Classical Mechanics

And Relativity

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2020-21 Semester 1

- Lectures with notes
- Notes sent earlier and
- Recorded lectures posted earlier
- Regular Tutorials (Most essential) and help sessions from your tutors

Online interactions in B slot: 9:30 AM to 11 AM
Exceptions and additional sessions will be notified

More on Plan

- Assignments
  - Formal assignment sheets
  - Problems posed during lectures and discussions
- Technical skill
- conceptual clarity
  - Partly discussed in tutorials/discussions
  - You may form groups with about 5 students in each group and discuss and submit joint solutions

- No single prescribed text book
  - Landau and Lifschitz Vol 1 (for NRCM)
  - Goldstein (for NRCM)
  - Landau and Lifschitz Vol 2 (for relativity) not in great depth

- Attendance automatic
- Submission of assignments not mandatory
- Grade based on Minor and Major performances
  - Minor of 2 hour duration
  - Major of three hour duration
- Both the exams will be open book open internet (tentative plan)

Huge syllabus

- Space Time Symmetries
- Kinematic symmteries
- Newton's Laws of Motion
- Galilean Relativity
    - Contact forces
    - Action at a distance
    - Galilean transformations (with limitations)
    - **The Galilean group***

- Constraints
- The action principle ıtem generalised coordinates
- Nöther symmetries and conservation laws
- Scaling symmetries and Virial theorem

- Central forces
- The $\frac{1}{r}$ potential
- Keplerian Orbits

- Simple harmonic Oscillator
- Coupled Oscillators and normal modes
- Transition to continuum mechanics

- Legendre transformations
- Hamilton's equations and Phase space
- Canonical transformations
- Poisson brackets: Dynamics and Symmetries

Topics Covered V
Rigid Body dynamics

- Moment of inertia tensor
- Body fixed and space fixed axes
- The Top and gyroscopic motion
- Rotating coordinate systems

Topics Covered VII
Hamilton Jacobi equations

- Separation of variables
- Glimpses of perturbation theory

- Incompatibility between Electrodynamics and Galilean transformations
- Michelson Morley and other experiments
- The Lorentz transformations
- Length contraction, time dilatation and relativity of simultaneity
- Four vectors and tensors
- Relativistic dynamics

- The three body problem
- deterministic chaos
- and what not

Scope of Classical Mechanics

- Classical: *not quantum*
  - Newtonian or Nonrelativistic classical mechanics
  - Relativistic Classical Mechancs

Quantum Domain: $\hbar \sim 10^{-34}$ j-s

Einsteinian domain:

$$c = 2.99792458 \times 10^{8} \text{ m/s}$$

Non Quantum $\implies$

$Angular\ momentum\ L \gg \hbar$

Nonrelativistic $\implies$

$Speeds\ v \ll c$

- Gravitational interaction between two neighbouring galaxies
- A star of one solar mass at a distance of 10^6 m from a black hole of 10^6 solar mass
- An electron, at room temperature, in a magnetic field of 1T
- An electron initially at rest, in an electrostatic field of 10^8 V/m
- A system of 10^6 hydrogen atoms in a container at a temperature $10^{-11}K$. Any constraint on container size?
- The interior of a neutron star

Non Relativistic CM

- Large mass
- Weak fields
- High temperatures
- Low pressures

Considering Every Thing

- No preferred location or origin (homogeneity)
- No preferred orientation (isotropy)
- No preferred event to mark time (homogeneity)

D =2+1

- The Euclidean plane $\mathbb{R}^2$
- A sphere of "radius" R
- A torus?
- Cyclic time (Circle)
- Time coordinate given by $\mathbb{R}^1$

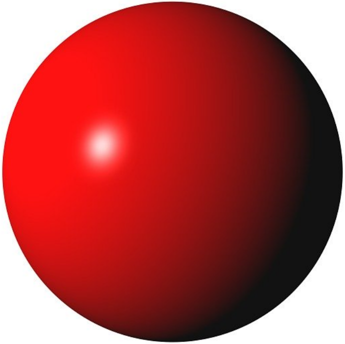


Figure: homogeneous and isotropic

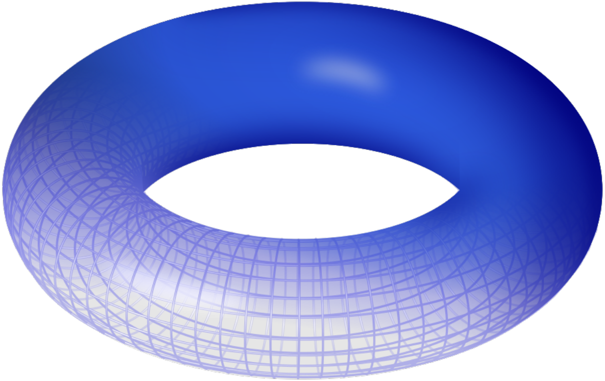


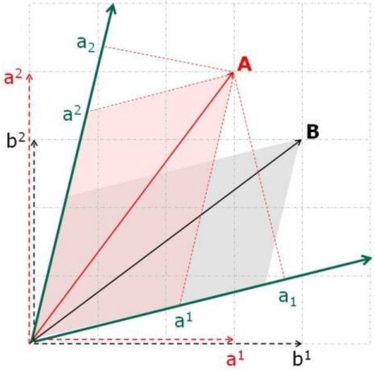
Figure: homogeneous and isotropic?

- Space is three dimensional
- Space is Euclidean. Global rectangular Cartesian coordinate system
- Time is homogeneous

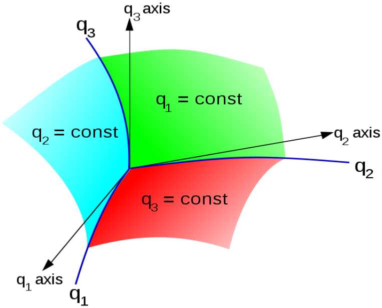
Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then,

$$\begin{aligned} d(P_1, P_2) &= \left\{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right\}^{\frac{1}{2}} \\ &\equiv ||\vec{r}_1 - \vec{r}_2|| \\ \angle(P_1, P_2) &= \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{||\vec{r}_1||\ ||\vec{r}_2||} \equiv \frac{\vec{r}_1 \cdot \vec{r}_2}{||\vec{r}_1||\ ||\vec{r}_2||} \end{aligned}$$

Oblique coordinate system



Credit:
http://www.mysearch.org.uk/website1/html/297.Components.html



Credit:
https://commons.wikimedia.org/w/index.php?curid=2074623

- Lengths of Rods (Distances between points)
- Rates of clocks
- Masses of bodies

Laws of Dynamics

- Forces
- Frames
- Newton's Laws

Basic hypothesis

- Forces are of physical origin
- Prior knowledge
  - Contact forces: Push and Pull
  - Muscular response
  - Intrinsic existence
- Forces are additive (vectors)

$$\vec{F}_t = \vec{F}_1 + \vec{F}_2$$

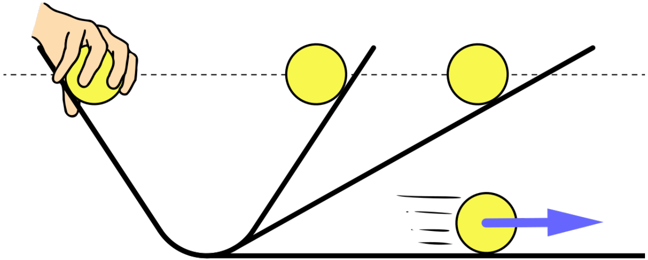
- Forces decrease with separations
- Remove the sources or
- Neutralise

Definition: A body is free if the net force acting on it vanishes

Inertial Frames (*IF*)

Special coordinate systems in which a free body does not accelerate

Corollary: Free bodies have only uniform velocity in *IF*



Credit: https://commons.wikimedia.org/wiki/File:Thought-experiment-on-inertia.svg

Static Transformations T

- Let $\mathscr{C}$ be an *IF*. Then $\bar{\mathscr{C}} = T\mathscr{C}$ is an *IF*.
- Special Case: The Affine Group
  - $\{\mathscr{A}\}$, the set of all linear transformations (translations + rotations + oblique) forms a group, preserving Galilean law of inertia.

$\vec{F}$: an applied force; $\vec{P}$ the momentum of the body.
Then,

$$\frac{d\vec{P}}{dt} = \vec{F}$$

- LHS is purely kinematic (effect)
- RHS is dynamics, known independently (Cause)

Examples

- Lorentz
- Hooke
- Gravitation
- Van der Waals
- Stokes

Inertia or Mass

$$\vec{P} = M(V)\vec{V}$$

The Fundamental Question

$$M(V) = ?$$

- Rate of moving clocks
- Length of moving rods
- Magnitude of uppermost speed (if any)

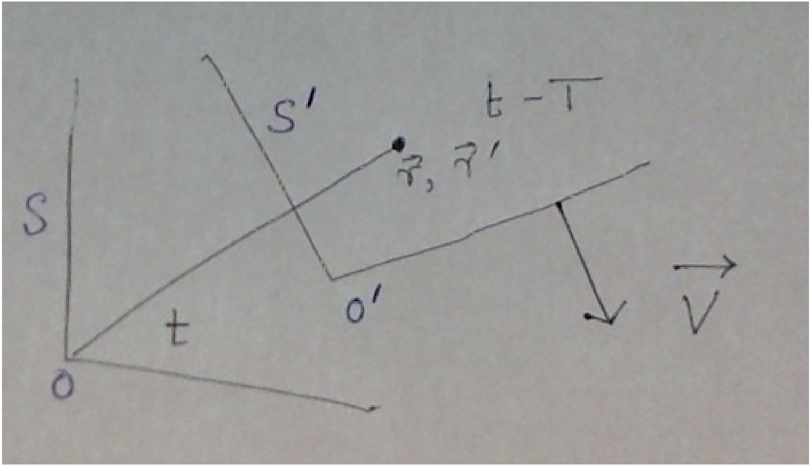
- Space is Eucliden
- Space is homogeneous and isotropic
- Time is isotropic
- Mass length and clock rates do not depend on the state of motion of bodies

Canonical coordinate systems: Rectangular Cartesisn

Affine transformations + transformations to moving frames: $S \to S'$

$$\begin{aligned} t' &= t - T \\ x'_i &= \mathscr{R}_{ij} x_j - R_j - V_i t \end{aligned}$$

Note the choice of the order


S
S'
t - T
$\vec{r}, \vec{r}'$
t
O
O'
$\vec{V}$

- Oscillator: $\vec{F}(\vec{r}) = -k(\vec{r} - \vec{R})$;
- Gravitation: $\vec{F}(\vec{r}) = -\frac{GMm}{|(\vec{r}-\vec{R}|^3}(\vec{r} - \vec{R})$

Facts and assumptions are consistent with Galilean Transformations between inertial frames

Principle of Galilean Relativity

Newton v/s Leibniz
https://plato.stanford.edu/entries/spacetime-theories/

Systems and Constraints

- Unconstrained systems
- Constrained systems
  - Classification
  - Implementation
  - generalised coordinates

N particles labelled $a = 1, \cdots, N$
Respective masses m_a
Mutual interactions
External Forces

Positions : $\vec{r}_a$; $a = 1, \dots, N$

Velocities : $\vec{v}_a$: $a = 1, \dots, N$

Forces : $\vec{F}_{a \to b}$ Force on b due to a

No self interaction.

$$\vec{F}_T^{(a)} = \sum_{b \neq a} \vec{F}_{b \to a} + \vec{F}_{ext}^{a}$$

$$
\begin{aligned}
\vec{F}_T^{(a)} &= \vec{F}_T^{(a)}(\{\vec{r}_b, \vec{v}_b\}); b = 1, \cdots N \\
\vec{F}_{b\to a} &= \vec{F}_{b\to a}(\vec{r}_{ba};\ \vec{v}_{ba}); b \neq a \\
\vec{F}^a_{ext} &= \vec{F}^a_{ext}(\vec{r}_a,\ \vec{v}_a)
\end{aligned}
$$

A set of coupled second order ODE.
Highly Nonlinear
$3N + 3N$ initial conditions

Configuration Space ($\mathscr{C}$)

degrees of freedom, $\mathrm{dof} = 3\mathrm{N} = \dim(\mathscr{C})$

$$\mathscr{C} = \mathbb{R}^{3N}$$

Extend to Momentum Space and Phase space

Examples

- Particle constrained to move within a a sphere
- A bead constrained to move on a rigid wire of some shape
- An insect on the surface of an expanding balloon
- Particle moving between two sticky walls
- Two bodies at the ends of a rigid rod
- Two particles stuck at the two ends of a thread of length L
- A rigid body
- A person on a unicycle (disc or coin on a rough surface)

Definition 1: A constraint is bilateral if it can be expressed as

$$f(t, \vec{r}_a, \vec{v}_a) = 0$$

Definition 2: A constraint is unilateral if it can be expressed as

$$f(t, \vec{r}_a, \vec{v}_a) \leq 0$$

Definition 3: A bilateral constraint is geometric, or finite If f is velocity independent

Definition 4: Else, kinematic

Important Class; constraints linear in velocities:

$$\sum_{1}^{N} \vec{c}_a \cdot \vec{v}a + D = 0$$

Let $f(\vec{r}_a, t) = 0$. Then,

$$\sum_a \nabla_{\vec{r}_a} f \cdot d\vec{r}_a + \frac{\partial f}{\partial t} dt = 0$$

But converse operation yields an arbitrary constant.
A differential constraint need not have finite form.
Integrability conditions (Discussed later)

$$(\text{Scleronomic, Rheonomic})$$
$$\times$$
$$(\text{Holonomic, Non-holonomic})$$
$$\times$$
$$(\textit{Conservative, Non-conservative})$$

Note: Italics in the last line

Scleronomic constraints

Definition: The constraints are time independent

Rheonomic constraints

Definition: The constraints are time dependent

Holonomic constraints

Definition: The coordinates can be given arbitrary infinitesmal displacements without violating constraints

The constraints finite, or integrable

They can be written in closed form

Their general form: $f(\{\vec{r}_a\}, t) = 0$. Remember this.

Definition III

Nonholonomic constraints

Definition: The constraints are not integrable. Not holonomic Not dealt with in detail; Only some examples

Examples

- two masses connected to a deformable rod
- Two masses connected to a rod expanding at a constant rate
- A disc rolling on a rough surface
- two molecules at the tips of a protein chain

Pfaffian Constraints

$$\sum_a \vec{l}_a^{\alpha} \cdot \vec{v}_a = 0; \alpha = 1, \cdots, k$$

Also First and Second class constraints

- $f(t, \vec{r}_a, \vec{v}_a) = 0 \implies$ redundant coordinates may be eliminated (not uniquely).
- *Generalised* coordinates. $\#$ GC $= \#$dof

Lecture 5
Free systems: Generalized coordinates

- Free systems ($D = 2 + 1$)
- Rectangular cartesian coordinates (x, y)
- Polar coordinates (r, θ)
- parabolic corrdinates ($x = \xi_1 \xi_2;\ y = \frac{1}{2}(\xi_1^2 - \xi_2^2)$)
  What is parabolic here?

Note: GC need not have dimension L^1

Example: Time independent Constraints

Geometric constraints ($D = 3 + 1$)

- Particle constrained to move along a line $ax + by + cz + d = 0$
- Particle constrained to move on a sphere $x^2 + y^2 + z^2 = R^2$
- Three particles at the vertices of an equilateral triangle, with one vertex held fixed.
- Same as above but with the centroid held fixed.

Recall Definition:

$$f(t, \{\vec{r}_a\}_1^N) = 0$$

- A particle on an expanding surface
- Two masses at the end of a rod with length of fixed time dependence
- Three bodies at the vertices of a dilating equilateral triangle

Possible displacements

m finite constraints:

$$f_\alpha(t, \{\vec{r}_a\}) = 0; \;\; \alpha = 1, \cdots m \implies$$

m differential constraints

$$\sum_a \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{d}r_a + \frac{\partial f_\alpha}{\partial t} dt = 0$$

Definition:$d\vec{r}_a$ are possible displacements. $\vec{v}_a = \frac{d\vec{r}_a}{dt}$ are possible velocities

Virtual displacements
Differences in possible displacements

Consider $\delta\vec{r}_a = d_1\vec{r}_i - d_2\vec{r}_a; a = 1, \cdots, N$. Then, with only positions involved,

$$\sum_a \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{\delta} r_a = 0$$

Definition: $\delta\vec{r}_a$ are called virtual displacements
Question: When is Virtual =Possible?

A particle at some latitude and longitude

- $d_1\vec{r}$ along the latitude
- $d_2\vec{r}$ along the longitude.

$\delta\vec{r} = d_1\vec{r} - d_2\vec{r}$ is virtual and possible

Sphere translating with velocity $\vec{u}$.

Family of possible velocities:

$$\vec{v} = \vec{v}_t - \vec{u}$$

Virtual $\neq$ Possible.
Repeat for expanding sphere

- System: N Particles. # coordinates: $3N$
- Constraints: $\# = m$ (linearly independent)
- $\#$ dof $= 3N - m$

- $\vec{F}_b$: Force acting on the b^{th} particle
- Newton: $\vec{a}_b \neq \frac{\vec{F}_b}{m_b}$
- Question: Constraints on $\vec{a}_b$???

Ans: Differentiate $f_\alpha(t, \{\vec{r}_a\}) = 0 \rightsquigarrow$

$$\sum_a \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{v}_a + \frac{\partial f_\alpha}{\partial t} = 0$$

Condition on Accelerations

$$\sum_a \left\{ \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{a}_a + \sum_b \frac{\partial^2 f_\alpha}{\partial \vec{r}_a \partial \vec{r}_b} \cdot \vec{v}_a \vec{v}_b + 2 \frac{\partial^2 f_\alpha}{\partial \vec{r}_i \partial t} \cdot \vec{v}_i \right\} + \frac{\partial^2 f_\alpha}{\partial t^2} = 0$$

Must be combined with external (Free) forces. Write

$$m_a \vec{a}_a = \vec{F}_a + \vec{R}_a$$

Question: How to determine the reactions?

Principle of Virtual Work

Additional solvability conditions

The Principle: Reaction forces do not do any work under virtual displacements:

$$
\begin{aligned}
\sum_a \vec{R}_a \cdot \delta \vec{r}_a &= 0 \\
\sum_a (m_i \vec{a}_a - \vec{F}_a) \cdot \delta \vec{r}_a &= 0
\end{aligned}
$$

Recall:

$$\sum_a \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{\delta} r_a = 0; \;\; \alpha = 1, \cdots k$$

Trick: Multiply each constraint by a Lagrange parameter

$$\sum_a \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial \vec{r}_a} \cdot \vec{\delta} r_a = 0$$

Now identify

$$\vec{R}_a = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial \vec{r}_a}$$

The full expression is

$$m_b\vec{a}_b = \vec{F}_b + \sum_{\alpha} \lambda_\alpha \frac{\partial f_\alpha}{\partial \vec{r}_b}$$

- Normal reaction
- Tension

An Example

A particle on a vertical circle of Radius R in G field

Equations:

$$\begin{aligned} x^2 + y^2 - R^2 &= 0 \\ \frac{d^2x}{dt^2} &= \lambda x \\ \frac{d^2y}{dt^2} &= \lambda y - g \end{aligned}$$

Solution worked out on board, and also in the relevant chapter that is posted.