Control of Jump Stochastic Systems via Characterization of Their Recurrent Patterns

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SooJean Han (한수진)

October 5th, 2022

California Institute of Technology

Caltech

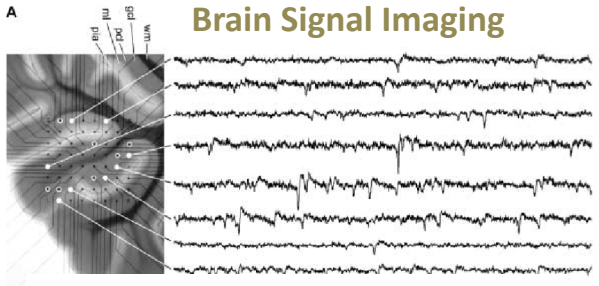
“Jump Stochastic Systems”

Systems with random and repetitive jumps.

Agile Quadrotor Flight

*O’Connell et al, 2022. *Neural-Fly Enables Rapid Learning for Agile Flight in Strong Winds*

Brain Signal Imaging



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Spacecraft Control

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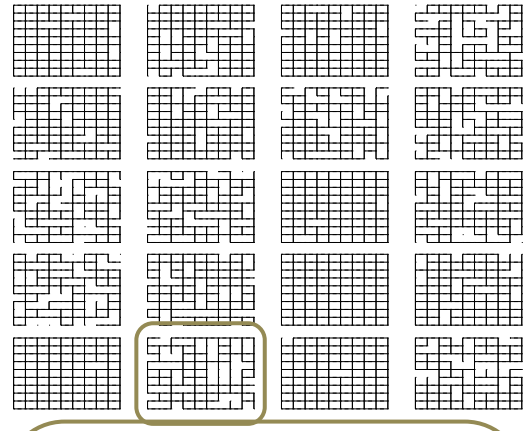
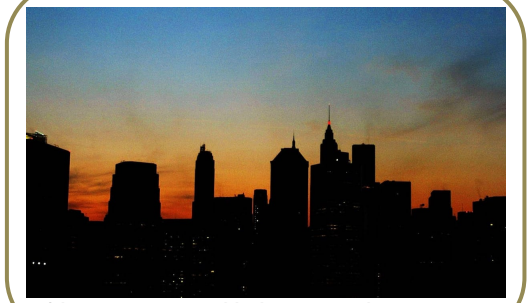
Power Grid





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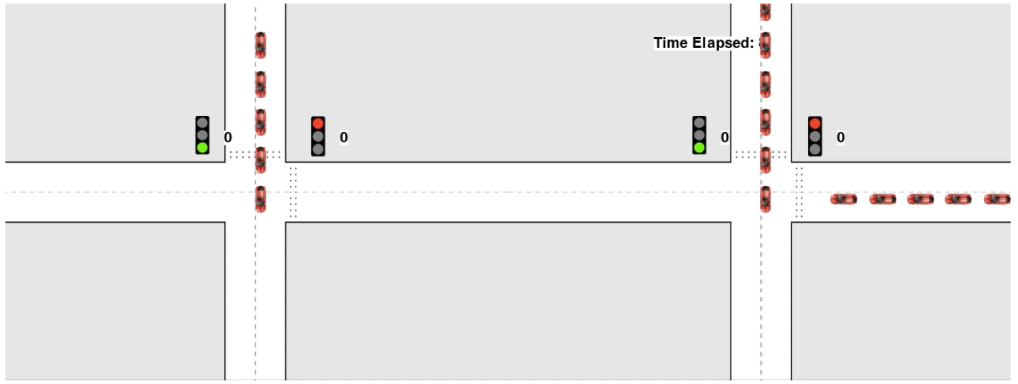
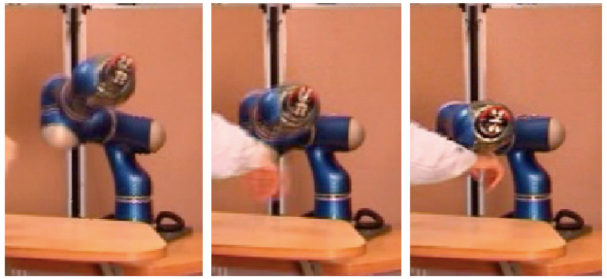
Vehicle Traffic Congestion



Robot Manipulator



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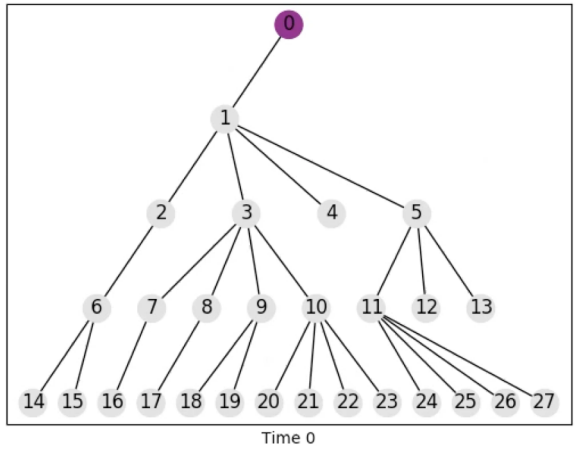
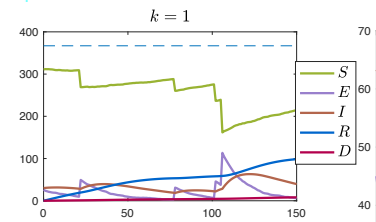
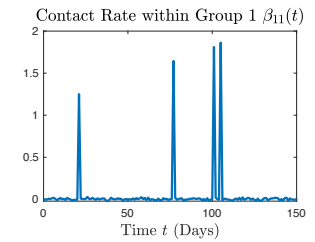
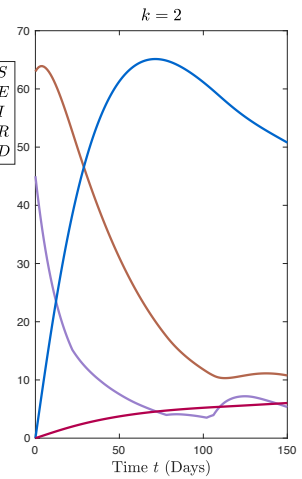
Epidemic Spread









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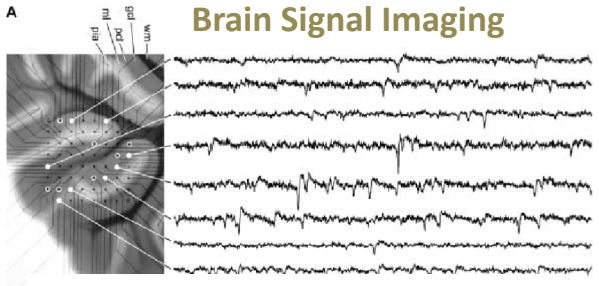

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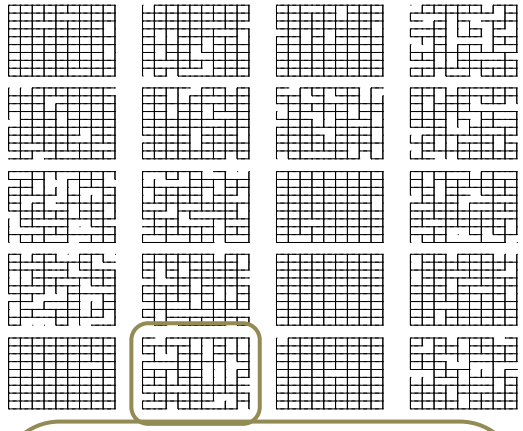
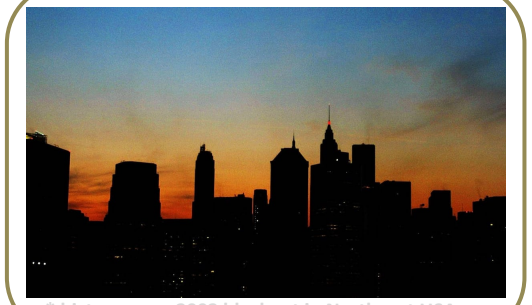
Fault-Tolerant Systems
(faults = external disturbances)

Power Grid





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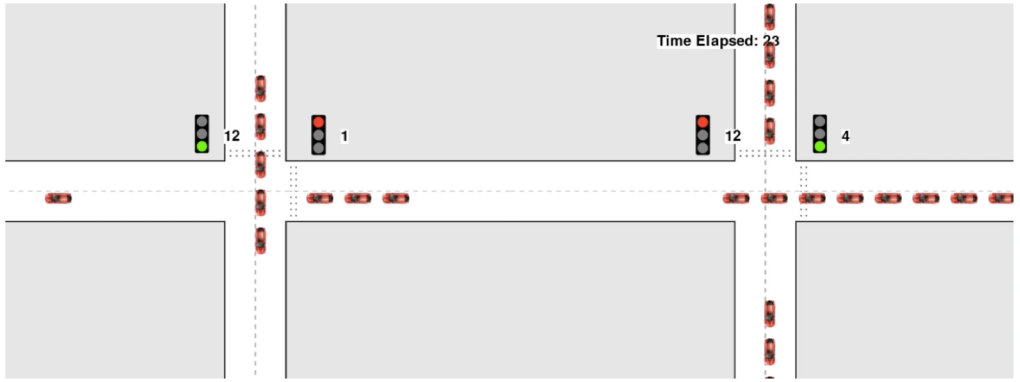

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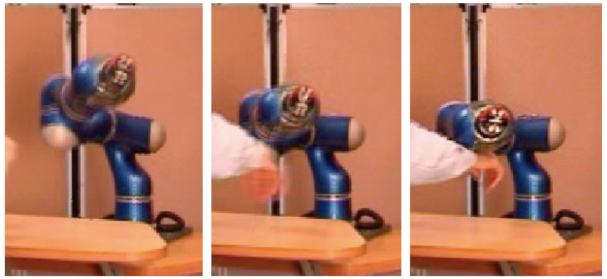

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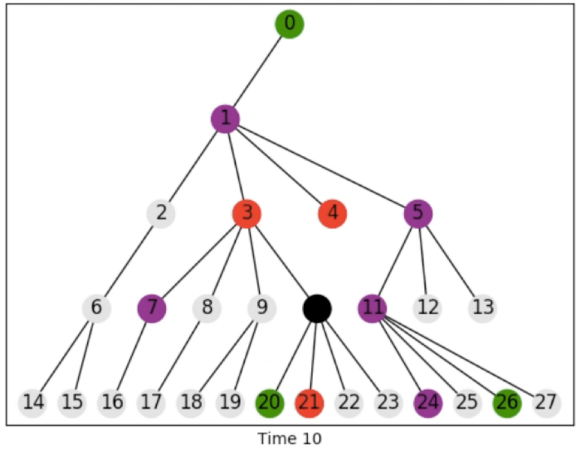

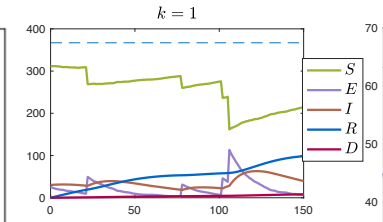

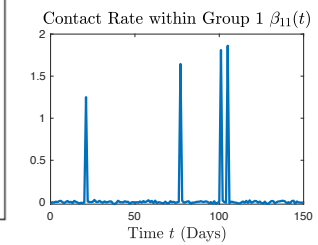

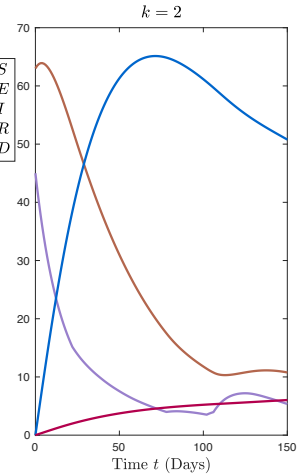

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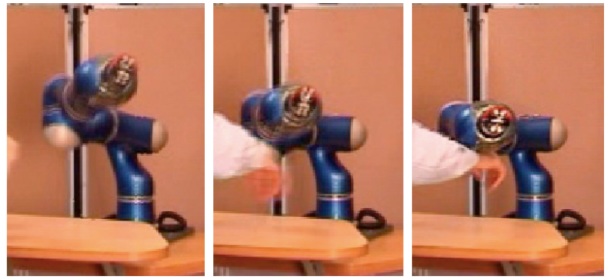

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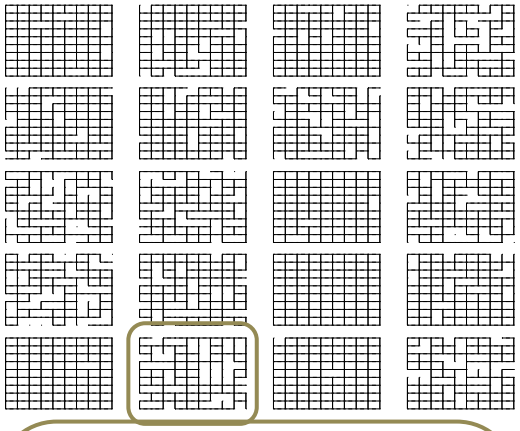

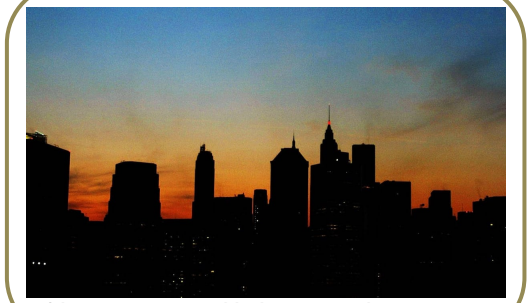

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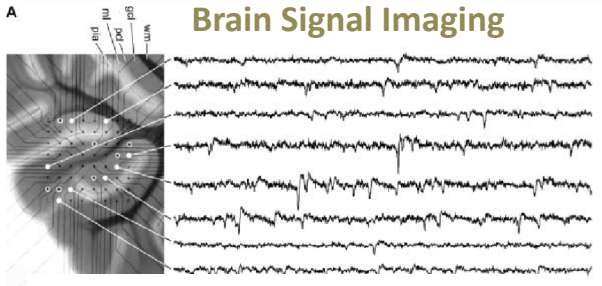

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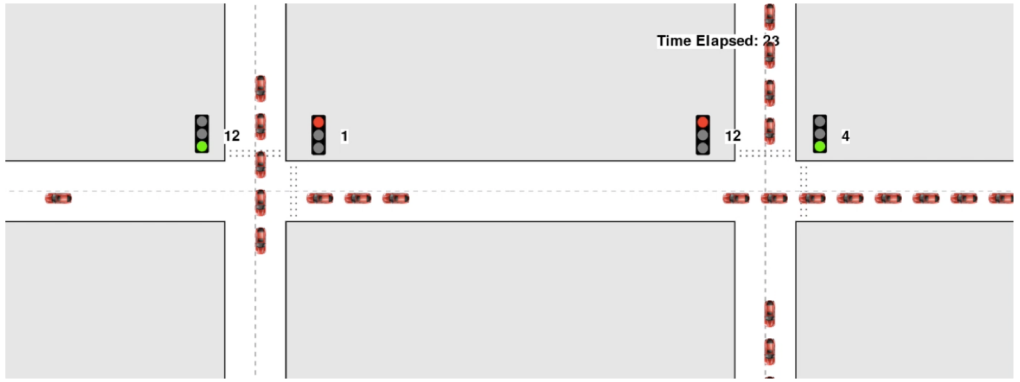
Event-Driven Systems
(jumps can be inherent)

Vehicle Traffic Congestion



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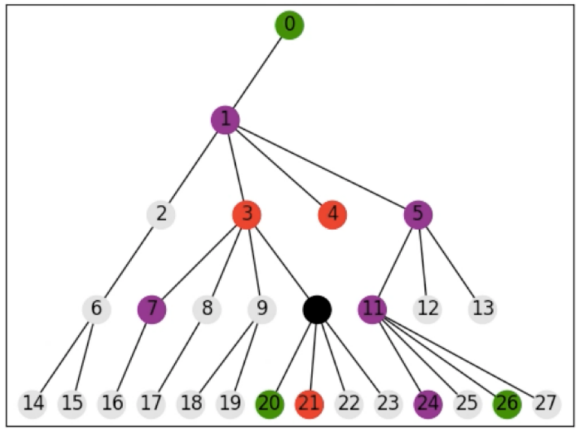
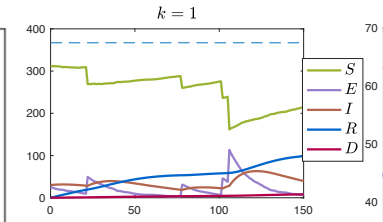
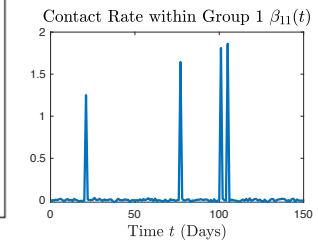
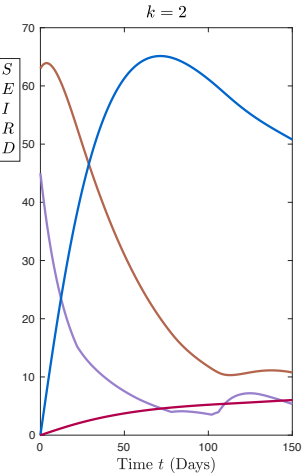








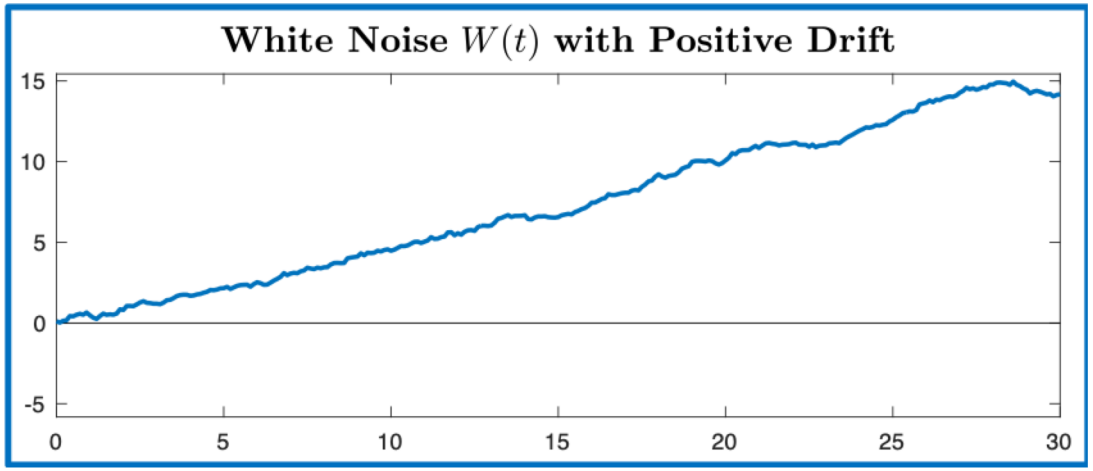
“Jump Stochastic Systems”

Poisson Shot Noise System:

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \xi(t,\mathbf{x})dN(t)$$

Gaussian White Noise System: Brownian motion

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \sigma(t,\mathbf{x})dW(t)$$



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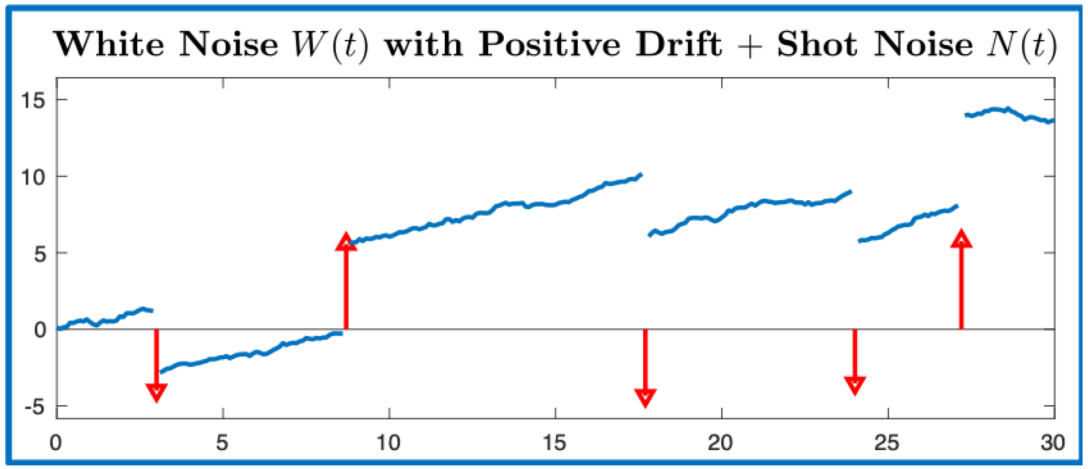

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“Jump Stochastic Systems”

Poisson Shot Noise System: compound Poisson

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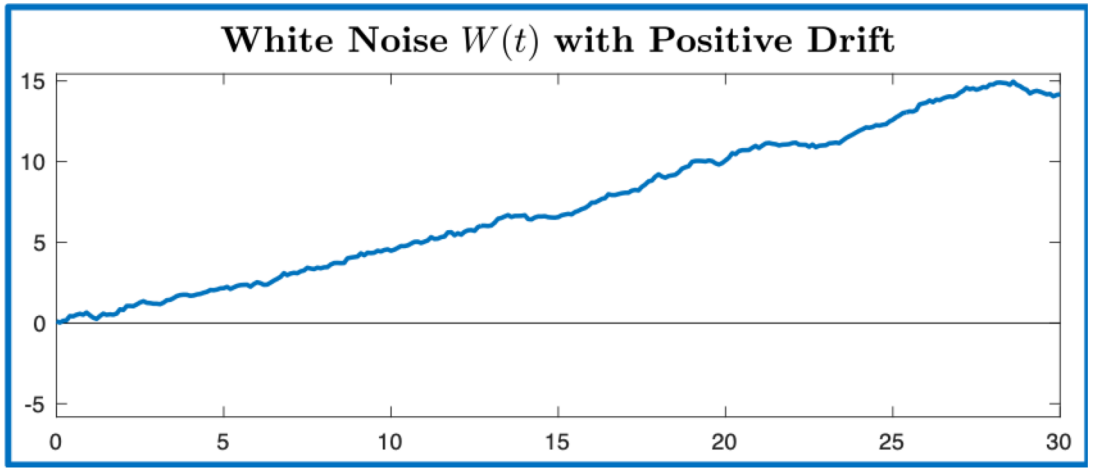

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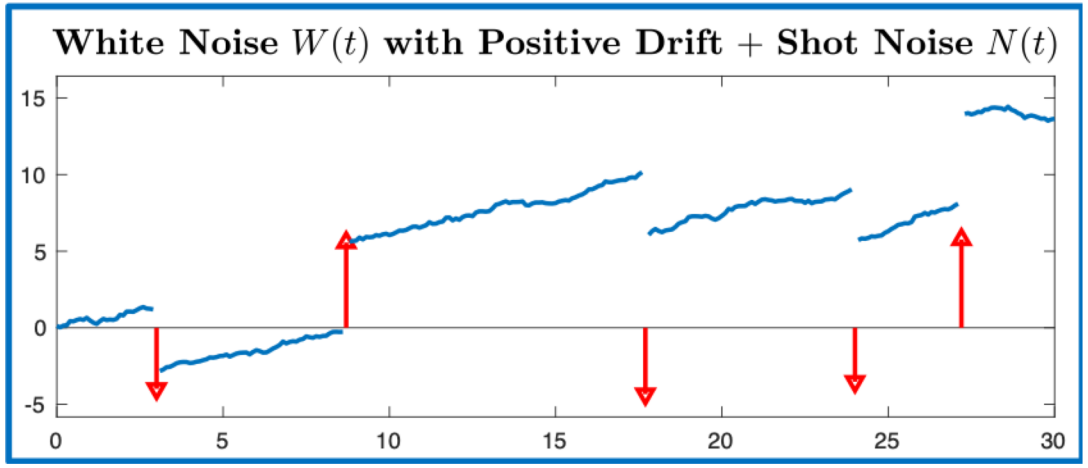

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Stochastic Systems”

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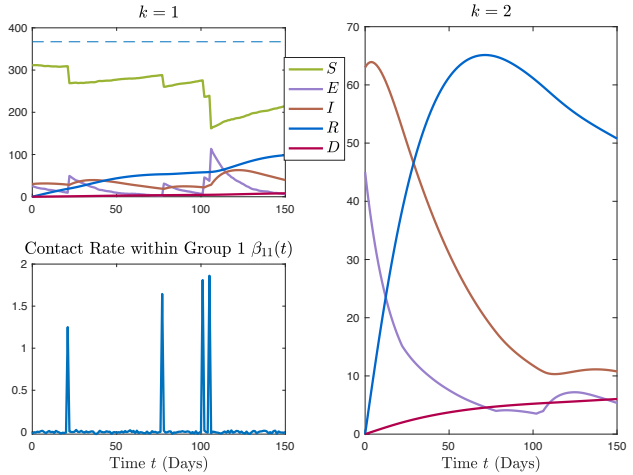

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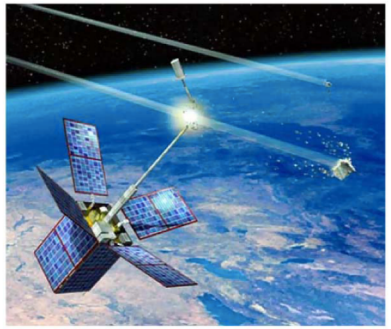

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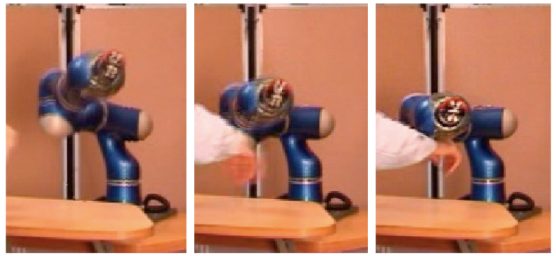

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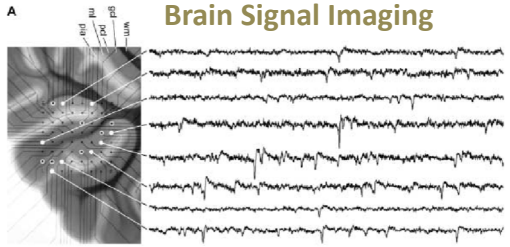

Many jump behaviors can be modeled with shot or Lévy noise

Poisson Shot Noise System: compound Poisson

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \xi(t,\mathbf{x})dN(t)$$

Lévy Noise System: Brownian motion + compound Poisson

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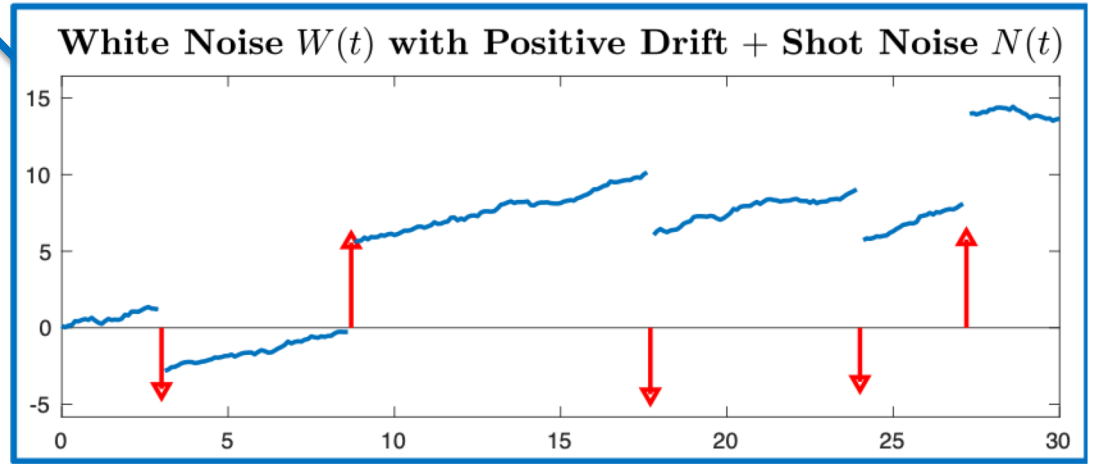

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Infinitesimal generator: $G \in \mathcal{C}^{(1,2)}$

$$\mathcal{L}G = \lim_{t \to 0} \frac{\mathbb{E}_{\mathbf{x}_0}\left[G(t, \mathbf{x}(t))\right] - G(0, \mathbf{x}_0)}{t}$$

Stochastic Stability

(e.g., Kushner 1967, Mao 1990, Applebaum 2009)

Preliminary Question: what conditions are required for shot/Lévy noise to keep the system **stable**?

Let $\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0$ be a solution trajectory of a shot/Lévy noise stochastic system.

Design Lyapunov function $V(t, \mathbf{x}(t))$ such that $\mathcal{L}V \leq -\beta V(t, \mathbf{x}(t))$ for some $\beta > 0$.

By Dynkin's formula*: $\mathbb{E}_{\mathbf{x}_0}[V(t, \mathbf{x}(t)] \leq V(0, \mathbf{x}_0)e^{-\beta t}$

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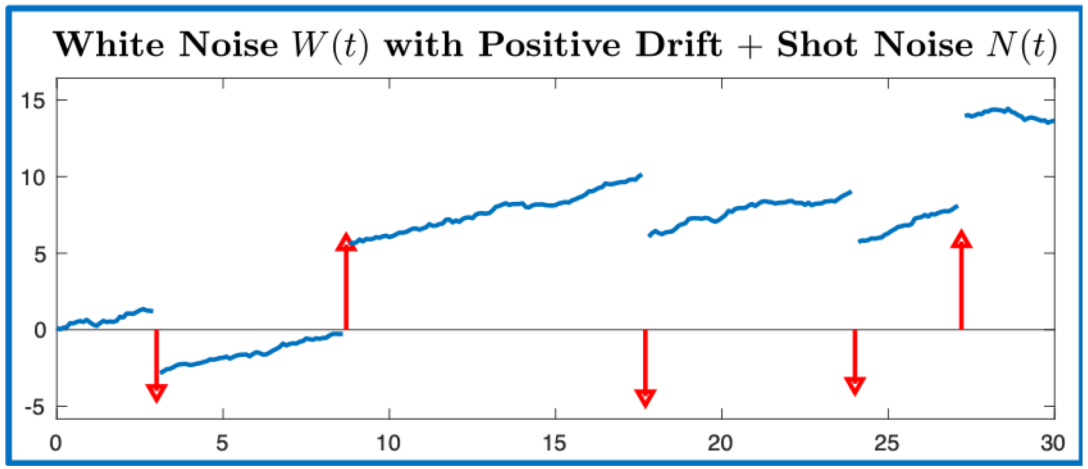

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* Eqn. (2-9) in Kushner 1967

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(e.g., Kushner 1967, Mao 1990, Applebaum 2009)

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***Stochastic incremental stability* generalizes this inequality.**

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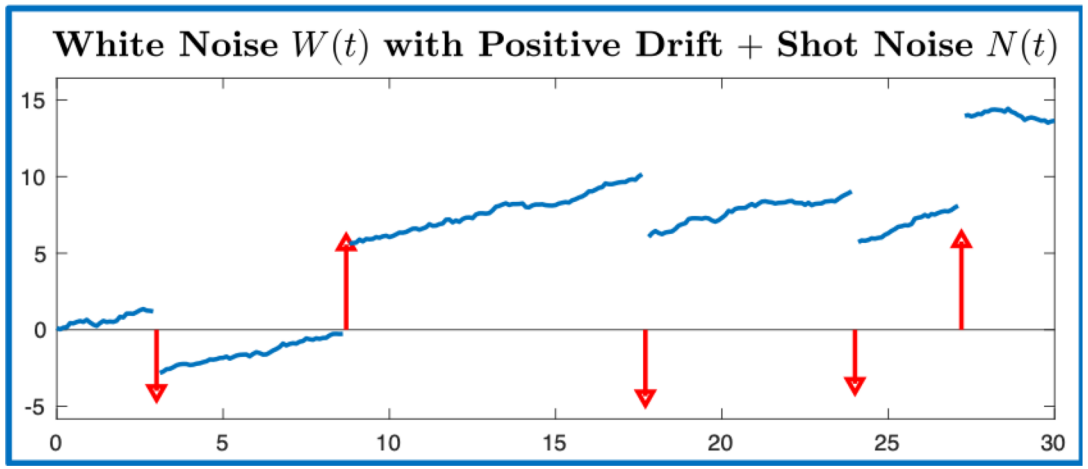

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* Pham, Tabareau, Slotine. *A Contraction Theory Approach to Stochastic Incremental Stability*. IEEE Transactions on Automatic Control, 2007.

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Control of Systems with Repeating Jump Patterns

Control for *jump stochastic systems* can be achieved by learning **patterns** in the system behavior.

At least 2 ways of improving performance:

- **memorize past patterns**
  → no re-computation of repeating control policies
- **predict future patterns**
  → schedule control policies in advance

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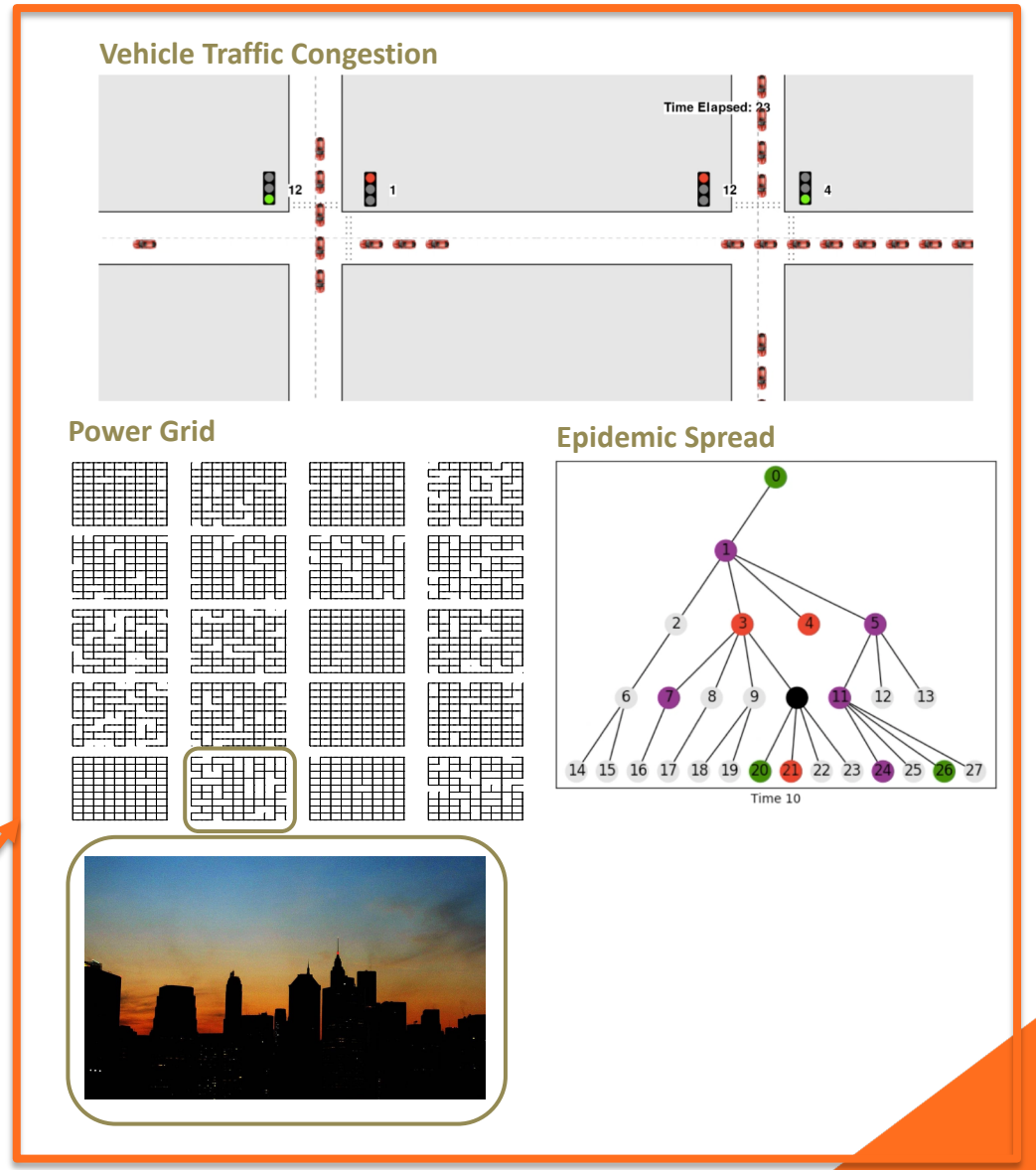
Systems that have jump patterns:

- Networks with time-varying topology (e.g., power grid)
  - **patterns** = power outages, line failures
- Vehicle traffic congestion
  - **patterns** = intersection snapshots
- Epidemic spread
  - **patterns** = contact-tracing data, travel routines



Discrete-Time Linear Markovian Jump Systems

$$T_n \triangleq \min\{s \in \mathbb{N} \,|\, N[s] = n\}$$

$$\|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

$$\{\xi_n\}_{n=1}^{\infty}, \quad \xi_n : \Omega \to \mathcal{X} \triangleq \{1, \cdots, M\}$$

$$\mathbf{x}[t+1] = A(\xi_{N[t]})\mathbf{x}[t] + B\mathbf{u}[t] + \mathbf{w}[t]$$

$$\mathbf{p}_{n+1}^\top = \mathbf{p}_n^\top P \qquad \text{Given } \xi_n = \varphi_n, \quad \xi_{n+1} = m \text{ w.p. } P[\varphi_n, m], m \in \mathcal{X}$$

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Assumption: Mode process $\{\xi_n\}$ operates on a timescale which is $\Delta T \in \mathbb{N}$ times longer than the timescale of the system, i.e. if $N[t] = n$, then $N[t + a\Delta T] = n + a$ for any $a \in \mathbb{N}$.

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- mode process $\{\xi_n\}$ unobservable.
- $\mathcal{X}$, matrices $\{A(1), \cdots, A(M)\}$ and B, initial mode ξ_0 are known.
- sparsity pattern of TPM P known, but transition probabilities unknown.

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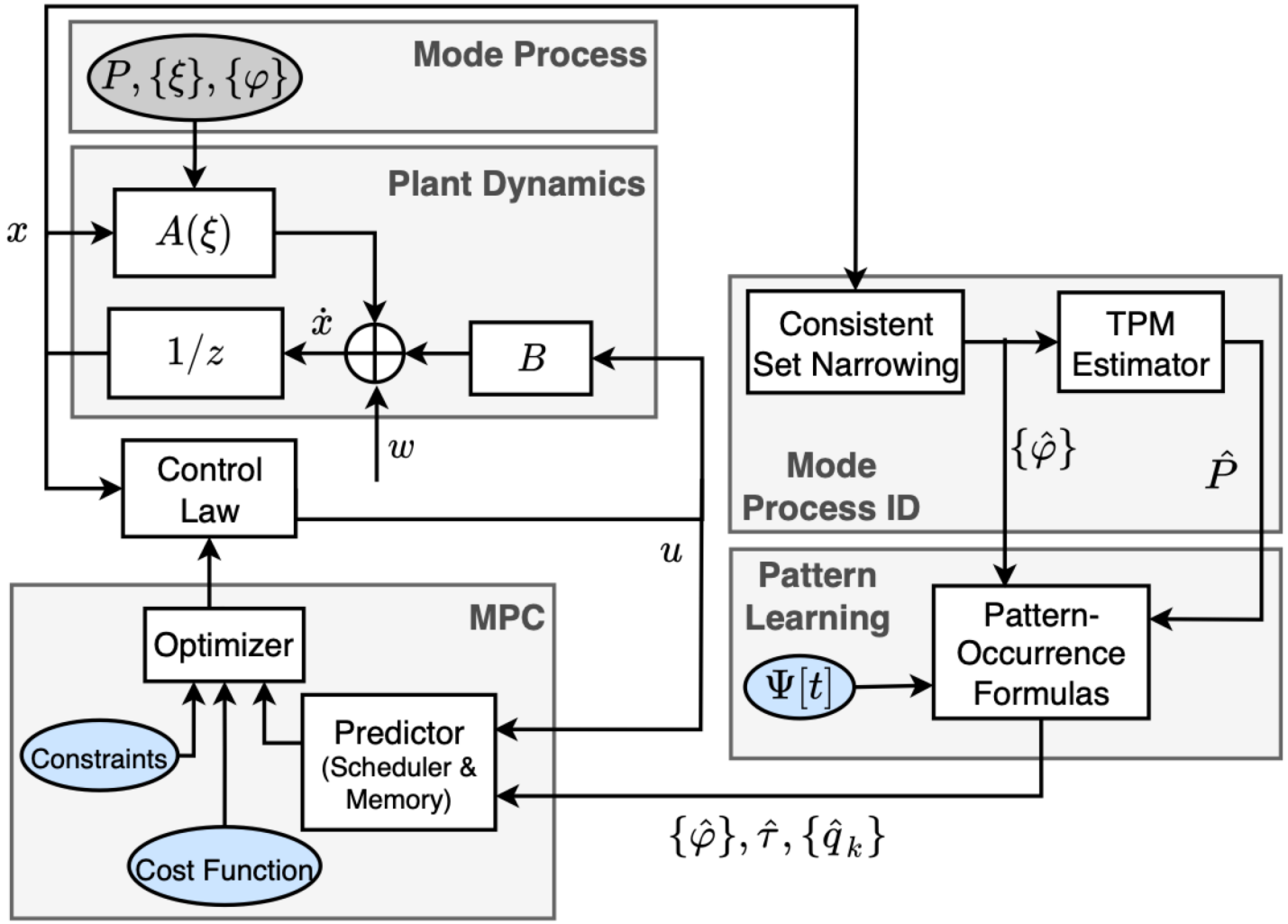
Proposed Controller Framework



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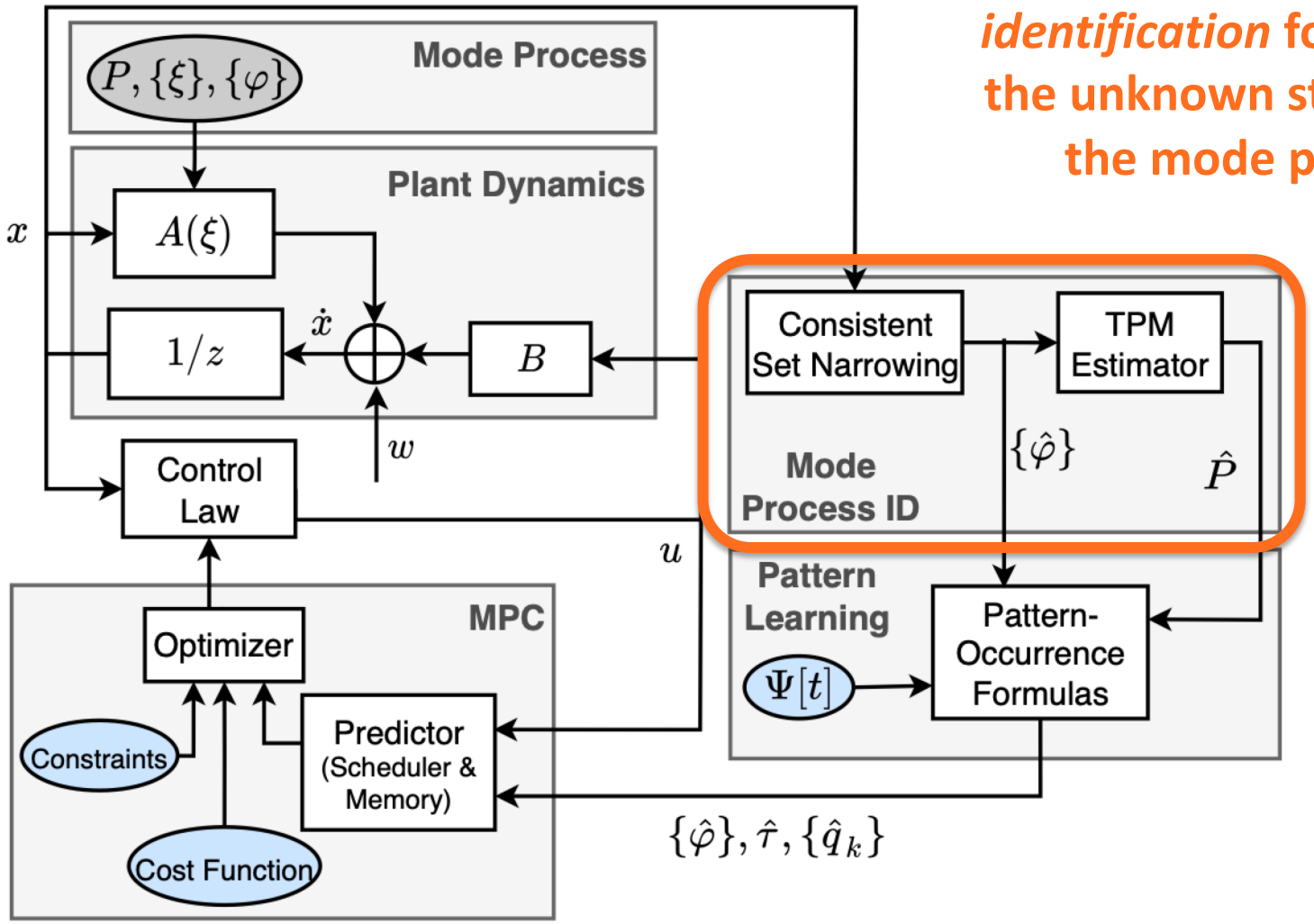
Mode Process Identification (ID)

Mode process identification **for learning the unknown statistics of the mode process**



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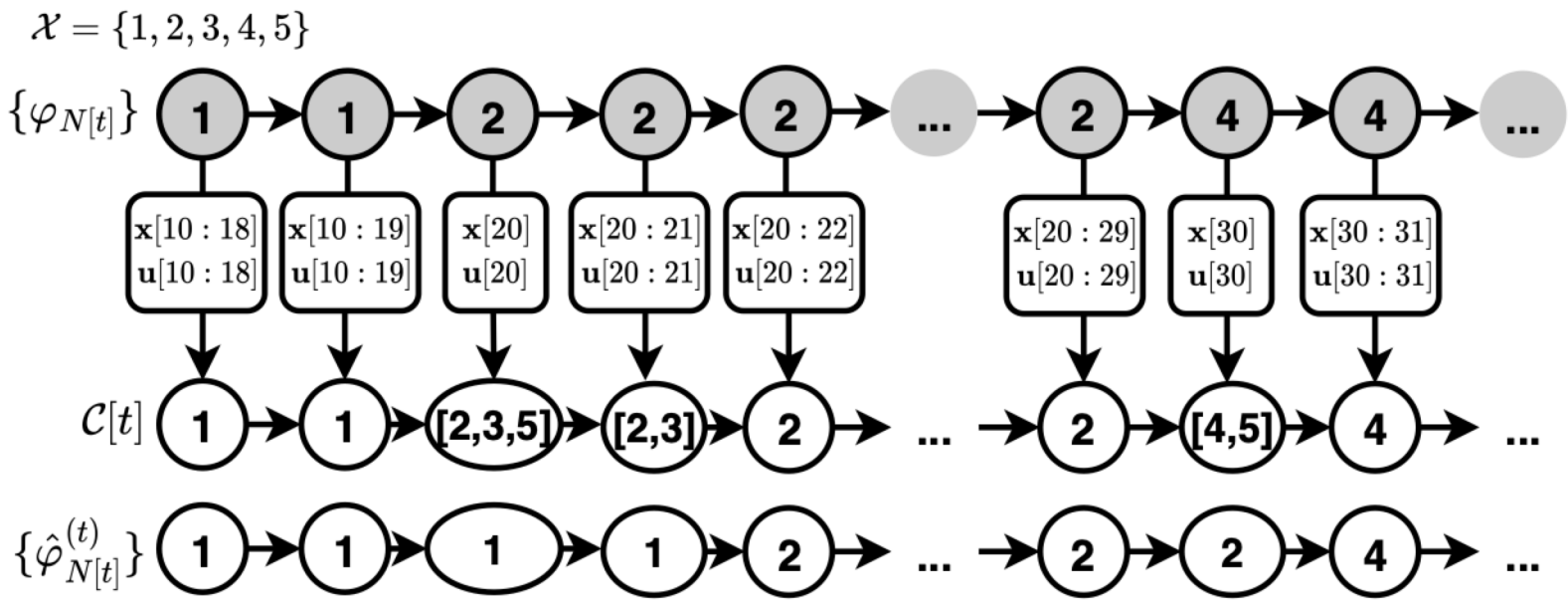
$\Delta T = 10$
$\mathrm{M} = 5$



Goal: Use observations $\mathbf{x}[T_{N[t]} : t], \mathbf{u}[T_{N[t]} : t]$ to estimate:

$\hat{P}^{(t)}$ of the true TPM P. (e.g., Metropolis-Hastings, Gibbs' sampling, or Baum-Welch)

$\hat{\varphi}^{(t)}_n$ of the true current mode φ_n. (e.g., Viterbi's algorithm)

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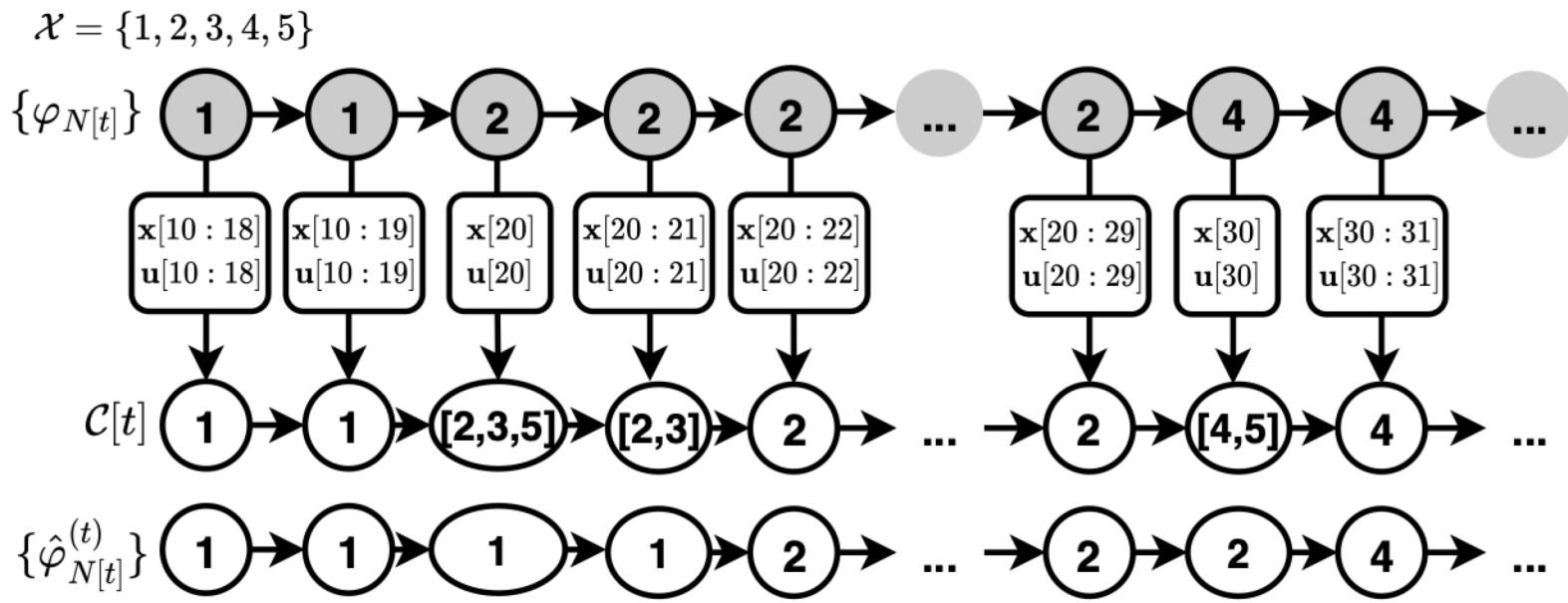
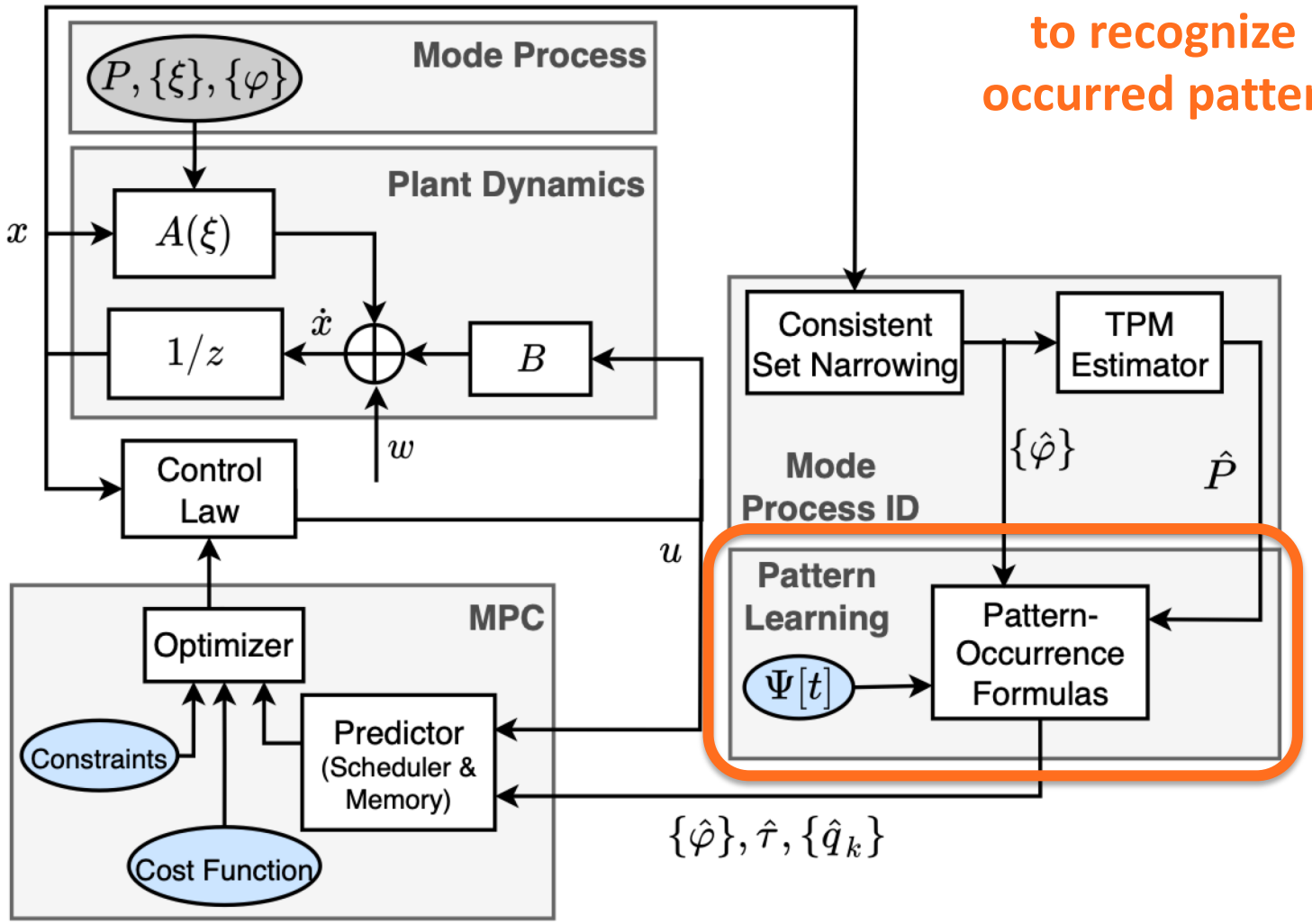


Consistent Set Narrowing:

$$\mathcal{C}[0] \triangleq \mathcal{X},\ \mathcal{C}[t] = \left\{ m \in \mathcal{C}[t-1] \mid \bigwedge_{r=T_n}^{t-1} \mathbb{1}\{\|\mathbf{x}[r+1] - A(m)\mathbf{x}[r] - B\mathbf{u}[r]\|_\infty \leq \overline{w}\} \right\}$$

$$\mathbf{x}[t+1] = A(\xi_{N[t]})\mathbf{x}[t] + B\mathbf{u}[t] + \mathbf{w}[t] \quad \|\mathbf{w}[t]\|_\infty \leq \overline{w}$$

Pattern-Learning in the Mode Process



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Definition 1 (Patterns). *Collection of patterns* $\Psi \triangleq \{\boldsymbol{\psi}_1, \cdots, \boldsymbol{\psi}_K\}$, where each $\boldsymbol{\psi}_k \triangleq (\psi_{k,1}, \cdots, \psi_{k,L})$ is a *(mode) pattern* with constant length $L \in \mathbb{N}$ and elements $\psi_{k,j} \in \mathcal{X}$.

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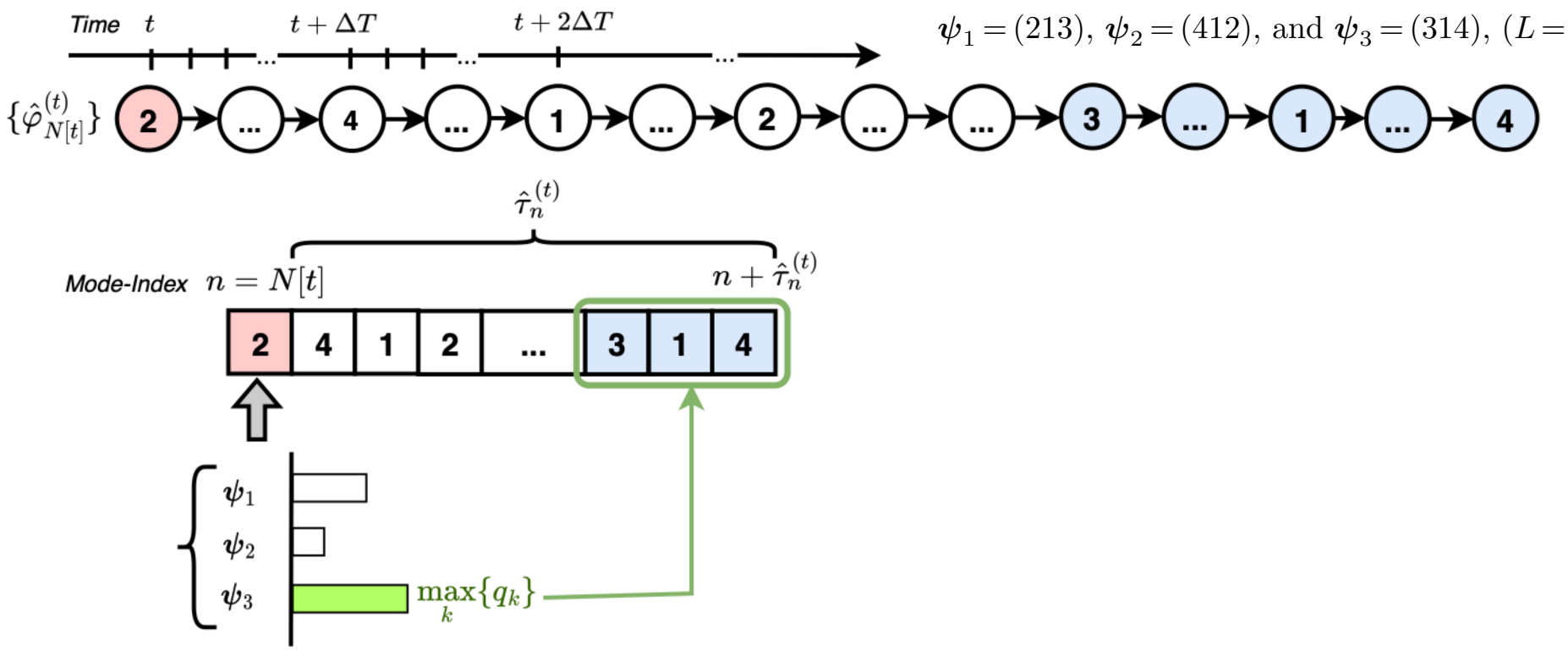

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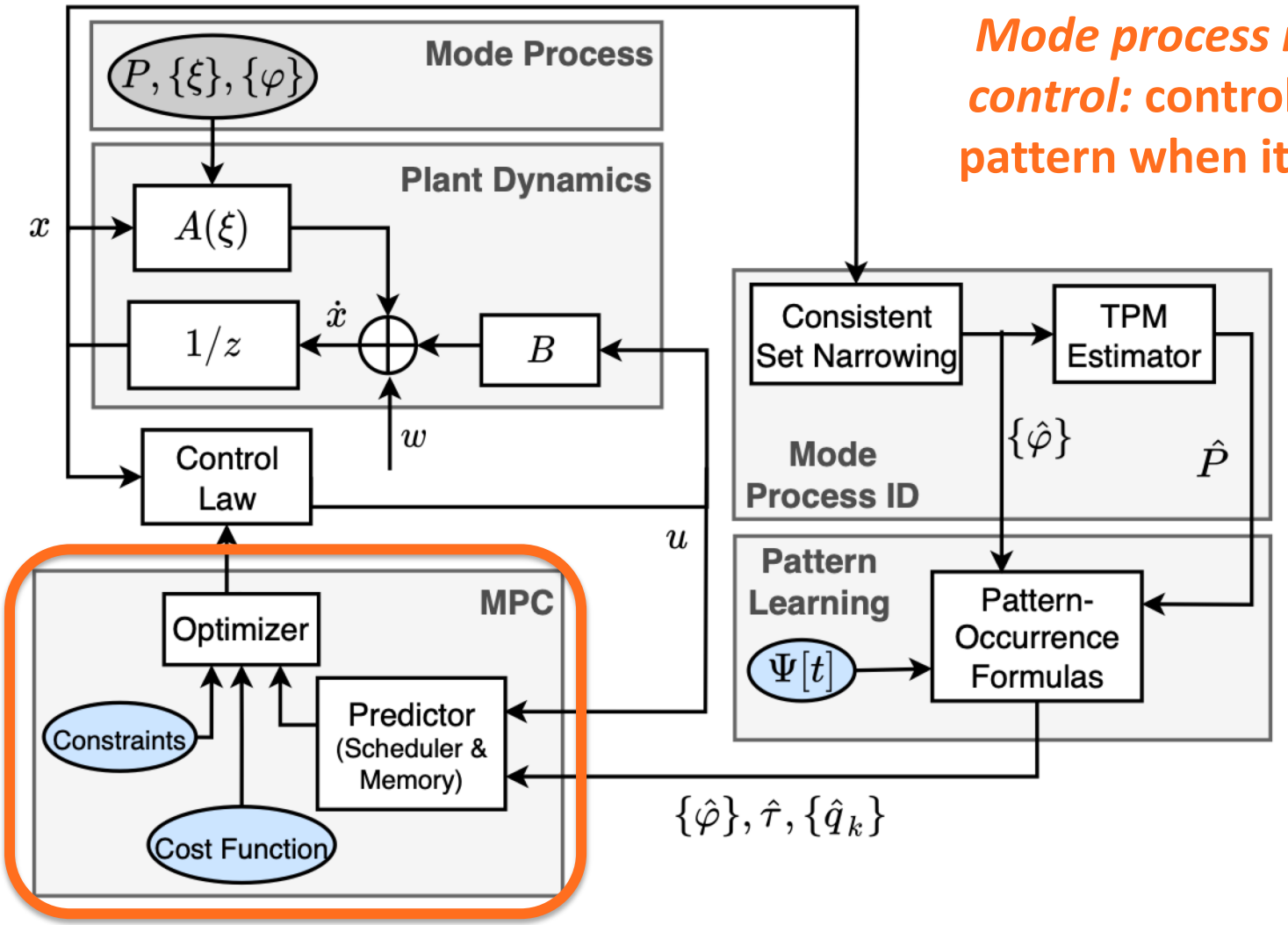
Mode Process Model Predictive Control (MPC)



Mode process model predictive control: **control law design for a pattern when it is first observed.**

(Implementation varies by application.)

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Mode Process Model Predictive Control (MPC)

Estimated current mode: $\hat{\varphi}_{N[t]}^{(t)} = m \in \mathcal{X}$. Constant-gain state-feedback control law $\mathbf{u}[t] = K(t,m)\mathbf{x}[t]$

E.g., obtained via minimizing standard LQR cost

$$J(t,m) = \sum_{s=t}^{T} \left(\mathbf{x}[s]^\top Q(m)\mathbf{x}[s] + \mathbf{u}[s]^\top R(m)\mathbf{u}[s]\right) + \mathbf{x}[T]^\top Q_f(m)\mathbf{x}[T],\ T \triangleq T_{N[t]+1} - 1$$

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Time-varying collection of patterns: choose set of feasible length-L future sequences of modes for some chosen *future horizon* $L \in \mathbb{N}$.

$$\Psi^{(t)} \subseteq \{\text{feasible } (\alpha_1, \cdots, \alpha_L) | \hat{P}^{(t)}[\hat{\varphi}^{(t)}_{N[t]}, \alpha_1] > 0, \alpha_i \in \mathcal{X}\}$$

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Let $k \triangleq \text{argmax}_{k'} \hat{q}^{(t)}_{k'}$ and mode-index $\tau \equiv \mathbb{E}[\hat{\tau}^{(t)}_n]$.

$$\mathbf{u}[s] = K(s, \psi_{k,1})\mathbf{x}[s], \quad s \in [t : T_{n+1} - 1]$$

$$\vdots$$

$$\mathbf{u}[s] = K(s, \psi_{k,L})\mathbf{x}[s], \quad s \in [T_{n+\tau} : T_{n+\tau+1} - 1]$$

Only apply control law for first predicted mode.

Then repeat at time T_{n+1}.

Mode Process Model Predictive Control (MPC)

$\Delta T = 10$
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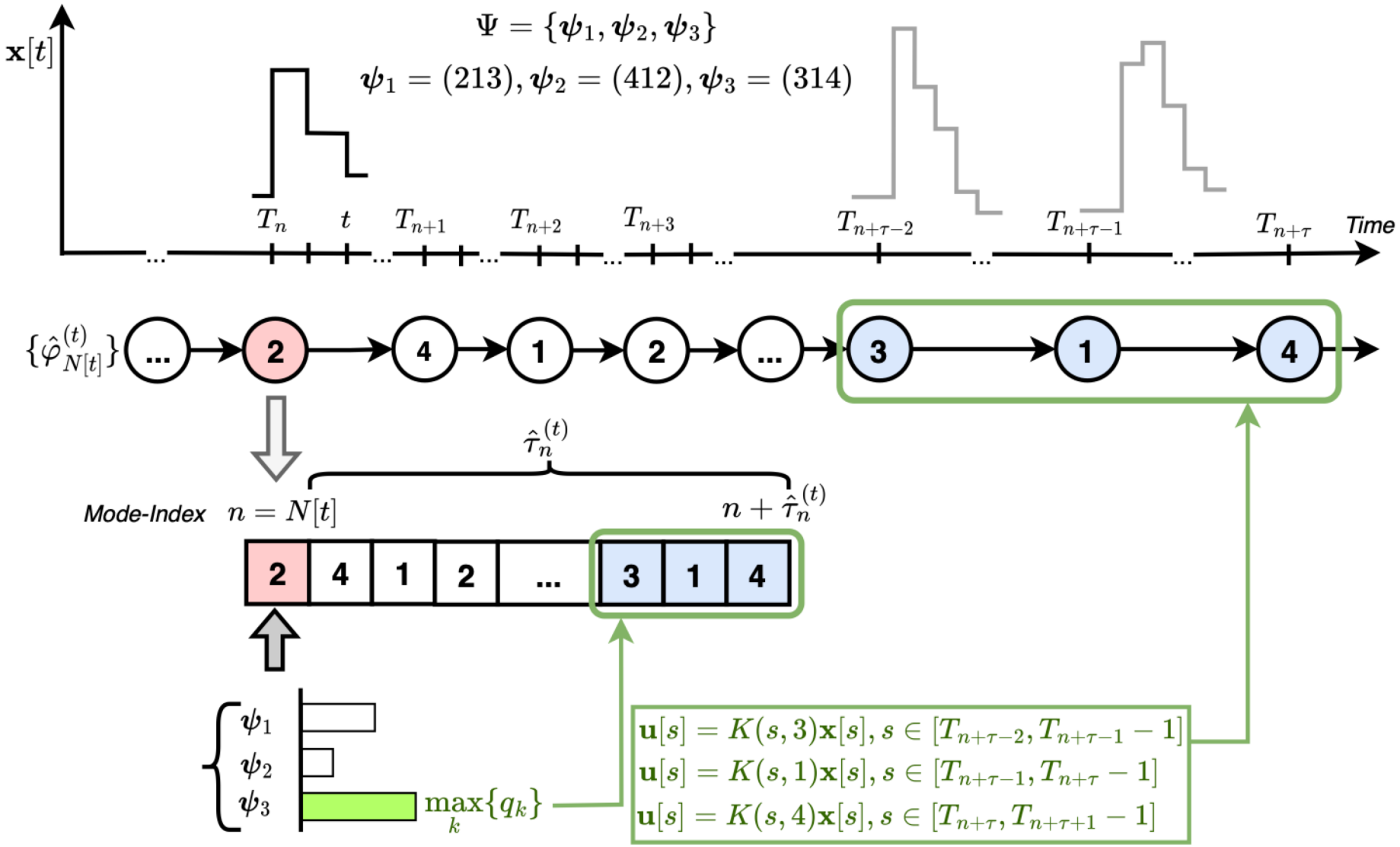

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Solving the Pattern-Occurrence Problem

Pattern-learning component **to recognize previously-occurred patterns of events.**

Problem (Pattern-Learning). Given Ψ, for each time $t \in \mathbb{N}$ and $n \triangleq N[t]$, derive estimates:

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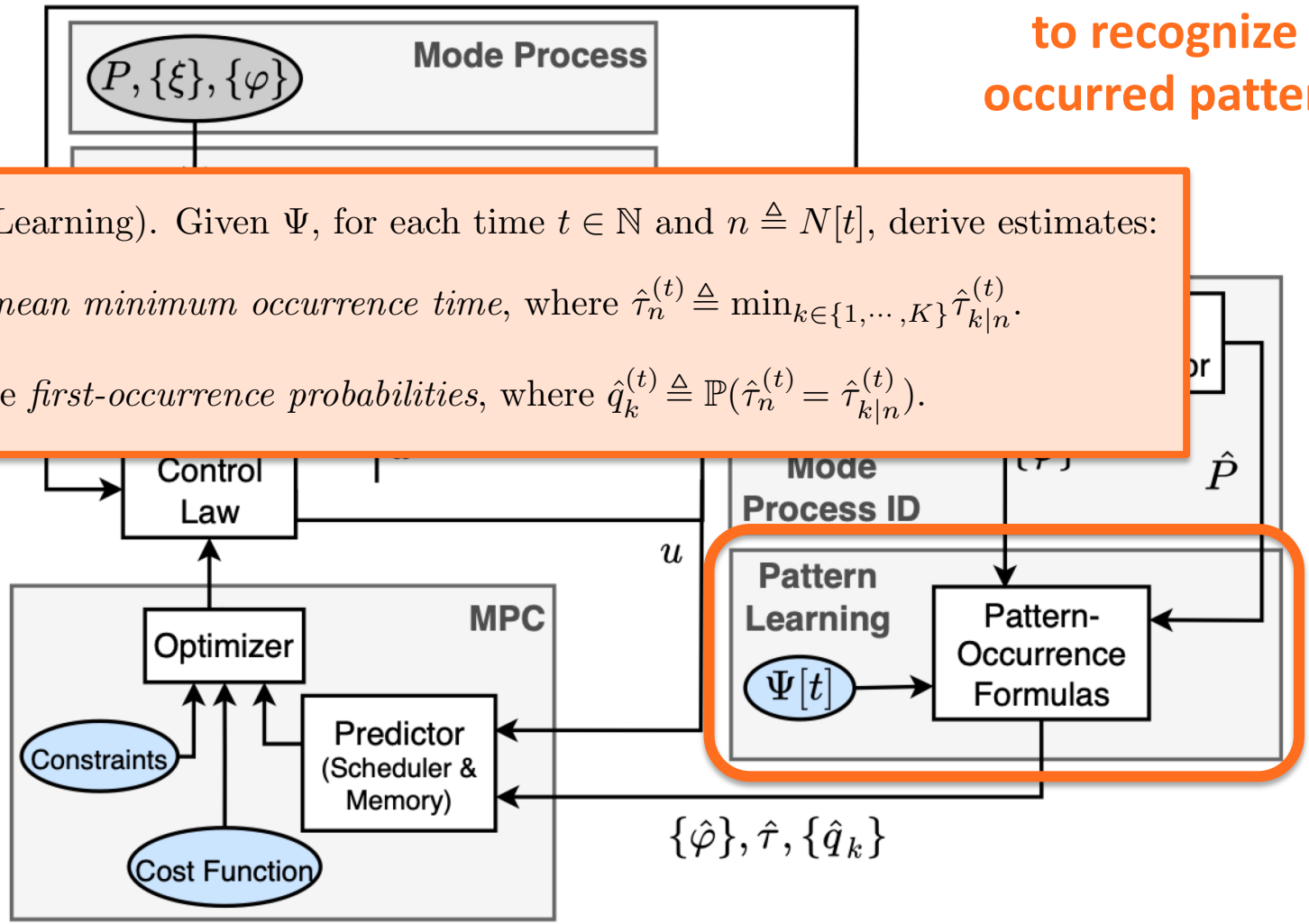

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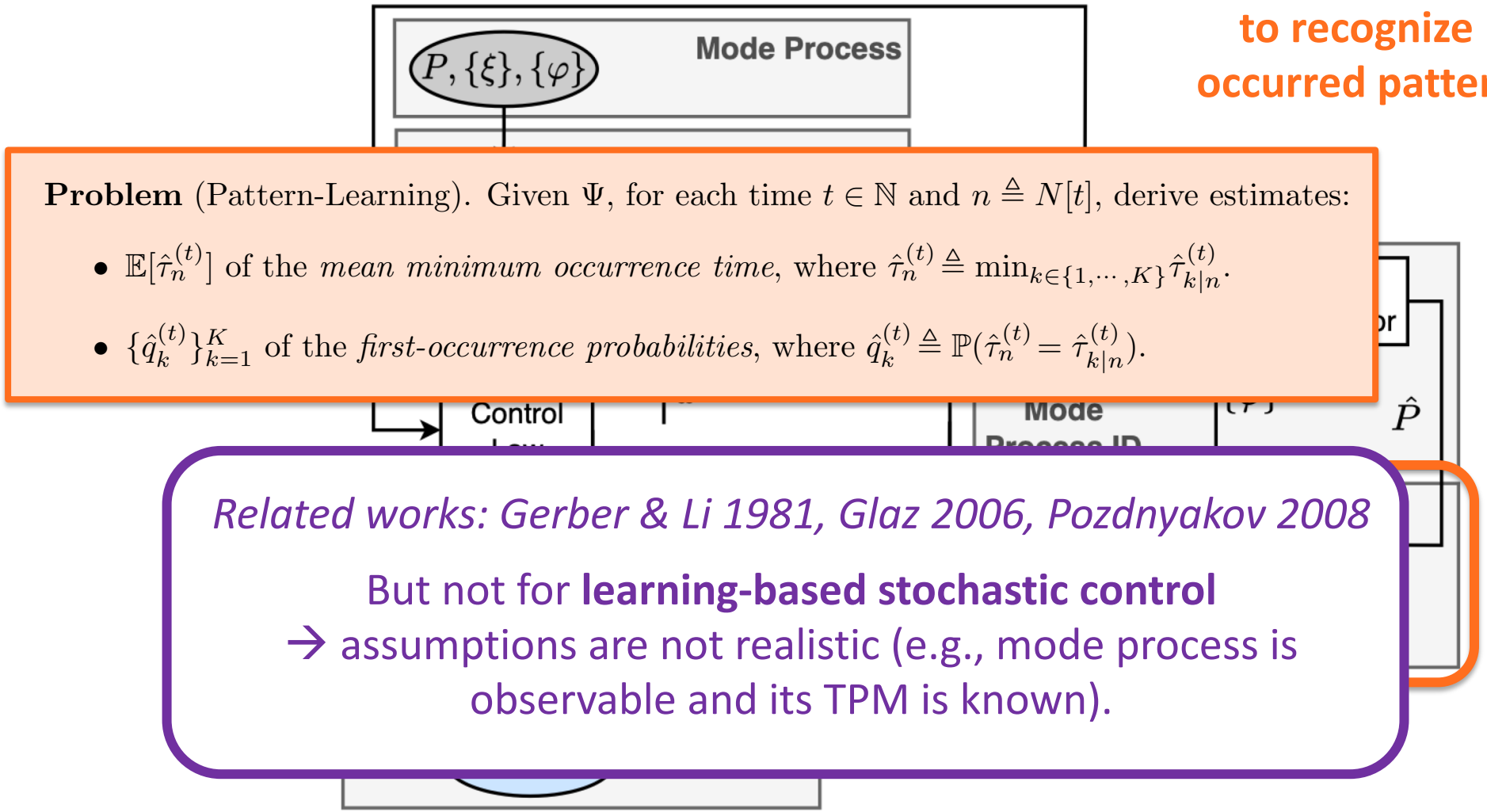

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→ Martingale Theory

Related works: Gerber & Li 1981, Glaz 2006, Pozdnyakov 2008

But not for **learning-based stochastic control**
→ assumptions are not realistic (e.g., mode process is observable and its TPM is known).

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Construction: Game Interpretation

Notation simplification: given $N[t]=n$, denote $\varphi_n \equiv \hat{\varphi}_n^{(t)}$, $P \equiv \hat{P}^{(t)}$, $\tau_{k|n} \equiv \hat{\tau}_{k|n}^{(t)}$, $\tau_n \equiv \hat{\tau}_n^{(t)}$, and $q_k \equiv \hat{q}_k^{(t)}$.

Definition 5 (Augmented Pattern)**.** In collection of patterns Ψ (see Definition 1), *augmented pattern* $\boldsymbol{\gamma}$ corresponding to pattern $\boldsymbol{\psi}_k \in \Psi$ is defined by prefixing two modes $m_1, m_2 \in \mathcal{X}$ such that the resulting sequence is feasible.

$$\text{Augmented collection } \Gamma \triangleq \{\text{feasible } (m_1, m_2) \circ \boldsymbol{\psi}_k \mid m_1, m_2 \in \mathcal{X}; \boldsymbol{\psi}_k \in \Psi\}, \qquad K_L \triangleq |\Gamma| \in \mathbb{N}$$

Example: Let $M = 4$ and $\mathcal{X} \triangleq \{1,2,3,4\}$.

TPM P is s.t. $P[m_1, m_2] = 0$ if $m_1 = m_2$ or $(m_1, m_2) \in \{(3,2),(2,3),(3,4),(4,3)\}$. $P = \begin{bmatrix} 0 & * & * & * \\ * & 0 & 0 & * \\ * & 0 & 0 & 0 \\ * & * & 0 & 0 \end{bmatrix}$

Collection $\Psi = \{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, \boldsymbol{\psi}_3\}$, $(K=3)$.

$\boldsymbol{\psi}_1 = (213)$, $\boldsymbol{\psi}_2 = (412)$, and $\boldsymbol{\psi}_3 = (314)$, $(L=3)$.

Augmented collection $\Gamma \triangleq \cup_{i=1}^{3} \Gamma_i$. $\Gamma_1 \triangleq \{\boldsymbol{\alpha} \circ \boldsymbol{\gamma}_1 | \boldsymbol{\alpha} \in \{(14),(21),(24),(31),(41)\}\}$

$\Gamma_2 \triangleq \{\boldsymbol{\alpha} \circ \boldsymbol{\gamma}_2 | \boldsymbol{\alpha} \in \{(12),(21),(31),(41),(42)\}\}$

$\Gamma_3 \triangleq \{\boldsymbol{\alpha} \circ \boldsymbol{\gamma}_3 | \boldsymbol{\alpha} \in \{(21),(31),(41)\}\}$ $\qquad (K_L = 13)$.

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Ending Strings

Collection $\Psi = \{\psi_1, \psi_2, \psi_3\}$, $(K = 3)$.
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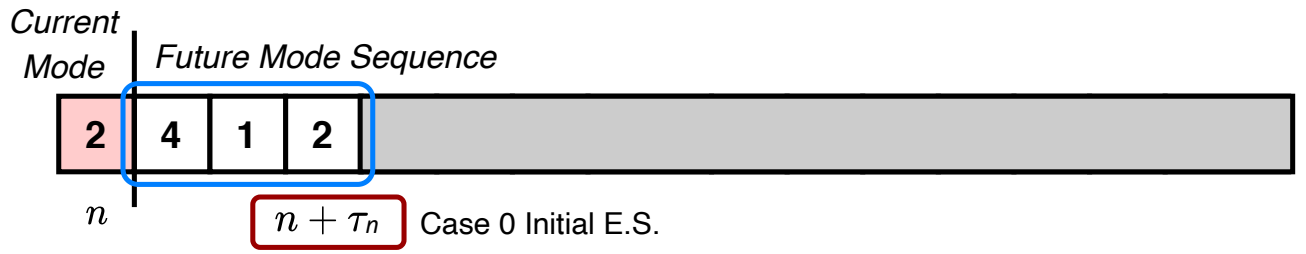
$\Gamma = \cup_{i=1}^{3} \Gamma_i = \{\gamma_1, \cdots, \gamma_8, \cdots, \gamma_{13}\}$
$\gamma_1 = (14) \circ \psi_1$, $\gamma_8 = (31) \circ \psi_2$, $\gamma_{13} = (41) \circ \psi_3$

Definition 6 (Ending String). Let the mode sequence $\{\xi_n, \xi_{n+1}, \cdots\}$ run until one pattern from Ψ is observed. An *ending string* associated with pattern $\psi_k \in \Psi$ terminates the mode process at mode-index $\tau_n > n$ if $\xi_{\tau_n - L + 1:\tau_n} = \psi_k$.

Case-0 initial-ending string $\boldsymbol{\beta}$: $\xi_{n+1:n+L} = \psi_k \Longrightarrow \boldsymbol{\beta} \triangleq \psi_k$. $(\mathcal{S}_I^{(0)} = \{\psi_2\}, K_I^{(0)} = 1, \tau_n = L)$.



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Case-1 *initial-ending string* $\boldsymbol{\beta}$:

$\xi_{n+1:n+L+1} = (m_1) \circ \psi_k \Longrightarrow \boldsymbol{\beta} \triangleq (m_1) \circ \psi_k$. $(\mathcal{S}_I^{(1)} = \{(1) \circ \psi_1, (1) \circ \psi_2, (1) \circ \psi_3, (2) \circ \psi_2\}, K_I^{(1)} = 4, \tau_n = L + 1)$.

Case 1 Initial E.S.

Ending Strings

Collection $\Psi = \{\psi_1, \psi_2, \psi_3\}$, $(K = 3)$.
$\psi_1 = (213)$, $\psi_2 = (412)$, and $\psi_3 = (314)$, $(L = 3)$.
Augmented collection:
$\Gamma = \cup_{i=1}^{3} \Gamma_i = \{\gamma_1, \cdots, \gamma_8, \cdots, \gamma_{13}\}$
$\gamma_1 = (14) \circ \psi_1$, $\gamma_8 = (31) \circ \psi_2$, $\gamma_{13} = (41) \circ \psi_3$

Definition 6 (Ending String). Let the mode sequence $\{\xi_n, \xi_{n+1}, \cdots\}$ run until one pattern from Ψ is observed. An *ending string* associated with pattern $\psi_k \in \Psi$ terminates the mode process at mode-index $\tau_n > n$ if $\xi_{\tau_n - L + 1:\tau_n} = \psi_k$.

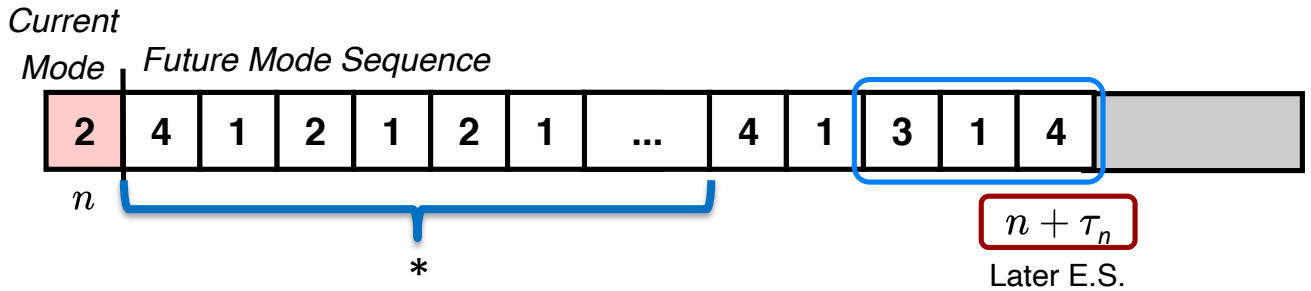
Case-0 initial-ending string $\boldsymbol{\beta}$: $\xi_{n+1:n+L} = \psi_k \Longrightarrow \boldsymbol{\beta} \triangleq \psi_k$. $(\mathcal{S}_I^{(0)} = \{\psi_2\}, K_I^{(0)} = 1, \tau_n = L)$.

Case-1 initial-ending string $\boldsymbol{\beta}$:

$\xi_{n+1:n+L+1} = (m_1) \circ \psi_k \Longrightarrow \boldsymbol{\beta} \triangleq (m_1) \circ \psi_k$. $(\mathcal{S}_I^{(1)} = \{(1) \circ \psi_1, (1) \circ \psi_2, (1) \circ \psi_3, (2) \circ \psi_2\}, K_I^{(1)} = 4, \tau_n = L + 1)$.

Later-ending string $\boldsymbol{\beta}$:

$\xi_{\tau_n - L + 1:\tau_n} = (*, m_1, m_2) \circ \psi_k \Longrightarrow \boldsymbol{\beta} \triangleq (m_1, m_2) \circ \psi_k$. $(\mathcal{S}_L = \{(*) \circ \gamma_\ell : \gamma_\ell \in \Gamma\}, K_L, \tau_n > n + L + 1)$.



$*$ is a placeholder for any feasible sequence of modes.

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Construction: Game Interpretation

$$\boldsymbol{\gamma}_\ell \triangleq (m_1, m_2) \circ \boldsymbol{\psi}_k$$

A *type-ℓ agent* accumulates rewards over mode process $\{\xi_n\}$ with the goal of observing augmented pattern $\boldsymbol{\gamma}_\ell \in \Gamma$.

1. If $\varphi_n = m_1$, type-ℓ agent n aims to observe the event $\{\xi_{n+1:n+L+1} = (m_2) \circ \boldsymbol{\psi}_k\}$.
2. Otherwise, if $\varphi_n \neq m_1$, type-ℓ agent n aims to observe the event $\{\xi_{n+1:n+L} = \boldsymbol{\psi}_k\}$.

For each $\ell \in \{1, \cdots, K_L\}$, a new *type-$\ell$ agent* n enters the game at mode-index $n \in \mathbb{N}$ with *initial reward* $c_\ell \in \mathbb{R}$.

Type-ℓ agent n total reward by mode-index $\overline{n} > n$:

$$r_{n,\overline{n}}^{(\ell)} = \mathbb{1}\{\varphi_n = m_2\} \sum_{j=1}^{L+2} B_{n,j}^{(1,\ell)} \mathbb{1}\{n + j \leq \overline{n}\} + \mathbb{1}\{\varphi_n \neq m_2\} \sum_{j=1}^{L+1} B_{n,j}^{(2,\ell)} \mathbb{1}\{n + j \leq \overline{n}\}$$

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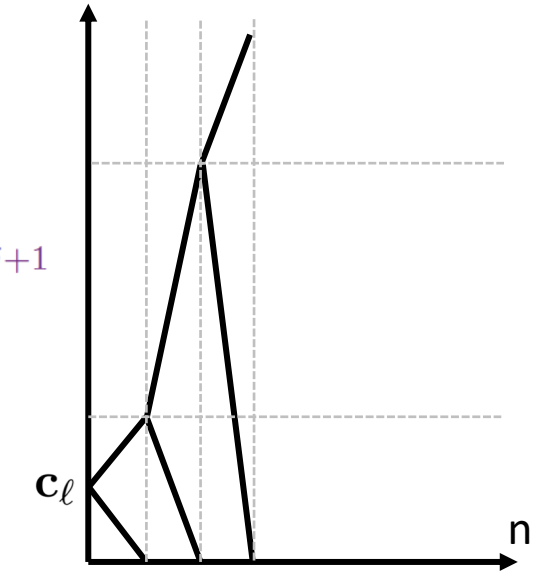
Betting strategies are defined such that the game is fair-odds.

$$B_{n,0}^{(1,\ell)} = B_{n,0}^{(2,\ell)} = c_\ell$$

$$B_{n,j}^{(1,\ell)} = \begin{cases} \left(P(\gamma_{\ell,j}, \gamma_{\ell,j+1})\right)^{-1} B_{n,j-1}^{(1,\ell)} - B_{n,j-1}^{(1,\ell)} & \text{if } \xi_{n+j-1} = \gamma_{\ell,j}, \xi_{n+j} = \gamma_{\ell,j+1} \\ 0 & \text{if } \sum_{jj=0}^{j-1} B_{n,jj}^{(1,\ell)} = 0 \\ -B_{n,j-1}^{(1,\ell)} & \text{else} \end{cases}$$

$B_{n,j}^{(2,\ell)}$ similarly.



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$$B^{(1,\ell)}_{n,0} = B^{(2,\ell)}_{n,0} = c_\ell$$

$$B^{(1,\ell)}_{n,j} = \begin{cases} (P(\gamma_{\ell,j}, \gamma_{\ell,j+1}))^{-1} B^{(1,\ell)}_{n,j-1} - B^{(1,\ell)}_{n,j-1} & \text{if } \xi_{n+j-1} = \gamma_{\ell,j}, \xi_{n+j} = \gamma_{\ell,j+1} \\ 0 & \text{if } \sum_{jj=0}^{j-1} B^{(1,\ell)}_{n,jj} = 0 \\ -B^{(1,\ell)}_{n,j-1} & \text{else} \end{cases}$$

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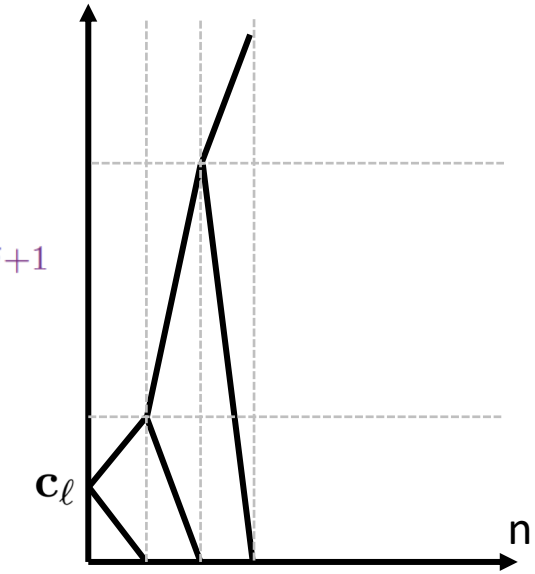
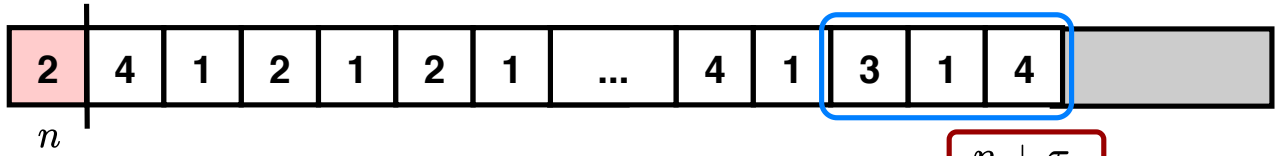


$$r^{(\ell)}_{n,\overline{n}} = \mathbb{1}\{\varphi_n = m_2\} \sum_{j=1}^{L+2} B^{(1,\ell)}_{n,j} \mathbb{1}\{n + j \le \overline{n}\} + \mathbb{1}\{\varphi_n \neq m_2\} \sum_{j=1}^{L+1} B^{(2,\ell)}_{n,j} \mathbb{1}\{n + j \le \overline{n}\}$$

$$\psi_3 = (314),\ \gamma_{13} = (41) \circ \psi_3$$

Type - 13 Agents



(Agent 1) 4 1 3 1 4

(Agent 2) 4 1 3 1 4

...

(Agent $\tau_n - 5$) 4 1 3 1 4

$$\frac{1}{P[4,1]P[1,3]P[3,1]P[1,4]} c_\ell - c_\ell$$

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First Quantity: Expected Minimum Time

$$\mathbb{E}[\tau] = \frac{1}{\sum_{\ell=1}^{K_L} c_\ell^*} \left[\left(1 - \sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s) \right) + \sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s) \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell^* \right]$$

$$\mathbb{P}(\boldsymbol{\beta}_s) = P[\varphi_0, \beta_1] \prod_{j=1}^{b_s - 1} P[\beta_j, \beta_{j+1}]$$

Stationarity of $\{\xi_n\} \Longrightarrow \tau \equiv \tau_n \ \forall\, n \in \mathbb{N}$.

$\mathbf{c}^* \in \mathbb{R}^{K_L}$ initial rewards chosen s.t.
$\sum_{\ell=1}^{K_L} W_{s\ell} c_\ell^* = 1 \ \ \forall\, s \in \{K_I + 1, \cdots, K_I + K_L\}$.

$W_{s\ell}$: *total gain* accumulated over all type-ℓ agents from observing (partial) occurrences of $\boldsymbol{\gamma}_\ell$ in ending string $\boldsymbol{\beta}_s$.

$\mathbb{E}[R_\tau^{(\ell)}]$: *expected type-$\ell$ cumulative net reward* over all type-ℓ agents by mode-index τ. $R_\tau^{(\ell)} \triangleq \sum_{n=1}^{\tau} r_{n,\tau}^{(\ell)}$

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$\mathbb{E}[R_{\overline{n}}]$: *expected cumulative net reward* over all agents by mode-index $\overline{n}$. $R_{\overline{n}} \triangleq \sum_{\ell=1}^{K_L} R_{\overline{n}}^{(\ell)}$

Proof: Apply Optional Stopping Theorem to martingale construction $\{R_{n\wedge\tau}^{(\ell)}\}_{n\in\mathbb{N}}$ and $\{R_{n\wedge\tau}\}_{n\in\mathbb{N}}$

Finite stopping time: $\mathbb{E}[\tau] < \infty$

Finite-valued martingale: $\mathbb{E}[|R_\tau^{(\ell)}|] < \infty$ and $\mathbb{E}[|R_\tau|] < \infty$

Uniform integrability: $\lim_{n\to\infty}\int_{\Omega_n} |R_n^{(\ell)}(\omega)|d\mathbb{P}(\omega) = 0 \quad \Omega_n \triangleq \{\omega \in \Omega | n < \tau\}$

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$$0 = \mathbb{E}[R_0] = \mathbb{E}[R_\tau] = \sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s) \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell^* + \left(1 - \sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s)\right) - \sum_{\ell=1}^{K_L} c_\ell^* \mathbb{E}[\tau]$$

Second Quantity: First-Occurrence Probabilities

$$q_k = \sum_{\boldsymbol{\beta}_s \in \mathcal{S}} \mathbb{P}(\boldsymbol{\beta}_s) \mathbb{1}\{\beta_{b_s - L+1:b_s} = \boldsymbol{\psi}_k\}$$

Proof: Use previous expression for $\mathbb{E}[\tau]$.

Rearrange expression for expected time.

$$\sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s) \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell = -\sum_{s=K_I+1}^{K_I+K_L} \mathbb{P}(\boldsymbol{\beta}_s) \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell + \sum_{\ell=1}^{K_L} c_\ell \mathbb{E}[\tau]$$

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Proof: Use previous expression for $\mathbb{E}[\tau]$.

Rearrange expression for expected time.

$$\underbrace{\sum_{s=1}^{K_I} \mathbb{P}(\boldsymbol{\beta}_s)}_{\text{known}} \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell = \underbrace{-\sum_{s=K_I+1}^{K_I+K_L} \mathbb{P}(\boldsymbol{\beta}_s)}_{\text{unknown}} \sum_{\ell=1}^{K_L} W_{s\ell} c_\ell + \sum_{\ell=1}^{K_L} c_\ell \mathbb{E}[\tau]$$

Solve for K_L unknowns by creating equations by choosing $\mathbf{c} \in \{[1, 0, \cdots, 0], \cdots, [0, 0, \cdots, 1]\}$

Use conditional probabilities: $q_k \triangleq \mathbb{P}(\tau = \tau_k) = \sum_{\boldsymbol{\beta}_s \in \mathcal{S}} \mathbb{P}(\boldsymbol{\beta}_s) \mathbb{P}(\tau = \tau_k \mid \boldsymbol{\beta}_s)$

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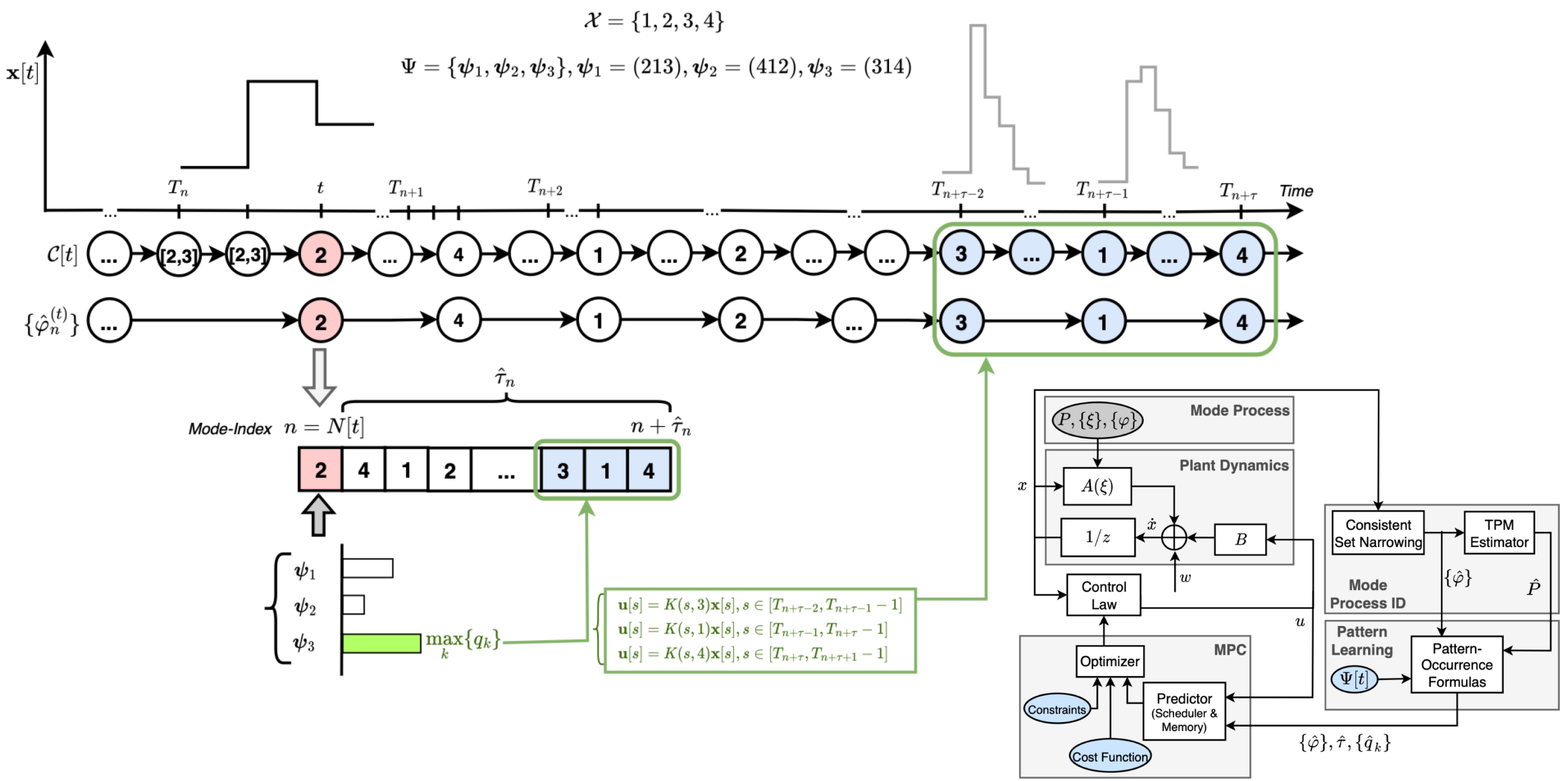
Proposed Controller Framework



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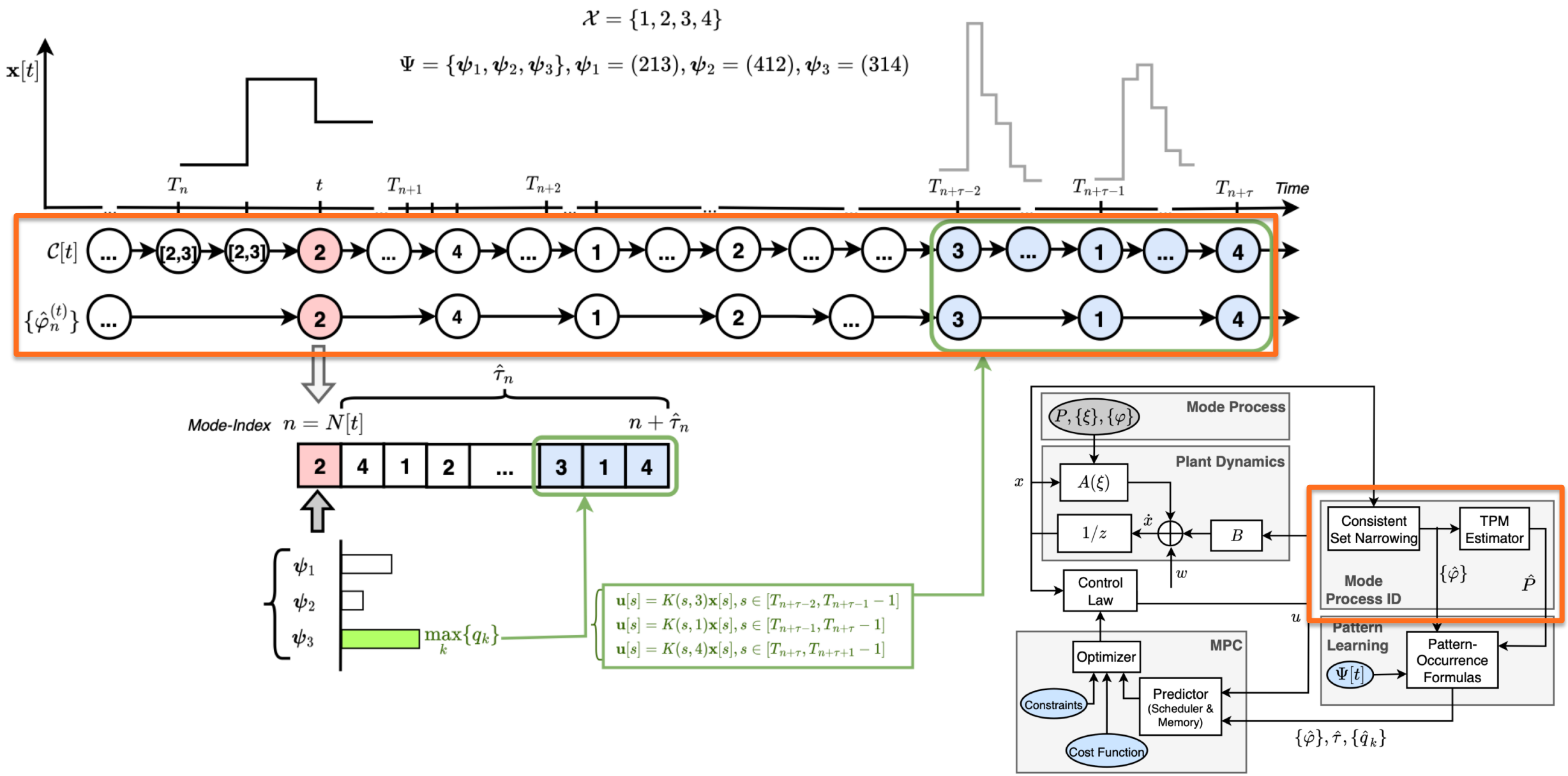

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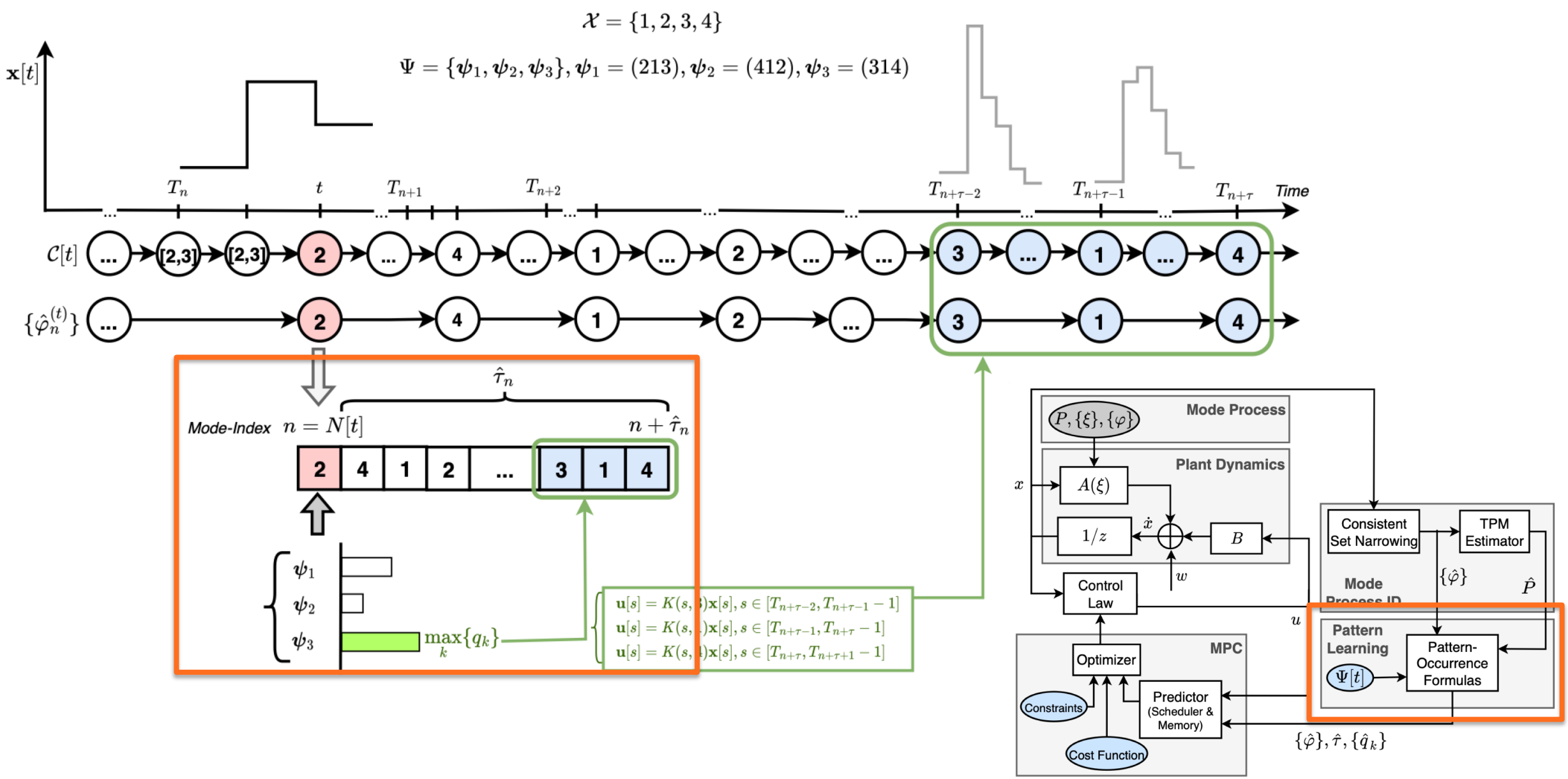

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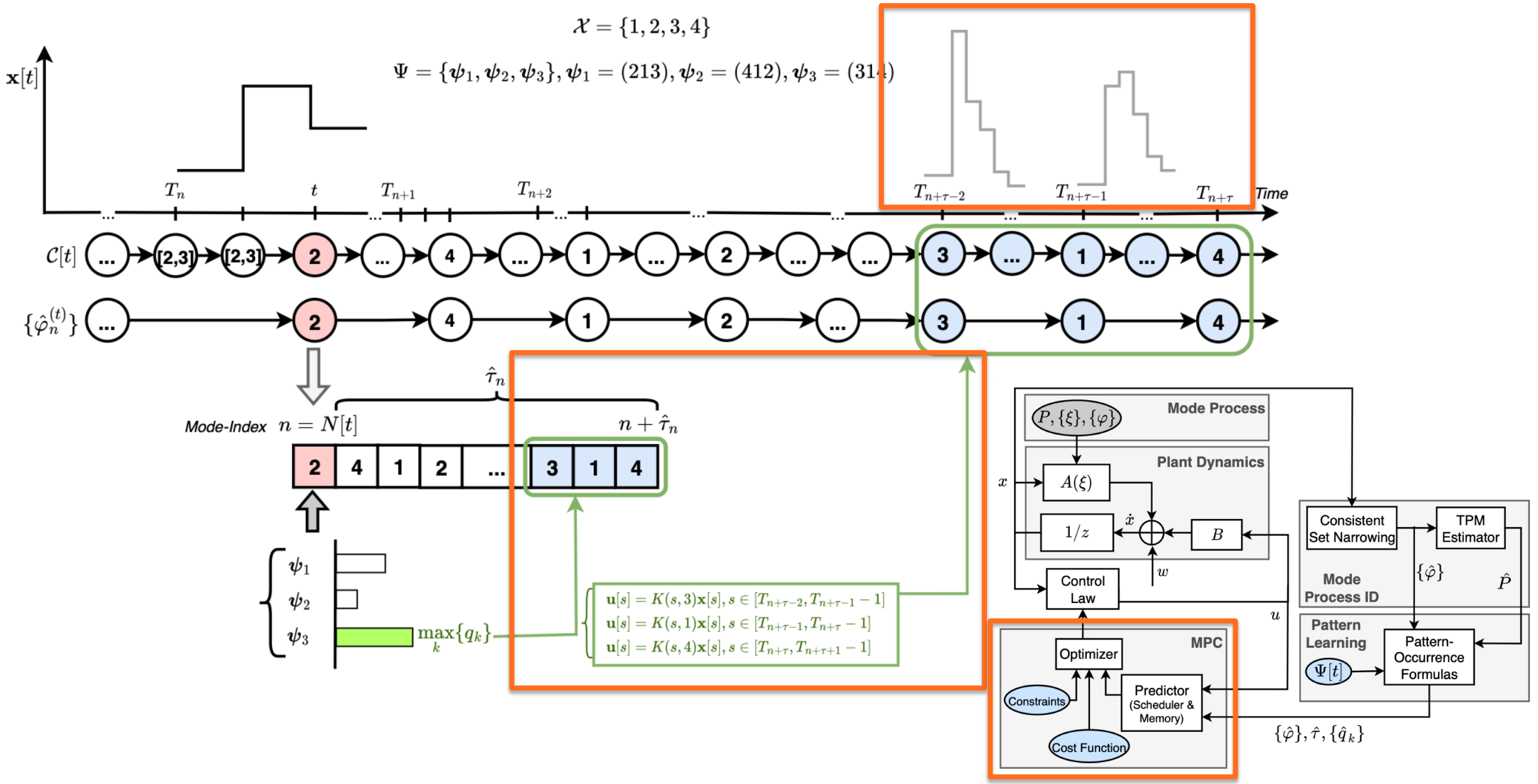

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Systems with Repeating Jump Patterns

Control for *jump stochastic systems* can be achieved by learning **patterns** in the system behavior.

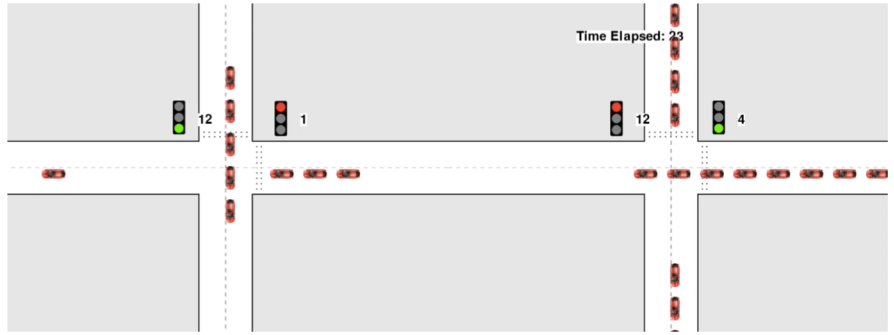
At least 2 ways of improving performance:

- **memorize past patterns**
  → **no re-computation of repeating control policies**
- **predict future patterns**
  → **schedule control policies in advance**

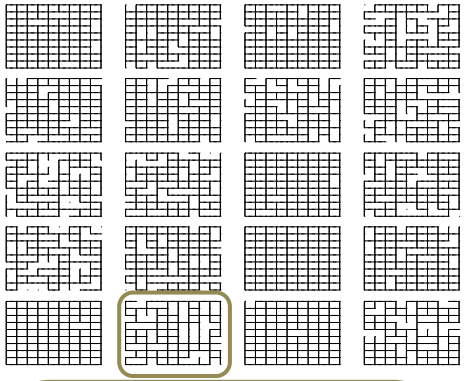
Systems that have jump patterns:

- Networks with time-varying topology (e.g., power grid)
  - **patterns** = power outages, line failures
- Vehicle traffic congestion
  - **patterns** = intersection snapshots
- Epidemic spread
  - **patterns** = contact-tracing data, travel routines

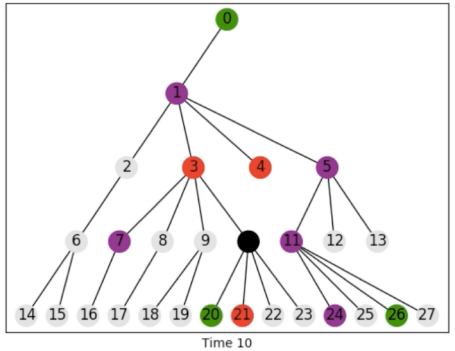
Vehicle Traffic Congestion

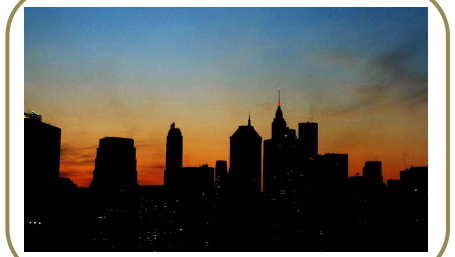


Power Grid



Epidemic Spread





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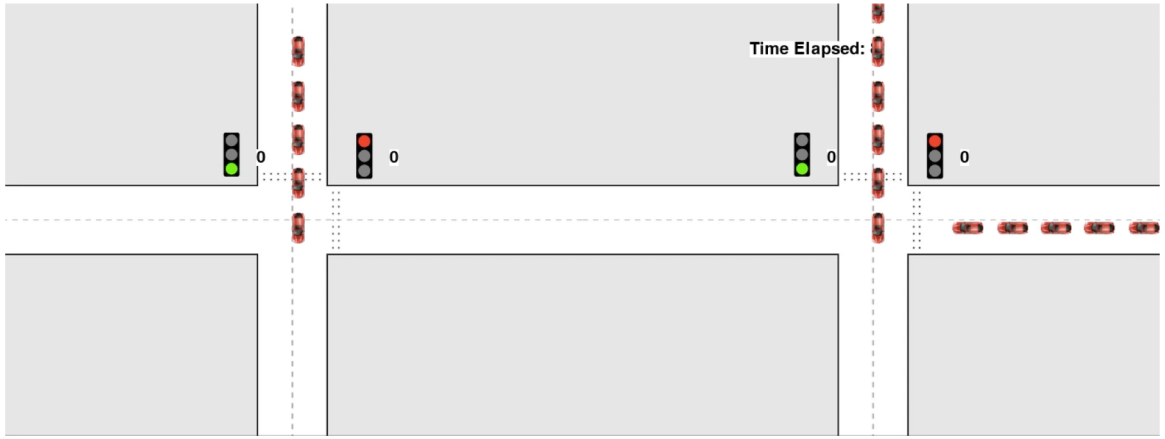
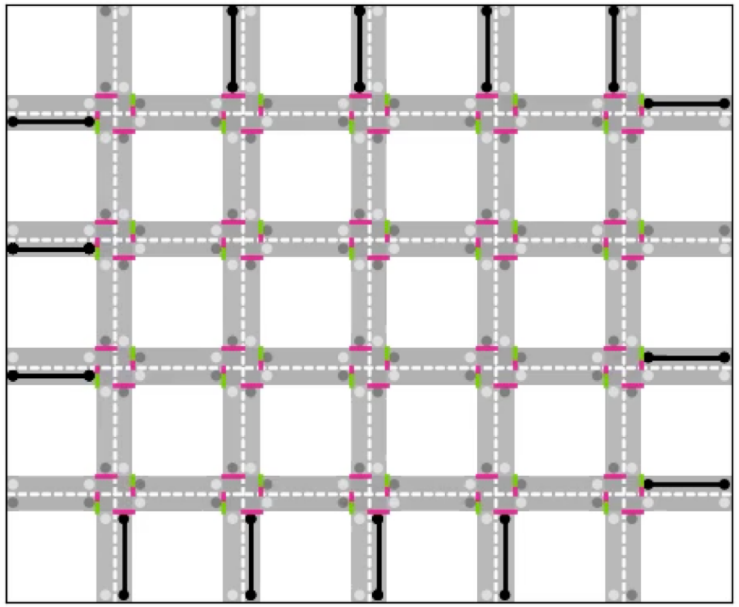
Systems with Repeating Jump Patterns

Vehicle traffic networks:

- *Pattern-Learning*: identify recurrent snapshots of the intersection to recycle light signal control actions.
- *Mode Process Controller*: rule-based table / optimization to minimize level of congestion as quickly as possible.





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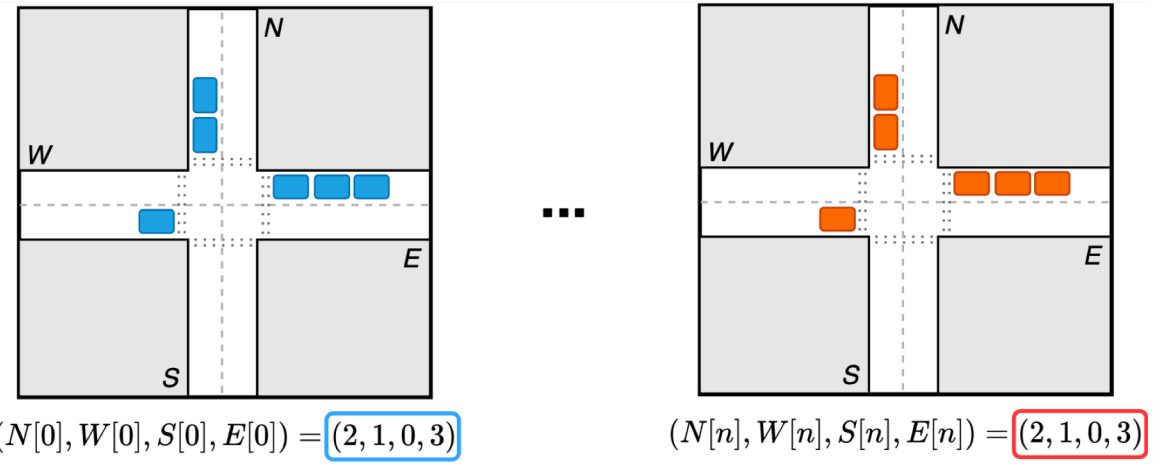

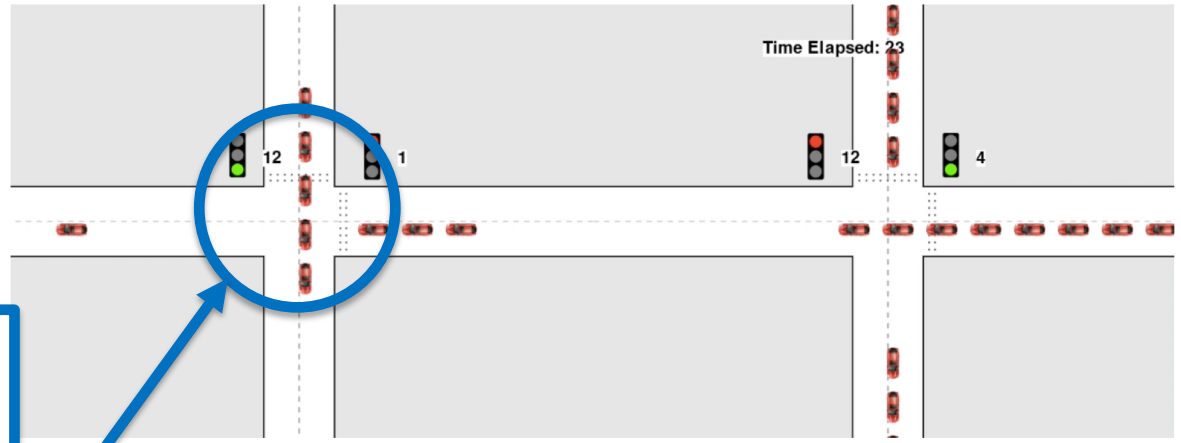

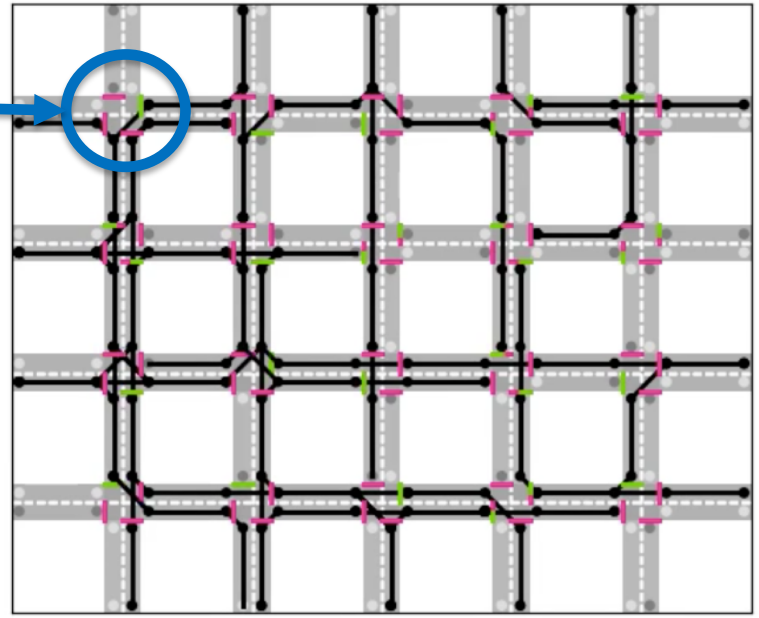

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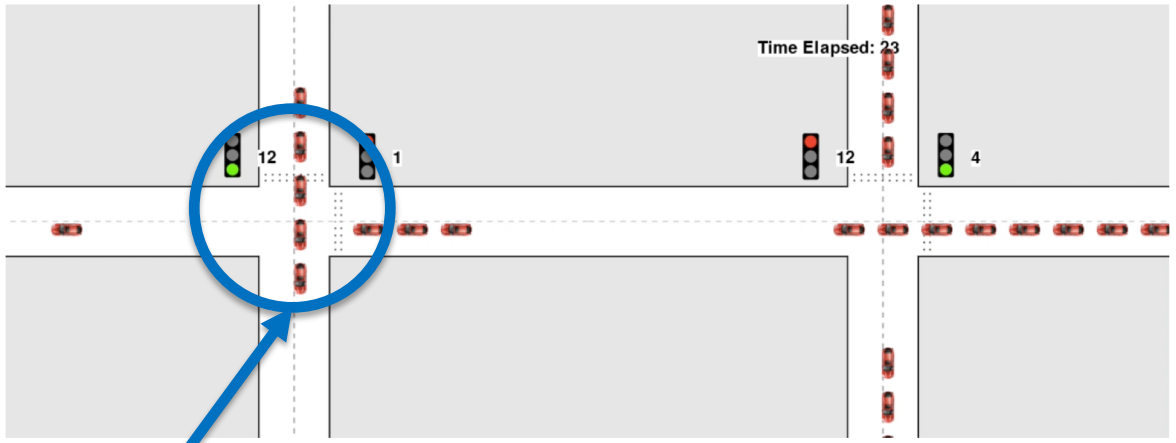
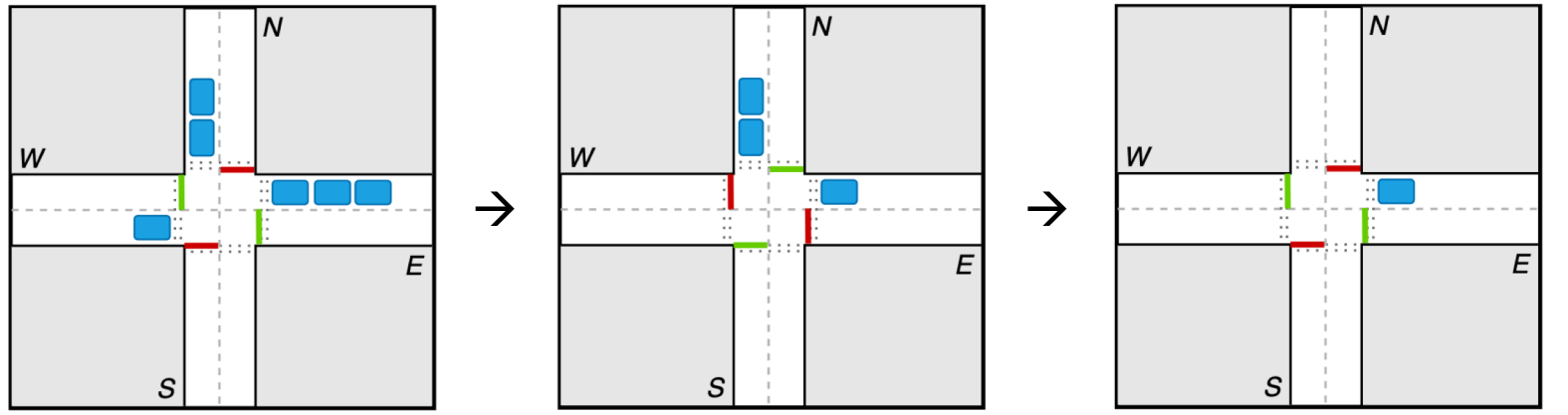
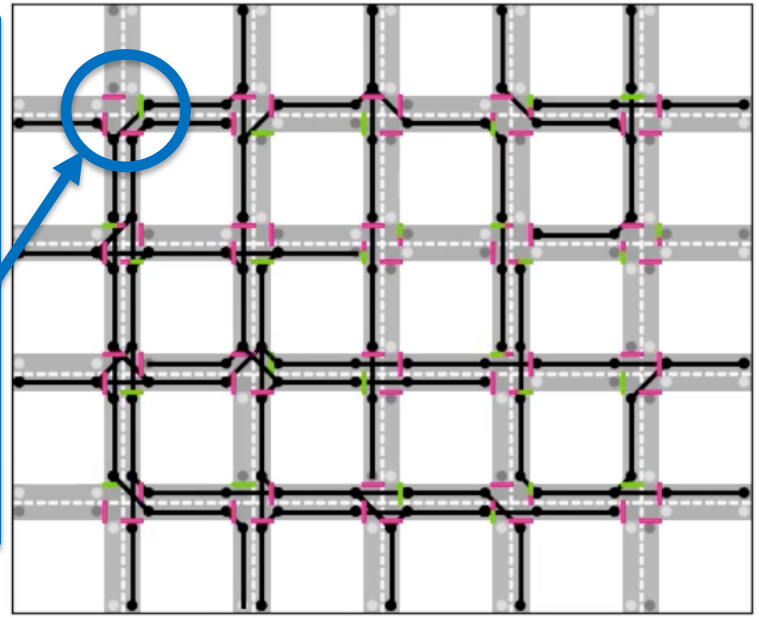






- Epidemic spread
  - patterns = contact-tracing data, travel routines

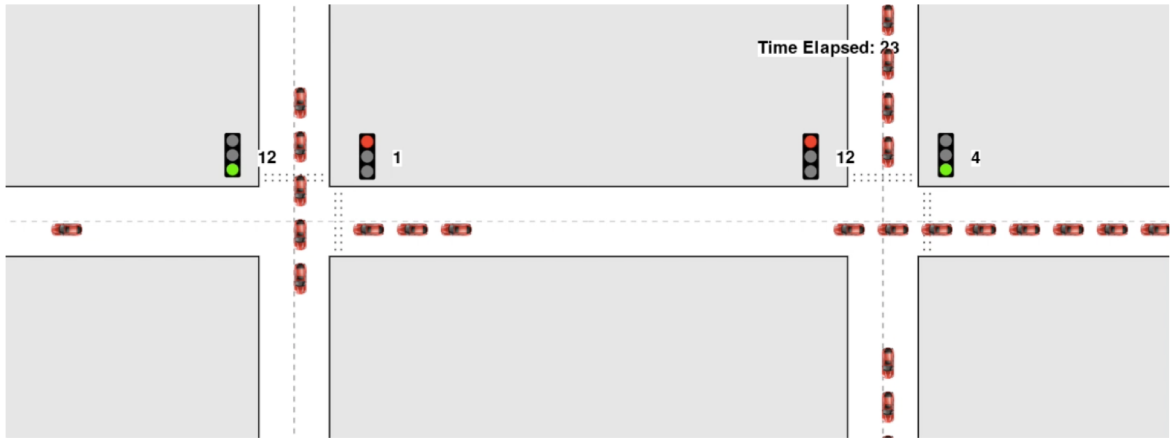
* SooJean Han, Johanna Gustafson, and Soon-Jo Chung. *Rule-Based Adaptive Control of an Isolated Intersection*, L4DC 2023, to be submitted.
* SooJean Han and Soon-Jo Chung. *Quantum Annealing for Iterative Traffic Flow Optimization over Vehicle Routes and Light Signals*, L4DC 2023, to be submitted.

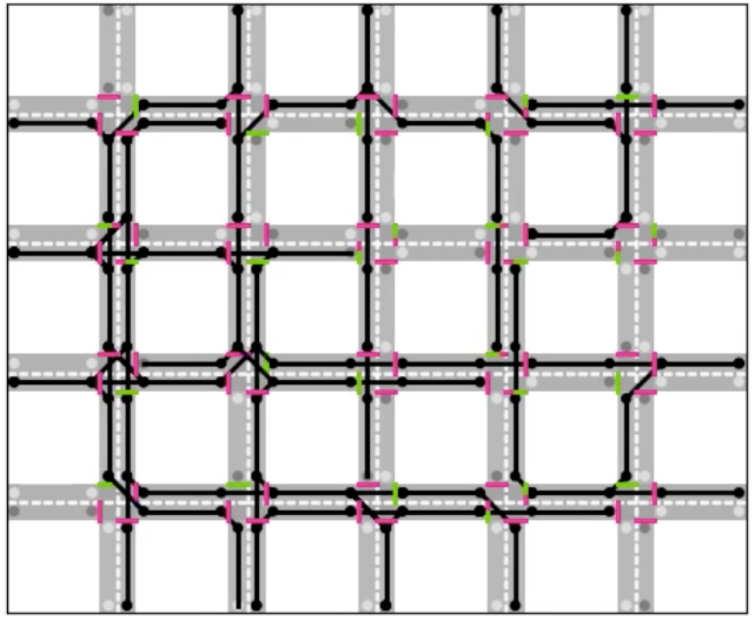
Systems with Repeating Jump Patterns

Vehicle traffic networks:

In urban intersection networks, structural patterns can also be exploited

- *Space* → rectangular grid, 4-way directions, constant distance between adjacent intersections.
- *Time* → traffic density evolution repeats due to human routine (e.g., on weekdays, traffic is heavy due to morning and evening commutes).





Systems that have jump patterns:

- Networks with time-varying topology (e.g., power grid)
  - **patterns** = power outages, line failures
- **Vehicle traffic congestion**
  - **patterns = intersection snapshots**
- Epidemic spread
  - **patterns** = contact-tracing data, travel routines

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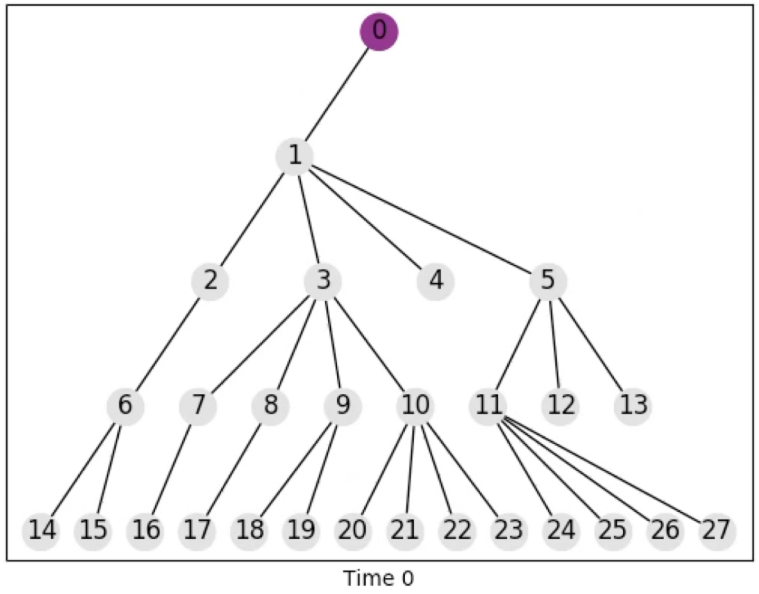
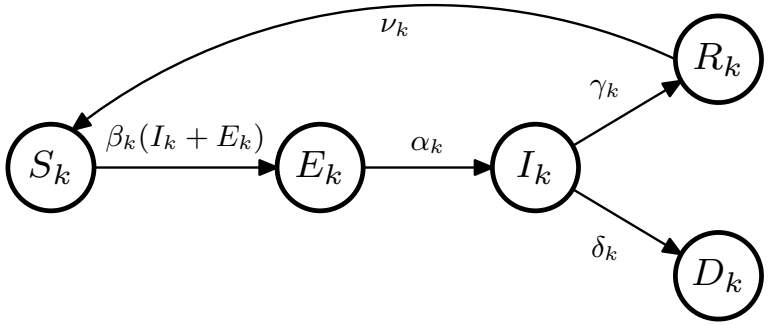
Systems with Repeating Jump Patterns

Epidemic spread:





$$\frac{dS_k(t)}{dt} = -\sum_{j=1}^{K} \frac{\beta_{kj}(I_j(t) + E_j(t))}{M_j(t)} S_k(t) + \nu_k R_k(t)$$

$$\frac{dE_k(t)}{dt} = \sum_{j=1}^{K} \frac{\beta_{kj}(I_j(t) + E_j(t))}{M_j(t)} S_k(t) - \alpha_k E_k(t)$$

$$\frac{dI_k(t)}{dt} = \alpha_k E_k(t) - (\gamma_k + \delta_k) I_k(t)$$

$$\frac{dR_k(t)}{dt} = \gamma_k I_k(t) - \nu_k R_k(t)$$

$$\frac{dD(t)}{dt} = \sum_{k=1}^{K} \delta_k I_k(t) \qquad M = \sum_{k=1}^{K} M_k(t),\ \forall t$$

Systems that have jump patterns:

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- Vehicle traffic congestion
  - **patterns** = intersection snapshots
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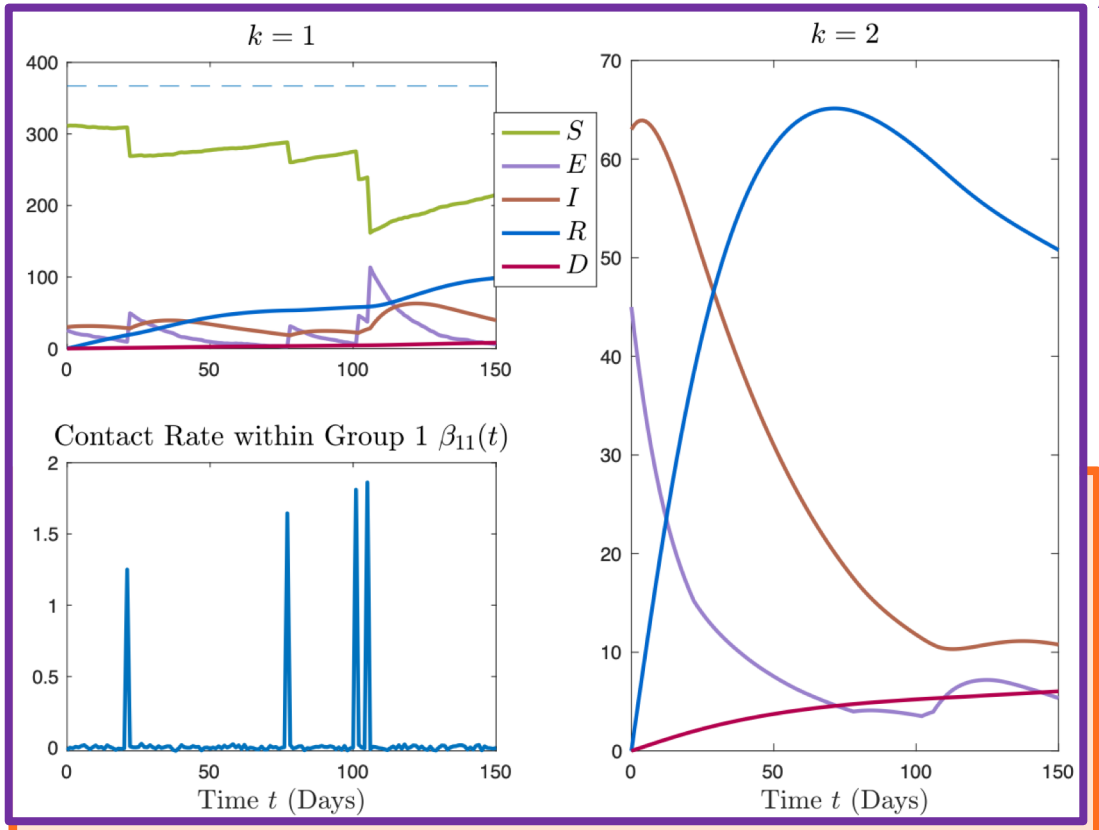
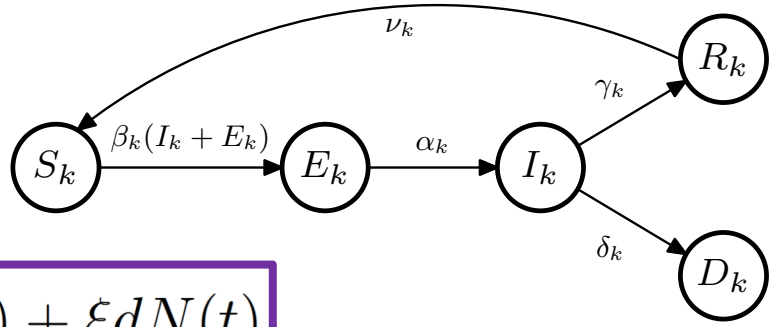
Systems with Repeating Jump Patterns

Epidemic spread:

Super-spreader phenomena: average rate of contact per day is stochastic with Poisson shot noise.



- **Epidemic spread**
  - **patterns = contact-tracing data, travel routines**



$$\beta dt = \beta_0 dt + \sigma dW(t) + \xi dN(t)$$

$$\frac{dS_k(t)}{dt} = -\sum_{j=1}^{K} \frac{\beta_{kj}(I_j(t) + E_j(t))}{M_j(t)} S_k(t) + \nu_k R_k(t)$$

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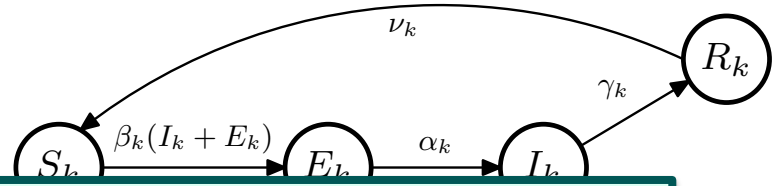
Systems with Repeating Jump Patterns

Epidemic spread:

Super-spreader phenomena: average rate of contact per day is stochastic with Poisson shot noise.

Emergence of variant strains / related viruses:
(e.g., Omicron / Monkeypox)

- All parameters change value (event-driven switching).
- The interarrival times between consecutive switches to a different virus → Poisson or renewal process.



$$\{\beta_{i,kj},\ \alpha_{i,k},\ \gamma_{i,k},\ \delta_{i,k},\ \nu_{i,k}\}_{i=1}^{A}$$

A total variant viruses (i.e., distinct modes).

$$\frac{dS_k(t)}{dt} = -\sum_{j=1}^{K} \frac{\beta_{kj}(I_j(t) + E_j(t))}{M_j(t)} S_k(t) + \nu_k R_k(t)$$

$$\frac{dE_k(t)}{dt} = \sum_{j=1}^{K} \frac{\beta_{kj}(I_j(t) + E_j(t))}{M_j(t)} S_k(t) - \alpha_k E_k(t)$$

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$$\frac{dD(t)}{dt} = \sum_{k=1}^{K} \delta_k I_k(t) \qquad M = \sum_{k=1}^{K} M_k(t),\ \forall t$$

Systems that have jump patterns:

- Networks with time-varying topology (e.g., power grid)
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Systems with Repeating Jump Patterns

$$\mathcal{G}(m) \triangleq (\mathcal{V}, \mathcal{E}(m))$$
$$m \in \{1, \cdots, M\}$$

Pattern-Learning: identify recurrent topologies of the network to recycle closed-loop responses.

Mode Process Controller: perform disturbance-rejection using an approach based on *system-level synthesis (SLS)*[*].

Systems that have jump patterns:

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Application: Networks with Time-Varying Topology

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$$\mathbf{x}_i[t+1] = A_{ii}(\xi_{N[t]})\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i(\xi_{N[t]})} A_{ij}(\xi_{N[t]})\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t], \quad i \in \mathcal{V}$$

$$\mathcal{N}_i(m) \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}(m)\}$$

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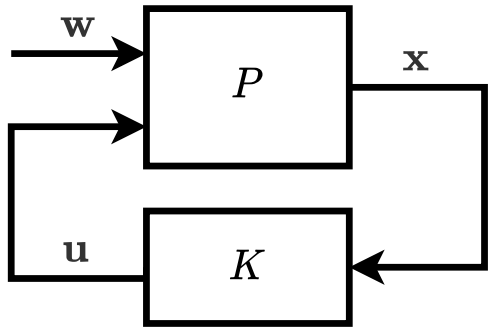

$$\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$$

$$\mathbf{x}[t+1] = A\mathbf{x}[t] + B\mathbf{u}[t] + \mathbf{w}[t]$$

State-feedback control: $\mathbf{u} = K\mathbf{x}$.

Closed-loop system z-transform:

$$\mathbf{x} = (zI - A - BK)^{-1}\mathbf{w}$$

Optimizing this transfer function by solving for K is difficult when (A, B) is large-scale.

$$\mathbf{x}_i[t+1] = A_{ii}\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i} A_{ij}\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t], \quad i \in \mathcal{V}$$

$$\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$$

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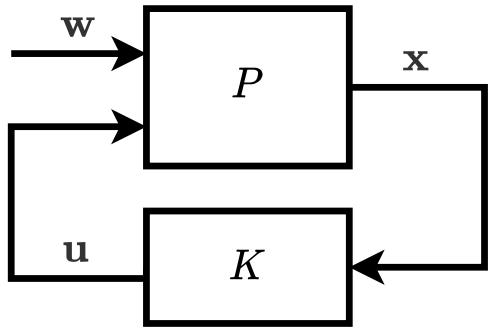

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SLS: Design for the entire closed-loop response.

$$\mathbf{\Phi} \triangleq \{\mathbf{\Phi}_x, \mathbf{\Phi}_u\} \text{ s.t. } \begin{bmatrix}\mathbf{x}\\ \mathbf{u}\end{bmatrix} = \begin{bmatrix}\mathbf{\Phi}_x\\ \mathbf{\Phi}_u\end{bmatrix}\mathbf{w}$$

$$\mathbf{\Phi}_x \leftrightarrow (zI - A - B_2K)^{-1}$$

$$\mathbf{\Phi}_u \leftrightarrow K(zI - A - B_2K)^{-1}$$

Optimizing over $\{\mathbf{\Phi}_x, \mathbf{\Phi}_u\}$ instead of K.

$$\mathbf{x}_i[t+1] = A_{ii}\mathbf{x}_i[t] + \sum_{j\in\mathcal{N}_i} A_{ij}\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t], \quad i \in \mathcal{V}$$

$$\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$$

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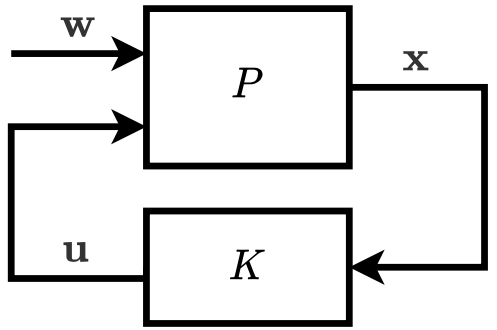

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$$\mathbf{\Phi} \triangleq \{\mathbf{\Phi}_x, \mathbf{\Phi}_u\} \text{ s.t. } \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \mathbf{w}$$

Implementation in terms of original signals: finite-horizon case

$$\hat{\mathbf{x}}[t] = \sum_{k=2}^{T} \Phi_x[k]\hat{\mathbf{w}}[t+1-k]$$

$$\hat{\mathbf{w}}[t] = \mathbf{x}[t] - \hat{\mathbf{x}}[t]$$

$$\mathbf{u}[t] = \sum_{k=1}^{T} \Phi_u[k]\hat{\mathbf{w}}[t+1-k]$$

$$\mathbf{x}_i[t+1] = A_{ii}\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i} A_{ij}\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t], \quad i \in \mathcal{V}$$

$$\mathcal{N}_i \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}\}$$

z-transform

$$\mathbf{\Phi}_x \equiv \mathbf{\Phi}_x(z) \triangleq \sum_{i=1}^{\infty} \Phi_x[i]\frac{1}{z^i}$$

$$\mathbf{\Phi}_u \equiv \mathbf{\Phi}_u(z) \triangleq \sum_{i=1}^{\infty} \Phi_u[i]\frac{1}{z^i}$$

Local implementation over $\mathcal{G}$: $\mathbf{\Phi}^{(i)} \triangleq \{\Phi_x^{(i)}[s], \Phi_u^{(i)}[s]\}_{s=1}^{T} \quad \forall\, i \in \mathcal{V}.$

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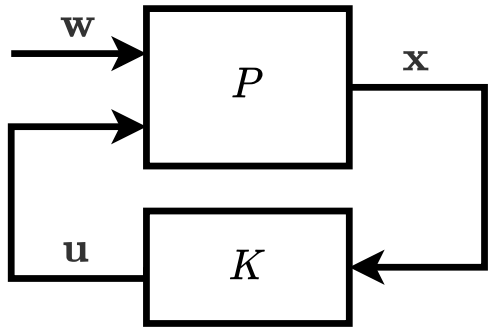

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$$m \in \{1, \cdots, M\}$$

$$\mathbf{\Phi} \triangleq \{\mathbf{\Phi}_x, \mathbf{\Phi}_u\} \text{ s.t. } \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}_x \\ \mathbf{\Phi}_u \end{bmatrix} \mathbf{w}$$

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$$\mathbf{\Phi}_x \equiv \mathbf{\Phi}_x(z) \triangleq \sum_{i=1}^{\infty} \Phi_x[i] \frac{1}{z^i}$$

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Time-varying local implementation over dynamic topology $\mathcal{G}(m)$:

$$\mathbf{\Phi}_m^{(i,t)} \triangleq \{\Phi_{x,m}^{(i,t)}[s], \Phi_{u,m}^{(i,t)}[s]\} \quad \forall\, m \in \{1, \cdots, M\}, i \in \mathcal{V}.$$

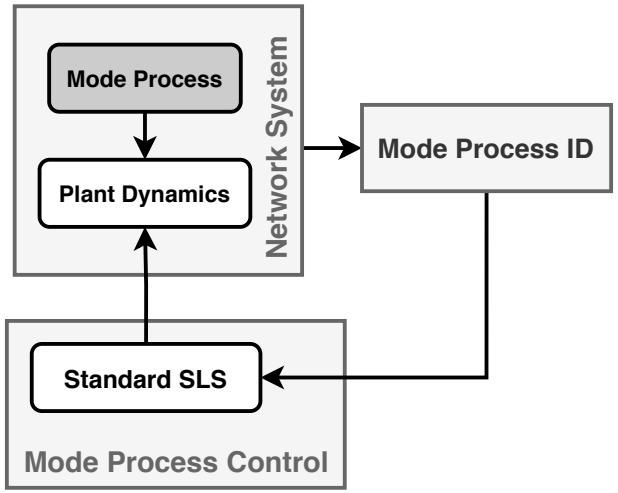
$$\mathbf{x}_i[t+1] = A_{ii}(\xi_{N[t]})\mathbf{x}_i[t] + \sum_{j \in \mathcal{N}_i(\xi_{N[t]})} A_{ij}(\xi_{N[t]})\mathbf{x}_j[t] + B_i\mathbf{u}[t] + \mathbf{w}_i[t], \quad i \in \mathcal{V}$$

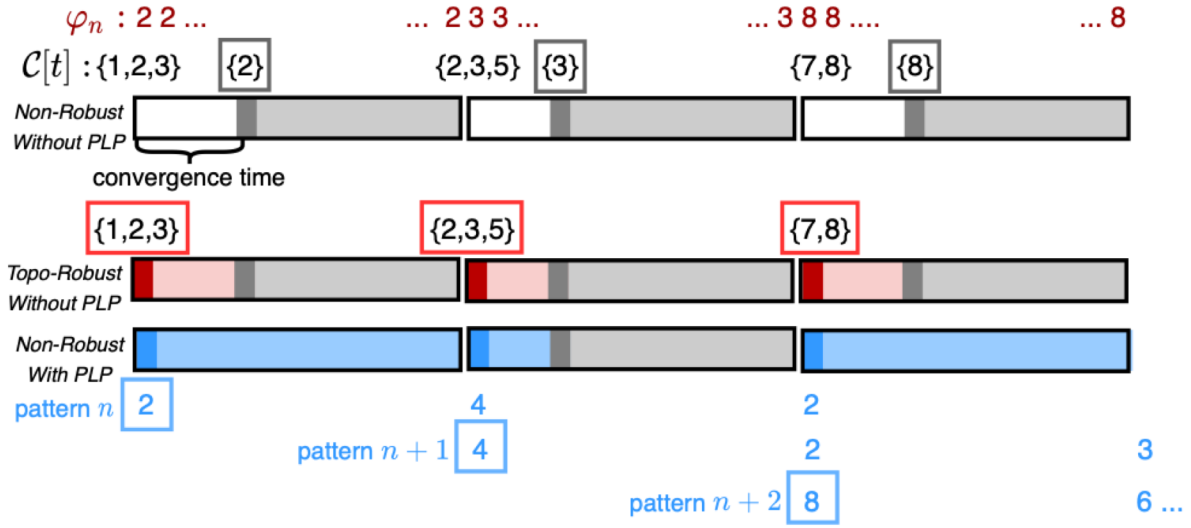
$$\mathcal{N}_i(m) \triangleq \{j \in \mathcal{V} : (i,j) \in \mathcal{E}(m)\}$$

* SooJean Han. *Localized Learning of Robust Controllers for Networked Systems with Dynamic Topology.* L4DC, 2020.

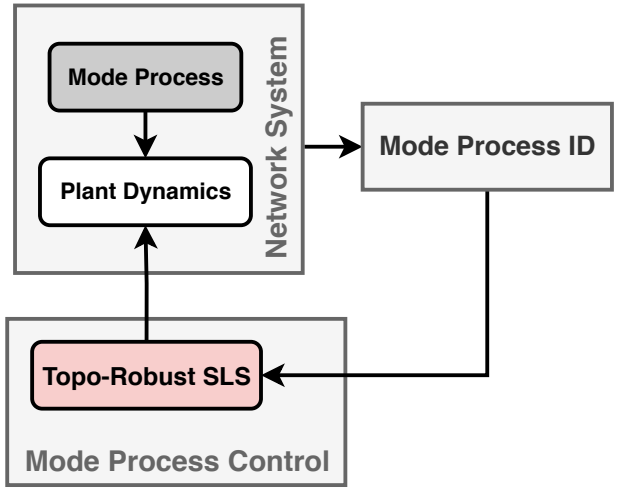
Three Controller Frameworks

Non-Robust w/o PLP:

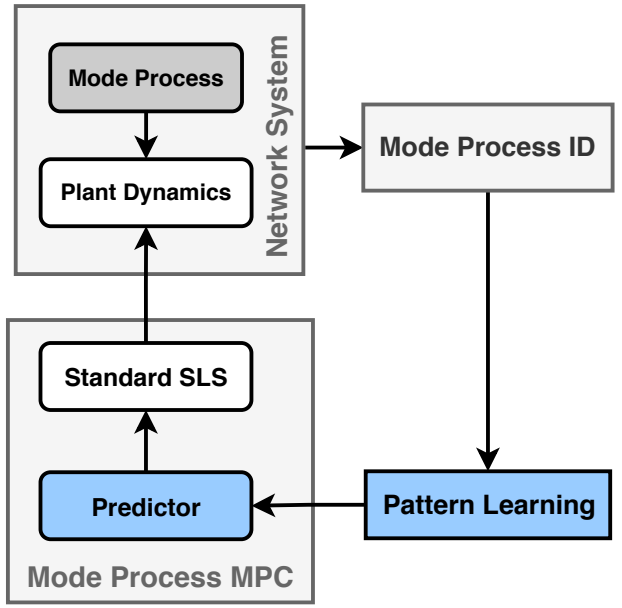




Topologically-Robust w/o PLP:



Non-Robust w/ PLP:



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Three Controller Frameworks

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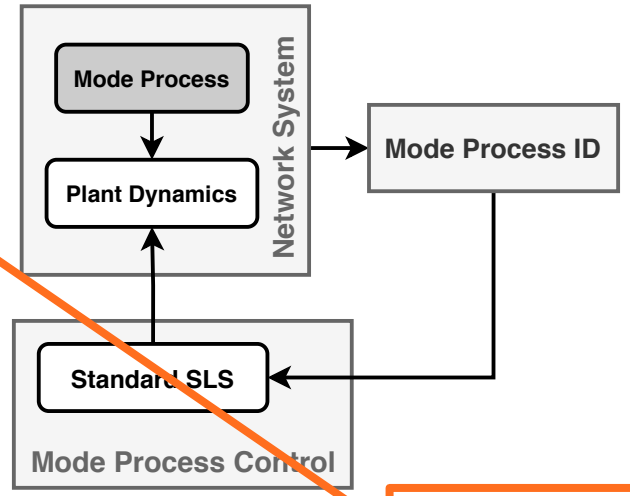
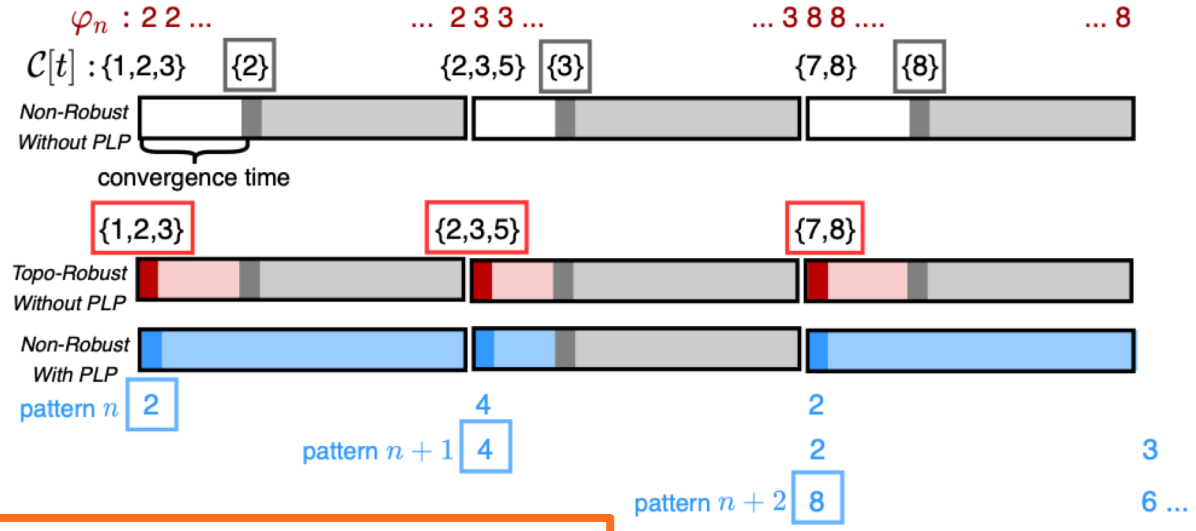
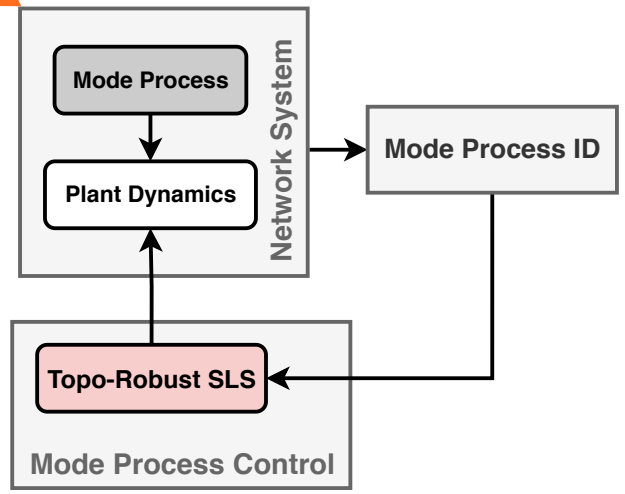




("PLP" = pattern learning and prediction)

Topologically-Robust w/o PLP:



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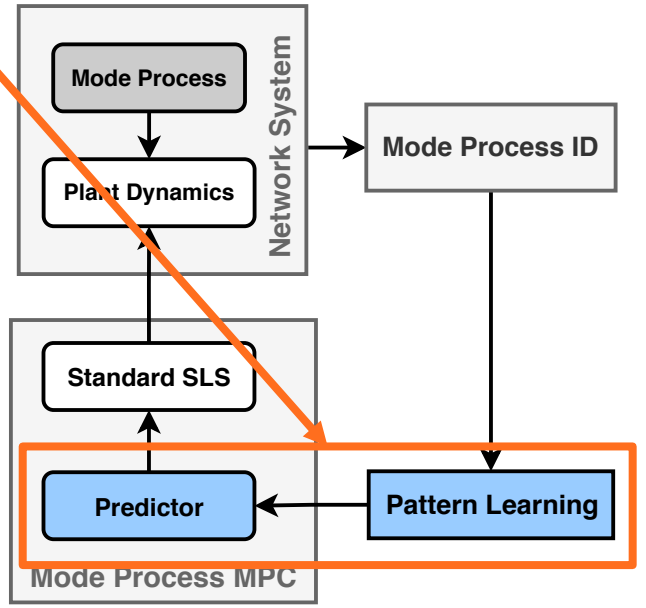

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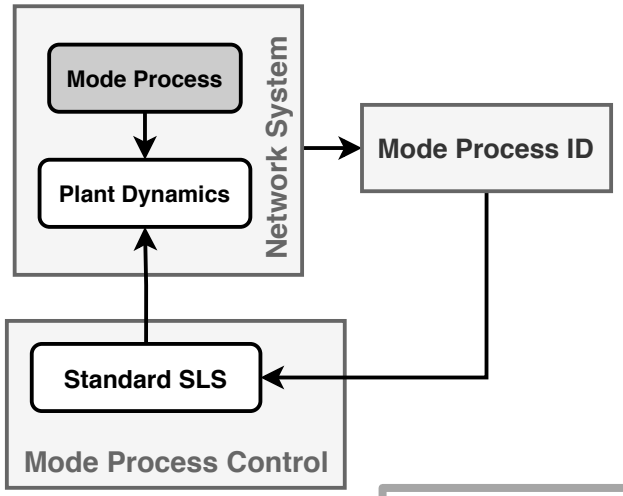
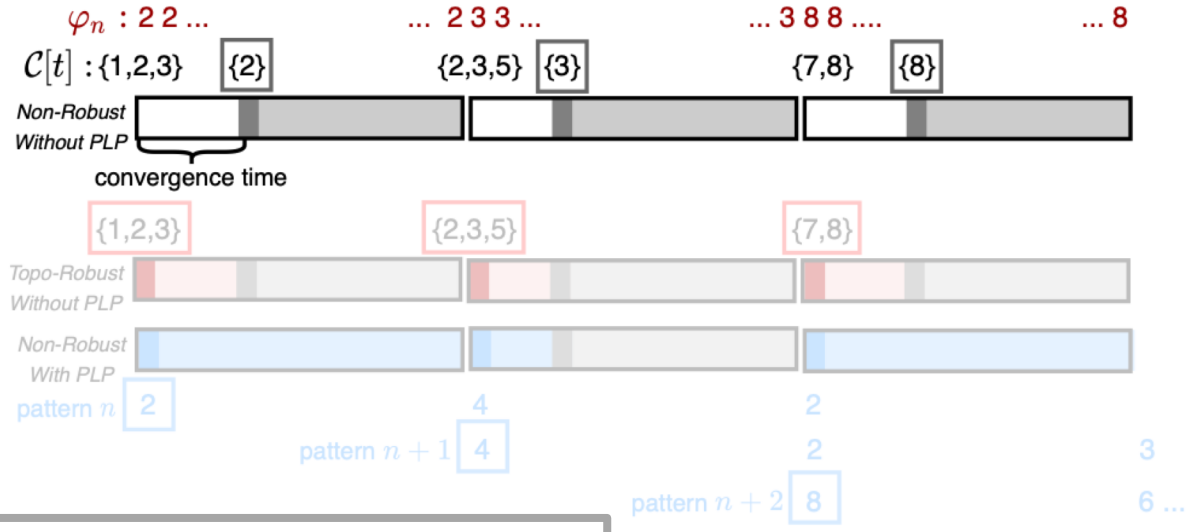
Three Controller Frameworks

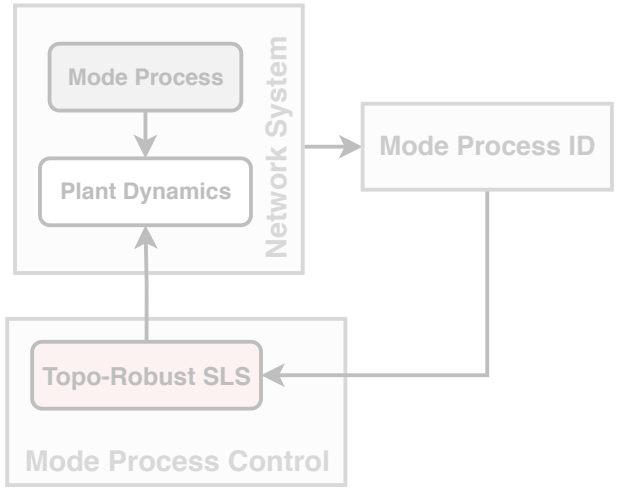
Non-Robust w/o PLP (NR):



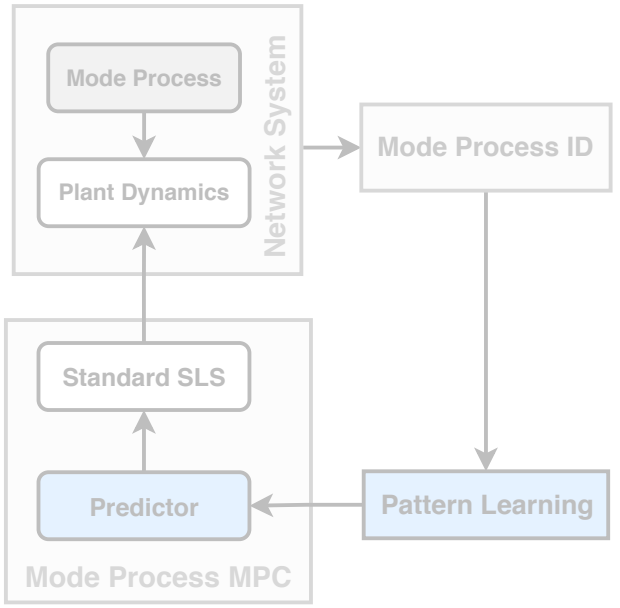


("PLP" = pattern learning and prediction)

Topologically-Robust w/o PLP:



Non-Robust w/ PLP:



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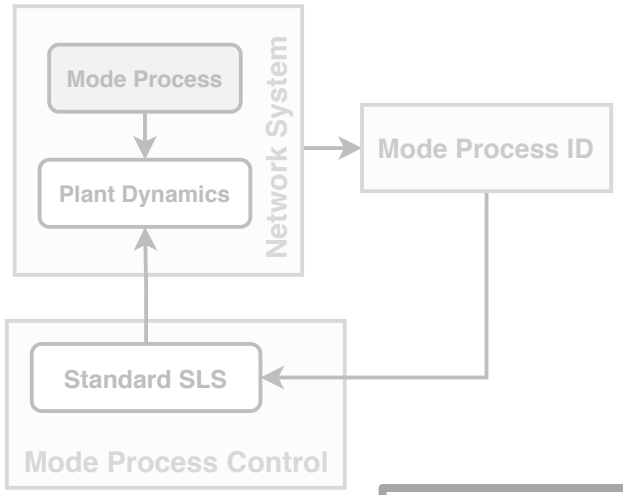
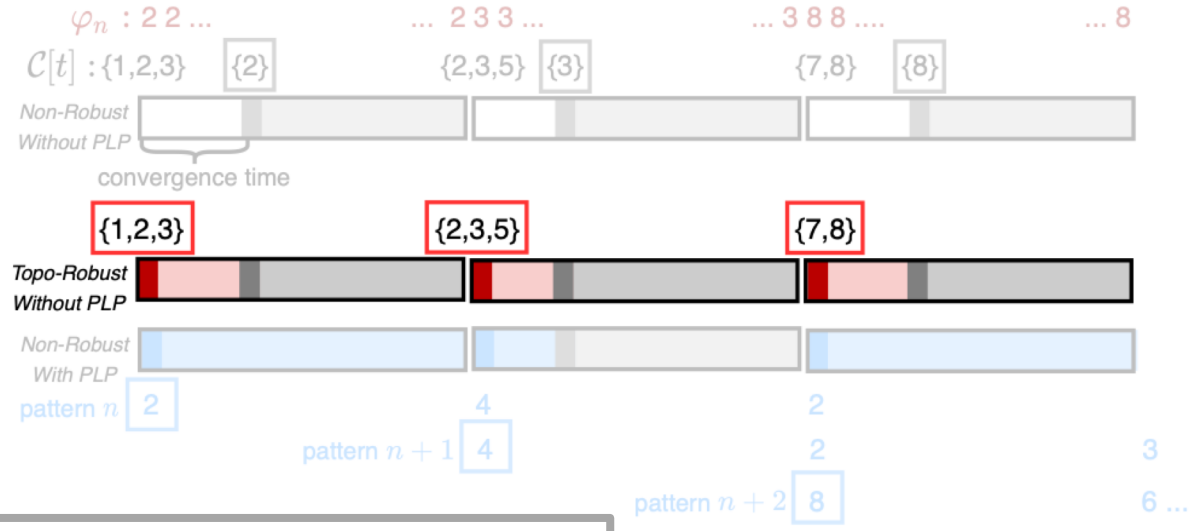
Three Controller Frameworks

Non-Robust w/o PLP (NR):





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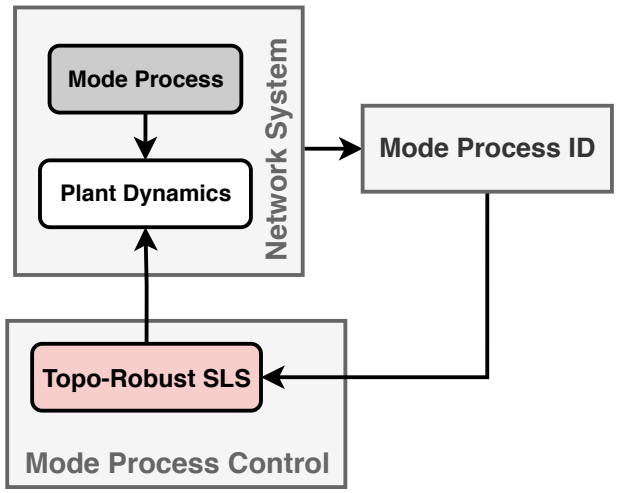
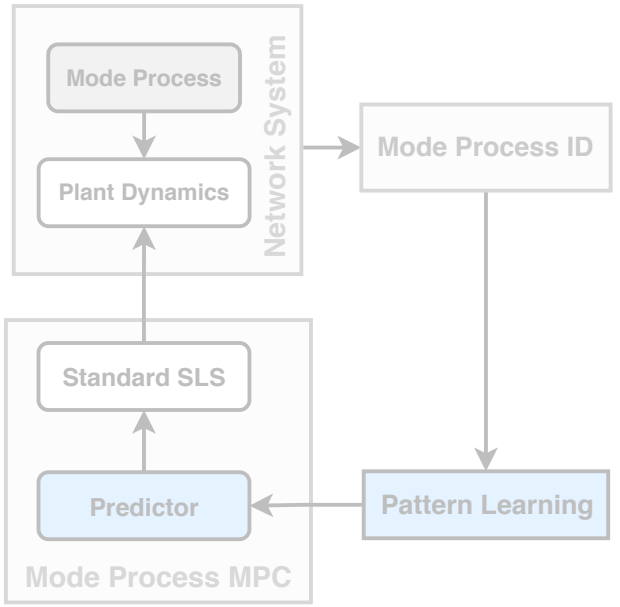
Topologically-Robust w/o PLP (TR):



Non-Robust w/ PLP:



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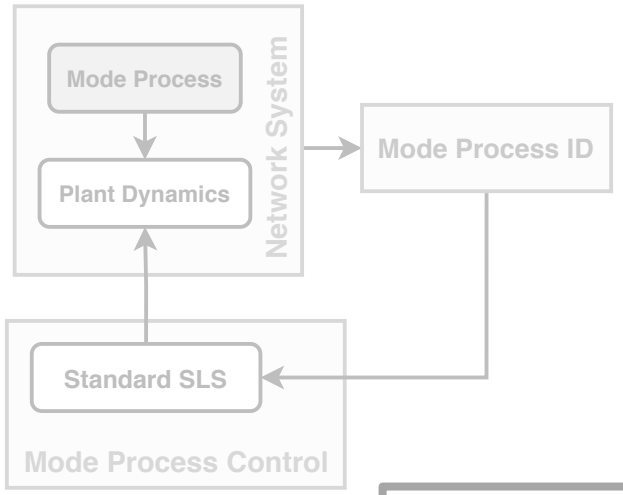
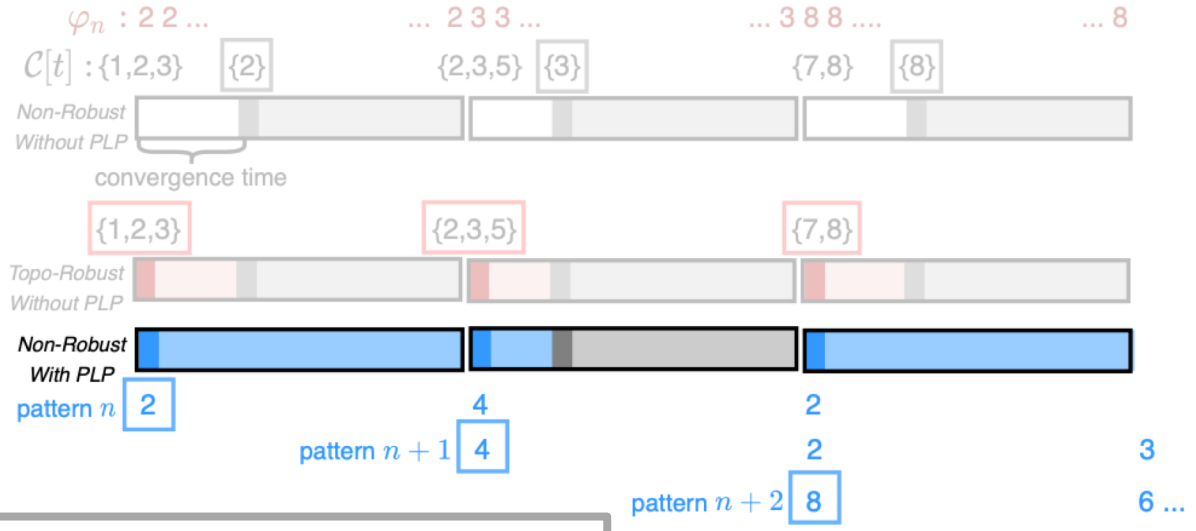
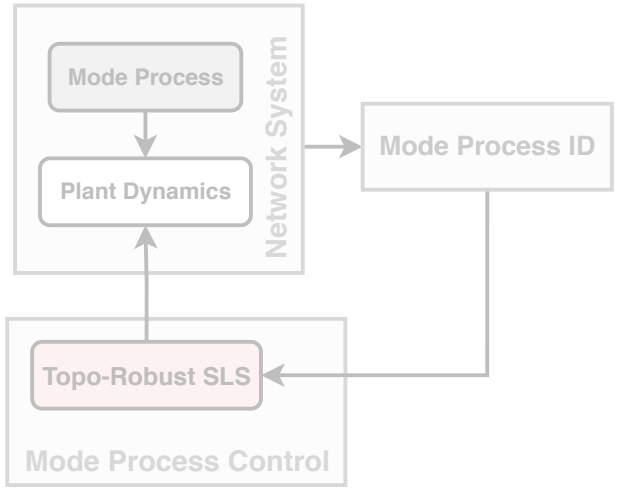


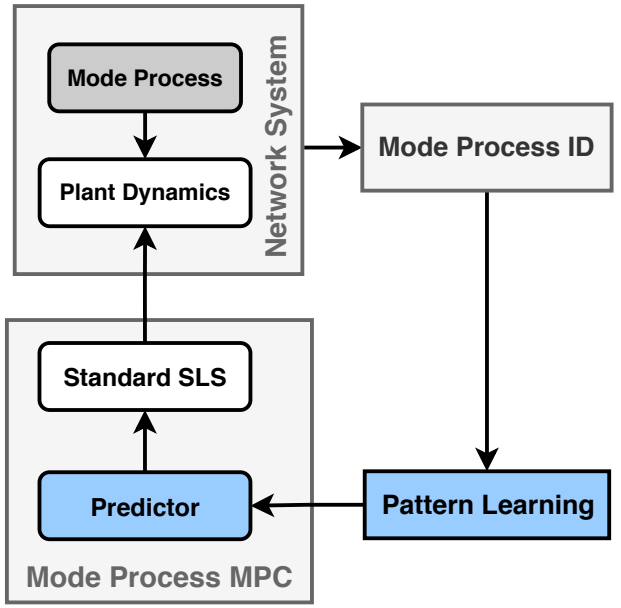


("PLP" = pattern learning and prediction)

Topologically-Robust w/o PLP (TR):



Non-Robust w/ PLP (NRPLP):



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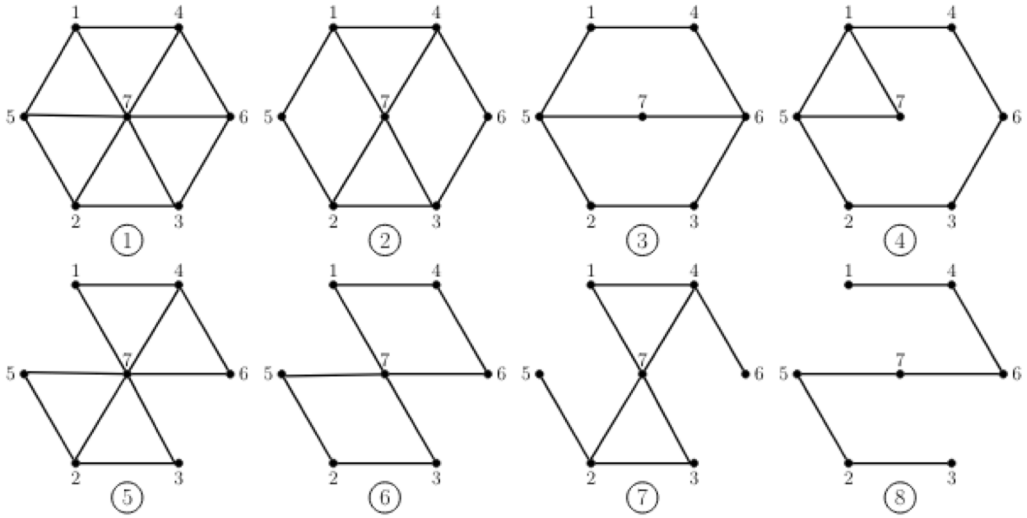
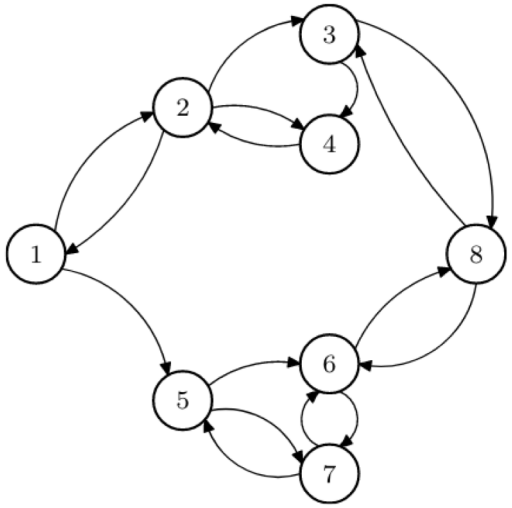
Two Network Systems

$$\mathcal{G}(m) \triangleq (\mathcal{V}, \mathcal{E}(m))$$
$$m \in \{1, \cdots, M\}$$

Hexagon System:

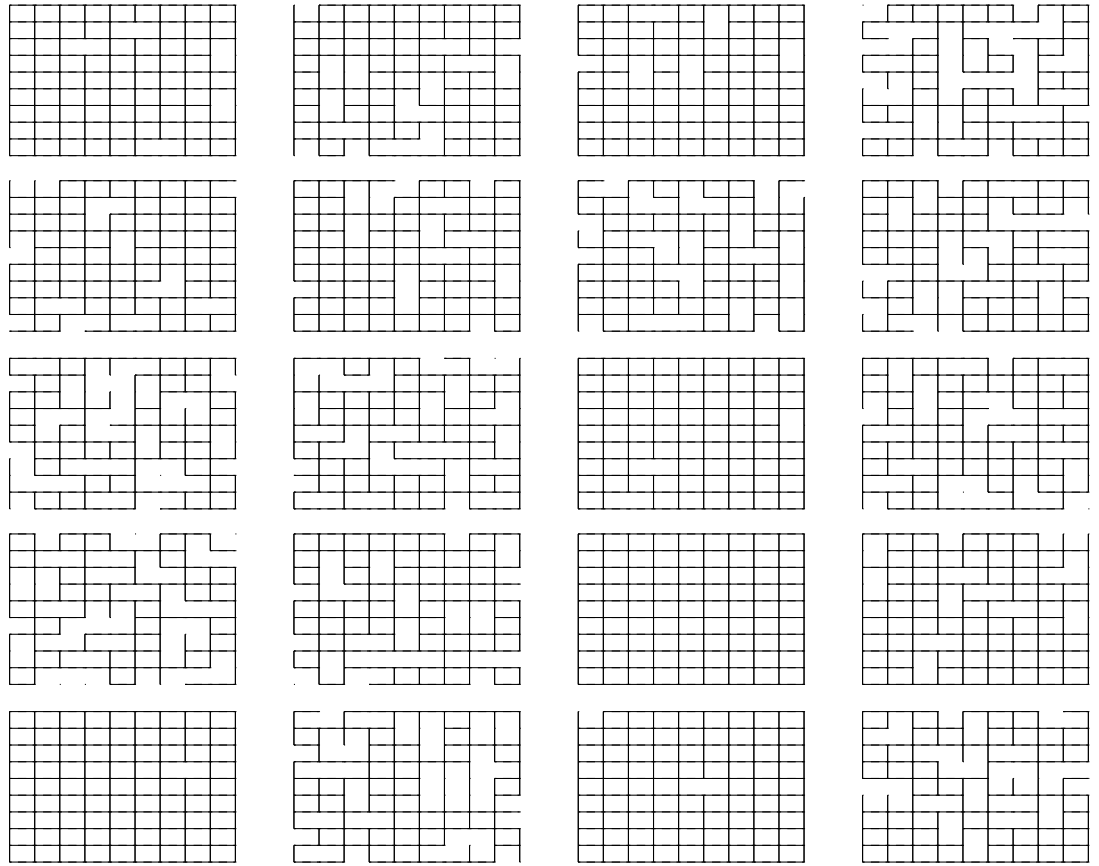
7 nodes, M = 8 modes





Rectangular Grid System:

10x10 = 100 nodes, M = 20 modes



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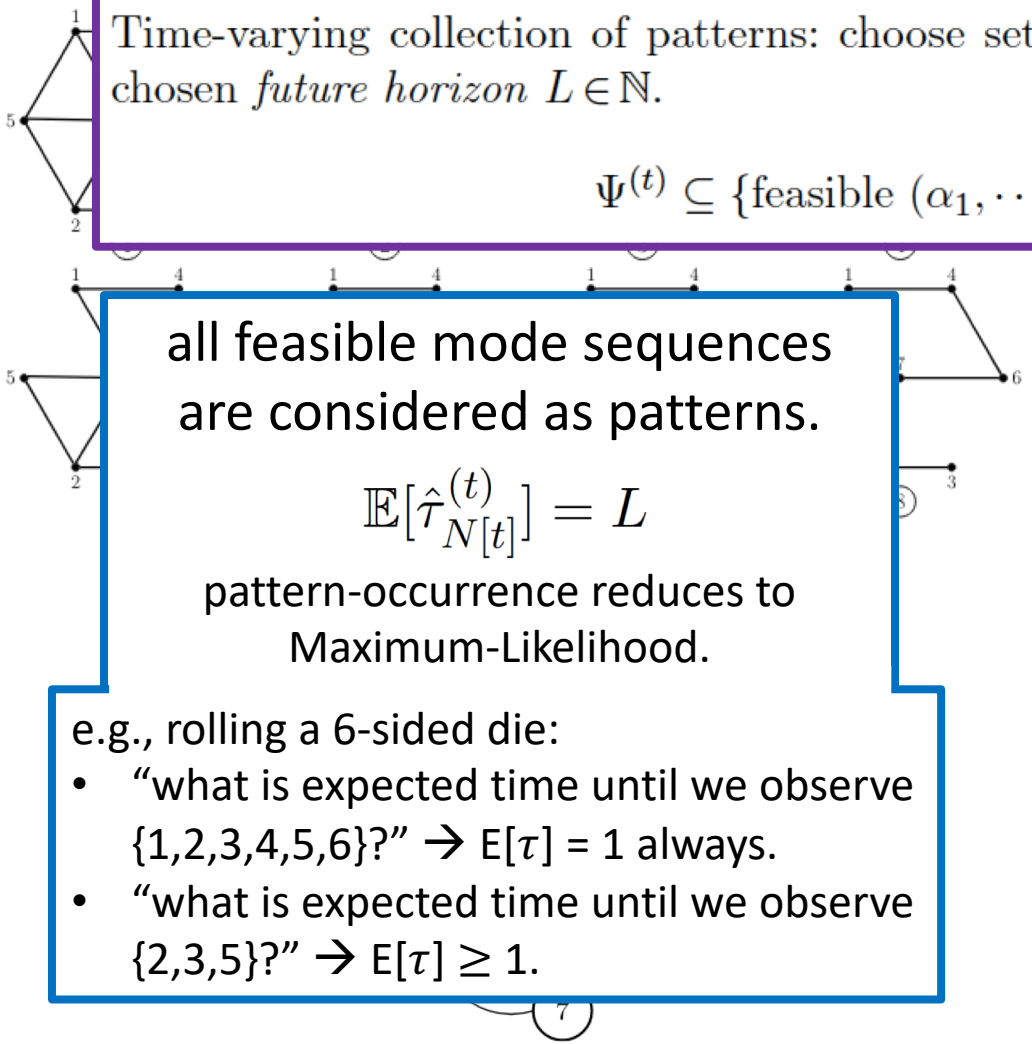

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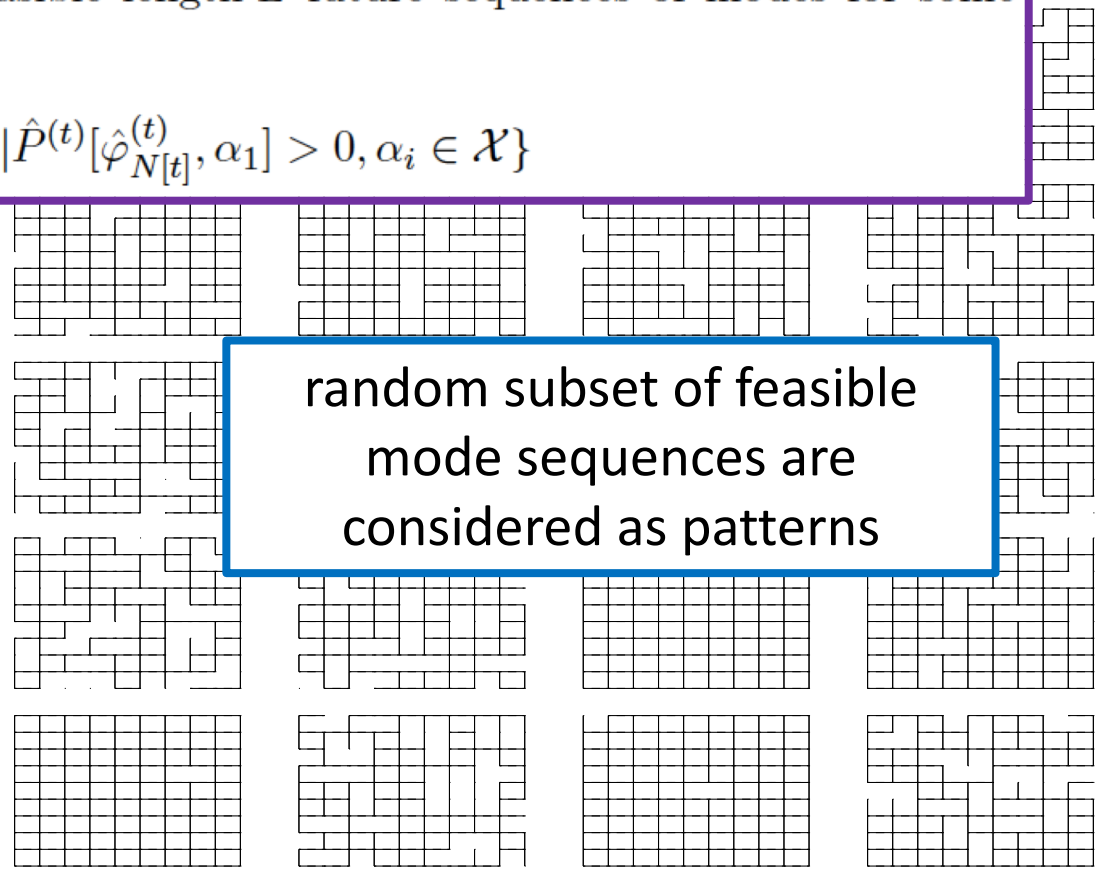

Recall for mode process MPC...

Time-varying collection of patterns: choose set of feasible length-L future sequences of modes for some chosen *future horizon* $L \in \mathbb{N}$.

$$\Psi^{(t)} \subseteq \{\text{feasible } (\alpha_1, \cdots, \alpha_L) | \hat{P}^{(t)}[\hat{\varphi}^{(t)}_{N[t]}, \alpha_1] > 0, \alpha_i \in \mathcal{X}\}$$

all feasible mode sequences are considered as patterns.

$$\mathbb{E}[\hat{\tau}^{(t)}_{N[t]}] = L$$

pattern-occurrence reduces to Maximum-Likelihood.

e.g., rolling a 6-sided die:

- "what is expected time until we observe {1,2,3,4,5,6}?" → E[τ] = 1 always.
- "what is expected time until we observe {2,3,5}?" → E[τ] ≥ 1.

random subset of feasible mode sequences are considered as patterns

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Application: Networks with Time-Varying Topology

LQR Cost: $$\frac{1}{T_{\text{sim}}} \sum_{t=1}^{T_{\text{sim}}} \mathbf{x}[t]^\top Q \mathbf{x}[t] + \mathbf{u}[t]^\top R \mathbf{u}[t]$$

LQR Cost	NR	TR	NRPLP
Hexagon	32.0972	43.7638	31.1688
Grid	467.8021	473.0120	467.0066

Error Norm: (disturbance-rejection)

$$\frac{1}{T_{\text{sim}}} \sum_{t=1}^{T_{\text{sim}}} \|\mathbf{x}[t]\|_2$$

Error Norm	NR	TR	NRPLP
Hexagon	2.9032	1.5984	1.6029
Grid	7.6266	5.5633	5.5349

Prop(ortion of Time**) Match(**ing Control Law Used**) :**

$$\frac{1}{T_{\text{sim}}} \sum_{t=1}^{T_{\text{sim}}} \mathbb{1}\{\hat{\varphi}_{N[t]}^{(t)} = \varphi_{N[t]}\}$$

Prop. Match	NR	NRPLP
Hexagon	0.4304	0.615
Grid	0.1533	0.16

Runtime	NR	TR	NRPLP
Hexagon	11.8314	67.2254	2.2689
Grid	101.3741	X	38.5824

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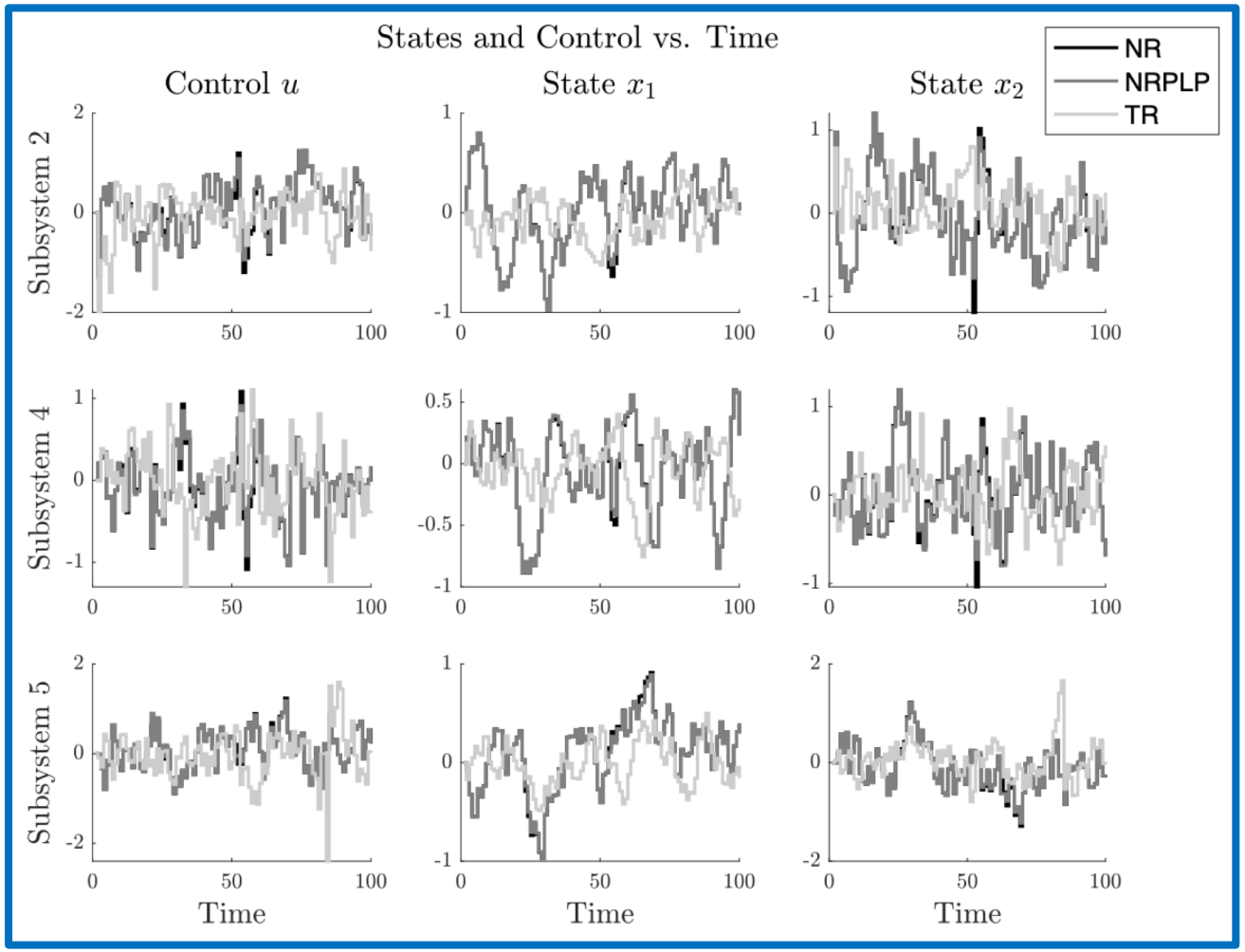

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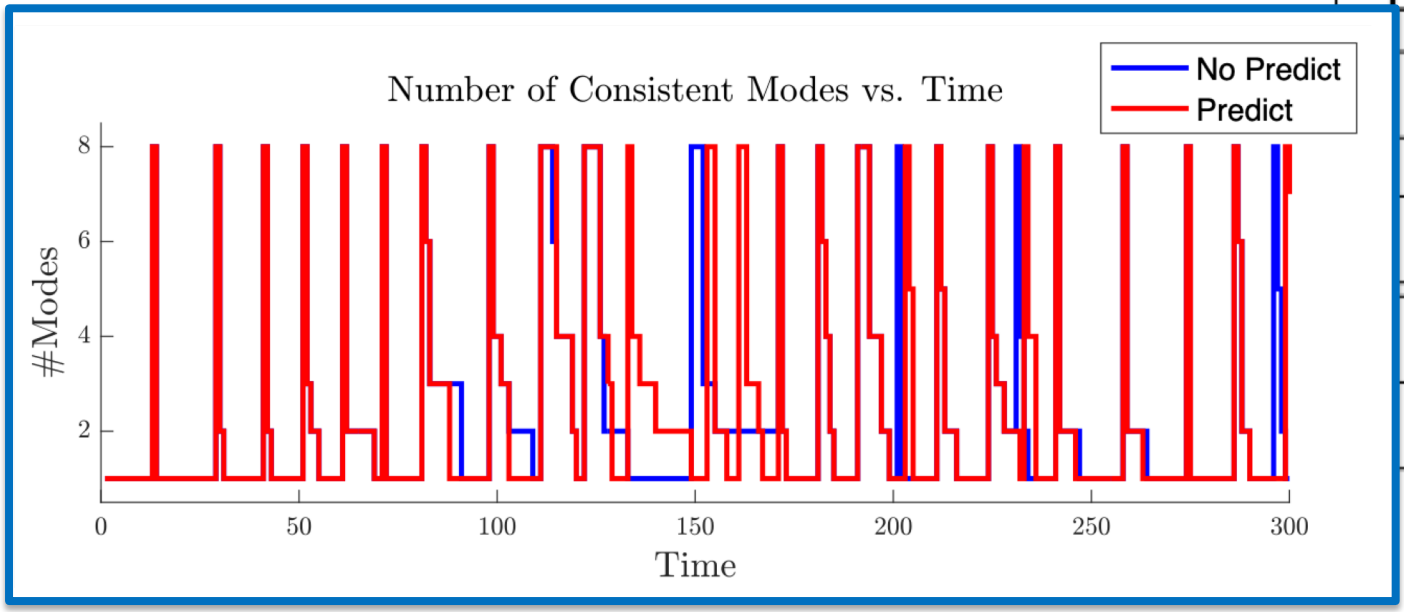

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$$\mathcal{C}[0] \triangleq \mathcal{X},\ \mathcal{C}[t] = \left\{ m \in \mathcal{C}[t-1] \mid \bigwedge_{r=T_n}^{t-1} \mathbb{1}\{\|\mathbf{x}[r+1] - A(m)\mathbf{x}[r] - B\mathbf{u}[r]\|_\infty \leq \overline{w}\} \right\}$$

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Application: Networks with Time-Varying Topology

At least 2 ways of improving performance:

- **memorize past patterns**
  → no re-computation of repeating control policies
- **predict future patterns**
  → schedule control policies in advance

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Implementation Details + Runtime:

- SLS is localized. Mode process ID and pattern learning are centralized.
- Unknown topologies → SLS computes a new law for every new mode observed.
- NR and TR (without PLP) does not store the SLS control law into memory when it is learned. → "*memorize past patterns → no re-computation of control law needed for those patterns*".
- Topology-robust SLS solves for a single control law to stabilize multiple topologies simultaneously → longer runtime than standard SLS.

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Proportion of Time Spent Using Matching Control Law:

- NRPLP consistently uses the matching control law more often than NR → pattern-learning can be an additional mode estimation algorithm. → "*predict future patterns → schedule control policies in advance*".
- Increasing the number of patterns in the pattern collection → more accurate consistent mode estimates.
- Decreasing the number of patterns in the pattern collection → increases expected minimum occurrence time → PLP more closely resembles the performance without PLP.

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Error Norm: (disturbance-rejection):
$$\frac{1}{T_{\text{sim}}} \sum_{t=1}^{T_{\text{sim}}} \|\mathbf{x}[t]\|_2$$

Controller Performance + Effort:

- LQR cost for TR is largest although error norm is one of the smallest → there is more control effort that goes into it.
- Stabilization performance of NRPLP is just as good as TR.
- *Implication*: For a good choice of pattern collections, appending PLP to a non-robust controller could be used as an efficient alternative to a robust controller.

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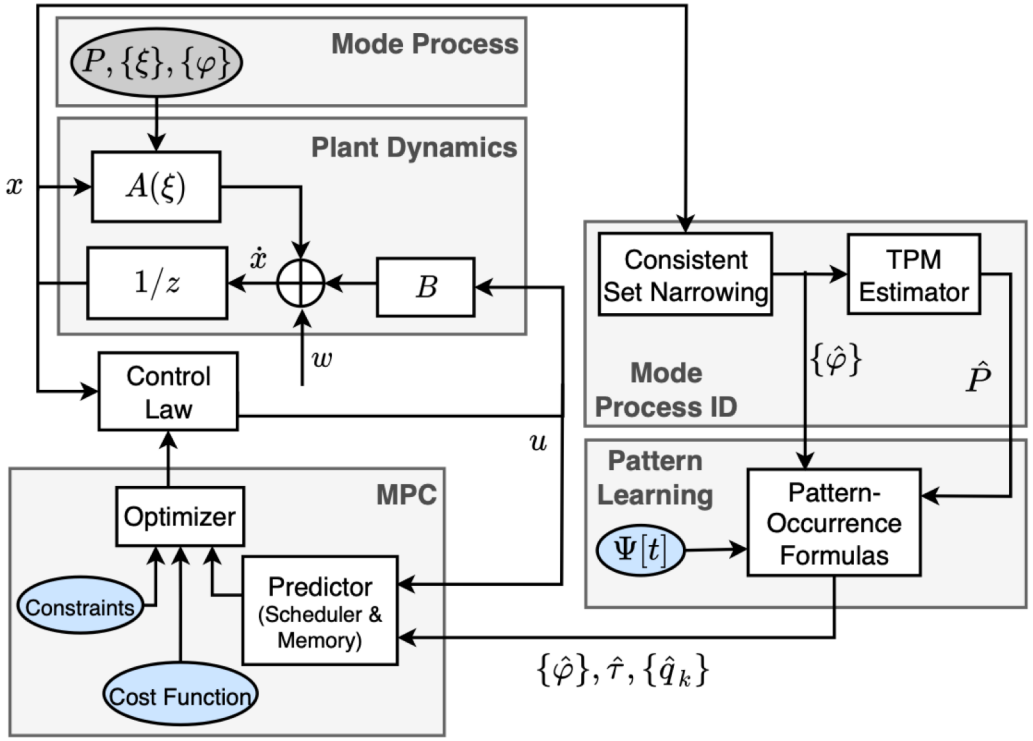
Summary: Jump Stochastic Systems

Control/estimation for *jump stochastic systems* can be achieved by learning **patterns** in the system behavior.

Learning patterns improves performance in at least 2 ways:

- **memorize past patterns**
  → **no re-computation of repeating control policies**
- **predict future patterns**
  → **schedule control policies in advance**



Systems that have jump patterns:

- Networks with time-varying topology (e.g., power grid)
  - **patterns** = power outages, line failures
- Vehicle traffic congestion
  - **patterns** = intersection snapshots
- Epidemic spread
  - **patterns** = contact-tracing data, travel routines

* SooJean Han and Soon-Jo Chung. *Incremental Nonlinear Stability Analysis of Stochastic Systems Perturbed by Lévy Noise*, International Journal of Robust and Nonlinear Control, 2022.
* SooJean Han. *Localized Learning of Robust Controllers for Networked Systems with Dynamic Topology*. L4DC, 2020.
* SooJean Han, Soon-Jo Chung, and John C. Doyle. *Predictive Control of Linear Discrete-Time Markovian Jump Systems via the Analysis of Recurrent Patterns*, Automatica, submitted 2022.

Thank you!

감사합니다!

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