Quantum Mechanics
PHL555

Problem Set 5

Projection and Unitary Operators

Consider a vector space $\mathcal{V}_N$ of dimension N. We can let $N \to \infty$ if necessary. Let $\{|v_i\rangle;\ i = 1, 2, \cdots N\}$ be an orthonormal basis in $\mathcal{V}_N$. Let $\mathcal{V}_k;\ k < N$ be a proper subspace of $\mathcal{V}_N$.

1. Let $\{|w_i\rangle;\ i = 1, 2, \cdots k\}$ be an orthonormal basis in $\mathcal{V}_k$.

2. Construct the projection operator, P_k for $\mathcal{V}_k$ in the basis provided by $|w_i\rangle$.

3. Verify explicitly that

   (i) P_k is hermitian

   (ii) P_k is idempotent: $P_k^2 = P_k$.

   (iii) P_k has eigenvalues $0, 1$.

   (iv) The eigenvectors with eigenvalues 1 belong to w_k

   (v) The eigenvectors belonging to zero lie in the orthogonal space.

4. Expand the basis vectors, $|w_i\rangle$ (in $\mathcal{V}_k$) in terms of the larger set of basis vectors $|v_i\rangle$. Express P_k in the new basis. Which of the above properties which you have verified holds in the new basis

5. In case you are befuddled, let us take a a concrete example. Let $N = 3$, and $k = 2$. Let the vector spaces be real. Let $|v_i\rangle$ be the unit vectors $\{\hat{i},\ \frac{1}{\sqrt{2}}(\hat{j} + \hat{k}),\ \frac{1}{\sqrt{2}}(\hat{j} - \hat{k})\}$ in terms of the standard orthogonal basis. Let $\{\hat{i},\ \hat{j}\}$ be the basis in the two dimensional subspace. Now work out the exercise for this special case, and then prove the general results Let us turn our attention to Unitary operators.

6. Verify that a unitary operator, U is represented by a unitary matrix in an orthonormal basis.

7. Verify that all eigenvalues of a unitary operator are of unit length, pure phases.

8. Suppose that $\exp(i\phi_i)$ are the eigenvalues. Use this to show that U can always be written as $U = \exp(iH)$, where H is hermitian. What is the ambiguity in the identification of H?