Poisson Brackets and their properties

1 Introduction

In these notes I will introduce Poisson brackets (PB) and discuss its properties. PB are of central importance: they allow us to derive equations of motion for any obseervable; they help us to identify conserved quantities with relative ease, and finally, they provide away to make transition to quantum mechanics.

PB have an inbuilt syplectic structure (also present in Hamilton's equations). But that discussion is beyond the scope of this course (for babes in woods). First we start with Hamilton's equations

2 Hamilton's equations

Recall that we defined

$$H(q,p) = \dot{q}(q,p)p \ - L(q,\dot{q}(q,p)); p = \frac{\partial L}{\partial \dot{q}}$$

Pay attention to the notation. All coordinates are denoted collectively by q and so is it with velocities and momenta. The summation over coordinates is implicit in the RHS. Finally, I have explicitly shown that we have expressed the generalised velocities as functions of coordinates and momenta. Thus LHS is just a function of coordinates and momenta. The same notation will be employed everywhere

You may employ the relation (derived in the class)

$$dH(q,p) = \dot{q}dp - \dot{p}dq \tag{1}$$

and infer Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2}$$

the solutions of which provide complete information on the dynamics. As a corrolary, we obtained the important result

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} \tag{3}$$

which tells us that H is a constant of motion if it does not have an explicit dependence of time.

3 Equation of motion for any observable

Hamilton's equations (2) give us the equations for the velocity and rate of change of momentum. It does much more: it allow us to determine the equations of motion for any observable.
Definition: An observable is any real valued function of generalised coordinates and momenta.
Exercise Show that the acceleration has the expression

$$\ddot{q}(q,p) = \frac{\partial^2 H}{\partial q \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial q}$$

Let $A(q,p)$ be an observable (think of energy, angular momentum, etc). Then (without employing collective notation), and making use of Eq 1, we get

$$\begin{aligned} \frac{dA}{dt} &= \sum_i \left\{ \frac{\partial A}{\partial q_i}\dot{q}_i + \frac{\partial A}{\partial p_i}\dot{p}_i \right\} \\ &= \sum_i \left\{ \frac{\partial A}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial H}{\partial q_i} \right\} \\ &\equiv -\left\{ H, A \right\}_{\mathrm{PB}} \end{aligned} \tag{4}$$

where, in the last line, I have introduced a special case of what are known as Poisson brackets. More generally, we define the Poisoon bracket of A with respect to B by

$$\left\{ A, B \right\}_{\mathrm{PB}} = \sum_i \left\{ \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial p_i}\frac{\partial A}{\partial q_i} \right\} \tag{5}$$

4 Constants of motion

Eq 4 gives us an algebraic method of verifying if an observable A is a constant of motion. The prescription is simply to evaluate the PB of the Hamiltonian with A. To make further progress we need a few properties of PB.

5 Properties of Poisson brackets

There are several properties which I list. They ones are not proved but left as exercises. They simply involve substitutions.

1. *Scalar multilpication*: $\left\{kA, B\right\}_{\mathrm{PB}} = k\left\{A, B\right\}_{\mathrm{PB}}$ where k is a constant
2. *Linearity*: $\left\{A + B, C\right\}_{\mathrm{PB}} = \left\{A, C\right\}_{\mathrm{PB}} + \left\{B, C\right\}_{\mathrm{PB}}$
3. Antisymmetry: $\left\{A, B\right\}_{\mathrm{PB}} = -\left\{B, A\right\}_{\mathrm{PB}}$; Corollary: $\left\{A, A\right\}_{\mathrm{PB}} = 0$.

4. *Jacobi identity*: $\left\{A,\left\{B,C\right\}\right\}_{\text{PB}}+\left\{B,\left\{C,A\right\}\right\}_{\text{PB}}+\left\{C,\left\{A,B\right\}\right\}_{\text{PB}}=0$

Note that these properties are automatically satified by vectors in three dimensions if we replace the brackets by the cross products. Example: $\left\{A,B\right\}_{\text{PB}}\rightarrow\vec{A}\times\vec{B}$. This connection is not accidental. You would know more of it in a course on group theory.

Jacobi identity is powerful. I did not fully show its power. First let me state the following result:

Theorem If two observables A, B aere constants of motion, so is their Poisson Bracket.

Proof: As done in the class, just employ Jacobi identity ($C = H$).

To see its power, let us pose the following question: Is it possible that only L_z is conserved but not the other two components of angular momentum? The answer is indeed YES. The simplest example is a particle in the Earth's uniform gravitational field. *proof*: Exercise. I now pose the next question: Is it possible that two components, say L_y, L_z are conserved, but not L_x? The answer from Lagrangian dynamics would require some algebra. And you may not be particularly wiser at the end of the day. But with PB, the answer is elegant. You just need to evaluate $\left\{L_y,L_z\right\}_{\text{PB}}$.

Exercise: Verify that the PB $\left\{L_y,L_z\right\}_{\text{PB}}=L_x$.

Thus we see that this the situation that we entertained is impossible. It must strike you that this result must have something to do intimately with the properties of rotations. Indeed it is. But this is not the occasion to discuss that. That again is a topic for a course like *group theory and quantum mechanics.*

6 Levi Civita symbols

There are two fundamental symbols in physics. The first is the Kronecker delta which you are all familiar with.

$$\delta_{ij}=\begin{cases}1 \text{ if } i=j\\ 0 \text{ if } i\neq j\end{cases} \tag{6}$$

This may be used to express the inner product of two vectors. Thus, $\vec{A}\cdot\vec{B}=\sum_{ij}\delta_{ij}A_iB_j$. An equally important, and an even more useful symbol is the Levi Civita symbol involving all the three coordinates (for our purposes).Let us first introduce the notation ϵ_{ijk} where the subscripts take values $1,2,3$. We impose the following properties:

1. ϵ_{ijk} is completely antisymmetric in all its indices. For example, $\epsilon_{ijk}=-\epsilon_{jik}$ where I have interchanged i,j. This interchange is called transposition. What do we conclude? the symbols ijk must all be distinct. Hence they must be some permutation of $(1,2,3)$.

2. There are obviously six permutations. We further classify them into two subsets. The permutations (213), (132), (321) are obtained by a single (odd number) permutation. They are called odd permutations. The other three, that is $(123), (231), (312)$ are obtained by two exchanges or transpositions. These are called even permutations.

We now define the Levi-Civita symbol

Definition: The Levi - Civita symbol is a completely anti symmetric function in all its three indices, with the assignment

$$\epsilon_{ijk} = \begin{cases} +1 \text{ if(ijk)is an even permutation of(123)} \\ -1 \text{ if(ijk)is an odd permutation of(123)} \end{cases} \tag{7}$$

You need to just do these two exercises given below.

Exercise 1: Show that yo can write the i^{th} component

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k$$

Exercise 2: Prove the identity

$$\epsilon_{ijk} \epsilon_{ilm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{jl}$$

Use these exercises and properties of PB listed above. You can easily prove that

$$\left\{ L_i, L_j \right\}_{\text{PB}} = \epsilon_{ijk} L_k$$

which concludes what we wished to prove.