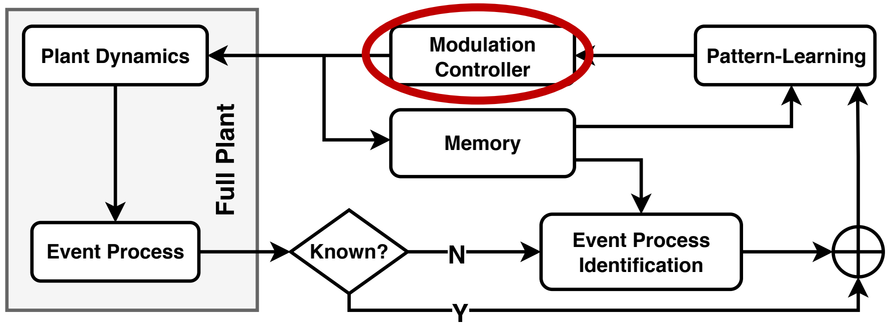
$$\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$$



Modulation control for control action design for a pattern when it is first recognized. Implementation varies by application.

* SooJean Han and Soon-Jo Chung. *Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*, IEEE Control Systems Letters (L-CSS). Submitted 2022.

Modulation Controller



$$\left(\mathbb{R}^{N_x}\right)^s \triangleq \mathbb{R}^{N_x} \times \cdots \times \mathbb{R}^{N_x}$$

$$\mathbf{u}[t] = U(\mathbf{x}[T_n : t], \hat{\varphi}_n), \quad U : \cup_{s=1}^{\Delta T} \left(\mathbb{R}^{N_x}\right)^s \times \mathbb{R}^{n_x} \to \mathbb{R}^{N_u}$$

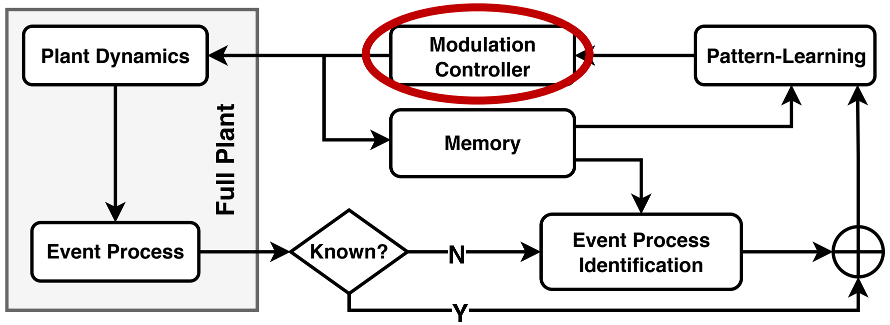
Anticipate future pattern of events $\boldsymbol{\xi}_{n+\tau}^{-r+1} = \boldsymbol{\psi} \triangleq (\psi_1, \cdots \psi_r)$

Create future control inputs $\mathbf{u}[T_{\overline{n}} : T_{n+\tau+1} - 1] = \mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}])$ $\quad \overline{n} \triangleq \max(n, n+\tau-r+1)$

$$\mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}) \triangleq [U_1(\mathbf{x}[T_{\overline{n}} : T_{\overline{n}+1} - 1]), \cdots, U_r(\mathbf{x}[T_{n+\tau} : T_{n+\tau+1} - 1])]$$

$$U_j(\mathbf{x}[T_{\overline{n}+j-1} : T_{\overline{n}+j} - 1]) \triangleq [U(\mathbf{x}[T_{\overline{n}+j-1}], \psi_j), \cdots, U(\mathbf{x}[\overline{n}+j-1 : T_{\overline{n}+j} - 1], \psi_j)]$$

Modulation Controller



$$\left(\mathbb{R}^{N_x}\right)^s \triangleq \mathbb{R}^{N_x} \times \cdots \times \mathbb{R}^{N_x}$$

$$\mathbf{u}[t] = U(\mathbf{x}[T_n : t], \hat{\varphi}_n), \quad U : \cup_{s=1}^{\Delta T} \left(\mathbb{R}^{N_x}\right)^s \times \mathbb{R}^{n_x} \to \mathbb{R}^{N_u}$$

Anticipate future pattern of events $\boldsymbol{\xi}_{n+\tau}^{-r+1} = \boldsymbol{\psi} \triangleq (\psi_1, \cdots \psi_r)$

Create future control inputs $\mathbf{u}[T_{\overline{n}} : T_{n+\tau+1} - 1] = \mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}])$ $\quad \overline{n} \triangleq \max(n, n+\tau-r+1)$

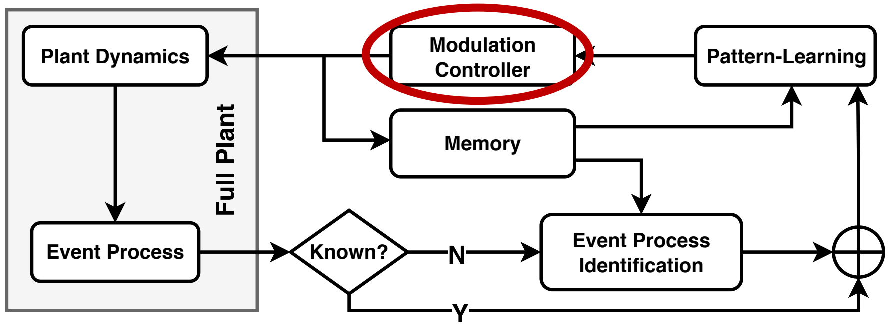
$$\mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}) \triangleq [U_1(\mathbf{x}[T_{\overline{n}} : T_{\overline{n}+1} - 1]), \cdots, U_r(\mathbf{x}[T_{n+\tau} : T_{n+\tau+1} - 1])]$$

$$U_j(\mathbf{x}[T_{\overline{n}+j-1} : T_{\overline{n}+j} - 1]) \triangleq [U(\mathbf{x}[T_{\overline{n}+j-1}], \psi_j), \cdots, U(\mathbf{x}[\overline{n}+j-1 : T_{\overline{n}+j} - 1], \psi_j)]$$

Update pattern-to-control law table $\mathcal{U}$, $\mathcal{U}[\boldsymbol{\psi}](\cdot) = \mathbf{U}(\cdot, \boldsymbol{\psi})$

For anticipated future occurrences of $\boldsymbol{\psi}$ over $T_{n'} : T_{n'+r-1} - 1$ for $n' > \overline{n}$, maintain future control inputs $\mathbf{u}[T_{n'} : T_{n'+r-1} - 1] = \mathcal{U}[\boldsymbol{\psi}](\mathbf{x}[T_{n'} : T_{n'+r-1} - 1)$ for future states $\mathbf{x}[T_{n'} : T_{n'+r-1} - 1]$.

Modulation Controller



$$\left(\mathbb{R}^{N_x}\right)^s \triangleq \mathbb{R}^{N_x} \times \cdots \times \mathbb{R}^{N_x}$$

$$\mathbf{u}[t] = U(\mathbf{x}[T_n : t], \hat{\varphi}_n), \quad U : \cup_{s=1}^{\Delta T} \left(\mathbb{R}^{N_x}\right)^s \times \mathbb{R}^{n_x} \to \mathbb{R}^{N_u}$$

Anticipate future pattern of events $\boldsymbol{\xi}_{n+\tau}^{-r+1} = \boldsymbol{\psi} \triangleq (\psi_1, \cdots \psi_r)$

Create future control inputs $\mathbf{u}[T_{\overline{n}} : T_{n+\tau+1} - 1] = \mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}])$ $\quad \overline{n} \triangleq \max(n, n + \tau - r + 1)$

$$\mathbf{U}(\mathbf{x}[T_{\overline{n}} : T_{n+\tau+1} - 1], \boldsymbol{\psi}) \triangleq [U_1(\mathbf{x}[T_{\overline{n}} : T_{\overline{n}+1} - 1]), \cdots, U_r(\mathbf{x}[T_{n+\tau} : T_{n+\tau+1} - 1])]$$

$$U_j(\mathbf{x}[T_{\overline{n}+j-1} : T_{\overline{n}+j} - 1]) \triangleq [U(\mathbf{x}[T_{\overline{n}+j-1}], \psi_j), \cdots, U(\mathbf{x}[\overline{n} + j - 1 : T_{\overline{n}+j} - 1], \psi_j)]$$

Update pattern-to-control law table $\mathcal{U}$, $\mathcal{U}[\boldsymbol{\psi}](\cdot) = \mathbf{U}(\cdot, \boldsymbol{\psi})$

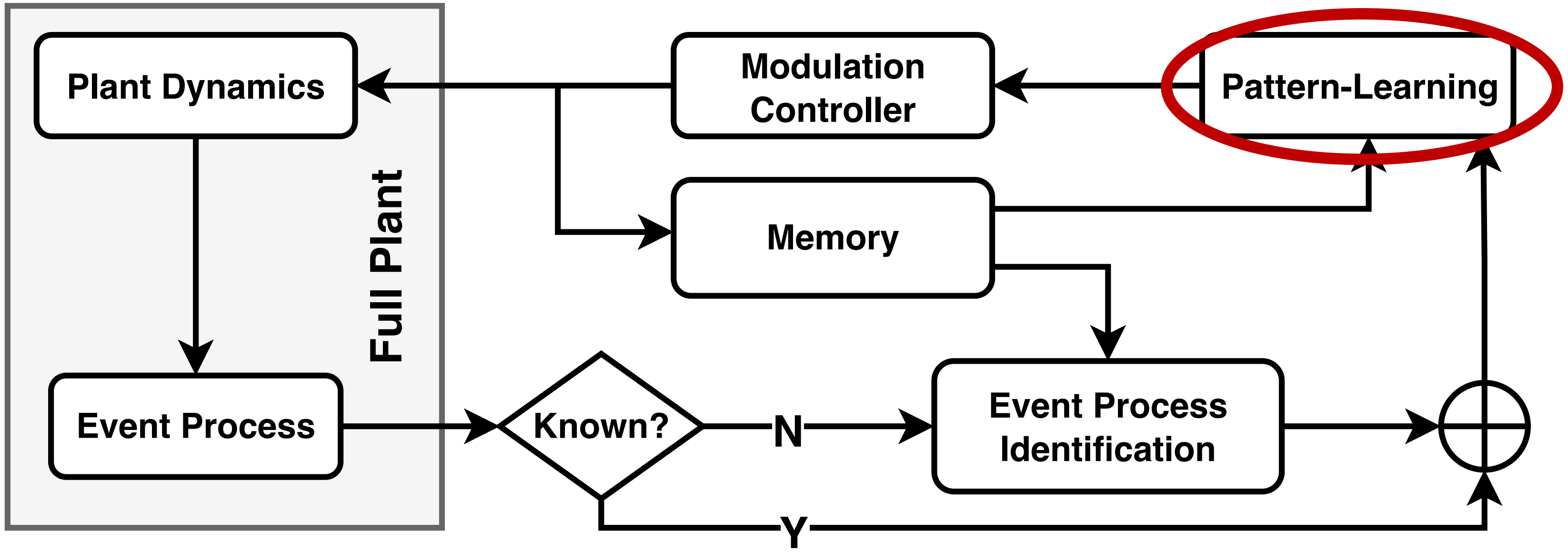
For anticipated future occurrences of $\boldsymbol{\psi}$ over $T_{n'} : T_{n'+r-1} - 1$ for $n' > \overline{n}$, maintain future control inputs $\mathbf{u}[T_{n'} : T_{n'+r-1} - 1] = \mathcal{U}[\boldsymbol{\psi}](\mathbf{x}[T_{n'} : T_{n'+r-1} - 1)$ for future states $\mathbf{x}[T_{n'} : T_{n'+r-1} - 1]$.

How far into the future do we have to anticipate the occurrence of a pattern?

Pattern Learning

$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$
$$\xi_n : \Omega \to \mathcal{X} \triangleq \{\zeta_1, \cdots, \zeta_M\}$$
$$P \triangleq [P(\zeta, \zeta')] \in \mathbb{R}^{M \times M}$$
$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

$$\mathbf{x}[t+1] = f(t, \mathbf{x}[t], \xi_{N[t]}) + g(t)\mathbf{u}[t] + \mathbf{w}[t]$$



Pattern-learning component to recognize previously-occurred patterns/sequences of states. → **Martingale Theory**

* **SooJean Han and Soon-Jo Chung. *Control of Stochastic Discrete-Event Systems via Martingale Analysis of Pattern Predictability*, IEEE Control Systems Letters (L-CSS). Submitted 2022.**

Pattern Learning

Definition 1 (Patterns)**.** *Collection of patterns* $\Psi \triangleq \{\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_K\}$, each $\boldsymbol{\psi}_k \triangleq (\psi_1^{(k)}, \cdots, \psi_{r_k}^{(k)})$ is *pattern* with length $r_k \in \mathbb{N}$ and elements $\psi_r^{(k)} \in \mathcal{X}$. $\square$

Pattern Learning

Definition 1 (Patterns)**.** *Collection of patterns* $\Psi \triangleq \{\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_K\}$, each $\boldsymbol{\psi}_k \triangleq (\psi_1^{(k)}, \cdots, \psi_{r_k}^{(k)})$ is *pattern* with length $r_k \in \mathbb{N}$ and elements $\psi_r^{(k)} \in \boldsymbol{\mathcal{X}}$. $\square$

Definition 2 (Pattern Occurrence Times)**.** Given current event-index $n^* \in \mathbb{N}$, with (estimated) past history $\boldsymbol{\xi}_{n^*}^{-h} = \varphi_{n^*-h:n^*}$ in event sequence $\{\xi_n\}$, define the following stopping times for any $h < \min(n^*, r_k)$:

$$\tau_{k|(n^*,h)} \triangleq \min\{n \in \mathbb{N} \,|\, \boldsymbol{\xi}_{n^*-h+n}^{-r_k+1} = \boldsymbol{\psi}_k,\ \boldsymbol{\xi}_{n^*}^{-h} = \varphi_{n^*-h:n^*}\}$$

Pattern Learning

Definition 1 (Patterns). *Collection of patterns* $\Psi \triangleq \{\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_K\}$, each $\boldsymbol{\psi}_k \triangleq (\psi_1^{(k)}, \cdots, \psi_{r_k}^{(k)})$ is *pattern* with length $r_k \in \mathbb{N}$ and elements $\psi_r^{(k)} \in \mathcal{X}$. $\square$

Definition 2 (Pattern Occurrence Times). Given current event-index $n^* \in \mathbb{N}$, with (estimated) past history $\boldsymbol{\xi}_{n^*}^{-h} = \varphi_{n^*-h:n^*}$ in event sequence $\{\xi_n\}$, define the following stopping times for any $h < \min(n^*, r_k)$:

$$\tau_{k|(n^*,h)} \triangleq \min\{n \in \mathbb{N} \mid \boldsymbol{\xi}_{n^*-h+n}^{-r_k+1} = \boldsymbol{\psi}_k,\ \boldsymbol{\xi}_{n^*}^{-h} = \varphi_{n^*-h:n^*}\}$$

Definition 3 (Time and Probability of First Occurrence). For the collection Ψ and any $h < \min(n^*, \min_k r_k)$:

$$\tau_{(n^*,h)} \triangleq \min_{k \in \{1,\cdots,K\}} \tau_{k|(n^*,h)},\ q_k \triangleq \mathbb{P}(\tau_{(n^*,h)} = \tau_{k|(n^*,h)})$$

Expected Time until Occurrence:
what is the expected time elapsed until the occurrence of patterns from the collection?

Probability of First Occurrence:
what is the probability that Pattern k is the first among all K patterns in the collection to occur?

Pattern Learning

Definition 1 (Patterns). *Collection of patterns* $\Psi \triangleq \{\psi_1, \cdots, \psi_K\}$, each $\psi_k \triangleq (\psi_1^{(k)}, \cdots, \psi_{r_k}^{(k)})$ is *pattern* with length $r_k \in \mathbb{N}$ □

Related works: Gerber & Li 1981, Glaz 2006, Pozdnyakov 2008

But not for learning-based stochastic control.

Assumed:

- Full knowledge of random process which generates the patterns
- No prior history of past events has been observed

Definition 2 (P h (estimated) past history $\xi_{n^*}^{-h} = \varphi_{n^*}$ ny $h < \min(n^*, r_k)$:

Definition 3 (Tim $< \min(n^*, \min_k r_k)$:

$$\tau_{(n^*,h)} \triangleq \min_{k \in \{1,\cdots,K\}} \tau_{k|(n^*,h)}, \quad q_k \triangleq \mathbb{P}(\tau_{(n^*,h)} = \tau_{k|(n^*,h)})$$

Expected Time until Occurrence:
what is the expected time elapsed until the occurrence of patterns from the collection?

Probability of First Occurrence:
what is the probability that Pattern k is the first among all K patterns in the collection to occur?

Pattern Learning

An *ending string* associated with pattern $\boldsymbol{\psi}_k \in \Psi$ occurs at event-index $\tau_{(n,H)}$ if $\boldsymbol{\xi}_{\tau_{(n,H)}}^{-r_k+1} = \boldsymbol{\psi}_k$.

Pattern Learning

For any $\boldsymbol{\psi}_{k_1}, \boldsymbol{\psi}_{k_2} \in \Psi$, $v_{k_2|k_1} \triangleq \max\{r < \min(r_{k_1}, r_{k_2}) \,|\, \psi^{(k_2)}_{1:r} = \psi^{(k_1)}_{r_{k_1}-r+1:r_{k_1}}\}$ is the *overlap*.

An *ending string* associated with pattern $\boldsymbol{\psi}_k \in \Psi$ occurs at event-index $\tau_{(n,H)}$ if $\boldsymbol{\xi}^{-r_k+1}_{\tau_{(n,H)}} = \boldsymbol{\psi}_k$.

overlap-ending string $\varphi_{n-H:n} \circ \psi^{(k)}_{v_k+1:r_k}$: if $\tau_{(n,H)} < n + r_k - 1$ and $\boldsymbol{\xi}^{r_k - v_k}_n = \psi^{(k)}_{v_k+1:r_k}$ with overlap $v_k \in \mathbb{N}$

$\mathcal{S}^{(-H)}_I, K^{(-H)}_I$

1 2 4 2 1 7 $\boxed{2\ 3\ 6\ 2\ 3}$ 5 7 8...

n

Pattern Learning

For any $\psi_{k_1}, \psi_{k_2} \in \Psi$, $v_{k_2|k_1} \triangleq \max\{r < \min(r_{k_1}, r_{k_2}) \,|\, \psi^{(k_2)}_{1:r} = \psi^{(k_1)}_{r_{k_1}-r+1:r_{k_1}}\}$ is the *overlap*.

$\boxed{\text{An } \textit{ending string} \text{ associated with pattern } \psi_k \in \Psi \text{ occurs at event-index } \tau_{(n,H)} \text{ if } \xi^{-r_k+1}_{\tau_{(n,H)}} = \psi_k.}$

overlap-ending string $\varphi_{n-H:n} \circ \psi^{(k)}_{v_k+1:r_k}$: if $\tau_{(n,H)} < n + r_k - 1$ and $\xi^{r_k - v_k}_n = \psi^{(k)}_{v_k+1:r_k}$ with overlap $v_k \in \mathbb{N}$

$\mathcal{S}^{(-H)}_I, K^{(-H)}_I$

initial-ending string β: • $\xi^{r_k-1}_n = \psi_k \implies \beta \triangleq \psi_k$ $\mathcal{S}^{(0)}_I, K^{(0)}_I$

1 2 4 2 1 7 2 3 $\boxed{2\ 3\ 6\ 2\ 3}$ 8…

n

Pattern Learning

For any $\boldsymbol{\psi}_{k_1}, \boldsymbol{\psi}_{k_2} \in \Psi$, $v_{k_2|k_1} \triangleq \max\{r < \min(r_{k_1}, r_{k_2}) \,|\, \psi^{(k_2)}_{1:r} = \psi^{(k_1)}_{r_{k_1}-r+1:r_{k_1}}\}$ is the *overlap*.

$\boxed{\text{An } \textit{ending string} \text{ associated with pattern } \boldsymbol{\psi}_k \in \Psi \text{ occurs at event-index } \tau_{(n,H)} \text{ if } \boldsymbol{\xi}^{-r_k+1}_{\tau_{(n,H)}} = \boldsymbol{\psi}_k.}$

overlap-ending string $\varphi_{n-H:n} \circ \psi^{(k)}_{v_k+1:r_k}$: if $\tau_{(n,H)} < n + r_k - 1$ and $\boldsymbol{\xi}^{r_k - v_k}_n = \psi^{(k)}_{v_k+1:r_k}$ with overlap $v_k \in \mathbb{N}$ $\mathcal{S}^{(-H)}_I, K^{(-H)}_I$

initial-ending string $\boldsymbol{\beta}$:
- $\boldsymbol{\xi}^{r_k-1}_n = \boldsymbol{\psi}_k \implies \boldsymbol{\beta} \triangleq \boldsymbol{\psi}_k$ $\mathcal{S}^{(0)}_I, K^{(0)}_I$
- $\boldsymbol{\xi}^{r_k}_n = (\zeta) \circ \boldsymbol{\psi}_k \implies \boldsymbol{\beta} \triangleq (\zeta) \circ \boldsymbol{\psi}_k$ $\mathcal{S}^{(+1)}_I, K^{(+1)}_I$

1 2 4 2 1 7 2 3 5 $\boxed{2\ 3\ 6\ 2\ 3}$ 8 …

n

Pattern Learning

For any $\psi_{k_1}, \psi_{k_2} \in \Psi$, $v_{k_2|k_1} \triangleq \max\{r < \min(r_{k_1}, r_{k_2}) \,|\, \psi^{(k_2)}_{1:r} = \psi^{(k_1)}_{r_{k_1}-r+1:r_{k_1}}\}$ is the *overlap*.

For each pattern $\psi_k \in \Psi$ define *augmented patterns* $(\zeta, \eta) \circ \psi_k \in \Gamma$ for $\zeta, \eta \in \mathcal{X}$. $|\Gamma| \triangleq K_L \in \mathbb{N}$

An *ending string* associated with pattern $\psi_k \in \Psi$ occurs at event-index $\tau_{(n,H)}$ if $\xi^{-r_k+1}_{\tau_{(n,H)}} = \psi_k$.

overlap-ending string $\varphi_{n-H:n} \circ \psi^{(k)}_{v_k+1:r_k}$: if $\tau_{(n,H)} < n + r_k - 1$ and $\xi^{r_k - v_k}_n = \psi^{(k)}_{v_k+1:r_k}$ with overlap $v_k \in \mathbb{N}$ $\mathcal{S}^{(-H)}_I, K^{(-H)}_I$

initial-ending string β:
- $\xi^{r_k-1}_n = \psi_k \implies \beta \triangleq \psi_k$ $\mathcal{S}^{(0)}_I, K^{(0)}_I$
- $\xi^{r_k}_n = (\zeta) \circ \psi_k \implies \beta \triangleq (\zeta) \circ \psi_k$ $\mathcal{S}^{(+1)}_I, K^{(+1)}_I$

later-ending string $(*, \zeta, \eta) \circ \psi_k$: $\tau_{(n,H)} > n + r_k$ and $\xi^{-r_k+1}_{\tau_{(n,H)}} = (\zeta, \eta) \circ \psi_k$ $\mathcal{S}_L \triangleq \{(*) \circ \gamma_\ell \,|\, \gamma_\ell \in \Gamma\}, K_L$

1 2 4 2 1 7 2 3 5 7… 4 $\boxed{2\ 3\ 6\ 2\ 3}$ 8 …

n

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*}\left(\left(1 - \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i)\right) + \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell)\right)}$$

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i \in \{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*} \left(\left(1 - \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \right) + \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell) \right)}$$

Gambler interpretation: Type-ℓ gambler bets on $\boldsymbol{\gamma}_\ell \triangleq (\zeta, \eta) \circ \boldsymbol{\psi}_k$, starting with initial reward $c_\ell \in \mathbb{R}$.

1. If $\xi_n = \zeta$, type-ℓ gambler n bets on the event $\{\xi_{n+1}, \cdots, \xi_{n+r_k+1}\} = (\eta) \circ \boldsymbol{\psi}_k$.
2. Otherwise, if $\xi_n \neq \zeta$, type-ℓ gambler n bets on the event $\{\xi_{n+1}, \cdots, \xi_{n+r_k}\} = \boldsymbol{\psi}_k$.

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i \in \{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*} \left(\left(1 - \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \right) + \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell) \right)}$$

Gambler interpretation: Type-ℓ gambler bets on $\boldsymbol{\gamma}_\ell \triangleq (\zeta, \eta) \circ \boldsymbol{\psi}_k$, starting with initial reward $c_\ell \in \mathbb{R}$.

1. If $\xi_n = \zeta$, type-ℓ gambler n bets on the event $\{\xi_{n+1}, \cdots, \xi_{n+r_k+1}\} = (\eta) \circ \boldsymbol{\psi}_k$.

2. Otherwise, if $\xi_n \neq \zeta$, type-ℓ gambler n bets on the event $\{\xi_{n+1}, \cdots, \xi_{n+r_k}\} = \boldsymbol{\psi}_k$.

Type-ℓ gambler n total reward by time $t \in \mathbb{N}$:

$$R_{n,t}^{(\ell)} = \mathbb{1}\{\xi_n = \eta\} \sum_{j=1}^{r_k+2} B_{n,j}^{(1,\ell)} \mathbb{1}\{n+j \le t\} + \mathbb{1}\{\xi_n \neq \eta\} \sum_{j=1}^{r_k+1} B_{n,j}^{(2,\ell)} \mathbb{1}\{n+j \le t\}$$

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i \in \{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*} \left(\left(1 - \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \right) + \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \gamma_\ell) \right)}$$

Gambler interpretation: Type-ℓ gambler n total reward by time $t \in \mathbb{N}$:

$$R_{n,t}^{(\ell)} = \mathbb{1}\{\xi_n = \eta\} \sum_{j=1}^{r_k+2} B_{n,j}^{(1,\ell)} \mathbb{1}\{n+j \leq t\} + \mathbb{1}\{\xi_n \neq \eta\} \sum_{j=1}^{r_k+1} B_{n,j}^{(2,\ell)} \mathbb{1}\{n+j \leq t\}$$

Betting strategies defined such that the game is fair-odds

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*}\left(\left(1 - \sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\right) + \sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\sum_{\ell=1}^{K_L} W_{i\ell}c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell)\right)}$$

Gambler interpretation: Type-ℓ gambler n total reward by time $t \in \mathbb{N}$:

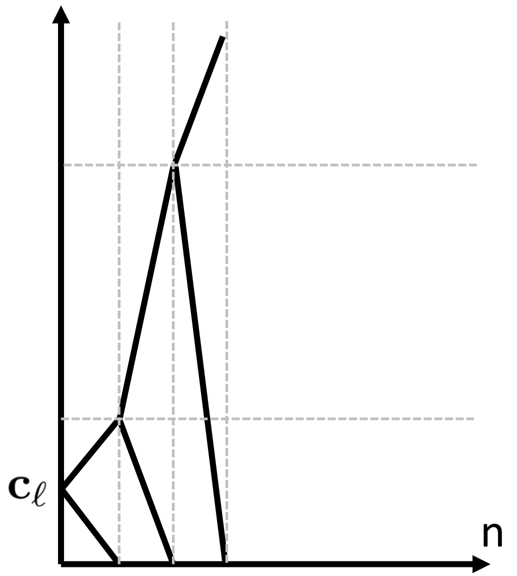
$$R_{n,t}^{(\ell)} = \mathbb{1}\{\xi_n = \eta\}\sum_{j=1}^{r_k+2} B_{n,j}^{(1,\ell)}\mathbb{1}\{n+j\le t\} + \mathbb{1}\{\xi_n \ne \eta\}\sum_{j=1}^{r_k+1} B_{n,j}^{(2,\ell)}\mathbb{1}\{n+j\le t\}$$

Betting strategies defined such that the game is fair-odds

$$B_{n,0}^{(1,\ell)} = B_{n,0}^{(2,\ell)} = c_\ell$$

$$B_{n,j}^{(1,\ell)} = \begin{cases} \left(P(\gamma_{\ell,j},\gamma_{\ell,j+1})\right)^{-1} B_{n,j-1}^{(1,\ell)} - B_{n,j-1}^{(1,\ell)} & \text{if } \xi_{n+j-1} = \gamma_{\ell,j}, \xi_{n+j} = \gamma_{\ell,j+1} \\ 0 & \text{if } \sum_{jj=0}^{j-1} B_{n,jj}^{(1,\ell)} = 0 \\ -B_{n,j-1}^{(1,\ell)} & \text{else} \end{cases}$$

$B_{n,j}^{(2,\ell)}$ similarly.



Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*} \left(\left(1 - \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \right) + \sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i) \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell) \right)$$

Gambler interpretation: Type-ℓ gambler n total reward by time $t \in \mathbb{N}$:

$$R_{n,t}^{(\ell)} = \mathbb{1}\{\xi_n = \eta\} \sum_{j=1}^{r_k+2} B_{n,j}^{(1,\ell)} \mathbb{1}\{n+j \leq t\} + \mathbb{1}\{\xi_n \neq \eta\} \sum_{j=1}^{r_k+1} B_{n,j}^{(2,\ell)} \mathbb{1}\{n+j \leq t\}$$

Cumulative reward over all type-ℓ gamblers: $C_\tau^{(\ell)} \triangleq \sum_{n=-H+1}^{\tau} R_{n,\tau}^{(\ell)} \implies C_h^{(\ell)} \triangleq \sum_{i=1}^{K_I+K_L} W_{i\ell} c_\ell, h \leq \tau$

Cumulative reward over all types of gamblers: $C_\tau \triangleq \sum_{\ell=1}^{K_L} C_\tau^{(\ell)} \implies C_h \triangleq \sum_{\ell=1}^{K_L} C_h^{(\ell)}, h \leq \tau.$

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*}\left(\left(1-\sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\right)+\sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\sum_{\ell=1}^{K_L}W_{i\ell}c_\ell^* - \sum_{\ell=1}^{K_L}c_\ell^* R(\varphi_{-H:0},\boldsymbol{\gamma}_\ell)\right)}$$

Gambler interpretation: Type-ℓ gambler n total reward by time $t \in \mathbb{N}$:

$$R_{n,t}^{(\ell)} = \mathbb{1}\{\xi_n = \eta\}\sum_{j=1}^{r_k+2} B_{n,j}^{(1,\ell)}\mathbb{1}\{n+j\le t\} + \mathbb{1}\{\xi_n \neq \eta\}\sum_{j=1}^{r_k+1} B_{n,j}^{(2,\ell)}\mathbb{1}\{n+j\le t\}$$

Cumulative reward over all type-ℓ gamblers: $C_\tau^{(\ell)} \triangleq \sum_{n=-H+1}^{\tau} R_{n,\tau}^{(\ell)} \implies C_h^{(\ell)} \triangleq \sum_{i=1}^{K_I+K_L} W_{i\ell}c_\ell, h \le \tau$

Cumulative reward over all types of gamblers: $C_\tau \triangleq \sum_{\ell=1}^{K_L} C_\tau^{(\ell)} \implies C_h \triangleq \sum_{\ell=1}^{K_L} C_h^{(\ell)}, h \le \tau.$

Current event-index $n^* \triangleq 0$ with past event-history $\varphi_{-H:0}$

$$\boldsymbol{\psi}_k \in \Psi \implies \tau_k \equiv \tau_{k|(0,H)}, \tau \equiv \tau_{(0,H)}$$

$$\boldsymbol{\gamma}_\ell \in \Gamma \triangleq (\zeta,\eta)\circ\boldsymbol{\psi}_k \implies \tau^a_{\ell|(n^*,h))}, \tau^a_{(n^*,h)}$$

$$\tau_{k|(n^*,h)} \triangleq \min\{n\in\mathbb{N} \mid \boldsymbol{\xi}_{n^*-h+n}^{-r_k+1} = \boldsymbol{\psi}_k,\ \boldsymbol{\xi}_{n^*}^{-h} = \varphi_{n^*-h:n^*}\}$$

$$\tau_{(n^*,h)} \triangleq \min_{k\in\{1,\cdots,K\}} \tau_{k|(n^*,h)}$$

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*}\left(\left(1 - \sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\right) + \sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\sum_{\ell=1}^{K_L} W_{i\ell}c_\ell^* - \sum_{\ell=1}^{K_L} c_\ell^* R(\varphi_{-H:0}, \gamma_\ell)\right)}$$

Cumulative reward over all type-ℓ gamblers: $C_\tau^{(\ell)} \triangleq \sum_{n=-H+1}^{\tau} R_{n,\tau}^{(\ell)} \implies C_h^{(\ell)} \triangleq \sum_{i=1}^{K_I+K_L} W_{i\ell}c_\ell, h \le \tau$

Cumulative reward over all types of gamblers: $C_\tau \triangleq \sum_{\ell=1}^{K_L} C_\tau^{(\ell)} \implies C_h \triangleq \sum_{\ell=1}^{K_L} C_h^{(\ell)}, h \le \tau.$

Proof: Apply Optional Stopping Theorem to martingale construction $\{C_{h\wedge\tau}\}_{h\in\mathbb{N}}$ and $\{C_{h\wedge\tau}^{(\ell)}\}_{h\in\mathbb{N}}$

Finite stopping time: $\mathbb{E}[\tau] < \infty \quad \tau \triangleq \min_k \tau_k, \quad \tau_k = \min_{\gamma_\ell\in\Gamma_k} \tau_\ell^a$

Finite-valued martingale:

$$C_{h\wedge\tau_\ell^a} = c_\ell R(\varphi_{-H:0} \circ \boldsymbol{\xi}_1^{h\wedge\tau_\ell^a}, \gamma_\ell) - c_\ell\,(h\wedge\tau_\ell^a + H - 1) \implies \mathbb{E}[C_{\tau_\ell^a}^\ell] < \infty \implies \mathbb{E}[C_\tau] < \infty$$

Uniform integrability (Doob's convergence theorem): $\lim_{h\to\infty}\int_{\Omega_h^{(\ell)}} |C_h^{(\ell)}(\omega)|d\mathbb{P}(\omega) = 0, \quad \Omega_h^{(\ell)} \triangleq \{\omega \in \Omega | h < \tau_\ell^a\}$

Question One: Expected Minimum Time

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i\in\{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*}\left(\left(1-\sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\right)+\sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\sum_{\ell=1}^{K_L}W_{i\ell}c_\ell^* - \sum_{\ell=1}^{K_L}c_\ell^* R(\varphi_{-H:0},\boldsymbol{\gamma}_\ell)\right)}$$

Proof ctd...

With conditions satisfied, we can now apply Optional Stopping Theorem (OST).

By martingale construction:

$$\mathbb{E}[C_\tau] = \sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\sum_{\ell=1}^{K_L}W_{i\ell}c_\ell^* + \left(1-\sum_{i=1}^{K_I}\mathbb{P}(\boldsymbol{\beta}_i)\right) - \sum_{\ell=1}^{K_L}c_\ell^*(\mathbb{E}[\tau]-H+1)$$

By OST:

$$\boxed{\mathbb{E}[C_\tau] = \mathbb{E}[C_0]} = \sum_{\ell=1}^{K_L}c_\ell^* R(\varphi_{-H:0},\boldsymbol{\gamma}_\ell) - (-H+1)\left(\sum_{\ell=1}^{K_L}c_\ell^*\right)$$

Question Two: First Occurrence Probabilities

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i \in \{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$q_k \triangleq \mathbb{P}(\tau = \tau_k)$$

$$\boxed{q_k = \sum_{\boldsymbol{\beta}_i \in \mathcal{S}} \mathbb{P}(\boldsymbol{\beta}_i) \mathbb{1}\{\beta_{d_i - r_k + 1 : d_i} = \boldsymbol{\psi}_k\} \quad \boldsymbol{\beta}_i \triangleq (\beta_1, \cdots, \beta_{d_i}) \in \mathcal{S}}$$

Question Two: First Occurrence Probabilities

$$\boldsymbol{\beta}_i \in \mathcal{S} \triangleq \bigcup_{i \in \{-H,0,1\}} \mathcal{S}_I^{(i)} \cup \mathcal{S}_L$$

$$\boxed{q_k = \sum_{\boldsymbol{\beta}_i \in \mathcal{S}} \mathbb{P}(\boldsymbol{\beta}_i) \mathbb{1}\{\beta_{d_i - r_k + 1:d_i} = \boldsymbol{\psi}_k\} \quad \boldsymbol{\beta}_i \triangleq (\beta_1, \cdots, \beta_{d_i}) \in \mathcal{S}}$$

Proof: Rearrange expression for expected time.

$$\sum_{\ell=1}^{K_L} c_\ell R(\varphi_{-H:0}, \boldsymbol{\gamma}_\ell) - \underbrace{\sum_{i=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_i)}_{\text{known}} \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell = \underbrace{\sum_{i=K_I+1}^{K_I+K_L} \mathbb{P}(\boldsymbol{\beta}_i)}_{\text{unknown}} \sum_{\ell=1}^{K_L} W_{i\ell} c_\ell - \sum_{\ell=1}^{K_L} c_\ell \mathbb{E}[\tau]$$

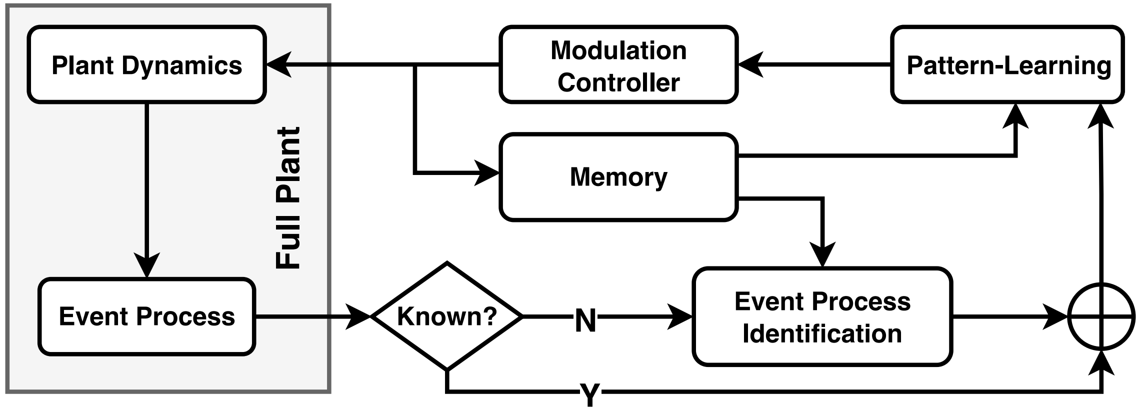
Solve for K_L unknowns by creating equations by choosing $\mathbf{c} \in \{[1, 0, \cdots, 0], \cdots, [0, 0, \cdots, 1]\}$

Use conditional probabilities: $q_k \triangleq \mathbb{P}(\tau = \tau_k) = \sum_{\boldsymbol{\beta}_i \in \mathcal{S}} \mathbb{P}(\boldsymbol{\beta}_i) \mathbb{P}(\tau = \tau_k \,|\, \boldsymbol{\beta}_i)$

Proposed Controller Framework

Algorithm 1 Two-Part Controller Framework

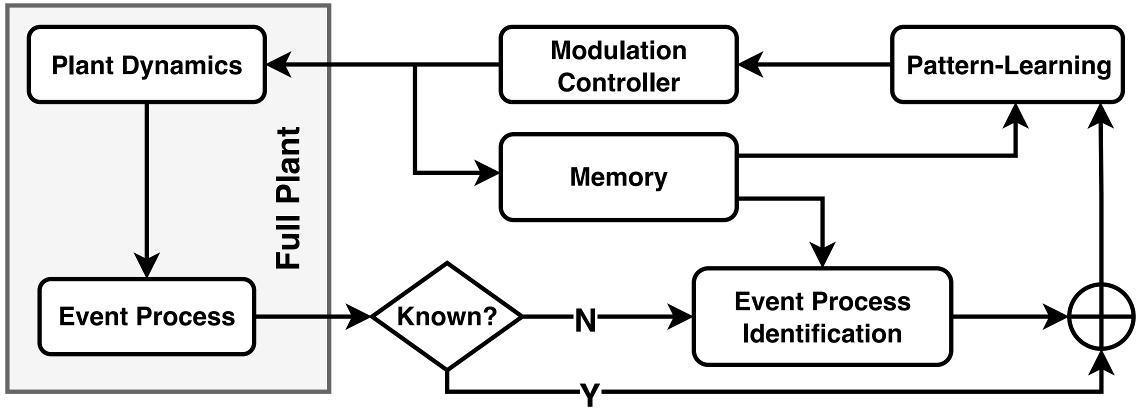
Initialize $\mathcal{C}_0 = \mathcal{X}$.
for $t = 1 : T_{\text{sim}}$ **do**
    Propagate true dynamics and event process one step.
    Update consistent set $\mathcal{C}_t$.
    Update TPM $\hat{P}_{N[t]}$ and sequence $\hat{\boldsymbol{\xi}}^{-H}_{N[t],t}$.
    for $\psi_k \in \Psi$ **do**
        if $\psi_k \notin \mathcal{U}$ **then**
            Synthesize $\mathcal{U}[\psi_k](\cdot) \leftarrow \mathbf{U}(\cdot, \psi_k)$.
        end if
    end for
    Update estimates $\hat{\tau}^{(t)}$ and $\{\hat{q}_k^{(t)}\}$.
    Apply pre-computed control sequences.
end for



Proposed Controller Framework

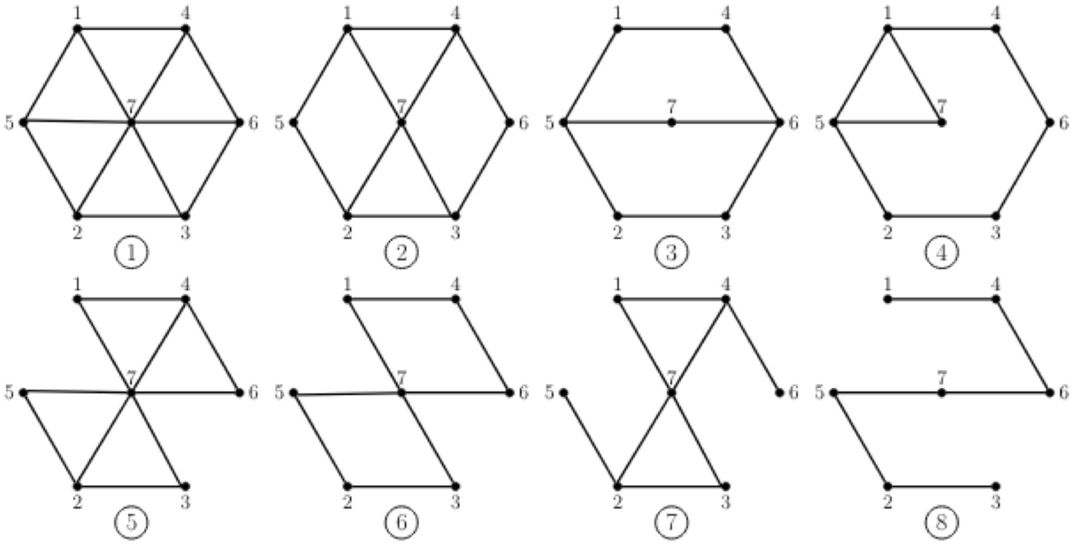
Algorithm 1 Two-Part Controller Framework

Initialize $\mathcal{C}_0 = \mathcal{X}$.
for $t = 1 : T_{\text{sim}}$ **do**
  Propagate true dynamics and event process one step.
  Update consistent set $\mathcal{C}_t$.
  Update TPM $\hat{P}_{N[t]}$ and sequence $\hat{\boldsymbol{\xi}}^{-H}_{N[t],t}$.
  for $\psi_k \in \Psi$ **do**
    if $\psi_k \notin \mathcal{U}$ **then**
      Synthesize $\mathcal{U}[\psi_k](\cdot) \leftarrow \mathbf{U}(\cdot, \psi_k)$.
    end if
  end for
  Update estimates $\hat{\tau}^{(t)}$ and $\{\hat{q}_k^{(t)}\}$.
  Apply pre-computed control sequences.
end for



$$\mathbf{u}[T_{N[t]+(\hat{\tau}^{(t)}-r_{k^*}+1)} : T_{N[t]+\hat{\tau}^{(t)}} - 1] = \mathcal{U}[\psi_{k^*}](\mathbf{x}[T_{N[t]+(\hat{\tau}^{(t)}-r_{k^*}+1)} : T_{N[t]+\hat{\tau}^{(t)}} - 1])$$

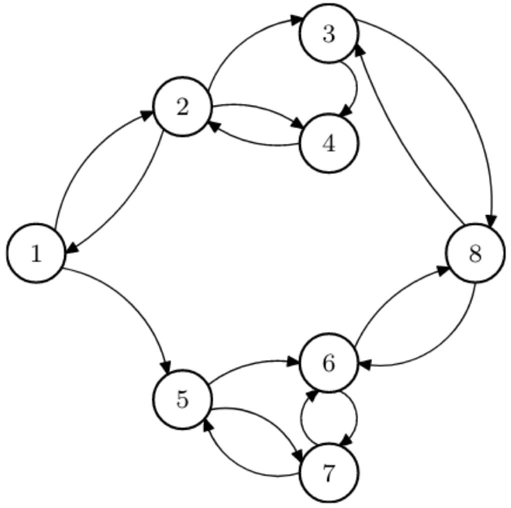
Application: Network Control



Pattern-Learning: identify reoccurring topologies of the local network to recycle closed-loop responses.

$$\mathcal{G}(m) \triangleq (\mathcal{V}, \mathcal{E}(m))$$
$$m \in \{1, \cdots, M\}$$

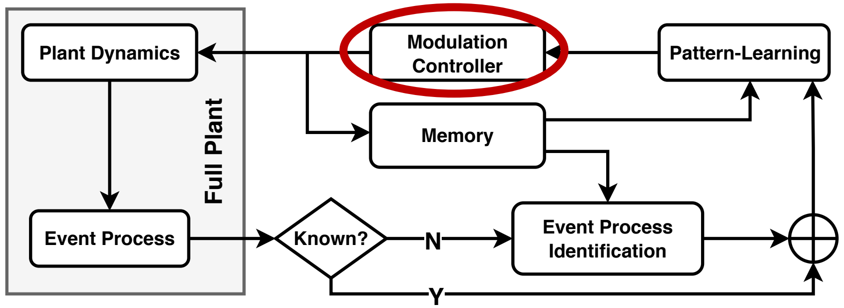
Modulation Controller: perform disturbance-rejection using iterative, localized, robust, and adaptive *system-level synthesis (SLS)*.



$$\mathbf{x}_i[t+1] = A_{ii}(\xi_{N[t]})\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i(\xi[N[t]])} A_{ij}(\xi[N[t]])\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t]$$

$$\mathcal{N}_i(m) \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}(m)\}$$

Application: Network Control



$$\mathbf{x}[0:t] = \mathbf{\Phi}_x \mathbf{w}[0:t]$$
$$\mathbf{u}[0:t] = \mathbf{\Phi}_u \mathbf{w}[0:t]$$

$$\Delta_s^j\left(A, B, \mathbf{\Phi}_{x,ji}^{(t)}, \mathbf{\Phi}_{u,ji}^{(t)}\right) \triangleq \mathbf{\Phi}_{x,ji}^{(t)}[s+1] - \sum_{\ell \in \mathcal{N}(j)} A_{j\ell}\mathbf{\Phi}_{x,\ell i}^{(t)}[s] - B_j \mathbf{\Phi}_{u,ji}^{(t)}[s]$$

Solve $\{\mathbf{\Phi}_{x,i}^{(t)}, \mathbf{\Phi}_{u,i}^{(t)}\}$ for each subsystem i:

$$\min \sum_{s=1}^{T} \left\| Q\mathbf{\Phi}_{x,i}^{(t)}[s] + R\mathbf{\Phi}_{u,i}^{(t)}[s] \right\|_1 + r_c \epsilon$$

$$\text{s.t.} \quad \left\| \sum_{j \in \mathcal{N}(i)} \Delta_s^j\left(A', B, \mathbf{\Phi}_{x,ji}^{(t)}, \mathbf{\Phi}_{u,ji}^{(t)}\right) \right\| \leq c_i \rho^{s-1} \forall s \leq T-1 \qquad \rho, \lambda_t^{(i)}, c_i, r_c, \epsilon > 0$$

$$A' = \hat{A}^i[t] + \sum_{\substack{\ell=1 \\ D_\ell \in \mathcal{A}^{(i)}}}^{|\mathcal{A}^{(i)}|} \xi_\ell D_\ell, \ \forall \boldsymbol{\xi} \in \mathcal{C}_t^{(i)}, \quad c_i \sum_{s=1}^{T} \rho^{s-1} \leq \lambda_t^{(i)} + \epsilon, \quad \epsilon > 0$$

*** SooJean Han. *Localized Learning of Robust Controllers for Networked Systems with Dynamic Topology*, Proceedings of the 2nd Conference on Learning for Dynamics and Control (L4DC), 2020.**
*** Dimitar Ho and John C. Doyle. *Scalable robust adaptive control from the system level perspective*, arXiv:1904.00077, 2019.**

Application: Network Control

Robust + scalable via localized computation.

Each node $i \in \{1, \cdots, N_s\}$:

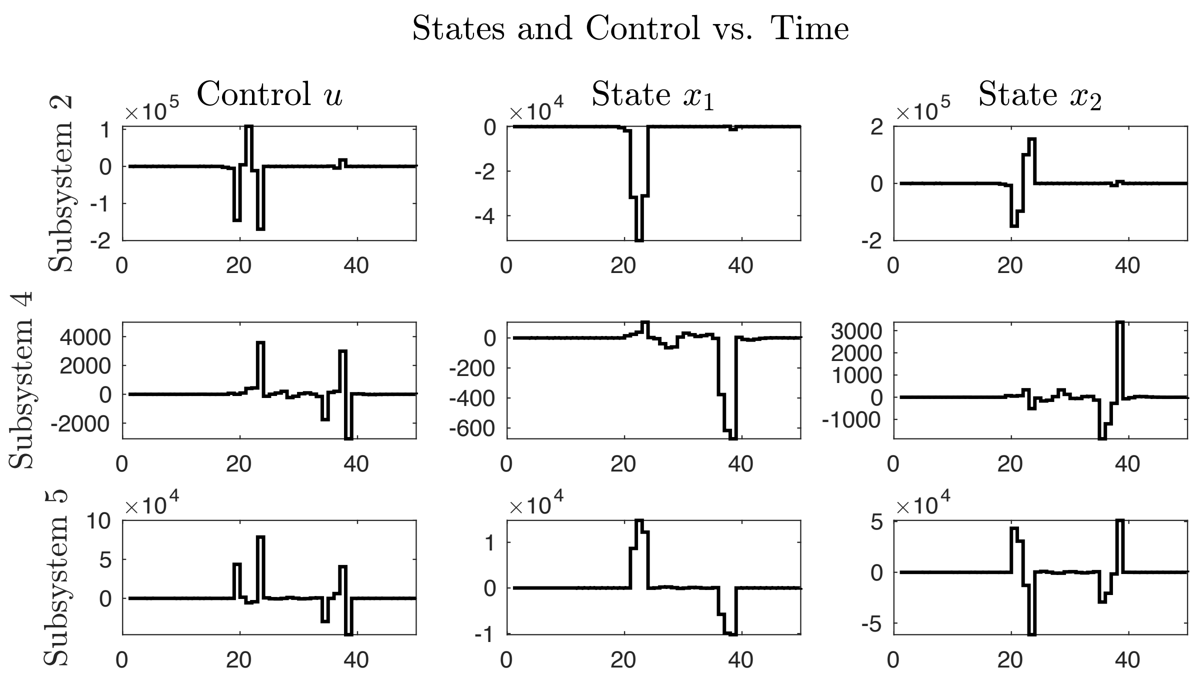
- pattern-prediction quantities $\hat{\tau}^{(i,t)}$, $\{\hat{q}_k^{(i,t)}\}_{k=1}^{K}$
- pattern-to-control law table $\mathcal{U}^{(i)}$

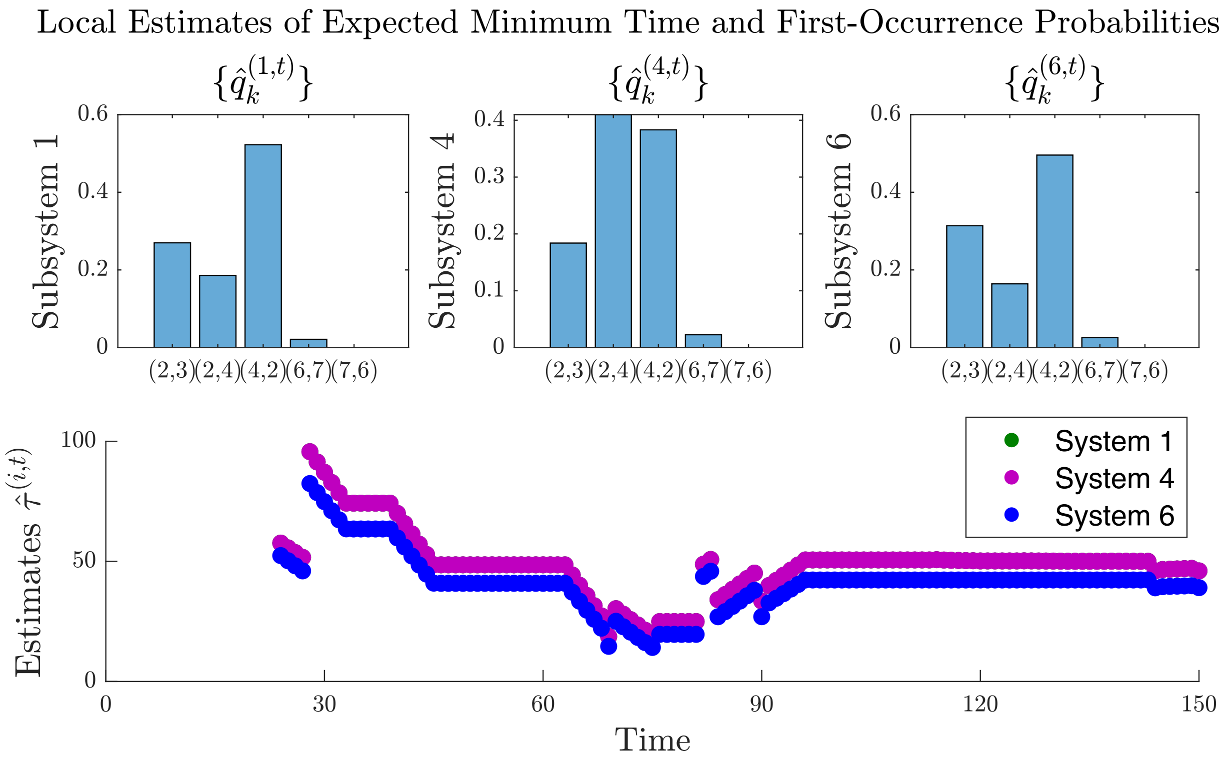
Application: Network Control

Robust + scalable via localized computation.

Each node $i \in \{1, \cdots, N_s\}$:

- pattern-prediction quantities $\hat{\tau}^{(i,t)}$, $\{\hat{q}_k^{(i,t)}\}_{k=1}^{K}$
- pattern-to-control law table $\mathcal{U}^{(i)}$



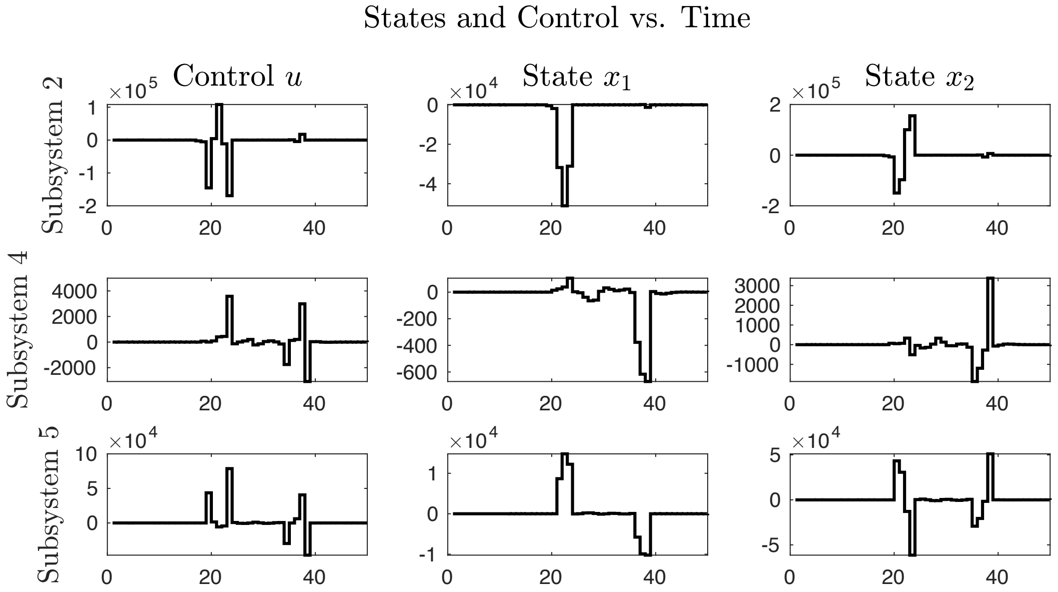


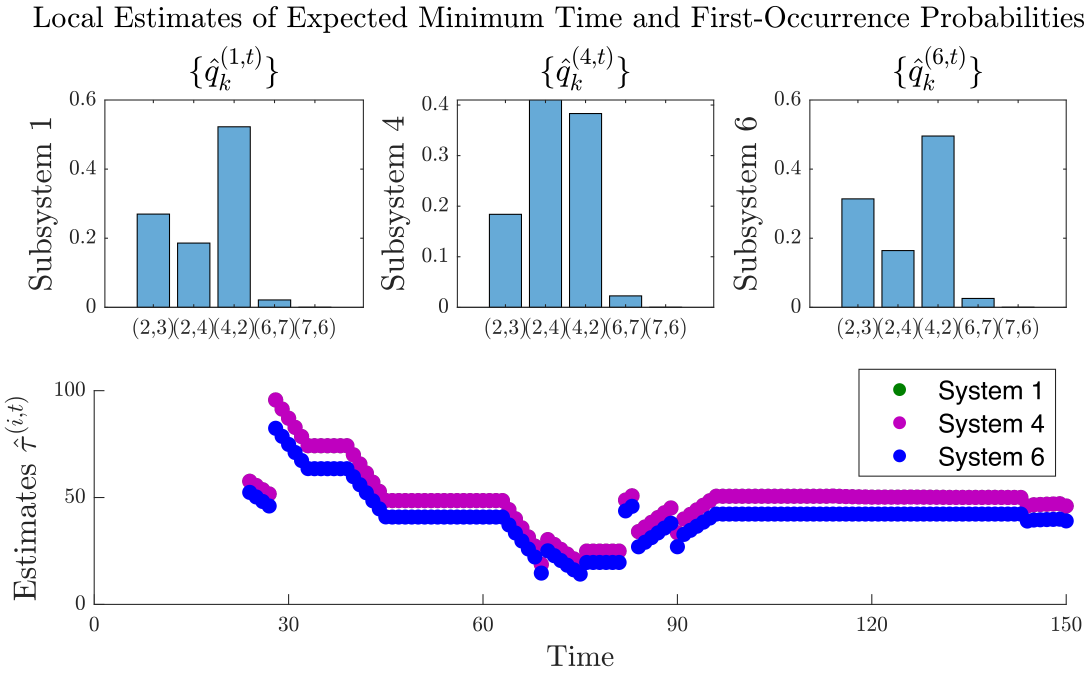
Application: Network Control

Robust + scalable via localized computation.

Each node $i \in \{1, \cdots, N_s\}$:

- pattern-prediction quantities $\hat{\tau}^{(i,t)}$, $\{\hat{q}_k^{(i,t)}\}_{k=1}^{K}$
- pattern-to-control law table $\mathcal{U}^{(i)}$





Future Work:

- More optimal event process identification → reduced impact from topology switches?
- More complex applications.