

## CIS 399 – Homework 2

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### Problem 1

Show that  $(1 - FNR) = \frac{A}{A+C}$ .

From the definition of FNR, we know that  $(1 - FNR)$  can be rewritten as

$$1 - \frac{C}{A+C} \tag{1}$$

Simplifying (1),

$$\begin{aligned} 1 - \frac{C}{A+C} &= \frac{1}{1} - \frac{C}{A+C} \\ &= \frac{1(A+C)}{1(A+C)} - \frac{C}{A+C} \\ &= \frac{A+C}{A+C} - \frac{C}{A+C} \\ &= \frac{A+C-C}{A+C} \\ &= \frac{A}{A+C} \end{aligned}$$

Thus, from the above,  $(1 - FNR) = \frac{A}{A+C}$ . QED.

### Problem 2

Show that  $(1 - PPV) = \frac{B}{A+B}$ .

From the definition of PPV, we know that  $(1 - PPV)$  can be rewritten as

$$1 - \frac{A}{A+B} \tag{2}$$

Simplifying (2),

$$\begin{aligned}
 1 - \frac{A}{A+B} &= \frac{1}{1} - \frac{A}{A+B} \\
 &= \frac{1(A+B)}{1(A+B)} - \frac{A}{A+B} \\
 &= \frac{A+B}{A+B} - \frac{A}{A+B} \\
 &= \frac{A+B-A}{A+B} \\
 &= \frac{B}{A+B}
 \end{aligned}$$

Thus, from the above,  $(1 - PPV) = \frac{B}{A+B}$ . QED.

### Problem 3

Show that  $\frac{BR}{1-BR} = \frac{A+C}{B+D}$ .

From the definition of BR, BR can be rewritten as

$$\frac{A+C}{|P|} \quad (3)$$

Substituting (3),  $\frac{BR}{1-BR}$  becomes

$$\frac{\frac{A+C}{|P|}}{1 - \frac{A+C}{|P|}} \quad (4)$$

From the definition of  $|P|$ , (4) can be further rewritten as

$$\frac{\frac{A+C}{A+B+C+D}}{1 - \frac{A+C}{A+B+C+D}} \quad (5)$$

Simplifying (5),

$$\begin{aligned}
 \frac{\frac{A+C}{A+B+C+D}}{1 - \frac{A+C}{A+B+C+D}} &= \frac{\frac{(A+C)(A+B+C+D)}{A+B+C+D}}{1(A+B+C+D) - \frac{(A+C)(A+B+C+D)}{A+B+C+D}} \\
 &= \frac{A+C}{(A+B+C+D) - (A+C)} \\
 &= \frac{A+C}{B+D}
 \end{aligned}$$

Thus, from the above,  $\frac{BR}{1-BR} = \frac{A+C}{B+D}$ . QED.

### Problem 4

Using the results from Problems 1, 2, and 3, show that:

$$FPR = \left(\frac{BR}{1-BR}\right)\left(\frac{1-PPV}{PPV}\right)(1-FNR)$$

From Problem 3, we know  $\frac{BR}{1-BR}$  can be rewritten as

$$\frac{A+C}{B+D} \quad (6)$$

From Problem 2, we know  $(1 - PPV)$  can be rewritten as

$$\frac{B}{A + B} \quad (7)$$

From the definition of PPV, we know PPV can be rewritten as

$$\frac{A}{A + B} \quad (8)$$

From Problem 1, we know  $(1 - FNR)$  can be rewritten as

$$\frac{A}{A + C} \quad (9)$$

Substituting (6), (7), (9), and (10)  $(\frac{BR}{1-BR})(\frac{1-PPV}{PPV})(1 - FNR)$  becomes

$$(\frac{A + C}{B + D})(\frac{\frac{B}{A+B}}{\frac{A}{A+B}})(\frac{A}{A + C}) \quad (10)$$

Simplifying (10),

$$\begin{aligned} (\frac{A + C}{B + D})(\frac{\frac{B}{A+B}}{\frac{A}{A+B}})(\frac{A}{A + C}) &= (\frac{A + C}{B + D})(\frac{\frac{B(A+B)}{A+B}}{\frac{A(A+B)}{A+B}})(\frac{A}{A + C}) \\ &= (\frac{A + C}{B + D})(\frac{B}{A})(\frac{A}{A + C}) \\ &= \frac{(A + C)BA}{(B + D)A(A + C)} \\ &= \frac{BAA + BAC}{(AB + AD)(A + C)} \\ &= \frac{BAA + BAC}{AAB + AAD + CAB + CAD} \\ &= \frac{BA(A + C)}{A(AB + AD + CB + CD)} \\ &= \frac{B(A + C)}{AB + AD + CB + CD} \\ &= \frac{B(A + C)}{(A + C)(B + D)} \\ &= \frac{B}{B + D} \end{aligned}$$

From the definition of FPR, we know FPR can be rewritten as

$$\frac{B}{B + D}$$

This is equivalent to the simplified version of  $(\frac{BR}{1-BR})(\frac{1-PPV}{PPV})(1 - FNR)$ , as shown above. Thus,  $FPR = (\frac{BR}{1-BR})(\frac{1-PPV}{PPV})(1 - FNR)$ . QED.

### Problem 5

The statement from Problem 4 holds for the entire population as well as for each group individually. Suppose that all three fairness notions are satisfied by our hypothesis, i.e.  $FPR_1 = FPR_2$ ,  $FNR_1 = FNR_2$ , and  $PPV_1 = PPV_2$ . Further, assume that all of these values, as well as the base rates, are neither 0 nor 1. Show that this implies that the base rates of the groups must be equal.

From the given, we know that  $FPR_1 = FPR_2$ . From Problem 4, we know FPR can be rewritten as

$$FPR = \left(\frac{BR}{1 - BR}\right)\left(\frac{1 - PPV}{PPV}\right)(1 - FNR) \quad (11)$$

From the given, we know that (11) holds for the entire population, as well as for each group individually. Thus, it holds that for groups 1 and 2,

$$\left(\frac{BR_1}{1 - BR_1}\right)\left(\frac{1 - PPV_1}{PPV_1}\right)(1 - FNR_1) = \left(\frac{BR_2}{1 - BR_2}\right)\left(\frac{1 - PPV_2}{PPV_2}\right)(1 - FNR_2) \quad (12)$$

From the given, we know that  $FNR_1 = FNR_2$  and that  $PPV_1 = PPV_2$ . Let the variable  $s$  represent  $FNR_1$  and  $FNR_2$ , since they are equal. Similarly, let the variable  $z$  represent  $PPV_1$  and  $PPV_2$ , since they are equal. Thus, (12) can be rewritten as

$$\left(\frac{BR_1}{1 - BR_1}\right)\left(\frac{1 - z}{z}\right)(1 - s) = \left(\frac{BR_2}{1 - BR_2}\right)\left(\frac{1 - z}{z}\right)(1 - s) \quad (13)$$

Simplifying (13), (and keeping in mind that none of these values are 0),

$$\begin{aligned} \left(\frac{BR_1}{1 - BR_1}\right)(1 - s) &= \left(\frac{BR_2}{1 - BR_2}\right)(1 - s) \\ \frac{BR_1}{1 - BR_1} &= \frac{BR_2}{1 - BR_2} \\ BR_1(1 - BR_2) &= BR_2(1 - BR_1) \\ BR_1 - BR_1BR_2 &= BR_2 - BR_1BR_2 \\ BR_1 &= BR_2 \end{aligned}$$

Thus, the base rates of the groups must be equal, since  $BR_1 = BR_2$ , as shown above. QED.

### Problem 6

Show that if our hypothesis makes no mistakes (i.e.  $B_1 = B_2 = 0$  and  $C_1 = C_2 = 0$ ), all three fairness notions will be satisfied, regardless of the base rates for each group.

We must show that  $FPR_1 = FPR_2$ ,  $FNR_1 = FNR_2$ , and  $PPV_1 = PPV_2$ , regardless of base rates.

1. Proof that  $FPR_1 = FPR_2$ : From the definition of FPR, we know that FPR for groups 1 and 2 respectively can be rewritten as

$$\frac{B_1}{B_1 + D_1} \quad (14)$$

$$\frac{B_2}{B_2 + D_2} \quad (15)$$

From the given, we know that  $B_1 = B_2 = 0$ . Hence, (14) and (15) can be rewritten respectively as

$$\frac{0}{0 + D_1} \quad (16)$$

$$\frac{0}{0 + D_2} \quad (17)$$

However, both (16) and (17) simplify to 0. Thus,  $FPR_1 = 0$  and  $FPR_2 = 0$ , and  $FPR_1 = FPR_2$ . QED.

2. Proof that  $FNR_1 = FNR_2$ : From the definition of FNR, we know that FNR for groups 1 and 2 respectively can be rewritten as

$$\frac{C_1}{A_1 + C_1} \quad (18)$$

$$\frac{C_2}{A_2 + C_2} \quad (19)$$

From the given, we know that  $C_1 = C_2 = 0$ . Hence, (18) and (19) can be rewritten respectively as

$$\frac{0}{A_1 + 0} \quad (20)$$

$$\frac{0}{A_2 + 0} \quad (21)$$

However, both (20) and (21) simplify to 0. Thus,  $FNR_1 = 0$  and  $FNR_2 = 0$ , and  $FNR_1 = FNR_2$ . QED.

3. Proof that  $PPV_1 = PPV_2$ : From the definition of PPV, we know that PPV for groups 1 and 2 respectively can be rewritten as

$$\frac{A_1}{A_1 + B_1} \quad (22)$$

$$\frac{A_2}{A_2 + B_2} \quad (23)$$

From the given, we know that  $B_1 = B_2 = 0$ . Hence, (22) and (23) can be rewritten respectively as

$$\frac{A_1}{A_1 + 0} \quad (24)$$

$$\frac{A_2}{A_2 + 0} \quad (25)$$

However, both (24) and (25) simplify to 1. Thus,  $PPV_1 = 1$  and  $PPV_2 = 1$ , and  $PPV_1 = PPV_2$ . QED.

From parts 1, 2, and 3, it is shown that all three fairness notions are satisfied, regardless of base rates. QED.