Volatility indices under the risk-neutral measure

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1 Chang et al (2013) Appenndix A: Extracting option-implied moments

2 Andersen, Bondarenko (2007): Construction and Interpretation of the model-free implied volatility

2.1 Introduction

different volatility indices

- historic volatility (HV): use past realized volatility to estimate future realized volatility
- Black-Scholes implied volatility (BSIV): use the BS model, enter option prices, solve for volatility
- Corridor implied volatility (CIV)
- Model-free implied volatility (MFIV): extract volatility from option prices using only no-arbitrage arguments

Note: I remember having read elsewhere (sorry unscientific quoting) that even though the indexes are called "volatility" indexes, they rather measure the return variation/quadratic variation than the volatility of the underlying stochastic process, which is unobserved. I think however they are a good estimate for the volatility.

Discussion MFIV

- "aims to measure the expected integrated variance, or more generally return variation, over the coming month, evaluated under the so-called risk-neutral or pricing (Q) measure"
- no assumptions made regarding underlying return dynamics in contrast to BSIV
- however, needs strikes spanning full range of possible values for underlying asset - data requirements often not met, hence approximations have to be made
- MFIV will often differ from return variation under P, hence not a pure volatility forecast but adds uncertainty surrounding that forecast → will contain a premium compensating for exposure to equity-index volatility

Some dimensions that all volatility indices have to meet

- risk-neutral density must satisfy NoA constraints, e.g. risk-neutral density must be positive
- how to handle tails of the distribution?
  - CBOE's VIX: truncate the tail (according to the authors, that makes VIX more an imperfect CIV index than a MFIV measure)
  - alternative: extend the risk-neutral density in the tails

2.2 Theoretical Background: Barrier variance and corridor variance contracts

Notation

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Option-prices can be computed using risk-neutral density (RND) $h_t(F_t)$

$$P_t(K) = E_t^Q[(K - F_T)^+] = \int_0^\infty (K - F_T)^+ h_t(F_T)\,\mathrm{d}F_T$$
$$C_t(K) = E_t^Q[(F_T - K)^+] = \int_0^\infty (F_T - K)^+ h_t(F_T)\,\mathrm{d}F_T$$

RND is as in Breeden-Litzenberger (1978)

$$h_t(F_T = \frac{\partial^2 P_t(K)}{\partial K^2}|_{K=F_T} = \frac{\partial^2 C_t(K)}{\partial K^2}|_{K=F_T}$$

According to Carr-Madan (1998) we can write the paypoff $g(F_T)$ (assuming finite second derivatives a.e.) as

$$x) + g'(x)(F_T - x) + \int_0^{F_0} g''(K)P_0(K)\,\mathrm{d}K + \int_{F_0}^\infty g''(K)C_0(K)\,\mathrm{d}K$$

set $x = F_0$ and take the expectation

$$E_0^Q[d(F_T)] = g(F_0) + \int_0^\infty g''(K)M_0(K)\,\mathrm{d}K \tag{1}$$

with $M_0(K) = min(P_t(K), C_t(K))$ (hence the OTM option).

In the current setting, F_t is a martingale under Q. Assume the diffusion

$$\frac{\mathrm{d}F_t}{F_t} = \sigma\,\mathrm{d}W_t \tag{2}$$

By Itos Lemma we get for the payoff

$$g(F_T) = g(F_0) + \int_0^T g'(F_t)\,\mathrm{d}F_t + \frac{1}{2}\int_0^T g''(F_t)F_t)^2\sigma_t^2\,\mathrm{d}t$$

Hence

$$E_0^Q[g(F_T)] = g(F_0) + \frac{1}{2}E_0^Q\left[\int_0^T g''(F_t)F_t^2\sigma_t^2\,\mathrm{d}t\right] \tag{3}$$

setting 1 and 3 equal we obtain

$$E_0^Q\left[\int_0^T g''(F_t)F_t^2\sigma_t^2\,\mathrm{d}t\right] = 2\int_0^\infty g''(K)M_0(K)\,\mathrm{d}K \tag{4}$$

2.2.1 Introduction of Barrier ocntracts

Contract for deriving MFIV

Consider a contract that pays at time T the realized variance only when the futures price lies below the barrier

$$BIVAR_B(0,T) = \int_0^T \sigma_t^2 I_t(B)\,\mathrm{d}t$$

with

$$I_t = I_t(B) = 1[F_t \leq B]$$

If $B \to \infty$ the payoff approaches the standard integrated variance

$$IVAR(0,T) = \int_0^T \sigma_t^2\,\mathrm{d}t$$

Suppose that the function $g(F_T)$ is choosen as

$$g(F_T) = g(F_T; B) = \left(1 - ln\frac{F_T}{B} + \frac{F_T}{B} - 1\right) I_T$$

The NoA value of the contract can be derived from equation 4 as

$$BVAR_0(B) = E_0^Q\left[\int_0^T \sigma_t^2 I_t dt\right] = 2\int_0^B \frac{M_0(K)}{K^2}\,\mathrm{d}K$$

The square-root can be interpreted as the option-implied barrier volatility

$$BIV_0(B) = \sqrt{\int_0^B \frac{M_0(K)}{K^2}\,\mathrm{d}K}$$

Note: So actually the volatility index is a standard deviation?

In the limiting case of $B = \infty$ the barrier implied volatiltiy coincides with the $MFIV_0$ (developed by Dupire (1993), Neuberger (1994)).

Contract for deriving CIV

The contract which pays corridor variance can be constructed from two barrier variance contracts with different barriers, here B_1 and B_2 lower and upper barrier

$$CIVAR_{B_1,B_2}(0,T) = \int_0^T \sigma_t^1 I_t(B_1, B_2)\,\mathrm{d}t$$

with

$$I_t(B_1, B_2) = I_t = 1[B_1 \leq F_t \leq B_2]$$

Hence the ocntracts pays the *corridor variance* when the futures price lies between the barriers. The value of the contract is

$$CVAR_0(B_1, B_2) = E_0^Q\left[\int_0^\infty \sigma_t^2 I_t\,\mathrm{d}t\right] = 2\int_{B_1}^{B_2} \frac{M_0(K)}{K^2}\,\mathrm{d}K$$

In the limiting case where $\Delta(B_1, B_2) \to 0$ the value of the contract approches the value of a contract that pays the future variance only *along* a strike.

$$SVAR_0(B) = \lim_{\Delta B \to 0} \frac{B}{\Delta B} CVAR_0(B, B + \Delta B) = 2\frac{M_0(B)}{B}$$