Volatility indices under the risk-neutral measure

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1 Chang et al (2013) Appenndix A: Extracting option-implied moments

2 Andersen, Bondarenko (2007): Construction and Interpretation of the model-free implied volatility

2.1 Introduction

different volatility indices

- historic volatility (HV): use past realized volatility to estimate future realized volatility
- Black-Scholes implied volatility (BSIV): use the BS model, enter option prices, solve for volatility
- Corridor implied volatility (CIV): extract volatility from option prices but truncate the strike prices used at a barrier
- Model-free implied volatility (MFIV): extract volatility from option prices using only no-arbitrage arguments and a full range of strike prices

Note: I remember having read elsewhere (sorry unscientific quoting) that even though the indexes are called "volatility" indexes, they rather measure the return variation/quadratic variation than the volatility of the underlying stochastic process, which is unobserved. I think however they are a good estimate for the volatility.

Discussion MFIV

- "aims to measure the expected integrated variance, or more generally return variation, over the coming month, evaluated under the so-called risk-neutral or pricing (Q) measure"
- no assumptions made regarding underlying return dynamics in contrast to BSIV
- however, needs strikes spanning full range of possible values for underlying asset - data requirements often not met, hence approximations have to be made
- MFIV will often differ from return variation under P, hence not a pure volatility forecast but adds uncertainty surrounding that forecast → will contain a premium compensating for exposure to equity-index volatility

Note: So MFIV contains a premium because it not only measure the uncertainty of the returns, but also the uncertainty regarding the return volatility? (see also Carr, Wu, 2008, summary in theoryquestion channel)

Some dimensions that all volatility indices have to meet

- risk-neutral density must satisfy NoA constraints, e.g. risk-neutral density must be positive
- how to handle tails of the distribution?
  - CBOE's VIX: truncate the tail (according to the authors, that makes VIX more an imperfect CIV index than a MFIV measure)
  - alternative: extend the risk-neutral density in the tails

2.2 Theoretical Background: Barrier variance and corridor variance contracts

Notation

- $P_t(K), C_t(K)$ .. call and put with strike K
- F_t .. S&P 500 future contract at time t: they use $r_f = 0$ but use forward instead of spot prices, using the relationship $F_t = e^{r_f(T-t)} S_t$
- $k = K/F_t$ .. moneyness

2.2.1 Derive relationship for NoA pricing of contracts

Option-prices can be computed using risk-neutral density (RND) $h_t(F_t)$

$$P_t(K) = E_t^Q[(K - F_T)^+] = \int_0^\infty (K - F_T)^+ h_t(F_T)\,\mathrm{d}F_T$$
$$C_t(K) = E_t^Q[(F_T - K)^+] = \int_0^\infty (F_T - K)^+ h_t(F_T)\,\mathrm{d}F_T$$

RND is as in Breeden-Litzenberger (1978)

$$h_t(F_T) = \frac{\partial^2 P_t(K)}{\partial K^2}\Big|_{K=F_T} = \frac{\partial^2 C_t(K)}{\partial K^2}\Big|_{K=F_T}$$

According to Carr-Madan (1998) we can write the paypoff $g(F_T)$ (assuming finite second derivatives a.e.) as

$$g(F_T) = g(x) + g'(x)(F_T - x) + \int_0^x g''(K)(K - F_T)^+\,\mathrm{d}K + \int_x^\infty g''(K)(F_T - K)^+\,\mathrm{d}K$$

set $x = F_0$ and take the expectation and setting $x = F_0$

$$E_0^Q[g(F_T)] = g(F_0) + \int_0^{F_0} g''(K)P_0(K)\,\mathrm{d}K + \int_{F_0}^\infty C_0(K)\,\mathrm{d}K \tag{1}$$
$$\Leftrightarrow E_0^Q[g(F_T)] = g(F_0) + \int_0^\infty g''(K)M_0(K)\,\mathrm{d}K \tag{2}$$

with $M_0(K) = min(P_t(K), C_t(K))$ (hence the OTM option).

Note: No expectation on the RHS because RHS is all F_0 measurable?

In the current setting, F_t is a martingale under Q. Assume the diffusion

$$\frac{\mathrm{d}F_t}{F_t} = \sigma_t\,\mathrm{d}W_t \tag{3}$$

By Itos Lemma we get for the payoff

$$g(F_T) = g(F_0) + \int_0^T g'(F_t)\,\mathrm{d}F_t + \frac{1}{2}\int_0^T g''(F_t)F_t^2\sigma_t^2\,\mathrm{d}t$$

Hence taking expectations

$$E_0^Q[g(F_T)] = g(F_0) + \frac{1}{2}E_0^Q\left[\int_0^T g''(F_t)F_t^2\sigma_t^2\,\mathrm{d}t\right] \tag{4}$$

Note: In the paper the second expectation is E_t^Q is that a typo?

setting **??** and **??** equal we obtain

$$E_0^Q\left[\int_0^T g''(F_t)F_t^2\sigma_t^2\,\mathrm{d}t\right] = 2\int_0^\infty g''(K)M_0(K)\,\mathrm{d}K \tag{5}$$

2.2.2 Introduction of Barrier contracts

Contract for deriving MFIV
$\rightarrow$ *show how we can derive the MFIV from a barrier volatility contract*

Consider a contract that pays at time T the realized variance only when the futures price lies below the barrier

$$BIVAR_B(0,T) = \int_0^T \sigma_t^2 I_t(B)\, \mathrm{d}t$$

with

$$I_t = I_t(B) = 1[F_t \leq B]$$

If $B \rightarrow \infty$ the payoff approaches the standard integrated variance

$$IVAR(0,T) = \int_0^T \sigma_t^2 \, \mathrm{d}t$$

Suppose that the function $g(F_T)$ is choosen as

$$g(F_T) = g(F_T; B) = \left(-ln\frac{F_T}{B} + \frac{F_T}{B} - 1 \right) I_T$$

The NoA value of the contract can be derived from equation **??** as

$$BVAR_0(B) = E_0^Q \left[\int_0^T \sigma_t^2 I_t dt \right] = 2 \int_0^B \frac{M_0(K)}{K^2} \, \mathrm{d}K$$

The square-root can be interpreted as the option-implied barrier volatility

$$BIV_0(B) = \sqrt{\int_0^B \frac{M_0(K)}{K^2} \, \mathrm{d}K}$$

Note: this is the formula in our paper to approximate $E_t^Q[RV]$, eq (6),(7). Hence we are missing the "proof" of the step that RV can be estimated with the integrated variance, for this check for example: Andersen, Bollerslev (2002): Parametric and Nonparametric volatility measurement.

In the limiting case of $B = \infty$ the barrier implied volatiltiy coincides with the $MFIV_0$ (developed by Dupire (1993), Neuberger (1994)).

Contract for deriving CIV
$\rightarrow$ *show how we can derive the CV from a corridor volatility contract*

The contract which pays corridor variance can be constructed from two barrier variance contracts with different barriers, here B_1 and B_2 lower and upper barrier

$$CIVAR_{B_1,B_2}(0,T) = \int_0^T \sigma_t^2 I_t(B_1, B_2)\, \mathrm{d}t$$

with

$$I_t(B_1, B_2) = I_t = 1[B_1 \leq F_t \leq B_2]$$

Hence the ocntracts pays the *corridor variance* when the futures price lies between the barriers. The value of the contract is

$$CVAR_0(B_1, B_2) = E_0^Q \left[\int_0^\infty \sigma_t^2 I_t \, \mathrm{d}t \right] = 2 \int_{B_1}^{B_2} \frac{M_0(K)}{K^2} \, \mathrm{d}K$$

In the limiting case where $\Delta(B_1, B_2) \rightarrow 0$ the value of the contract approches the value of a contract that pays the future variance only *along* a strike.

$$SVAR_0(B) = \lim_{\Delta B \rightarrow 0} \frac{B}{\Delta B} CVAR_0(B, B + \Delta B) = 2 \frac{M_0(B)}{B}$$