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Appendix C

The Fourier Transformation

C.1 Theoretical Background

- A general introduction to the Forier transformation and how it is typically integrated.
- We have a signal $y(t) = \sum_{j=1}^{5} y_j \sin(2\pi f_j t + \phi_j)$
  - ↳ y_j is amplitude
  - ↳ f_j is frequency
  - ↳ ϕ_j is phase of the jth sine wave
- Any function $y(t)$ can be written as a sum of sine waves.
- We can express $y(t) = \sum_{j=1}^{5} y_j \sin(2\pi f_j t + \phi_j)$ as an integral over frequency f
- This is normally written as: $y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{-2\pi i f t}\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega/2\pi)\, e^{-i\omega t}\, d\omega\,,$
  - ↳ $i = \sqrt{-1}$
- The most general cases are complex
- It is most common to refer $y(t)$ as a function of time, and transform $Y(f)$ into frequency.
- The forward and inverse Fourier Transformations are written as: $Y(f) = \int_{-\infty}^{\infty} y(t)\, e^{2\pi i f t}\, dt = \int_{-\infty}^{\infty} y(t)\, e^{i\omega t}\, dt\,,$

  and we use the relation: $\int_{-\infty}^{\infty} e^{i\omega(t-t')} d\omega = 2\pi\delta(t-t')$

Why do we want to change our functions to begin with?

If a electronically recorded sound system can be written as a Fourier Sum, then we can decompose it into the sum of "pure" tones.

The frequencies of these tones in the original sound are also the direct measure of the fourier tranformations frequencies.

For many physical situations, the sum of the indiuidual responses of each component is just the total response.

C.2 Discrete Fourier Transform

This section will tell us how exactly to compute a Fourier transformation.

- If we are given $y(t)$ how do we get to $Y(f)$?
- If we are given a signal in the form of $y(t)$ then we can get $Y(f)$ using the integral

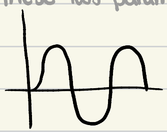
$$y(t) = \int_{-\infty}^{\infty} Y(f)\, e^{-2\pi i f t}\, df = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega/2\pi)\, e^{-i\omega t}\, d\omega\,,$$

- The equation: $\sum_{n=0}^{N-1} e^{2\pi i n(m-m')/N} = N\delta_{m,m'}$, means if we perform a forward transformation, then a inverse transformation we get the original function.
- The sampling theorem is from sampling only twice each period of a sine wave, captures the Fourier Component.

C.3 Fast Fourier Transform (FFT)

- The FFT has made many important calculations more feasible. It is used in X-rays, and tomography
- FFT makes previous impractical calculations possible.
- The FFT returns the transformed output values in the same data arrays as used for the input.
- Most of this section was heavy on examples and creating pseudo code.

C.4 Examples: Sampling Interval and Number of Data Points

- We are in control of two parameters in a discrete Fourier transform: sampling interval and the number of points to sample.
- Making practical decisions involving these two parameters is important if we want to obtain useful transforms.
- The simplest signal is a pure sign wave 
- A FFT with an arbitrary phase often yields a nonzero sine and cosine components; these help describe the signal.
- If sampling time does not match the frequencies of the Fourier components, the FFT has a more complicated appearance.

C.5 Examples: Aliasing

- We were introduced to the sampling theorem in the last section.
- What if the frequencies of the two components are not below the Nyquist frequency $\left(\frac{1}{2\Delta t}\right)$.
- If we have all data points, how do we know what is the true frequency?
- Aliasing is the folding back of frequencies that are above the Nyquist frequency.
- We should arrange for the Nyquist frequency to be higher than any Fourier components that we expect to be present in the signal.
- The sampling interval is typically the time step of the simulation, and this should always be small compared to the characteristic time scales of the problem.
  ↳ all fourier components lie below the Nyquist frequency.

C.6 Power Spectrum

- We mainly care about frequencies and relative amplitudes of the Fourier transformation, not really their phases.
  - The power spectrum of function $y(t)$ can be defined as the Fourier transformation of its autocorrelation

$$Corr[y](\tau) = \int_{-\infty}^{\infty} y(t)^* y(t+\tau) d\tau \,, \tag{C.13}$$

and the power spectrum is given by

$$PS[y](f) = \int_{-\infty}^{\infty} y(t)^* y(t+\tau) e^{2\pi i f \tau} d\tau = |Y(f)|^2 \,. \tag{C.14}$$