Deriving Quantified Invariants Using Algorithmic Learning*

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18 Dec 2009
ROPAS Show & Tell

* Joint work with Yungbum, Cristina, Prof. Wang, and Prof. Yi

Invariants

Invariants

Problem : Find an invariant I for the Loop

$$\{\delta\}\ \texttt{while}\ \rho\ \texttt{do}\ S\ \texttt{end}\ \{\epsilon\}$$

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Invariant must satisfy the following conditions:

(A) $\delta \Rightarrow I$ (I holds when entering the loop)

(B) $I \wedge \rho \Rightarrow Pre(I, S)$ (I holds at each iteration)

precondition that guarantees I after execution of the statement S

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Note that we have $(I \wedge \neg\rho \Rightarrow \epsilon) \equiv (I \Rightarrow \epsilon \vee \rho)$

$$\delta \Rightarrow I \Rightarrow \epsilon \vee \rho$$

under approximation of an invariant

over approximation of an invariant

The CDNF Algorithm

The CDNF Algorithm

Nader Bshouty

The CDNF Algorithm

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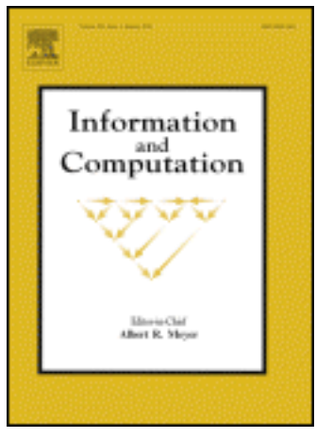
UNIVERSITY OF CALGARY

The CDNF Algorithm

Nader Bshouty



Journal of Information and Computation
1995

The CDNF Algorithm

Nader Bshouty

Exact Learning Boolean Functions via the Monotone Theory

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The University of Calgary

Calgary, Alberta, Canada T2N 1N4

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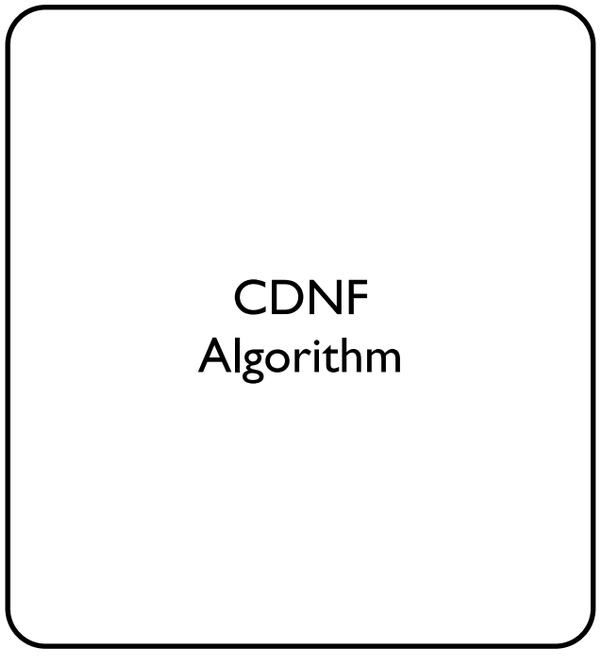
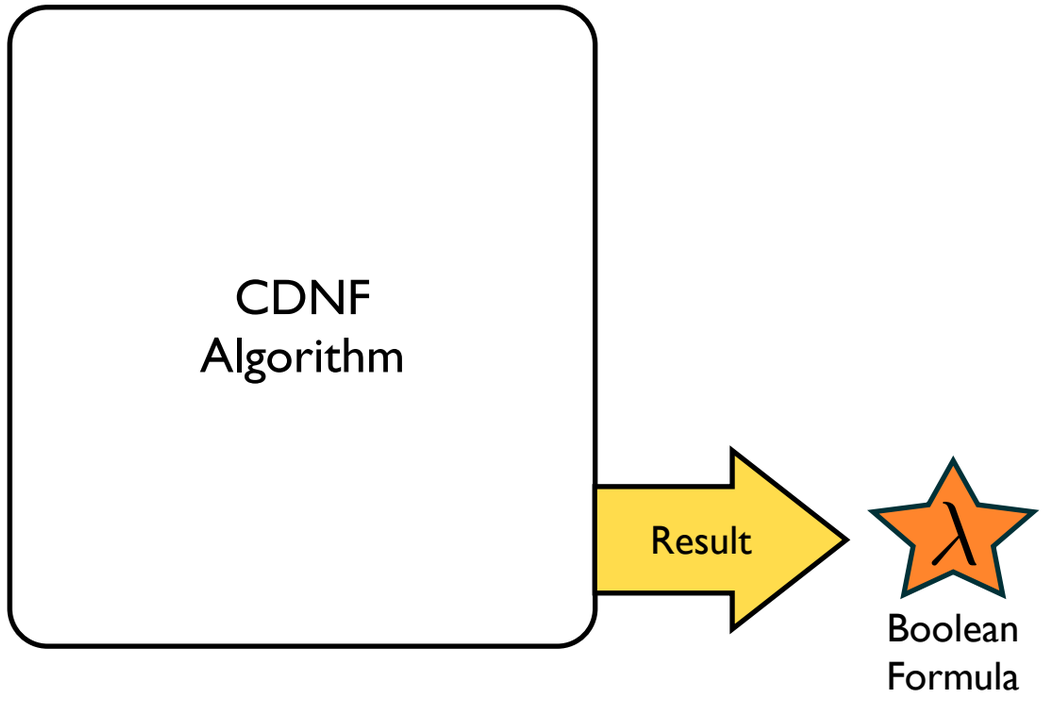
The CDNF Algorithm



Learning

Actively a Boolean formula
from membership and equivalence queries
in polynomial time

The CDNF Algorithm



Actively Learning a Boolean formula

from membership and equivalence queries

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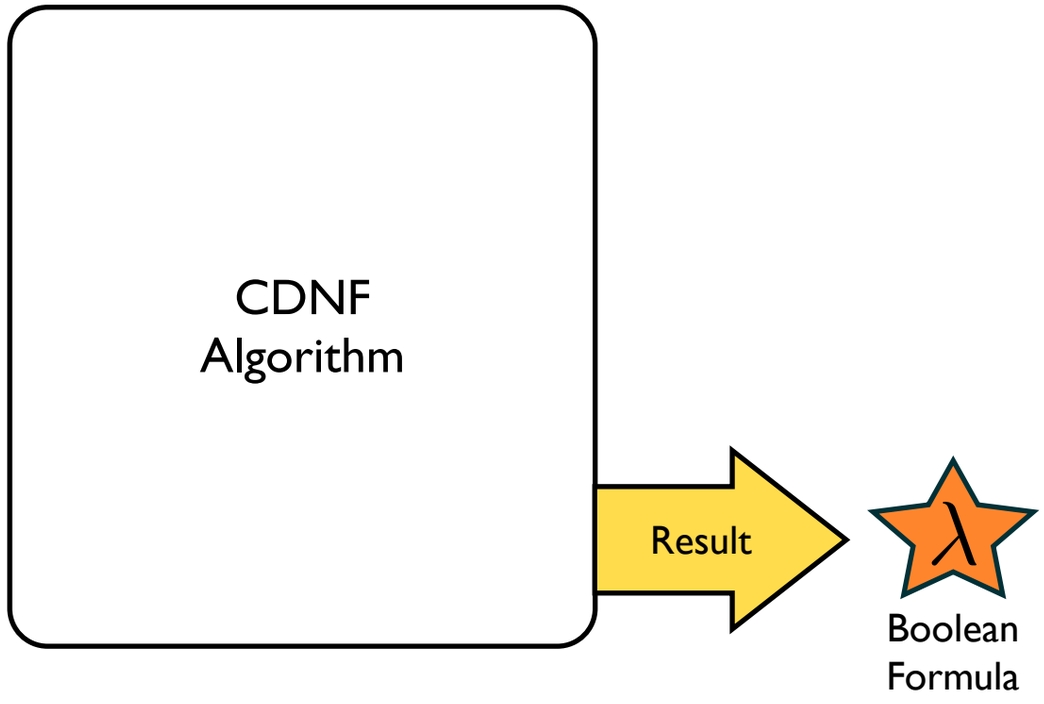

Actively Learning a Boolean formula

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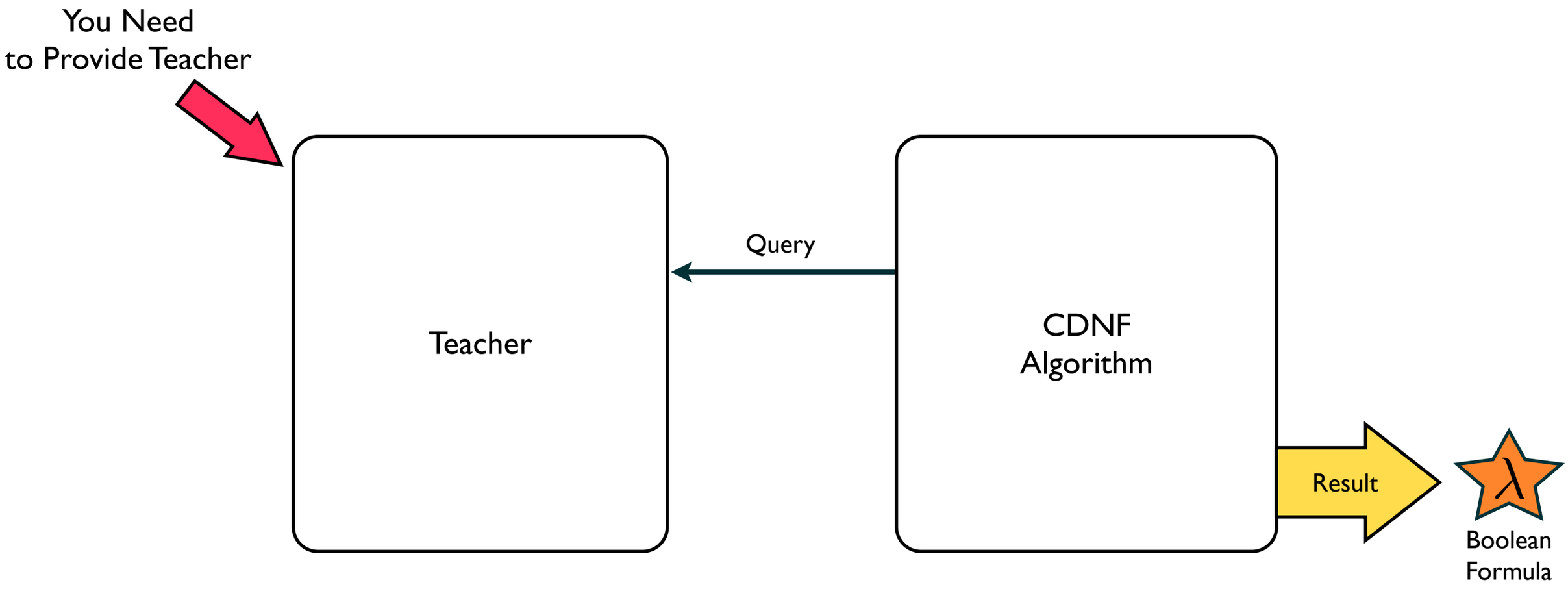

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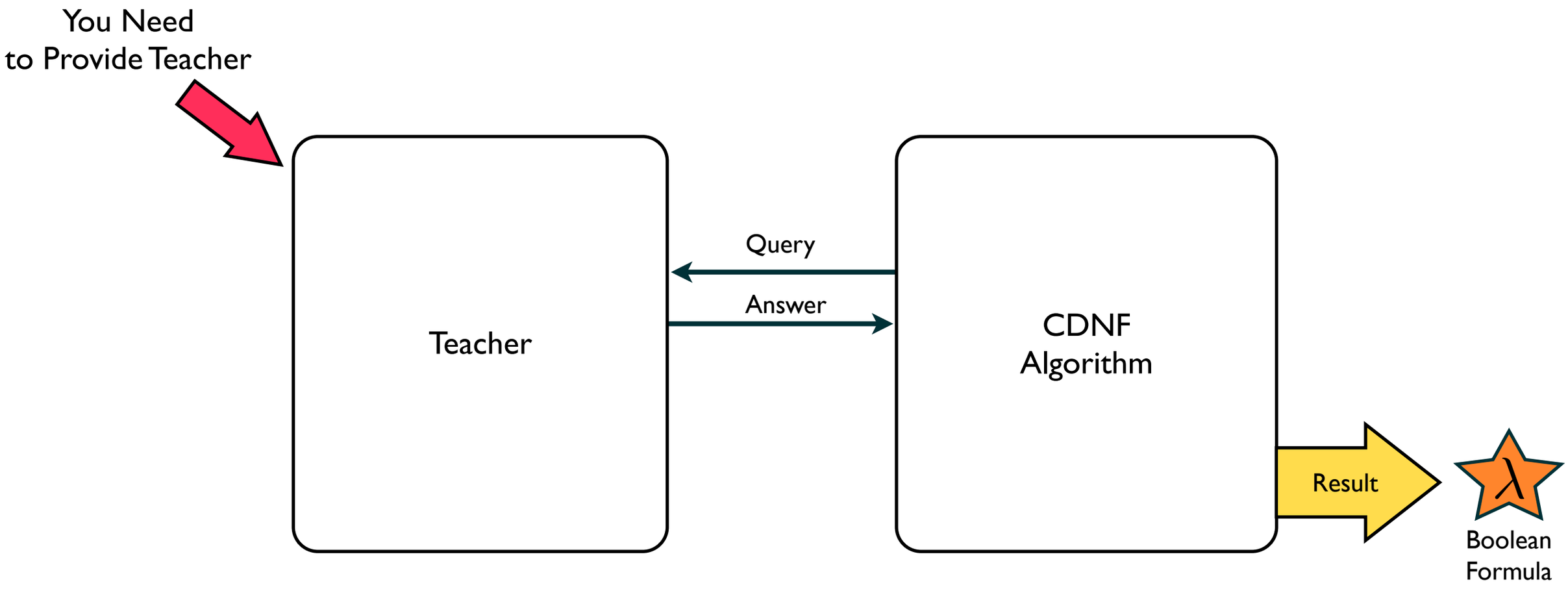

Actively Learning a Boolean formula
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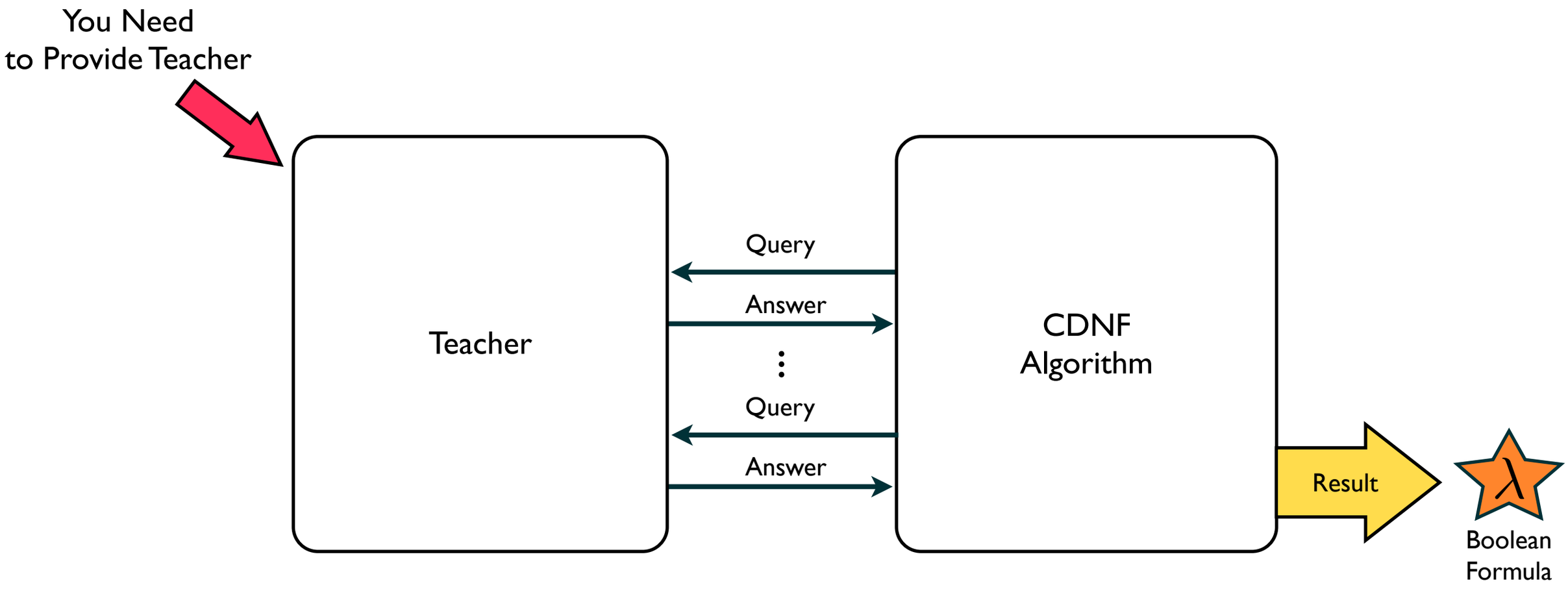


Actively Learning a Boolean formula
from membership and equivalence queries
in polynomial time

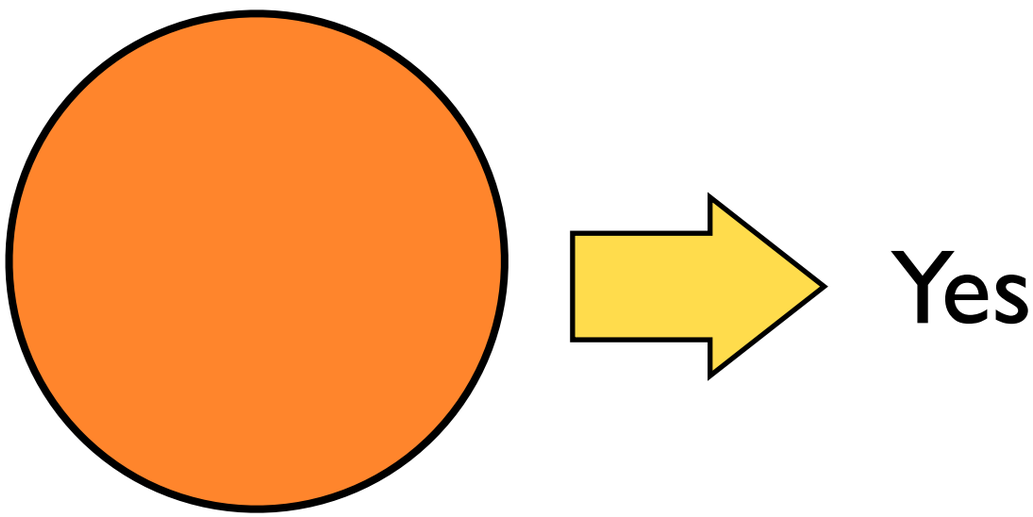
Two Types of Query

$$\mu \overset{?}{\models} \lambda$$

1. Membership Query $MEM(\mu)$ asks if the truth assignment μ satisfies λ.

$$MEM(\mu) = Yes \quad \text{if } \mu \models \lambda$$
$$MEM(\mu) = No \quad \text{if } \mu \not\models \lambda$$

Two Types of Query

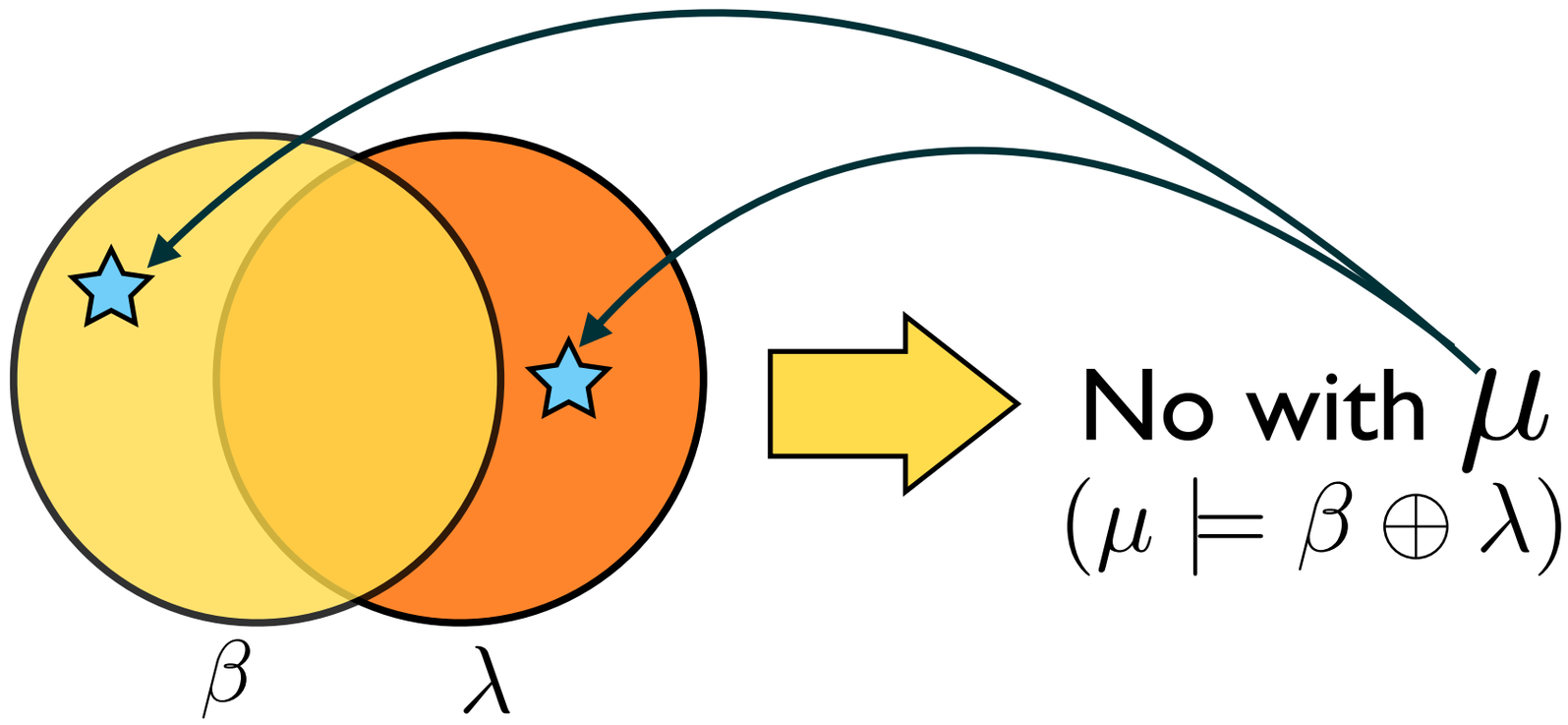


$$\beta \equiv \lambda$$

2. Equivalence Query $EQ(\beta)$ asks
   if the Boolean formula β is equivalent to λ,
   If not, the teacher returns a truth assignment
   as a counterexample γ.

$$EQ(\beta) = Yes \qquad \text{if } \beta \equiv \lambda$$

Two Types of Query



2. Equivalence Query $EQ(\beta)$ asks
   if the Boolean formula β is equivalent to λ,
   If not, the teacher returns a truth assignment
   as a counterexample μ.

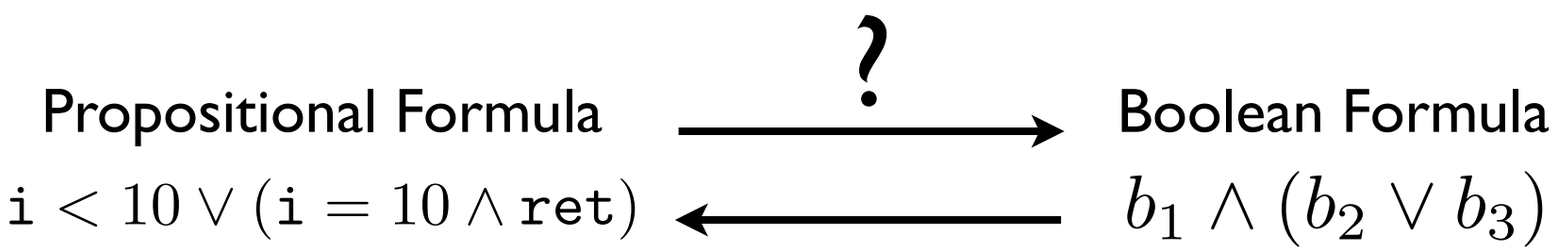
$$EQ(\beta) = Yes \qquad \text{if } \beta \equiv \lambda$$
$$EQ(\beta) = No \text{ with } \mu \qquad \text{if } \beta \not\equiv \lambda \wedge (\mu \models \beta \oplus \lambda)$$

Deriving Propositional Invariants using Algorithmic Learning

Predicate Abstraction

Problem:

We want to find a Propositional invariant
while the CDNF algorithm finds a Boolean formula.

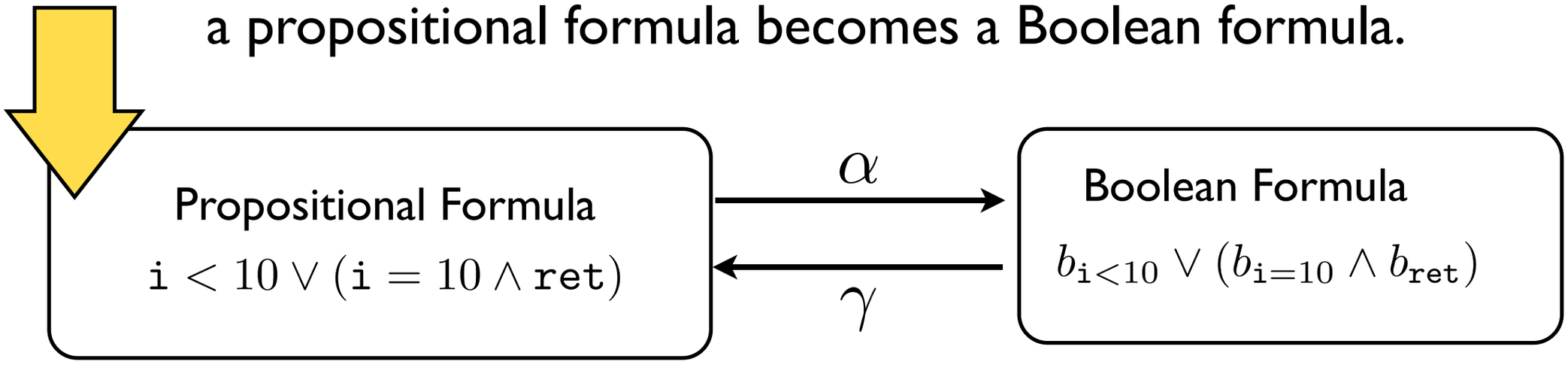


Predicate Abstraction

Solution:

Use predicate abstraction!
If we see atomic propositions as Boolean variables,
a propositional formula becomes a Boolean formula.

$\underline{\iota}, \overline{\iota}$



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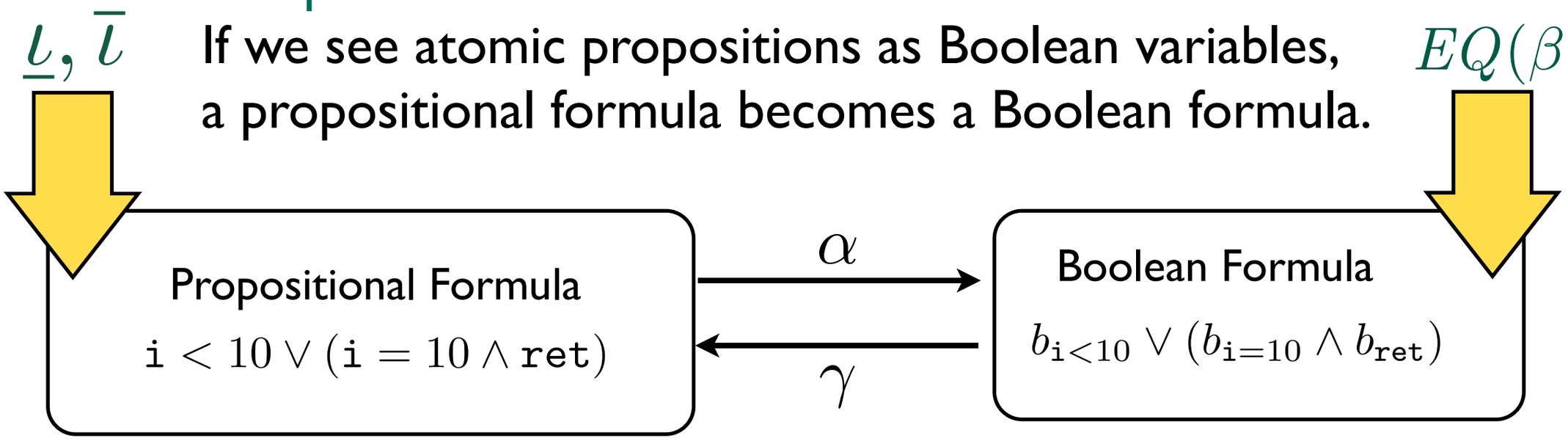

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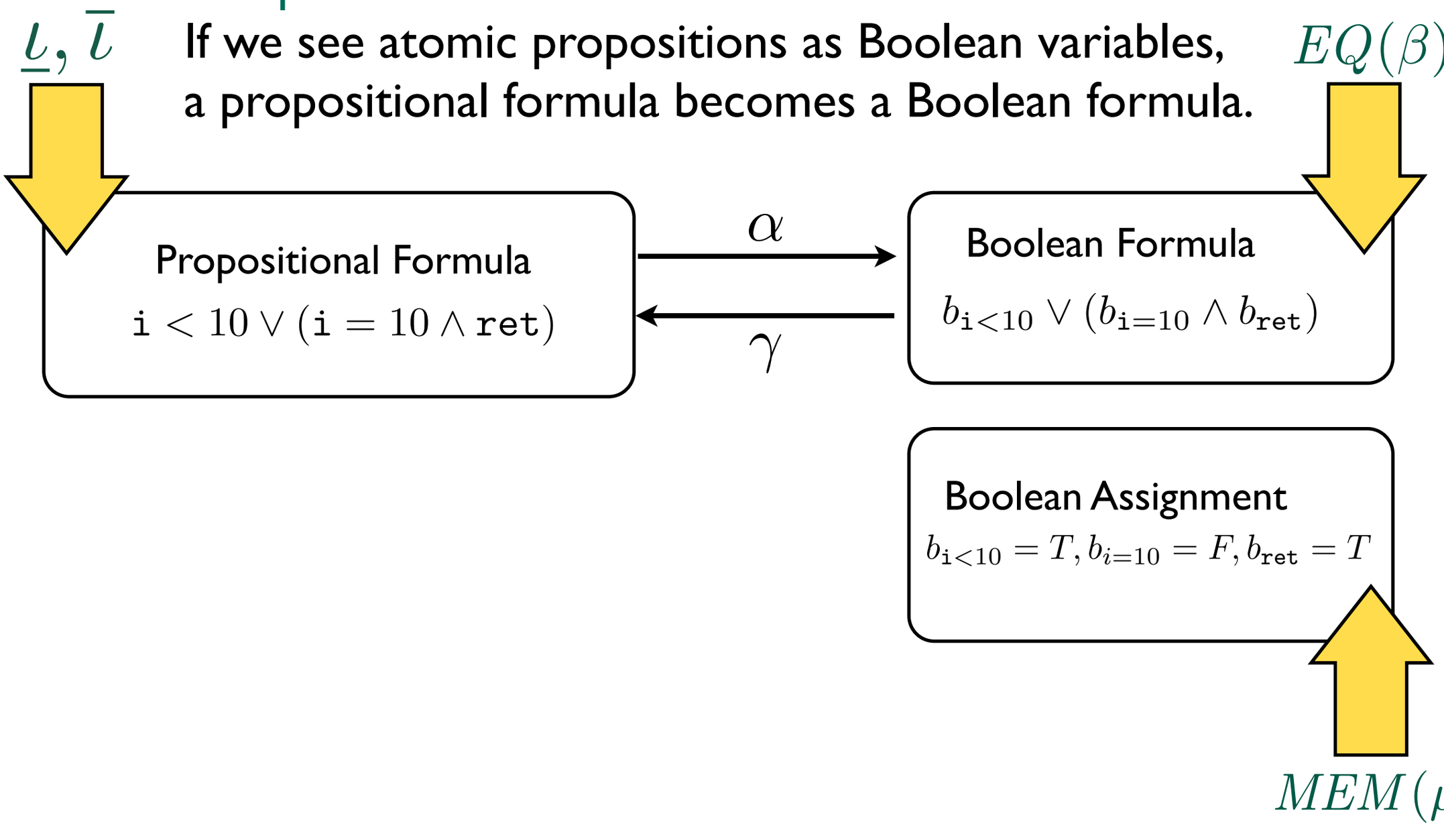

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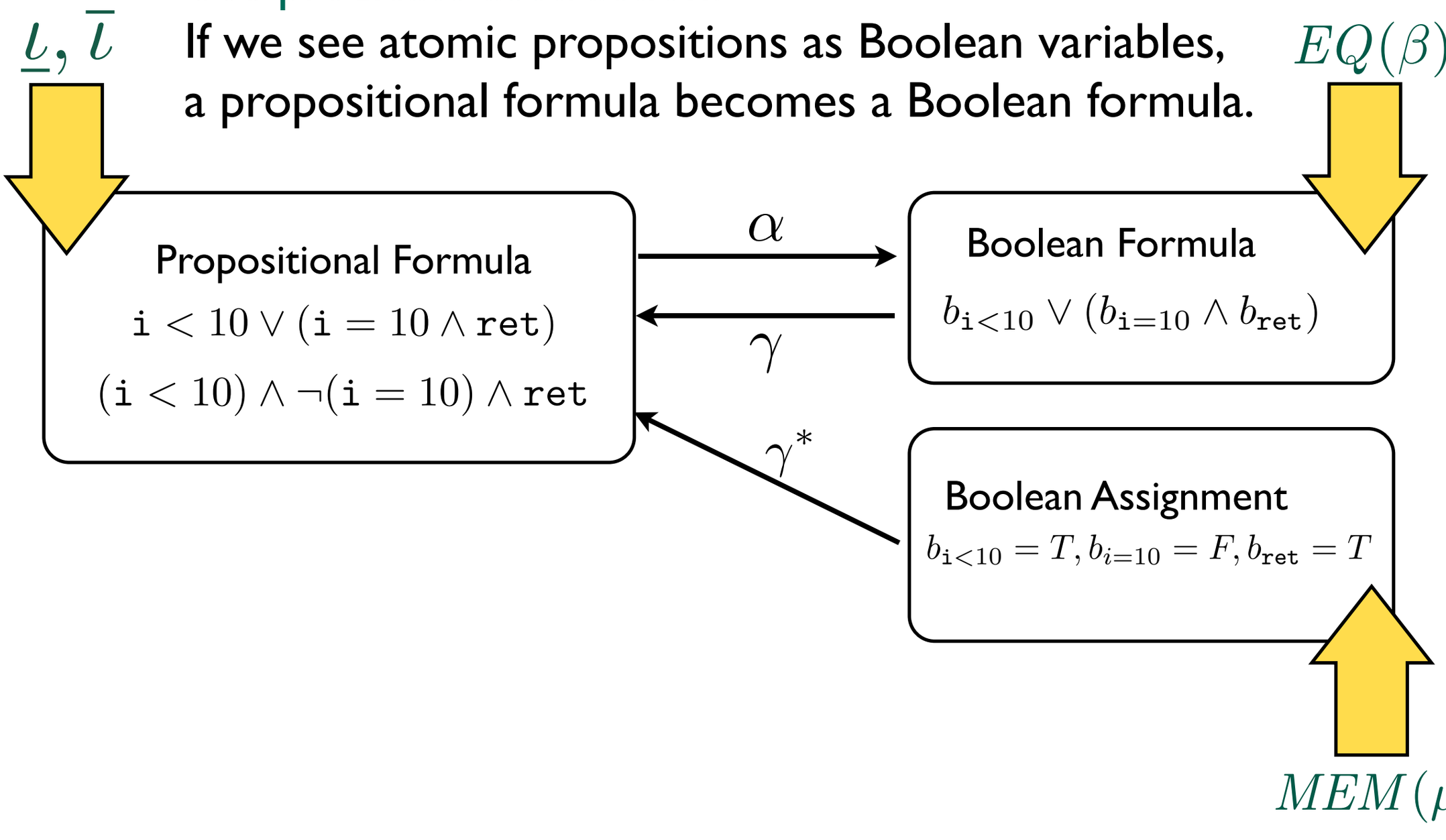

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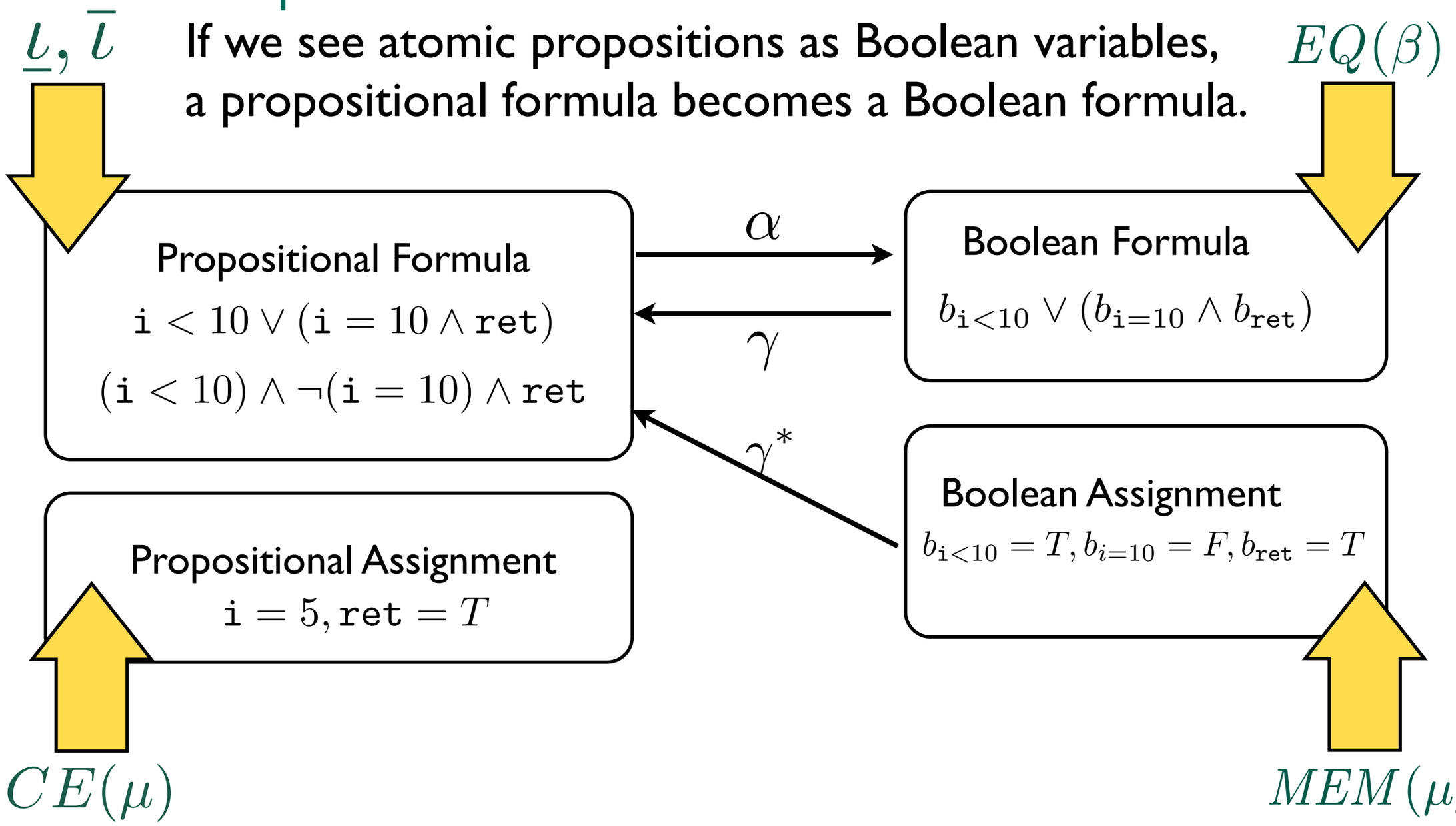

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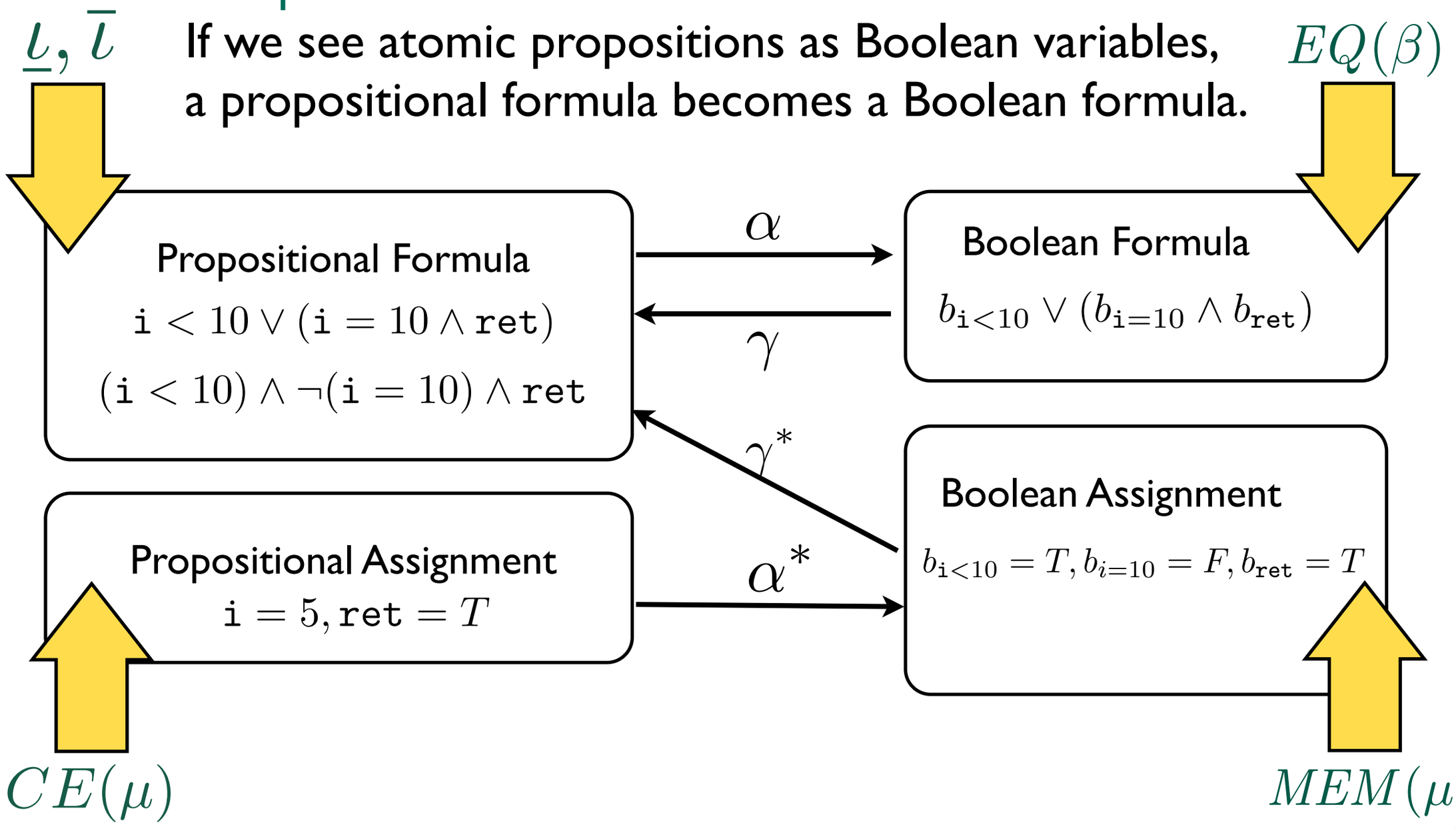
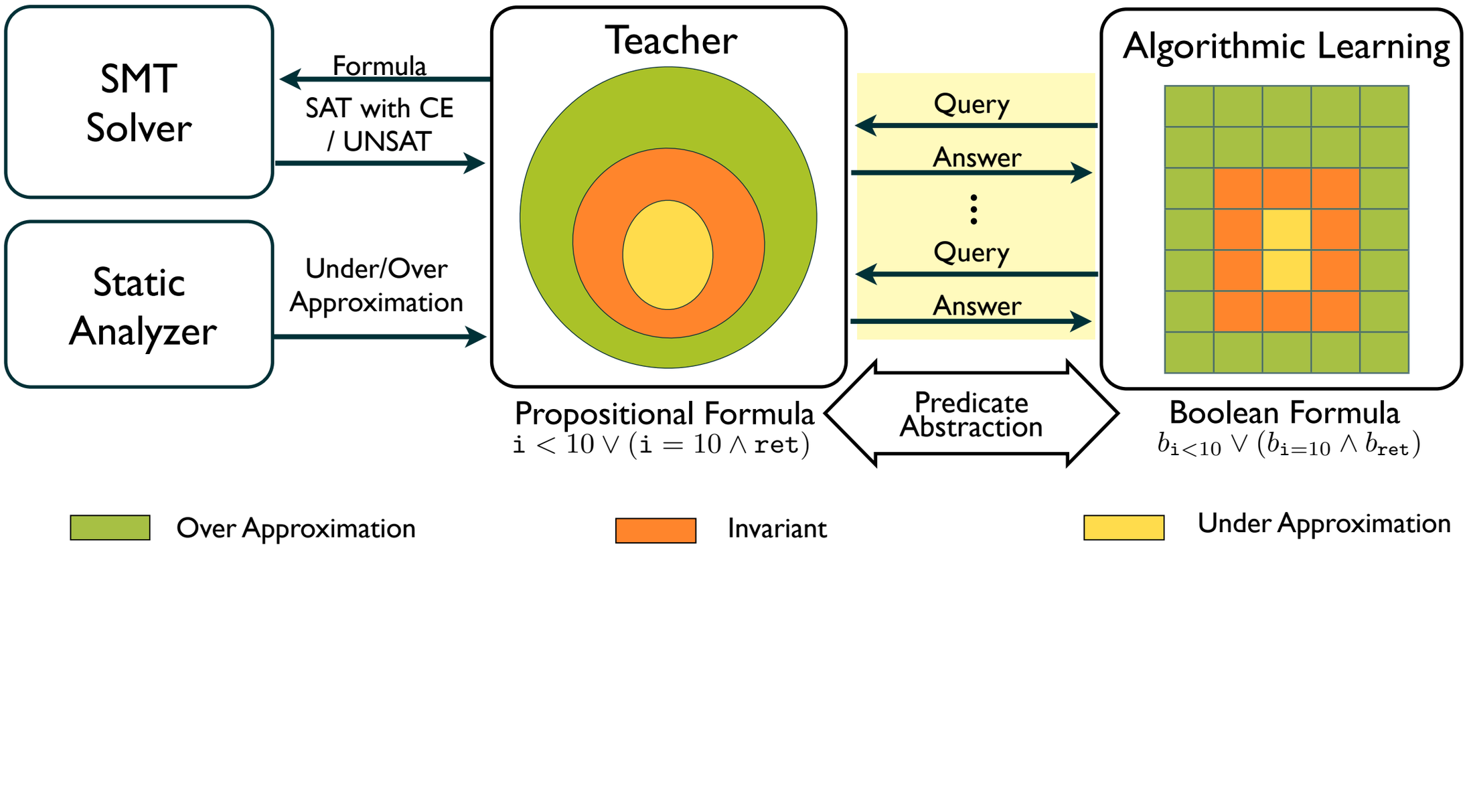


Overview


SMT Solver
Formula
SAT with CE / UNSAT
Static Analyzer
Under/Over Approximation
Teacher
Query
Answer
Query
Answer
Algorithmic Learning
Propositional Formula
$\mathtt{i} < 10 \vee (\mathtt{i} = 10 \wedge \mathtt{ret})$
Predicate Abstraction
Boolean Formula
$b_{\mathtt{i}<10} \vee (b_{\mathtt{i}=10} \wedge b_{\mathtt{ret}})$
Over Approximation
Invariant
Under Approximation

Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma(\beta)$ is an invariants (check the three conditions), then "Yes"

(A) $\delta \Rightarrow I$ (I holds when entering the loop)

(B) $I \wedge \rho \Rightarrow Pre(I, S)$ (I holds at each iteration)

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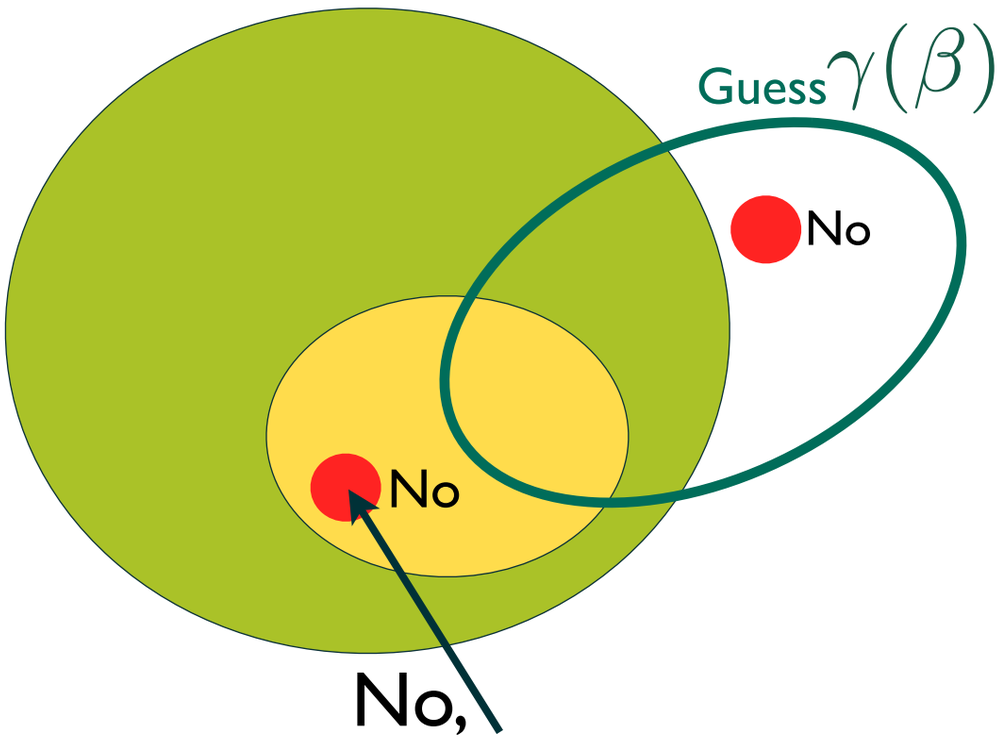
Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma(\beta)$ is an invariants (check the three conditions), then "Yes"
2. Try to find a counter example.

Case 1



with found a counterexample μ.

$$\mu \models \gamma(\beta) \oplus \underline{\iota}$$

Over Approximation Under Approximation

Resolving Equivalence Query $EQ(\beta)$

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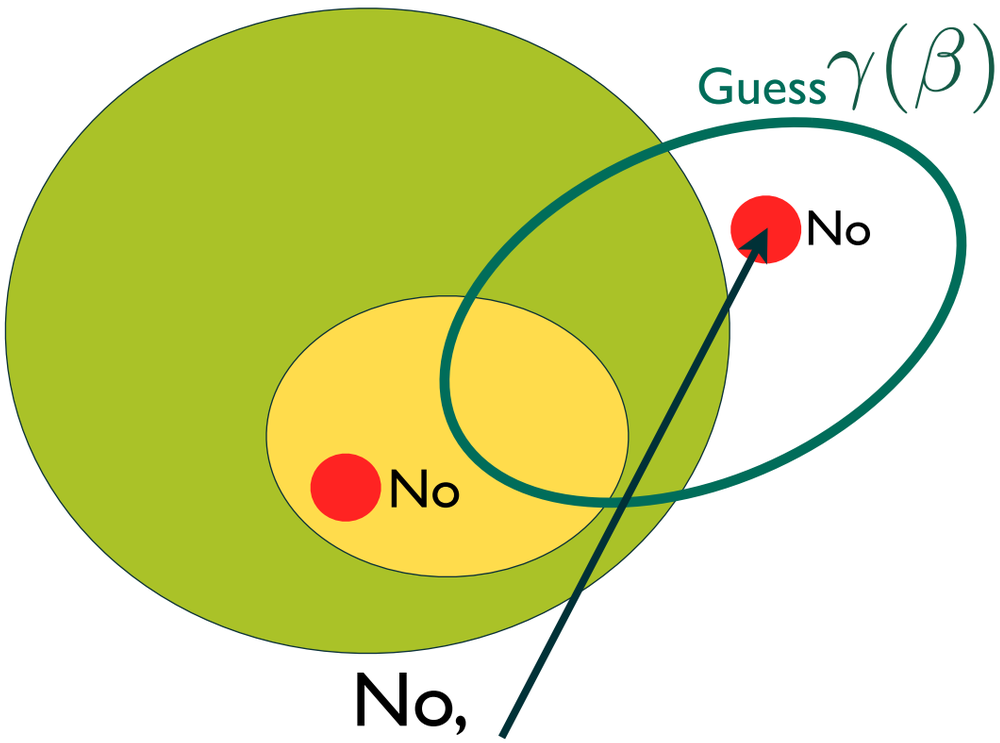

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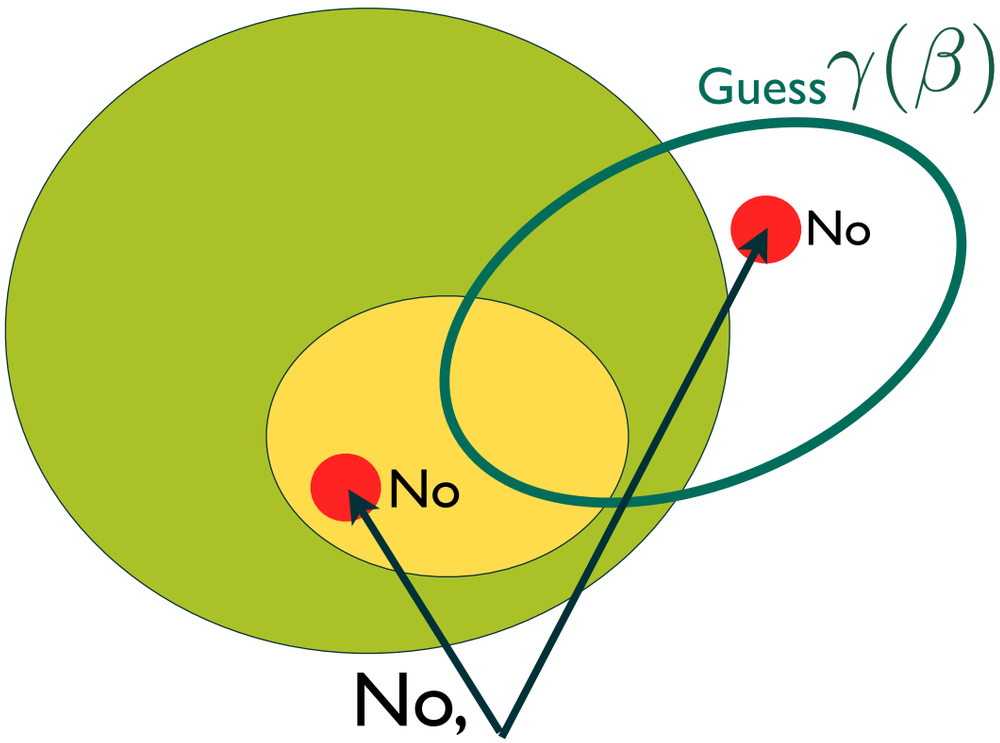

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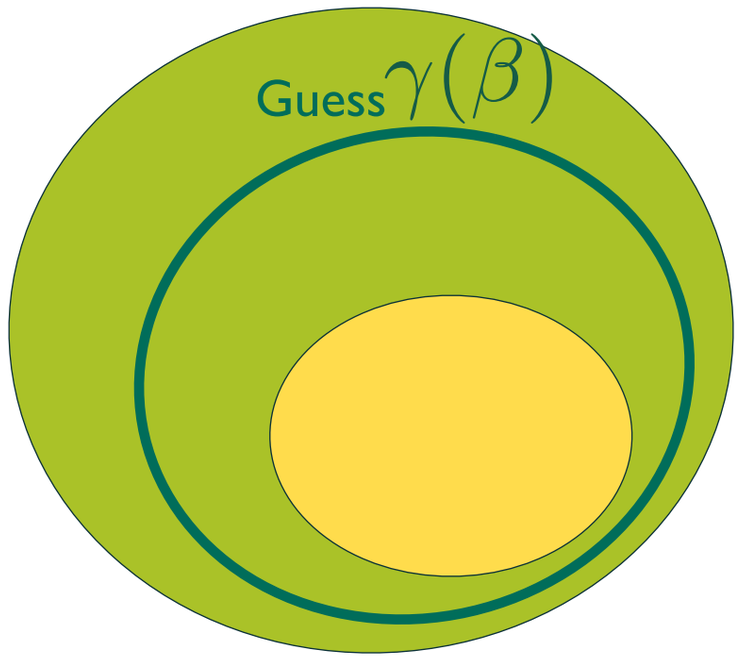
Case 2 $\underline{\iota} \Rightarrow \gamma(\beta) \Rightarrow \bar{\iota}$



Cannot find a counterexample.

Over Approximation Under Approximation

Resolving Equivalence Query $EQ(\beta)$

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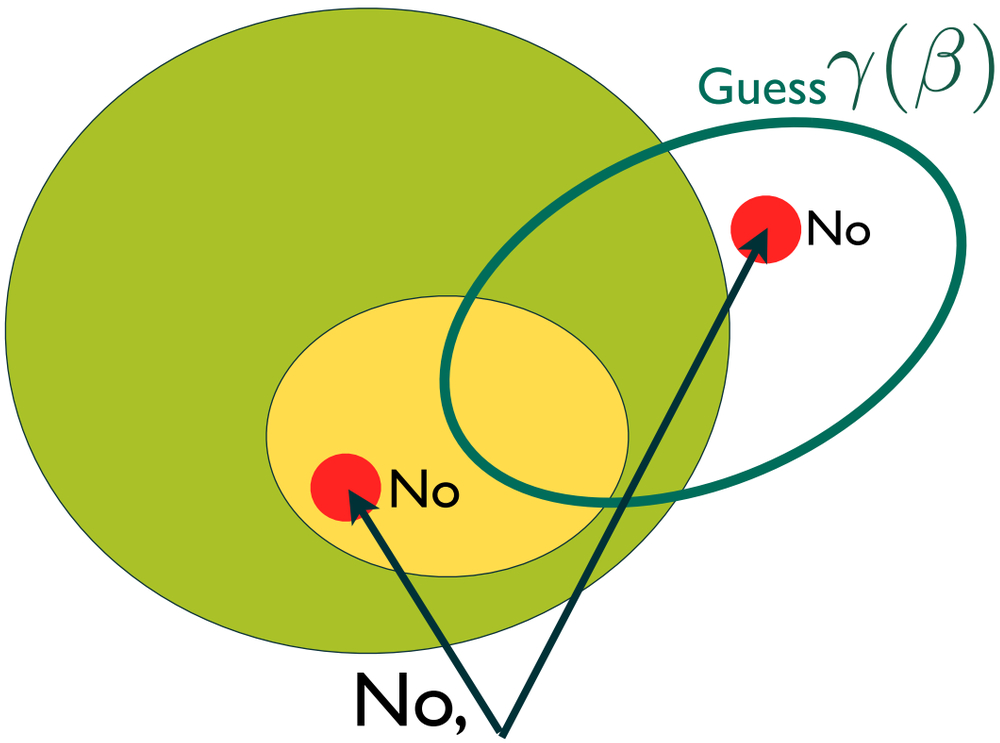

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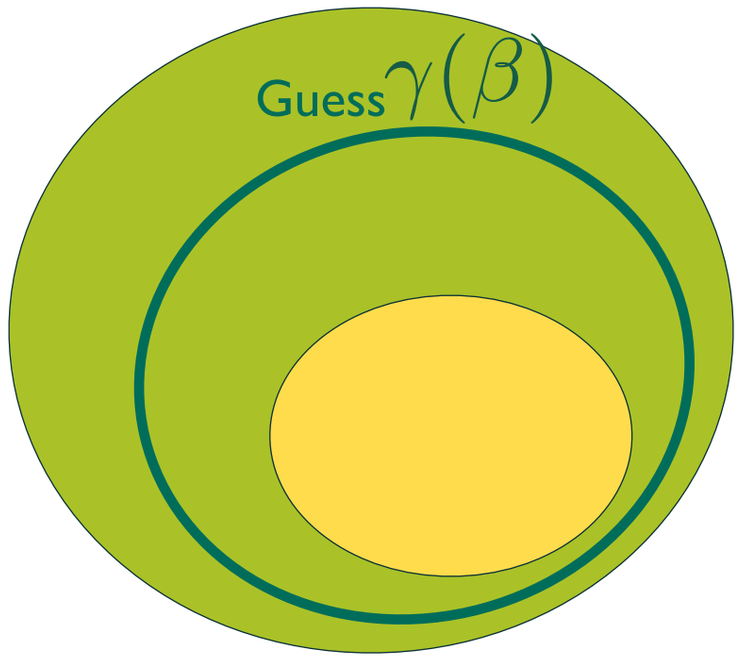

Restart!

Cannot find a counterexample.

Over Approximation Under Approximation

Resolving Membership Query $MEM(\mu)$

Use $\gamma^*(\mu)$ to answer membership query.

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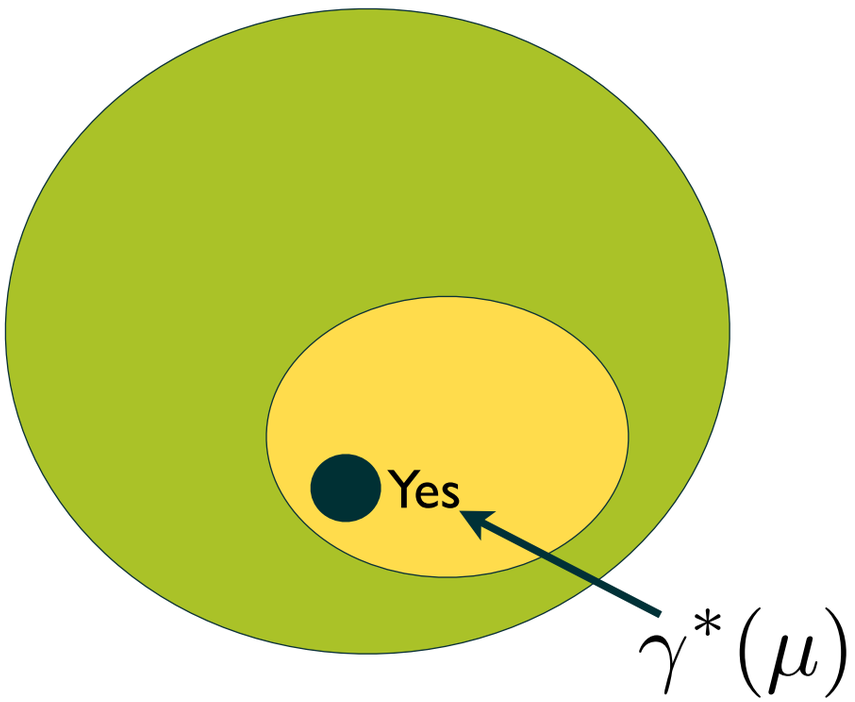

Answer Yes

$\gamma^*(\mu) \Rightarrow \underline{\iota}$

Resolving Membership Query $MEM(\mu)$

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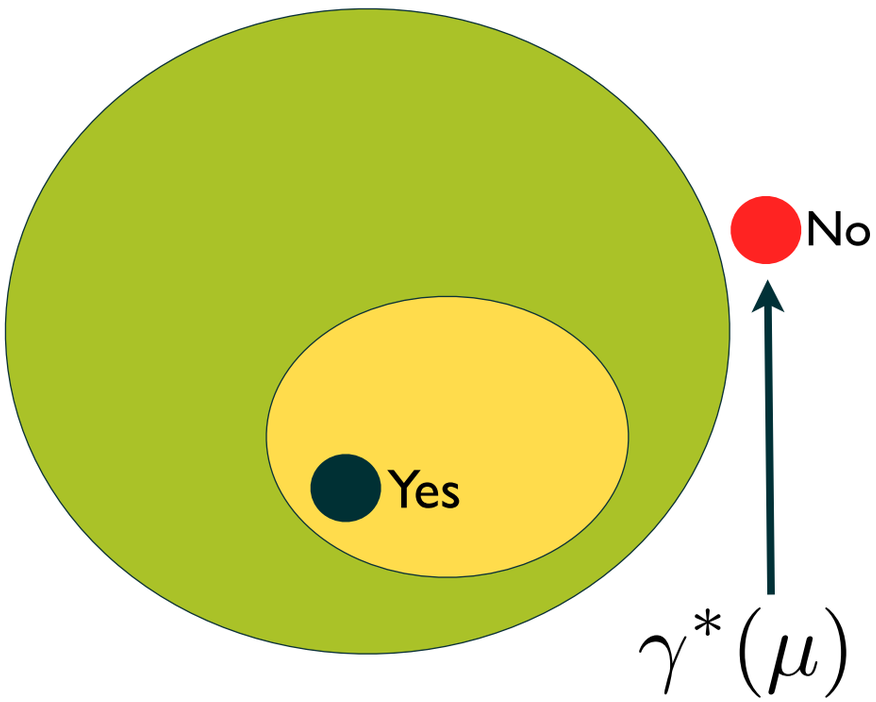

Answer No

$\gamma^*(\mu) \not\Rightarrow \bar{\iota}$

Resolving Membership Query $MEM(\mu)$

Use $\gamma^*(\mu)$ to answer membership query.

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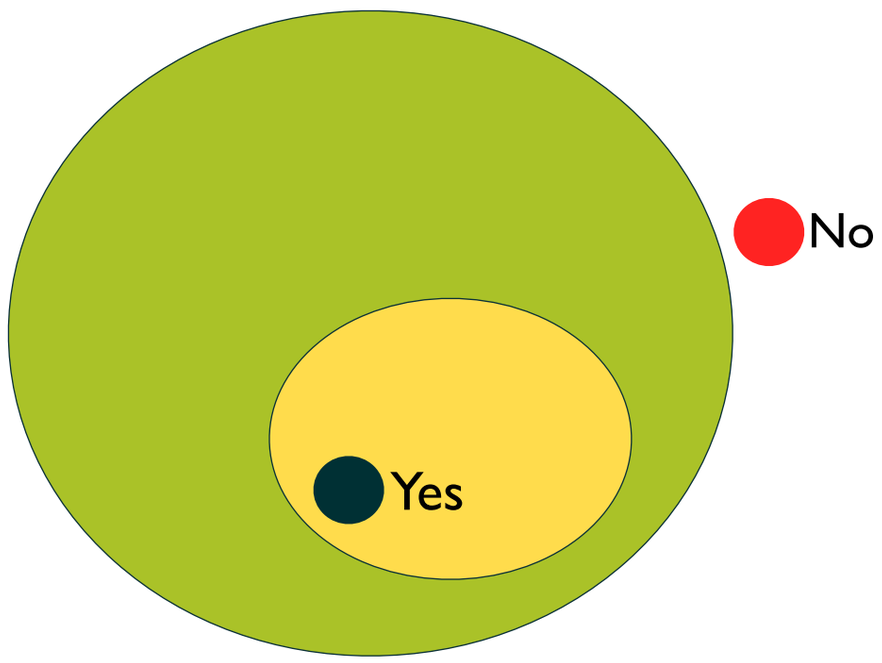
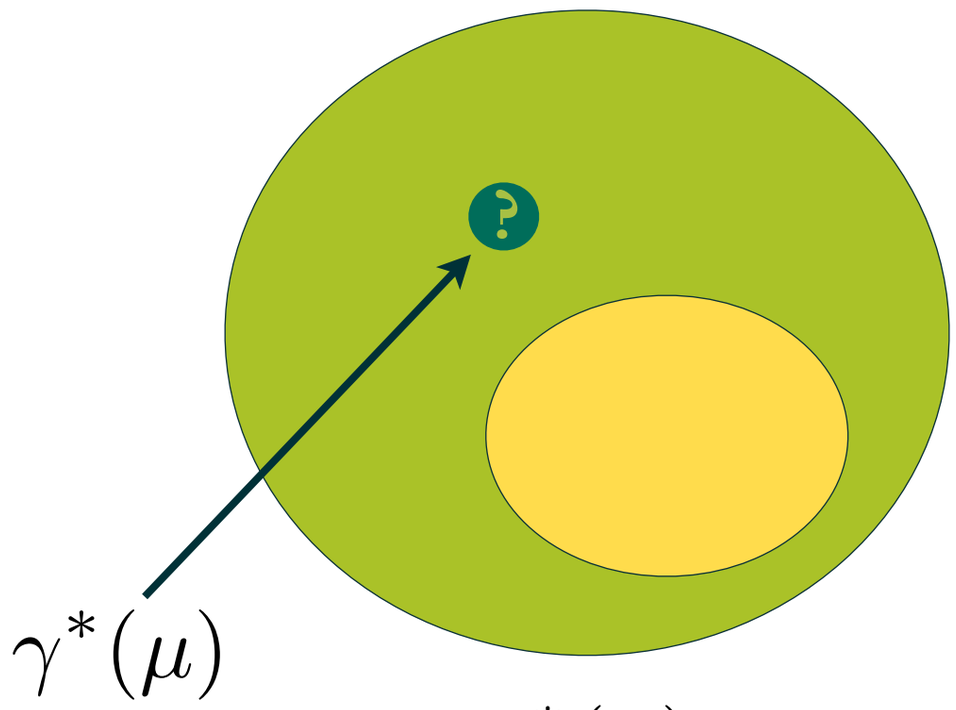


Answer Yes/No

Case 2



$\underline{\iota} \Rightarrow \gamma^*(\mu) \Rightarrow \bar{\iota}$

Resolving Membership Query $MEM(\mu)$

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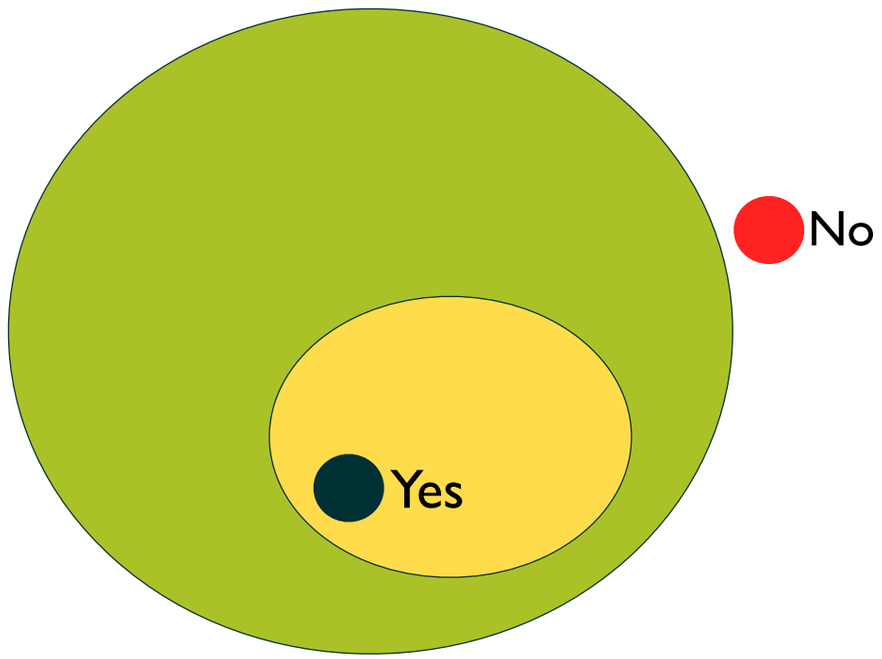

Answer Yes/No

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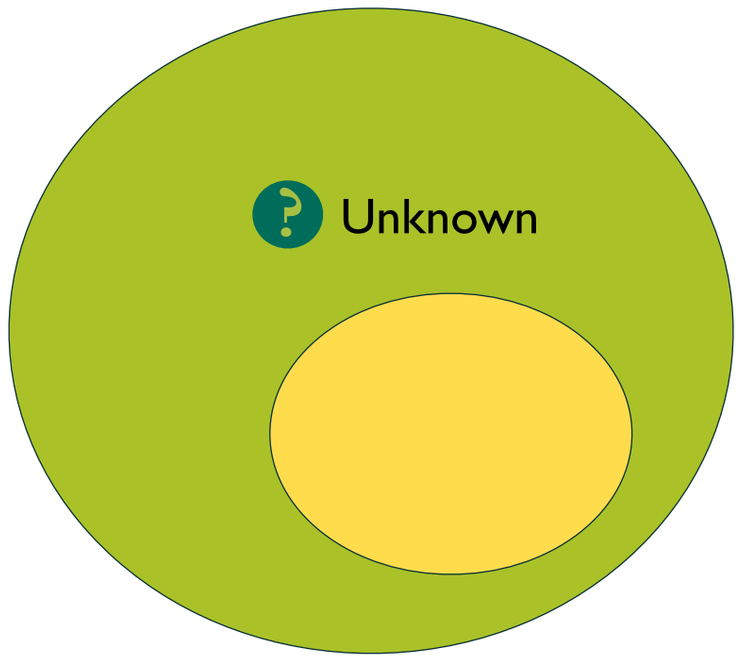

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Use $\gamma^*(\mu)$ to answer membership query.

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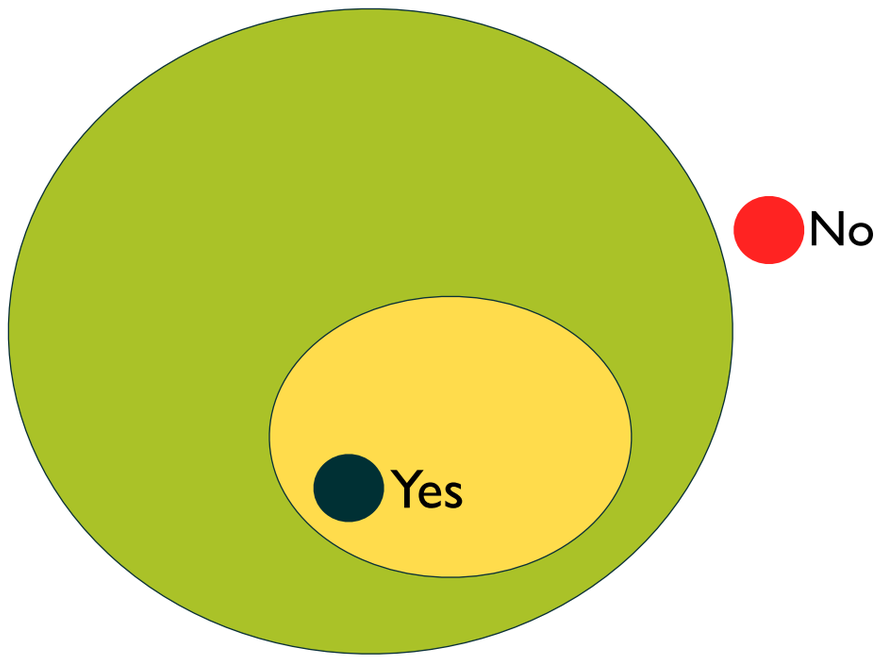
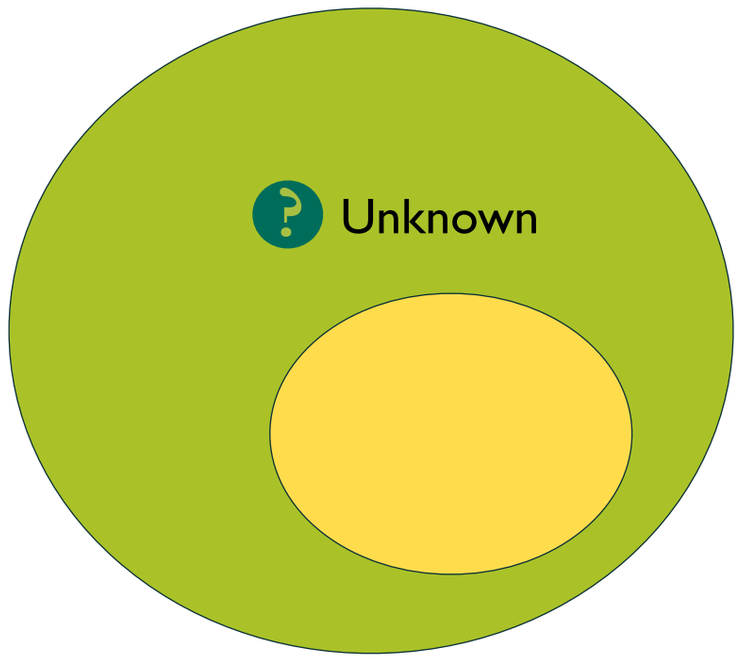


Answer Yes/No

Case 2



Random Answer!

Q: Random Answer, Sound?

Q: Random Answer, Sound?

Yes, it is sound. Found invariants are always real invariants.

Why Sound?

When resolving equivalence query $EQ(\beta)$

(A) $\delta \Rightarrow I$ (I holds when entering the loop)

(B) $I \wedge \rho \Rightarrow Pre(I, S)$ (I holds at each iteration)

(C) $I \wedge \neg\rho \Rightarrow \epsilon$ (I gives ϵ after leaving the loop)

1. If $\gamma(\beta)$ is an invariants (check the three conditions), then "Yes"

We always **check** the conditions before say "Yes".

SMT solver is **sound** and **complete** for propositional formulae.

Deriving Quantified Invariants using Algorithmic Learning

Problems

1. Incompleteness in SMT

2. Predicate Abstraction between
   First-order formulae and Boolean formulae

3. Query Resolution

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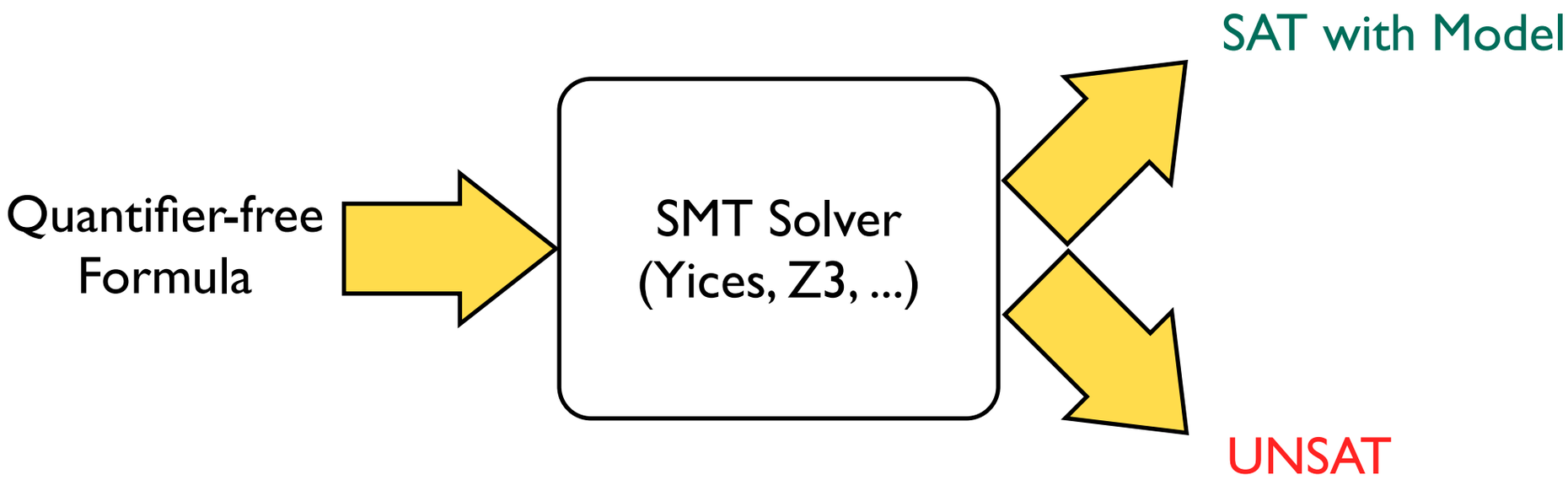
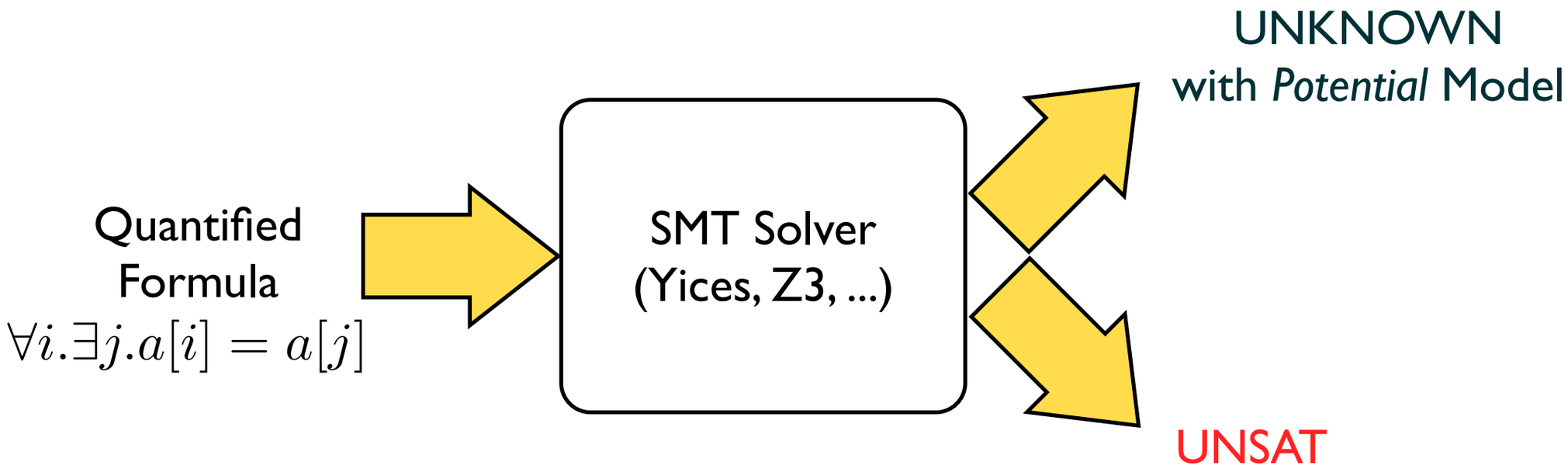
Incompleteness in SMT



SMT solver is sound and complete for the quantifier-free formulae

Incompleteness in SMT



SMT solver is sound but incomplete for the quantified formulae

Problems

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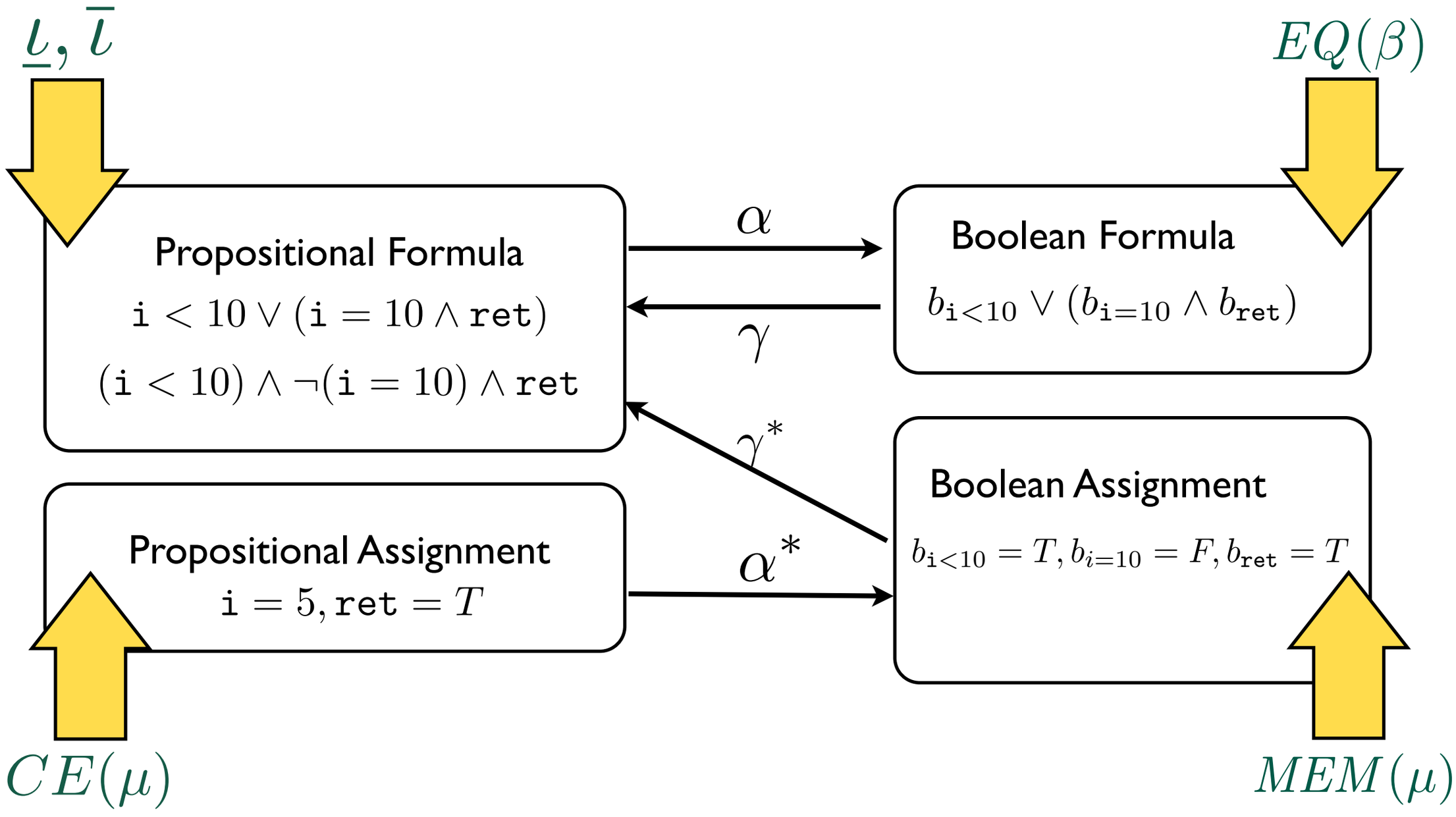
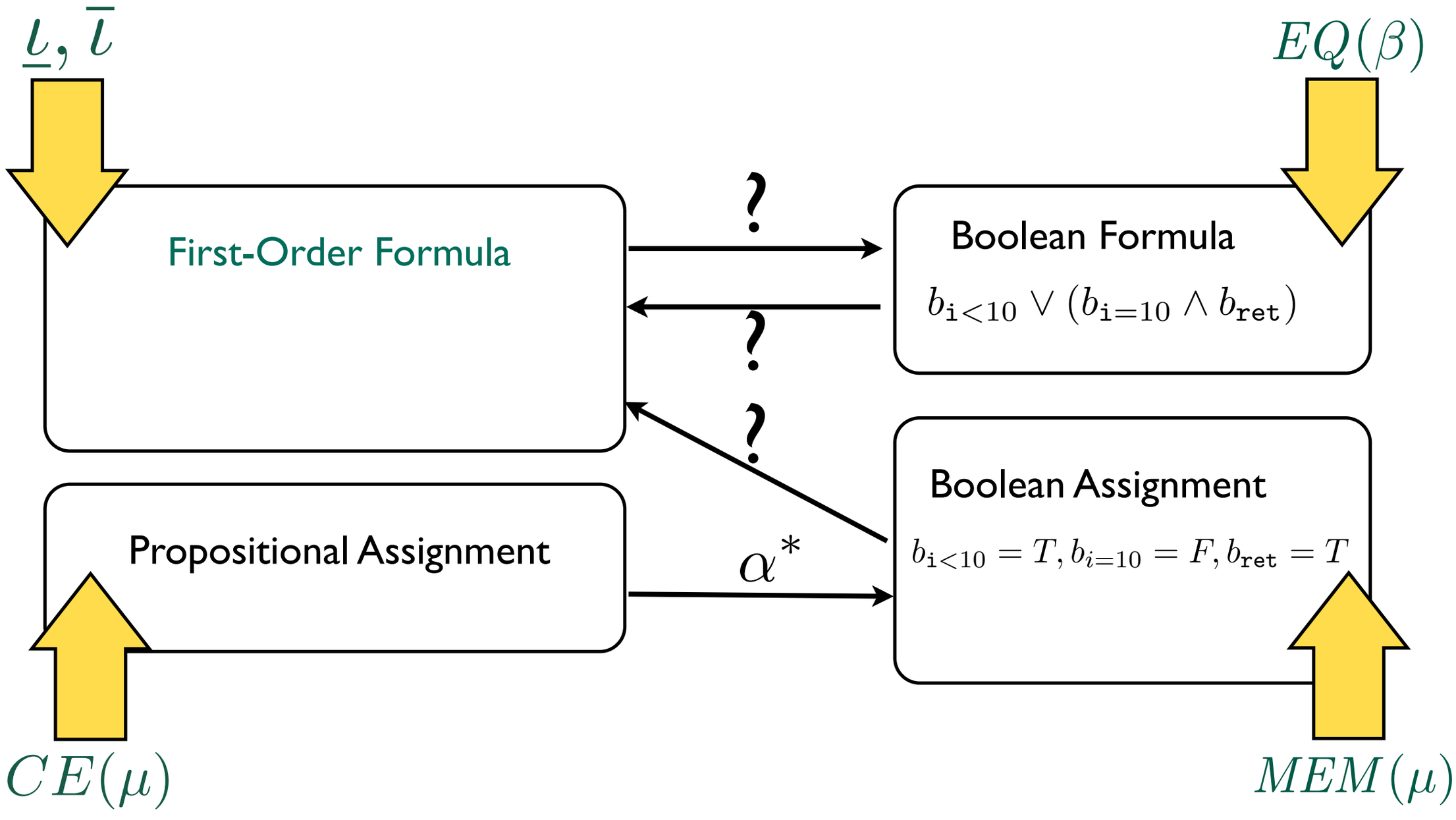
Predicate Abstraction in VMCAI'10

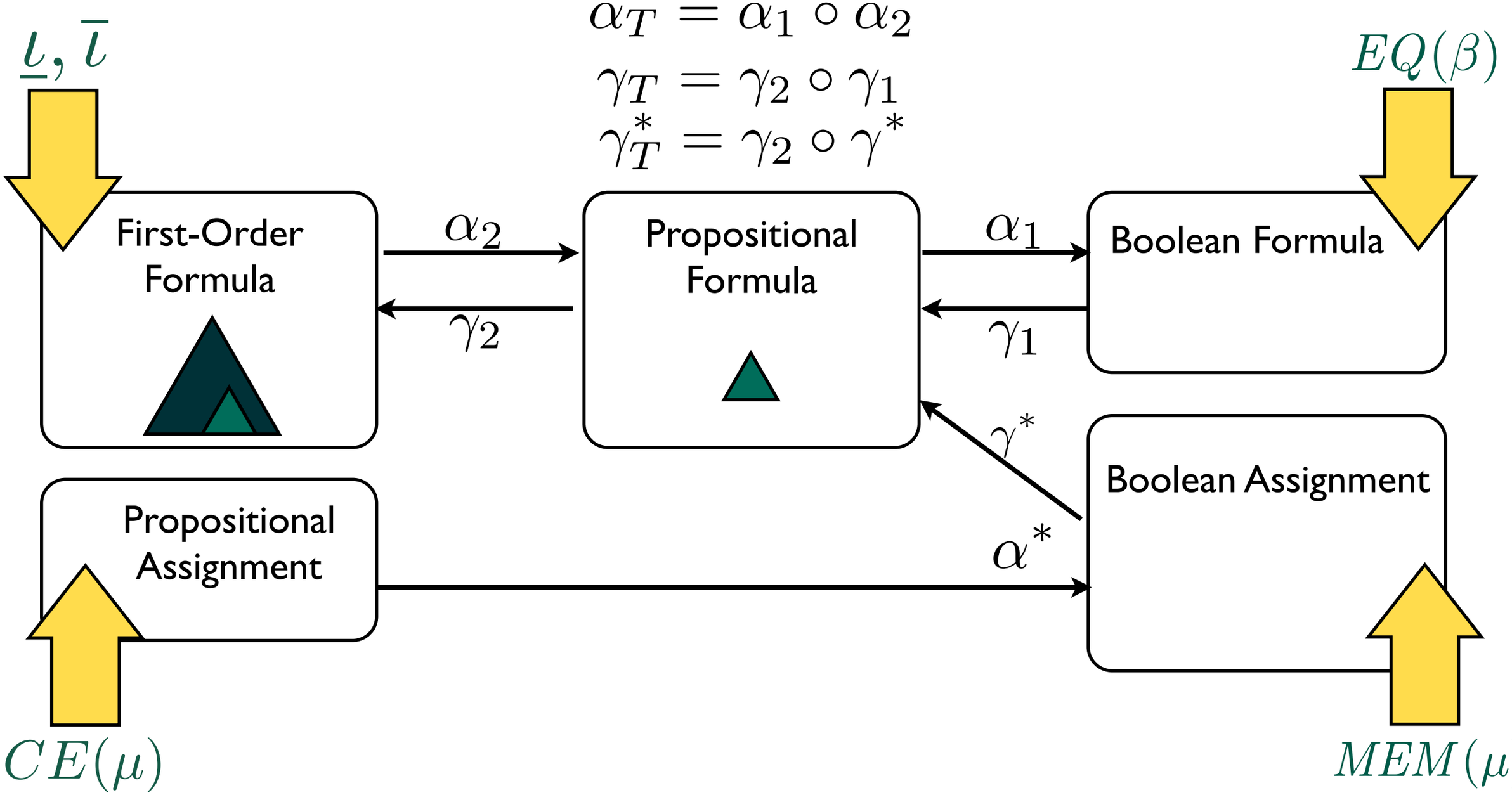


Need New Predicate Abstraction



Predicate Abstraction in Template Based Approach

$$\alpha_T = \alpha_1 \circ \alpha_2$$
$$\gamma_T = \gamma_2 \circ \gamma_1$$
$$\gamma_T^* = \gamma_2 \circ \gamma^*$$



Predicate Abstraction in Template Based Approach

Predicate Abstraction does not established with any arbitrary templates.

$$T[x] = x \vee true$$

$$(p_1 \Rightarrow p_2) \Rightarrow (T[p_1] \Rightarrow T[p_2])$$
$$(p_1 \Rightarrow p_2) \Rightarrow (T[p_2] \Rightarrow T[p_1])$$

$$T[x] = x \wedge false$$

$$(p_1 \Rightarrow p_2) \Rightarrow (T[p_1] \Rightarrow T[p_2])$$
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Def 3 Well-formed template

A template t with unknown x is well-formed with respect to $p_1 \in \mathsf{Prop}_A$ and a valuation S of the free variables under the following conditions ($t[p]$ is a shorthand for $t[x \mapsto p]$):

- $t[p_1] = p_1$ *is well-formed*
- $t[p_1] = \forall I.t'[p_1]$ *is well-formed if* $t'[p_1]$ *is well-formed*
- $t[p_1] = t'[p_1] \vee p$ *is well-formed if* $S \models t'[p_1] \not\Rightarrow p$ *and* $t'[p_1]$ *is well-formed*
- *if* $t[p_1] = t'[p_1] \wedge p$ *is well-formed if* $S \models t\_pos'[p_1] \wedge p$ *and* $t'[p_1]$ *is well-formed*
- $t[p_1] = \neg t'[p_1]$ *is well-formed if* $t'[p_1]$ *is well-formed*

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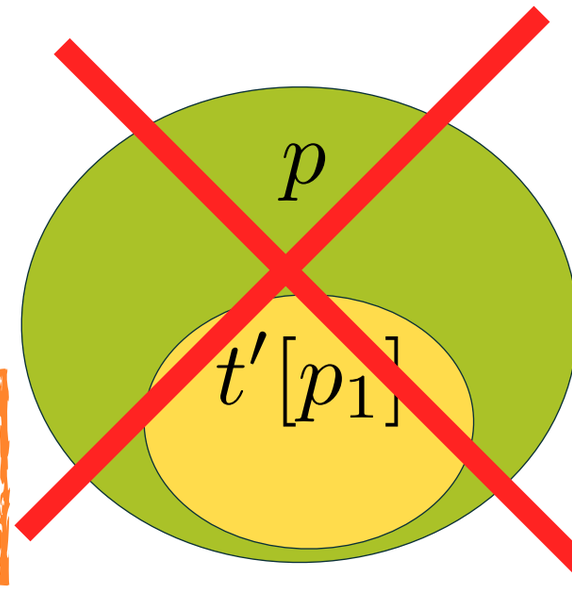
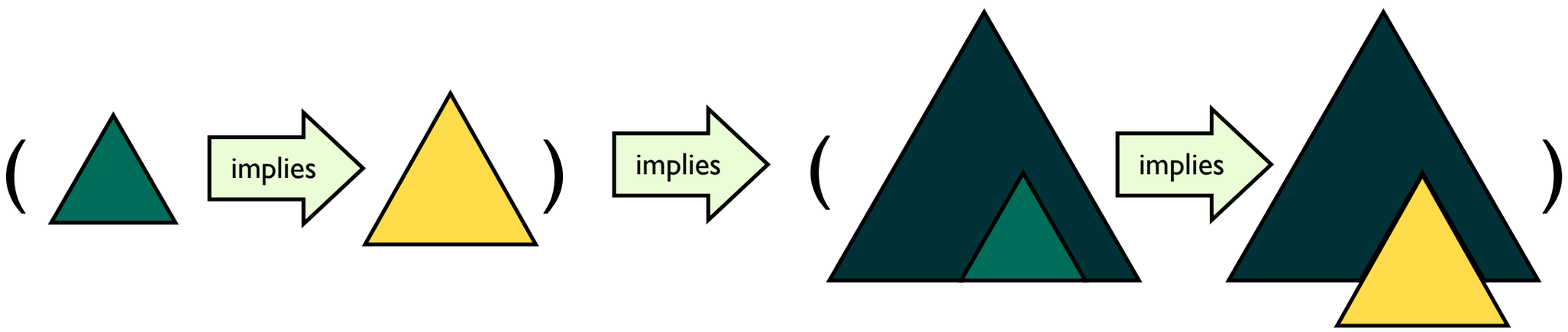
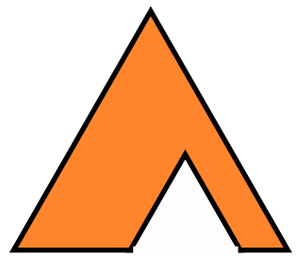


Positive Template 

Def 1 Positive unknown *(this definition applies to general templates with more than one hole)*
Given a template t, an unknown v is said to be positive if, given any substitution σ_v that maps all unknown variables in t different from v to some set of predicates, the following holds:

$$\forall \sigma_v, p_1, p_2 : (p_1 \Rightarrow p_2) \Rightarrow (t\sigma_v[v \mapsto p_1] \Rightarrow t\sigma_v[v \mapsto p_2]).$$

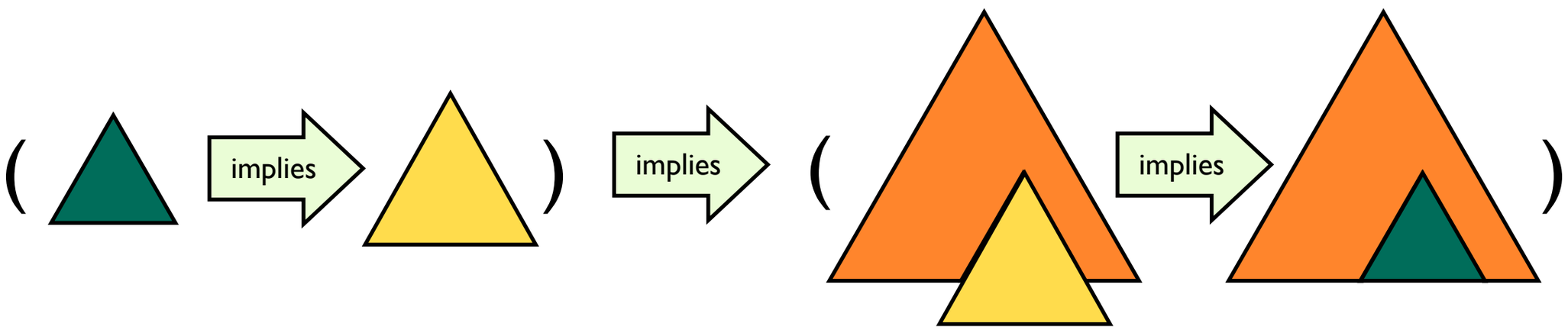


Negative Template 

Def 2 Negative unknown

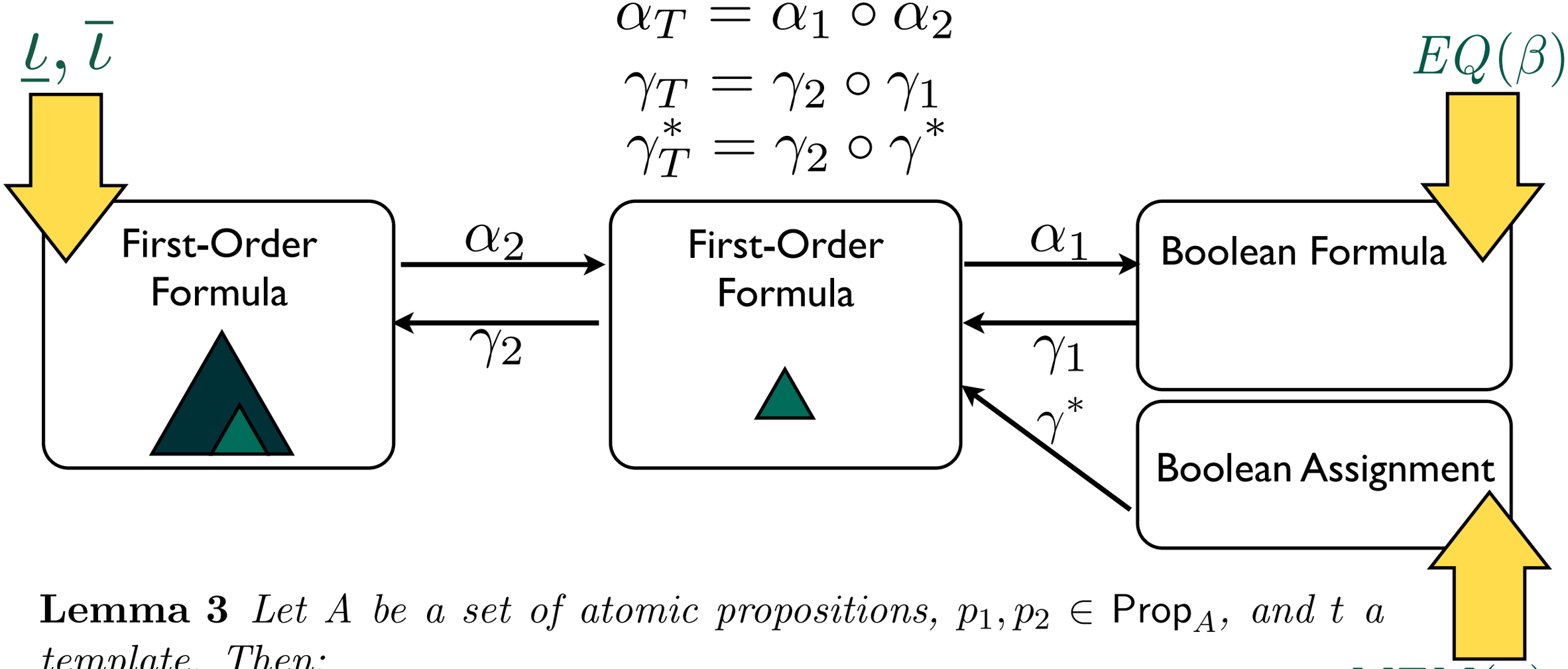
Given a template t, an unknown v is said to be negative if, given any substitution σ_v that maps all unknown variables in t different from v to some set of predicates, the following holds:

$$\forall \sigma_v, p_1, p_2 : (p_1 \Rightarrow p_2) \Rightarrow (t\sigma_v[v \mapsto p_2] \Rightarrow t\sigma_v[v \mapsto p_1]).$$



Predicate Abstraction

$$\alpha_T = \alpha_1 \circ \alpha_2$$
$$\gamma_T = \gamma_2 \circ \gamma_1$$
$$\gamma_T^* = \gamma_2 \circ \gamma^*$$



Lemma 3 *Let A be a set of atomic propositions, $p_1, p_2 \in \mathsf{Prop}_A$, and t a template. Then:*

$S \models t[p_1] \Rightarrow t[p_2]$ *implies* $S \models p_1 \Rightarrow p_2$, *if t has positive unknown and it is well−formed with respect to p_1 and and model S,*
$S \models t[p_1] \Rightarrow t[p_2]$ *implies* $S \models p_2 \Rightarrow p_1$, *if t has positive unknown and it is well−formed with respect to p_2 and and model S.*

where S gives valuations for the free variables.

Predicate Abstraction

Lemma 3 *Let A be a set of atomic propositions, $p_1, p_2 \in \mathsf{Prop}_A$, and t a template. Then:*

$$S \models t[p_1] \Rightarrow t[p_2] \text{ implies } S \models p_1 \Rightarrow p_2, \text{ if } t \text{ has positive unknown and it is well-formed with respect to } p_1 \text{ and and model } S,$$
$$S \models t[p_1] \Rightarrow t[p_2] \text{ implies } S \models p_2 \Rightarrow p_1, \text{ if } t \text{ has positive unknown and it is well-formed with respect to } p_2 \text{ and and model } S.$$

where S gives valuations for the free variables.

This is weaker than we desired: It checks satisfiability, not validity!

Limitation:

UNSAT means "NOT Valid",
however SAT does not always mean "Valid".

Consequence:

We can say "No" for sure at membership query resolution,
however we cannot say "Yes".

Problems

1. Incompleteness in SMT

2. Predicate Abstraction between
   First-order formulae and Boolean formulae

3. Query Resolution

Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma_T(\beta)$ is an invariants (check the three conditions), then "Yes"

(A) $\delta \Rightarrow I$ (I holds when entering the loop)

(B) $I \wedge \rho \Rightarrow Pre(I, S)$ (I holds at each iteration)

(C) $I \wedge \neg\rho \Rightarrow \epsilon$ (I gives ϵ after leaving the loop)

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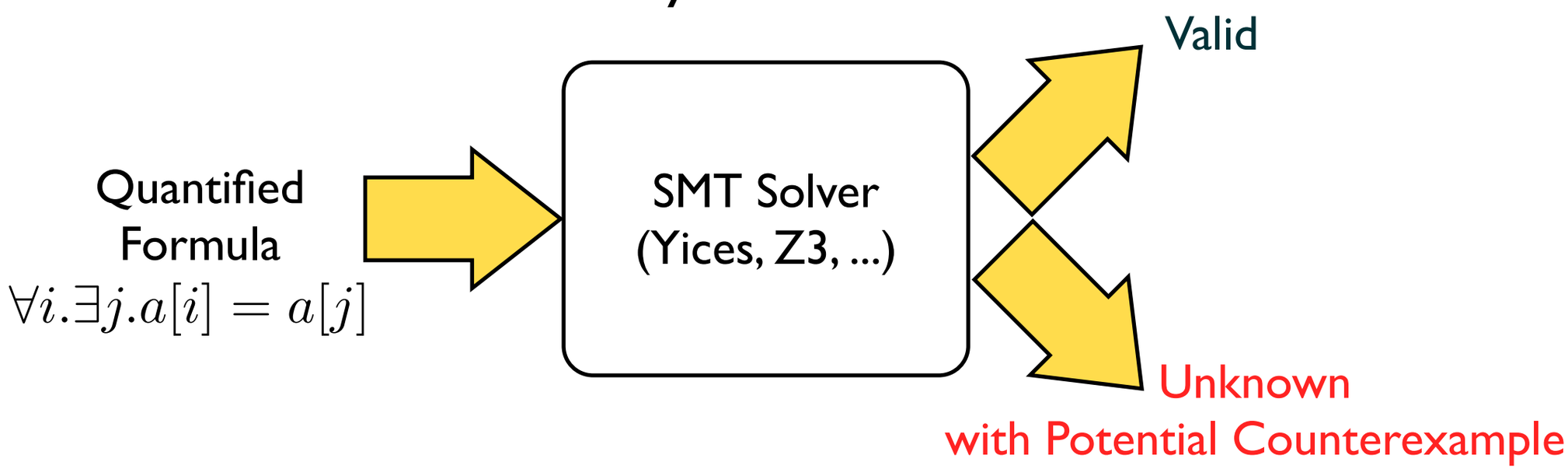
Because of the incompleteness of SMT solver,
we have two cases for validity check.



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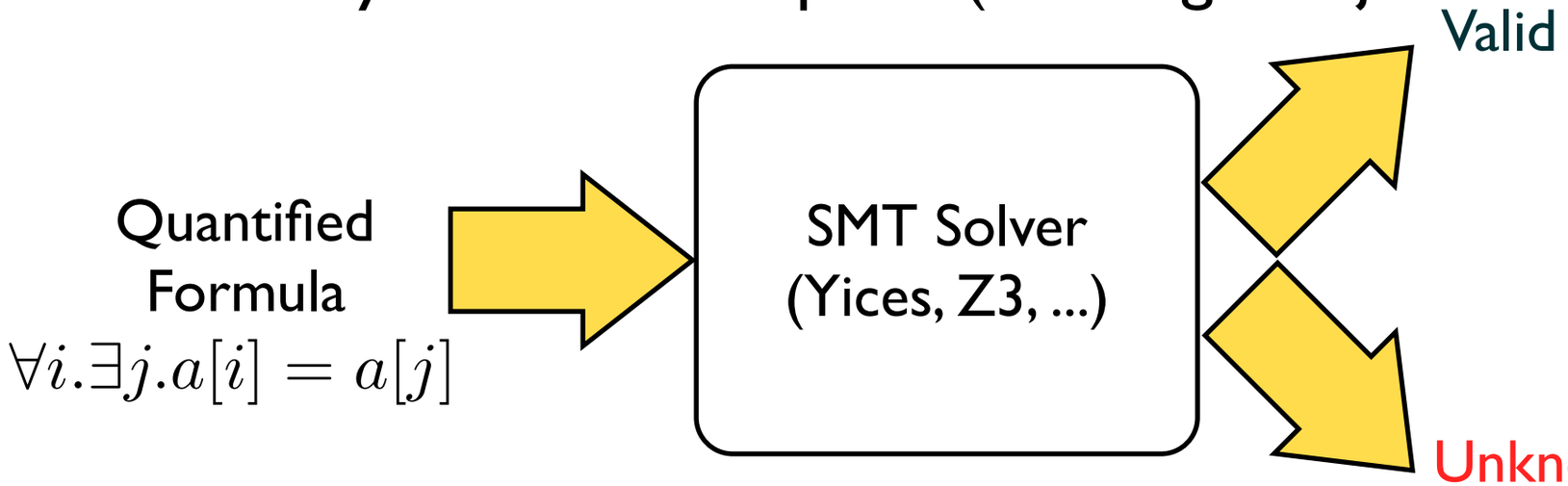
Consequences:

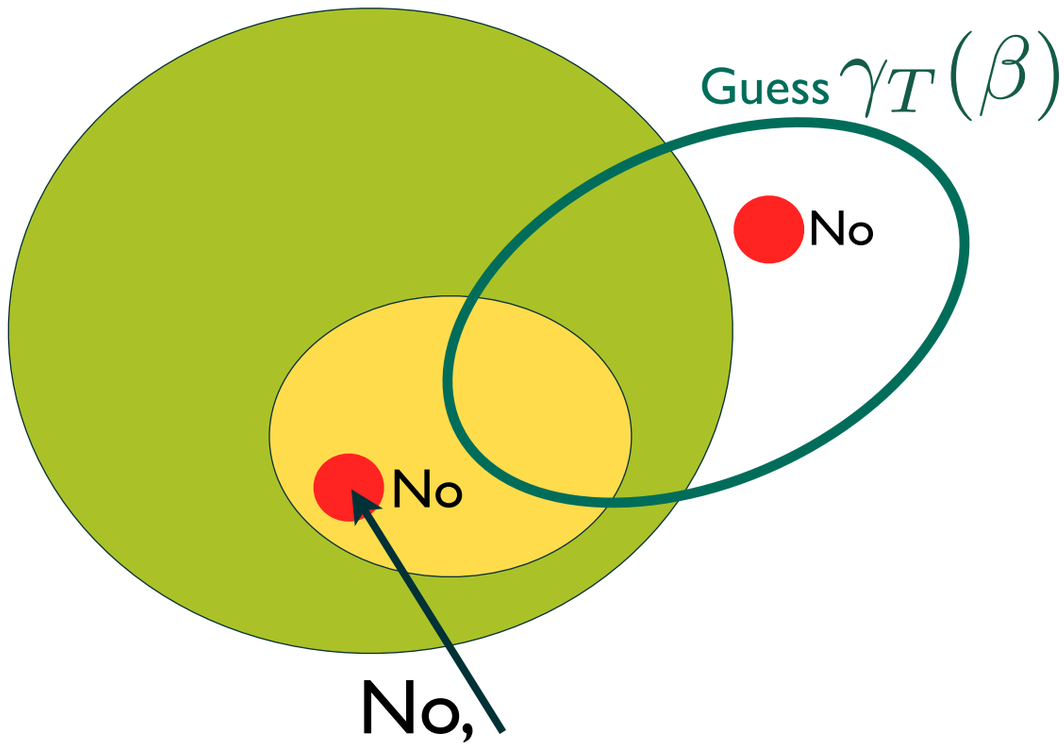
1. The system is sound (Found invariant is the real one)
2. The system is incomplete (We might reject the true invariant.)



Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma_T(\beta)$ is an invariants (check the three conditions), then "Yes"
2. Try to find a counter example.

Case 1



with found a potential counterexample μ.

$$\mu \models \gamma_T(\beta) \oplus \underline{\iota}$$

Over Approximation Under Approximation

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1. If $\gamma_T(\beta)$ is an invariants (check the three conditions), then "Yes"
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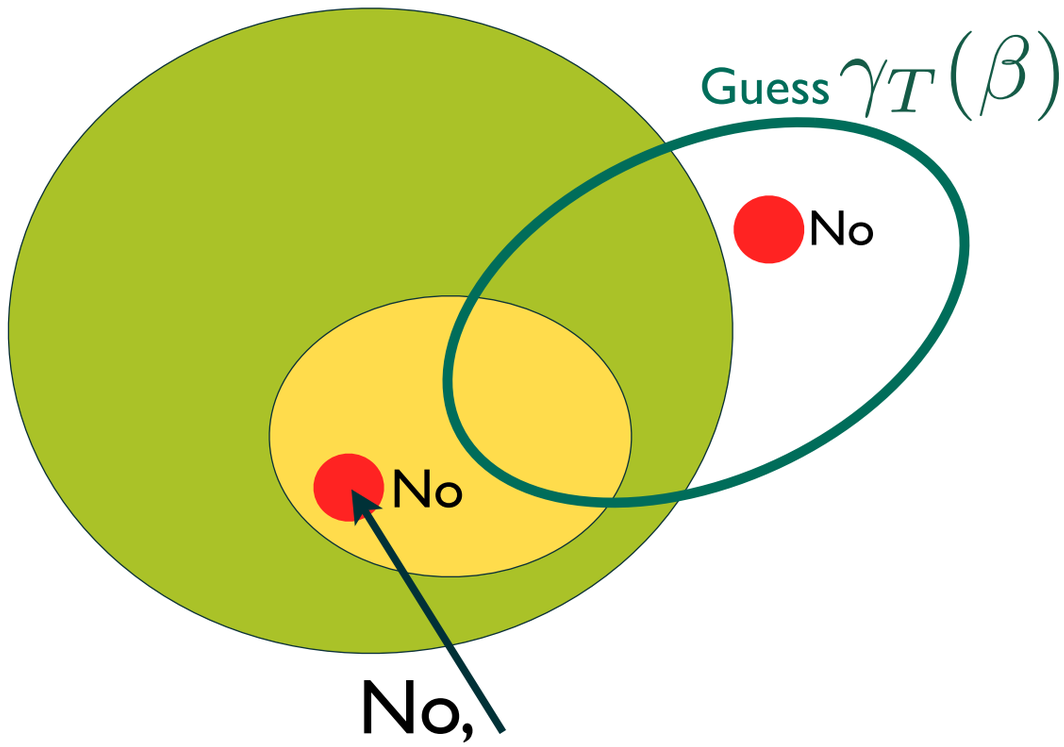

Potential counterexample:

Could be real => Use it?
Could be false => Restart?

with found a potential counterexample μ.

$$\mu \models \gamma_T(\beta) \oplus \underline{\iota}$$

Over Approximation Under Approximation

Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma_T(\beta)$ is an invariants (check the three conditions), then "Yes"
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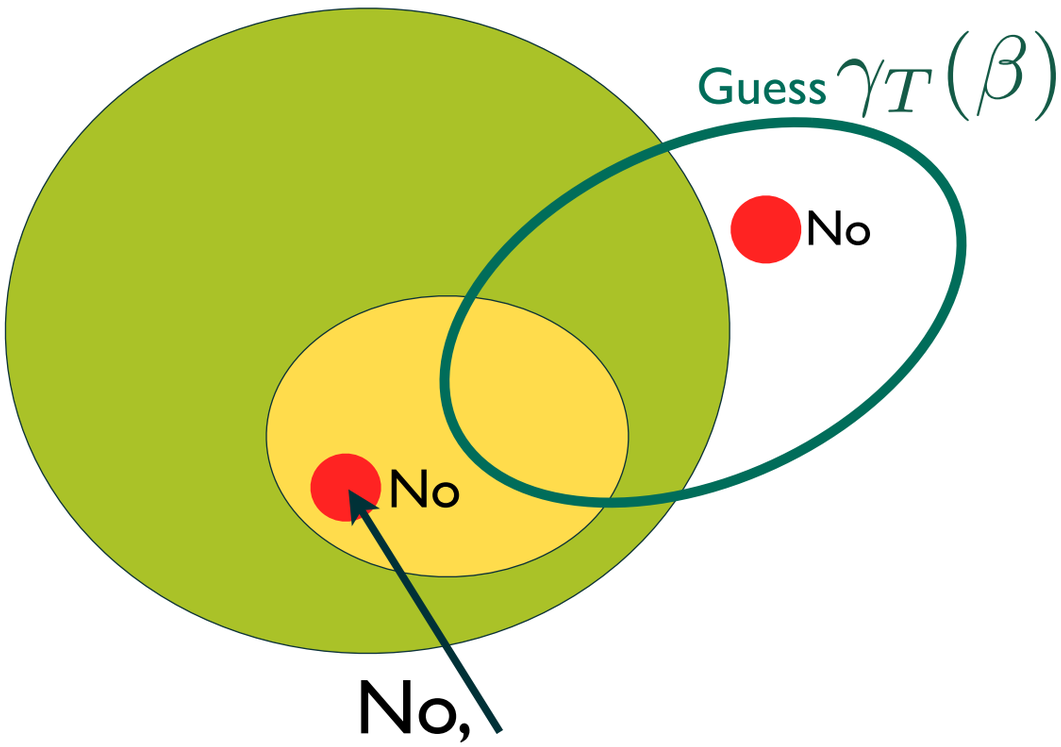

Potential counterexample:

Just use it!,
It does not affect the soundness.

with found a potential counterexample μ.

$$\mu \models \gamma_T(\beta) \oplus \underline{\iota}$$

Over Approximation Under Approximation

Resolving Equivalence Query $EQ(\beta)$

1. If $\gamma_T(\beta)$ is an invariants (check the three conditions), then "Yes"
2. Try to find a counter example.

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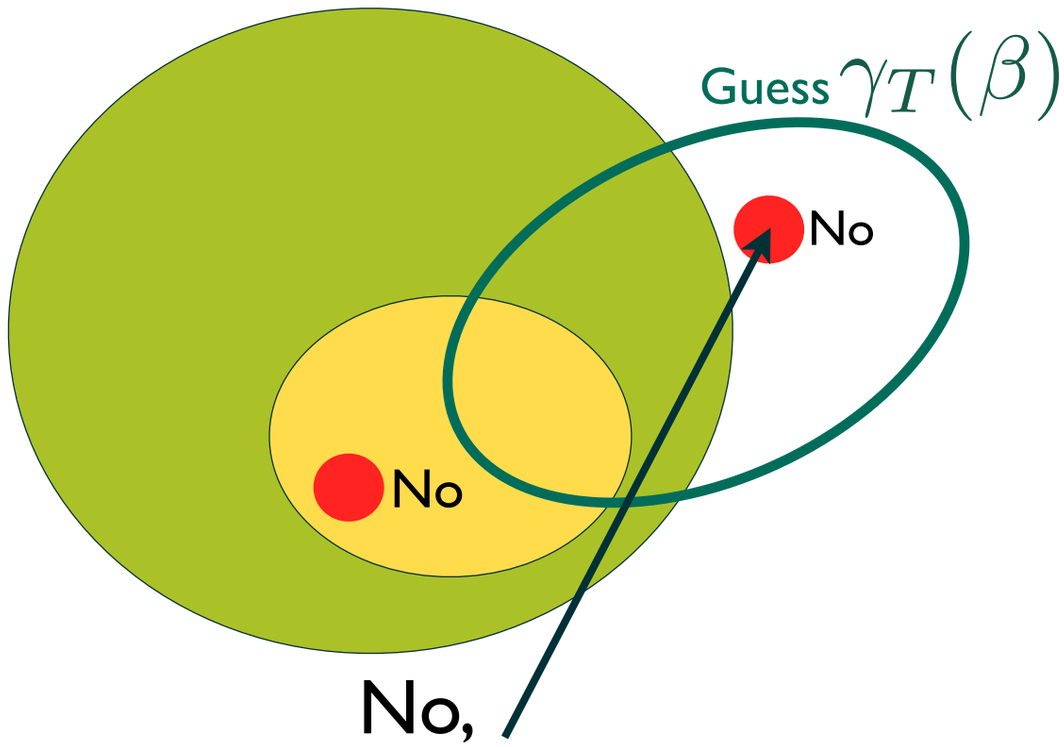

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$$\mu \models \gamma_T(\beta) \oplus \bar{\iota}$$

Over Approximation Under Approximation

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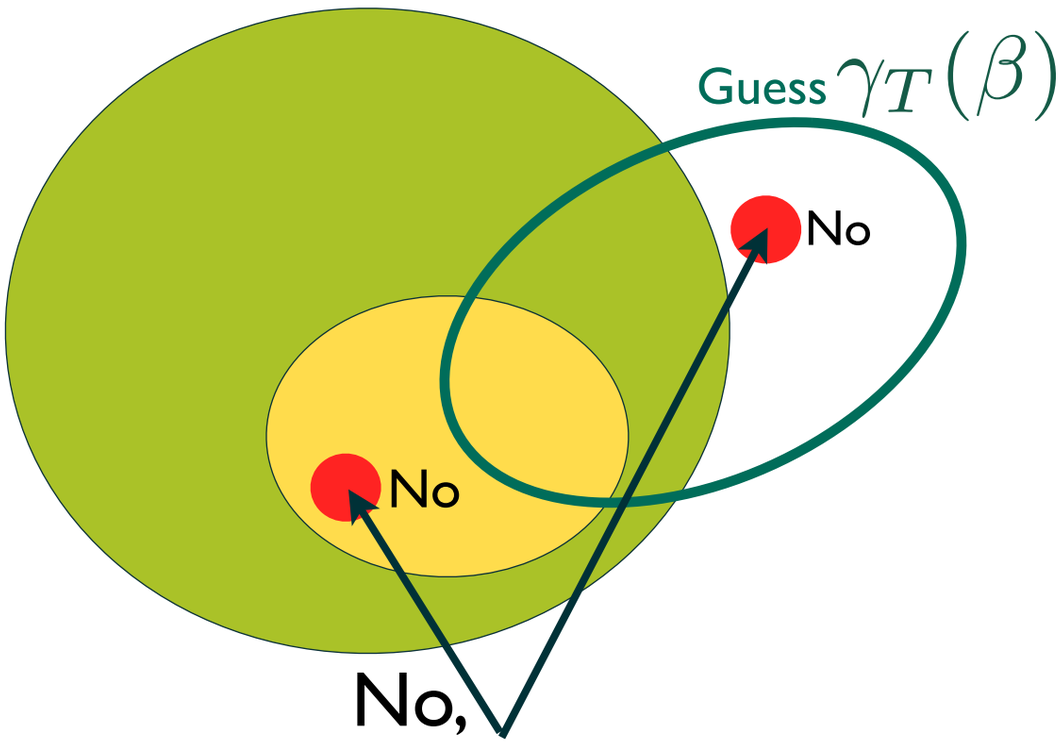
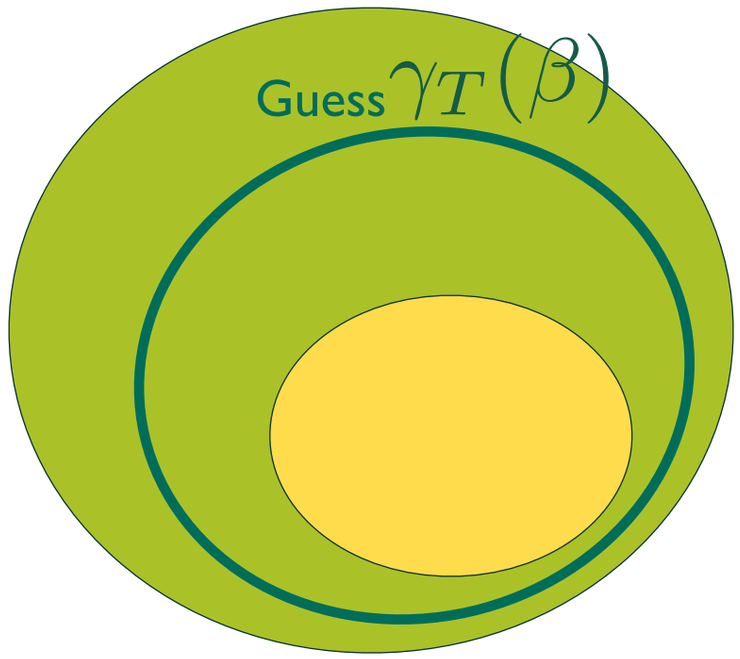


with found a counterexample μ.

Case 2 $\underline{\iota} \Rightarrow \gamma_T(\beta) \Rightarrow \bar{\iota}$



Restart!

Cannot find a counterexample.

Over Approximation Under Approximation

Resolving Membership Query $MEM(\mu)$

1. Check the well-formedness of the template T with $\gamma^*(\mu)$

   => Reject not well-formed template.

Resolving Membership Query $MEM(\mu)$

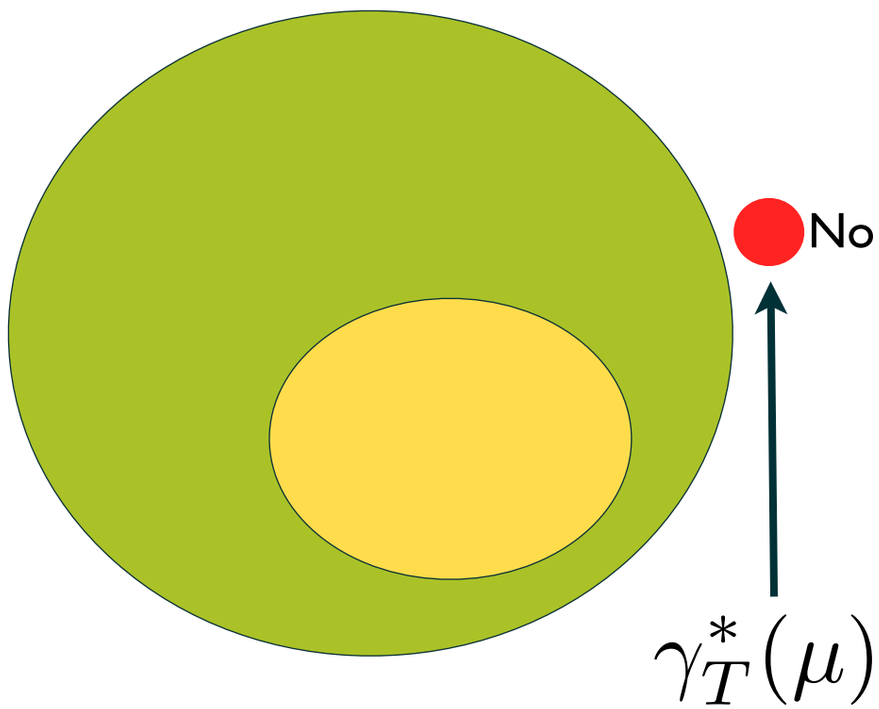
2. Use $\gamma_T^*(\mu)$ to answer membership query.
   Depends on the polarity of the template.

Resolving Membership Query $MEM(\mu)$

2. Use $\gamma_T^*(\mu)$ to answer membership query.

For positive template T

Case 1



Answer No

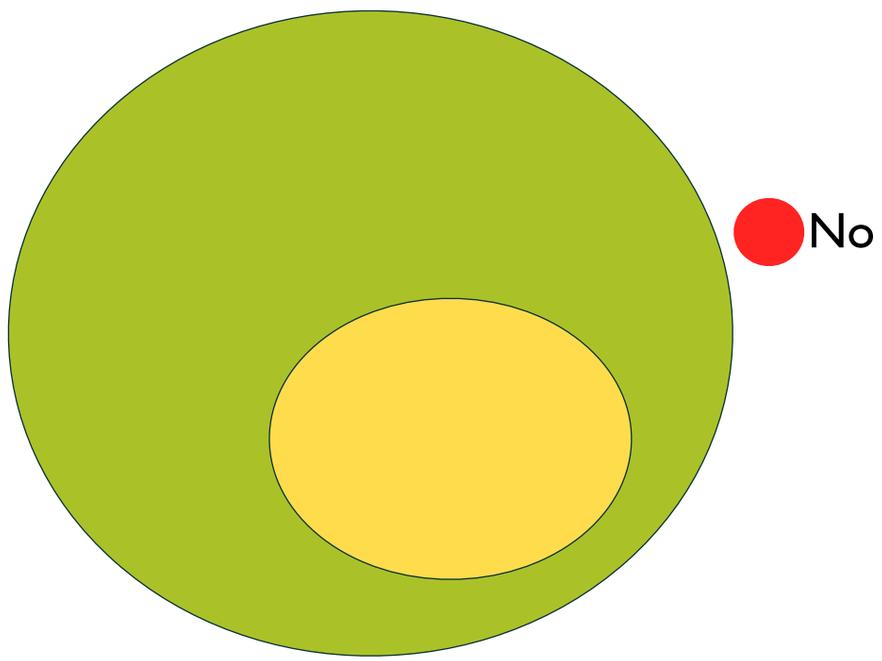
$S \models \gamma_T^*(\mu) \not\Rightarrow \bar{\iota}$

Resolving Membership Query $MEM(\mu)$

2. Use $\gamma_T^*(\mu)$ to answer membership query.

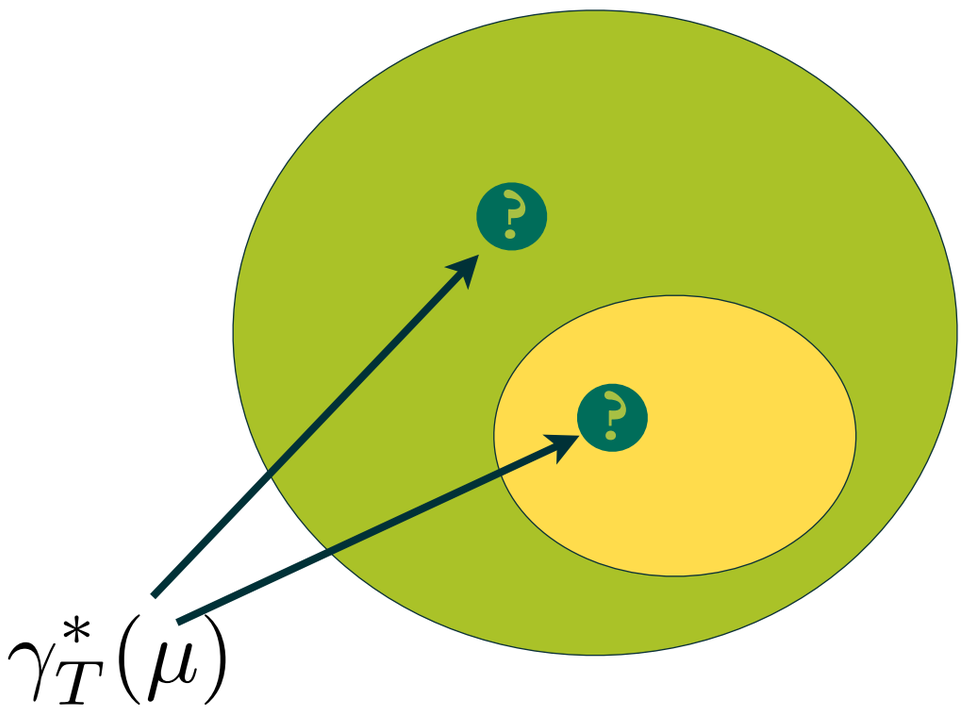
For positive template T

Case 1



Answer No

Case 2



Resolving Membership Query $MEM(\mu)$

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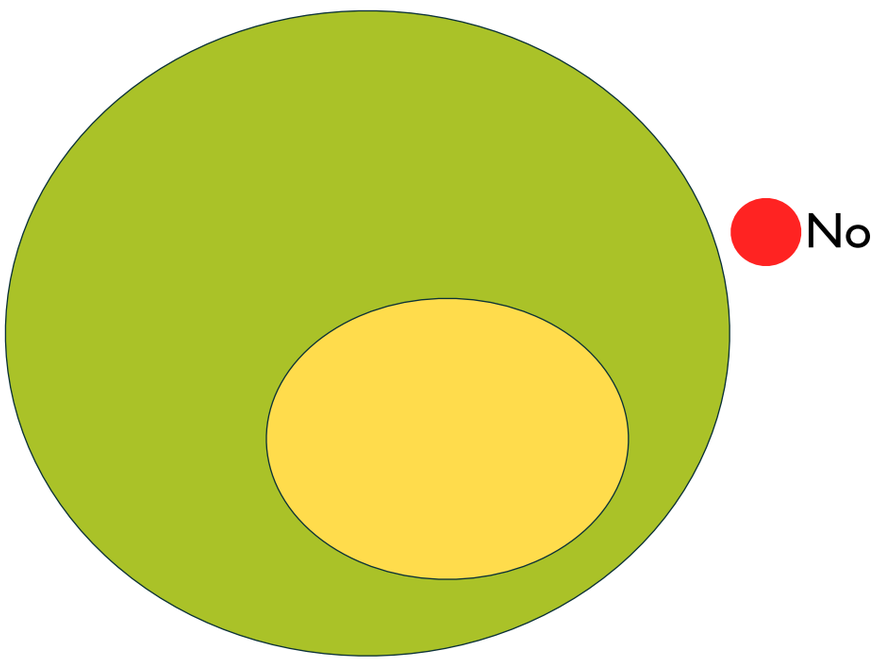
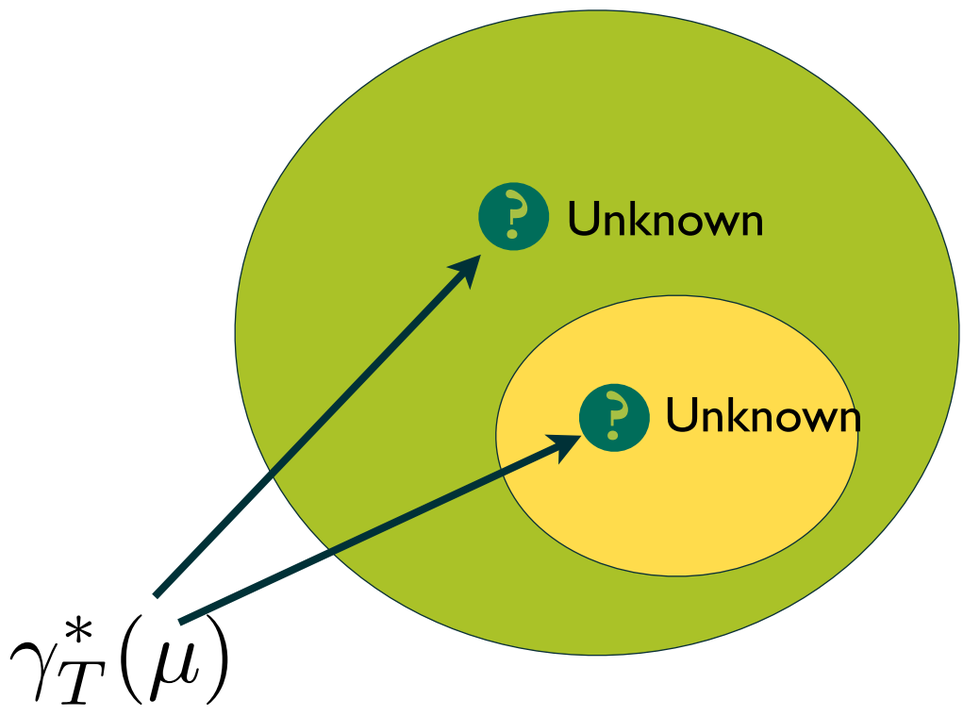


Answer No

Case 2



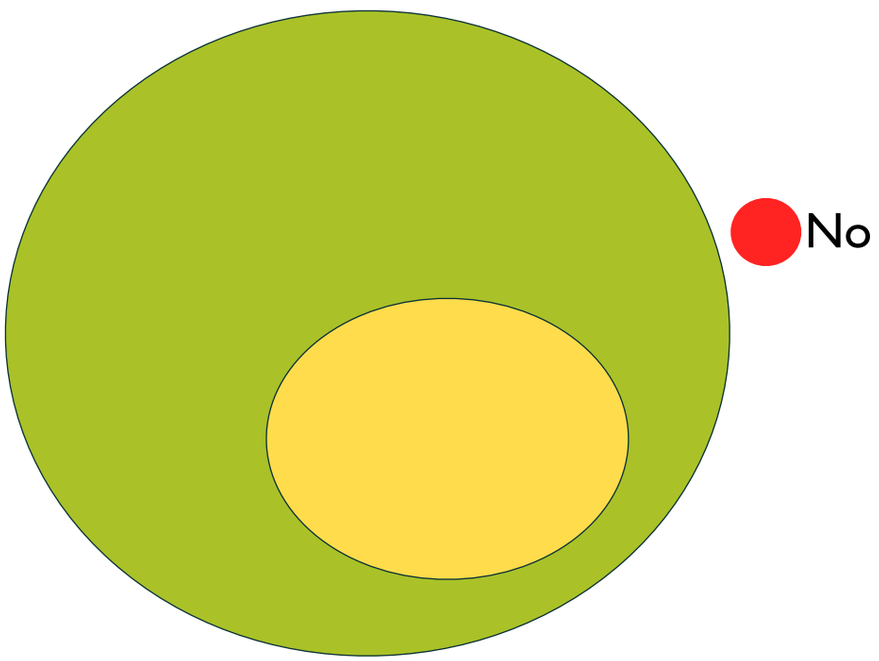
Cannot Answer!

Resolving Membership Query $MEM(\mu)$

2. Use $\gamma_T^*(\mu)$ to answer membership query.

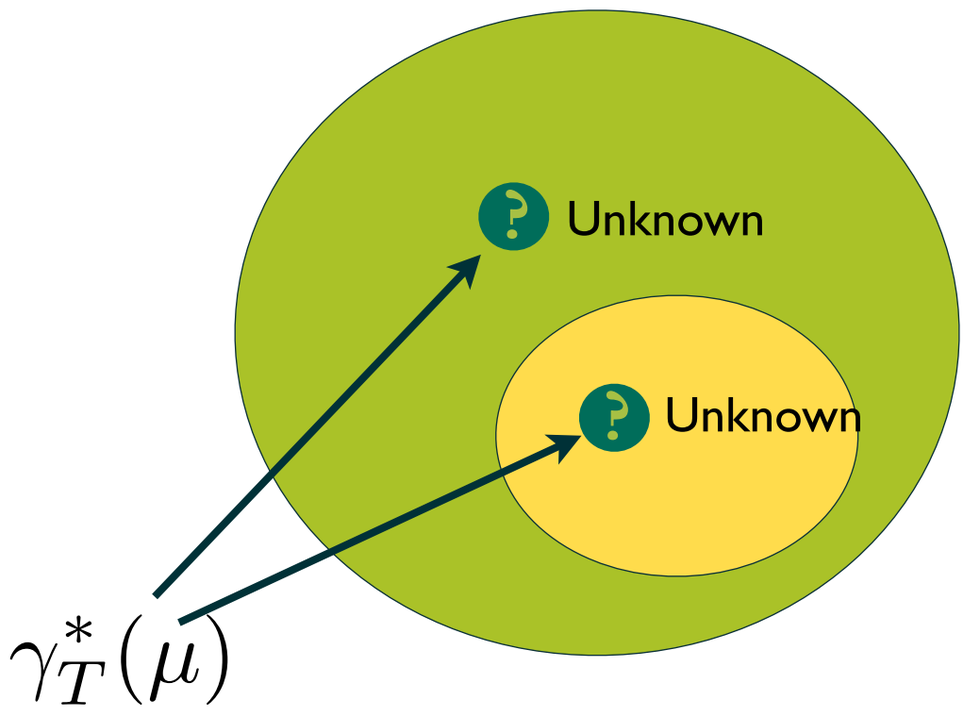
For positive template T

Case 1



Answer No

Case 2



Random Answer!

Implementation

- Basic examples work:
  - Array initialization program
  - Maximal element finding program
- Working on more complex examples
  - Selection Sort
  - Binary Search

Thank You