Intro to Modern Control – Project (Part I)

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Introduction

The system that we have chosen for our analysis is the quarter car model for an active suspension.

In automotive suspensions, passive suspensions (with no actuators) are popular because of economic benefits, but they have significant disadvantages in terms of performance. Specifically, there is a trade-off among the three parameters which are used to define suspension performance: the road roughness isolation capability, the suspension deflection and the tire deflection. In passive suspension systems, improvement of a certain parameter comes at the cost of the other two being worsened [1].

In the case of active suspensions, there are electronic actuators placed in the suspension system, as shown in the quarter car model in Fig. 1. With specific control laws, the actuator can be used to improve on the performance of passive systems. In this project, open and closed loop optimal control strategies will be used to improve the performance of the system.

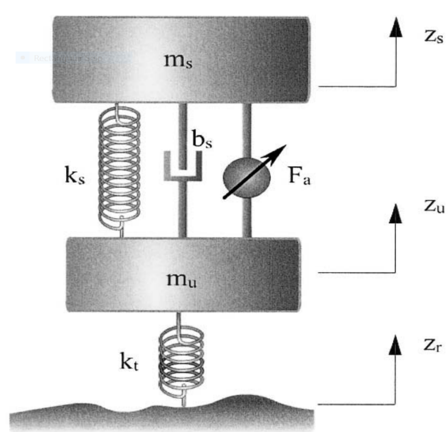

Figure 1: Quarter Car model [1].

Part I of the project deals with the system model description, basic properties of the system, and open loop optimal control.

System Model

As shown in Fig. 1, the variables z_s, z_u and z_r represent the absolute positions of the sprung mass (m_s), unsprung mass (m_u) and road profile respectively. F_a is the actuator force, k_s and b_s are the suspension stiffness and damping parameters respectively, and k_t is the tire stiffness. The tire damping is assumed to be negligible. The values of these system parameters are those used by Wong [2]. The chosen values are as follows:

$$
\begin{aligned}
m_s &= 453.5\ \mathtt{kg} \\
m_u &= 45.25\ \mathtt{kg} \\
k_s &= 15000\ \mathtt{N/m} \\
k_t &= 176000\ \mathtt{N/m} \\
b_s &= 1400\ \mathtt{Ns/m}
\end{aligned}
$$

The equations of motion for the quarter car model can therefore be written as follows:

$$
m_s\ddot{z}_s + b_s(\dot{z}_s - \dot{z}_u) + k_s(z_s - z_u) = F_a \tag{1}
$$

$$
m_u\ddot{z}_u - b_s(\dot{z}_s - \dot{z}_u) - k_s(z_s - z_u) + k_t(z_u - z_r) = -F_a \tag{2}
$$

Writing the same in state-space form, the system can be represented as:

$$
\dot{\mathbf{x}} = A\mathbf{x} + BF_a + L\dot{z}_r \tag{3}
$$

where

$$
\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \tag{4}
$$

$$
x_1 = z_s - z_u \tag{5}
$$

$$
x_2 = \dot{z}_s \tag{6}
$$

$$
x_3 = z_u - z_r \tag{7}
$$

$$
x_4 = \dot{z}_u \tag{8}
$$

$$
A = \begin{bmatrix} 0 & 1 & 0 & -1 \\ -\frac{k_s}{m_s} & -\frac{b_s}{m_s} & 0 & \frac{b_s}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k_s}{m_u} & \frac{b_s}{m_u} & -\frac{k_t}{m_u} & -\frac{b_s}{m_u} \end{bmatrix} \tag{9}
$$

$$
B = \begin{bmatrix} 0 & \frac{1}{m_s} & 0 & -\frac{1}{m_u} \end{bmatrix}^T \tag{10}
$$

$$
L = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix}^T \tag{11}
$$

Stability

The characteristic polynomial of the system is

$$P(\lambda) = \lambda^4 + \frac{b_s(m_s + m_u)}{m_s m_u}\lambda^3 + \frac{k_s m_s + k_t m_s + k_s m_u}{m_s m_u}\lambda^2 + \frac{b_s k_t}{m_s m_u}\lambda + \frac{k_s k_t}{m_s m_u} \quad (12)$$

From Routh-Hurwitz criterion, stability conditions for the system are

$$\frac{b_s(m_s + m_u)}{m_s m_u} > 0 \quad (13)$$

$$k_s(m_s^2 + m_u^2) + k_t m_s^2 + 2k_s m_s m_u > 0 \quad (14)$$

$$k_t m_s^2 > 0 \quad (15)$$

As system parameters m_s, m_u, b_s, k_t and k_s are all positive, both the stability conditions are always satisfied. Hence, the system is BIBO stable. This is expected because the system is a series combination of two mass-spring-damper systems.

System Response

There are two typical conditions which have been investigated in this project. The first is a response under static conditions. In this case, the system is loaded, and at time $t = 0$, this load is removed. Therefore, the case becomes an initial condition problem with no road input. Simulations for this case with initial deflection of -5`cm` in the sprung mass are shown in Fig. $2-5$. It can be seen that the system settles to the equilibrium position (taken to be zero in this case) after some oscillations.

The second case is a dynamic case with road input. In this case, the road profile is selected to be a sinusoidal input with an amplitude of 5`cm` and a frequency of 1`Hz`. The initial conditions of the states are specified to be zero in this case. The simulation results are shown in Fig. 6–9. It is to be noted that in both the cases presented, there is no actuator force in the passive suspension.

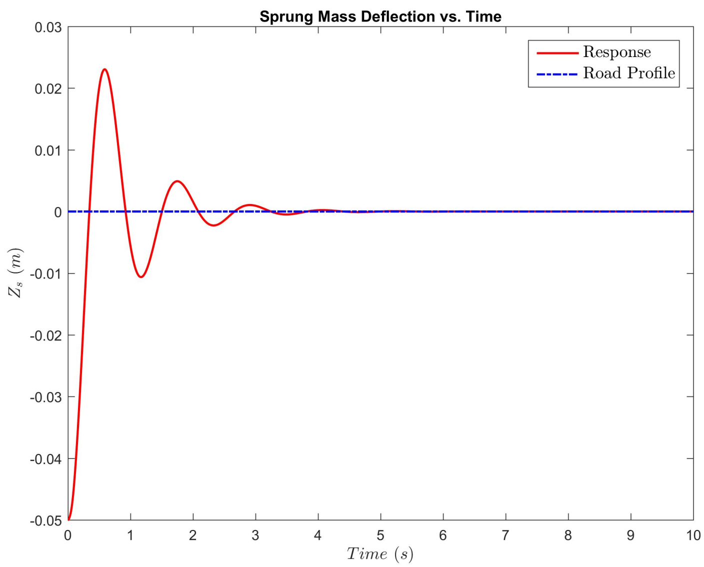

Figure 2: Sprung mass deflection for static case.

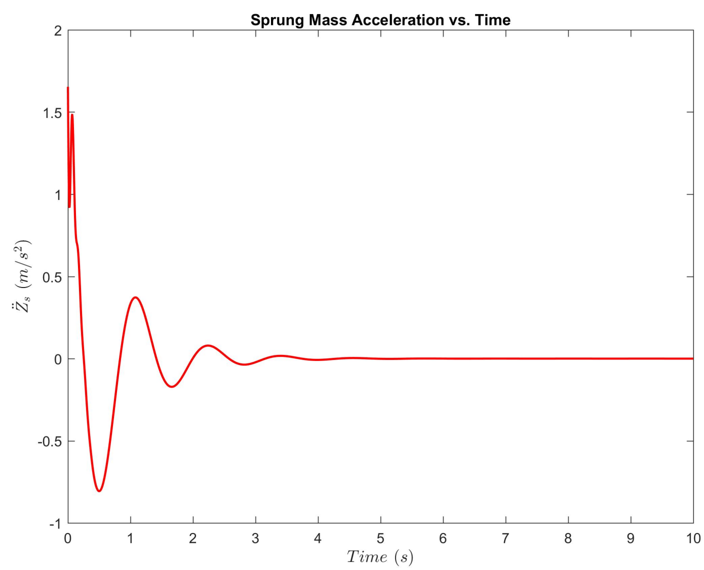

Figure 3: Sprung mass acceleration for static case.

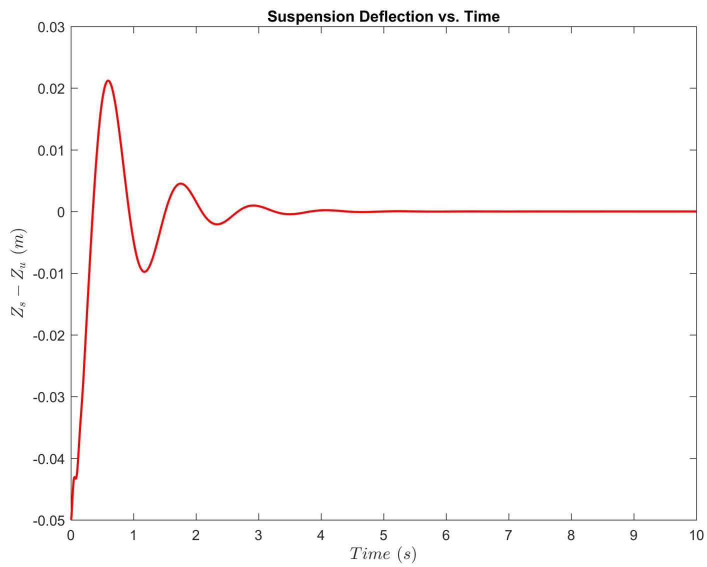

Figure 4: Suspension deflection for static case.

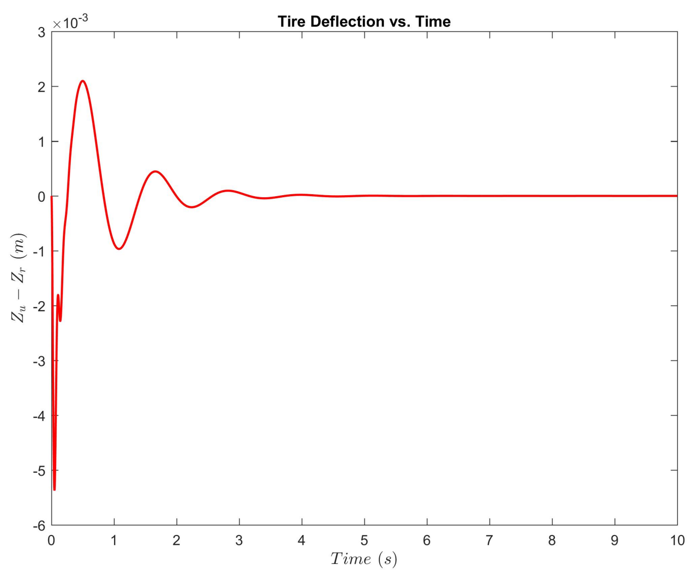

Figure 5: Tire deflection for static case.

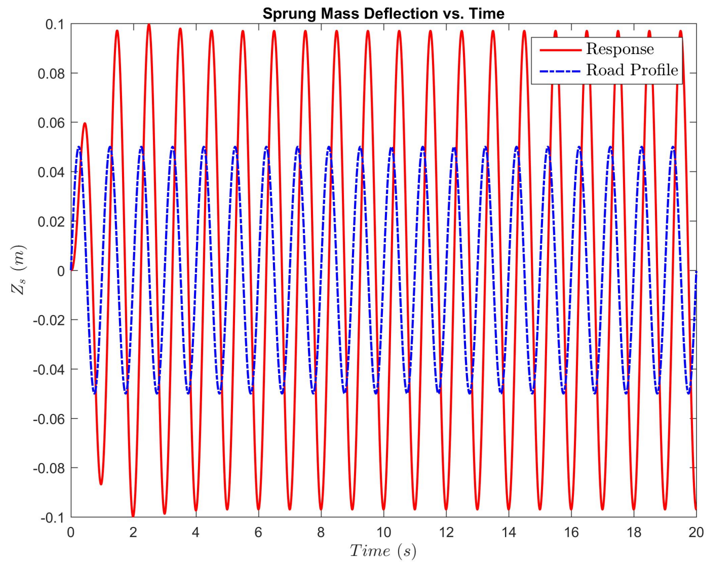

Figure 6: Sprung mass deflection for dynamic case.

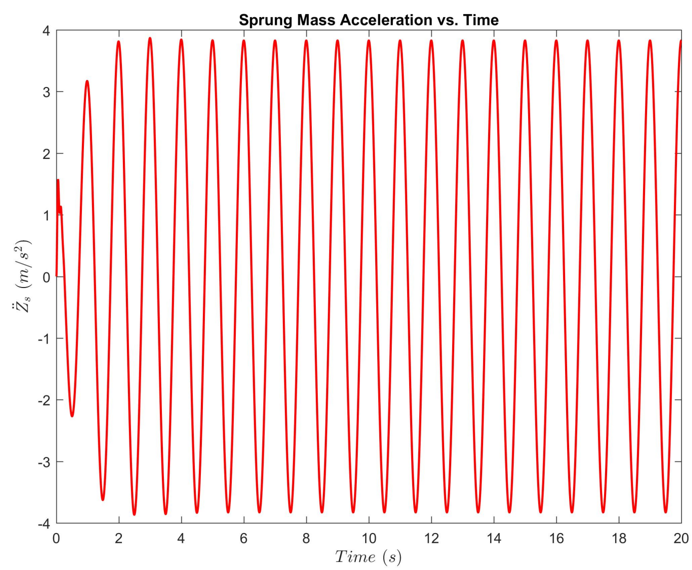

Figure 7: Sprung mass acceleration for dynamic case.

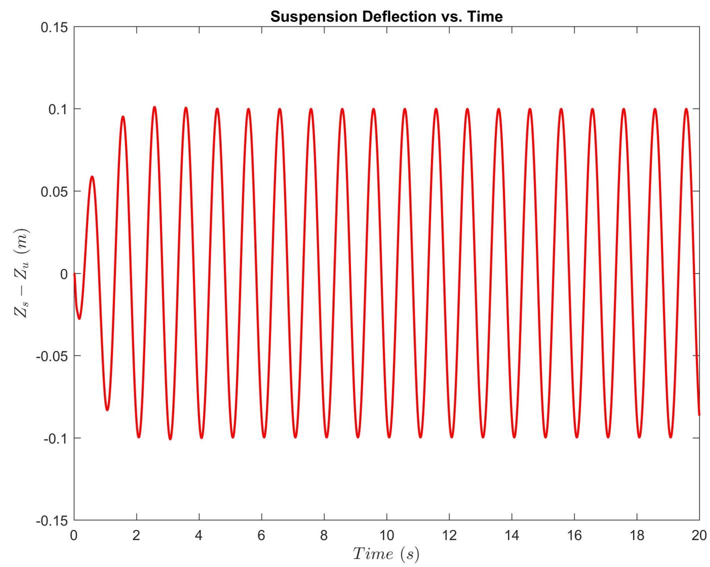

Figure 8: Suspension deflection for dynamic case.

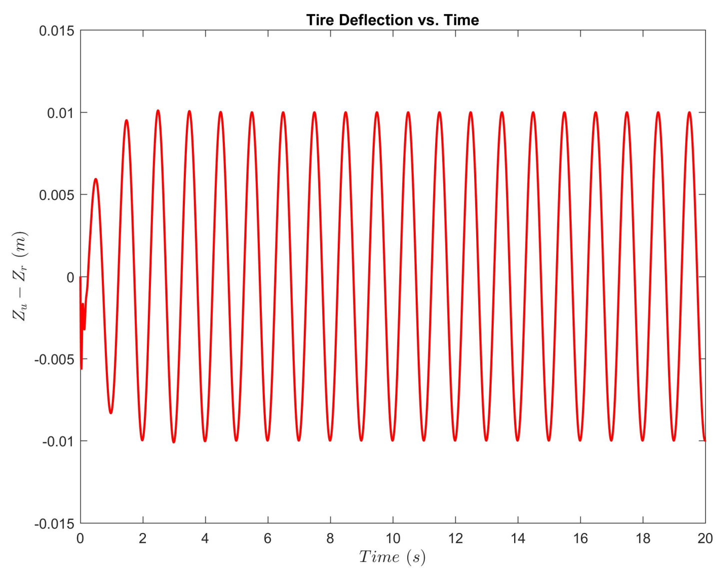

Figure 9: Tire deflection for dynamic case.

Eigenvalues, Eigenvectors and Canonical Forms

For the parameters used for simulations, the open loop eigenvalues of the system are as follows:

$$\lambda_{1-4} = \begin{bmatrix} -15.6805 + 62.3981i \\ -15.6805 - 62.3981i \\ -1.3326 + 5.4133i \\ -1.3326 - 5.4133i \end{bmatrix}$$

The corresponding eigenvectors are can be calculated using the equation,

$$[\lambda_i I - A]\, v_i = 0 \tag{16}$$

where v_i are the eigenvectors. The eigenvectors can be found using MATLAB as follows:

$$T = \begin{bmatrix} 0.0032 + 0.0155i & 0.0032 - 0.0155i \\ -0.0171 - 0.0443i & -0.0171 + 0.0443i \\ -0.0038 - 0.0151i & -0.0038 + 0.0151i \\ 0.9986 & 0.9986 \end{bmatrix} \cdots$$
$$\begin{matrix} -0.0456 - 0.1574i & -0.0456 + 0.1574i \\ -0.9832 & 0.9832 \\ -0.0035 - 0.0138i & 0.0035 + 0.0138i \\ 0.0701 + 0.0371i & 0.0701 - 0.0371i \end{matrix} \Bigg]$$

The modified transformation matrix can be written as:

$$\bar{T} = \begin{bmatrix} 0.0032 & 0.0155 & -0.0456 & -0.1574 \\ -0.0171 & -0.0443 & 0.9832 & 0 \\ -0.0038 & -0.0151 & 0.0035 & -0.0138 \\ 0.9986 & 0 & 0.0701 & 0.0371 \end{bmatrix}$$

Now, using the equation $\Lambda = \bar{T}^{-1} A \bar{T}$, the canonical form Λ can be shown as:

$$\Lambda = \begin{bmatrix} -15.6805 & 62.3981 & 0 & 0 \\ -62.3981 & -15.6805 & 0 & 0 \\ 0 & 0 & -1.3326 & 5.4133 \\ 0 & 0 & -5.4133 & -1.3326 \end{bmatrix}$$

which is nothing but modified diagonal form of the eigenvalue matrix.

Controllability and Observability

In the actively controlled suspension, since F_a is the controlled input, the controllability depends on the matrix pair (A, B). The controllability matrix is written as:

$$\begin{aligned}\zeta &= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0.0243 & -0.8270 & -66.6770 \\ 0.0022 & -0.0750 & 1.7491 & 233.1922 \\ 0 & -0.0221 & 0.7520 & 68.4261 \\ -0.0221 & 0.7520 & 68.4261 & -5261.8372 \end{bmatrix}\end{aligned}$$

The rank of the controllability matrix, ζ, is 4, and therefore, the system is controllable with the actuator shown in Fig. 1.

The observability of the system depends on the outputs of the system. If the suspension deflection and the velocity of the sprung mass are available as outputs, the output matrix, C, can be written as:

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Using this matrix, the observability matrix can be written as:

$$O = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -33.0761 & -3.0871 & 0 & 3.0871 \\ -364.5678 & -34.0263 & 3889.5028 & 34.0263 \\ 1125.4573 & 71.9666 & -12007.2853 & 71.9666 \\ 12404.9027 & 793.2231 & -132345.4923 & 3096.2796 \\ -26236.7078 & -1323.3020 & 279914.3280 & -10683.9832 \end{bmatrix}$$

The observability matrix, O, is full rank, and therefore, the system is observable.

Open Loop Optimal Control

The open loop optimal controller can be formulated using the Pontryagin function [3]. For the open loop optimal control, the same state equation is valid, i.e., equation 3. However, an additional quadratic cost function is defined as,

$$J = \frac{1}{2}\int_0^{t_f} \left(\ddot{z}_s + \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2\right) dt \tag{17}$$

where $x_{1,2,3,4}$ are the states. The cost function puts weightage on the three performance parameters defined earlier (sprung mass acceleration, suspension deflection and tire deflection), as well as on the sprung and unsprung mass velocities. Using the state-space formulation (equation 3), the above cost function can be reformulated as:

$$J = \frac{1}{2}\int_0^{t_f} \left(\mathbf{x}^T Q \mathbf{x} + 2\mathbf{x}^T N F_a + F_a^T R F_a\right) dt \tag{18}$$

where

$$R = \frac{1}{m_s^2} \tag{19}$$

$$N = \begin{bmatrix} -\frac{k_s}{m_s^2} & -\frac{b_s}{m_s^2} & 0 & \frac{b_s}{m_s^2} \end{bmatrix}^T \tag{20}$$

$$Q = \begin{bmatrix} \frac{k_s^2}{m_s^2} + \rho_1 & \frac{b_s k_s}{m_s^2} & 0 & -\frac{b_s k_s}{m_s^2} \\ \frac{b_s k_s}{m_s^2} & \frac{b_s^2}{m_s^2} + \rho_2 & 0 & -\frac{b_s^2}{m_s^2} \\ 0 & 0 & \rho_3 & 0 \\ -\frac{b_s k_s}{m_s^2} & -\frac{b_s^2}{m_s^2} & 0 & \frac{b_s^2}{m_s^2} + \rho_4 \end{bmatrix} \tag{21}$$

Therefore, the Pontryagin function, H, can be defined as follows:

$$H = \frac{1}{2}\left(\mathbf{x}^T Q\mathbf{x} + 2\mathbf{x}^T N F_a + F_a^T R F_a\right) + \boldsymbol{\lambda}^T \left(A\mathbf{x} + BF_a + L\dot{z}_r\right) \tag{22}$$

From equation 22, the state and co-state equations can be derived as:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \boldsymbol{\lambda}} \tag{23}$$

$$= A\mathbf{x} + BF_a + L\dot{z}_r \tag{24}$$

$$\dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \mathbf{x}} \tag{25}$$

$$= -Q\mathbf{x} - NF_a - A^T\boldsymbol{\lambda} \tag{26}$$

The stationarity equation is defined as:

$$0 = \frac{\partial H}{\partial F_a} \tag{27}$$

$$= RF_a + B^T\boldsymbol{\lambda} + N^T\mathbf{x} \tag{28}$$

$$\implies F_a = -R^{-1}B^T\boldsymbol{\lambda} - R^{-1}N^T\mathbf{x} \tag{29}$$

Using equation 29 in equation 24 and equation 26, the state and co-state equations can be written as:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\boldsymbol{\lambda}} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}N^T & -BR^{-1}B^T \\ -(Q - NR^{-1}N^T) & NR^{-1}B^T - A^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix} + \begin{bmatrix} L\dot{z}_r \\ \mathbf{0} \end{bmatrix} \tag{30}$$

Now that the state and co-state equations have been defined, the boundary conditions have to be defined for simulation. For the initial time, the states of the system, x_{1-4}, are defined as shown earlier in the 'System Response' section. For the static case, the system is given an initial condition due to unloading, whereas in the dynamic case, the system states start from zero. Therefore,

$$\mathbf{x}_0 = \begin{bmatrix} x_{1,0} & 0 & 0 & 0 \end{bmatrix}^T \tag{31}$$

The optimal controller also has co-states, and therefore, needs four more boundary conditions. These are defined at the final time, $t = t_f$, by the equation,

$$\left(\frac{\partial S}{\partial \mathbf{x}} - \boldsymbol{\lambda}\right)^T \delta\mathbf{x} \quad = \quad 0 \tag{32}$$

$$\implies \boldsymbol{\lambda}|_{t_f} \quad = \quad \left.\frac{\partial S}{\partial \mathbf{x}}\right|_{t_f} \tag{33}$$

Since there is no contribution of the final states in the cost function, $S = 0$, and therefore, $\boldsymbol{\lambda}|_{t_f} = 0$. This is the two-point boundary value problem.

For the open loop optimal control strategy, the same test cases are used for simulations. Results for the static case with initial deflection are presented in Fig. 10–13. For the case with sinusoidal road input, the optimal control results are shown in Fig. 14–17.

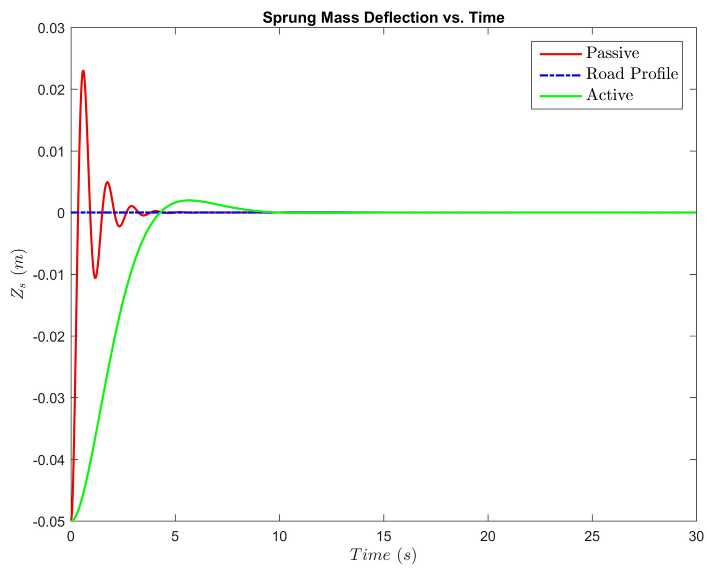

Figure 10: Sprung mass deflection for static case with active control.

As seen in both the cases, the active suspension with open loop optimal control leads to better performance than the passive suspension system, as characterized by attenuation in all three performance parameters. In the static test cases, the response of the active suspension is slow as the cost function J, defined in equation 17, also has some weightage on the unsprung and sprung mass velocities. In the dynamic case, the responses change at the end because of the defined boundary conditions.

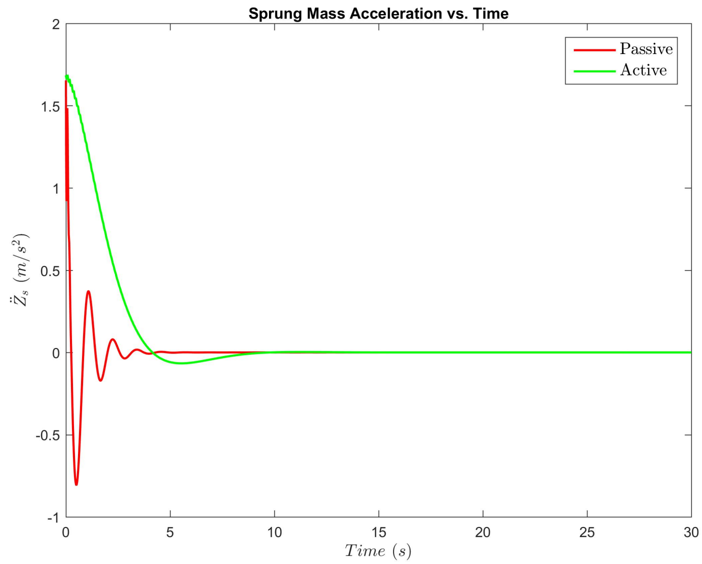

Figure 11: Sprung mass acceleration for static case with active control.

The co-states are shown in Fig. 18. Because the problem is defined as a finite time optimal control problem, the boundary values of the co-states have to be satisfied at the final time, $t = t_f$. Therefore, the two-point boundary value solution causes changes in values in all the states near the final time in order to satisfy the co-state boundary values, as seen in Fig. 14–17. If this problem is reformulated as an infinite time optimal control problem, the states do not diverge.

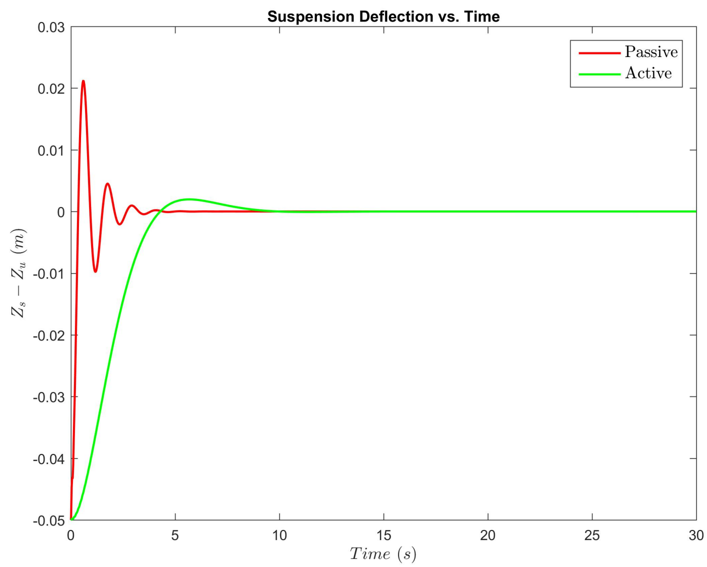

Figure 12: Suspension deflection for static case with active control.

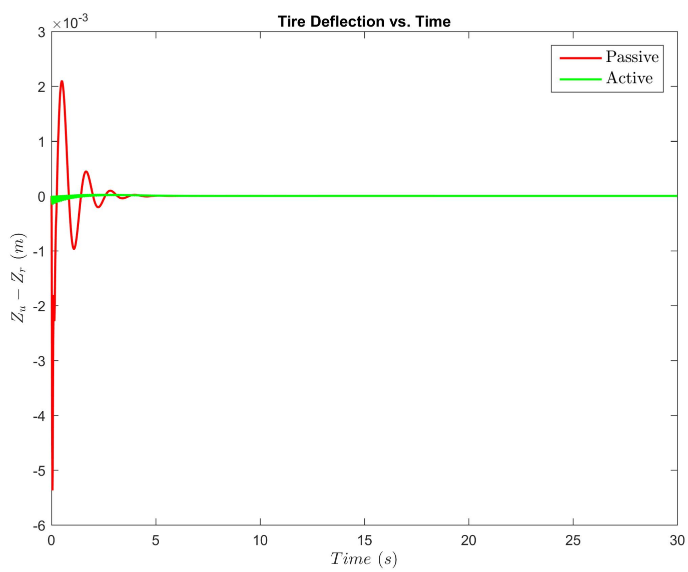

Figure 13: Tire deflection for static case with active control.

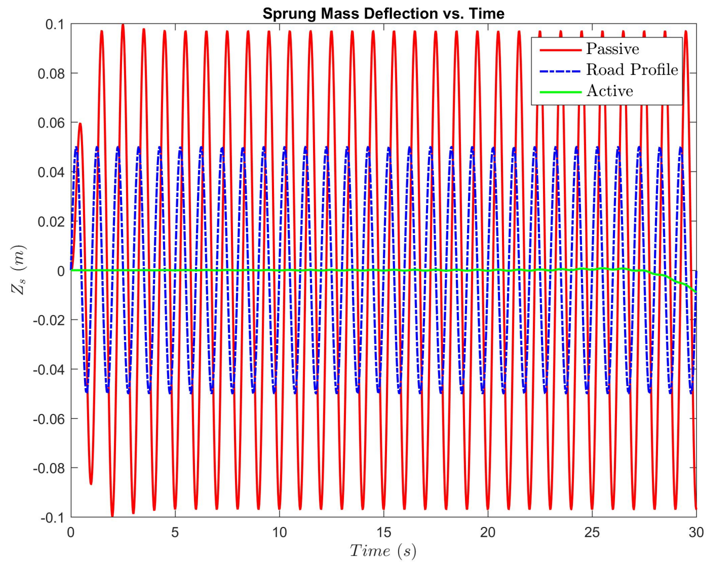

Figure 14: Sprung mass deflection for dynamic case with active control.

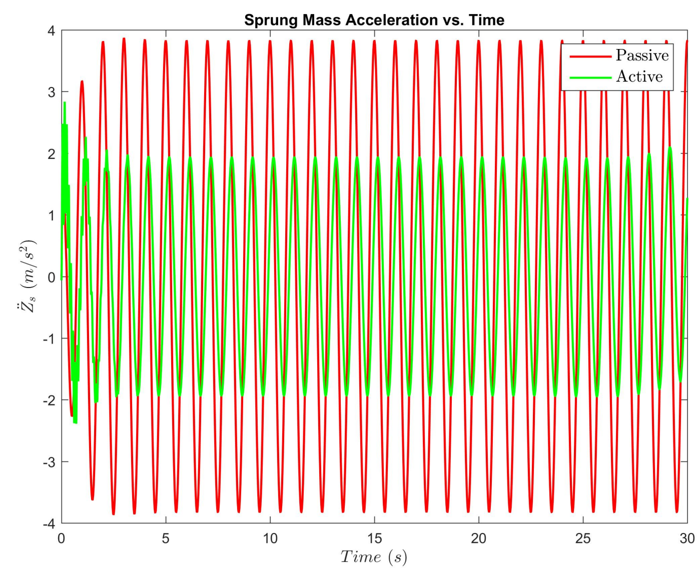

Figure 15: Sprung mass acceleration for dynamic case with active control.

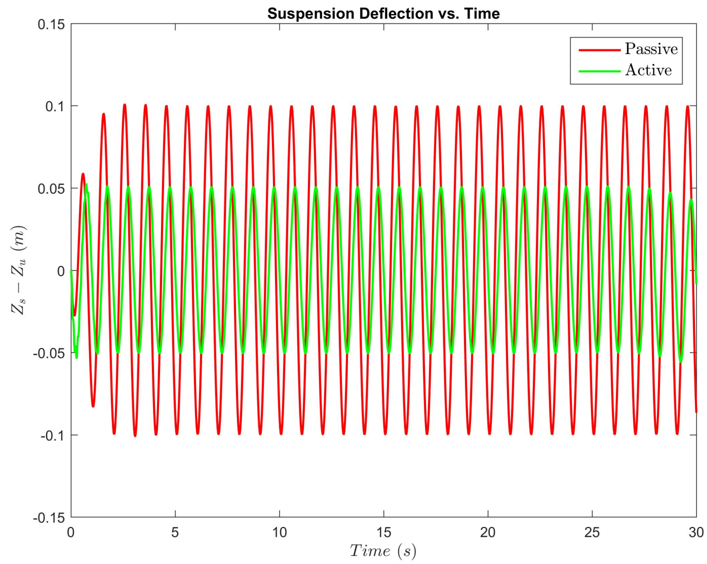

Figure 16: Suspension deflection for dynamic case with active control.

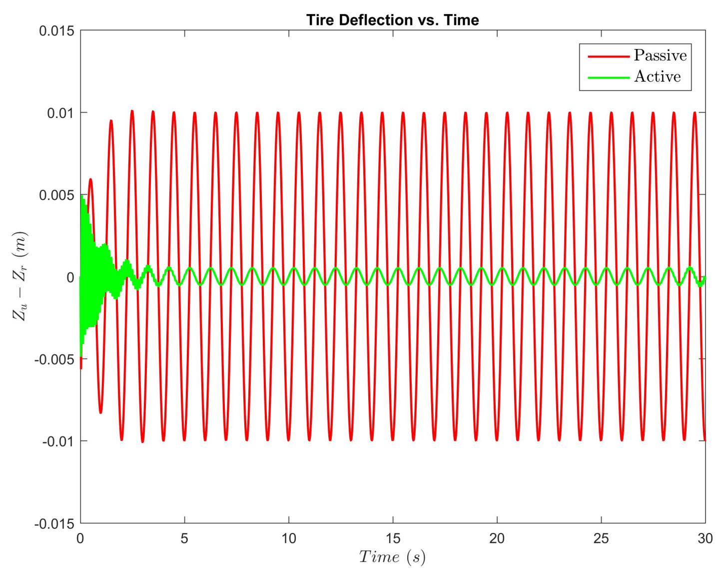

Figure 17: Tire deflection for dynamic case with active control.

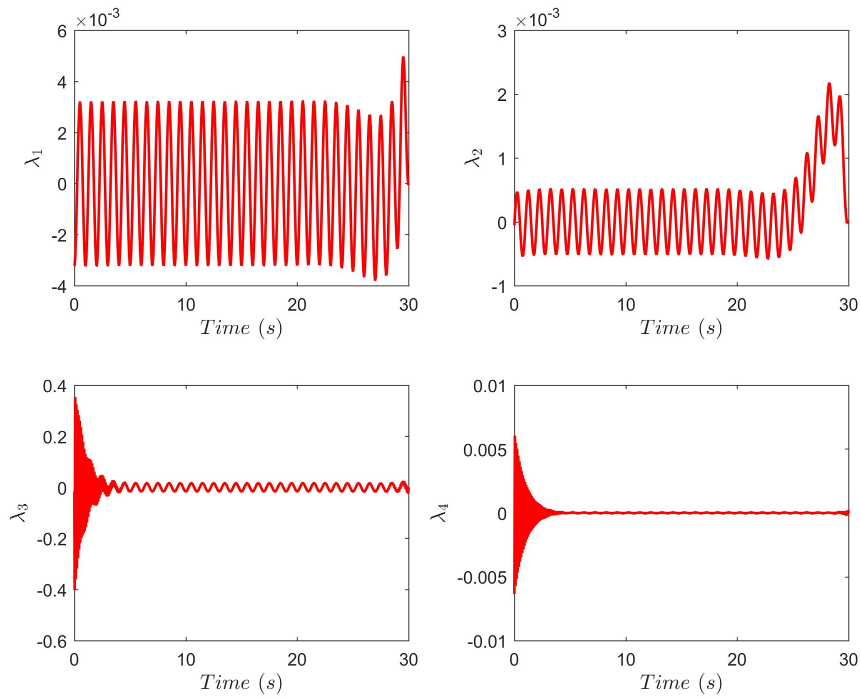

Figure 18: Co-states for dynamic case with active control.

Codes

```
clear
clc

%% Parameters

global ks kt ms mu bs bt w Amp A B C L rho1 rho2 rho3 rho4 R N Q ...
    Rinv x10 x20 x30 x40

kt = (704*10^3)/4; % N/m
ks = 15*10^3; % N/m
bs = 1400; % Ns/m  Check
bt = 0; % Ns/m
mu = 181/4; % kg
ms = 1814/4; % kg
rho1 = 0.400;
rho2 = 0.040;
rho3 = 0.400;
rho4 = 0.040;
Amp = 0.05;
w = 1*2*pi;
t0 = 0;
tf = 20;
steps = 20000;

A = [0 1 0 -1; -ks/ms -bs/ms 0 bs/ms; 0 0 0 1; ks/mu bs/mu -kt/mu ...
    -(bs+bt)/mu ];
B = [0;1/ms;0;-1/mu];
L = [0;0;-1;0];
C = [1 0 0 0;0 1 0 0];

R = 1/ms^2;
Rinv = 1/R;
N = [-ks/ms^2; -bs/ms^2; 0; bs/ms^2];
Q = [(ks^2/ms^2 + rho1)  bs*ks/ms^2           0      -bs*ks/ms^2;
     bs*ks/ms^2          (bs^2/ms^2 + rho2)    0      -bs^2/ms^2;
     0                   0                    rho3   0;
     -bs*ks/ms^2         -bs^2/ms^2           0      (bs^2/ms^2 + ...
         rho2)];

%% Q2

% x1 = zs-zu
% x2 = zsdot
% x3 = zu-zr
% x4 = zudot

zs0 = 0;
zu0 = 0;
zsdot0 = 0;
zudot0 = 0;
zr0 = 0;

x10 = zs0-zu0;
```

```
x20 = zsdot0;
x30 = zu0-zr0;
x40 = zudot0;

x0 = [x10; x20; x30; x40];

tspan = [t0 tf];
[T,Y] = rk4fixed(@car,tspan,x0,steps);

lengthPass = size(T,1);
zacclPass = zeros(1,lengthPass);

for i = 1:lengthPass
    [xdot, zsddotPass] = car(T(i),Y(i,:)');
    zacclPass(1,i) = zsddotPass;
end

u = 0;
ZR = Amp*sin(w*T);

zu = Y(:,3) + ZR;
zs = Y(:,1) + zu;

fig = figure(1);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,zs,'-r','LineWidth',1.5)
hold on
plot(T,ZR,'-.b','LineWidth',1.5)
title('Sprung Mass Deflection vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_s\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
legend('Response','Road Profile')
set(legend,'Interpreter','Latex','FontSize',12)
print('Passive-SMD-Dynamic','-djpeg','-r300')

fig = figure(2);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,zacclPass,'-r','LineWidth',1.5)
title('Sprung Mass Acceleration vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$\ddot{Z}_s\hspace{0.05in}(m/s^2)$','Interpreter','Latex','FontSize',12)
print('Passive-SMA-Dynamic','-djpeg','-r300')

fig = figure(3);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,Y(:,1),'-r','LineWidth',1.5)
title('Suspension Deflection vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_s - ...
    Z_u\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
print('Passive-SD-Dynamic','-djpeg','-r300')
```

```matlab
fig = figure(4);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,Y(:,3),'-r','LineWidth',1.5)
title('Tire Deflection vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_u - ...
    Z_r\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
print('Passive-TD-Dynamic','-djpeg','-r300')

%% Q3

disp('Eigen Values of A');
[EigVec,EigVal] = eig(A);
EigVal
eig(A)
disp('Eigen Vectors of A');
EigVec

sigma1 = real(EigVal(1,1));
omega1 = imag(EigVal(1,1));
sigma2 = real(EigVal(3,3));
omega2 = imag(EigVal(3,3));

disp('Canonical Form (Two Imaginary)');
Canon = [-sigma1 omega1 0 0;-omega1 -sigma1 0 0;0 0 -sigma2 ...
    omega2;0 0 -omega2 -sigma2]

%% Q4

Co = ctrb(A,B);
disp('Controllability Matrix');
Co
Corank = rank(Co)
disp('Rank of Controllability Matrix');
Corank = rank(Co)
OB = obsv(A,C);
disp('Obersvability Matrix');
OB
disp('Rank of Observability Matrix');
Obrank = rank(OB)

%% Q5

solinit = bvpinit(linspace(t0,tf,steps),[0 0 0 0 0 0 0 0]);
sol = bvp4c(@OLoptimalControl,@OLcontrolBC,solinit);

length = size(sol.x,2);
xdotMat = zeros(length,8);
Force = zeros(1,length);
zaccl = zeros(1,length);

for i = 1:length
    [xdot, F, zsddot] = OLoptimalControl(sol.x(i),sol.y(:,i));
    xdotMat = xdot;
```

```
    Force(1,i) = F;
    zaccl(1,i) = zsddot;
end

fig = figure(5);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,zs,'-r','LineWidth',1.5)
hold on
plot(T,ZR,'-.b','LineWidth',1.5)
plot(sol.x,(sol.y(1,:) + sol.y(3,:) + ...
    ZR'),'Color','green','LineWidth',1.5)
title('Sprung Mass Deflection vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_s\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
legend('Passive','Road Profile','Active ')
set(legend,'Interpreter','Latex','FontSize',12)
print('Active-SMD-Dynamic','-djpeg','-r300')

fig = figure(6);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,zacclPass,'-r','LineWidth',1.5)
hold on
plot(T,zaccl,'-g','LineWidth',1.5)
title('Sprung Mass Acceleration vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$\ddot{Z}_s\hspace{0.05in}(m/s^2)$','Interpreter','Latex','FontSize',12)
legend('Passive','Active ')
set(legend,'Interpreter','Latex','FontSize',12)
print('Active-SMA-Dynamic','-djpeg','-r300')

fig = figure(7);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,Y(:,1),'-r','LineWidth',1.5)
hold on
plot(T,sol.y(1,:),'-g','LineWidth',1.5)
title('Suspension Deflection vs. Time')
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_s - ...
    Z_u\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
legend('Passive','Active ')
set(legend,'Interpreter','Latex','FontSize',12)
print('Active-SD-Dynamic','-djpeg','-r300')

fig = figure(8);
set(fig,'Position',[1800 -320 1200 1000])
clear title
clear legend
plot(T,Y(:,3),'-r','LineWidth',1.5)
hold on
plot(T,sol.y(3,:),'-g','LineWidth',1.5)
title('Tire Deflection vs. Time')
```

```
xlabel('$Time\hspace{0.05in}(s)$','Interpreter','Latex','FontSize',12)
ylabel('$Z_u - ...
    Z_r\hspace{0.05in}(m)$','Interpreter','Latex','FontSize',12)
legend('Passive','Active')
set(legend,'Interpreter','Latex','FontSize',12)
print('Active-TD-Dynamic','-djpeg','-r300')

%%

function [y,zsddot] = car(t,x)

global ks bs ms w Amp A B L

x1 = x(1);
x2 = x(2);
x3 = x(3);
x4 = x(4);

F = 0;

zrdot = Amp*w*cos(w*t);

y = A*x + B*F + L*zrdot;
zsddot = (F - ks*x1 - bs*(x2 - x4))/ms;
end

%%

function [xdot, F, zsddot]=OLoptimalControl(t,x)

global ks ms bs w Amp A B L N Q Rinv

x1 = x(1);
x2 = x(2);
x3 = x(3);
x4 = x(4);

X = [x1; x2; x3; x4];

zrdot = Amp*w*cos(w*t);

lambda1 = x(5);
lambda2 = x(6);
lambda3 = x(7);
lambda4 = x(8);

lambda = [lambda1; lambda2; lambda3; lambda4];

% Stationarity equation

F = -Rinv*B'*lambda;

% State and Co-state equations

Xdot = A*X - (B*Rinv*B')*lambda + L*zrdot -(B*Rinv)*X'*N;
lambdadot = -Q*X + (N*Rinv*B' - A')*lambda + N*Rinv*X'*N;
```

```
xdot = [Xdot;lambdadot];

zsddot = (F - ks*x1 - bs*(x2 - x4))/ms;

%%

function residual = OLcontrolBC(ya,yb)

global x10

residual = [(ya(1) - x10); ya(2); ya(3); ya(4); yb(5); yb(6); ...
    yb(7); yb(8)];
```

References

[1] Rajamani, Rajesh, *Vehicle Dynamics and Control*. Mechanical Engineering Series, Springer Science & Business Media, 2011.

[2] Wong, J. Y., *Theory of Ground Vehicles*. John Wiley & Sons, 2001 (3rd edition).

[3] Lewis, F. L. and Syrmos, V. L., *Optimal Control*. John Wiley & Sons, 1995.