

Introduction to Linear Algebra

- Gilbert Strang

13

Linear Transformations

(Legendre and Hermite
Polynomials)

ROLL
OUR
TIGERS

FAILURE

WILL NEVER
OVERTAKE ME
IF MY DETERMINATION
TO SUCCEED
IS STRONG ENOUGH.

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Legendre Polynomials

$\{x = (x)\}, x = (x), P_0(x) = (1), P_1(x) = (x), P_2(x) = \dots$

Any polynomial $p(x) = \sum_{i=0}^n a_i x^i$ of degree n is a linear combination of $\{1, x, x^2, \dots, x^n\}$ which forms a basis of the space of all polynomials of degree not higher than n .

These basis polynomials can be orthogonalized by the Gram-Schmidt process, which converts a set of independent functions $q_0(x), q_1(x), \dots, q_n(x)$ into a set of orthogonal functions $P_0(x), P_1(x), \dots$

$\dots, P_n(x)$:

$$P_k(x) = q_k(x) - \sum_{i=1}^{k-1} \frac{\langle q_k(x), P_i(x) \rangle}{\langle P_i(x), P_i(x) \rangle} P_i(x)$$

(so) that $\langle P_i(x), P_j(x) \rangle = 0$ for all $i \neq j$.

Here, $\langle P_i(x), P_j(x) \rangle$ is the inner product of $P_i(x)$ and $P_j(x)$ defined as:

$$\langle P_i(x), P_j(x) \rangle = \int_{-1}^1 P_i(x) P_j(x) dx$$

Given $q_0(x) = 1$, $q_1(x) = x$, $q_2(x) = x^2$, $q_3(x) = x^3$,
 $\dots, q_n(x) = x^n$.

W. moga f. $\int_{-1}^1 x^2 dx = (x)^3 \Big|_{-1}^1 = 0$ boimonggo puk.

$$P_0(x) = q_0(x) = 1$$

$$P_1(x) = x - \frac{\int_{-1}^1 x^2 dx}{\int_{-1}^1 dx} = x$$

berilanggung ad yas elisiongut $\int_{-1}^1 x^2 dx = 0$

$$P_2(x) = x^2 - \frac{\int_{-1}^1 x^2 \cdot x dx}{\int_{-1}^1 x dx} = x^2$$

$\int_{-1}^1 x dx = 0$ berilanggung $\int_{-1}^1 x \cdot x dx = 0$

$$= x^2 - \frac{\frac{1}{3}[x^3]_{-1}^1}{[x]_{-1}^1} x - 0 = x^2 - \frac{1}{3} \cdot \frac{2}{2}$$

$$= x^2 - \frac{1}{3}(3x^2 - 1) = x^2 - \frac{1}{3}(3x^2 - 1) = x^2 - \frac{1}{3}(3x^2 - 1)$$

$$P_3(x) = x^3 - \frac{\int_{-1}^1 x^3 dx}{\int_{-1}^1 dx} - \frac{\int_{-1}^1 x^3 \cdot x dx}{\int_{-1}^1 x \cdot x dx} - \frac{\int_{-1}^1 x^3 (x^2 - \frac{1}{3}) dx}{\int_{-1}^1 (x^2 - \frac{1}{3}) (x^2 - \frac{1}{3}) dx} \quad (x^2 = \frac{1}{3})$$

(x) f. subong rami uti $\int_{-1}^1 (x^2 - \frac{1}{3})(x^2 - \frac{1}{3}) dx = 0$

$$= x^3 - 0 - \frac{\frac{1}{3}[x^5]_{-1}^1}{\frac{1}{3}[x^3]_{-1}^1} x - 0$$

$$(x) \quad \text{Given } x^3 - \frac{3}{5} \cdot \frac{2}{2} x = x^3 - \frac{3}{5}x = \frac{1}{5} [5x^3 - 3x]$$

$$\begin{aligned} P_4(x) &= x^4 - \frac{\int x^4 dx}{\int dx} - \frac{\int x^4 \cdot x dx}{\int x^2 dx} - \frac{\int x^4 (x^2 - \frac{1}{3}) dx}{\int (x^2 - \frac{1}{3})^2 dx} \\ &\quad \left(\begin{array}{l} \frac{1}{5} \int x^4 (x^3 - \frac{3}{5}x) dx \\ \left(x^3 - \frac{3}{5}x \right) \end{array} \right) \left(x^3 - \frac{3}{5}x \right) \\ &= x^4 - \frac{\frac{1}{5} [x^5]_1^x}{[x]_1^x} - 0 + \frac{\frac{1}{7} [x^7]_1^x - \frac{1}{3 \cdot 5} [x^5]_1^x}{\frac{1}{5} [x^5]_1^x - \frac{2}{3 \cdot 3} [x^3]_1^x + \frac{1}{9} [x]_1^x} \left(x^2 - \frac{1}{3} \right) \end{aligned}$$

- 0

$$\begin{aligned} &= x^4 - \frac{1}{5} - \frac{\frac{2}{7} - \frac{2}{15}}{\frac{2}{5} - \frac{4}{9} + \frac{2}{9}} \left(x^2 - \frac{1}{3} \right) \Rightarrow \frac{2}{5} - \frac{2}{9} \\ &= x^4 - \frac{1}{5} - \frac{30 - 14}{105} \cdot \frac{45}{48} \left(x^2 - \frac{1}{3} \right) \end{aligned}$$

$$\begin{aligned} &= \cancel{x^4 - \frac{1}{5}} - \cancel{\frac{16}{105} \times \frac{35}{48}} \left(x^2 - \frac{1}{3} \right) \\ &= x^4 - \frac{1}{5} - \frac{16}{105} \times \frac{35}{48} \left(x^2 - \frac{1}{3} \right) \\ &= x^4 - \frac{1}{5} - \frac{6}{7} x^2 + \frac{2}{7} = x^4 - \frac{6}{7} x^2 + \frac{3}{35} \\ &= \frac{1}{35} (35x^4 - 30x^2 + 3) // \end{aligned}$$

The orthogonal polynomials $P_0(x), P_1(x), \dots, P_n(x)$ are the Legendre polynomials.

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = x^2 - \frac{1}{3} = \frac{1}{3}(3x^2 - 1)$$

$$P_3(x) = x^3 - \frac{3}{5}x = \frac{1}{5}(5x^3 - 3x)$$

$$P_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35} = \frac{1}{35}(35x^4 - 30x^2 + 3)$$

$$\left(\frac{1}{2} - \frac{3s}{2}\right) \frac{\frac{8}{21} - \frac{6}{5}}{\frac{5}{7} + \frac{1}{7} - \frac{3}{2}} = \frac{1}{2} - \frac{3s}{2}$$

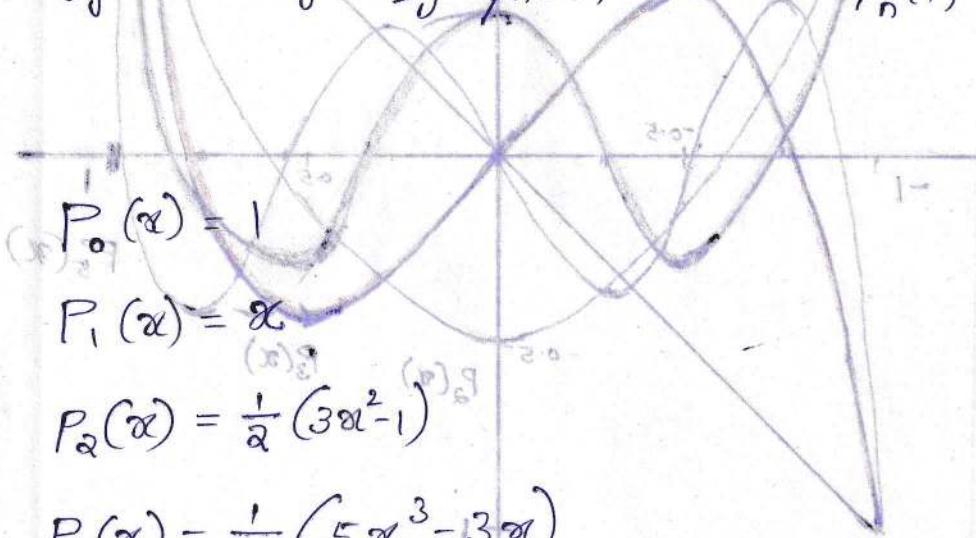
$$\left(\frac{1}{2} - \frac{3s}{2}\right) \frac{\frac{24}{84} - \frac{10-3s}{20}}{\frac{84}{20} - \frac{20}{2}} = \frac{1}{2} - \frac{3s}{2}$$

* $P_n(x)$ is an even function if n is even (or)
an odd function if n is odd.

$$P_n(-x) = (-1)^n P_n(x)$$

Instead of normalization, the orthogonal Legendre polynomials are subject to standardization.

The Legendre polynomials are standardized by dividing by $P_n(1)$, so that $P_n(1) = 1$.



$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

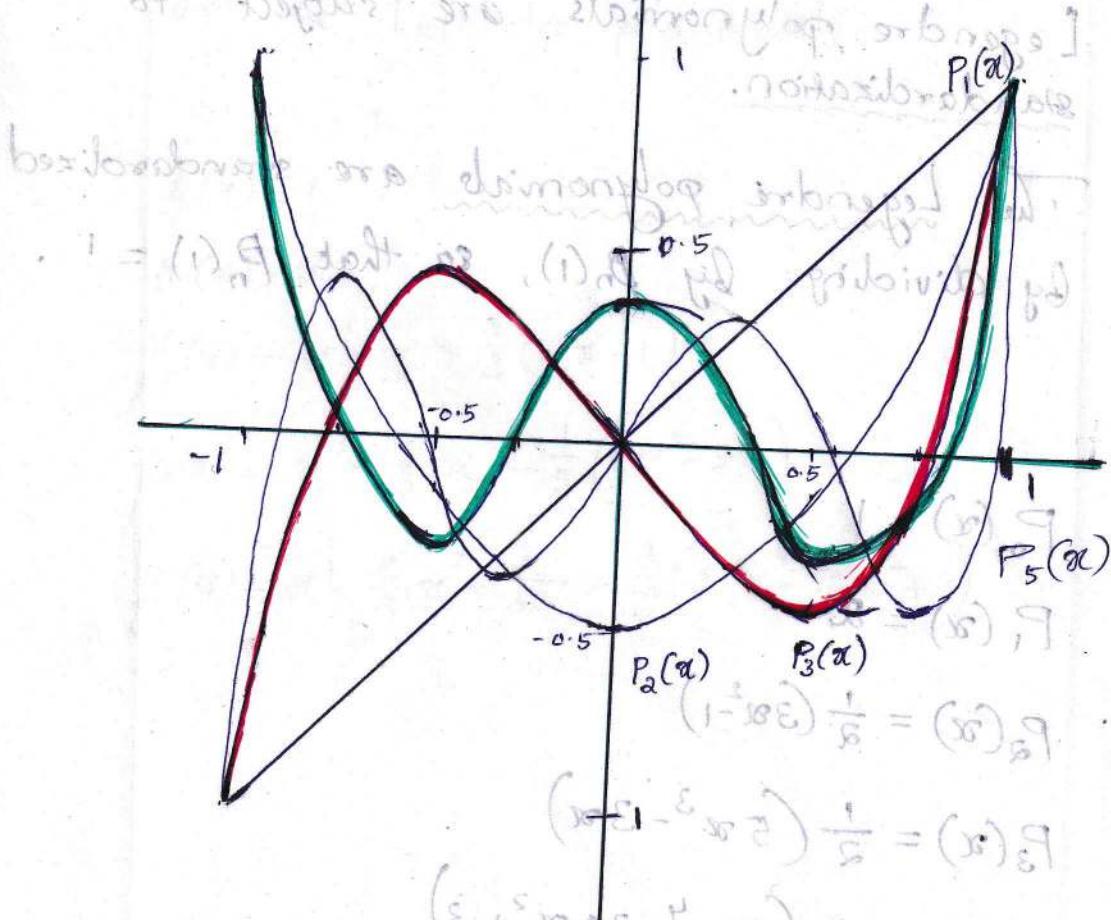
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

Parity of odd terms

Legendre polynomials are orthogonal over the interval

and therefore are eigenvalues of the operator

orthogonal over the interval



$$(1-10\varepsilon)^{\frac{1}{5}} = (x)_5$$

$$(x^2 - \varepsilon_{\infty}^2)^{\frac{1}{5}} = (x)_5$$

$$(\varepsilon + 10.4\varepsilon - 10\varepsilon^2)^{\frac{1}{5}} = (x)_5$$

$$(x^2 + \varepsilon_{\infty}^2)^{\frac{1}{5}} = (x)_5$$

Legendre Polynomials

□ Legendre's differential equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad (1)$$

n is constant

$$x^m D(1-x) m \sum_{n=0}^{\infty} f_n x^n = "f"$$

$$y'' - \frac{2x}{1-x^2} y' + \frac{n(n+1)}{1-x^2} y = 0 \quad (2)$$

Assume that the differential equation has a solution, which can be written as the Maclaurin series :

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{1}{2!} y''(0)x^2 + \frac{1}{3!} y'''(0)x^3 + \dots \\ &= \sum_{m=0}^{\infty} \frac{y^{(m)}(0)}{m!} x^m \\ &= \sum_{m=0}^{\infty} a_m x^m \end{aligned}$$

$$y = \sum_{m=0}^{\infty} Q_m x^m$$

$$y' = \sum_{m=1}^{\infty} m Q_m x^{m-1}$$

für $m > 1$

$$y'' = \sum_{m=2}^{\infty} m(m-1) Q_m x^{m-2}$$

$$(1-x^2) \sum_{m=2}^{\infty} = \frac{(1+x)x}{m(m-1)} Q_m x^{m-2} - 2x \sum_{m=1}^{\infty} m Q_m x^{m-1}$$

mit weiteren Rechnungen + k $\sum_{m=0}^{\infty} Q_m x^m = 0$
ist es möglich die Koeffizienten zu bestimmen
 $k = n(n+1)$

$$\sum_{m=2}^{\infty} m(m-1) Q_m x^{m-2} = \sum_{m=2}^{\infty} m(m-1) Q_m x^{m-2}$$

$$- \sum_{m=1}^{\infty} 2m Q_m x^m + \sum_{m=0}^{\infty} k Q_m x^m = 0$$

$$2Q_1 + \sum_{m=2}^{\infty} k Q_m = 0$$

$$\text{Put } s+m-a \Rightarrow m=s+a$$

$$\sum_{s=0}^{\infty} (s+a)(s+1) a_{s+2} x^s - \sum_{s=0}^{\infty} (s+a)(s+1) a_{s+2} x^{s+2}$$

$$-\sum_{s=0}^{\infty} 2(s+a) a_{s+2} x^{s+1} + \sum_{s=0}^{\infty} s k a_{s+2} x^{s+1} = 0$$

~~all~~ terms: $2 a_2 + n(n+1) a_0 = 0 \rightarrow a_2 = \frac{-n(n+1)}{2} a_0$

x^3 terms: $3 \cdot 2 a_3 + [-2 a_1 + n(n+1)] a_1 = 0$

 $\rightarrow a_3 = \frac{-(n-1)(n+2)}{3 \cdot 2} a_1$

x^s terms:

$$(s+2)(s+1) a_{s+2} + [-s(s-1) - 2s + n(n+1)] a_s = 0$$

$$a_{s+2} = \frac{-(n-s)(n+s+1)}{(s+2)(s+1)} a_s$$

$s = 0, 1, \dots$

→ recurrence relation (or) recursion formula

$$Q_2 = \frac{-n(n+1)}{2!} Q_0$$

$$Q_4 = \frac{-(n-2)(n+3)}{4!} Q_0$$

$$= \frac{(n-2)n(n+1)(n+3)}{4!} Q_0$$

$$\boxed{\frac{(1+n)n}{s} - sP}$$

4!

$$Q_3 = \frac{-(n-1)(n+2)}{3!} Q_1$$

$$Q_5 = \frac{(n-3)(n+4)}{5 \cdot 4} Q_3$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} Q_1$$

$$0 = P \left[(1+n)n + sP s - \right] + sP s \cdot e$$

$$\boxed{P - \frac{(s+n)(1-n)}{s \cdot e} = sP}$$

$$0 = P \left[(1+n)n + 2s - (1-n)s - \right] + sP (1+n)(s+n)$$

$$\boxed{sP \frac{(1+n)(s+n)(s-n)}{(1+n)(s+n)} = sP}$$

$\therefore 110 = ?$

Number of visitors (n) visitors in summer \leftarrow

$$y(x) = \sum_{m=0}^{\infty} a_m x^m$$

nonzero w/t

$$0 \neq 1 = \begin{vmatrix} 0 & y_1(x) & y_2(x) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} = (0)(0)(0)$$

closure,

$$y_1(x) = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-1)n(n+1)(n+3)}{4!} x^4 - \dots$$

$$[d^n] y_2(x) = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \dots$$

independent if and only if the two terms cancel.

These series converge for $|x| < 1$.

$y_1(x)$ contains even powers of x only, while

$y_2(x)$ contains odd powers of x only.

$\rightarrow y_1$ and y_2 are independent.

$\therefore y(x) = a_0 y_1(x) + a_1 y_2(x)$ is a general solution of the Legendre's differential equation on the interval $-1 < x < 1$.

The Wronskian of y_1 and y_2 at $t = 0$ is:

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

LA \oplus

$\Rightarrow y_1$ and y_2 are independent

* Let f & g be differentiable on $[a, b]$.

① If $W(f, g)(t_0) \neq 0$ for some $t_0 \in [a, b]$

then $f(t)$ and $g(t)$ are linearly independent on $[a, b]$

② If f and g are linearly dependent

on $[a, b]$, then $W(f, g)(t) = 0$ for

all $t \in [a, b]$

independent on a, b \Leftarrow

$$\text{as } (0)_{ab,D} + (0)_{ab,D} = (0)_{ab}$$

which implies f and g linearly independent

$\therefore 1 > 0 > 1 \rightarrow$ linearly independent with no overlaps

Polynomial solutions

The reduction of power series to polynomials is a great advantage because then we have solutions for all α , without convergence restrictions. For special functions arising as solutions of ODEs this happens quite frequently leading to various important families of polynomials.

For Legendre's equation this happens when the parameter 'n' is a non-negative integer, then the RHS becomes zero in the recurrence relation. ~~so~~ for $s = n$.

$$a_{n+2} = 0, a_{n+4} = 0, a_{n+6} = 0, \dots$$

Hence,

If n is even, $y_1(\alpha)$ reduces to a polynomial of degree ' n '.

If n is odd, $y_2(\alpha)$ reduces to a polynomial of degree n .

These polynomials, multiplied by some constants
are called Legendre polynomials, $P_n(x)$.

Legendre polynomials are the most important class of orthogonal polynomials.
They are used in solving boundary value problems involving spherical harmonics.
They are also used in quantum mechanics for calculating the angular momentum of particles.

The Legendre polynomials are defined as follows:
Let θ be the polar angle of a point in spherical coordinates.
The Legendre polynomial of degree n is given by
$$P_n(\cos \theta) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$\therefore P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

For example, if $x = \cos \theta$, then $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$, $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.
The degree of the polynomial $P_n(x)$ is n .

stants

$$a_{s+2} = \frac{-(n-s)(n+s+1)}{(s+2)(s+1)} a_s$$

using 2nd

$$a_s = \frac{-(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2}$$

$$\left[\frac{-(s-2)(s-a)(s-n)}{(s-a)(s-n)} + \frac{(s-a)a}{(1-nb)b} + 16 \right] nD =$$

$$a_{n-2} = \frac{-n(n-1)}{2(2n-1)} a_n$$

$$a_{n-4} = \frac{-(n-2)(n-3)}{4(2n-3)} a_{n-2} = \frac{(-1)^2 n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} a_n$$

$$a_{n-6} = \frac{-(n-4)(n-5)}{6(2n-5)} a_{n-4} = \frac{(-1) \cdot n(n-1)(n-2)(n-3)(n-4)(n-5)}{2 \cdot 4 \cdot 6 \cdot (2n-1)(2n-3)(2n-5)} a_n$$

$$\frac{! (nB)}{! (n) \cdot 6} = \frac{1 \cdot 3 \cdot 5 \cdots (n-15)(n-13)}{12} = 10$$

$1 = 10$, next new

$$a_{n-2m} = (-1)^m \frac{n(n-1)(n-2) \cdots (n-(2m-1))}{2 \cdot 4 \cdot 6 \cdots 2m \cdot (2n-1)(2n-3) \cdots (2n-(2m-1))} a_n$$

~~(s-a)(s-n)~~ ~~(1-nb)b~~ ~~16~~ ~~! (n)~~ ~~! (n)~~

This yields the polynomial solution:

$$y = Q_n x^n + Q_{n-2} x^{n-2} + Q_{n-4} x^{n-4} + \dots$$

$$= Q_n \left[x^n + \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-3)(2n-4)} x^{n-4} \dots \right]$$

The Legendre polynomials $P_n(x)$ are

defined by choosing

$$Q_n = \frac{(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1}{n!} = \frac{(2n)!}{2^n (n!)^2}$$

such that, $P_n(1) = 1$

$$P_n(x) = \frac{(1-n)(-n) \dots (s-n)(1-n)}{2^n (n!)^2} x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 (2n-3)(2n-4)} x^{n-4} \dots$$

$$P_n(x) = \frac{(2n)!}{2^n (n!)^2} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \dots \right]$$

$$= \frac{(2n)!}{2^n (n!)^2} \left[x^n - \sum_{l=1}^{M_n} \frac{(-1)^l n(n-1)(n-2) \dots (n-2l+1)}{2^l l! (2n-1)(2n-3) \dots (2n-2l+3)} x^{n-2l} \right]$$

$$P_n(x) = \sum_{l=0}^{M_n} (-1)^l \frac{(2n-2l)!}{2^l l! (n-l)! (n-2l)!} x^{n-2l}$$

is called the Legendre polynomial of degree n .

$$M_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$$P_n(x) = \sum_{k=0}^{M_n} (-1)^k \frac{(2n-2k)!}{2^k k! (n-k)! (n-2k)!} x^{n-2k}$$

where $M_n = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

$$\underline{P_0(x) = 1}$$

$$P_1(x) = \frac{2!}{2} x^1 = x$$

$$P_2(x) = \frac{\frac{4!}{2} \cdot 3}{4! \cdot 2! \cdot 2!} x^2 - \frac{2!}{4} \cdot 2$$

$$= \frac{3}{2} x^2 - \frac{1}{2} = \frac{1}{2} (3x^2 - 1)$$

$$a_n = \frac{(2n-1)(2n-3)\dots\cdot 5\cdot 3\cdot 1}{n!} = \frac{(2n)!}{2^n(n!)^2}$$

Proof
10/12/2020.

Generating Function

Consider the function,

$$g(x, t) = \frac{1 - 2xt + t^2}{1 - xt} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\text{Expand } (1 - 2xt + t^2)^{-\frac{1}{2}} = (1 - (2x-t)t)^{-\frac{1}{2}} \text{ in powers}$$

of t for $|t| < 1$ using binomial expansion:

~~$$[1 - (2x-t)t]^{-\frac{1}{2}} = 1 + \frac{1}{2}(2x-t)$$~~

~~$$(1 + m) = 1 + m + \frac{m(m-1)}{2!} + \frac{m(m-1)(m-2)}{3!} + \dots$$~~

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots$$

$m \notin \mathbb{W}$,

m : \pm ve (or) fraction : infinite # of terms.

$$\left[1 - t(2\alpha - t)\right]^{\frac{1}{2}} = 1 + \left(\frac{-1}{2}\right)(-t(2\alpha - t)) + \frac{1}{2!} \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) t^2 (2\alpha - t)^2$$

$$+ \frac{1}{3!} \left(\frac{-1}{2}\right) \left(\frac{-3}{2}\right) \left(\frac{-5}{2}\right) t^3 (2\alpha - t)^3 + \dots$$

$$= 1 + \frac{1}{2} t(2\alpha - t) + \frac{1 \times 3}{2 \times 4} t^2 (2\alpha - t)^2$$

$$+ \frac{1 \times 3 \times 5}{2 \times 4 \times 6} t^3 (2\alpha - t)^3 + \dots$$

using binomial

$$= 1 + \frac{1}{2} t(2\alpha - t) + \frac{3}{8} t^2 (2\alpha - t)^2 + \frac{5}{16} t^3 (2\alpha - t)^3 + \dots$$

~~$$= 1 + \alpha t - \frac{t^2}{2} + \frac{3}{8} t^2 (4\alpha^2 + t^2 - 4\alpha t) + \dots$$~~

~~$$\frac{5}{16} t^3 (8\alpha^3 - t^3 - 12\alpha^2 t + 16\alpha t^2)$$~~

$$+ \frac{(s-m)(1-m)}{15} t^5 + \frac{(1-m)}{15} t^6 + 8m + 1 = (s+1)$$

~~$$= 1 + \alpha t - \frac{t^2}{2} + \frac{3}{8} \alpha^2 t^2 + \frac{3}{8} t^4 - \frac{3}{2} \alpha t^3$$~~

~~$$+ \frac{5}{8} \alpha^3 t^3 - \frac{5}{16} t^5 - \frac{15}{4} t^6 \alpha^2 + \dots$$~~

$$= 1 + \alpha t + \frac{1}{2} (3\alpha^2 - 1) t^2 + \frac{1}{2} (5\alpha^3 - 3\alpha) t^3 + \dots$$



Le

Equating the coefficient of t^n for
 $n = 0, 1, 2, \text{ and } 3$. with $g(x, t) = \sum_{n=0}^{\infty} P_n(x) t^n$
we find the 1st few Legendre polynomials
are given by :

$$P_0(x) = 1$$

$$P_1(x) = x$$

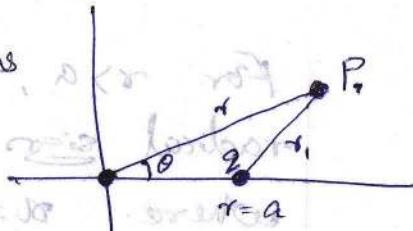
$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$\Rightarrow g(x, t) = \frac{1}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)^{-\frac{1}{2}}$$

is called the generating function for
Legendre polynomials.

Ex:- The generating function is useful in solving physical problems involving the potential associated with any inverse square force.



* Electrostatic potential due to a charge 'q' displaced from origin.

Consider an electric charge q placed non-the z-axis's at $r=a$.

$$\left(\frac{q}{r}\right) (\text{area}) \cdot q \sum_{r=a}^{\infty} \frac{1}{r^2 \sin \theta} = V$$

The electrostatic potential at a non-axial point due to this charge is given by

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{r_1}$$

$$r_1 = \sqrt{q^2 + a^2 - 2aq \cos \theta}$$

[Law of cosines]

$$V = \frac{q}{4\pi\epsilon_0 r} \cdot \frac{1}{\sqrt{q^2 + a^2 - 2aq \cos \theta}}$$

$$= \frac{q}{4\pi\epsilon_0 r} \cdot \frac{1}{\sqrt{1 + \left(\frac{q}{r}\right)^2 + 2\left(\frac{q}{r}\right) \cos \theta}}$$

For $r > a$, and the expression under the radical sign may be written as $(1-2at+t^2)^{-\frac{1}{2}}$
 where $a = \cos \theta$ and $t = \frac{a}{r}$; $|t| < 1$.

substituting value of t in above
 we get $P_n(\cos \theta)$ as
 value of $\cos \theta$

$$V = \frac{q}{4\pi\epsilon_0 r} \sum_{n=0} P_n(\cos \theta) \left(\frac{a}{r} \right)$$

P expands into no terms
 $\rightarrow n = 0$ upto $2m - 1$ only

for $n = 0$ is the first term of series and
 for $n = 1$ is second term of series

$$\frac{1}{r} \cdot \frac{1}{\partial \vec{r} H} = V$$

now we have
 from above

$$\cos \theta \partial \vec{r} H - \frac{1}{r} \partial \vec{r} H = 0$$

$$\frac{1}{\cos \theta \partial \vec{r} H + \frac{1}{r} \partial \vec{r} H} \cdot \frac{P}{\partial \vec{r} H} = V$$

$$\frac{1}{\cos \left(\frac{a}{r} \right) \partial \vec{r} H + \left(\frac{P}{r} \right) + 1} \cdot \frac{P}{\partial \vec{r} H} = V$$

□ Recurrence relations

The generating function for Legendre Polynomials is:

$$g(\alpha, t) = \frac{1}{\sqrt{1 - 2\alpha t + t^2}} = \sum_{n=0}^{\infty} P_n(\alpha) t^n$$

$$\frac{\partial g}{\partial t} = \frac{-1}{2} \frac{-2\alpha + 2t}{(1 - 2\alpha t + t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(\alpha)$$

$$\frac{\alpha - t}{(1 - 2\alpha t + t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1} P_n(\alpha)$$

$$\frac{\alpha - t}{\sqrt{1 - 2\alpha t + t^2}} = (1 - 2\alpha t + t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(\alpha)$$

$$(\alpha - t) \sum_{n=0}^{\infty} P_n(\alpha) t^n = (1 - 2\alpha t + t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(\alpha)$$

$$(1 - 2\alpha t + t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(\alpha) + (t - \alpha) \sum_{n=0}^{\infty} P_n(\alpha) t^n = 0$$

$$\sum_{n=0}^{\infty} \left[(n+1) P_n(\alpha) t^{n+1} - (an+1) \alpha P_n(\alpha) t^n + n P_n(\alpha) t^{n-1} \right] = 0$$

all terms containing t^n
will disappear.

To collect the coeff. of t^n , we replace
~~n by $n-1$~~ in the ~~1st~~ term and by
~~n+1~~ in the last term. Then equating
the resultant expression to zero, we
get:

$$(*) \text{ for } \sum_{n=0}^{\infty} (t^n + t^{n-1}) = \frac{1+t+ts}{1-ts}$$

$$n P_{n-1}(\alpha) - (an+1) n P_n(\alpha) + (n+1) P_{n+1}(\alpha) = 0$$

$$(an+1) \alpha P_n(\alpha) = (n+1) P_{n+1}(\alpha) + n P_{n-1}(\alpha)$$

$$(x) \text{ for } \sum_{n=0}^{\infty} (t^n + t^{n-1}) = \frac{1-t}{1-ts}$$

$$(x) \text{ for } \sum_{n=0}^{\infty} (t^n + t^{n-1}) = 1 + (x) \sum_{n=0}^{\infty} (-ts)^n$$

$$1 = f(x) + \sum_{n=0}^{\infty} (x-1) + (x) \sum_{n=0}^{\infty} t^n + ts \sum_{n=0}^{\infty} (-ts)^n$$

$n=1$,

$$3x P_1(x) = 2 P_2(x) + P_0(x)$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) \cancel{=} 2x^2$$

$$3x^2 = 2 P_2(x) + 1 \Rightarrow P_2(x) = \frac{1}{3} (3x^2 + 1)$$

$$\sum_{n=0}^{\infty} f(n) q^n = \frac{1}{1 - f(q)}$$

$$\sum_{n=0}^{\infty} f(n) q^n = \frac{1}{1 - f(q)}$$

$$\sum_{n=0}^{\infty} f(n) q^n = \sum_{n=0}^{\infty} f_n$$

$$\sum_{n=0}^{\infty} f(n) q^{(k+1)} = \left[f(q) + f(q) q + \dots \right] \sum_{n=0}^{\infty}$$

$$\sum_{n=0}^{\infty} f(n) q^n + f(n) q^n =$$

~~P(x)~~

$$(x)_0 q + (x)_1 q^2 = (x) q \text{ vs}$$

$$g(x,t) = \frac{t}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x) t^n$$

$$\frac{\partial g}{\partial x} = \frac{\left(\frac{-1}{2}\right)(-2t)}{\frac{1}{2}(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(x) t^n$$

$$\frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} P'_n(x) t^n$$

$$\frac{t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2) \sum_{n=0}^{\infty} P'_n(x) t^n$$

$$t \sum_{n=0}^{\infty} P_n(x) t^n = (1-2xt+t^2) \sum_{n=0}^{\infty} P'_n(x) t^n$$

$$\begin{aligned} \sum_{n=0}^{\infty} \left[2x P'_n(x) + P_n(x) \right] t^{n+1} &= \sum_{n=0}^{\infty} (1+t^2) P'_n(x) t^n \\ &= \sum_{n=0}^{\infty} t^n P'_n(x) + \sum_{n=0}^{\infty} P'_n(x) t^{n+2} \end{aligned}$$

Collecting coefficients of t^{n+1} from both sides and equating them. $(*) \nabla_{\alpha} P_n(t) = (1+\alpha\beta)$

$$\alpha \nabla_{\alpha} P_n(x) + P_n'(x) = P_{n+1}'(x) + P_{n-1}'(x)$$

$$(*) \nabla_{\alpha} P_n(x) + (1+\alpha\beta) P_n(x) = (1+\alpha\beta) P_{n+1}(x) + (1+\alpha\beta) P_{n-1}(x)$$

$$(*) \nabla_{\alpha} P_n(x) + (1+\alpha\beta) P_n(x) = (1+\alpha\beta) P_{n+1}(x)$$

Both sides of the equation are homogeneous in α .

$$\cancel{(*) \nabla_{\alpha} P_n(x)} + \cancel{(1+\alpha\beta) P_n(x)} = \cancel{(1+\alpha\beta) P_{n+1}(x)}$$

$$(*) \nabla_{\alpha} P_n(x) = (1+\alpha\beta) P_{n+1}(x)$$

t^{n+2}

1st recurrence relation,

$$(2n+1)\alpha P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating w.r.t α and multiply by
the result

$$(2n+1)P_n(x) + (2n+1)\alpha P'_n(x) = (n+1)P'_{n+1}(x) + nP'_{n-1}(x)$$

$$2(2n+1)\alpha P'_n(x) = 2(n+1)P'_{n+1}(x) + 2n P'_{n-1}(x) - 2(2n+1)P_n(x)$$

2nd recurrence relation is multiplied by $2n+1$,

$$\cancel{2(2n+1)\alpha P'_n(x) + P_n(x)} = \cancel{P'_{n+1}(x)} + \cancel{P'_{n-1}(x)}$$

$$2(2n+1)\alpha P'_n(x) + (2n+1)P_n(x) = (2n+1)P'_{n+1}(x) + (2n+1)P'_{n-1}(x)$$

Substituting for $\alpha(2n+1) \propto P'_n(\alpha)$,

$$\begin{aligned} \alpha(2n+1) \underline{\underline{P'_{n+1}(\alpha)}} + 2n \underline{\underline{P'_{n-1}(\alpha)}} - \alpha(2n+1) P_n(\alpha) + \alpha(2n+1) P_n(\alpha) = \\ = (2n+1) \underline{\underline{P'_{n+1}(\alpha)}} + (2n+1) \underline{\underline{P'_{n-1}(\alpha)}} \end{aligned}$$

$$P'_{n+1}(\alpha) - P'_{n-1}(\alpha) = (2n+1) P_n(\alpha)$$

□ Orthogonality Relations

Legendre polynomials are orthogonal. This enables us to express a given function defined on the interval $(-1, 1)$ in a series of Legendre polynomials.

The m^{th} and n^{th} order Legendre polynomials $P_m(x)$ and $P_n(x)$ respectively satisfy the equations.

$$(1-x^2) P_m''(x) - 2x P_m'(x) + m(m+1) P_m(x) = 0$$

$$(1-x^2) P_n''(x) - 2x P_n'(x) + n(n+1) P_n(x) = 0$$

Multiplying the 1st eqn. by $P_n(x)$ and the 2nd equation by $P_m(x)$, and subtract

$$(1-x^2) \left[P_n P_m'' - P_m P_n'' \right] - 2x \left[P_n P_m' - P_m P_n' \right] =$$

$$= [n(n+1) - m(m+1)] P_m P_n$$

$$\frac{d}{d\alpha} \left[(-\alpha^2) (P_n P_m^1 - P_m P_n^1) \right] = [n(n+1) - m(m+1)] P_m P_n$$

no terms having a common set of odd terms

Since (-1) disappears with no benefit
Sing. over range α from $\alpha=-1$ to $\alpha=+1$,

$$\sqrt{1-\alpha^2}$$

$$[n(n+1) - m(m+1)] \int_{-1}^{+1} P_m(\alpha) P_n(\alpha) d\alpha = (-\alpha^2) \left[P_n P_m^1 - P_m P_n^1 \right]_{-1}^{+1}$$

$(-\alpha^2)$ vanishes for both the limits ($\alpha=\pm 1$)

$$\rightarrow \text{RHS} = 0$$

$$0 = (m) \cancel{q} (1+m) n + (n) \cancel{q} (1+n) m - (m) \cancel{q} (n-1)$$

$$[n(n+1) - m(m+1)] \int_{-1}^{+1} P_m(\alpha) P_n(\alpha) d\alpha = 0$$

therefore, $P_m(\alpha) P_n(\alpha)$ no terms

when $m \neq n$,

$$\boxed{\int_{-1}^{+1} P_m(\alpha) P_n(\alpha) d\alpha = 0}$$

$$\left\{ \sum_{n=0}^{\infty} P_n(\alpha) \frac{x^n}{n!} \right\} \left\{ \sum_{m=0}^{\infty} P_m(\alpha) \frac{t^m}{m!} \right\} = \frac{1}{\sqrt{1-2\alpha x+t^2}}$$

$$\frac{1}{\sqrt{1-2\alpha x+t^2}} = \sum_{n=0}^{\infty} P_n(\alpha) t^n$$

$$\frac{1}{\sqrt{1-\alpha s x+s^2}} = \sum_{m=0}^{\infty} P_m(\alpha) s^m$$

Multiplying these equations & integrating w.r.t α ,
between $\alpha = -1$ and $\alpha = +1$

$$\int_{-1}^{+1} \frac{1}{\sqrt{(1-\alpha x+t^2)(1-\alpha s x+s^2)}} d\alpha =$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_{-1}^{+1} P_m(\alpha) P_n(\alpha) d\alpha \{ s^m t^n \}$$

RHS survives for terms with $m=n$ only.

$$\int_{-1}^{+1} \frac{dx}{1-2tx+t^2} = \sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} [P_n(x)]^2 dx \right\} t^{2n}$$

Substitute $1-2tx+t^2 = u \Rightarrow -2t dx = du$

$$dx = -\frac{du}{2t}$$

$$x = -1 : u = 1 + at + t^2 = (1+t)^2$$

$$x = +1 : u = 1 - at + t^2 = (1-t)^2$$

$$\begin{aligned} I &= \int_{-1}^{+1} \frac{dx}{1-2tx+t^2} = \frac{1}{2t} \int_{(1-t)^2}^{(1+t)^2} \frac{du}{u} \\ &= \frac{1}{2t} \left[\log|u| \right]_{(1-t)^2}^{(1+t)^2} \\ &= \frac{1}{2t} \ln \left(\frac{(1+t)^2}{(1-t)^2} \right) = \frac{1}{t} \ln \left(\frac{1+t}{1-t} \right) \end{aligned}$$

who can has error not? answer 2H9

$$\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots$$

$$\log(1-t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \dots$$

$$\begin{aligned}\frac{1}{t} \log\left(\frac{1+t}{1-t}\right) &= \frac{2}{t} \left[t + \frac{t^3}{3} + \frac{t^5}{5} + \dots \right] \\ &= 2 \left[1 + \frac{t^2}{3} + \frac{t^4}{5} + \dots \right] \\ &= 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{2n+1}\end{aligned}$$

$$\sum_{n=0}^{\infty} \left\{ \int_{-1}^{+1} [P_n(x)]^2 dx \right\} t^{2n} = \sum_{n=0}^{\infty} \left(\frac{2}{2n+1} \right) t^{2n}$$

Equating the coeffs of t^{2n} ,

$$\int_{-1}^{+1} [P_n(x)]^2 dx = \frac{2}{2n+1}$$

$$\int_{-1}^{+1} P_m(x) P_n(x) dx = \frac{2}{2n+1} \delta_{nm}$$

called,

the Kronecker's delta,

$$\delta_{nm} = \begin{cases} 0 & \text{when } n \neq m \\ 1 & \text{when } n = m \end{cases}$$

→ Legendre polynomials of different orders

are orthogonal.

$$\int_a^b P_m(x) P_n(x) dx = \frac{2}{b-a} \left[P_m(x) \right]_a^b$$

□ Completeness of Legendre polynomials

We define a set of orthogonal functions is complete if there is no other function orthogonal to all of them.

We'll expand a function in a series of Legendre polynomials, which form a complete orthogonal set on $(-1, 1)$. This completeness means that any well-behaved function $f(x)$ can be approximated to any desired accuracy by a series of $P_k(x)$ through the relation:

$$f(x) = \sum_{k=0}^{\infty} A_k P_k(x) \quad , -1 \leq x \leq 1$$

$$\int_{-1}^{+1} f(x)P_m(x)dx = A_m \int_{-1}^{+1} P_m(x)P_m(x)dx = A_m$$

$$\int_{-1}^{+1} f(x)P_0(x)dx = A_0 \int_{-1}^{+1} P_0(x)P_0(x)dx = A_0$$

$$\int_{-1}^{+1} P_m(x) f(x) dx = \sum_{k=0}^{\infty} A_k \int_{-1}^{+1} P_m(x) P_k(x) dx$$

$\xrightarrow{\text{using orthogonality}}$

$$A_m = \frac{2}{2m+1} \int_{-1}^{+1} P_m(x) f(x) dx$$

~~(or)~~ $A_m = \frac{2}{2m+1} \sum_{k=0}^{\infty} A_k S_{km}$

$$(or) A_m = \frac{2}{2m+1} \sum_{k=0}^{\infty} A_k S_{km}$$

$$A_m = \frac{2}{2m+1} \int_{-1}^{+1} P_m(x) f(x) dx$$

(or)

$$A_k = \frac{2}{2k+1} \int_{-1}^{+1} P_k(x) f(x) dx$$

Ex:

Expand the function, $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & -1 < x \leq 0 \end{cases}$

in a series of the form $\sum_{k=0}^{\infty} A_k P_k(x)$

Ans:

$$A_k = \frac{2^{k+1}}{\pi} \int_{-1}^{+1} P_k(x) f(x) dx$$

$$= \frac{2^{k+1}}{\pi} \left[\int_{-1}^0 P_k(x) f(x) dx + \int_0^1 P_k(x) f(x) dx \right]$$

$$= \frac{2^{k+1}}{\pi} \int_0^1 P_k(x) dx$$

$$P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{1}{2}(3x^2 - 1), P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$A_0 = \frac{1}{2} \int_0^1 P_0(x) dx = \frac{1}{2} \int_0^1 1 dx = \frac{1}{2}$$

$$A_1 = \frac{3}{2} \int_0^1 P_1(x) dx = \frac{3}{2} \int_0^1 x dx = \frac{3}{4}$$

$$A_2 = \frac{5}{2} \int_0^1 P_2(x) dx = \frac{5}{2} \int_0^1 \frac{1}{2}(3x^2 - 1) dx = \frac{5}{4} \left[x^3 - x \right]_0^1 = 0$$

$$A_3 = \frac{7}{4} \int_{-1}^1 P_3(\alpha) d\alpha = \frac{7}{4} \int_{-1}^1 (5\alpha^3 - 3\alpha) d\alpha$$

$$= \frac{7}{4} \left[\frac{5\alpha^4}{4} - \frac{3\alpha^2}{2} \right]_{-1}^1 = \frac{7}{4} \left[\frac{5}{4} - \frac{3}{2} \right]$$

$$= \frac{7}{4} \cdot \frac{-1}{4} = -\frac{7}{16}$$

$$\left. \text{res}(s) \right| + \left. \text{res}(s) \right| \frac{1+2s}{s} = 1A$$

$$P(s) = \frac{1}{2} P_0(s) + \frac{3}{4} P_1(s) - \frac{7}{16} P_3(s) + \dots$$

$$\left. \text{res}(s) \right| \frac{1+2s}{s}$$

$$(s-1)^{\frac{1}{2}} \cdot (s-2)^{\frac{1}{2}} \cdot (1-s)^{\frac{1}{2}} = (s-1)^{\frac{1}{2}} \cdot (s-2)^{\frac{1}{2}} \cdot (s-1)^{\frac{1}{2}}$$

$$\frac{1}{s} = \text{res} \left(\frac{1}{s} \right) - \text{res} \left(s-1 \right) \left(s-2 \right) \left(s-1 \right) \frac{1}{s} = 0A$$

$$\frac{s}{s} = \text{res} \left(\frac{s}{s} \right) \left(\frac{s}{s} = \text{res} \left(s-1 \right) \left(s-2 \right) \left(s-1 \right) \frac{s}{s} = 0A \right)$$

$$0 = \left[(s-1)^{\frac{1}{2}} \cdot (s-2)^{\frac{1}{2}} \cdot (s-1)^{\frac{1}{2}} \right] \left(\frac{s}{s} = \text{res} \left(s-1 \right) \left(s-2 \right) \left(s-1 \right) \frac{s}{s} = 0A \right)$$

□ Rodrigues' formula

Consider the function, $v = \frac{1+b}{x^2-1} e^{bx}$

$$v = \frac{1+b}{x^2-1} (x^2-1)^n$$

$$\frac{dv}{dx} = n(x^2-1)^{n-1} \cdot 2x = 2nx(x^2-1)^{n-1}$$

$$(x^2-1) \frac{dv}{dx} = 2nx(x^2-1)^n = 2nv$$

i.e; v satisfies the differential equation

$$(1-x^2) \frac{dv}{dx} + 2nxv = 0$$

Differentiating again w.r.t x ,

$$(1-x^2) \frac{d^2v}{dx^2} - 2x \frac{dv}{dx} + 2n \frac{dv}{dx} + 2nv = 0$$

$$(1-x^2) \frac{d^2v}{dx^2} + 2(n-1) \frac{dv}{dx} + 2nv = 0$$

Differentiating again w.r.t α

$$(1-\alpha^2) \frac{d^{2+1}}{d\alpha^{2+1}} V - 2\alpha \frac{d^{1+1}}{d\alpha^{1+1}} V + 2(n-1) \frac{dV}{d\alpha} + 2(n-2)\alpha \frac{dV}{d\alpha^{1+1}}$$
$$+ 2n\alpha \frac{dV}{d\alpha} = 0$$

Compar

$$(1-\alpha^2) \frac{d^{2+1}}{d\alpha^{2+1}} V + 2\alpha(n-1) \frac{d^{1+1}}{d\alpha^{1+1}} V + (1+1)(2n-1) \frac{dV}{d\alpha} = 0$$
$$\Rightarrow (1-\alpha^2) \frac{d^{2+1}}{d\alpha^{2+1}} V + \frac{2n\alpha(1-\alpha)}{\alpha b} = 0$$

neglecting terms off α we get V

$$(1-\alpha^2) \frac{d^{2+r}}{d\alpha^{2+r}} V + 2\alpha(n-r-1) \frac{d^{1+r}}{d\alpha^{1+r}} V + (r+1)(an-r) \frac{d^r V}{d\alpha^r} = 0$$

is now being eliminated

When $r=n$, $\frac{vb}{ab} n \text{ rms} + \frac{vb}{ab} \text{ rms} - \frac{vb}{ab}(n-1)$

$$(1-\alpha^2) \frac{d^{n+2}}{d\alpha^{n+2}} V - 2\alpha \frac{d^{n+1}}{d\alpha^{n+1}} V + n(n+1) \frac{d^n V}{d\alpha^n} = 0$$
$$\Rightarrow (1-\alpha^2) \frac{d^{n+2}}{d\alpha^{n+2}} V + \frac{vb}{ab}(n-1) V + \frac{vb}{ab}(n-1)$$

$$(1-\alpha^2) \frac{d^2}{d\alpha^2} \left(\frac{d^n v}{d\alpha^n} \right) - 2\alpha \frac{d}{d\alpha} \left(\frac{d^n v}{d\alpha^n} \right) + n(n+1) \frac{d^n v}{d\alpha^n} = 0$$

Comparing it with Legendre's differential equation:

$$(1-\alpha^2) \frac{d^2 y}{d\alpha^2} - 2\alpha \frac{d y}{d\alpha} + n(n+1)y = 0$$

$\rightarrow \frac{d^n v}{d\alpha^n}$ satisfies the Legendre's equation

$$P_n(\alpha) = C \frac{d^n v}{d\alpha^n} \quad , \quad C: \text{constant}$$

$$\boxed{P_n(\alpha) = C \frac{d^n v}{d\alpha^n} (\alpha^2 - 1)^n} \quad \longleftarrow$$

To determine this constant C , we have to consider terms with the highest power of x on both sides.

The term with the highest power of x in the expression for $P_n(x)$ is $\frac{(2n)!}{2^n(n!)^2} x^n$.

$$\begin{aligned}\frac{(2n)!}{2^n(n!)^2} x^n &= C \cdot \frac{d^n}{dx^n} x^n \\ &= C \cdot a_n(a_{n-1})(a_{n-2}) \dots [a_{n-(n-1)}] x^n \\ &= C \cdot \frac{(2n)!}{n!} x^n\end{aligned}$$

$$\Rightarrow C = \boxed{\frac{1}{2^n n!}} = \boxed{\frac{1}{2^n n!}}$$

Hence, (a) go with this method

$$P_n(x) = \frac{1}{(2^n)n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

→ Rodrigues' formula for $P_n(x)$

$$(1 - x^2 + x^2 - 1) \frac{\frac{d}{dx} \frac{1}{x^2}}{x^2} =$$

$$(x^2 + x^2 - 1) \frac{\frac{d}{dx} \frac{1}{x^2}}{x^2} =$$

$$(x^2 + x^2 - 1) \frac{\frac{d}{dx} \frac{1}{x^2}}{x^2} =$$

$$(x^2 - 1) \frac{1}{x^2} =$$

$$(x^2 - 1) \frac{1}{x^2} = [x^2 - 1] \cdot \frac{1}{x^2} =$$

Ex:- Obtain the value of $P_3(x)$ using
Rodrigues' formula

Ans:

$$P_n(x) = \frac{1}{(2^n) n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_3(x) = \frac{1}{(2^3) 3!} \frac{d^3}{dx^3} (x^2 - 1)^3$$

$$= \frac{1}{48} \frac{d^3}{dx^3} (x^6 - 3x^4 + 3x^2 - 1)$$

$$= \frac{1}{48} \frac{d^2}{dx^2} (6x^5 - 12x^3 + 6x)$$

$$= \frac{1}{48} \frac{d}{dx} (30x^4 - 36x^2 + 6)$$

$$= \frac{1}{48} (120x^3 - 72x)$$

$$= \frac{1}{48} \times 24 [5x^3 - 3x] = \underline{\underline{\frac{1}{2}(5x^3 - 3x)}}$$

□ Hermite Polynomials

Hermite's differential equation,

$$\frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0$$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$y = \sum_{j=0}^{\infty} a_j x^j$$

$$y' = \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$y'' = \sum_{j=2}^{\infty} + j(j-1) a_j x^{j-2}$$

$$0 = 2a_1 + 2a_3 + 2a_5 + \dots$$

$$0 = 2a_1 + 2a_3 + 2a_5 + \dots$$

$$\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} - 2x \sum_{j=1}^{\infty} j a_j x^{j-1}$$

$$+ 2x \sum_{j=0}^{\infty} a_j x^j = 0$$

$$\boxed{0 = (\text{some terms}) + \sum_{j=1}^{\infty} 2j a_j x^j - \sum_{j=1}^{\infty} 2j a_j x^j}$$

$$+ \sum_{j=0}^{\infty} 2a_2 a_j x^j = 0$$

$$\boxed{\sum_{j=2}^{\infty} j(j-1) a_j x^{j-2} + \sum_{j=1}^{\infty} 2(2-j) a_j x^j + 2a_2 a_0 x^0 = 0}$$

x^0 terms: $2a_2 + 2a_2 a_0 = 0$

$$\boxed{a_2 = \frac{-2a_2}{2 \times 1} a_0}$$

x^1 terms: ~~$3 \times 2 a_3 + 2(2-1) a_1 = 0$~~

$$3 \times 2 a_3 + 2(2-1) a_1 = 0$$

$$\boxed{a_3 = \frac{-2(2-1)}{3 \times 2} a_1}$$

(without α^2 terms : $4 \times 3 a_4 + 2(2-2)a_2 = 0$)

$$a_4 = \frac{-2(2-2)}{4 \times 3} a_2 = \frac{(-2) 2(2-2)}{4!} a_0$$

$$\begin{matrix} i & j \\ 2 & 0 \end{matrix} = (-2)^2$$

α^3 terms : $5 \times 4 a_5 + 2(2-3)a_3 = 0$

$$a_5 = \frac{-2(2-3)}{5 \times 4} a_3 = \frac{(-2) 2(2-2)(2-3)}{5!} a_0$$

α^4 terms : $6 \times 5 a_6 + 2(2-4)a_4 = 0$

$$a_6 = \frac{-2(2-4)}{6 \times 5} a_4 = \frac{(-2) 2(2-2)(2-3)(2-4)}{6!} a_0$$

$$a_{j+2} = \frac{-2(2-j)}{(j+1)(j+2)} a_j$$

$i=0$

The general solution of the homogeneous differential equation is:

$$\frac{(s-5)(s-6)}{12} = \frac{(s-5)s}{12} = s(s-5)$$

$$y(x) = \sum_{j=0}^{\infty} a_j x^j$$

$$= e^{(s-5)x} + c_1 x e^{(s-5)x}$$

$$= a_0 y_1(x) + a_1 y_2(x)$$

$$\frac{(s-5)(s-6)}{12} = \frac{(s-5)s}{12} = s(s-5)$$

where,

$$y_1(x) = 1 + \frac{(-2)^2}{2!} x^2 + \frac{(-2)^3 (2-2)}{4!} x^4 + \frac{(-2)^3 (2-2)(2-4)}{6!} x^6$$

$$y_2(x) = x + \frac{(-2)(2+1)}{3!} x^3 + \frac{(-2)^2 (2-1)(2-3)}{5!} x^5 + \dots$$

If ω is a non-zero negative integer
 $y(\omega)$ series will be an infinite series

The Wronskian of y_1 and y_2 at 0 is :

$$W(y_1, y_2)(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y'_1(0) & y'_2(0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

$\Rightarrow y_1$ and y_2 are independent

situation arise when $\alpha = 0$

When $\alpha = 0$, $l = 0, 1, 2, \dots$

series will

then the series correspond to $y_l(\alpha)$
will terminate at the term:

so amount is $\frac{(-\alpha)^l (2l)(2l-2) \dots (2l-(2l-2))}{(2l)!} \alpha^{2l}$

(α)H terms are of the form of α^{2l}

$$\begin{aligned} &= \frac{(-\alpha)^l (2l)(2l-2) \dots 2}{(2l)!} \times \alpha^{2l} \\ &= \frac{(-1)^l l!}{(2l)!} \alpha^{2l} \end{aligned}$$

all subsequent terms will have their
numerators equal to zero.

The series correspond to $y_l(\alpha)$ will however
remain an infinite series.

For the series to appear more systematic,
we choose

$$a_0 = \frac{(-1)^l (\alpha l)!}{l!}$$

$\alpha = 0, 1, 2, \dots$

αl must be even

$\therefore \alpha$ must be even

The polynomial thus obtained is known as
the Hermite polynomial of degree αl , $H_{\alpha l}(x)$.

$$H_{\alpha l}(x) = a_0 \left[1 + \frac{(-\alpha)(\alpha l)}{\alpha l!} x^2 + \frac{(-\alpha)(\alpha l)(\alpha l - 2)}{4!} x^4 + \frac{(-\alpha)^3 (\alpha l)(\alpha l - 2)(\alpha l - 4)}{6!} x^6 \right.$$

$$+ \cdots \cdots + \frac{(-\alpha)^{l-1} (\alpha l)(\alpha l - 2) \cdots 4}{(2l - 2)!} x^{2l-2} =$$

$$\left. + \frac{(-\alpha)^l (\alpha l)(\alpha l - 2) \cdots 2}{(\alpha l)!} x^{2l} \right]$$

αl must be even

revised this. (03/08/2023)
• uses of large numbers
• uses of small numbers
• uses of large numbers
• uses of small numbers

$$= \frac{(-1)^l (2l)!}{l!} \left[1 + \frac{(-1)(2l)}{2!} (2x)^2 + \frac{(-1)^3 l(l-1)}{4!} (2x)^4 + \right.$$

$$\left. + \frac{(-1)^6 l(l-1)(l-2)}{6!} (2x)^6 + \dots \right]$$

$$H_{2l}(x) = \frac{(-1)^l (2l)!}{(2l)!} + \frac{(-1)^{l-1} (2l)!}{2! (l-1)!} + \dots$$

$$+ \frac{(-1)(2l)!}{(2l-2)!} (2x)^{2l-2} + (2x)^{2l}$$

$! (1+16)$

$$\frac{(2l)!}{! (1+16)} l (1-)$$

$$x^{2l}$$

$$\frac{((x_0)(G-1))!}{l!} (-1)^{l+1} + \frac{(x_0)(G-1)!}{(l+1)!} + \frac{1}{(l+1)!} =$$

When $2l$ is a ~~plus~~ odd integer.

$$\frac{((x_0)!)^l}{l!} (-1)^{l+1}; l=0, 1, 2, \dots$$

then the series corresp. to $y_2(x)$
will terminate at the term

$$\frac{(-2)^l (2l)(2l-1) \cdots (2l-(2l-1))}{(2l+1)!} x^{2l+1} = \frac{(-2)^l}{(2l+1)!}$$

$$\frac{(-2)^l (2l+1)!}{(2l+1)!}$$

$$\frac{(-1)^l \cdot 2^l \cdot 2^l \cdot l! \cdot x^{2l+1}}{(2l+1)!}$$

$$\frac{(-1)^l l! (2x)^{2l+1}}{(2l+1)! 2}$$

This
deg.

H_{2l+1}

dimin

last

for

$= 2$

$$+ \frac{(-1)^{\ell} (1+\alpha)^{\ell-1} (1)}{(1-\alpha)^{\ell+1}} + \dots = (x) H_{\alpha, \ell}$$

Choosing, $a_1 = 2 \frac{(-1)^{\ell} (\alpha\ell+1)!}{(\alpha\ell+1)!}$

$$+ \frac{(-1)^{\ell} (\alpha\ell+1)!}{(\alpha\ell+1)!} + \dots$$

This leads to Hermite's polynomial of degree $(\alpha\ell+1)$:

$$H_{\alpha, \ell}(x) = a_1 \left[x + \frac{(-\alpha)(\alpha\ell)}{3!} x^3 + \frac{(-\alpha)^2 (\alpha\ell)(\alpha\ell-2)}{5!} x^5 + \right.$$

$$\left. \frac{(-\alpha)^3 (\alpha\ell)(\alpha\ell-2)(\alpha\ell-4)}{7!} x^7 + \dots \right]$$

$$+ \frac{(-\alpha)^{\ell-1} (\alpha\ell)(\alpha\ell-2) \dots 4}{(\alpha\ell-1)!} x^{\alpha\ell-1} + \frac{(-\alpha)^{\ell} (\alpha\ell)(\alpha\ell-2) \dots 2}{(\alpha\ell+1)!} x^{\alpha\ell+1}$$

$$= \frac{(-1)^{\ell} (\alpha\ell+1)!}{\ell!} \left[x + \frac{(-\alpha)(\alpha\ell)}{3!} x^3 + \frac{(-\alpha)^2 (\alpha\ell)(\alpha\ell-2)}{5!} x^5 \right.$$

$$\left. + \frac{(-\alpha)^3 (\alpha\ell)(\alpha\ell-2)(\alpha\ell-4)}{7!} x^7 + \dots \right]$$

$$+ \frac{(-\alpha)^{\ell-1} (\alpha\ell)(\alpha\ell-2) \dots 4}{(\alpha\ell-1)!} x^{\alpha\ell-1} + \frac{(-\alpha)^{\ell} (\alpha\ell)(\alpha\ell-2) \dots 2}{(\alpha\ell+1)!} x^{\alpha\ell+1}$$

$$H_{2l+1}(x) = \frac{(-1)^l (2l+1)!}{l!} x + \frac{(-1)^{l-1} (2l+1)!}{3! (l-1)!} (2x)^3 +$$

$$\frac{!((1+16)^{(1)}(1))}{6=0, \text{ pricash}} + \frac{(-1)(2l+1)!}{(2l-1)!} (2x)^{2l-1} + (2x)^{2l+1}$$

\circ binomial coefficients of ideal fit
 $(1+16)$ sample

- * The constants a_0 and a_1 are taken in this particular manner, because Hermite polynomials of degree n is defined in such a way that the terms containing the highest power of x is $(2x)^n$.

$$\left[\frac{(6-16)(16)^5(6-)}{12} + \frac{(16)(6-)}{15} + \dots \right] \frac{!((1+16)^{(1)})}{13} =$$

$$\left[\frac{(6-16)(6-16)(16)^5(6-)}{14} + \dots \right] \frac{!((1+16)^{(1)})}{14} +$$

Taking $x=0$ where x is any integer including zero, and define the Hermite polynomial of degree n as:

$$I = (x)_n H$$

n is even

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!} (2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4}$$

$$+ \dots + \frac{(-1)^{\frac{n}{2}} n!}{(\frac{n}{2})!} (2x)^{\frac{n}{2}} = (x)_n H$$

n is odd

$$H_n(x) = (2x)^n - \frac{n(n-1)}{1!} (2x)^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2!} (2x)^{n-4}$$

$$+ \dots + \frac{(-1)^{\frac{n-1}{2}} n!}{(\frac{n-1}{2})!} (2x)^{\frac{n-1}{2}}$$

Topic: First few Hermite polynomials are:

Hermite

$$H_0(x) = 1$$

$$H_1(x) = 2x$$

$$H_2(x) = 4x^2 - 2$$

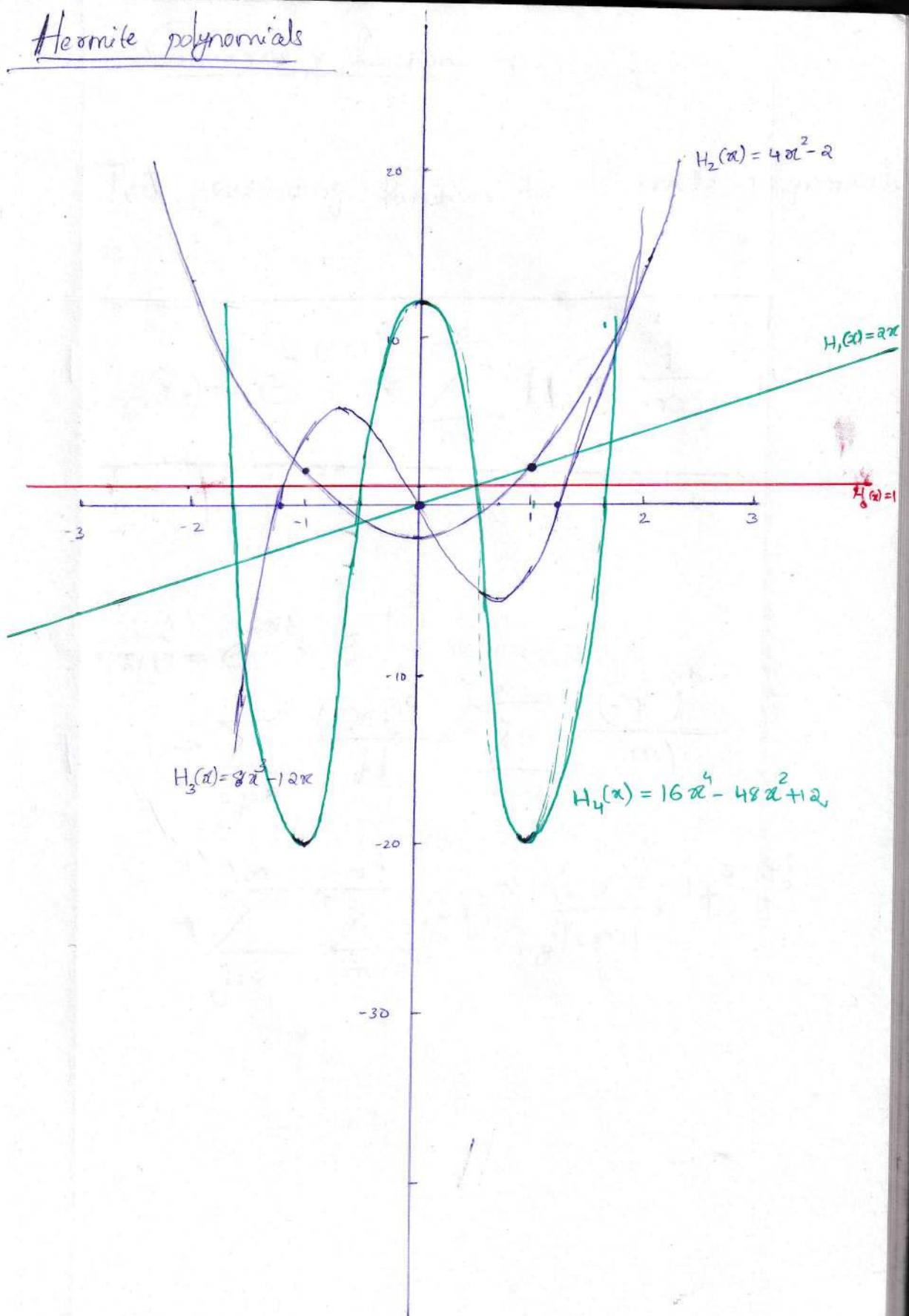
$$H_3(x) = 8x^3 - 12x$$

$$H_4(x) = 16x^4 - 48x^2 + 12$$

$$H_5(x) = \frac{32x^5 - 160x^3 + 120x}{1} - (x) = (x) H$$

$$(x) - \frac{(n+1)}{1(n+1)} + \dots +$$

Hermite polynomials



□ Generating function, $n = m+j$

The generating function for Hermite polynomials
is:

$$g(x, t) = e^{2xt - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

$$\begin{aligned} g(x, t) &= e^{2xt} \times e^{-t^2} \\ &= \sum_{j=0}^{\infty} \frac{(2xt)^j}{j!} \times \sum_{m=0}^{\infty} \frac{(-t^2)^m}{m!} \\ &= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} (-1)^m \frac{(2x)^j}{j! m!} t^{2m+j} \end{aligned}$$

Put $j+2m = n$, without further waste

Obviously this equality can be satisfied for various combinations of the values of j and m .

$$\begin{array}{l} j=0, m=0; \\ \text{if } j=\frac{n}{2}, m=0; \\ j=n-4, m=2; \end{array} \quad \left[\begin{array}{l} \text{so } j=100 \\ 0=n \end{array} \right] = S = (1, \infty) \mathbb{C}$$

and so on.

If n is even,

$$\begin{array}{l} j=0, m=\frac{n}{2}; \\ \text{if } j=\frac{n}{2}, m=0; \\ 0=j \end{array} \quad \left[\begin{array}{l} \text{so } j=100 \\ 0=n \end{array} \right] = S = (1, \infty) \mathbb{C}$$

If n is odd,

$$\begin{array}{l} j=1, m=\frac{n-1}{2}; \\ \text{if } j=\frac{n-1}{2}, m=0; \\ 0=j \end{array} \quad \left[\begin{array}{l} \text{so } j=100 \\ 0=n \end{array} \right] =$$

The coeff. of t^n is:

$$\frac{(2x)^n}{n! \cdot 0!} - \frac{(2x)^{n-2}}{(n-2)! \cdot 1!} + \frac{(2x)^{n-4}}{(n-4)! \cdot 2!} - \dots$$
$$\left. \begin{array}{l} \left\{ \begin{array}{l} \frac{(-1)^{\frac{n}{2}}}{0! \left(\frac{n}{2}\right)!} \text{ for even } n \\ \frac{(-1)^{\frac{n-1}{2}} (2x)}{1! \left(\frac{n-1}{2}\right)!} \text{ for odd } n \end{array} \right. \end{array} \right.$$

Coeff. of $\frac{t^n}{n!}$, i.e., $H_n(x)$ will be obtained by multiplying the above expression by $n!$.

Ex: 7

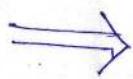
$H_n(x)$ and $H_n(-x)$

$$\begin{array}{|c|c|} \hline & \frac{x+t-t^2}{2!} \\ \hline & \times \\ \hline & \times \\ \hline & \times \\ \hline & \times \\ \hline \end{array}$$

We'll change x to $-x$, and t to $-t$.

$$e^{2xt-t^2} = \sum_n H_n(-x) \frac{(-t)^n}{n!}$$
$$= \sum_n (-1)^n H_n(-x) \frac{t^n}{n!}$$

$$e^{2xt-t^2} = \sum_n H_n(x) \frac{t^n}{n!}$$



$$H_n(x) = (-1)^n H_n(-x)$$

$$H_n(-x) = (-1)^n H_n(x)$$

□ Recurrence Relation

$$e^{2\alpha t - t^2} = \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!}$$

$$e^{2\alpha t - t^2} = \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!}$$

$(2\alpha)_n H_{n+1}(\alpha) - (\alpha)_n H_n(\alpha) = (2\alpha)_{n+1} H_n(\alpha)$

Differentiating w.r.t t ,

$$(2\alpha - \alpha t) e^{2\alpha t - t^2} = \sum_{n=1}^{\infty} H_n(\alpha) \frac{t^n}{(n-1)!}$$

$$(2\alpha - \alpha t) \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!} = \sum_{n=1}^{\infty} H_n(\alpha) \frac{t^{n+1}}{(n-1)!}$$

Equating coeff. of t^{n+1} from the 2 sides,

$$2\alpha \frac{H_{n+1}(\alpha)}{(n+1)!} - \alpha \frac{H_n(\alpha)}{n!} = \frac{H_{n+2}(\alpha)}{(n+1)!}$$

$$H_{n+2}(\alpha) = 2\alpha H_{n+1}(\alpha) - \alpha(n+1) H_n(\alpha)$$

→ (1)

Ex:-

$$H_0(\alpha) = 1 \quad \& \quad H_1(\alpha) = 2\alpha$$

Ans:

$$H_{n+2}(\alpha) = 2\alpha H_{n+1}(\alpha) - 2(n+1) H_n(\alpha)$$

$n=0$,

$$H_2(\alpha) = 2\alpha H_1(\alpha) - 2 H_0(\alpha)$$

$$= 2\alpha(2\alpha) - 2 = 4\alpha^2 - 2$$

$$H_3(\alpha) = 2\alpha H_2(\alpha) - 4 H_1(\alpha) = 2\alpha(4\alpha^2 - 2) - 4(2\alpha)$$

$$= 8\alpha^3 - 12\alpha$$

$$H_4(\alpha) = 2\alpha H_3(\alpha) - 6 H_2(\alpha) = 16\alpha^4 - 48\alpha^2 + 12$$

$$\frac{(x)_n H}{(1+\alpha)^n} = \frac{(x)_n H}{(1+\alpha)^n} s = \frac{(x)_n H}{(1+\alpha)^{n+s}}$$

(*)

$$(x)_n H (1+\alpha)^s - (x)_n H s \alpha^n = (x)_n H$$

$$e^{\alpha t - t^2} = \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!}$$

Partially differentiating w.r.t α ,

$$\alpha t e^{\alpha t - t^2} = \sum_{n=0}^{\infty} H'_n(\alpha) \frac{t^n}{n!}$$

$$\alpha t \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!} = \sum_{n=0}^{\infty} H'_n(\alpha) \frac{t^n}{n!}$$

$$\alpha \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty} H'_n(\alpha) \frac{t^n}{n!}$$

~~WNAH~~ Equating the coeff. of t^{n+1}

$$\frac{\alpha H_n(\alpha)}{(n+1)!} = \frac{H'_{n+1}(\alpha)}{(n+1)!}$$

$$\alpha H_n(\alpha) = \frac{H'_{n+1}(\alpha)}{(n+1)!} \implies H'_{n+1}(\alpha) = \alpha (n+1) H_n(\alpha)$$

$$H'_{n+1}(\alpha) = \alpha (n+1) H_n(\alpha)$$

Combining eq^{ns} $\sum_{n=0}^{\infty} H_{n+1}(x) + \sum_{n=0}^{\infty} H_n(x) = 0$

$$H_{n+2}(x) = 2\alpha H_{n+1}(x) - 2(n+1) H_n(x)$$

$$H'_{n+1}(x) = 2(n+1) H_n(x)$$

$$H_{n+2}(x) = 2\alpha H_{n+1}(x) - H'_{n+1}(x)$$

$$H_{n+1}(x) = 2\alpha H_n(x) - H'_n(x)$$

Differentiating w.r.t α ,

$$H'_{n+1}(x) = 2H_n(x) + 2\alpha H'_n(x) - H''_n(x) = 2(n+1) H_n(x)$$

$$H''_n(x) - 2\alpha H'_n(x) + 2n H_n(x) = 0$$

$$(H''_n(x) - 2\alpha H'_n(x)) B = (H'_n(x) - 2n H_n(x)) B$$

It signifies
~~compute with~~ Hermite's differential equation (for the
integral or 0 value of η) for $H_n(x)$.

It means that if the polynomials $H_n(x)$
satisfy the recurrence relations ① and ②
they must satisfy Hermite's differential equation.

$$H'' + \frac{4x}{n} H' + \dots + (n)_2 H \frac{-12}{16} + (n)_1 H = \left(\frac{x^2 - 16}{4}\right) \frac{6}{16}$$

$$H'' + \frac{4x}{n} H' + \dots + (n)_2 H \frac{-12}{16} + (n)_1 H = \left(\frac{x^2 - 16}{4}\right) \frac{6}{16}$$

$$+ (n)_1 H + \frac{(1-x)H(1-x)}{1(1+x)} + (n)_0 H = \left(\frac{x^2 - 16}{4}\right) \frac{6}{16}$$

Rodrigues' formula

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}$$

On line

$$= 1 + tH_1(x) + \frac{t^2}{2!} H_2(x) + \dots + \frac{t^n}{n!} H_n(x) + \dots$$

Partial differentiation w.r.t 't' yields,

$$\frac{\partial}{\partial t} \left(e^{2xt-t^2} \right) = H_1(x) + \frac{\partial t}{2!} H_2(x) + \dots + \frac{\partial t^{n-1}}{n!} H_n(x) + \dots$$

$$\frac{\partial^2}{\partial t^2} \left(e^{2xt-t^2} \right) = H_2(x) + \frac{3 \times 2}{3!} H_3(x) + \dots + \frac{n(n-1)t^{n-2}}{n!} H_n(x) + \dots$$

$$\frac{\partial^n}{\partial t^n} \left(e^{2xt-t^2} \right) = H_n(x) + \frac{(n+1)n(n-1)\dots 2}{(n+1)!} t H_{n+1}(x) + \dots$$

$$A + t = 0,$$

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n} \left(e^{2xt-t^2} \right) \right]_{t=0}$$

~~$$e^{2xt-t^2} = e^{2xt} e^{-t^2}, e^{-t^2} = e^{-x^2} e^{-(x-t)^2}$$~~

$$H_n(x) = e^{x^2} \left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2} \right]_{t=0}$$

$e^{-(x-t)^2}$ is a function of $(x-t)$, and for such a function the partial derivative w.r.t t can be obtained from the partial derivative w.r.t x by just changing the sign.

So, for the n th order partial derivative the sign will change 'n' times.

$$\therefore H_n(x) = e^{x^2} (-1)^n \left[\frac{\partial^n}{\partial x^n} e^{-(x-t)^2} \right]_{t=0}$$

$$= (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

\Rightarrow Rodrigues' formula for the Hermite polynomials.

□ Orthogonality relations

no terms in α & t in $H_n(\alpha)$ give zero
orthogonal basis

$$e^{2\alpha t - t^2} = \sum_{n=0}^{\infty} H_n(\alpha) \frac{t^n}{n!}$$

change α to u & express the expansion as
a power series in u :

$$e^{2\alpha u - u^2} = \sum_{m=0}^{\infty} H_m(\alpha) \frac{u^m}{m!}$$

Using these equations and the resultant of
by e^{-u^2} gives:

$$e^{-x^2 + 2\alpha t - t^2 + 2\alpha u - u^2} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-x^2} H_n(\alpha) H_m(\alpha) \frac{t^n u^m}{n! m!}$$

$$e^{\alpha tu} e^{-\frac{1}{2}(x^2 + t^2 + u^2 - 2\alpha t - 2\alpha u - 2tu)} = e^{\alpha tu} e^{-\frac{1}{2}(\alpha - t - u)^2}$$

$$(e^{\alpha tu} e^{-\frac{1}{2}(\alpha - t - u)^2})^2 = (1_{\text{Hilb}})$$

Integrating w.r.t. x from $-\infty$ to $+\infty$ and interchanging the order of the summation and integration.

Let

$$e^{\text{at}u} \int_{-\infty}^{+\infty} e^{-\frac{(x-t-u)^2}{n}} dx = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{n}} H_n(x) H_m(x) \frac{t^m u^m}{n! m!}$$

$$\text{Put } z = x - t - u \Rightarrow dz = dx$$

$$\int_{-\infty}^{+\infty} e^{-\frac{z^2}{n}} dz = \Gamma\left(\frac{n}{2}\right) = \sqrt{\pi} =$$

check
on 22
(17)

Now we have

$$\text{Set } z^2 = k \Rightarrow 2z dz = dk \text{ with } z > 0$$

$$\frac{1}{(m!)^2} \int_{-\infty}^{+\infty} H_n(x) e^{-\frac{x^2}{n}} dx = \int_0^{\infty} e^{-\frac{z^2}{n}} dz = \int_0^{\infty} e^{-k} \frac{dk}{2\sqrt{k}}$$

$$= \int_0^{\infty} (e^{-\frac{z^2}{n}} - e^{-k}) dk = \Gamma\left(\frac{n}{2}\right)$$

$$\Gamma(n+1) = \int_0^{\infty} x^n e^{-x} dx = n!$$

$$\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx = \frac{\frac{m+1}{2} \times \frac{n+1}{2}}{2 \sqrt{\frac{m+n+2}{2}}}$$

Let $m=n=0$,

$$\int_0^{\pi/2} dx = \frac{\pi}{2} = \frac{\Gamma_2 \times \Gamma_2}{2 \sqrt{1}} = \frac{(\Gamma_2)^2}{2 \times 0!} = \frac{(\Gamma_2)^2}{2}$$

$$\Gamma_2 = \sqrt{\pi} = \pi b(0) H(0) H(0)$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-z^2} dz = \Gamma_2 = \sqrt{\pi}$$

$$m n \delta^m \pi \cdot 10^n \delta = \pi b(0) H(0) H(0)$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx \frac{t^n u^m}{n! m!} = \sqrt{\pi} e^{at+bu}$$

$$at+bu = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{(at+bu)^n}{n!} = \sqrt{\pi} \left[1 + \frac{at+bu}{1!} + \frac{(at+bu)^2}{2!} + \dots \right]$$

\rightarrow written above this $(at+bu)$ bracket

For
unity
from

Equating the coeff. of $t^n u^m$

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 0 \quad \text{if } n \neq m$$

$$\text{So and } \int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = \frac{\pi}{\sqrt{n!}} \delta_{nm}$$

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \frac{\pi}{\sqrt{n!}} \delta_{nm}$$

Combining the results,

$$\int_{-\infty}^{+\infty} e^{-x^2} H_n(x) H_m(x) dx = 2^n n! \frac{\pi}{\sqrt{n!}} \delta_{nm}$$

$$S \pi = \frac{n!}{\sqrt{n!}} \delta_{nm} \text{ Kronecker delta}$$

The Hermite polynomials of different degrees are orthogonal to each other on the interval $(-\infty, +\infty)$ with weight function e^{-x^2} .

For Legendre polynomials the weight function is unity and the range of integration varies from -1 to +1.

□ Normalized Hermite functions

$$S \cdot \bar{H} = (16) \psi$$

Many formulas are simpler when expressed in terms of unnormalized Hermite polynomials. However, the unnormalized functions increase very rapidly with 'N', so we shall use only the normalized Hermite functions Ψ_n :

$$\Psi_n(x) = e^{-x^2/2} \cdot H_n(x) \cdot \frac{1}{\pi^{1/4} \cdot 2^n \cdot \sqrt{n!}}$$

The normalized Hermite functions $\Psi_n(x)$ have the orthogonality property that

$$\int_{-\infty}^{+\infty} \Psi_n(x) \Psi_m(x) dx = \delta_{mn}$$

$$\Psi_0(\alpha) = \frac{1}{\pi} e^{-\frac{\alpha^2}{2}}$$

besserges nicht solange $\alpha < 0$ erlaubt, wenn
dann $\Psi_1(\alpha) = \sqrt{\alpha} \cdot \frac{1}{\pi} \alpha e^{-\frac{\alpha^2}{2}}$

zweitens 2. Ordnung, bedenken, dass es
 $\Psi_2(\alpha) = \left(\sqrt{2} \alpha^2 - \frac{1}{\sqrt{2}} \right) \frac{1}{\pi} e^{-\frac{\alpha^2}{2}}$

$$\Psi_3(\alpha) = \left(\sqrt{\frac{4}{3}} \alpha^3 - \sqrt{3} \alpha \right) \frac{1}{\pi} e^{-\frac{\alpha^2}{2}}$$

$$\Psi_4(\alpha) = \left(\sqrt{\frac{5}{3}} \alpha^4 - \sqrt{6} \alpha^2 + \frac{\sqrt{6}}{4} \right) \frac{1}{\pi} e^{-\frac{\alpha^2}{2}}$$

$$\Psi_5(\alpha) = \left(\frac{2}{15} \sqrt{15} \alpha^5 - \frac{2}{3} \sqrt{15} \alpha^3 + \frac{\sqrt{15}}{2} \alpha \right) \frac{1}{\pi} e^{-\frac{\alpha^2}{2}}$$

(x, Ψ) erlaubt sonst bedenken, ob
nicht physikal. Voraussetzung, ob nicht

$$\boxed{\partial = i\hbar (\vec{p}) \Psi(x) \Psi}$$

Recursion relations

$$H_n'(x) = 2x H_n(x) - H_{n+1}(x)$$

$$\left[e^{\frac{x^2}{2}} \psi_n'(x) + x e^{\frac{x^2}{2}} \psi_n(x) \right] \cancel{2\sqrt{n!} \cdot \cancel{x^4}} = 2x e^{\frac{x^2}{2}} \psi_n(x) \cancel{2\sqrt{n!} \cdot \cancel{x^4}} \\ - e^{\frac{x^2}{2}} \psi_{n+1}'(x) \cdot \cancel{2\sqrt{(n+1)!} \cdot \cancel{x^4}}$$

$$\psi_n'(x) + x \psi_n(x) = 2x \psi_n(x) - \psi_{n+1}(x) \times \sqrt{2(n+1)}$$

$$\boxed{\psi_n'(x) = x \psi_n(x) - \sqrt{2(n+1)} \psi_{n+1}(x)}$$

$$H_{n+1}' = 2(n+1) H_n \implies H_n' = 2n H_{n-1}'(x)$$

$$\left[e^{\frac{x^2}{2}} \psi_n'(x) + x e^{\frac{x^2}{2}} \psi_n(x) \right] \cancel{2\sqrt{n!} \cdot \cancel{x^4}} = \\ 2n e^{\frac{x^2}{2}} \psi_{n-1}(x) \times \cancel{2\sqrt{2(n-1)!} \cdot \cancel{x^4}}$$

$$\psi_n'(x) + x \psi_n(x) = \sqrt{2n} \psi_{n-1}(x)$$

$$\boxed{\psi_n'(x) = -x \psi_n(x) + \sqrt{2n} \psi_{n-1}(x)}$$

$$\frac{d\Psi_n}{d\alpha} = \alpha\Psi_n(\alpha) - \sqrt{\alpha(n+1)}\Psi_{n+1}(\alpha)$$

$$\frac{d\Psi_n}{d\alpha} = -\alpha\Psi_n(\alpha) + \sqrt{\alpha n}\Psi_{n-1}(\alpha), \quad n \geq 0$$

$$(\alpha)\psi \text{ op } = \cancel{\alpha} \cancel{\psi} \left[(\alpha)\psi \cancel{\text{ op}} + (\alpha)\psi \cancel{\alpha} \right]$$

$$(\alpha)\psi - (\alpha)\psi \cancel{\alpha} = (\alpha)\psi \cancel{\text{ op}} + (\alpha)\psi \cancel{\alpha}$$

These must be initiated by the derivatives of Ψ_0 and Ψ_1 , which are $(\alpha)\Psi_0 = (\alpha)_0\Psi$

$$\begin{aligned} \frac{d\Psi_0}{d\alpha} &= \frac{-1}{\sqrt{\alpha}}\Psi_1 = -\alpha\Psi_0 \\ &= -\sqrt{\alpha}\Psi_0 = -\alpha\pi e^{-\alpha^2/2} \quad \left. \begin{array}{l} \Psi_1 = \sqrt{2}\alpha\Psi_0 \\ \Psi_0 = e^{-\alpha^2/2} \end{array} \right\} \Psi_1 = \sqrt{2}\alpha\Psi_0 \\ &= -\sqrt{\alpha}\pi \left[(\alpha)\psi \cancel{\text{ op}} + (\alpha)\psi \cancel{\alpha} \right] \end{aligned}$$

$$\begin{aligned} \frac{d\Psi_1}{d\alpha} &= -\alpha\Psi_1(\alpha) + \sqrt{\alpha}\Psi_0(\alpha) \\ &= -\sqrt{2}\alpha^2\Psi_0(\alpha) + \sqrt{2}\Psi_0(\alpha) \\ &= \sqrt{2}\left[1-\alpha^2\right]\Psi_0(\alpha) \\ &= \sqrt{2}\pi^{-\alpha^2/2} e^{-\alpha^2/2} \underline{(1-\alpha^2)} \end{aligned}$$

□ Raising & Lowering Operators

$$a = (x)_n H \quad a^* = (x)_n H^* \quad a^* a = (x)_n H^* H = (x)_n^2 H$$

The raising operator a , when applied to a Hermite function of degree n , gives a result which is proportional to the Hermite function of the next highest degree. Similarly, the lowering operator a^* reduces the degree of a Hermite function (by one).

$$R = \left(\frac{d}{dx} - x \right) \quad \text{- raising operator}$$

$$R \Psi_n = -\sqrt{2(n+1)} \Psi_{n+1} = (x)_n \Psi \left(x - \frac{b}{2\sigma b} \right)$$

$$L = \left(\frac{d}{dx} + x \right) \quad \text{- lowering operator}$$

$$L \Psi_n = \sqrt{n} \Psi_{n-1}$$

$$H_n''(\alpha) - 2\alpha H_n'(\alpha) + 2n H_n(\alpha) = 0$$

$$\text{thus } \cancel{\Psi_n''} + \alpha \cancel{\Psi_n'} + \cancel{\alpha^2 \Psi_n} + \cancel{-2\alpha \Psi_n} + \cancel{2n \Psi_n} = 0$$

$$-\cancel{2\alpha \Psi_n} - \cancel{2\alpha \Psi_n} + \cancel{2n \Psi_n} = 0$$

$$\Psi_n''(\alpha) - \alpha^2 \Psi_n(\alpha) = -(2n+1) \Psi_n(\alpha)$$

$$\text{rotating basis} \rightarrow \left(x - \frac{b}{\omega b} \right) = \xi$$

$$\frac{d^2 \Psi_n}{dx^2} - \alpha^2 \Psi_n(\alpha) = -(2n+1) \Psi_n(\alpha)$$

$$\left(\frac{d^2}{dx^2} - \alpha^2 \right) \Psi_n(\alpha) = n \Psi_n(\alpha) = -(2n+1) \Psi_n(\alpha)$$

$$\text{rotating basis} \rightarrow \text{The Hermite eigenoperator} \left(x + \frac{b}{\omega b} \right) = \xi$$

$$n = \frac{d^2}{dx^2} - \alpha^2 = \frac{1}{2} [R L + L R]$$

$$\alpha \Psi_n = \frac{1}{2} [L - R] \Psi_n$$

$$= \sqrt{\gamma_2} \Psi_{n-1} + \sqrt{(n+1)/2} \Psi_{n+1}$$

$$\frac{d}{dx} \Psi_n = \frac{1}{2} [L + R] \Psi_n$$

$$= \sqrt{\gamma_2} \Psi_{n-1} - \sqrt{(n+1)/2} \Psi_{n+1}$$

Ex:- 1D Harmonic Oscillator

The time independent Schrödinger eqⁿ-

$$\frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi(x) = E \psi(x)$$

~~left of box is 0 up to bottom w/~~
~~work w/ no sum of do sum up down~~
~~& nothing w/ down~~
Conservative force: A conservative force may be
defined as one for which the work done
in moving b/w 2 points A and B is
independent of the path taken b/w the
2 points.

The implication of "conservative" in this
context is that, you could move it from
A to B by one path and return to A
by another path with no net loss of
energy - any closed return path to A
takes net zero work.

The energy of an object which is subject
only to that conservative force is dependent
upon its position and not upon the path by
which it reached that position. This makes it
possible to define a potential energy function which
depends upon position only..

so potential is proportional to displacement \vec{r}

$$(iii) \Psi E = (i\hbar)\Psi^* \frac{\partial}{\partial r} \psi + \frac{\Psi^* p^2}{2m} + \frac{\Psi^* q^2}{2m} \vec{r}^2$$

The potential energy U is equal to the work you must do to move an object from the $U=0$ reference point to the position r .
 i.e. The reference point at which you assign the value $U=0$ is arbitrary, so may be chosen for convenience, like choosing the origin of coordinates in a coordinate system.

$$\text{A. of } U = - \int F(x) dx$$

Work is done by force F against $-dU$

$F(x) = -\frac{dU}{dx}$

Work is done by force F against dU

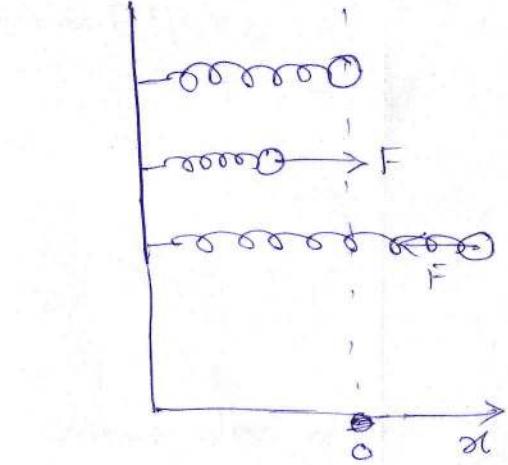
$$F = -kx$$

Hooke's law

$$U = - \int_0^x -kx dx$$

$$= \frac{1}{2} kx^2$$

$$= \frac{1}{2} m\omega^2 x^2$$



$$\omega = \sqrt{\frac{k}{m}} : \text{angular freq. of the oscillator}$$

The Hamiltonian of the particle is:

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2} k \hat{x}^2 = \frac{\hat{P}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

\hat{x} : position operator

$\hat{P} = -i\hbar \frac{\partial}{\partial x}$: momentum operator

Time-independent Schrödinger equation,

$$H|\psi\rangle = E|\psi\rangle$$

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi(x) = E \psi(x)$$

$\rightarrow \frac{\partial^2}{\partial t^2} \psi(x) + \left(\frac{\hbar^2}{m} \omega^2 x^2 + \frac{E}{\hbar^2} \right) \psi(x) = 0$

$$\psi = \Psi \left(\beta - i\frac{t}{\hbar \omega} \right) + \frac{\psi_0}{\beta b}$$

We'll rewrite eq. in a dimensionless form by introducing a new independent variable through the relation $\beta = (\beta')\psi$

$$\xi = \left(\frac{m\omega}{\hbar} \right)^{1/2} x \quad \text{and} \quad \frac{\psi_0}{\beta b} = \frac{\psi_0}{\beta b}$$

define a parameter γ as:

$$E = \left(\gamma + \frac{1}{2} \right) \hbar \omega \quad \frac{\psi_0}{\beta b} = \frac{\psi_0}{\beta b}$$

$$\frac{\hbar^2}{\beta^2} \frac{d^2\psi}{dx^2} + \left[\gamma \left(1 - \frac{1}{\beta^2} \right) + \frac{\hbar^2}{\beta^2} \omega^2 x^2 - \frac{\hbar^2}{\beta^2 b^2} \right] \psi = 0$$

$$\frac{d\psi}{dx} = \frac{d\psi}{d\xi} \cdot \frac{d\xi}{dx} = \left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{d\psi}{d\xi}$$

$$\frac{d}{dx} \left(\frac{d\psi}{dx} \right) = \left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{d}{dx} \left(\frac{d\psi}{d\xi} \right) = \left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{d^2\psi}{d\xi^2} \cdot \frac{d\xi}{dx}$$

$$= \left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{d^2\psi}{d\xi^2}$$

$$\frac{-\hbar^2}{2m} \left(\frac{\partial \psi}{\partial x} \right) \frac{d^2 \psi}{dx^2} + \frac{1}{2} m \omega^2 \left(\frac{\hbar}{m \omega} \right) \epsilon_p^2 \psi = \left(\lambda + \frac{1}{2} \right) \hbar \omega \psi$$

Multiplying by $\frac{-2}{\hbar \omega}$,

$$\frac{d^2 \psi}{dx^2} + \left(2\lambda + 1 - \epsilon_p^2 \right) \psi = 0$$

Let's take, $\psi(\epsilon_p) = e^{-\epsilon_p^2/2} f(\epsilon_p)$

$$\begin{aligned} \frac{d \psi}{d \epsilon_p} &= e^{-\epsilon_p^2/2} \frac{df}{d \epsilon_p} - \epsilon_p e^{-\epsilon_p^2/2} f(\epsilon_p) \\ \frac{d^2 \psi}{d \epsilon_p^2} &= e^{-\epsilon_p^2/2} \frac{d^2 f}{d \epsilon_p^2} - \epsilon_p e^{-\epsilon_p^2/2} \frac{d f}{d \epsilon_p} - e^{-\epsilon_p^2/2} f + \epsilon_p^2 e^{-\epsilon_p^2/2} f \\ &= \left[\frac{d^2 f}{d \epsilon_p^2} - 2 \epsilon_p \frac{df}{d \epsilon_p} + (\epsilon_p^2 - 1) f \right] e^{-\epsilon_p^2/2} \end{aligned}$$

$$\frac{\partial b}{\partial b} \frac{\psi'' b}{\psi b} \left(\frac{\omega m}{d} \right) = \left(\frac{\psi b}{\partial b} \right) \frac{b}{\partial b} \left(\frac{\omega m}{d} \right) = \left(\frac{\psi b}{\partial b} \right) \frac{b}{\partial b}$$

$$\frac{\psi'' b}{\partial b} \left(\frac{\omega m}{d} \right) =$$

Substituting,

$$\left[\frac{d^2 f}{d \xi^2} - 2\xi \frac{df}{d\xi} + (\xi^2 - 1)f + (2\lambda + 1 - \xi^2)f \right] e^{-\xi^2/2} = 0.$$

~~bottom exponents~~

$$\vartheta_r D \sum_{n=0}^{\infty} = \vartheta_r D \sum_{n=0}^{\infty} \varphi^n = (\varphi)^r$$

$$\frac{d^2 f}{d \xi^2} - 2\xi \frac{df}{d\xi} + 2\lambda f(\xi) = 0$$

\rightarrow Hermite differential eq.

for ~~$r \in \mathbb{Z}$~~ , where ' λ ' is
0 or +ve integers.

Whether ' λ ' is an integer ?

Frobenius method

$$f(\varphi) = \varphi^s \sum_{n=0}^{\infty} a_n \varphi^n = \sum_{n=0}^{\infty} a_n \varphi^{n+s}$$

$$f'(\varphi) = \varphi^{s-1} \sum_{n=0}^{\infty} n a_n \varphi^n = \sum_{n=0}^{\infty} n a_n \varphi^{n+s-1}$$

$$f''(\varphi) = \varphi^{s-2} \sum_{n=0}^{\infty} n(n-1) a_n \varphi^n = \sum_{n=0}^{\infty} n(n-1) a_n \varphi^{n+s-2}$$

$$f'''(\varphi) = \varphi^{s-3} \sum_{n=0}^{\infty} n(n-1)(n-2) a_n \varphi^n = \sum_{n=0}^{\infty} n(n-1)(n-2) a_n \varphi^{n+s-3}$$

S' neglect no in & related W

Substituting conditions of uniqueness for $s=2$

\Rightarrow to find unique and non-zero

$$\sum_{r=0}^{\infty} \left[(r+s)(r+s-1) a_r \epsilon^r - 2 \left\{ (r+s)-2 \right\} a_r \epsilon^{r+s} \right] = 0$$

For lowest power of ϵ

$$\sum_{r=0}^{\infty} \left[(r+s)(r+s-1) a_r - 2 \epsilon^2 \left\{ r+s-2 \right\} a_r \right] \epsilon^r = 0$$

For $r=2$

is valid for all ϵ .

~~lowest power of ϵ~~

\therefore Coeff. of each power of ϵ can be equated to zero.

The lowest power of ϵ is $r+s-2$ when $r=0$.

For $r=0$, and $r=1$ we get two indicial equations.

$$\begin{aligned} s(s-1) a_0 &= 0 & \left. \begin{aligned} a_0 &\neq 0 \Rightarrow s=0 \quad (i) \\ a_1 &\neq 0 \Rightarrow s=0 \quad (ii) \end{aligned} \right\} -1 \end{aligned}$$

$s = -1$ is not acceptable condition since it introduces new singularity at $\epsilon_0 = 0$.

$\epsilon_0 = 0$ is not a singular point from the form of the Hermite differential eqn.

$$\theta = \frac{1}{2} \left[\text{erf} \left(\frac{\epsilon_0 - 2\pi r}{\sqrt{2}} \right) - \text{erf} \left(\frac{\epsilon_0 + 2\pi r}{\sqrt{2}} \right) \right] \xrightarrow[r \rightarrow \infty]{\theta = 0}$$

No real solution

~~erf function~~

lower than zero

root of biquad

$\theta = r$ and $s = -2\pi r$ is a pair of lower bound

boundary, but if $r < 0$ then $\theta > 0$

exception

$$1 \text{ (r)} \quad \theta = 0 \iff \theta = \omega \quad \theta = \omega(1-2)^2$$

$$1 - \text{ (r)} \quad \theta = 0 \iff \theta = \omega \quad \theta = \omega(1+2)^2$$

In general for φ ,

$$(\gamma+s+\alpha)(\gamma+s+1) a_{\gamma+2} = \alpha [\gamma+s-\lambda] a_\gamma$$

$$\boxed{\frac{a_{\gamma+2}}{a_\gamma} = \frac{\alpha(\gamma+s-\lambda)}{(\gamma+s+\alpha)(\gamma+s+1)}}$$

For $s=0$ starting with $a_0 (q_1)$ generates all the even (odd) numbered coeff. that go with the even (odd) powers of φ ,

$$a_2 = \frac{-\alpha\lambda}{2} a_0, \quad a_4 = \frac{-\alpha(\gamma-2)}{12} = \frac{(-\alpha)^2 \lambda (\lambda-2)}{24} a_0,$$

$$a_3 = \frac{-\alpha(\gamma-1)}{6} a_1, \quad a_5 = \frac{-\alpha(\gamma-3)}{20} a_3 = \frac{(-\alpha)^2 (\gamma-1)(\gamma-3)}{120} a_1, \quad \dots$$

$$f(\epsilon)_{\text{even}} = a_0 + a_2 \epsilon^2 + a_4 \epsilon^4 + \dots$$

$$f(\epsilon)_{\text{odd}} = a_1 \epsilon + a_3 \epsilon^3 + a_5 \epsilon^5 + \dots$$

$$f(\epsilon) = \underbrace{a_0}_{\text{Bd}} \left[a_0 + a_2 \epsilon^2 + a_4 \epsilon^4 + \dots \right] + \left(a_1 \epsilon + a_3 \epsilon^3 + a_5 \epsilon^5 + \dots \right)$$

The full solution $f(\epsilon)$ can be determined from 2 constants a_0 and a_1 , which is expected for a 2nd order equation.

However, not all the solutions so obtained are normalizable!

At very large r , the recursion formula behaves

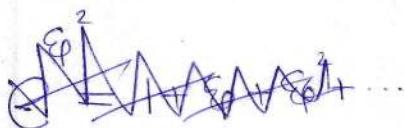
$$\frac{a_{r+2}}{a_r} = \frac{2 \left[1 + \frac{s}{r} - \frac{\lambda}{r} \right]}{r \left[1 + \frac{s+2}{r} \right] \left[1 + \frac{s+1}{r} \right]} \xrightarrow[r \rightarrow \infty]{} \frac{2s}{\lambda}$$

$$\Rightarrow \text{odd soln.} \quad \text{odd soln.} \quad \Psi$$

$$+ \hat{P}_0 D + \hat{P}_1 D + \dots = (\hat{P}) f$$

For,

$$+ \hat{P}_0 D^2 + \hat{P}_1 D^3 + \dots = (\hat{P}) f$$



$$\left(+ \hat{P}_0 D e^{q^2} + \hat{P}_1 D^2 e^{q^2} + \hat{P}_2 D^3 e^{q^2} + \dots \right) = \sum_{r=0}^{\infty} b_r e^{q^r}$$

$$\text{where, } b_r = \frac{1}{(\gamma_r)!}$$

Behaviour of $f(\hat{P})$ is like this. Not with

if b_r is \hat{P} then
Ratio of two adjacent coeff. is more
of roots of λ so not large

$$\text{so } \frac{b_{r+2}}{b_r} = \frac{(\gamma_r)!}{(\gamma_{r+2})!} \underset{\text{for large } r}{\underset{\text{as } r \rightarrow \infty}{\longrightarrow}} \frac{1}{\frac{\gamma_r}{2} + 1} \underset{\text{so it becomes}}{\underset{r \rightarrow \infty}{\longrightarrow}} \frac{2}{\gamma}$$

assumed, which is true at r goes from \mathbb{N}

Hence, asymptotically, $f(\hat{P})$ is behaving like

$$e^{\frac{q^2}{\gamma}} \quad \text{for large } \hat{P}$$

$$\frac{\left[\frac{\gamma}{2} - \frac{2}{\gamma} + 1 \right] \gamma}{\left[\frac{1+\gamma}{\gamma} + 1 \right] \left[\frac{6\gamma^2}{\gamma} + 1 \right] \gamma} = \frac{6\gamma^2}{\gamma^2}$$

i.e; if γ is a -ve integer

$$\Psi = e^{-\frac{q^2}{\gamma}} f(\hat{P}) \text{ goes like } e^{-\frac{q^2}{\gamma}}$$

The wavefunction ψ approaches infinity for $x \rightarrow \pm \infty$

Since, $|\psi(x)|^2 dx$ defines the probability of finding the particle b/w x and $x+dx$, it follows that the probability of finding the oscillating particle away from the origin will be infinite. It is not physically admissible.

A value of γ that is neither a +ve integer nor zero is not permitted.

For $\lambda = n$, is zero or above the integer.

we can get a particular polynomial solution of the Hermite differential equation - the Hermite polynomial of degree n .

$$f(\xi) = N H_n(\xi) \quad \text{with } (\xi^2 + a^2) = n E$$

The solution of time-independent Schrödinger equation can be written as:

$$\psi = N e^{-\frac{\xi^2}{2}} H_n(\xi)$$

This form of the solution is well behaved; it tends to zero as $\xi \rightarrow \pm\infty$.

The allowed values of λ correspond to $n=0, 1, 2, \dots$. This implies that the linear harmonic oscillator has a discrete set of equidistant energy levels:

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega, \quad n=0, 1, 2, \dots$$

The corresp. wavefunction is:

$$\Psi_n(x) = N_n e^{-\frac{1}{2} \frac{m\omega}{\hbar} x^2} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right)$$

n : vibrational quantum #

harmonic oscillator with boundary conditions $x \rightarrow \infty$ at max displacement

Ψ_n

$$\int_{-\infty}^{+\infty} |\psi_n(x)|^2 dx = 1 \quad , \quad \epsilon_p = \left(\frac{m\omega}{\hbar}\right)^{1/2} \alpha$$

$$d\epsilon_p = \left(\frac{m\omega}{\hbar}\right)^{1/2} dx$$

$$\Rightarrow N_n^2 \left(\frac{\hbar}{m\omega}\right)^{1/2} \int_{-\infty}^{+\infty} e^{-\epsilon_p^2} H_n(\epsilon_p) H_n(\epsilon_p) d\epsilon_p = 1$$

$$\int_{-\infty}^{+\infty} e^{-\epsilon_p^2} H_n(\epsilon_p) H_m(\epsilon_p) d\epsilon_p = \delta_{nm}$$

$$N_n = \left[\left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{1}{2^n n! \pi^{1/2}} \right]^{1/2}$$

The normalized wavefunctions are given by,

$$\psi_n(x) = \left[\left(\frac{m\omega}{\hbar} \right)^{1/2} \frac{1}{2^n n! \pi^{1/2}} \right]^{1/2} e^{-\frac{1}{2} \frac{m\omega}{\hbar} x^2} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$

$$\Psi_n = N_n e^{-\frac{\epsilon^2}{2}} H_n(\epsilon)$$

$$\frac{d\Psi_n}{d\epsilon} = N_n e^{-\frac{\epsilon^2}{2}} \left[-\epsilon H_n(\epsilon) + \frac{dH_n}{d\epsilon} \right]$$

$$= N_n e^{-\frac{\epsilon^2}{2}} \left[-\epsilon H_n(\epsilon) + 2n H_{n-1} \right]$$

$$= -\epsilon \Psi_n + N_n 2n e^{-\frac{\epsilon^2}{2}} H_{n-1}$$

$$N_n e^{-\frac{\epsilon_0^2}{2}} H_{n-1} = \left[\left(\frac{m\omega}{\hbar} \right)^{Y_2} \frac{1}{2^n n! \pi^{Y_2}} \right]^{\frac{1}{2}} e^{-\frac{\epsilon_0^2}{2}} H_{n-1}$$

$$= \left[\left(\frac{m\omega}{\hbar} \right)^{Y_2} \frac{1}{2^{n-1} (n-1)! \pi^{Y_2}} \right]^{\frac{1}{2}} \sqrt{2} \sqrt{n} e^{-\frac{\epsilon_0^2}{2}} H_{n-1}$$

$$= N_{n-1} e^{-\frac{\epsilon_0^2}{2}} H_{n-1} \times \sqrt{2n}$$

$$= \sqrt{2} \cdot \sqrt{n} \Psi_{n-1}$$

$$\frac{1}{\sqrt{2}} \left[\frac{d\Psi_n}{d\epsilon} + \epsilon \Psi_n \right] = \sqrt{n} \Psi_{n-1}$$

$$\hat{a}\Psi_n = \frac{1}{\sqrt{2}} \left(\epsilon + \frac{d}{d\epsilon} \right) \Psi_n = \sqrt{n} \Psi_{n-1}$$

$\hat{a} = \epsilon + \frac{d}{d\epsilon}$: Lowering (annihilation) operators.

$$\frac{d\psi_n}{d\epsilon} = N_n e^{-\frac{\epsilon^2}{4\epsilon_0}} \left[-\epsilon \dot{H}_n(\epsilon) + \frac{dH_n}{d\epsilon} \right]$$

$$\frac{dH_n}{d\epsilon} = \partial_\epsilon H_n(\epsilon) - H_{n+1}(\epsilon)$$

$$\begin{aligned} \frac{d\psi_n}{d\epsilon} &= N_n e^{-\frac{\epsilon^2}{4\epsilon_0}} \left[-\epsilon H_n(\epsilon) + \partial_\epsilon H_n(\epsilon) - H_{n+1}(\epsilon) \right] \\ &= N_n e^{-\frac{\epsilon^2}{4\epsilon_0}} \left[\epsilon \dot{H}_n(\epsilon) - H_{n+1}(\epsilon) \right] \end{aligned}$$

$$N_n e^{-\frac{\epsilon^2}{4\epsilon_0}} H_{n+1} = \left[\left(\frac{m\omega}{\hbar} \right)^{Y_2} \frac{1}{2 \cdot n! \cdot \pi^{Y_2}} \right]^{Y_2} \times \dot{H}_{n+1}(\epsilon)$$

$$= \left[\left(\frac{m\omega}{\hbar} \right)^{Y_2} \frac{1}{2^{n+1} (n+1)! \pi^{Y_2}} \right]^{Y_2} \sqrt{2(n+1)} e^{\frac{\epsilon^2}{4\epsilon_0}} \dot{H}_{n+1}(\epsilon)$$

$$= \sqrt{2(n+1)} \psi_{n+1}$$

$$\frac{d\psi_n}{d\epsilon} = \epsilon \psi_{n+1} - \sqrt{2(n+1)} \psi_{n+1}$$

\Rightarrow

$$\cancel{\frac{1}{\sqrt{2}} \left[\epsilon_0 \psi_n - \frac{d\psi_n}{d\phi} \right]} = \sqrt{n+1} \psi_{n+1} \quad \hbar = \frac{eVb}{2m}$$

$$\cancel{a_{n+1}^+ = \frac{1}{\sqrt{2}} \left[\epsilon_0 \psi_n - \frac{d\psi_n}{d\phi} \right]}$$

$$(13) \quad H = \cancel{(1) H_{PS} + (2) H_{HS} + (3) H_{SH}} \quad \hbar = \frac{eVb}{2m}$$

$$\frac{1}{\sqrt{2}} \left[\epsilon_0 \psi_n - \frac{d\psi_n}{d\phi} \right] = \sqrt{n+1} \psi_{n+1}$$

$$a_{n+1}^+ \psi_n = \frac{1}{\sqrt{2}} \left[\epsilon_0 - \frac{d}{d\phi} \right] \psi_n = \sqrt{n+1} \psi_{n+1}$$

$a_{n+1}^+ = \phi - \frac{d}{d\phi}$: raising (creation) operators.

$$(aa^+ - a^+a)\Psi_n = a\sqrt{n+1}\Psi_{n+1} - a^+\sqrt{n}\Psi_{n-1}$$

$$= (n+1)\Psi_{n+1} - n\Psi_{n-1} = \Psi_n$$