PYL743

Problem Set 1 – Basic Ideas

All groups are necessarily finite in this problem set.

1. Verify that the identity element of a group G is unique.

2. Verify that the inverse of any group element is unique.

3. Show that any group of order 4 is necessarily Abelian.

4. Consider the set of nonzero real numbers. Let the product be defined as $xy = x^y$. Does this satisfy group properties?

5. Consider the space of all $M \times N$ matrices defined over (i) integers, rational numbers, real numbers, and complex numbers. Show that they form a group in each case under addition.

6. Consider the set of all $N \times N$ matrices of nonzero determinant. Consider each case enumerated above and find which of them form a group under standard matrix multiplication.

7. Consider square matrices of some fixed dimension. Define their product to be $M \star N = [M, N]$. Do they form a group with some reasonable restriction on matrices if necessary?

8. Write down the multiplication for the group of symmetries of a square.

9. Write down the group multiplication table for the group of symmetries of an equilateral triangle.

10. In each of the two groups above, identify all the subgroups, normal subgroups, and the corresponding factor groups.

11. Show that any group of prime order is necessarily abelian, and cyclic.

12. Write down the group multiplication table for the permutation group S_3 of three symbols. List all the subgroups and identify the one that is isomorphic to the group of symmetries of an equilateral triangle.

13. Determine all the normal subgroups of the permutation group S_4. Hence determine all possible (inequivalent) homomorphic mappings from S_4 to other groups.

14. Determine all distinct classes of the permutation group S_3.

15. Verify explicitly that the set of all even permutations forms a subgroup of the symmetric group S_n.

16. Show that the symmetric group S_3 is isomorphic to the group of symmetries of an equilateral triangle.

17. Show that any group of order 2 is isomorphic to the group $\mathbb{Z}_2 = \{+1, -1\}$.

18. Show that any group of order 4 is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.