Week_3

정유진

Two parameter models

① Normal data with noninformative prior distribution

Likelihood : $y_i|\mu,\sigma^2 \sim N(\mu,\sigma^2)$ $\Rightarrow$ likelihood

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}}\cdot \exp\left[-\frac{1}{2\sigma^2}(y_i-\mu)^2\right]$$

$$= \underbrace{\left(\frac{1}{\sqrt{2\pi}}\right)^{n}}_{상수}\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\cdot \exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right]$$

$$\propto \sigma^{-n}\cdot \exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right]$$

prior: $p(\mu,\sigma^2) \propto (\sigma^2)^{-1}$ $\longleftarrow$ $p(\mu,\sigma^2) = p(\mu)\cdot p(\sigma^2)$ (improper prior)

$\propto 1\cdot \frac{1}{\sigma^2}$

$p(\sigma^2) \propto \frac{1}{\sigma^2}$

$p(\log\sigma^2) \propto 1$

$p(\mu) \propto 1$

posterior: $p(\mu,\sigma^2|y) = \dfrac{p(\mu,\sigma^2)\,p(y|\mu,\sigma^2)}{\int p(y|\mu,\sigma^2)\cdot p(\mu,\sigma^2)\,d\mu\, d\sigma^2}$ → constant $\propto$ $p(\mu,\sigma^2)\cdot p(y|\mu,\sigma^2)$

$$= \sigma^{-n}\cdot\sigma^{-2}\cdot \exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right] = \sigma^{-n-2}\cdot \exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right]$$

→ $\sum(y_i-\bar{y}+\bar{y}-\mu)^2$

$= \sum(y_i-\bar{y})^2 + \sum(\bar{y}-\mu)^2 + 2\underbrace{\sum(y_i-\bar{y})}_{0}(\bar{y}-\mu)$

$$= \sigma^{-n-2}\cdot \exp\left[-\frac{1}{2\sigma^2}\left\{\sum(y_i-\bar{y})^2 + n(\bar{y}-\mu)^2\right\}\right]$$

$s^2 = \dfrac{\sum(y_i-\bar{y})^2}{n-1}$ (Sample variance of y_i's)

$$= \sigma^{-n-2}\cdot \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2 + n(\bar{y}-\mu)^2\right\}\right].$$

$s^2, \bar{y}$: sufficient statistic

$$p(\theta|y_1,\ldots,y_n,\sigma^2) = p(\theta|\sigma^2)p(y_1,\ldots,y_n|\theta,\sigma^2)/p(y_1,\ldots,y_n|\sigma^2)$$
$$\propto p(\theta|\sigma^2)p(y_1,\ldots,y_n|\theta,\sigma^2)$$
$$\propto \exp\{-\frac{1}{2\tau_0^2}(\theta-\mu_0)^2\}\exp\{-\frac{1}{2\sigma^2}\sum(y_i-\theta)^2\}.$$

Adding the terms in the exponents and ignoring the -1/2 for the moment, we have

$$\frac{1}{\tau_0^2}(\theta^2-2\theta\mu_0+\mu_0^2)+\frac{1}{\sigma^2}(\sum y_i^2-2\theta\sum y_i+n\theta^2) = a\theta^2-2b\theta+c, \text{ where}$$

$$a=\frac{1}{\tau_0^2}+\frac{n}{\sigma^2},\quad b=\frac{\mu_0}{\tau_0^2}+\frac{\sum y_i}{\sigma^2},\quad \text{and } c=c(\mu_0,\tau_0^2,\sigma^2,y_1,\ldots,y_n).$$

Now let's see if $p(\theta|\sigma^2,y_1,\ldots,y_n)$ takes the form of a normal density:

$$p(\theta|\sigma^2,y_1,\ldots,y_n) \propto \exp\{-\frac{1}{2}(a\theta^2-2b\theta)\}$$
$$=\exp\{-\frac{1}{2}a(\theta^2-2b\theta/a+b^2/a^2)+\frac{1}{2}b^2/a\}$$
$$\propto \exp\{-\frac{1}{2}a(\theta-b/a)^2\}$$
$$=\exp\left\{-\frac{1}{2}\left(\frac{\theta-b/a}{1/\sqrt{a}}\right)^2\right\}.$$

FCB 5-2.

$\mu_n = \frac{b}{a} = \tau_n^2 \cdot b$

$\tau_n^2 = \frac{1}{a}$

Posterior: $p(\mu,\sigma^2|y) = \underbrace{p(\mu|\sigma^2,y)}_{①}\cdot\underbrace{p(\sigma^2|y)}_{②}$

① $p(\mu|\sigma^2,y)$ = μ 모르고 σ^2 알 때 posterior distribution

(복습)

- $\mu \sim N(\mu_0, \tau_0^2) \propto \exp[-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2]$
- $y|\mu \sim N(\mu,\sigma^2) \propto \exp[-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2]$

$$p(\mu|y) \propto p(\mu)\cdot p(y|\mu) = \exp[-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2]$$
$$= \exp[-\frac{1}{2\tau_0^2}(\mu^2-2\mu_0\mu+\cancel{\mu_0^2})-\frac{1}{2\sigma^2}\Sigma(\cancel{y_i^2}-2\mu y_i+\mu^2)]$$
$$\propto \exp[-\frac{1}{2}(\frac{\mu^2}{\tau_0^2}-\frac{2\mu_0}{\tau_0^2}\mu-\frac{2\mu\underbrace{\Sigma y_i}_{=n\bar{y}}}{\sigma^2}+\frac{n\mu^2}{\sigma^2})]$$
$$= \exp[-\frac{1}{2}\{(\frac{1}{\tau_0^2}+\frac{n}{\sigma^2})\mu^2+(-\frac{2\mu_0}{\tau_0^2}-\frac{2n\bar{y}}{\sigma^2})\mu\}]$$

$a=\frac{1}{\tau_n^2}$ $\quad b=\frac{2\mu_n}{\tau_n^2}$

$\Rightarrow \tau_n^2=\frac{1}{a}, \quad \mu_n = -\frac{1}{2}\tau_n^2\cdot b$

$\therefore \tau_n^2 = \frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}$

$\mu_n = -\frac{1}{2}\cdot\frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\cdot(-\frac{2\mu_0}{\tau_0^2}-\frac{2n\bar{y}}{\sigma^2}) = \left(\frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\right)\mu_0+\left(\frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\right)\bar{y}$

가중평균의 형태!

$$\mu|y \sim N\left(\frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\mu_0+\frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\bar{y},\ \frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\right)$$

★ noninformative prior.

prior의 분산→∞ precision→0

$\Rightarrow \frac{1}{\tau_0^2}\to 0$

$\therefore \mu|\sigma^2,y \sim N(\bar{y},\frac{\sigma^2}{n})$

$\boxed{\chi^{-2}(v,\tau^2) = \Gamma^{-1}(\frac{v}{2},\frac{v\tau^2}{2})}$

② $p(\sigma^2|y) = \int p(\mu,\sigma^2|y)\,d\mu \propto \int \sigma^{-n-2}\exp[-\frac{1}{2\sigma^2}\{(n-1)s^2+n(\bar{y}-\mu)^2\}]\,d\mu$

$$= \sigma^{-n-2}\exp(-\frac{1}{2\sigma^2}(n-1)s^2)\cdot\underbrace{\int\exp[-\frac{n}{2\sigma^2}(\mu-\bar{y})^2]\,d\mu}_{=\sqrt{\frac{2\pi\sigma^2}{n}}\text{ : normalizing constant.}}$$

$$= \sigma^{-n-2}\exp(-\frac{1}{2\sigma^2}(n-1)s^2)\cdot\sqrt{\frac{2\pi\sigma^2}{n}}$$
$$\propto \sigma^{-(n+1)}\exp(-\frac{1}{2\sigma^2}(n-1)s^2) = (\sigma^2)^{-\frac{(n+1)}{2}}\exp(-\frac{1}{2\sigma^2}(n-1)s^2)$$

$\therefore p(\sigma^2|y) \sim \text{Inv-}\chi^2(n-1,s^2)$

$= \text{Inv-gamma}(\frac{n-1}{2},\frac{(n-1)\cdot s^2}{2})$

inverse gamma pdf

$$\frac{\beta^\alpha}{\Gamma(\alpha)}y^{-\alpha-1}e^{-\frac{\beta}{y}}$$

$\alpha=\frac{v}{2},\ \beta=\frac{v\tau^2}{2}$

scaled inverse chisquare pdf

Scale-Inv-$\chi^2(v,\tau^2)$

$$\frac{(\tau^2\cdot\frac{v}{2})^{\frac{v}{2}}}{\Gamma(\frac{v}{2})}\cdot x^{-(1+\frac{v}{2})}\cdot\exp[-\frac{v\tau^2}{2x}]$$

Posterior: $= p(\mu|\sigma^2, y)\cdot p(\sigma^2|y) \Rightarrow N\left(\mu \mid \bar{y}, \frac{\sigma^2}{n}\right)\cdot \text{Inv-}\chi^2(\sigma^2 \mid n-1, s^2)$

$\Rightarrow$ conditional posterior $dist^n$ $\times$ marginal posterior $dist^n$

$\sigma^2 \sim \sigma^2|y = \text{Inv-}\chi^2(\sigma^2|n-1, s^2) \Rightarrow \mu \sim \mu|\sigma^2, y = N\left(\mu|\bar{y}, \frac{\sigma^2}{n}\right) \Rightarrow (\mu, \sigma^2) \sim \mu, \sigma^2 | y$

Sampling from the joint posterior distribution

It is easy to draw samples from the joint posterior distribution: first draw σ^2 from (3.5), then draw μ from (3.3). We also derive some analytical results for the posterior distribution, since this is one of the few multiparameter problems simple enough to solve in closed form.

* Analytic form of the marginal posterior distribution of μ.

$$p(\mu|y) = \int_0^\infty p(\mu, \sigma^2|y)\, d\sigma^2 \propto \int_0^\infty \sigma^{-(n+2)} \exp\left[-\frac{1}{2\sigma^2}\{(n-1)s^2 + n(\bar{y}-\mu)^2\}\right] d\sigma^2$$

$A = (n-1)s^2 + n(\bar{y}-\mu)^2$

$z = \frac{A}{2\sigma^2}$, $\sigma^2 = \frac{A}{2z}$, $J = \left|\frac{d\sigma^2}{dz}\right| = \frac{A}{2}\cdot z^{-2}$

$$\Rightarrow \int_0^\infty \left(\frac{A}{2z}\right)^{-\frac{(n+2)}{2}} \cdot \exp(-z)\cdot \frac{A}{2}\cdot z^{-2}\, dz$$

$$\propto A^{-\frac{n}{2}} \int_0^\infty \cancel{A^{-1}}\cdot \cancel{A}\cdot \exp(-z)\cdot z^{\frac{n}{2}-1}\, dz = A^{-\frac{n}{2}} \underbrace{\int_0^\infty z^{\frac{n}{2}-1} \exp(-z)\, dz}_{=\Gamma\left(\frac{n}{2}, 1\right)} \propto A^{-\frac{n}{2}}$$

$$A^{-\frac{n}{2}} = \left[(n-1)s^2 + n(\bar{y}-\mu)^2\right]^{-\frac{n}{2}} \propto \left[1 + \frac{(\bar{y}-\mu)^2}{(n-1)\left(\frac{s^2}{n}\right)}\right]^{-\frac{n}{2}}$$

$\therefore\ \mu|y \sim t_{n-1}\left(\bar{y}, \frac{s^2}{n}\right)$

— t 분포 (mean μ, scale σ^2, df ν)

$$p(\theta) = \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{\nu\pi}\sigma}\boxed{\left(1 + \frac{1}{\nu}\left(\frac{\theta-\mu}{\sigma}\right)^2\right)}^{-(\nu+1)/2}$$

* Posterior predictive distribution - Noninformative prior

$$p(\tilde{y}|y) = \iint p(\tilde{y}|\mu,\sigma^2)\cdot p(\mu,\sigma^2|y)\,d\mu\,d\sigma^2 = \iint p(\tilde{y}|\mu,\sigma^2)\cdot p(\mu|\sigma^2,y)\cdot p(\sigma^2|y)\,d\mu\,d\sigma^2$$

$$= \int \boxed{\int p(\tilde{y}|\mu,\sigma^2)\cdot p(\mu|\sigma^2,y)\,d\mu}\; p(\sigma^2|y)\,d\sigma^2$$

$$= \int p(\tilde{y}|\sigma^2,y)\cdot p(\sigma^2|y)\,d\sigma^2 \qquad \int p(\tilde{y},\mu|\sigma^2,y)\,d\mu = p(\tilde{y}|\sigma^2,y)$$

①

$$p(\tilde{y}|\sigma^2,y) = \int p(\tilde{y}|\mu,\sigma^2)\cdot p(\mu|\sigma^2,y)\,d\mu \;\propto \int \exp\left[-\frac{1}{2\sigma^2}(\tilde{y}-\mu)^2\right]\cdot\exp\left[-\frac{n}{2\sigma^2}(\mu-\bar{y})^2\right]d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\{(\tilde{y}-\mu)^2 + n(\mu-\bar{y})^2\}\right]d\mu = \int \exp\left[-\frac{1}{2\sigma^2}\{\tilde{y}^2 - 2\mu\tilde{y} + \mu^2 + n\mu^2 - 2n\mu\bar{y} + n\bar{y}^2\}\right]d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\{(n+1)\mu^2 - 2(\tilde{y}+n\bar{y})\mu + \tilde{y}^2 + n\bar{y}^2\}\right]d\mu$$

$\mu|\sigma^2,y$
$\sim N(\bar{y}, \frac{\sigma^2}{n})$

$$= \int \exp\left[-\frac{n+1}{2\sigma^2}\left\{\mu^2 - \frac{2(\tilde{y}+n\bar{y})}{n+1}\mu + \frac{\tilde{y}^2+n\bar{y}^2}{n+1}\right\}\right]d\mu$$

$\sigma^2|y$
$\sim \chi^{-2}(n-1, s^2)$
$= \Gamma^{-1}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$

$$= \int \exp\left[-\frac{n+1}{2\sigma^2}\left\{\left(\mu - \frac{\tilde{y}+n\bar{y}}{n+1}\right)^2 - \left(\frac{\tilde{y}+n\bar{y}}{n+1}\right)^2 + \frac{\tilde{y}^2+n\bar{y}^2}{n+1}\right\}\right]d\mu$$

$$\Downarrow \quad + \frac{n\tilde{y}^2 + n\bar{y}^2 - 2n\tilde{y}\bar{y}}{(n+1)^2}$$

$$= \underbrace{\int \exp\left[-\frac{n+1}{2\sigma^2}\left(\mu - \frac{\tilde{y}+n\bar{y}}{n+1}\right)^2\right]d\mu}_{\text{normalizing constant의 역수}} \cdot \exp\left[-\frac{n}{2\sigma^2}\cdot\left(\frac{(\tilde{y}-\bar{y})^2}{n+1}\right)\right]$$

$$= \sqrt{\frac{2\pi\sigma^2}{n+1}}\cdot\exp\left[-\frac{n}{2\sigma^2(n+1)}(\tilde{y}-\bar{y})^2\right]$$

$$\propto \exp\left[-\frac{n}{2\sigma^2(n+1)}(\tilde{y}-\bar{y})^2\right]$$

$$\therefore\; \tilde{y}|\sigma^2,y \sim N(\bar{y}, (1+\tfrac{1}{n})\sigma^2)$$

② $p(\tilde{y}|y)$

$$= \int p(\tilde{y}|\sigma^2,y)\cdot p(\sigma^2|y)\,d\sigma^2 = \int\left[\left(\frac{n}{2\pi(n+1)\sigma^2}\right)^{\frac{1}{2}}\exp\left(-\frac{n}{2\sigma^2(n+1)}\cdot(\tilde{y}-\bar{y})^2\right)\cdot(\sigma^2)^{-\frac{n-1}{2}+1}\cdot\exp\left(-\frac{(n-1)s^2}{2\sigma^2}\right)\right]d\sigma^2$$

$$\propto \int (\sigma^{-2})^{\frac{1}{2}}\cdot(\sigma^{-2})^{\frac{n-1}{2}+1}\cdot\exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + s^2(n-1)\right\}\right]d\sigma^2$$

$$= \int \underbrace{\sigma^{-\frac{n}{2}-1}\cdot\exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + s^2(n-1)\right\}\right]}_{\Gamma^{-1}\left(\frac{n}{2},\,\frac{1}{2}\left[\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + s^2(n-1)\right]\right)\text{의 normalizing constant의 역수}}d\sigma^2$$

$$= \frac{\Gamma\left(\frac{n}{2}\right)}{\left(\frac{1}{2}\left[\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + s^2(n-1)\right]\right)^{+\frac{n}{2}}}$$

$$\propto \left[\frac{n}{n+1}(\tilde{y}-\bar{y})^2 + s^2(n-1)\right]^{-\frac{n}{2}} \propto \left[\frac{(\tilde{y}-\bar{y})^2}{(n-1)(1+\frac{1}{n})s^2} + 1\right]^{-\frac{(n-1)+1}{2}}$$

$\therefore\; v = n-1$
mean: $\bar{y}$
scale: $(1+\frac{1}{n})s^2$

$$\therefore\; \tilde{y}|y \sim t(n-1, \bar{y}, (1+\tfrac{1}{n})s^2)$$

② Normal data with a conjugate prior distribution.

Likelihood: $p(y|\mu,\sigma^2) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \cdot \sigma^{-n} \cdot \exp\left[-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right]$

Prior: $p(\mu,\sigma^2) = p(\mu|\sigma^2)\cdot p(\sigma^2)$

$$\begin{cases} \mu|\sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{k_0}\right) \\ \sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) = \text{Inv-gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right) \end{cases}$$

$$p(\mu|\sigma^2)\cdot p(\sigma^2) \propto \left(\frac{\cancel{k_0}}{\sigma^2}\right)^{\frac{1}{2}} \exp\left(-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right) \cdot \frac{\cancel{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\nu_0/2}}}{\cancel{\Gamma(\nu_0/2)}} \cdot \left(\frac{1}{\sigma^2}\right)^{\nu_0/2+1} \cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left[-\frac{1}{2\sigma^2}\left\{k_0(\mu-\mu_0)^2+\nu_0\sigma_0^2\right\}\right]$$

$$\Rightarrow N\text{-Inv-}\chi^2\left(\mu_0, \frac{\sigma_0^2}{k_0}; \nu_0, \sigma_0^2\right)$$

Posterior: $p(\mu,\sigma^2|y) \propto p(y|\mu,\sigma^2)\cdot p(\mu,\sigma^2) \propto (\sigma^2)^{-\frac{n}{2}} \cdot \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right]$

$$\times (\sigma^2)^{-\frac{1}{2}} \cdot (\sigma^2)^{-\frac{\nu_0}{2}-1} \exp\left[-\frac{1}{2\sigma^2}\left\{k_0(\mu-\mu_0)^2+\nu_0\sigma_0^2\right\}\right]$$

$$\Rightarrow N\text{-Inv-}\chi^2\left(\mu_n, \frac{\sigma_n^2}{k_n}; \nu_n, \sigma_n^2\right)$$

$\mu_n = \frac{k_0}{k_0+n}\mu_0 + \frac{n}{k_0+n}\bar{y}$ (k_0: prior sample size) → combine the prior information and the data information

$k_n = k_0 + n$

$\nu_n = \nu_0 + n$ interpretation) prior sample size

$$\underbrace{\nu_n\sigma_n^2}_{\text{posterior sum of squares}} = \underbrace{\nu_0\sigma_0^2}_{\text{prior sum of squares}} + \underbrace{(n-1)s^2}_{\text{sample sum of squares}} + \underbrace{\frac{k_0 n}{k_0+n}(\bar{y}-\mu_0)^2}_{\text{additional uncertainty conveyed by the difference between sample mean and prior mean}}.$$

* The conditional posterior distribution $p(\mu|\sigma^2, y)$

$$\mu|\sigma^2,y \sim N\left(\frac{\frac{k_0}{\sigma^2}}{\frac{k_0}{\sigma^2}+\frac{n}{\sigma^2}}\mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{k_0}{\sigma^2}+\frac{n}{\sigma^2}}\bar{y},\ \frac{1}{\frac{k_0}{\sigma^2}+\frac{n}{\sigma^2}}\right)$$

* The marginal posterior distribution $p(\sigma^2|y)$

$$p(\sigma^2|y) \propto p(\sigma^2)\cdot p(y|\sigma^2) = p(\sigma^2)\cdot\int p(y|\mu,\sigma^2)\cdot p(\mu|\sigma^2)\,d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot\int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left[-\frac{1}{2\sigma^2}\{(n-1)s^2+n(\bar{y}-\mu)^2\}\right]\cdot\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}\exp\left[-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\int\exp\left[-\frac{1}{2\sigma^2}n(\bar{y}-\mu)^2-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$=\int\exp\left[-\frac{1}{2\sigma^2}\{n(\bar{y}-\mu)^2+k_0(\mu-\mu_0)^2\}\right]d\mu$$

$$=\int\exp\left[-\frac{1}{2\sigma^2}\{n\mu^2-2n\bar{y}\mu+n\bar{y}^2+k_0\mu^2-2k_0\mu_0\mu+k_0\mu_0^2\}\right]d\mu$$

$$=\int\exp\left[-\frac{1}{2\sigma^2}\{(n+k_0)\mu^2-2(n\bar{y}+k_0\mu_0)\mu+k_0\mu_0^2+n\bar{y}^2\}\right]d\mu$$

$$=\int\exp\left[-\frac{n+k_0}{2\sigma^2}\left\{\mu^2-2\frac{(n\bar{y}+k_0\mu_0)}{n+k_0}\mu+\left(\frac{(n\bar{y}+k_0\mu_0)}{n+k_0}\right)^2\right\}-\frac{1}{2\sigma^2}\left\{k_0\mu_0^2+n\bar{y}^2-\frac{(n\bar{y}+k_0\mu_0)^2}{n+k_0}\right\}\right]d\mu$$

$$k_0\mu_0^2+n\bar{y}^2-\frac{n^2\bar{y}^2+2k_0\mu_0n\bar{y}+k_0^2\mu_0^2}{n+k_0}$$

$$=\frac{nk_0\mu_0^2+\cancel{n^2\bar{y}^2}+\cancel{k_0^2\mu_0^2}+nk_0\bar{y}^2-\cancel{n^2\bar{y}^2}-2k_0\mu_0n\bar{y}-\cancel{k_0^2\mu_0^2}}{n+k_0}$$

$$=\frac{nk_0\mu_0^2-2k_0\mu_0n\bar{y}+nk_0\bar{y}^2}{n+k_0}$$

$$=\frac{nk_0(\mu_0^2-2\mu_0\bar{y}+\bar{y}^2)}{n+k_0}$$

$$=\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}$$

$$=\int\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu\cdot\exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right)$$

$$=\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}}\cdot\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot\exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)\cdot\exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right)$$

$$\cdot\underbrace{\int\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu}_{N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0},\ \frac{\sigma^2}{n+k_0}\right)\ \text{Normalizing constant}} = \frac{n+k_0}{\sigma^2}\propto\left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}}$$

$$=(\sigma^2)^{-\frac{\nu_0+n}{2}-1}\cdot\exp\left[-\frac{1}{2\sigma^2}\underbrace{\left\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}}_{=\nu_n\sigma_n^2}\right]$$

$$=(\sigma^2)^{-\frac{\nu_0+n}{2}-1}\cdot\exp\left[-\frac{\nu_n}{2\sigma^2}\underbrace{\left\{\frac{1}{\nu_n}\left\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right\}\right\}}_{=\sigma_n^2}\right]$$

$$\therefore\ \sigma^2|y\sim\text{Inv-}\chi^2(\nu_0+n,\sigma_n^2)$$
$$=\text{Inv-}\chi^2(\nu_n,\sigma_n^2)$$

Sampling: $p(\mu,\sigma^2|y)=N\left(\mu\,\middle|\,\mu_n,\frac{\sigma^2}{k_0+n}\right)\cdot\text{Inv-}\chi^2(\nu_n,\sigma_n^2)$

$$\sigma^2\sim\text{Inv-}\chi^2(\nu_n,\sigma_n^2),\quad \mu\sim N\left(\mu\,\middle|\,\mu_n,\frac{\sigma^2}{k_0+n}\right)\ \Rightarrow\ (\mu,\sigma^2)\sim(\mu,\sigma^2|y)$$