classmate

soorajss1729@gmail.com

+91-9400635788

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NAME: SOORAJ.S STD.: ______ SEC.: ______ ROLL NO.: ______ SUB.: ______

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10

QUANTUM ERROR CORRECTION

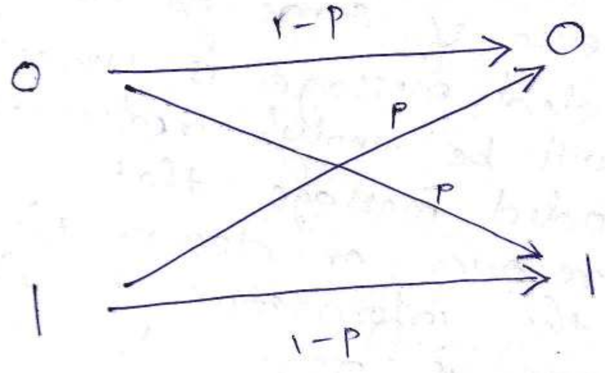
The key idea about error correcting is that,

If we wish to protect a message against the effects of noise, then we should encode the message by adding some redundant information to the message. That way, even if some of the information in the encoded message is currepted by noise, there will be enough redundancy in the encoded message that it is possible to recover or decode the message so that all the information in the original message is recovered.

Example :

Suppose,
we wish to send a bit from one location to another throt a noisy classical communication channel.

The effect of the noise in the channel is to flip the bit being transmitted with probability $p>0$, while with probability $1-p$ the bit is transmitted without error. Such a channel is known as a binary symmetric channel.



A simple means of protecting the bit against the effects of noise in the binary symmetric channel is to replace the bit we wish to protect with 3 copies of itself:

$$0 \longrightarrow 000$$

$$1 \longrightarrow 111$$

The bit strings 000 and 111 are sometimes referred to as the logical 0 and logical 1, since they play the role of 0 and 1, respectively.

We now send all 3 bits thro' the channel
At the receiver's end of the channel
3 bits are output, and the receiver has to decide what the value of the original bit was. Suppose,

001 were output from the channel.
Provided the probability p of a bit flip is not too high, it is very likely that the 3rd bit was flipped by the channel, and that 0 was the bit that was sent.

This type of decoding is called majority voting, since the decoded o/p from the channel is whatever value, 0 or 1, appears more times in the actual channel o/p. Majority voting fails if 2 or more of the bits sent thro' the channel were flipped, and succeeds otherwise.

Probability of a bit flip $= 3$

Probability that 2 or more of the bits are flipped $= P(2 \text{ bit flips}) + P(3 \text{ bit flips})$

$$= \underline{\underline{3p^2(1-p) + p^3}} = \underline{3p^2 - 2p^3}$$

$\therefore$ The probability of error, $P_e = 3p^2 - 2p^3$

Without encoding, the probability of an error was p, so the code makes the transmission more reliable provided $P_e < p$, which occurs whenever $p < 1/2$.

$P_e < p \implies 3p^2 - 2p^3 < p$

$2p^3 - 3p^2 + p = p(2p^2 - 3p + 1) > 0$

$p = \frac{3 \pm 1}{4} = 1$ (or) $\frac{1}{2}$

$p(p - \frac{1}{2})(p-1) > 0$

Case 1 $\quad p > 0$ & $p - \frac{1}{2} > 0$ & $p - 1 > 0$

$p > 0$ & $p > \frac{1}{2}$ & $p > 1 \implies p > 1$

Not possible

Case 2 $\quad p - \frac{1}{2} > 0$ & $p < 0$ & $p - 1 < 0$

$p > \frac{1}{2}$ & $p < 0$ & $p < 1$

$p > \frac{1}{2}$ & $p < 0 \implies$ Not possible p.

Case 3. $\quad p > 0$ & $p - \frac{1}{2} < 0$ & $p - 1 < 0$

$p > 0$ & $p < \frac{1}{2}$ & $p < 1$

$p > 0$ & $p < \frac{1}{2}$

$\implies \underline{\underline{p < \frac{1}{2}}}$

The type of code just described is called a repetition code, since we encode the message to be sent by repeating it a # of times.

The 3 qubit bit flip code

We have 3 formidable difficulties to develop quantum error-correcting codes to protect quantum states against the effects of noise,

(i) No cloning - Implementing the repetition code quantum mechanically by duplicating the quantum state 3 or moretimes, is forbidden by the no-clouning theorem. Even if cloning were possible, it coould not be possible to measure & compare the 3 quantum states o/p from the channel.

(ii) Errors are continuous - A continuum of different errors may occur on a single qubit. Determing which error occurred in order to correct it would appear to require infinite precision, and therefore infinite resources.

(iii) Measurement destroys quantum information -

In classical error-correction we observe the o/p from the channel, and decide what decoding procedure to adopt. Observation in quantum mechanics generally destroys the quantum state under observation, and makes recovery impossible.

Suppose,

we send qubits thro' a channel which leaves the qubits untouched with probability $1-p$, and flips the qubits with probability p.

ie.,

with probability p the state $|\psi\rangle$ is taken to the state $X|\psi\rangle$, where X is the usual Pauli σ_x operator, or bit flip operator. This channel is called the bit flip channel.

Bit flip code

Suppose,

we encode the single qubit state $a|0\rangle + b|1\rangle$ in 3 qubits as $a|000\rangle + b|111\rangle$.

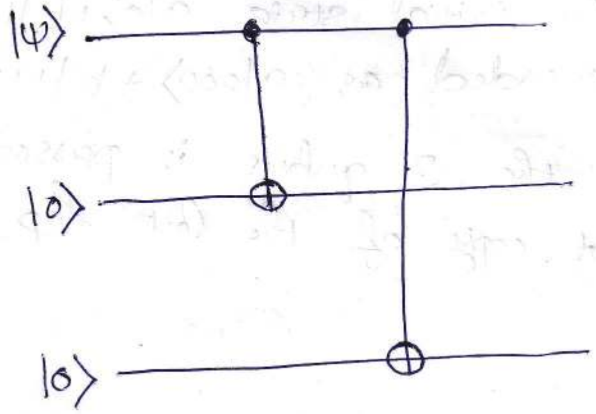
$$|0\rangle \longrightarrow |0_L\rangle \equiv |000\rangle$$

$$|1\rangle \longrightarrow |1_L\rangle \equiv |111\rangle$$

where the superpositions of basis states are taken to corresponding superpositions of encoded states.

The notation $|0_L\rangle$ and $|1_L\rangle$ indicates that these are the logical $|0\rangle$ and logical $|1\rangle$ states, not the physical zero and one states.

A circuit performing this encoding is given as,



* Encoding circuit for the 3 qubit bit flip code

$$|\psi_1\rangle = |\psi\rangle \otimes |0\rangle \otimes |0\rangle = (a|0\rangle + b|1\rangle) \otimes |0\rangle \otimes |0\rangle$$
$$= a|0\rangle \otimes |0\rangle \otimes |0\rangle + b|1\rangle \otimes |0\rangle \otimes |0\rangle$$

$$|\psi_2\rangle = a|0\rangle \otimes |0\rangle \otimes |0\rangle + b|1\rangle \otimes X|0\rangle \otimes |0\rangle$$
$$= a|0\rangle \otimes |0\rangle \otimes |0\rangle + b|1\rangle \otimes |1\rangle \otimes |0\rangle$$

$$|\psi_3\rangle = a|0\rangle \otimes |0\rangle \otimes |0\rangle + b|1\rangle \otimes |1\rangle \otimes X|0\rangle$$
$$= a|0\rangle \otimes |0\rangle \otimes |0\rangle + b|1\rangle \otimes |1\rangle \otimes |1\rangle$$
$$= a|000\rangle + b|111\rangle$$

Now,

Suppose the initial state $a|0\rangle + b|1\rangle$ has been perfectly encoded as $a|000\rangle + b|111\rangle$.
Each of the 3 qubits is passed through an independent copy of the bit flip channel.

Suppose a bit flip occurred on one or fewer of the qubits. There is a simple 2 stage error-correction procedure which can be used to recover the correct quantum state in this case:

<u>Stage 1</u> : <u>Error detection (or) syndrome diagnosis</u>

We perform a measurement which tells us what error, if any, occurred on the quantum state. The measurement result is called the <u>error syndrome</u>.

For the bit flip channel there are 4 error syndromes, corresponding to the 4 projection operators :

$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111|$ no error

$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011|$ bit flip on qubit one

$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101|$ bit flip on qubit two

$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110|$ bit flip on qubit three

Suppose for example that a bit flip occurs on qubit one, so the corrupted state is $a|100\rangle + b|011\rangle$.

$\Rightarrow \langle\psi|P_1|\psi\rangle = 1$

$\Rightarrow$ the outcome of the measurement result (the error syndrome) is certainly 1.

The syndrome measurement does not cause any change to the state : it is $a|100\rangle + b|011\rangle$ both before and after syndrome measurement.

The syndrome contains only information about what error has occurred, and does not allow us to infer anything about the value of a or b, ie., it contains no information about the state being protected. This is a generic feature of syndrome measurements, since to obtain information about the identity of a quantum state it is in general necessary to perturb that state.

Stage 2: We use the value of the error syndrome to tell us what procedure to use to recover the initial state.

Ex:- If the error syndrome was 1, indicating a bit flip on the 1st qubit, then we flip that qubit again, recovering the original state $a|000\rangle + b|111\rangle$ with perfect accuracy.

The 4 possible error syndromes and the recovery procedure in each case are:

0 (no error) – do nothing

1 (bit flip on 1st qubit) – flip the 1st qubit again

2 (bit flip on 2nd qubit) – flip the 2nd qubit again

3 (bit flip on 3rd qubit) – flip the 3rd qubit again

For each value of the error syndrome it is easy to see that the original state is recovered with perfect accuracy, given that the corresp. error occurred.

This error-correction procedure works perfectly, provided bit flip occurs on one or fewer of the 3 qubits. This occurs with probability,

$$= P(\text{No bit flip}) + P(\text{bit flip on one qubit})$$

$$= (1-p)^3 + 3p(1-p)^2$$

$$= (1-p)^2(1-p+3p)$$

$$= (1-p)^2(1+2p)$$

$$= (1-2p+p^2)(1+2p)$$

$$= 1+\cancel{2p}-\cancel{2p}-4p^2+p^2+2p^3$$

$$= 1-3p^2+2p^3$$

The probability of an error remaining uncorrected is $= 1-(1-3p^2+2p^3)$

$$= 3p^2-2p^3$$

just as for the classical repetition code.

$\Rightarrow$ Provided $p < 1/2$ the encoding and decoding improve the reliability of storage of the quantum state.

☐ Improving the error analysis

This error analysis is not completely adequate. The problem is that not all errors and states in quantum mechanics are created equal: quantum states live in a continuous space, so it is possible for some errors to corrupt a state by a tiny amount, while others mess it up completely.

Ex:-

the bit flip 'error' X, which does not affect the state $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ at all, but flips the $|0\rangle$ state so it becomes a $|1\rangle$. In the former case we would not be worried about a bit flip error occurring, while in the latter case we would obviously be very worried.

To address this problem $\Longrightarrow$ Fidelity

The fidelity b/w a pure and a mixed state is given by,

$$F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle}$$

which is the square root of the overlap

? b/w $|\psi\rangle$ and ρ.

The object of quantum error-correction is to increase the fidelity with which quantum information is stored (or communicated) up near the maximum possible fidelity of 1.

Compare : min. fidelity achieved by the 3 qubit bit flip code with the fidelity when no error-correction is performed.

Suppose the quantum state of interest is $|\psi\rangle$.

<u>case 1</u> - Without using the error-correcting code

The state of the qubit after being sent through the channel is,

$$\rho = (1-p)|\psi\rangle\langle\psi| + p X |\psi\rangle\langle\psi| X$$

The fidelity is given by,

$$F = \sqrt{\langle\psi|\rho|\psi\rangle}$$

$$= \sqrt{\langle\psi|\left((1-p)|\psi\rangle\langle\psi| + pX|\psi\rangle\langle\psi|X\right)|\psi\rangle}$$

$$= \sqrt{(1-p) + p\langle\psi|X|\psi\rangle\langle\psi|X|\psi\rangle}$$

$$= \sqrt{(1-p) + p\left|\langle\psi|X|\psi\rangle\right|^2}$$

The 2^{nd} term under the square root is non-negative, and equal to 0 when $|\psi\rangle = |0\rangle$.

$\Rightarrow$ The minimum fidelity is, $F_{min} = \sqrt{1-p}$

ILA(8)

case 2 – The 3 qubit error-correcting code is used to protect the state $|\psi\rangle = a|0_L\rangle + b|1_L\rangle$

The quantum state after both the noise and error-correction is,

$$\rho = \left[(1-p)^3 + 3p(1-p)^2\right]|\psi\rangle\langle\psi| + \cdots$$

The omitted terms represent contributions from bit flips on 2 or 3 qubits.

All the omitted terms are +ve operators,

$\langle\psi|A|\psi\rangle \geq 0$ for all $|\psi\rangle$ for a +ve operator A.

ILA(8)

$\therefore$

$$F = \sqrt{\langle\psi|\rho|\psi\rangle} \geq \sqrt{(1-p)^3 + 3p(1-p)^2}$$

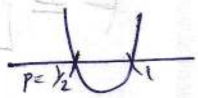
$$= \sqrt{1 - 3p^2 + 2p^3}.$$

ie, The fidelity is at least $\sqrt{1-3p^2+2p^3}$

$$p-(3p^2-2p^3) = 2p^3-3p^2+p = p(2p^2-3p+1) = 0$$

$$\Delta = \sqrt{9-8} = 1$$

$$p = \frac{3 \pm 1}{4} = 1 \text{ (or) } 1/2$$



$$p-(3p^2-2p^3) > 0 \qquad \text{for } p < 1/2$$

$$p > 3p^2-2p^3$$

$$-p < -3p^2+2p^3 \implies 1-p < 1-3p^2+2p^3$$

$$\sqrt{1-p} < \sqrt{1-3p^2+2p^3} \qquad \text{for } p < 1/2$$

$\therefore$

The fidelity of storage for the quantum state is improved provided $p < 1/2$.

Ex:10.2 The action of the bit flip channel can be described by the quantum operation $\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$. Show that this may be given an alternate operator-sum representation, as $\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_-$ where P_+ and P_- are projectors onto the $+1$ and -1 eigenstates of X, $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle-|1\rangle}{\sqrt{2}}$, respectively. This later representation can be understood as a model in which the qubit is left alone with probability $1-2p$, and is 'measured' by the environment in the $|+\rangle, |-\rangle$ basis with probability $2p$.

Ans: $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues & eigenvectors:

$$\lambda_- = -1 \quad , \quad |\lambda_-\rangle = |-\rangle$$

$$\lambda_+ = +1 \quad , \quad |\lambda_+\rangle = |+\rangle$$

$$X = |+\rangle\langle+| - |-\rangle\langle-| = P_+ - P_-$$

$$I = |+\rangle\langle+| + |-\rangle\langle-| = P_+ + P_-$$

$$\mathcal{E}(\rho) = (1-p)\rho + pX\rho X$$

$$= (1-2p)\rho + p\rho + pX\rho X$$

$$= (1-2p)\rho + pX\rho X + pI\rho I$$

$$= (1-2p)\rho + p(P_+ - P_-)\rho(P_+ - P_-) + p(P_+ + P_-)\rho(P_+ + P_-)$$

$$= (1-2p)\rho + 2p\, P_+\rho P_+ + 2p\, P_-\rho P_-$$

This is equivalent to measuring the state in the $|+\rangle, |-\rangle$ state with probability $2p$.

Syndrome measurement – another way

Suppose that instead ~~of~~ measuring the 4 projectors P_0, P_1, P_2, P_3, we performed two measurements –

1^{st}: the observable $Z_1 Z_2$ (ie; $Z \otimes Z \otimes I$)

2^{nd}: the observable $Z_2 Z_3$ (ie; $I \otimes Z \otimes Z$)

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \implies \lambda = \pm 1, \; |\lambda_+\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; |\lambda_-\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Each of these observables has eigenvalues ± 1.

$\therefore$

Each measurement provides a single bit of information, for a total of 2 bits of information – 4 possible syndromes, just as in the earlier description.

The 1st measurement, of $Z_1 Z_2$, can be thought of as comparing the 1st and 2nd qubits to see if they are the same.

$$Z_1 Z_2 = Z \otimes Z \otimes I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes I$$

$$= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes I$$

$$= (|0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| - |0\rangle\langle 0| \otimes |1\rangle\langle 1| - |1\rangle\langle 1| \otimes |0\rangle\langle 0|) \otimes I$$

$$= (|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$$

which corresp. to a projective measurement with projectors $(|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I$ and $(|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I$.

$\Rightarrow$ Measuring $Z_1 Z_2$ can be thought of as comparing the values of the 1st and 2nd qubits, giving +1 if they are the same, and -1 if they are different.

Similarly,

measuring $Z_2 Z_3$ compares the values of the 2nd and 3rd qubits, giving +1 if they are the same, and −1 if they are different.

Combining these 2 measurements results can determine whether a bit flip occurred on one of the qubits or not, and if so, which one:

if both measurement results give +1 then with high probability no bit flip has occurred; If measuring $Z_1 Z_2$ gives +1 and measuring $Z_2 Z_3$ gives −1 then with high probability just the 3rd qubit flipped;

if measuring $Z_1 Z_2$ gives −1 and measuring $Z_2 Z_3$ gives +1 then with high probability just the 1st qubit flipped;

Finally, if both measurements give −1 then with high probability just the 2nd qubit flipped.

What's crucial to the success of these measurements is that neither measurement gives any information about the amplitudes 'a' and 'b' of the encoded quantum state, and thus neither measurement destroys the superpositions of quantum states that we wish to preserve using the code.

Ex: 10.3

Ans:

Ex: 10.3 Show by explicit calculation that measuring Z_1Z_2 followed by Z_2Z_3 is equivalent, up to labeling of the measurement outcomes, to measuring the 4 projectors in the sense that both procedures result in the same measurement statistics and in the same measurement states.

post-measurement states.

Ans:

☐ 3 qubit phase flip code

The bit flip code is interesting, but it does not appear to be that significant an innovation over classical error-correcting codes, and leaves many problems open (Ex:- many kinds of errors other than bit flips can happen to qubits).

A more interesting noisy quantum channel is the phase flip error model for a single qubit

- the qubit is left alone with probability $1-p$, and with probability p the relative phase of the $|0\rangle$ and $|1\rangle$ states is flipped.

ie., the phase flip operator Z is applied to the qubit with probability $p>0$, so that the state $a|0\rangle+b|1\rangle$ is taken to the state $a|0\rangle-b|1\rangle$ under the phase flip.

There is no classical equivalent to the phase flip channel, since classical channels don't have any property equivalent to phase.

Phase flip channel $\longrightarrow$ bit flip channel.

Suppose we work in the qubit basis $|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, $|-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$. W.r.t this basis the operator Z takes $|+\rangle$ to $|-\rangle$ and vice versa,

i.e., it acts just like a bit flip w.r.t the labels '+' to '−'!

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

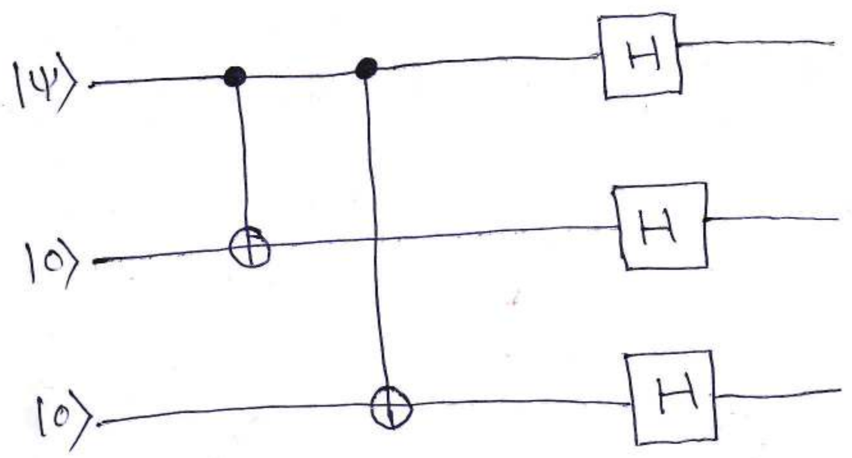
$\Rightarrow$ This suggests using the states $|0_L\rangle = |+++\rangle$ and $|1_L\rangle = |---\rangle$ as logical zero and one states for protection against phase flip errors.

All the operations needed for error-correction — encoding, error-detection and recovery — are performed just as for the bit flip channel, but wrt the $|+\rangle, |-\rangle$ basis instead of the $|0\rangle, |1\rangle$ basis.

To accomplish this basis change we simply apply the Hadamard gate and its inverse (also the Hadamard gate) at appropriate points in the procedure.

$$H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle \qquad \& \qquad H|+\rangle = |0\rangle$$

$$H|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle \qquad\qquad H|-\rangle = |1\rangle$$

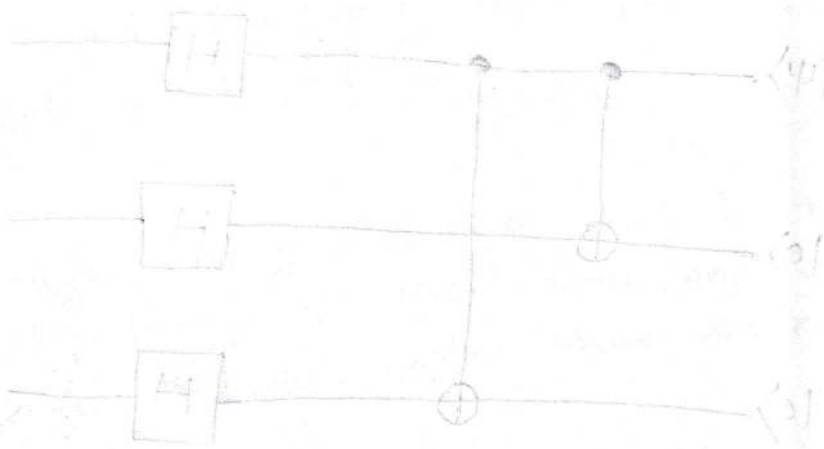


* Encoding circuits for the phase flip code.

Encoding : 1st we encode in 3 qubits exactly as for the bit flip channel
2nd apply a Hadamard gate to each qubit.

Error detection: achieved by applying the same projective measurements as in the bit flip case, but conjugated by Hadamard gates.

$$P_j \longrightarrow P_j' \equiv H^{\otimes 3} P_j H^{\otimes 3}$$



Equivalently, syndrome measurement may be performed by measuring the observables

$$H^{\otimes 3} Z_1 Z_2 H^{\otimes 3} = (H \otimes H \otimes H)(Z \otimes Z \otimes I)(H \otimes H \otimes H)$$

$$= HZH \otimes HZH \otimes HIH$$

$$= X \otimes X \otimes I = X_1 X_2$$

and $H^{\otimes 3} Z_2 Z_3 H^{\otimes 3} = X_2 X_3$.

Measurement of the observables X_1X_2 and X_2X_3 corresponds to comparing the sign of the qubits one and two, and two and three, respectively, in the sense that measurement of X_1X_2 gives $+1$ for states like $|+\rangle|+\rangle \otimes (\cdot)$ or $|-\rangle|-\rangle \otimes (\cdot)$, and -1 for states like $|+\rangle|-\rangle \otimes (\cdot)$ or $|-\rangle|+\rangle \otimes (\cdot)$.

Recovery : Hadamard-conjugated recovery operation from the bit flip code

Ex:-

Suppose we detected a flip in the sign of the 1st qubit from $|+\rangle$ to $|-\rangle$.
Then we recover by applying $HX_1H = Z_1$ to the 1st qubit.

This code for the phase flip channel has the same characteristics as the code for the bit flip channel. In particular, the minimum fidelity for the phase flip code is the same as that for the bit flip code, and we have the same criteria for the code producing an improvement over the case with no error-correction.

⇒ These 2 channels are unitarily equivalent, since there is a unitary operator U (in this case the Hadamard gate) such that the action of one channel is the same as the other, provided the 1st channel is preceded by U and followed by $U^\dagger$.

These operations may be trivially incorporated into the encoding & error-correction operations

Ex:-10.4 Consider the 3 qubit bit flip code.

Suppose we had performed the error syndrome measurement by measuring the 8 orthogonal projectors corresp. to projections onto 8 computational basis states.

(a) Write out the projectors corresp. to this measurement and explain how the measurement result can be used to diagnose the error syndrome: either no bits flipped or bit # j flipped where $j = 1,2,3$.

(b) Show that the recovery procedure works only for computational basis states.

(c) What's the min. fidelity for the error-correction procedure?

□ The Shor code

- simple quantum code which can protect against the effects of an arbitrary error on a single qubit.
- combination of 3 qubit phase flip and bit flip codes.

1^{st}: encode each of these qubits using the 3 qubit phase flip code

$$|0\rangle \longrightarrow |{+}{+}{+}\rangle \,,\quad |1\rangle \longrightarrow |{-}{-}{-}\rangle$$

2^{nd}: encode each of these qubits using the 3 qubit bit flip code

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \longrightarrow \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

$$|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} \longrightarrow \frac{|000\rangle - |111\rangle}{\sqrt{2}}$$

The result is a 9 qubit code, with codewords given by,

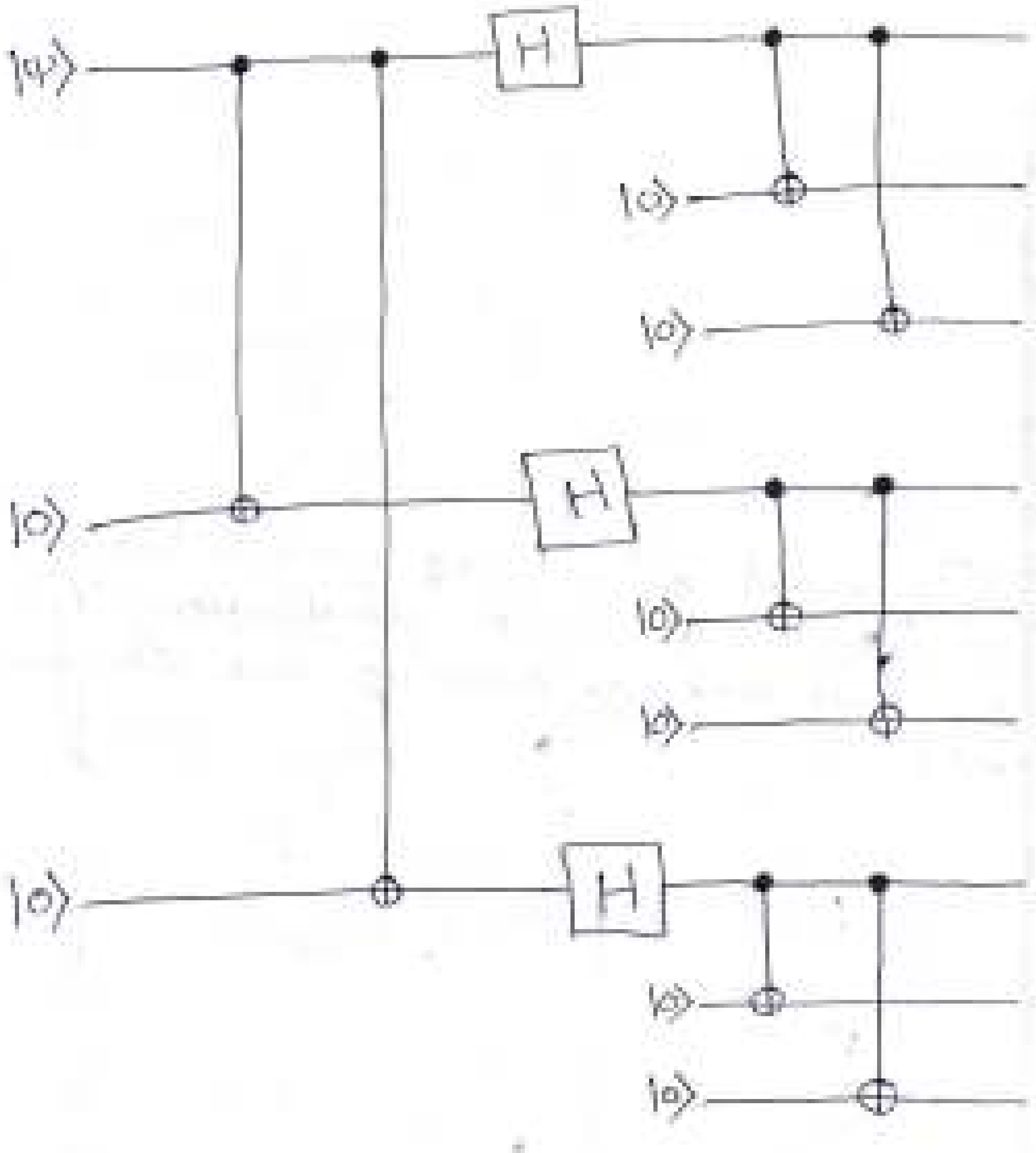
$$|0\rangle \to |0_L\rangle = \frac{(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)}{2\sqrt{2}}$$

$$|1\rangle \to |1_L\rangle = \frac{(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle)}{2\sqrt{2}}$$

$$\alpha|0\rangle+\beta|1\rangle \to \alpha|0_L\rangle+\beta|1_L\rangle =$$

$$= \frac{\alpha}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)$$

$$+ \frac{\beta}{2\sqrt{2}}(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle)$$



* Encoding circuit for the Shor 9 qubit code.

- The Shor code is able to protect against phase flip and bit flip errors on any qubit.

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bit flip

Suppose a bit flip occurs on the 1st qubit. As for the bit flip code, we perform a measurement of Z_1Z_2 comparing the 1st two qubits, and find that they are different. We conclude that a bit flip error occurred on the 1st or 2nd qubit. Next, we compare the 2nd and 3rd qubit by performing a measurement of Z_2Z_3. We find that they are the same, so it could not have been the 2nd qubit which flipped.

$\Rightarrow$ the 1st qubit must have flipped, and recover from the error by flipping the 1st qubit again, back to its original state.

In a similar way, we can detect and recover from the effects of bit flip errors on any 9 qubits in the code.

i.e.,
Measure parities with 3 qubit blocks: $Z_1Z_2, Z_2Z_3; Z_4Z_5, Z_5Z_6; Z_7Z_8, Z_8Z_9$

If X error is detected in i-th qubit, correct by gate X_i.

(If bit-flips in different 3-qubit blocks, they all can be corrected)

Phase flip

Suppose a phase flip occurs on the 1st qubit.

Such a phase flip flips the sign of the 1st block of qubits, changing $|000\rangle+|111\rangle$ to $|000\rangle-|111\rangle$, and vice versa.

The state becomes,

$$\frac{\alpha}{2\sqrt{2}}\Big(|000\rangle-|111\rangle\Big)\otimes\Big(|000\rangle+|111\rangle\Big)\otimes\Big(|000\rangle+|111\rangle\Big)$$

$$+\frac{\beta}{2\sqrt{2}}\Big(|000\rangle+|111\rangle\Big)\otimes\Big(|000\rangle-|111\rangle\Big)\otimes\Big(|000\rangle-|111\rangle\Big)$$

A phase flip on any of the 1st three qubits has this effect.

The key idea here is to detect which of the 3 blocks of 3 qubits has experienced a change of sign.

Syndrome measurement begins by comparing the sign of the 1st and 2nd blocks of 3 qubits.

For example, $(|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle)$ has the sign $(-)$ in both blocks of qubits, while $(|000\rangle - |111\rangle) \otimes (|000\rangle + |111\rangle)$ has different signs.

When a phase flip occurs on any of the 1st three qubits, we find that the signs of the 1st and 2nd blocks are different. The 2nd and final stage of syndrome measurement is to compare the sign of the 2nd and 3rd blocks of qubits. We find that these are the same, and conclude that the phase must have flipped in the 1st block of 3 qubits.

We recover from this by flipping the sign in the 1st block of 3 qubits back to its original value.

We can recover from a phase flip on any of the 9 qubits in a similar manner.

ie.,

Measure parities of phases of 3 qubit blocks :

$X_1X_2X_3X_4X_5X_6$, $X_4X_5X_6X_7X_8X_9$

($X_1X_2X_3$ changes sign of wavefunction if $000 - 111$ and does nothing if $000 + 111$.

If phase flip error is detected in j^{th} 3-qubit block, correct by applying Z-gate to any qubit in this block.

<u>Note:</u> This step is insensitive to bit flips

Ex:-

$$X_1X_2X_3\left(|001\rangle - |110\rangle\right) = |110\rangle - |001\rangle = -\left(|001\rangle - |110\rangle\right)$$

$\Rightarrow$ We can first detect errors, then correct.

bit & phase flip

bit & phase flip

Suppose both bit and phase flip errors occur on the 1^{st} qubit, ie., the operator ZX_1 is applied to that qubit.

The procedure for detecting a bit flip error will detect a bit flip on the 1^{st} qubit, and correct it, and the procedure for detecting a phase flip error will detect a phase flip on the 1^{st} block of 3 qubits, and correct it.

bit & phase flip

Suppose both bit and phase flip errors occur on the i^{th} qubit, i.e. the operator Z_iX_i is applied to that qubit.

The procedure for detecting a bit flip error will detect a bit flip on the i^{th} qubit, and correct it, and the procedure for detecting a phase flip error will detect a phase flip on the i^{th} block of 3 qubits, and correct it.

Ex: 10.5

Show that the syndrome measurement for detecting phase flip errors in the Shor code corresponds to measuring the observables $X_1X_2X_3X_4X_5X_6$ and $X_4X_5X_6X_7X_8X_9$.

Ex: 10.6 Show that recovery from a phase flip on any of the 1st 3 qubits may be accomplished by applying the operator $Z_1Z_2Z_3$.

Shor code — arbitrary errors

Shor code protects against completely arbitrary errors, provided they only affect a single qubit!

The error can be tiny - a rotation about the z-axis of the Bloch sphere by $\pi/263$ radians, say, or it can be an apparently disastrous error like removing the qubit entirely and replacing it with garbage!

This is an extraordinary fact that the apparent continuum of errors that may occur on a single qubit can all be corrected by correcting only a discrete subset of those errors; all other possible errors being corrected automatically by this procedure!

Suppose noise of an arbitrary type is occurring on the 1^{st} qubit only.

We describe noise by a trace-preserving quantum operation $\mathcal{E}$.

Analyze error-correction by expanding $\mathcal{E}$ in an operator-sum representation with operation elements $\{E_i\}$.

Suppose,
the state of the encoded qubit is $|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$ before the noise acts, then after the noise has acted the state is,

$$\mathcal{E}(|\psi\rangle\langle\psi|) = \sum_i E_i |\psi\rangle\langle\psi| E_i^\dagger$$

Focus on the effect error-correction has on a single term in this sum, say $E_i|\psi\rangle\langle\psi|E_i^\dagger$.

E_i is an operator on the 1st qubit alone, which can be expanded in the Pauli matrix basis,

ie, expanded as a linear combination of the identity I, the bit flip X_1, the phase flip Z_1, and the combined bit and phase flip $-iY = X_1 Z_1$:

$$E_i = e_{i0} I + e_{i1} X_1 + e_{i2} Z_1 + e_{i3} X_1 Z_1$$

The (un-normalized) quantum state $E_i|\psi\rangle$ can thus be written as a superposition of 4 terms, $|\psi\rangle, X_1|\psi\rangle, Z_1|\psi\rangle, X_1 Z_1|\psi\rangle$.

$$E_i|\psi\rangle = e_{i0}|\psi\rangle + e_{i1} X_1|\psi\rangle + e_{i2} Z_1|\psi\rangle + e_{i3} X_1 Z_1|\psi\rangle$$

Measuring the error syndrome collapses this superposition into one of the 4 states:
$|\psi\rangle, X_1|\psi\rangle, Z_1|\psi\rangle$ or $X_1 Z_1|\psi\rangle$ from which recovery may then be performed by applying the appropriate inversion operation, resulting in the final state $|\psi\rangle$.

The same is true for all the other operation elements E_i.

Thus,
error-correction results in the original state $|\psi\rangle$ being recovered, despite the fact that the error on the 1st qubit was arbitrary.

⇒ A fundamental & deep fact about quantum error-correction

- by correcting just a discrete set of errors the bit flip, phase flip and combined bit-phase flip, in this example, a quantum error-correcting code is able to automatically correct an apparently larger (continuous!) class of errors.

other

$E_i = e_i I$

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The Stabilizer Formalism

$X|0\rangle = |1\rangle$
$X|1\rangle = |0\rangle$

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

EPR state of two qubits

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

$X_1X_2|\psi\rangle = |\psi\rangle$ and $Z_1Z_2|\psi\rangle = |\psi\rangle$

$\Longrightarrow$ the state $|\psi\rangle$ is stabilized by the operators X_1X_2 and Z_1Z_2.

- The state $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ is the unique quantum state (up to a global phase) which is stabilized by the operators X_1X_2 and Z_1Z_2.

The basic idea of the stabilizer formalism is that many quantum states can be more easily described by working with the operators that stabilize them than by working explicitly with the state itself.

It turns out that many quantum codes (including CSS codes and the Shor code) can be much more compactly described using stabilizers than in the state vector description. Even more importantly, errors on the qubits and operations such as the Hadamard gate, phase gate, and even the controlled-NOT gate and measurements in the computational basis are all easily described using the stabilizer formalism!

Pauli group G_n on n qubits. For a single qubit, the Pauli group is defined to consist of all the Pauli matrices, together with multiplicative factors $\pm 1, \pm i$:

$$G_1 \equiv \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}. \quad (10.81)$$

This set of matrices forms a group under the operation of matrix multiplication.

The general Pauli group on n qubits is defined to consist of all n-fold tensor products of Pauli matrices, and again we allow multiplicative factors $\pm 1, \pm i$.

A group $(G, \cdot)$ is a non-empty set G with a binary group multiplication operation '$\cdot$', with the following properties:

(i) Closure : $g_1 \cdot g_2 \in G$ for all $g_1, g_2 \in G$

(ii) Associativity : $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$ for all $g_1, g_2, g_3 \in G$

(iii) Identity : there exists $e \in G$ such that
$\forall g \in G : g \cdot e = e \cdot g = g$

(iv) Inverses : For all $g \in G$, there exists $g^{-1} \in G$ such that $g g^{-1} = e$ and $g^{-1} g = e$

• Let $(G, *)$ be a group. A non-empty subset H of G is called a subgroup of G if, i.e. $\boxed{H \leq G}$

(i) $a * b \in H \ \forall \ a, b \in H$

i.e. $*$ is a binary operation on H

(ii) $(H, *)$ is itself a group.

Given $H \subseteq G$,
$H \leq G \Longleftrightarrow a, b \in H \Rightarrow ab^{-1} \in H$

Ex:- $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ and $(\mathbb{C}, +)$

Suppose S is a subgroup of G_n and define V_S to be the set of n qubit states which are fixed by every element of S. V_S is the vector space stabilized by S, and S is said to be the stabilizer of the space V_S, since every element of V_S is stable under the action of elements in S.

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