Optimal Control of Active Suspension

- Ajinkya Jain and Soovadeep Bakshi

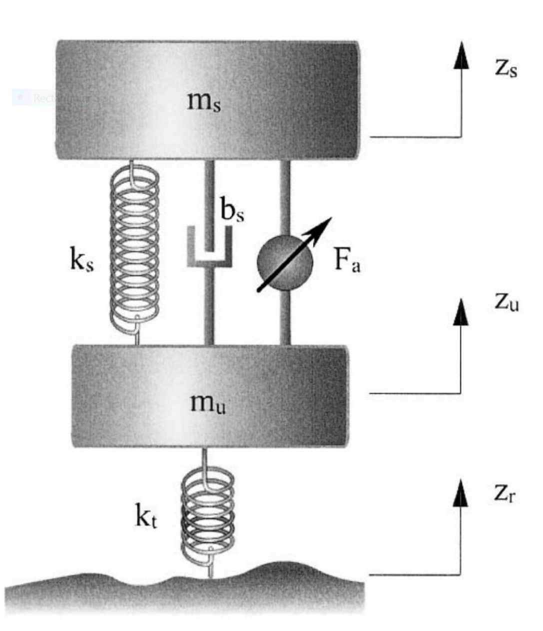
Motivation

Introduction: Physical System

- Quarter Car Suspension Model:

$$m_s\ddot{z}_s + b_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) \;=\; F_a$$

$$m_u\ddot{z}_u - b_s(\dot{z}_s - \dot{z}_u) - k_s(z_s - z_u) + k_t(z_u - z_r) \;=\; -F_a$$



Rajamani, Rajesh. *Vehicle Dynamics And Control.* 1st ed. Springer, 2014.

Physical System

System Parameters:

- Sprung Mass: $m_s = 453.5 \text{ kg}$
- Unsprung Mass: $m_u = 45.25 \text{ kg}$
- Suspension Stiffness $k_s = 15000 \text{ N/m}$
- Suspension Damping $b_s = 1400 \text{ Ns/m}$
- Tire Stiffness $k_t = 176000 \text{ N/m}$

The tire damping is generally small (~10 Ns/m), and is therefore neglected!

Wong, J. (2008). *Theory of Ground Vehicles*. Hoboken, NJ: John Wiley & Sons.

System Modeling

- Fourth order system
- State-Space representation:

  - ➢ Suspension Deflection $x_1 = z_s - z_u$
  - ➢ Sprung Mass Velocity $x_2 = \dot{z}_s$
  - ➢ Tire Deflection $x_3 = z_u - z_r$
  - ➢ Unsprung Mass Velocity $x_4 = \dot{z}_u$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$$

System Modeling

$$\dot{\mathbf{x}} = A\mathbf{x} + BF_a + L\dot{z}_r$$

$$A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & \frac{1}{m_s} & 0 & -\frac{1}{m_u} \end{bmatrix}^T \qquad L = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^T$$

Controllability

$$
\begin{aligned}
\zeta &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0.0243 & -0.8270 & -66.6770 \\ 0.0022 & -0.0750 & 1.7491 & 233.1922 \\ 0 & -0.0221 & 0.7520 & 68.4261 \\ -0.0221 & 0.7520 & 68.4261 & -5261.8372 \end{bmatrix}
\end{aligned}
$$

Rank = 4: Controllable!

Observability

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -33.0761 & -3.0871 & 0 & 3.0871 \\ -364.5678 & -34.0263 & 3889.5028 & 34.0263 \\ 1125.4573 & 71.9666 & -12007.2853 & 71.9666 \\ 12404.9027 & 793.2231 & -132345.4923 & 3096.2796 \\ -26236.7078 & -1323.3020 & 279914.3280 & -10683.9832 \end{bmatrix}$$

Rank = 4: Observable!

Stability Analysis

- Characteristic Equation:

$$P(\lambda) = \lambda^4 + \frac{b_s(m_s + m_u)}{m_s m_u}\lambda^3 + \frac{k_s m_s + k_t m_s + k_s m_u}{m_s m_u}\lambda^2 + \frac{b_s k_t}{m_s m_u}\lambda + \frac{k_s k_t}{m_s m_u}$$

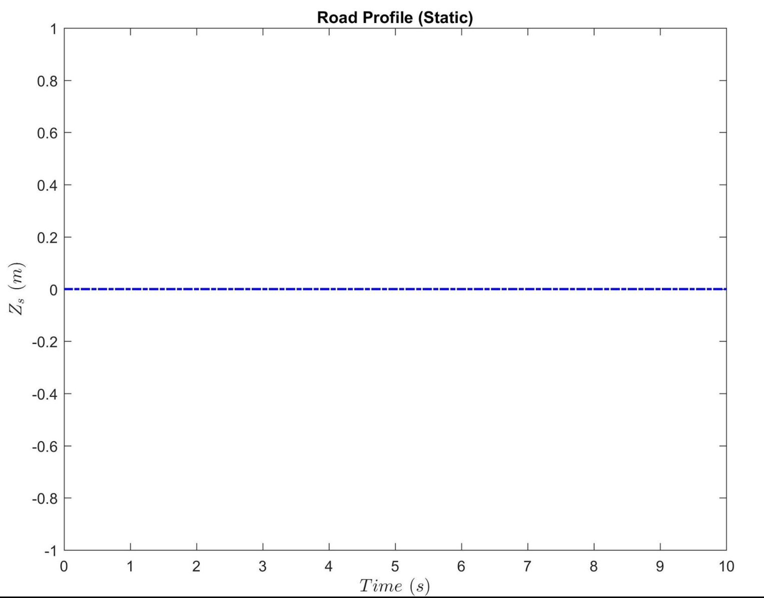
- Stability Conditions: Routh-Hurwitz Criterion

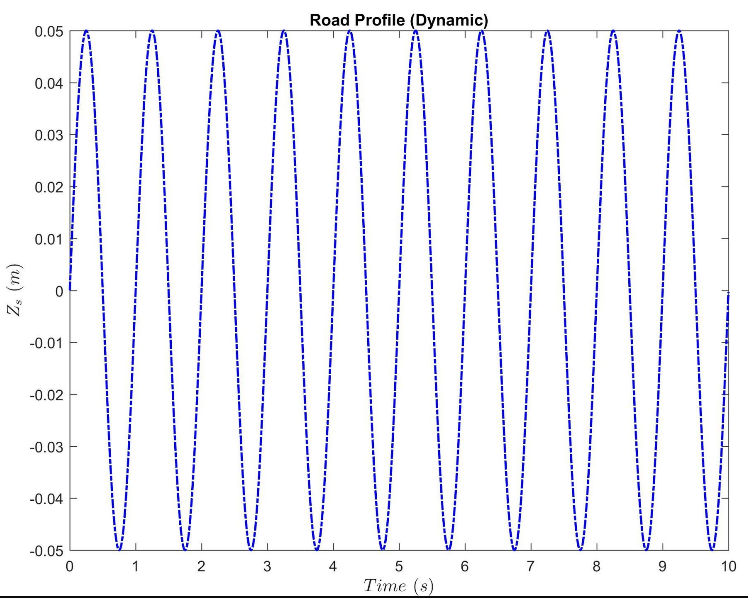
$$\frac{b_s(m_s + m_u)}{m_s m_u} > 0 \qquad k_s(m_s^2 + m_u^2) + k_t m_s^2 + 2k_s m_s m_u > 0 \qquad k_t m_s^2 > 0$$

System **BIBO Stable!**

Passive System Analysis

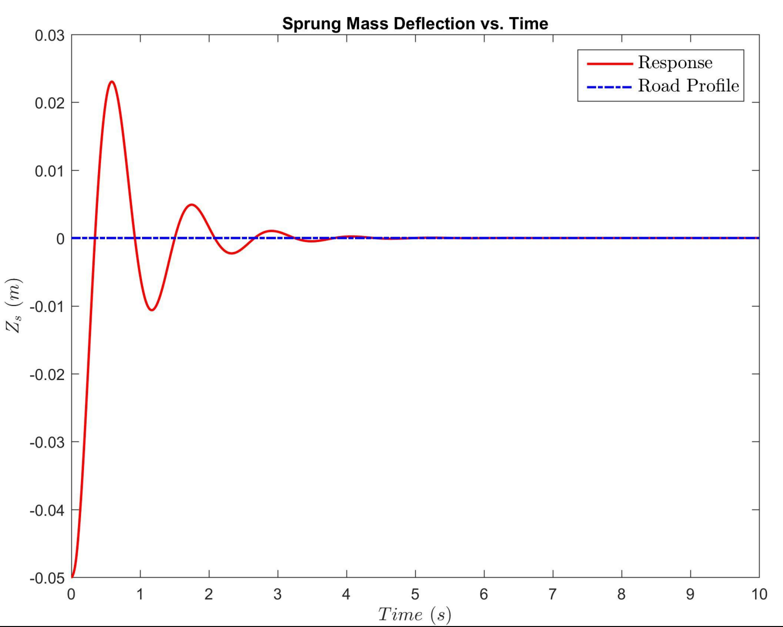
- Input Road Profiles:

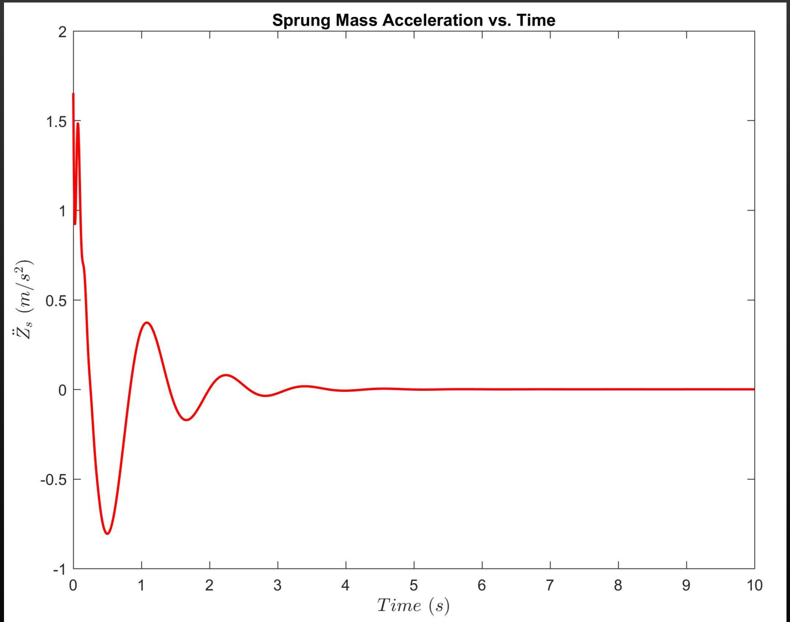




Passive System Analysis

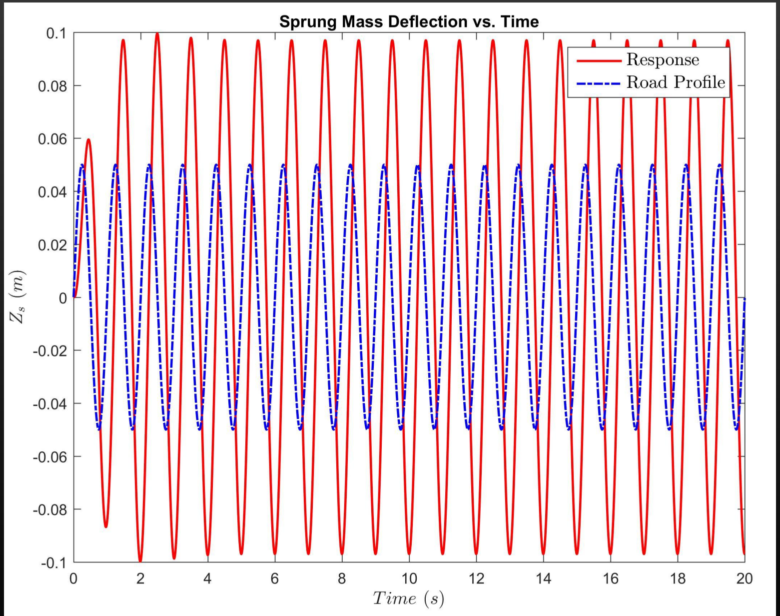
- Static Input: Flat Road

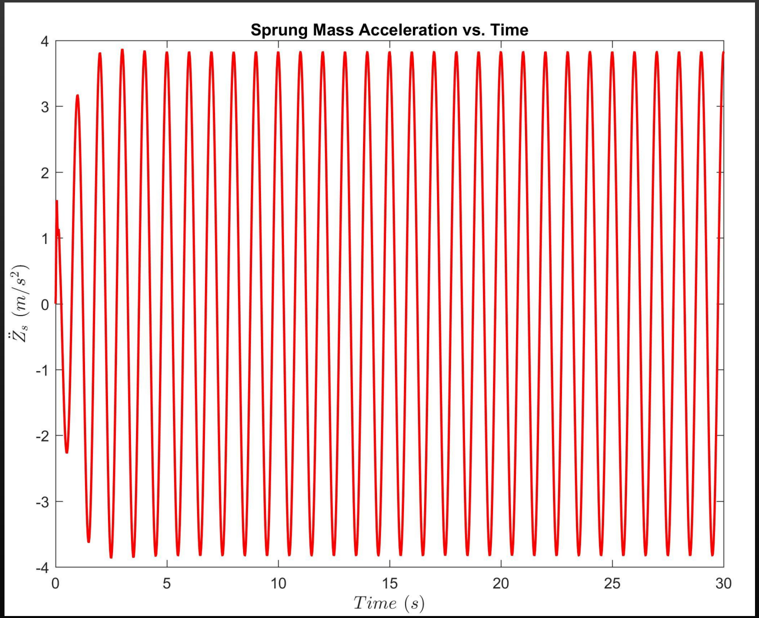




Passive System Analysis

- Dynamic Input: Road with continuous bumps!





Optimal Controller

- Performance Index

$$J = \frac{1}{2}\int_0^{t_f} \left(\ddot{z}_s^2 + \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2\right) dt$$

$$\rho_1 = 0.4 \quad \rho_2 = 0.04 \quad \rho_3 = 0.4 \quad \rho_4 = 0.04$$

$$J = \frac{1}{2}\int_0^{t_f} \left(\mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T N F_a + F_a^T R F_a\right) dt$$

Optimal Controller

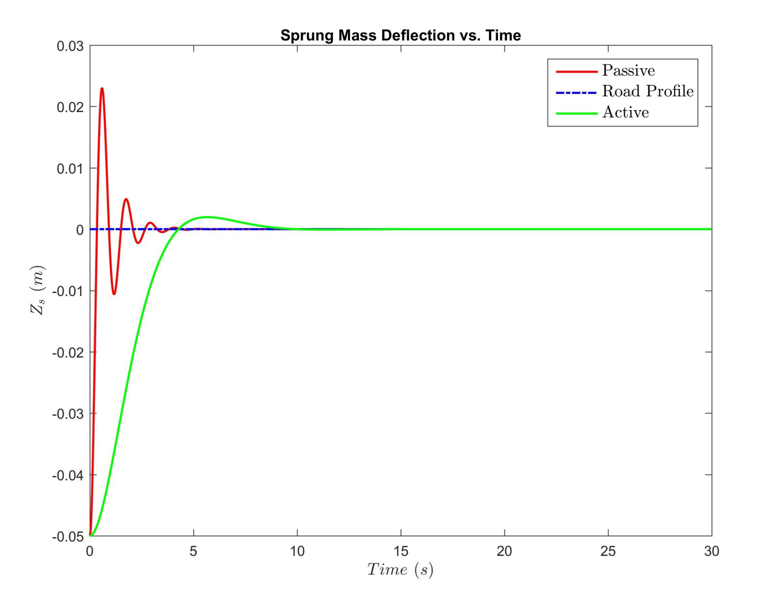
- Performance Index

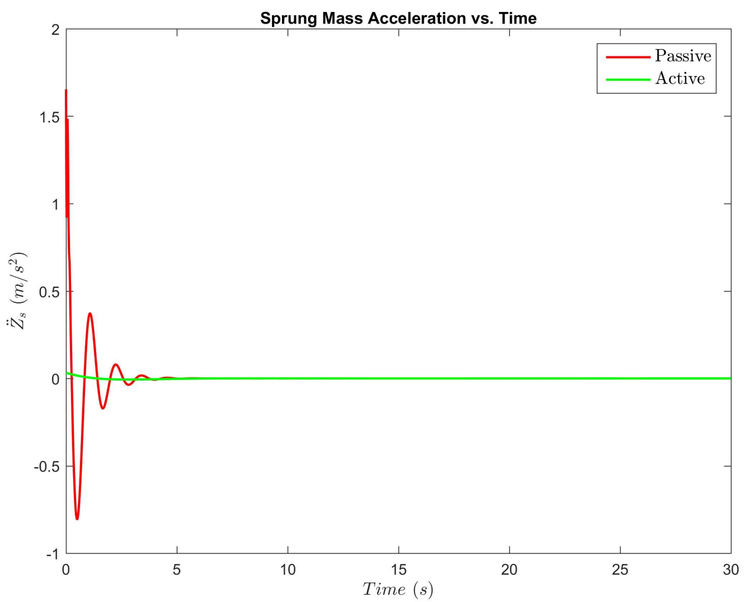
$$R = \frac{1}{m_s^2} \qquad N = \begin{bmatrix} -\frac{k_s}{m_s^2} & -\frac{b_s}{m_s^2} & 0 & \frac{b_s}{m_s^2} \end{bmatrix}^T$$

$$Q = \begin{bmatrix} \frac{k_s^2}{m_s^2} + \rho_1 & \frac{b_s k_s}{m_s^2} & 0 & -\frac{b_s k_s}{m_s^2} \\ \frac{b_s k_s}{m_s^2} & \frac{b_s^2}{m_s^2} + \rho_2 & 0 & -\frac{b_s^2}{m_s^2} \\ 0 & 0 & \rho_3 & 0 \\ -\frac{b_s k_s}{m_s^2} & -\frac{b_s^2}{m_s^2} & 0 & \frac{b_s^2}{m_s^2} + \rho_4 \end{bmatrix}$$

Open Loop Controller

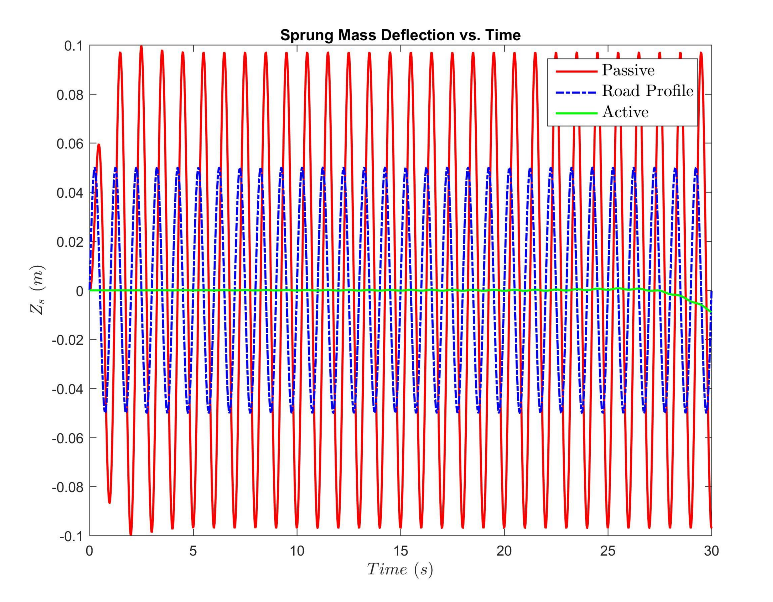
- Final time fixed, Final state free: Static Input

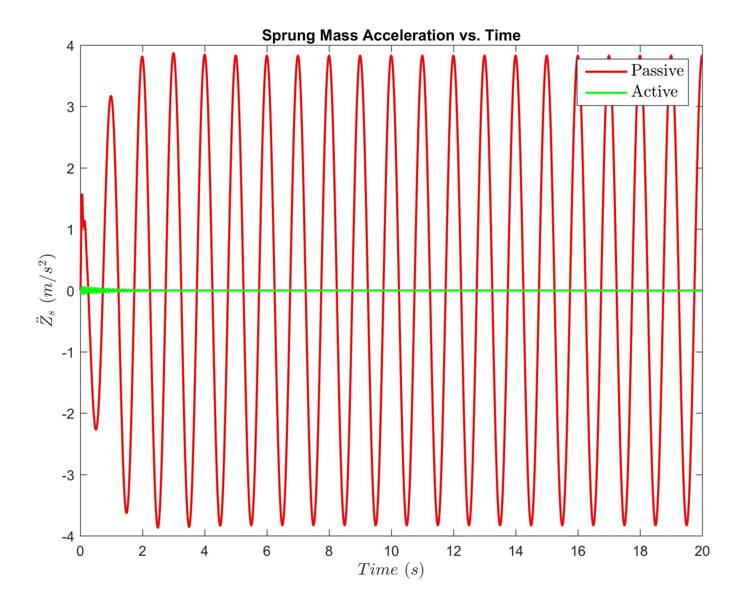




Open Loop Controller

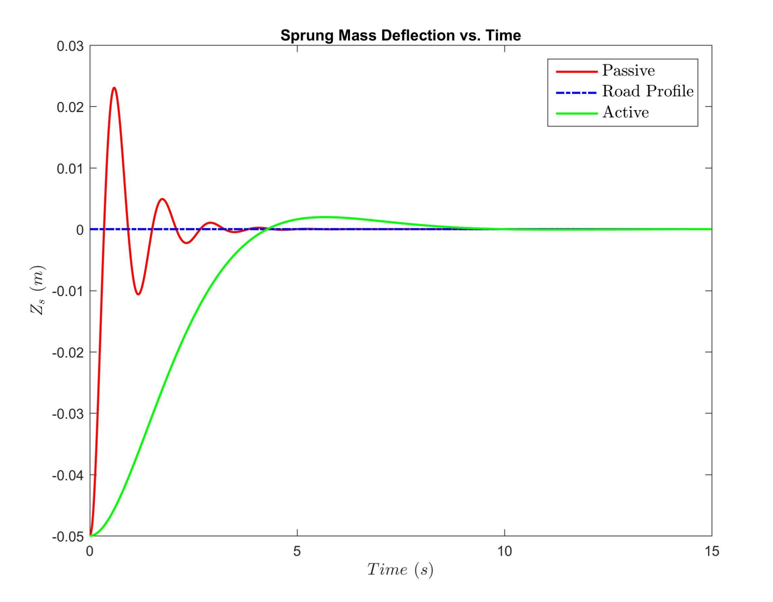
- Final time fixed, Final state free; Dynamic Input

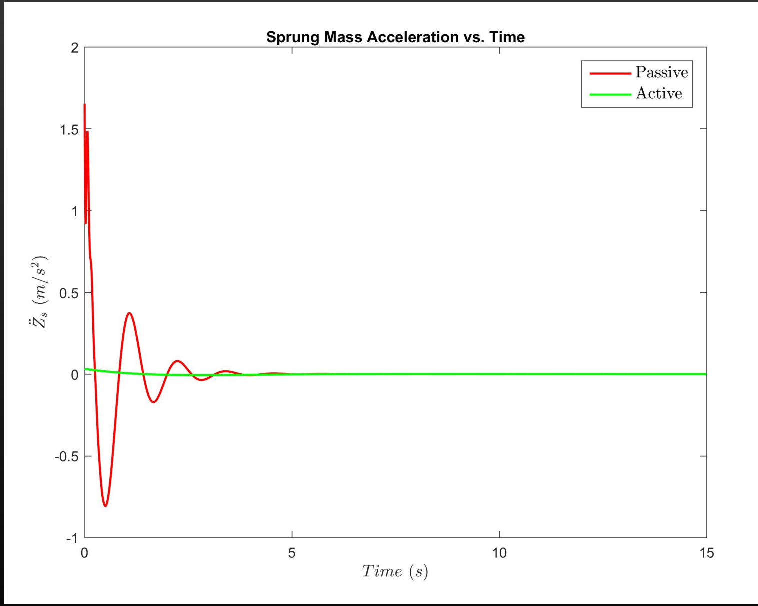




Open Loop Controller

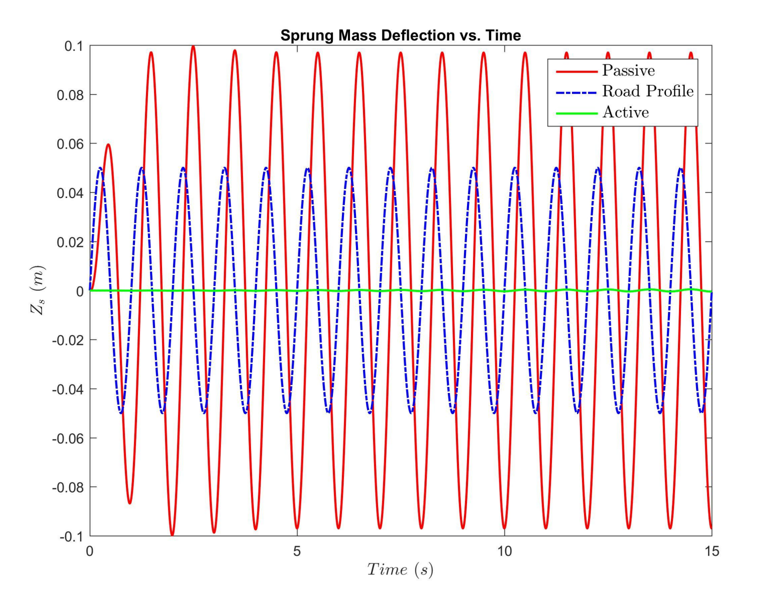
- Final state fixed, final time fixed (all states driven to zero)

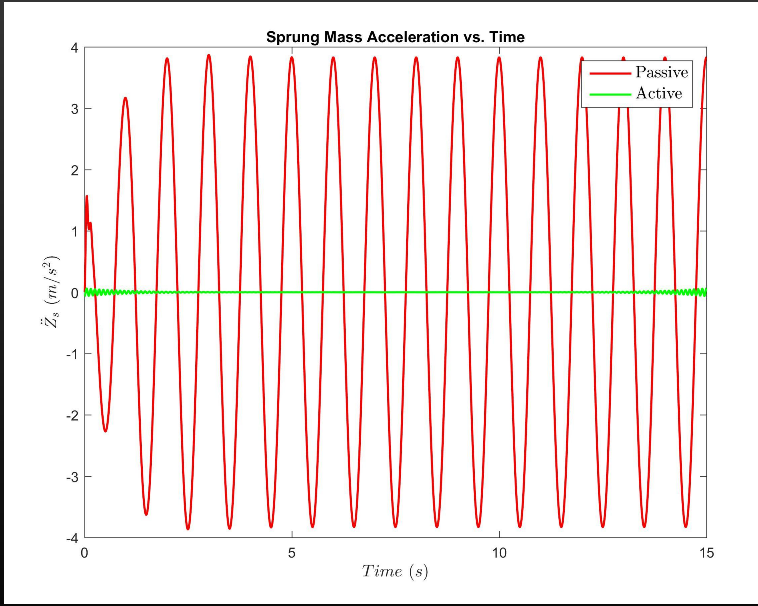




Open Loop Controller

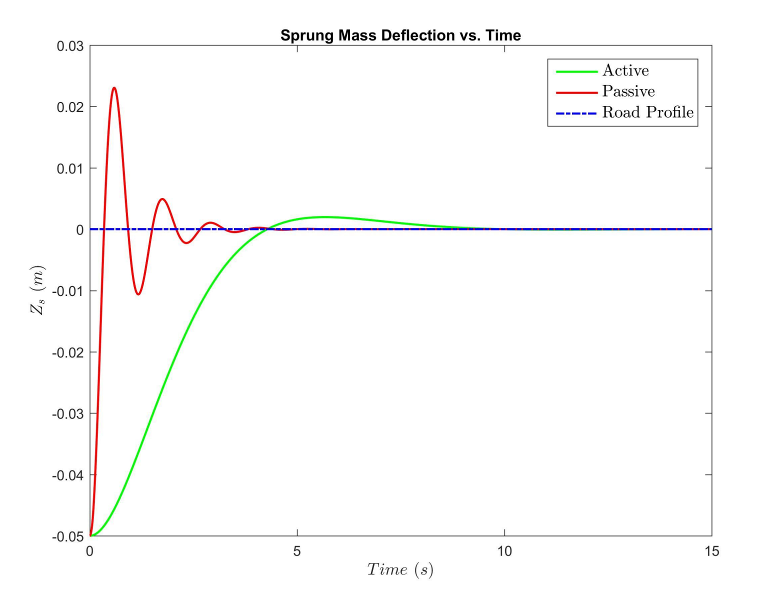
- Final state fixed, final time fixed

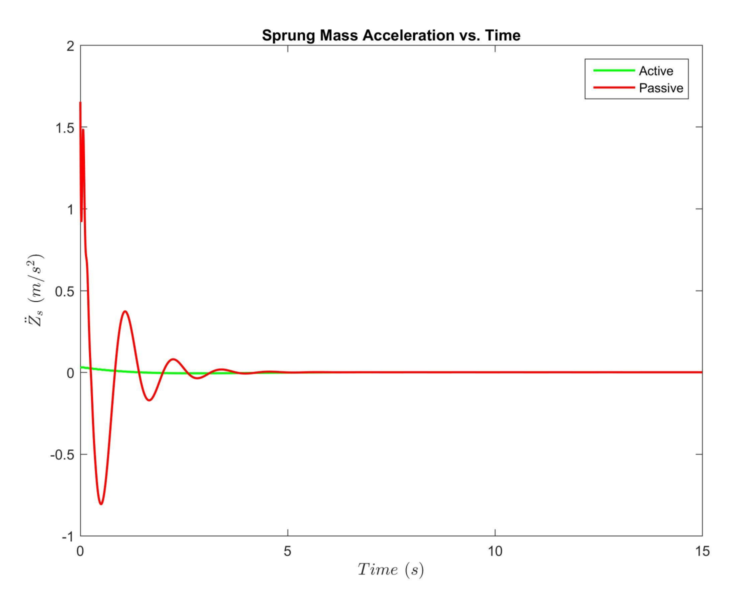




Closed Loop Controller

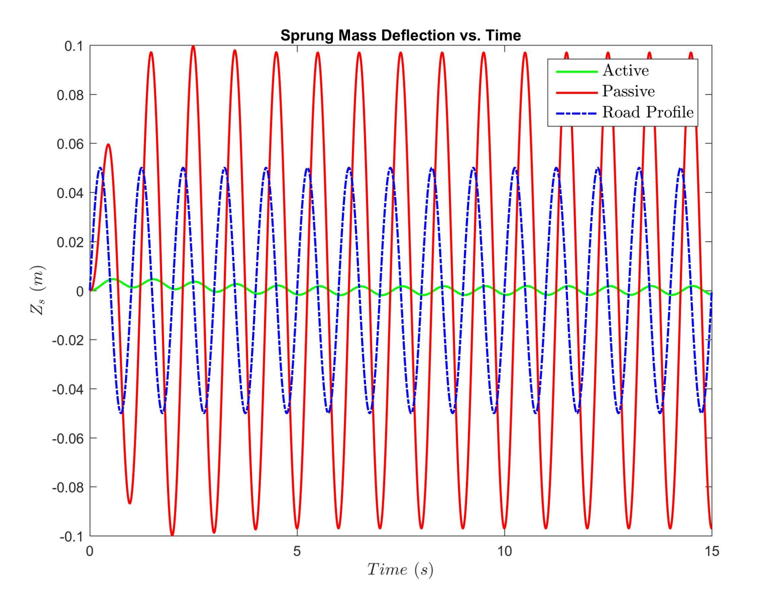
- LQR - infinite time with static input

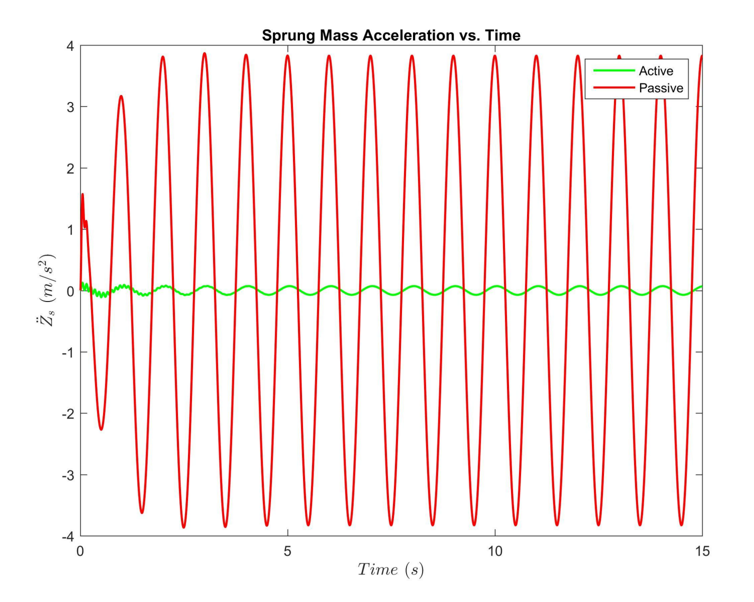




Closed Loop Controller

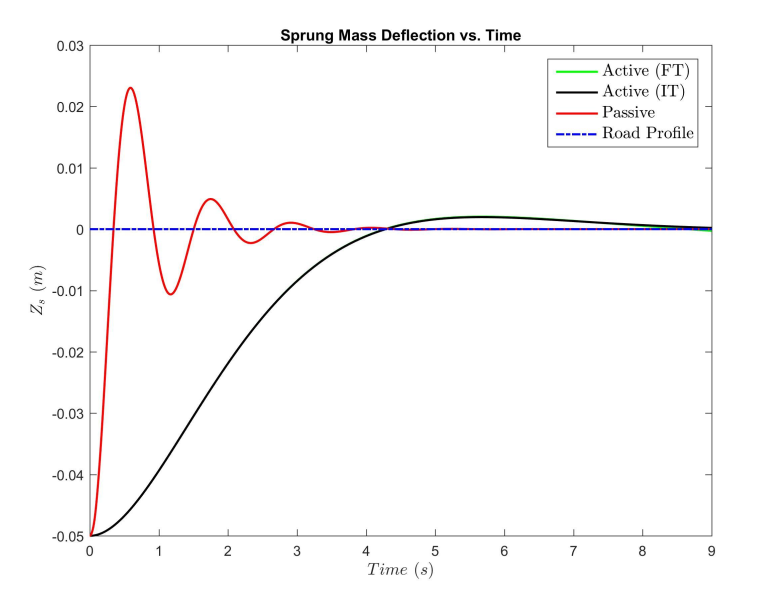
- LQR - infinite time with dynamic input

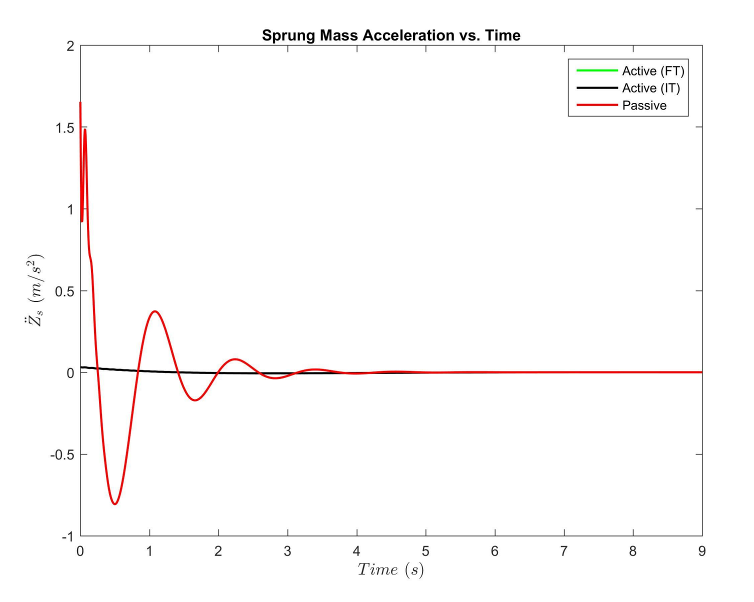




Closed Loop Controller

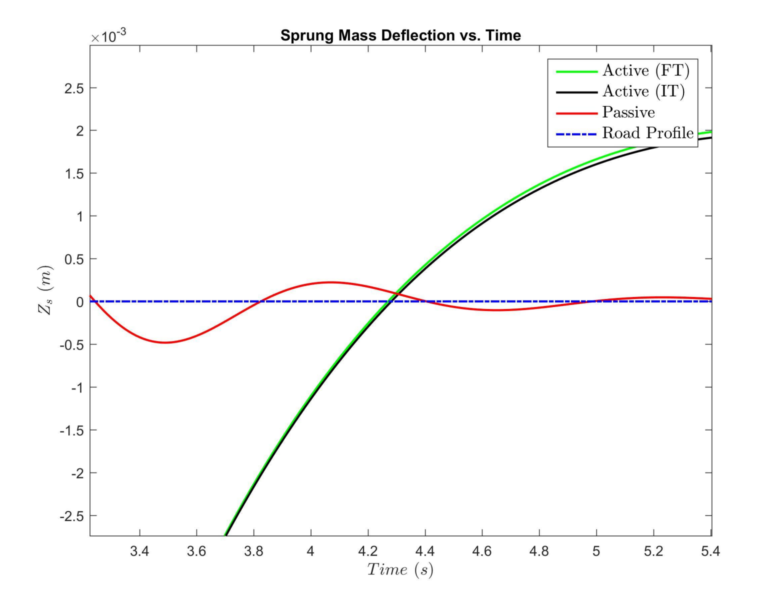
- LQR - finite time with static input



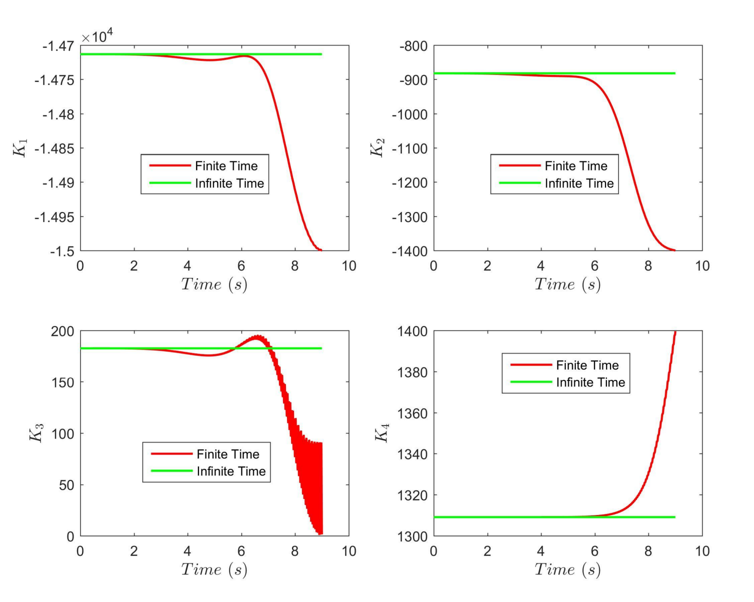


Closed Loop Controller

- LQR - finite time with static input



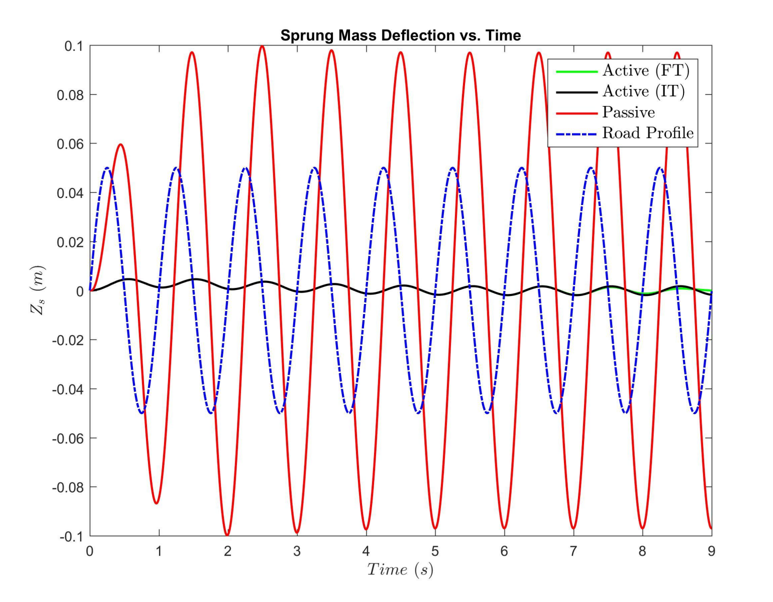
Finite time LQR is faster!

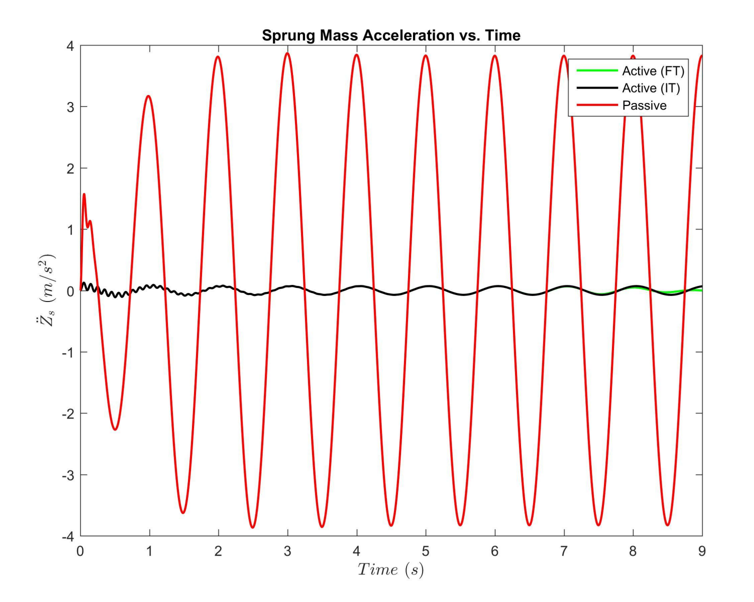


Gain updates

Closed Loop Controller

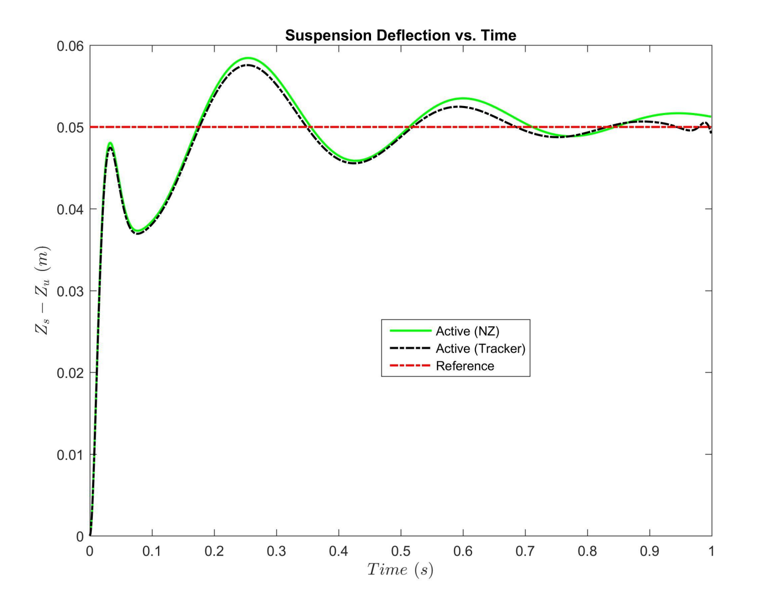
- LQR - finite time with Dynamic Input



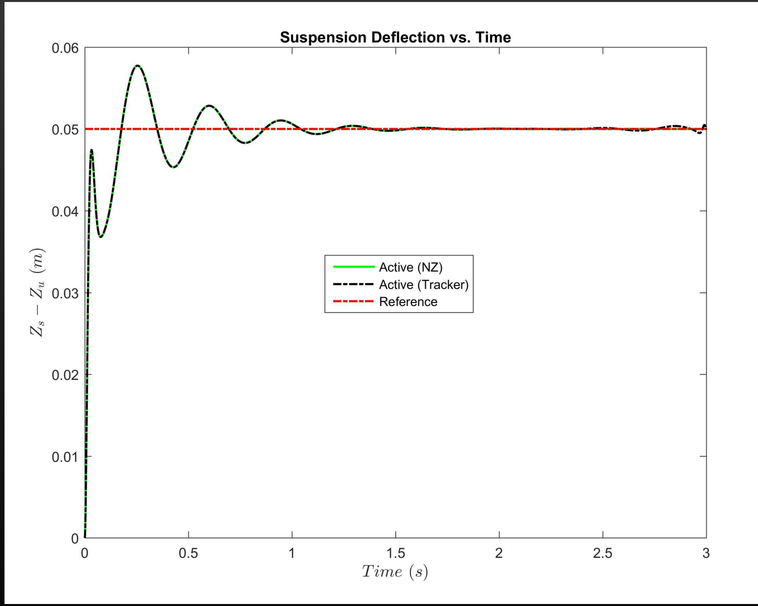


Closed Loop Controller

- Non-zero set point (static road input, ρ_1= 100000)



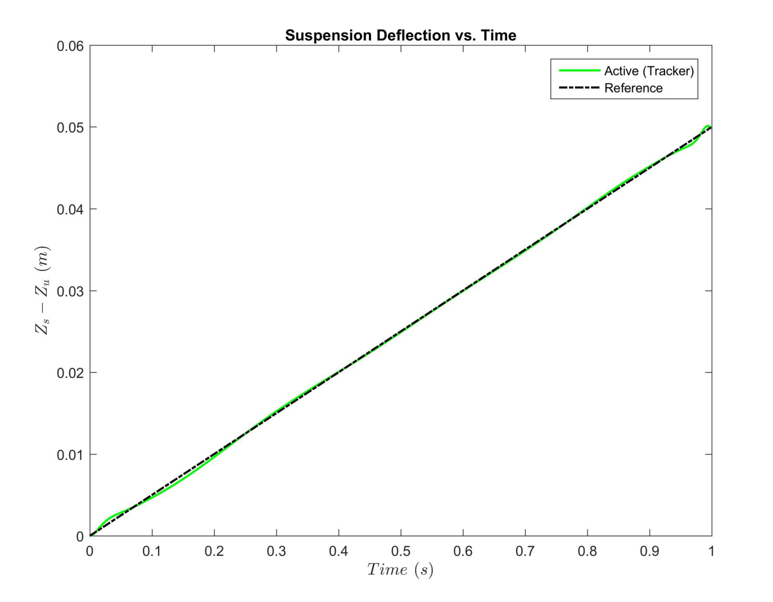
$t_f = 1$



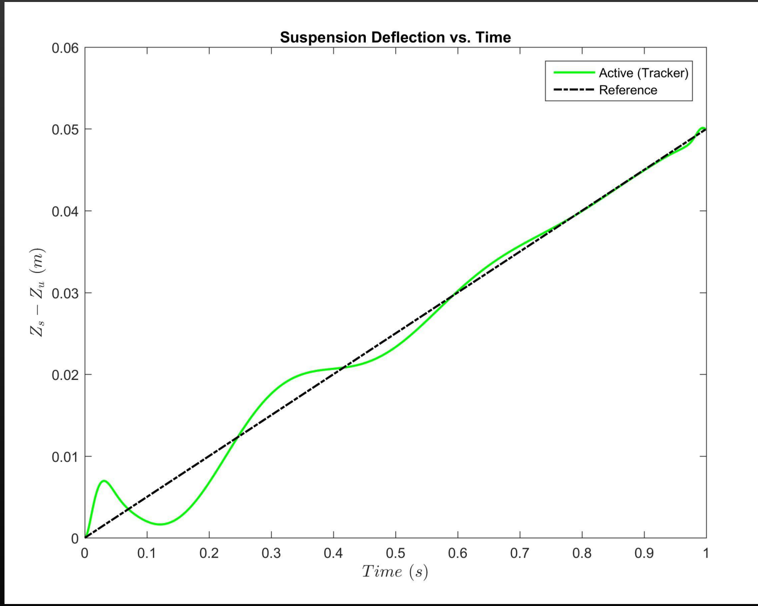
$t_f = 3$

Closed Loop Controller

- Finite time Tracker



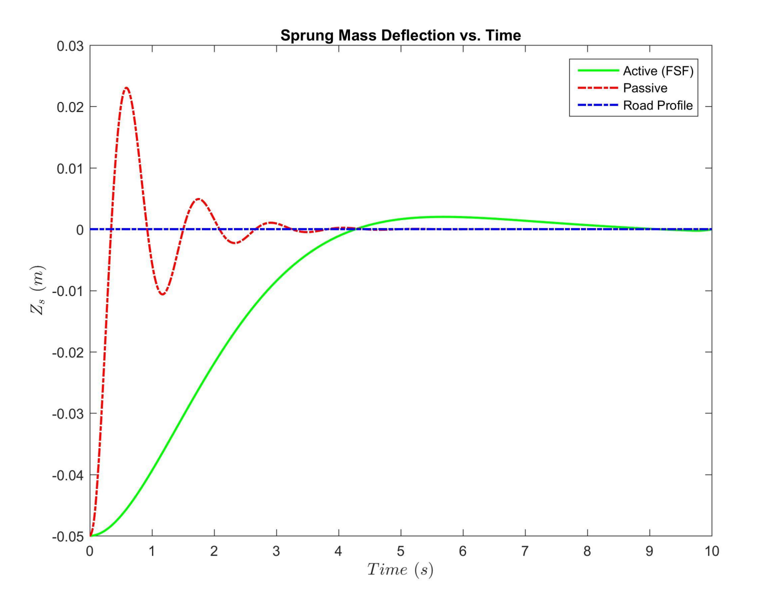
Static

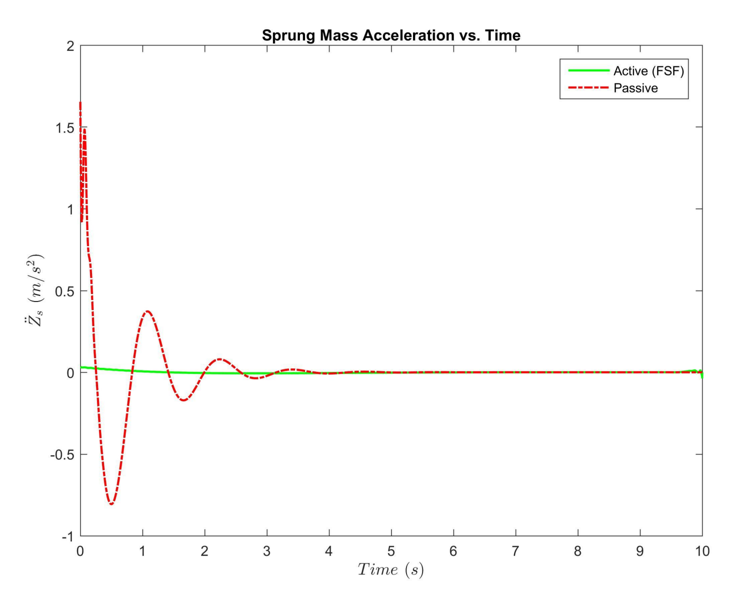


Dynamic

Closed Loop Controller

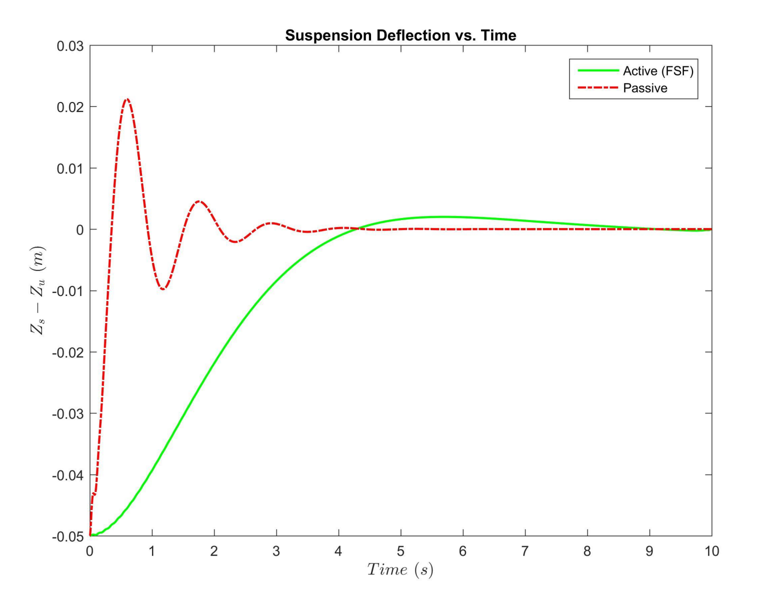
- Final state fixed

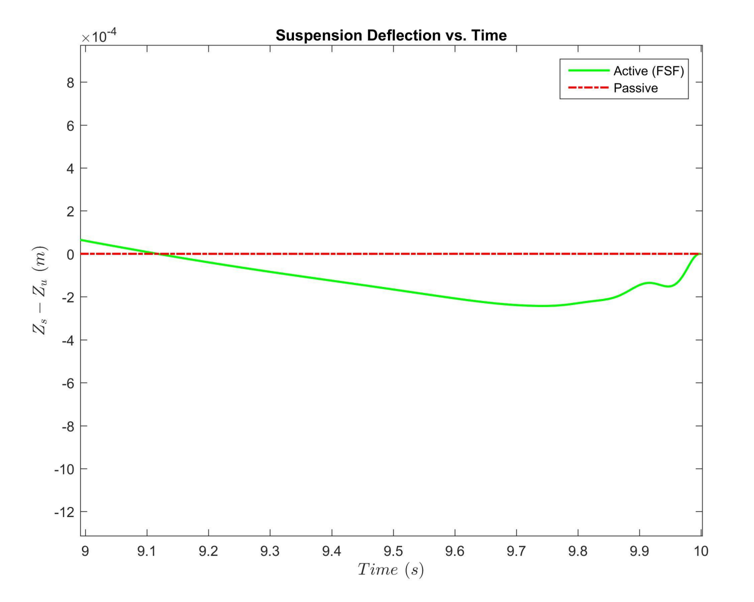




Closed Loop Controller

- Final state fixed





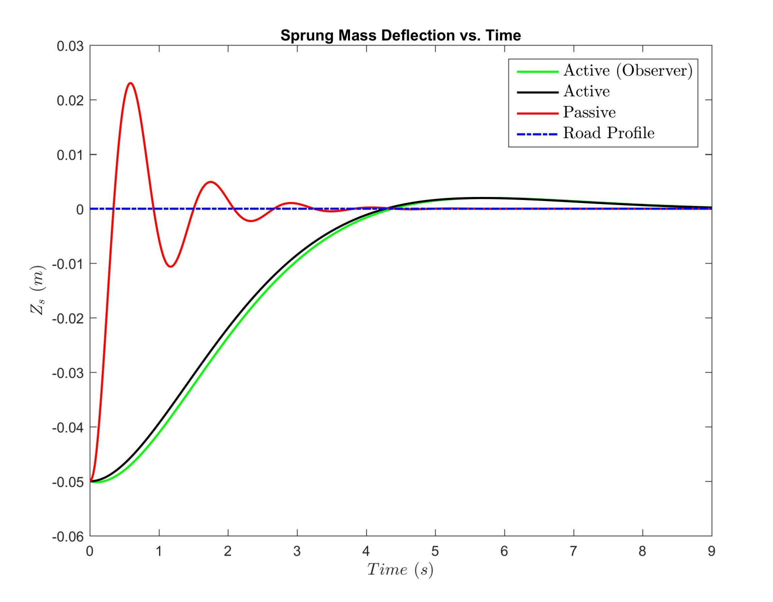
Suspension deflection driven to zero!

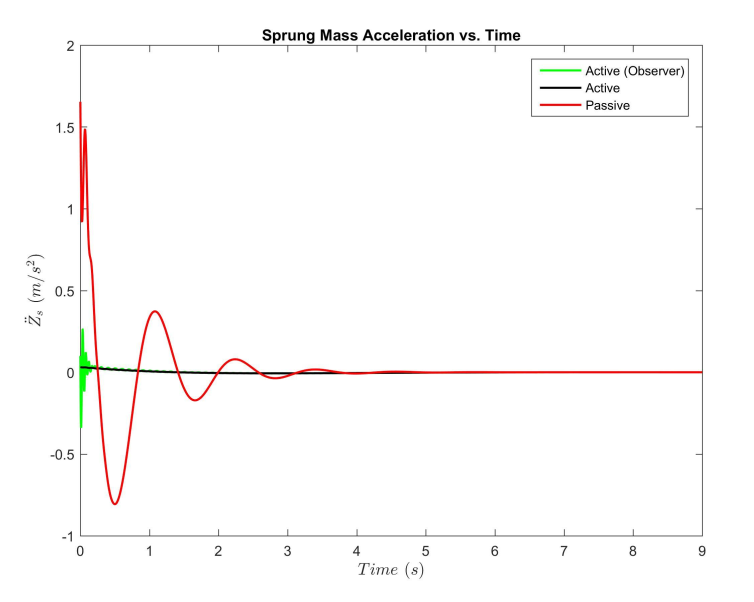
Reduced Order

- Observation Matrix: $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$
- Measured States = 2 $[x_1, x_2]$
  - ➢ Suspension Deflection x_1
  - ➢ Sprung Mass Velocity x_2
- Estimated States = 2 $[x_3, x_4]$
- Estimator Eigenvalues = 20*(Closed Loop Eigenvalues)

Reduced Order

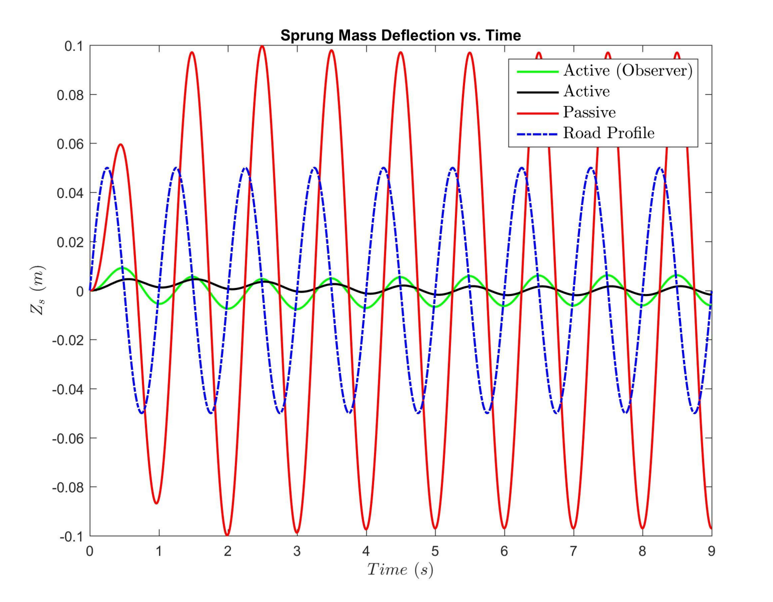
- Closed-loop LQR : Infinite

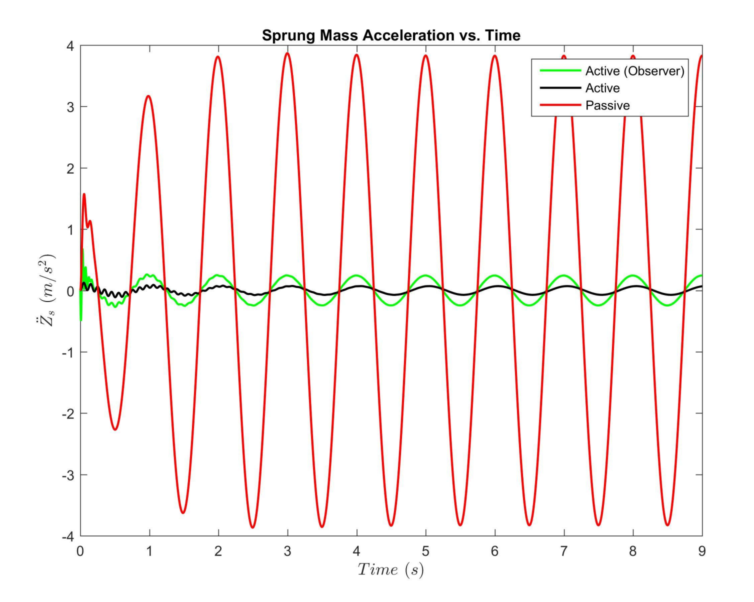




Reduced Order

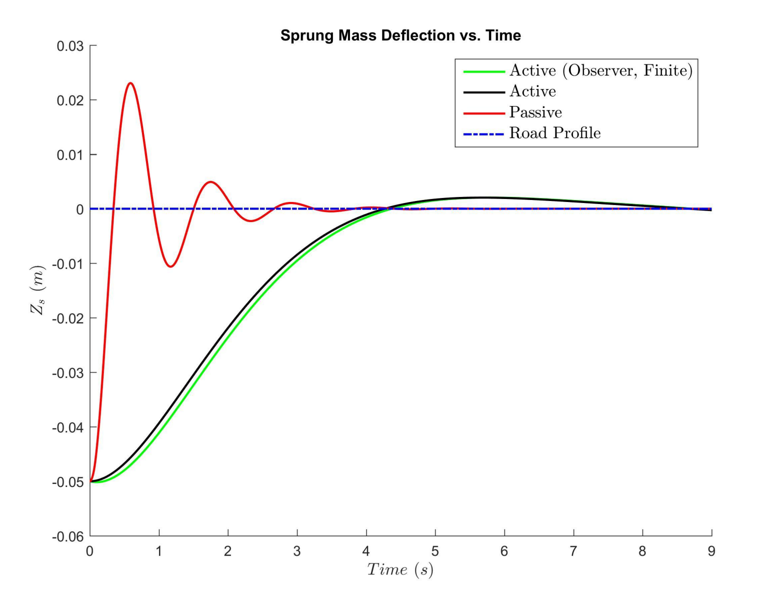
- Closed-loop LQR : Infinite

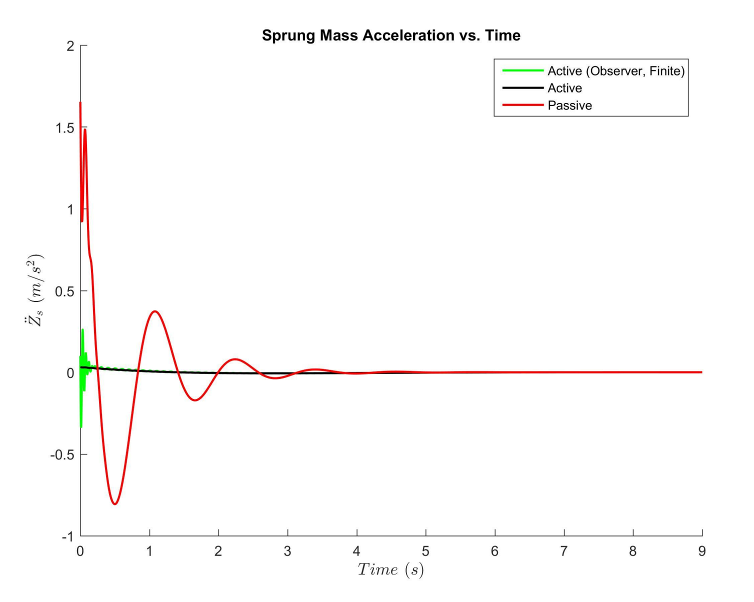




Reduced Order

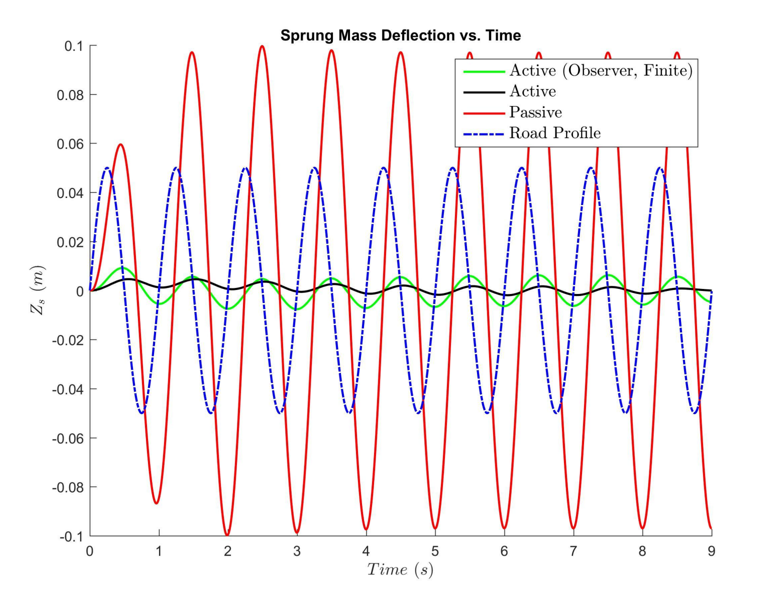
- Closed-loop LQR : Finite time

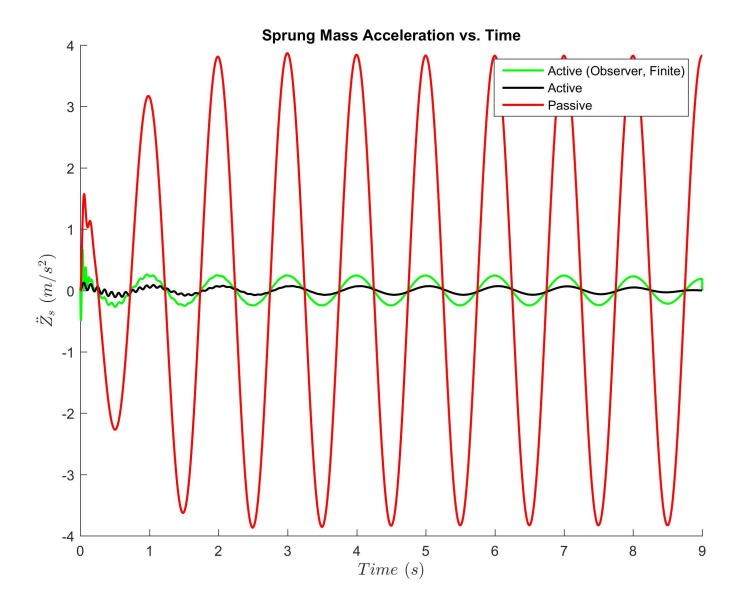




Reduced Order

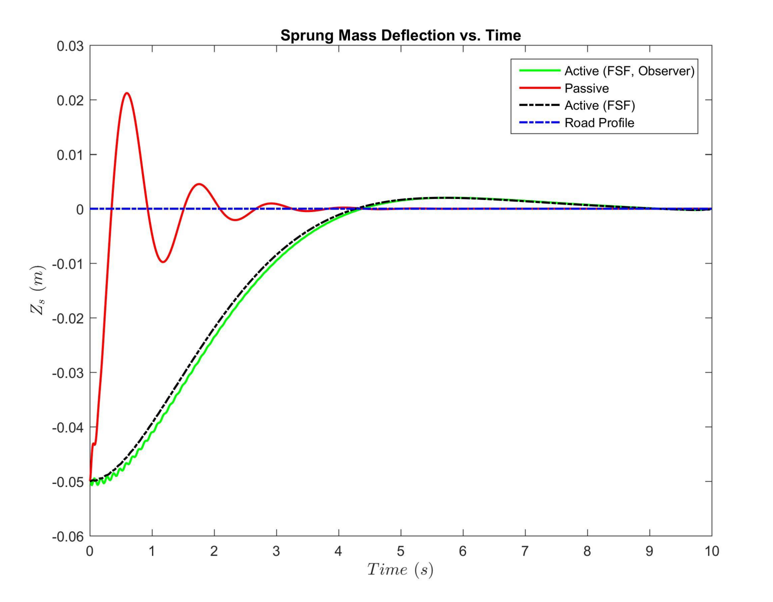
- Closed-loop LQR : Finite time

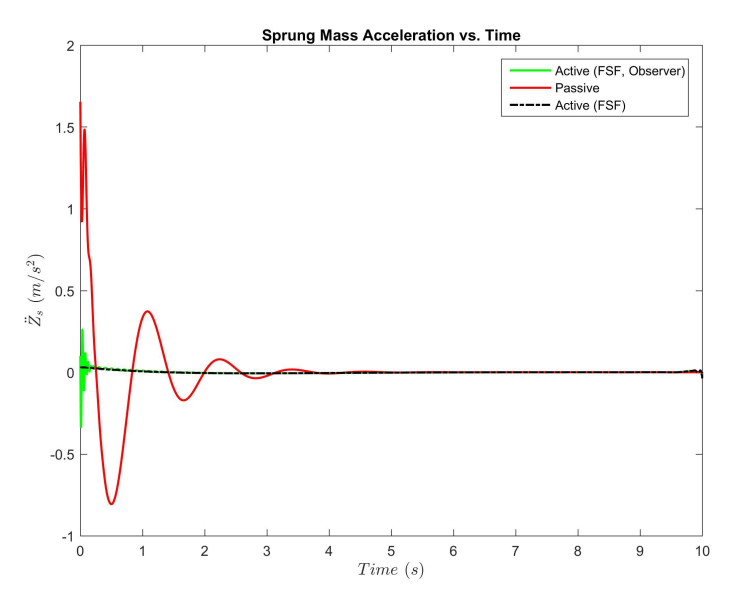




Reduced Order

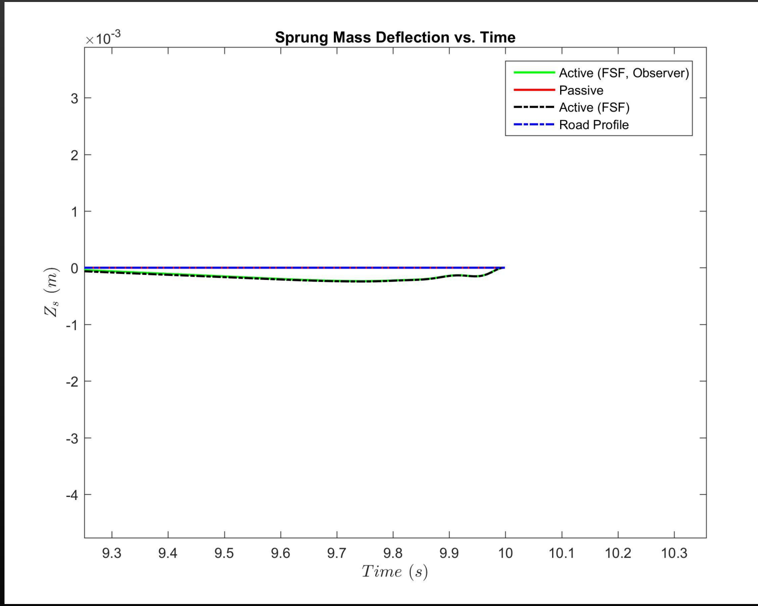
- Closed-loop LQR : Final State Fixed Finite time





Reduced Order

- Closed-loop LQR : Final State Fixed Finite time



Thank You!

Questions?