Incremental Nonlinear Stability Analysis for Stochastic Systems Perturbed by Lévy Noise

SooJean Han and Soon-Jo Chung, *Senior Member, IEEE*

***Abstract*— This paper presents a theoretical framework for characterizing incremental stability of nonlinear stochastic systems perturbed by noise processes of two types, Poisson shot noise, and Lévy noise, both of which are understudied in the controls community compared to white noise despite being an equally prevalent phenomenon. The practical utilities of this study include the design of stochastic controllers and observers which do not immediately resort to model-free synthesis from scratch simply because the noise process is non-Gaussian. Our main contributions are two theorems, one for each type of noise, that show that trajectories of the stochastic system arising from distinct initial conditions and noise sample paths are able to exponentially converge to within a steady-state bounded error ball of each other in the mean. We show the convergence rate for shot noise systems is the same as the unperturbed system, but with a tradeoff between the shot noise intensity and the size of the error ball. Furthermore, the convergence rate and the error ball for the Lévy noise system is shown to be nearly a direct addition of the respective quantities for the shot and white noise systems separately; this result is analogous to the representation of Lévy noise as an affine combination of white and shot noise through the well-known Lévy-Khintchine formula. We numerically demonstrate our results using a 1D shot noise, linear system and a 5D Lévy noise, nonlinear system.**

***Index Terms*— Nonlinear control systems, Poisson processes, Random processes, Robust stability, Stability analysis, Stability criteria, Stochastic processes, Stochastic systems, Uncertain systems**

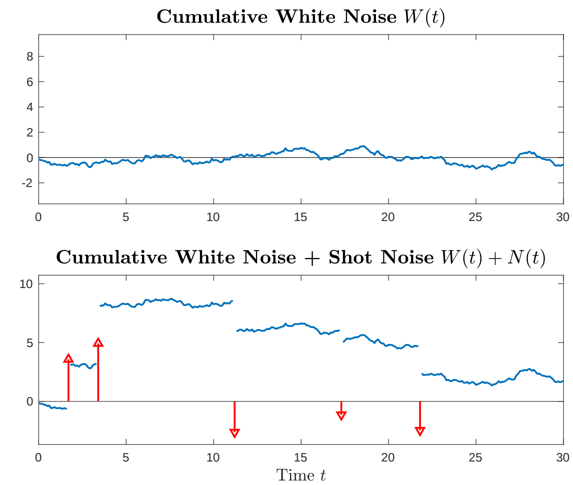

Fig. 1: In the top, a pure white noise process. In the bottom, a white noise process superimposed atop shot noise impulses. As a result of the shot noise, the effect over time is more severe (larger drift) than pure white noise.

I. Introduction

Model-based design methods of controllers for stochastic systems usually aim for robustness against additive white Gaussian noise (AWGN); it is typically the initial go-to model when stochastic noise is considered in systems. There are many applications such as vision-based localization/mapping [1], spacecraft navigation [2], and motion-planning [3], where such a noise model works very well in practice. Consequently, there have been numerous extensive work devoted to analyzing the stability of and controlling systems with AWGN perturbations, including the classical Linear Quadratic Gaussian (LQG) model [4], [5], the path integral approach [6], convex optimization-based approaches [7], [8], as well as a number of reinforcement learning-based approaches [9], [10].

However, there is a major lack of generality with this assumption of white noise. White noise is typically small in magnitude, continuous in the sense that large drifts occur steadily over time, and affects a system persistently for a measurable duration of time. It is unable to characterize a broad class of sudden impulsive perturbations, often referred to as *shot noise* [11] or *jump noise*, which arises in real-world scenarios almost just as frequently as white noise does. Some examples of shot noise in practical settings are the large fluctuations in stock prices in financial economics [12] and signal-processing neuronal spikes arising from brain activity in neuroscience [13]. Furthermore, one can observe an even broader class of general *Lévy noise* processes, which can be expressed as an affine combination of both white and shot noise.

In this paper, we design a theoretical framework for nonlinear stability analysis of stochastic systems perturbed by shot and Lévy noise. In particular, we ask a standard stochastic stability question posed in [14]: can trajectories of the system, arising from different sample paths of noise and different initial conditions within a bounded set, be bounded in some region after a sufficiently elapsed time? This question has been answered previously in [15], [16] for white noise systems; now we focus on addressing the question first for only shot noise, then combine our results with the white noise case to address it for Lévy noise.

The primary benefit of answering this question is that many methods of stochastic controller and observer design, previously suitable for only white noise processes, can now be expanded to account for a much larger class of noise which includes severe jumps. For instance, the observer from [16], designed specifically to be stable against white noise, can now be enhanced to be stable against shot noise phenomena such as massive errors from proprioceptive sensors. Stochastic controller design also has potential to be improved; while many analytical synthesis methods have been devoted to the Gaussian white noise case, there is little to no methodological process for handling non-Gaussian noise, and oftentimes, machine-learning-based methods [17]–[19] are used to learn the noise process without fitting it to a particular distribution. Another downside of learning the noise model from scratch is the massive amounts of offline training and limited training time. A result which guarantees stability within a bounded error ball for a stochastic system perturbed by shot or Lévy noise is essential to the development of a model-based analytical design approach which bypasses the above-mentioned limitations. Much like how an unknown noise process of a system can be fit

SooJean Han is with the Department of Computing and Mathematical Sciences, California Institute of Technology, Pasadena, CA 91125, USA. Email: soojean@caltech.edu.

Soon-Jo Chung is with the Graduate Aerospace Laboratory, California Institute of Technology, Pasadena, CA 91125, USA. Email: sjchung@caltech.edu.

according to a Gaussian mixture model, recognizing that shot noise can be represented by a Poisson distribution and that Lévy noise is a linear combination of the two types offers a much broader model to fit an unknown non-Gaussian noise process to. The analytically-designed noise model can then be used as a baseline which can be improved upon via learning techniques in order to efficiently design controllers instead of resorting to computationally-intensive methods from the start.

The contributions of this paper are as follows. We define the notion of stochastic contraction for shot and Lévy noise systems, then show that solution trajectories of the system perturbed by different sample paths of shot or Lévy noise processes globally exponentially converge to within a bounded error ball of each other in *expectation*. We emphasize the existence of a tradeoff between the intensity of the shot noise and the size of the error ball; because the largest perturbations would arise from the shot noise component of the Lévy noise process, our stability results can be alternatively interpreted as conditions imposed on the noise process such that the mean time between consecutive jumps (i.e., *interarrival time*) is large enough so that the system may be stabilized. Furthermore, although the Lévy noise case is a direct combination of the white noise and the shot noise cases, there are two major distinctions on our treatment for the shot noise systems and previous results on white noise systems [15], [16]. First, stochastic contraction for shot noise cannot be proven using the same Lyapunov-like function that was used to prove stochastic contraction for white noise. Second, instead of comparing two noise-perturbed trajectories against each other, we compare a noise-perturbed trajectory against an unperturbed one.

A. Related Work

Lyapunov-Sense Stability: Traditional characterizations of deterministic and stochastic system stability are defined in the Lyapunov sense [14], [20]. For deterministic systems, we have the well-known direct and indirect Lyapunov methods, which can be found in any standard control theory textbook (e.g. [21], [22]). For white-noise stochastic systems, sufficient stability conditions can be derived for nonlinear systems [23] or systems which are affine in control input [24]. Although Gaussian white noise is the most prevalent model for stochastic dynamics, there have also been works on characterizing stability for stochastic systems perturbed by non-Gaussian noise beyond the seminal work of [14], which laid out the foundations of Lyapunov-based stochastic stability theory. Asymptotic stability of systems driven by Lévy noise is developed in [25] and exponential stability is studied in [26].

Incremental Stability: Formally defined in Definition 3, incremental stability generalizes Lyapunov-sense stability be considering the convergence of solution trajectories towards some desired trajectory rather than an equilibrium point or a limit cycle. Systems that satisfy the incremental stability property have guaranteed global exponential convergence towards the desired trajectory. Applications of incremental stability arise in numerous settings such as cooperative control over multi-agent swarm systems [27] and phase synchronization in directed networks [28], [29]. For deterministic systems, there has been an extensive amount of work characterizing incremental stability for nonlinear dynamics [30], [31]. The work of [15] took the first step in extending incremental stability to Gaussian white noise stochastic systems, and [16] extended this theory to more general state-dependent metrics, which is useful in the construction of nonlinear observers or controllers. However, to the authors' knowledge, incremental stability for dynamics perturbed by non-Gaussian noise processes has not been considered; this paper intends to take the first step in doing so for shot and Lévy noise systems.

Hybrid Systems: We emphasize a poignant distinction between the literature of hybrid or jump-Markov systems versus our considered setting of systems perturbed by a general class of jump-discontinuous noise processes. In shot or Lévy noise systems, the jumps arise solely from the noise process, independently of the open-loop dynamics, unlike hybrid systems where the switches (i.e., jumps) arise as an inherent property in the open-loop dynamics. Despite this important distinction, the two settings can still be closely related to one another through two concepts. First, stability analysis techniques are primarily focused on handling the jump-discontinuities of the system. Literature towards this direction of research for hybrid systems include Lyapunov-sense conditions for asymptotic stability [32], [33] and characterizations of incremental stability was studied in [34]. Second, *dwell time* can be related to the interarrival time by viewing it as a form of stability criterion which ensures that the system has sufficient time to converge to within a bounded error ball of a desired state in between consecutive switching/jumping phases. One notable example which utilizes dwell-time criterion for nonlinear systems is in [35], where it is shown that input-to-state induced norms should be bounded uniformly between switches. In terms of applications, dwell-time criterion for attaining exponential stability has been shown to be effective for robotic systems, in particular walking locomotion and flapping flight [36] as well as autonomous vehicle steering [37].

B. Paper Organization

We begin in Sec. II by setting up the specific form of SDE we consider throughout the work. Sec. III discusses background material needed to understand our main theorems, with a heavy focus on relating it to our problem setting rather than a direct relay of standard results. In Sec. IV, we present our main stochastic contraction theorems, Sec. IV-A for shot noise systems and Sec. IV-B for Lévy noise systems, and discuss their implications. In Section V, we use two examples to numerically demonstrate what we show in Sec. IV. Finally, we conclude the paper in Sec. VI.

II. Setup

We consider systems that can be expressed as SDEs of the following form:

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \sigma(t,\mathbf{x})dW(t) + \xi(t,\mathbf{x})dN(t) \tag{1}$$

where

- $f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a deterministic function in $\mathcal{C}^{(1,2)}$, i.e., f is once-differentiable in time and twice-differentiable in state.
- $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is the variation of the white noise which belongs in $\mathcal{C}^{(1,1)}$, while $W : \mathbb{R}^+ \to \mathbb{R}^d$ is a d-dimensional Brownian motion process. Collectively, the $\sigma(t,\mathbf{x})dW(t)$ denotes the additive *white noise* of the system.
- $\xi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^{n\times \ell}$, which also belongs in $\mathcal{C}^{(1,1)}$ describe the jumps that occur, and $N(t)$ is the ℓ-dimensional standard Poisson process with intensity λ. Collectively, the $\xi(t,\mathbf{x})dN(t)$ denotes the additive *shot noise* of the system. The meaning and notation associated with this term is discussed formally in Sec. III-A. Throughout this paper, we use the terms "shot", "jump", and "Poisson" noise interchangeably.

Note that if $\sigma(t,\mathbf{x}) \equiv 0$, we have the following *shot-noise SDE*

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \xi(t,\mathbf{x})dN(t) \tag{2}$$

while if $\xi(t,\mathbf{x}) \equiv 0$, we recover the *white-noise SDE*:

$$d\mathbf{x}(t) = f(t,\mathbf{x})dt + \sigma(t,\mathbf{x})dW(t) \tag{3}$$

We focus mostly on the shot noise case, before combining it with white noise to discuss the Lévy noise case.

III. Background Review

In this section, we briefly review the relevant background material for our main results, Theorem 1 and Corollary 1. Sec. III-A reviews the Poisson random measure and the Poisson integral to familiarize the reader with the notation, then presents an extended version of the well-known Campbell's formula to time-varying functions, which will be used in the proofs to both Theorem 1 and Corollary 1. In Sec. III-B, previous results on deterministic incremental stability and white noise stochastic contraction are mentioned, then new analogous definitions of stochastic contraction for shot and Lévy noise systems are created. We emphasize that the discussion in Sec. III-B, while perhaps standard to the reader, is important because it is used as motivation for choosing a Lyapunov-like function which is different from the norm-square form conventionally used to prove exponential convergence of many systems.

A. Poisson Random Measures and Stochastic Integration

The definition of the Poisson random measure and the Poisson integral that we use throughout the paper closely follows that of Theorem 2.3.5 in [38] and Definition 3.1 in [39].

Definition 1 (Poisson Random Measure). Let $E \subseteq \mathbb{N}^\ell$. We define random measure $N([0,t] \times E)$ on *jump space* E until some time $t > 0$ with *intensity measure* $\text{Leb}\times\nu$, where Leb denotes the standard Lebesgue measure (the measure in time) and ν is the probability measure on the jump space E (describing the distribution of the jumps). We denote the *intensity (parameter)* for the Poisson process corresponding to the intensity measure ν as λ. One can think of λ as the average number of arrivals over time. N is called a *Poisson random measure* if the following are satisfied:

1) if $E_1, \cdots, E_n$ are pairwise disjoint subsets of E, then $N([0,t] \times E_1), \cdots, N([0,t] \times E_n)$ are independent.
2) for each $E_i \subseteq E$, the random measure $N([0,t] \times E_i)$ has a corresponding Poisson process with intensity parameter $\lambda_i := (\nu(E_i)/\nu(E))\lambda$.

We denote the time of the ith arrival with random variable T_i. □

Definition 2 (Poisson Integral). Let $\xi : [0,\infty) \times \mathbb{R}^n \times E \to \mathbb{R}$ be a predictable, bounded Borel-measurable function and N be a Poisson random measure on $[0,t] \times E$ with intensity measure $\text{Leb} \times \nu$. We define the *Poisson integral* of ξ as follows:

$$I_\xi := \int_{[0,t]\times E} \xi(s, \mathbf{x}(s), y) N(ds, dy) \tag{4}$$

where $\mathbf{x}(t)$ is the solution to (2). □

We make use of the well-known *Campbell's formula* for the proof of our main results. The formula for time-invariant functions $\xi(y)$ is often presented in the literature, e.g., Section 3.2 of [40] and Proposition 2.7 of [41]. The lemma below is a simple extension of the formula to functions which are both time-varying and dependent on some underlying SDE dynamics $\xi(t, \mathbf{x}(t), y)$.

Lemma 1 (Campbell's Formula). Let $\xi : [0,\infty) \times \mathbb{R}^n \times E \to \mathbb{R}$ be a predictable, locally-bounded Borel-measurable function and $N([0,t] \times E)$ denote the Poisson random measure with intensity λ over the jump space E. If ξ also satisfies the integrability condition $\int_{[0,t]\times E} |\xi(s, \mathbf{x}(s), y)| ds\nu(dy) < \infty$ almost-surely, then

$$\mathbb{E}[I_\xi] = \int_{[0,t]\times E} \mathbb{E}[\xi(s, \mathbf{x}(s), y)] ds\nu(dy) \tag{5}$$

Remark 1. In Sec. IV, we make use of Lemma 1 when the Poisson process is standard, i.e., it is a simple counting random measure. Following the construction of (4):

$$\begin{aligned}\int_0^t \xi(s, \mathbf{x}(s)) dN(s) &= \sum_{0<t\leq T} \xi(t, \Delta\mathbf{x}(t)) \mathbb{1}\{\Delta N(t) \neq 0\} \\ &= \sum_{i=1}^{N(t)} \xi(T_i, \Delta\mathbf{x}(T_i))\end{aligned} \tag{6}$$

where $\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-)$, $\Delta N(t) = N(t) - N(t-)$, and the indicator in the second line of the equation determines whether or not a jump occurred at time t. We denote this special case by $N(t)$ without the argument y, since we have essentially isolated the jump as the multiplicative factor $\xi(T_i, \mathbf{x}(T_i))$ for each i; all we need to represent is the time of each arrival and the cumulative number of arrivals until time t. This representation has an intuitive interpretation: if we think of the noise process as a sequence of impulses where the ith impulse arrives at time T_i, then integrating a function ξ with respect to it over an interval of time $[0,t]$ would only pick out the values of ξ at $T_i \in [0,t]$. □

B. Previous Results on Notions of Stability

Definition 3 (Incremental Stability). For any nonlinear function $f \in \mathcal{C}^{(1,2)}$, the deterministic, noiseless system $\dot{\mathbf{x}} = f(t, \mathbf{x})$ is said to be *incrementally (globally exponentially) stable* if there exist constants $\kappa, \alpha > 0$ such that

$$\|\mathbf{x}(t) - \mathbf{y}(t)\|_m \leq \kappa \|\mathbf{x}_0 - \mathbf{y}_0\|_m e^{-\alpha t} \tag{7}$$

for any norm $m \geq 1$, all $t \geq 0$, and all solution trajectories $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of the system with respective initial conditions $\mathbf{x}_0$ and $\mathbf{y}_0$. We assume $\mathbf{x}_0 \neq \mathbf{y}_0$, otherwise the two trajectories are exactly the same for all t and (7) is trivially satisfied with equality. □

We denote $\delta\mathbf{z}$ to be the infinitesimal displacement length between the two trajectories, and represent it as a path integral by parametrizing it using a measure parameter $\mu \in [0,1]$:

$$\begin{aligned}\mathbf{y}(t) - \mathbf{x}(t) &= \int_{\mathbf{x}}^{\mathbf{y}} \delta\mathbf{z} = \int_0^1 \left(\frac{\partial\mathbf{z}}{\partial\mu}\right) d\mu \\ \implies \|\mathbf{y}(t) - \mathbf{x}(t)\| &\leq \int_{\mathbf{x}}^{\mathbf{y}} \|\delta\mathbf{z}\| = \int_0^1 \left\|\frac{\partial\mathbf{z}}{\partial\mu}\right\| d\mu\end{aligned} \tag{8}$$

Its rate of change can be approximated by the dynamics $\delta\dot{\mathbf{z}} = (\partial_{\mathbf{x}} f)\delta\mathbf{z}$, where $\partial_{\mathbf{x}} f$ denotes the derivative of $f(t, \mathbf{x})$ being taken with respect to the second argument. Oftentimes, performing a differential coordinate transform from $\delta\mathbf{z}$ to $\Theta\delta\mathbf{z}$, where $\Theta \in \mathbb{R}^{n\times n}$, makes it easier to verify (7).

The transformed virtual dynamics become

$$(d/dt)\delta\mathbf{z} = F\delta\mathbf{z}, \quad F := (\dot{\Theta} + \Theta(\partial_{\mathbf{x}} f))\Theta^{-1} \tag{9}$$

In many cases, $\Theta := \Theta(t, \mathbf{x})$ is dependent on time and state and we can assume the existence of bounds:

$$\begin{aligned}\underline{h} &= \inf_{t,\mathbf{x}} \lambda_{\min}(\Theta), \ \overline{h} = \sup_{t,\mathbf{x}} \lambda_{\max}(\Theta) \\ h' &= \sup_{t,\mathbf{x},i} \|(\partial_x\Theta)_i\|, h'' = \sup_{t,\mathbf{x},i,j} \left\|(\partial_x^2\Theta)_{ij}\right\|\end{aligned} \tag{10}$$

Definition 4 (Basic Contraction). An equivalent way to say that a system is incrementally stable is to say that it is *contracting* with respect to a differential coordinate transform $\Theta(t, \mathbf{x})$ and convergence rate $\alpha > 0$. For instance, the inequality (7) implies that the system $\dot{\mathbf{x}}(t) = f(t, \mathbf{x})$ is *contracting* under $\Theta(t, \mathbf{x}) = I$ with rate α. But instead of using Definition 3 directly, it is oftentimes easier to prove that a system is contracting by analyzing the time-derivative of a choice of Lyapunov-like function, and invoking the Comparison

Lemma (see Sec. 9.3 of [21]). For deterministic systems, one common choice is the modified path length:

$$V(t, \mathbf{z}, \delta\mathbf{z}) = \int_0^1 \|\Theta(t, \mathbf{z}(\mu, t))\delta\mathbf{z}\| \, d\mu \tag{11}$$

If there exists a $\Theta(t, \mathbf{x})$ and $\alpha > 0$ such that the following condition is satisfied:

$$\dot{\Theta}(t, \mathbf{x}) + \Theta(t, \mathbf{x}) \left(\frac{\partial f}{\partial \mathbf{x}}\right) \leq -\alpha\Theta(t, \mathbf{x}) \tag{12}$$

then the system is contracting. Another common choice is the norm-squared Lyapunov-like function:

$$V(t, \mathbf{z}, \delta\mathbf{z}) = \int_0^1 \delta\mathbf{z}^T S(t, \mathbf{z}(\mu, t))\delta\mathbf{z} d\mu \tag{13}$$

where $S(t, \mathbf{x}) := \Theta^T\Theta(t, \mathbf{x})$. If there exists such a $S(t, \mathbf{x})$ and $\alpha > 0$ so that the following condition is satisfied:

$$\left(\frac{\partial f}{\partial \mathbf{x}}\right)^T S(t, \mathbf{x}) + S(t, \mathbf{x}) \left(\frac{\partial f}{\partial \mathbf{x}}\right) + \dot{S}(t, \mathbf{x}) \leq -2\alpha S(t, \mathbf{x}) \tag{14}$$

then the system is contracting. □

Remark 2. The two conditions (12) and (14) are exactly the same: both imply that all system trajectories converge globally exponentially to a single trajectory with a convergence rate equal to α. Yet, there is a very specific reason why we chose to distinguish between the two forms. Standard contraction results for deterministic systems, e.g. Definition 2 of [42], present contraction in terms of only (14), derived from using the Lyapunov-like function (13). As shown in Theorem 3 of [15] and Lemma 2 of [16], (13) can also be used to prove contraction in white noise systems (3). However, for shot noise SDEs (2), such a Lyapunov-like function cannot be used to prove exponential convergence due to (16): Fig. 2 intuitively shows how the square of differences can be unbounded in the simple 1D case with identity metric. Therefore, our results Theorem 1 and Corollary 1 uses the path length form (11) instead. □

Incremental stability for deterministic systems has been established as a concept of convergence between different solution trajectories starting from different initial conditions [42]. In the stochastic setting, the difference between trajectories also arises from using different noise processes. Stochastic contraction for white noise systems (3) is defined in Definition 2 of [15]. We define a similar notion of stochastic contraction for the shot noise system (2) and the Lévy noise SDE (1).

Definition 5 (Stochastically Contracting)**.** The system (1) is said to be *stochastically contracting* if:

1) the unperturbed system $\dot{\mathbf{x}} = f(t, \mathbf{x})$ is contracting with some differential coordinate transform $\Theta(t, \mathbf{x})$ and convergence rate α.
2) there exist constants $\gamma, \eta > 0$ such that $\sup_{t,\mathbf{x}} \|\sigma(t, \mathbf{x})\|_F \leq \gamma$ and $\sup_{t,\mathbf{x}} \|\xi(t, \mathbf{x})\|_F \leq \eta$.

Separately, (3) is stochastically contracting under the same two conditions above without η and ξ in condition 2, while (2) removes mention of γ and σ. □

IV. Stochastic Contraction Theorems for Shot and Lévy Noise

In this section, we present the two main results of this paper. In Sec. IV-A, we state Theorem 1, a stochastic contraction criterion for shot noise system (2), then prove it using the path integral Lyapunov-like function form (11). We also draw attention to a tradeoff between

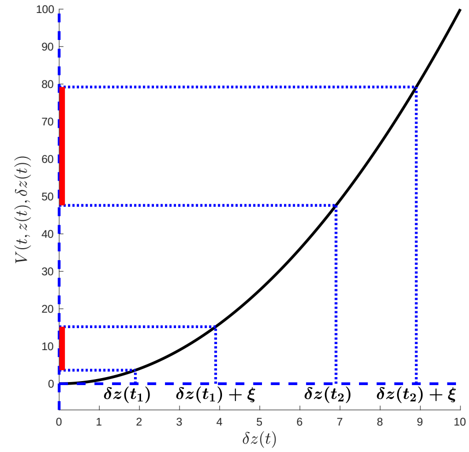

Fig. 2: Scalar example of the norm-squared Lyapunov function (13) violating (16). Although the height of the jumps, indicated on the x-axis, are η at both times t_1 and t_2, the difference of $V(t, z(t), \delta z(t))$ at time t_2 is larger than the difference at time t_1, as shown by the red lines on the y-axis. Thus, there is no way to bound the squared difference.

the intensity λ of the shot noise and the size of the steady-state error ball, which allows our derived stability criteria to be interpreted as a condition on the class shot noise process with mean interarrival time large enough for the system to converge within the bounded error ball of each other between consecutive jumps. In Sec. IV-B, the stochastic contraction theorem for Lévy noise systems is presented and proven via combination of the white and the shot noise cases. However, in order to take advantage of the affine representation of Lévy noise prescribed by the Lévy-Khintchine Decomposition Theorem, we need a derivation of the white noise case which is different from what was presented in the previous literature [15], [16]. Namely, the path integral Lyapunov-like function (11) is used instead of (13), and a noise-perturbed trajectory is compared against an unperturbed one. Appropriate modifications to the white noise case are made before presenting the Lévy noise case in combination.

A. Shot Noise Case

Consider two trajectories of a system: $\mathbf{x}(t)$ a solution of (2), and $\mathbf{y}(t)$ a solution of the deterministic $\dot{\mathbf{y}} = f(t, \mathbf{y})$. We define the parameter $\mu \in [0, 1]$ such that

$$\begin{aligned} \mathbf{z}(\mu = 0, t) &= \mathbf{x}(t), & \mathbf{z}(\mu = 1, t) &= \mathbf{y}(t) \\ \xi_{\mu=0}(t, \mathbf{z}) &= \xi(t, \mathbf{x}), & \xi_{\mu=1}(t, \mathbf{z}) &= 0 \\ N_{\mu=0}(t, \mathbf{z}) &= N(t), & N_{\mu=1}(t, \mathbf{z}) &= 0 \end{aligned} \tag{15}$$

with virtual system $d\mathbf{z}(\mu, t) = f(t, \mathbf{z}(\mu, t))dt + \xi_\mu(t, \mathbf{z}(\mu, t))dN_\mu(t)$ and the virtual dynamics $d\delta\mathbf{z}(\mu, t) = F\delta\mathbf{z}(\mu, t)dt + \delta\xi_\mu(t, \mathbf{z})dN_\mu$ with the generalized Jacobian F from (9), where $\delta\xi_\mu = \left[\frac{\partial \xi_{\mu,1}}{\partial \mathbf{z}}\delta\mathbf{z}, \cdots, \frac{\partial \xi_{\mu,\ell}}{\partial \mathbf{z}}\delta\mathbf{z}\right]$ and $\xi_\mu := [\xi_{\mu,1}, \cdots, \xi_{\mu,\ell}]$, where $\xi_{\mu,i}$ is the ith column of ξ_μ.

Theorem 1 (Shot Noise Contraction Theorem)**.** Suppose that (2) is stochastically contracting in the sense of Definition 5 under a differential coordinate transform $\Theta(t, \mathbf{x})$ which satisfies (10). Further assume that the initial conditions adhere to some probability distribution $p(\mathbf{z}_0) = p(\mathbf{x}_0, \mathbf{y}_0)$, where p is either compactly-supported, or is a distribution with finite second moment. Moreover, for the Lyapunov-like path length function (11), suppose there exists a continuously differentiable function $c_{\mathbf{z}_0} : \mathbb{R}^+ \to \mathbb{R}^+$, which is bounded for each fixed initial condition $\mathbf{z}_0$ such that

$$\mathbb{E}_{\mathbf{z}_0}\left[V(t,\mathbf{z}(t),\delta\mathbf{z}(t)) - V(t-,\mathbf{z}(t-),\delta\mathbf{z}(t-))\right] \leq c_{\mathbf{z}_0}(t) \quad (16)$$

and such that $\mathbb{E}[c_{\mathbf{z}_0}(t)] := \int c_{\mathbf{z}_0}(t)dp(\mathbf{z}_0) \leq c(t)$ for some continuously differentiable, bounded, deterministic function $c : \mathbb{R}^+ \to \mathbb{R}^+$. With the setup and assumptions above, we have the following inequality:

$$\mathbb{E}[\|\mathbf{y}-\mathbf{x}\|] \leq \frac{1}{\underline{h}}\mathbb{E}[\|\mathbf{y}_0 - \mathbf{x}_0\|]e^{-\beta_s t} + \frac{\kappa_s(t)}{\underline{h}} \quad (17)$$

where

$$\beta_s := \alpha \quad (18a)$$

$$\kappa_s(t) := \frac{\eta\lambda}{\beta_s}\left(c(t) - c(0)e^{-\beta_s t} - \int_0^t c'(s)e^{-\beta_s(t-s)}ds\right) \quad (18b)$$

λ is the intensity of the Poisson process $N(t)$, $\underline{h}$ is defined in (10), $c'(t)$ is the derivative of $c(t)$, α is the deterministic contraction rate, and η is the bound on the magnitude of the jumps $\xi(t,\mathbf{x})$ described by Definition 5.

Proof: We apply Itô's formula to our Lyapunov-like function (11). The version of the formula for scalar processes is standard, e.g. see Theorem 32 of [43] or Theorem 3.7 of [44], and an extension to multiple dimensions is straightforward. (11) becomes

$$\begin{aligned} V(t,\mathbf{z},\delta\mathbf{z}) &= V(0,\mathbf{z}_0,\delta\mathbf{z}_0) + \int_0^t \partial_t V(s,\mathbf{z},\delta\mathbf{z})ds \\ &+ \int_0^t \sum_{i=1}^n \left[\partial_{z_i}V(s,\mathbf{z},\delta\mathbf{z})f_i(s,\mathbf{z}) + \partial_{\delta z_i}V(s,\mathbf{z},\delta\mathbf{z})\left(F\delta\mathbf{z}\right)_i\right] \\ &+ \sum_{s\leq t}\left(V(s,\mathbf{z}(s),\delta\mathbf{z}(s)) - V(s-,\mathbf{z}(s-),\delta\mathbf{z}(s-))\right) \end{aligned} \quad (19)$$

where F is the generalized Jacobian from (9), and the subscript of i in $\xi_{\mu,i}$ (and other similar notation) denotes the ith component of the respective vector. Note that these are dimension $1 \times d$, and the Poisson dN_μ is dimension $d \times 1$, so the overall product is a scalar, as expected. Note that the terms of Itô's formula which correspond to the continuous part of the process disappear because (2) has none, simplifying the expression considerably.

A bound on the first three terms are derived directly from deterministic contraction of an unperturbed system $\dot{\mathbf{x}} = f(t,\mathbf{x})$, and we can use the condition (12). A bound on the last term comes from the fourth condition (16) and by taking the expected value $\mathbb{E}_{\mathbf{z}_0}$ so that Campbell's formula can be applied. Applying $\mathbb{E}_{\mathbf{z}_0}$ across the entire equation (19), and combining the upper bounds on the deterministic and shot noise terms, we get:

$$\begin{aligned} &\mathbb{E}_{\mathbf{z}_0}\left[V(t,\mathbf{z},\delta\mathbf{z})\right] - V(0,\mathbf{z}_0,\delta\mathbf{z}_0) \\ &\qquad \leq -\alpha\int_0^t \mathbb{E}_{\mathbf{z}_0}[V(s,\mathbf{z},\delta\mathbf{z})]ds + \eta\lambda c_{\mathbf{z}_0}(t) \end{aligned} \quad (20)$$

where α is the contraction rate of the unperturbed system.

We can then obtain a bound on the solution $\mathbb{E}_{\mathbf{z}_0}[V(t,\mathbf{z},\delta\mathbf{z})]$ using the Comparison lemma. Integrating (20) over the probability distribution $p(\mathbf{z}_0)$ and using the fact that $\underline{h}\mathbb{E}\left[\|\mathbf{y}(t)-\mathbf{x}(t)\|\right] \leq \mathbb{E}\left[\mathbb{E}_{\mathbf{z}_0}[V(t,\mathbf{z},\delta\mathbf{z})]\right]$ where $\mathbb{E}\left[\|\mathbf{y}-\mathbf{x}\|\right] := \int \|\mathbf{y}-\mathbf{x}\|\, dp(\mathbf{x}_0,\mathbf{y}_0)$ gives us our desired bound (17), with β_s as in (18a) and $\kappa_s(t)$ as in (18b). An additional step using integration-by-parts is taken to explicitly show the dependence on $1/\beta_s$, which intuitively tells us that a slower convergence rate corresponds to a larger bounded error ball. ■

Remark 3. Note that both the shot noise system and the deterministic system have contraction rate α. This is because in between consecutive jumps, the shot noise system behaves exactly as the deterministic system. However, the difference between the two is the nonzero bounded error ball $\kappa_s(t)$ (18b). Furthermore, note that $\kappa_s(t) \propto \lambda$, and the interarrival times $T_i - T_{i-1} \sim \text{Exp}(1/\lambda)$; indeed, shorter interarrival times correspond to a larger error ball due to the more rapid accumulation of deviations away from the nominal trajectory. We explore this tradeoff numerically using the example in Sec. V-A. □

B. Lévy Noise Case

Recall that a process $L(t)$ is said to be a *Lévy process* if all paths of L are right-continuous and left-limit (rcll), $\mathbb{P}(L(0) = 0) = 1$, and L has stationary and independent increments [39], [45]. Note that both Gaussian white noise and Poisson shot noise separately are Lévy processes. As such, a linear combination of white and Poisson noise, such as in Fig. 1, is also a Lévy process. In particular, we can invoke the *Lévy-Khintchine Decomposition Formula*, stated formally in Theorem 1.6 of [39] or Theorem 2.7 of [44], to decompose any Lévy noise process into an affine combination of white and shot noise. Analogously, the white noise contraction theorem [16] and Theorem 1 further direct us into a contraction condition for stochastic systems perturbed by Lévy noise.

Consider two trajectories of a system: $\mathbf{x}(t)$ a solution of (1), and $\mathbf{y}(t)$ a solution of the deterministic $d\mathbf{y}(t) = f(t,\mathbf{y})dt$. We define the parameter $\mu \in [0,1]$ such that (15) holds and

$$\begin{aligned} \sigma_{\mu=0}(t,\mathbf{z}) &= \sigma(t,\mathbf{x}), & \sigma_{\mu=1}(t,\mathbf{z}) &= 0 \\ W_{\mu=0}(t,\mathbf{z}) &= W(t), & W_{\mu=1}(t,\mathbf{z}) &= 0 \end{aligned} \quad (21)$$

The virtual system for this setting amounts to $d\mathbf{z}(\mu,t) = f(t,\mathbf{z}(\mu,t))dt + \sigma_\mu(t,\mathbf{z}(\mu,t))dW_\mu(t) + \xi_\mu(t,\mathbf{z}(\mu,t))dN_\mu(t)$ and the virtual dynamics $d\delta\mathbf{z} = F\delta\mathbf{z}dt + \delta\sigma_\mu dW_\mu + \delta\xi_\mu dN_\mu$, with again F as in (9).

Analogous to the shot noise parameters (18), denote β_w, κ_w and β_ℓ, κ_ℓ to be the contraction rate and steady-state error bound for the white noise SDE (3) and the Lévy noise SDE, respectively. Although one possible exact form of β_w and κ_w can be found in [15], [16], we do not use those versions for the following two reasons. First, there is a discrepancy between the Lyapunov-like functions used to prove contraction for the white noise case in [15], [16] and the shot noise case; this is detailed in Remark 2. Second, both [15], [16] consider the difference between two noise-perturbed trajectories – $\mathbf{x}(t)$, solution to (3) with noise term $\sigma_1(t,\mathbf{x})dW_1(t)$, and $\mathbf{y}(t)$, solution to (3) with $\sigma_2(t,\mathbf{x})dW_2(t)$ – instead of one noise-perturbed trajectory against the deterministic trajectory. With these two differences addressed, our new parameters for the white noise case become:

$$\beta_w := \alpha - \frac{h''\gamma^2}{2\underline{h}}, \quad \kappa_w := \frac{h'\gamma^2}{2\beta_w}\left(1 - e^{-\beta_w t}\right) \quad (22)$$

We will formally derive how (22) was obtained as a part of the proof to our main corollary.

Corollary 1 (Stochastic Contraction Theorem for General Lévy Noise)**.** With the setup described above, suppose (1) is stochastically contracting in the sense of Definition 5. Furthermore, we impose the same assumptions as in Theorem 1: the existence of a continuously-differentiable $c(t)$ from the construction of (16), metric bounds (10), and the distribution on the initial conditions $p(\mathbf{z}_0) = p(\mathbf{x}_0,\mathbf{y}_0)$ with either compact support or finite second moment. Then the following inequality is satisfied:

$$\mathbb{E}[\|\mathbf{y}-\mathbf{x}\|] \leq \frac{1}{\underline{h}}\mathbb{E}\left[\|\mathbf{y}_0 - \mathbf{x}_0\|\right]e^{-\beta_\ell t} + \frac{\kappa_\ell(t)}{\underline{h}} \quad (23)$$

where

$$\beta_\ell := \alpha - \frac{h''\gamma^2}{2\underline{h}} \tag{24a}$$

$$\kappa_\ell(t) := \frac{\eta\lambda}{\beta_\ell}\left(c(t) - c(0)e^{-\beta_\ell t} - \int_0^t c'(s)e^{-\beta_\ell(t-s)}ds\right) + \frac{h'\gamma^2}{2\beta_\ell}\left(1 - e^{-\beta_\ell t}\right) \tag{24b}$$

λ is the intensity of the Poisson process $N(t)$, $\underline{h}, h', h''$ are defined in (10), $c'(t)$ is the derivative of $c(t)$, α is the deterministic contraction rate, and γ and η are defined in the second condition of Definition 5.

Proof: Applying Itô's formula to (13):

$$V(t,\mathbf{z},\delta\mathbf{z}) = V(0,\mathbf{z}_0,\delta\mathbf{z}_0) + \int_0^t \partial_t V(s,\mathbf{z},\delta\mathbf{z})ds + \int_0^t \sum_{i=1}^n \left[\partial_{z_i} V(s,\mathbf{z},\delta\mathbf{z}) f_i(s,\mathbf{z}) + \partial_{\delta z_i} V(s,\mathbf{z},\delta\mathbf{z})\left(F\delta\mathbf{z}\right)_i\right] + \tag{25a}$$

$$\int_0^t \sum_{i=1}^n \left[\partial_{z_i} V(s,\mathbf{z},\delta\mathbf{z})\sigma_{\mu,i}(\mathbf{z}) + \partial_{\delta z_i} V(s,\mathbf{z},\delta\mathbf{z})\delta\sigma_{\mu,i}(\mathbf{z})\right] dW_\mu(s)$$

$$+ \frac{1}{2}\left[\int_0^t \sum_{i,j=1}^n \frac{\partial^2 V}{\partial\delta z_i \partial\delta z_j}(s,\mathbf{z}(s-),\delta\mathbf{z}(s-))d\left[\delta\mathbf{z}_i,\delta\mathbf{z}_j\right]^c \right. \tag{25b}$$

$$+ \int_0^t \sum_{i,j=1}^n \frac{\partial^2 V}{\partial z_i \partial\delta z_j}(s,\mathbf{z}(s-),\delta\mathbf{z}(s-))d\left[\mathbf{z}_i,\delta\mathbf{z}_j\right]^c \tag{25c}$$

$$\left. + \int_0^t \sum_{i,j=1}^n \frac{\partial^2 V}{\partial z_i \partial z_j}(s,\mathbf{z}(s-),\delta\mathbf{z}(s-))d\left[\mathbf{z}_i,\mathbf{z}_j\right]^c\right] \tag{25d}$$

$$+ \sum_{s\le t}\left(V(s,\mathbf{z}(s),\delta\mathbf{z}(s)) - V(s-,\mathbf{z}(s-),\delta\mathbf{z}(s-))\right) \tag{25e}$$

where F is the generalized Jacobian from (9), and the brackets $[\cdot]^c$ denote the continuous-part quadratic variation. As in the proof of Theorem 1, a bound on the first three terms are derived directly from deterministic contraction of an unperturbed system $\dot{\mathbf{x}} = f(t,\mathbf{x})$, and we can use the condition (12). With $\mathbb{E}_{\mathbf{z}_0}[\cdot]$ taken, the bound on the last term (25e) comes from Campbell's formula and (16). When we take expectation over the entire equation, the white noise term between (25a) and (25b) disappears to 0 due to being a martingale with zero mean.

Simplifying the quadratic variation terms (25b) to (25d) requires computing the partial derivatives of V. Using matrix multiplication, the submultiplicativity property, and the fact that $\Theta(t,\mathbf{z})$ is independent of $\delta\mathbf{z}$, we obtain the following relationships for each fixed time t:

$$\frac{\partial^2 V}{\partial\delta z_i \partial\delta z_j} d\left[\delta\mathbf{z}_i,\delta\mathbf{z}_j\right]^c = 0, \text{ since } \frac{\partial^2 V}{\partial\delta z_i \partial\delta z_j} = 0$$

$$\frac{\partial^2 V}{\partial z_i \partial\delta z_j} d\left[\mathbf{z}_i,\delta\mathbf{z}_j\right]^c \le \int_0^1 \|\partial_z\Theta\| \sum_{k=1}^d \sigma_{\mu,ik}\delta\sigma_{\mu,jk} d\mu \le h'\gamma^2$$

$$\frac{\partial^2 V}{\partial z_i \partial z_j} d\left[\mathbf{z}_i,\mathbf{z}_j\right]^c \le \int_0^1 \left\|\partial_z^2\Theta\right\| \left\|\frac{\partial\mathbf{z}}{\partial\mu}\right\| \sum_{k=1}^d \sigma_{\mu,ik}\sigma_{\mu,jk} d\mu \le \frac{h''\gamma^2}{\underline{h}} V$$

where γ is the white-noise bound described in Definition 5.

Combining the bounds of each individual term in (25) and taking the expected value with respect to fixed initial condition $\mathbf{z}_0$ yields the following inequality:

$$\mathbb{E}_{\mathbf{z}_0}\left[V(t,\mathbf{z},\delta\mathbf{z})\right] - V(0,\mathbf{z}_0,\delta\mathbf{z}_0) \le -\left(\alpha - \frac{h''\gamma^2}{2\underline{h}}\right)\int_0^t \mathbb{E}_{\mathbf{z}_0}[V(s,\mathbf{z},\delta\mathbf{z})]ds + \frac{h'\gamma^2}{2} + \eta\lambda c_{\mathbf{z}_0}(t) \tag{26}$$

As in the proof to Theorem 1, we obtain a bound on the solution $\mathbb{E}_{\mathbf{z}_0}[V(t,\mathbf{z},\delta\mathbf{z})]$ using the Comparison lemma. Then we integrate (26) over the probability distribution $p(\mathbf{z}_0)$ and use the fact that $\underline{h}\mathbb{E}\left[\|\mathbf{y}(t) - \mathbf{x}(t)\|\right] \le \mathbb{E}\left[\mathbb{E}_{\mathbf{z}_0}[V(t,\mathbf{z},\delta\mathbf{z})]\right]$ to get our desired bound (23), with β_ℓ set as in (24a) and $\kappa_\ell(t)$ as in (24b). ■

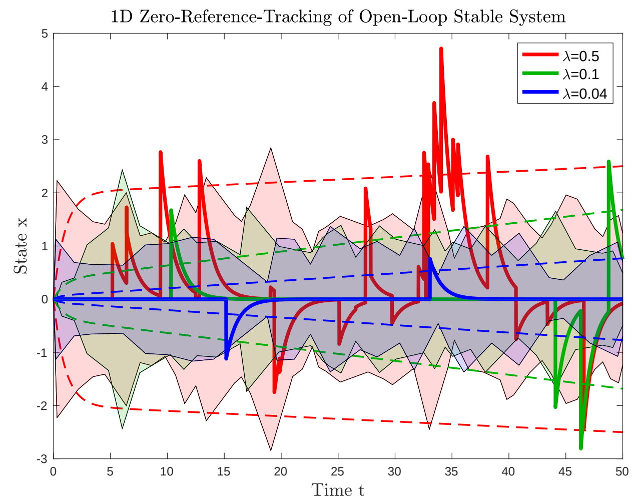

Fig. 3: Three system responses of a stable linear system with additive shot noise for varying values of λ. For each corresponding color of λ, the dashed line shows the outline of the envelope captured by the theoretical error bound derived from Theorem 1, and the transparently-shaded region is the empirical bound created from taking the maximum envelope over multiple Monte-Carlo trial trajectories.

Note that with the modified white noise parameters (22), β_ℓ and $\kappa_\ell(t)$ for the combined SDE are nearly a direct summation of the parameters (22) and (18). Specifically, β_ℓ requires an extra α term to be removed otherwise the convergence rate due to the deterministic part of the system would be counted twice:

$$\beta_\ell = \beta_s + \beta_w - \alpha = \beta_w \tag{27}$$

and $\kappa_\ell(t)$ is an exact summation of the two parts, with contraction rate β_ℓ used in place of β_w or β_s. Again, because $\beta_s = \alpha$, $\beta_\ell = \beta_w$ because in between consecutive jumps, the shot noise system (2) behaves exactly like as $\dot{\mathbf{x}} = f(t,\mathbf{x})$.

V. Numerical Examples

A. 1D Reference Tracking

Consider the system

$$dx(t) = ax(t)dt + \xi(t,x(t))dn(t) \tag{28}$$

where $n(t)$ is a 1D Poisson process with rate $\lambda > 0$, and $|\xi(x(t))| \le \eta$ for each t and a fixed constant $\eta > 0$. For a less restrictive example, we can impose the system to be controllable, then design a state-feedback controller of the form $u = -kx$ such that $a - k < 0$, but we choose $a < 0$ so that the system is stable from the start for its notational simplicity. The contraction metric $S(t,z)$ can be chosen to be the identity 1 and deterministic contraction rate $\alpha := |a|$. We parameterize z using μ, as before, and restrict $z(t)$ to be nonnegative ($z(t) \geq 0$ for all $t > 0$).

We simulate (28) using LQR to track the zero line (i.e., perform disturbance rejection). For a fixed $\eta > 0$, $\xi(t, x(t))$ is the random variable which takes value $\eta > 0$ with probability p, and $-\eta$ with probability $1 - p$. The theoretical bound of $\kappa_s(t)$ provided by Theorem 1 is

$$\kappa_s(t) = \eta\lambda \left[\left(\frac{1}{\beta_s^2} + \frac{1}{\beta_s} \right) e^{\beta_s t} - \frac{\beta_s t - \beta_s - 1}{\beta_s^2} \right] \tag{29}$$

We visualize our results in Fig. 3 for the deterministic contraction rate $\alpha := |a| = 2$ and $\eta = 4$. Three simulated trajectories of the noise-perturbed system, with $\lambda = 1/2$ (red), $1/10$ (green), $1/25$ (blue), are shown in solid lines. Plotted in dashed lines with their respective colors is the upper-bound of the envelope captured by (29) for each respective value of λ, and the light, transparent shaded region is the empirical κ_s bound created by generating 20 Monte-Carlo trial trajectories and taking the maximum envelope overall. Due to the symmetry in jump heights ξ, the positive envelope is taken and reflected across the time-axis. Note that the tradeoff described in Remark 3 is apparent: a larger λ corresponds to a wider envelope and more jumps occur for the fixed duration of time.

The envelopes illustrated capture the bound around the zero line that the trajectories converge towards; it is entirely feasible for the shot noise perturbations to bring trajectories outside of the envelope. That being said, the simulations demonstrate the conservativeness of the theoretical κ_s bound from Theorem 1; in this simple case, even trajectories corresponding to $\lambda = 0.5$ converge to within a very small distance about zero. Moreover, we observe the effects of Remark 3 in this experiment; to ensure sufficient convergence of the system trajectory to the reference, the bound relies on longer interarrival times $T_i - T_{i-1}$, which correspond to smaller intensity λ.

B. Dubins' Car

Consider the Lévy noise-perturbed nonlinear dynamics of the 5-dimensional simplified Dubins' car model [46]:

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{v} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v\cos(\theta) + w_1 + +n_1 \\ v\sin(\theta) + w_2 + n_2 \\ \omega \\ a + w_3 + n_3 \\ \gamma + w_4 + n_4 \end{bmatrix} \tag{30}$$

where $w(t) = (w_1, w_2, w_3, w_4)^T$ is the white noise part, $n(t) = (n_1, n_2, n_3, n_4)^T$ is the shot noise part. The control inputs are the translational acceleration a and angular acceleration γ. The interpretation of the noise process is as follows: noise in the positional components x and y arise as a result of road bumps and interference with obstacles, while noise in the control inputs arise due to faulty potentiometer readings which could affect the feedback loop. The covariance of the white noise component is $3I$, while the jumps are sampled uniformly from $[-10, 10]$ for the x and y positions and from $[-2, 2]$ for the control inputs. We fix $\lambda = 1/10$ and simulate 50 different Monte-Carlo trajectories of the system. For the first one second, we supply a constant linear acceleration $a = 5$ and constant angular acceleration $\gamma = \frac{\pi}{4}$.

We illustrate this experiment in Fig. 4. On the left, three sample trajectories are plotted in 3D space, with positional components x and y plotted against time t; the solid bright red line corresponds to the noiseless Dubins' car tracing out the circle over time, which we take as our reference. On the right, we project this space down to just the x and y-positions, and create an empirical (projected) κ_s bound by taking the convex hull over the 50 trajectories. The largest bounded set is partitioned into three regions, defined by varying shades of blue.

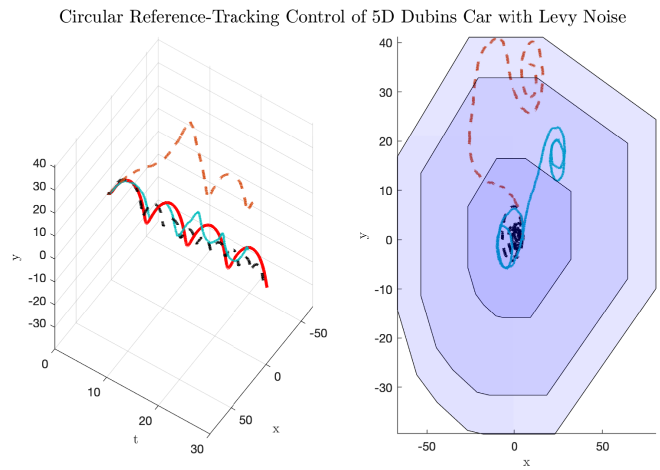

Fig. 4: Three sample open-loop white and shot noise perturbed trajectories corresponding to each of three disjoint partitions of the maximum norm ball (lightest shade of blue). The dashed-black trajectory lies fully within the smallest set, the solid-blue lies fully in the medium-sized set, and the dashed-orange trajectory remains in the largest bounded set. The solid-red line represents the trajectory of the unperturbed system.

The three trajectories in both plots correspond to one another by color and by line style: the orange-dashed trajectory is a sample among the 50 which belongs in the largest bounded set, the turquoise-solid trajectory is contained in the medium region, while the black-dashed trajectory is entirely within the smallest region. Overall, the fraction of trajectories that has its maximum deviation in each region for this particular experiment is given by $\left(\frac{25}{50}, \frac{16}{50}, \frac{9}{50}\right)$ lying in the smallest, medium, and largest regions, respectively. Since a majority of the trajectories lie in the smallest region, this shows that under different sample paths of the Lévy noise process, the system trajectory is able to remain reasonably bounded within the norm ball of the unperturbed trajectory, prescribed by Corollary 1.

VI. CONCLUSION

We have designed incremental stability criteria for nonlinear shot noise stochastic systems (2) in Theorem 1 and Lévy noise stochastic systems (1) in Corollary 1. Solution trajectories corresponding to different initial conditions from within a bounded set and different realizations of the noise process were shown to converge to within a bounded error ball of each other in the mean, and the proofs were done using a path length Lyapunov-like function (11) which was different from the conventional norm-square design (13). While the convergence rate for (2) is equal to that of the unperturbed system $\dot{\mathbf{x}} = f(t, \mathbf{x})$ because the shot noise system behaves exactly as the deterministic system in between consecutive jumps, the steady-state error ball $\kappa_s(t)$ defined in (18b) is nonzero. Moreover, Remark 3 shows that shorter interarrival times between jumps correspond to a larger error ball due to the faster accumulation of disturbances which drive the stochastic trajectory away from the nominal. This tradeoff is illustrated via simulation using a 1D simple linear system perturbed by three different intensities of shot noise in Sec. V-A. Furthermore, the convergence rate and steady-state error ball of (1) is shown to be nearly a direct combination of the parameters for the white noise case (22) and the shot noise case (18) as a result of the Lévy-Khintchine formula. This necessitated a change in the white noise contraction analysis from previous literature [15], [16] in two ways: 1) (11) is used instead of (13), and 2) a noise-perturbed trajectory is compared against an unperturbed one. In Sec. V-B, we empirically determine the steady-state bounded error ball of a 5D nonlinear system perturbed by Lévy noise to demonstrate these results.

We emphasize that the benefits of our work are two-fold: 1) the phenomenon of jumps in noise is understudied in the controls community compared to Gaussian white noise despite being equally prevalent, and 2) it segways into a methodological design process for stochastic controllers and observers that counter noise processes which include instantaneous, large-magnitude jumps, without necessarily resorting to computationally-intensive model-free techniques simply because the noise process is non-Gaussian. The study of an application in support of point 2) is the subject of future work.

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