

Endu

Introduction to Linear Algebra  
- Gilbert Strang



Orthogonality

Eigenvalues & Eigenvectors



# SNOWBOARDING

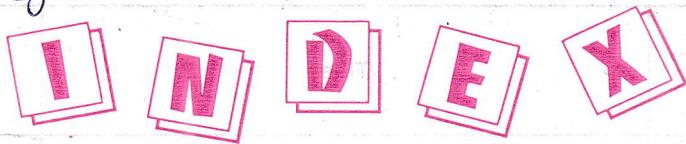
## EXTREME SPORT

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S. No.	Date	Title	Page No.	Teacher's Sign / Remarks
		<p style="text-align: center;"><u>INTRODUCTION TO LINEAR ALGEBRA</u></p> <p style="text-align: center;">- Gilbert Strang, MIT (5<sup>th</sup> Edition)</p>		

24. Find a basis for the subspace  $S$  in  $\mathbb{R}^4$   
 spanned by all solutions of  
 $\alpha_1 + \alpha_2 + \alpha_3 - \alpha_4 = 0$

Ans:  $\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = 0$

$S = N(A) : \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(B) Find a basis for the orthogonal complement  $S^\perp$

Ans:  $C(A^T) \perp N(A)$

Basis for  $S^\perp$  is  $(1, 1, 1, -1)$

② Find  $b_1$  in  $S$  and  $b_2$  in  $S^\perp$  so that  
 $b_1 + b_2 = b = (1, 1, 1, 1)$

Ans:  $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \alpha c_1 + \alpha_0 c_0 = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

 $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccccc|c} 0 & -1 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 2 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

$$b_1 = x_{1n} = \left[ \begin{array}{c} 1 \\ 1 \\ -1 \\ 1 \end{array} \right] + 0 + 2 \left[ \begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array} \right] = \pm (a) \quad \text{and} \quad b_2 = x_{2n} = \left[ \begin{array}{c} 1 \\ 1 \\ 1 \\ 0 \end{array} \right]$$

$$\left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & y_2 & -y_2 & 0 \\ 0 & 0 & 3/2 & -y_2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & y_2 & -y_2 & 0 \\ 0 & 0 & 1 & -y_2 & 0 \\ 0 & 0 & 0 & 4/3 & 2 \end{array} \right]$$

$$\left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & y_2 & -y_2 & 0 \\ 0 & 0 & 1 & -y_3 & 0 \\ 0 & 0 & 0 & 1 & 3/2 \end{array} \right] \xrightarrow{\text{Row operations}} \left[ \begin{array}{ccccc|c} 1 & -1 & -1 & 0 & -y_2 \\ 0 & 1 & y_2 & 0 & 3/4 \\ 0 & 0 & 1 & 0 & y_2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & 3y_2 \\ 0 & 0 & 0 & 1 & 3/2 \end{array} \right] \rightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & y_2 \\ 0 & 1 & 0 & 0 & y_2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3/2 \end{array} \right]$$

oder zwei linke Zeilen tauschen

Zeile 1:  $b_1 = \alpha_1 n = \frac{1}{2}$

Zeile 2:  $b_2 = \alpha_2 n = \frac{1}{2}$

Zeile 3:  $b_3 = \alpha_3 n = \frac{3}{2}$

Zeile 4:  $b_4 = \alpha_4 n = \frac{3}{2}$

$\Rightarrow$   $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_2 \\ y_2 \\ 3/2 \end{bmatrix}$

$$b_2 = \alpha_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} y_2 \\ y_2 \\ y_2 \\ -y_2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

rechts stehendes Gleichungssystem mit  
 $a = 2d - b_2$  reduzieren

25. If  $ad - bc > 0$ , the entries in  $A = QR$  are

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{\begin{bmatrix} a & -c \\ c & a \end{bmatrix}}{\sqrt{a^2 + c^2}} \cdot \frac{\begin{bmatrix} a^2 + c^2 & ab + cd \\ 0 & ad - bc \end{bmatrix}}{\sqrt{a^2 + c^2}}$$

26.

Write  $A = QR$  when  $a, b, c, d = 2, 1, 1, 1$  and also  $1, 1, 1, 1$ . Which entry of  $R$  becomes zero when the columns are dependent & Gram-Schmidt breaks down?

Ans: Non-singular example.

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix}$$

Singular example:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

The Gram-Schmidt process breaks down when  $ad - bc = 0$ .

30. The 1<sup>st</sup> 4 wavelets are in the columns  
in this wavelet matrix W:

$$W = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

What is special about the columns?

Find the inverse wavelet transform  $W^{-1}$ ?

Ans: The columns of the wavelet matrix  $W$  are  
orthonormal.

$$\therefore W^{-1} = W^T$$

32. If  $u$  is a unit vector, then  $Q = I - 2uu^T$

is a reflection matrix. (Householder reflection matrix).

Find  $Q_1$  from  $u = (0, 1)$  and  $Q_2$  from  $u = \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Draw the reflections when  $Q_1$  and  $Q_2$  multiply the vectors  $(1, 2)$  and  $(1, 1, 1)$

Ans:

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

reflects across x-axis

$$Q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \times \frac{1}{2}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

reflects across y+z=0 plane

33. Find all matrices that are both orthogonal &

lower triangular

Ans:

$$Q = \begin{bmatrix} q_{11} & 0 & 0 & \cdots & 0 \\ q_{21} & q_{22} & 0 & \cdots & 0 \\ q_{31} & q_{32} & q_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_{m1} & q_{m2} & q_{m3} & \cdots & q_{mn} \end{bmatrix}$$

$$Q^T Q = I$$

$$\begin{bmatrix} q_{11} & q_{21} & q_{31} \\ 0 & q_{22} & q_{32} \\ 0 & 0 & q_{33} \end{bmatrix} \begin{bmatrix} q_{11} & 0 & 0 \\ q_{21} & q_{22} & 0 \\ q_{31} & q_{32} & q_{33} \end{bmatrix} = \begin{bmatrix} q_{11}^2 + q_{21}^2 + q_{31}^2 & q_{21} q_{31} & q_{31} q_{32} \\ q_{21} q_{31} & q_{22}^2 + q_{32}^2 & q_{32} q_{33} \\ q_{31} q_{32} & q_{32} q_{33} & q_{33}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$q_{11}^2 + q_{21}^2 = q_{22}^2 = q_{33}^2 = 1$$

$\Rightarrow$   $\pm 1$  on the main diagonal & zeros elsewhere.

## 6

EIGEN VALUES & EIGEN VECTORSLINEAR ALGEBRA

Part 1 :  $A\mathbf{x} = \mathbf{b}$

Balance, equilibrium, steady state.

Part 2 : is about change - time enters the picture  
 cont. time is a differential equation  $\frac{du}{dt} = Au$  (or)  
 time steps is a difference equation  $U_{k+1} = AU_k$ .

These equations are not solved by elimination.

Suppose, the solution vector  $u(t)$  stays in the direction of a fixed vector  $\mathbf{x}$ . Then we need to only find the # (changing with time) that multiplies  $\mathbf{x}$ .

(A) # is easier than a vector.

$\Rightarrow$  We want "eigenvectors"  $\mathbf{x}$  that don't change direction when you multiply by  $A$ .

29/07/2019 V. W. Bhat, EASY 130/1

Suppose,

you need the 100<sup>th</sup> power  $A^{100}$ .

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}, A^2 = \begin{bmatrix} 0.70 & 0.45 \\ 0.30 & 0.55 \end{bmatrix}, A^3 = \begin{bmatrix} 0.650 & 0.525 \\ 0.350 & 0.475 \end{bmatrix}$$

$A^{100}$ 's columns are very close to the eigenvector  $(0.6, 0.4)$ .

$$A^{100} \approx \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

$A^{100}$  was found by using the eigenvalues of  $A$ , not by multiplying 100 matrices.

$$\lambda_1 = 1 \quad \& \quad \lambda_2$$

$$A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = 1 \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} \Rightarrow A^{100} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$$

symbolic link lost at conversion time. It's

the  $A^{100}$  flattening of vector argument

\* Almost all vectors change direction when they are multiplied by  $A$ . Certain exceptional vectors  $\alpha$  are in the same direction as  $A\alpha$ . Those are the eigenvectors.

The basic equation is  $A\alpha = \lambda\alpha$ . The #  $\lambda$  is an eigenvalue of  $A$ .

The eigenvalue  $\lambda$  tells whether the special vector  $\alpha$  is stretched or shrunk or reversed or left unchanged, when it is multiplied by  $A$ .  
The eigenvalue  $\lambda$  could be zero. Then  $A\alpha = 0\alpha$  means that this eigenvector  $\alpha \in N(A)$ .

If  $A = I$ , every vector has  $A\alpha = I\alpha = \alpha$ .

All vectors are eigenvectors of  $I$ . All eigenvalues,  $\lambda = 1$ .

$\lambda = 0 \rightarrow  A  = 0$ , $A$ is singular
$A\vec{\alpha} = 0\vec{\alpha}$
$\vec{\alpha} \in N(A)$

Ex:1.

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

Dns.  $\det \begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\frac{1}{2})$

For  $\lambda=1 \Rightarrow \lambda=\frac{1}{2}$

$\det(A-\lambda I) = 0$

$\Rightarrow$  The eigenvectors  $\alpha_1, \alpha_2 \in N(A-I)$  &  $N(A-\frac{1}{2}I)$

$$A\alpha_1 = A \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \alpha_1$$

$$A\alpha_2 = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2}\alpha_2$$

\* All other vectors are combinations of the 2 eigenvectors

$$\alpha_1 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \alpha_1 + 0.2 \alpha_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix}$$

$$A\alpha_1 = A \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \alpha_1 + \frac{1}{2}(0.2)\alpha_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$$

$$A^{99} \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = \alpha_1 + \left(\frac{1}{2}\right)^{99} 0.2 \alpha_2 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} \text{very small vector} \end{bmatrix}$$

The eigenvector  $\alpha_1$  is a "steady state" that doesn't change (because  $\lambda_1=1$ ). The eigenvector  $\alpha_2$  is a "decaying mode" that virtually disappears (because  $\lambda_2=\frac{1}{2}$ ). The higher the power of  $A$ , the more closely its columns approach the steady state.

$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$  is a Markov matrix

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = 0.6 + 0.2 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{0}$$

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} = 0.6 + 0.2 + 0.2 = \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \mathbf{0}$$

Ex: 2 The projection matrix,  $P = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$  has eigenvalue  $\lambda_1 = 1$  and  $\lambda_2 = 0$ .

Ques: Eigenvectors,  $\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ;  $\alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$P\alpha_1 = \alpha_1 \quad (\text{Steady state})$$

$$P\alpha_2 = 0 \quad (\text{Nullspace})$$

- 1. Markov matrix: each column of  $P$  adds to 1.  
So,  $\lambda=1$  is an eigenvalue
- 2.  $P$  is singular:  $\lambda=0$  is an eigenvalue
- 3.  $P$  is symmetric: eigenvectors  $(1,1)$  and  $(1,-1)$  are  $\perp$ .

### The Only eigenvalues

$P$ : projection matrix  $\rightarrow \lambda_1 = 0, P\alpha_1 = 0\alpha_1$  fills up the nullspace  
 $\lambda_2 = 1, P\alpha_2 = \alpha_2$  fills up the column space

Nullspace is projected to 0. The column space projects onto itself. The projection keeps the column space and destroys the nullspace.

$$V = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow PV = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ ist } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ ein entkoppeltes Profil}$$

(drei Phasen)

Projektion

Bei 3 Stufen der Komplexität kann es zu einer automatischen Verarbeitung von 3 Phasen führen.

Autorenprojektionen sind in drei Phasen unterteilt:

Projektion (P) nutzt weniger Zeit als die eigene Projektion (Q).

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Autorenprojektionen sind in drei Phasen unterteilt:

Autorenprojektionen sind in drei Phasen unterteilt:

Ex:3

Projekt  
habe ei

Permutations have all  $|A|=1$ , motion of line. It's a reflection across a perpendicular line to the line.

Ex:3. The reflection matrix  $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  has eigenvalues 1 and -1.

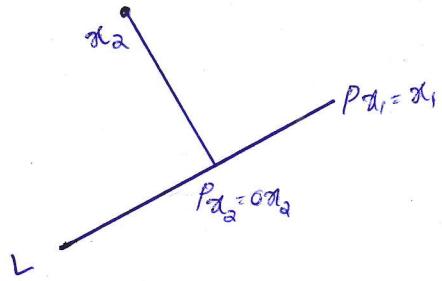
$R$  is a reflection & at the same time a permutation.

$$\lambda_1 = 1, \alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

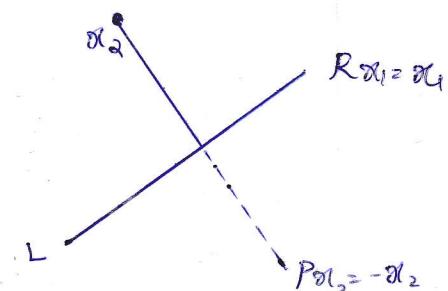
$$\lambda_2 = -1, \alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvectors for  $R$  are the same as for  $P$ , because, reflection = 2 (projection) - I :

$$R = 2P - I : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Projection onto L  
have eigenvalues 1 and 0



Reflection across line L  
 $R$  have eigenvalues 1 and -1

- When a matrix is shifted by  $I$ , each  $\lambda$  is shifted by  $1$ . No changes in eigenvectors.

- Eq<sup>n</sup> for the Eigenvalues  $\lambda$  of the matrix  $A$  is  
the equation of the characteristic polynomial.

$$A\alpha = \lambda \alpha \implies (A - \lambda I)\alpha = 0$$

$\Rightarrow$  The eigenvectors make up the  $N(A - \lambda I)$ .

If  $(A - \lambda I)\alpha = 0$  has a non-zero solution,  
 $(A - \lambda I)$  is not invertible  $\Rightarrow \det(A - \lambda I) = 0$ .

- \* The #  $\lambda$  is an eigenvalue of  $A$  iff  $A - \lambda I$  is singular. i.e.,  $\det(A - \lambda I) = 0$

$\det(A - \lambda I)$  is a polynomial function of the variable  $\lambda$  of the degree  $n$ .  $\square$

∴ Its term of degree  $n$  is always  $(-1)^n \lambda^n$ .

$$\det(A - \lambda I) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

$$= (-1)^n \left[ \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) \lambda^{n-1} + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \cdots + \lambda_{n-1} \lambda_n) \lambda^{n-2} - \cdots - (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \right]$$

OM(23)

$$\det(A - \lambda I) = (-1)^n \left[ \lambda^n - \lambda^{n-1} \sum_i \lambda_i + \lambda^{n-2} \sum_{ij} \lambda_i \lambda_j - \cdots - (-1)^n \prod_{i=1}^n \lambda_i \right]$$

$$= (-1)^n \left[ \lambda^n - \lambda^{n-1} \cdot \text{tr}(A) + \cdots + (-1)^n \cdot \det(A) \right]$$

□ Determinant & Trace  $\rightarrow$  ~~is multiplicative~~  
~~longer~~

on (2) Bad news: Elimination does not preserve the  $\lambda$ 's

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \xrightarrow{\text{Row 2} - 2 \cdot \text{Row 1}} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \rightarrow \lambda = 0 \quad (\text{or}) \quad \lambda = 2$$

~~$R = U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}, \lambda = 0, \lambda = 1$~~

Good news:

$$\text{tr}(A) = \sum_{i=1}^n \lambda_{ii} = \sum_{i=1}^n \lambda_i$$

$$\det(A) = \prod_{i=1}^n \lambda_i$$

\* Eigenvalues of a triangular matrix lie along  $\square$   
its diagonal.

Proof

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = 0$$

$$\Rightarrow \lambda \in \{a_{11}, a_{22}, \dots, a_{nn}\} = 0 \Rightarrow$$

$$\text{If } \sum_{i=1}^n a_{ii} = \lambda \sum_{i=1}^n b_i \text{ then }$$

□ Imaginary Eigenvalues

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = Q$$

$$\begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix}$$

$\sin(90^\circ) = 1$

Ex:5  $\text{Rot}(90^\circ) = Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  has no real eigenvalues.

$$\lambda_1 = i \quad \& \quad \lambda_2 = -i$$

$$\lambda_1 + \lambda_2 = \text{tr}(Q) = 0$$

$$\det(Q) = \lambda_1 \lambda_2 = 1 \quad \& \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

After a rotation, no real vector  $Qx_0$  stays in the same direction as  $x_0$ . ( $x_0 = 0$  is useless)  
as  $\det(Q) \neq 0$

$$Q = \text{Rot}(90^\circ)$$

$$Q^2 = \text{Rot}(180^\circ) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

Its eigenvalues are  $\lambda = -1$  and  $-1$ .

Squaring  $Q$  will square each  $\lambda$ , so we must have

$$\lambda^2 = -1$$

∴ The eigenvalues of the  $90^\circ$  rotation matrix  $Q$  are  $+i$  and  $-i$ , because  $i^2 = -1$ .

$$Q_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{euler angle representation } \square$$

$$\det(Q_2 - \lambda I) = 0 \implies \boxed{\lambda^2 + 1 = 0}$$

Eigenvalues are:  $\lambda_1 = i$ ,  $\lambda_2 = -i \Rightarrow$  pure

Complex  
eigenvectors:

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix} \quad \text{for } \lambda = i$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{for } \lambda = -i$$

- $Q$  is orthogonal matrix  $\implies |\lambda| = 1 \iff \lambda$

It's because of the eulerAngles

- $Q$  is skew symmetric  $\implies \lambda$  is purely imaginary

$\therefore$  rotation matrix is not diagonalizable

It's something but it's not

rotation matrix is not diagonalizable

- A symmetric matrix ( $S^T = S$ ) can be compared to a real #.
- A skew symmetric matrix ( $A^T = -A$ ) can be compared to an imaginary #.
- An orthogonal matrix ( $Q^T Q = I$ ) corresponds to a complex # with  $|z| = 1$

Eigenvalues of  $AB$  &  $A+B$

Gives? On eigenvalue  $\lambda$  of  $A$  times an eigenvalue  $\beta$  of  $B$  usually does not give an eigenvalue of  $AB$ .

**False proof:**  $AB\alpha = A\beta\alpha = \beta A\alpha = \beta\lambda\alpha$

seems like  $\beta\lambda$  is an eigenvalue of  $AB$ !

Note —

This proof is correct, when  $\alpha$  is an eigenvector for  $A$  and  $B$ .

Similarly,

the eigenvalues of  $A+B$  are generally not  $\lambda+\beta$ .

for ex. if  $A$  and  $B$  are  $2 \times 2$  matrix

$$VS = \lambda A$$

$$(V+A)S = VAS + AAS = V(\lambda A) + A(V\lambda) = V(\lambda A) + (\lambda V)A$$

$V$  &  $A$   $\neq$  commutes not of  $V\lambda = \lambda V$

and  $V$  &  $B$  defines rules  $B = VB$

then  $V$  &  $B$  also  $V \leftarrow VB \leftarrow$

Suppose,  $\alpha$  really is an eigenvector for both  $A$  and  $B$ . Then,

$$AB\alpha = \lambda_1 B\alpha \quad \& \quad BA\alpha = \lambda_2 A\alpha$$

When all  $n$  eigenvectors are shared, we can multiply eigenvalues.

\*  $A$  and  $B$  share the same  $n$  independent eigenvectors iff  $AB = BA$ .

### Proof

Assume that  $AB = BA$  and that  $v$  is an eigenvector of  $A$ ,

$$Av = \lambda v$$

$$A(Bv) = (AB)v = (BA)v = B(Av) = B(\lambda v) = \lambda(Bv)$$

$\therefore Bv$  is an eigenvector of  $A$

$\therefore Bv$  is a scalar multiple of  $v$

$$\Rightarrow Bv = \mu v \Rightarrow v \text{ is also an eigenvector of } B.$$

(OR)

If A & B have the same eigenvectors.

$$A = P D_A P^{-1} \text{ and } B = P D_B P^{-1}$$

$$\begin{aligned} AB &= P D_A P^{-1} P D_B P^{-1} = P D_A D_B P^{-1} = P D_B D_A P^{-1} \\ &= P D_B P^{-1} P D_A P^{-1} = BA \end{aligned}$$

- \* Two diagonalizable matrices are simultaneously diagonalizable iff they commute.

on (23)

## Heisenberg's uncertainty principle

In quantum mechanics, the position matrix  $P$  and the momentum matrix  $Q$  do not commute.

In fact,  $\cancel{P} \cancel{Q} \cancel{Q} \cancel{P} =$   
 $Q P - P Q = I$

To have  $P\alpha=0$  at the same time as  $Q\alpha=0$  would require  $\alpha=I\alpha=0$ .

⇒ If we knew the position exactly, we could not also know the momentum exactly.

S.1(a)

Find the eigenvectors & eigenvalues of  $A, A^2, A^T$   
and  $A^T + 4I$ .

**6.1(B)** How can you estimate the eigenvalues of any  $A$ ?

Gershgorin Circle theorem

~~check  
ILAC(i)~~

Every eigenvalue of  $A$  must be "near" at least one of the entries  $a_{ii}$  on the main diagonal.

For  $\lambda$  to be "near"  $a_{ii}$  means that  $|a_{ii} - \lambda|$  is no more than the sum  $R_i$  of all other  $|a_{ij}|$  in that row  $i$  of the matrix. Then

$R_i = \sum_{j \neq i} |a_{ij}|$  is the radius of a circle centered

at  $a_{ii}$  from its own bound  $R_i = R$  radius.

- \* Every  $\lambda$  is in the circle around one or more diagonal entries  $a_{ii}$ :  $|a_{ii} - \lambda| \leq R_i$

Reasoning: If  $\lambda$  is an eigenvalue, then  $A - \lambda I$  is not invertible. Then  $A - \lambda I$  can not be diagonally dominant. So at least one diagonal entry  $a_{ii} - \lambda$  is not larger than the sum  $R_i$  of all other entries  $|a_{ij}|$  in row  $i$ .

Ex:  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

First circle:  $|z-a| \leq |b| = R_1$

$2^{\text{nd}}$  circle:  $|z-d| \leq |c| = R_2$

innermost circle  $|z-a| \leq |b| = R_1$

These are circles in the complex plane, since  $a$  could  
certainly be complex.

$$A = \begin{bmatrix} d_1 & 1 & 2 \\ 2 & d_2 & 1 \\ -1 & 2 & d_3 \end{bmatrix}$$

$|z-d_1| \leq 1+2=3=R_1$

$|z-d_2| \leq 2+1=3=R_2$

$|z-d_3| \leq 1+2=3=R_3$

All eigenvalues of this 'A' lie in a circle of radius  $R=3$  around one or more of the diagonal entries  $d_1, d_2, d_3$ .

$$R = |R_{\max}| = \text{maximum length}$$

# Diagonalizing a Matrix

- When  $\vec{x}$  is an eigenvector, multiplication by ' $A$ ' is just multiplication by a number ' $\lambda$ '.  
All the difficulties of matrices are swept away.  
Instead of an interconnected system, we can follow the eigenvectors separately. It is like having a diagonal matrix with no off-diagonal interconnections.

→ The matrix  $A$  turns into a diagonal matrix  $\Lambda$  when we use the eigenvectors properly.

Chlorophytum comosum (A. Nels.) Schultes

Diagonalization: Suppose  $A_{n \times n}$  has  $n$  linearly independent eigenvectors  $\alpha_1, \dots, \alpha_n$ . Put them into the columns of an eigenvector matrix  $X$ . Then  $X^{-1}AX$  is the eigenvalue matrix  $\Lambda$ :

$$X^T A X = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

The matrix A is diagonalized.

$$AX = A \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{bmatrix} = \begin{bmatrix} A\alpha_1 & A\alpha_2 & \dots & A\alpha_n \end{bmatrix}$$

$$\text{If } \text{not diagonal} = \begin{bmatrix} \lambda_1 \alpha_1 & \lambda_2 \alpha_2 & \dots & \lambda_n \alpha_n \end{bmatrix}$$

Rücknahme der  $\lambda$  möglichkeit bei  $A$

$$\text{we can write} = \begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} = X\Lambda$$

so that each column contains only  $\alpha_i$

$$AX = X\Lambda \Rightarrow X^{-1}AX = \Lambda \Rightarrow A = X\Lambda X^{-1}$$

$X$  kann bis zu  $n$  linear unabh. Spalten haben

• Eigenvectors sind linear unabh.

The matrix  $X$  has an inverse, because its columns (the eigenvectors of  $A$ ) were assumed to be linearly independent.

$\Rightarrow$  Without  $n$  independent eigenvectors, we can't write  $X$  with columns as  $\alpha_1, \alpha_2, \dots, \alpha_n$

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_n \end{bmatrix} = \Lambda = X\Lambda X^{-1}$$

Eigenvectors in  $A$  wieder herstellen

•  $A$  and  $\Lambda$  have the same eigenvalues  $\lambda_1, \dots, \lambda_n$ .  
The eigenvectors are different.

• The eigenvectors are not unique because

•  $\text{adj}(X)$  is the inverse of  $X$ .

$$A^k = X \Lambda X^{-1} X \Lambda X^{-1} \cdots X \Lambda X^{-1} = X \Lambda^k X^{-1}$$

•  $\text{adj}(X)$  is also unique.

- on(23)  
proof*
- Suppose, the eigenvalues  $\lambda_1, \dots, \lambda_n$  are all different. Then it is automatic that the eigenvectors  $\alpha_1, \dots, \alpha_n$  are independent.  
The eigenvector matrix  $X$  will be invertible.  
 $\Rightarrow$  Any matrix that has no repeated eigenvalues can be diagonalized.

- We can multiply eigenvectors by any non-zero constant.

$A(c\alpha) = c(A\alpha)$  is still true

• Some matrices have too few eigenvectors. Those matrices can not be diagonalized.

Ex:  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  has 2 linearly independent eigenvectors but no enough eigenvectors to form a basis for  $\mathbb{R}^2$ .

\* There is no connection b/w invertibility and diagonalizability:

- Invertibility is concerned with the eigenvalues ( $\lambda = 0$  or  $\lambda \neq 0$ )
- Diagonalizability is concerned with the eigenvectors (too few or enough for  $X$ ).

\* Eigen vectors  $\alpha_1, \alpha_2, \dots, \alpha_j$  that correspond to distinct (all different) eigenvalues are linearly independent. An  $n \times n$  matrix that has  $n$  different eigenvalues (no repeated  $\lambda$ 's) must be diagonalizable.

### Proof

Suppose,

$$c_1\alpha_1 + c_2\alpha_2 = 0$$

$$\text{and } A(c_1\alpha_1 + c_2\alpha_2) = c_1\lambda_1\alpha_1 + c_2\lambda_2\alpha_2 = 0$$

$$c_1\lambda_1\alpha_1 + c_2\lambda_2\alpha_2 = 0.$$

$$(\lambda_1 - \lambda_2)c_1\alpha_1 = 0 \text{ (since } \alpha_1 \neq 0)$$

$$\lambda_1 \neq \lambda_2 \text{ and } \alpha_1 \neq 0 \implies c_1 = 0$$

$$\text{Similarly } c_2 = 0.$$

Only the combination with  $c_1 = c_2 = 0$  gives

$$c_1\alpha_1 + c_2\alpha_2 = 0.$$

$\therefore$  Eigen vectors  $\alpha_1$  and  $\alpha_2$  must be independent.

This proof extends directly to  $j$  eigenvectors.

(QD) By induction

to  
early  
as

A = 3 2 0 0 0

6 8

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

different values of A



values of A

$$[1] \Rightarrow [0 0] \text{ or }$$

$$XAX = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ or } A$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = X^TAX = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A = 0 \text{ or } A = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Ex:9 . Powers of A

The Markov matrix  $A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$

$$\lambda_1 = 1, \lambda_2 = 0.5$$

$$\alpha_1 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix} = X \Lambda X^{-1}$$

$$A^k = X \Lambda^k X^{-1} = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & (0.5)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$$k \rightarrow \infty : A^\infty = \begin{bmatrix} 0.6 & 1 \\ 0.4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0.4 & -0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 \\ 0.4 & 0.4 \end{bmatrix}$$

- $A^k \rightarrow$  zero matrix when all  $|z_i| < 1$ .



$X^{-1}$

## □ Similar Matrices : Same Eigenvalues

Suppose the eigenvalue matrix  $\Lambda$  is fixed.

As we change the eigenvector matrix  $X$ , we get a whole family of different matrices

$$A = X\Lambda X^{-1} \text{ - all with the same eigenvalues in } \Lambda.$$

All those matrices  $A$  (with the same  $\Lambda$ ) are called similar.

This idea extends to matrices that can't be diagonalized. We choose one constant matrix  $C$  (not necessarily  $\Lambda$ ). And we look at the whole family of matrices  $A = BCB^{-1}$ , allowing all invertible matrices  $B$ . Those matrices  $A$  and  $C$  are called similar.

$C$  might not be diagonal.

Columns of  $B$  might not be eigenvectors.

We only require  $B$  is invertible.

Similar matrices  $A$  &  $C$  have the same eigenvalues.

\* All the matrices  $A = BCB^{-1}$  are similar.  
 They all share the eigenvalues of  $C$

Q.M(23)

Proof

① Suppose,  $C\alpha = \lambda\alpha$

$$(BCB^{-1})(B\alpha) = B(C\alpha) = B\lambda\alpha = \lambda(B\alpha)$$

$\Rightarrow$  same  $\lambda$

$$\begin{aligned} ② P_A(\lambda) &= \det(A - \lambda I) = \det(BCB^{-1} - \lambda BIB^{-1}) \\ &= \det(B(C - \lambda I)B^{-1}) = \det(B) \det(C - \lambda I) \det(B^{-1}) \\ &= \det(B) \det(B^{-1}) \det(C - \lambda I) = \det(BB^{-1}) \det(C - \lambda I) \\ &= \det(I) \det(C - \lambda I) = \det(C - \lambda I) = P_C(\lambda) \end{aligned}$$

$\Rightarrow$  The characteristic polynomials of  $A$  and  $B$   
 are equal

$\therefore$  Equal eigenvalues.

## Fibonacci Numbers

The Fibonacci sequence

0, 1, 1, 2, 3, 5, 8, 13, ... comes from

$$F_{k+2} = F_{k+1} + F_k$$

Problem : Find the Fibonacci number  $F_{100}$

Let,  $U_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$

$$U_{k+1} = AU_k$$

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} = \overline{F_{k+1}}$$

$$U_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}$$

$$U_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} U_k$$

$$U_k = AU_k$$

$$U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, U_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, U_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, U_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \dots$$

$$U_{100} = A^{100} U_0 = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda-\lambda & 1 \\ 1 & 1-\lambda-\lambda \end{bmatrix} = \begin{bmatrix} 1-2\lambda & 1 \\ 1 & 1-2\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 1 = 0$$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{5}}{2}$$

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \approx 1.618 ; \quad \lambda_2 = \frac{1-\sqrt{5}}{2} \approx -0.618$$

$$(A - \lambda_1 I)x = 0 \Rightarrow \begin{bmatrix} (1-\lambda_1)t_1 + t_2 \\ b_1 - \lambda_1 t_2 \end{bmatrix} = 0$$

$$(1-\lambda_1)t_1 + \lambda_1 t_2 = 0$$

$$b_1 - \lambda_1 t_2 = 0$$

$$b_1 - (\lambda_1^2 + \lambda_1 + 1) = 0 \quad \& \quad t_2 = \frac{b_1}{\lambda_1}$$

$$x_1 = \text{span} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}, \quad x_2 = \text{span} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$U_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{\begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} - \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}}{\lambda_1 - \lambda_2} = \frac{\alpha_1 - \alpha_2}{\lambda_1 - \lambda_2}$$

$$U_{100} = A^{100} U_0 = A^{100} \frac{\alpha_1}{\lambda_1 - \lambda_2} - A^{100} \frac{\alpha_2}{\lambda_1 - \lambda_2}$$

$$= \frac{1}{\lambda_1 - \lambda_2} \left[ A^{100} \alpha_1 - A^{100} \alpha_2 \right] = \frac{1}{\lambda_1 - \lambda_2} \left[ (\lambda_1)^{100} \alpha_1 - (\lambda_2)^{100} \alpha_2 \right]$$

$$U_{100} = \frac{(\lambda_1)^{100} \alpha_1 - (\lambda_2)^{100} \alpha_2}{\lambda_1 - \lambda_2} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$\alpha_1 = \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix} \quad \alpha_2 = \begin{bmatrix} \alpha_2 \\ 1 \end{bmatrix}$$

$$U_{100} = \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix}$$

$$U_{100} = \frac{\lambda_1^{100}}{\lambda_1 - \lambda_2} \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix} - \frac{(\lambda_2)^{100}}{\lambda_1 - \lambda_2} \begin{bmatrix} \alpha_2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\lambda_1)^{100} - (\lambda_2)^{100}}{\lambda_1 - \lambda_2} \\ \frac{(\lambda_2)^{100} - (\lambda_2)^{100}}{\lambda_1 - \lambda_2} \end{bmatrix} = \begin{bmatrix} F_{101} \\ F_{100} \end{bmatrix}$$

$$\lambda_1 - \lambda_2 = \sqrt{5} \quad \& \quad (\lambda_2)^{100} \approx 0$$

100th Fibonacci #,

$$F_{100} = \frac{\lambda_1^{100} - \lambda_2^{100}}{\lambda_1 - \lambda_2} = \text{nearest integer to } \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{100}$$

Every  $F_k$  is a  $\checkmark$  whole #.

\* The ratio  $\frac{F_{101}}{F_{100}}$  must be very close to the limiting

ratio  $\frac{1 + \sqrt{5}}{2}$ . — golden mean.  $\approx 1.61$

## Matrix Powers, $A^k = \underline{U}$

Fibonacci's example is a typical difference equation

$$\boxed{A^k X = U} \quad \boxed{U_{k+1} = AU_k}$$

Each step multiplies by  $A$ .

The solution is  $\boxed{U_k = A^k u_0}$

$$AU_0 = (X \wedge X^{-1})^k U_0 = (X \wedge X^{-1})(X \wedge X^{-1}) \dots \dots (X \wedge X^{-1}) U_0$$

$$= (X \wedge X^{-1})^k U_0$$

$$U_{k+1} = A U_k$$

- ① Write  $U_0$  as a combination  $c_1 \alpha_1 + \dots + c_n \alpha_n$  of the eigenvectors.

$$U_0 = Xc = [v_1 \ v_2 \ \dots \ v_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow c = X^{-1} U_0$$

- ② Multiply each eigenvector  $\alpha_i$  by  $(\lambda_i)^k$ .

- ③ Add up the pieces  $c_i (\lambda_i)^k \alpha_i$  to find the solution,

$$U_k = A^k U_0 = (X \Lambda^k X^{-1}) U_0$$

Solution for  $U_{k+1} = AU_k$

$$U_k = A^k U_0 = c_1(\lambda_1)^k \alpha_1 + \dots + c_n(\lambda_n)^k \alpha_n$$

$$= X \Lambda^k X^{-1} U_0 = X \Lambda^k C = [\alpha_1 \dots \alpha_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Ex:3. Start from  $u_0 = (1, 0)$ . Compute  $A^k u_0$  for this faster Fibonacci

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \text{ has } \lambda_1 = 2 \text{ & } \alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \quad \text{and} \quad \alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$F_{k+2} = F_{k+1} + 2F_k$$

~~Their growth~~ The new #'s start with 0, 1, 1, 3. They grow faster because of  $\lambda=2$

$$\text{Qn: } u_{k+1} = A^k u_k \Rightarrow u_k = A^k u_0$$

$$u_0 = c_1 \alpha_1 + c_2 \alpha_2$$

$$u_k = c_1 (\lambda_1)^k \alpha_1 + c_2 (\lambda_2)^k \alpha_2$$

$$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow c_1 = c_2 = \frac{1}{3}$$

$$u_k = A^k u_0 = \frac{1}{3} 2^k \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3} (-1)^k \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$$

Ex:

8

this

$$F_k = \frac{1}{3} (2^k - (-1)^k)$$

Ex: Fourier series built from the eigenvectors  $e^{ikx}$

$$g \frac{d}{dx}$$

$$g(x=0)$$

$$\frac{1}{3}$$

$$+1$$

## □ Non diagonalizable Matrices

Suppose  $\lambda$  is an eigenvalue of  $A$ .

① Eigenvectors (geometric) : There are non-zero solutions to  $A\mathbf{x} = \lambda \mathbf{x}$

② Eigenvectors (algebraic) : The determinant of  $(A - \lambda I)$  is zero.

There are 2 ways to count the multiplicity of eigenvalues: Always  $A \cdot M \geq G \cdot M$  for each  $\lambda$

① (Geometric Multiplicity = GM)

- count the independent eigenvectors for  $\lambda$ :

$$G \cdot M = \dim(N(A - \lambda I))$$

② (Algebraic Multiplicity = AM)

- counts the repetitions of  $\lambda$  among the eigenvalues.

Look at the  $n$  roots of  $\det(A - \lambda I) = 0$

If  $A$  has  $\lambda = 4, 4, 4$ , then that eigenvalue has  
 $AM = 3$  and  $GM = 1, 2$ , or 3.

Ex:-  $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$  has  $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$   
 $\Rightarrow \lambda = 0, 0$ , Lett  
1 eigen vector  
 $AM = 2$   
 $GM = 1$

$\Rightarrow$  This shortage of eigen vectors when  $GM$  is below  $AM$  means that ' $A$ ' is not diagonalizable.

Ex:-  $A = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$  has  $\begin{vmatrix} 5-\lambda & 1 \\ 0 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25 = 0$   
 $(\lambda - 5)^2 = 0$   
 $\underline{\lambda = 5, 5}$   
 $AM = 2$  &  $GM = 1$

Ex:-  $\begin{bmatrix} 6 & -1 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 7 & 2 \\ -2 & 3 \end{bmatrix}$

$\Rightarrow$  These matrices are not diagonalizable.

6.2 (A)

The Lucas numbers are like the Fibonacci numbers except they start with  $L_1 = 1$  and  $L_2 = 3$ . Using the same rule  $L_{k+2} = L_{k+1} + L_k$

the next Lucas numbers are  $4, 7, 11, 18$ .

Show that the Lucas number  $L_{100} = \lambda_1^{100} + \lambda_2^{100}$

Ques:

$$L_{k+2} = L_{k+1} + L_k$$

$$L_{k+1} = L_{k+1}$$

$$U_{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} U_k = AU_k$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2} \Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2}, \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\lambda_1(1-\lambda) t_1 + \lambda_2 t_2 = 0$$

$$t_1 - \lambda_1 t_2 = 0$$

$$\frac{t_1 [\lambda_1 - \lambda_2]}{t_1} = 0$$

$$\alpha_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$\alpha_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$U_1 = \begin{bmatrix} L_2 \\ L_1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1 \alpha_1 + c_2 \alpha_2 = c_1 \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\begin{aligned} c_1 \lambda_1 + c_2 \lambda_2 &= 3 \\ c_1 \lambda_2 + c_2 \lambda_2 &= \lambda_2 \\ c_1 (\lambda_1 - \lambda_2) &= 3 - \lambda_2 \end{aligned} \Rightarrow c_1 = \lambda_1 \text{ & } c_2 = \lambda_2$$

$$U_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 + \lambda_2^2 \\ \lambda_1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} \text{tr}(A^2) \\ \text{tr}(A) \end{bmatrix}$$

6.2(B)

$$U_{100} = \begin{bmatrix} L_{101} \\ L_{100} \end{bmatrix} = A^{99} U_1$$

$$= c_1(\lambda_1)^{99} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + c_2(\lambda_2)^{99} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$= (\lambda_1)^{100} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + (\lambda_2)^{100} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

Ans:

$$L_{100} = \lambda_1^{100} + \lambda_2^{100}$$

$$\begin{bmatrix} R \\ 1 \end{bmatrix} = \lambda_1^{100} \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} + \lambda_2^{100} \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} R \\ 1 \end{bmatrix} = \lambda_1^{100} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2^{100} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \lambda_1^{100} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + \lambda_2^{100} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$R = \lambda_1^{100} R + \lambda_2^{100} R$$

$\lambda_1 =$

$A$

The  
ort

6.2(B) Find the inverse & the eigenvalues and the determinant of this matrix  $A$ :  $A = X$

Describe an eigenvector matrix  $X$  that gives  $X^T A X = \Lambda$ .

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$$

Dns:  $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} + 5 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -B + 5I$

$$\lambda: 0, 0, 0, 4$$

$\lambda: 5, 5, 5, 1$

$$\lambda_1 = 1, \quad v_1 = c(1, 1, 1, 1)$$

$A^T = A$  ( $A$  is symmetric) : 1 eigenvectors.

The nicest eigenvector matrix  $X$  is the symmetric orthogonal Hadamard matrix  $H$ .

for example

$$X = H \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = H \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = H$$

using with  $X$

$$A = X A' X^{-1}$$

Eigenvalues of  $A'$ :  $1, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}$

The eigenvectors are not changed.

$$A^{-1} = X \Lambda^{-1} X^{-1}$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}^{-1} = A$$

$\Sigma = (\text{diag}) \text{wide}$

$\Sigma = \text{wide}$

$\Sigma = \text{wide}$

$\Sigma = \text{wide}$

$\Sigma = \text{wide}$

$(\text{wide}) \times \text{wide}$

product of  $\Sigma$  & (eigenvectors of  $A'$ )  $A^{-1}$

Identify with  $X$  which obtained from  $H$  with Schmidt's theorem

## □ System of Differential Equations

$$\frac{du}{dt} = \lambda u \xrightarrow{\text{solution}} u(t) = u(0) e^{\lambda t}$$

↳ The solutions that start from the #  $u(0)$  at time  $t=0$ .

If we have an  $n \times n$  system of ODE's. The unknown is a vector  $\vec{u}$ . It starts from the initial vector  $\vec{u}(0)$ , which is given. We expect  $n$  exponents  $e^{\lambda t}$  in  $u(t)$  with  $\lambda$ 's.

Systems of  $m$  equations

$$\frac{d\vec{u}}{dt} = A\vec{u}$$

starting from the vector

$$u(0) = \begin{bmatrix} u_1(0) \\ \vdots \\ u_n(0) \end{bmatrix} \quad \text{at } t=0$$

We'll need  $n$  constants to match the  $n$ -components of  $u(0)$ .

1<sup>st</sup> job — find  $n$  "pure exponential solutions"

$$u = e^{\lambda t} \text{ by using } A\vec{u} = \lambda \vec{u}$$

$A$  is a constant matrix.

In other linear equations, ' $A$ ' changes as ' $t$ ' changes.

In non-linear systems, ' $A$ ' changes as ' $u$ ' changes.

$$\text{order of } A = \frac{\partial f}{\partial u}$$

$$0 = \begin{bmatrix} f_1(u) \\ f_2(u) \\ \vdots \\ f_n(u) \end{bmatrix}$$

- All shear
- The
- If

□ Solution of  $\frac{d\vec{u}}{dt} = A\vec{u}$

Our pure exponential solution will be  $e^{\lambda t}$  times a fixed vector  $\vec{x}$ .

Guess -  $\lambda$  is an eigenvalue of  $A$ , and  $\vec{x}$  is the eigenvector

Prove - Substitute  $\vec{u}(t) = e^{\lambda t} \vec{x}$  into the equation

$$\frac{du}{dt} = A\vec{u} \Rightarrow \cancel{\text{when}} \lambda e^{\lambda t} \vec{x} = A e^{\lambda t} \vec{x} \\ \Rightarrow \underline{\lambda \vec{x} = A \vec{x}}$$

choose  $u = e^{\lambda t} \vec{x}$     {  $\frac{du}{dt} = \lambda e^{\lambda t} \vec{x}$  agrees with  $Au = A e^{\lambda t} \vec{x}$   
 when  $A\vec{x} = \lambda \vec{x}$

- All components of this special solution  $u = e^{\lambda t} \vec{x}$  share the same  $e^{\lambda t}$ .
- The solution grows when  $\lambda > 0$ , it decays when  $\lambda < 0$ .

• If  $\lambda \in \mathbb{C}$ ,

$\operatorname{Re}(\lambda)$  decides growth or decay

$\operatorname{Im}(\lambda)$  gives oscillation  $e^{i\omega t}$  like a sine wave.

Ex:1 Solve  $\frac{du}{dt} = Au = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} u$  starting from

$$u(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Ans: This is a vector equation for  $u$ .

It contains 2 scalar equations for the components  $y$  and  $z$ . They are "coupled together" because the matrix  $A$  is not diagonal.

$$\frac{du}{dt} = Au \Rightarrow \frac{d}{dt} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix}$$

$$\Rightarrow \frac{dy}{dt} = z \quad \text{and} \quad \frac{dz}{dt} = y$$

\* The idea of eigenvectors is to combine those eq's in a way that gets back to 1 by 1 problems.

$$\frac{dy}{dt} + \frac{dz}{dt} = \frac{d}{dt}(y+z) = \cancel{\frac{dy}{dt}} + \cancel{\frac{dz}{dt}} \quad \left| \begin{array}{l} \frac{dy}{dt} - \frac{dz}{dt} = \frac{d}{dt}(y-z) = -(y-z) \\ (y+z) = c_1 e^t \\ (y-z) = c_2 e^{-t} \end{array} \right.$$

$$(y+z) = c_1 e^t \quad (y-z) = c_2 e^{-t}$$

$$\begin{bmatrix} y \\ z \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

$$\alpha_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The pure exponential solutions  $u_1$  and  $u_2$

$$u_1(t) = e^{\lambda_1 t} \alpha_1 = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad u_2(t) = e^{\lambda_2 t} \alpha_2 = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\text{Complete solution: } u(t) = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + De^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} Ce^t + De^{-t} \\ Ce^t - De^{-t} \end{bmatrix}$$

$$u(0) = (u_1(0), u_2(0)) \quad \text{decides } C \text{ and } D$$

$$u(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = C \begin{bmatrix} 1 \\ 1 \end{bmatrix} + D \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \underline{C=3, D=1}$$

$$\frac{du}{dt} = Au$$

① Write  $u(0)$  as a combination  $c_1\alpha_1 + \dots + c_n\alpha_n$

of the eigenvectors of  $A$

$$u(0) = Xc = [\alpha_1 \dots \alpha_n] \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

② Multiply each eigenvector  $\alpha_i$  by its growth factor  $e^{\lambda_i t}$

③ The solution is the same combination of these new solutions  $e^{\lambda_i t}\alpha_i$ :

$$\frac{du}{dt} = Au \quad : \quad u(t) = c_1 e^{\lambda_1 t} \alpha_1 + \dots + c_n e^{\lambda_n t} \alpha_n$$

- If 2  $\lambda$ 's are equal, with only one eigenvector, another solution is needed. (It'll be  $t e^{\lambda t} \alpha$ )

?

A Defective Eigenvalue

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

A has repeated eigenvalues.

Say  $A$  is  $2 \times 2$ .

To form the general solution, we need 2 linearly independent solutions, but we only have one:

$$\vec{x}_1(t) = e^{\lambda t} \vec{x}_1$$

When we faced similar problem in the 2nd order linear case, we were able to work around it by multiplying the solution by  $t$ .

Let's try it here.

$$\vec{u}' = A\vec{u}$$

$$e^{\lambda t} \vec{x}_1 + \underline{\lambda t e^{\lambda t} \vec{x}_1} = A t e^{\lambda t} \vec{x}_1$$

$$\vec{x}_1 = 0 \implies \vec{x}_1 = 0$$

$$\lambda \vec{x}_1 e^{\lambda t} = A \vec{x}_1 e^{\lambda t} \implies (A - \lambda I) \vec{x}_1 = 0$$

$\alpha_1$  is an eigenvector of  $A$  &  $\lambda_1 = 0$ .

N.P  $\Rightarrow$  We need another approach

Lets guess

$$u(t) = e^{\lambda t} \alpha_1 + t e^{\lambda t} \alpha_2$$

$$u' = Au$$

$$\lambda e^{\lambda t} \alpha_1 + e^{\lambda t} \alpha_2 + \underline{\lambda t e^{\lambda t} \alpha_2} = A \left( e^{\lambda t} \alpha_1 + \underline{t e^{\lambda t} \alpha_2} \right)$$

$$(\lambda \alpha_1 + \alpha_2 + \underline{\lambda t \alpha_2}) e^{\lambda t} = (\lambda \alpha_1 + \underline{t \lambda \alpha_2}) e^{\lambda t}$$

$$A \alpha_2 = \lambda \alpha_2 \implies (A - \lambda I) \alpha_2 = 0$$

$$\lambda \alpha_1 + \alpha_2 = A \alpha_1 \implies (A - \lambda I) \alpha_1 = \alpha_2$$

$(A - \lambda I) \alpha_2 = 0$  :  $\alpha_2$  is an eigenvector of  $A$ .

$$(A - \lambda I) \alpha_1 = \alpha_2$$

(or)

$$(A - \lambda I)^2 \alpha_1 = 0$$

:  $\alpha_2(t) = e^{\lambda t} \alpha_1 + t e^{\lambda t} \alpha_2$  will be a solution to the differential equation  
 $u' = Au$ .

A vector  $\alpha_1$  satisfying  $(A - \lambda I)\alpha_1 = \alpha_2$  (and) is called a generalized eigenvector.

Ex: 2

$$5x^2D + 4x^2D = (-1)x$$

$$5A + 4A = 0$$

$$(5I + 4A)A = 5I + 4A + 4A = 9A$$

$$A(5I + 4A) = 5A + 4A^2$$

$$Ax = 0 \quad (\text{if } x \neq 0)$$

$$Ax = 0 \quad (\text{if } x \neq 0)$$

An eigenvector is not for  $(E_A - A)$

$$Ax = 0 \quad (\text{if } x \neq 0)$$

other things are not available

$$Ax = 0 \quad (\text{if } x \neq 0)$$

(a)

$$Ax = 0 \quad (\text{if } x \neq 0)$$

Ex:2 Solve  $\frac{du}{dt} = Au$  knowing the eigenvalues  $\lambda = 1, 2, 3$

of  $A$ :

$$\frac{du}{dt} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} u \quad \text{starting from } u(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix}$$

Eigenvectors are:  $\alpha_1 = (1, 0, 0)$ ,  $\alpha_2 = (1, 1, 0)$ ,  $\alpha_3 = (1, 1, 1)$

Step 1:-  $u(0) = \begin{bmatrix} 9 \\ 7 \\ 4 \end{bmatrix} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3$

Step 2: The factors  $e^{\lambda t}$  give exponential solutions

$$e^{t\alpha_1} \quad \text{and} \quad e^{2t\alpha_2} \quad \text{and} \quad e^{3t\alpha_3}$$

Step 3: Combination that starts from  $u(0)$  is:

$$u(t) = 2e^{t\alpha_1} + 3e^{2t\alpha_2} + 4e^{3t\alpha_3}.$$

## 2<sup>nd</sup>-order equations

$$my'' + by' + ky = 0$$

$$\frac{dy}{dt} = \frac{dy^2}{dt^2} + \frac{dy}{dt} + \frac{y^2}{t}$$

standard form

homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0$$

characteristic equation

$$r^2 + pr + q = 0$$

real roots

complex roots

repeated roots

imaginary roots

transient solution

stationary solution

total solution

superposition principle

initial value problem

homogeneous case

non-homogeneous case

particular solution

Substitute  $y = e^{\lambda t}$ ,

$$m \frac{d^2y}{dt^2} + b \frac{dy}{dt} + ky = 0 \implies (m\lambda^2 + b\lambda + k) e^{\lambda t} = 0$$

$$m\lambda^2 + b\lambda + k = 0$$

The equation for  $y$  has 2 pure solutions,

$$y_1 = e^{\lambda_1 t} \text{ and } y_2 = e^{\lambda_2 t}$$

$c_1 y_1 + c_2 y_2$  give the complete solution unless  $\lambda_1 = \lambda_2$ .

Linear Algebra: We turn the scalar equation (with  $y$ ) into a vector equation for  $y$  and  $y'$

$$y'' + by' + ky = 0 \Leftrightarrow y'' + by' + ky = 0$$

Let  $m=1$ ,

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}$$

$$\frac{dy}{dt} = y'$$

$$\frac{dy'}{dt} = -ky - by$$

$$\frac{du}{dt} = \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k & -b \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

Solve  $\frac{du}{dt} = Au$  by eigenvalues of  $A$ .

$$\begin{vmatrix} -\lambda & 1 \\ -k & -b-\lambda \end{vmatrix} = \det(A - \lambda I) = \lambda^2 + b\lambda + k = 0$$

$$-\lambda t_1 + t_2 = 0 \Rightarrow \alpha_1 = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}, \alpha_2 = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

$$u(t) = c_1 e^{\lambda_1 t} \alpha_1 + c_2 e^{\lambda_2 t} \alpha_2 = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix}$$

1<sup>st</sup> component of  $u(t)$  ::  $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

2<sup>nd</sup> component of  $u(t)$  ::  $\frac{dy}{dt} = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$

$$2^{\text{nd}} \text{ component of } u(t) :: \frac{dy}{dt} = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}$$

↑  
velocity

$$\begin{bmatrix} 5 \\ 8 \end{bmatrix} = N$$

$$A_{2 \times 2} = \begin{bmatrix} 0 & 1 \\ -k & b \end{bmatrix} : \text{companion matrix}$$

a companion to the 2<sup>nd</sup> order equation with  $y''$

A<sup>2</sup> has eigenvalues of  $\lambda = \frac{-b \pm \sqrt{b^2 - 4k^2}}{2}$

$$0 = \det(A - \lambda I) = (b - \lambda)(b - \lambda) = \begin{vmatrix} b-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ is } 0, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ is } 0 \leftarrow \text{eigenbasis}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \xrightarrow{\lambda_1 R} \begin{bmatrix} 1 \\ \lambda_1 R \end{bmatrix} \xrightarrow{\lambda_2 R} \begin{bmatrix} 1 \\ \lambda_1 R + \lambda_2 R \end{bmatrix} = (A) v$$

Ex:3 Motion around a circle with  $y'' + y = 0$   
and  $y = \cos t$ .

$$y'' + y = -\frac{d^2y}{dt^2} + y = -\frac{d^2}{dt^2} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y \\ y' \end{bmatrix}$$

Ans:  $m=1$ , stiffness  $k=1$ , and  $d=0$ : no damping.

Calculus Put  $y = e^{\lambda t}$  into  $y'' + y = 0$

$$\lambda^2 + 1 = 0 \implies \lambda_1 = i, \lambda_2 = -i$$

$$y(t) = C_1 e^{it} + C_2 e^{-it} = \cos(C_1 + C_2) + i \sin(C_1 - C_2)$$

$$y(0) = 1, \quad y'(0) = 0,$$

$$y'(t) = -\sin(C_1 + C_2) + i \cos(C_1 - C_2)$$

$$1 = (C_1 + C_2)$$

$$0 = i(C_1 - C_2)$$

$$\underline{2C_1 = 1 \Rightarrow C_1 = \frac{1}{2}, C_2 = \frac{1}{2}}$$

$$\left. \begin{array}{l} y(t) = \frac{e^{it} + e^{-it}}{2} = \cos t \\ \hline \end{array} \right\}$$

Algebra

$$\frac{du}{dt} = \frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = Au$$

$$\lambda_1 = i, \quad \lambda_2 = -i$$

$A$  is antisymmetric

$$\alpha_1 = (1, i), \quad \alpha_2 = (1, -i)$$

but  $\vec{v}(t)$  has about a linear trend @  $t=0$ .

$$u(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{x_1 + x_2}{2} = \frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$u(t) = \frac{1}{2}e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} + \frac{1}{2}e^{-it} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} = \begin{bmatrix} g(t) \\ h(t) \end{bmatrix}$$

$u = (\cos t, -\sin t)$  goes around a circle of radius 1 + 0,  $t=0$

(0,0) starts  $(\cos 0, -\sin 0) = (1, 0)$

$$\int_{00}^{00} \frac{d^2 \vec{r}}{dt^2} = M \cdot \left\{ \begin{array}{l} (\cos \omega, 0) = 1 \\ (\omega \sin \omega, 0) = \omega \end{array} \right. \quad \frac{d^2 \vec{r}}{dt^2} = \frac{1}{M} \cdot F_{ext} = 0.08$$

$$nA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{\partial}{\partial t} & \frac{\partial}{\partial b} \\ 0 & \frac{\partial}{\partial b} & \frac{\partial}{\partial t} \end{bmatrix} =$$

symmetric in  $\vec{v}$   $\Rightarrow$  stable

$$(1-0) \otimes (0,0,0) = 0$$