QFF-Non-Reversible

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Abstract

We refine the proof that for a nearly reversible Markov chain, the error term arising in the Chebyshev recurrence of Quantum Fast-Forwarding (QFF) grows only linearly in the number of steps. Specifically, we consider the modified quantum walk operator $U' = (V^*)^\dagger SV$ that arises when the Markov chain is close (but not identical) to being reversible, and show how the operator-norm bound $\|(U')^2 - I\| \leq \epsilon$ translates into a linear-in-t bound on the norm of the deviation from the ideal Chebyshev evolution.

0.1 Introduction

Quantum Fast-Forwarding (QFF) leverages a quantum walk to simulate a classical Markov chain quadratically faster than classical random walks. In the exactly reversible setting, one exploits the fact that the quantum walk operator U acts like the Chebyshev recurrence on the *flat* subspace:

$$\Pi_\flat U^t |v, \flat\rangle = T_t(D)|v, \flat\rangle,$$

where $D = \sqrt{P \circ P^T}$ is the so-called *discriminant* of the transition matrix P, and T_t is the t^{th} Chebyshev polynomial of the first kind.

0.1.1 Nearly Reversible Markov Chains

In many applications, the chain is only nearly reversible. One then defines a modified quantum walk operator

$$U' = (V^*)^\dagger SV,$$

where V^* is related to the time-reversed chain. When reversibility is exact, $(V^*)^\dagger = V^\dagger$ and the operator is Hermitian. But for nearly reversible chains,

$$(U')^2 = I + \Delta, \quad \|\Delta\| \leq \epsilon,$$

so U' is approximately Hermitian. In the QFF setting, we would like to show that this small deviation Δ only yields a linear accumulation of error over t steps.

0.2 Setup and Approximate Recurrence

0.2.1 Exact Chebyshev Dynamics (Reversible Case)

In the exactly reversible scenario, one proves that on the flat subspace,

$$U^2 = I, \quad \Pi_\flat W^t |v, \flat\rangle = T_t(D)|v, \flat\rangle,$$

and the Chebyshev polynomials satisfy the recurrence

$$T_{t+1}(x) = 2xT_t(x) - T_{t-1}(x).$$

Thus,

$$\Pi_\flat W^{t+1}|v, \flat\rangle = 2D\left[\Pi_\flat W^t|v, \flat\rangle\right] - \Pi_\flat W^{t-1}|v, \flat\rangle.$$

where $W = R_\flat U = (2\Pi_\flat - I)U$ and $D = \Pi_\flat U$. The operator U effectively enforces that $\Delta \equiv 0$ in this idealized scenario.

0.2.2 Nearly Reversible Case

We define the generalized discriminant matrix

$$D' := \sqrt{P \circ (P^*)^\top},$$

where P^* is the time-reversed Markov chain with respect to the stationary distribution π, given by

$$P^*_{ji} = \frac{\pi_i P_{ij}}{\pi_j}$$

This yields the elementwise expression

$$D'_{ij} = P_{ij} \cdot \sqrt{\frac{\pi_i}{\pi_j}},$$

which corresponds exactly to the similarity transformation

$$D' = \Pi^{1/2} P \Pi^{-1/2},$$

where $\Pi = \mathrm{diag}(\pi)$. Thus, D' is similar to P, and they share the same spectrum. However, D' is not symmetric in general unless P satisfies detailed balance. In the reversible case, D' becomes symmetric and coincides with the classical discriminant matrix $D = \sqrt{P \circ P^\top}$. In nearly reversible regimes, this construction can still be useful for approximating symmetric operators in spectral or quantum walk-based analyses.

Lemma 1. Let D' be the discriminant matrix defined above. Then,

$$\|D'\| \leq 1,$$

where $\|\cdot\|$ denotes the spectral norm (operator norm induced by the Euclidean norm).

Proof. Define the matrix $\Pi^{1/2} = \mathrm{diag}(\sqrt{\pi_1}, \sqrt{\pi_2}, \ldots, \sqrt{\pi_n})$, so that:

$$D' = \Pi^{1/2} P \Pi^{-1/2}.$$

This shows that D' is similar to P, and hence they have the same eigenvalues:

$$\mathrm{Spec}(D') = \mathrm{Spec}(P).$$

In particular, their spectral norms are equal:

$$\|D'\| = \|P\|.$$

Since P is a stochastic matrix (its rows sum to 1), its spectral norm satisfies $\|P\| \leq 1$. Therefore,

$$\|D'\| \leq 1.$$

□

Let's define

$$y_t := \Pi_\flat (W')^t |v, \flat\rangle, \quad x_t := T_t(D')|v, \flat\rangle.$$

where $W' = R_\flat U' = (2\Pi_\flat - I)U'$. We want to show

$$y_t = x_t + \Delta_t, \quad \|\Delta_t\| = O(t\epsilon).$$

From $(U')^2 = I + \Delta$ with $\|\Delta\| \leq \epsilon$, one obtains a local error at each step. More precisely, there is a vector e_{t+1} (satisfying $\|e_{t+1}\| \leq C\epsilon$) such that

$$y_{t+1} = 2D' y_t - y_{t-1} + e_{t+1}.$$

Proof. Using the operator identity:

$$\Pi_b (W')^t = \Pi_b R_b U'(2\Pi_b - I)U'(W')^{t-2},$$

we rewrite the recurrence as:

$$\begin{aligned}\Pi_\flat (W')^t &= 2\,\underbrace{\Pi_\flat R_\flat}_{\Pi_\flat}\,U'\,\underbrace{\Pi_\flat U'}_{\Pi_\flat W'}(W')^{t-2} - \underbrace{\Pi_\flat R_\flat}_{\Pi_\flat}(U')^2 (W')^{t-2}\\ &= 2\Pi_\flat U'(\Pi_\flat (W')^{t-1}) - \Pi_\flat (W')^{t-2} - \Pi_\flat \Delta (W')^{t-2}.\end{aligned}$$

In the second line, we used the facts that $\Pi_\flat U' = \Pi_\flat R_\flat U' = \Pi_\flat W'$ (since $R_\flat$ reflects back into the $\Pi_\flat$-subspace), and that $(U')^2 = I + \Delta$, which introduces the extra perturbation term $\Pi_\flat \Delta (W')^{t-2}$. Thus, applying both sides to $|v, \flat\rangle$, we obtain the full perturbed recurrence:

$$\begin{aligned}y_t &= 2\Pi_\flat U' y_{t-1} - y_{t-2} - \Pi_\flat \Delta (W')^{t-2}|v, \flat\rangle\\ &= 2D' y_{t-1} - y_{t-2} + e_t,\end{aligned}$$

where $e_t := -\Pi_\flat \Delta (W')^{t-2}|v, \flat\rangle$ and $y_t := \Pi_\flat (W')^t |v, \flat\rangle$.

We now bound the final term in the recurrence:

$$e_t = -\Pi_b \Delta (W')^{t-2}|v, \flat\rangle.$$

To proceed, we first bound the norm of the walk operator $W' = R_b U'$. Since $(U')^2 = I + \Delta$, we can write

$$U' = \sqrt{I + \Delta} = I + \tfrac{1}{2}\Delta + O(\|\Delta\|^2),$$

which implies

$$\|U'\| \leq 1 + C_1 \varepsilon,$$

for some constant C_1, assuming $\|\Delta\| \leq \varepsilon \ll 1$. As R_b is a reflection, we have $\|R_b\| = 1$, so

$$\|W'\| = \|R_b U'\| \leq 1 + C_1 \varepsilon.$$

By submultiplicativity,

$$\|(W')^{t-2}\| \le (1 + C_1\varepsilon)^{t-2} \le e^{C_1\varepsilon(t-2)} \approx 1 + O(t\varepsilon).$$

Now, taking norms:

$$\|e_t\| = \| - \Pi_b \Delta (W')^{t-2} |v, \flat\rangle\| \le \|\Pi_b\| \cdot \|\Delta\| \cdot \|(W')^{t-2}\| \cdot \||v, \flat\rangle\|.$$

Using $\|\Pi_b\| \le 1$, $\|\Delta\| \le \varepsilon$, and $\||v, \flat\rangle\| = 1$, we obtain:

$$\|e_t\| \le \varepsilon(1 + O(t\varepsilon)) = \varepsilon + O(t\varepsilon^2).$$

Thus, as long as $t = O(1/\varepsilon^2)$, we get:

$$\boxed{\|e_t\| = O(\varepsilon)},$$

and the recurrence

$$y_{t+1} = 2Dy_t - y_{t-1} + e_{t+1}$$

holds with

$$\|e_{t+1}\| \le C\varepsilon.$$

□

Now, substitute $y_t = x_t + \Delta_t$:

$$x_{t+1} + \Delta_{t+1} = 2D' y_t - y_{t-1} + e_{t+1}.$$

The ideal recurrence (Chebyshev) is:

$$x_{t+1} = 2D' x_t - x_{t-1},$$

Subtracting the ideal (Chebyshev) sequence x_t from both sides then yields a recurrence for the error:

$$\Delta_{t+1} = 2D'\Delta_t - \Delta_{t-1} + e_{t+1}$$

This equation captures the deviation of the perturbed system from the ideal Chebyshev recurrence. The goal is to show $\|\Delta_t\| \le O(t\epsilon)$.

0.3 Bounding Error Accumulation

We outline three common ways to see that $\|\Delta_t\|$ remains at most $O(t\epsilon)$.

0.3.1 Unrolling the Recurrence

Since

$$\Delta_{t+1} = 2D'\Delta_t - \Delta_{t-1} + e_{t+1},$$

we iteratively expand (*unroll*) the recurrence. Each e_j is multiplied by factors involving D'. Provided $\|D'\| \le 1$ on the relevant subspace, these factors are bounded by 1 in norm (one can also use the property $|T_t(x)| \le 1$ for $x \in [-1, 1]$). Thus,

$$\|\Delta_{t+1}\| \le \sum_{j=1}^{t+1} \|(\text{bounded operators}) \cdot e_j\| \le (t+1) \max_j \|e_j\| \le (t+1)O(\epsilon),$$

showing linear growth in t.

Proof. We are given the recurrence:

$$\Delta_{t+1} = 2D'\Delta_t - \Delta_{t-1} + e_{t+1},$$

with initial conditions $\Delta_0 = \Delta_1 = 0$, and local error terms satisfying $\|e_j\| \leq \epsilon$. We aim to bound $\|\Delta_{t+1}\|$.

We compute step by step:

$$
\begin{aligned}
\Delta_2 &= 2D'\Delta_1 - \Delta_0 + e_2 = 0 + 0 + e_2 = e_2, \\
\Delta_3 &= 2D'\Delta_2 - \Delta_1 + e_3 = 2D'e_2 + 0 + e_3 = 2D'e_2 + e_3, \\
\Delta_4 &= 2D'\Delta_3 - \Delta_2 + e_4 = 2D'(2D'e_2 + e_3) - e_2 + e_4 \\
&= 4D'^2 e_2 + 2D'e_3 - e_2 + e_4, \\
\Delta_5 &= 2D'\Delta_4 - \Delta_3 + e_5 \\
&= 2D'(4D'^2 e_2 + 2D'e_3 - e_2 + e_4) - (2D'e_2 + e_3) + e_5 \\
&= 8D'^3 e_2 + 4D'^2 e_3 - 2D'e_2 + 2D'e_4 - 2D'e_2 - e_3 + e_5 \\
&= (8D'^3 - 4D')e_2 + (4D'^2 - I)e_3 + 2D'e_4 + e_5.
\end{aligned}
$$

Continuing in this way, we find that:

$$
\Delta_{t+1} = \sum_{j=1}^{t+1} A_j e_j,
$$

where each A_j is a polynomial in D' constructed recursively. By Lemma 1, D' is a bounded operator with $||D'|| \leq 1$, and since the recurrence only applies a constant number of algebraic operations at each step, we obtain $\|A_j\| \leq \alpha$ for some constant α independent of t.

Taking norms:

$$
\|\Delta_{t+1}\| \leq \sum_{j=1}^{t+1} \|A_j e_j\| \leq \sum_{j=1}^{t+1} \|A_j\| \cdot \|e_j\| \leq \sum_{j=1}^{t+1} \alpha\epsilon = (t+1)\alpha\epsilon.
$$

Therefore, we conclude:

$$\boxed{\|\Delta_{t+1}\| \leq O(t\epsilon)}.$$

□

0.3.2 Induction

Assume $\|\Delta_s\| \leq \alpha s \epsilon$ for $s < t$. From

$$\Delta_t = 2D'\Delta_{t-1} - \Delta_{t-2} + e_t,$$

we deduce

$$\|\Delta_t\| \leq 2\|D'\| \cdot \|\Delta_{t-1}\| + \|\Delta_{t-2}\| + \|e_t\| \leq 2\alpha(t-1)\epsilon + \alpha(t-2)\epsilon + \epsilon.$$

This is at most $\alpha t \epsilon$ for sufficiently large α. Thus, by induction, $\|\Delta_t\| \leq \alpha t \epsilon$.

0.3.3 Proof of the Bound on $\|\Delta_t\|$

We now rigorously establish the bound $\|\Delta_t\| \leq O(t\epsilon)$ using the updated recurrence:

$$\Delta_t = 2D'\Delta_{t-1} - \Delta_{t-2} + e_t, \quad \text{where } \|e_t\| \leq C\epsilon.$$

Proof. Taking the norm of both sides of the recurrence relation:

$$\begin{aligned} \|\Delta_t\| &\leq 2\|D'\| \cdot \|\Delta_{t-1}\| + \|\Delta_{t-2}\| + \|e_t\| \\ &\leq 2\alpha(t-1)\epsilon + \alpha(t-2)\epsilon + \epsilon \\ &\quad \text{(by Lemma 1 and using } \|e_t\| \leq \epsilon) \end{aligned}$$

We proceed by mathematical induction.

Base Case: Assume that for small values of t, say $t = 0$ and $t = 1$, we have

$$\|\Delta_0\| \leq C_0\epsilon, \quad \|\Delta_1\| \leq C_0\epsilon,$$

for some constant C_0.

Inductive Hypothesis: Assume that for all $k < t$, we have

$$\|\Delta_k\| \leq Ck\epsilon,$$

for some constant C independent of t.

Inductive Step. Applying the inductive hypothesis:

$$\|\Delta_{t-1}\| \leq C(t-1)\epsilon, \quad \|\Delta_{t-2}\| \leq C(t-2)\epsilon.$$

on the error term:

$$\|\Delta_t\| \leq 2C(t-1)\epsilon + C(t-2)\epsilon + C_1\epsilon.$$

$$\|\Delta_t\| = C(3t-4)\epsilon + C_1\epsilon.$$

For sufficiently large t, this is bounded by:

$$\|\Delta_t\| \leq \alpha t\epsilon, \quad \text{for some constant } \alpha.$$

Thus, by induction, the error bound grows at most linearly in t:

$$\|\Delta_t\| = O(t\epsilon).$$

□

0.4 Conclusion

We have shown that when $\|(U')^2 - I\| \le \epsilon$, the deviation Δ_t between the ideal Chebyshev state x_t and the actual state y_t obeys

$$\|\Delta_t\| = O(t\epsilon).$$

This ensures that if the chain is only slightly nonreversible, the QFF evolution still stays close to the ideal reversible Chebyshev dynamics up to a time t that can be taken fairly large, thus preserving the quantum speedup for practical time scales.

0.5 Hypothesis

Hypothesis: If the Markov chain P is exactly reversible, then the error term vanishes, i.e., $\Delta_t = 0$, and the quantum fast-forwarding procedure achieves the ideal relation

$$(\text{Quantum Walk})^{\sqrt{t}} = P^t.$$

In the nearly reversible case, where the deviation from reversibility is such that

$$\|\Delta_t\| \le O(t\epsilon),$$

the evolution on the flat subspace is instead given by

$$\Pi_\flat (U')^t |v, \flat\rangle = T_t(D')|v, \flat\rangle + \Delta_t |v, \flat\rangle.$$

Since the ideal eigenvalue function is $T_t(\cos\theta) = \cos(t\theta)$, the additive error of order $t\epsilon$ translates into an effective multiplicative error in the simulated evolution. Inverting the perturbed transformation yields

$$(\text{Quantum Walk})^{\sqrt{t}} \approx P^t \exp(-O(t\epsilon)).$$

0.6 Discussion: Penalizing the Speedup in the Exponent

In the reversible case, the Chebyshev recurrence

$$T_t(\cos\theta) = \cos(t\theta)$$

implies that the quantum walk operator exactly simulates the t-step evolution P^t using roughly $\sqrt{t}$ quantum walk steps.

In the nearly reversible case, the modified operator U' is not exactly Hermitian. More precisely,

$$(U')^2 = I + \Delta, \quad \|\Delta\| \leq \epsilon,$$

so that each application of U' introduces a local error of size at most ϵ. These local errors accumulate over t steps, resulting in a total error bounded by

$$\|\Delta_t\| \leq O(t\epsilon).$$

Because the Chebyshev polynomial satisfies $T_t(\cos\theta) = \cos(t\theta)$ in the ideal case, the perturbed eigenvalue transformation due to the local error becomes:

$$\cos(t\theta) + O(t\epsilon).$$

To understand how this impacts the evolution, we interpret the perturbed cosine as arising from a small shift in the phase. Using the Taylor expansion,

$$\cos((t-\delta)\theta) = \cos(t\theta) + \delta\theta\sin(t\theta) + \frac{(\delta\theta)^2}{2}\cos(t\theta) - \cdots,$$

we observe that a first-order perturbation $O(t\epsilon)$ in $\cos(t\theta)$ corresponds to a shift δ satisfying

$$\delta\theta\sin(t\theta) = O(t\epsilon).$$

Assuming $\sin(t\theta) = \Theta(1)$, this implies

$$\delta = O(t\epsilon)$$

Hence, we may equivalently write

$$\cos(t\theta) + O(t\epsilon) \approx \cos(t - O(t\epsilon))\theta),$$

i.e., the additive error in the eigenvalue function manifests as a small multiplicative time shift in the phase. This means the effective transformation corresponds to applying the evolution for time $t' + O(t'\epsilon)$.

To compare with the classical evolution simulated by D^t, we analyze both eigenvalue functions using Taylor expansion:

$$\cos^t(\theta) = 1 - \frac{t\theta^2}{2} + O(t^2\theta^4), \quad \cos(t'\theta) = 1 - \frac{t'^2\theta^2}{2} + O(t'^4\theta^4).$$

Equating the two up to second order in θ, we get

$$\frac{t\theta^2}{2} \approx \frac{t'^2\theta^2}{2} \quad \Rightarrow \quad t' = \sqrt{t}.$$

Thus, the perturbed evolution effectively corresponds to applying a cosine function with a phase scaled by $\sqrt{t + O(t\epsilon)}$.

When we invert this Chebyshev-based transformation to simulate the classical evolution P^t, the additive error in the cosine function manifests as a multiplicative error in the exponent. That is, the ideal relation

$$(\text{Quantum Walk})^{\sqrt{t}} = P^t$$

is now modified to:

$$(\text{Quantum Walk})^{\sqrt{t+O(t\epsilon)}} \approx (\text{Quantum Walk})^{\sqrt{t}(1+O(\epsilon))} \approx (\text{Quantum Walk})^{\sqrt{t}+O(\sqrt{t}\epsilon)} \approx P^t.$$

This means that the square-root speedup is penalized in the exponent by an additive term proportional to $\sqrt{t}\epsilon$. To maintain a quantum advantage, it becomes essential that $t\epsilon \ll 1$; otherwise, the exponential penalty can significantly degrade the performance of the quantum fast-forwarding scheme.