Optimal Control and Estimation Theory

Abstract

This manuscript is a collection of lecture notes on optimal control theory, filtering, and state estimation theory. Based on material from the course Ae103A/CDS112 as taught by Professor Soon-Jo Chung in Winter 2020 at the California Institute of Technology.

1 January 14th: Overview of Optimal Control

Given system dynamics

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

with *state* $\mathbf{x} : [t_0, t_f] \to\in \mathbb{R}^n$, *control input* $\mathbf{u} : [t_0, t_f] \to \mathcal{U} \subset \mathbb{R}^m$, *initial time* t_0, and *terminal time* t_f. For simplicity, we will often denote $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{u} \in \mathcal{U}$. Assume $\mathcal{U}$ is a compact subset of $\mathbb{R}^m$.

Assumptions:

1. system is *well-posed*: for all initial conditions $(t_0, \mathbf{x}_0)$ and every $\mathbf{u} \in \mathcal{U}$, there exists a unique solution trajectory $\mathbf{x}(t)$ on the time interval $[t_0, t_f]$ (fixed or free-endpoint).
2. f is *Lipschitz-continuous*: for all bounded sets $D \subset \mathbb{R} \times \mathbb{R}^n \times \mathcal{U}$, there exists a constant $C > 0$ such that $|f(t, \mathbf{x}_1, \mathbf{u}) - f(t, \mathbf{x}_2, \mathbf{u})| \leq C|\mathbf{x}_1 - \mathbf{x}_2|$ for all $(t, \mathbf{x}_1, \mathbf{u}), (t, \mathbf{x}_2, \mathbf{u}) \in D$.

Suppose we are also given the following *cost functional*.

$$J(\mathbf{u}) := \int_{t_0}^{t_f} \mathcal{L}\left(t, \mathbf{x}(t), \mathbf{u}(t)\right) dt + K(t_f, \mathbf{x}_f) \tag{2}$$

where $\mathcal{L}$ and K are called the *running cost* and *terminal cost*, respectively. Both the terminal time t_f and the *terminal state* $\mathbf{x}_f := \mathbf{x}(t_f)$ can be either free or fixed.

The optimal control problem is posed as follows: find the control input $\mathbf{u}^* \in \mathcal{U}$ which minimizes or maximizes the cost functional subject to the system dynamics (1) and the terminal conditions as constraints.

Introduction to Calculus of Variation:

$$f(x^* + \alpha d) = f(x^*) + \left[\frac{\partial f}{\partial x}\Big|_{x=x^*} d\right] \alpha + \mathcal{O}(\alpha)$$

$$\lim_{\alpha \to 0} \frac{\mathcal{O}(\alpha)}{\alpha} = 0$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} d \approx \Delta f = \text{Differential/Increments} = f(x^* + \alpha d) - f(x^*)$$

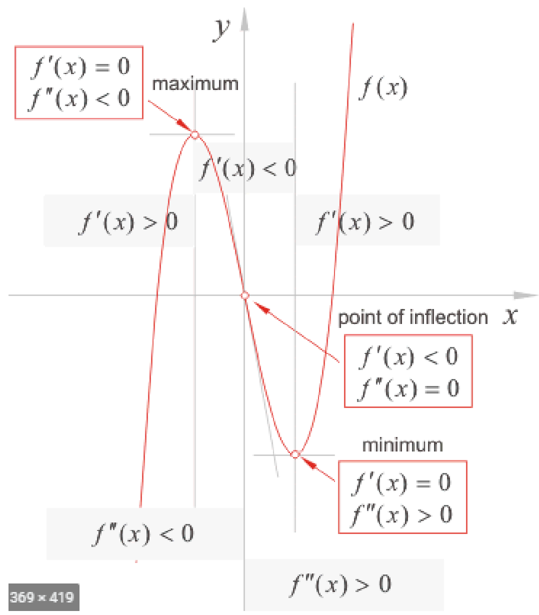


Figure 1: An example of an optimization problem for 1 dimensional system.

Variation:

$$\Delta J = \text{increment} = J(y^* + \alpha\eta) - J(y^*)$$

where $y = y(t)$ and J is a functional, i.e. function of functions.

$$J(y^* + \alpha\eta) = J(y^*) + \delta J\big|_{y^*}\alpha + \mathcal{O}(\alpha)$$

Optimal Control Theory:

$$\text{minimize } J = \min \textstyle\int_{t_0}^{t_f} g(y(t), t)dt$$

$$\delta J = \int_{t_0}^{t_f} \delta g(y(t), t)dt + g(y(t_f), t_f)\delta t_f + g(y(t_0), t_0)\delta t_0$$

Since t_0 is always given, $g(y(t_0), t_0)\delta t_0 = 0$. Aside, in Kirk notation: $y = y^* + \delta y$, which leads to $\delta g = \frac{\partial g}{\partial y}\delta y$.

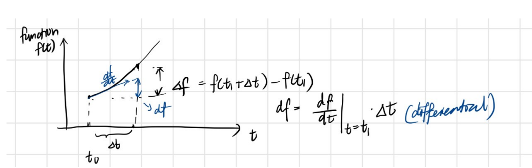


Figure 2: An pictorial representation of the idea of variation.

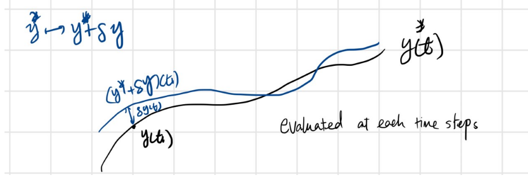


Figure 3: An pictorial representation of functional: cost function of functions $y(t)$.

Review of first order optimally:

$$\min f(x) \text{ where } x \in \mathbf{R}^n \text{ and } f : \mathbf{R}^n \to \mathbf{R}$$

Local optimal x^* is defined as: x^* is a local minimum if

$$\exists \epsilon > 0, \quad \|x^* - x\| < \epsilon \quad s.t. \quad f(x)7 \geq f(x^*)$$

In term of functional $J : V \to R$, $V = \mathcal{C}^1([a,b], \mathbf{R})$, $t \in [a,b] \to \mathbf{R}$ where $\mathcal{C}^1$ indicate finite-amount of discontinuities. The local optimal condition is:

$$\exists \epsilon > 0, \quad \|x^* - x\| < \epsilon \quad s.t. \quad J(y(x))7 \geq J(y(x^*))$$

Different Types of Norm:

$$\mathcal{L}_\infty : \text{0-norm} : \|x\|_0 = \max_{t \in [a,b]} |x(t)|$$

$$\text{1-norm} : \|x\|_1 = \max_{t \in [a,b]} |x(t)| + \max_{t \in [a,b]} |\dot{x}(t)|$$

$$\mathcal{L}_p : \text{p-norm} : \|x\|_p = (\int_a^b |x(t)|^p dt)^{1/p}$$

$$\text{Strong extremum} : \|x^* - x\|_0 < \epsilon \qquad \text{Weak extremum} : \|x^* - x\|_1 < \epsilon$$

Functional:

$$J(y(t) + \alpha\eta(t)) = g(\alpha) = g(y) + g'(0)\alpha + \mathcal{O}(\alpha) \text{ where } g'(0) = \delta J|_y(\eta)$$

Function:

$$f(x + \alpha d) = g(\alpha) = g(y) + g'(0)\alpha + \mathcal{O}\alpha$$

where $g'(0) = \bigtriangledown f(x)d$.
We want to prove $g'(0) = 0$
Proof by Contradiction: assume $g'(0) \neq 0$. Then, $g(\alpha) = g(0) + g'(0)\alpha + \mathcal{O}(\alpha)$. By rearranging, $g(\alpha) - g(0) < g'(0)\alpha + |\mathcal{O}(\alpha)|$. For sufficiently small ϵ, $\|\alpha| < \epsilon \to |g'(0)\alpha > |\mathcal{O}(\alpha)|$, choose α such that $g(\alpha) < g(0)$ reaches a contradiction!
Euler-Lagrange equation:
Derivation of Euler-Lagrange equation:

$$J(x(t)) = \int_{t_0}^{t_f} g(\dot{x}, x, t)dt$$

$$\delta J(x(t)) = \int_{t_0}^{t_f} [g_x \delta x + g_{\dot{x}} \delta \dot{x}] dt$$
$$L = K(q, \dot{q}) - V(q) \quad q \in \mathbf{R}^n$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$
$$J(x(t)) = \int_{t_0}^{t_f} g(\dot{x}, x, t) dt = \int_{t_0}^{t_f} [g_x] \delta x dt - \int_{t_0}^{t_f} \frac{d}{dt} [g_{\dot{x}}] \delta x dt$$

Thus we get the Euler-Lagrange equation

$$\frac{d}{dt} g_{\dot{x}} - g_x = 0$$

2 January 15th: Optimization Methods Review

Agenda

1. Finite dimensional optimization
   - sufficient & necessary conditions
   - geometric
2. Infinite Dimensional optimization
   - variation
   - function spaces/norms

Consider:

$$\dot{x} = f(t, x, u), \text{ where } x \in \mathbb{R}^n \text{ and } u \in \mathbb{U} \subset \mathbb{R}^m$$

.

And a cost function:

$$J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t)) \, dt + K(t_f, x_f)$$

Local minimum

$f : \mathbb{R}^n \to \mathbb{R}$, $D \subset \mathbb{R}^n$, 1-1 Euclidean norm, $x^* \in D$

There is a local minimum of f over D if:

$$\exists \, \varepsilon > 0 \text{ s.t. } \forall x \in D, |x - x| < \varepsilon$$

then $f(x^*) \leq f(x)$.

Unconstrained Optimality

First Order Necessary Condition for a local minimum:

$$\boxed{\nabla f(x^*) = 0}$$

(i.e. x^* stationary point)

$$(\nabla f(x^*))_j = \frac{\partial f(x^*)}{\partial x_j}$$

(i.e. x^* is a local min $\Rightarrow$ necessary condition, but x^* is a local min $\not\Leftarrow$ necessary condition)

Second Order Necessary Condition for Unconstrained Optimality:

$$\nabla^2 f(x^*) \geq 0 \text{ when } f \in \mathcal{C}^2$$

$$(\nabla^2 f(x^*))_{ij} = \frac{\partial^2}{\partial x_i \partial x_j} f(x^*)$$

$$\forall z \in \mathbb{R}^2, z^* A z \geq 0 \iff A \geq 0$$

Second Order Sufficient Condition:

sufficient condition $\Rightarrow x^*$ is a local minimum, but
sufficient condition $\not\Leftarrow x^*$ is a local minimum

$$\nabla f(x^*) = 0$$

$$\nabla^2 f(x^*) > 0 \text{ when } f \in \mathcal{C}^2$$

if $x^* \in \text{Int}(D)$, then x^* is a local minimum.

Weirstrauss Theorem:

If f continuous and if D is compact, then $\exists$ a global minimum of f over D.

compact $\Leftrightarrow$ closed & bounded

Constrained Optimization:

$$D = \{x : h_1(x) ... h_m(x) = 0\}$$

$$h_i = \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$$

Regular point: $\{\nabla h_i(x^*)\}$ are linearly independent.

$$
\begin{aligned}
\mathbb{R}^2 : f(x,y) &= x + y \\
h(x,y) &= x^2 + y^2 - 1
\end{aligned}
$$

$$
\begin{aligned}
\nabla f(x^*) &|| \nabla h(x^*) \\
\nabla f(x^*) + \lambda^* h(x^*) &= 0
\end{aligned}
$$

3 Janurary 20th: Euler-Lagrange Equations and the Pontryagin Maximum Principle

3.1 Euler-Lagrange Equations

For the n-dimensional case, the basic *Calculus of Variations (CoV) problem* is formulated as follows: we want to find an optimal trajectory $\mathbf{x}^* : [0,T] \to \mathbb{R}^n$ that minimizes the cost function

$$
J(\mathbf{x}) = \int_0^T \mathcal{L}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t)) dt
$$

among all $\mathcal{C}^1$ functions $\mathbf{x} : [0,T] \to \mathbb{R}^n$ which satisfy $\mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{x}_T$ (fixed-endpoint problem).

Theorem 1 (Euler-Lagrange Equations)**.** *The optimal trajectory* $\mathbf{x}^*(t)$ *satisfies the **Euler-Lagrange Equations**:*

$$
\frac{d}{dt} \nabla_v \mathcal{L}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) = \nabla_x \mathcal{L}(t, \mathbf{x}^*(t), \dot{\mathbf{x}}^*(t)) \tag{EL}
$$

Proof. All admissible perturbations $\mathbf{y} : [0,T] \to \mathbb{R}^n$ should satisfy the boundary conditions $\mathbf{y}(0) = \mathbf{y}(T) = 0$, i.e, deviations of trajectories away from the nominal $\mathbf{x}^*(t) + \alpha \mathbf{y}$ for $\alpha \in \mathbb{R}$ such that the original boundary conditions governing $\mathbf{x}^*$ are still satisfied.

Parameterize the cost J in terms of α: $c(\alpha) = J(\mathbf{x}^* + \alpha \mathbf{y})$, for $\alpha \in \mathbb{R}$. Since $\mathbf{x}^*$ (corresponding to $\alpha = 0$) is known to be a minimizer of this function, we have $c(\alpha) \geq c(0)$ for all $\alpha \in \mathbb{R}$ and $c'(0) = 0$. Substituting in the expression for c:

$$
\begin{aligned}
c(\alpha) &= \int_0^T \mathcal{L}(t, \mathbf{x}^*(t) + \alpha \mathbf{y}(t), \dot{\mathbf{x}}^*(t) + \alpha \dot{\mathbf{y}}(t)) dt \\
c'(\alpha) &= \int_0^T \left(\sum_{i=1}^n \mathcal{L}_{x_i}(t, \mathbf{x}^* + \alpha \mathbf{y}, \dot{\mathbf{x}}^* + \alpha \dot{\mathbf{y}}) y_i(t) + \sum_{i=1}^n \mathcal{L}_{v_i}(t, \mathbf{x}^* + \alpha \mathbf{y}, \dot{\mathbf{x}}^* + \alpha \dot{\mathbf{y}}) \dot{y}_i(t) \right) dt
\end{aligned}
$$

where $\mathcal{L}_{x_i}$ denotes the partial derivative of $\mathcal{L}$ taken with respect to x_i, likewise v_i.

Note that because the optimum occurs when $\alpha = 0$, we have that $c'(0) = 0$. Substituting this in yields:

$$
0 = c'(0) = \int_0^T \left(\sum_{i=1}^n \mathcal{L}_{x_i}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) y_i(t) + \sum_{i=1}^n \mathcal{L}_{v_i}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) \dot{y}_i(t) \right) dt \tag{3}
$$

For fixed component index $1 \leq j \leq n$, choose $\mathbf{y}$ such that

$$\mathbf{y}(t) = \begin{cases} 0 & \text{if } i \neq j \\ \eta(t) & \text{if } i = j \end{cases}$$

where η is an arbitrary function. Then (3) reduces to

$$0 = \int_0^T \left(\mathcal{L}_{x_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*)\eta(t) + \mathcal{L}_{v_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*)\dot{\eta}(t) \right) dt \tag{4}$$

Use integration-by-parts (IBP) to reduce the second term of the sum:

$$\int_0^T \mathcal{L}_{v_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*)\dot{\eta}(t) = \mathcal{L}_{v_j}\eta(t)\Big|_0^T - \int_0^T \eta(t) \cdot \frac{d}{dt}\mathcal{L}_{v_j} dt$$

Recall that $\eta(0) = \eta(T) = 0$. Substituting this back into (4), our final result is then

$$0 = \int_0^T \left(\mathcal{L}_{x_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) - \frac{d}{dt}\mathcal{L}_{v_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) \right) \eta(t) dt$$

and because $\eta(t)$ is arbitrarily chosen, this further implies that

$$\mathcal{L}_{x_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) - \frac{d}{dt}\mathcal{L}_{v_j}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) = 0$$

By varying index j, we get

$$\nabla_x \mathcal{L}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*) = \frac{d}{dt}\nabla_v \mathcal{L}(t, \mathbf{x}^*, \dot{\mathbf{x}}^*)$$

which is our desired result. □

For a free-endpoint problem, there's an additional constraint that needs to be satisfied:

$$\nabla_v \mathcal{L}(t_f, \mathbf{x}(t_f), \dot{\mathbf{x}}(t_f)) = 0$$

3.2 Pontryagin Maximum Principle

To motivate some new definitions, take the specific scalar case $n = 1$. Two special cases arise from the Lagrangian $\mathcal{L}(t, x, \dot{x})$.

1. **No dependence on state x**: $\mathcal{L} = \mathcal{L}(t, \dot{x})$. Then (EL) becomes

$$\frac{d}{dt}\mathcal{L}_v(t, \dot{x}) = 0 \Longrightarrow \mathcal{L}_v = c, \ c \in \mathbb{R}$$

   The quantity $p := \mathcal{L}_v$ is referred to as the *momentum*.

2. **No dependence on time** t: $\mathcal{L} = \mathcal{L}(x, \dot{x})$. Expanding out the LHS of (EL), and multiplying across by $\dot{x}$, we get:

$$\begin{aligned} 0 &= -\mathcal{L}_x(x, \dot{x}) + \mathcal{L}_{vx}(x, \dot{x}) \cdot \dot{x} + \mathcal{L}_{vv} \cdot \ddot{x} \\ \implies 0 &= -\mathcal{L}_x \dot{x} + \mathcal{L}_{vx} \dot{x}^2 + \mathcal{L}_{vv} \ddot{x} \dot{x} = \frac{d}{dt} (\mathcal{L}_v \dot{x} - \mathcal{L}) \\ \implies \mathcal{L}_v \dot{x} - \mathcal{L} &= c, \ c \in \mathbb{R} \end{aligned}$$

   The quantity $H(t, x, \dot{x}, p) := \mathcal{L}_v \dot{x} - \mathcal{L} = p \cdot \dot{x} - \mathcal{L}$ is referred to as the *Hamiltonian.*

In general ($n \geq 1$ dimensions), we can write the momentum as

$$\mathbf{p}(t) := \nabla_v \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{5}$$

and the Hamiltonian as

$$H(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) = \langle \mathbf{p}, \dot{\mathbf{x}} \rangle - \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) \tag{6}$$

Note further that $\mathcal{L}$ is now back to the general function of t and $\mathbf{x}$. Thus, the momentum and Hamiltonian need not be constant if $(\mathbf{x}, \mathbf{p})$ satisfy (EL), but all concepts stay the same.

Theorem 2 (Hamilton Canonical Equations)**.** *Let* $\mathbf{x} \in \mathbb{R}^n$ *solve the Euler-Lagrange equations (EL). Then* $(\mathbf{x}, \mathbf{p})$ *solves the Hamilton canonical equations:*

$$\dot{\mathbf{x}}(t) = \nabla_p H(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{p}(t)) \tag{7a}$$
$$\dot{\mathbf{p}}(t) = -\nabla_x H(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{p}(t)) \tag{7b}$$

The second equation (7b) is often called the adjoint equations or costate equations.

Proof. For the first equation, take the gradient across the definition of the Hamiltonian (6) with respect to $\mathbf{p}$:

$$\nabla_p H(t, \mathbf{x}, \dot{\mathbf{x}}, \mathbf{p}) = \dot{\mathbf{x}} - \underbrace{\nabla_p \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}})}_{=0}$$

For the second equation, we have

$$\begin{aligned} \dot{\mathbf{p}}(t) &= \frac{d}{dt} \nabla_v \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) \text{ by definition (5)} \\ &= \nabla_x \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) \text{ by (EL)} \\ &= -\nabla_x H(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{p}(t)) \text{ by taking gradient across (6) wrt } \mathbf{x} \end{aligned}$$

We have one additional condition if we take the gradient of (6) wrt $\dot{\mathbf{x}}$:

$$\nabla_v H(t, \mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{p}(t)) = \mathbf{p} - \nabla_v \mathcal{L}(t, \mathbf{x}, \dot{\mathbf{x}}) = 0 \text{ by definition (5)}$$

□

3.3 Optimal Control Problem and the Maximum Principle

Now suppose we want to solve an optimal control problem of the form described in Section 1 with cost functional

$$J(\mathbf{u}) := \int_{t_0}^{t_f} \mathcal{L}\left(\mathbf{x}(t), \mathbf{u}(t)\right) dt + K(t_f, \mathbf{x}_f)$$

We now modify definition (6) as the *dynamical systems Hamiltonian* for the context of optimal control problems:

$$H(\mathbf{x}, \mathbf{p}, \mathbf{u}) := \langle \mathbf{p}, f(\mathbf{x}, \mathbf{u}) \rangle - \mathcal{L}(\mathbf{x}, \mathbf{u}) \tag{8}$$

Theorem 3 (Maximum Principle for Basic Fixed-Endpoint Problem)**.** *Let $\mathbf{u}^* : [t_0, T] \to \mathcal{U}$ be an optimal control (i.e., $J(\mathbf{u}^*) \leq J(\mathbf{u})\, \forall \mathbf{u} \in \mathcal{U}$). Let $\mathbf{x}^* : [t_0, T] \to \mathbb{R}^n$ be the corresponding optimal state trajectory. Then there exists a function $\mathbf{p}^* : [t_0, T] \to \mathbb{R}^n$ such that:*

1. *$(\mathbf{x}^*, \mathbf{p}^*)$ satisfies (7) with boundary conditions $\mathbf{x}^*(t_0) = \mathbf{x}_0, \mathbf{x}^*(T) = \mathbf{x}_T$, and terminal condition $\mathbf{p}^*(T) = \nabla_x K(T, \mathbf{x}_T)$.*
2. *$H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*) = \max_{\mathbf{u} \in \mathcal{U}} H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u})$, and $H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*)$ is constant for $t \in [t_0, T]$. (Note: dynamical systems Hamiltonian is not an explicit function of t.)*

One can also consider what happens in the case where the endpoint is free. Define a *target set* $\mathcal{S} = [t_0, \infty) \times \mathcal{S}_1$ where $\{\mathbf{x} \in \mathbb{R}^n \mid h_1(\mathbf{x}) = \cdots h_{n-k}(\mathbf{x}) = 0\}$, $k \leq n$, and $h_i \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$.

Theorem 4 (Maximum Principle for Basic Free-Endpoint Problem)**.** *Let $\mathbf{u}^* : [t_0, T] \to \mathcal{U}$ be an optimal control. Let $\mathbf{x}^* : [t_0, T] \to \mathbb{R}^n$ be the corresponding optimal state trajectory. Then there exists a function $\mathbf{p}^* : [t_0, T] \to \mathbb{R}^n$ such that:*

1. *$(\mathbf{x}^*, \mathbf{p}^*)$ satisfies (7) with boundary conditions $\mathbf{x}^*(t_0) = \mathbf{x}_0, \mathbf{x}^*(t_f) \in \mathcal{S}_1$, and terminal condition $\mathbf{p}^*(T) = \nabla_x K(T, \mathbf{x}_T)$.*
2. *$H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*) = \max_{\mathbf{u} \in \mathcal{U}} H(t, \mathbf{x}^*, \mathbf{p}^*, \mathbf{u})$, and $H(\mathbf{x}^*, \mathbf{p}^*, \mathbf{u}^*)$ is constant for $t \in [t_0, t_f]$.*
3. *The vector $\mathbf{p}^*(t_f)$ is orthogonal to the tangent space to $\mathcal{S}_1$ at $\mathbf{x}^*(t_f)$:*

$$\langle \mathbf{p}^*(t_f), \mathbf{d} \rangle = 0, \;\; \forall \mathbf{d} \in T_{\mathbf{x}^*(t_f)} \mathcal{S}_1$$
$$T_{\mathbf{x}^*(t_f)} \mathcal{S}_1 := \{\mathbf{d} \in \mathbb{R}^n \mid \langle \nabla_x h_i(\mathbf{x}^*(t_f)), \mathbf{d} \rangle = 0, i = 1, \cdots, n-k\}$$

Alternatively stated: $\mathbf{p}^*(t_f)$ is a linear combination of the gradients $\nabla_x h_i(\mathbf{x}^*(t_f))$.

The last condition 4. is known as the *transversality condition.*

The difference between the two Maximum Principles is only in the boundary conditions for the Hamilton canonical equations (7).

January 20th: Alternative Version

Agenda:
1. Euler-Lagrange Equation: n-dimensional extension
2. Hamiltonian Canonical Equations
3. Optimal Control $\dot{x} = f(x, u)$, Maximally Principle
4. Examples

Basic Calculus of Variation (CoV):

$$\mathcal{C}^1 : x^* : [0, T] \to \mathbf{R}^n$$

that minimizes $J(x) = \int_0^T L(t, x(t), \dot{x}(t)$, s.t. $x(0) = x_0$, $x(T) = T$.
Euler-Lagrange:

$$\frac{d}{dt}(\nabla_v L(t, x^*, \dot{x}^*)) = \nabla_x L(t, x^*, \dot{x}^*))$$

Proof. $y : [0, T] \to \mathbb{R}^n$, s.t. $y(0) = y(T) = 0$.

$$x(t) = x^*(t) + \alpha y(t) \qquad \alpha \in \mathbb{R}$$

$$C(\alpha) = \int_0^T L(t, x^* + \alpha y, \dot{x}^* + \alpha \dot{y})dt$$

$$C'(0) = \int_0^T \sum_{i=1}^n L_{x_i}(t, x^*, \dot{x}^*)y_i + \sum_{i=1}^n L_{v_i}(t, x^*, \dot{x}^*)\dot{y}_i dt = 0$$

and $y(t) = 0$ if $i \neq j$ and $y(t) = \eta(t)$ if $i = j$ Apply integration be parts, we can

$$L_{x_j}(t, x^*, \dot{x}^*) = \frac{d}{dt} L_{v_i}(t, x^*, \dot{x}^*) \qquad \forall j = 1, \ldots \mathbb{R}$$

n = 1: two cases,
1. $L(t, \dot{x})$ no dependence on x.

$$\frac{d}{dt} L_v(t, \dot{x}) = 0 \quad \to \quad L_{v_i}(t, \dot{x}) = c \in \mathbb{R}$$

2. $L(x, \dot{x})$ no dependence on t.

$$\frac{d}{dt} L_v(x, \dot{x}) = 0 = L_{vx}(x, \dot{x})\dot{x} + L_{vv}(x, \dot{x})\ddot{x}$$

$$\frac{d}{dt}(L_v \dot{x} - L) = 0$$

Thus, the Hamiltonian $L_v \dot{x} - L = c \in \mathbb{R}$

In general:
Momentum:

$$P(t) = \nabla_v L(t, x, \dot{x}) \tag{9}$$

Hamiltonian:

$$H(t, x, \dot{x}, P) =< P, \dot{x} > -L(t, x, \dot{x}) \tag{10}$$

Combining those two we get the Hamiltonian canonical equations: $x \in \mathbb{R}^n$ satisfies Euler-Lagrange:

$$P^*(t) = \nabla_v L(t, x^*, \dot{x}^*)$$

Then (x^*, P^*) satisfy:

$$\dot{x}^*(t) = \nabla_p H(t, x, \dot{x}, P) \tag{11}$$

$$P^*(t) = -\nabla_x H(t, x, \dot{x}, P) \tag{12}$$

Proof.
From equation (2), we get:

$$\nabla_p H(t, x^*, \dot{x}^*, P^*) = \dot{x}^* - \nabla_p L(t, x^*, \dot{x}^*)$$

Note, $L(t, x^*, \dot{x}^*) = 0$,thus:

$$\nabla_p H(t, x^*, \dot{x}^*, P^*) = \dot{x}^*$$

Further, combining equation (1) and the Euler-Lagrange equation:

$$\dot{P}(t) = \frac{d}{dt}\nabla_v L(t, x^*, \dot{x}^*) = \nabla_x L(t, x^*, \dot{x}^*)$$

$$\nabla_x H(t, x^*, \dot{x}^*, P^*) = -\nabla_x L(t, x^*, \dot{x}^*)$$

Finally,

$$\nabla_v H(t, x^*, \dot{x}^*, P^*) = P - \nabla_x L(t, x^*, \dot{x}^*)$$

Optima control:

$$\dot{x} = f(x, u)$$

The dynamical system is autonomous (no dependent on t), with $x(t_0) = x_0$, $x \in \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$, and U is a compact set.
Assumptions: 1. Well Posted $\forall(t_0, x_0)$, $u \in U$, There exist a unique x, such that $\dot{x}(t) = f(x, u)$ on $t \in [t_0, t_f]$.
2. f is Lipschitz continuous.
Then,

$$J(x, u) = \int_{t_0}^{t_f} L(x(t), u(t))dt + K(t_f, x_f)$$

$$H(x, P, u) =< P, f(x, u) > -L(x, u)$$

1. Fixed-Endpoint Maximum Principle:
Let $u^* : [t_0, T] \to U$ be an optimal control and $\dot{x}^* = f(x^*, u^*)$, then there exist $P^* : [t_0, T] \to \mathbb{R}^n$ such that:
i. (x^*, p^*) satisfies Hamiltonian canonical equations with boundary $x^*(t_0) = x_0$ and $x^*(T) = x_T$ which x_T is fixed. The terminal conditions is: $P^*(T) = \nabla_x K(T, x_T)$.
ii. $H(x^*, P^*, u^*) = \max_{u \in U} H(x^*, P^*, u)$ and constant for $[t_0, t_f]$.

2. Free-Endpoint Maximum Principle:
Let $u^* : [t_0, T] \to U$ be an optimal control and $\dot{x}^* = f(x^*, u^*)$, then there exist $P^* : [t_0, t_f] \to \mathbb{R}^n$, where t_f is free, such that:

i. (x^*, p^*) satisfies Hamiltonian canonical equations with boundary $x^*(t_0) = x_0$ and $x^*(T) = S$. The terminal conditions is: $P^*(t_f) = \nabla_x K(t_f, x_f)$. S is the tangent set defined as:

$$S = \{x \in \mathbb{R}^n \big| h_1(x) = \cdots = h_{n-k}(x) = 0, \quad h_i(x) \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}, \quad k \leq n\}$$

ii. $H(x^*, P^*, u^*) = \max_{u \in U} H(x^*, P^*, u)$ and constant for $[t_0, t_f]$.
iii. $P^*(t_f) \perp T_{x^*(tf)} S$ which gives $< P^*(t_f), \nabla_x h_i(x) >= 0 \quad \forall i = 1, \ldots, k$. This condition is also known as the transversality condition.

Example 1. (Control of Production in Factory)
$x(t)$: Output at time t.
$u(t)$: Fraction of output reinvested at time t. Note $0 \leq u(t) \leq 1$.
Dynamics:

$$\dot{x}(t) = u(t)x(t)$$

where $x(0) = x_0$, and $t \in [0, T]$. To maximize total consumption until time T:

$$J(x, u) = \int_0^T (1 - u(t))x(t)dt$$

Hamiltonian:

$$H(x, P, u) = Pux - (1 - u)x$$

Hamiltonian canonical equations gives:

$$\dot{x} = ux$$

$$\dot{P}(t) = 1 - u(P + 1)$$

with terminal condition $P(T) = 0$.
Apply Maximum Principle we get:

$$H(x, P, u) = \max_{0 \leq u(t) \leq 1} \{Pux - (1 - u)x\} = -x + \max_{0 \leq u(t) \leq 1} \{ux(P + 1)\}$$

Thus, the optimal control is $u^*(t) = 1$ if $P(t) > 1$, and $u^*(t) = 0$ if $P(t) \leq 1$. Moreover,,

$$\dot{P}(t) = 1 - u(P + 1)$$

where $P(T) = 0$, $0 \leq t \leq T$. Combining the condition of $u^*(t)$ we get: $\dot{P}(t) = 1$, and $P(T) = 0$. Solving boundary value problem and get:

$$P(t) = t - T$$

This gives us the switching time for the control: Thus we get:

$$u^*(t) = 1 \text{ if } t \in [0, T - 1)$$

$$u^*(t) = 0 \text{ if } t \in [T - 1, T]$$

Example 2. (Distance between 2 sets)

$$\dot{x}(t) = u(t)$$

for $u \in D = \{u \in \mathbb{R}^2 u_1^2 + u_2^2 = 1\}$

$$J(x,u) = -tf$$

Hamiltonian:

$$H(x,P,u) =< P, u > -1$$

Hamiltonian canonical equations gives:

$$\dot{P}(t) = -\nabla_X H(x,P,u) = 0$$

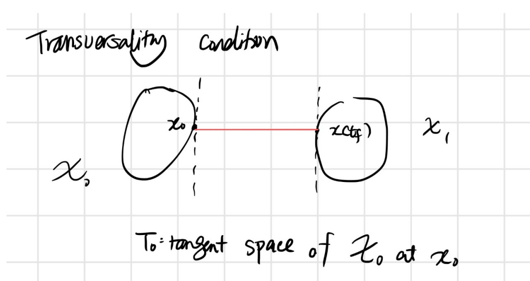
As a result, $P(t) = P^{(0)} \in \mathbb{R}^2$.
Apply Maximum Principle we get:

$$H(x,P,u^*) \max_{u(t)\in D} \{P_1u_1 + P_2u_2 - 1\}$$

Thus, $u^*(t) = \frac{1}{\|P^{(0)}\|}P^{(0)} = \dot{x}(t)$. Solving for $x(t)$, we get:

$$x(t) = \frac{1}{\|P^{(0)}\|}P^{(0)}t + c$$

The solution is a straight line illustrated with the figure below:



Geometric Interpretation of the transversailty condition:

$$P^*(t_f) \perp T_{x(t_f)}$$

P is constant leads to $P^*(t_0) = P^{(0)} \perp T_{x(t_f)}$.
We then conclude:

$$P^{(0)} \perp T_0$$

.

4 January 21st: Optimal Control Problem

4.1 COV applied to optimal control

Consider the following optimization problem:

$$\min_{\boldsymbol{u}(t)} J = \int_{t_o}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt + h(\boldsymbol{x}_f, t_f) \tag{13}$$

such that

$$\begin{aligned} \dot{\boldsymbol{x}} &= a(\boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{x}(t_o) &= \boldsymbol{x}_0 \\ \boldsymbol{x}(t_f) &= \boldsymbol{x}_f \end{aligned} \tag{14}$$

where, $\boldsymbol{x}_o$ and t_o are give and t_f and $\boldsymbol{x}_f$ both or either of them could be free or $\boldsymbol{x}_f$ can be some function of t_f i.e. $\boldsymbol{x}_f = \theta(t_f)$ which can be written as follows

$$\boldsymbol{m}(\boldsymbol{x}_f, t_f) = 0 \tag{15}$$

So the optimization is constrained by 14 and 15. Using Lagrange multipliers we obtain following augmented cost function

$$J_a = h(\boldsymbol{x}_f, t_f) + \boldsymbol{\nu}^T \boldsymbol{m}(\boldsymbol{x}_f, t_f) + \int_{t_o}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{p}^T(\boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t) - \dot{\boldsymbol{x}}(t))dt \tag{16}$$

where $\boldsymbol{\nu}$ and $\boldsymbol{p}$ are Lagrange multipliers. Say

$$H = g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + \boldsymbol{p}^T \boldsymbol{a}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{17}$$

and

$$w(t_f, \boldsymbol{x}_f) = h(\boldsymbol{x}_f, t_f) + \boldsymbol{\nu}^T m(\boldsymbol{x}_f, t_f) \tag{18}$$

where H is called Hamiltonian. Then 16 is written as

$$J_a = w(t_f, \boldsymbol{x}_f) + \int_{t_o}^{t_f} H - \boldsymbol{p}^T \dot{\boldsymbol{x}}(t)dt \tag{19}$$

Using the necessary condition for optimality we have (using Leibniz rule with the fact that t_f is a variable and t_o is constant given)

$$\delta J_a = \boldsymbol{w}_{\boldsymbol{x}_f} \delta \boldsymbol{x}_f + w_{t_f} \delta t_f + \int_{t_o}^{t_f} \boldsymbol{H}_{\boldsymbol{x}} \delta \boldsymbol{x} + \boldsymbol{H}_{\boldsymbol{u}} \delta \boldsymbol{u} + \boldsymbol{a}^T \delta \boldsymbol{p} - \dot{\boldsymbol{x}}^T \delta \boldsymbol{p} - \boldsymbol{p}^T \delta \dot{\boldsymbol{x}} dt + H(t_f)\delta t_f - \boldsymbol{p}(t_f)^T \dot{\boldsymbol{x}}(t_f) \delta t_f = 0 \tag{20}$$

Lets evaluate the term $-\int_{t_o}^{t_f} \boldsymbol{p}^T \delta \dot{\boldsymbol{x}} dt$ by using integration by parts

$$-\int_{t_o}^{t_f} \boldsymbol{p}^T \delta \dot{\boldsymbol{x}} dt = -[\boldsymbol{p}^T \delta \boldsymbol{x}]_{t_o}^{t_f} + \int_{t_o}^{t_f} \left(\frac{d}{dt}\boldsymbol{p}^T\right) \delta \boldsymbol{x} dt = -\boldsymbol{p}(t_f)^T \delta \boldsymbol{x}(t_f) + \int_{t_o}^{t_f} \dot{\boldsymbol{p}}^T \delta \boldsymbol{x} dt \tag{21}$$

The variation of $\boldsymbol{x}_f$ consists of two parts. First, the variation in $\boldsymbol{x}$ at time t_f i.e. $\delta \boldsymbol{x}(t_f)$ and second, the variation in t_f which gives variation in $\boldsymbol{x}_f$ by amount $\dot{\boldsymbol{x}}(t_f)\delta t_f$. Hence we have $\delta \boldsymbol{x}_f = \delta \boldsymbol{x}(t_f) + \dot{\boldsymbol{x}}(t_f)\delta tf$. From this we obtain $\delta \boldsymbol{x}(t_f) = \delta \boldsymbol{x}_f - \dot{\boldsymbol{x}}(t_f)\delta t_f$, putting this in 21 we obtain

$$-\int_{t_o}^{t_f} \boldsymbol{p}^T \delta \dot{\boldsymbol{x}} dt = -\boldsymbol{p}(t_f)^T \delta \boldsymbol{x}_f + \boldsymbol{p}(t_f)^T \dot{\boldsymbol{x}}(t_f) \delta t_f + \int_{t_o}^{t_f} \dot{\boldsymbol{p}}^T \delta \boldsymbol{x} dt \tag{22}$$

Note that $\delta x(t_o) = 0$. Putting 22 in 20 we get

$$\begin{aligned}\boldsymbol{w}_{\boldsymbol{x}_f}\delta\boldsymbol{x}_f + w_{t_f}\delta t_f + \int_{t_o}^{t_f} \boldsymbol{H_x}\delta\boldsymbol{x} + \boldsymbol{H_u}\delta\boldsymbol{u} + \boldsymbol{a}^T\delta\boldsymbol{p} - \dot{\boldsymbol{x}}^T\delta\boldsymbol{p}dt \\ -\boldsymbol{p}(t_f)^T\delta\boldsymbol{x}_f + \boldsymbol{p}(t_f)^T\dot{\boldsymbol{x}}(t_f)\delta t_f + \int_{t_o}^{t_f} \dot{\boldsymbol{p}}^T\delta\boldsymbol{x}dt + H(t_f)\delta t_f - \boldsymbol{p}(t_f)^T\dot{\boldsymbol{x}}(t_f)\delta t_f = 0\end{aligned} \tag{23}$$

After simplification and collecting terms consist of δx_f, δt_f, δx, δu, and δp, we obtain

$$(\boldsymbol{w}_{\boldsymbol{x}_f} - \boldsymbol{p}(t_f)^T)\delta\boldsymbol{x}_f + (w_{t_f} + H(t_f))\delta t_f + \int_{t_o}^{t_f} (\boldsymbol{H_x} + \dot{\boldsymbol{p}}^T)\delta\boldsymbol{x} + (\boldsymbol{a}^T - \dot{\boldsymbol{x}}^T)\delta\boldsymbol{p} + \boldsymbol{H_u}\delta\boldsymbol{u}dt = 0 \tag{24}$$

Since 24 has to be true for all values of δx_f, δt_f, δx, δu, and δp, following necessary conditions are obtained

$$\dot{\boldsymbol{x}} = \boldsymbol{a} \tag{25}$$

$$\dot{\boldsymbol{p}} = -H_x^T \tag{26}$$

$$\boldsymbol{H_u} = 0 \tag{27}$$

$$w_{t_f} + H(t_f) = 0 \tag{28}$$

$$\boldsymbol{w_{x_f}} - \boldsymbol{p}(t_f)^T = 0 \tag{29}$$

If t_f is free then 28 should be satisfied otherwise it is required. If $\boldsymbol{x}_f$ is free then $\boldsymbol{x}_f$ is free then 29 is required else not.

The original optimization problem 13 with constrains 14 and 15 has been simplified to following boundary value problem.

$$\begin{bmatrix}\dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}}\end{bmatrix} = \begin{bmatrix}\boldsymbol{a} \\ -\boldsymbol{H_x}^T\end{bmatrix} \tag{30}$$

with

$$\begin{aligned}\boldsymbol{x}(t_o) &= \boldsymbol{x}_o \\ \boldsymbol{p}(t_f) &= \boldsymbol{w_{x_f}}^T\end{aligned} \tag{31}$$

4.1.1 Example

Problem statement
Let's look at one example

$$\begin{aligned}\min_u J = 0.5\alpha t_f^2 + 0.5\int_0^{t_f} bu^2dt \\ y(0) = 10; \\ \dot{y}(0) = 0; \\ y(t_f) = 0; \\ \dot{y}(t_f) = 0\end{aligned} \tag{32}$$

Where t_f is free. Following are the state space dynamics:

$$\begin{aligned} \boldsymbol{x} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \end{aligned} \tag{33}$$

where

$$\begin{aligned} \boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned} \tag{34}$$

Solution

The Hamiltonian is given by

$$H = 0.5bu^2 + \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^T \begin{bmatrix} 0 \\ u \end{bmatrix} = 0.5bu^2 + p_1 x_2 + p_2 u \tag{35}$$

by 26 we obtain

$$\begin{aligned} \dot{p}_1 = 0 \\ \dot{p}_2 = -p_1 \end{aligned} \tag{36}$$

This gives following solution for p:

$$\begin{aligned} p_1 = c_1 \\ p_2 = -c_1 t + c_2 \end{aligned} \tag{37}$$

And by 27 we obtain $H_u = bu + p_2 = 0$ this gives by 37

$$u = -p_2/b = \frac{c_1 t - c_2}{b} \tag{38}$$

Using the condition 25 we obtain:

$$\begin{aligned} \dot{x_2} = u = (c_1 t - c_2)/b \\ \dot{x_1} = x_2 \end{aligned} \tag{39}$$

This gives the solution for x_2 and x_1 which are as follows:

$$\begin{aligned} x_2(t) = \frac{c_1 t^2}{2b} - \frac{c_2 t}{b} + c3 = \dot{y} \\ x_1(t) = \frac{c_1 t^3}{6b} - \frac{c_2 t^2}{2b} + c_3 t + c_4 = y \end{aligned} \tag{40}$$

using the initial value condition given in 32, obtain $c_3 = 0$ and $c_4 = 10$. Using the final value condition given in 32 we obtain:

$$\begin{aligned} x_1(t_f) = y(t_f) = \frac{c_1 t_f^3}{6b} - \frac{c_2 t_f^2}{2b} + 10 = 0 \\ x_2(t_f) = \dot{y}(t_f) = \frac{c_1 t^2}{2b} - \frac{c_2 t}{b} = 0 \end{aligned} \tag{41}$$

Solving above equations for c_1 and c_2 we obtain:

$$\begin{aligned} c_1 &= \frac{120b}{t_f^3} \\ c_2 &= \frac{60b}{t_f^2} \end{aligned} \tag{42}$$

So now the only unknown remains is t_f, to obtain that we use the last necessary condition remain to us i.e 28.

Notice that in the current problem we have $w(t_f, x_f) = h(t_f) = 0.5 * \alpha t_f^2$, this gives

$$w_{t_f} = h_{t_f} = \alpha t_f \tag{43}$$

by 28 we obtain

$$w_{t_f} + H(t_f) = h_{t_f} + H(tf) = 0 \tag{44}$$

note that $H(t_f) = 0.5b(u(t_f))^2 + p_1(t_f)x_2(t_f) + p_2(t_f)u(t_f) = u(t_f)(0.5bu(t_f) + p_2(t_f))$, putting this in 44 we obtain

$$\alpha t_f + u(t_f)(0.5bu(t_f) + p_2(t_f)) = 0 \tag{45}$$

putting $u(t_f) = \frac{c_1 t_f - c_2}{b}$ and $p_2(t_f) = -c_1 t_f + c_2$. Putting c_1 and c_2 from 42 and putting the resultant expression of $u(t_f)$ and $p_2(t_f)$ in the above equation and solving for t_f we obtain

$$t_f = \left(\frac{1800b}{\alpha}\right)^{1/5} \tag{46}$$

This completes the solution of the problem.

4.2 References

1. Liberzon 3 and 4
2. Kirk section 4.4 and 5.8

5 January 23rd: Comprehensive Example: Linear Quadratic Regulator

5.1 LQR

Consider the following linear time varying system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}(t)x + \boldsymbol{B}(t)\boldsymbol{u} \\ \boldsymbol{z}(t) &= \boldsymbol{C_z}(t)\boldsymbol{x} \end{aligned} \tag{47}$$

with following cost function

$$J = 0.5\boldsymbol{x}^T(t_f)\boldsymbol{P}_{t_f}\boldsymbol{x}(t_f) + 0.5\int_{t_0}^{t} \boldsymbol{z}^T(t)\boldsymbol{R_{zz}}\boldsymbol{z}(t) + \boldsymbol{u}^T(t)\boldsymbol{R_{uu}}\boldsymbol{u}(t)dt \tag{48}$$

where $\boldsymbol{R_{zz}}(t) > 0$, $\boldsymbol{R_{xx}}(t) > 0$, and $\boldsymbol{R_{uu}}(t) \geq 0$. Note that $\boldsymbol{R_{xx}} = \boldsymbol{C_z}^T\boldsymbol{R_{zz}}\boldsymbol{C_z}$.

$\boldsymbol{A}(t)$ is continuous where $\boldsymbol{B}(t)$, $\boldsymbol{C_{zz}}(t)$, $\boldsymbol{R_{zz}}(t)$, $\boldsymbol{R_{xx}}(t)$, and $\boldsymbol{R_{uu}}(t)$, are piecewise continuous.

The Hamiltonian of the problem is given by

$$H = 0.5\boldsymbol{x}^T(t)\boldsymbol{R_{xx}}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R_{uu}}\boldsymbol{u}(t) + \boldsymbol{p}^T(\boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}) \tag{49}$$

Using the necessary conditions for the minimization we have

$$\dot{\boldsymbol{p}} = -\boldsymbol{H_x^T} = -\boldsymbol{R_{xx}}\boldsymbol{x} - \boldsymbol{A}^T\boldsymbol{p}(t) \tag{50}$$

The condition $\boldsymbol{H_u} = 0$ gives$\boldsymbol{u}^T\boldsymbol{R_{uu}} + \boldsymbol{p}^T\boldsymbol{B} = 0$ which gives

$$\boldsymbol{u} = -\boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T\boldsymbol{p}(t) \tag{51}$$

and finally the condition $\boldsymbol{w_{x_f}} = \boldsymbol{p}^T(tf)$, where $w(t_f, \boldsymbol{x_f}) = 0.5\boldsymbol{x}^T(t_f)\boldsymbol{P}_{t_f}\boldsymbol{x}(t_f)$, gives

$$\boldsymbol{p}(tf) = \boldsymbol{P_{t_f}}\boldsymbol{x}(t_f) \tag{52}$$

Putting 51 in 47 we get

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)x - \boldsymbol{B}(t)\boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T\boldsymbol{p}(t) \tag{53}$$

Note that the 53 and 50 forms systems of equation, which can be written as follows

$$\begin{bmatrix} \dot{\boldsymbol{x}} \\ \dot{\boldsymbol{p}} \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{A} & -\boldsymbol{B}(t)\boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T \\ \hline -\boldsymbol{C_z}^T\boldsymbol{R_{zz}}\boldsymbol{C_z} & -\boldsymbol{A}^T \end{array}\right] \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{p} \end{bmatrix} \tag{54}$$

with boundary condition given by 52.

The solution such problem is given by state transition matrix given as follows

$$\begin{bmatrix} \boldsymbol{x}(t_f) \\ \boldsymbol{p}(t_f) \end{bmatrix} = \left[\begin{array}{c|c} \boldsymbol{F_{11}(t_f, t)} & -\boldsymbol{F_{12}(t_f, t)} \\ \hline \boldsymbol{F_{21}(t_f, t)} & -\boldsymbol{F_{22}(t_f, t)} \end{array}\right] \begin{bmatrix} \boldsymbol{x}_t \\ \boldsymbol{p}_t \end{bmatrix} \tag{55}$$

Solving the above equation we obtain $\boldsymbol{p^*}(t) = \boldsymbol{P}(t)\boldsymbol{x}^*(t)$. Now, we need to get the dynamics of $\boldsymbol{P}(t)$ for which we have $\dot{\boldsymbol{p}} = \dot{\boldsymbol{P}}x + \boldsymbol{P}\dot{\boldsymbol{x}}$. Using 54 we obtain Differential Ricatti Equation (DRE).

$$-\dot{\boldsymbol{P}} = \boldsymbol{PA} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{C_z}^T\boldsymbol{R_{zz}}\boldsymbol{C_z} - \boldsymbol{PBR_{uu}}^{-1}\boldsymbol{B}^T\boldsymbol{P} \tag{56}$$

In steady state $\dot{\boldsymbol{P}} = 0$.

5.1.1 Conditions for steady state

1. independent of $\boldsymbol{P_{t_f}} > 0$, the steady state unique $\boldsymbol{P_{ss}}$ of DRE exists $\boldsymbol{P_{ss}} \geq 0$ if $[\boldsymbol{A}, \boldsymbol{C_z}]$ is detectable, $[A, B]$ stabilizable.
2. $\boldsymbol{P_{ss}} > 0$ exists, if $[\boldsymbol{A}, \boldsymbol{C_z}]$ is observable, $[A, B]$ stabilizable.

5.1.2 Example

Problem statement

Consider

$$J = \int_0^1 (u-x)^2 dt \qquad (57)$$
$$\dot{x} = u$$

free terminal state (x_{t_f} is free).

Solution

Hamiltonian is given by $H = (u-x)^2 + pu$. This give

$$\dot{p} = -H_x^T = -2(x-u) \qquad (58)$$

and by $H_u = 0$ we get $2(u-x) + p = 0$, which implies $p = 2(x-u)$, using this in 58 we get

$$\dot{p} = -p \qquad (59)$$

This gives the solution $p(t) = e^{-t}p(0)$. The condition $p(t_f) = 0$ implies $p(0) = 0$. We know that $p = 2(x-u)$, using $p(t) = 0$ we have the solution

$$u = x \qquad (60)$$

This gives the following solution for the states

$$\dot{x} = x \qquad (61)$$

5.2 Question: Liberzon 3.6

$$\begin{aligned} J &= \int_{t_o}^{t_f} L(t, x, \dot{x}) dt \\ n &= 2, k = 1 \\ \dot{x_1} &= f(x_1, x_2, u) \\ \dot{x_2} &= u \end{aligned} \qquad (62)$$

The Hamiltonian is given by

$$H = p_1 f(x_1, x_2, u) + p_2 u - L \qquad (63)$$

Using the condition $\dot{p} = -H_x^T$ we obtain

$$\begin{aligned} \dot{p}_1 &= -p_1 f_{x_1} + L_{x_1} + L_{\dot{x}_1} f_{x_1} \\ \dot{p}_2 &= -p_1 f_{x_2} + L_{x_2} - L_{\dot{x}_1} f_{x_2} \end{aligned} \qquad (64)$$

by $H_u = 0$ we get

$$p_1 f_u + p2 - L_{\dot{x}_2} f_u = 0 \qquad (65)$$

Now, using Lagrange multiplier we have augmented Lagrangian as follows

$$\bar{L} = L + \lambda(\dot{x}_1 - f(x_1, x_2, \dot{x}_2) \tag{66}$$

Using $\lambda = p_1 - L_{\dot{x}_1}$ it can be shown that

$$\frac{d}{dt}\bar{L}_{\dot{x}_1} = \dot{p}_1 = -p_1 f_{x_1} + L_{x_1} + L_{\dot{x}_1} f_{x_1} = \bar{L}_{x_1} \tag{67}$$

similarly it can show that

$$\frac{d}{dt}\bar{L}_{\dot{x}_2} = \bar{L}_{x_2} \tag{68}$$

5.3 References

1. Kirk 5.2
2. Liberzon 3

6 January 27th: Fixed and Free-Endpoint Problems

Statement of Maximum Principle: for fixed and free endpoint conditions

Fixed Endpoints:

$$\dot{x} = f(x, u, t), \;\; x \in \mathbb{R}^n$$
$$J = \int_{t_0}^{t_f} L\left(t, x, \dot{x}\right) dt$$

optimal control (optimal in a global sense now, not just a local sense):

$$u^* : [t_0, t_f] \rightarrow \mathcal{U}$$

corresponding optimal trajectory:

$$x^* : [t_0, t_f] \rightarrow \mathbb{R}^n$$

Theorem:

$$\exists \;\; p^* : [t_0, t_f] \rightarrow \mathbb{R}^n, \quad \exists p_0^* \leq 0$$
$$\text{s.t. } (p_0^*, p^*(t)) \neq (0, 0) \;\;\; \forall t \in [t_0, t_f]$$

(note: here p^* is the costate and p_0^* is the abnormal multiplier

(1) x^* and p^* satisfy:

$$\dot{x}^* = H_p(x^*, u^*, p^*, p_0^*) \leftarrow \mathbb{R}^n$$
$$\dot{p}^* = -H_x(x^*, u^*, p^*, p_0^*) \leftarrow \mathbb{R}^n$$

with boundary conditions:

$$x^*(t_0) = x_0, \quad x^*(t_f) = x_f$$

$$H(x,u,p,p_0) = \langle p, f(x,u)\rangle + p_0 L(x,u)$$

(2) For each fixed $t, u \to H(x^*(t), u, p^*(t), p_0^*)$ has a global max at: $u = u^*(t)$,

$$\text{i.e. } H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x^*(t), u, p^*(t), p_0^*)$$

$$\text{holds for all } t \in [t_0, t_f] \text{ and } u \in \mathcal{U}$$

(3) $H(x^*(t), u^*(t), p^*(t), p_0^*) = 0 \;\; \forall\, t \in [t_0, t_f]$

$p_0^* \leftarrow$ abnormal multiplier
$p_0^* = 0 \to$ degenerate case
$p_0^* \neq 0 \to$ can recover earlier results

Free Endpoints:
handles t_f free and x_f free cases

u^* optimal, corresponding x^*

$$\exists\, p^*, p_0^* \text{ s.t. } (p^*, p_0^*) \neq (0,0)\, \forall\, t$$

Note: costate still holds, optimality still holds.
(1),(2), (3) all still hold, <u>and</u> you get a new condition:

(4) $p^*(t_f)$ is orthogonal to the tangent space to $\mathcal{S}$ at $x^*(t_f)$:

$$\mathbb{S} = [t_0, \infty) \times \mathcal{S}_1, \;\; \mathcal{S}_1 \subset \mathbb{R}^n$$

$\equiv \langle p^*(t_f), d\rangle = 0$
$\equiv p^*(t_f)$ is a linear combination of $\nabla h : (x^*(t_f))$

7 February 4th: Comprehensive Example: Switching Curves

<u>example</u>: changing curves

$$y(s) = G(s)u(s)$$

$$G(s) = \frac{1}{s^2}$$

$$|u(t)| \leq u_m$$

Goal: determine an admissible set so as to minimize this terminal cost:

$$J(u) = \int_{t_0}^{t_f} (1 + b|u(t)|)\, dt \qquad \text{where } b > 0$$

$$y(t_f) = \dot{y}(t_f) = 0, \qquad t_f \text{ free}$$
$$y(0) = y_0$$
$$\dot{y}(0) = \dot{y}_0$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x_1 := y \qquad \dot{x}_1 = x_2$$
$$x_2 := \dot{y} \qquad \dot{x}_2 = u$$

<u>Hamiltonian</u>:

$$H = (1 + b|u|) + p_1 x_2 + p_2 u$$

$\Rightarrow$ costate equations:

$$\dot{p}_1 = -H_{x_1} = 0 \qquad p_1(t) = c_1 \in \mathbf{R}$$
$$\dot{p}_2 = -H_{x_2} = -p_1 \qquad p_2(t) = -c_1 t + c_2$$

Pontryagin states optimal control is obtained where H is minimized.

$$\min_{u \,\in\, \mathcal{U}} 1 + b|u| + p_1 x_2 + p_2 u = 1 + p_1 x_2 + \min_{|u| \,\leq\, u_m} (b|u| + p_2 u)$$

Note that $(b|u| + p_2 u) =: \widetilde{H}$ and $\mathcal{U}$ is the set of admissible states.

Consider the cases:
1) <u>$p_2 > b > 0$:</u>

(e.g. $p_2 = 2$ and $b = 1$)

$$u^*(t) = -u_m$$

2) <u>$p_2 < -b < 0$:</u>

$$u^*(t) = +u_m$$

3) <u>$|p_2| = 0$:</u>

$$u^*(t) = 0$$

So overall we have:

$$u^* = \begin{cases} -u_m & \text{when} \quad p_2(t) > b \\ 0 & \text{when} \quad -b \le p_2(t) \le b \\ +u_m & \text{when} \quad p_2(t) < -b \end{cases}$$

Now let's take a look at the boundary conditions (BC):

$$x_2(t_f) = 0 \Rightarrow H(t_f) = 0 \Rightarrow$$

$$1 + b|u(t_f)| + p_2(t_f)u(t_f) + p_1(t_f)x_1(t_f) = 0$$

Check consistency:

1)

$$u = u_m \Rightarrow 1 + bu_m + p_2(t_f)u_m = 0 \Rightarrow$$

$$p_2(t_f) = -b - \frac{1}{u_m} < -b$$

2)

$$u = -u_m \Rightarrow 1 + bu_m - p_2(t_f)u_m = 0 \Rightarrow$$

$$p_2(t_f) = b + \frac{1}{u_m} > b$$

We still don't know what the terminal control should be (control input at the terminal time...)

$$u(t_f) = u_m \text{ or } u(t_f) = -u_m?$$

First let's consider the case where $u(t_f) = u_m$.

$$p_2(t) = -c_1 t + c_2$$

we want to know what the trajectory looks like for these conditions.

Let's look at the case of switching going on in the control law:

(assume $c_1 \neq 0$)

<u>for $t \in [t_1, t_f)$</u>

$$p_2(t_1) = -b, \quad p_2(t_f) = -b - \frac{1}{u_m}$$

$$\Rightarrow t_1 = t_f - \frac{1}{c_1 u_m}$$

$$p_2(t_2) = b - \frac{1}{u_m} \qquad\qquad \Rightarrow t_2 = t_f - \frac{1}{c_1}\left(2b + \frac{1}{u_m}\right)$$

$$\ddot{y} = u_m \ \Rightarrow \dot{y}(t) = u_m t + c_3 \ \Rightarrow y(t) = \frac{1}{2}u_m t^2 + c_2 t + c_4$$

$$\dot{y}(t_f) = y(t_f) = 0 \ \Rightarrow c_3 = -u_m t_f \ \Rightarrow c_4 = \frac{1}{2}u_m t_f^2$$

$$\begin{aligned} y(t) &= \frac{1}{2}u_m t^2 - u_m t_f t + \frac{1}{2}u_m t_f t_f^2 = \frac{1}{2}u_m (t - t_f)^2 \\ \dot{y}(t) &= u_m t - u_m t_f = u_m (t - t_f) \end{aligned}$$

$$\Rightarrow \boxed{y(t) = \frac{1}{2u_m}\dot{y}^2}$$

<u>for $t \in [t_2, t_1)$:</u> "coasting phase"

$$\begin{aligned} u^*(t) &= 0 \ \ \text{control input is zero.} \\ y(t_1) &= \frac{1}{2}u_m (t_1 - t_f)^2 = \frac{1}{2u_m c_1^2} \ \text{from (1)} \\ \Rightarrow \dot{y}(t_1) &= -\frac{1}{c_1} \end{aligned}$$

Thus, $\dot{y}(t)$ is constant in $t \in [t_2, t_1)$. This implies $\dot{y}(t_2) = -\frac{1}{c_1}$.

Finally, by (1) and (2):

$$y(t_2) = (2b + \frac{1}{2u_m})\frac{1}{c_1^2}$$

Note that $\frac{1}{c_1^2} = \dot{y}(t_2)^2$.

<u>in general</u>:

$$\Rightarrow \boxed{y(t) = (2b + \frac{1}{2u_m})\dot{y}(t)^2}$$

<u>for $t \in [t_0, t_2)$</u>

$y_0, \dot{y}_0$ given

$\dot{y}(t) = -u_m + c_5 \Rightarrow c_5 = \dot{y}_0$

$y(t) = -\frac{1}{2}u_m t^2 + c_5 t + c_6 \Rightarrow c_6 = y_0$

$\dot{y}(t) = -u_m t_2 + \dot{y}_0 = -\frac{1}{c_1}$

$y(t_2) = -\frac{1}{2}u_m t_2 + \dot{y}_0 t_2 + y_0$

$\Rightarrow$

$$\boxed{c_1 = \frac{2(b + \frac{1}{u_m})}{t_f - \frac{1}{u_m}\dot{y}_0}}$$

8 February 6th: Principle Of Optimality and Dynamic Programming

8.1 Shortest Path Problem

Google Maps uses a variation of Dijkstra algorithm and A^* algorithm which is a generalization of Dijkstra to find the shortest path.

Below, there is a toy-example of a shortest path-problem.

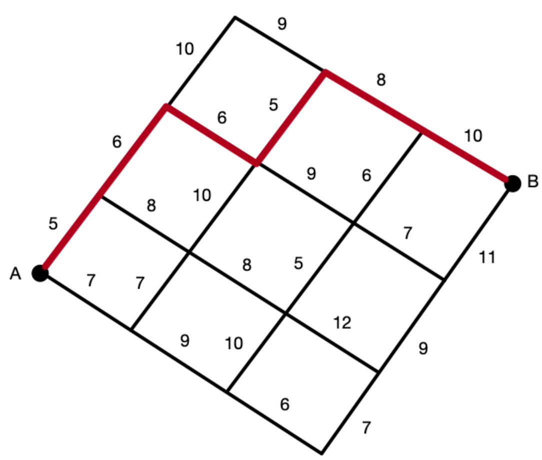


Figure 4: Simplified shortest path problem, with red the shortest path

In this problem, if we write down all the possibilities, then the number of paths scales with $\frac{(2n)!}{(n!)^2}$. So for instance, for this grid, we get 20 possibilities.

We apply the principle of optimality to reduce the number of paths. The main idea is as following:

suppose the optimal solution pases through some intermediate point (x_1, t_1), then the optimal solution of the optimal problem at (x_0, y_0) must be a continuation of the same path.

To apply the principle of optimality to the toy-problem, we start with the final point B and we back-propagate. For example, $\min((6+10),(7+11)) = 16$. Generally, we can write as following:

$$min(cost(x,a) + cost^*(a, target)) \tag{69}$$

with $a \in$ the set of all intermediare points.

The principle of optimality reduced the number of paths. The principle of optimality is the basic principle of dynamic programming (D.P.) The scaling of the problem becomes:

$$(n+1)^2 - 1 \tag{70}$$

Thus, for our toy-problem, the number of paths reduces to 15.

8.2 Classical Control Problem Formulation

$$\begin{aligned} \min \quad & J = h(x(t_f)) + \int_{t_0}^{t_f} g(x,u,t)dt \\ \text{s.t.} \quad & \dot{x} = a(x,u,t) \end{aligned} \tag{71}$$

with $x(t_0) =$ fixed, t_f is also fixed.

We need to discritize all possible values and generate a large mesh, as in the Figure 5

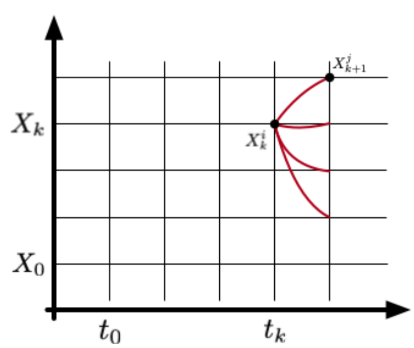


Figure 5: Large mesh of possible values *** not sure why this picture is so bad

$$J(x_k^i) = \min_{x_{k+1}^j} \left(g(x_k^i, u_k^{ij}, t_k)\Delta t + J^*(x_{k+1}^j) \right) \tag{72}$$

Here, $J^*(x_{k+1}^j)$ is called the cost-to-go and x_{k+1}^j is the state at the next timestep at point j. $g(x_k^i, u_k^{ij}, t_k)$ is the transition cost. Below, there is a transition cost, discretized:

$$\int_{t_k}^{t_{k+1}} g(x,u,t)dt = g(x_k,u_k,t_k)\Delta t \tag{73}$$

Also, if we discretize the dynamics, we can figure out u_k^{ij}, the control input to move from i to j. This problem might require interpolation.

8.3 Bellman Equation

$$J^*(x_k^i) = \min_{u_k^{ij}} \left(g(x_k^i, u_k^{ij}, t_k)\Delta t + J^*(x_{k+1}^j, t_{k+1}) \right) \tag{74}$$

Direct implementation of D.P. in real-world applications is usually prohibited by the "curse of dimensionality", which is a term introduced by Bellman. *** formulate the "curse of dim." better

8.4 Discrete LQR (Linear Quadratic Regulator)

The cost function is:

$$J = \frac{1}{2}x_N^T H x_N + \frac{1}{2}\sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k) \tag{75}$$

We apply the principle of optimality, stating that $J_N^* = \frac{1}{2}x_N^T H x_N$.

$$J_{N-1}^* = \min_{u_{N-1}}(x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1}) + J_N^* \tag{76}$$

$$J_{N-1}^* = \frac{1}{2}\min_{u_{N-1}}(x_{N-1}^T Q_{N-1} x_{N-1} + u_{N-1}^T R_{N-1} u_{N-1} + x_N^T H x_N) \tag{77}$$

Recall the following:

$$x_N = A_{N-1}X_{N-1} + B_{N-1}u_{N-1} \tag{78}$$

$$\begin{aligned} \frac{\partial(u^T R u)}{\partial u} &= 2u^T R \\ \frac{\partial(Ru)}{\partial u} &= R \\ \frac{\partial^2(u^T R u)}{\partial u^2} &= R > 0 \end{aligned} \tag{79}$$

If we apply partial derivative, we get:

$$\frac{\partial J_{N-1}(x_{N-1})}{\partial u_{N-1}} = 0 \tag{80}$$

This result is a necesary and sufficient condition because in this case, the Hessian is P.D.

The final input is as following. Note that the Optimal Controller for discrete LTV is a linear controller

$$u^*_{N-1} = -(R_{N-1} + B^T_{N-1} H B_{N-1})^{-1} B^T_{N-1} H A_{N-1} x_{N-1} = -F_{N-1} X_{N-1} \tag{81}$$

For $P = H$, the optimal cost is:

$$J^*_{N-1} = \frac{1}{2} x^T_{N-1} P_{N-1} x_{N-1} \tag{82}$$

By induction, we can get:

$$\begin{aligned} u^*_k &= -F_k X^*_k \\ P_k &= Q_k + F^T_k R_k F_k + (A_k - B_k F_k)^T P_{k+1} (A_k - B_k F_k) \end{aligned} \tag{83}$$

and we can show for $k-1$, $\frac{\partial J^*_{k-1}}{\partial u_{k-1}}$. Find F_{k-1}, then we can reversibly compute and back-propagate. $P_k \to P_{ss}$, where P_{ss} is steady-state. If $(CA, \sqrt{Q})$, with $Q = C^T C$ is observable, then $P_k = P_{k+1} = P_{ss}$,

8.5 HJB (Hamilton–Jacobi–Bellman) Equation

$$\begin{aligned} J = \min_{u \in U} \quad & h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x, u, t) dt \\ \text{s.t.} \quad & \dot{x} = a(x, u, t) \end{aligned} \tag{84}$$

We need to constraint the final solution to a surface $m(x_f, t_f) = 0$ or give a value of t_f.

Problems with HJB:

- HJB cannot handle collision avoidance issues
- the function needs to be at least C^1
- using Pontryagin's minimumm principle, we have to solve a 2-point boundary problem, but HJB requires us to solve partial differential equations

We apply the principle of optimality and start by some intermediate point $t \leq t_f$

$$J^*(x(t), t) = \min_{u \in U} \int_t^{t+\Delta t} g dt + J^*(x(t + \Delta t)) \tag{85}$$

$J^*(x(t_f), t_f) = h(x_f, t_f)$ - from the boundary conditions

Using Taylor, we expand

$$J^*(x(t + \Delta t), t + \Delta t) \approx J^*(x(t), t) + J_t^* \Delta t + J_x^*(x(t + \Delta t) - x(t)) + H.O.T. \tag{86}$$

$$J^*(x(t + \Delta t), t + \Delta t) \approx J^*(x(t), t) + J_t^* \Delta t + J_x^*(a \Delta t) + H.O.T. \tag{87}$$

$$J^*(x(t), t) = \min_{u \in U} (g \Delta t + J^*(x(t), t) + J_t^* \Delta t + J_x^* a \Delta t) \tag{88}$$

Thus, we have the PDE

$$0 = \min_{u \in U} (g + J_t^* + J_x^* a(x, u, t)) \tag{89}$$

with $J^*(x(t_f), t_f) = h(x_f, t_f)$

$$-J_t^* = \min_{u \in U} (H) \tag{90}$$

with $H = g + J_x^* a(x, u, t)$ is the Hamiltonian and $J_x^* \approx P^T$ (from PMP).

9 February 11th: More Dynamic Programming Examples

9.1 Bellman DP example

In this lecture note we would look at one example of Bellman DP equation. For recap, following is the Bellman DP equation

$$J^*(\boldsymbol{x}_k) = \min_{\boldsymbol{u}_k} [g_d(\boldsymbol{x}_k, \boldsymbol{u}_k)] + J^*(\boldsymbol{x}_{k-1}) \tag{91}$$

where $J^*(\boldsymbol{x}_{k-1}$ is cost to go and $\min_{\boldsymbol{u}_k} [g_d(\boldsymbol{x}_k, \boldsymbol{u}_k)]$ forms the transient cost.

9.1.1 Example

Problem statement:

Consider the following optimization problem

$$\min_u J = x^2(T) + 2\int_0^T u^2(t)dt \tag{92}$$

such that

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu(t) \\ 0 \leq x(t) &\leq 1.5 \\ -1 \leq u(t) &\leq 1 \\ 0 \leq t &\leq T \end{aligned} \tag{93}$$

Before we can use 91 we have to discrete the dynamics with step size Δt. This means the time is divided into $N = T/\Delta t$ equal increments. Following are the discretized dynamics

$$x_{k+1} = (1 + a\Delta t)x_k + b\Delta t u(t) \tag{94}$$

similarly the cost function is discretized to be as following:

$$J = x^2(N) + 2\Delta t \sum_{k=0}^{N-1} u^2(k) \tag{95}$$

For this example consider $a = 0$, $b = 1$, $T = 2$, $\Delta t = 1$, $N = 2$. So the problem is:

$$\min_{u(0),u(1)} J = x^2(2) + 2u^2(0) + 2u^2(1) \tag{96}$$

subject to constrains

$$\begin{aligned} x(k+1) = x(k) + u(k) &\qquad k = \{0, 1\} \\ 0 \leq x(k) \leq 1.5 &\qquad k = \{0, 1, 2\} \\ -1 \leq u(k) \leq 1 &\qquad k = \{0, 1\} \end{aligned} \tag{97}$$

In this problem we will assume that the states are quantized as $x(k) = 0, 0.5, 1.0, 1.5$, and input is quantized as $u(k) = -1, -0.5, 0, 0.5, 1.0$. The corresponding BP equation is

$$J^*(x^*(k)) = \min_{u(k)} 2u^2(k) + J^*(x^*(k+1)) \qquad k = \{0, 1\} \tag{98}$$

where $J(x(2)) = x^2(2)$.

Solution Note that the there are no constrain on the initial and final values of the state $x(k)$. So we would make the tables so that it would give allow us to discover the optimal path from any given initial and/or final conditions.

We start from the end time i.e k = 2. The cost at k = 2 is given by table 1.

Table 1: J(x_2) for different x(2)

x(2)	$J(x_2) = x^2(2)$
0	0
0.5	0.25
1	1
1.5	2.25

Now we need to find $J^*(x_1^*) = min_{u(1)} 2u^2(1) + J^*(x_2^*)$.

Table 2: J(x(1)) for all possible x(1) and u(1)

<table>
<tr><th>x(1)</th><th>u(1)</th><th>$x(2) = x(1) + u(1)$</th><th>$J(x_1) = 2u^2(1) + J(x(2))$</th><th>$J^*(x_1^*)$</th><th>$u^*(1)$</th></tr>
<tr><td rowspan="3">0</td><td>1</td><td>1</td><td>3</td><td rowspan="3">0</td><td rowspan="3">0</td></tr>
<tr><td>0.5</td><td>0.5</td><td>0.75</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="4">0.5</td><td>1</td><td>1.5</td><td>4.25</td><td rowspan="4">0.25</td><td rowspan="4">0</td></tr>
<tr><td>0.5</td><td>1</td><td>1.5</td></tr>
<tr><td>0</td><td>0.5</td><td>0.25</td></tr>
<tr><td>-0.5</td><td>0</td><td>0.5</td></tr>
<tr><td>1.0</td><td>0.5</td><td>1.5</td><td>2.75</td><td rowspan="4">0.75</td><td rowspan="4">-0.5</td></tr>
<tr><td></td><td>0</td><td>1</td><td>1</td></tr>
<tr><td></td><td>-0.5</td><td>0.5</td><td>0.75</td></tr>
<tr><td></td><td>-1</td><td>0</td><td>2</td></tr>
<tr><td>1.5</td><td>0</td><td>1.5</td><td>2.25</td><td rowspan="3">1.5</td><td rowspan="3">-0.5</td></tr>
<tr><td></td><td>-0.5</td><td>1</td><td>1.5</td></tr>
<tr><td></td><td>-1</td><td>0.5</td><td>2.25</td></tr>
</table>

From table 2 it is clear that if $x(1) = 0$ then the optimal input is $u^*(1) = 0$ and the corresponding cost is $J^*(x_1) = 0$, similarly if the state at time instant 1 is $x(1) = 0.5$ then the optimal input $u^*(1) = 0$ and the corresponding cost is $J^*(x_1^*) = 0.25$. Similar conclusion can be obtained for other possibilities

As mentioned before there is no constrain on $x(2)$ and there is no initial condition $x(0)$. Now we complete the solution by providing the final table for $J(x(0))$.

Table 3: J(x(0)) for all possible x(0) and u(0)

<table>
<tr><th>x(0)</th><th>u(0)</th><th>$x(1) = x(0) + u(0)$</th><th>$J(x_0) = 2u^2(0) + J(x(1))$</th><th>$J^*(x_0)$</th><th>$u^*(0)$</th></tr>
<tr><td rowspan="3">0</td><td>1</td><td>1</td><td>3</td><td rowspan="3">0</td><td rowspan="3">0</td></tr>
<tr><td>0.5</td><td>0.5</td><td>0.75</td></tr>
<tr><td>0</td><td>0</td><td>0</td></tr>
<tr><td rowspan="4">0.5</td><td>1</td><td>1.5</td><td>4.25</td><td rowspan="4">0.25</td><td rowspan="4">0</td></tr>
<tr><td>0.5</td><td>1</td><td>1.5</td></tr>
<tr><td>0</td><td>0.5</td><td>0.25</td></tr>
<tr><td>-0.5</td><td>0</td><td>0.5</td></tr>
<tr><td>1.0</td><td>0.5</td><td>1.5</td><td>2.75</td><td>0.75</td><td>-0.5</td></tr>
</table>

	0	1	1		
	-0.5	0.5	0.75		
	-1	0	2		
1.5	0	1.5	2.25		
	-0.5	1	1.5	1.5	-0.5
	-1	0.5	2.25		

From the above table the optimal path for this system is $x(0) = 0 \to x(1) = 0 \to x(2) = 0$. For which $u^*(0) = u^*(1) = 0$ and total cost $J^*(x(0)) = 0$.

Let us say we have been give initial constrain $x(1) = 1$. Then the optimal inputs would be $u^*(0) = -0.5$ which will give $x(1) = 0.5$ from which we would apply input $u^*(1) = 0$ this would give total optimal cost of $J^*(x(0)) = 0.75$.

9.2 HJB equation

We have seen HJB equation in lecture of Feb 6, 2020. For recap, for the process is given by:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{a}(\boldsymbol{x}(t)), \boldsymbol{u}(t), t) \tag{99}$$

with cost minimization of follwing cost function

$$J = h(\boldsymbol{x_f}, t_f) + \int_{t_0}^{t_f} g(\boldsymbol{x}(t), \boldsymbol{u}(t), t)dt \tag{100}$$

The HJB equation is given as follows:

$$-J_t^*(\boldsymbol{x}(t), t) = \min_{\boldsymbol{u}\in\mathcal{U}}[g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + J_{\boldsymbol{x}}^*(\boldsymbol{x}(t), t)[\boldsymbol{a}(\boldsymbol{x}(t)), \boldsymbol{u}(t), t)]] \tag{101}$$

where J^* denote the optimal cost and $[g(\boldsymbol{x}(t), \boldsymbol{u}(t), t) + J_{\boldsymbol{x}}^*(\boldsymbol{x}(t), t)[\boldsymbol{a}(\boldsymbol{x}(t)), \boldsymbol{u}(t), t)]$ is Hamiltonian H.

9.3 HJB example

Consider the following system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{u} \tag{102}$$

such that $\boldsymbol{A} + \boldsymbol{A}^T = 0$ and $||\boldsymbol{u}|| \leq 1$. The cost is defined as follows

$$J = \int_0^{t_f} 1dt = t_f \tag{103}$$

so $H = 1 + J_x^*(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{u})$, to the input which minimizes H is $\boldsymbol{u}^* = -\frac{(J_x^*)^T}{||J_x^*||}$. The HJB equation becomes

$$-J_t^* = 1 + J_x^*[\boldsymbol{A}\boldsymbol{x}] - ||J_x^*|| \tag{104}$$

A possible solution for his equation is $J^*(\boldsymbol{x}) = ||\boldsymbol{x}||$. The HJB equation is satisfied by the proposed solution. So the optimal input is given by as follows

$$\boldsymbol{u}^* = -\frac{(J_x^*)^T}{||J_x^*||} = -\frac{\boldsymbol{x}^T}{||\boldsymbol{x}||} / \frac{||\boldsymbol{x}^T||}{||\boldsymbol{x}||} = -\frac{\boldsymbol{x}^T}{||\boldsymbol{x}||} \tag{105}$$

9.4 LQR revisited

Consider the LQR problem which is as follows:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x} + \boldsymbol{B}(t)\boldsymbol{u}(t) \tag{106}$$

with cost function as follows

$$J = 0.5\boldsymbol{x}^T(t_f)\boldsymbol{h}\boldsymbol{x}(t_f) + 0.5\int_{t_0}^{t} \boldsymbol{x}^T(t)\boldsymbol{R_{xx}}\boldsymbol{x}(t) + \boldsymbol{u}^T(t)\boldsymbol{R_{uu}}\boldsymbol{u}(t)dt \tag{107}$$

where $R_{uu} \geq 0$ and $R_u u > 0$. The hamiltonian is given as follows:

$$H = 0.5(\boldsymbol{x}^T(t_f)\boldsymbol{R_{xx}}\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{R_{uu}}\boldsymbol{u}) + J_x^*(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}). \tag{108}$$

differentiating Hamiltonian with respect to $\boldsymbol{u}$ we get

$$\begin{aligned} H_u = \boldsymbol{u}^T\boldsymbol{R_{uu}} + J_x^*(\boldsymbol{B}) = 0 \\ \boldsymbol{u}^* = -\boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T(t)(J_x^*(\boldsymbol{x},t))^{T^S} \end{aligned} \tag{109}$$

Now guess $J^* = 0.5 * \boldsymbol{x}^T\boldsymbol{P}(t)\boldsymbol{x}$. From HJB equation we have

$$\begin{aligned} J_t &= 0.5 * \boldsymbol{x}^T\dot{\boldsymbol{P}}(t)\boldsymbol{x} \\ J_x &= \boldsymbol{x}^T\boldsymbol{P}(t) \\ -J_t &= \min_{\boldsymbol{u}} H \\ J(\boldsymbol{x}(t_f), t_f) &= 0.5\boldsymbol{x}^T(t_f)\boldsymbol{h}\boldsymbol{x}(t_f) \end{aligned} \tag{110}$$

solving above equation will yield LQR Riccati equation, which is as follows

$$\begin{aligned} -\dot{\boldsymbol{P}} = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^T\boldsymbol{P} + \boldsymbol{R_{xx}} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T\boldsymbol{P} \\ \boldsymbol{u}^* = \boldsymbol{R_{uu}}^{-1}\boldsymbol{B}^T\boldsymbol{P} \end{aligned} \tag{111}$$

with terminal condition

$$\boldsymbol{P}(t_f) = \boldsymbol{h} \tag{112}$$

9.5 Reference

1. Kirk section 3.5

February 11th: Alternative Version

Bellman Equation

$J^*(x_k) = min_{x_k,u_k}\ g_d(x_k, u_k)$, with $g_d(x_k, u_k)$ being the stage cost.

Example - Kirk p.58 - 59

Consider a system described by the first-order differential equation;

$\frac{d[x(t)]}{dt} = ax(t) + bu(t)$,

where $x(t)$ and $u(t)$ are the state and control variables respectively, and a and b are constants.

The admissible values of state and control variables are constrained by;

$0.0 \leqslant x(t) \leqslant 1.5$ and $|u(t)| \leqslant 1$

The performance measure (cost) to be minimized is;

$J = x^2(T) + \lambda \int_0^T u^2(t)\ dt$

Since $t \in [0, 2]$, specified final time $T = 2$.

For $N = 2$, $\Delta t = 1$ and $t_k = [0, 1, 2]$

For $\Delta x = 0.5$, $x_k = [0, 0.5, 1, 1.5]$ and for $\Delta u = 0.5$, $u_k = [-1, -0.5, 0, 0.5, 1]$

Then the discrete system dynamics would be;

$x_{k+1} = (1 + a\Delta t)x_k + \Delta t u_k$

Eg. set $a = 0, \Delta t = 1$, then $x_{k+1} = x_k + u_k$

x_2^j	$J_2^* = x_2^2$
0	0
0.5	0.25
1	1
1.5	2.25

x_1^i	x_2^j
0	0
0.5	0.5
1	0.5
1.5	1

u (1)	$x_2^j = x_1^j + u(1)$			
x_1^i, x_2^j	0	0.5	1	1.5
0	0	0.5	1	$\cancel{1.5}$
0.5	-0.5	0	0.5	1
1	-1	-0.5	0	0.5
1.5	$\cancel{-1.5}$	-1	-0.5	0

$J^*(x_2) = min\, x^2{}_{(t=2)}$

$J_2^* + \Delta J_{12}^{ij}$	x_2^j			
x_1^i	0	0.5	1	1.5
0	0	0.5, 0.75	2, 3	×
0.5	0.5	0, 0.25	0.5, 1.5	2, 4.25
1	2	0.5, 0.75	0, 1	0.5, 2.75
1.5	×	2, 2.25	0.5, 1.5	0, 2.25

$J^*(x_1) = min_{u(1)}(2\,u^2(1) + J^*(x_2))$

$g_d\, u(2) = 2\,u^2(1)$

Figure 6: Iterations

Hamilton-Jacobi-Bellman) HJB Equation

$J^*(x(t),t) = min_{u\in\mathbb{U}}\ (g(x,u,t)\Delta t + J^*(x(t),t) + J_t{}^*\Delta t + J_x{}^* f(x,u,t)\Delta t)$

Then,

$-J_t{}^* = min_{u\in\mathbb{U}}\ (g(x,u,t) + J_x{}^* f(x,u,t))$

with the Hamiltonian $H = g(x,u,t) + J_x{}^* f(x,u,t)$,

$-J_t{}^* = min_{u\in\mathbb{U}}\ (H)$

Example 1

$\dot{x} = Ax + u, A + A^T = 0 \Rightarrow A = -A^T$ (skew-symmetric)

Since $|u(t)| \leqslant 1$ and $J = \int_0^{t_f} 1\ dt = t_f$;

$H = g(x,u,t) + J_x{}^*(Ax+u) = 1 + J_x{}^*(Ax+u)$

(**Note**: $-J_t{}^* = 1 + J_x{}^*(Ax) = ||J_x{}^*||$)

Recall, $x^T x = ||x||_2{}^2$, then $J_x(J_x)^T = ||J_x{}^*||^2$

Then, for minimum H, $u^* = -\frac{(J_x{}^*)^T}{||J_x{}^*||}$

Candidate; $J_x{}^* = ||x|| = \sqrt{x^T x}$

Therefore, $J_x{}^* = \frac{x^T}{\sqrt{x^T x}} = \frac{x^T}{||x||}$

(**Note**: Set $J_t{}^* = 0$ and HJB equation can be verified)

i.e., $u^* = -\frac{x}{||x||}; ||\frac{x^T}{||x||}|| = 1$

Linear Quadratic Regulator (LQR) revisited..

$\dot{x} = A(t)x + B(t)u(t)$

The continuous-time cost function is given as;

$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \frac{1}{2}\int_{t_o}^{t_f} x^T R_{xx} x + u^T R_{uu} u\ dt$

Then the Hamiltonian $H = \frac{1}{2}(x^T R_{xx} x + u^T R_{uu} u) + J_x{}^*(Ax + Bu)$

For unconstrained control input; $H_u = 0$.

Then, $u^T R_{uu} + J_x{}^*(B) = 0$, leading to $u^* = -R_{uu}{}^{-1} B^T(t)(J_x{}^*)^T(x,t)$

Guess; $J^*(x,t) = \frac{1}{2}x^T P(t)x$

Then, $J^*(x(t_f),t_f) = \frac{1}{2}x^T(t_f)Hx(t_f)$ with terminal condition $P(t_f) = H$

Then derivatives w.r.t. time and state;

$J_t{}^*(x,t) = \frac{1}{2}x^T\dot{P}(t)x$

$J_x{}^*(x,t) = x^T P(t)$

Leading to $J_t{}^*(x,t) = min(H)$ and $u^* = -R_{uu}{}^{-1}B^T Px$.

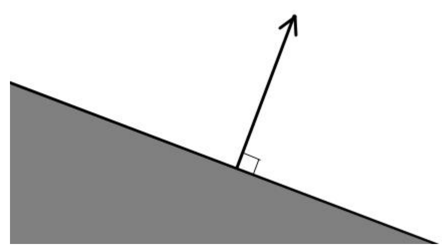
(**Note**: LQR Differential Riccati Equation $:= -\dot{P} = PA + A^T P + R_{xx} - PBR_{uu}{}^{-1}B^T P$)

10 February 17th: Convex Optimization Background

Definition: Convex Set The set $\mathcal{S}$ is convex if $\forall x_1, x_2 \in \mathcal{S}$ then $\forall \epsilon \in [0,1] \quad \epsilon x_1 + (1-\epsilon)x_2 \in \mathcal{S}$.

Definition: Conic (Nonnegative) Combination For two vectors x_1 and x_2, the associated cone S is defined as $\{x | x = \sum_{i=1}^{2} \alpha_i x_i, \ \ \alpha_i \geq 0\}$

Definition: Hyperplane A hyperplane is a planar surface of dimension R^{n-1} that bisects a space in $\mathbb{R}^n$. This can be defined as $\{x | a^\intercal x = b\}$ where $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$. Hyperplanes can be used to define halfspaces: $\{x | a^\intercal x \leq b\}$



Definition: Ellipsoid With the center of the ellipsoid defined as: $x_c \in \mathbb{R}^n$

$$\epsilon = \{x \in \mathbb{R}^n | (x - x_c)^\intercal P^{-1}(x - x_c) \leq r\}$$

Definition: Polyhedron A polyhedron can be defined using multiple hyperplanes For this we use the notation that $\preceq$ indicates row-wise inequalities.

$$\{x \in \mathbb{R}^n | Ax \preceq b, Cx = d\}$$

or alternatively:

$$\cup_i^m \{x | a_i^\intercal x \leq b_i\}$$

10.1 Separating Hyperplane Theorem

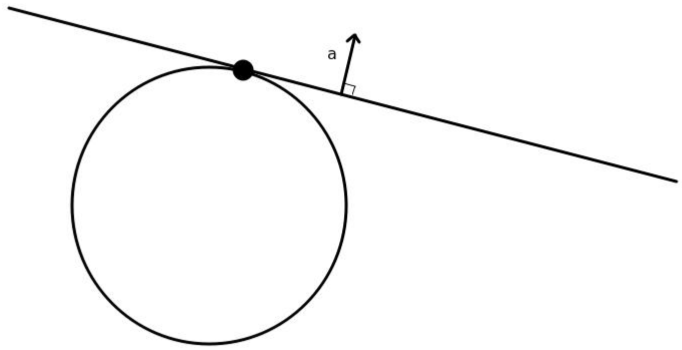
Let C, D be nonempty convex sets. If $C \cap D = \emptyset$. Then $\exists a, b$ such that:

$$\forall x_c \in C, \ \ a^\intercal x_c \leq b$$

$$\forall x_d \in D, \ \ a^\intercal x_d > b$$

10.2 Supporting Hyperplane

C nonempty, closed, convex. Then $\exists a \neq 0$ s.t. $a^\intercal x \leq a^\intercal x_0 \;\; \forall x_0 \in \partial C$



The Optimization Problem:

$$\begin{aligned} \min_x \quad & f(x) \\ \text{s.t.} \; & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_j^\intercal x = b_j, \quad j = 1, \dots, \ell \\ & c_k^\intercal x \leq d_k, \quad k = 1, \dots, p \end{aligned}$$

Typical Well-Studied Optimization Problems:

- **Linear Program (LP):** $f(x) = g^\intercal x$, for $g \in \mathbb{R}^n, x \in \mathbb{R}^n$ with affine contraints. Minimum occurs on the boundary or is the entire set.
- **Quadratic Program (QP):** $f(x) = \frac{1}{2}x^\intercal P x + q^\intercal x + r$ for $P \in \S_+^n, q \in \mathbb{R}^n, r \in \mathbb{R}$
- **Quadratically Constrained Quadratic Program (QCQP)** A quadratic program with the constraints: s.t. $\frac{1}{2}P_i x + q_i^\intercal x + r_i \leq 0$ and $Ax = b$
- **Second Order Cone Problem (SOCP)** $\min_x g^\intercal x$ with the constraint: s.t. $||A_i x + b_i||^2 \leq c_i^\intercal x + d_i, \;\; i = 1, dots, m$
- **Semidefinite Program (SDP)** $\min_x g^\intercal x$ with the constraint: s.t. $\sum_{i=1}^n x_i F_i + G \preceq 0$ for $F_i, G \in \mathbb{S}_+^k$

10.3 Common Transformations

1. Introduce Slack Variables:

$$\begin{aligned}
&\min_{x} \quad f(x) \\
&\text{s.t. } a_i^\intercal x = b_i, \quad i = 1, \dots, \ell \\
&\qquad \downarrow \text{ with the addition of a slack variable} \\
&\qquad \downarrow \\
&\min_{x,\xi} \quad f(x) + ||\xi|| \\
&\text{s.t. } a_i^\intercal x + \xi_i = b_i, \quad i = 1, \dots, \ell \\
&\qquad \xi_i \geq 0, \quad i = 1, \dots, \ell
\end{aligned}$$

2. Epigraph Form (Dual)

$$\begin{aligned}
&\min_{x} \quad f(x) \\
&\text{s.t. } x \in C \\
&\qquad \downarrow \\
&\qquad \downarrow \\
&\min_{t} \quad t \\
&\text{s.t. } f(x) - t \leq 0 \quad \forall x \in C
\end{aligned}$$

3. LP $\rightarrow$ SDP

$$\begin{aligned}
&\min_{x} \quad f(x) \\
&\text{s.t. } Ax \leq b \\
&\qquad \downarrow \\
&\qquad \downarrow \\
&\min_{x} \quad c^\intercal x \\
&\text{s.t. } \operatorname{diag}(Ax - b) \preceq 0
\end{aligned}$$

4. SOCP $\rightarrow$ SDP

$$\begin{aligned}
&\min_{x} \quad f(x) \\
&\text{s.t. } ||A_i x + b_i||^2 \leq c_i^\intercal x + d_i \qquad\qquad \downarrow \\
&\qquad \downarrow \\
&\min_{x} \quad c^\intercal x \\
&\text{s.t. } \begin{bmatrix} (c_i^\intercal + d_i)I & A_i x + b_i \\ (A_i x + b_i)^\intercal & I \end{bmatrix} \succeq 0
\end{aligned}$$

Which comes from the Schurr Complement

$$Q(x) = Q(x)^\intercal, S(x), R(x) = R(x)^\intercal$$

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^\intercal & R(x) \end{bmatrix} \succ 0 \implies R(x) \succ 0$$

and

$$Q(x) - S(x)R(x)^{-1}S(x)^\intercal \succ 0$$

10.4 Example: Eigenvalue Minimization Problem

$$\begin{aligned}
&\min_x \quad \lambda_{max}(A(x)) \\
&\quad\ A(x) \in \mathbb{S}^n \\
&\quad\ \downarrow \\
&\min_t \quad t \\
&\text{s.t. } A(x) \succeq tI \iff A(x) - tI \succeq 0, \forall x
\end{aligned}$$

10.5 Example: Matrix Norm Minimization

$$\begin{aligned}
&\min_x \quad \lambda_{\max}(A(x)^\intercal A(x)) \\
&\qquad \downarrow \\
&\min_t \quad t^2 \\
&\text{s.t.} \quad A(x)^\intercal - tI \preceq 0 \\
&\qquad \downarrow \\
&\min_t \quad t \\
&\text{s.t.} \quad \begin{bmatrix} tI & A(x) \\ A^\intercal(x) & tI \end{bmatrix} \succeq 0
\end{aligned}$$

February 17th: Alternative Version

Convex Sets

Def: Set C is **convex** if $\forall x_1, x_2 \in C$
$\{\alpha_1 x_1 + \alpha_2 x_2 | \alpha_1 + \alpha_2 = 1, \alpha_i \geqslant 0\} \subseteq C$

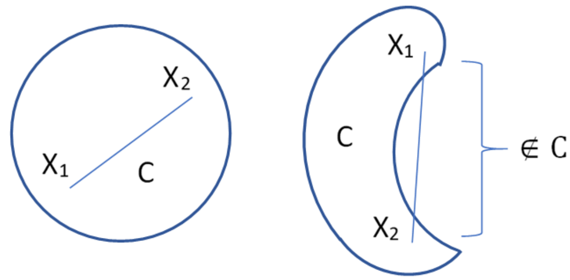


Figure 7: Convex Sets

Cone

Def: A **conic** (non-negative) combination of x_1, x_2
Cone: K= $\{x = \alpha_1 x_1 + \alpha_2 x_2, \forall \alpha_i \geqslant 0\}$

Hyperplane

Def: **Hyperplane** $\{x | a^T x = b\}$,with constant $a \in \mathbb{R}^n / \{0\}, b \in \mathbb{R}$
Halfspace $= \{x | a^T x \leqslant b\}$

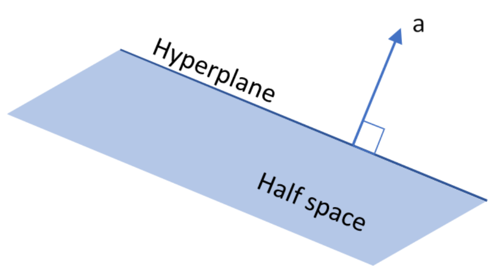


Figure 8: Hyperplane and Halfspace

Ellipsoid

Center $x_c \in \mathbb{R}^n$, $r = 1$

Def: **Ellipsoid**: $\epsilon = \{x \in \mathbb{R}^n | (x - x_c)^T P^{-1} (x - x_c) \leqslant r\}$
P; positive definite, symmetric matrix
If $r = 1, P = \mathbb{I}$, the ellipsoid becomes an Euclidean Ball with radius r.

Polyhedron

Def: **Polyhedron**: $\epsilon = \{x \in \mathbb{R}^n | Ax \preccurlyeq b, Cx = d\}$

$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, C \in \mathbb{R}^{p \times n}, d \in \mathbb{R}^p$

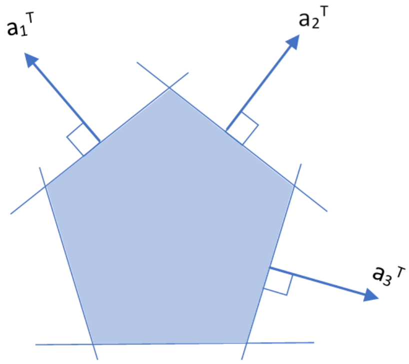


Figure 9: Polyhedron

Def:

- $\$^n$:= set of symmetric matrices $\subset \mathbb{R}^{n\times n}$
- $\$^n{}_+$:= set of positive semi definite matrices $\subset \$^n$
- $\$^n{}_{++}$:= set of positive definite matrices $\subset \$^n{}_+$

Theorem: Separating Hyperplane

If C, D non-empty, convex, disjoint sets ($C \cap D = \emptyset$),
then, $\exists a \neq 0$ such that $a^T x \leqslant b, \forall x \in C, a^T y > b, \forall y \in D$

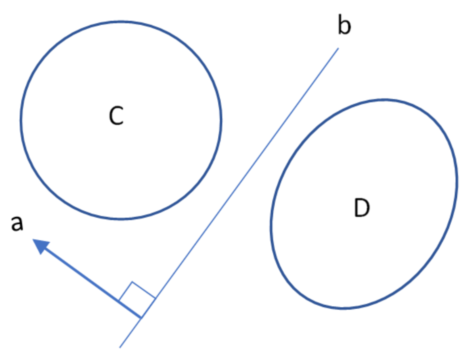


Figure 10: Separating Hyperplane

Theorem: Supporting Hyperplane

If C is a non-empty,closed, convex set,
then, $\exists a \neq 0$ such that $a^T x \leqslant a^T x_o, \forall x_o \in \partial C$

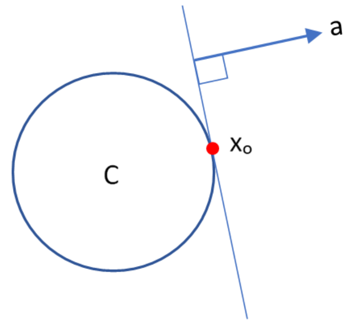


Figure 11: Supporting Hyperplane

Standard Convex Optimization Problem Form

$min_x\ f(x)$
such that,

- $f_i(x) \leqslant 0, i = 1, ...m$
- $a_j{}^T x = b_j, j = 1, ...l$
- $c_k{}^T x \leqslant d_k, k = 1, ...p$

Linear Program (LP):

$f(x) = g^T x$ (all constraints are affine (linear))

Quadratic Program (QP):

$P \in \$^n{}_+, q \in \mathbb{R}^n, r \in \mathbb{R}$
$f(x) = \frac{1}{2} x^T P x + q^T x + r$

Quadratically Constrained Quadratic Program (QCQP):

$min_x\ \frac{1}{2} x^T P x + q^T x + r$
such that,
$\frac{1}{2} x^T P_i x + q_i{}^T x + r_i \leqslant 0, Ax = b$

11 February 18th: Probability and Random Processes Review

Adapted from Anthony Fragoso's Lecture Slides.

11.1 Random Variable

Probabilities transfer from sample space to set of real numbers.

The *probability distribution* extends the probability measure to the image of X, and is described by *cumulative distribution function*

$$F_x(x) = P(X \leq x)$$

A random variable X is *discrete* if it takes at most a countable number of values. It is characterized by a *probability mass function*

$$F_x(x) = \sum_{x_i \leq x} f_x(x_i)$$

A continuous RV has continuous cumulative distribution function. It has a *probability density function*

$$F_x(x) = \int_{-\infty}^{x} f_x(t)dt$$

11.2 Joint Random Variables

A *joint probability distribution* of two random variables X and Y gives probability that each RV falls in a certain range, given by the joint CDF

$$F_X Y(x, y) = P(X \leq x, Y \leq y)$$

- For X, Y discrete, the joint probability mass function is given by

$$f_X Y(x, y) = P(X = x \text{ and } Y = y)$$

- For X, Y continuous, the joint probability density function is given by

$$f_{XY}(x, y) = \frac{d^2 F_{XY}(x, y)}{dxdy}$$

X and Y are *independent* iff

$$F_{XY}(x, y) = F_X(x) F_Y(y)$$

or

$$f_{XY} = f_x f_y$$

11.3 Conditional Probability Distributions

The *conditional probability distribution* of X given Y is the probability distribution of X when the value of Y is known.

- For X, Y either discrete or continuous, the conditional pmf/pdf is

$$f_{X|Y}(x|Y=y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

Conditional probabilities adhere to *Bayes rule*:

$$f_{X|Y} = \frac{f_{Y|X}f_X}{f_Y}$$

If X and Y are independent,

$$f_{X|Y} = \frac{f_Y f_X}{f_Y} = f_X$$

The *marginal probability distribution* is distribution of X without specifying Y. For continuous X and Y,

$$f_X = \int f_{XY}(x,y)dy$$

11.4 Expectation, Variance, and Characteristic Functions

The *expected value E[X]* of a random variable X is the weighted mean of its values.

- For discrete RV:

$$E[g(X)] = \sum_i g(x_i)f_X(x_i)$$

- For continuous RV,

$$E[g(X)] = \int g(x)f_X(x)$$

The spread of X about its mean is its *variance* given by:

$$Var(X) = E[(X-E[X])^2] = E[X^2 - 2XE[X] + E[X]^2] = E[X^2] - E[X]^2$$

(Expectation of a constant/expectation is that constant/expectation)

For continuous X, the *characteristic function* can generate different moments of X useful in computing variance and expectation.

$$\phi_X(t) = E[e^{jtX}] = \int_{-\infty}^{\infty} f_X e^{jtX}dx$$

such that,

$$E[X^k] = \frac{\phi^{(k)}(0)}{j^k}$$

11.5 Gaussian Random Variables

A continuous *Gaussian or normal* variable

$$X \sim N(mean = m_x, var = (stan.dev. = \sigma_x)^2)$$

has the pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{1}{2\sigma_X^2}(x-m_x)^2}$$

Considering a linear transformation of the random variable X, $Z = aX + b$ the pdf is given by

$$Z \sim N(am_x + b, a^2(\sigma_x)^2)$$

If the continuous variable $W = X + Y$, W is normal whose mean and variance is the sum of those of X and Y.

$$W \sim N(m_x + m_y, (\sigma_y)^2 + (\sigma_x)^2)$$

11.6 Covariance, Orthogonality, and Correlation

The *covariance* of two continuous, jointly distributed RV X and Y is given by:

$$Cov(X, Y) = E[(X - m_x)(Y - m_y) = E[XY] - E[X]E[Y]$$

taken over the joint distribution.

Orthogonality arises when

$$E[XY] = 0$$

If X and Y are independent (i.e. $f_{XY} = f_X f_Y$), then they are uncorrelated ($E[XY] = E[X]E[Y]$). The converse is not necessarily true except for Gaussian.

The *correlation coefficient*

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

is a normalized estimate of the two variables' linear relationship.

- If $X = Y, \rho = 1$ (positive)
- If $X = -Y, \rho = -1$ (negative)
- If X and Y are independent, $\rho = 0$ (uncorrelated)

11.7 Multivariate Gaussian

Let $X_1, X_2, ..., X_N$ be RV with mean m_n.

We are now sampling random vectors $X = [X_1, X_2, ..., X_N]^T$, with vector of means $m = [m_1, m_2, ..., m_N]^T$, and *covariance matrix* C such that

$$C_{ij} = E[(X_i - E[X_i])(X_j - E[X_j])]$$

In a similar form to the univariate case, X is a *Gaussian random vector* if the joint pdf of X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}^N det(C)} e^{-\frac{1}{2}[x-m]^T C^{-1}[x-m]}$$

Similarly, if we have a linear transformation $Y = AX + B$, where A is a matrix and B is a vector:

$$Y \sim N(Am + B, ACA^T)$$

11.8 Random Processes

A *random process* is a collection of RV indexed by an index set $T : X_{tt \in T}$

The index set can be used as a time index, where every X_t is a distinct RV in time.

A *realization* of random process is the time series, or *sample path*, of a particular run of the random process. Random process statistic describe the *ensemble* of possible random processes.

11.9 Ergocity

A random process is *ergodic* if the mean of any realization $\hat{X}_t$ in time is identical to the ensemble mean.

$$E[X_t] = \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{X}_{t'} dt'$$

In other words, time series data can be used instead of ensemble statistics to characterize the process.

Random processes can also be characterized by their joint ensemble finite-dimensional distributions between elements. k-th order distribution:

$$F_{X_1 X_2 ... X_k}, F_{X_2 X_4 ... X_{2k}}, etc$$

11.10 Autocorrelation

The *autocorrelation function* detects correlation between all pairs of elements in the random process

$$R_{XX}(t_1, t_2) = E[X_{t_1} X_{t_2}] = \int \int x_{t_1} x_{t_2} f_{x_{t_1} x_{t_2}}(x_{t_1}, x_{t_2}) dx_{t_1} dx_{t_2}$$

(Note: Expectations are ensemble averages of the distribution of each RV)

If X_t is ergodic, autocorrelation can be determined from a single realization $\hat{X}_t$ (because time series yields ensemble statistics)

$$R_{XX}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T \hat{X}_t \hat{X_{t+\tau}} dt$$

11.11 Stationarity

A RV is *strict-sense stationary SSS* if $\forall n \in \mathbb{N}, \tau \in \mathbb{R}, (t_1, t_2, ..., t_n) \in \mathbb{R}$

$$F(X_{t_1}, X_{t_2}, ..., X_{t_n}) = F(X_{t_1+\tau}, X_{t_2+\tau}, ..., X_{t_n+\tau})$$

In other words, all joint finite-dimensional distributions are invariant to time shifts.

A RV is *wide-sense stationary WSS* if $\forall \tau \in \mathbb{R}$ and $t_2 - t_1 = \tau$

- $$E[X_{t_1}] = E[X_{t_1+\tau}] \forall t_1$$
- $$R_{XX}(t_1, t_2) = R_{XX}(0, t_2 - t_1) \equiv R_{XX}(\tau)$$
- $Var(X_t)$ is finite

(Note: if for all first and second order joint distributions are invariant to time-shift, the process is sufficiently WSS.) (Note: SSS implies WSS.)

11.12 Example

$$X_t = A \sim N(0, \sigma^2)$$

is SSS (all ensemble finite-dimensional distributions time-invariant) but not ergodic.

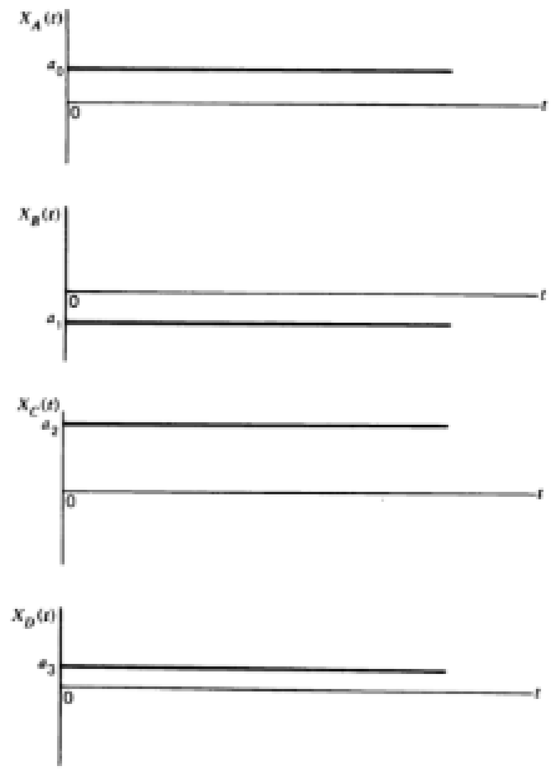


Figure 12:

11.13 Cross-correlation

Let $\{X_t\}$ and $\{Y_t\}$ be random processes. Cross-correlation function describes correlation between each pair of elements in $\{X_t\}$ and$\{Y_t\}$:

$$R_{XY}(t_1, t_2) = E[X_{t_1} Y_{t_2}]$$

Suppose $\{X_t\}$ and $\{Y_t\}$ are jointly WSS. R_{XY} is dependent only the timeshift $\tau = t_2 - t_1$:

$$R_{XY}(\tau) = E[X_t Y_{t+\tau}]$$

and

$$R_{YX}(\tau) = E[X_{t+\tau} Y_t] = E[X_t Y_{t-\tau}] = R_{XY}(-\tau)$$

If $Z_t = X_t + Y_t$, where X_t and Y_t are jointly stationary, the autocorrelation given by:

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{YX}(\tau) + R_{XY}(\tau) + R_{YY}(\tau)$$

If X and Y are zero-mean uncorrelated,

$$R_{ZZ} = R_{XX} + R_{YY}$$

11.14 Gaussian Processes

A random process $\{X_t\}$ is *Gaussian* if each of its finite-dimensional distributions is a multivariate Gaussian RV.

A Gaussian random process is WSS iff it is SSS - Gaussian RV defined only by their mean and variance.

12 February 20th: Methods of Least-Squares

One of the simplest estimation problems is in determining the parameters of a (linear) regression model when observing a sequential stream of data inputs and output pairs. Recall that we can use least-squares methods to determine these unknown parameters when we are given a collection of data. We begin with a review of the original least-squares problem, as well as two extensions: the weighted least-squares problem and the stochastic least-squares problem

12.1 Linear Least-Squares

We will derive the normal equations in the general case where inputs $\mathbf{x}$ are $(d+1)$-dimensional ($\mathbf{x}^T \in \mathbb{R}^{(d+1)}$) and outputs y are scalars. The format of the input $\mathbf{x}$ is $\begin{bmatrix} 1 & \overline{\mathbf{x}} \end{bmatrix}$, where $\overline{\mathbf{x}}$ is the actual d-dimensional input to the system and the 1 is appended to the front simply to account for bias in our model.

Suppose we are given n of these input-output data pairs. We can stack all the datapoints together to form an n-dimensional vector of outputs $\mathbf{y}$ and a $n \times (d+1)$ matrix of inputs X. The stacked regression model becomes:

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{x}_1 \\ 1 & \mathbf{x}_2 \\ \vdots & \vdots \\ 1 & \mathbf{x}_n \end{bmatrix} \underbrace{\begin{bmatrix} b \\ \mathbf{a} \end{bmatrix}}_{\theta} \implies \mathbf{y} = X\theta$$

where $\theta \in \mathbb{R}^{d+1}$.

Our objective is to determine θ so that we minimize the following least-squares cost function:

$$J(\theta) = \sum_{i=1}^{n} (y_i - \mathbf{x}_i\theta)^2$$

The cost function simplifies to:

$$J(\theta) = (\mathbf{y} - X\theta)^T(\mathbf{y} - X\theta) = \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T X\theta + \theta^T X^T X\theta$$

for which we can find the minimizing coefficient θ simply by differentiating with respect to it and setting the equation to 0:

$$\nabla_\theta J(\theta) = -2X^T\mathbf{y} + 2(X^T X)\theta := 0 \Longrightarrow \theta = (X^T X)^{-1}(X^T\mathbf{y}) \tag{113}$$

The expression of θ given by (113) is typically referred to as the *normal equations*.

(Note: The derivative of $\mathbf{y}^T X\theta$ with respect to θ is the coeeficient matrix transposed: $X^T\mathbf{y}$ (to make the dimensions consistent with θ). The derivative of $\theta^T X^T X\theta$ can be determined through the product rule: $X^T X\theta + (\theta^T X^T X)^T$, and again we must make the dimensions consistent with θ.)

12.2 Weighted Linear Least-Squares

In some cases the observations may not be equally reliable. Therefore, it is convenient to use Weighted Least Square.

Observation: I will change notations to be consistent with the notations from class.

$$\begin{aligned} \mathbf{y} = X\theta &\rightarrow \mathbf{y} = H\mathbf{x} \\ J(\theta) = (\mathbf{y} - X\theta)^T(\mathbf{y} - X\theta) &\rightarrow J = \frac{1}{2}(\mathbf{y} - H\hat{\mathbf{x}})^T(\mathbf{y} - H\hat{\mathbf{x}}) \end{aligned} \tag{114}$$

For this case and with the aforementioned notations, the cost function to be minimized becomes:

$$J_w(\hat{\mathbf{x}}) = \frac{1}{2}(\mathbf{y} - H\hat{\mathbf{x}})^T W(\mathbf{y} - H\hat{\mathbf{x}}) \tag{115}$$

where W is a diagonal weight matrix (e.g. if w_1 component in W matrix is bigger than w_2 component means that we trust y_1 more than y_2.

The weighted least square estimate solution is:

$$\hat{\mathbf{x}}_{WLS} = (H^T W H)^{-1} H^T W Y \tag{116}$$

February 20th: Alternative Version

12.2.1 Linear least square with prior estimate

Let x_0 be the prior estimate and $y = Hx + v$, with $v - N(0, R)$ We define the cost functional as:

$$J = \frac{1}{2}(x - x_0)^T P_0^{-1}(x - x_0) + \frac{1}{2}(y - Hx)^T R^{-1}(y - Hx) \tag{117}$$

Here, P_0 reflects confidence in your estimate x_0.

P_0^{-1} vs. R^{-1} is a tradeoff between the prior estimate and new measurement.

We compute $\frac{\partial J}{\partial x} = 0$ as follows:

$$2x^T P_0^{-1} - 2x_0 P_0^{-1} - 2y^T R^{-1} H + 2x^T H^T R^{-1} H = 0 \tag{118}$$

We update the state as follows:

$$x_1 = x_0 + P_1 H^T R^{-1}(y - Hx_0) \tag{119}$$

In Eq. 119, $y - Hx_0$ is called the innovation and Hx_0 is the predicted measurement.

Observation: In fact, the Kalman Filter Gain comes from Eq.119.

Observation: for very good measurements R^{-1} will be large, thus we are getting more trust in our measurements. For bad measurements, the innovation decreases.

P_1^{-1} is the measurement covariance matrix update and is defined as:

$$P_1^{-1} = P_0^{-1} + H^T R^{-1} H \tag{120}$$

For a poor sensor, R is large, thus, as mentioned earlier R^{-1} will be small, thus P_1^{-1} will be small and P_1 will be large.

In Figure 13, we visualize the propagation steps.

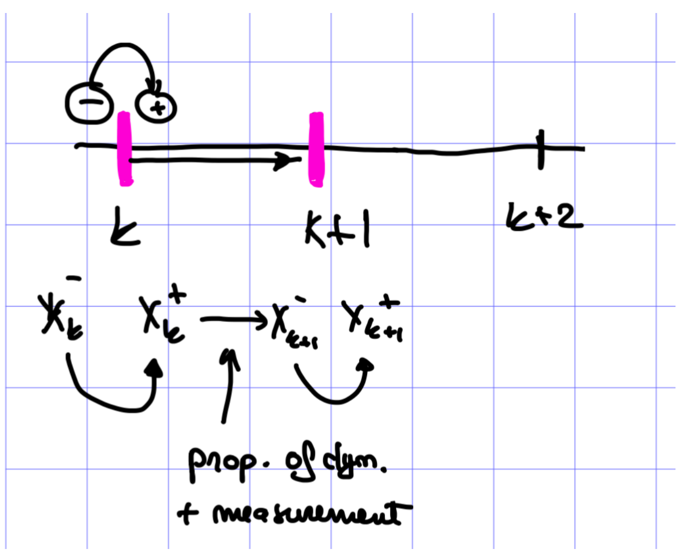


Figure 13: Propagation steps

12.2.2 Nonlinear least square with prior estimate

Here, $y = h(x) + v$, with $v - N(0, R)$.

$$J_{nl} = \frac{1}{2}(x - x_0)^T P_0^{-1}(x - x_0) + \frac{1}{2}(y - H(x))^T R^{-1}(y - H(x)) \tag{121}$$

We compute $\frac{\partial J_{nl}}{\partial x} = 0$ as follows, and we solve for x

$$P_0^{-1}(x - x_0) - H_x R^{-1}(y - h(x)) = 0 H_x = \frac{\partial h}{\partial x}|_x \tag{122}$$

13 February 25th: Stochastic Least-Squares

Stochastic Least-Squares

With linear measurement relationship;
$y = Hx + v, v \sim \mathbb{N}(0, R)$
y: measurement vector
v: measurement noise
H: provides the noise-free relationship between measurement and state vector x

$\overline{x} = E[x]$
$\overline{y} = E[y|x] = Hx$
CoVariance Matrix: $CoV(y|x) = E[(y - \overline{y})(y - \overline{y})^T|x] = E[vv^T] = R$
Likelihood Function: $P(y|x) = \frac{1}{2\pi^{\frac{n}{2}}|R|^{\frac{1}{2}}} e^{-J}$

With Cost Function: $J = \frac{1}{2}(y - Hx)^T R^{-1}(y - Hx)$
and $\hat{x} = arg\ max\ e^{-J} = arg\ min\ J$
(**Note**: $y = hx + v = Hx, \hat{x} = H^{-1}y$)

Thus,
Maximum Likelihood Estimate: $\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$

$log(P(y_1|x), P(y_2|x), P(y_3|x)) = \sum_{i=1}^{m} J_i$

Consistency and Unbiasedness

1. *Consistency*: depends on the no. of independent measurements (N)

For Error: $e = x - \hat{x}$ and Mean Square Error: $E[ee^T]$;

The estimator is a Constant Filter if the following limit is achieved.

$\lim_{N\to\infty} E[(x-\hat{x})^T(x-\hat{x})] = 0$

2.*Biasedness*: if $E[x-\hat{x}] = 0, \forall N$, the estimator $\hat{x}$ is unbiased

Computing the bias;

$E[x-\hat{x}] = E[x - H^{-1}(Hx - v)] = -H^{-1}E[v] = 0$

i.e. the Stochastic Least-Square is unbiased. Similarly, WLS is unbiased for zero mean noise (s)

Checking for the Consistency of WLS and Stochastic LS;

$\lim_{N\to\infty} E[(x-\hat{x})^T(x-\hat{x})] = \lim_{N\to\infty} E[||e||^2]$

But, $||e||^2 = e^Te = trace(ee^T)$,then, $E[||e||^2] = trace(ee^T) = trace(CoV(e))$

Recall, $CoV(e) = E[(x-\hat{x})(x-\hat{x})^T] = E[H^{-1}vv^T(H^{-1})^T] = (H^TR^{-1}H)^{-1}$

Then, $\lim_{N\to\infty} E[||e||^2] = \lim_{N\to\infty} trace(CoV(e))$

If this limit (error from R) goes to zero, then the measurement is consistent.

Influence of Measurement Structure

The appropriateness of the geometry of measurement sensors is indicated by the value of Geomtric Dilution of Precision (DoP), which is dependent on the measurement structure.

Consider,

$y = Hx = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} x + v$, with $y_i = x + v_i$ and $y \in \mathbb{R}^{N\times n}$, having the following measurement structure;

- $E[v_i] = 0$
- $E[v_iv_i^T] = \sigma^2$
- $E[v_iv_j^T] = 0$ for $i \neq j$
- $R = \sigma^2\mathbb{I}$

Then, $\lim_{N\to\infty} CoV(e) = \lim_{N\to\infty} \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^T \frac{\mathbb{I}}{\sigma^2} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix}^{-1} = \lim_{N\to\infty} [\frac{N}{\sigma^2}]^{-1} = \lim_{N\to\infty} \frac{\sigma^2}{N} = 0$

This indicates that the Stochastic Least-Square is consistent.

Least-Squares with Prior Estimate

1.*Linear Least-Square with Prior Estimate*

- Prior Estimate: x_o
- Measurement:$y = Hx + v$
- Measurement Noise: $v \sim \mathbb{N}(0, R)$

Then, the Cost Functional J is defined as follows;

$J_l = \frac{1}{2}(x - x_o)^T P_o^{-1}(x - x_o) + \frac{1}{2}(y - Hx)^T R^{-1}(y - Hx)$

Here, P_o reflects the confidence in the prior estimate x_o. And $P_o^{-1}(x = x_o)$ versus $R^{-1}(y = Hx)$ is a trade-off between prior estimate and new measurement.

Minimizing the Cost Functional J_l by $\frac{\partial J_l}{\partial x} = 0$;

$2x^T P_o^{-1} = 2(x_o^T P_o^{-1}) - 2(y^T R^{-1} H) + 2(x^T H^T R^{-1} H) = 0$

And the updated state x_1 is obtained as follows;

$x_1 = x_o + P_1 H^T R^{-1}(y - Hx_o)$

where, Hx_o is referred to as the predicted measurement and $(y - Hx_o)$ as the innovator.

(**Note**: Kalman Gain is derived from the updated state equation, and accurate measurements lead to a high R^{-1}, resulting in a high innovator)

Measurement Covariance Update P_1^{-1} is defined as follows;

$P_1^{-1} = P_o^{-1} + H^T R^{-1} H$
With $P_1 = (P_o^{-1} + H^T R^{-1} H)^{-1}$

Here, as the accuracy of measurement increases, R^{-1} will increase, leading to a high P_1.

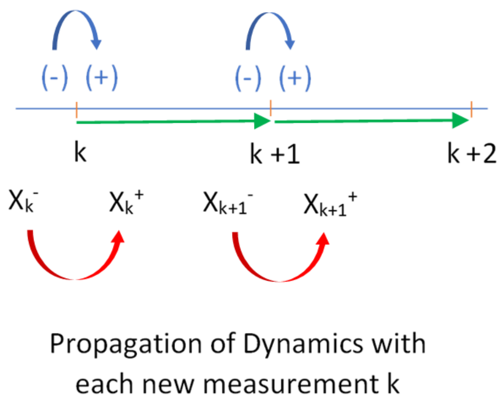


Figure 14: Propagation Steps

2.*Nonlinear Least-Square with Prior Estimate*

- Prior Estimate: x_o
- Measurement:$y = hx + v$
- Measurement Noise: $v \sim \mathbb{N}(0, R)$

Then, the Cost Functional J is defined as follows;

$J_{nl} = \frac{1}{2}(x - x_o)^T P_o^{-1}(x - x_o) + \frac{1}{2}(y - h(x))^T R^{-1}(y - h(x))$

Minimizing the Cost Functional J_{nl} by $\frac{\partial J_{nl}}{\partial x} = 0$;

$P_o^{-1}(x - x_o) - H_x R^{-1}(y - h(x)) = 0$

Then, solving for x; $H_x = \frac{\partial h(x)}{\partial x}|_x$

February 25th: Alternative Version

We have the following linear measurement relationship:

$$\mathbf{y} = H\mathbf{x} + v \tag{123}$$

where $\mathbf{y}$ is the noisy measurement vector, v is the measurement vector and H is a matrix that gives the ideal (noiseless) connection between the measurement and the state vector $\mathbf{x}$.

We assume that the noise has a Gaussian distribution $v - \mathbb{N}(0, R)$, where R is the measurement covariance matrix and is defined as below. Using $\bar{\mathbf{y}} = E[y|x] = H\mathbf{x}$, we have

$$\text{Cov}(y|x) = E[(\mathbf{y} - \bar{\mathbf{y}})(\mathbf{y} - \bar{\mathbf{y}})^T|x] = E[vv^T] = R \tag{124}$$

We have the following likelihood function that we want to maximimize

$$P(y|x) = \frac{1}{\sqrt{(2\pi)^n |R|}} e^{-J} \tag{125}$$

$$\hat{\mathbf{x}} = \arg\max e^{-J} = \arg\min J \tag{126}$$

Thus, the estimate is:

$$\hat{\mathbf{x}} = (H^T R^{-1} H)^{-1} H^T R^{-1} y \tag{127}$$

13.1 Consistency and Unbiasedness

A. **Consistency** depends on N (no of independent measurements).

If the next limit is zero, then the estimator is said to be a consistent filter.

$$\lim_{N\to\infty} E[(\mathbf{x} - \hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}})] = 0 \tag{128}$$

where the error is $e = \mathbf{x} - \hat{\mathbf{x}}$ and $E[ee^T]$ is the mean square error.

B. **Bias**: if $E[\mathbf{x} - \hat{\mathbf{x}}] = 0\ \forall N$, then the estimator is unbiased.

We now check if the properties hold for stochastic least squares. Using Eq. 127, we compute the bias:

$$E[\mathbf{x} - \hat{\mathbf{x}}] = E[x - H^{-1}(Hx - v)] = -H^{-1}E[v] = 0 \tag{129}$$

Thus, the stochastic least square is unbiased.

Similar, WLS is unbiased for zero mean noise.

Example, if a gyroscope has bias, then we can introduce bias as an additional term in the state variable and estimate it.

Now, let's check consistency for WLS and Stochastic LS.

$$\lim_{N\to\infty} E[(\mathbf{x} - \hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}})] = \lim_{N\to\infty} E[\|e\|^2] \tag{130}$$

Recall $\|e\|^2 = e^T e = \text{Tr}(ee^T)$ and $\text{Cov}(e) = E[(\mathbf{x} - \hat{\mathbf{x}})^T(\mathbf{x} - \hat{\mathbf{x}})]$. Thus, Eq. 130 becomes:

$$\lim_{N\to\infty} E[\|e\|^2] = \lim_{N\to\infty} \mathrm{Tr}(\mathrm{Cov}(e)) \tag{131}$$

where $\mathrm{Cov}(e) = E[H^{-1}vv^T(H^{-1})^T] = (H^TR^{-1}H)^{-1}$. This basically shows how much error we have from R. As a result, if $\lim_{N\to\infty} \mathrm{Tr}(e) = 0$, then the measurement is consistent.

The structure of the measurement influences as well. Geometric Dilution of Precision (DoP) quantifies how good is the geometry of your sensors.

If $y = Hx = \begin{bmatrix} 1 \\ 1 \\ . \\ . \\ 1 \end{bmatrix} + v \to y_i = x + v_i$, with the measurement noise having the following structure:

$$\begin{aligned} E[v_i] &= 0 \\ E[v_iv_i^T] &= \sigma^2 \\ E[v_iv_j^T] &= 0, i \neq j \end{aligned} \tag{132}$$

Then we can write the covariance as following:

$$\lim_{N\to\infty} \mathrm{Cov}(e) = \lim_{N\to\infty} \left(\begin{bmatrix} 1 \\ 1 \\ . \\ 1 \end{bmatrix}^T \frac{1}{\sigma} I \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \end{bmatrix} \right)^{-1} = \lim_{N\to\infty} \left(\frac{N}{\sigma^2}\right)^{-1} = \lim_{N\to\infty} \frac{\sigma^2}{N} = 0 \tag{133}$$

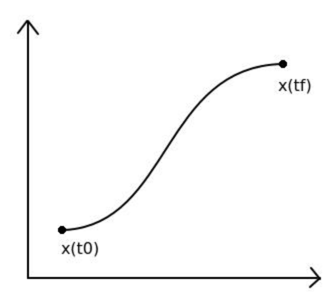
Thus, the stochastic least square is consistent.

14 February 27th: Stochastic Propagation of Dynamics

The system dynamics are given by:

$$x_k = A_{d_{k-1}} x_{k-1} + B_{d_{k-1}} u_{k-1} + w_{k-1}$$

where w is Guassian white noise.



$$
\begin{aligned}
\mathbb{E}[w_k] &= 0 \text{ white sequence, zero mean sequence} \\
\mathbb{E}[w_{k1} w_{k2}^\intercal] &= Q_{k1}\Delta(k_1 - k_2) \text{ where } \Delta(a) = 1 \iff a = 0 \\
\mathbb{E}[x_0] & m_0 \\
P_0 &= \mathbb{E}[(x_0 - m_0)(x_0 - m_0)^\intercal] \\
\mathbb{E}[(x_0 - m_0) w_k^\intercal] &= 0 \quad \forall k
\end{aligned}
$$

14.1 Propagate the Mean and Covariance Matrix

- Mean:

$$
\begin{aligned}
\mathbb{E}[x_k] = m_k &= \mathbb{E}[A_d x_{k-1} + B_d u_{k-1} + w_{k-1}] \\
&= A_d m_{k-1} + B_d u_{k-1}
\end{aligned}
$$

- Covariance:

$$
\begin{aligned}
P_{k-1} &\to P_k \\
P_k &= \mathbb{E}[(x_k - m_k)^\intercal (x_k - m_k)] \\
&= \mathbb{E}[A_d x_{k-1} + B_d u_{k+1} + \cdots]
\end{aligned}
$$

$$
\begin{aligned}
\text{Let } \hat{A} &= A_d x_{k-1} + \cancel{B_d u_{k-1}} + w_{k-1} - A_d m_{k-1} - \cancel{B_d uk} - 1 \\
&= A_d(x_{k-1} - m_{k-1}) + w_{k-1} \\
P_k &= \mathbb{E}[\hat{A}\hat{A}^\intercal] \\
&= \mathbb{E}[(A_d(x_{k-1} - m_{k-1}) + w_{k-1})\hat{A}^\intercal] \\
&= A_d P_{k-1} A_d^\intercal + Q_{k-1} \text{ where } Q_{k-1} = \operatorname{cov}(w_{k-1} w_{k-1}^\intercal)
\end{aligned}
$$

14.2 Relationship Between Q_c and Q_k

Ignoring the deterministic input u

$$
\begin{aligned}
x(t) &= e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-t_0)} B u(\tau) d\tau \\
&= \Phi(t, t_0) x_0 + \int_{t_0}^{t} \Phi(t, t_0) e^{A(t-\tau)} B u(\tau) d\tau \\
x_k &= \varphi_T(x_{k-1}), x(t_k) = \varphi_{t_k - t_{k-1}}(x(t_{k-1})) \\
&= A_{d_{k-1}} x_{k-1} + w_{k-1} \\
x(t_k) &= A_{d_{k-1}} x_{k-1} + w_{k-1}
\end{aligned}
$$

$$
\begin{aligned}
Q_{k-1} &= \mathbb{E}\left[\left(\int_{k-1}^{k} \Phi(t_k,\tau)B_w w(\tau)d\tau\right)\left(\int_{k-1}^{k} \Phi(t_k,\gamma)B_w w(\gamma)d\gamma\right)^{\intercal}\right] \\
&= E\left[\int\int_{k-1}^{k} \Phi(t_k,\tau)B_w w(\tau)w(\gamma)^{\intercal}B_w^{\intercal}\Phi(t_k,\gamma)^{\intercal}d\tau d\gamma\right] \\
&= \int\int_{k-1}^{k} \Phi(t_k,\tau)B_w\mathbb{E}[w(\tau)w(\gamma)^{\intercal}]B_w^{\intercal}\Phi(t_k,\gamma)^{\intercal}d\tau d\gamma \\
&= \int\int_{k-1}^{k} \Phi(t_k,\tau)B_w Q_c B_w^{\intercal}\Phi(t_k,\gamma)^{\intercal}d\tau d\gamma \\
&\quad \text{as } \Delta t \to 0, \Phi \to I \\
&\approx \int B_w Q_c B_w^{\intercal} d\tau \\
&\approx B_w Q_c B_w^{\intercal}\Delta t = Q_{k-1}
\end{aligned}
$$

14.3 Continuous Time System

$$\dot{x} = Ax + B_w w$$
$$y = C_y x + \nu$$

where w is the process noise and ν is the measurement noise.

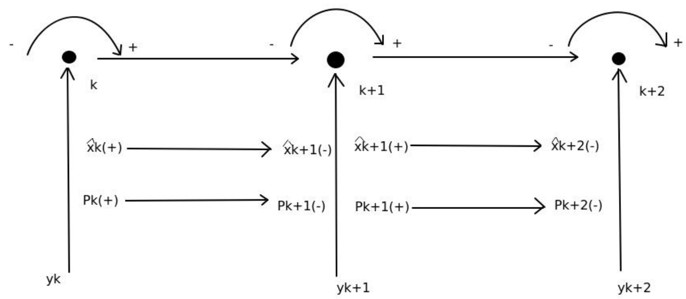
$$\mathbb{E}[w(t_1)w(t_2)^{\intercal}] = Q_c(t_1)\delta(t_1 - t_2)$$
$$\mathbb{E}[\nu(t_1)\nu(t_2)^{\intercal}] = R_c(t_1)\delta(t_1 - t_2)$$
$$\mathbb{E}[w] = \mathbb{E}[v] = 0$$

14.4 Discrete Time System

$$y_k = Cx_k + \nu_k$$

Average value $\nu_k = \frac{1}{\Delta t}\int_0^{\Delta t} \nu(t)dt$

$$
\begin{aligned}
R_k &= \mathbb{E}[\nu_k \nu_k^{\intercal}] \\
&\leq \frac{1}{\Delta t^2}\int\int_0^{\Delta t} \mathbb{E}[\nu(\gamma)\nu(\tau)^{\intercal}]d\tau d\gamma \\
&= \frac{1}{\Delta t^2}\int_0^{\Delta t} R_c d\tau = \frac{R_c \Delta t}{\Delta t^2} = \frac{R_c}{\Delta t} \\
R_k &\leq \frac{R_c}{\Delta t}
\end{aligned}
$$



$$
\begin{aligned}
J_k &= \mathbb{E}[(x_k - \hat{x}_k^{\intercal}(x_k - \hat{x}_k^{(+)})] \\
&= ||x_k - \hat{x}_k^{(+)}||_2^2 = \text{ trace}\left(P_k^{(+)}\right)
\end{aligned}
$$

$$\frac{d}{dA}\text{trace}(AB) = B^{\intercal}$$

$$\frac{d}{dA}\text{trace}(ABA^{\intercal}) = 2AB$$

Blending:

$$
\begin{aligned}
\hat{x}_k^{(+)} &= \hat{x}_k^{(-)} + L_k(y_k - C_d\hat{x}_k^{(-)} \\
P_k^{(+)} &= (1 - L_kC_d)P_k^{(-)}(I - L_kC_d)^{\intercal} + L_kR_kL_k^{\intercal} \\
&\frac{\partial\text{trace}}{\partial L_k}(P_k) = 0
\end{aligned}
$$

$$
\begin{aligned}
L_k &= P_k^{(-)}C_d^{\intercal}[C_dP_k^{(-)}C_d^{\intercal} + R_k] \\
&= P_k^{(+)}C_d^{\intercal}R_k^{-1}
\end{aligned}
$$

15 March 2nd: Filtering Methods

15.1 General Bayesian Filtering

$$
\begin{aligned}
x_{k+1} &= f(x_k, w_k), \quad w_k, v_k \text{ I.I.D.} \\
y_k &= h(x_kv_k) \\
a_k &:= y_{1:k} = \{y_1, \ldots, y_k\}, \quad p(x_0) \text{ is known}
\end{aligned}
$$

Filtering problem:

$$\text{"prior" } p(x_k|a_{k-1}) \to p(x_{k+1}|a_k)$$

Assume Markovian:

$$
\begin{aligned}
&=^{(1)} p(x_{k+1}|x_k, \ldots, x_0, a_k = p(x_{k+1}|x_k) \\
&=^{(2)} p(y_k|x_k, \ldots, x_0, a_{k-1}) = p(y_k|x_k)
\end{aligned}
$$

Two Steps:

<u>Step One: Measurement Update</u>

$$p(x_k|a_{k-1}) \to p(x_k|a_k, y_k) \text{ ”posterior”}$$

Bayes rule:

$$\begin{aligned} p(x_k|a_k) &= p(x_k|y_k, a_{k-1}) \\ &= \frac{p(x_k, y_k|a_{k-1})}{p(y_l|a_{k-1})} \\ &= \frac{p(y_k|x_k)p(x_k|a_{k-1})}{p(y_k|a_{k-1})} \end{aligned}$$

<u>Step Two: Prediction</u>

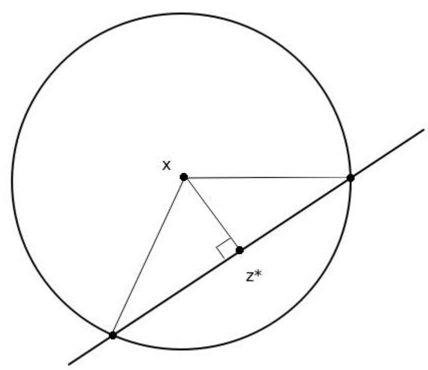
$$p(x_k|a_k) \to p(x_{k+1}|a_k)$$

Chapman-Kolmogorov Equation with Markov Assumption Simplification:

$$p(x_{k+1}|a_k) = \int_{\mathbb{R}^n} p(x_{k+1}|x_k)p(x_k|a_k)dx_k$$

$$\ell^2 = \mathcal{L}^2(\Omega, F, \mathbb{P}) \text{ s.t. } x \in \mathcal{L}^2 \implies \mathbb{E}[||x||^2] < \infty$$

$\mathcal{V} \subset \mathcal{L}^2$ is a closed linear subspace of $\mathcal{L}^2$. Our goal is to find some z^* in $\mathcal{V}$ that best approximates: z^* s.t. $\mathbb{E}[||x - z^*||^2] \leq \mathbb{E}[||x - z||^2] \quad \forall z \in \mathcal{V}$



z^* minimizes the mean squared error (MMSE)), so $z^* \perp (x - z^*)$.

15.2 Orthagonality Principle

For $x \in \mathcal{L}^2$

- There exists a unique $z^* \in \mathcal{V}$ which is the MMSE
- $y \in \mathcal{L}^2$ then $y = z^*$ iff

1. $y \in \mathcal{V}$
2. $(x-y) \perp z \quad \forall z \in \mathcal{V}$ $\mathbb{E}[(x-z)^\intercal z] = 0$ $e = x - z^*$ is the error of the estimate

- $\mathbb{E}[||x - z^*||^2] = \mathbb{E}[||x||^2] - \mathbb{E}[||z^*||^2]$

Given $y \in \mathbb{R}^m$, take space $\mathcal{V} = \{c_0 + c_1 y_1 + \cdots | c_m y_m\}$. Find $z^* \in \mathcal{V}$, $z^* = Ay + b$ where A is all $c_{i \neq 0}$ and b is c_0. Thus we have that $e = x - Ay - b \in \mathbb{R}^n$ and $z^* = Ay + b$.

1. $e_i \perp \mathbf{1}, \quad i = 1, \ldots, n \implies E[e] = 0 \implies$ unbiased
   $\mathbb{E}[x] - A\mathbb{E}[y] - b = 0$ so $b = \mathbb{E}[x] - A\mathbb{E}[y]$
2. $e_i \perp y_i, \quad i = 1, \ldots, n \quad j = 1, \ldots, m$

$$
\begin{aligned}
\operatorname{cov}(e, y) = 0 &\implies \operatorname{cov}(x - Ax - b, y) = 0 \\
&\implies \operatorname{cov}(x, y) - A\operatorname{cov}(y) = 0 \\
A &= \operatorname{cov}(x, y)\operatorname{cov}(y)^{-1}
\end{aligned}
$$

Thus we find that: $z^* = \hat{\mathbb{E}}[x|y] = \mathbb{E}[x] + \operatorname{cov}(x, y)\operatorname{cov}(y)^{-1}(y - \mathbb{E}[y])$.

15.3 Linear Innovation Sequence

This can be used instead of the matrix inversion lemma. Instead of $a_k = \{y_1, \ldots, y_k\}$ consider: $\tilde{a}_k = \{\tilde{y}_1, \ldots, \tilde{y}_k\}$ where $\tilde{y}_i = y_i - \hat{\mathbb{E}}[y_i | \hat{a}_{i-1}] \quad \forall i = 1, \ldots, k$. This is similar to Gram-Shmidt production of orthogonal bases:

$$
\hat{E}[x|\tilde{a}_k] = \bar{x} + \sum_{i=1}^{k} \mathbb{E}[x - \bar{x} | \tilde{y}_i]
$$

Simplifies to:

$$
= \bar{x} + \sum_{i=1}^{k} \operatorname{cov}(x, \tilde{y}_i)\operatorname{cov}(\tilde{y}_i)^{-1}\tilde{y}_i
$$

15.4 Discrete Time Kalman Filter (DTKF) Derivation

$$
\begin{aligned}
x_{k+1} &= A_k x_k + w_k, & x_0 &\sim N(\bar{x}_0, \Sigma_0) \\
y_k &= C_k x_k + v_k & w_k &\sim N(0, Q) \\
& & v_k &\sim N(0, R)
\end{aligned}
$$

where each are pairwise uncorrelated, $\operatorname{cov}(x_0, w_k) = 0 \;\; \forall i \neq j$.

For brevity, we use the notation: $\tilde{a}_k = \{\tilde{y}_1, \ldots, \tilde{y}_k\}$.

Update the estimate of $\hat{x}$ using the most recent measurement, y_k:

$$\hat{x}_k^{(-)} = \hat{E}[x_k|\tilde{a}_{k-1}]$$
$$\hat{x}_k^{(+)} = \hat{E}[x_k|\tilde{a}_k]$$

Update the estimate of the covariance after the most recent measurement, y_k:

$$P_k^{(-)} = \mathrm{cov}(x_k - \hat{x}_k^{(-)})$$
$$P_k^{(+)} = \mathrm{cov}(x_k - \hat{x}_k^{(+)})$$

Goal:

$$\left(\hat{x}_k^{(-)}, P_k^{(-)}\right) \to \left(\hat{x}_k^{(+)}, P_k^{(+)}\right)$$

Two Step Process: Measurement Update:

$$\left(\hat{x}_k^{(-)}, P_k^{(-)}\right) \to \left(\hat{x}_k^{(+)}, P_k^{(+)}\right)$$

$$\begin{aligned}
\tilde{y}_k &= y_k - \hat{\mathbb{E}}[y|\tilde{a}_{k-1}] \\
&= y_k - (C_k \hat{x}_k^{(-)})
\end{aligned}$$

$$\begin{aligned}
\hat{x}_k^{(+)} &= \hat{E}[x_k|\tilde{a}_k] \\
&= \hat{E}[x_k|\tilde{y}_k, \tilde{a}_{k-1}] \\
&= \hat{x}_k^{(-)} + \mathrm{cov}(x_k, \tilde{y}_k)\mathrm{cov}(\tilde{y}_k)^{-1}\tilde{y}_k
\end{aligned}$$

$$L_k = \mathrm{cov}(x_k, \tilde{y}_k)\mathrm{cov}(\tilde{y}_k)^{-1}\tilde{y}_k$$

$$\begin{aligned}
P_k^{(+)} &= \mathrm{cov}((x_k - \hat{x}_k^{(-)}) - L_k\tilde{y}_k) \\
&= \mathrm{cov}(x_k - \hat{x}_k^{(-)}) - \mathrm{cov}(x_k - \hat{x}_k^{(-)} - \tilde{y}_k)L_k^{\intercal} - L_k\mathrm{cov}(\tilde{y}, x) + L_k\mathrm{cov}(\tilde{y}_k)L_k^{\intercal} \\
&= \mathrm{cov}(x_k - \hat{x}_k^{(-)}) - 2L_k^{\intercal}\mathrm{cov}(x_k - \hat{x}_k^{(-)}, \tilde{y}_k) + L_k\mathrm{cov}(\tilde{y}_k)L_k^{\intercal} \\
&= P_k^{(-)} - L_k\mathrm{cov}(\tilde{y}_k)L_k^{\intercal}
\end{aligned}$$

(1)

$$\begin{aligned}
\mathrm{cov}(x_k, \tilde{y}_k) &= \mathrm{cov}(x_k, y_k - C_k x_k^{(-)} \\
&= \mathrm{cov}(x_k, C_k(x_k - \hat{x}_k^{(-)}) + v_k) \\
&= \mathrm{cov}(x_k, x_k - \hat{x}_k^{(-)})C_k^{\intercal} \\
&= P_k^{(-)}C_k^{\intercal}
\end{aligned}$$

$$\begin{aligned}
\mathrm{cov}(\hat{x}_k^{(-)}, x_k - \hat{x}_k^{(-)}) &= 0 \\
\text{Orthogonality} \implies &= 0
\end{aligned}$$

(2)

$$\begin{aligned}
\operatorname{cov}(\tilde{y}_k) &= \operatorname{cov}(C_k(x_k - \hat{x}_k^{(-)}) + v_k) \\
&= C_k P_k^{(-)} C_k^\intercal + Rs
\end{aligned}$$

$$\begin{aligned}
\hat{x}_k^{(+)} &= \hat{x}_k^{(-)} + L_k(\tilde{y}_k - C_k \hat{x}_k^{(-)}) \\
P_k^{(+)} &+ P_k^{(-)} - L_k(C_k P_k^{(-)} C_k^\intercal + R) L_k^\intercal \\
&\text{where } L_k = P_k^{(-)} C_k^\intercal (C_k P_k C_k^\intercal + R)^{-1}
\end{aligned}$$

Prediction Step:

$$\left(\hat{x}_k^{(-)}, P_k^{(-)}\right) \to \left(\hat{x}_k^{(+)}, P_k^{(+)}\right)$$

$$\begin{aligned}
\hat{x}_{k+1}^{(-)} &= \hat{\mathbb{E}}[x_{k+1} | \tilde{a}_k] \\
&= \hat{\mathbb{E}}[A_k x_k + w_k | \tilde{a}_k] \\
&= A_k \hat{\mathbb{E}}[x_k | \tilde{a}_k] + \cancel{\hat{\mathbb{E}}[w_k | \tilde{a}_k]} \\
&= A_k \hat{x}_k^{(+)}
\end{aligned}$$

$$\begin{aligned}
P_{k+1}^{(-)} &= \operatorname{cov}(x_{k+1} - \hat{x}_{k+1}^{(-)}) \\
&= \operatorname{cov}(A_k x_k + w_k - A_k \hat{x}_k^{(+)}) \\
&= \operatorname{cov}(A_k(x_k - \hat{x}_k^{(+)}) + w_k)
\end{aligned}$$

Thus we have that:

$$\begin{aligned}
P_{k+1}^{(-)} &= A_k P_k^{(+)} A_k^\intercal + Q \\
\hat{x}_{k+1}^{(-)} &= A_k \hat{x}_k^{(+)}
\end{aligned}$$

15.5 Continuous Time Kalman Filter

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_w w(t) \\
y(t) &= Cx(t) + v(t)
\end{aligned}$$

Pairwise uncorrelated:

$$\begin{aligned}
x_0 &\sim N(0, \Sigma_0) \\
\mathbb{E}[w(s)w(t)] &= Q\delta(t-s) \\
\mathbb{E}[v(s)v(t)] &= R\delta(t-s) sa_t = \{y(s) : 0 \leq s < t\}
\end{aligned}$$

Goal: Find $(\hat{x}(t), \Sigma(t))$ s.t. $\min J$ where $J = \mathbb{E}[(x - \hat{x}(t))^\intercal (x - \hat{x}(t))] = \mathrm{tr}(\Sigma(t))$.

Next we derive the ODEs $\dot{\hat{x}}(t), \dot{\Sigma}(t)$. Given the mean update step: $\dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - C\hat{x}(t))$.

$$\begin{aligned}
\text{Error: } e(t) &= x(t) - \hat{x}(t) \\
\dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\
&= (A - LC)e(t) + B_w w(t) - Lv(t)
\end{aligned}$$

$$\begin{aligned}
\dot{\Sigma}(t) &= \mathbb{E}[\dot{e}e^\intercal + e\dot{e}^\intercal] \quad \text{where } \Sigma(t) = \mathbb{E}[ee^\intercal] \\
&= (A - LC)\Sigma^\intercal(t) + \Sigma(t)(A^\intercal - C^\intercal L^\intercal) + \mathbb{E}[B_w w e^\intercal] + \mathbb{E}[e w^\intercal B^\intercal w] - \mathbb{E}[Lve^\intercal] - \mathbb{E}[ev^\intercal L^\intercal]
\end{aligned}$$

Recall the standard solutions to our ODEs:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad x(t_0) = x_0$$

Has the unique solution: $x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$.

Let $\Phi(s,t) = e^{A(t-s)}$ be our state transition matrix such that for $e(t)$, $\Phi(t_0,t) = e^{(A-LC)(t-t_0)}$. So our solution for e is:

$$\begin{aligned}
e(t) &= \Phi(0,t)e(0) + \int_0^t \Phi(s,t)B_w w(s)ds - \int_0^t \Phi(s,t)Lv(s)ds \\
&= 0 \int_0^t \Phi(s,t)B_w w(s)ds - \int_0^t \Phi(s,t)Lv(s)ds
\end{aligned}$$

$$e(0) = x_0 - \hat{x}(0) = \mathbb{E}[x_0] \text{ (unbiased) } = 0$$

$$\begin{aligned}
\mathbb{E}[ew^\intercal(t)Bw^\intercal] &= \ldots \cancel{\mathbb{E}[x_0 w^\intercal]} \cdots + \int_0^t \Phi(s,t)B_w \mathbb{E}[w(s)w^\intercal(t)]B_w^\intercal ds - \int_0^t \ldots \cancel{\mathbb{E}[v(s)w^\intercal(t)}\ldots ds \\
&= \int_0^t \Phi(s,t)B_w Q B_w^\intercal \delta(t-s)ds
\end{aligned}$$

Using the Fourier Transform Identity: $\int_a^b f(x)\delta(b-x)dx = \frac{1}{2}f(b)$ this becomes:

$$\begin{aligned}
&= \frac{1}{2}\Phi(t,t)B_w Q B_w^\intercal \\
&= I
\end{aligned}$$

Repeat this for the three other terms and you'll ultimately arrive at the continuous time covariance equation:

$$\dot{\Sigma}(t) = (A - LC)\Sigma^\intercal(t) + \Sigma(t)(A - LC)^\intercal + B_w Q B_w^\intercal + LRL^\intercal$$

March 2nd: Alternative Version

15.6 General Bayesian Filtering

Let $x_k \in \mathbb{R}^n$, $w_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^m$, and $v_k \in \mathbb{R}^m$ represent the state, process noise, measurement, and measurement noise of the system at time k, respectively. Note that w_k and v_k are iid. The update equations are given by

$$\begin{aligned} x_{k+1} &= f(x_k, w_k) \\ y_{k+1} &= h(x_k, v_k). \end{aligned}$$

We now define the *Linear Innovations Sequence* as $a_k := y_{1:k} = \{y_1, \ldots, y_k\}$. For the following discussion, we will assume the system is Markovian, meaning

1. $p(x_{k+1}|x_k, ..., x_0, a_k) = p(x_{k+1}|x_k)$
2. $p(y_{k+1}|x_k, ..., x_0, a_{k-1}) = p(y_k|x_k)$

15.7 Filtering Problem

The **filtering problem** is to determine $p(x_{k+1}|a_k)$ from $p(x_k|a_{k-1})$, also known as the prior. This process consists of two steps: measurement update, followed by prediction, as outlined below. Measurement Update: $p(x_k|a_{k-1}) \to p(x_k|a_k)$ This process comes directly from Bayes rule, i.e.

$$\begin{aligned} p(x_k|a_k) &= p(x_k|y_k, a_{k-1}) \\ &= \frac{p(x_k, y_k|a_{k-1})}{p(y_k|a_{k-1})} \\ &= \frac{p(y_k|x_k)p(x_k|a_{k-1})}{p(y_k|a_{k-1})} \end{aligned}$$

Prediction: $p(x_k|a_k) \to p(x_{k+1}|a_k)$ From Chapman-Kolmogorov we have

$$p(x_{k+1}|a_k) = \int_{\mathbb{R}^n} p(x_{k+1}|x_k)p(x_k|a_k)dx_k.$$

15.8 Orthogonality Principle

The goal is to find a $z^* \in \mathcal{V} \subset \mathscr{L}^2$ such that $\underbrace{\mathbb{E}[\|x - z^*\|^2]}_{MSE} \leq \mathbb{E}[\|x - z\|^2] \ \forall \ z \in \mathcal{V}$. A depiction of this task can be seen in Figure 15.

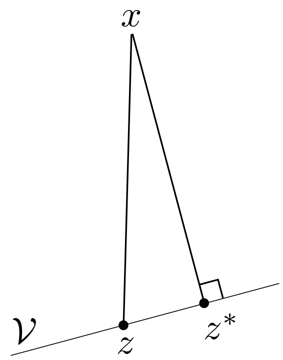


Figure 15: Orthogonality Principle. z^* is value which minimizes the Mean Squared Error value, meaning it is the closest point to x on $\mathcal{V}$.

From the orthogonality principle, we have that for $x \in \mathscr{L}^2$:

1. $\exists$ unique $z^* \in \mathcal{V}$ which is the minimizing Mean Squared Estimator
2. For $y \in \mathscr{L}^2$, $y = z^*$ iff
   (a) $y \in \mathcal{V}$
   (b) $(x - y) \perp z \; \forall \; z \in \mathcal{V} \implies \mathbb{E}[(x-y)z^\top] = 0$
3. $\mathbb{E}[\|x - z^*\|^2] = \mathbb{E}[\|x\|^2] - \mathbb{E}[\|z^*\|^2]$

Given $y \in \mathbb{R}^m$, take $\mathcal{V} = \{c_0 + c_1 y_1 + \cdots + c_m y_m | c_i \in \mathbb{R}^n\}$. Then, there exists $A \in \mathbb{R}^{n \times m}, b \in \mathbb{R}^n$ such that $z^* = Ay + b$. Defining the error term as $e := x - Ay - b$, we have that

1. $e_i \perp 1 \quad i = 1, \ldots, n$
2. $e_i \perp y_j \quad i = 1, \ldots, n; j = 1, \ldots, m$

which implies

$$\begin{aligned} &\mathbb{E}[e] = 0 \qquad (\textit{unbiased}) \\ \implies &\mathbb{E}[x] - A\mathbb{E}[y] - b = 0 \\ \implies &b = \mathbb{E}[x] - A\mathbb{E}[y]. \end{aligned}$$

Additionally, for $Cov(e, y) \in \mathbb{R}^{n \times m}$ we have

$$\begin{aligned} 0 &= Cov(e, y) \\ &= Cov(x - Ay - b, y) \\ &= Cov(x - y) - ACov(y) \\ \implies A &= Cov(x - y)Cov(y)^{-1}. \end{aligned}$$

With all of this, we can state

$$\hat{\mathbb{E}}[x|y] := z^* = \mathbb{E}[x] + Cov(x, y)Cov(y)^{-1}(y - \mathbb{E}[y]).$$

Tying this back to the linear innovation sequence, consider $\tilde{a}_k = \{\tilde{y}_1, \ldots, \tilde{y}_k\}$ where $\tilde{y}_i = y_i - \hat{\mathbb{E}}[y_i|\tilde{a}_{i-1}]$. Note that $\mathbb{E}[\tilde{y}_i] = 0, \tilde{y}_i \perp \tilde{y}_j, i \neq j$. This leads to the important conclusion

$$\hat{\mathbb{E}}[x|\tilde{a}_k] = \overline{x} + \sum_{i=1}^{k} \underbrace{\hat{\mathbb{E}}[x - \tilde{x}|\tilde{y}_i]}_{Cov(x,\tilde{y}_i)Cov(\tilde{y}_i)^{-1}\tilde{y}_i}$$

15.9 Discrete Time Kalman Filter

Under the same Markov assumption as before, we have

$$\begin{aligned} x_{k+1} &= A_k x_k + w_k \\ y_k &= c_k x_k + v_k \end{aligned}$$

where $x_0 \sim \mathcal{N}(\overline{x}_0, \Sigma_0), w_k \sim \mathcal{N}(0, Q), v_k \sim \mathcal{N}(0, R)$ are piecewise uncorrelated. Using the Linear Innovations sequence from before, $a_k := \{\tilde{y}_1, \ldots, \tilde{y}_k\}$, we have

Estimates of x_k	Error Covariances
$\hat{x}_k^{(-)} = \hat{\mathbb{E}}[x_k|\tilde{a}_{k-1}]$	$P_k^{(-)} = Cov(x_k - \hat{x}_k^{(-)})$
$\hat{x}_k^{(+)} = \hat{\mathbb{E}}[x_k|\tilde{a}_k]$	$P_k^{(+)} = Cov(x_k - \hat{x}_k^{(+)})$

Our goal is $(\hat{x}_k^{(-)}, P_k^{(-)}) \to (\hat{x}_{k+1}^{(-)}, P_{k+1}^{(-)})$. <u>Measurement Update: $(\hat{x}_k^{(-)}, P_k^{(-)}) \to (\hat{x}_k^{(+)}, P_k^{(+)})$</u>

$$\tilde{y}_k = y_k - \underbrace{\hat{\mathbb{E}}[y_k|\tilde{a}_{k-1}]}_{c_k\hat{x}_k^{(-)}}$$

$$\begin{aligned} \hat{x}_k^{(+)} &= \hat{\mathbb{E}}[x_k|\tilde{a}_k] \\ &= \hat{\mathbb{E}}[x_k|\tilde{y}_k, \tilde{a}_{k-1}] = \hat{x}_k^{(-)} + \underbrace{Cov(x_k, \tilde{y}_k)Cov(\tilde{y})^{-1}}_{L_k}\tilde{y}_k \end{aligned}$$

$$\begin{aligned} P_k^{(+)} &= Cov(x_k - \hat{x}_k^{(+)}) \\ &= Cov((x_k - \hat{x}_k^{(-)}) - L_k\tilde{y}_k) \\ &= \underbrace{Cov(x_k - \hat{x}_k^{(-)})}_{P_k^{(-)}} - Cov(x_k - \hat{x}_k^{(-)}, \tilde{y}_k)L_k^\top - L_k Cov(\tilde{y}_k, x_k - \hat{x}_k^{(-)}) + L_k Cov(\tilde{y}_k)L_k^\top \\ &= P_k^{(-)} - Cov(x_k, \tilde{y}_k)L_k^\top - L_k Cov(\tilde{y}_k, x_k) + L_k Cov(\tilde{y}_k)L_k^\top \\ &= P_k^{(-)} - 2Cov(\tilde{y}_k, x_k)L_k^\top + L_k Cov(\tilde{y}_k)L_k^\top \\ &= P_k^{(-)} - L_k Cov(\tilde{y}_k)L_k^\top \end{aligned}$$

$$
\begin{aligned}
Cov(x_k, \tilde{y}_k) &= Cov(x_k, y_k - C_k\hat{x}_k^{(-)}) = Cov(x_k, C_k(x_k - \hat{x}_k^{(-)}) + v_k) \\
&Cov(x_k, x_k - \hat{x}_k^{(-)})C_k^\top \\
&= P_k^{(-)}C_k^\top
\end{aligned}
$$

where the last equality holds because $Cov(x_k^{(-)}, x_k - \hat{x}_k^{(-)}) = 0$ by the orthogonality principle.

$$
\begin{aligned}
Cov(\tilde{y}_k) &= Cov(C_k(x_k - \hat{x}_k^{(-)}) + v_k) \\
&= C_k P_k^{(-)} C_k^\top + R
\end{aligned}
$$

From this we have the important conclusion that

$$
\begin{aligned}
\hat{x}_k^{(+)} &= \hat{x}_k^{(-)} + L_k(y_k - C_k\hat{x}_k^{(-)}) \\
P_k^{(+)} &= P_k^{(-)} + L_k(C_k P_k^{(-)} C_k^\top + R)L_k^\top
\end{aligned}
$$

<u>Prediction: $(\hat{x}_k^{(+)}, P_k^{(+)}) \to (\hat{x}_{k+1}^{(-)}, P_{k+1}^{(-)})$</u>

$$
\begin{aligned}
\hat{x}_{k+1}^{(-)} &= \hat{\mathbb{E}}[x_{k+1}|\tilde{a}_k] \\
&= \hat{\mathbb{E}}[A_k x_k + w_k|\tilde{a}_k] \\
&= A_k \underbrace{\hat{\mathbb{E}}[x_k|\tilde{a}_k]}_{\hat{x}_k^{(+)}} + \cancelto{0}{\hat{E}[w_k|\tilde{a}_k]}
\end{aligned}
$$

$$
\begin{aligned}
P_{k+1}^{(-)} &= Cov(x_{k+1} - \hat{x}_{k+1}^{(-)}) \\
&= Cov(Ak(x_k - \hat{x}_k^{(+)}) + w_k) \\
&= A_k P_k^{(+)} A_k^\top + Q
\end{aligned}
$$

15.10 Continuous Time Kalman Filter

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + b_w w(t) \\
y(t) &= Cx(t) + v(t) \\
a_t &= \{y(s) : 0 \le s \le t\}
\end{aligned}
$$

$$
\text{pairwise uncorrelated} \begin{cases} x_o \sim \mathcal{N}(0, \epsilon_0) \\ \mathbb{E}[w(s)w(t)] = Q\delta(t-s) \\ \mathbb{E}[v(s)v(t)] = R\delta(t-s) \end{cases}
$$

The goal is to find $(\hat{x}(t), \Sigma(t))$ where $\Sigma(t) = \mathbb{E}[(x-\hat{x})(x-\hat{x})^\top)]$ such that $J = \mathbb{E}[\dot{e}e^\top + e\dot{e}^\top] = trace(\Sigma(t))$ is minimized. The error term $e(t)$ is given by

$$\begin{aligned} e(t) &:= x(t) - \hat{x}(t) \\ \dot{e}(t) &= \dot{x}(t) - \dot{\hat{x}}(t) \\ &= (A - LC)e(t) + B_w w(t) - Lv(t) \end{aligned}$$

Assume

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + L(y(t) - C\hat{x}(t)) \\ \dot{\Sigma}(t) &= \mathbb{E}[(x-\hat{x})(x-\hat{x})^\top)] \\ &= (A - LC)\Sigma^\top(t) + \Sigma(t)(A^\top - C^\top L^\top) + \mathbb{E}[B_w w e^\top] + \mathbb{E}[e w^\top B_w^\top] - \mathbb{E}[Lve^\top] - \mathbb{E}[ev^\top L^\top] \end{aligned}$$

Recall that $\dot{x} = Ax(t)$, giving rise to

$$x(t) = \underbrace{e^{A(t-t_0)}}_{\Phi(t_0,t)} x_0 + \int_{t_0}^{t_f} \underbrace{e^{A(t-\tau)}}_{\Phi(\tau,t)} Bu(\tau)d\tau$$

For $e(t)$, $\Phi(t_o + t) = e^{(a-LC)(t-t_0)}$, which implies

$$e(t) = \cancelto{0}{\Phi(0,t)e(0)} + \int_{t_0}^{t} \Phi(s,t)B_w w(s)ds - \int_{t_0}^{t} \Phi(s,t)Lv(s)ds$$

$$\dot{\Sigma}(t) = (A - LC)\Sigma^\top(t) + \sigma(t)(A - LC)^\top + B_w Q B_w^\top + LRL^\top$$

16 March 3rd: Kalman Filtering Methods

This lecture covered Discrete Time Kalman Filter and Continuous Time Kalman Filter. The continuous time dynamics of our interest is:

$$\dot{x} = Ax + B_w w$$

where $E(w) = 0$ and $E(w(t)w(t)^T = Qc\delta(t - T)$.
The discrete time dynamics is:

$$x_k = A_d x_{k-1} + w_{k-1}$$

$$y_k = C_d x_k + v_k$$

defined $Q_k = \int_0^{\Delta t} \Phi(\Delta t, t) B_w Q_c B_w^T \Phi(\Delta t, t)^T dt \approx B_w Q_c B_w^T \Delta t$. $\Phi(t_1, t_2) = e^{A(t_1 - t_2)}$, and $E(v_i v_j^T) = R_k \Delta_{ij}$.

Discrete Time Kalman Filter:

$$\hat{x}_k^+ = \hat{x}_k^- + L_k(y_k - C_d\hat{x}_k^-)$$

where $y_k - C_d\hat{x}_k^-$ is termed as the innovation.

$$J = E((\tilde{x}_k^+)^T\tilde{x}_k) = trace(P_k)$$

where $x = x - \hat{x}$, and by the first order necessary condition: $\frac{\partial J}{\partial L_k} = 0$.

Note the following trace properties: $\frac{d(trace(AB)}{dA} = B^T$ and $\frac{d(trace(ACA^T)}{dA} = AC$. Thus,

$$L_k = R_k^- C_d^T(C_dP_k^-C_d^T + R_k)^{-1}$$

$$P_k^+ = E((x_k - \hat{x}_k^+)(x_k - \hat{x}_k^+)^T) = (I - L_kC_d)P_k^-(I - L_kC_d)^T + L_kP_kL_k^T$$

SubstitutingL_k and get,

$$P_k^+ = (I - L_kC_d)P_k^-$$

Another useful symmetric form is:

$$P_k^+ = P_k^- - P_k^-C_d^T(C_dP_k^-C_d^T + R_k)^{-1}C_dP_k^-$$

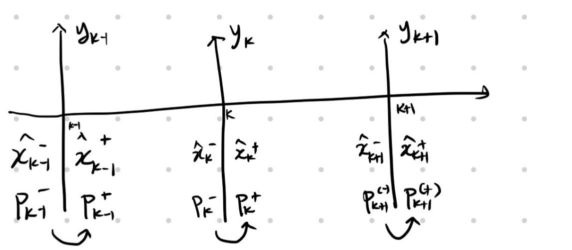


Figure 16: An pictorial representation of update sequence.

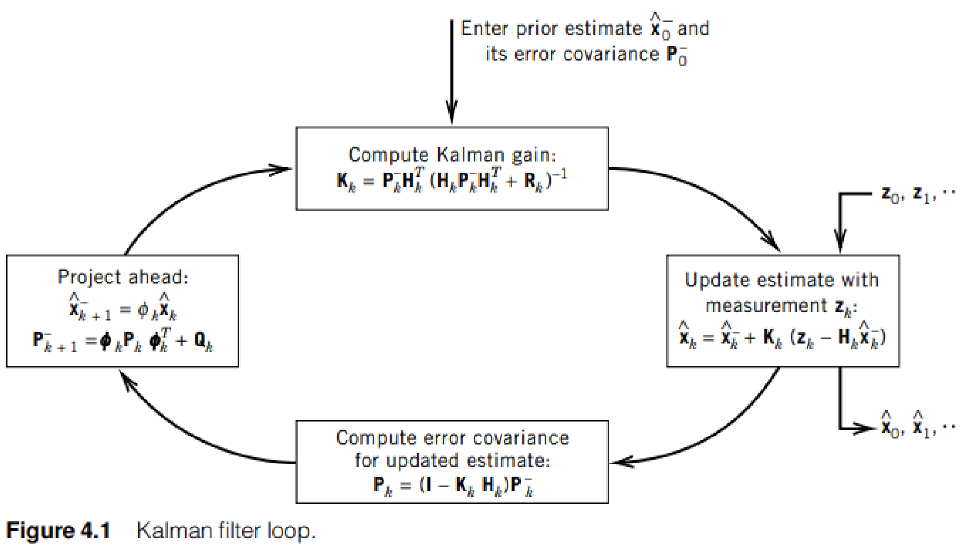


Figure 17: An pictorial representation of discrete time Kalman filter loop.

Matrix Inversion Lemma:

$$(A^{-1} + B^T C^{-1} B)^{-1} = A - AB^T (BAB^T + C)^{-1} BA$$

Let $A \to P_k^-$, $B \to C_d$, $C \to R_k$, we can get the covariance update in the following form:

$$(P_k^+)^{-1} = (P_k^-)^{-1} + C_d^T R_k {-} 1 C_d$$

known as the least squares with prior estimate, and further we can use this form of P_k^+ to get the Kalman filter gain as:

$$L_k = P_k^+ C_d^T R_k^{-1}$$

Steady State: Need to differentiate P_{ss}^- and P_{ss}^+. If steady state, then we have $P_{ss} = P_k = P_{k+1}$. Thus,

$$P_{ss}^- = A_d P_{ss}^+ Ad + Q_k$$

$$P_{ss}^+ = P_{ss}^- - P_{ss}^- C_d^T (C_d P_{ss}^- C_d^T + R_k)^{-1} C_d P_{ss}^-$$

Continuous Time Kalman Filter (1st method using Discrete time KF): Convert Discrete time Kalman filter to continuous time Kalman filter by taking $\Delta t \to 0$, which leads to $R_k \Delta t \to R$, $P_k^- \to P(t)$, and $C_d \to C_y$. Moreover,

$$\frac{L_k}{\Delta t} = L(t) = P C_y^T R^{-1}$$

$$\dot{\hat{x}} = A\hat{x} + L(t)(y - C_y \hat{x})$$

$$\dot{P} = \frac{P_{k+1}^- - P_k^-}{\Delta t}$$

The Riccatti Equation for continuous time Kalman filter is:

$$\dot{P} = AP + PA^T + B_w Q_c B_w^T - P C_y^T R^{-1} C_y P$$

Continuous Time Kalman Filter (2nd method): Let $\tilde{x} = e = x - \hat{x}$, and taking time derivative:

$$\dot{e} = (A - LC)e + B_w w - Lv$$

$$P = cov(e)$$

$$\dot{P} = (A - LC)P + P(A - LC)^T + B_w Q_c B_w^T + LRL^T$$

$$\min(E(e^T e)) = \min(trace(P))$$

$$\min(P) = \min(\int \dot{P} dt) = \min(trace(\dot{P}))$$

Combining everything, we shall arrive at the following minimization problem:

$$\min(trace((A - LC)P + P(A - LC)^T + B_w Q_c B_w^T + LRL^T)$$

Using first order necessary condition for optimal solution by solving: $\frac{\partial trace(\dot{P})}{\partial L} = 0$, we can get the continuous time Kalman filter gain as:

$$L = P C_y^T R^{-1}$$

and the Riccatti equation:

$$\dot{P} = AP + PA^T + B_w Q_c B_w^T - P C_y^T R^{-1} C_y P$$

Table comparing LQR and KF:

LQR	KF
B	C^T
C_z	B_w
Q_{zz}	Q_c
R	R
P	P
K	L^T

Thus, LQR and KF are duals of each other.

17 March 5th: Particle Filtering

Adapted from Karena Cai's blogpost at: https://medium.com/@cai.karena/math-you-need-to-estimate-the-weather-stocks-robots-and-more-d4bb76dd18fe

17.1 Introduction to Particle Filtering

Particle filtering is a technique where we use Monte Carlo methods to iterative estimate the state of the system at every time step following the model:

$$x_{t+1} = f(x_t) + n_t$$

(x is the state of the system and n is additive noise).

17.2 Example: 1-D car

Assume the following model:

$$\begin{aligned} x_{t+1} &= x_t * \triangle t + n_t \\ v_{t+1} &= v_t + w_t \\ e_t &= [n_t; w_t] \sim N(0, \sigma_1^2) \end{aligned}$$

(where n and w are Gaussian white noise, x is the position, and v is velocity).

17.3 Measurement Model

Nonlinear measurements have the form:

$$y_{t+1} = h(x_{t+1}) + \eta_{t+1}$$

$$\eta_t \sim N(0, \sigma_2^2)$$

For a GPS signal of our car, measurements are:

$$y_{t+1} = x_{t+1} + \eta_{t+1}$$

Another sensor that notes the distance to an object with defined location:

$$y_{t+1} = |(x_{t+1} - p_x)| + \eta_{t+1}$$

(where p is the position of the defined object).

The GPS signals will have some uncertainty.

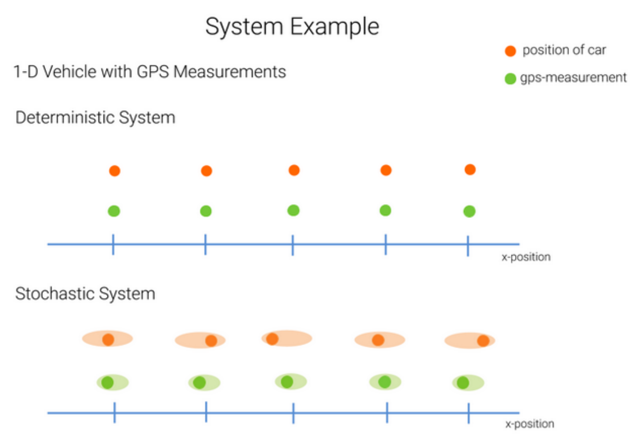


Figure 18: Orange and light green indicate the range of noise in the GPS signal and in the dynamics of the car.

We want to find the "posterior" probability distribution of the car being at a certain position given a string of measurements, namely:

$$p(x_t | y_{0:t})$$

Use Baye's Theorem to calculate posterior.

$$p(x_t | y_{0:t}) = \frac{p(y_t | x_t) p(x_t | y_{0:t-1})}{p(y_t | y_{0:t-1})}$$

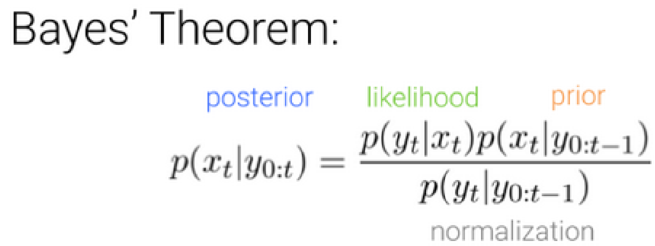


Figure 19

The normalization factor ensures the posterior takes on values between 0 and 1.

The *prior distribution* is our guess on where the state of the system will be as it evolves to the next time step.

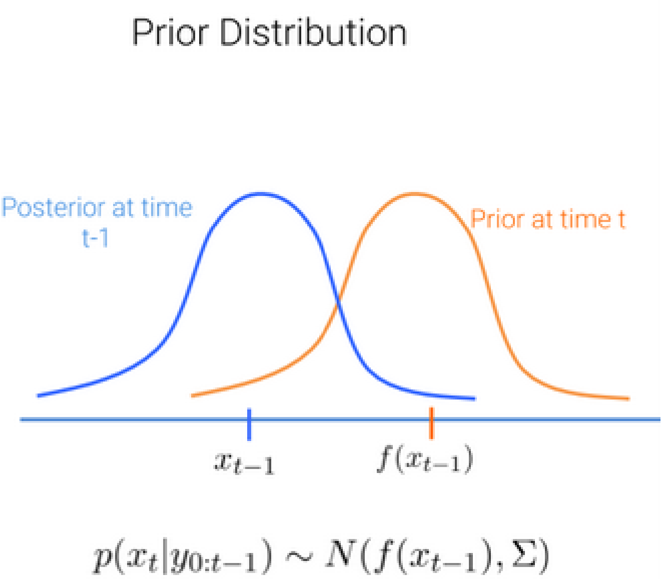


Figure 20: Orange is the blue propagated forward a time step by the dynamics.

The *likelihood function* tells us the probability of receiving a particular measurement y at position x.

17.4 Particle Filtering

Particle filtering is a specific Monte-Carlo method to iteratively derive the posterior distribution using Baye's Rule.

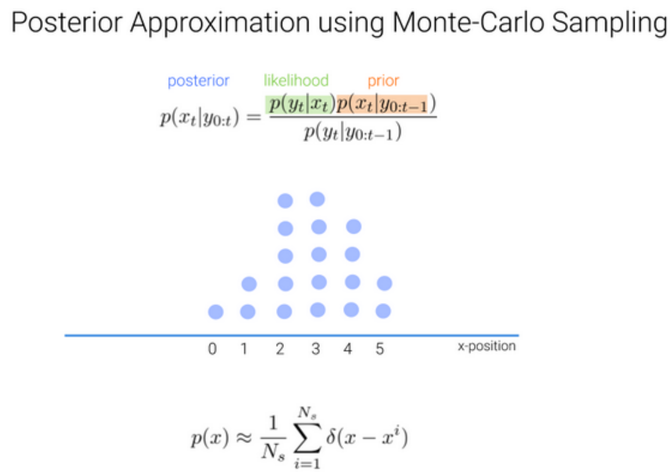


Figure 21: Formula states that the number of points in a bin divided by the total number of points approximately gives the probability of being in that pin.

We cannot sample directly from the posterior, but we can sample from the prior distribution and weigh the probabilities by the likelihood function.

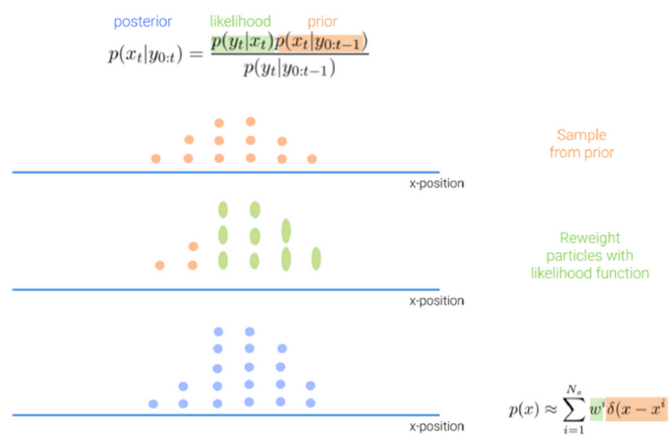


Figure 22: Stretched particles mean they have more weight, thus adding more particles to the bin that particle was in.

17.5 PF Algorithm

Step 1: Sample n particles from some initial Gaussian distribution estimating the initial state of the system at time 0.

Step 2: Propagate every point x forward through the noisy dynamics. The new distribution of particles is the prior distribution.

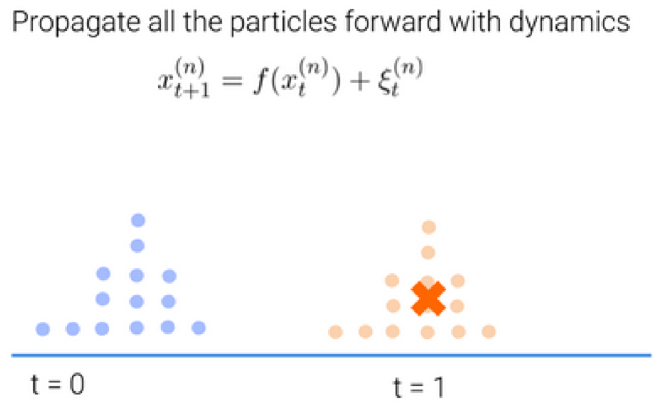


Figure 23: Red X marks the mean, or the postion the car is most likely in.

Step 3: Find the weights of all the particles. Then redistribute the particles such that all of them have equal weights. This will reshape the prior.

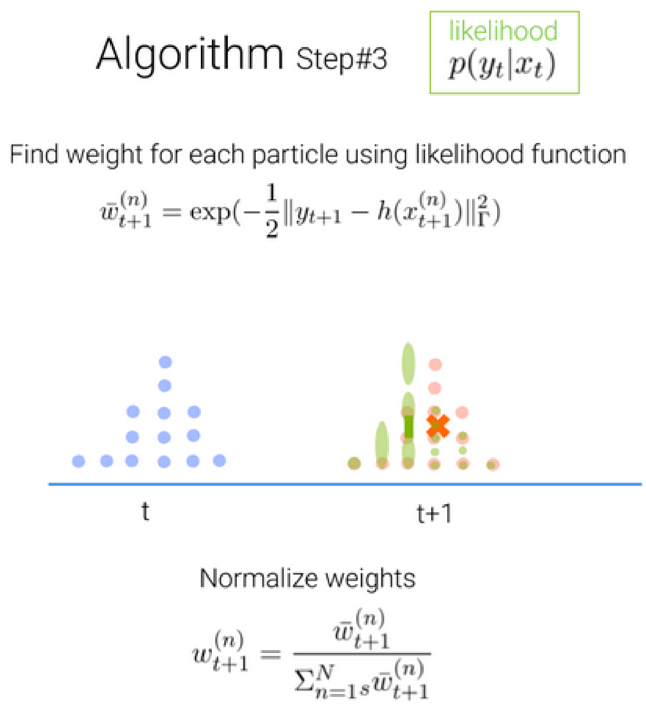


Figure 24: Finding the weights of all the particles in the prior.

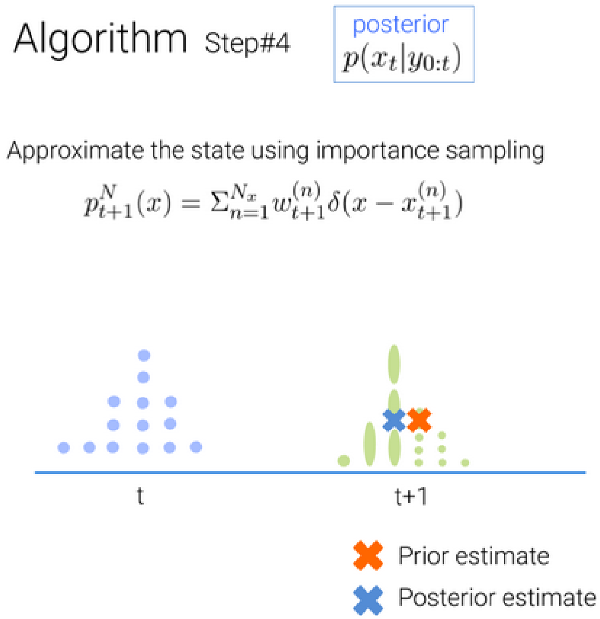


Figure 25: Reshape the distribution such that all the particles have equal weights, but each bin maintains the same mass as after the weighing.

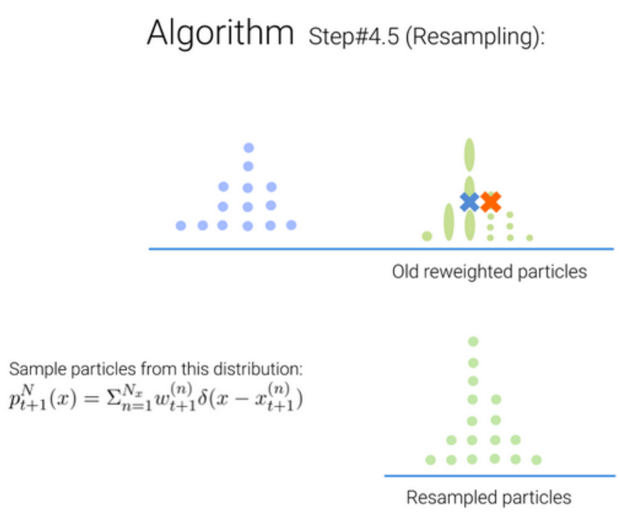


Figure 26: Treating the mass of the bin as a relative probability, sample n particles again such that each one has the same weight.

17.6 Particle degeneration

By reshaping the distribution such that all particles have equal weights again, we can avoid particle degeneration, where one particle has all the weight.

Figure 27

17.7 Resources

More rigorous introduction: [Link]