Chapter 4: Lagrangians and equations of motion

1 Introduction

These notes arise because of our discussions today (16 Nov) in which many questions were raised. I had not anticipated such nice questions. Though I was answering, many a time there there was miscommunicaion and more confusion.

So let us lay to rest all the matters, once and for all.

And this is an important chapter. So read it very carfully.

Many questions were asked about equations given arbitrary lagrangians. One of them was just a function of x. Now let me clarify the situation.

2 Structure of The Lagrangian

Our Euler lagrange equations were derived under the assumption that both x, v can be varied ndpendently. It is sufficint to take just one coordinate for our purpose. Hence, L has to necessarily have dependence on v.

Let us return to the actions. Suppose it were just a function of x. Then,

$$S = \int_{t_1}^{t_t} dt L(x(t))$$

because of which, the variational prnciple gives

$$\delta S = 0 \implies \int_{t_1}^{t_t} dt \frac{dL(x)}{dx} \delta x = 0$$

Since th limits of integration are arbitrary and so is the infinitesmal δx, the Euler lagrange equation will simply be

$$\frac{dL}{dx} = 0$$

which is simply an algebraic equation. There is no differential equation. If there were n such coordinates, we sould get s many equations:

$$\frac{\partial L}{\partial x_i} = 0; i = 1, \cdots, n$$

So there is no dynamics at all.

Suppose next that the lagrangian depends linearly on v, apart from a dependence on x. For simplicity, consider one dimension. Then we use the usual E-L equation which we have derived. Notice that

$$\frac{dL}{dv} = f(x); \;\; \frac{\partial L}{\partial x_i} = g(x, v)$$

where g is a linear function of v. So we get the equation

$$\frac{df(x)}{dt} = v\frac{df}{dx} = g(x, v)$$

which will be a first order differential equation in x. This does not serve our purpose. Again, tehre is no dynamics. terms Hence, the minimum requirement is that L have a quadratic depndence (It can also have linear in addition) to get a seccond order differential equation.

There is one important physical consequence: Newtonian relativity does not admit massless particles.

3 More general lagrangians

What if the lagrangian depended on accelerations also? That is, we allowed independent variation of acclerations also. Remember that there are acceleration dependent forces like back reaction because of radiation. Let us see what are the resulting Euler Lagrange equations.

We now have three independent variations: δx, δv, δa, where a is the acceleration. The codition is that there are endpoint variation of both position and velocity. We know how to handle the first two variations. Let us look at the accleration term. We get, for the acceleration dependent term,

$$\delta S_a = \int dt \frac{\partial L}{\partial a}\delta a$$

We need to convert δa to δx by repeated integration by parts. So we write

$$\delta a = \frac{d}{dt^2}\delta x$$

and integrate by parts. I indicate the steps. Let $\frac{\partial L}{\partial a} \equiv G$. Then,

$$G\frac{d}{dt}\delta v \rightarrow \underline{\frac{d}{dt}\Big\{G\delta v\Big\}} - \frac{dG}{dt}\delta v \rightarrow -\underline{\frac{d^2}{dt^2}\Big\{G\delta x\Big\}} + \frac{d^2G}{dt^2}\delta x$$

where the terms underlined vanish because they are total derivative terms. Remembering the form of G, and including the other terms involving x, v, we get the equation

$$\frac{d^2}{dt^2}\left\{\frac{\partial L}{\partial a}\right\} - \frac{d}{dt}\left\{\frac{\partial L}{\partial v}\right\} + \frac{\partial L}{\partial x} = 0$$

So the lesson is that one cannot make blind use of the same equation every where. Now you know what the E-L equation will be even for higher derivative dependence.

4 Rotational Symmetry of the lagrangian

Let me give a more persuasive argument regarding conservation of angular momentum. The steps are the same, but the reasoning is more explicit. Let under a rotation, $\vec{r} \to \vec{r}' = \mathcal{R}(\hat{n}, \theta)\vec{r}$ where $\mathcal{R}$ is the rotation operation. Let $\Delta\vec{r} = \vec{r} - \vec{r}'$ be the difference. We use the notation $\vec{\theta} = \theta\hat{n}$. In a similar manner, $\Delta\vec{r} = \vec{v} - \vec{v}'$ for velocities. Both of them have the same transformation properties.

Rotational symmetry means that the lagrangian, after the transformation (rotation), is independent of $\vec{\theta}$. So, if tehre are N particles, labelled by $\alpha = 1, \cdots, N$,

$$\Delta L = 0 = L(\vec{r}_\alpha', \vec{v}_\alpha', t) - L(\vec{r}_\alpha, \vec{v}_\alpha, t)$$

for *all* possible rotations. In particular, it should be valid for infinitesmal rotations. Under and infinitesmal rotation $\Delta\vec{\theta}$, we have

$$\vec{r}' = \vec{r} + \vec{r} \times \Delta\vec{\theta}; \ \vec{v}' = \vec{v} + \vec{v} \times \Delta\vec{\theta}$$

A Taylor expansion gives us the result that

$$\Delta L = \sum_{\alpha}^{N} \left\{ \frac{\partial L}{\partial \vec{r}_\alpha} \vec{r}_\alpha \times \Delta\vec{\theta} + \frac{\partial L}{\partial \vec{v}_\alpha} \vec{v}_\alpha \times \Delta\vec{\theta} \right\}$$

What remains is to bring the free quantity $\vec{\theta}$ out by rearranging the terms. Secondly we make use of the Euler Lagrange equations. Check that you get a total time derivative to be zero, i.e.,

$$\frac{d}{dt}\left[\sum_{\alpha=1}^{N} \left\{ \vec{r}_\alpha \times \frac{\partial L}{\partial \vec{v}_\alpha} \right\} \right] \cdot \Delta\vec{\theta} \equiv \frac{d\vec{L}}{dt} \cdot \Delta\vec{\theta} = 0$$

Employing the usual arguments, we conclude that the quantity in the paranthesis $\left[\cdots\right]$ is a constant in time. That is, it is conserved. And, of course, it is nothing but the total angular momentum.

Was the reasoning given in the class incorrect? No it is not and let me repeat that argument too. Imagine a sequence of coodinate N systems $C_1, C_2, \cdots C_N$ where any two adjecent coordinate systems are different only by an infinetesmal relative orientation. For N sufficiently large, C_N will differ from C_1 by a finite relative orientation.

What was the argument in the class? If L remains the same in C_i and C_{i+1}, then angular momentum is conserved, obviously, in both the coordinate systems. Now employ recursive reasoning between C_{i+1} and C_{i+2} and so on. Angular momentum is conserved in all the coordinate systems.

We are not asserting that the angular momentum is the same in all the coordinate systems. That would be an absurd statement.

The lesson is that invariance under infinitesmal transformation is sufficient to guarantee the conservation.

5 Compatibility with space translations

Linear momentum conservation is compatible with space translation (trivial exercise). The question, raised in the class, is whether it is true of $\vec{L}$ too. The answer is easy to find. Under a space translation,

$$\vec{r}_\alpha \to \vec{r}_\alpha + \vec{a}; \ \vec{p}_\alpha \to \vec{p}_\alpha$$

Thus

$$\vec{L} \to \vec{L} + \vec{a} \times \vec{P}$$

where $\vec{P}$ is the total momentum of the system. Clearly the conservation holds under space translation if and only if the total momentum of the system ic constant, i.e., there is no external force.

6 Spin

Let us conclude this short chapter by defining spin. It is simply the total angular momentum in the centre of mass frame of the system. Remember that the centre of mass frame is defined by the one in which the total momentum of the system, $\vec{P} = 0$. It is easy to check that spin is invariant under space translations. Moreover, a point particle in classical mechanics can never possess nonvanishing spin. The mystery of quantum mechanics is that even point particles can possess spin.