Representation and Description

- *Representation* and Description
  - Representing regions in 2 ways:
    - Based on their external characteristics (its boundary):
      - Shape characteristics
    - Based on their internal characteristics (its region):
      - Regional properties: color, texture, and …
    - Both

- Representation and *Description*
  - Describes the region based on a selected representation:
    - Representation → boundary or textural features
    - Description → length, orientation, the number of concavities in the boundary, statistical measures of region.

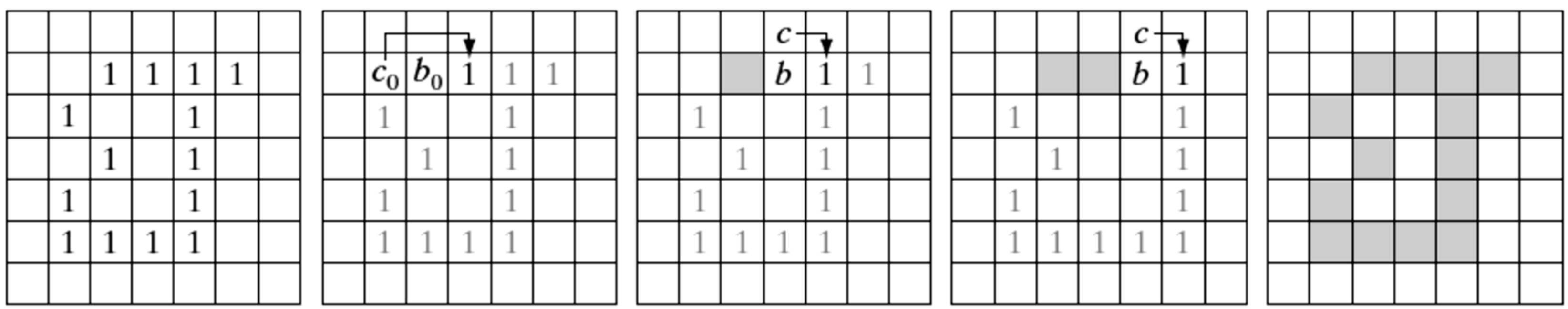
- Invariant Description:
  - Size (Scaling)
  - Translation
  - Rotation

- Boundary (Border) Following:
  - We need the boundary as a ***ordered sequence*** of points.

- Moore Boundary Tracking Algorithm:
  1. Let the starting point, b_0 , be the *uppermost*, *leftmost* point in the image that is labeled 1. Denote by c_0 the *west* neighbor of b_0 . Clearly c_0 always will be a background point. Then, examine the 8 neighbors of b_0 , starting at c_0 and proceeding in clockwise direction. Let b_1 denote the first pixel encountered whose value is 1, and let c_1 be the points immediately preceding b_1 in the sequence. Store the location of b_0 and $b_{1.}$
  2. Let b=b_1 and c=c_1.
  3. Let the 8 neighbors of b , starting at c and proceeding in a clockwise direction be denoted by n_1,n_2,… , n_{k} . Find the first n_k whose value is 1.
  4. Let b=n_k and c=$n_{k\text{-}1}$.
  5. Repeat steps 3 and 4 until b=b_0 and the next boundary point found is $b_{1.}$

Representation and Description

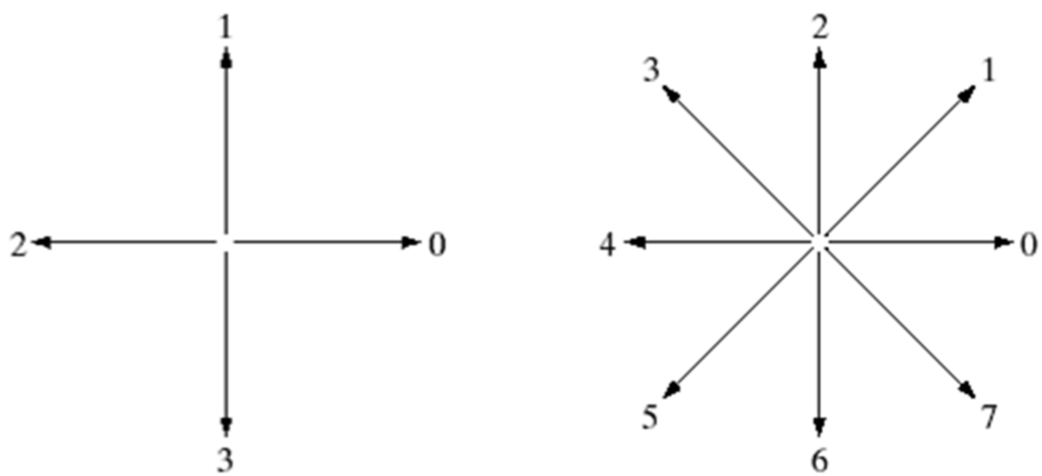
- Example:



a b c d e

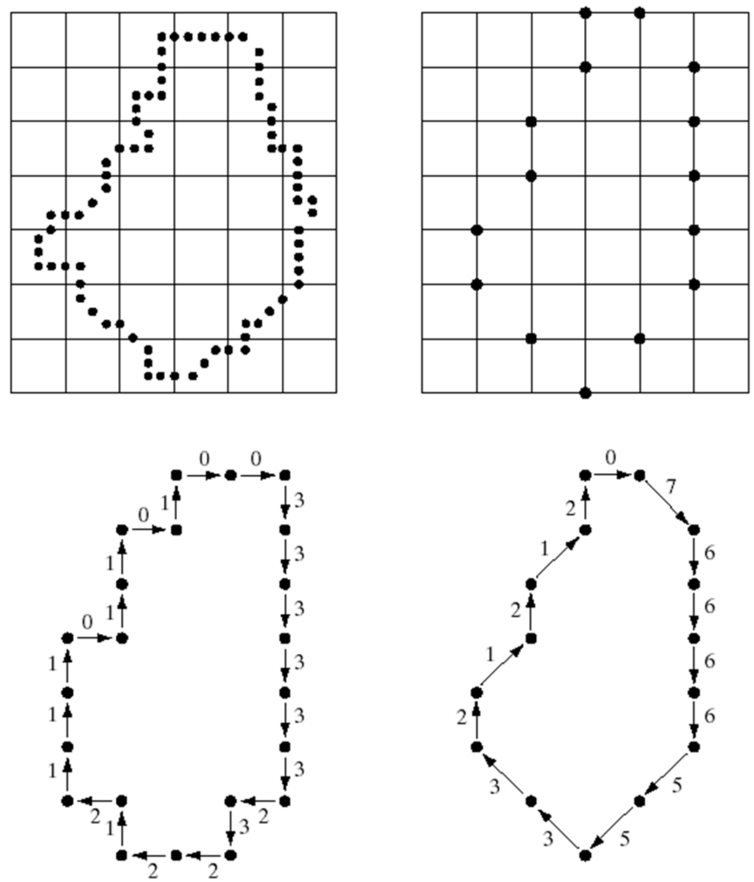
FIGURE 11.1 Illustration of the first few steps in the boundary-following algorithm. The point to be processed next is labeled in black, the points yet to be processed are gray, and the points found by the algorithm are labeled as gray squares.

- Freeman Chain Code:
  - Code the 4 or 8 connectivity



- Chain Code Problems:
  - Long Code
  - Low noise Robustness
    - Solution: Resampling
  - Starting Point:
    - Solution: Rotary/Circular shift until forms a minimum integer
      - 10103322→ 01033221
  - Angle normalization:
    - First difference (Counterclockwise)
      - 10103322 → 3133030 or 33133030 (transition between last and first)
    - Useful for integer multiple of used chain code (45° or 90°)

- Example:
  - Resampling
  - 4 and 8 chain codes



Representation and Description

- Example:

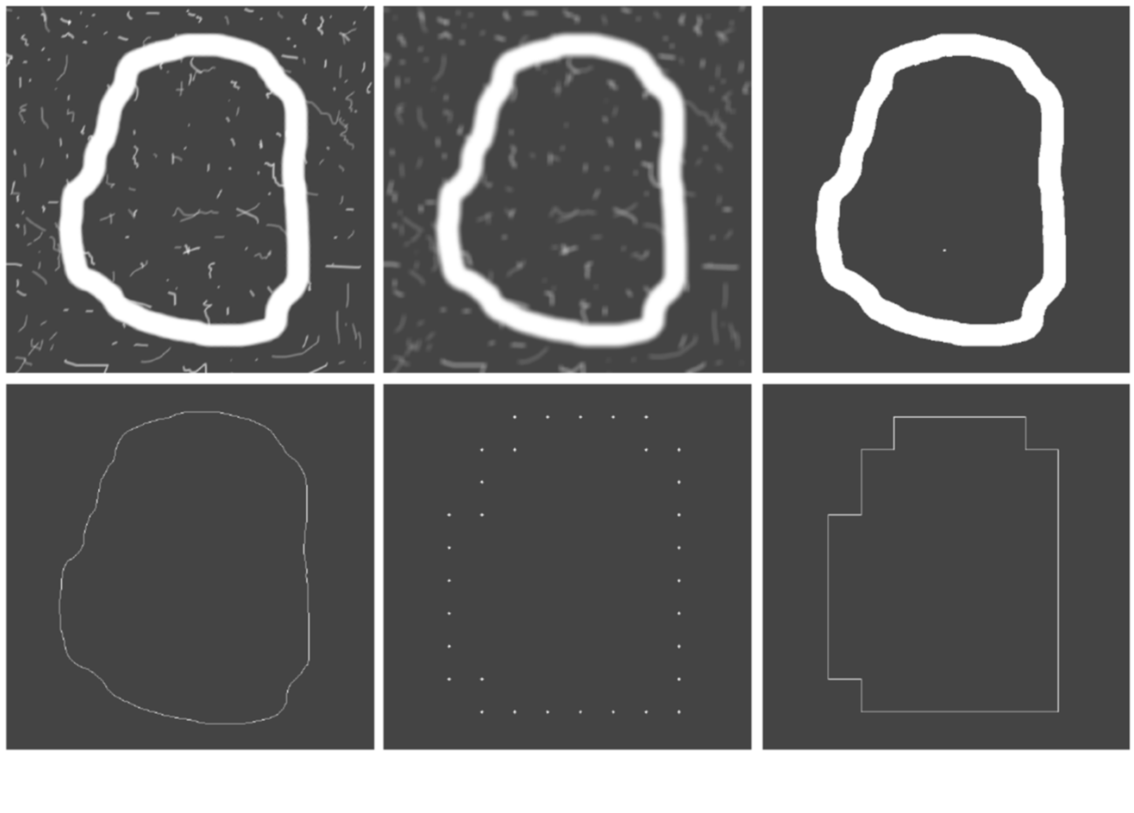


FIGURE 11.5 (a) Noisy image. (b) Image smoothed with a 9 × 9 averaging mask. (c) Smoothed image, thresholded using Otsu's method. (d) Longest outer boundary of (c). (e) Subsampled boundary (the points are shown enlarged for clarity). (f) Connected points from (e).

- Polygon Approximation:
  - Minimum-Perimeter Polygons
    - Read Pages 801-807.

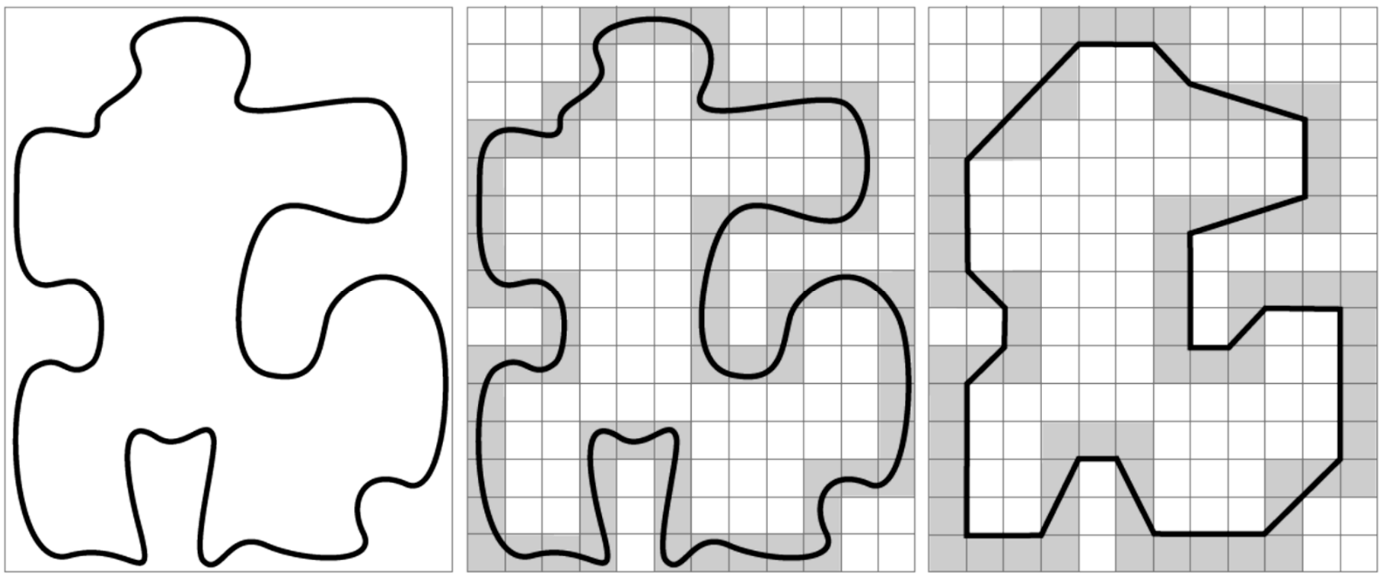
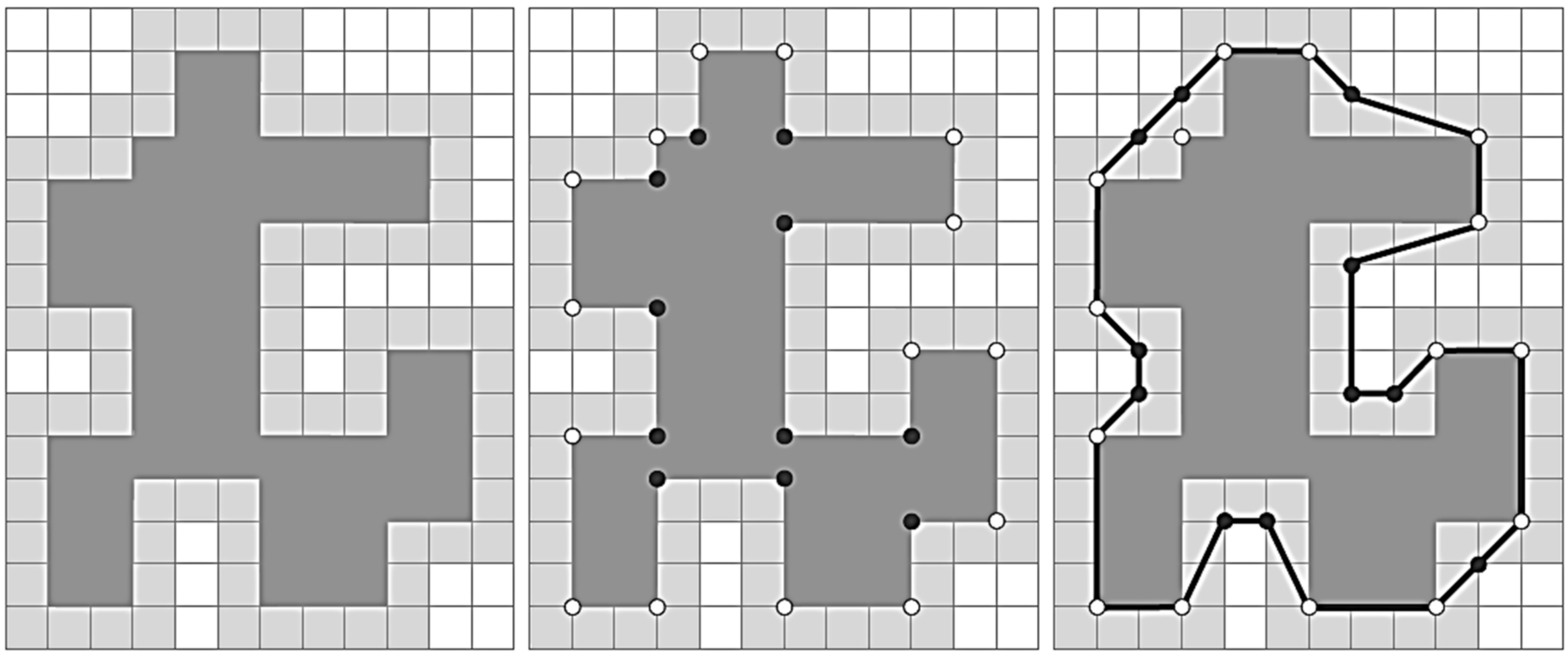


FIGURE 11.6 (a) An object boundary (black curve). (b) Boundary enclosed by cells (in gray). (c) Minimum-perimeter polygon obtained by allowing the boundary to shrink. The vertices of the polygon are created by the corners of the inner and outer walls of the gray region.

- Example (Cont.):



a b c

FIGURE 11.7 (a) Region (dark gray) resulting from enclosing the original boundary by cells (see Fig. 11.6). (b) Convex (white dots) and concave (black dots) vertices obtained by following the boundary of the dark gray region in the counterclockwise direction. (c) Concave vertices (black dots) displaced to their diagonal mirror locations in the outer wall of the bounding region; the convex vertices are not changed. The MPP (black boundary) is superimposed for reference.

- Example:

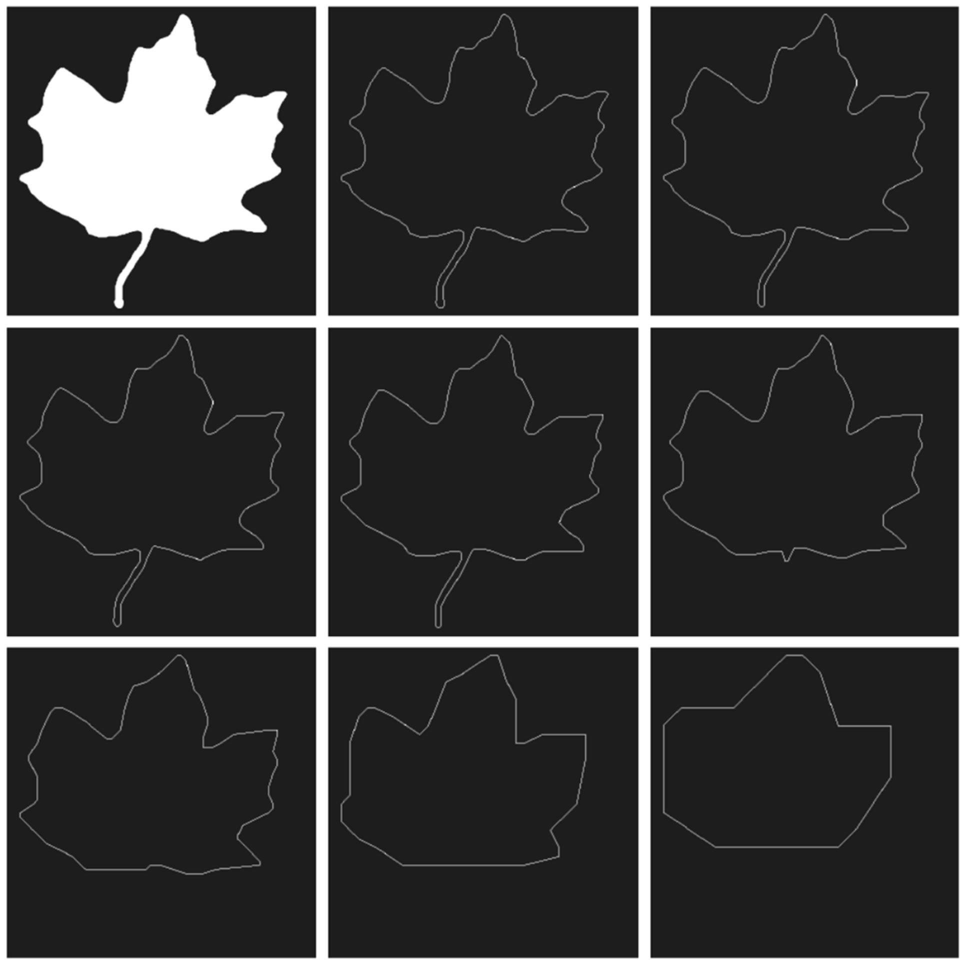
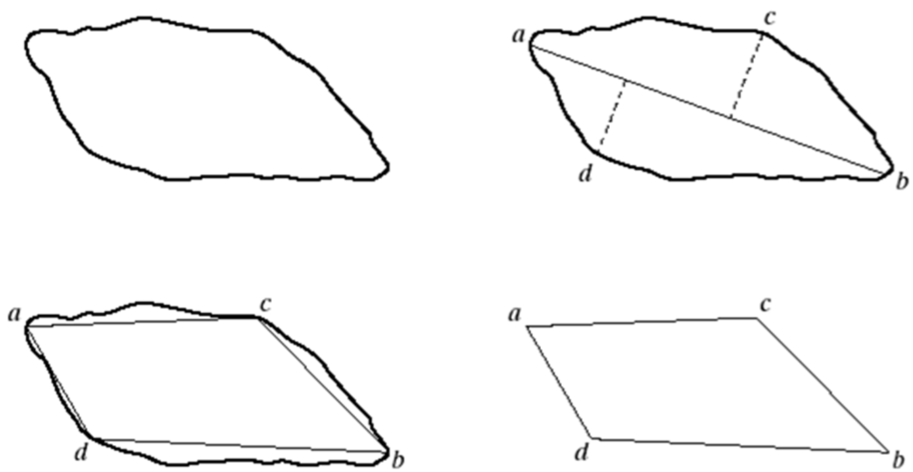


FIGURE 11.8 (a) 566 × 566 binary image. (b) 8-connected boundary. (c) through (i), MMPs obtained using square cells of sizes 2, 3, 4, 6, 8, 16, and 32, respectively (the vertices were joined by straight lines for display). The number of boundary points in (b) is 1900. The numbers of vertices in (c) through (i) are 206, 160, 127, 92, 66, 32, and 13, respectively.

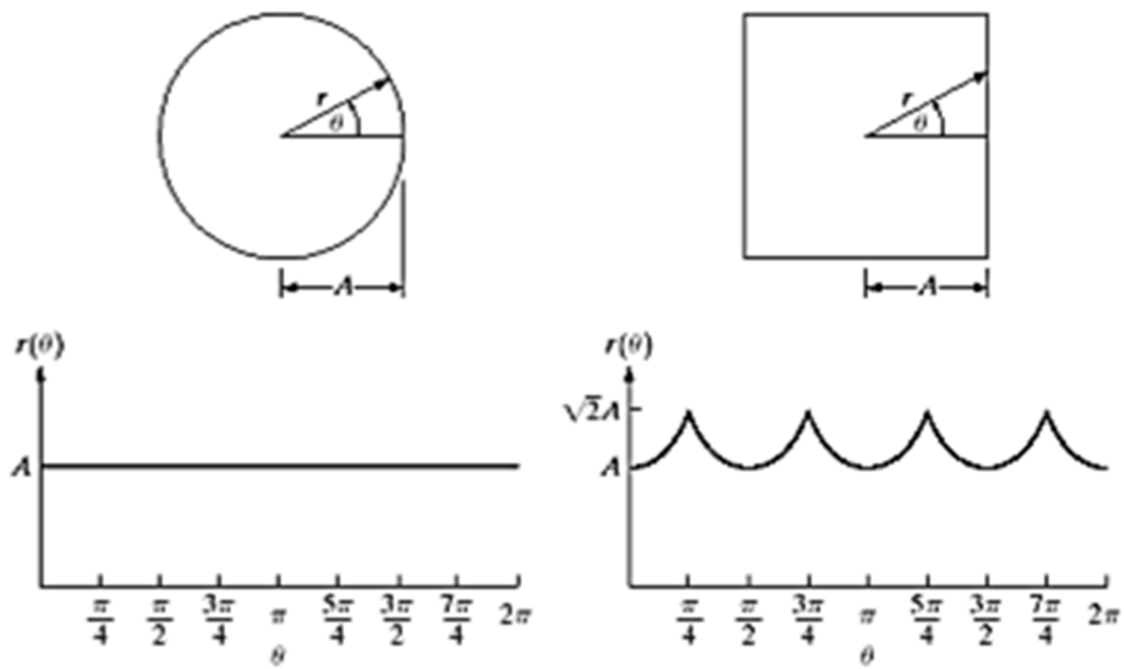
- Polygon Approximation:
  - Merging:
    - Start from a seed point
    - Continue on a line based on local average error (e.g. linear regression)
    - Stop if error exceeds a threshold
    - Continue from the last point
    - ....
  - No guarantee for corner detection

- Polygon Approximation:
  - Splitting:
    - Segments to two parts based on a criteria (e.g. maximum internal distance)
    - Check each segment for splitting based on another criteria (e.g. linearity error)
    - ….



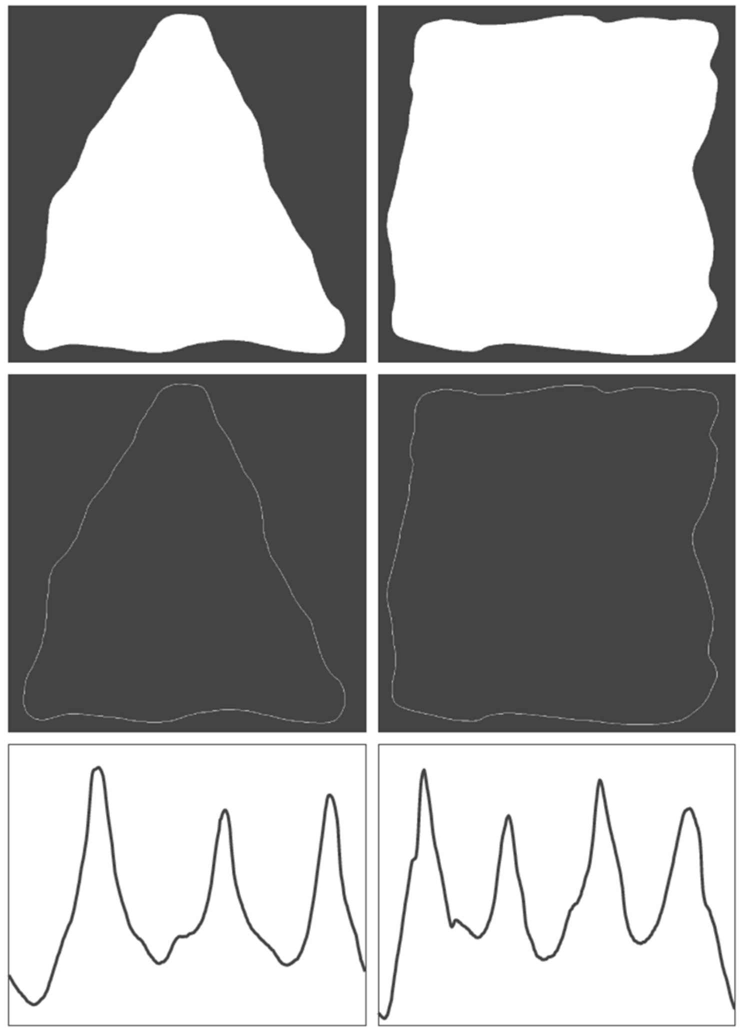
- Signatures:
  - A 1-D functional representation of a boundary
    - Distance vs. Angle (in the polar representation):
      - Invariant to translation
      - Non-Invariant to rotation (may be achieved by start point selection)
        - » Farthest point from centroid
        - » The point on eigen axis
        - » Use chain code solution for the start point
    - Line tangent angle
    - Histogram of tangent angle

- Example:

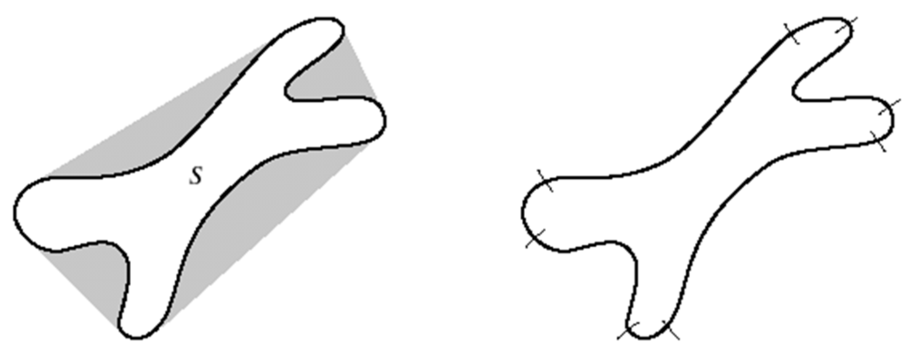


Representation and Description

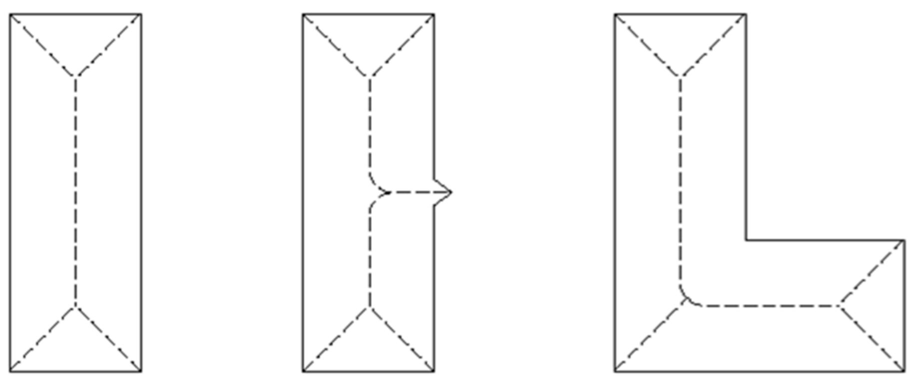
- Example:



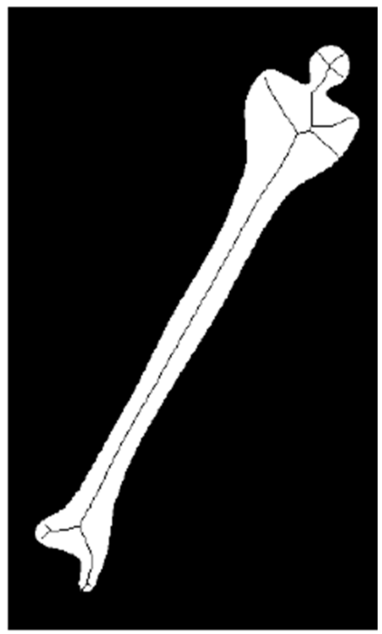
- Boundary Segments:
  - Convex Hull: Smallest convex set containing the S
  - Convex Deficiency: The set difference D=H-S
  - Problem of noise:
    - Smoothing (K-neighbour average)
    - Polygon approximation
      - Convex Hull of Polygon



- Boundary Segments:
- Skeletonization / Thinning:
  - Medial Axis Transform (MAT)
    - Point p is belong to medial (Region R and Border B):
      - Has more than one closest neighbor in B
  - Chapter 9 and Page 813

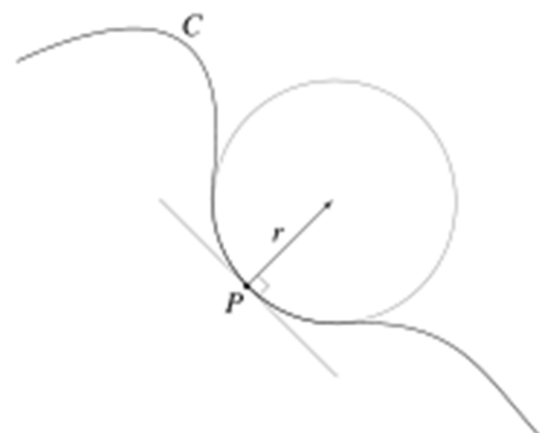


- Example of Thinning:

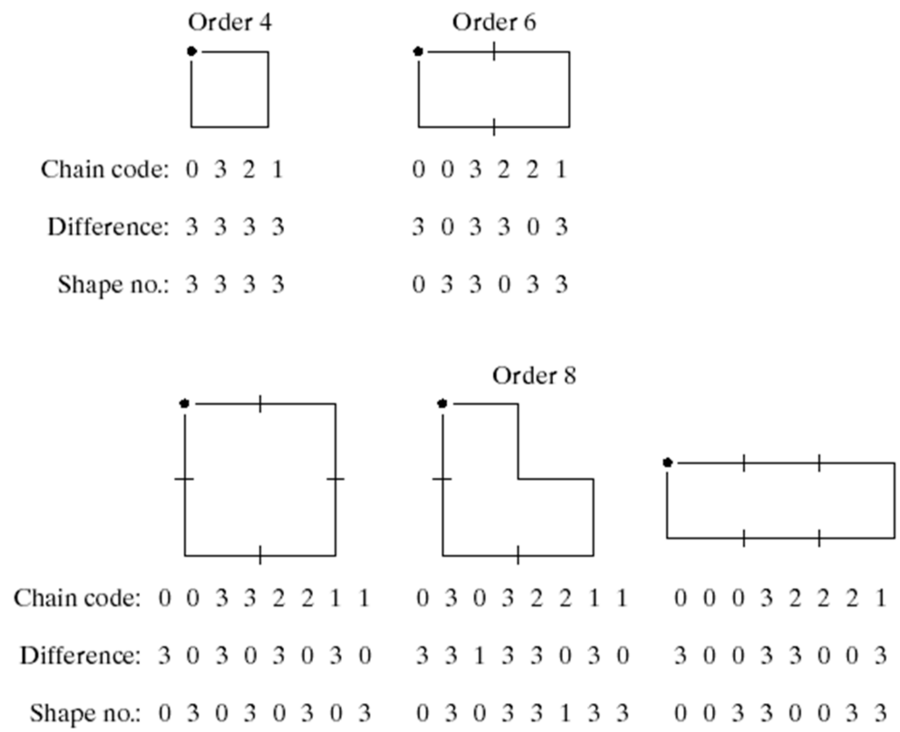


- Boundary Descriptor:
  - Simple one:
    - Diameter: $\mathrm{Diam}(B) = \max_{i,j}\left\{D\left(p_i, q_j\right)\right\}$
      - Lead to Major axis and minor axis (Perpendicular to major)
      - Basic Rectangle
    - Curvature

$$k = \frac{1}{R} = \frac{x'y'' - x''y'}{\left[\left(x'\right)^2 + \left(y'\right)^2\right]^{1.5}}$$

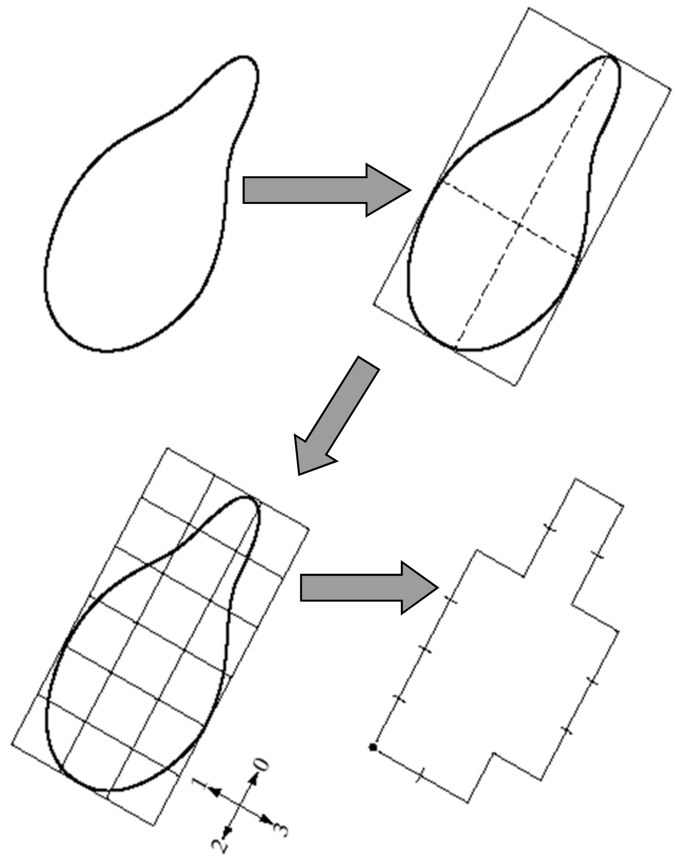


- Shape Number
  - Smallest integers of first difference circular chain code.



Representation and Description

- Example:



Chain code: 0 0 0 0 3 0 0 3 2 2 3 2 2 2 1 2 1 1

Difference: 3 0 0 0 3 1 0 3 3 0 1 3 0 0 3 1 3 0

Shape no.: 0 0 0 3 1 0 3 3 0 1 3 0 0 3 1 3 0 3

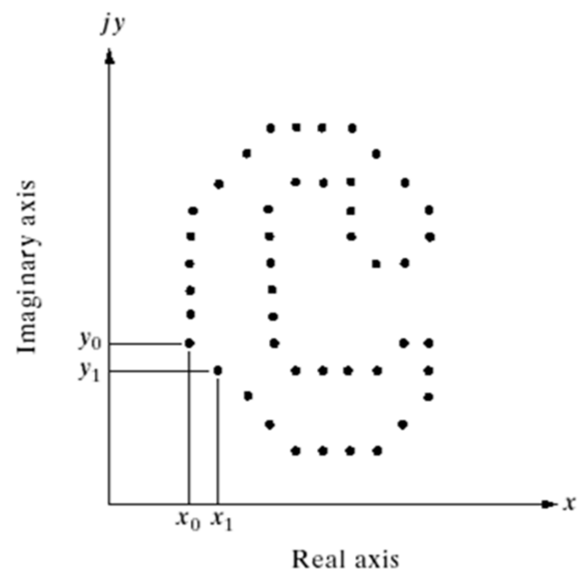
- Fourier Descriptors:

$$s(k) = x(k) + jy(k)$$

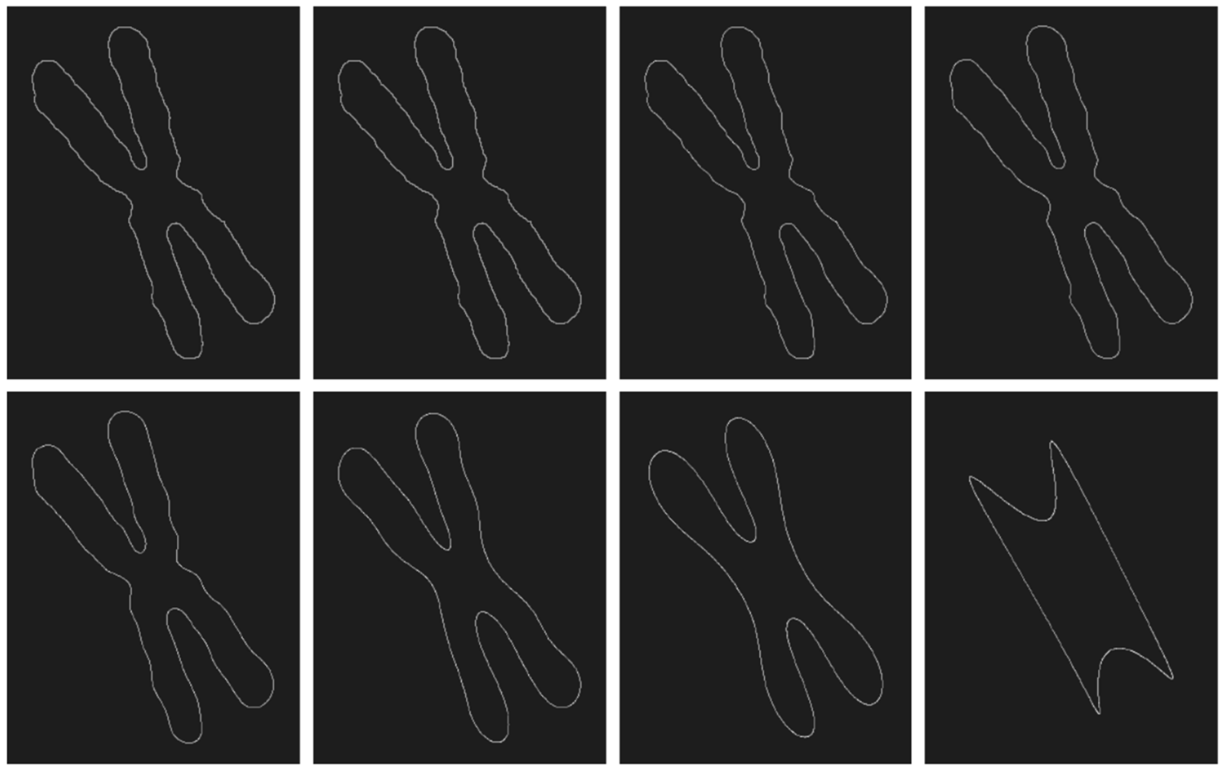
$$a(u) = \sum_{k=0}^{K-1} s(k) \exp\left(-j2\pi \frac{uk}{K}\right)$$

$$s(k) = \frac{1}{K} \sum_{u=0}^{K-1} a(u) \exp\left(+j2\pi \frac{uk}{K}\right)$$

$$\hat{s}(k) = \frac{1}{P} \sum_{u=0}^{P-1} a(u) \exp\left(+j2\pi \frac{uk}{K}\right)$$



- Example:



a b c d
e f g h

FIGURE 11.20 (a) Boundary of human chromosome (2868 points). (b)–(h) Boundaries reconstructed using 1434, 286, 144, 72, 36, 18, and 8 Fourier descriptors, respectively. These numbers are approximately 50%, 10%, 5%, 2.5%, 1.25%, 0.63%, and 0.28% of 2868, respectively.

- Problem of invariance:
  - Rotation

$$a_r(u)=\frac{1}{K}\sum_{k=0}^{K-1}s(k)e^{j\theta}\exp\left(-j2\pi\frac{uk}{K}\right)=a(u)e^{j\theta}$$

  - Normalization will solve!

$$\hat{a}_s(u)=\frac{a_s(u)}{a_s(u_0)},\quad \frac{a_s(u)}{\sum_i a_s(u_i)},\quad \frac{a_s(u)}{\sqrt{\sum_i\left|a_s(u_i)\right|^2}},\ldots.$$

- Problem of invariance:
  - Translation

$$\Delta_{xy} = \Delta_x + j\Delta_x$$

$$s_t(k) = s(k) + \Delta_{xy} = \left[x(k) + \Delta_x\right] + j\left[y(k) + \Delta_y\right]$$

$$a_t(u) = \Delta_{xy} + \delta(u)$$

  - Translation to centroid will solve!

- Problem of invariance:
  - Scaling

$$s_s(k) = \alpha s(k) \Rightarrow a_s(u) = \alpha a(u)$$

  - Normalization will solve

$$\hat{a}_s(u) = \frac{a_s(u)}{a_s(u_0)}, \quad \frac{a_s(u)}{\sum_i a_s(u_i)}, \quad \frac{a_s(u)}{\sqrt{\sum_i |a_s(u_i)|^2}}, \ldots$$

- Problem of invariance:
  - Starting Point

$$s_p(k) = s(k-k_0) = \left[x(k-k_0), y(k-k_0)\right]$$

$$\therefore a_p(u) = a(u)\exp\left(-\frac{j2\pi k_0 u}{K}\right)$$

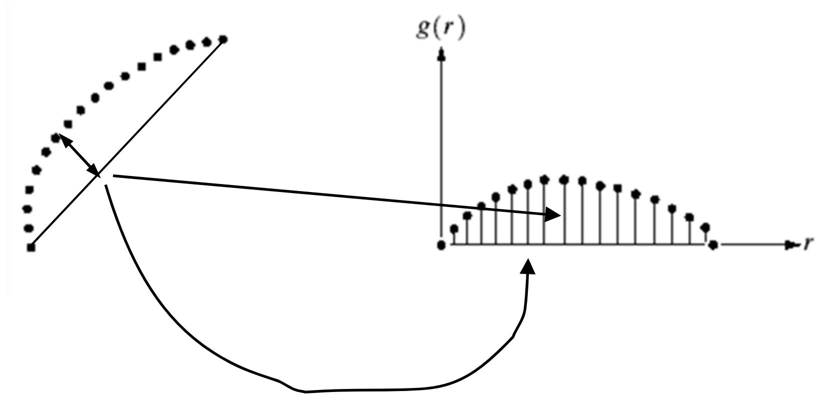
  - Normalization will solve

$$\hat{a}_p(u) = \frac{a_p(u)}{a_p(u_0)}, \quad \frac{a_p(u)}{\sum_i a_p(u_i)}, \quad \frac{a_p(u)}{\sqrt{\sum_i \left|a_p(u_i)\right|^2}}, \ldots.$$

- Properties in a single table:

Transformation	Boundary	Fourier Descriptor
Identity	$s(k)$	$a(u)$
Rotation	$s_r(k) = s(k)e^{j\theta}$	$a_r(u) = a(u)e^{j\theta}$
Translation	$s_t(k) = s(k) + \Delta_{xy}$	$a_t(u) = a(u) + \Delta_{xy}\delta(u)$
Scaling	$s_s(k) = \alpha s(k)$	$a_s(u) = \alpha a(u)$
Starting point	$s_p(k) = s(k - k_0)$	$a_p(u) = a(u)e^{-j2\pi k_0 u/K}$

- Statistical Moments:
  - Boundary Representation



$$\left(r, v = g(r)\right)$$

v as random variable

$$m = \sum_{i=0}^{A-1} v_i p(v_i)$$

$$\mu_n(v) = \sum_{i=0}^{A-1} (v_i - m)^n \, p(v_i)$$

$$\left(r, v = g(r)\right)$$

normalized $g(r)$ as histogram

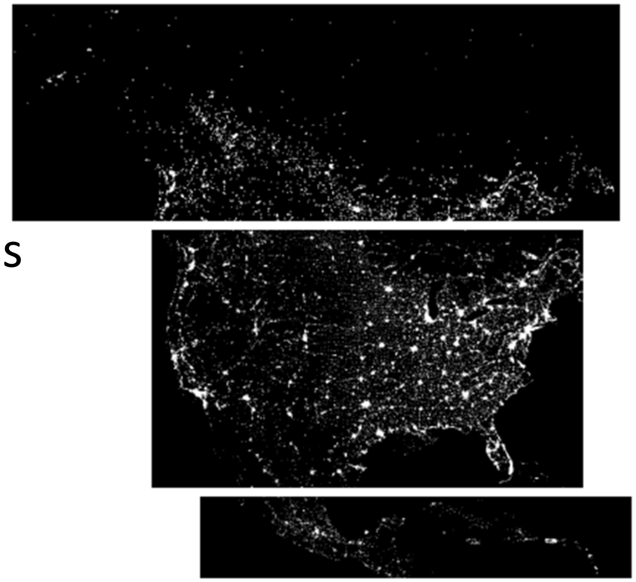
$$m = \sum_{i=0}^{A-1} r_i g(r_i)$$

$$\mu_n(v) = \sum_{i=0}^{A-1} (r_i - m)^n \, g(r_i)$$

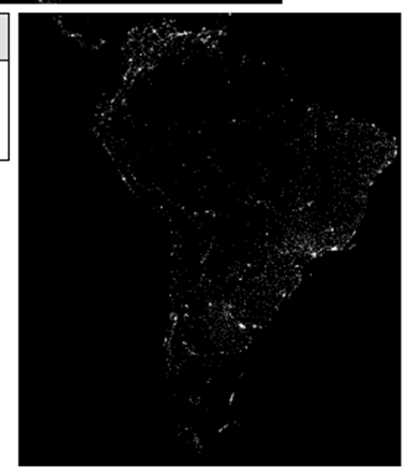
- Regional Descriptor:
  - The simple one:
    - Area (Number of pixels)
    - Perimeter (Length of boundary)
    - Compactness (Perimeter^2/Area)
    - Circularity: Ratio of the area to the area of a circle with same perimeter
    - Mean, median, max, min, ratio pixels above/below … from intensity data.

Representation and Description

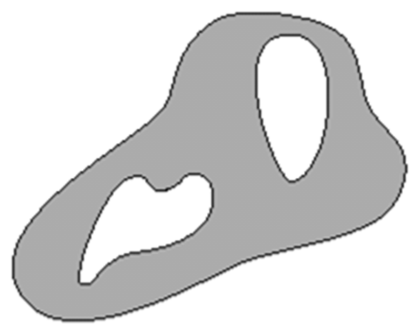
- Example:
  - Normalized area:
    - Ratio of light pixels to total light pixels



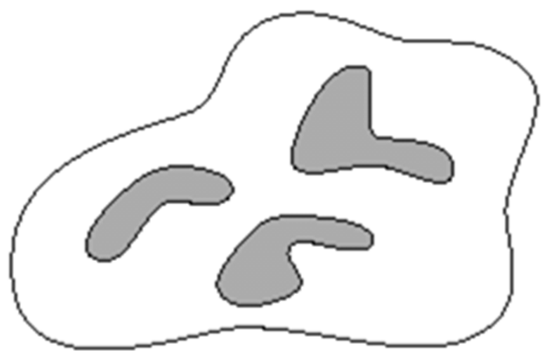
Region no. (from top)	Ratio of lights per region to total lights
1	0.204
2	0.640
3	0.049
4	0.107



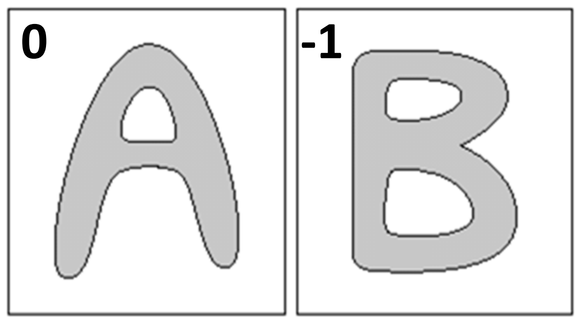
- Topology Descriptor:
  - Number of holes (H):
    - Invariants to several operators.



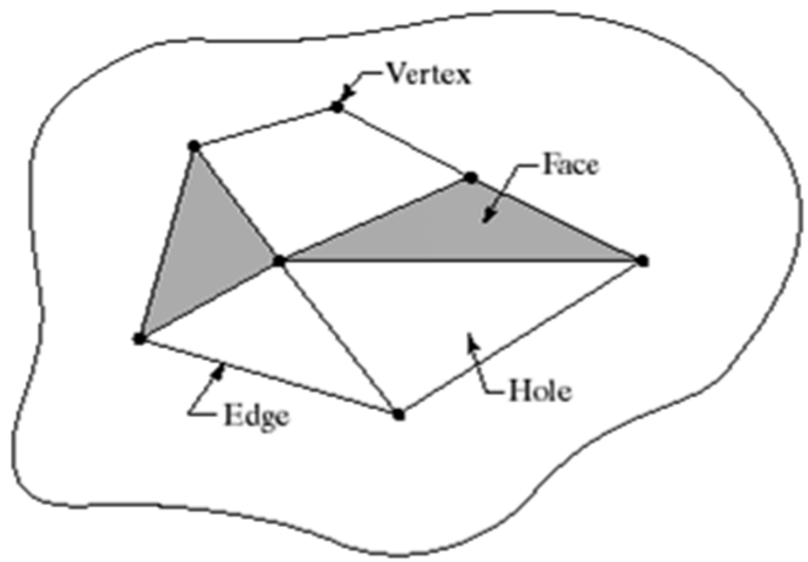
- Topology Descriptor:
  - Number of connected components (C) :
    - Invariants to several operators.



- Topology Descriptor:
  - Euler number (E=C-H) :
    - Invariants to several operators.



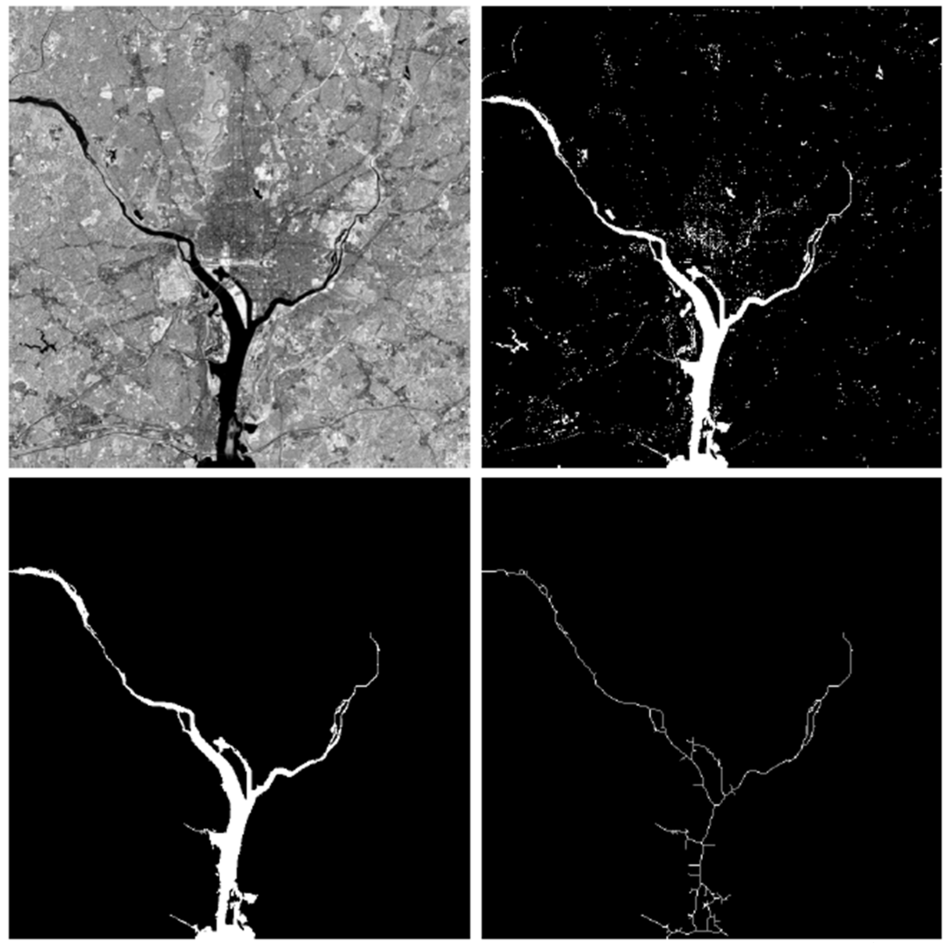
- Topology Descriptor:
  - Polygonal net:
    - V: # of vertices (7)
    - Q: # of edges (11)
    - F: # of faces (2)
    - **E=C-H=V-Q+F**
      - C=1 , H=3
      - E=-2



Representation and Description

- Example:

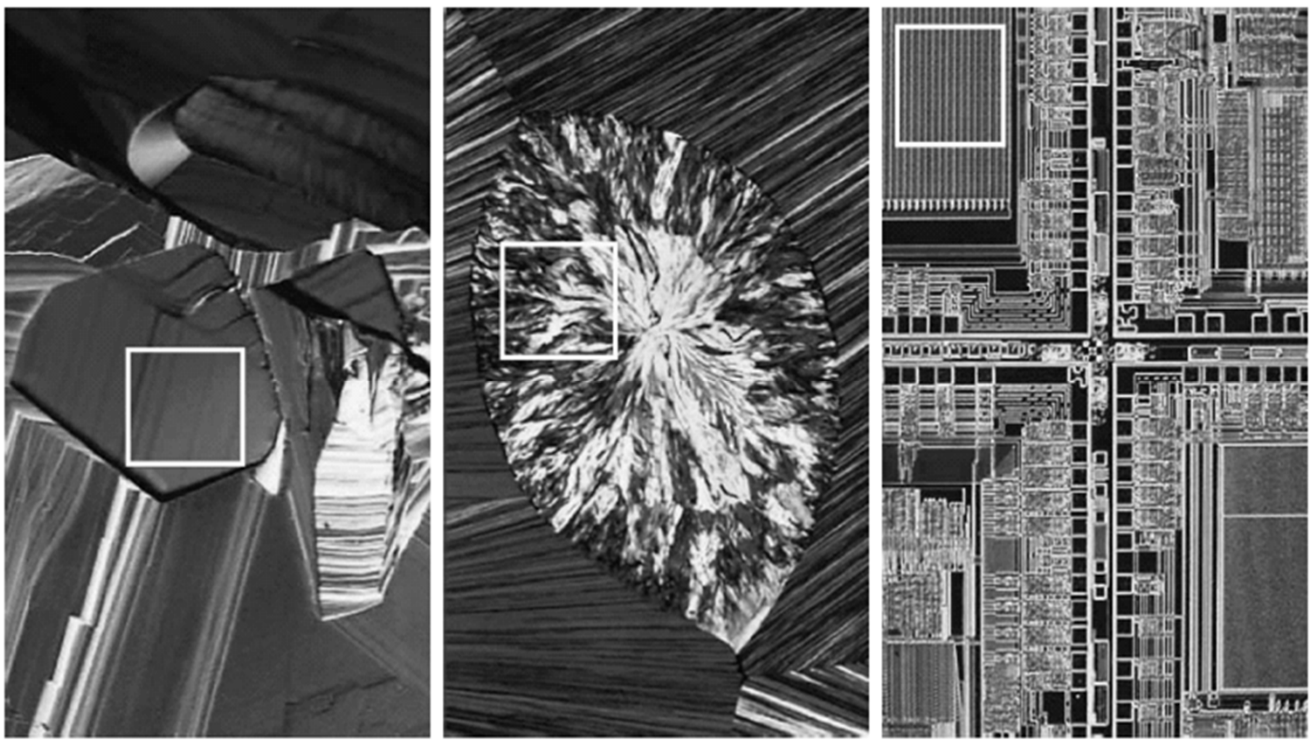
Original Threshold



Largest Connected Region Skeleton

- Texture:
  - No formal definition

Smooth Coarse Regular



- Statistical Approaches
  - 1^{st} order grey level statistics
    - From normalised histogram
      - One pixel gray level repeat n times
  - 2^{nd} order grey level statistics
    - From GLCM (Grey Level Co-ocurrence Matrix)
      - Repeatation of two pixels in a pre-defined neighbourhood
    - Needs:
      - A Positioning Operator, **P**.
      - GLCM(i,j): # of times that points with gray level Z_i occure relative to points with gray level Z_j

- Texture feature from 1st order statistics:

$$\mu_n(z) = \sum_{i=0}^{L-1} (z_i - m)^n P(z_i), \quad m = \sum_{i=0}^{L-1} z_i P(z_i)$$

$$R(z) = 1 - \frac{1}{1+\sigma_z^2} : \text{Gray Level Contrast (Normalized)}$$

$$\mu_3(z) : \text{Skewness}$$

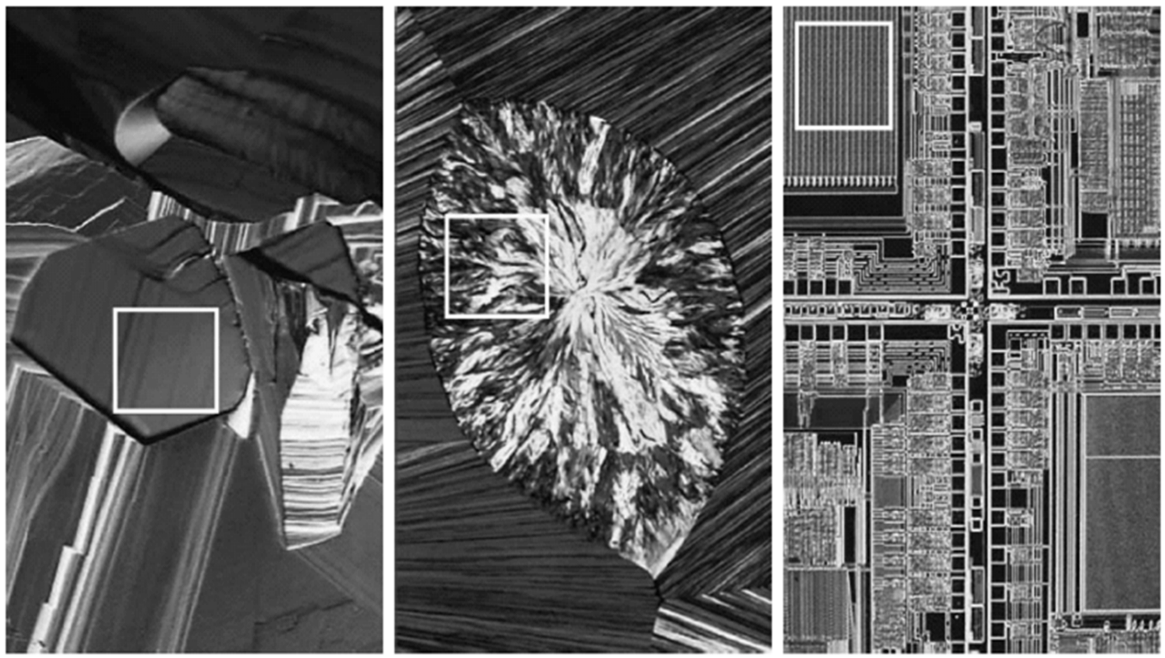
$$\mu_4(z) : \text{Kurtosis, Flatness}$$

$$U(z) = \sum_{i=0}^{L-1} p^2(z_i) : \text{Uniformity}$$

$$e(z) = -\sum_{i=0}^{L-1} p(z_i) \log\left(p(z_i)\right) : \text{Entropy}$$

- Example:

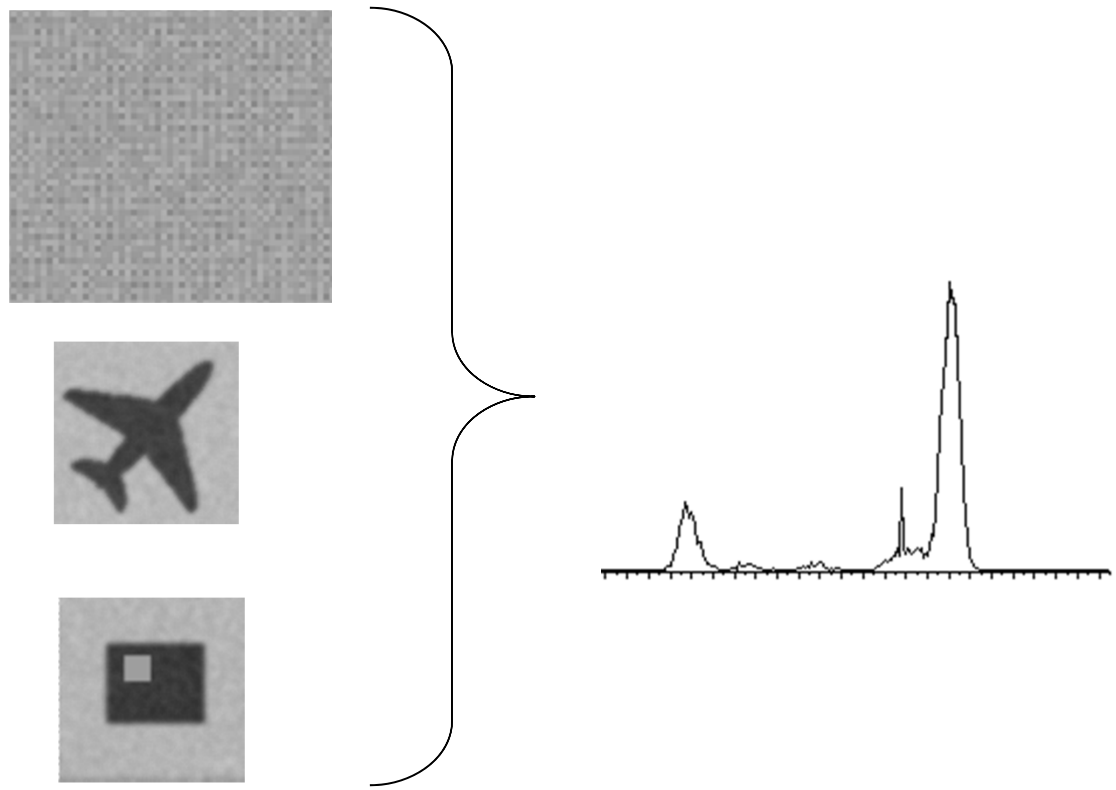
Smooth Coarse Regular



Texture	Mean	Standard deviation	R (normalized)	Third moment	Uniformity	Entropy
Smooth	82.64	11.79	0.002	−0.105	0.026	5.434
Coarse	143.56	74.63	0.079	−0.151	0.005	7.783
Regular	99.72	33.73	0.017	0.750	0.013	6.674

Representation and Description

- Problem with 1st order histogram
  - Lack of spatial information



- Gray Level Co-Occurence Matrix (GLCM):

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & 1 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 = 0 & z_2 = 1 & z_3 = 2 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} : \text{one pixel to right one below}$$

$$\mathbf{G} = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix} \Rightarrow \mathbf{P} = \frac{1}{16}\begin{bmatrix} 4 & 2 & 1 \\ 2 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

- Gray Level Co-Occurence Matrix (GLCM):

1	1	7	5	3	2
5	1	6	1	2	5
8	8	6	8	1	2
4	3	4	5	5	1
8	7	8	7	6	2
7	8	6	2	6	2

Image f

	1	2	3	4	5	6	7	8
1	1	2	0	0	0	1	1	0
2	0	0	0	0	1	1	0	0
3	0	1	0	1	0	0	0	0
4	0	0	1	0	1	0	0	0
5	2	0	1	0	1	0	0	0
6	1	3	0	0	0	0	0	1
7	0	0	0	0	1	1	0	2
8	1	0	0	0	0	2	2	1

Co-occurrence matrix $\mathbf{G}$

- Gray Level Co-Occurence Matrix (GLCM):

N_g : # of gray levels

$$n = \sum_{i=1}^{N_g} \sum_{j=1}^{N_g} g_{ij} \Rightarrow p_{ij} = \frac{g_{ij}}{n}$$

$$m_r = \sum_{i=1}^{N_g} i \sum_{j=1}^{N_g} p_{ij}, \quad m_c = \sum_{j=1}^{N_g} j \sum_{i=1}^{N_g} p_{ij}$$

$$\sigma_r^2 = \sum_{i=1}^{N_g} (i - m_r)^2 \sum_{j=1}^{N_g} p_{ij}, \quad \sigma_c^2 = \sum_{j=1}^{N_g} (j - m_c)^2 \sum_{i=1}^{N_g} p_{ij}$$

- Texture feature from GLCM

$$\underset{i,j}{Max}(p_{ij}) : \text{Maximum probability (G1)}$$

$$\sum_i \sum_j \frac{(i-m_r)(j-m_c)}{\sigma_r \sigma_c} : \text{Correlation (G2)}$$

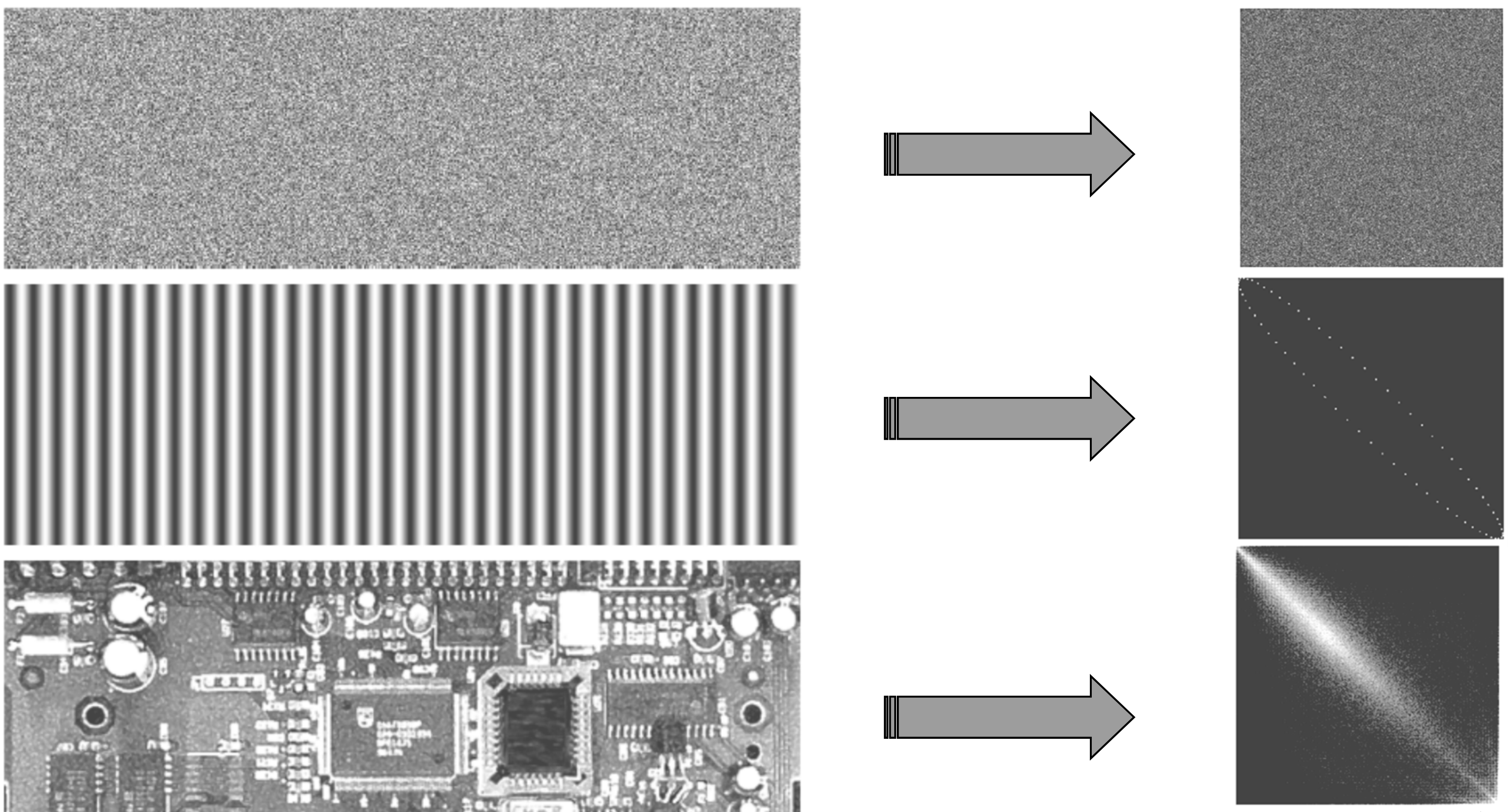
$$\sum_i \sum_i (i-j)^2 p_{ij} : \text{Contrast (G3)}$$

$$\sum_i \sum_j p_{ij}^2 : \text{Uniformity}$$

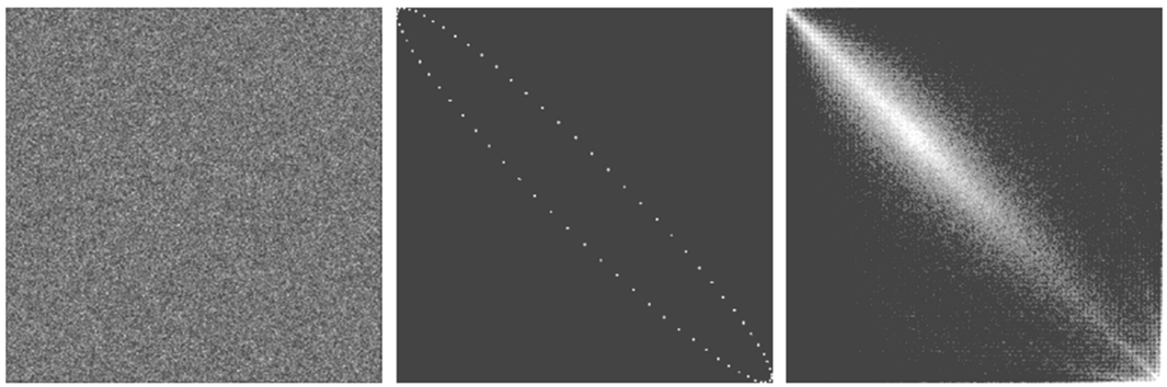
$$\sum_i \sum_i \frac{p_{ij}}{1+|i-j|} : \text{Homogeneity}$$

$$-\sum_i \sum_j p_{ij} \log_2(p_{ij}) : \text{Entropy}$$

- Example:
  - *One pixel immediately to he right*



- Example:



Normalized Co-occurrence Matrix	Descriptor					
	Max Probability	Correlation	Contrast	Uniformity	Homogeneity	Entropy
$\mathbf{G}_1/n_1$	0.00006	−0.0005	10838	0.00002	0.0366	15.75
$\mathbf{G}_2/n_2$	0.01500	0.9650	570	0.01230	0.0824	6.43
$\mathbf{G}_3/n_3$	0.06860	0.8798	1356	0.00480	0.2048	13.58

- Effect of Horizontal offset:
  - Correlation index
  - Horizontal distance between neighbors

Noisy Sin Circuit board

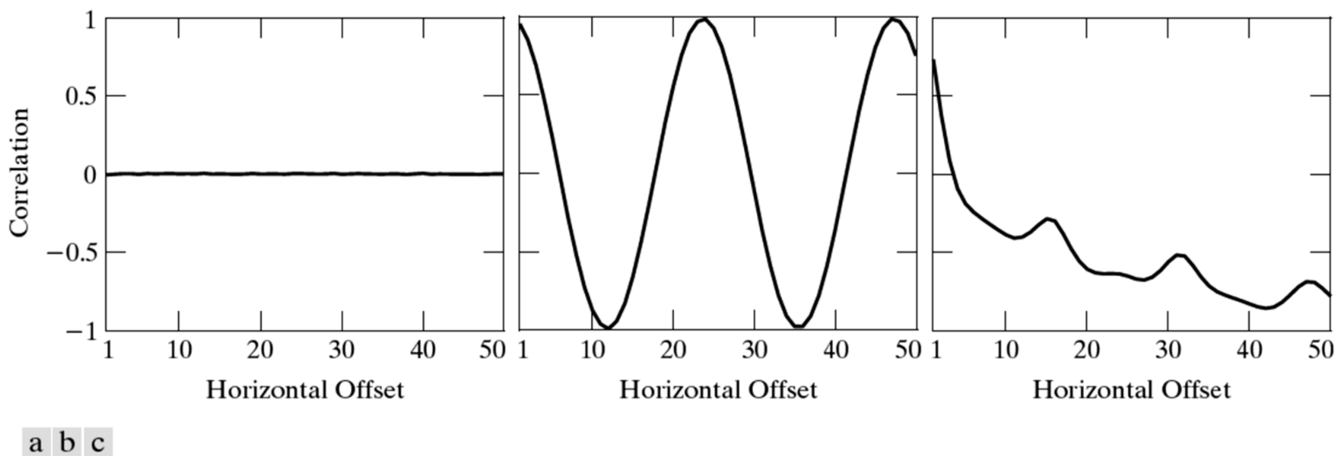
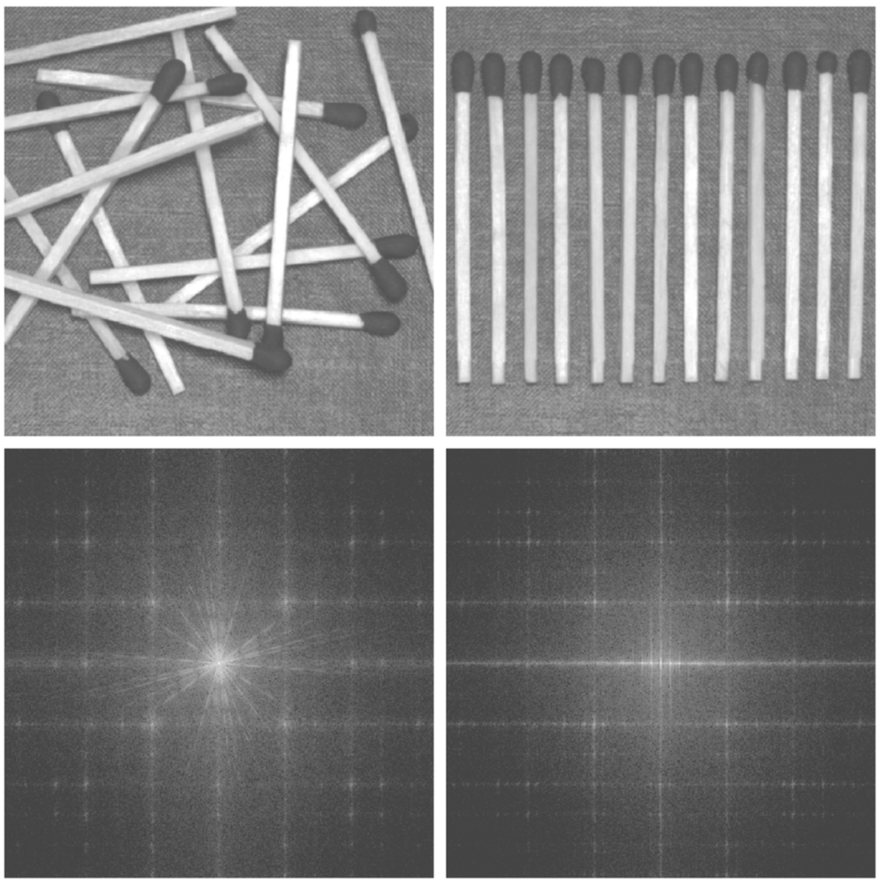


FIGURE 11.32 Values of the correlation descriptor as a function of offset (distance between "adjacent" pixels) corresponding to the (a) noisy, (b) sinusoidal, and (c) circuit board images in Fig. 11.30.

- Spectral approaches:
  - Peaks and Frequencies related to periodicity:
  - $S(r,\theta)$, $S_r(\theta)$,$S_\theta(r)$:

$$\left.\begin{aligned} S(r) &= \sum_{\theta=0}^{\pi} S_\theta(r) \\ S(\theta) &= \sum_{r=1}^{R} S_r(\theta) \end{aligned}\right\} \Rightarrow \begin{bmatrix} S(r) & S(\theta) \end{bmatrix}$$

- Spectral approaches:

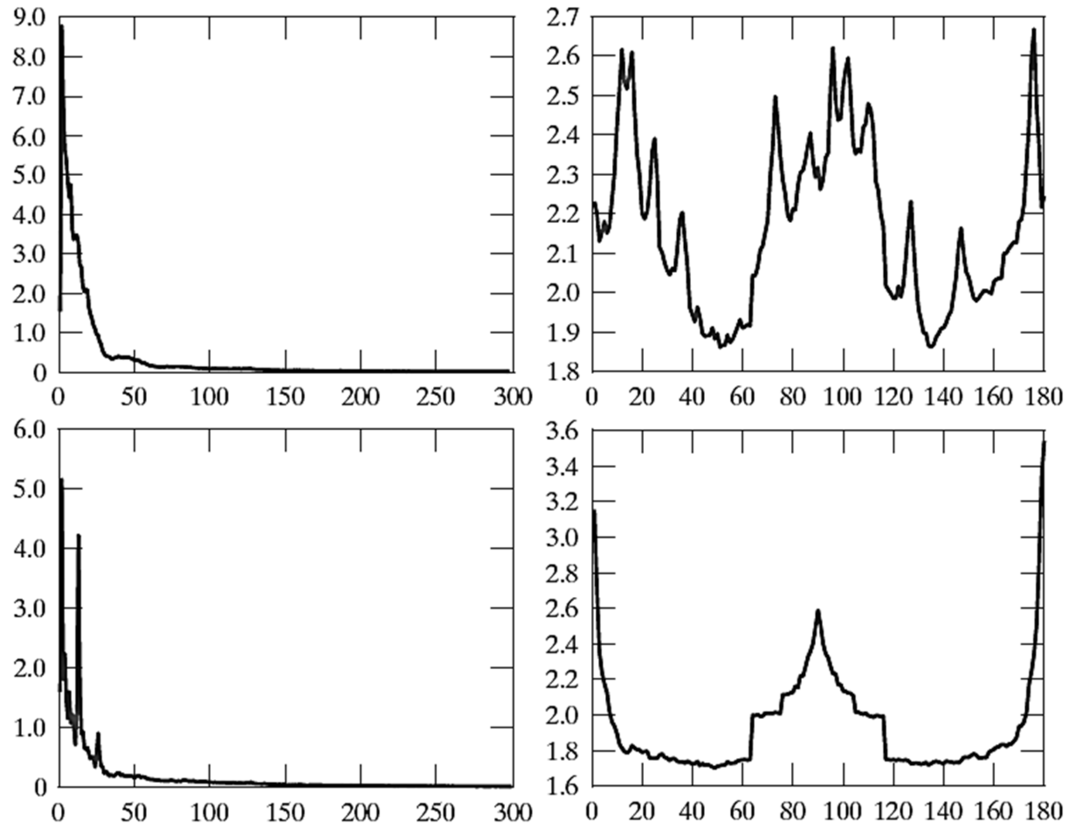


a	b
c	d

FIGURE 11.35
(a) and (b) Images of random and ordered objects. (c) and (d) Corresponding Fourier spectra. All images are of size 600 × 600 pixels.

Representation and Description

- Spectral approach:



a b
c d

FIGURE 11.36 Plots of (a) $S(r)$ and (b) $S(\theta)$ for Fig. 11.35(a). (c) and (d) are plots of $S(r)$ and $S(\theta)$ for Fig. 11.35(b). All vertical axes are $\times 10^5$.

- Moments of Image as a 2D pdf:

Moments of order of (p+q)

$$m_{pq} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x^p y^q f(x, y) \, dxdy$$

Central moments of order of (p+q)

$$\mu_{pq} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \overline{x})^p (y - \overline{y})^q f(x, y) \, dxdy$$

$$\overline{x} = \frac{m_{10}}{m_{00}}; \overline{y} = \frac{m_{01}}{m_{00}}$$

- Digital Data:

Moments of order of (p+q)

$$m_{pq} = \sum_x \sum_y x^p y^q f(x,y)$$

$$\mu_{pq} = \sum_x \sum_y (x-\overline{x})^p (y-\overline{y})^q f(x,y), \quad \overline{x} = \frac{m_{10}}{m_{00}}; \overline{y} = \frac{m_{01}}{m_{00}}$$

$$\mu_{00} = \sum_x \sum_y f(x,y) = m_{00}$$

$$\mu_{10} = \sum_x \sum_y (x-\overline{x})^1 (y-\overline{y})^0 f(x,y) = m_{10} - \frac{m_{10}}{m_{00}} m_{00} = 0$$

$$\mu_{01} = \sum_x \sum_y (x-\overline{x})^0 (y-\overline{y})^1 f(x,y) = m_{01} - \frac{m_{01}}{m_{00}} m_{00} = 0$$

- Moments:

2nd ordens centrale momenter $p+q=2$

$\mu_{11}, \mu_{20}, \mu_{02}$

3rd ordens centrale momenter $p+q=3$

$\mu_{21}, \mu_{12}, \mu_{30}, \mu_{03}$

Normalized central momens

$$\eta_{pq} = \frac{\mu_{pq}}{\mu_{00}^{\gamma}} \quad \text{and} \quad \gamma = \frac{p+q}{2} + 1$$

- A set of invariant moments (7 by Hu)

- Hu Moments:

$$\phi_1 = \eta_{20} + \eta_{02}$$

$$\phi_2 = \left(\eta_{20} - \eta_{02}\right)^2 + 4\eta_{11}^2$$

$$\phi_3 = \left(\eta_{30} - 3\eta_{12}\right)^2 + \left(3\eta_{12} - \eta_{03}\right)^2$$

$$\phi_4 = \left(\eta_{30} + \eta_{12}\right)^2 + \left(\eta_{21} + \eta_{03}\right)^2$$

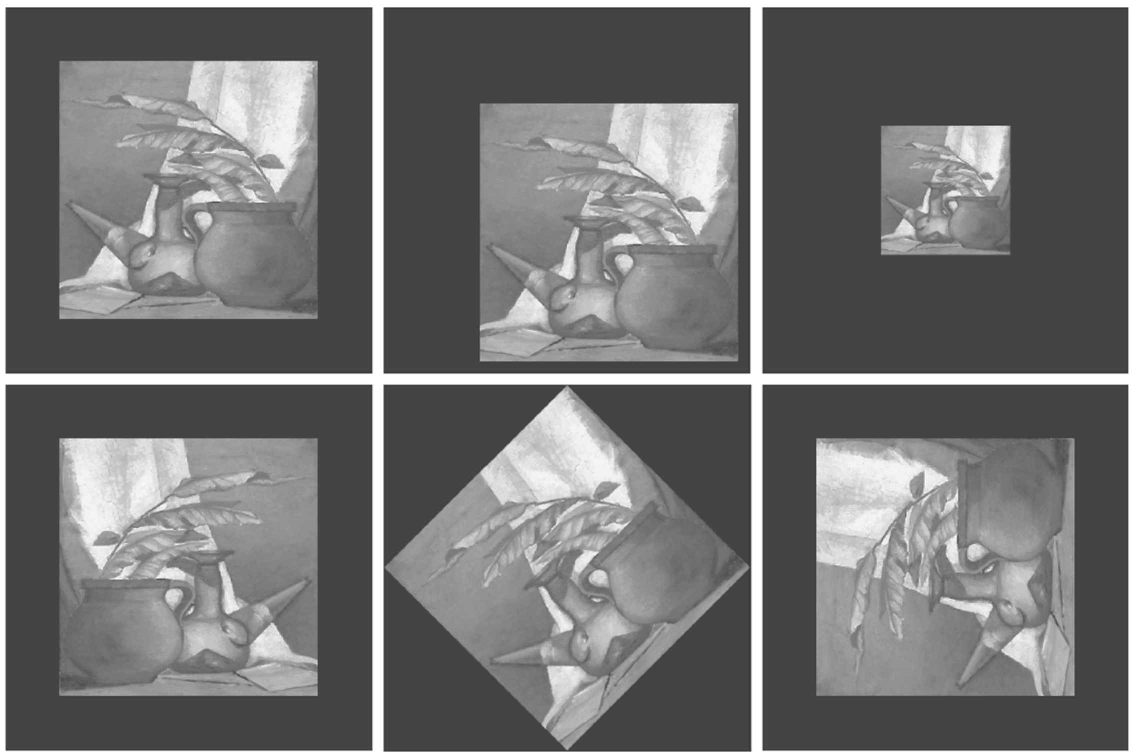
$$\begin{aligned}\phi_5 = &\left(\eta_{30} - 3\eta_{12}\right)\left(\eta_{30} + \eta_{12}\right)\left[\left(\eta_{30} + \eta_{12}\right)^2 - 3\left(\eta_{21} + \eta_{03}\right)^2\right] \\ &+ \left(3\eta_{21} - \eta_{03}\right)\left(\eta_{21} + \eta_{03}\right)\left[3\left(\eta_{30} + \eta_{12}\right)^2 - \left(\eta_{21} + \eta_{03}\right)^2\right]\end{aligned}$$

$$\phi_6 = \left(\eta_{20} - \eta_{02}\right)\left[\left(\eta_{30} + \eta_{12}\right)^2 - \left(\eta_{21} - \eta_{03}\right)^2 4\eta_{11}\left(\eta_{30} + \eta_{12}\right)\left(\eta_{21} + \eta_{03}\right)\right]$$

$$\begin{aligned}\phi_7 = &\left(3\eta_{21} - \eta_{03}\right)\left(\eta_{30} + \eta_{12}\right)\left[\left(\eta_{30} + \eta_{12}\right)^2 - 3\left(\eta_{30} + \eta_{12}\right)^2\right] \\ &+ \left(\eta_{30} - 3\eta_{12}\right)\left(\eta_{21} + \eta_{03}\right)\left[3\left(\eta_{30} + \eta_{12}\right)^2 - \left(\eta_{21} + \eta_{03}\right)^2\right]\end{aligned}$$

$$\phi_8 = \eta_{11}\left[\left(\eta_{30} + \eta_{12}\right)^2 - \left(\eta_{03} + \eta_{21}\right)^2\right] - \left(\eta_{20} - \eta_{02}\right)\left(\eta_{30} + \eta_{12}\right)\left(\eta_{03} + \eta_{21}\right)$$

- Example:



a b c
d e f

FIGURE 11.37 (a) Original image. (b)–(f) Images translated, scaled by one-half, mirrored, rotated by 45° and rotated by 90°, respectively.

- Results of invariant moments:

Moment Invariant	Original Image	Translated	Half Size	Mirrored	Rotated 45°	Rotated 90°
ϕ_1	2.8662	2.8662	2.8664	2.8662	2.8661	2.8662
ϕ_2	7.1265	7.1265	7.1257	7.1265	7.1266	7.1265
ϕ_3	10.4109	10.4109	10.4047	10.4109	10.4115	10.4109
ϕ_4	10.3742	10.3742	10.3719	10.3742	10.3742	10.3742
ϕ_5	21.3674	21.3674	21.3924	21.3674	21.3663	21.3674
ϕ_6	13.9417	13.9417	13.9383	13.9417	13.9417	13.9417
ϕ_7	−20.7809	−20.7809	−20.7724	20.7809	−20.7813	−20.7809

TABLE 11.5 Moment invariants for the images in Fig. 11.37.

- Complex Zernike Moments:

$$A_{mn} = \frac{m+1}{\pi} \iint_{x^2+y^2 \le 1} f(x,y) V_{mn}^{*}(x,y)\, dxdy$$

$$m \in \mathbb{Z}_{\ge 0}, \quad n \in \mathbb{Z}, \quad m - |n| = even, \ |n| \le m$$

- Where V_{mn} are Zernike polynomials:

$$V_{mn}(r,\theta) = R_{mn}(r) e^{jn\theta}, \quad (r,\theta):\ \text{over unit disc}$$

$$R_{mn}(r) = \sum_{s=0}^{\frac{m-|n|}{2}} (-1)^s F(m,n,s,r)$$

$$F(m,n,s,r) = \frac{(m-s)!}{s!\left(\frac{m+|n|}{2}-s\right)!\left(\frac{m-|n|}{2}-s\right)!} r^{m-2s}$$

- Some Zernike Polynomials:

$$R_{m,n} = R_{m,-n}$$

$$R_{0,0} = 1$$

$$R_{1,1} = r$$

$$R_{2,0} = 2r^2 - 1$$

$$R_{2,2} = r^2$$

$$R_{3,1} = 3r^3 - 2r$$

$$R_{3,3} = r^3$$

- How to Compute Zernike Polynomials:
  - Scale and Translation invariance transform:
    - Shift to center of gravity ($m_{10} = m_{01} = 0$)
    - Fix m_{00} at a predetermined value (= β):

$$g(x,y) = f\left(\frac{x}{\alpha} + \bar{x}, \frac{y}{\alpha} + \bar{y}\right), \quad \alpha = \sqrt{\frac{\beta}{m_{00}}}$$

  - Map Image to the unit disc using polar coordinates.
  - The center of the image is the origin of the unit disc
  - Those pixels falling outside the unit disc are not used in the computation.

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

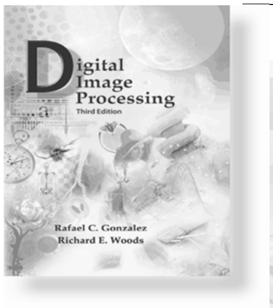
- How to Compute Zernike Polynomials:

$$A_{m,n} = \frac{m+1}{\pi} \sum_x \sum_y g(x,y) V^*_{m,n}(x,y)$$

- Legendre Moments:

$$\lambda_{m,n} = \frac{(2m+1)(2n+1)}{4} \int_{-1}^{1}\int_{-1}^{1} P_m(x) P_n(y)\, dxdy, \quad m,n \in \mathbb{Z}_{\geq 0}$$

$$P_k(r) = \frac{1}{2^k k!} \frac{d^k}{dx^k} \left(x^2 - 1\right)^k$$



- Fourier-Mellin Moments:
  - Fourier-Mellin Transform:

$$F_{m,n} = \int_0^{2\pi}\int_0^{\infty} r^m f(r,\theta) e^{-in\theta} r dr d\theta, \quad m,n \in \mathbb{Z}$$

  - Fourier-Mellin Moments:

$$\Phi_{m,n} = \frac{1}{2\pi a_m}\int_0^{2\pi}\int_0^{\infty} f(r,\theta) Q_m(r) e^{-in\theta} r dr d\theta, \quad m,n \in \mathbb{Z}$$

$Q_m(r)$: orthogonal polynomial of degree m over the raneg $0 \le r \le 1$

$$\int_0^1 Q_m(r) Q_l(r) r dr = a_m \delta[m-k]$$

- Principal Component Analysis:
  - Consider a multi (value/Channel/Spectral) images:

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$$

$$\mathbf{m}_x = E\{\mathbf{x}\} \approx \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_k, \quad K: \text{\# of observations}$$

$$\mathbf{C}_x = E\left\{(\mathbf{x}-\mathbf{m}_x)(\mathbf{x}-\mathbf{m}_x)^T\right\} \approx \frac{1}{K}\sum_{k=1}^{K}\mathbf{x}_k\mathbf{x}_k^T - \mathbf{m}_x\mathbf{m}_x^T$$

$$\mathbf{C}_x: \text{ Real Positive Definiter Matrix}$$

$$\mathbf{C}_x v = \lambda v: \quad \lambda_i \geq 0, \quad v_i \perp v_j, i \neq j$$

$$\mathbf{A} = \left[v_1 | v_2 | \cdots v_n |\right]^T, \quad \lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \Rightarrow A^T = A^{-1}$$

- Karhunen–Loève or Hotelling Transform:
  - Analysis:

$$\mathbf{y} = \mathbf{A}(\mathbf{x} - \mathbf{m_X}) \Rightarrow \mathbf{m_y} = \mathbf{0}, \quad \mathbf{C_y} = \mathbf{A}\mathbf{C_x}\mathbf{A}^{\mathrm{T}} = diag\left(\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}\right)$$

  - Complete Synthesis:

$$\mathbf{x} = \mathbf{A}^T\mathbf{y} + \mathbf{m_X}$$

  - Optimal Synthesis:

$\mathbf{A}_k$ = Form Using k eigenvectors correspondinf to k-largest eigenvalues

$$\hat{\mathbf{x}} = \mathbf{A}_k^T\mathbf{y} + \mathbf{m_X}$$

$$e_{rms} = E\left\{\|\mathbf{x} - \hat{\mathbf{x}}\|^2\right\} = \sum_{j=1}^{n} \lambda_j - \sum_{j=1}^{k} \lambda_j = \sum_{j=k+1}^{n} \lambda_j$$

$\therefore$ PCA Analysis

Representation and Description

- Example:

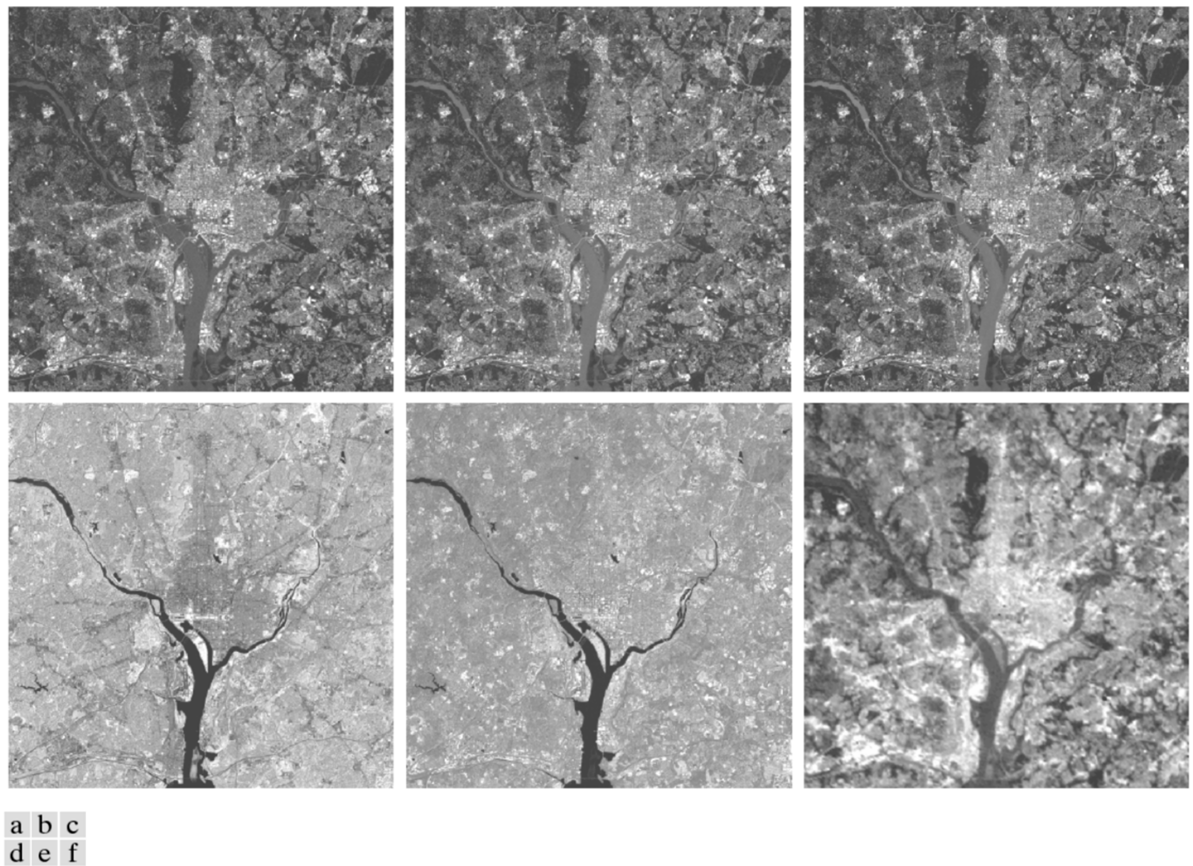


FIGURE 11.38 Multispectral images in the (a) visible blue, (b) visible green, (c) visible red, (d) near infrared, (e) middle infrared, and (f) thermal infrared bands. (Images courtesy of NASA.)

- Data Preparation:

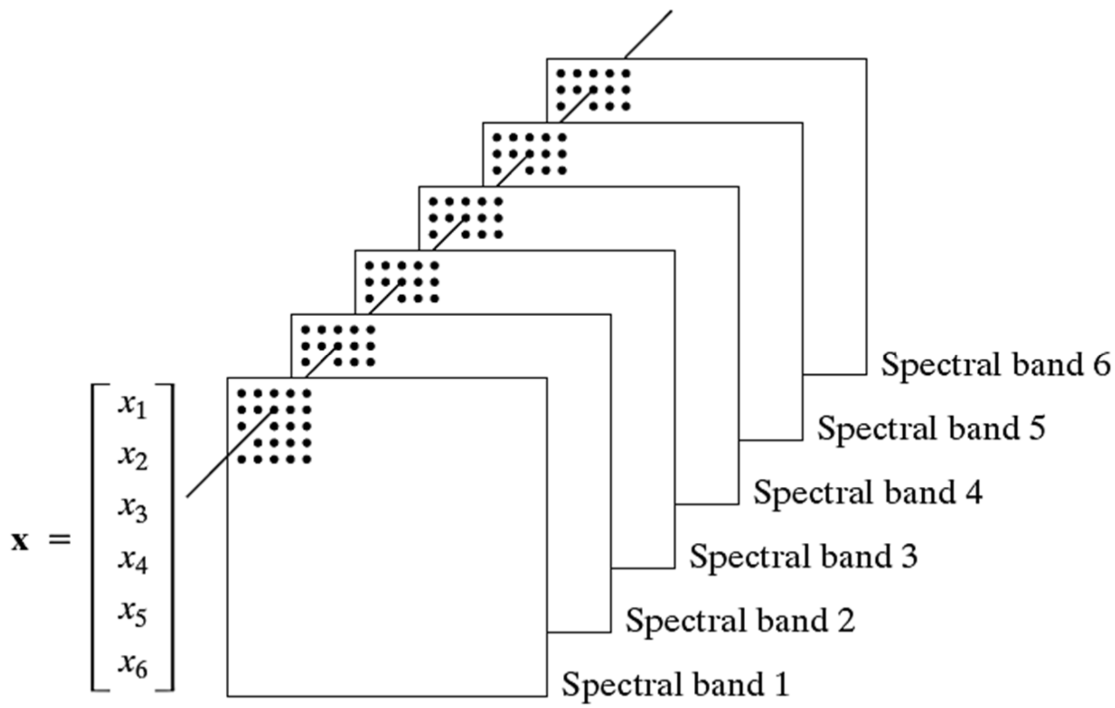
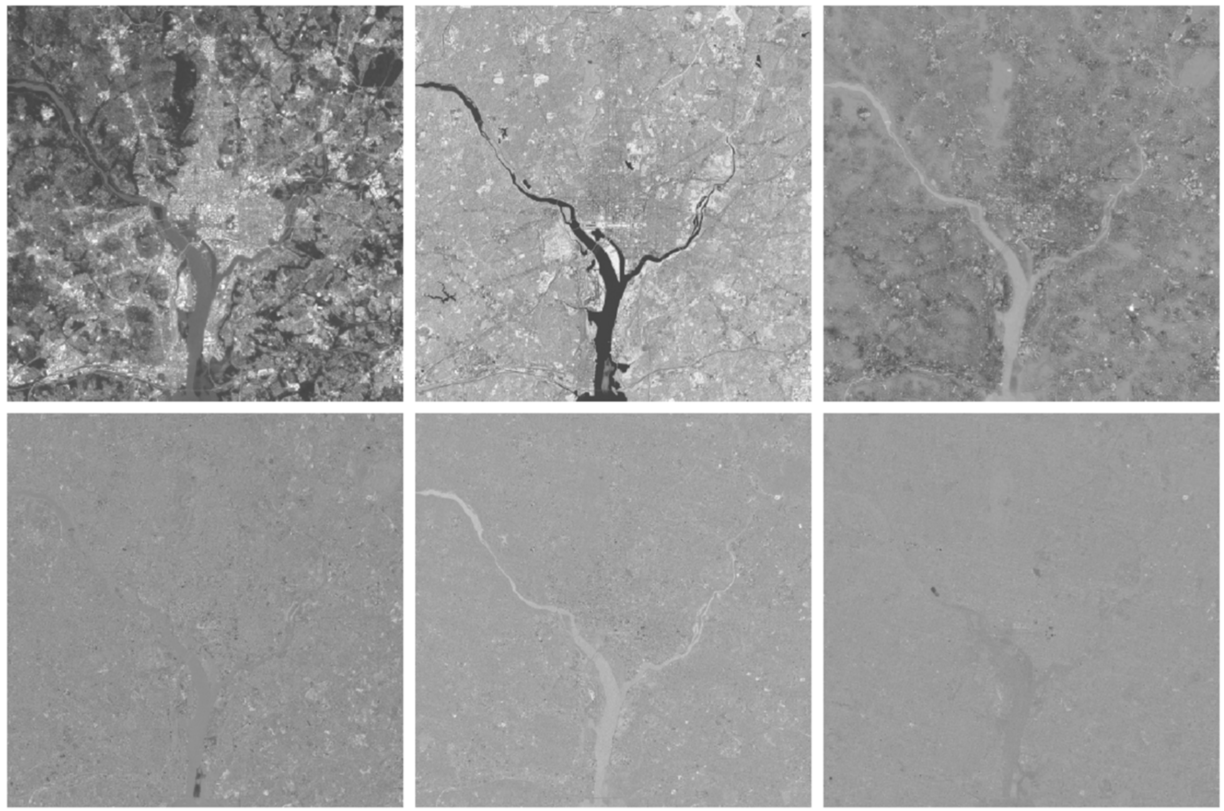


FIGURE 11.39 Formation of a vector from corresponding pixels in six images.

- Eigenvalues Analysis:

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
10344	2966	1401	203	94	31

- PCA Analysis:
  - Eigneimages



a b c
d e f

FIGURE 11.40 The six principal component images obtained from vectors computed using Eq. (11.4-6). Vectors are converted to images by applying Fig. 11.39 in reverse.

Representation and Description

- Example:
  - Using 2 components

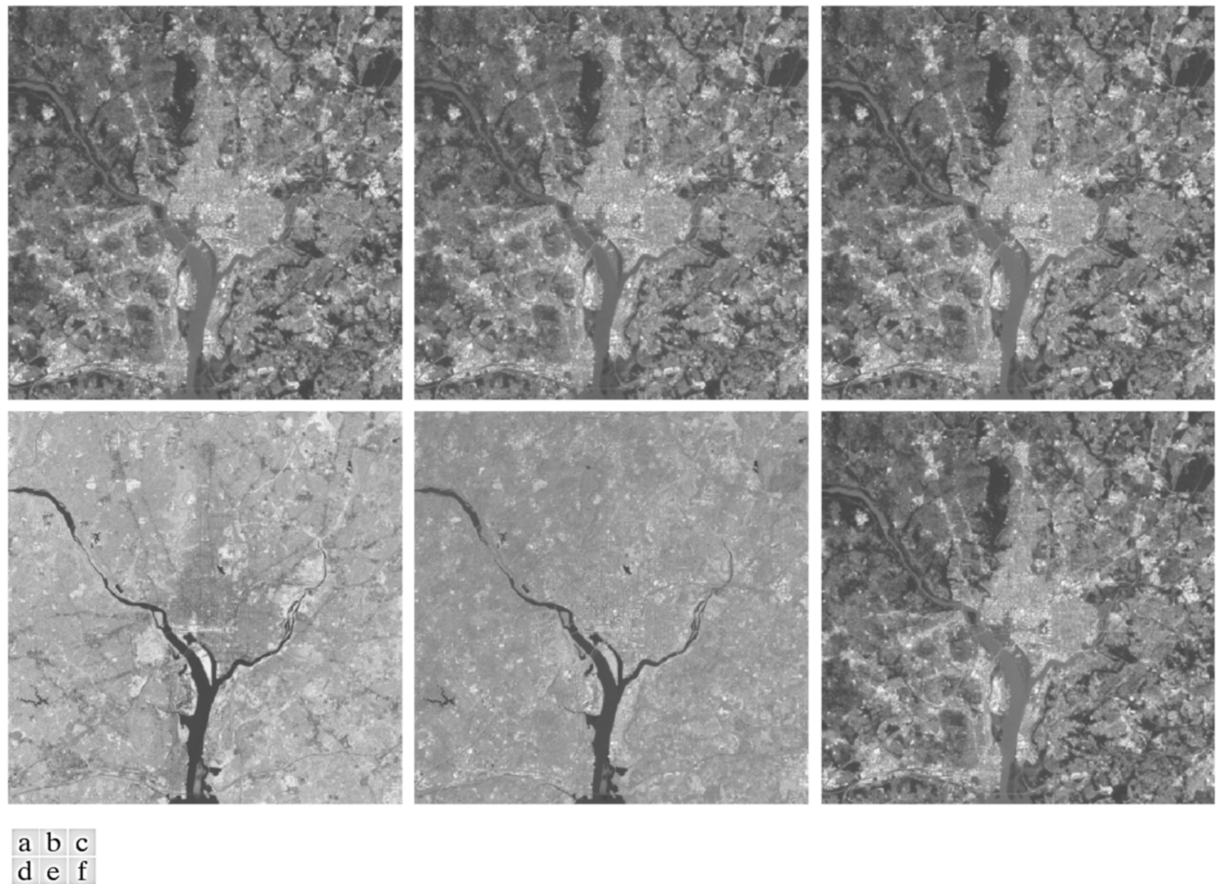


FIGURE 11.41 Multispectral images reconstructed using only the two principal component images corresponding to the two principal component images with the largest eigenvalues (variance). Compare these images with the originals in Fig. 11.38.

- Example:
  - Difference

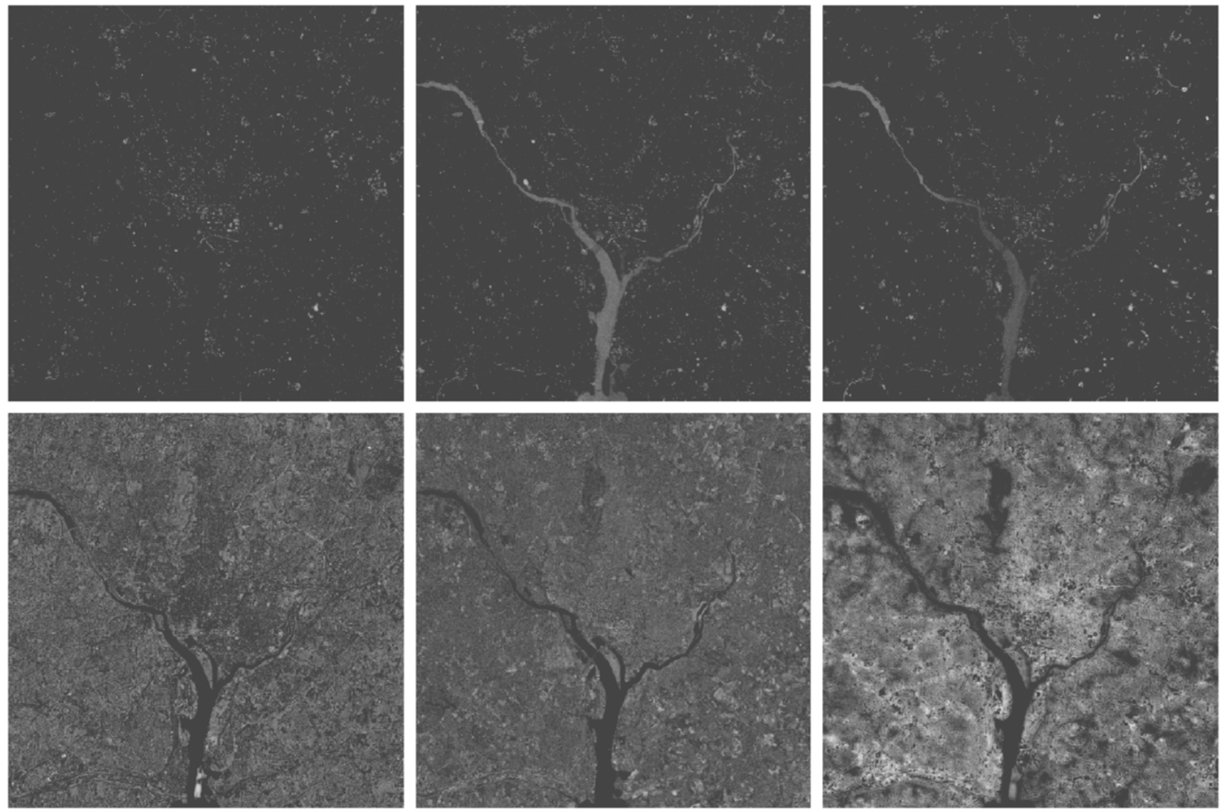


FIGURE 11.42 Differences between the original and reconstructed images. All difference images were enhanced by scaling them to the full [0, 255] range to facilitate visual analysis.

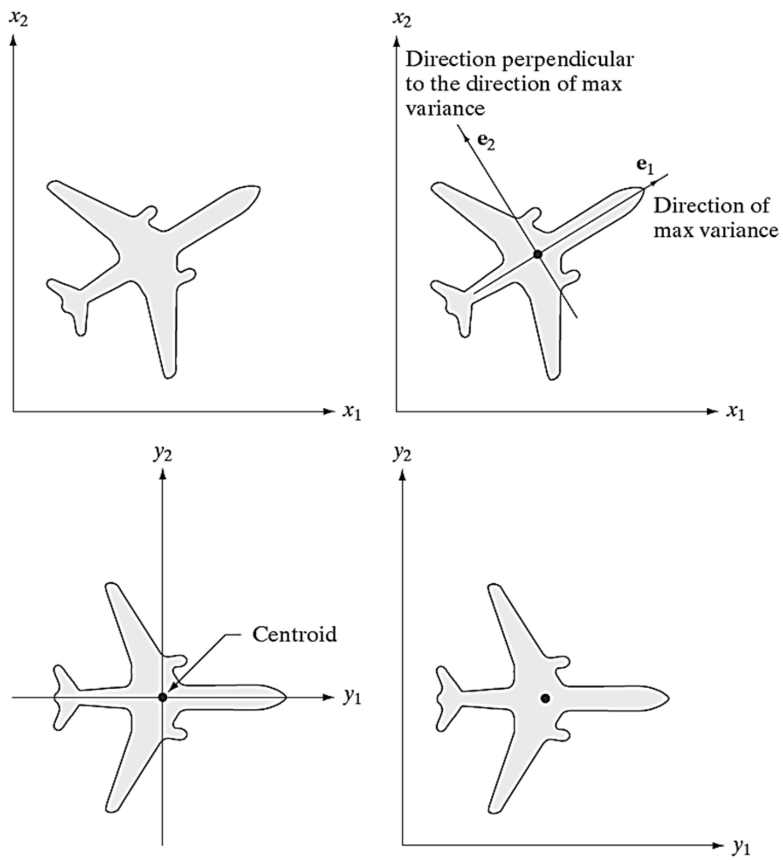
- PCA in Boundary description:

$$\mathbf{m_x} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\mathbf{C_x} = \begin{bmatrix} 3.333 & 2.00 \\ 2.00 & 3.333 \end{bmatrix}$$

$$\mathbf{e}_1 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$\mathbf{e}_2 = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$$



a b
c d

FIGURE 11.43 (a) An object. (b) Object showing eigenvectors of its covariance matrix. (c) Transformed object, obtained using Eq. (11.4-6). (d) Object translated so that all its coordinate values are greater than 0.

- PCA in Boundary description:

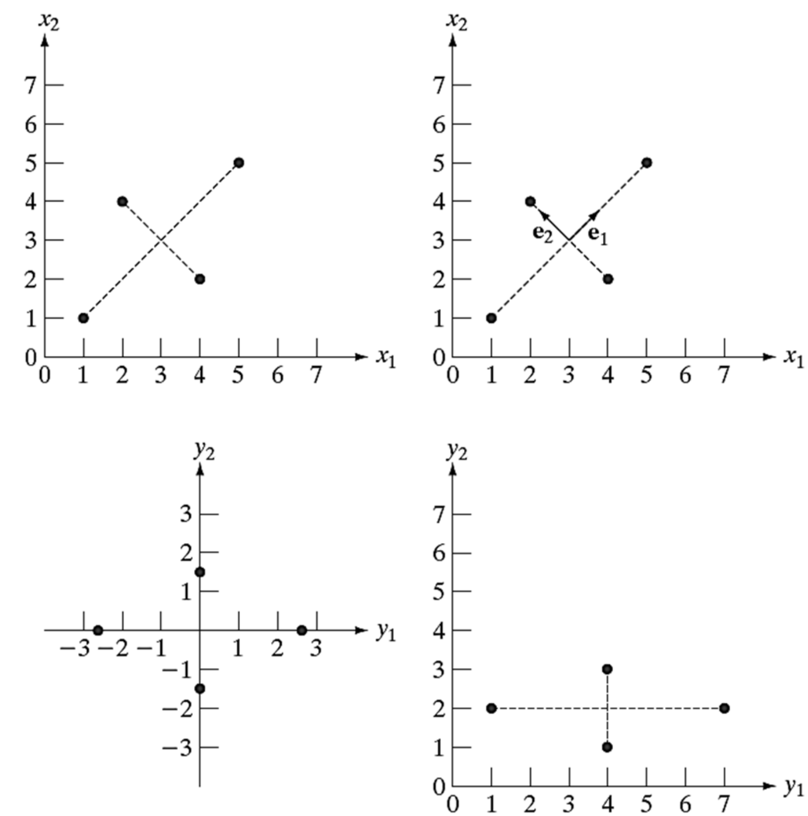


FIGURE 11.44 A manual example. (a) Original points. (b) Eigenvectors of the covariance matrix of the points in (a). (c) Transformed points obtained using Eq. (11.4-6). (d) Points from (c), rounded and translated so that all coordinate values are integers greater than 0. The dashed lines are included to facilitate viewing. They are not part of the data.