Learning-Based Control of Stochastic Discrete-Event Systems via Martingale and Renewal Theory Analysis of Pattern Occurrence

SooJean Han*and Soon-Jo Chung†

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Abstract

Stability of stochastic systems that are perturbed by external Poisson shot noise can be analyzed. In reality, however, random impulsive behavior in stochastic systems can come in a multitude of distributions and from different sources, e.g. due to external noise or due to abrupt switches as an inherent feature of the dynamics. In this paper, we extend our analysis to discrete-time stochastic systems with repeated patterns of impulse behavior generated from Markov processes. Many modern methods of control for large-scale stochastic systems of this type can be decomposed as a hierarchical controller composed of two parts. The first part is a pattern-learning component which recognizes specific state sequences and stores in memory the corresponding control action that needs to be taken when the sequence has occurred, while the second part is a modulation control component which computes the optimal control action for a pattern when it has occurred for the first time. In this paper, we focus on the first part: analyzing patterns of the trajectories from the network state in order to approximate the behavior of its external process. We survey various methods of deriving closed-form solutions to problems related to pattern-occurrence. We focus specifically on two cases: 1) when patterns are generated from an i.i.d. sequence, and 2) when patterns are generated from a Markov chain. We leverage concepts from renewal and martingale theory to derive analytical closed-form expressions for the mean duration time between consecutive occurrences, and first occurrence probabilities; this provides the theoretical backbone of the pattern-learning component. We demonstrate application of this two-part controller framework to a simple power grid network with time-varying topology.

1 Introduction

We studied a property of incremental stability of nonlinear stochastic systems perturbed by Poisson shot noise [1]. Although many existing model-based controller synthesis methods exist primarily for Gaussian stochastic systems, our analysis in [1] suggested the possibility of expanding such methods to Poisson shot noise perturbations. Subsequently, we may extend model-based controller synthesis methods to more general distributions of random impulsive behavior. For discrete time and event systems, these extensions come more naturally because impulse behavior tends to occur as repeated patterns. Common methods of fault diagnosis and prediction for deterministic systems invoke a labeled transition representation for their ease in identifying repeated sequences of states over time [2, 3]; common discrete systems which are

*SooJean Han and Soon-Jo Chung are with the Department of Computing and Mathematical Sciences, California Institute of Technology, Pasadena CA 91125. Email: soojean@caltech.edu

†Soon-Jo Chung is also with the Graduate Aerospace Laboratory, California Institute of Technology. Email: sjchung@caltech.edu.

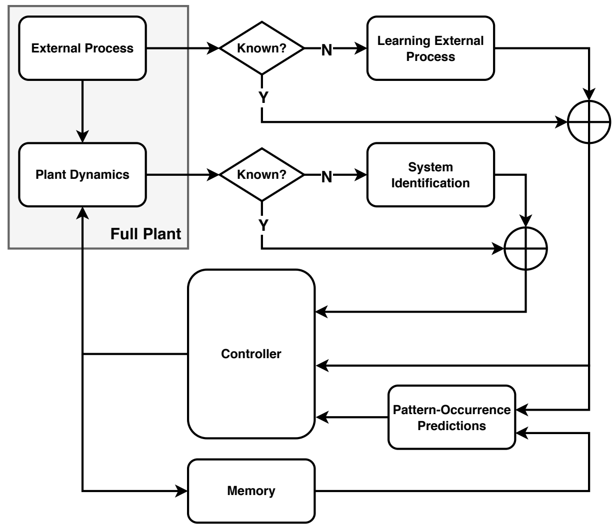

Figure 1: A flow diagram representation of the two-part hierarchical controller

stochastic in nature are queueing-based systems, where repetition arises from the time-varying number of entities in the queue.

Some learning-based methods of control for stochastic systems may still be inefficient without using knowledge of repetitive states in the system. One way of reducing extraneous online computation of control laws is to preserve the system's past trajectories and its corresponding control law into memory, so that control laws for environments which have been observed before do not need to be re-computed. This insight is not new to the engineering community; in fact, a number of state-of-the-art machine learning algorithms today are being developed for the purposes of efficient control by retaining memory of past trajectories; some notable methods are recurrent neural networks, long short-term memory networks (LSTMs), and imitation learning [4]. While there has been no shortage of numerical demonstrations suggesting that the use of memory improves the performance of state-of-the-art controllers, there has been little mathematical theory to support them. Explicit model-based characterizations are especially useful in the robust control against specific state patterns which cause faults in the system.

In this paper, we generalize our setup in [1] by considering discrete stochastic systems with repeated patterns of impulse behavior which can be modeled as Markov processes. We use renewal theory and martingale theory to derive mathematical expressions for certain quantities, such as the mean duration between consecutive pattern occurrences, and the probability of a pattern being the first to occur among a specific group of patterns. Under minimal simplifying assumptions, many approaches towards learning-based control with memory can be characterized by a modular framework shown in Figure 1, composed of the following two key parts:

- a *pattern-learning component* which recognizes specific state patterns and stores in memory the corresponding control action applied to the system when a pattern has occurred
- the *modulation control component* which computes the control action to be taken when a pattern is first observed

Designing the pattern-learning component based on the theoretical foundation described above enables intelligent design of the present based on an anticipated future. We demonstrate example applications of this controller to an extension of the our previous work in [5], which presented an iterative, robust, localized *system-level synthesis (SLS)* approach to the control of a power grid network, and apply the pattern-learning component for fault-tolerance and prediction when the network topology varies over time with link deletions and additions.

2 The Prediction of Patterns

3 Independent, Identically-Distributed Sequences

3.1 Problem Formulation

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random vector values, $X_n = (X_{1,n}, \cdots, X_{n_x,n})$, such that $X_n : \Omega \to \mathbb{R}^{n_x}$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_n\}_{n=1}^{\infty}$ defined by $\mathcal{F}_n \triangleq \sigma(X_0, X_1, \cdots, X_n)$. We henceforth denote to all sequences of the form $\{\cdot\}_{n=1}^{\infty}$ using the shorthand notation $\{\cdot\}$, e.g., $\{X_n\}_{n=1}^{\infty} \equiv \{X_n\}$ and $\{\mathcal{F}_n\}_{n=1}^{\infty} \equiv \{\mathcal{F}_n\}$. We fix $\{X_n\}$ to be generated from a sequence of i.i.d. random variables taking values from a discrete, finite set $\mathcal{X}$, with probability distribution $p(x) \triangleq \mathbb{P}(X_n = x)$ for all $n \in \mathbb{N}$ and $x \in \mathcal{X}$.

Definition 1 (Patterns)**.** Denote finite, ordered sequences of $\{X_n\}_{n=1}^{\infty}$ as $A \triangleq (\mathbf{a}_1, \cdots, \mathbf{a}_m)$, $B \triangleq (\mathbf{b}_1, \cdots, \mathbf{b}_r)$, and $B_j \triangleq (\mathbf{b}_1^{(j)}, \cdots, \mathbf{b}_{r_j}^{(j)})$ for any $j \in \{1, \cdots, K\}, K \in \mathbb{N}$, where the $\mathbf{a}_i, \mathbf{b}_\ell^{(j)}, \mathbf{b}_k \in \mathcal{X}$ and $m, r, r_1, \cdots, r_K \in \mathbb{N}$. We refer to these specific sequences as *patterns (of interest)* throughout the remainder of the paper. □

For occurrences of the pattern A in the sequence $\{X_n\}$, we can associate an integer-valued stochastic process $\{N(t), t \geq 0\}$ to the sequence of random variables $\{S_n\}$ such that $N(t) = \max\{n > 0 | T_n \leq t\}$, where $T_0 \triangleq 0$, $T_n \triangleq \sum_{m=1}^{n} S_m$ for $n \geq 1$ are the arrival times of the process. Then $N(t)$ is a *renewal process* and occurrences of A in $\{X_n\}$ are called *renewals*. In particular, if $\{R_n\}$ is another random variable sequence such that a reward of R_n is gained at the time of the nth renewal, then such types of renewal processes are referred to as *renewal reward processes*.

For the patterns described in Definition 1, define the following stopping times: $\tau_{0,B}$ is the time until the first occurrence of pattern B, τ_B is the time between consecutive occurrences of pattern B, and $\tau_{B|A}$ is the time until the first occurrence of pattern B after pattern A has already been observed.

$$\begin{aligned} \tau_{0,B} &\triangleq \min\{n \in \mathbb{N} \mid (X_n, \cdots, X_{n+k-1}) = B\} \\ \tau_B &\triangleq \min\{n \in \mathbb{N} \mid (X_{n+\tau_{0,B}}, \cdots, X_{n+\tau_{0,B}+k-1}) = B\} \\ \tau_{B|A} &\triangleq \min\{n \in \mathbb{N} \mid (X_{n+\tau_A}, \cdots, X_{n+\tau_A+k-1}) = B\} \end{aligned} \tag{1}$$

Here, $\tau_{0,A}, \tau_A, \tau_{A|B}$ are defined similarly. For the collection of patterns $\mathcal{B} \triangleq \{B_1, \cdots, B_M\}$, we specifically write the shorthand notations

$$\tau_j \triangleq \tau_{B_j|A}, \quad i, j \in \{1, \cdots, M\} \tag{2}$$

and further define

$$\tau \triangleq \min_{j \in \{1, \cdots, M\}} \tau_j, \quad \alpha_j \triangleq \mathbb{P}(\tau = \tau_j) \tag{3}$$

Definition 2 (Pattern Overlap)**.** For a sequence $\{X_n\}_{n=1}^{\infty}$ taking values $x_1, x_2, \cdots$ from a certain probability distribution, we say that a pattern $(a_1, a_2, \cdots, a_m)$ has an *overlap* of size $k < m$ if

$$k \triangleq \max\{\ell < m \mid (a_1, \cdots, a_\ell) = (a_{m-\ell+1}, \cdots, a_m)\}$$

□

Problem 1 (Pattern-Occurrence)**.** We are interested in addressing the following:

1. characterize the distributions of $\tau_{0,B}, \tau_B$, and $\tau_{B|A}$, especially their expected values.
2. characterize the distributions of τ and $\{\alpha_j\}_{j=1}^{M}$, especially $\mathbb{E}[\tau]$ and the α_j.

a formula for the case of multiple patterns is discussed in [6]. The setting where $\{X_n\}$ is generated from a Markov chain is discussed in [7], but only for scalars and when the chain takes only two values, i.e. $|\mathcal{X}| = 2$.

3.2 A Geometric Distribution Argument

In this problem, we are interested in the expected value of τ_A. Within the specific context of the pattern occurrence problem, we refer to a "renewal" as the event when the pattern string A repeats itself. Hence, the interarrival times of such a renewal process are distributed in the same manner as τ_A.

The following case-by-case argument, inspired by [8], derives $\mathbb{E}[\tau_{0,A}], \mathbb{E}[\tau_A]$ and $\mathbb{E}[\tau_{B|A}]$ by relating them to Geometric random variables. There are two cases that need to be considered, depending on the amount of overlap there is in the pattern we are interested in, where an overlap is defined in Definition 2.

- ***Case 1: there is no pattern overlap.*** In this case, it is intuitive that the occurrence time T can be treated as a geometric random variable. Hence,

$$\mathbb{E}[\tau_A] = \frac{1}{\prod\limits_{i=1}^{m} p(a_i)} \tag{4}$$

  since $\prod_{i=1}^{m} p(a_i)$ is the probability of observing exactly the sequence $(a_1, \cdots, a_m)$. For a more comprehensive derivation of this formula, note that $T > n+m$ iff the pattern does not occur for the first n values, and the next m values are $(a_1, \cdots, a_m)$. That is:

$$\{\tau_A > n+m\} \iff \{\tau_A > n \text{ and } (X_{n+1}, \cdots, X_{n+m}) = (a_1, \cdots, a_m)\}$$

  In terms of probabilities:

$$\mathbb{P}(\tau_A > n+m) = \mathbb{P}(\tau_A > n)\mathbb{P}((X_{n+1}, \cdots, X_{n+m}) = (a_1, \cdots, a_m)) = \mathbb{P}(\tau_A > n) \prod_{i=1}^{m} p(a_i)$$

  Note that by the definition of expected value:

$$1 = \sum_{n=0}^{\infty} \mathbb{P}(\tau_A > n+m) = \prod_{i=1}^{m} p(a_i) \sum_{n=0}^{\infty} \mathbb{P}(\tau_A > n) = \prod_{i=1}^{m} p(a_i)\mathbb{E}[\tau_A]$$

  Indeed, dividing through by $\prod_{i=1}^{m} p(a_i)$ yields (4).

- ***Case 2: there is a pattern overlap of size k, which does not have an overlap itself.*** Suppose the pattern $(a_1, \cdots, a_m)$ has an overlapping subsequence $(a_1, \cdots, a_k)$ which itself does not have an overlap. Note that if $(a_1, \cdots, a_k)$ has an overlap, then we can repeat Case 2 by induction.

  Define $\tau_{A,y}$ to be the next time after the first occurrence of the augmented pattern $(a_1, \cdots, a_m, y)$ it takes to observe the augmented pattern again. Then it is easy to see that $\tau_{A,y} > \tau_A$, that is:

$$\tau_{A,y} = \tau_A + \Delta\tau \tag{5}$$

where $\Delta\tau$ is the time after the next occurrence of the original pattern $(a_1, \cdots, a_m)$ it takes to observe the next occurrence of the augmented pattern. Note that the computation of $\mathbb{E}[\tau_{A,y}]$ is a simple instance of Case 1, since the addition of y to the original sequence removes the overlap. Hence:

$$\mathbb{E}[\tau_{A,y}] = \frac{1}{p(y)\prod_{i=1}^{m} p(a_i)}$$

We can compute $\mathbb{E}[\Delta\tau_A]$ by conditioning on the next value X, assuming the sequence $(a_1, \cdots, a_m)$ has already been observed.

$$\mathbb{E}[\Delta\tau | X = z] = \begin{cases} 1 + \mathbb{E}[\tau_{A,y} | a_1, \cdots, a_{k+1}] & \text{if } z = a_{k+1} \\ 1 + \mathbb{E}[\tau_{A,y} | a_1] & \text{if } z = a_1 \\ 1 & \text{if } z = y \\ 1 + \mathbb{E}[\tau_{A,y}] & \text{if } z \notin \{a_1, a_{k+1}, y\} \end{cases}$$

from which we can construct equations and compute $\mathbb{E}[\tau_A]$. Namely:

$$\mathbb{E}[\Delta\tau_A] = 1 + p(a_{k+1})\mathbb{E}[\tau_{A,y}|a_1, \cdots, a_{k+1}] + p(a_1)\mathbb{E}[\tau_{A,y}|a_1] + (1 - p(a_{k+1}) - p(a_1) - p(y))\mathbb{E}[\tau_{A,y}]$$

where $p(y)$ denotes the probability of observing y in a trial. Note that we can directly write, by definition,

$$\begin{aligned} \mathbb{E}[\tau_{A,y}] &= \mathbb{E}[\tau(a_1, \cdots, a_{k+1})] + \mathbb{E}[\tau_{A,y}|a_1, \cdots, a_{k+1}] \\ \mathbb{E}[\tau_{A,y}] &= \mathbb{E}[\tau(a_1)] + \mathbb{E}[\tau_{A,y}|a_1] \end{aligned}$$

where $\tau(a_1, \cdots, a_j)$ represents the time it takes to observe the next occurrence of the pattern $(a_1, \cdots, a_j)$ after its first occurrence. Since neither of the sequences $(a_1, \cdots, a_{k+1})$ nor (a_1) have overlaps, we can use the first case to compute:

$$\mathbb{E}[\tau(a_1, \cdots, a_{k+1})] = \frac{1}{\prod_{i=1}^{k+1} p(a_i)}, \quad \mathbb{E}[\tau(a_1)] = \frac{1}{p(a_1)}$$

In combination with (5), we get:

$$\begin{aligned} \mathbb{E}[\tau_{A,y}] = \mathbb{E}[\tau_A] + 1 + p(a_{k+1})\left(\mathbb{E}[\tau_{A,y}] - \mathbb{E}[\tau(a_1, \cdots, a_{k+1})]\right) + p(a_1)\left(\mathbb{E}[\tau_{A,y}] - \mathbb{E}[\tau(a_1)]\right) \\ + (1 - p(a_{k+1}) - p(a_1) - p(y))\mathbb{E}[\tau_{A,y}] \end{aligned}$$

$$\implies p(y)\mathbb{E}[\tau_{A,y}] = \mathbb{E}[\tau_A] + \cancel{1} - p(a_{k+1})\mathbb{E}[\tau(a_1, \cdots, a_{k+1})] - \cancel{p(a_1)\mathbb{E}[\tau(a_1)]}$$

$$\implies \mathbb{E}[\tau_A] = \cancel{p(y)} \cdot \frac{1}{\cancel{p(y)}\prod_{i=1}^{m} p(a_i)} + p(a_{k+1}) \cdot \frac{1}{\prod_{i=1}^{k+1} p(a_i)}$$

In conclusion, where a specific pattern $(a_1, \cdots, a_m)$ has an overlap of size k

$$\mathbb{E}[\tau_A] = \frac{1}{\prod_{i=1}^{m} p(a_i)} + \frac{1}{\prod_{i=1}^{k} p(a_i)}$$

3.3 Interpretation via Martingale Theory

In contrast to Section 3.2, the pattern-occurrence problems can be solved from an alternative perspective using martingales, which we adapt from [9].

We first make the following notations. For two patterns A and B defined in Definition 1:

$$\delta_{i,j}(A,B) \triangleq \begin{cases} \frac{1}{\mathbb{P}(X=b_j)} & \text{if } a_i = b_j \\ 0 & \text{else} \end{cases} \tag{6}$$

for all $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, r\}$.

Now suppose, at each time instant $n \in \mathbb{N}$, a gambler arrives and bets money on the event $\{(X_n, X_{n+1}, \cdots, X_{n+r-1}) = B\}$, using betting strategy $\{Y_{n,j}\}_{j=0}^{r-1}$, starting from $Y_{n,0} = 1$. Each gambler's strategy is defined such that the game has fair odds, i.e.

$$Y_{n,j+1} \triangleq \begin{cases} \left(\frac{1}{\mathbb{P}(X=b_{j+1})}\right) Y_{n,j} - Y_{n,j} & \text{if } X_{n+r-j} = b_{j+1} \\ -Y_{n,j} & \text{if } X_{n+r-j} \neq b_{j+1} \end{cases} \tag{7}$$

for each $j \in \{0, \cdots, r-1\}$. When a gambler first loses everything he has, he leaves the game and never returns. Hence, gambler n's total earnings by some time $\ell \in \mathbb{N}$ is given by $R_{n,\ell} = \sum_{j=0}^{r-1} Y_{n,j} \mathbb{1}\{n+j \leq \ell\}$, given by

$$R_{n,\ell} = \begin{cases} \left(\prod_{\substack{j\in\{0,\cdots,r-1\} \\ n+j\leq\ell}} \mathbb{P}(X=b_j) \right)^{-1} - 1 & \text{if } \forall j \in \{0,\cdots,r-1\} \text{ s.t. } n+j \leq \ell,\ \delta_{n+j,j+1}(X^n, B) > 0 \\ -1 & \text{if } \exists j \in \{0,\cdots,m-1\} \text{ s.t. } n+j \leq \ell,\ \delta_{n+j,j+1}(X^n, B) = 0 \end{cases} \tag{8}$$

where we denote $X^n \triangleq (X_1, \cdots, X_n)$. We set up the betting strategy of each gambler in this way so that the accumulated net reward over all gamblers $\{\sum_{n=1}^{\ell} R_{n,\ell}\}_{\ell=1}^{\infty}$ of the game is a martingale. Then by the optional stopping theorem, we can have that $\mathbb{E}[\sum_{n=1}^{\ell\wedge\tau_B} R_{n,\ell\wedge\tau_B}] = 0$. This makes it easy to compute $\mathbb{E}[\tau_B]$, as we demonstrate concretely using the example below.

Example 1 (4-Sided Die)**.** Suppose we roll a 4-sided die with the sequence of possible outcomes $X_1, X_2, \cdots$ distributed as

$$\mathbb{P}(X_n = k) = \begin{cases} p_1 \triangleq \frac{1}{4} & \text{if } k = 1 \\ p_2 \triangleq \frac{1}{3} & \text{if } k = 2 \\ p_3 \triangleq \frac{1}{4} & \text{if } k = 3 \\ p_4 \triangleq \frac{1}{6} & \text{if } k = 4 \end{cases}$$

for all $n \in \mathbb{N}$. Suppose we are interested in observing the pattern $B \triangleq (b_1, \cdots, b_r) \triangleq (2,3,4,2,3)$ of length $k = 5$. Using the formula from Section 3.2 and the fact that B has an overlap of size 2, we get

$$\mathbb{E}[\tau_B] = \left(\frac{1}{3\cdot 4\cdot 6\cdot 3\cdot 4}\right)^{-1} + \left(\frac{1}{3\cdot 4}\right)^{-1} = 876$$

We can obtain the same number via the martingale interpretation. Suppose the current sequence of die outcomes until time $\ell = 11$ is given by $X^\ell = (1,2,4,2,3,1,2,3,4,2,3)$, then only two gamblers remain by time 11: gambler 7 and gambler 10. The other gamblers, $1,2,3,4,5,6,8,9,11$ leave the game at timesteps $1,3,3,5,5,6,8,9,11$, respectively. Using (8), we have total earnings

$$\sum_{n=1}^{\tau_B} R_{n,\tau_B} = W_B(\tau_B) - \tau_B \tag{9}$$

where $W_B(\tau_B)$ denotes the total winnings accumulated over all the gamblers, who bet on observing pattern B, until time τ_B. Using the numerical values and the specific X^n defined above, the expected winnings of gambler 7 is given by $1/(p_2p_3p_4p_2p_3) = 864$ at time 11, while the expected winnings of gambler 10 is given by $1/(p_2p_3) = 12$; the other gamblers have left the game with winnings 0. Hence:

$$0 = \mathbb{E}\left[\sum_{n=1}^{\tau_B} R_{n,\tau_B}\right] = \mathbb{E}[W_B(\tau_B)] - \mathbb{E}[\tau_B] = (864+12) - \mathbb{E}[\tau_B] \tag{10}$$

where the left-hand side of the above is equal to zero because it is a martingale by the optional stopping theorem. Therefore, algebraic manipulation yields $\mathbb{E}[\tau_B] = 864 + 12 = 876$, which matches with the result obtained using the method of Section 3.2. $\square$

Note that $\mathbb{E}[\tau_{B|A}]$ can be computed similarly to $\mathbb{E}[\tau_B]$, using (9):

$$\sum_{n=\tau_A+1}^{\tau_B} R_{n,\tau_B} = \sum_{n=1}^{\tau_B} R_{n,\tau_B} - \sum_{n=1}^{\tau_A} R_{n,\tau_B} = W_B(\tau_B) - \tau_B - W_B(\tau_A) + \tau_A = W_B(\tau_B) - W_B(\tau_A) + \tau_{B|A} \tag{11}$$

since $\tau_{B|A} = \tau_B - \tau_A$, and $W_B(\tau_A)$ denotes the total winnings obtained by all the gamblers who gamble to view outcome B until time τ_A. Below, we elaborate on the 4-sided die example to discuss the computation of $W_B(\tau_A)$.

Example 2 (4-Sided Die Continued)**.** In continuation of the setup of Example 1, suppose that we have already observed a die outcome of $A \triangleq (2,4,2,3)$, with length $m = 4$. Thus, a total of 4 gamblers play a game throughout A, and obtain various rewards depending on (partial) observance of B. Based on (8) for this particular sequence, the only gambler who has positive expected winnings the third gambler, who wins $1/(p_2p_3) = 12$; all the other gamblers have 0 expected winnings. Thus, $\mathbb{E}[W_B(\tau_A)] = 12$, and so

$$0 = \mathbb{E}\left[\sum_{n=\tau_A+1}^{\tau_B} R_{n,\tau_B}\right] = \mathbb{E}[W_B(\tau_B)] - \mathbb{E}[W_B(\tau_A)] + \mathbb{E}[\tau_{B|A}] \implies \mathbb{E}[\tau_{B|A}] = 876 - 12 = 864$$

$\square$

For the sake of easier notation, we further denote

$$A \Diamond B = \prod_{i=1}^{\min(m,r)} \delta_{i,i}(A,B) + \prod_{i=1}^{\min(m-1,r)} \delta_{i+1,i}(A,B) + \cdots + \delta_{\min(m,r),1}(A,B) \tag{12}$$

Essentially, (12) conveys the total winnings obtained by a gambler who is betting to observe (partial) occurrences of B in the sequence A. As a result, we can rewrite the expressions for $\mathbb{E}[\tau_B]$ and $\mathbb{E}[\tau_{B|A}]$ defined above using (12), described in the following theorem.

Theorem 1 (Expected Waiting Time Until a Sequence)**.** The expected waiting time for a sequence B is given by

$$\mathbb{E}[\tau_B] \triangleq B \Diamond B \tag{13}$$

Given a starting sequence A, the expected waiting time for a sequence B is given by

$$\mathbb{E}[\tau_{B|A}] \triangleq B \Diamond B - A \Diamond B \tag{14}$$

The proof of Theorem 1 basically hinges on the fact that the stopped process $\{\sum_{n=1}^{\ell\wedge\tau} R_{n,\ell\wedge\tau}\}_{\ell=1}^{\infty}$, for some stopping time τ, is a martingale. Formally, we require the following important theorem for martingales.

Lemma 1 (Doob's Martingale Convergence)**.** Let $\{X_n\}_{n=1}^{\infty}$ be a martingale on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, P)$, i.e. $X_m = \mathbb{E}[X_n|\mathcal{F}_m]$ for $m \le n$. Let τ be a stopping time for the martingale. If $\mathbb{E}[X_\tau] < \infty$ and X_n is uniformly-integrable, i.e.

$$\liminf_{n\to\infty} \int_{\{\omega\in\Omega:|\tau(\omega)|>n\}} |X_n(\omega)|dP(\omega) = 0$$

then $\mathbb{E}[X_\tau] = \mathbb{E}[X_0]$.

Proof of Theorem 1. By the notation introduced in (7) and (8):

$$R_{n,\ell\wedge\tau_{B|A}} = \sum_{j=0}^{d-1} Y_{n,j}\mathbb{1}\{n+j \le \ell\wedge\tau_{B|A}\}, \qquad \sum_{n=m-1}^{d} R_{n,\ell\wedge\tau_{B|A}} = AX^d\lozenge B - (d+m),\ d \ge \tau_{B|A} \tag{15}$$

where AX^d denotes the concatenated sequence $(a_1, \cdots, a_m, X_1, \cdots, X_d)$. We have that $\{Y_{n\wedge\tau_{B|A}}\}$ is a martingale as well.

Recall that $\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}} = B\lozenge B - \tau_{B|A}$. Because $\mathbb{E}[\tau_{B|A}] < \infty$, we have that $\mathbb{E}[|\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}}|] \le B\lozenge B - \tau_{B|A} < \infty$, and on the event set $\{\tau_{B|A} > d\}$ for some $d \in \mathbb{N}$, $|\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}}| \le |AX^d\lozenge B| + d \le B\lozenge B + \tau_{B|A}$. We can apply Doob's martingale convergence lemma to get

$$\lim_{d\to\infty} \int_{\{\omega\in\Omega:\tau_{B|A}(\omega)>d\}} |\sum_{n=1}^{d} R_{n,d}(\omega)|dP(\omega) \le \lim_{d\to\infty} \int_{\{\omega\in\Omega:\tau_{B|A}(\omega)>d\}} (B\lozenge B + \tau_{B|A})dP(\omega) = 0 \tag{16}$$

Because $\{\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}}\}$ is a martingale, $\mathbb{E}[\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}}] = \mathbb{E}[R_{1,1}] = A\lozenge B$ since $X^0 = ()$ and the above formula shows that $\mathbb{E}[\sum_{n=1}^{\tau_{B|A}} R_{n,\tau_{B|A}}] = B\lozenge B - \mathbb{E}[\tau_{B|A}]$ and so $\mathbb{E}[\tau_{B|A}] = B\lozenge B - A\lozenge B$. This yields our desired results. $\square$

Based on Theorem 1, we have two additional important results.

Theorem 2. Suppose we are given the i.i.d. sequence as before and the patterns described in Definition 1. Then

$$\sum_{j=1}^{K} p_j(B_j\lozenge B_i) = \mathbb{E}[\tau] + A\lozenge B_i \tag{17}$$

Furthermore, if B_i and B_j are not subsequences of each other, then the odds that B_j precedes B_i in $X_1, X_2, \cdots$ are

$$(B_i\lozenge B_i - B_i\lozenge B_j) : (B_j\lozenge B_j - B_j\lozenge B_i) \tag{18}$$

Proof. For the first result, note

$$\mathbb{E}[\tau_i] = \mathbb{E}[\tau] + \mathbb{E}[\tau_i - \tau] = \mathbb{E}[\tau] + \sum_{j=1}^{K} p_j\mathbb{E}[\tau_i - \tau_j|\tau = \tau_j] \tag{19}$$

Note that by Theorem 1, $\mathbb{E}[\tau_i] = B_i \lozenge B_i - A \lozenge B_i$ and $\mathbb{E}[\tau_{i|j}] = B_i \lozenge B_i - B_j \lozenge B_i$. Substituting both into (19), we get

$$B_i \lozenge B_i - A \lozenge B_i = \mathbb{E}[\tau] + \sum_{j=1}^{K} p_j (B_i \lozenge B_i - B_j \lozenge B_i) \implies \mathbb{E}[\tau] = A \lozenge B_i - \sum_{j=1}^{K} p_j (B_j \lozenge B_i) \tag{20}$$

which is exactly the desired equation. The second result follows directly from Theorem 1. This concludes the proof. □

We can represent (17) as a matrix system of equations. Define $e_{ji} \triangleq \mathbb{E}[\tau_{i|j}] = B_i \lozenge B_i - B_j \lozenge B_i$ and $e_i \triangleq \mathbb{E}[\tau_i] = B_i \lozenge B_i - A \lozenge B_i$ for all $i, j \in \{1, \cdots, K\}$. Then

$$\left[\begin{array}{c|cccc} 0 & 1 & 1 & \cdots & 1 \\ \hline 1 & e_{11} & e_{21} & \cdots & e_{K1} \\ 1 & e_{12} & & & \\ \vdots & \vdots & & \ddots & \\ 1 & e_{1K} & & & e_{KK} \end{array}\right] \begin{bmatrix} \mathbb{E}[\tau] \\ p_1 \\ p_2 \\ \vdots \\ p_K \end{bmatrix} = \begin{bmatrix} 1 \\ e_1 \\ e_2 \\ \vdots \\ e_K \end{bmatrix} \tag{21}$$

Solving for the unknown variables $\mathbb{E}[\tau], p_1, \cdots, p_K$ is now a matter of matrix inversion and multiplication.

Theorem 3. For the specific choice of $A \triangleq (a_1, a_2, \cdots, a_m)$ and $B \triangleq (a_2, \cdots, a_m)$:

$$\mathbb{P}(\tau_A = \tau_B) = \frac{B \lozenge B - B \lozenge A}{A \lozenge A - A \lozenge B - B \lozenge A + B \lozenge B} \tag{22}$$

Proof. Note that $\tau_A \geq \tau_B$ always because B is an ending subsequence of A. Denote $X^{\tau_B} \triangleq (X_1, \cdots, X_\ell, a_2, \cdots, a_m)$. Theorem 1 shows that the process $\{\sum_{n=1}^{\ell} R_{n, \ell \wedge \tau_{B|A}}\}_{\ell=1}^{\infty}$ is a martingale. Thus

$$\mathbb{E}[X^{\tau_B} \lozenge A - \tau_B] = 0, \quad \mathbb{E}[X^{\tau_B} \lozenge B - \tau_B] = 0$$

The second equality is obvious because there is always exactly one occurrence of B in the sequence X^{τ_B}. For the first equality, $X^{\tau_B} \lozenge A - \tau_B$ can be thought of as the total rewards of all the gamblers who are betting for sequence A until time τ_B, which is clearly a martingale process.

Combining the two together, note:

$$X^{\tau_B} \lozenge A - X^{\tau_B} \lozenge B = \begin{cases} A \lozenge A - A \lozenge B & \text{if } \tau_A = \tau_B \\ B \lozenge A - B \lozenge B & \text{if } \tau_A > \tau_B \end{cases}$$

because $\tau_A = \tau_B$ if X_ℓ in the sequence X^{τ_B} is equal to a_1, and $\tau_A > \tau_B$ otherwise. Using conditional expectations, we get:

$$0 = \mathbb{E}\left[X^{\tau_B} \lozenge A - X^{\tau_B} \lozenge B\right] = (A \lozenge A - A \lozenge B)\mathbb{P}(\tau_A = \tau_B) - (B \lozenge A - B \lozenge B)\mathbb{P}(\tau_A > \tau_B)$$

Rearranging the terms yields the desired expression. □

3.4 Formulas for the Generating Functions

While the approach described in Section 3.3 derives formulas for the expected values of our interested quantities, we are also interested in the probability distributions, namely, their moment-generating functions.

We can extend the martingale argument to obtain expressions for the generating functions of $\tau_{B|A}, \tau_B$, as inspired by [10].

First, it is useful to associate the following Markov chain embedding to the pattern-occurrence problem. Let $\{Z_n\}_{n=1}^{\infty}$ be a Markov chain with state-space $\mathcal{Z} \triangleq \{0, 1, \cdots, r\}$, such that the state at time $n \in \mathbb{N}$ is

$$Z_n \triangleq \max\{j \in \mathbb{N} : X_{n-j+1} = b_1, \cdots, X_n = b_j, (X_0, X_1, \cdots, X_{m-1}) = A\}$$

Note that for all intermediate states of the chain $j \in \{1, \cdots, r-1\}$, transitions only occur to either $j+1$ with probability $p(b_{j+1})$ or 0 with probability $1 - p(b_{j+1})$. Moreover, we can write

$$\tau_{B|A} \triangleq \min\{n \in \mathbb{N} : Z_n = r\} \tag{23}$$

A corresponding Markov chain for rewriting $\tau_{0,B}$ can be derived similarly.

Associated with each $\delta_{i,j}(A, B)$ for all $i \in \{1, \cdots, m\}$, $j \in \{1, \cdots, r\}$, defined from (6), define

$$\epsilon_{i,j}(z, A, B) \triangleq (1/z)\delta_{i,j}(A, B) \tag{24}$$

for $z \in (0, 1]$. Further define

$$A\blacklozenge B(z) = \prod_{i=1}^{\min(m,r)} \epsilon_{i,i}(z, A, B) + \prod_{i=1}^{\min(m-1,r)} \epsilon_{i+1,i}(z, A, B) + \cdots + \epsilon_{\min(m,r),1}(z, A, B) \tag{25}$$

Unlike the previous analysis in Section 3.3, we shift indices to define a gambler's rewards in the following way. Define the reward $\tilde{R}_{n,\ell}$ of gambler $n \in \{-m+1, \cdots, 0\} \cup \mathbb{N}$ by time $\ell \in \mathbb{N}$ to be

$$\tilde{R}_{n,\ell} = \begin{cases} 1 & \text{if } \ell < n \\ \begin{cases} \left(\prod_{\substack{j \in \{0, \cdots, k-1\} \\ n+j \leq \ell}} \mathbb{P}(X = b_j)\right)^{-1} & \begin{array}{l}\text{if } (A \circ X^{\ell})_{m+n-1:\ell+m} \\ \qquad = B_{1:\ell-n+1}\end{array} \\ 0 & \text{else} \end{cases} & \text{if } n \leq \ell < n + r \\ \tilde{R}_{n,n+r-1} & \text{if } \ell \geq n + r \end{cases} \tag{26}$$

Each gambler's initial fortune is still 1. In contrast to the rewards $\{R_{n,\ell}\}_{\ell=1}^{\infty}$ defined in Section 3.3 for computing $\mathbb{E}[\tau_{B|A}]$ and $\mathbb{E}[\tau_B]$, the $\{\tilde{R}_{n,\ell}\}_{\ell=1}^{\infty}$ defined here does not penalize losses. When a gambler loses a turn, the gambler simply wins nothing, but he leaves the game when he earns nothing for the first time. Hence, there is no penalty of -1 in each entry of $\tilde{R}_{n,\ell}$ compared to $R_{n,\ell}$. However, $\{\tilde{R}_{n,\ell}\}_{\ell=1}^{\infty}$ is still a martingale because the game is still fair odds.

Example 3. Consider an i.i.d. sequence of coin tosses with state-space $\mathcal{X} \triangleq \{H, T\}$. Let $A = (H, T)$ with $m = 2$, $B = (H, T, H)$ with $r = 3$. Suppose the sequence observed so far is given by $AX^{\ell} \triangleq (H, T, T, H, T, H)$ with $\ell = \tau_{B|A} = 4$. Using these specific values, the construction of (26) becomes clear.

$$\begin{aligned} \tilde{R}_{-1,4} &= \tilde{R}_{-1,1} = 0 \text{ since the last 3 terms of } AX^1 = (H, T, T) \text{ don't match with } B \\ \tilde{R}_{0,4} &= \tilde{R}_{0,2} = 0 \text{ since the last 3 terms of } AX^2 = (T, T, H) \text{ don't match with } B \\ \tilde{R}_{2,4} &= \frac{1}{p_H^2 p_T} \\ \tilde{R}_{3,4} &= 0 \text{ since last 2 terms of } AX^4 \text{ don't match first 2 terms of } B \end{aligned}$$

$$\tilde{R}_{4,4} = \frac{1}{p_H}$$
$$\tilde{R}_{5,4} = \tilde{R}_{6,4} = \cdots = 1$$

Essentially, the nonzero and nonunit values of $\tilde{R}_{n,\ell}$ follow the same construction as the nonzero values of $R_{n,\ell}$. □

A version of Theorem 1 for deriving the generating functions of $\tau_{B|A}$ and τ_B can be described as follows.

Theorem 4 (Generating Function of the Waiting Time Until a Sequence)**.** Under the setup described above,

$$\mathbb{E}[z^{\tau_{B|A}}] = \frac{1 + (1-z)A\blacklozenge B(z)}{1 + (1-z)B\blacklozenge B(z)} \tag{27}$$

Proof. Construct the appropriate martingale process

$$Y_\ell \triangleq \sum_{n=-m+1}^{\infty} z^{n-1}\tilde{R}_{n,\ell} \tag{28}$$

for $\ell \in \mathbb{N}$ and $z \in (0,1)$. Note $\{Y_\ell\}$ is a martingale because we've see that $\{\tilde{R}_{n,\ell}\}$ is a martingale, and linear combinations of martingales defined on the same filtration are martingales. Note that $\tau_{B|A}, \tau_B$ are stopping times for $\{Y_\ell\}$ and that Y_ℓ are bounded for all $\ell \in \mathbb{N}$ and each fixed $z \in (0,1)$ because

$$Y_\ell \le \left(\prod_{j=1}^{r} \frac{1}{p(b_j)}\right) \cdot \sum_{n=-m+1}^{\infty} z^{n-1} = \left(\prod_{j=1}^{r} \frac{1}{p(b_j)}\right)\left(\frac{z^{-m}}{1-z}\right) \text{ by geometric series} \tag{29}$$

We can therefore invoke the Optional Stopping Theorem for martingales to conclude

$$\mathbb{E}[Y_{\tau_{B|A}}] = Y_0 \tag{30}$$

and similarly for Y_{τ_B}. Substitute the definition of $\tilde{R}_{n,\ell}$ into Y_ℓ with $\ell = \tau_{B|A}$. From Example 3, it is clear that $Y_{\tau_{B|A}}$ can be split apart into three different sums on a case-by-case basis:

$$\begin{aligned} Y_{\tau_{B|A}} &= \sum_{n=-m+1}^{\tau_{B|A}-r} z^{n-1} \cdot 0 + \sum_{n=\tau_{B|A}-r}^{\tau_{B|A}} z^{n-1}\tilde{R}_{n,\tau_{B|A}} + \sum_{n=\tau_{B|A}+1}^{\infty} z^{n-1} \\ &= z^{\tau_{B|A}} \sum_{n=1}^{r} z^{-n}\tilde{R}_{n,\tau_{B|A}} + \frac{z^{\tau_{B|A}}}{1-z} \text{ by geometric series} \\ &= z^{\tau_{B|A}}\left(B\blacklozenge B(z) + \frac{1}{1-z}\right) \end{aligned} \tag{31}$$

On the other hand:

$$Y_0 = \sum_{n=-m+1}^{0} z^{n-1}\tilde{R}_{n,\tau_{B|A}} + \frac{1}{1-z} = A\blacklozenge B(z) + \frac{1}{1-z} \tag{32}$$

By the Optional Stopping Theorem, we have

$$\mathbb{E}[z^{\tau_{B|A}}]\left(B\blacklozenge B(z) + \frac{1}{1-z}\right) = A\blacklozenge B(z) + \frac{1}{1-z} \tag{33}$$

and rearranging the terms to isolate $\mathbb{E}[z^{\tau_{B|A}}]$ yields the desired result (27). □

Define $g_{ij}(z) \triangleq \mathbb{E}[z^{\tau_{j|i}}]$ to be the generating function of $\tau_{j|i}$ and $g_i \triangleq \mathbb{E}[z^{\tau_i}]$, the generating function of τ_i. Similar to (21), we can derive a system of equations for the generating functions as well. Note that

$$\begin{aligned} g_i \triangleq \mathbb{E}[z^{\tau_i}] = \mathbb{E}[z^{\tau+(\tau_i-\tau)}] &= \sum_{j=1}^{K} \mathbb{E}\left[z^{\tau_j} + (\tau_i - \tau_j)\mathbb{1}\{\tau = \tau_j\}\right] \\ &= \sum_{j=1}^{K} \mathbb{E}\left[z^{\tau_i - \tau_j}\right] \mathbb{E}\left[z^{\tau}\mathbb{1}\{\tau = \tau_j\}\right] = \sum_{j=1}^{K} g_{ji}\mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_j\}] \end{aligned} \tag{34}$$

where $\mathbb{1}\{\tau = \tau_j\}$ denotes the event where pattern B_j is the first in $\mathcal{B}$ to occur first, for any $j \in \{1, \cdots, K\}$. In matrix form:

$$\left[\begin{array}{c|cccc} -1 & 1 & 1 & \cdots & 1 \\ \hline 1 & g_{11}(z) & g_{21}(z) & \cdots & g_{K1}(z) \\ 1 & g_{12}(z) & & & \\ \vdots & \vdots & & \ddots & \\ 1 & g_{1K}(z) & & & g_{KK}(z) \end{array}\right] \begin{bmatrix} \mathbb{E}[z^{\tau}] \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_1\}] \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_2\}] \\ \vdots \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_K\}] \end{bmatrix} = \begin{bmatrix} 0 \\ g_1(z) \\ g_2(z) \\ \vdots \\ g_K(z) \end{bmatrix} \tag{35}$$

Using Theorem 4, we can simplify $g_{ij}(z)$ and $g_i(z)$ from (35) in terms of known quantities. Note that:

$$\begin{aligned} \left(\frac{1}{1-z} + B_i\blacklozenge B_i(z)\right)\mathbb{E}[z^{\tau_i}] &= \left(\frac{1}{1-z} + A\blacklozenge B_i(z)\right) \\ \left(\frac{1}{1-z} + B_i\blacklozenge B_i(z)\right)\mathbb{E}[z^{\tau_{i|j}}] &= \left(\frac{1}{1-z} + B_j\blacklozenge B_i(z)\right) \end{aligned}$$

Hence, (35) becomes

$$\left[\begin{array}{c|cccc} -1 & 1 & 1 & \cdots & 1 \\ \hline \frac{1}{1-z} & B_1\blacklozenge B_1(z) & B_1\blacklozenge B_2(z) & \cdots & B_1\blacklozenge B_K(z) \\ \frac{1}{1-z} & B_2\blacklozenge B_1(z) & & & \\ \vdots & \vdots & & \ddots & \\ \frac{1}{1-z} & B_K\blacklozenge B_1(z) & & & B_K\blacklozenge B_K(z) \end{array}\right] \begin{bmatrix} \mathbb{E}[z^{\tau}] \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_1\}] \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_2\}] \\ \vdots \\ \mathbb{E}[z^{\tau}\mathbb{1}\{\tau = \tau_K\}] \end{bmatrix} = \begin{bmatrix} 0 \\ A\blacklozenge B_1(z) + \frac{1}{1-z} \\ A\blacklozenge B_2(z) + \frac{1}{1-z} \\ \vdots \\ A\blacklozenge B_K(z) + \frac{1}{1-z} \end{bmatrix} \tag{36}$$

3.5 Martingale Theory Interpretation Extended to Multiple Teams

While [10] used a Markov chain embedding method to construct a linear system of equations to solve $\mathbb{E}[\tau]$, we demonstrate an alternative method based on [6], which uses a martingale construction. One advantage of using martingales over a Markov chain embedding is that the martingale method allows for an easier extension to patterns generated by a Markov chain.

Let $\mathcal{B} \triangleq \{B_1, \cdots, B_K\}$, where the B_j are patterns from Definition 1. Introduce M teams of gamblers, with team $j \in \{1, \cdots, K\}$ betting on the event that pattern B_j is the first among all K patterns to occur in the sequence. Team j begins with an initial betting amount of $\$c_j$ and bets according to the strategy described in Section 3.3.

Let C_n be the net gain of the casino by time $n \in \mathbb{N}$. Clearly:

$$C_\tau \triangleq \begin{cases} \left(\sum_{j=1}^{K} c_j\right)\tau - [(B_1 \Diamond B_1)c_1 + \cdots + (B_1 \Diamond B_K)c_K] & \text{if } \tau = \tau_1 \\ \vdots & \vdots \\ \left(\sum_{j=1}^{K} c_j\right)\tau - [(B_K \Diamond B_1)c_1 + \cdots + (B_K \Diamond B_K)c_K] & \text{if } \tau = \tau_K \end{cases} \tag{37}$$

Note that the process $\{C_n\}$ is a zero-mean martingale with stopping time τ, and $\mathbb{E}[\tau] < \infty$ with appropriate assumptions. Thus

$$0 = \mathbb{E}[C_\tau] = \left(\sum_{j=1}^{K} c_j\right)\mathbb{E}[\tau] - \begin{bmatrix} p_1 & p_2 & \cdots & p_K \end{bmatrix} \underbrace{\begin{bmatrix} B_1 \Diamond B_1 & B_1 \Diamond B_2 & \cdots & B_1 \Diamond B_K \\ B_2 \Diamond B_1 & & & \\ \vdots & & \ddots & \\ B_K \Diamond B_1 & & & B_K \Diamond B_K \end{bmatrix}}_{=:M_{\mathcal{B}}} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_K \end{bmatrix} \tag{38}$$

Although the variables $(p_1, p_2, \cdots, p_K)^T$ are unknown too, we can avoid their computation by choosing $\mathbf{c}^* \triangleq (c_1^*, \cdots, c_K^*)^T$ such that $M_{\mathcal{B}}\mathbf{c}^* = \mathbf{1}^K$, where $\mathbf{1}^K$ is the k-dimensional vector of all ones. Note that any other choice of $\mathbf{c}^*$ would work for (38), but the specific choice of $\mathbf{c}^*$ simplifies the formula considerably. We have

$$\mathbb{E}[\tau] = \frac{1}{\sum_{j=1}^{K} c_j^*} \tag{39}$$

3.6 First Occurrence Probabilities: Connecting Two Approaches

Recall the matrix equations derived using the Markov chain embedding method from Section 3.4.

$$\mathbb{E}[\tau_i] = \mathbb{E}[\tau] + \sum_{j=1}^{K} \alpha_j \mathbb{E}[\tau_{i|j}], \quad \mathbb{E}[\tau_i] = B_i \Diamond B_i - A \Diamond B_i$$

We emphasize that here $\tau_i \triangleq \tau_{B_i|A}$, that is, we seek the time to observe B_i *assuming* an initial sequence of A has already occurred. Moreover, the definition of $\tau_{i|j}$ is defined differently in the sense that it does not refer to the duration between consecutive occurrences, i.e. $\tau_{i|i} = 0$ instead of τ_i.

In contrast, the formulas using the martingale interpretation from Section 3.5 only deal with first occurrence times with no initial sequence. Nevertheless, the structure of the formulas are the same.

$$\mathbb{E}[\tau_{0,i}] = \mathbb{E}[\tau] + \sum_{j=1}^{K} \alpha_j \mathbb{E}[\tau_{i|j}], \quad \mathbb{E}[\tau_i] = B_i \Diamond B_i$$

where $\tau_{0,i} \triangleq \tau_{0,B_i}$. Using the $\Diamond$ notation:

$$B_i \Diamond B_i = \mathbb{E}[\tau] + \sum_{j=1}^{K} \alpha_j (B_i \Diamond B_i - B_j \Diamond B_i) \quad \Longrightarrow \quad 0 = \mathbb{E}[\tau] - \sum_{j=1}^{K} \alpha_j B_j \Diamond B_i$$

From Section 3.5, we have

$$0 = \mathbb{E}[C_\tau] = \sum_{j=1}^{K} c_j \mathbb{E}[\tau] - \sum_{i=1}^{K}\sum_{j=1}^{K} (B_i \Diamond B_j) c_j$$

and we chose an appropriate $\mathbf{c}^*$ so that we could solve for $\mathbb{E}[\tau]$ without needing to solve for $\mathbb{P}(E_i) = \alpha_i$. However, we can choose specific $\mathbf{c} = \{[1, 0, \cdots, 0], \cdots, [0, 0, \cdots, 1]\}$ to obtain the system of equations shown in Section 3.4. Since $\mathbb{E}[\tau]$ is computed first and known, we can simplify the matrix equations as

$$M_{\mathcal{B}}^T \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_K \end{bmatrix} = \begin{bmatrix} \mathbb{E}[\tau] \\ \mathbb{E}[\tau] \\ \vdots \\ \mathbb{E}[\tau] \end{bmatrix}$$

4 Markov Chain Sequences

4.1 Problem Formulation

Now, we consider the previous pattern-occurrence problems for the case where $\{X_n\}$ is generated by an irreducible Markov chain over the state-space $\mathcal{X} \triangleq \mathcal{X}_1 \times \cdots \times \mathcal{X}_{n_x}$, where $X_{i,n} \in \mathcal{X}_i$ for all $n \in \mathbb{N}$ and each $\mathcal{X}_i$ is a finite discrete set. Let the transition probability matrix for $\{X_n\}$ be $P \triangleq \{P(X, X')\}$ with stationary distribution $\{\pi(X)\}$ and initial probability distribution $\{p_0(X)\}$ for any $X \in \mathcal{X}$.

Lemma 2 (Mean Return Time to a State)**.** For the Markov chain $\{X_n\}$ described above, let $f_i \triangleq \min\{n \geq 1 \mid X_n = X_0 = i\}$ to be the minimum return time to state i given the chain is started at $X_0 = i$. If X_n is a positive-recurrent Markov chain, i.e. $\mathbb{E}[f_i] < \infty$ for all $i \in \mathbb{R}$, then a unique stationary distribution $\{\pi_i\}$ exists and $\mathbb{E}[f_i] = 1/\pi_i$ for all $i \in \mathbb{R}$.

4.2 Interpretation via Martingale Theory: Two States

Primary reference: [7]

We consider the specific case where $\mathcal{X} = \{H, T\}$ takes only two values, with transition probabilities $p_{HH}, p_{HT}, p_{TH}, p_{TT}$ and initial probability distribution $\{q_0(H), q_0(T)\}$. For pattern A defined in Definition 1, we are specifically interested in computing $\mathbb{E}[\tau_{0,A}]$. The sequence $\{X_n\}$ can terminate at time $\tau_{0,A}$ according to any one of the three following ending scenarios.

1. Pattern A occurs from the start.
2. $(X_{\tau_{0,A}-m}, \cdots, X_{\tau_{0,A}}) = \{H\} \circ A$.
3. $(X_{\tau_{0,A}-m}, \cdots, X_{\tau_{0,A}}) = \{T\} \circ A$.

We initialize two gamblers. Both gamblers observe the outcome X_n of trial n, but begin betting from time $n + 1$; this means no bets are made on outcome X_1. Each gambler uses one of the following two betting strategies:

1. Strategy 1: at time $n+1$, the gambler bets \$$c_1$ on outcome $\{X_{n+1} = a_1\}$ and continues betting until he sees $\{(X_{n+1}, \cdots, X_{n+m}) = A\}$.

2. Strategy 1+: at time $n+1$, the gambler bets \$$c_2$ on
   - outcome $\{X_{n+1} = a_1\}$ and continues betting until he sees $\{(X_{n+1}, \cdots, X_{n+m}) = A\}$ if $X_n \neq a_1$.
   - outcome $\{X_{n+1} = a_2\}$ and continues betting until he sees $\{(X_{n+1}, \cdots, X_{n+m-1}) = (a_2, \cdots, a_m)\}$ if $X_n = a_1$.

We emphasize that this strategy only works for when when A is not a consecutive run of all H or all T, otherwise Strategy 1+ cannot be applied.

Define weight matrix $W \triangleq [W_{ij}] \in \mathbb{R}^{3\times 2}$ to be such that \$$W_{ij}c_j$ is the amount won by team $j \in \{1, 2\}$ when the game terminates with ending scenario $i \in \{1, 2, 3\}$. To define the entries of the weight matrix, first define the following Markov chain version of the δ_i from (6) and the corresponding $\Diamond$ notation:

$$\delta_i^{(1)}(A, X^h) \triangleq \begin{cases} \left(P(X_{h-i}, a_1) \prod_{j=2}^{i} P(a_{j-1}, a_j) \right)^{-1} & \text{if } a_j = X_{h-i+j} \; \forall j \in \{1, \cdots, i\} \\ 0 & \text{else} \end{cases} \tag{40a}$$

$$\delta_i^{(2,1)}(A, X^h) \triangleq \begin{cases} \left(P(X_{h-i}, a_1) \prod_{j=2}^{i} P(a_{j-1}, a_j) \right)^{-1} & \text{if } a_j = X_{h-i+j} \; \forall j \in \{1, \cdots, i\} \text{ and } X_{h-i} \neq a_1 \\ 0 & \text{else} \end{cases} \tag{40b}$$

$$\delta_i^{(2,2)}(A, X^h) \triangleq \begin{cases} \left(\prod_{j=2}^{i+1} P(a_{j-1}, a_j) \right)^{-1} & \text{if } a_j = X_{h-i+j-1} \; \forall j \in \{1, \cdots, i+1\} \\ 0 & \text{else} \end{cases} \tag{40c}$$

then the entries of W are specifically constructed as

$$W_{i1} = \sum_{i=1}^{\min(r_i-1, m)} \delta_i^{(1)}(A, G_i), \quad W_{i2} = \sum_{i=1}^{\min(r_i-1, m)} \delta_i^{(2,1)}(A, G_i)) + \sum_{i=1}^{\min(r_i-1, m-1)} \delta_i^{(2,2)}(A, G_i) \tag{41}$$

where $G_1 \triangleq A, G_2 \triangleq \{H\} \circ A$, and $G_3 \triangleq \{T\} \circ A$, and $r_i \in \mathbb{N}$ is the length of G_i.

Define E_i to be the event that the game ends on scenario i. Then the casino's net gain is

$$C_{\tau_{0,A}} = (c_1 + c_2)(\tau_{0,A} - 1) - \sum_{i=1}^{3} \mathbb{1}\{E_i\} \sum_{j=1}^{2} W_{ij} c_j \tag{42}$$

where there are $\tau_{0,A} - 1$ copies of $(c_1 + c_2)$ positive gain accumulated since neither gambler bets at initial time 1.

Again choose $\mathbf{c}^* \triangleq (c_1^*, c_2^*)^T$ such that $\sum_{j=1}^{2} W_{ij} c_j = 1$ for the ending scenarios $i \in \{2, 3\}$ (excluding the initial ending $i = 1$). Using the fact that $\{C_n\}$ is a zero-mean martingale and that $\tau_{0,A}$ is a finite-mean stopping time:

$$\mathbb{E}[\tau_{0,A}] = 1 + \frac{1}{c_1^* + c_2^*} \left(\mathbb{P}(E_1)(W_{11} c_1^* + W_{12} c_2^*) + (1 - \mathbb{P}(E_1)) \right) \tag{43}$$

Note that $\mathbb{P}(E_1)$ is easy to compute:

$$\mathbb{P}(E_1) = \frac{1}{q_0(a_1) \prod_{i=1}^{m-1} P_{a_i,a_{i+1}}} \tag{44}$$

Now for the two-state Markov chain, suppose we consider a collection of patterns $\mathcal{B} \triangleq \{B_1, \cdots, B_K\}$, where the B_j are defined in Definition 1. Construct a new set of patterns using the following two-step procedure to determine all possible ending scenarios:

1. For each pattern $B_j \in \mathcal{B}$, add $\{H\} \circ B_j$ and $\{T\} \circ B_j$.
2. Discard patterns which are impossible to observe as an ending scenario, e.g. if $B_1 = HT$ and $B_2 = HH$, then $\{H\} \circ B_1 = HHT$ is impossible to observe because we cannot observe such a pattern before the occurrence of B_2.

After performing this two-step extension to all $B_j \in \mathcal{B}$, accumulate all the extended patterns into $\mathcal{B}'$. To count the number of ending scenarios, we specify two types of patterns:

- The set of all *matched* patterns is defined $\mathcal{B}_m \triangleq \{B_j \in \mathcal{B} : \{H\} \circ B_j \in \mathcal{B}' \text{ and } \{T\} \circ B_j \in \mathcal{B}'\}$. Let $K_m \triangleq |\mathcal{B}_m|$.
- The set of all *unmatched* patterns is defined $\mathcal{B}_u \triangleq \mathcal{B}/\mathcal{B}_m$. Let $K_u \triangleq |\mathcal{B}_u|$.

Note that there are a total of $K + K_u + 2K_m$ ending scenarios: K initial ending scenarios, K_u unmatched ending scenarios, and $2K_M$ matched ending scenarios $\{H\} \circ B_j$ and $\{T\} \circ B_j$ for each $B_j \in \mathcal{B}_m$. We specifically enumerate the extended patterns of $\mathcal{B}' \triangleq \{B'_1, \cdots, B'_{K_u+2K_m}\}$, where pattern B'_k has length $r'_k \in \mathbb{N}$.

Gambling strategies for each of the possible ending patterns in $\mathcal{B}$:

1. For each of the K_u unmatched endings $j \in \{1, \cdots, K_u\}$, associate one gambler using Strategy 1 starting with the initial bet $\$c_j$.
2. For each of the K_m matched endings $j \in \{K_u + 1, \cdots, K_u + 2K_m\}$, associate two gamblers, the first gambler uses Strategy 1 with initial bet $\$c_j$, while the second gambler uses Strategy 1+ with initial bet $\$c_j$.

The weight matrix $W \triangleq [W_{\ell k}] \in \mathbb{R}^{(K+K_u+2K_m)\times(K_u+2K_m)}$ is now defined similarly to the single pattern case, where $\$W_{\ell k} c_k$ is the amount that is gained by gambler $k \in \{1, \cdots, K_u + 2K_m\}$ when ending scenario $\ell \in \{1, \cdots, K + K_u + 2K_m\}$ is observed. The individual elements of this new weight matrix are

$$W_{\ell k} \triangleq \tag{45}$$

$$\begin{cases} \sum_{i=1}^{\min(r_\ell-1, r_k)} \delta_i^{(1)}(B_k, B_\ell) \text{ if } 1 \le \ell \le K, 1 \le k \le K_u + K_m \\ \sum_{i=1}^{\min(r_\ell-1, r_{k-K_m})} \delta_i^{(2,1)}(B_{k-K_m}, B_\ell) + \sum_{i=1}^{\min(r_\ell-1, r_{k-K_m}-1)} \delta_i^{(2,2)}(B_{k-K_m}, B_\ell) \\ \qquad\qquad \text{if } 1 \le \ell \le K, K_u + K_m < k \le K_u + 2K_m \\ \sum_{i=1}^{\min(r'_{\ell-K}-1, r_{k-K_m})} \delta_i^{(1)}(B_{k-K_m}, B'_{\ell-K}) \text{ if } K+1 \le \ell \le K + K_u + 2K_m, 1 \le k \le K_u + K_m \\ \sum_{i=1}^{\min(r'_{\ell-K}-1, r_{k-K_m})} \delta_i^{(2,1)}(B_{k-K_m}, B'_{\ell-K}) + \sum_{i=1}^{\min(r'_{\ell-K}-1, r_{k-K_m}-1)} \delta_i^{(2,2)}(B_{k-K_m}, B'_{\ell-K}) \\ \qquad\qquad \text{if } K+1 \le \ell \le K + K_u + 2K_m, K_u + K_m \le k \le K_u + 2K_m \end{cases}$$

Again, $B_\ell \in \mathcal{B}$ are the initial ending scenarios, and $B'_k \in \mathcal{B}'$ (length $r'_k \in \mathbb{N}$) are the later ending scenarios with either $\{H\}$ or $\{T\}$ appended to the front.

As defined above, let $\tau \triangleq \min_{j\in\{1,\cdots,K\}} \tau_j$. The casino's net gain is then computed as

$$C_\tau = \sum_{j=1}^{K_u+2K_m} c_j(\tau-1) - \sum_{i=1}^{K} \mathbb{1}\{E_i\} \sum_{j=1}^{K_u+2K_m} W_{ij}c_j - \sum_{i=K+1}^{K+K_u+2K_m} \mathbb{1}\{E_i\} \sum_{j=1}^{K_u+2K_m} W_{ij}c_j$$

where we split the usual sum in the second term depending on whether the ending scenario is one of the initial scenarios or one of the later ending scenarios.

Choose $\mathbf{c}^* \triangleq (c_1^*, \cdots, c_{K_u+2K_m}^*)^T$ to be such that $\sum_{j=1}^{K_u+2K_m} W_{ij}c_j = 1$ for all $i \in \{K+1, \cdots, K+K_u+2K_m\}$. The casino's net gain then simplifies to

$$C_\tau = \sum_{j=1}^{K_u+2K_m} c_j^*(\tau-1) - \sum_{i=1}^{K} \mathbb{1}\{E_i\} \sum_{j=1}^{K_u+2K_m} W_{ij}c_j^* - \sum_{i=K+1}^{K+K_u+2K_m} \mathbb{1}\{E_i\} \tag{46}$$

and using the Optional Stopping Theorem:

$$\mathbb{E}[\tau] = 1 + \frac{1}{\sum_{j=1}^{K_u+2K_m} c_j^*} \left((1 - \sum_{i=1}^{K} P_i) + \sum_{i=1}^{K} P_i \left(\sum_{j=1}^{K_u+2K_m} W_{ij}c_j^* \right) \right) \tag{47}$$

where $P_i \triangleq \mathbb{P}(E_i)$ can be computed using the formula similar to (44).

4.3 Interpretation via Martingale Theory: Multiple States

Primary references: [11, 12]

Now consider the extension to general finite state-space $|\mathcal{X}| > 2$. For the sake of simplicity, enumerate the state space in the following way $\mathcal{X} \triangleq \{1, \cdots, M\}$ such that $M > 2$.

In contrast to the two-state Markov chain, we consider the previous two outcomes of the sequence $\{X_n\}$ to construct each ending scenario, instead of just one. Then there are a total of $M^2 + M + 1$ feasible ending scenarios for each pattern $B_j \in \mathcal{B}$.

1. Pattern B_j occurs from the start.

2. For some $y \in \mathcal{X}$, $\{(X_1, \cdots, X_{r_j+1}) = \{y\} \circ B_j\}$ occurs from the start.
3. For some $x, y \in \mathcal{X}$, $(X_{\tau-r_j-1}, \cdots, X_\tau) = \{x, y\} \circ B_j$.

In contrast to the two-state Markov chain, we now have $(M+1)K$ possible initial ending scenarios instead of just K. The number of later ending scenarios is M^2K.

Similar to the two-state Markov chain case, it is necessary to discard some patterns which are impossible to observe as an ending scenario. Furthermore, note that some appended patterns may not occur due to the structure of the Markov chain. Post clean-up, let $\mathcal{S}_I$ to be the set of feasible initial-ending scenarios with $K_I \triangleq |\mathcal{S}_I|$, and let $\mathcal{S}_L$ to be the set of feasible later-ending scenarios with $K_L \triangleq |\mathcal{S}_L|$.

Let $*\mathcal{S}_L \triangleq \{*B : B \in \mathcal{S}_L\}$ denote each ending scenario; we emphasize that the asterisk is simply there to emphasize that each scenario in $\mathcal{S}_L$ is a later-ending scenario. Assign a gambling team $j \in \{1, \cdots, K_L\}$ to each pattern $\{x, y\} \circ B_i \in \mathcal{S}_L$ where $i \in \{1, \cdots, K\}$. Depending on which ending scenario from $\mathcal{S}_I \cup *\mathcal{S}_L$ occurs, the gamblers accumulate rewards according to the following betting strategy:

1. If $X_n = x$, the team gambles $\$c_j$ on observing the pattern $\{y\} \circ B_i$ from time $n+1$ to time $n+r_i+1$.
2. If $X_n \neq y$, the team gambles $\$c_j$ on observing the pattern B_i from time $n+1$ to $n+r_i$.

To define the entries of the weight matrix, we note that we can no longer use (40). The multi-state Markov chain computation requires the following modification:

$$\delta_i^{(1)}(\{x, y\} \circ A, X^h) \triangleq \begin{cases} \left(P(x, y)P(y, a_1) \prod_{j=2}^{i-1} P(a_{j-1}, a_j)\right)^{-1} \\ \qquad \text{if } X_{h-i} = x, X_{h-i+1} = y, X_{h-i+j} = a_{j-1} \ \forall j \in \{2, \cdots, i\} \\ 0 \quad \text{else} \end{cases} \tag{48a}$$

$$\delta_i^{(2)}(\{x, y\} \circ A, X^h) \triangleq \begin{cases} \left(P(X_{h-i}, a_1) \prod_{j=2}^{i} P(a_{j-1}, a_j)\right)^{-1} & \text{if } a_j = X_{h-i+j} \ \forall j \in \{1, \cdots, i\} \text{ and } X_{h-i} \neq x \\ 0 & \text{else} \end{cases} \tag{48b}$$

The weight matrix $W \triangleq [W_{\ell k}] \in \mathbb{R}^{(K_I+K_L)\times K_L}$ is now defined as the amount that is gained by gambling team $k \in \{1, \cdots, K_L\}$ when ending scenario $\ell \in \{1, \cdots, K_I + K_L\}$ is observed. The individual elements of this new weight matrix are

$$W_{\ell k} = \sum_{i=1}^{\min(r_k-1, m_\ell+1)} \delta_i^{(1)}(\{x, y\} \circ F_\ell, G_k) + \sum_{i=1}^{\min(r_k-1, m_\ell)} \delta_i^{(2)}(\{x, y\} \circ F_\ell, G_k) \tag{49}$$

where $\{x, y\} \circ F_\ell \in \mathcal{S}_I \cup \mathcal{S}_L$ with F_ℓ having length $m_\ell \in \mathbb{N}$ for $\ell \in \{1, \cdots, K_I + K_L\}$, and $G_k \in \mathcal{S}_L$ with length $r_k \in \mathbb{N}$ for $k \in \{1, \cdots, K_L\}$.

The expression for the casino's net gain, and correspondingly $\mathbb{E}[\tau]$, follow similarly to the two-state Markov chain case.

$$C_\tau = \sum_{j=1}^{K_L} c_j(\tau - 1) - \sum_{i=1}^{K_I} \mathbb{1}\{E_i\} \sum_{j=1}^{K_L} W_{ij}c_j - \sum_{i=K_I+1}^{K_I+K_L} \mathbb{1}\{E_i\} \sum_{j=1}^{K_L} W_{ij}c_j$$

Choose $\mathbf{c}^* \triangleq (c_1^*, \cdots, c_{K_L}^*)^T$ to be such that $\sum_{j=1}^{K_L} W_{ij} c_j = 1$ for all $i \in \{K+1, \cdots, K+K_L\}$. Simplifying the casino's net gain then using the Optional Stopping Theorem:

$$\mathbb{E}[\tau] = 1 + \frac{1}{\sum_{j=1}^{K_L} c_j^*} \left((1 - \sum_{i=1}^{K_I} P_i) + \sum_{i=1}^{K_I} P_i \left(\sum_{j=1}^{K_L} W_{ij} c_j^* \right) \right) \tag{50}$$

where $P_i \triangleq \mathbb{P}(E_i)$ can be computed using the formula similar to (44).

Example 4 (Markov Chain with Three States)**.** Let $\mathcal{X} = \{1, 2, 3\}$ and $\mathcal{B} := \{B_1, b_2, B_3\}$ with $B_1 = 323, B_2 = 313, B_3 = 33$. The initial probability distribution $\{q_0(x)\}$ over $\mathcal{X}$ is uniform with probability 1/3 and the transition probability matrix is

$$P = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

First, using the brute-force approach, we obtain the following equations

$$\begin{bmatrix} 1-P_{11} & 0 & -P_{13} & 0 & 0 \\ 0 & 1-P_{22} & -P_{23} & 0 & 0 \\ 0 & 0 & 1 & -P_{31} & P_{32} \\ -P_{11} & 0 & 0 & 1 & 0 \\ 0 & -P_{22} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_0(1) \\ e_0(2) \\ e_0(3) \\ e_0(31) \\ e_0(32) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \tag{51}$$

where $e_0(*)$ denotes the expected time to reach one of the patterns in $\mathcal{B}$ when the sequence of state $*$ has been observed so far. This yields the solution $\mathbb{E}[\tau] = \sum_{i=1}^{3} q_0(i) e_0(i) = 8.4667$.

Now we use the approach outlined in this section. First, we have

$$\mathcal{S}_I \triangleq \mathcal{B} \cup \{1323, 2323, 1313, 2313, 133, 233\}$$
$$*\mathcal{S}_L \triangleq \{*11323, *22323, *11313, *22313, *1133, *2233\}$$

where $*$ represents any element in $\mathcal{X}$ which precedes a later ending scenario in $\mathcal{S}_L$. Then we have a 15×6 weights matrix. The K_L gamblers bet on each scenario of $\mathcal{S}_L \triangleq \{11323, 22323, 11313, 22313, 1133, 2233\}$. Some entries for the first gambling team, which bets on pattern 11323, are

$$W_{11} = \sum_{i=1}^{\min(2,4)} \delta_i^{(1)}(\{1,1\} \circ (323), 323) + \sum_{i=1}^{\min(2,3)} \delta_i^{(2)}(\{1,1\} \circ (323), 323) = \frac{1}{P_{23}} = 4$$

$$W_{10,1} = \sum_{i=1}^{\min(2,4)} \delta_i^{(1)}(\{1,1\} \circ (323), *11323) + \sum_{i=1}^{\min(2,3)} \delta_i^{(2)}(\{1,1\} \circ (323), *11323) = \frac{1}{P_{11}P_{13}P_{32}P_{23}} + \frac{1}{P_{23}} = \frac{268}{3}$$

□

4.4 First Occurrence Probabilities for Multi-State Markov Chains

Similar to the problem of Section 3.6, we are now interested in obtaining the probabilities $\alpha_j \triangleq \mathbb{P}(\tau = \tau_j)$ for $j \in \{1, \cdots, K\}$ for the Markov chain setting. Recall

$$0 = \mathbb{E}[C_\tau] = \sum_{j=1}^{K_L} c_j (\mathbb{E}[\tau] - 1) - \sum_{i=1}^{K_I} \mathbb{P}(E_i) \sum_{j=1}^{K_L} W_{ij} c_j - \sum_{i=K_I+1}^{K_I+K_L} \mathbb{P}(E_i) \sum_{j=1}^{K_L} W_{ij} c_j$$

Again, $\mathbb{E}[\tau]$ was computed without needing all the values $\mathbb{P}(E_i)$ by substituting in $\mathbf{c}^*$. Hence, to obtain $\{\alpha_j\}$ assuming $\mathbb{E}[\tau]$ is known, we can rearrange the above equation:

$$0 = \mathbb{E}[C_\tau] = \sum_{j=1}^{K_L} c_j(\mathbb{E}[\tau]-1) - \sum_{j=1}^{K_L}\left(\sum_{i=1}^{K_I+K_L}\mathbb{P}(E_i)W_{ij}\right)c_j \tag{52}$$

and choose one of K_L different basis vectors $\mathbf{c} \in \{[1,0,\cdots,0],\cdots,[0,0,\cdots,1]\}$ to get K_L different equations.

Note that here, E_i is the event that the ith ending scenario has occurred. Each α_j is computed by summing the appropriate subset of $\mathbb{P}(E_i)$, which can be thought of as the conditional probabilities enumerating each ending scenario associated with the pattern B_j. In Example 4, for instance,

$$\alpha_1 \triangleq \mathbb{P}(\tau = \tau_{323}) = \mathbb{P}(323) + \mathbb{P}(1323) + \mathbb{P}(2323) + \mathbb{P}(*1323) + \mathbb{P}(*2323)$$

and similarly for $\alpha_2 \triangleq \mathbb{P}(\tau = \tau_{313})$ and $\alpha_3 \triangleq \mathbb{P}(\tau = \tau_{33})$. Overall, there will be K_L equations with K_L of the $\mathbb{P}(E_i)$ unknown.

4.5 Formulas in Steady-State

When we have observed the Markov chain for long enough, we can assume steady-state and approximate the expected times $\mathbb{E}[\tau_{0,B}], \mathbb{E}[\tau_B], \mathbb{E}[\tau_{B|A}]$ using the stationary distribution $\{\pi(x)\}$. Using steady-state assumptions simplifies the expressions of Section 4.3 considerably.

Theorem 5 (Expected Waiting Times in Steady-State)**.** Under the patterns in Definition 1 and the notation of Section 3.1:

- When the pattern A does not contain any overlaps

$$\mathbb{E}[\tau_A] = \frac{1}{\pi_{a_1}\prod_{i=1}^{m-1} P_{a_i,a_{i+1}}} \tag{53}$$

  and the expected time until the first occurrence is

$$\mathbb{E}[\tau_{0,A}|X_0 = x_0] = \mu(x_0,a_1) - \mu(a_m,a_1) + \mathbb{E}[\tau_A] \tag{54}$$

- When the pattern A has an overlap $(a_1,\cdots,a_k)$ of size $k \in \mathbb{N}$, and $(a_1,\cdots,a_k)$ itself does not contain any overlaps:

$$\mathbb{E}[\tau_{0,A}|X_0 = a_0] = \mu(a_0,a_1) - \mu(a_k,a_1) + \frac{1}{\pi_{a_1}\prod_{i=1}^{k-1} P_{a_i,a_{i+1}}} + \mathbb{E}[\tau_A] \tag{55}$$

Before proving Theorem 5, we define a few notations. For any $z_1,\cdots,z_\ell \in \mathcal{X}$, let $\Delta T(z_1,\cdots,z_\ell)$ be additional number of transitions needed to observe $(a_1,\cdots,a_m)$ assuming $(z_1,\cdots,z_\ell)$ has already been observed. We define an associated Markov chain $\{Y_n\}$ with states given by the past m sequences of the original Markov chain X_n, i.e. $Y_n \triangleq (X_{n-m+1}, \cdots, X_n)$. It is easy to verify that Y_n has the memorylessness property. Denote the stationary probabilities of Y_n to be $\rho(z_1, \cdots, z_m)$ for any sequence of states $(z_1, \cdots, z_m) \in \mathcal{X}^m$. Note that

$$\rho(z_1, \cdots, z_m) \triangleq \frac{1}{\pi_{z_1} \prod_{i=1}^{m-1} P_{z_i, z_{i+1}}} \tag{56}$$

Let $\tau(z_1, \cdots, z_m)$ denote the number of transitions of the Markov chain Y_n between consecutive visits to the state $(z_1, \cdots, z_m), z_k \in \mathcal{X}$ for all $k = 1, \cdots, m$. Note that by Lemma 2:

$$\mathbb{E}[\tau(z_1, \cdots, z_m)] = \frac{1}{\rho(z_1, \cdots, z_m)} \tag{57}$$

Proof of (53) and (54). Suppose the pattern has no overlaps. Then by Lemma 2:

$$\mathbb{E}[\tau_A] = \mathbb{E}[\Delta T(x_m)] = \mathbb{E}[\tau(x_1, \cdots, x_m)] = \frac{1}{\rho(x_1, \cdots, x_m)} \tag{58}$$

and (53) follows from (56).

To compute (54), note that

$$\mathbb{E}[\tau_A] = \mathbb{E}[\Delta T(x_m)] = \mu(x_m, x_1) + \mathbb{E}[\Delta T(x_1)] \tag{59}$$

That is, the expected number of transitions between two consecutive occurrences of $(x_1, \cdots, x_m)$ is equivalent to the expected additional amount of time it takes to observe $(x_1, \cdots, x_m)$ given x_m has already been observed. This is then equivalent to the expected number of transitions it takes to observe x_1 given x_m, plus the expected number of transitions it takes to observe $(x_1, \cdots, x_m)$ given x_1.

Using (58), (59) becomes:

$$\mathbb{E}[\Delta T(x_1)] = \frac{1}{\rho(x_1, \cdots, x_m)} - \mu(x_m, x_1) \tag{60}$$

By the same token as (59), note that

$$\mathbb{E}[\tau_{0,A} | X_0 = x_0] = \mu(x_0, x_1) + \mathbb{E}[T(x_1)] \tag{61}$$

Substituting (60) into (61) yields the desired result (54). □

Proof of (55). Suppose the pattern $(x_1, \cdots, x_m)$ has an overlapping subsequence $(x_1, \cdots, x_k)$, $k \leq m$, which itself does not have an overlap.

In this case, we have by (58):

$$\mathbb{E}[\tau_A] = \mathbb{E}[\Delta T(x_{m-k+1}, \cdots, x_m)] = \mathbb{E}[\Delta T(x_1, \cdots, x_k)] = \mathbb{E}[\tau(x_1, \cdots, x_m)] = \frac{1}{\rho(x_1, \cdots, x_m)} \tag{62}$$

Note that the expression of $\mathbb{E}[\tau_A]$ here is the same as the case with no overlaps. However, there is a difference between its interpretation: $\mathbb{E}[\tau_A] = \mathbb{E}[\Delta T(x_m)]$ in the case without overlaps, but $\mathbb{E}[\tau_A] = \mathbb{E}[\Delta T(x_{m-k+1}, \cdots, x_m)]$ here.

Denote $T_0(x_1, \cdots, x_k)$ to be the time it takes to observe the sequence $(x_1, \cdots, x_k)$ for the first time. Then note that

$$\mathbb{E}[\tau_{0,A}|X_0 = x_0] = \mathbb{E}[\tau_{0,A}(x_1, \cdots, x_k)|X_0 = x_0] + \mathbb{E}[\Delta T(x_1, \cdots, x_k)] \tag{63}$$

That is, the expected number of transitions until the first occurrence of $(x_1, \cdots, x_m)$ is equivalent to the expected number of transitions it takes to observe $(x_1, \cdots, x_k)$, plus the expected additional time it takes to observe $(x_1, \cdots, x_m)$ after $(x_1, \cdots, x_k)$ has been observed.

Since $(x_1, \cdots, x_k)$ has no overlaps:

$$\mathbb{E}[\tau_{0,A}(x_1, \cdots, x_k)|X_0 = x_0] = \mu(x_0, x_1) - \mu(x_k, x_1) + \frac{1}{\pi_{x_1} \prod_{i=1}^{k-1} P_{x_i, x_{i+1}}} \tag{64}$$

Substituting (64) and (62) into (63) yields the desired result. $\square$

5 Case Study I: The Dynamic Power Grid Network

In this section, we demonstrate the framework of Figure 1 to the specific case study of a power grid network with time-varying topology, set up and extended from [5] in the following way. For each topology $m \in \{1, \cdots, M\}$, the network is described by a graph in the usual notation $\mathcal{G}(m) \triangleq (\mathcal{V}, \mathcal{E}(m))$, where $|\mathcal{V}| = N_s \in \mathbb{N}$ is the number of nodes.

Assumption 1. Topology changes occur in the set of links $\mathcal{E}$, but not the nodes $\mathcal{V}$, i.e. N_s remains constant over all time. $\square$

The discrete-time dynamics for each subsystem $i \in \{1, \cdots, N_s\}$ can be expressed as the following linear hybrid system:

$$\mathbf{x}_i[t+1] = A_{ii}(\xi[N[t]])\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i(\xi[N[t]])} A_{ij}(\xi[N[t]])\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}[t] \tag{65}$$

Here, $\mathbf{x}[t] \in \mathbb{R}^{N_x}, \mathbf{u}[t] \in \mathbb{R}^{N_u}, \mathbf{w}[t] \in \mathbb{R}^{N_x}$ for all $t \in \mathbb{N}$, and $N[t]$ is the number of phase switches that have been made by time t. The neighboring nodes of subsystem i are defined by $\mathcal{N}_i(m) \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}(m)\}$. The full network system is defined without subscripts i; that is, the full vector state is given by $\mathbf{x}[t] \triangleq [\mathbf{x}_1]^T \cdots \mathbf{x}_{N_s}^T]^T \in \mathbb{R}^{N_s N_x}$, and we denote the full matrix for each topology $m \in \{1, \cdots, M\}$ as $A(m) \triangleq [A_{ij}(m)] \in \mathbb{R}^{N_s N_x \times N_s N_x}$.

Assumption 2. For all $m \in \{1, \cdots, M\}$, the pair $(A(m), B)$ is controllable. Furthermore, B is fixed constant and known. $\square$

The external phase-switching mechanism of the dynamics have state $\xi[n] \in \{1, \cdots, M\}$ for a finite number $M \in \mathbb{N}$ total phases, and abides by an irreducible Markov chain with transition probability matrix $[P_{ij}] \in \mathbb{R}^{M \times M}$. Note the phase switches can be modeled as a sequence of tuples $\{\xi[n]\}_{n=1}^{\infty}$ which becomes the sequence $\{X_n\}$ generating our patterns of interest. Mathematically:

$$\xi[N[t]+1] = h(\xi[N[t]]) \triangleq \begin{cases} 1 & \text{w.p. } P_{\xi[N[t]],1} \\ \vdots & \\ M & \text{w.p. } P_{\xi[N[t]],M} \end{cases} \tag{66}$$

where $h : \{1, \cdots, M\} \to \{1, \cdots, M\}$.

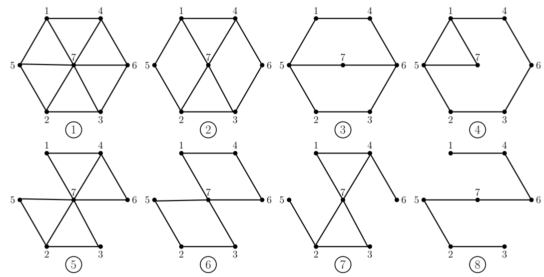

Figure 2: The states of the power grid topology switches between due to link failures and repairs over time.

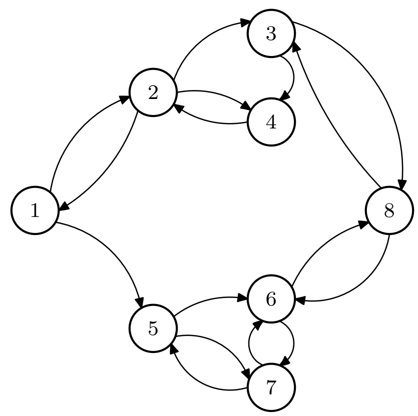

Figure 3: The true dynamics of the power grid state topology changes over time, represented as a Markov chain.

5.1 Topology-Adaptive System-Level Synthesis

The methodology proposed in [5] designs an iterative procedure for localized control, based on the novel *systems-level synthesis (SLS)* [13], which is robust and adaptive to topological changes. Controllers are implemented in a localized fashion: each node $i \in \{1, \cdots, N_s\}$ accumulates information from a local subsystem created by using the local $A_k^{(i)}$ and $B^{(i)}$ submatrices formed by grouping neighboring nodes $\mathcal{N}_i[t] \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$, and extracting the rows of the full matrices A_k and B corresponding to $\mathcal{N}_i[t]$. By Assumption 1, we construct a global combinatorial set $\mathcal{A} \subset \mathbb{R}^{N_x \times N_x}, |\mathcal{A}| \leq \binom{N_s}{2}$ by enumerating all possible single link modifications among the nodes $\mathcal{V}$. We prescribe a subset $\mathcal{A}^{(i)}$ of this global set to each subsystem $i \in \mathcal{V}$, corresponding to single link modifications which are relevant only to $\mathcal{N}_i[t]$.

Let $\hat{A}^i[t]$ be subsystem i's estimate of the local topology at time t. We assume the initial topology is known $\hat{A}^i[0] = A_k^{(i)}$, for some $k \in \{1, \cdots, K\}$; this means an initial distributed closed-loop controller response $\{\Phi_{k,x}^{(i)}[s], \Phi_{k,u}^{(i)}[s]\}$ are easily computed using the standard SLS method developed in [13]. Here, $s \in \{1, \cdots, T\}$ is the spectral component, and $T \in \mathbb{N}$ is the finite horizon chosen by design.

Define $\mathcal{P}_0^{(i)} := \{-1, 0, 1\}^{|\mathcal{A}^{(i)}|}$ to be the initial *set of consistent coefficients* corresponding to the enumerated set $\mathcal{A}^{(i)}$ of single-link basis matrices for subsystem i. For vector $\boldsymbol{\xi} \in \mathcal{P}_0^{(i)}$, a coefficient of $\xi_\ell = -1$ means that link ℓ has been deleted, $\xi_\ell = 1$ if new link ℓ has been added, and $\xi_\ell = 0$ if link ℓ remains unchanged.

As phase switches occur over time, each subsystem i tests all possible combinations of link deletions and additions from $\mathcal{A}^{(i)}$ as guesses for how the local topology has changed. We update the set of consistent coefficients $\mathcal{P}_t^{(i)}$ by identifying which coefficients remain consistent with an observation of a single timestep

$(\mathbf{x}[t], \mathbf{x}[t-1], \mathbf{u}[t-1])$ of the system dynamics (65)

$$\mathcal{P}_{t+1}^{(i)} := \left\{ \boldsymbol{\xi} \in \mathcal{P}_t^{(i)} \;\middle|\; \left\| \mathbf{x}[t+1] - \left(\hat{A}^i[t] + \sum_{\substack{\ell=1 \\ D_\ell \in \mathcal{A}^{(i)}}}^{|\mathcal{A}^{(i)}|} \xi_\ell D_\ell \right) \mathbf{x}[t] + B\mathbf{u}[t] \right\| \leq \eta \right\} \tag{67}$$

If $|\mathcal{P}_t^{(i)}| > 1$, it is necessary to construct a topologically-robust controller $\{\Psi_{x,i}^{(t)}[s], \Psi_{u,i}^{(t)}[s]\}$ for each subsystem i. Define the matrix

$$\Delta_s^j \left(A, B, \Psi_{x,ji}^{(t)}, \Psi_{u,ji}^{(t)} \right) := \Psi_{x,ji}^{(t)}[s+1] - \sum_{\ell \in \mathcal{N}(j)} A_{j\ell} \Psi_{x,\ell i}^{(t)}[s] - B_j \Psi_{u,ji}^{(t)}[s] \tag{68}$$

Here, $i, j = 1, \cdots, N_s$, $s = 1, \cdots, T$ for finite horizon $T \in \mathbb{N}$, t is the simulation timestep, and $A_{j\ell}$ is the submatrix of A relating subsystems $j, \ell \in \{1, \cdots, N_s\}$. To construct a topologically-robust controller $\{\Psi_{x,i}^{(t)}, \Psi_{u,i}^{(t)}\}$ for each subsystem i, we solve the following distributed optimization problem:

$$\min \sum_{s=1}^{T} \left\| Q\Psi_{x,i}^{(t)}[s] + R\Psi_{u,i}^{(t)}[s] \right\|_1 + r_c \epsilon \tag{69a}$$

$$\text{s.t.} \left\| \sum_{j \in \mathcal{N}(i)} \Delta_s^j \left(A', B, \Psi_{x,ji}^{(t)}, \Psi_{u,ji}^{(t)} \right) \right\| \leq c_i \rho^{s-1} \forall s \leq T-1,\ A' = \hat{A}^i[t] + \sum_{\substack{\ell=1 \\ D_\ell \in \mathcal{A}^{(i)}}}^{|\mathcal{A}^{(i)}|} \xi_\ell D_\ell,\ \forall \boldsymbol{\xi} \in \mathcal{P}_t^{(i)} \tag{69b}$$

$$c_i \sum_{s=1}^{T} \rho^{s-1} \leq \lambda_t^{(i)} + \epsilon, \quad \epsilon > 0 \tag{69c}$$

for design parameters $\rho, \lambda_t^{(i)}, c_i, r_c, \epsilon > 0$. If $|\mathcal{P}_t^{(i)}| = 1$, we can invoke standard SLS to compute $\{\Phi_{k,x}^{(i)}[s], \Phi_{k,u}^{(i)}[s]\}_{s=1}^{T}$ for subsystem i instead of solving (69), where $k \in \{1, \cdots, K\}$ is the consistent topology constructed from the coefficients of $\mathcal{P}_t^{(i)}$. Complete details of the robust, adaptive controller implementation can be found in [5].

5.2 Case Study: Dynamic Power Grid Network

We now implement the pattern learning component to improve controller performance by predicting specific phase transition patterns in the system (65) and (66). In the context of Figure 1, the "external process" refers to the phase-switching mechanism (66), while the "plant dynamics" refers to (65). The controller of Section 5.1 can be used as the modulation control component responsible for controlling power flow in the network despite disturbances due to line failures. We emphasize that, as a result of the localized implementation, both the "controller" and "pattern-occurrence prediction" blocks in Figure 1 will be implemented in a localized way for each subsystem $i \in \{1, \cdots, N_s\}$.

The specific A and B matrices in (65) for a single subsystem $i \in \{1, \cdots, N_s\}$ in the simplified discrete-time power grid is given by

$$\begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \end{bmatrix} [t+1] = \begin{bmatrix} 1 & \Delta t \\ -\frac{a_i}{c_i}\Delta t & 1 - \frac{b_i}{c_i}\Delta t \end{bmatrix} \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \end{bmatrix} [t] + \sum_{j \in \mathcal{N}(i)/\{i\}} \begin{bmatrix} 0 & 0 \\ \frac{a_{ij}}{c_i}\Delta t & 0 \end{bmatrix} \begin{bmatrix} x_1^{(j)} \\ x_2^{(j)} \end{bmatrix} [t] + \begin{bmatrix} 0 \\ \Delta t \end{bmatrix} (w_i[t] + u_i[t]) \tag{70}$$

Here, c_i is its inertia, b_i is a damping factor, w_i is the external disturbance, u_i is the control action, and Δt is the sampling time. The states are the relative phase angle $x_1^{(i)}$ between its rotor axis and external field, and its derivative, the frequency $x_2^{(i)}$.

In the pattern-learning component, we keep track of a specific sequence of transitions which are likely to cause faults in the system. It becomes important to approximate the expected duration between consecutive occurrences of such transitions, and compute the probability that some transitions will precede other transitions. Addressing both aspects require the theoretical foundation established in Section **??**. The utilization of the "learning external process" and "system identification" blocks in Figure 1 depends on the extent of our knowledge about the environment. In the case study of this power grid network, the sparsity patterns of the $\{A(1), \cdots, A(M)\}$ are unknown, parameters such as inertial and damping factor are known, and the true transition probability matrix $[P_{ij}]$ behind the phase-switching mechanism (66) is unknown.

When only the spatial structure of the matrices $\{A(1), \cdots, A(M)\}$ are unknown, it suffices to use the topology identification procedure described in Section 5.1 to identify the topology this correspond to the "system identification" block in Figure 1. Each subsystem $i \in \mathcal{V}$ thus maintains a local estimate $\hat{P}^{(i)}[t]$ of the transition probability matrix P, which it updates over time t as it learns the topology over time, by empirically computing the proportion of transitions that go between different topologies once it identifies which topology it is currently in. Each subsystem also stores and updates a pattern table $\mathcal{T}_i$ which maps topologies to closed-loop response mappings for its own corresponding subsystem, i.e. $\mathcal{T}_i[k] = \{\Phi_{k,x}^{(i)}[s], \Phi_{k,u}^{(i)}[s]\}_{s=1}^{T}$. The entries of $\mathcal{T}_i$ are updated if a topology has not been observed before, and a new controller response is synthesized for it. Each subsystem also maintains a local estimate of the expected waiting times $\hat{\tau}^{(i)}[t]$ and first occurrence probabilities $\{\hat{\alpha}_j^{(i)}[t]\}_{j=1}^{K}$ of the collection of patterns $\mathcal{B}$ from the pattern-occurrence problems (3), where $K \triangleq |\mathcal{B}|$. Given a time series of identified topologies, we use the previous state information to construct initial and later-ending strings to compute the $\{\alpha_j\}$.

Identifying and predicting the occurrence of fault patterns allows us to achieve two things: 1) decide the order of checking consistent coefficients in the $\mathcal{P}_t^{(i)}$, and 2) reduce the impact induced by abrupt changes in topology. In this case, we target specific combinations of line deletions over time, which are what we use to represent as the patterns in the power grid system. Oftentimes, a phase switch can cause none of the coefficients to be consistent $|\mathcal{P}_t^{(i)}| = 0$; it is then necessary to perform a reset of coefficients $\mathcal{P}_t^{(i)} = \mathcal{P}_0^{(i)}$, and perform one more round of updating the consistent set based on the observation $(\mathbf{x}[t], \mathbf{x}[t-1], \mathbf{u}[t-1])$. When estimates of the expected waiting times $\hat{\tau}^{(i)}[t]$ and first occurrence probabilities $\{\hat{\alpha}_j^{(i)}[t]\}_{j=1}^{K}$ of specific fault patterns $\mathcal{B}$ are available, however, it may not be necessary to reset coefficients to the full initial set $\mathcal{P}_0^{(i)}$. To improve the speed of topology identification, we can use $\{\hat{\alpha}_j^{(i)}[t-1]\}_{j=1}^{K}$ to filter out coefficients in $\mathcal{P}_0^{(i)}$ which are clearly inconsistent. We denote this filtering process with a function $U(\mathcal{P}_0^{(i)}, \{\hat{\alpha}_j^{(i)}[t-1]\}_{j=1}^{K})$.

The algorithm pseudocode of applying Figure 1 specifically to the system prescribed by (65) and (66) is given in Algorithm 1. The specific network we choose to study is a hexagonal arrangement of $N_s = 7$ nodes and $M = 8$ phases, as shown in Figure 2 and Figure 3. The full system response for a sample trajectory over $T_{\text{sim}} = 50$ time steps is shown in Figure 4. We designate the specific fault patterns to be any topology sequence which includes any single transitions with more than two total link changes; from Table 1, this means $\mathcal{B} \triangleq \{(2,3), (2,4), (4,2), (6,7), (7,6)\}$ with $K \triangleq |\mathcal{B}| = 5$. The estimated quantities are illustrated in Figure 5.

The estimated quantities are illustrated in Figure 5. The matrix norm difference between the true and the estimated transitions probability matrices constructed for each of the three subsystems in the top of Figure 5 decrease over time. This is consistent with the fact that each subsystem learns more about the

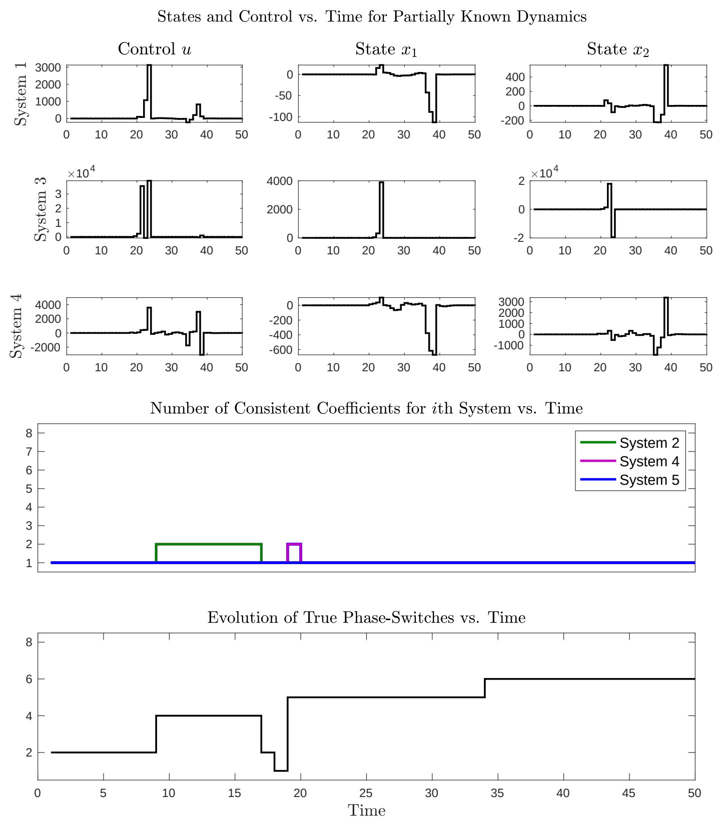

Figure 4: [Top] The evolution over time of the state trajectory and control law of three chosen subsystems $1, 3, 4$ under partially-known dynamics. [Bottom] The evolution over time of the number of consistent coefficients in the topology identification process of three other chosen subsystems $2, 4, 5$, along with the true phase-switching over time.

Transition Pairs	Link Changes	Total
$(1, 2)$	$(+0, -2)$	2
$(1, 5)$	$(+0, -2)$	2
$(2, 3)$	$(+2, -4)$	6
$(2, 4)$	$(+1, -3)$	4
$(3, 4)$	$(+1, -1)$	2
$(3, 8)$	$(+0, -2)$	2
$(5, 6)$	$(+0, -2)$	2
$(5, 7)$	$(+0, -2)$	2
$(6, 7)$	$(+2, -2)$	4
$(6, 8)$	$(+0, -2)$	2

Table 1: Possible transition pairs in Figure 3 and the links changed between them.

Algorithm 1 Two-Part Hierarchical Learning-Based Control for the Power Grid Network

Input: T_{sim}, $\hat{\mathbf{x}}[0]$, $\xi[0]$, $\{\mathcal{P}_0^{(i)}\}_{i=1}^{N_s}$, $\{\hat{P}^{(i)}[0]\}_{i=1}^{N_s}$

for Subsystem $i = 1 : N_s$ **do**
    Initialize $\mathcal{T}_i[\xi[0]] \leftarrow \{\Phi^{(i)}_{\xi[0],x}[s], \Phi^{(i)}_{\xi[0],u}[s]\}_{s=1}^{T}$.
end for
for $t = 2 : T_{\text{sim}}$ **do**
    Propagate true dynamics $\{$(65), (66), (70)$\}$ one step.
    for Subsystem $i = 1 : N_s$ **do**
        Update consistent set $\mathcal{P}_t^{(i)}$ using (67).
        if $|\mathcal{P}_t^{(i)}| = 1$ **then**
            Update transition probability matrix $\hat{P}^{(i)}[t]$.
            continue
        else if $|\mathcal{P}_t^{(i)}| = 0$ **then**
            if $\hat{\tau}^{(i)}[t-1]$ has elapsed **then**
                $\mathcal{C}^{(i)} \triangleq U(\mathcal{P}_0^{(i)}, \{\hat{\alpha}_j^{(i)}[t-1]\}_{j=1}^{|\mathcal{B}|})$.
            else
                $\mathcal{C}^{(i)} \triangleq \mathcal{P}_0^{(i)}$.
            end if
            Reset consistent set $\mathcal{P}_t^{(i)}$ using (67) on $\mathcal{C}^{(i)}$.
            if $|\mathcal{P}_t^{(i)}| = 1$ **then**
                Update transition probability matrix $\hat{P}^{(i)}[t]$.
                Get single element $\xi[t] \in \mathcal{P}_t^{(i)}$
                if $\xi[t] \in \mathcal{T}_i$ **then**
                    $\{\Phi^{(i)}_{\xi[t],x}[s], \Phi^{(i)}_{\xi[t],u}[s]\}_{s=1}^{T} \leftarrow \mathcal{T}_i[\xi[t]]$.
                else
                    Standard SLS: $\{\Phi^{(i)}_{\xi[t],x}[s], \Phi^{(i)}_{\xi[t],u}[s]\}_{s=1}^{T}$.
                    $\mathcal{T}_i[\xi[t]] \leftarrow \{\Phi^{(i)}_{\xi[t],x}[s], \Phi^{(i)}_{\xi[t],u}[s]\}_{s=1}^{T}$.
                end if
                continue
            end if
        end if
        Solve (69) for $\mathcal{P}_t^{(i)}$: $\{\Psi^{(t)}_{x,i}[s], \Psi^{(t)}_{u,i}[s]\}_{s=1}^{T}$.
        Update pattern-occurrence $\hat{\tau}^{(i)}[t], \{\hat{\alpha}_j^{(i)}[t]\}_{j=1}^{|\mathcal{B}|}$.
    end for
end for

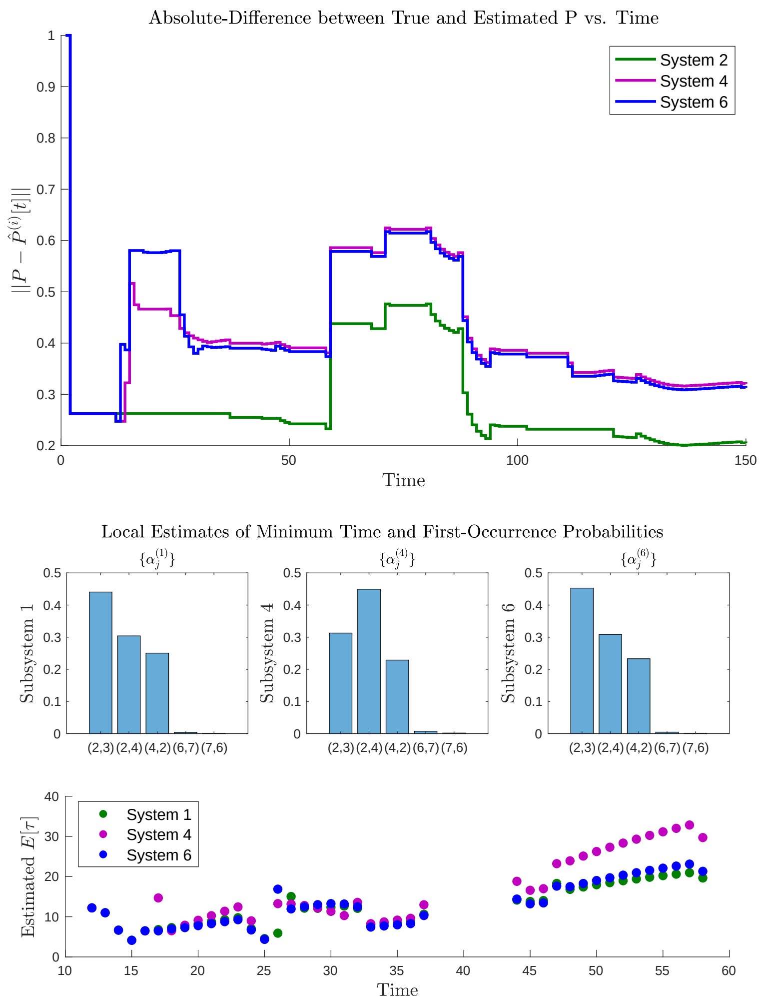

Figure 5: [Top] The evolution over time of the absolute-difference metric of the true transition probability matrix and the estimates of three chosen subsystems. [Bottom] The final estimates of the first-occurrence probabilities $\{\alpha_j^{(i)}[T_{\text{sim}}]\}$ and the estimated $\hat{\tau}^{(i)}[t]$ over time for three chosen subsystems $i \in \{1, 4, 6\}$.

underlying switching dynamics with longer time. Furthermore, as the system learns more about the possible number of topologies the network can switch amongst, the transition probabilities of the underlying Markov chain change in distribution. Hence, all three subsystem estimates of the expected next-occurrence time in the bottom of Figure 5, increase over time. There are skips in the interval of time from between timesteps 37 and 44 because each subsystem identifies topology transitions which are irrelevant to our patterns of interest $\mathcal{B}$.

6 Conclusion

In this paper, we presented a robust controller framework for discrete-time stochastic systems with partially known dynamics by learning repeated patterns in its random impulsive behavior. The controller framework is roughly split into two parts, 1) a pattern-learning component which learns and recognizes patterns, and 2) a modulation control component which computes the control law when a pattern is first observed. In the case where the impulse disturbances are generated from a Markov chain, we derive closed-form expressions for the mean duration between consecutive occurrences, and first occurrence probabilities of the patterns using martingale-based analysis. We demonstrate application of this controller framework by extending the iterative, localized, robust adaptive system-level synthesis (SLS) controller presented in [5] to a simple grid network which undergoes changes in topology due to link faults.

There are numerous extensions for potential future work, such as considering more general distributions of impulse sequences. Furthermore, determining a representation of patterns for applications which deal with a very large number of state configurations such that there are enough overlaps for analysis is a question of future work.

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