Uniqueness, Continuation, and Reflection of Analytic Functions

Math 307, Complex Analysis

Zhaoyang Li, Baoguanyan Kang

Duke Kunshan University

May 3, 2024

Outline

Identity Theorem

Theorem

If two functions f and g are analytic in the same domain D and $f(z) = g(z)$ on a subset of D that has a limit point z_0 in D, then $f(z) = g(z)$ everywhere in D.

A subset of D $\longrightarrow$ Everywhere in D

Step 1

Define new function:

Define $h(z) = f(z) - g(z)$. Since f and g are analytic in D, h is also analytic in D.

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Set S be a subset of D, where there exists a (z_n) in S, $(z_n) \to z_0$:

Since $f(z_n) = g(z_n)$ in S, we have $h(z_n) = 0$.

Step 2

Continuity of analytic function:

Since h is analytic, h is continuous. We have

$$\lim_{n\to\infty} h(z_n) = h(\lim_{n\to\infty} z_n) = h(z_0) = 0$$

Thus, z_0 is a zero of h(z).

Step 3

Isolated Zero Property

A zero z_0 of an analytic function h that is not identically zero is isolated.

Since $h(z_n) = 0$ for all $n \in \mathbb{N}$ in any neighborhood of z_0, z_0 is not an isolated zero.

Therefore, h is identically zero in D, which shows $f(z) = g(z)$ everywhere in D.

Importance of z_0 Being Inside D

Why z_0 on the boundary causes issues:

- The neighborhood of z_0 may partially lie outside D.
- We cannot ensure that h is analytic in any neighborhood of z_0.
- We cannot ensure $h(z_n) = 0$ for all $n \in \mathbb{N}$ in any neighborhood of z_0.

Analytic Continuation

Definition

If $f(z)$ is analytic in a domain D and $F(z)$ is analytic in a domain D' with $D' \supset D$ and $F(z) = f(z)$ in D, then we say that F is an analytic continuation of f.

Remark

- Targets analytic functions
- Aims to extend the domain beyond its original scope
- Serves as a mathematical technique

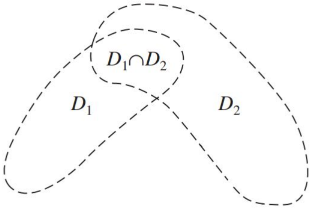
Analytic Continuation

Visualization

Given two domains D_1 and D_2, $D_1 \cap D_2 \neq \emptyset$

For f_1 is analytic in D_1, there *may* exist a analytic function f_2 in D_2, such that $f_2(z) = f_1(z)$ for each z in $D_1 \cap D_2$

f_2 is an **analytic continuation** of f_1 into D_2



Visualization of analytic continuation

Analytic Continuation

Example

The geometric series at zero

$$f(z) = 1 + z + z^2 + \dots$$

converges and is analytic in the open disk $D = \{z \mid |z| < 1\}$.

$$zf(z) = z + z^2 + z^3 + \dots$$

$$zf(z) - f(z) = 1 \implies f(z) = \frac{1}{1-z}, \quad z \in D$$

$F(z) = \frac{1}{1-z}$ is analytic in the larger domain $D' = \mathbb{C} \setminus \{1\}$.

Thus, F is an analytic continuation of f.

Analytic Continuation

Uniqueness of Analytic Continuation

If $F_1 \colon D' \to \mathbb{C}$ and $F_2 \colon D' \to \mathbb{C}$ are two analytic continuations of $f \colon D \to \mathbb{C}$, $F_1(z) = F_2(z)$ for all $z \in D'$.

Proof

An immediate consequence of the **Identity Theorem**

F_1 and F_2 are analytic in D.

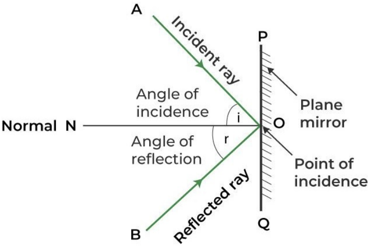
For every $z \in D \subseteq D'$, $F_1(z) = F_2(z) = f(z)$.

By the Identity Theorem, $F_1(z) = F_2(z)$ for all $z \in D'$.

Reflection Principle

Theorem

Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose **lower half plane is the reflection of the upper half plane** with respect to that axis. Then $\overline{f(z)} = f(\overline{z})$ for each z in D if and only if $f(x)$ is **real** for each point x on the segment.



Reflection of light

Reflection Principle

Proof ($\rightarrow$)

Setting $F(z) = \overline{f(\overline{z})}$, we want to prove $F(z)$ is analytic and $f(z) = F(z)$.

Let $f(z) = u(x, y) + iv(x, y), F(z) = U(x, y) + iV(x, y)$

$\overline{f(\overline{z})} = u(x, -y) - iv(x, -y)$

$U(x, y) = u(x, t), V(x, y) = -v(x, t)$ where $t = -y$

Since f is analytic, it satisfies the Cauchy–Riemann equations.

$$u_x = v_t, u_t = -v_x$$

$$U_x = u_x, V_y = -v_t \frac{dt}{dy} = v_t$$

Thus, $U_x = V_y$

Similarly, $U_y = u_t \frac{dt}{dy} = -u_t, V_x = -v_x$

Thus, $U_y = -V_x$

Since $F(z)$ satisfies the Cauchy–Riemann equations and those derivatives are continuous, $F(z)$ is analytic in D.

Since $f(x)$ is real on the segment of the real axis in D, $v(x, 0) = 0$.

$$F(x) = U(x, 0) + iV(x, 0) = u(x, 0) - iv(x, 0) = u(x, 0)$$

Therefore, $F(z) = f(z)$ at each point on the segment.

By the theorem: an analytic function defined on D is uniquely determined by its values along any line segment lying in D

Thus, $F(z) = f(z)$ in D.

Reflection Principle

Proof ($\leftarrow$)
$u(x,-y) - iv(x,-y) = u(x,y) + iv(x,y)$

If $(x,0)$ is a point on the segment of the real axis in D,
$u(x,0) - iv(x,0) = u(x,0) + iv(x,0)$

Therefore, $v(x,0) = 0$

$f(x)$ is real on the segment of the real axis lying in D.

Proof is done.

Reflection Principle

Example

Show $f(z) = e^x \cos y + ie^x \sin y$ follows the **reflection principle** that $\overline{f(z)} = f(\overline{z})$ for each z.

Reflection Principle

Example

Show $f(z) = e^x \cos y + ie^x \sin y$ follows the **reflection principle** that $\overline{f(z)} = f(\overline{z})$ for each z.

$$u(x, y) = e^x \cos y \quad \text{and} \quad v(x, y) = e^x \sin y.$$

Since $u_x = v_y$, $u_y = -v_x$, and these derivatives are continuous, $f(z)$ is analytic.
When $y = 0$, we have

$$f(x) = e^x \cos 0 + ie^x \sin 0 = e^x,$$

which is real for all $x \in \mathbb{R}$.
Since the domain of f is symmetric about the real axis and f is real on the real axis, we have

$$\overline{f(z)} = f(\overline{z})$$

for all $z \in \mathbb{C}$.

Reflection Principle

Verification

If $f(z) = e^x \cos y + ie^x \sin y$, then

$$\overline{f(z)} = e^x \cos y - ie^x \sin y,$$

and

$$f(\overline{z}) = e^x \cos(-y) + ie^x \sin(-y) = e^x \cos y - ie^x \sin y,$$

Thus, $\overline{f(z)} = f(\overline{z})$ for all $z \in \mathbb{C}$.

Reflection Principle

$f(x)$ **is real** $\rightarrow f(x)$ **is pure imaginary**

$\overline{f(z)} = f(\overline{z}) \rightarrow \overline{f(z)} = -f(\overline{z})$

Proof
Setting $F(z) = \overline{-f(\overline{z})}$, we want to prove $F(z)$ is analytic and $f(z) = F(z)$.

...

Reflection Principle

Reflection about the unit circle

The reflection of $z = x + iy$ in the unit circle is

$$\frac{1}{\overline{z}} = \frac{z}{|z|^2} = \frac{x+iy}{x^2+y^2}.$$

Reflection Principle

Reflection about the unit circle

The reflection of $z = x + iy$ in the unit circle is

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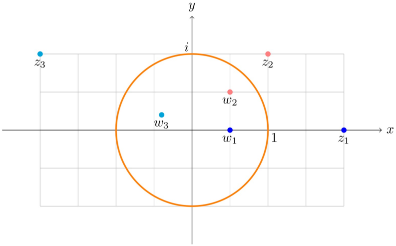
Geometric interpretation
The **product of the distances** from the **origin** to z and its reflection $\frac{1}{\bar{z}}$ is always equal to 1.

- If z lies outside the unit circle, its reflection will be inside the unit circle.
- If z lies inside the unit circle, its reflection will be outside the unit circle.

Reflection Principle

Example



Reflection about the unit circle

Remark

- For z on the unit circle:
  Its distance from the origin is 1, $z = \frac{1}{\bar{z}}$. That is, z is its own reflection in the unit circle.
- The center of the circle 0 reflects to the point at ∞.