Baire's Theorem

Math 308, Real Analysis

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Outline

Baire's Theorem

Theorem 3.5.4 The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

❓ How to deal with the 'countable union' of sets?

❓ What does it mean for a set to be 'nowhere-dense'?

❓ Why cannot $\mathbb{R}$ be made up of those kinds of sets?

F_σ Set and G_δ Set

Definition 3.5.1. A set $A \subseteq \mathbb{R}$ is called an F_σ set if it can be written as the countable union of closed sets. A set $B \subseteq \mathbb{R}$ is called a G_δ set if it can be written as the countable intersection of open sets.

Recall that,

- The union of a finite collection of closed sets is closed.
- The intersection of a finite collection of open sets is open.

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Recall that,

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F_σ set and G_δ set are neither closed sets nor open sets.

Example Related to F_σ Set and G_δ Set

Example 1

(a) A closed interval $[a, b]$ is a G_δ set.

$$[a, b] = \bigcap_{n=1}^{\infty} (a - \tfrac{1}{n}, b + \tfrac{1}{n})$$

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$$[a, b] = \bigcap_{n=1}^{\infty}(a - \tfrac{1}{n}, b + \tfrac{1}{n})$$

(b) The half-open interval $(a, b]$ is both a G_δ and an F_σ set.

$$(a, b] = \bigcap_{n=1}^{\infty}(a, b + \tfrac{1}{n}) = \bigcup_{n=1}^{\infty}[a + \tfrac{1}{n}, b]$$

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Example 2

Exercise 3.5.1. Argue that a set A is a G_δ set if and only if its complement is an F_σ set.

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($\rightarrow$):

A set A is a G_δ set if $A = \bigcap_{n=1}^{\infty} O_n$, where each O_n is an open set.

By De Morgan's Law,

$$\left(\bigcap_{n=1}^{\infty} O_n \right)^c = \bigcup_{n=1}^{\infty} O_n^c$$

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($\longleftarrow$):

A set A^c is an F_σ set if $A^c = \bigcup_{n=1}^{\infty} C_n$, where each C_n is a closed set.

By De Morgan's Law,

$$\left(\bigcup_{n=1}^{\infty} C_n\right)^c = \bigcap_{n=1}^{\infty} C_n^c$$

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Example 3

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(b) The set of irrationals I forms a G_δ set.

By De Morgan's Law, we get

$$\mathbb{I} = \mathbb{Q}^c = \bigcap_{n=1}^{\infty} [r_n, r_n]^c$$

Example Related to F_σ Set and G_δ Set

Example 4

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Each F_σ set can be expressed as:

$$F_\sigma = \bigcup_{n=1}^{\infty} C_n$$

where each C_n is a closed set.

A countable collection of F_σ sets can be written as:

$$F_\sigma^m = \bigcup_{n=1}^{\infty} C_n^m \quad \text{for each } m \in \mathbb{N}$$

Example Related to F_σ Set and G_δ Set

The countable union of these F_σ sets is:

$$\bigcup_{m=1}^{\infty} F_\sigma^m = \bigcup_{m=1}^{\infty} \left(\bigcup_{n=1}^{\infty} C_n^m \right) = \bigcup_{(m,n)\in\mathbb{N}\times\mathbb{N}} C_n^m$$

We know $\mathbb{N} \times \mathbb{N}$ is countable.

Thus, it is a countable union of closed sets, which is an F_σ set.

Example Related to F_σ Set and G_δ Set

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where each C_n is a closed set.

A finite number of F_σ sets can be written as:

$$F_\sigma^i = \bigcup_{n=1}^{\infty} C_n^i$$

for $i = 1, 2, \ldots, k$.

Example Related to F_σ Set and G_δ Set

The intersection of these F_σ sets is:

$$\bigcap_{i=1}^{k} F_\sigma^i = \bigcap_{i=1}^{k} \left(\bigcup_{n=1}^{\infty} C_n^i \right)$$

$$= \bigcup_{n_1=1}^{\infty} \bigcup_{n_2=1}^{\infty} \cdots \bigcup_{n_k=1}^{\infty} \left(C_{n_1}^1 \cap C_{n_2}^2 \cap \ldots \cap C_{n_k}^k \right)$$

Since the intersection of an arbitrary collection of closed sets is closed, $C_{n_1}^1 \cap C_{n_2}^2 \cap \ldots \cap C_{n_k}^k$ is closed.

Therefore, we have a countable union of closed sets, which is an F_σ set.

Theorem: Intersection of Countable Dense Open Sets

Definition A set $G \subseteq \mathbb{R}$ is *dense* in $\mathbb{R}$ if, given any two real numbers $a < b$, it is possible to find a point $x \in G$ with $a < x < b$.

Theorem 3.5.2. If $\{G_1, G_2, G_3, \dots\}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.

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Theorem 3.5.2. If $\{G_1, G_2, G_3, \ldots\}$ is a countable collection of dense, open sets, then the intersection $\bigcap_{n=1}^{\infty} G_n$ is not empty.

Hint: Apply the Nested Interval Property

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Hint: Apply the Nested Interval Property

Proof

- Construct nested intervals

  Since G_1 is open, there exists an open interval $(a_1, b_1) \subseteq G_1$.
  Let $I_1 = [c_1, d_1] \subseteq (a_1, b_1) \subseteq G_1$.

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  Since G_1 is open, there exists an open interval $(a_1, b_1) \subseteq G_1$.
  Let $I_1 = [c_1, d_1] \subseteq (a_1, b_1) \subseteq G_1$.

  Now suppose $I_n \subseteq G_n$.
  Since G_{n+1} is dense and $G_{n+1} \cap (c_n, d_n)$ is open, there exists an interval $(a_{n+1}, b_{n+1}) \subseteq G_{n+1} \cap (c_n, d_n)$.
  Letting $I_{n+1} = [c_{n+1}, d_{n+1}] \subseteq (a_{n+1}, b_{n+1})$ gives us a new closed interval.

Theorem: Intersection of Countable Dense Open Sets

- This gives us a collection of sets with $I_{n+1} \subseteq I_n$, $I_n \subseteq G_n$, $I_n \neq \emptyset$. We can apply the Nested Interval Property to conclude

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

and thus

$$\bigcap_{n=1}^{\infty} G_n \neq \emptyset$$

since each $I_n \subseteq G_n$.

Theorem: Interpretation of $\mathbb{R}$

Exercise 3.5.5. It is impossible to write $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ where for each $n \in \mathbb{N}$, F_n is a closed set containing no nonempty open intervals.

"No nonempty open intervals"

F_n **is a closed set containing no nonempty open intervals**

$=$ F_n is a closed set containing empty open intervals

$\neq$ F_n is empty

$\rightarrow$the elements in F_n might be isolated (For instance, $\mathbb{Q}$)

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By De Morgan's Law, $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ is equivalent to $\emptyset = \bigcap_{n=1}^{\infty} G_n$, where $G_n = F_n^c$.
Then, G_n must be open and dense.

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Then, G_n must be open and dense.

1. Since $G_n = F_n^c$, F_n is closed, G_n is open.

2. Let $a, b \in \mathbb{R}$ and $a < b$, since F_n contains no nonempty open intervals, $(a, b) \nsubseteq F_n$, then $(a, b) \bigcap G_n \neq \emptyset$

There exists $c \in (a, b) \bigcap G_n \subseteq (a, b)$ such that $c \in G_n$, which means that G_n is dense.

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According to Theorem 3.5.2 which tells that $\bigcap_{n=1}^{\infty} G_n \neq \emptyset$ for dense, open set G_n, contradicting with $\emptyset = \bigcap_{n=1}^{\infty} G_n$
Hence, it is impossible to write $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$

Example Related to Theorem

Exercise 3.5.6. Use the conclusion drawn from the previous theorem to show that **the set $\mathbb{I}$ of irrationals cannot be an F_σ set**(can be written as the countable union of closed sets).

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Recall $\mathbb{Q}$ can be written as $\mathbb{Q} = \bigcup_{n=1}^{\infty}[r_n, r_n]$ where r_n is the enumeration of $\mathbb{Q}$.
$\rightarrow \mathbb{Q}$ is a F_σ set

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By contradiction, assuming $\mathbb{I} = \bigcup_{n=1}^{\infty} F_n$
Then, $\mathbb{Q}$ and $\mathbb{I}$ must contain no nonempty intervals.

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Recall $\mathbb{Q}$ can be written as $\mathbb{Q} = \bigcup_{n=1}^{\infty}[r_n, r_n]$ where r_n is the enumeration of $\mathbb{Q}$.
$\rightarrow \mathbb{Q}$ is a F_σ set

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Then, $\mathbb{Q}$ and $\mathbb{I}$ must contain no nonempty intervals.

$\mathbb{Q}$ is the countable union of points

Between every two irrational numbers $a < b$, there must exist a rational number $c \in (a, b)$

Example related to the theorem

Assuming $\mathbb{R} = \mathbb{Q} \bigcup \mathbb{I} = \bigcup_{n=1}^{\infty} P_n$
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In the previous theorem we have shown that $\mathbb{R} \neq \bigcup_{n=1}^{\infty} P_n (P_n$ is the closed set containing no nonempty intervals), which is a contradiction.

Therefore, $\mathbb{I}$ is not an F_σ set.

Baire's Theorem

Theorem 3.5.4 The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

❓ How to deal with the 'countable union' of sets?

❓ What does it mean for a set to be 'nowhere-dense'?

❓ Why cannot $\mathbb{R}$ be made up of those kinds of sets?

The Property for a Dense Set

Definition: G is dense in $\mathbb{R}$ if and only if every point of $\mathbb{R}$ is a limit point of G.

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Example: $\mathbb{Q}$ is dense in $\mathbb{R}$

Every real number a is the limit point of $\mathbb{Q}$

$\rightarrow \overline{\mathbb{Q}} = \mathbb{R}$

Examples for the dense set

1. $A = \mathbb{Q} \bigcap [0, 5]$

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2. $\mathbb{I} =$ The set of irrationals
$\mathbb{I}$ is dense in $\mathbb{R}$ since $\bar{\mathbb{I}}$=$\mathbb{R}$

Nowhere-dense set

Definition: A set E is nowhere-dense if $\overline{E}$ contains no nonempty open intervals.

Example: $B = \frac{1}{n} : n \in \mathbb{N}$

B is nowhere-dense since $\overline{B} = B \bigcup \{0\}$, it contains no nonempty intervals since the elements in $\overline{B}$ are isolated.

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Property: A set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

Nowhere-dense set

A set E is nowhere-dense in $\mathbb{R}$ if and only if the complement of $\overline{E}$ is dense in $\mathbb{R}$.

Proof: First suppose E is nowhere-dense, want to show the complement of $\overline{E}$ is dense in $\mathbb{R}$.

Since E is nowhere-dense, according to the definition, $\overline{E}$ contains no nonempty open intervals.

for $a, b \in \mathbb{R}$, $a < b$, $(a, b) \nsubseteq \overline{E}$, so $(a, b) \bigcap \overline{E}^c \neq \emptyset$

$\rightarrow$We can find a $c \in (a, b) \bigcap \overline{E}^c \subseteq (a, b)$ and $c \in \overline{E}^c$, meaning $\overline{E}^c$ is dense.

Nowhere-dense det

A set E is nowhere-dense in R if and only if the complement of $\overline{E}$ is dense in R.

Then, suppose the complement of $\overline{E}$ is dense in $\mathbb{R}$, want to show E is nowhere-dense.

$\overline{E}^c$ is dense in $\mathbb{R}$, meaning for all $a < b$, there exists $c \in (a, b)$ such that $c \in \overline{E}^c$

Since $c \notin \overline{E}$ and $c \in (a, b)$, $(a, b) \nsubseteq \overline{E}$

Therefore, $\overline{E}$ contains no nonempty open intervals, meaning E is nowhere-dense by definition.

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Proof: By contradiction, assuming $E_1, E_2, E_3...$ are each nowhere-dense sets and satisfy $\mathbb{R} = \bigcup_{n=1}^{\infty} E_n$.

Since E_n are nowhere-dense, by definition, $\overline{E_n}$ contains no nonempty open intervals ($\overline{E_n}$ is closed).

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Recall from the Theorem mentioned before that it is impossible to write $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$ where for each $n \in \mathbb{N}$, F_n is a closed set containing no nonempty open intervals.

Therefore, $\bigcup_{n=1}^{\infty} \overline{E_n} \neq \mathbb{R}$.

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Therefore, $\bigcup_{n=1}^{\infty} \overline{E_n} \neq \mathbb{R}$.

Since $E_n \subseteq \overline{E_n}$, $\bigcup_{n=1}^{\infty} E_n \neq \mathbb{R}$

Proof Done!

Interpretation of Baire's Theorem

The set of real numbers $\mathbb{R}$ cannot be written as the countable union of nowhere-dense sets.

Example of nowhere-dense set: $\{a\}$ or $[a, a]$

$\mathbb{R} \neq \bigcup_{n=1}^{\infty} [a_n, a_n]$

$\mathbb{R}$ is an uncountable set

References

Abbott, S. (2015). *Understanding analysis.* Springer.