

Introduction to Linear Algebra  
- Gilbert Strang

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Linear Transformations  
(Fourier Series and Transform)

DRAW  
OUR  
TIGERS

**FAILURE**

WILL NEVER  
OVERTAKE ME  
IF MY DETERMINATION  
TO SUCCEED  
IS STRONG ENOUGH.

## INDEX

1Q

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## □ Orthogonal Bases for Function Space

3 leading even-odd bases for theoretical and numerical computations.

⑤ The Fourier Basis :

$$1, \sin x, \cos x, \sin 2x, \cos 2x, \dots$$

⑥ The Legendre Basis :

$$1, x, x^2 - \frac{1}{3}, x^3 - \frac{3}{5}x, \dots$$

The Chebyshev Basis :

$$1, x, 2x^2 - 1, 4x^3 - 3x, \dots$$

The Fourier Basis functions (sines & cosines) are all periodic. They repeat over every  $2\pi$  interval because  $\cos(\alpha + 2\pi) = \cos\alpha$  and  $\sin(\alpha + 2\pi) = \sin\alpha$ . So this basis is especially good for functions  $f(\alpha)$  that are themselves periodic:  $f(\alpha + 2\pi) = f(\alpha)$ .

The sine-cosine basis is also excellent for approximation. If we have a smooth periodic function  $f(\alpha)$ , then a few sines and cosines (low frequencies) are all we need. Jumps in  $f(\alpha)$  and noise in the signal are seen in higher frequencies (larger  $n$ ).

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0 \quad \text{for all } n, m$$

→ (Left) orthogonality of various harmonics int

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \begin{cases} 0, & \text{for } n \neq m \\ \pi, & \text{for } n = m \end{cases}$$

$$(\sin nx + \cos mx) \sum_{l=0}^{\infty} a_l =$$

→ Fourier Basis is orthogonal.

constant in angle mean w.r.t. in waves related  
(components straight)

~~Fourier series~~:

The Fourier series of a function  $f(x)$  is its expansion into sines and cosines:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

↳ trigonometry is used here

→ Fourier series is just linear algebra in function space (infinite dimensional)

$$\int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} \frac{a_0}{2} dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) dx \\
 &= 2\pi \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx dx \\
 &\quad \text{(using } \int_{-\pi}^{\pi} \cos nx dx = 0 \text{ and } \int_{-\pi}^{\pi} \sin nx dx = 0\text{)} \\
 &= 2\pi a_0 \\
 \implies \boxed{a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx}
 \end{aligned}$$

$$\begin{aligned}
 \int_{-\pi}^{\pi} f(x) \cos nx dx &= \int_{-\pi}^{\pi} \left[ a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right] \cos nx dx \\
 &= a_0 \int_{-\pi}^{\pi} \cos nx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos nx dx \\
 &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos nx dx \\
 &\quad \text{using } \int_{-\pi}^{\pi} \cos nx dx = 0 \\
 \int_{-\pi}^{\pi} f(x) \cos nx dx &= a_n \pi \\
 \implies \boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx} \quad \text{for } n = 1, 2, 3, \dots
 \end{aligned}$$

$$\int_{-\pi}^{\pi} f(x) \cos(mx) dx = Q_m \int_{-\pi}^{\pi} \cos(mx) \cos(mx) dx = Q_m \frac{\pi}{2}$$

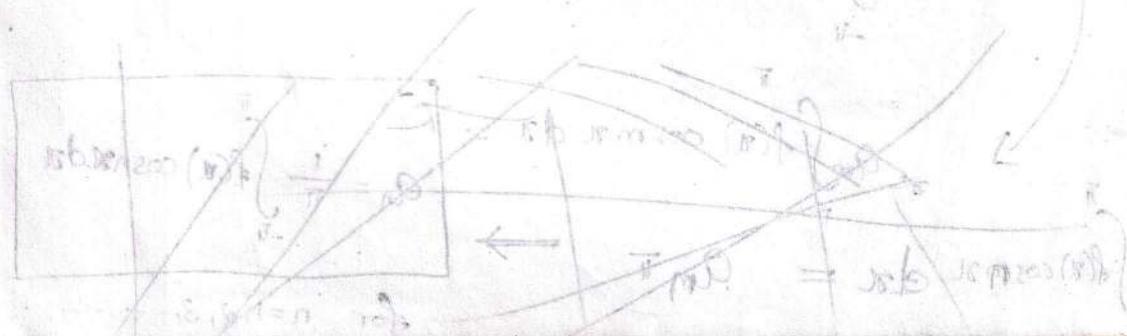
$$Q_m = \frac{\int_{-\pi}^{\pi} f(x) \cos(mx) dx}{\int_{-\pi}^{\pi} (\cos(mx))^2 dx} = \frac{(f(x), \cos(mx))}{(\cos(mx), \cos(mx))}$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx$$

$$Q_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) dx$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx) dx$$

for  $n=1, 2, \dots$



If  $f$  be a piecewise continuous function on  $[-\pi, \pi]$ . Then the Fourier series of  $f$  is the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where the coefficients  $a_n$  and  $b_n$  in this series are defined by

$$\frac{a_0}{2} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

and are called Fourier coefficients of  $f$ .

The Fourier series of a periodic function  $f(x)$  of period  $T$  is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2\pi n x}{L} + b_n \sin \frac{2\pi n x}{L} \right)$$

for some set of Fourier coefficients  $a_k$  &  $b_k$  defined by the integrals:

$$a_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos \left( \frac{2\pi n x}{L} \right) dx \quad \leftrightarrow \quad \frac{v_k^T b}{v_k^T v}$$

$$b_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \sin \left( \frac{2\pi n x}{L} \right) dx$$

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$\frac{1}{\sqrt{2\pi}}$

The length of a typical f(x) is

$$(f, f) = \int_{-\pi}^{\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + \dots)^2 dx$$

(Signal power = Signal energy)

$$= \int_{-\pi}^{\pi} (a_0^2 + a_1^2 \cos^2 x + b_1^2 \sin^2 x + a_2^2 \cos^2 2x + \dots) dx$$

[orthogonality]

$$\|f\|^2 = (f, f) = 2\pi a_0^2 + \pi (a_1^2 + b_1^2 + a_2^2 + \dots)$$

$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots$  is an orthonormal basis for our function space.

These are unit vectors.

We could combine them with coefficients  $a_0, a_1, a_2, \dots$  to yield a function  $F(x)$ .

function length = Vector length

$$\|F\|^2 = (F, F) = A_0^2 + A_1^2 + B_1^2 + A_2^2 + \dots$$

dimensions

The function has finite length exactly when the vector of coefficients has finite length.

Fourier series gives us a perfect match between the Hilbert spaces for functions and for vectors. The function is in  $L^2$  and, its

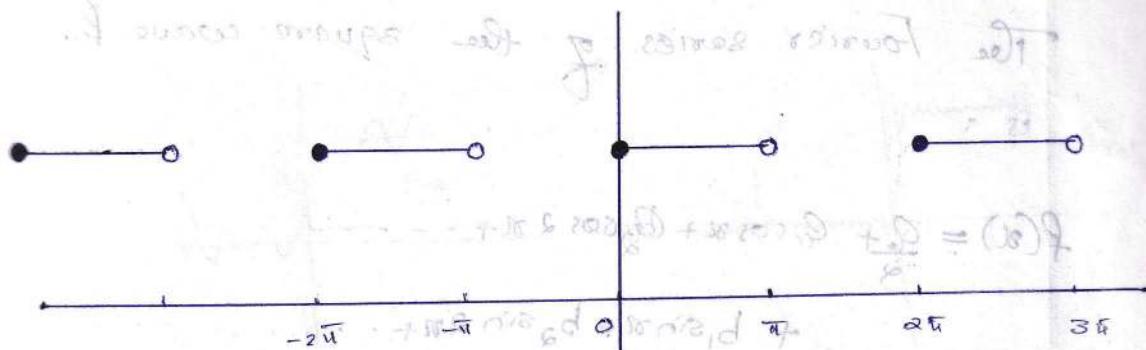
Fourier coeff. are in  $\ell^2$ .

$$\text{Fourier coeff. } \xrightarrow{\text{continuous function}} \xrightarrow{\text{discrete function}} \xrightarrow{\text{vector in } \ell^2}$$

- \* The function space contains  $f(x)$  exactly when the Hilbert space contains the vector  $v = (a_0, a_1, b_1, a_2, \dots)$  of Fourier coefficients of  $f(x)$ . Both must have finite length.

Ex:- Find the Fourier coeff. and Fourier series of the square-wave function  $f$  defined by,

$$f(x) = \begin{cases} 0 & -\pi \leq x < 0 \\ 1 & 0 \leq x < \pi \end{cases} \quad \text{and } f(x+2\pi) = f(x)$$



$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} 1 dx \right] = 0 + \frac{1}{\pi} [\pi] = 1.$$

$$= \frac{1}{\pi} \times \pi = 1.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} \cos(nx) dx \right]$$

$$= 0 + \frac{1}{\pi} \left[ \frac{\sin(nx)}{n} \right]_0^\pi = \frac{1}{\pi n} (0 - 0) = 0.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx = \frac{1}{\pi} \left[ \int_{-\pi}^0 0 dx + \int_0^{\pi} \sin(nx) dx \right]$$

$$= -\frac{1}{\pi} \left[ \frac{\cos(nx)}{n} \right]_0^\pi = -\frac{1}{\pi n} [\cos(n\pi) - \cos(0)]$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (\cos n\pi - 1) = \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{2}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

The Fourier series of the square wave function

is:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots$$

$$= \frac{1}{2} + 0 + 0 + \dots$$

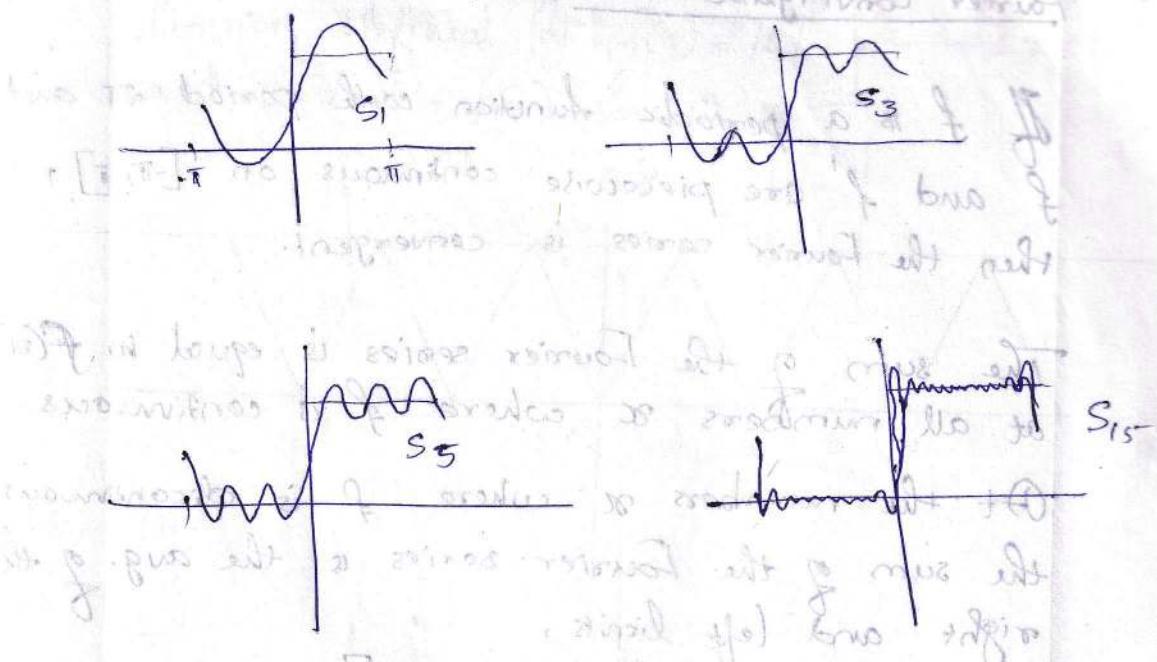
$$\begin{aligned} &+ \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \dots \\ &= \frac{1}{2} + \frac{2}{\pi} \sin x + \frac{2}{3\pi} \sin 3x + \frac{2}{5\pi} \sin 5x + \dots \end{aligned}$$

$$f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2}{(2k-1)\pi} \sin[(2k-1)x], \text{ where } k \in \mathbb{Z}$$

$$0 = \left( \frac{1}{\pi} \right) \frac{1}{\pi} = \left[ \frac{(2k-1)x}{\pi} \right] \frac{1}{\pi} + 0 = \frac{(2k-1)x}{\pi}$$

$$\frac{(2k-1)x}{\pi} = (2k-1)x \Rightarrow \frac{(2k-1)x}{\pi} = \frac{1}{\pi} \Rightarrow x = \frac{1}{2k-1}$$

$$\left[ (2k-1)x - (2k-1)x \right] \frac{1}{\pi} = \left[ \frac{(2k-1)x}{\pi} \right] \frac{1}{\pi}$$



$$S_n = \frac{1}{2} + \frac{2}{\pi} \left[ \sin(\alpha) + \frac{2}{3\pi} \sin(3\alpha) + \dots + \frac{2}{n\pi} \sin(n\alpha) \right]$$

As  $n$  increases,  $S_n(\alpha)$  becomes a better approximation to the square-wave function. It appears that the graph of  $S_n(\alpha)$  is approaching the graph of  $f(\alpha)$ , except where  $\alpha=0$  or  $\alpha$  is an integer multiple of  $\pi$ . i.e., it looks as if  $f$  is equal to the sum of its Fourier series except at the points where  $f$  is discontinuous.

Ex:-

## Fourier convergence theorem

If  $f$  is a periodic function with period  $2\pi$  and  $f$  and  $f'$  are piecewise continuous on  $[-\pi, \pi]$ , then the Fourier series is convergent.

The sum of the Fourier series is equal to  $f(x)$  at all numbers  $x$  where  $f$  is continuous.

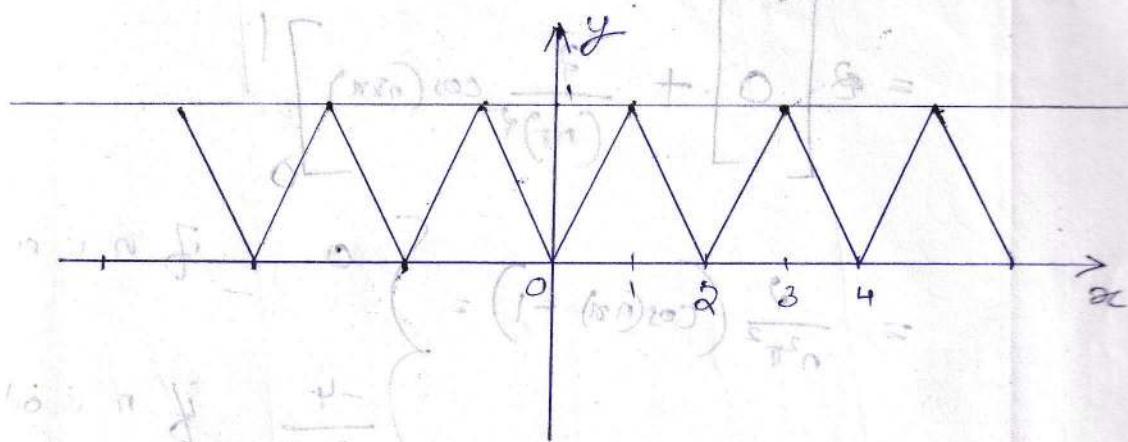
At the numbers  $x$  where  $f$  is discontinuous the sum of the Fourier series is the avg. of the right and left limits,

$$\frac{1}{2} [f(x^+) + f(x^-)]$$

## Gibbs phenomenon

It is a phenomenon in which the Fourier series overshoots the function value at a discontinuity. This overshoot is called the Gibbs phenomenon. It occurs because the Fourier series is a trigonometric sum of sine waves, which cannot perfectly represent a step function. The overshoot is most pronounced near the discontinuity and decreases as the number of terms in the series increases.

Ex:- Find the Fourier series of the triangular wave function defined by  $f(x) = |x|$  for  $-1 \leq x \leq 1$  and  $f(x+2) = f(x)$  for all  $x$



$$L=1.$$

$$a_n = \frac{2}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) \cos(n\pi x) dx$$

$$= \frac{2}{2} \int_0^1 |x| \cos(n\pi x) dx$$

$$= \int_0^1 -x \cos(n\pi x) dx + \int_0^1 x \cos(n\pi x) dx$$

$$= \left[ \frac{x \sin(n\pi x)}{n\pi} \right]_0^1 - \left( \int_0^1 \cos(n\pi t) dt \right) + \int_0^1 x \cos(n\pi x) dx \quad t = -x$$

$$= 2 \int_0^1 x \cos(n\pi x) dx \quad \begin{cases} y = \cos(n\pi x) \\ \text{is an even function} \end{cases}$$

$$= 2 \left[ x \frac{\sin(n\pi x)}{n\pi} \Big|_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi} dx \right]$$

$$a_n = 2 \left[ \frac{\sin(n\pi x)}{n\pi} + \frac{1}{(n\pi)^2} \cos(n\pi x) \right]$$

$$= 2 \left[ 0 + \frac{1}{(n\pi)^2} \cos(n\pi x) \right]$$

$$= \frac{1}{n^2\pi^2} (\cos(n\pi x) - 1) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_{-1}^1 |x| \sin(n\pi x) dx = 0$$

The Fourier series is:

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2} \cos(\pi x) - \frac{4}{9\pi^2} \cos(3\pi x) - \frac{4}{25\pi^2} \cos(5\pi x) - \dots$$

$$f(x) = \frac{1}{2} - \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi^2} \cos[(2k-1)\pi x] \quad \text{for all } x.$$

$$\left[ \sin \frac{(2k-1)\pi x}{\pi} \right] - \frac{(2k-1)\pi \cos x}{\pi} \Big|_0^\pi =$$

Ex:- The double angle formula in trigonometry is

$$\cos 2\alpha = 2\cos^2 \alpha - 1 \implies \cos^2 \alpha = \frac{1}{2} + \frac{1}{2} \cos 2\alpha.$$

Cor

f(G)

(B)

f(a)

where

C

C

## Complex form

$$\begin{aligned}
 f(\alpha) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(n\alpha) + b_n \sin(n\alpha) \right) \\
 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right) \\
 &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - i b_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + i b_n}{2} e^{-inx}
 \end{aligned}$$

$$f(\alpha) = \sum_{n=-\infty}^{+\infty} C_n e^{inx}$$

where,

$$C_0 = \frac{a_0}{2}, \quad C_n = \frac{a_n - i b_n}{2}, \quad C_{-n} = \frac{a_n + i b_n}{2}$$

$C_n$  complex Fourier coefficients.

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\alpha) e^{-inx} d\alpha, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\begin{aligned}
 & C_n = \int_{-\pi}^{\pi} f(x) e^{inx} dx = \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{inx} dx \\
 & \text{using } \int_a^b (c_0 + c_1 x + c_2 x^2 + \dots) dx = c_0 \frac{b-a}{2} + c_1 \frac{b^2-a^2}{4} + \dots \\
 & \text{we get } c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx
 \end{aligned}$$

$$\begin{aligned}
 C_n &= \int_{-\pi}^{\pi} f(x) e^{inx} dx = \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-inx} dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} c_n e^{inx} \right) e^{-inx} dx = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx} e^{-inx} dx = c_n
 \end{aligned}$$

$$\rightarrow C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

The complex form of the Fourier series of a periodic function  $f(x)$  of period  $T$  is:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{inx}{L}}$$

where, the Complex Fourier coefficient  $C_n$  is

$$C_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-\frac{inx}{L}} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Ex: Fourier series of

$$f(x) = \text{sgn}(x) = \begin{cases} -1, & -\pi \leq x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$$

Ans:

$$\begin{aligned} C_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \int_{-\pi}^0 (-1) e^{inx} dx + \frac{1}{2\pi} \int_0^{\pi} 1 e^{-inx} dx \\ &= \frac{1}{2\pi} \left[ \frac{e^{-inx}}{-in} \right]_{-\pi}^0 - \left[ \frac{e^{-inx}}{-in} \right]_0^{\pi} = \frac{1}{2\pi} \left[ 1 - e^{-i\pi n} - e^{-i0} + 1 \right] = \frac{-i}{2\pi n} \left[ 2 - \frac{(e^{i\pi n} + e^{-i\pi n})}{2} \right] \\ &= \frac{-i}{2\pi n} \left[ 2 - 2 \cos n\pi \right] = \frac{i}{\pi n} (\cos n\pi - 1) \end{aligned}$$

$$\boxed{C_n = \frac{i}{\pi n} \begin{cases} (-1)^n - 1 & \text{if } n = 2k \\ -\frac{2i}{(2k-1)\pi} & \text{if } n = 2k-1 \end{cases}}$$

The Fourier series of the function in complex form is:

$$f(x) = \text{sgn}(x) = \sum_{k=-\infty}^{+\infty} \frac{e^{i(2k-1)x}}{2k-1}$$

$$\boxed{f(x) = \sum_{k=-\infty}^{+\infty} \frac{e^{i(2k-1)x}}{2k-1}}$$

$$n=2k-1$$

$$f(x) = \text{sgn}(x) = \frac{-2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{e^{i(2k-1)x}}{2k-1} = \frac{-2i}{\pi} \sum_{k=-\infty}^{\infty} \frac{e^{inx}}{2k-1}$$

$$= \frac{-2i}{\pi} \sum_{n=1}^{\infty} \left[ \frac{e^{-inx}}{-n} + \frac{e^{inx}}{n} \right] = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{e^{-inx} - e^{inx}}{2in} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\alpha)}{n}$$

$$\begin{aligned} f(\pi/2) &= 1 \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2k-1)\pi/2)}{2k-1} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \end{aligned}$$

Ex: Using the complex form find the Fourier series of the function  $f(x) = x^2$ , defined on  $[0, 1]$ .

Ane:

$$\begin{aligned} c_n &= \frac{1}{2} \int_{-1}^1 x^2 e^{-inx} dx = \frac{1}{2} \int_0^1 x^2 e^{-inx} dx \\ c_n &= \frac{1}{2} \int_{-1/2}^{1/2} f(x) e^{-inx} dx = \frac{1}{2} \int_{-1}^1 x^2 e^{-inx} dx \\ &= \frac{1}{2} \left[ x^2 \frac{e^{-inx}}{-inx} \right]_{-1}^1 - \int_{-1}^1 x^2 \frac{e^{-inx}}{-inx} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left[ n^2 \frac{e^{-in\pi}}{-i\pi n} \right] + \frac{1}{2i(n\pi)} \left[ 2n \frac{e^{-in\pi}}{(-i\pi n)} - \frac{2e^{-in\pi}}{-i\pi n} \right] \\
&\approx \frac{1}{2} n^2 \frac{e^{-in\pi}}{-i\pi n} + \frac{2ne^{-in\pi}}{\cancel{2\pi^2 n^2}} \cdot \frac{1}{2} \cancel{\frac{2}{\pi^2 n^2}} \frac{e^{-in\pi}}{-i\pi n} \\
\text{mod } &\approx \frac{n^2 e^{-in\pi}}{-2i\pi n} + \frac{2ne^{-in\pi}}{2\pi^2 n^2} + \frac{2e^{-in\pi}}{2i\pi^3 n^3} \\
&= \frac{e^{in\pi}}{-2i\pi n} + \frac{2e^{-in\pi}}{2\pi^2 n^2} + \frac{2e^{-in\pi}}{2i\pi^3 n^3} + \frac{e^{in\pi}}{2i\pi n} + \frac{2e^{-in\pi}}{2\pi^2 n^2} \\
&\quad - \frac{2e^{-in\pi}}{2i\pi^3 n^3} \\
&= \frac{1}{\pi n} \frac{e^{in\pi} - e^{-in\pi}}{2i} + \frac{2}{\pi^2 n^2} \frac{e^{in\pi} + e^{-in\pi}}{2} \\
&\quad + \frac{2}{\pi^3 n^3} \frac{e^{in\pi} - e^{-in\pi}}{2i} \\
&= \frac{1}{\pi n} \frac{\sin(n\pi)}{2i} + \frac{2}{\pi^2 n^2} \cos(n\pi) - \frac{2}{\pi^3 n^3} \sin(n\pi) \\
c_n = & \frac{2}{n^2 \pi^2} (-1)^n
\end{aligned}$$

Ex's

$$\text{Coz} \left( \frac{1}{2} \int x^2 dx \right) = \frac{1}{2} \left[ \frac{x^3}{3} \right]_{-1}^1$$

$$= \frac{1}{6} [1+1] = \frac{1}{3}$$

Ans: f

The Fourier extension is complex form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n}{n\pi} e^{inx}$$

$$+ \sum_{n=1}^{\infty} \frac{b_n}{n\pi} e^{-inx}$$

$$(-1)^{-n} = \frac{1}{(-1)^n} = (-1)^n$$

$$f(x) = x = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{e^{inx} + e^{-inx}}{2}$$

$$(II) f(x) = \frac{a_0}{2} + (a_n) \cos \frac{n\pi x}{L} + (b_n) \sin \frac{n\pi x}{L}$$

$$= \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x)$$

$$\xrightarrow{(-1)^n = 1}$$

Ans: f

Ex:

$$f(z) = \frac{a \sin \alpha - i \cos \alpha}{1 - z \cos \alpha + z^2} A + f(z) \circ$$

$$1 = z \circ, 1 = A \leftarrow$$

$$\text{Ans: } f(z) = \frac{\frac{a}{2i} \frac{e^{iz} - e^{-iz}}{1 - 2i}}{1 - \cancel{z} \frac{e^{iz} + e^{-iz}}{2} + \cancel{z^2} - 1} \rightarrow \frac{1}{iz} = (B)$$

$$= \frac{1}{2i} \frac{a(e^{iz} - e^{-iz})}{1 - a(e^{iz} + e^{-iz}) + a^2} = |x|$$

$$\text{D}'\text{D} \sum_{n=1}^{\infty} = \frac{1}{2i} \frac{a(e^{iz} - e^{-iz})}{1 - a(e^{iz} + e^{-iz}) + a^2} > |x|$$

$$\text{D}'\text{D} \sum_{n=1}^{\infty} = \frac{1}{2i} \frac{a(e^{iz} - e^{-iz})}{1 - ae^{iz} - ae^{-iz} + a^2} = \frac{a(e^{iz} - e^{-iz})}{1 - ae^{iz} - ae^{-iz} + a^2}$$

$$= \frac{1}{2i} \frac{a(e^{iz} - e^{-iz})}{(1 - ae^{-iz})(1 - ae^{iz})} = (B)$$

$$= \frac{1}{2i} \left[ \frac{A}{1 - ae^{iz}} + \frac{B}{1 - ae^{-iz}} \right] \sum_{n=1}^{\infty}$$

$$a(e^{iz} - e^{-iz}) = A(1 - ae^{-iz}) + B(1 - ae^{iz})$$

$$= A - Aae^{-iz} + B - Bae^{iz}$$

$$\therefore A + B = 0 \quad \text{and} \quad A - B = 0 \Rightarrow (A + B)(1 - a) = 0$$

$$\underline{B = -A}$$

$$a(e^{ix} - e^{-ix}) = a A \left( e^{ix} - e^{-ix} \right)$$

$$\Rightarrow A = 1, B = -1$$

$$f(\alpha) = \frac{1}{2i} \left( \frac{1}{1 - ae^{i\alpha}} - \frac{1}{1 - ae^{-i\alpha}} \right)$$

$$|ae^{i\alpha}| = |a| |e^{i\alpha}| = |a| < 1$$

$$|ae^{-i\alpha}| < 1$$

$$\frac{1}{1 - ae^{i\alpha}} = 1 + ae^{i\alpha} + a^2 e^{2i\alpha} + \dots = \sum_{n=0}^{\infty} a^n e^{in\alpha}$$

$$\frac{1}{1 - ae^{-i\alpha}} = 1 + ae^{-i\alpha} + a^2 e^{-2i\alpha} + \dots = \sum_{n=0}^{\infty} a^n e^{-in\alpha}$$

$$f(\alpha) = \frac{1}{2i} \left[ \sum_{n=0}^{\infty} a^n \left( e^{in\alpha} - e^{-in\alpha} \right) \right] \frac{1}{iB}$$

$$\sum_{n=0}^{\infty} a^n \sin(n\alpha) = \frac{1}{2i} (e^{iB\alpha} - e^{-iB\alpha})$$

$$= (B-i)(B+i) \left( \frac{1}{2i} (e^{iB\alpha} - e^{-iB\alpha}) \right)$$

$$A = 2$$

$$\text{Ex: } f(t) = \cosh(\frac{t}{\pi} - 1) \quad \text{for } 0 \leq t \leq 1$$

even function  $f(t)$  is periodic with period  $\pi$ .  $\sum_{n=-\infty}^{\infty} = (\text{?})f$

$$c_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(t) e^{-i n \pi t} dt = \frac{1}{\pi} \int_0^\pi e^{-i n \pi t} \cosh(t-1) dt$$

$$= \int \cosh(t) dt = \sinh(t) + C$$

$$\sinh(t) dt = \cosh(t) + C$$

$$\begin{aligned} I &= \frac{1}{2} \left[ e^{-i n \pi t} \sinh(t-1) - \int (-i \pi n) e^{-i n \pi t} \sinh(t-1) dt \right] \\ &= \frac{1}{2} \left[ e^{-i n \pi t} \sinh(t-1) + i \pi n \left[ e^{-i n \pi t} \cosh(t-1) - \int (-i \pi n) e^{-i n \pi t} \cosh(t-1) dt \right] \right] \end{aligned}$$

$$2I = \frac{1}{2} e^{-i n \pi t} \sinh(t-1) + i \pi n e^{-i n \pi t} \cosh(t-1) - \pi^2 n^2 \times 2I$$

$$I(2 + 2\pi^2 n^2) = \frac{1}{2} e^{-i n \pi t} \sinh(t-1) + i \pi n e^{-i n \pi t} \cosh(t-1)$$

$$I = \frac{1}{1 + \pi^2 n^2} \times \frac{1}{2} \left[ e^{-i n \pi t} (\sinh(t-1) + i \pi n e^{-i n \pi t} \cosh(t-1)) \right]$$

$$= \frac{1}{2(1 + \pi^2 n^2)} \left[ \sinh(1) + i \pi n \cosh(1) - \sinh(-1) - i \pi n \cosh(-1) \right]$$

$$= \frac{\sinh(1)}{1 + \pi^2 n^2}$$

The complex Fourier series is: (1)

It has side components in  $\frac{2\pi n t}{T}$  which have

$$f(t) = \sum_{n=-\infty}^{+\infty} C_n e^{j 2\pi n t / T}$$

$$+ b_n(\omega) \text{ where } b_n(\omega) = \sum_{n=-\infty}^{+\infty} \frac{\sinh(i)}{1 + \omega^2} e^{j 2\pi n t / T} S(\omega) + \frac{1}{2} \delta(\omega)$$

$$2 + j\omega \cos - j\omega \sin \Rightarrow$$

$$2 + j\omega \cos - j\omega \sin \Rightarrow$$

$$(1-i) \cos - i \sin - \left( 1+i \right) \cos - i \sin \Rightarrow \frac{1}{\sqrt{2}} = 1$$

$$(1+i) \cos - i \sin - \left( 1-i \right) \cos - i \sin \Rightarrow \frac{1}{\sqrt{2}} = 1$$

$$\text{This is } 1 - (1-i) \cos - i \sin + (1-i) \cos - i \sin \Rightarrow \frac{1}{\sqrt{2}} = 1$$

$$(1-i) \cos - i \sin \Rightarrow \cos + (-i) \sin \Rightarrow \frac{1}{\sqrt{2}} = (\cos + i \sin) \Rightarrow$$

$$\left[ (1-i) \cos - i \sin + (1-i) \cos - i \sin \right] \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1$$

$$\left[ (1-i) \cos - i \sin + (1-i) \cos - i \sin + (1) \cos - i \sin \right] \cdot \frac{1}{(\sqrt{2})^2} = 1$$

## Fourier Transforms

drives system without phase errors

involves time  $\rightarrow$  and frequencies

Fourier transforms as a limit of Fourier series

if series removed and at unequal length

Fourier transforms (FTs) are an extension of Fourier series that can be used to describe non periodic functions on an infinite interval.

The key idea is to see that a non-periodic function can be viewed as a periodic one, but taking the limit of  $L \rightarrow \infty$ .

$$\text{and } f(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right)$$

and  $\omega_0$  corresponding bewalls of bins

dimensions needed for storage with

$$\frac{\pi L}{\omega_0} = \Delta \theta \text{ in } (1 + \alpha \sin \omega_0 t)$$

~~Fourier series~~ Fourier series only includes modes with wavenumbers  $k_n = \frac{2n\pi}{L}$  with adjacent modes separated by  $\delta k = \frac{2\pi}{L}$

What happens to our Fourier series if we let  $L \rightarrow \infty$  ?  
 If  $L \rightarrow \infty$  then the separation between adjacent modes becomes zero and the series converges to a continuous function.

Consider the complex Fourier series for  $f(x)$ :  
~~the~~  $\sum_{n=-\infty}^{+\infty} C_n e^{ik_n x}$  where  $i^2 = -1$

$$f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{ik_n x}$$

where,

$$C_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ik_n x} dx$$

and the allowed wavenumbers are  $k_n = \frac{2n\pi}{L}$ .

The separation of adjacent wavenumbers (i.e., for  $n \rightarrow n+1$ ) is  $\delta k = \frac{2\pi}{L}$

~~sk~~

~~sk~~  $\delta k = \frac{2\pi}{L}$  : As  $L \rightarrow \infty$ , the modes become more numerous and more finely separated in  $k$ .

In the limit, we are then interested in the variation of  $C$  as a function of the continuous variable  $k$ .

- The factor  $\frac{1}{L}$  outside the integral looks problematic for taking the limit  $L \rightarrow \infty$ , but this can be evaded by defining a new quantity :

$$\tilde{f}(k) = L \times C(k) = \lim_{L \rightarrow \infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(\alpha) e^{-ik\alpha} d\alpha$$

$L \rightarrow \infty, \delta k \rightarrow 0, \lim_{\substack{L \rightarrow \infty \\ \delta k \rightarrow 0}} n \delta k = k$

$$= \int_{-\infty}^{\infty} f(\alpha) e^{-ik\alpha} d\alpha$$

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26-43

from around when we, we find it, we find it

The functions  $\tilde{f}(k)$ , [ $\tilde{f}$  tilde], more commonly  
'f twiddle';  $f_k$  is another common notation  
is the Fourier transform of the non-periodic  
function  $f(x)$ .  $f(x)$  is not periodic

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

rotates with a  
sine wave

graph of  $\tilde{f}(k)$  between  $x$  and  $k$

fitting over

stolen -

$$x \in \mathbb{R} \setminus \{0\} \quad \tilde{f}(k) = (\mathcal{F}f)(k) = (\mathcal{F}f)^*(k)$$

$d = d\theta \pi$  and  $0 < d\theta, 0 < \pi$

stolen -

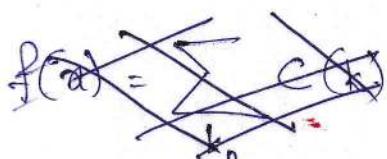
$$x \in \mathbb{R} \setminus \{0\} \quad \tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$\left[ \int_{-\infty}^{\infty} f(x) e^{-ikx} dx \right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} dk$$

We need the inverse Fourier transform, this gives us back the function  $f(x)$  if we know  $\hat{f}$ .

We just need to rewrite the Fourier series,

Mode spacing  $\delta k = \frac{2\pi}{L}$



$$f(x) = \lim_{L \rightarrow \infty} \sum_{n=-\infty}^{+\infty} c(n\delta k) e^{in\delta k x}$$

$$= \lim_{L \rightarrow \infty} \sum_{n=-\infty}^{+\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} c(\delta k) e^{inx} d\delta k \cdot \delta k$$

$$\lim_{\delta k \rightarrow 0} \sum g(k) \delta k = \int g(k) dk$$

$$k = n\delta k : -\infty \rightarrow +\infty \iff n : -\infty \rightarrow +\infty$$

$$f(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(k) e^{ik\omega} dk$$

is the inverse Fourier transform.

~~With inverse Fourier transform we can see what was f(x) by taking it back to our original narrow distribution of time shift.~~

$$\frac{\partial f}{\partial t} = f'' \text{ provides info.}$$

~~$\frac{\partial f}{\partial t} = f''$~~

$$\text{so from } \frac{\partial f}{\partial t} = f'' \text{ we get } \sum_{n=-\infty}^{\infty} n i \delta(n) = (f'')(t).$$

$$\text{so from } \frac{\partial f}{\partial t} = f'' \text{ we get } \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} n i \delta(n) =$$

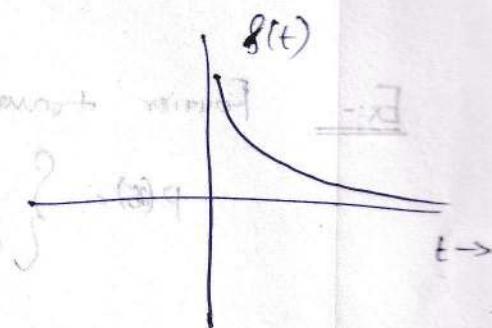
$$ik(f(t)) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} n i \delta(n)$$

$$ik(f(t)) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} n i \delta(n) \Rightarrow$$

Ex:- Find the Fourier transform of the one-sided exponential function

$$f(t) = \begin{cases} 0 & , t < 0 \\ e^{-\alpha t} & , t > 0 \end{cases}$$

where  $\alpha$ : the constant



Ans:  $\tilde{f}(k) = \int_{-\infty}^{\infty} e^{-\alpha x} e^{-ikx} dx$

$$= \left[ \int_0^{\infty} e^{-i(\alpha+k)x} dx \right] = \left[ \frac{e^{-i(\alpha+k)x}}{-i(\alpha+k)} \right]_0^{\infty}$$

$$= \frac{1}{\alpha + ik}$$

This real function has a complex Fourier transform

$$\tilde{f}(e^{-\alpha x} u(x)) = \frac{1}{\alpha + ik}, \alpha > 0$$

where,  $u(t)$  is the Heaviside unit step function:

$$u(t) = \begin{cases} 0 & , t < 0 \\ 1 & , t > 0 \end{cases}$$

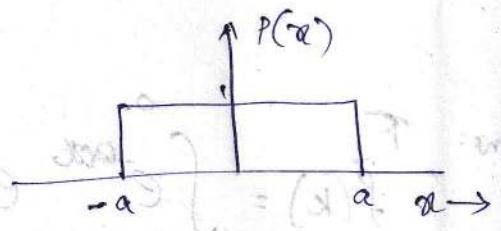
Fourier transform of a rectangular signal with width  $a$  and height  $b$

$$P(\omega) = \int_{-\infty}^{\infty} p(x) e^{-j\omega x} dx = b \int_{-a/2}^{a/2} e^{-j\omega x} dx = b \left[ \frac{e^{-j\omega x}}{-j\omega} \right]_{-a/2}^{a/2} = b \left( \frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-j\omega} \right) = b \frac{2 \sin(\omega a/2)}{\omega}$$

Ex:- Fourier transform of the rectangular pulse

$$p(x) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

Ans:



$$\tilde{f}(k) = \int_{-a}^a p(x) e^{-j k x} dx = \left[ \frac{e^{-j k x}}{-j k} \right]_{-a}^a = \frac{e^{-j k a} - e^{j k a}}{-j k} = \frac{2 \sin(k a)}{k}$$

$$= \frac{2 \sin(k a)}{k} = \frac{2 a \cdot \sin(k a)}{(k a) \pi}$$

Resultant signal is a rectangular pulse with a sinc function.

$$P(\omega) = \int_{-\infty}^{\infty} p(x) e^{-j\omega x} dx = b \int_{-a/2}^{a/2} e^{-j\omega x} dx = b \left[ \frac{e^{-j\omega x}}{-j\omega} \right]_{-a/2}^{a/2} = b \left( \frac{e^{-j\omega a/2} - e^{j\omega a/2}}{-j\omega} \right) = b \frac{2 \sin(\omega a/2)}{\omega}$$

Interger & half is with different signs

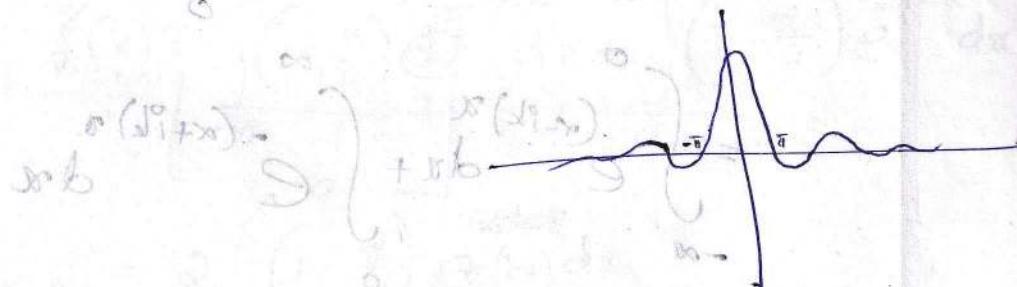
nothing

$$0 > f \quad \left\{ \begin{array}{l} x^2 \\ -x^2 \end{array} \right\} \quad (x)^2$$

$-0 < x$  means

For  $a=1$ ,

$$f_1(k) = \frac{\sin k}{k} \quad \text{even function.}$$



$$\left[ \frac{\sin(\Omega_i + \alpha)}{\Omega_i + \alpha} \right] + \left[ \frac{\sin(\Omega_i - \alpha)}{\Omega_i - \alpha} \right]$$

$$\frac{0.5}{\Omega_i + \alpha} + \frac{1}{\Omega_i - \alpha}$$

Ex:- Fourier transform of the 2 sided exponential function

$$f(x) = \begin{cases} e^{\alpha x}, & t < 0 \\ e^{-\alpha x}, & t > 0. \end{cases}$$

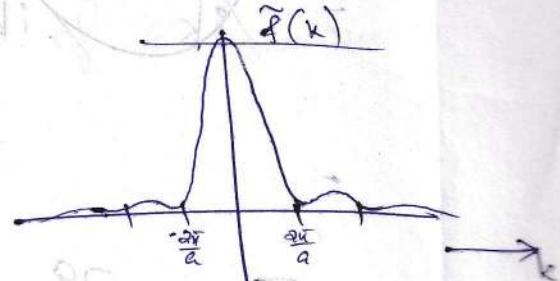
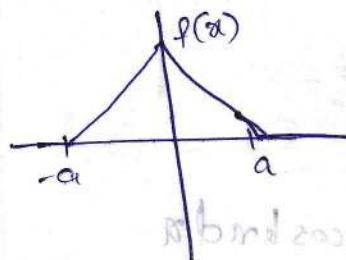
where  $\alpha > 0$ .

Ans:

$$\begin{aligned} \tilde{f}(k) &= \int_{-\infty}^0 e^{\alpha x} e^{-ikx} dx + \int_0^{\infty} e^{-\alpha x} e^{-ikx} dx \\ &= \int_{-\infty}^0 e^{(\alpha - ik)x} dx + \int_0^{\infty} e^{-(\alpha + ik)x} dx \\ &= \left[ \frac{e^{(\alpha - ik)x}}{\alpha - ik} \right]_{-\infty}^0 + \left[ \frac{e^{-(\alpha + ik)x}}{-(\alpha + ik)} \right]_0^{\infty} \\ &= \frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} = \frac{2\alpha}{\alpha^2 + k^2} // \end{aligned}$$

Ex: Fourier transform of the unit "triangle" function of width  $2a$ :

$$f(\alpha) = \begin{cases} 1 + \frac{\alpha}{a} & -a \leq \alpha < 0 \\ 1 - \frac{\alpha}{a} & 0 \leq \alpha \leq a \end{cases}$$



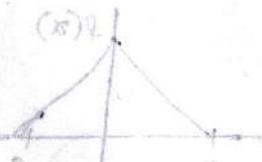
$$\text{Dm: } \tilde{f}(k) = \int_{-a}^0 \left(1 + \frac{\alpha}{a}\right) e^{-ik\alpha} d\alpha + \int_0^a \left(1 - \frac{\alpha}{a}\right) e^{-ik\alpha} d\alpha$$

$$= 2 \int_0^a \left(1 - \frac{\alpha}{a}\right) e^{-ik\alpha} d\alpha$$

$$\begin{aligned} &= \frac{2}{a} \int_0^a \alpha e^{-ik\alpha} d\alpha - \frac{2}{a} \int_0^a e^{-ik\alpha} d\alpha \\ &= \frac{2}{-ik} \left[ e^{-ik\alpha} \right]_0^a - \frac{2}{a} \left[ \frac{e^{-ik\alpha}}{-ik} \right]_0^a \\ &= \frac{2}{-ik} \left[ e^{-ika} - 1 \right] - \frac{2}{a} \left[ \frac{e^{-ika}}{-ik} - \frac{1}{-ik} \right] \end{aligned}$$

$$= \frac{2(\alpha - 1)}{ik} - \frac{2}{a} \left[ \frac{ae^{-ik\alpha}}{-ik} + \frac{e^{ik\alpha}}{k^2} - \frac{2e^{-ik\alpha}}{ak^2} + \frac{2}{ak^2} \right]$$

Ex:-



$$= \left[ \frac{2 \sin(k\alpha)}{k} \right]_0^\alpha - \frac{2}{a} \int \alpha \cos(k\alpha) d\alpha$$

$$= \frac{2 \sin(ka)}{k} - \frac{2}{a} \left[ \alpha \frac{\sin(ka)}{k} - \int \frac{\sin(ka)}{k} d\alpha \right]$$

$$= \frac{\alpha \sin(ka)}{k} - \frac{2}{a} \left[ \frac{\alpha \sin(ka)}{k} + \frac{\cos(ka)}{k^2} \right]_0^\alpha$$

$$= \frac{\alpha \sin(ka)}{k} - \frac{2}{a} \left[ \frac{\alpha \sin(ka)}{k} + \frac{\cos(ka)}{k^2} - \frac{1}{k^2} \right]$$

$$= \frac{2}{ak^2} \left[ 1 - \cos ka \right] = 2 \times \frac{2 \sin^2 \left( \frac{ka}{2} \right)}{ak^2}$$

$$= \frac{4 \sin^2 \left( \frac{ka}{2} \right)}{a^2 k^2}$$

Ans: I

OM  
Appendix

$\int_{-\infty}^{+\infty} e^{-\frac{x^2}{2a^2}} dx = \sqrt{\pi} a$

Ex:- Fourier transform of a Gaussian

$$f(x) = e^{-\frac{x^2}{2a^2}}$$

Ans:  $I = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2a^2}} dx = \sqrt{2\pi a^2} = \sqrt{2\pi} a$

on  
22  
Appendix

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2a^2}} \cdot e^{-ikx} dx$$

$$= \int_{-\infty}^{+\infty} e^{-\left(\frac{x^2}{2a^2} + ikx\right)} dx$$

$$\begin{aligned} \frac{x^2}{2a^2} + ikx &= \frac{1}{2a^2} (x^2 + 2ika^2 x) + \frac{1}{2a^2} (k^2 a^4) + ikx - ikx \\ &= \frac{1}{2a^2} (x + ika^2)^2 + \frac{1}{2a^2} (k^2 a^4) + ikx - ikx \\ &= \frac{1}{2a^2} (x + ika^2)^2 + \frac{k^2 a^4}{2} \end{aligned}$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} e^{-\frac{(x+ika^2)^2}{2a^2}} \cdot e^{-\frac{k^2a^2}{2}} dx$$

$$= e^{-\frac{k^2a^2}{2}} \int_{-\infty}^{+\infty} e^{-\frac{u^2}{2a^2}} du$$

$\underbrace{\sqrt{2\pi}a}_{(x)}$

Set  
 $u = x + ik a^2$   
 $du = dx$

$$D \tilde{f}(k) = \frac{\sqrt{2\pi}a}{\sqrt{2\pi}a} e^{-\frac{k^2a^2}{2}}$$

$\cancel{D \tilde{f}(k) = \frac{\sqrt{2\pi}a}{\sqrt{2\pi}a} e^{-\frac{k^2a^2}{2}}}$

$$D \tilde{f}(k) = \left( s_{ik} + \frac{s_{ik}}{s_{ik} + s_{ik}} \right) - \left( s_{ik} + \frac{s_{ik}}{s_{ik} + s_{ik}} \right)$$

$\cancel{D \tilde{f}(k) = \left( s_{ik} + \frac{s_{ik}}{s_{ik} + s_{ik}} \right) - \left( s_{ik} + \frac{s_{ik}}{s_{ik} + s_{ik}} \right)}$

$$s_{ik} - s_{ik} + \left( s_{ik} + s_{ik} \right) \frac{1}{s_{ik} + s_{ik}} = s_{ik} + \frac{s_{ik}}{s_{ik}}$$

$$s_{ik} - s_{ik} + \left( s_{ik} + s_{ik} \right) \frac{1}{s_{ik}} + \left( s_{ik} + s_{ik} \right) \frac{1}{s_{ik}} =$$

$$\frac{s_{ik} s_{ik}}{s_{ik}} + \left( s_{ik} + s_{ik} \right) \frac{1}{s_{ik}}$$

Ex:-

## The delta function

### The Dirac delta function

Q. 8

Q. 9

Q. 10

Q. 11

Q. 12

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Q. 244

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Q. 246

Q. 247

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Q. 250

Q. 251

Q. 252

Q. 253

Q. 254

Q. 255

Q. 256

Q. 257

Q. 258

Q. 259

Q. 260

Q. 261

Q. 262

Q. 263

Q. 264

Q. 265

Q. 266

Q. 267

Q. 268

Q. 269

Q. 270

Q. 271

Q. 272

Q. 273

Q. 274

Q. 275

Q. 276

Q. 277

Q. 278

Q. 279

Q. 280

Q. 281

Q. 282

→ The Dirac delta function  $\delta(x)$  is a

"generalized function" (But, strictly speaking, not a function) which satisfy ① & ②

with the condition that the integral is

② must be interpreted according to eq. ③

where the functions  $\delta_\epsilon(x)$  satisfy eq. ③ & ④

old definition

infinitesimal interval  
arbitrary width

arbitrary width

There are infinite functions  $\delta_\epsilon(x)$  which

satisfy eq's ③ & ④.

With increasing of  $\epsilon$  it becomes more and more

Ex: Exponential width  $\propto e^{-x^2/\epsilon^2}$  of a Gaussian

$$* \delta_\epsilon(x) = \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2}$$

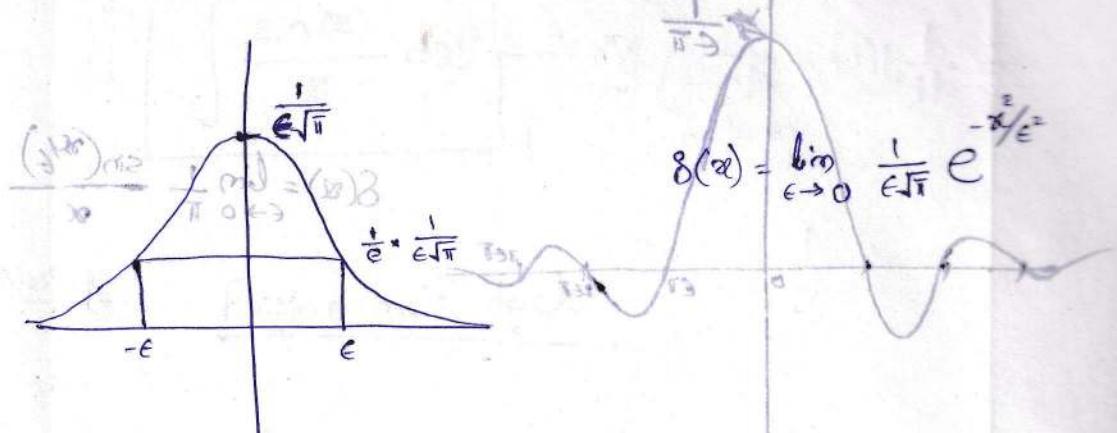
$$\text{rb}(\infty), \delta \quad \left. \begin{array}{l} \text{only} \\ \text{for } |x| \leq \epsilon/2 \end{array} \right\} = \text{rb}(\infty)\delta$$

$$* \delta_\epsilon(x) = \begin{cases} \infty & \text{for } |x| \leq \epsilon/2 \\ 0 & \text{for } |x| > \epsilon/2 \end{cases}$$

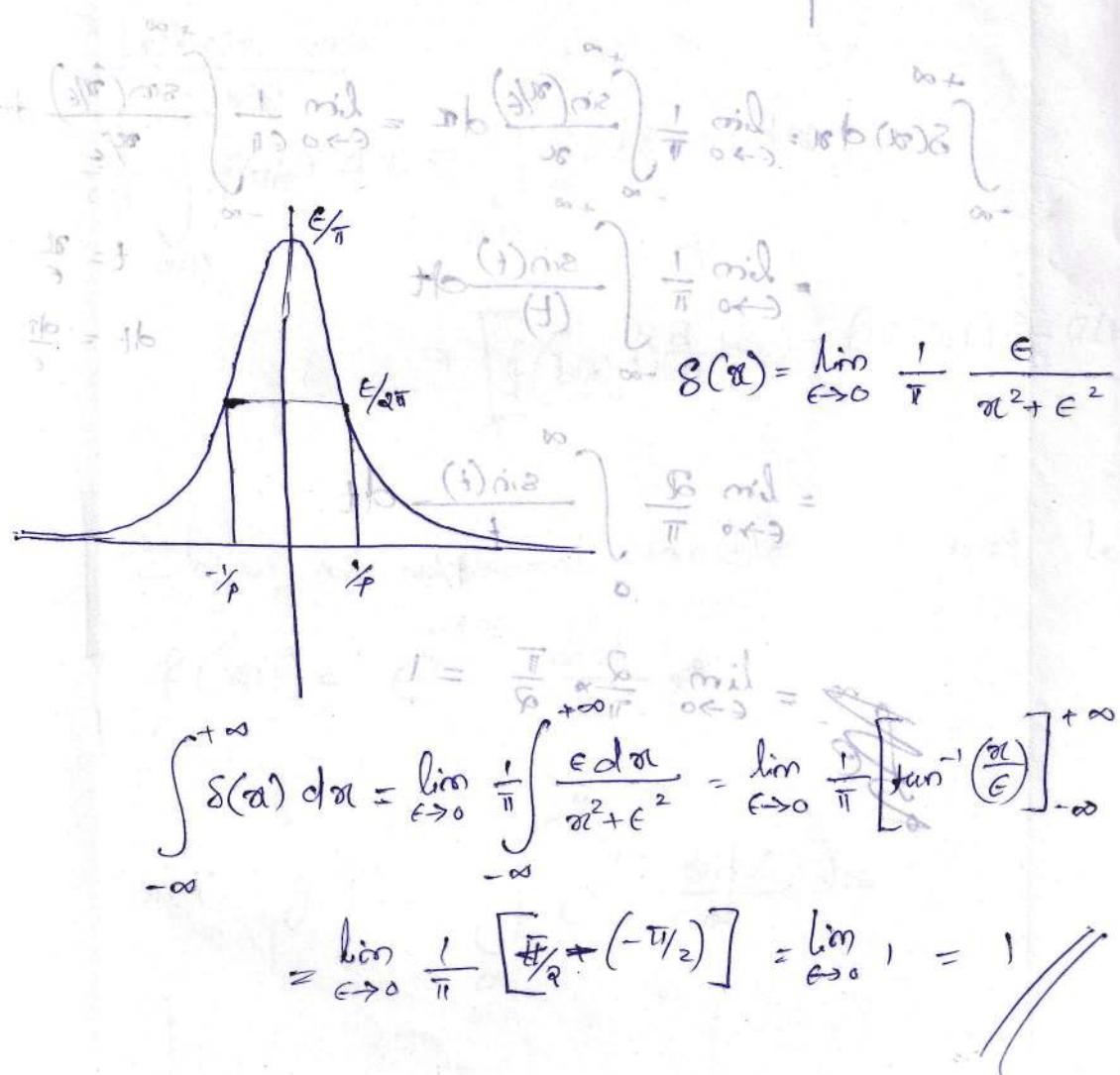
$$* \delta_\epsilon(x) = \frac{1}{\sqrt{\pi}} \frac{e^{-x^2/\epsilon^2}}{x^2 + \epsilon^2} \quad \text{Lorentzian}$$

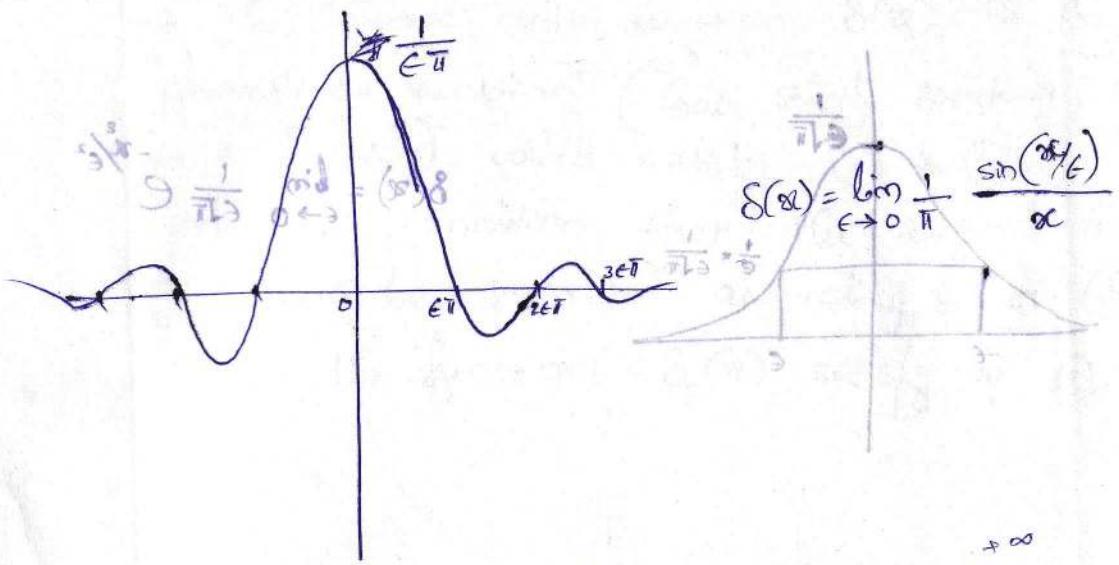
$$* \delta_\epsilon(x) = \frac{1}{\pi} \frac{\sin(x/\epsilon)}{x} \quad \text{Dirichlet}$$

$$\left. \begin{array}{l} \text{rb}(\infty), \delta \\ \text{and} \\ \text{Dirichlet} \end{array} \right\} = \text{rb}(\infty)\delta$$



$$\delta(\alpha) = \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\pi}} e^{-\alpha^2/\epsilon^2}$$





Method: 1

$$\begin{aligned}
 \int_{-\infty}^{+\infty} \delta(x) dx &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(\pi/x)}{x} dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon} \int_{-\infty}^{+\infty} \frac{\sin(\pi/x/\epsilon)}{x/\epsilon} dx \\
 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(t)}{(t/\epsilon)} dt \\
 &\quad \text{with } t = \frac{\pi x}{\epsilon}, \quad dt = \frac{\pi}{\epsilon} dx \\
 &= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_0^\infty \frac{\sin(t)}{t} dt \\
 &= \lim_{\epsilon \rightarrow 0} \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{t} dt = 1
 \end{aligned}$$

~~$\int_{-\infty}^{+\infty} \delta(x) dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(\pi/x)}{x} dx = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi \epsilon} \int_{-\infty}^{+\infty} \frac{\sin(\pi/x/\epsilon)}{x/\epsilon} dx$~~

~~$= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\sin(t)}{(t/\epsilon)} dt$~~

~~$\quad \text{with } t = \frac{\pi x}{\epsilon}, \quad dt = \frac{\pi}{\epsilon} dx$~~

~~$= \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \int_0^\infty \frac{\sin(t)}{t} dt$~~

~~$= \lim_{\epsilon \rightarrow 0} \frac{2}{\pi} \int_0^{\pi/2} \frac{1}{t} dt = 1$~~

Leibniz

$$\frac{d}{dt} \int_a^b \alpha(t)$$

Defi

$f(t)$

$(f(t))_{t=a}$

and

$$\boxed{\int \frac{\sin(x)}{x} dx}$$

Method 1

Feynman's trick

Leibniz rule

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x,t) dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t) dx +$$

$$[f(b(t),t) \frac{d}{dt} b(t) - f(a(t),t) \frac{d}{dt} a(t)]$$

Define an (additional) variable  $t$  and let

$$f(x,t) = e^{-tx} \frac{\sin x}{x}$$

$$\text{and } g(t) = \int e^{-tx} \frac{\sin x}{x} dx$$

$$\left[ \left( \frac{1}{1+t} + \frac{1-t}{1-t} \right) - 0 \right] \frac{1}{i\pi} =$$

$$\frac{d}{dt} g(t) = \frac{d}{dt} \int_0^\infty f(\alpha t) d\alpha \frac{\text{(using)} \frac{d}{dt}}{\text{at}}$$

$$= \frac{d}{dt} \int_0^\infty e^{-t\alpha} \frac{\sin \alpha}{\alpha} d\alpha$$

$$= \int_0^\infty \frac{d}{dt} e^{-t\alpha} \cdot \frac{\sin \alpha}{\alpha} d\alpha$$

$$= - \int_0^\infty e^{-t\alpha} \frac{\sin \alpha}{\alpha} d\alpha$$

$$= - \int_0^\infty e^{-t\alpha} \frac{e^{i\alpha} - e^{-i\alpha}}{2i} d\alpha$$

(1)  $\lim_{t \rightarrow 0^+}$

$$= \frac{-1}{2i} \int_0^\infty \left[ e^{-\alpha(t-i)} - e^{-\alpha(t+i)} \right] d\alpha$$

$$= \frac{-1}{2i} \left[ \frac{1}{t-i} - \frac{1}{t+i} \right]$$

$$= \frac{-1}{2i} \left[ 0 - \left( \frac{-1}{t-i} + \frac{1}{t+i} \right) \right]$$

$\lim_{t \rightarrow 0^+}$   
 range of  
 integrals

$$= \frac{-1}{2i} \left[ \frac{1}{t-i} - \frac{1}{t+i} \right]$$
~~$$= \frac{-1}{2i} \left[ \frac{t+i - t-i}{t^2 + 1} \right] = \frac{-1}{t^2 + 1}$$~~

$$\frac{dg(t)}{dt} = \frac{d}{dt} \left[ \int_0^\infty f(\alpha) e^{-t\alpha} \cdot \frac{\sin \alpha}{\alpha} d\alpha \right] = \frac{-1}{t^2 + 1}$$

$$g(t) = \int \frac{-dt}{t^2 + 1} = -\tan^{-1}(t) + C$$

~~requires proof~~

$$\lim_{t \rightarrow \infty} g(t) = 0 \implies C = \lim_{t \rightarrow \infty} \tan^{-1}(t) = \pi/2.$$

~~Interchange of operations: limits  
of integrals~~

$$g(t) = -\tan^{-1}(t) + \pi/2$$

$$g(0) = \frac{\pi}{2} = \int_0^\infty \frac{\sin \alpha}{\alpha} d\alpha$$

Method: 2

$$\int_0^\infty e^{-\alpha t} dt = \frac{1}{\alpha}$$

A2

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \int_0^\infty e^{-\alpha t} \sin x dx dt$$

$$= \int_0^\infty \int_0^\infty e^{-\alpha t} \frac{e^{ix} - e^{-ix}}{2i} dx dt$$

$$= \frac{1}{2i} \int_0^\infty \int_0^\infty \left[ e^{-\alpha(t-i)} - e^{-\alpha(t+i)} \right] dx dt$$

$$= \frac{1}{2i} \int_0^\infty \left[ \frac{-1}{t-i} e^{-\alpha(t-i)} + \frac{1}{t+i} e^{-\alpha(t+i)} \right] dt$$

$$= \frac{1}{2i} \int_0^\infty \left[ 0 - \left( \frac{-1}{t-i} + \frac{1}{t+i} \right) \right] dt$$

$$= \frac{1}{2i} \int_0^\infty \frac{2i}{t^2+1} dt$$

$$\text{A4} \quad = \frac{\pi}{2} \int_0^{\infty} \frac{dt}{t^2 + 1} = \left[ \tan^{-1} t \right]_0^{\infty}$$
$$= \frac{\pi}{2}$$

Direct delta  
continuous

$$\int_{-\infty}^{+\infty} f(x) dx = \lim_{\epsilon \rightarrow 0} \int_{-\epsilon/2}^{\epsilon/2} \frac{1}{\epsilon} dx = \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} dx \stackrel{\epsilon \rightarrow 0}{\rightarrow} 1$$

$\delta^3 = |x|$  rob       $\delta^1 =$  measure  
 $\delta^3 < |x|$  rob       $0 = (x) \delta$

$\delta^3 + \delta^1$

$$= b(x_0 - x) \delta(x) + b(x_0 + x) \delta(x)$$


---

Shifting property of Dirac delta function

For any function  $f(x)$  continuous at  $x_0$ ,

It shows the value of  $f(x)$  at  $x_0$ .

$$\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = f(x_0)$$

Integrating over the interval of  $x$  and  $x_0$ .

This gives a sense of measure to the Dirac delta function. — it measures the value of  $f(x)$  at the point  $x_0$ .

Proof

$$\delta(x) \stackrel{def}{=} \lim_{\epsilon \rightarrow 0} \delta_\epsilon(x) \quad \left\{ \begin{array}{l} \text{for } |x| < \epsilon/2 \\ \text{for } |x| > \epsilon/2 \end{array} \right. \quad \left\{ \begin{array}{l} \text{at } x = 0 \\ \text{as } \epsilon \rightarrow 0 \end{array} \right.$$

where,

$$\delta_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & \text{for } |x| \leq \epsilon/2 \\ 0 & \text{for } |x| > \epsilon/2 \end{cases}$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x - x_0) dx = \lim_{\epsilon \rightarrow 0} \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} f(x) \delta_\epsilon(x - x_0) dx$$

$$= \lim_{\epsilon \rightarrow 0} \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} f(x) \cdot \frac{1}{\epsilon} dx$$

normalizing factor for point P

Now to evaluate (if normalizing)

Over this very small range of  $x$ , the function  $f(x)$  can be thought to be constant and can be taken out of the integral.

normalizing factor for point P

$$= f(x_0) \lim_{\epsilon \rightarrow 0} \int_{x_0 - \epsilon/2}^{x_0 + \epsilon/2} \frac{1}{\epsilon} dx$$

$$= f(x_0) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \times \epsilon = f(x_0)$$

$$= f(x_0) \lim_{\epsilon \rightarrow 0} 1 = f(x_0).$$

$$(x)\delta = (x_0)\delta |_{D|}$$

$$\frac{(x)\delta}{|D|} = (x_0)\delta$$

Any function  $\delta(x-x_0)$  that satisfies the shifting property is the Dirac delta function.

$$0 = \int_{-\infty}^{+\infty} f(x) \delta(x) dx = (x_0)\delta |_{D|}$$

$$\boxed{\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0)} \quad \textcircled{i}$$

$$0 \neq 0 \quad \text{not} \quad \int_{-\infty}^{+\infty} f(x) \delta(x) dx = (x_0)\delta |_{D|} \leftarrow$$

ii

$$\text{Def. of } \delta(x) \quad \left\{ \begin{array}{l} \text{if } x=0 \\ \text{if } x \neq 0 \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l} \text{if } x=0 \\ \text{if } x \neq 0 \end{array} \right. = \quad \left\{ \begin{array}{l} \text{if } x=0 \\ \text{if } x \neq 0 \end{array} \right.$$

Scaling property.  $\exists \alpha \in \mathbb{R}$  s.t.  $\forall x \in \mathbb{R}$   $\delta(\alpha x) = |\alpha| \delta(x)$

$$\delta(\alpha x) = |\alpha| \delta(x) \quad \text{with, } \alpha \neq 0$$

without loss of generality we can assume  $\alpha > 0$

Proof.

$$\text{i) } |\alpha| \delta(\alpha x) = \begin{cases} |\alpha|(+\infty) & \text{for } \alpha x = 0 \\ |\alpha|(0) & \text{for } \alpha x \neq 0 \end{cases}$$

$$\Rightarrow |\alpha| \delta(\alpha x) = \begin{cases} +\infty & \text{for } x=0 \\ 0 & \text{for } x \neq 0 \end{cases}$$

(ii) For  $a > 0$ ,

$$\int_{-\infty}^{+\infty} |a| \delta(ax) dx = \int_{-\infty}^{+\infty} |a| \delta(t) \frac{dt}{a}$$

$t = ax \quad \text{put}$   
 $-dt = adx$

$$= \int_{-\infty}^{+\infty} \delta(t) dt = 1$$

$x: -\infty \rightarrow +\infty$   
 $t = ax: -\infty \rightarrow +\infty$

$$\left. \frac{1}{|a|} \right|_{-\infty}^{+\infty} = \text{rb}(x_0 - s_0) \delta(x_0)$$

For  $a < 0$ ,

$$\int_{-\infty}^{+\infty} |a| \delta(ax) dx = \int_{-\infty}^{+\infty} |a| \delta(t) \frac{dt}{a}$$

$dx: -\infty \rightarrow +\infty$   
 $t = ax: +\infty \rightarrow -\infty$

$$= \int_{-\infty}^{+\infty} -a \cdot \delta(t) \cdot \frac{dt}{a}$$

$\text{rb}(x_0 - s_0) \delta(x_0) = -a \delta(x_0 - s_0) \delta(x_0)$

$$= \int_{-\infty}^{+\infty} \delta(t) dt = 1$$

$+ \text{rb}(x_0 - s_0) \delta(x_0 - s_0)$

$$\Rightarrow |a| \delta(ax) = \delta(x)$$

$$\Rightarrow \cancel{\text{rb}(x_0 - s_0) \delta(x_0 - s_0)}$$

$$\boxed{\delta(-x) = \delta(x)}$$

$$\delta(\alpha^2 - a^2) = \frac{1}{2|a|} (\delta(\alpha-a) + \delta(\alpha+a))$$

$\int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha = \frac{1}{2|a|} (f(a) - f(-a))$

$$\int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha$$

$\Rightarrow$  SC

$$\int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha = \frac{1}{2|a|} (f(a) - f(-a))$$

$\alpha = \sqrt{y} \Rightarrow d\alpha = \frac{dy}{2\sqrt{y}}$

Set  ~~$\alpha^2 = y$~~

$$\int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha$$

$$\begin{aligned} \int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha &= \int_{-\infty}^0 f(\alpha) \delta(\alpha^2 - a^2) d\alpha + \int_0^{+\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha \\ &= \int_0^{\infty} f(-\alpha) \delta(\alpha^2 - a^2) d\alpha + \int_0^{\infty} f(\alpha) \delta(\alpha^2 - a^2) d\alpha \end{aligned}$$

$$= \int_0^{\infty} f(-\sqrt{y}) \delta(y - a^2) \frac{dy}{2\sqrt{y}} + \int_0^{\infty} f(\sqrt{y}) \delta(y - a^2) \frac{dy}{2\sqrt{y}}$$

$$(xy)\delta = (x-y)\delta$$

$$\delta(\alpha^2 - a^2)$$

$$\delta(x^2 - a^2) = \frac{1}{2|a|} (\delta(x-a) + \delta(x+a))$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x^2 - a^2) dx = \frac{1}{2|a|} (f(a) - f(-a))$$

$$\frac{\partial b(\beta) \delta(p)}{\partial p} = \partial b(\infty) \delta(p)$$

$$x = \sqrt{y} \Rightarrow dx = \frac{dy}{2\sqrt{y}}$$

~~$\frac{\partial^2 y}{\partial x^2}$~~   ~~$2 \cdot \partial x \cdot dy$~~

$$\int_{-\infty}^{+\infty} f(x) \delta(x^2 - a^2) dx = \int_{-\infty}^0 f(x) \delta(x^2 - a^2) dx + \int_0^{+\infty} f(x) \delta(x^2 - a^2) dx$$

$$= \int_0^\infty f(-x) \delta(x^2 - a^2) dx + \int_0^\infty f(x) \delta(x^2 - a^2) dx$$

$$= \int_0^\infty f(-\sqrt{y}) \delta(y - a^2) \frac{dy}{2\sqrt{y}} + \int_0^\infty f(\sqrt{y}) \delta(y - a^2) \frac{dy}{2\sqrt{y}}$$

$$(x) \delta = (x) \delta$$

$$= \frac{1}{2} \int_0^\infty \left[ f(-\sqrt{x}) + f(\sqrt{x}) \right] \delta(x-a^2) dx$$

$$\int_{-\infty}^{+\infty} f(a) \delta(x-a^2) dx = \frac{1}{2} \left[ \frac{f(-a) + f(a)}{|a|} \right] = \frac{f(a) + f(-a)}{2|a|}$$

aber rechtschrieben

$$\Rightarrow \delta(x^2 - a^2) = A \delta(x-a) + B \delta(x+a)$$

$$\int_{-\infty}^{+\infty} f(x) \left[ A \delta(x-a) + B \delta(x+a) \right] dx = A f(a) + B f(-a)$$

$$\frac{f(a) + f(-a)}{2|a|} = A f(a) + B f(-a)$$

$$A = B = \frac{1}{2|a|}$$

$$\delta(x^2 - a^2) = \frac{1}{2|a|} (\delta(x-a) + \delta(x+a))$$

In general,  $\int_{-\infty}^{\infty} f(x) \delta(g(x)) dx = \int_{-\infty}^{\infty} f(x) \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} dx =$

$$\left( \delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|} \right) \Leftrightarrow \int_{-\infty}^{+\infty} f(x) \delta(g(x)) dx = \sum_i \frac{f(x_i)}{|g'(x_i)|}$$

composition rule

$$(p \circ g) \delta_1 + (p \circ g) \delta_2 A = (p \circ g) \delta \Leftrightarrow$$

$$(p \circ g) \delta_1 + (p \circ g) \delta_2 A = \int_{-\infty}^{+\infty} [(p \circ g) \delta_1 + (p \circ g) \delta_2 A] (p \circ g) \delta dx$$

$$(p \circ g) \delta_1 + (p \circ g) \delta_2 A = \frac{(p \circ g) \delta_1 + (p \circ g) \delta_2}{|p \circ g|}$$

$$\frac{1}{|p \circ g|} \delta = A$$

$$((p \circ g) \delta_1 + (p \circ g) \delta_2) \frac{1}{|p \circ g|} = (p \circ g) \delta$$

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} \delta(x - x_0) e^{-ikx} dx = e^{-ikx_0}$$

when  $x_0=0$ ,  $\tilde{f}(k)=1$

Taking the inverse FT

$$\begin{aligned}\delta(x - x_0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx_0} e^{ikx} dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x_0)} dk\end{aligned}$$

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$$

## □ Discrete Fourier Transform (DFT)

of signals  $\mathbf{x}$  is

$$[x_0] \quad \dots \quad [x_{N-1}] = [x] \quad \text{for } N = 8$$

The DFT is the equivalent of the continuous Fourier transform for functions that are sampled discretely with  $N$  equally spaced points  $x_n$  separated by  $\Delta x = \frac{L}{N}$  such that  $x_n = n \Delta x = n \left(\frac{L}{N}\right)$ .

The Fourier transform of the original signal  $f(x)$  would be

$$\tilde{f}(k) = \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

(T7Q) professor wrote something

Let  $N$  samples be

$$f[x_n] = f[0], f[1], \dots, f[N-1]$$

It's a function of  $\alpha$  in T7Q with  
points  $x_n$  are called mesh points.  
 $L = N \Delta \alpha \rightarrow \Delta \alpha = \frac{L}{N}$   
 $\Delta \alpha = \frac{L}{N} = n \left( \frac{L}{N} \right)$   
 $k_m = m \Delta k$ , where  $\Delta k = \frac{\pi}{L}$  &  $k_m = \frac{m\pi}{L}$

Given function  $f(\alpha)$  is zero outside the  
region  $\alpha \in [0, N]$ , we can express its  
Fourier transform as:

$$\tilde{f}(k) = \int_0^N f(\alpha) e^{-ik\alpha} d\alpha$$

The  
prop  
valu  
%

The integral can be approximated by summing over the N-sampled values:

$$\begin{aligned}
 \tilde{f}(k_m) &\approx \sum_{n=0}^{N-1} f(x_n) e^{-ik_m x_n} \Delta x \\
 &= \sum_{n=0}^{N-1} f(x_n) e^{-ik_m x_n} \cdot \left(\frac{L}{N}\right) \\
 &= \frac{L}{N} \sum_{n=0}^{N-1} f(x_n) e^{-ik_m x_n} \\
 &\quad S[x_n] + \sum_{n=0}^{N-1} = [x_n] \neq
 \end{aligned}$$

The discrete Fourier transform (DFT) of the sequence  $x_n$  is given by

$$f[k_m] = \sum_{n=0}^{N-1} f[x_n] e^{-ik_m x_n}$$

The value of DFT is (up to the constant of proportionality  $\frac{L}{N}$ ) an approximation of the value of the Fourier transform at the frequency  $\frac{n}{L}$ .

je determinanten niet begrijpt dat  
zonder behulp van de rekenmachine

$$e^{-ik_m \Delta x} = e^{-i \frac{2m\pi}{L} \cdot n \Delta x} = e^{-i \frac{2m\pi}{N} \cdot n \cdot \frac{k}{N}}$$
$$\left(\frac{1}{N}\right) \cdot e^{-i \frac{2m\pi}{N} \cdot n} = e^{i \frac{2m\pi}{N} \cdot n}$$

$$f[k_m] = \sum_{n=0}^{N-1} f[x_n] e^{-ik_m \Delta x}$$
$$= \sum_{n=0}^{N-1} f[x_n] e^{-i \frac{2m\pi}{N} \Delta x}$$
$$= \sum_{n=0}^{N-1} f[x_n] e^{-i \frac{2m\pi}{N} n}$$

$$S[x_n] + \sum_{n=0}^{N-1} = [x_n] f$$

zo (TIC) moet voor elke  $n$  de  $i \frac{2m\pi}{N} n$  tekenen en dan de  $f[x_n]$  ophangen  
zo dat de  $i \frac{2m\pi}{N} n$  de  $i \frac{2m\pi}{N} n$  is en de  $f[x_n]$  de  $f[x_n]$  is

Similarly,

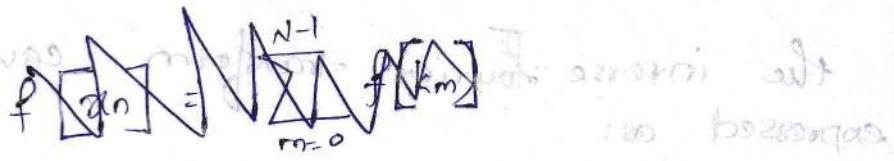
the inverse Fourier transform can be expressed as:

$$f(x) = \frac{1}{2\pi} \int_0^N \tilde{f}(k) e^{ikx} dk = [ab] f$$

the integral can be approximated by summing over the  $N$ -sampled values:

$$\begin{aligned} f(x_0) &\approx \frac{1}{2\pi} \sum_{m=0}^{N-1} f(k_m) e^{ik_m x_0} \delta k \\ &= \frac{1}{2\pi} \sum_{m=0}^{N-1} f(k_m) e^{ik_m x_0} * \frac{2\pi}{L} \\ &= \frac{1}{L} \sum_{m=0}^{N-1} f(k_m) e^{ik_m x_0} [ab] f \end{aligned}$$

Substitute,  $f(k_m) = \frac{L}{N} f[k_m]$



$$f[an] = \sum_{m=0}^{N-1} f[km] e^{ik_m a_0}$$

$$f[an] = \frac{1}{N} \sum_{m=0}^{N-1} f[km] e^{ik_m a_0}$$

(B)

The inverse DFT is:

$$f[an] = \frac{1}{N} \sum_{m=0}^{N-1} f[km] e^{-ik_m a_0}$$

$$[an] + \frac{1}{N} = (an) +$$

$$f[\alpha_n] = \frac{1}{N} \sum_{m=0}^{N-1} f[k_m] e^{ik_m \alpha_n}$$
$$= \frac{1}{N} \sum_{m=0}^{N-1} f[k_m] e^{i \frac{2m\pi}{N} \alpha_n}$$
$$= \frac{1}{N} \sum_{m=0}^{N-1} f[k_m] e^{i \frac{2m\pi}{N} n}$$

□ ~~ABM~~ Change of Basis

infinitesimal change of basis  
DFT of  $\alpha_m$  in rotation  $\frac{2\pi m}{N}$  of  $f$

$$f[\alpha_m] = \sum_{n=0}^{N-1} f[\alpha_n] e^{\frac{2\pi i m n}{N}}$$

The  $I-DFT$  is:

$$f[\alpha_n] = \frac{1}{N} \sum_{m=0}^{N-1} f[\alpha_m] e^{-\frac{2\pi i m n}{N}}$$

$$\begin{bmatrix} 5 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ \vdots \\ -1/2 \end{bmatrix}$$

Reck

A discrete function of finite duration  
consisting of  $N$ -samples can be thought  
of as a vector in  $\mathbb{C}^N$ .

$$\Theta[\omega] \neq \sum_{n=0}^{N-1} = [\omega_n]$$

$$\begin{aligned}
 f[\alpha_n] &= f[0] e_0 + f[1] e_1 + \dots + f[N-1] e_{N-1} \\
 &\stackrel{n \text{ samples}}{\approx} S[\omega] \neq \sum_{n=0}^{N-1} \frac{1}{\sqrt{N}} = [\omega_n]
 \end{aligned}$$

$$= \sum_{k=0}^{N-1} f[k] \delta[\alpha_n - k]$$

$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{bmatrix} = \begin{bmatrix} f[0] \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f[1] \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f[2] \\ \vdots \\ 0 \end{bmatrix} + \dots + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ f[N-1] \end{bmatrix}$$

Real space basis functions:  $e_{\alpha_k} = \delta(\alpha - \alpha_k)$

$$c_k = (e_{\alpha_k}, f(\alpha)) = \int_{-\infty}^{+\infty} f(\alpha) \delta(\alpha - \alpha_k) d\alpha = f(\alpha_k)$$

$$\begin{aligned} (e_{\alpha_0}, e_{\alpha_1}) &= \int_{-\infty}^{+\infty} \delta(\alpha - \alpha_0) \delta(\alpha - \alpha_1) d\alpha \\ &= \int_{-\infty}^{+\infty} \delta(\alpha - \alpha_0) \delta(\alpha - \alpha_1) d\alpha = \delta(\alpha_0 - \alpha_1) \end{aligned}$$

$\Rightarrow e_{\alpha_k} = \delta(\alpha - \alpha_k)$  are orthonormal.

The DFT is used to represent the vector in  $\mathbb{C}^N$ , corresponds to a discrete function of finite length consisting of  $N$  samples, as a linear combination of vectors  $V_k \in \mathbb{C}^N$  of the form:

$$V_k = \left( 1, e^{i \frac{2\pi k}{N}}, e^{-i \frac{2\pi k}{N}}, \dots, e^{-i \frac{2\pi (N-1)k}{N}} \right)$$

and now TAC will be,  $R = 0, 1, \dots, N-1$   
with respect to  $k$  from 0 to  $N-1$   
need only in robust D-polymerase

$$\langle V_l, V_k \rangle = \sum_{n=0}^{N-1} e^{i \frac{2\pi ln}{N}} e^{-i \frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} e^{i \frac{2\pi (l-k)n}{N}}$$

$$= \frac{1 - e^{i \frac{2\pi (l-k)N}{N}}}{1 - e^{i \frac{2\pi (l-k)}{N}}} = \begin{cases} 1 & \text{for } l \neq k \\ N & \text{for } l = k \end{cases}$$

$$f[k_m] = \sum_{n=0}^{N-1} f[x_n] e^{-i k_m x_n} = \sum_{n=0}^{N-1} f[x_n] e^{-i \frac{2\pi m}{N} n} \quad | n = 0, 1, \dots, N-1$$

$$f[x_n] = \frac{1}{N} \sum_{m=0}^{N-1} f[k_m] e^{i k_m x_n} = \frac{1}{N} \sum_{m=0}^{N-1} f[k_m] e^{i \frac{2\pi m}{N} n}$$

so we know what happens if how is DFT

$$\text{DFT}(\tilde{x}_n) = \tilde{f}(k) = \tilde{c}_k = \sum_{n=0}^{N-1} f(x_n) e^{-i \frac{2\pi k}{N} n}$$

so it's equal to  $\tilde{f}(k)$  so it's equal to  $\tilde{c}_k$

so it's equal to  $\tilde{f}(k)$  so it's equal to  $\tilde{c}_k$

so it's equal to  $\tilde{f}(k)$  so it's equal to  $\tilde{c}_k$

$$\tilde{f}(k) = \langle x, v_k \rangle$$

The vectors  $\{v_k\}_{k=0}^{N-1}$  form an orthogonal

basis for  $\mathbb{C}^N$ , and the DFT can be understood as computing the coefficients for representing a vector in this basis.

Let  $\tilde{c}_k$  be the coefficients of  $x$  in the basis  $\{v_k\}_{k=0}^{N-1}$  so that

$$x = \tilde{c}_0 v_0 + \tilde{c}_1 v_1 + \dots + \tilde{c}_{N-1} v_{N-1}$$

$$\langle x, v_k \rangle = c_k \langle v_k, v_k \rangle + \sum_{n=1}^{N-1} = [x]_k$$

$$[x]_k + \sum_{n=1}^{N-1} =$$

$$\sum_{n=1}^{N-1} = [x]_k + \sum_{n=1}^{N-1} = [x]_k$$

$$\langle v_k, v_k \rangle = N \quad \text{and} \quad \langle \tilde{f}_1, v_k \rangle = \tilde{f}(k)$$

$$\Rightarrow \tilde{c}_k = \frac{\tilde{f}(k)}{N}$$

circle around  $\alpha = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{f}(k) v_k$  is

$$\alpha = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{f}(k) v_k$$

Discrete Inverse Fourier transform form (IDFT).

(ii)  $\mathcal{F} : \langle V, \langle \cdot, \cdot \rangle \rangle$  bnd  $W = \langle W, \langle \cdot, \cdot \rangle \rangle$

$$\frac{(x)_E}{n} \rightarrow \tilde{x}$$

The  $N$ -point discrete Fourier transform  
( $N$ -pt DFT) is a linear map from  $\mathbb{C}^N$  to  $\mathbb{C}^N$ .

$$Y(k) = \sum_{n=0}^{N-1} x(n) e^{-j 2 \pi k n / N}$$

$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N] \end{bmatrix}$$

color

and

$$f[x_n] = \sum_{m=0}^{N-1} \tilde{f}[k_m] e^{\frac{i \cdot 2\pi m}{N} n} = \sum_{m=0}^{N-1} \tilde{c}_k e^{\frac{i \cdot 2\pi m}{N} n}$$

$\rightarrow \mathbb{C}^N$

$$\begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ \vdots \\ f[N-1] \end{bmatrix} = \tilde{c}_0 \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \tilde{c}_1 \begin{bmatrix} 1 \\ e^{\frac{i \cdot 2\pi}{N}} \\ e^{\frac{i \cdot 4\pi}{N}} \\ \vdots \\ e^{\frac{i \cdot 2\pi(N-1)}{N}} \end{bmatrix} + \tilde{c}_2 \begin{bmatrix} 1 \\ e^{\frac{i \cdot 4\pi}{N}} \\ e^{\frac{i \cdot 8\pi}{N}} \\ \vdots \\ e^{\frac{i \cdot 4\pi(N-1)}{N}} \end{bmatrix} + \dots$$

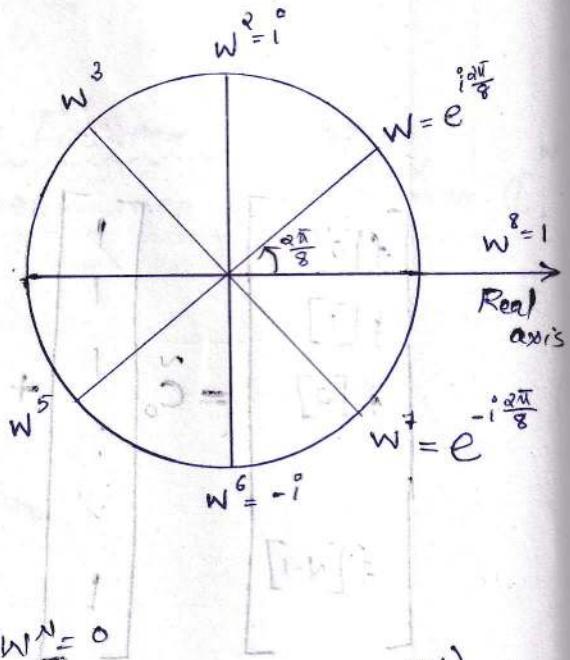
$$= \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W & W^2 & \cdots & W^{N-1} \\ 1 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_{N-1} \end{bmatrix} = F C$$

where,  $W = e^{\frac{i \cdot 2\pi}{N}}$ , is the  $N^{\text{th}}$  root of unity

$$\text{and } W^N = W^{2N} = \dots = 1$$

Ex:-

There are 8 solutions to  $z^8 = 1$ . Their spacing is  $45^\circ = \frac{2\pi}{8}$  rad.



$$W^N = 1 \Rightarrow W^{N-1} = 0 \quad 1 - W^N = 0$$
$$\cancel{W^N} = (W-1) \quad 1 - W^N = (1-W)(1+W+W^2+\dots+W^{N-1}) = 0$$
$$\Rightarrow 1 + W + W^2 + \dots + W^{N-1} = 0 \quad \text{& } W^N = 1$$

→ The columns of  $F_w$  are orthogonal.

Find  $\mathbf{g}$  from  $\mathbf{h}_w$  with  $\mathbf{g} = \frac{1}{\sqrt{n}} \mathbf{h}_w$

$$\mathbf{I} = \dots = \frac{1}{\sqrt{n}} \mathbf{h}_w^T \mathbf{h}_w = \frac{1}{n}$$

$$F_C = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & w^{N-1} & w^{2(N-1)} & \dots & w^{(N-1)^2} \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_{N-1} \end{bmatrix}$$

$w$

$$= \begin{bmatrix} \tilde{c}_0 + \tilde{c}_1 + \tilde{c}_2 + \dots + \tilde{c}_{N-1} \\ \tilde{c}_0 + \tilde{c}_1 w + \tilde{c}_2 w^2 + \dots + \tilde{c}_{N-1} w^{N-1} \\ \tilde{c}_0 + \tilde{c}_1 w^2 + \tilde{c}_2 w^4 + \dots + \tilde{c}_{N-1} w^{2(N-1)} \\ \vdots \\ \tilde{c}_0 + \tilde{c}_1 w^{N-1} + \tilde{c}_2 w^{2(N-1)} + \dots + \tilde{c}_{N-1} w^{(N-1)^2} \end{bmatrix}$$

\*  $F$  is symmetric ( $F^T = F$ ), but not Hermitian.

$\frac{1}{\sqrt{2}}F$  is a unitary matrix, i.e.,  $\left(\frac{1}{\sqrt{2}}F^H\right)\left(\frac{1}{\sqrt{2}}F\right) = I$

$$F^{-1} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & w & w^2 & \dots & w^{N-2} & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-2)} & w^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & w^{(N-1)} & w^{2(N-1)} & \dots & w^{(N-1)^2} & w \end{bmatrix} = \frac{\bar{F}}{N}$$

estimate for  $\hat{f}_j$  and  $(\hat{f}_j - f_j)$  variance

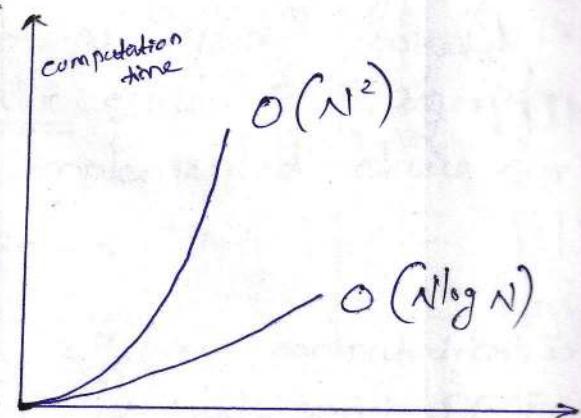
$$\hat{f}_j = \left(\hat{f}_{j1}\right) \left(\hat{f}_{j2}\right) \dots \left(\hat{f}_{jN}\right)$$

for

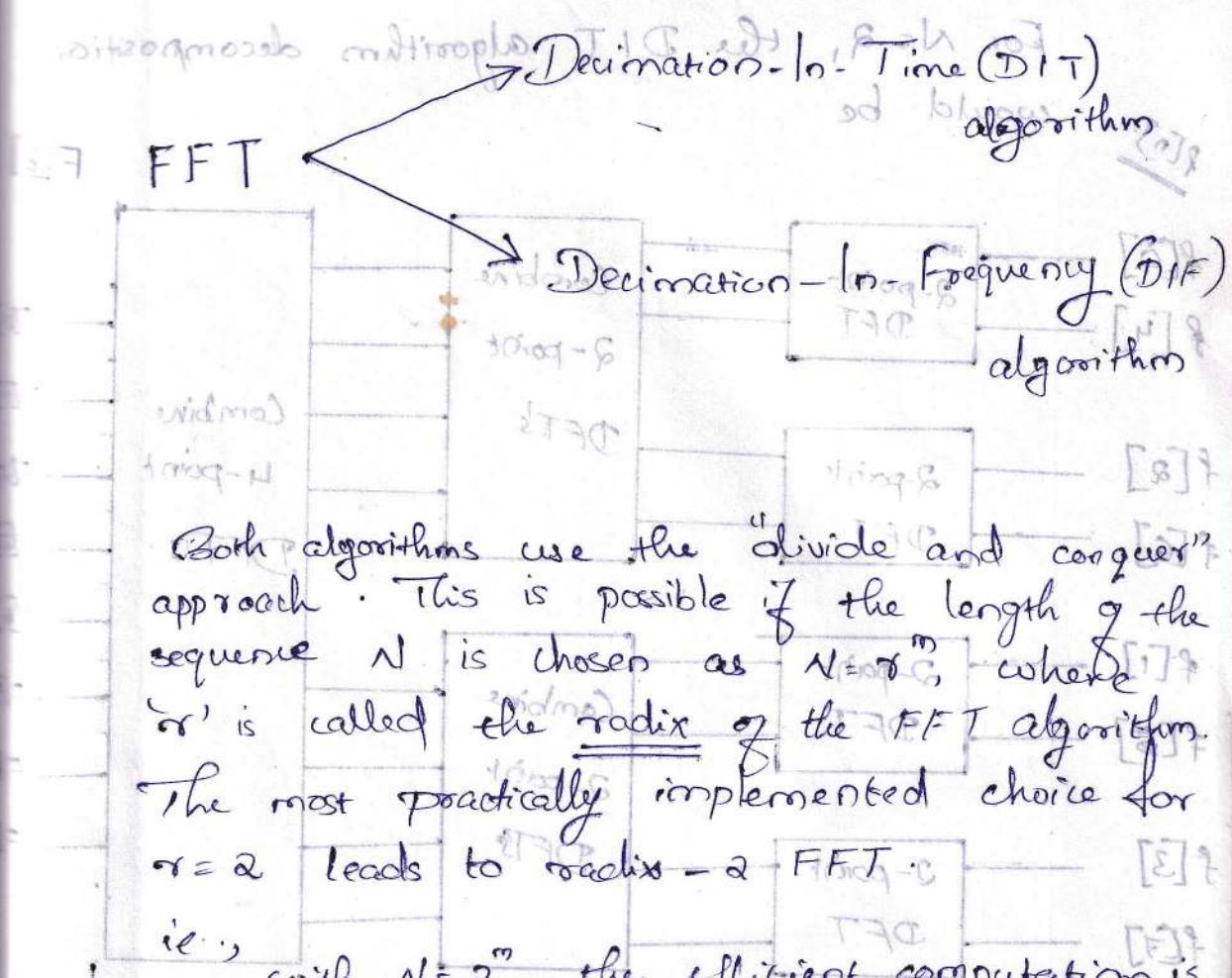
## Fast Fourier Transform (FFT)

The Fast Fourier Transform (FFT) is an efficient  $O(N \log N)$  algorithm for calculating DFTs.

- originally discovered by Gauss in the early 1800's
- rediscovered by Cooley and Tukey at IBM in the 1960's.

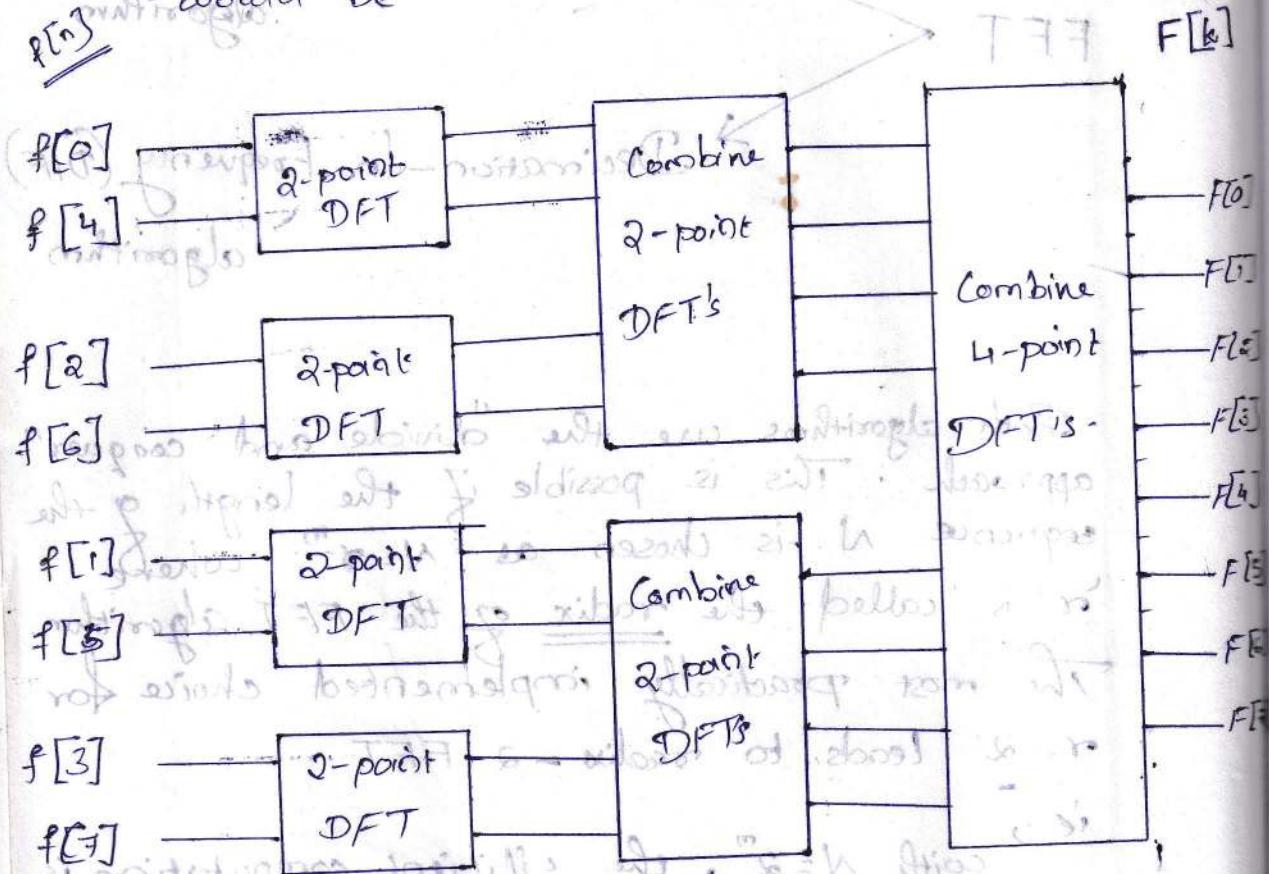


$N$	10	100	1000	$10^6$	$10^9$
$N^2$	100	$10^4$	$10^6$	$10^{12}$	$10^{18}$
$N \log N = \log N^2$	1	200	3000	$6 \times 10^6$	$9 \times 10^9$



Both algorithms use the "divide and conquer" approach. This is possible if the length of the sequence  $N$  is chosen as  $N = 2^m$ , where ' $r$ ' is called the radix of the FFT algorithm. The most practically implemented choice for  $r = 2$  leads to radix - 2 FFT. i.e., with  $N = 2^m$ , the efficient computation is achieved by breaking the  $N$ -point DFT into two  $\frac{N}{2}$ -point DFTs, then breaking each  $\frac{N}{2}$ -point DFT into two  $\frac{N}{4}$ -point DFTs and continuing this process until 2-point DFTs are obtained.

(For  $N=8$ , the DIT algorithm decomposition would be)



$T_{\text{FFT}} = \frac{N}{2} \log_2 N + N$

$T_{\text{FFT}} = \frac{N}{2} \log_2 N + N$

## A Decimation-In-Time (DIT) algorithm

$$W[n] = \sum_{k=0}^{N-1} f[k] W[k]$$

The time-domain sequence  $f[n]$  is decimated into two  $\frac{N}{2}$ -point sequences, one composed of even indexed values of  $f[n]$ , and the other composed of odd indexed values of  $f[n]$ . i.e.,

$$g[n] = f[2n]$$

$$h[n] = f[2n+1]$$

The  $N$ -point DFT of  $f[n]$  is given by

$$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j \frac{2\pi k n}{N}}$$

$$= \sum_{n=0}^{N-1} f[n] W_N^n$$

Twiddle factor

$$F[k_m] = \sum_{n=0}^{N-1} f[n] W_N^{nm}$$

unit - of - non - periodic

$$= \sum_{n, \text{even}} f[n] W_N^{nm} + \sum_{n, \text{odd}} f[n] W_N^{nm}$$

bottom row

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_N^{2nm} + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_N^{(2n+1)m}$$

top row

example taking  $\frac{N}{2}$  out  $W_N^{2nm} \times W_N^m$

odd bus  $\rightarrow$   $f[2n+1]$  is zero because  $n$  is even

center

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_N^{2nm} + \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_N^{2nm} \cdot W_N^m$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_N^{2nm} + W_N^m \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_N^{2nm}$$

for now

$$W_N^{2nm} = e^{-j \frac{2\pi n}{N} (2n)} = e^{-j \frac{2\pi n}{N/2} n} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_{N/2}^{nm} + W_N^m \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_{N/2}^{nm}$$

rotate subint

$$F[k_m] = \sum_{n=0}^{\frac{N}{2}-1} g[n] W_N^{nm} + W_N^n \sum_{n=0}^{\frac{N}{2}-1} h[n] W_N^{nm}$$

~~$[m] H [n] W + [m] R = [m]$~~

$$= G[k_m] + W_N^n H[k_m]$$

~~$[m] H [n] W - [m] R = [m]$~~

where,

$G[k_m]$  and  $H[k_m]$  are the  $\frac{N}{2}$ -point DFTs  
of  $g[n]$  and  $h[n]$  respectively.

$$W_N^{m+\frac{N}{2}} = e^{-i \frac{2(m+\frac{N}{2})\pi}{N}} = e^{-i \frac{2m\pi}{N}}$$

$$e^{i\pi} = -e^{i\frac{2m\pi}{N}} = -W_N^m$$

~~for perfect overlap of opposite phases and \*~~

$$W_{N/2}^{n(m+\frac{N}{2})} = e^{-i \frac{2(n+\frac{N}{2})\pi}{N/2}} = e^{-i \frac{2n\pi}{N/2}} e^{-i 2\pi n} = e^{-i \frac{2n\pi}{N/2}} = 1$$

$$= W_{N/2}^{nm}$$

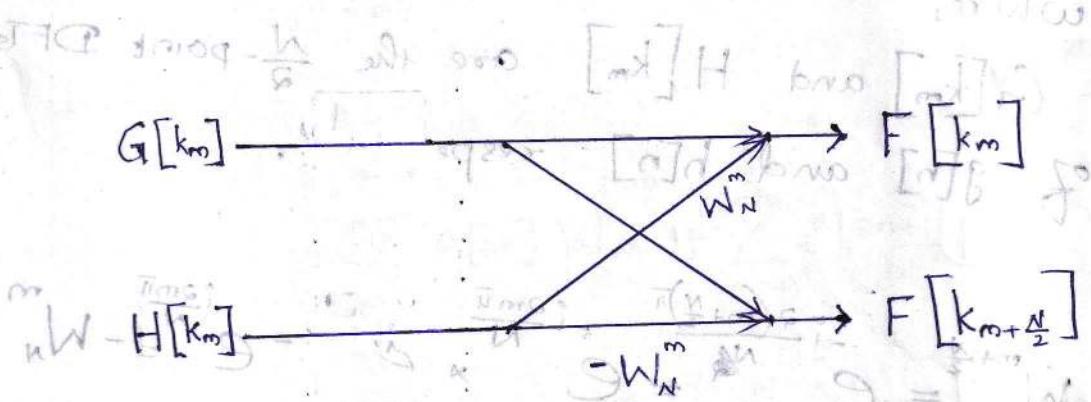
$$F[k_{m+\frac{N}{2}}] = \sum_{n=0}^{\frac{N}{2}-1} g[n] W_{N/2}^{nm} + W_N^n \sum_{n=0}^{m+\frac{N}{2}-1} h[n] W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} g[n] W_{N/2}^{nm} + W_N^n \sum_{n=0}^{\frac{N}{2}-1} h[n] W_{N/2}^{nm}$$

$$= G[k_m] - W_N^n H[k_m]$$

$$F[k_m] = G[k_m] + W_N H[k_m]$$

$$F[k_{m+\frac{N}{2}}] = G[k_m] - W_N H[k_m]$$



\* The butterfly stage of Cooley-Tukey DIT FFT

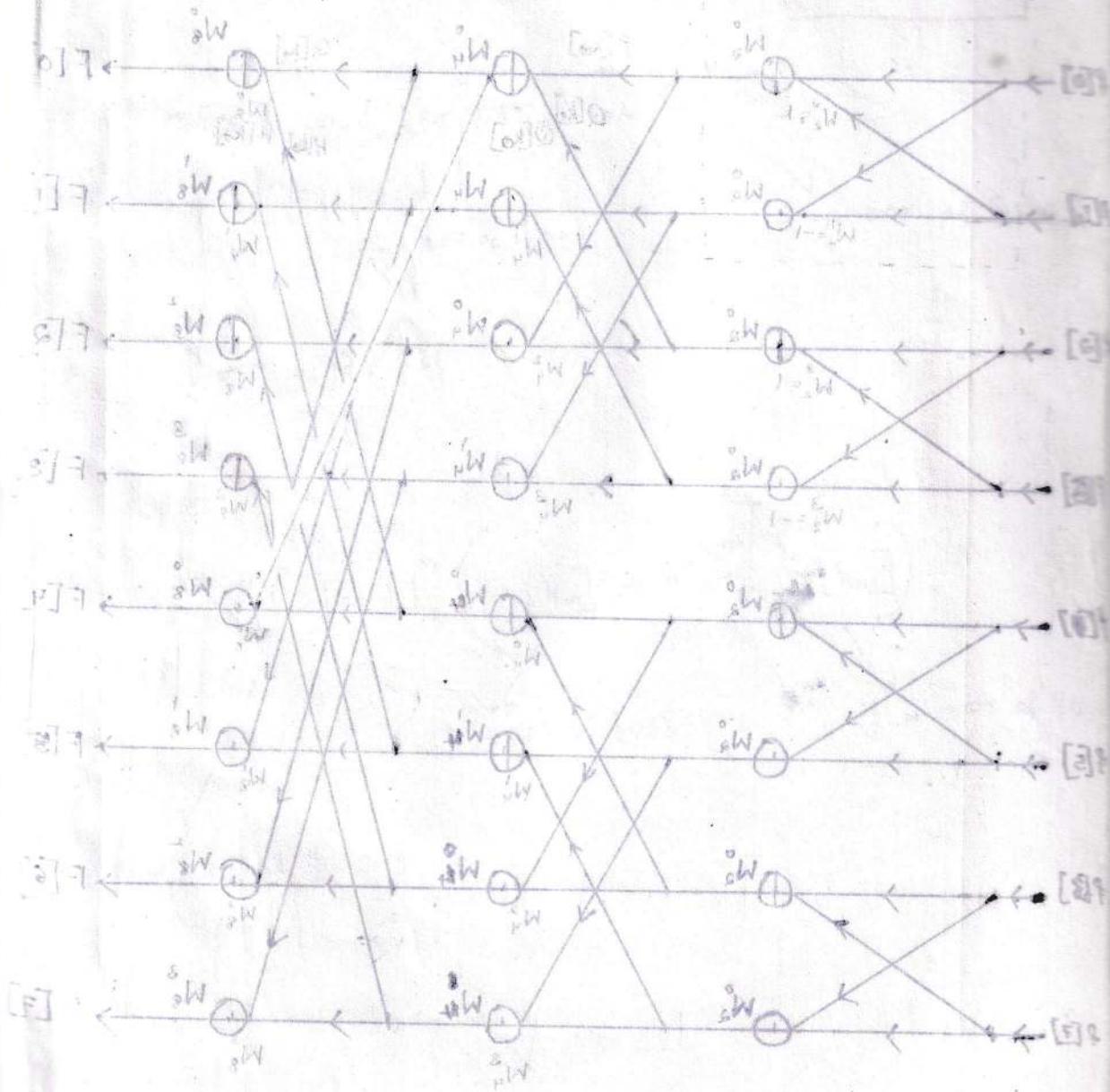
$$\text{Diagram showing butterfly stages: } G[k_m] \rightarrow F[k_m] \text{ and } G[k_m] \rightarrow F[k_{m+\frac{N}{2}}]$$

$$\sum_{n=0}^{\frac{N}{2}-1} W_n e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{\frac{N}{2}-1} W_n e^{-j\frac{2\pi}{N}(k+\frac{N}{2})n} = F[k_m]$$

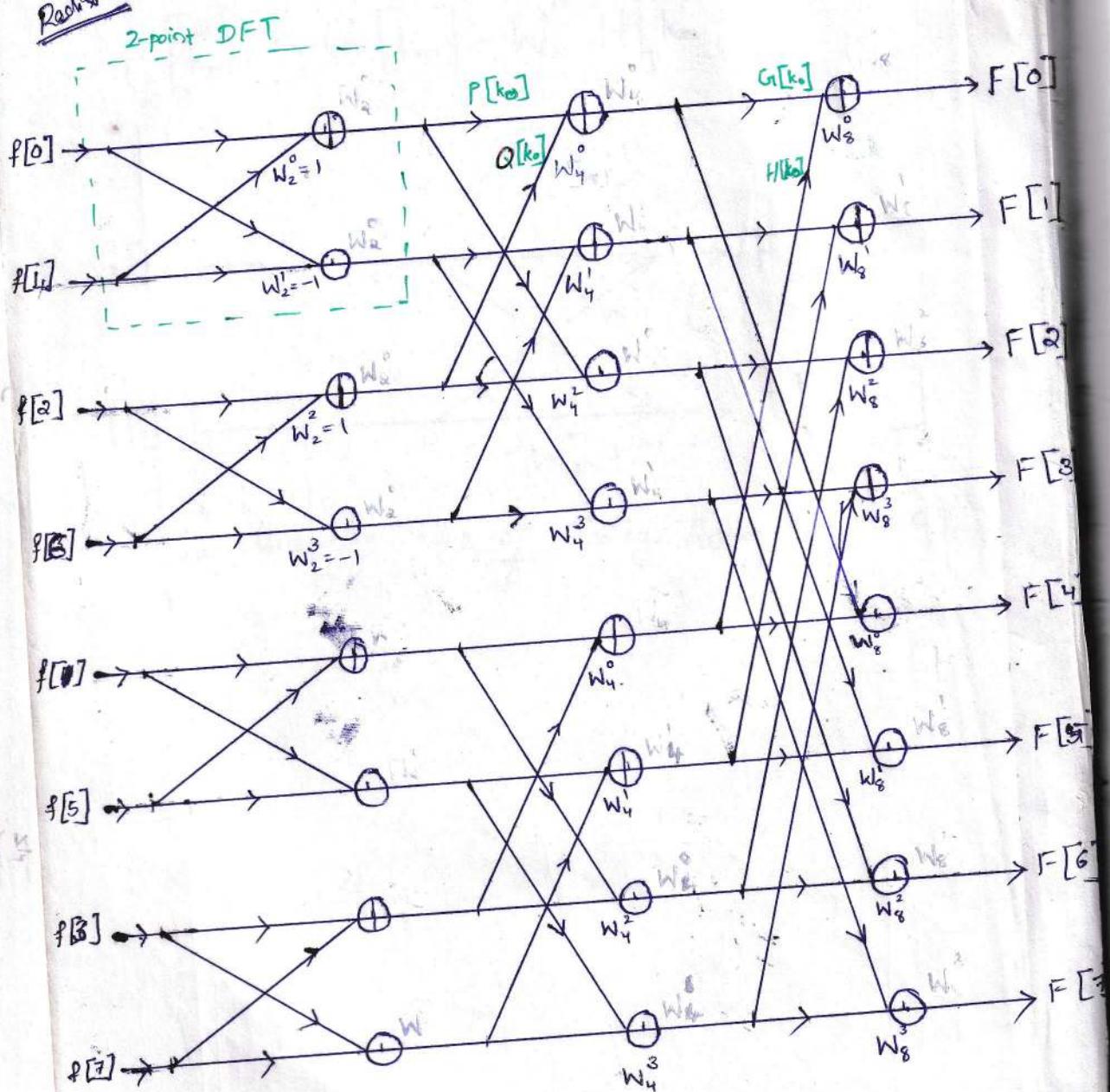
$$\sum_{n=0}^{\frac{N}{2}-1} W_n e^{-j\frac{2\pi}{N}kn} - \sum_{n=0}^{\frac{N}{2}-1} W_n e^{-j\frac{2\pi}{N}(k+\frac{N}{2})n} = F[k_{m+\frac{N}{2}}]$$

$$[m] H [n] W - [m] G =$$

The process is repeated till we get 2-point DFTs.



\* The overall butterfly diagram for DIT FFT  
algorithm for  $N=8$  is:



$$F[k_m] = G[k_m] + W_N^m H[k_m]$$

$\downarrow$

$\frac{N}{2}$ -pt DFT of  $\{z_{0,2,4,6}\}$

$\downarrow$

$N$ -pt DFT of  $\{z_{0,1,2,3,4,5,6,7}\}$ .

$$W_N^m = e^{-\frac{2\pi i m}{N}}$$

$$G[k_m] = P[k_m] + W_{N/2}^m Q[k_m]$$

$\downarrow$

$\frac{N}{4}$ -pt DFT of  $\{z_{0,4}\}$

$\downarrow$

$\frac{N}{2}$ -pt DFT of  $\{z_{0,2,4,6}\}$

+

$$W_N^m H[k_m] = W_N^m \left( R[k_m] + W_{N/2}^m S[k_m] \right)$$

$\downarrow$

$\frac{N}{4}$ -pt DFT of  $\{z_{1,3}\}$

$\downarrow$

$\frac{N}{4}$ -pt DFT of  $\{z_{5,7}\}$

$\downarrow$

$\frac{N}{2}$ -pt DFT of  $\{z_{1,3,5,7}\}$

F

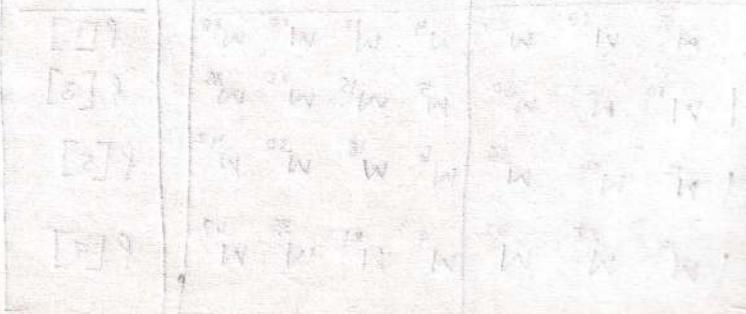
$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N} mn} = \sum_{n=0}^{N-1} f[n] W_N^{nm}$$

$$= \sum_{\substack{n=0 \\ (\text{even } n)}}^{\frac{N}{2}-1} f[2n] W_N^{2nm} + \sum_{\substack{n=0 \\ (\text{odd } n)}}^{\frac{N}{2}-1} f[2n+1] W_N^{(2n+1)m}$$

$$W_N^{2nm} = e^{-j\frac{2\pi}{N} m(2n)} = e^{-j\frac{2\pi}{N} nm} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_{N/2}^{nm} + W_N^m \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_{N/2}^{nm}$$

→ Any DFT can be constructed from the sum of 2 smaller DFTs. This implies that we can attack the problem using the "divide & conquer" approach.



$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-j\frac{2\pi}{N} k_m n} = \sum_{n=0}^{N-1} f[n] W_N^n$$

$$W_N^{nm} = e^{-2\pi i m (2n)/N} = e^{-2\pi i mn/N} = W_{N/2}^{nm}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[2n] W_{N/2}^{nm} + W_n^m \sum_{n=0}^{\frac{N}{2}-1} f[2n+1] W_{N/2}^{nm}$$

→ Any DFT can be constructed from the sum of 2 smaller DFTs. This implies that we can attack the problem using the "divide & conquer approach.

Ex:-

 $N=8$  point DFTApp  
block

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \\ F(4) \\ F(5) \\ F(6) \\ F(7) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W & W^2 & W^3 & W^4 & W^5 & W^6 & W^7 \\ 1 & W^2 & W^4 & W^6 & W^8 & W^{10} & W^{12} & W^{14} \\ 1 & W^3 & W^6 & W^9 & W^{12} & W^{15} & W^{18} & W^{21} \\ 1 & W^4 & W^8 & W^{12} & W^{16} & W^{20} & W^{24} & W^{28} \\ 1 & W^5 & W^{10} & W^{15} & W^{20} & W^{25} & W^{30} & W^{35} \\ 1 & W^6 & W^{12} & W^{18} & W^{24} & W^{30} & W^{36} & W^{42} \\ 1 & W^7 & W^{14} & W^{21} & W^{28} & W^{35} & W^{42} & W^{49} \end{bmatrix} \begin{bmatrix} f[0] \\ f[1] \\ f[2] \\ f[3] \\ f[4] \\ f[5] \\ f[6] \\ f[7] \end{bmatrix}$$

Reordering the matrix to the DIT split in the equation, separating the even & odd index samples:

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \\ F[4] \\ F[5] \\ F[6] \\ F[7] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & W^2 & W^4 & W^6 & W & W^3 & W^5 & W^7 \\ 1 & W^4 & W^8 & W^{12} & W^2 & W^6 & W^{10} & W^{14} \\ 1 & W^6 & W^{12} & W^{18} & W^3 & W^9 & W^{15} & W^{21} \\ 1 & W^8 & W^{16} & W^{24} & W^4 & W^{12} & W^{20} & W^{28} \\ 1 & W^{10} & W^{20} & W^{30} & W^5 & W^{15} & W^{25} & W^{35} \\ 1 & W^{12} & W^{24} & W^{36} & W^6 & W^{18} & W^{30} & W^{42} \\ 1 & W^4 & W^{28} & W^{42} & W^7 & W^{21} & W^{35} & W^{49} \end{bmatrix} \begin{bmatrix} f[0] \\ f[2] \\ f[4] \\ f[6] \\ f[1] \\ f[3] \\ f[5] \\ f[7] \end{bmatrix}$$

Applying the same reordering to each  $4 \times 4$  blocks:

$$\begin{bmatrix} F[0] \\ F[1] \\ F[2] \\ F[3] \\ F[4] \\ F[5] \\ F[6] \\ F[7] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W^4 & W^2 & W^6 \\ 1 & W^8 & W^4 & W^{12} \\ 1 & W^{12} & W^6 & W^{18} \\ 1 & W^{16} & W^8 & W^{24} \\ 1 & W^{20} & W^{10} & W^{30} \\ 1 & W^{24} & W^{12} & W^{36} \\ 1 & W^{28} & W^4 & W^{12} \end{bmatrix} \begin{bmatrix} f[0] \\ f[4] \\ f[2] \\ f[6] \\ f[1] \\ f[5] \\ f[3] \\ f[7] \end{bmatrix}$$

$$W^n = W^{n+Nk}$$

$$W^n = -W^{n+\frac{N}{2}}$$

$$W^{nk} = 1$$

$$\begin{cases} W^N = 1 \\ W^{\frac{N}{2}} = e^{-\frac{9\pi}{2}} = -1 \end{cases}$$

$$\begin{bmatrix}
 F[0] \\
 F[1] \\
 F[2] \\
 F[3] \\
 F[4] \\
 F[5] \\
 F[6] \\
 F[7]
 \end{bmatrix}
 = \begin{bmatrix}
 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & -1 & \omega^2 & -\omega^2 & \omega & -\omega & \omega^3 & -\omega^3 \\
 1 & 1 & -1 & -1 & \omega^2 & \omega^2 & -\omega^2 & -\omega^2 \\
 1 & -1 & -\omega^2 & \omega^2 & \omega^3 & -\omega^3 & \omega & -\omega \\
 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
 1 & -1 & \omega^2 & -\omega^2 & -\omega & \omega & -\omega^3 & \omega^3 \\
 1 & 1 & -1 & -1 & -\omega^2 & -\omega^2 & \omega^2 & \omega^2 \\
 1 & -1 & -\omega^2 & \omega^2 & -\omega^3 & \omega^3 & -\omega & \omega
 \end{bmatrix}
 \begin{bmatrix}
 f[0] \\
 f[4] \\
 f[2] \\
 f[6] \\
 f[i] \\
 f[5] \\
 f[3] \\
 f[7]
 \end{bmatrix}$$

Index  
 0  
 1  
 2  
 3  
 4  
 5  
 6  
 7

$$W = W$$

$$W^- = W$$

$$= W$$

If we represent the indices of the  $f[x_n]$  vector as binary numbers, they correspond to the bit-reversed representation of the original indices.

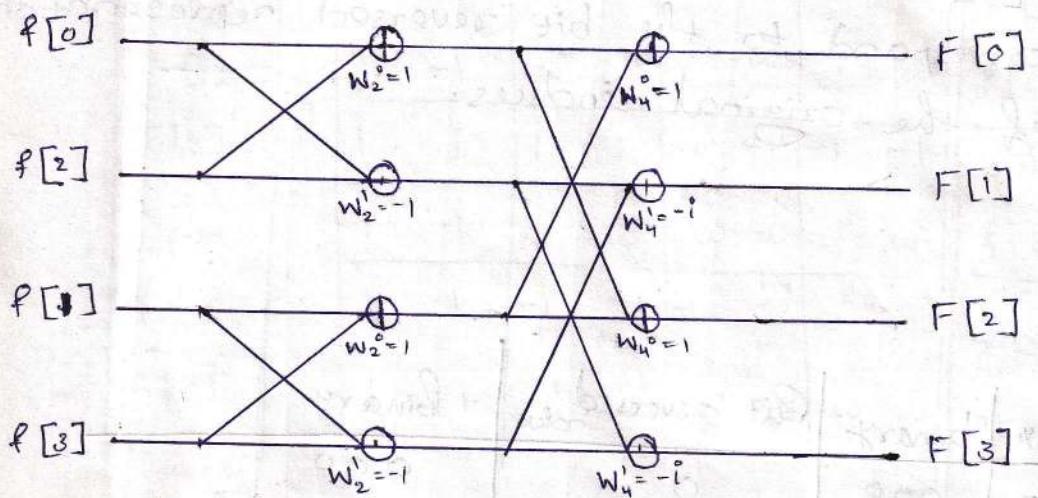
Index	binary	Bit reversed index	Binary
0	000	0	000
1	001	4	100
2	010	2	010
3	011	6	110
4	100	1	001
5	101	5	101
6	110	3	011
7	111	7	111

Ex:-

XXXXXXXXXXXX

DIT FFT algorithm for  $N=4$

By entries  
that



$$F_4 =$$

F-factors  
for FFT

$$F[k_m] = G[k_m] + W_N^m H[k_m]$$

$$F[k_{m+\frac{N}{2}}] = G[k_m] - W_N^m H[k_m]$$

$$F_4 =$$

We want to multiply  $F$  times  $G$  as quickly as possible. Normally a matrix times a vector takes  $n^2$  separate multiplications — the matrix has  $n^2$  entries.

If the matrix has zero entries then multiplication can be skipped. But the Fourier matrix has no zeroes.

Combines the half sized output in a way that the same full size  $F = FG$ .

By using the special pattern  $W^{ik}$  for its entries,  $F$  can be factored in a way that produces many zeros. This is the FFT.

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix} \quad \& \quad F_2 = \begin{bmatrix} 1 & 1 \\ 1 & i^2 \\ 1 & i \\ 1 & i^2 \end{bmatrix}$$

Factors }  
for FFTs }

$$F_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & i \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -i \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & D_2 \\ I_2 & -D_2 \end{bmatrix} \begin{bmatrix} F_2 & 0 \\ 0 & F_2 \end{bmatrix}$$

even-odd  
permutation

Combines the two half sized outputs in a way that produces the same full size output

$$g = f_{\text{rc}}$$

 Performs half sized transforms.  
 $F_2$  &  $F_2$  on the even  $f[n]$  & odd  $f[n]$ 's.

$$\text{When } N = 1024 = 2^{10} \implies \frac{N}{2} = 512$$

-2<sup>11</sup>i / 1024

$$W = e$$

$$F_{1624} = \begin{bmatrix} I_{512} & D_{512} \\ I_{512} & -D_{512} \end{bmatrix} \begin{bmatrix} F_{512} \\ O \end{bmatrix} \begin{bmatrix} O & F_{512} \end{bmatrix} \begin{bmatrix} \text{even-odd} \\ \text{permutation} \end{bmatrix}$$

$D_{512}$ : diagonal matrix with entries  
 $(1, w, w^2, \dots, w^{511})$

## The Full FFT By Recursion

We reduced  $F_N$  to  $F_{N/2}$ . Keep going to  $F_{N/4}$

- Every  $F_{512}$  lead to  $F_{256}$ . Then 256 leads to 128. That is recursion.

$$\begin{bmatrix} F_{512} & 0 \\ 0 & F_{512} \end{bmatrix} = \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \cdot \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \cdot \begin{bmatrix} I & D \\ I & -D \end{bmatrix} \cdot \begin{bmatrix} F & & & \\ & F & & \\ & & F & \\ & & & F \end{bmatrix} \cdot \begin{bmatrix} I & D \\ I & -D \end{bmatrix}$$

pick 0,4,8  
pick 2,6,10  
pick 1,5,9  
pick 3,7,11

- \* The final count for size  $N=2^l$  is reduced from  $N^2$  (for DFT) to  $\frac{N}{2} \log_2 N = \frac{1}{2} N l$  (for FFT)

### Reasoning

There are  $l = \log_2 N$  levels.

Each level has  $\frac{N}{2}$  multiplications from the diagonal D's

$$\implies \text{Final count} = \frac{N}{2} \times l = \frac{1}{2} N l = \underline{\underline{\frac{1}{2} N \log_2 N}}$$

### B Decimation-In-Frequency (DIF) algorithm

Here,

we decimate the DFT sequence  $F[k_m]$  into smaller and smaller subsequences (instead of the time-domain sequence  $f[n]$ ).

$$F[k_m] = \sum_{n=0}^{N-1} f[n] e^{-j \frac{2\pi k_m n}{N}}$$

$$= \sum_{n=0}^{N-1} f[n] W_N^{k_m n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[n] W_N^{k_m n} + \sum_{n=\frac{N}{2}}^{N-1} f[n] W_N^{k_m n}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} f[n] W_N^{k_m n} + \sum_{n=0}^{\frac{N}{2}-1} f\left[n + \frac{N}{2}\right] W_N^{(n+\frac{N}{2})k_m n}$$

$$\begin{aligned} W_N^{(n+\frac{N}{2})k_m n} &= e^{j \frac{2\pi k_m n}{N} (n+\frac{N}{2})} = e^{j \frac{2\pi k_m n}{N} n} e^{j \frac{2\pi k_m n}{N} \frac{N}{2}} = e^{j \frac{2\pi k_m n}{N} n} \\ W_N^{(n+\frac{N}{2})k_m n} &= (-1)^m W_N^{k_m n} \end{aligned}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] + (-1)^m f\left[n + \frac{N}{2}\right] \right\} W_N^{k_m n}$$

$$m = 0, 1, \dots, M-1$$

$$W_{N/2}^{n(m)} e^{-\frac{i 2(\omega m)\pi}{N} n} = e^{-\frac{i \omega m \pi}{N/2} n} = W_{N/2}^{nm}$$

$$W_N^{n(m+1)} = e^{-\frac{i 2(\omega m)\pi}{N} n} \times e^{-\frac{i \omega \pi}{N} n} = W_{N/2}^{nm} W_N^n$$

Orni [Decimating  $F[km]$  into even and odd  
It's g indexed samples, will be two rolls

$$F[k_{am}] = \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] + f\left[n + \frac{N}{2}\right] \right\} W_{N/2}^{nm}$$

$$\Rightarrow [n]_{k=0,1,\dots,\frac{N}{2}-1} = \left[ \frac{N}{2}-1 \right]$$

$$F[k_{am+1}] = \sum_{n=0}^{\frac{N}{2}-1} \left\{ f[n] - f\left[n + \frac{N}{2}\right] \right\} W_n W_{N/2}^{nm}$$

$$uW[n] \sum_{k=0}^{\frac{N}{2}-1} + uW[n] \sum_{k=0,1,\dots,\frac{N}{2}-1} =$$

$$W\left[\frac{n}{2}+n\right] \sum_{o=n}^{\frac{N}{2}} + uW[n] \sum_{o=n}^{\frac{N}{2}} =$$

$$\text{Let, } g[n] = f[n] + f\left[n + \frac{N}{2}\right]; \quad 0 \leq n \leq \frac{N}{2}-1$$

$$h[n] = \left\{ f[n] - f\left[n + \frac{N}{2}\right] \right\} W_N^n; \quad 0 \leq n \leq \frac{N}{2}-1$$

$$uW \left\{ \left[ \frac{n}{2} \right] \left[ \frac{n}{2} + n \right] + [n] \right\} \sum_{o=n}^{\frac{N}{2}} =$$

$$1, N, \dots, 1, 0 = m$$

Substituting,

[5] 7

[6] 7

[7] 7

[8] 7

$$G[k_m] = F[k_{m+n}], \quad 0 \leq k \leq \frac{N}{2}$$

$$\text{and } H[k_m] = F[k_{m+n+1}], \quad 0 \leq k \leq \frac{N}{2}$$

[1] 7

[2] 7

[3] 7

For  $N=8$ ,

$$g[0] = f[0] + f[4]$$

$$g[1] = f[1] + f[5]$$

$$g[2] = f[2] + f[6]$$

$$g[3] = f[3] + f[7]$$

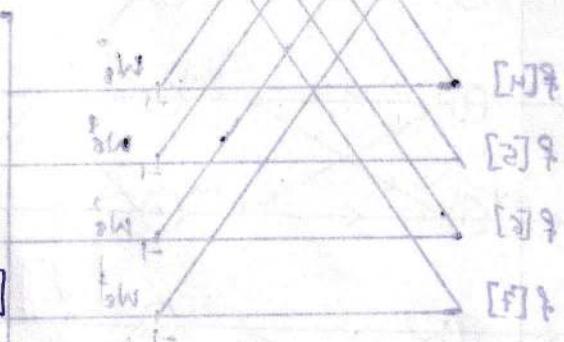
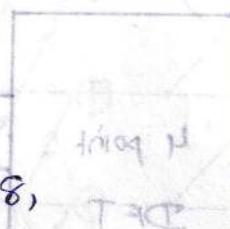
$$h[0] = \{f[0] + f[4]\} W_8^0$$

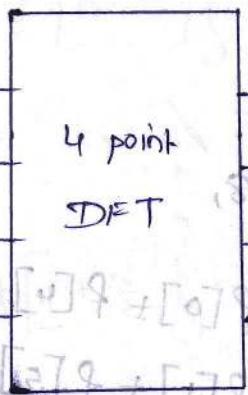
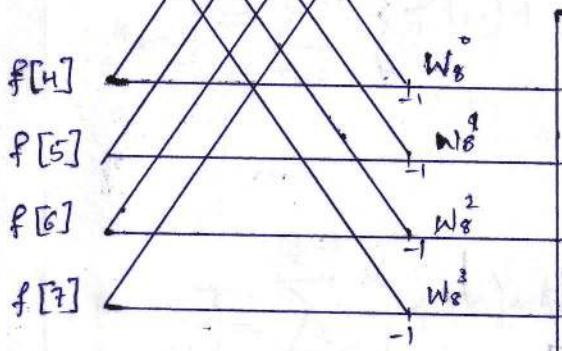
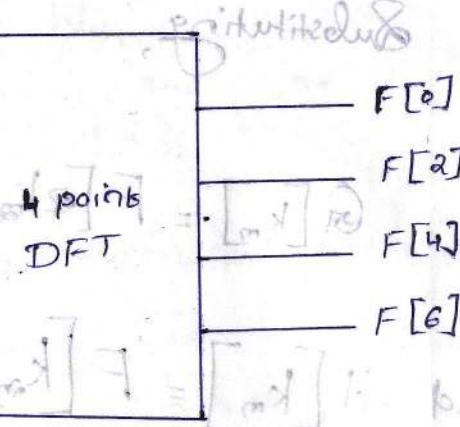
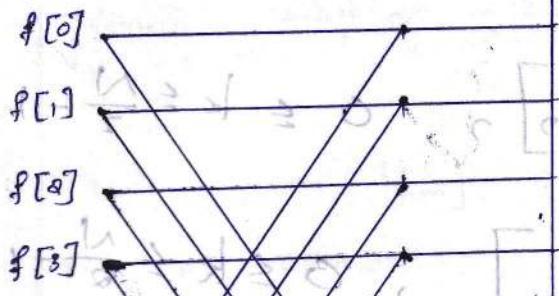
$$h[1] = \{f[1] + f[5]\} W_8^1$$

$$h[2] = \{f[2] + f[6]\} W_8^2$$

$$h[3] = \{f[3] - f[7]\} W_8^3$$

Diagram illustrating the butterfly structure for  $N=8$ . The diagram shows a tree-like structure where each node is the sum of two nodes from the previous level. The levels are labeled  $t_0, t_1, t_2, t_3$  from top to bottom. The nodes at each level are labeled  $t_{0,0}, t_{0,1}, t_{0,2}, t_{0,3}$ ,  $t_{1,0}, t_{1,1}, t_{1,2}, t_{1,3}$ ,  $t_{2,0}, t_{2,1}, t_{2,2}, t_{2,3}$ , and  $t_{3,0}, t_{3,1}, t_{3,2}, t_{3,3}$ .





The above process of elimination is repeated for  $G[km]$  and  $H[km]$  until we reach a 2-point sequence.

$$\begin{aligned} & \text{for } G[km] \text{ and } H[km] \text{ we reach a } \\ & \text{2-point sequence.} \end{aligned}$$

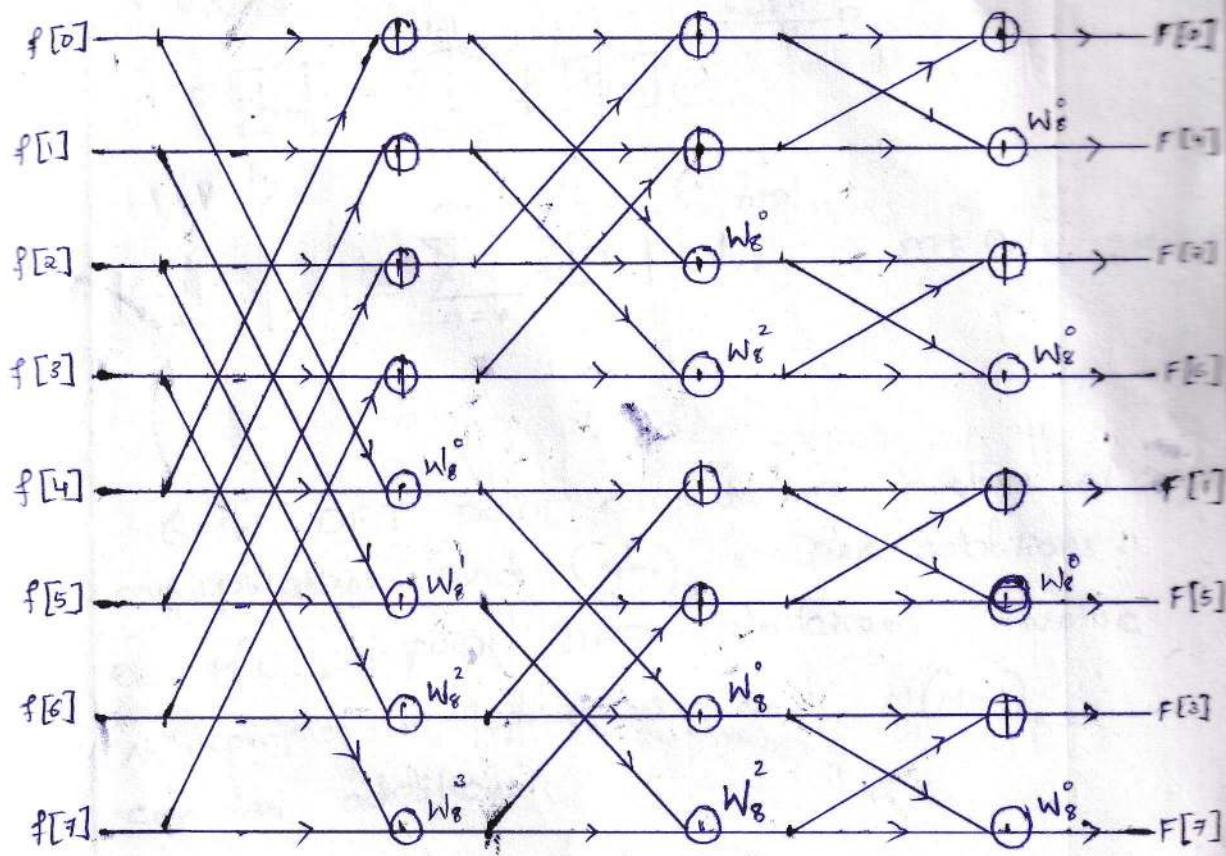
$$W \{ [2]f + [1]g \} = [0]d$$

$$W \{ [2]f - [1]g \} = [1]d$$

$$W \{ [2]f + [2]g \} = [2]d$$

$$W \{ [2]f - [2]g \} = [3]d$$

\* The final butterfly diagram for Decimation-In-Frequency (DIF) FFT algorithm for  $N=8$  is:



## Efficiency of FFT algorithm T 7.7

An  $N$ -point DFT is given by,

~~FFT~~

$$F[k_m] = \sum_{n=0}^{N-1} f[x_n] e^{-j \frac{2\pi m}{N} n}$$

$$f = \sum_{n=0}^{N-1} f[x_n] W_N^n ; m = 0, 1, \dots, N-1$$

So = ~~Efficient~~ does in ~~most~~ efficient algorithms

Each DFT point computation involves  $N$  complex multiplications and  $(N-1)$  complex additions.

Do, the  $N$ -point DFT calculations involve  $N^2$  complex multiplications and  $N(N-1)$  complex additions.

Ex:- 1024 point sequence

$\approx 10^6$  complex multiplication & addition

## FFT multiplies T.F.F of function $\square$

Ex:-

# of stages in a butterfly diagram =  $\log_2 N$

Approx  
to 24

# of butterflies in each stage =  $\frac{N}{2}$

→  
20  
co  
co

# of complex multiplications in each stage  
in  $n \times n$  is  $\frac{n^2}{2}$  Butterflies = 1

\* O is  
saving  
incre

# of complex additions in each butterfly = 2

comes in ~~estimating number of log. T.F.F~~ and

~~number of additions (1-n) times~~ ~~number of additions~~

~~number of additions T.F.F + log - n additions~~

$\therefore (1-n)n$  times ~~number of additions~~  $= \frac{n^2}{2} \log_2 N$

Total # of complex multiplications =  $n \log_2 N$

Total # of complex additions =  $n \log_2 N$

additions & multiplications are  $O(n \log_2 N)$

Ex:- 1024 point sequence

Approx. 5120 complex multiplications and  
10240 complex additions.

→ approx. 100 times less addition and  
200 times less multiplication than direct  
computation of DFT.

- \* As the # of ifp samples increases, the savings in the # of computations also increase.

P.S. 9.3

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

TFA and a brief ↗

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & i \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -i & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & i^2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} I_2 & D_2 & F_2 & 0 \\ I_2 - D_2 & 0 & F_2 & \text{even-odd permutations} \end{bmatrix}$$

easier to invert w/ odd permutations

2. Invert the 3 factors in eq. to find a fast factorization of  $F^{-1}$ .

Ans:

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & i^2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & i^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & -i & 0 & i \end{bmatrix} = \frac{F^H}{4}$$

3.  $F$  is symmetric. So transpose the equation to find a new FFT

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

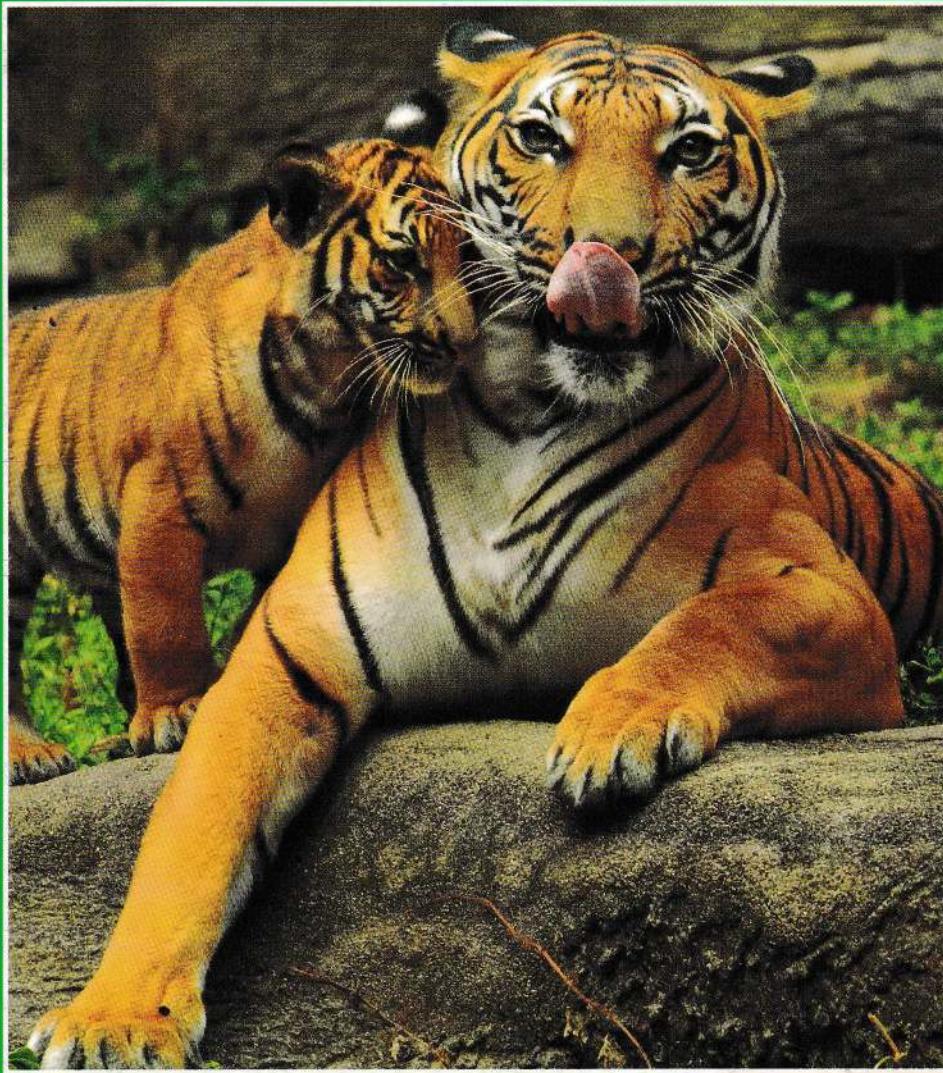
Ans:

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & i^2 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & i^2 & 0 & -i & 0 & i \end{bmatrix}$$

Permutation  
Post

4. All entries in the factorization of  $F_6$  involve powers of  $w_6 = 6^{\text{th}}$  root of 1

$$F = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & i & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & i & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = F_6$$



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