A_n is Simple for $n \geq 5$

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1 Introduction

We will show that A_n is simple for $n \geq 5$. A group is simple when it is nontrivial and there are no normal subgroups besides the trivial group and the group itself. To say n must be greater than 5, we first must look at A_1 through A_4. We know A_1 and A_2 are trivial and therefore not simple groups. Next, A_3 is simple because it has order 3, but A_4 has a normal subgroup, $\{(1), (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, and as a result is not a simple group.

We will prove that A_n is simple for $n \geq 5$ by first proving five lemmas, then the theorem.

2 Preliminary Lemmas

Lemma 1. *For $n \geq 3$, A_n is generated by 3-cycles.*

Proof. The identity $e = (1) = (1\ 2\ 3)(1\ 3\ 2)$ is a product of 3-cycles. Let σ be a non identity element in A_n, $\sigma = \tau_1\tau_2...\tau_r$ where σ is a product of transpositions.

We know that $\text{sign}(\sigma) = 1$ and $\text{sign}(\tau_1\tau_2...\tau_r) = (-1)^r$, thus r must be even.

Now, write the right side as successive transpositions, $\tau_i\tau_i + 1$, where i is odd. Now, we will look at each case of transposition products in S_n:

Case 1: τ_i and $\tau_i + 1$ are equal.

We see that $\tau_i\tau_i + 1 = (1) = (123)(132)$. Therefore, $\tau_i\tau_i + 1$ is the product of two 3-cycles.

Case 2: τ_i and $\tau_i + 1$ have exactly one element in common.

Let the common element be a, so let $\tau_i = (ab)$ and $\tau_i + 1 = (ac)$ where $b \neq c$. From this we have $\tau_i\tau_i + 1 = (ab)(ac) = (acb) = (abc)(abc)$. Therefore, $\tau_i\tau_i + 1$ is the product of two 3-cycles.

Case 3: τ_i and $\tau_i + 1$ are disjoint.

Let $\tau_i = (ab)$ and $\tau_i + 1 = (cd)$. Then $\tau_i\tau_i + 1 = (ab)(cd) = (ab)(bc)(bc)(cd) = (bca)(cdb) = (abc)(bcd)$. Therefore, $\tau_i\tau_i + 1$ is the product of two 3-cycles.

□

Lemma 1.1. *Conjugacy is an equivalence relation.*

Proof. Let $g_1, g_2, g_3, x_1, x_2 \in G$ be arbitrary.

$g_1 = eg_1e^{-1}$, so conjugacy is reflexive.

If $g_1 = x_1g_2x_1^{-1}$, then $g_2 = x_1^{-1}g_1(x_1^{-1})^{-1}$, so conjugacy is symmetric.

If $g_1 = x_1g_2x_1^{-1}$ and $g_2 = x_2g_3x_2^{-1}$, then $g_1 = x_1(x_2g_3x_2^{-1})x_1^{-1} = (x_1x_2)g_3(x_1x_2)^{-1}$, so conjugacy is transitive.

Being reflexive, symmetric, and transitive, conjugacy is an equivalence relation. □

Lemma 1.2. *For $n \geq 5$, all 3-cycles in A_n are conjugate in A_n.*

Proof. Given a 3-cycle (abc),

$$(123) = (1a)(2b)(3c)(abc)(3c)(2b)(1a) = ((1a)(2b)(3c))(abc)((1a)(2b)(3c))^{-1}.$$

If $(1a)(2b)(3c)$ is in A_n, (abc) and (123) are conjugate in A_n. Otherwise,

$$(123) = ((45)(1a)(2b)(3c))(abc)((45)(1a)(2b)(3c))^{-1},$$

so (abc) and (123) are conjugate in A_n. In either case, we find all 3-cycles are conjugate in A_n to (123) and thus to each other. □

Lemma 1.3. *For $n \geq 5$, the conjugate of all 3-cycles in A_n are 3-cycles.*

Proof. Consider $\tau, \sigma \in A_n$, where τ is a 3-cycle (abc). Given $x \in \{a, b, c\}$,

$$\sigma\tau\sigma^{-1}(\sigma(x)) = \sigma(\tau(x)).$$

Thus σ contains the cycle $(\sigma(a)\sigma(b)\sigma(c))$. It remains to show that elements of $\{1, 2, ..., n\} \setminus \{\sigma(a), \sigma(b), \sigma(c)\}$ remain fixed under $\sigma\tau\sigma^{-1}$. Consider such an element n. σ is bijective and $\sigma^{-1}(\{\sigma(a), \sigma(b), \sigma(c)\}) = \{a, b, c\}$, so $\sigma^{-1}(n) \notin \{a, b, c\}$. Thus τ fixes $\sigma^{-1}(n)$ and $\sigma\tau\sigma^{-1}(n) = \sigma\sigma^{-1}(n) = n$, completing the proof. □

Lemma 2. *A_5 and A_6 are simple.*

Proof. If N is a normal subgroup of A_n, the conjugacy classes in A_n contained in N partition N since conjugacy is an equivalence relation and given $\sigma \in N$, $\sigma \in \{\pi\sigma\pi^{-1} \mid \pi \in A_n\} \subseteq N$. The conjugacy classes of A_5 and A_6 are given in the following tables:

Table 1: A_5 Conjugacy Classes

Representative	e	(12345)	(21345)	(12)(34)	(123)
Order	1	12	12	15	20

By Lagrange's theorem, any subgroup of A_5 or A_6 must have an order dividing 60 or 360 respectively. However, if N is a normal subgroup of A_5 or A_6, its order must be the sum of distinct entries including 1 (since N contains e) in the corresponding tables. However, the only such orders possible are 1 and 60, so N must be either trivial or non-proper. Thus A_5 and A_6 are simple. □

Table 2: A_6 Conjugacy Classes

Representative	e	(123)	(123)(456)	(12)(34)	(12345)	(23456)	(1234)(56)
Order	1	40	40	45	72	72	90

3 A_n is simple for $n > 6$

Proof. Suppose $N \trianglelefteq A_n$ be a non-trivial subgroup for $n > 6$. Let σ be a non-identity element of N, i.e., $\sigma(l) \neq l$ for some $l \in \{1, 2, \cdots, n\}$. Let $\tau = (i\ j\ k)$ where $i, j, k \neq l$ and $\sigma(l) \in \{i, j, k\}$. Then,

$$\begin{aligned} \tau\sigma\tau^{-1}(l) &= \tau(\sigma(l)) \neq \sigma(l) \\ \therefore \quad \tau\sigma\tau^{-1} &\neq \sigma \end{aligned} \tag{1}$$

Let $\phi = \tau\sigma\tau^{-1}\sigma^{-1}$ then $\phi \neq (1)$ since $\tau\sigma\tau^{-1} \neq \sigma$. Also, $\tau\sigma\tau^{-1} \in N$ since $\tau \in A_n$ and $\sigma \in N \trianglelefteq A_n$.

$$\sigma, \tau\sigma\tau^{-1} \in N \implies \phi = (\tau\sigma\tau^{-1})\sigma^{-1} \in N \tag{2}$$

Now,

$$\phi = \tau\sigma\tau^{-1}\sigma^{-1} = \tau(\sigma\tau^{-1}\sigma^{-1}) \tag{3}$$

Using lemma 1.3

$$\tau^{-1} \text{ is a 3-cycle} \implies \sigma\tau^{-1}\sigma^{-1} \text{ is also a 3-cycle} \tag{4}$$

That means, $\phi = \tau(\sigma\tau^{-1}\sigma^{-1}) \in N$ is a product of two 3-cycles. Therefore, ϕ permutes at most 6 numbers in $\{1, \cdots, n\}$. Let H be the copy of A_6 inside A_n corresponding to the even permutations of these 6 numbers (augmented to 6 numbers arbitrarily if ϕ permutes fewer than 6 numbers), i.e., $H \cong A_6$. Since ϕ is a product of two 3-cycles, it is an even permutation on these 6 numbers. Therefore,

$$\begin{aligned} \phi \in H \quad \text{and} \quad \phi \in N \quad &and \quad \phi \neq (1) \\ \therefore \quad \phi \in N \cap H \implies &N \cap H \text{ is non-trivial} \end{aligned} \tag{5}$$

Now, given $N \trianglelefteq A_n$ we have $gng^{-1} \in N \quad \forall \quad g \in A_n, n \in N$. For any $h \in H \leq A_n$ and $n \in N \cap H$,

$$\begin{aligned} h \in A_n \quad and \quad n \in N \quad and \quad N \trianglelefteq A_n \\ \therefore \quad hnh^{-1} \in N \end{aligned} \tag{6}$$

$$\begin{aligned} h \in H \implies h^{-1} \in H \quad and \quad n \in N \cap H \implies n \in H \\ \therefore \quad hnh^{-1} \in H \end{aligned} \tag{7}$$

From equations 6 and 7, $hnh^{-1} \in N \cap H \quad \forall \quad h \in H,\ n \in N \cap H$

$$\therefore \quad N \cap H \trianglelefteq H \tag{8}$$

Therefore, from equations 5 and 8, $N \cap H$ is non-trivial and $N \cap H \trianglelefteq H$. Since $H \cong A_6$, which is simple, that only contains the normal subgroups (1) and H. Therefore, $N \cap H \in \{(1), H\}$, and given $N \cap H$ is non-trivial, $N \cap H = H$, and hence $H \subseteq N$.

A_6 contains all the even permutations of our 6 numbers and any 3-cycle is an even permutation. Therefore, A_6 contains 3-cycles. Then,

$$H \cong A_6 \implies H \text{ contains 3-cycles} \tag{9}$$

$$\therefore \quad H \subseteq N \implies N \text{ contains 3-cycles} \tag{10}$$

i.e., each non-trivial subgroup $N \trianglelefteq A_n$ contains a 3-cycle. Then, by lemma 1.3, N contains all 3-cycles. That means, using lemma 1, N contains all elements that generate A_n. Since $N \trianglelefteq A_n$, N must contain all the possible products of the elements that generate A_n. Therefore, N must contain every element of A_n. That means, $N \subseteq A_n$. Also, since $N \trianglelefteq A_n$ we have $N \subseteq A_n$. Combining both gives $N = A_n$, i.e., any non-trivial normal subgroup of A_n for $n > 6$ is A_n itself.[1] $\square$

References

[1] That hilarious paper (1824)

[2] Judson, T. W. (2021). Abstract Algebra: Theory and Applications. Stephen F. Austin State University.