ORBITS NEAR A BLACK HOLE

by

Li Zhaoyang, Jesse Campbell

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1. Introduction

A black hole is an immensely dense celestial body with the gravitational force so strong that it can consume everything in its vicinity, including light. Formed from the collapse of massive stars that have exhausted their nuclear fuel, black holes are the final step in stellar evolution.

The study of black holes has developed immensely throughout the 19th and 20th centuries. Einstein's theory of general relativity offered a new perspective on gravity as the curvature of spacetime caused by mass and energy, and went beyond the framework of Newtonian mechanics. Later, the first specific solution of Einstein's field equations, the Schwarzschild metric, was discovered. These advancements continue to persist up to the recent detection of gravitational waves, and the capture of the first image of a black hole in 2022. One open problem facing scientists today is uncovering the internal structure of black holes, where we currently face a lack of precise understanding and observation.

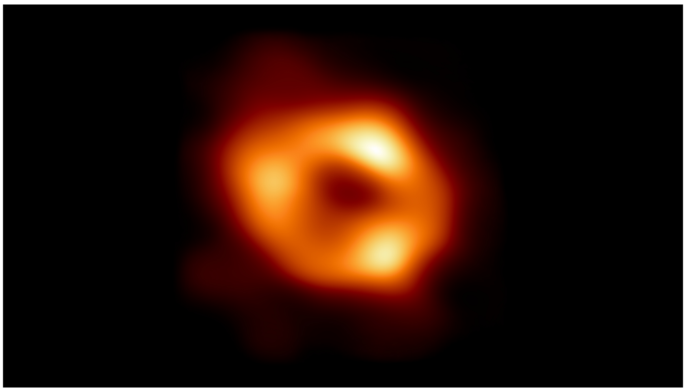

FIGURE 1. Picture of Sagittarius A*, the center of the Milky Way, as captured by the James Webb telescope in 2022. Sagittarius A* is a supermassive black hole with mass millions of times greater than the Sun.

One of the features of a black hole is its event horizon. When an object enters this boundary, it is trapped by the powerful central gravitational field and can no longer escape. In this paper, our attention is on orbits beyond the event horizon, near the black hole. Utilizing the Schwarzschild metric, we derive the effective potential equation for a non-rotating black hole within a stationary, spherically symmetric system. Furthermore, we analyze the circular orbits and dynamics of the system using the energy method, leading to a discussion about the precession of orbits predicted by the Schwarzschild solution.

Exploring the orbits near black holes deepens our comprehension of the motion under extreme gravitational conditions in the universe. It explains the spatial context near black holes and provides insights into the relationship between forces, energy, and motion in four-dimensional spacetime. All in all, black holes remain at the forefront of astrophysics.

This paper relies heavily on the lecture notes 19 and 20 from Professor Yacine Ali-Haimoud of New York University [**AH19b**] [**AH19a**] and chapter 11.8 from David Nolte's book "Introduction to Modern Dynamics". [**Nol15**]

2. Black Hole Background Knowledge

Stellar-mass black holes with three times to a dozen times the Sun's mass are scattered throughout the Milky Way galaxy. A stellar-mass black hole forms when a massive star exhausts its nuclear fuel and undergoes gravitational collapse. As the core of the collapsed massive star becomes incredibly dense, it creates a singularity - a point of infinite density and gravity - and generates an immensely strong gravitational field. Observed from an external perspective, the event horizon marks the boundary of a black hole. Once an object crosses the event horizon, it gets drawn into the black hole's gravitational pull without any possibility of escape, including light. It also represents the region where the gravitational pull becomes so intense that the escape velocity exceeds the speed of light.

2.1. The Schwarzschild Radius. — The Schwarzschild radius (R_s) defines the critical radius at which an object must be compressed to form a black hole. It also represents the radius of the event horizon of a non-rotating black hole, where the escape velocity matches the speed of light.

To find the escape velocity, we apply the conservation of mechanical energy at initial and final states, which is given by,

$$U_{\mathrm{i}} + K_{\mathrm{i}} = U_{\mathrm{f}} + K_{\mathrm{f}}$$

At the initial state, an object has a speed of v at a distance r from the center of a massive body. Finally, at an infinite distance from the massive body (where the object escapes), both the potential energy and the kinetic energy are equal to zero. Then, the equation becomes,

$$-\frac{GMm}{r} + \frac{1}{2}mv^2 = 0$$

$$r = \left.\frac{2GM}{v^2}\right|_{v=c}$$

Therefore, we obtain the Schwarzschild radius that,

$$R_s = \frac{2GM}{c^2}$$

where c is the speed of light, G is the gravitational constant, M is the mass of the massive body.

For a human, the Schwarzschild radius is of the order of 10^{-23} cm, much smaller than the nucleus of an atom; for a typical star such as the Sun, it is about 3 km.

2.2. Einstein's Field Equations. — Einstein's field equations describe the fundamental interaction between gravity and the curvature of spacetime by matter and energy, given by a set of highly nonlinear partial differential equations,

$$R_{\mu v} - \frac{1}{2}Rg_{\mu v} = 8\pi T_{\mu v}$$

where $R_{\mu v}$ is the Ricci curvature tensor, R is the Ricci curvature scalar, $g_{\mu v}$ is the metric tensor, $T_{\mu v}$ is the stress-energy tensor, and $\mu, v = 0, 1, 2, 3$ represent one time component and three spatial components. In the following discussion of the Schwarzschild metric, we avoid delving into the computation of physical quantities, considering that this project focuses on differential equations and dynamical systems.

3. Derivation of the Schwarzschild Metric

In order to understand metrics intuitively, we draw an analogy to using a ruler to measure the distance between two points in space. In four-dimensional spacetime, the metric serves as a tool to measure the intervals between two events, expressing both spatial and temporal relationships.

The Schwarzschild metric describes the spacetime structure around a non-rotating black hole, which is a special solution derived from Einstein's field equations. It rests on three assumptions. First, that the system is spherically symmetric, which means redundant degrees of freedom exist. Second, the system is under vacuum conditions so that no matter is distributed in space except for a mass point at the origin. Third, the system does not change over time.

3.1. Interval in a Flat Spacetime. — In flat spacetime, there is no gravity or gravitational fields. The spacetime is assumed to be flat and linear.

Suppose there are two points, P and Q, in a three-dimensional Cartesian coordinate system, with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. The distance formula is given by,

$$|PQ|^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Similarly, the spacetime interval (ds^2) in four-dimensional flat spacetime, called the Minkowski (Lorentz) metric, is given by,

$$ds^2 = -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2$$

where c is the speed of light, dt is the time interval, and dx, dy, dz are spatial coordinate intervals. The negative sign ensures Lorentz invariance that the speed of light remains constant in different inertial reference frames. In special relativity, the spacetime interval between two events is an invariant with respect to reference frames.

Furthermore, the spacetime interval in spherical coordinate system is,

$$ds^2 = -(cdt)^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{1}$$

with the relationship of $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.

3.2. Interval in a Curved Spacetime. — According to Einstein's general theory of relativity, the extremely high mass and energy of a black hole will cause the curvature of spacetime, affecting the trajectories of objects.

Recall that the Schwarzschild metric is a special solution derived from Einstein's field equation. These three assumptions greatly simplify the calculation: spherical symmetry and stability determine that terms dependent on time and angles vanish, and no terms associated with matter energy in vacuum. Based on that, we finally obtain $g_{\mu v}$, which is a matrix given by,

$$g_{\mu v} = \begin{bmatrix} -(1 - \frac{2GM}{c^2 r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2GM}{c^2 r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \tag{2}$$

We rewrite the Schwarzschild metric in spherical polar coordinates, namely,

$$ds^2 = -\left(1 - \frac{2M}{c^2 r}\right)c^2 dt^2 + \frac{dr^2}{1 - \frac{2M}{c^2 r}} + r^2 d\Omega^2 \text{where } d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$$

where ds^2 is the spacetime interval, c is the speed of light, G is the gravitational constant, M is the mass of the spherical symmetric object, dt is the time interval, dr is the radial coordinate interval, and $d\theta$ and $d\phi$ are the spherical coordinate intervals.

4. Effective Potential Equation

Notice that in (2), the coordinates t and ϕ disappear, which implies symmetry about time and angles, giving two conserved quantities. Since G and c are both constants, we set them equal to 1. These two conserved quantities are the energy of a static unit mass,

$$E = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$$

and the angular momentum of a static unit mass.

$$L = r^2 \sin^2\theta \frac{d\phi}{d\tau}$$

Here, we introduce the concept of proper time, denoted as τ, to aid in explaining the motion of objects. In the theory of relativity, time is relative, implying that different observers, depending on their states of motion or gravitational fields, will measure distinct time intervals. The proper time of an object expresses the time measured in its own reference frame, equal to the time measured by an observer relatively at rest with respect to that object.

Because angular momentum is conserved, the motion occurs in a plane. In other words, in spherical polar coordinates, $\theta = \frac{\pi}{2}$ remains constant.

The dot product of the four-velocity vector according to the Minkowski (Lorentz) metric equals -1.

$$-\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 = -1$$

Notice that $\sin^2\theta = 1$. Substitute E and L.

$$-\left(1 - \frac{2M}{r}\right)^{-1} E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \left(\frac{L^2}{r^2}\right) = -1$$

Regarding $\frac{dr}{d\tau}$ as velocity,

$$\frac{E^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - 1\right]$$

we obtain the following expressions for total energy $\mathcal{E}$ and effective potential V_{eff}.

$$V_{\text{eff}(r)} = \frac{1}{2}\left(1 - \frac{2M}{r}\right)\left(1 + \frac{L^2}{r^2}\right) - \frac{1}{2}$$

$$\mathcal{E} = \frac{E^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r)$$

Recall that the Schwarzschild radius is given by,

$$R_s = \frac{2GM}{c^2}$$

and we account for m to get another form of the effective potential equation.

$$V_{\text{eff}}(r) \equiv -\frac{GmM}{r} + \frac{L^2}{2mr^2} - \frac{R_s L^2}{2mr^3} \tag{3}$$

5. Orbits in the Schwarzschild Metric

We have derived the famous Schwarzschild metric and seen the equation for effective potential of geodesics in the equatorial plane.

In this section, we will use this equation to find the location and characteristics of the circular orbits around a central mass and compare it to Newtonian motion. We will also show that bound orbits do not close, but rather have a procession.

5.1. Circular Orbits. — First, let us consider a simpler case of the equations for motion in a Newtonian system. Here, the Newtonian effective potential is given by,

$$V_{\text{eff}}(r) \equiv -\frac{GmM}{r} + \frac{L^2}{2mr^2}$$

which you can see is the same as the general relativity case in equation (3) without the term involving r^{-3}.

One key observation that we will justify is that **circular orbits are such that** $V'_{\text{eff}}(r) = 0$. This is because the effective potential is the sum of the gravitational potential energy and the centrifugal potential energy on an object. So, when $V'_{\text{eff}}(r) = 0$, the two forces are balanced.

Another way of seeing this is through the law of conservation of energy. The kinetic energy in polar coordinates is given by,

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

and the gravitational potential energy is given by,

$$PE_g = -\frac{GmM}{r}$$

Now, notice that the equation for V_{eff} can be found hiding in the equation for total energy.

$$E = KE + PE_g = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{GmM}{r} = \frac{1}{2}m\dot{r}^2 + \frac{(mr^2\dot{\theta})^2}{2mr^2} - \frac{GmM}{r}$$

However, we have that the angular momentum is given by $L = mr^2\dot{\theta}$, therefore,

$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}$$

We know that circular orbits correspond to $\dot{r} = 0$ and that energy is a conserved quantity, so the following relation is satisfied,

$$E = V_{\text{eff}}(r) \to 0 = V'_{\text{eff}}(r)$$

Using this fact, we can find the circular orbits of the Newtonian system. Differentiating, we find,

$$V'_{\text{eff}}(r) = \frac{GmM}{r^2} - \frac{L^2}{mr^3} = 0 \iff r = \frac{L^2}{Gm^2M}$$

We can now apply this same concept to the more complex expression for the effective potential in the general relativity case.

$$V'_{\text{eff}}(r) = \frac{GmM}{r^2} - \frac{L^2}{mr^3} + \frac{3R_sL^2}{2mr^4} = 0 \iff GmMr^2 - \frac{L^2}{m}r + \frac{3R_2L^2}{2m} = 0$$

Solving this quadratic equation, we find,

$$r = \frac{1}{2GmM} \cdot \frac{L^2}{m}\left(1 \pm \sqrt{1 - \frac{6R_sGm^2M}{L^2}}\right)$$

Dividing by c^2, we can substitute the equation for the Schwarzschild radius in the denominator.

$$r_{+,-} = \frac{1}{R_sm} \cdot \frac{L^2}{mc^2}\left(1 \pm \sqrt{1 - \frac{6R_sGm^2M}{L^2}}\right) \tag{4}$$

By the property of the square root, we find that orbits only exist for

$$L > m\sqrt{6R_sGM}$$

5.2. Stability of Orbits. — Now that we have the radii of the circular orbits, we can determine their stability. A circular orbit can either be unstable or stable. We can analyze the stability by looking at the effective force F_{eff}, which in the case where our effective potential is a function of one variable, is given by,

$$F_{\text{eff}}(r) = -V'_{\text{eff}}(r) \rightarrow F'_{\text{eff}}(r) = -V''_{\text{eff}}(r)$$

If the change in the effective force is negative, than any small change in the radius will feel a force cback to the orbit, therefore the orbit is stable. The opposite is true of a positive change in effective force.

$$F'_{\text{eff}}(r) = \frac{2GmM}{r^3} - \frac{3L^2}{mr^4} + \frac{6R_sL^2}{mr^5}$$

Plugging in, we find that $F(r_+) > 0$ and $F(r_-) < 0$. From this we can conclude that r_+ is the radius of the stable orbit of the system.

5.3. Innermost Stable Circular Orbit (ISCO). — The innermost stable circular orbit is self-explanatory. Formally, it is "the smallest stable circular orbit in which a test particle can stably orbit a massive object in general relativity." [**Wik23**]

Since we have determined that r_+ is the radius of the stable orbit of the black hole system, we will find the minimum value of r_+, which occurs when,

$$L^2 = 6GMR_sm^2 = 3c^2R_s^2m^2$$

Plugging this into the equation for r_+, we find,

$$r_{\text{ISCO}} = 3R_s$$

The physical interpretation of this is that no particle can have a stable circular orbit with a radius smaller than three times the Schwarzschild radius. Anything closer will always spiral into the black hole.

5.4. Energy Method. — The energy method is a way of visualizing the phrase portrait of a dynamical system which has a conserved quantity. In this case, total energy is conserved, so we can examine the graph of effective potential $V_{\text{eff}}(r)$ for given constant total energy curves.

Here, notice that the local maximum of the effective potential function corresponds to a saddle point in the phase portrait and a local minimum in the effective potential corresponds to a center.

It can be seen that the saddle point demarcates the orbits around the black hole, such as the orbit labeled "circular orbit", as well as initial conditions that accelerate into the black hole, which are those that approach the Schwarzschild radius.

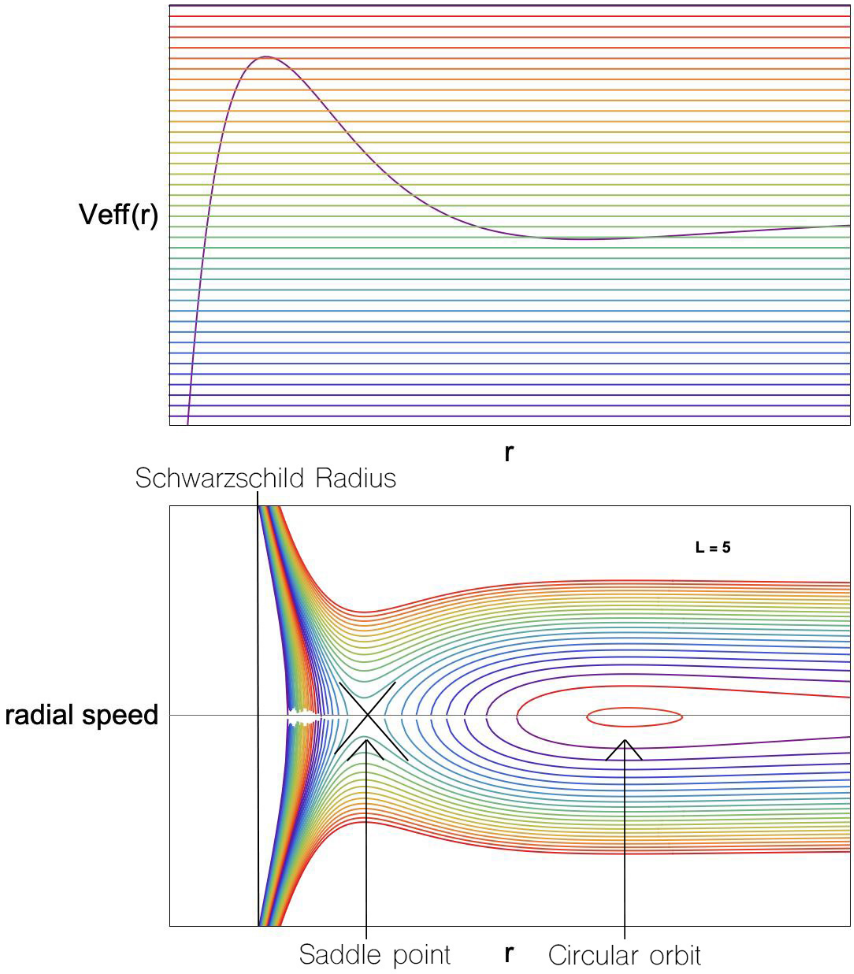

FIGURE 2. Effective potential with $L = 5$ and $M = m = c = G = 1$ and phase portrait on a log scale. Note that the bottom plot has a larger range on the horizontal axis, so the plots do not align exactly. This purposefully done to show the Schwarzschild radius. [**Efs23**]

It should be noted that particles on orbits that approach the Schwarzschild radius are consumed by the black hole.

The line that begins at the top left of the saddle point and ends at the bottom right is the stable direction, which corresponds to orbits moving to the right in the top half of the plane and orbits moving to the left (towards the Schwarzschild radius) on the bottom half of the plane. Likewise, the other direction in the saddle is the unstable direction.

5.5. Precession of Bound Orbits. — Bound orbits are those with negative total energy. These orbits are either circular or elliptical in shape. A particle in a bound orbit cannot escape unless some energy is supplied which makes the total energy positive.

For a bound orbit, the radius is bound within a range $r_{\min} \leq r \leq r_{\max}$, which are called the *pericenter* and *apocenter*, respectively.

What we will show here is that if we choose the angle ϕ such that $r(\phi) = r_{\min}$ and $\frac{dr}{d\tau}(\phi = 0) = 0$, the next point at which $r(\tau) = r_{\min}$ is not $\phi = 2\pi$. This means that bound orbits do not close, but have a precession.

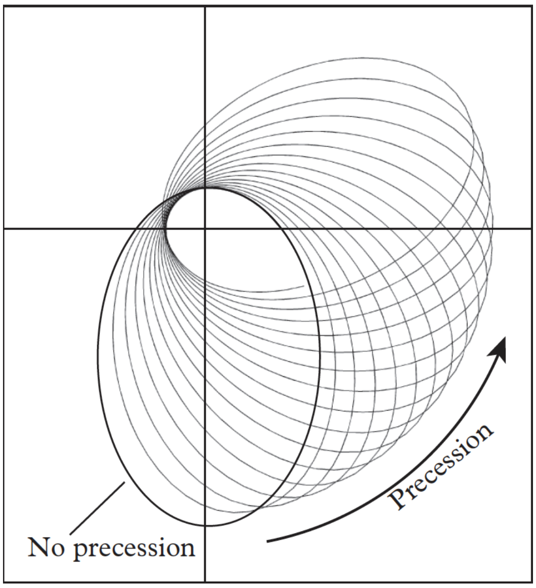

FIGURE 3. Diagram depicting the precession of orbits. The bolded orbit corresponds to the non-general relativity case, such as in the Kepler problem.

For the sake of simplicity, we will take the constants $m = G = c = 1$, thereby leaving us with the following equations,

$$\frac{d\varphi}{d\tau} = \frac{L}{r^2}\ , \mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \text{ and } V_{\text{eff}} = -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

Now, let us relate the radial velocity to the angular velocity.

$$\frac{1}{d\tau} = \frac{L}{r^2}\frac{1}{d\varphi} \to \frac{dr}{d\tau} = \frac{L}{r^2}\frac{dr}{d\varphi}$$

Plugging into the energy equation,

$$\mathcal{E} = \frac{1}{2}\frac{L^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + V_{\text{eff}}(r)$$

Our goal now is to transform this expression into a differential equation such that we can solve for $r(\varphi)$ and analytically approximate the precession of orbits. To do this, we will begin by substituting the coordinate r with $u \equiv \frac{L^2}{Mr}$. This given the following expression,

$$r = \frac{L^2}{Mu} \to \frac{dr}{du} = -\frac{L^2}{Mu^2} = -\frac{L^2}{M}\frac{M^2r^2}{L^4} = -\frac{Mr^2}{L^2}$$

Plugging this into the energy equation,

$$\begin{aligned}\mathcal{E} &= \frac{1}{2}\frac{L^2}{r^4}\left(\frac{dr}{du}\frac{du}{d\varphi}\right)^2 + V_{\text{eff}}\left(\frac{L^2}{Mr}\right) = \frac{1}{2}\frac{L^2}{r^4}\left(\frac{M^2r^4}{L^4}\right)\left(\frac{du}{d\varphi}\right)^2 + V_{\text{eff}}(u) \\ &= \frac{1}{2}\left(\frac{M}{L}\right)^2\left(\frac{du}{d\varphi}\right)^2 + V_{\text{eff}}(u)\end{aligned} \tag{5}$$

Now, we will simplify the expression $V_{\text{eff}}(u)$, namely,

$$\begin{aligned}V_{\text{eff}}(u) &= -\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3} = -\frac{M^2}{L^2}u + \frac{M^2}{2L^2}u^2 - \left(\frac{M}{L}\right)^4 u^3 \\ &= \left(\frac{M}{L}\right)^2\left(-u + \frac{1}{2}u^2 - \left(\frac{M}{L}\right)^2 u^3\right)\end{aligned} \tag{6}$$

Combining (5) and (7), we find,

$$\mathcal{E} = \left(\frac{M}{L}\right)^2\left(\frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 - u + \frac{1}{2}u^2 - \left(\frac{M}{L}\right)^2 u^3\right) \tag{7}$$

We will now use the relationship between specific orbital energy $\mathcal{E}$ and *eccentricity* e to rewrite the left hand side.

$$e = \sqrt{1 + \frac{2\mathcal{E}L^2}{M^2}} \implies -\frac{1}{2}(1 - e^2) = \left(\frac{L^2}{M^2}\right)\mathcal{E}$$

The eccentricity describes how closely an orbit follows a perfect circle. An eccentricity $e = 0$ is a perfect circle, whereas for $0 < e < 1$, the orbits become increasingly longer ellipses. $e = 1$ corresponds to a parabolic orbit, and $e > 1$ corresponds to a hyperbolic orbit.

In our case, recall that bound orbits have a negative specific orbital energy, therefore by this relation we would expect eccentricity to be $0 \leq e < 1$.

Rewriting (7),

$$-\frac{1}{2}(1-e^2) = \left(\frac{L}{M}\right)^2 \mathcal{E} = \frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 - u + \frac{1}{2}u^2 - \left(\frac{M}{L}\right)^2 u^3 \tag{8}$$

Differentiating u with respect to φ, we have,

$$\frac{d^2u}{d\varphi^2} + u - 1 = \left(\frac{M}{L}\right)^2 3u^2 \tag{9}$$

Alas, we have a nonlinear, second-order, non-homogeneous ordinary differential equation in u.

One key observation is that, given our Schwarzschild radius $R_s = 2M$, a particle that is orbiting near the event horizon will have a radius close to R_s, that is $r \approx 2M$.

Furthermore, the specific angular momentum of a particle in a circular orbit is given by,

$$L = \frac{2\pi rv}{1 - \frac{2M}{r}}$$

Therefore it is clear that when $r \approx 2M$, the denominator becomes very small and L becomes arbitrarily large. For this reason, we will assume that $0 < (M/L)^2 \ll 1$.

If we call this small quantity ϵ, we can use a perturbation method to approximate the solution $u(\varphi)$, which we will now do.

Rewriting (9),

$$\frac{d^2u}{d\varphi^2} + u - 1 = 3\epsilon u^2 \tag{10}$$

Now, let us rewrite u as a function of φ and ϵ so that we can Taylor expand it about $\epsilon = 0$ up to linear order, namely,

$$u(\varphi, \epsilon) \approx u(\varphi, 0) + \epsilon \frac{\partial u(\varphi, 0)}{\partial \epsilon} = u_0(\varphi) + \epsilon u_1(\varphi) \tag{11}$$

Now, we have transformed our difficult problem into two simpler subproblems. First, we will find a solution for $u_0(\varphi)$ by solving the second-order non-homogeneous equation,

$$\frac{d^2u_0}{d\varphi^2} + u_0 - 1 = 0$$

Which yields the following solution,

$$u_0(\varphi) = c_1 \cos\varphi + c_2 \sin\varphi + 1$$

Let $u_0 = 1 + A\cos\varphi$, then plugging into (8),

$$-\frac{1}{2}(1-e^2) = \frac{1}{2}(A^2\sin^2\varphi) - (1 + A\cos\varphi) + \frac{1}{2}(1 + 2A\cos\varphi + A^2\cos^2\varphi)$$

Simplifying using the fact that $\sin^2\varphi = 1 - \cos^2\varphi$,

$$-\frac{1}{2}(1-e^2) = -\frac{1}{2}(1-A^2) \implies A^2 = e^2$$

In our case, we will take $A = e$ since it corresponds to positive angular momentum. Finally, our equation for $u_0(\varphi)$ is given by,

$$u_0(\varphi) = 1 + e\cos\varphi \tag{12}$$

Next, we will find a solution to $u_1(\varphi)$. Again, we can simply use the definition, which will leave us with a non-homogeneous, linear, second-order equation in u_1.

$$\begin{aligned} u_1(\varphi) = \frac{\partial u(\varphi,0)}{\partial\epsilon} &= \frac{\partial^2}{\partial\varphi^2}\frac{\partial u(\varphi,0)}{\partial\epsilon} + \frac{\partial u(\varphi,0)}{\partial\epsilon} = 3u(\varphi,0)^2 \\ &= \frac{\partial^2 u_1}{\partial\varphi^2} + u_1 = 3(1 + e\cos\varphi)^2 \end{aligned}$$

The solution to this equation can be found exactly and is given by,

$$u_1(\varphi) = (3 + e^2) + (C - 3 - e^2)\cos\varphi + e^2\sin^2\varphi + 3e\varphi\sin\varphi$$

where C is a constant of integration that can be solved for using the condition that $\frac{du_1}{d\varphi}(\varphi = 0) = 0$

One note about this solution is that is cannot be correct for large φ since the term $3e\varphi\sin\varphi$ will increase to an arbitrarily large number. For this reason, we say that solution is only valid for $0 \leq \varphi \leq 2\pi$. After this period, we will set φ to be our new angle and restart the calculations.

Finally, we can solve for the precession of the orbits. In specific, we know that each pericenter passage satisfies the same conditions as the pericenter passage when $\varphi = 0$. That is, the radial velocity is 0. We can denote the angle of the next pericenter passage as $2\pi + \Delta\varphi$. Differentiating the approximate solution for u,

$$u'(2\pi + \Delta\varphi) = u_0'(2\pi + \Delta\varphi) + \left(\frac{M}{L}\right)^2 u_1'(2\pi + \Delta\varphi) = 0 \tag{13}$$

We then write out the Taylor expansion of $u'(\varphi)$ about 2π,

$$u'(\theta) \approx u_0'(2\pi) + \left(\frac{M}{L}\right)^2 u_1'(2\pi) + (\theta - 2\pi)\left(u_0''(2\pi) + \left(\frac{M}{L}\right)^2 u_1''(2\pi)\right)$$

Letting $\theta = 2\pi + \Delta\varphi$,

$$\Delta\varphi \approx -\frac{u_0'(2\pi) + (M/L)^2 u_1''(2\pi)}{u_0''(2\pi) + (M/L)^2 u_1''(2\pi)}$$

However it is easily verifiable that $u_0'(2\pi) = u_1''(2\pi) = 0$, therefore,

$$\Delta\varphi \approx -\left(\frac{M}{L}\right)^2 \frac{u_1''(2\pi)}{u_0''(2\pi)} = 6\pi\left(\frac{M}{L}\right)^2$$

6. Conclusion

In this paper, we have examined the physical intuition of black holes; the Schwarzschild solutions to Einstein's field equations for a spherically symmetric, non-rotating black hole; and the properties of orbits around Schwarzschild black holes.

It is interesting to note that the Schwarzschild solution applied to a broader class of orbits beyond those around black holes. In fact, the Schwarzschild solution can be applied in any case where the mass of one body is much greater than the other, so the larger mass can be viewed as stationary.

For example, before the advent of general relativity, the Newtonian physical model explained the ellipitcal orbit of planets in the solar system, and was mostly consistent. However, this model failed to explain the precession of Mercury's orbit around the Sun.

This phenomenon was so puzzling to scientists in the 19th century that astronomers such as *Urbain Le Verrier* hypothesized the existence of an unseen planet or massive ring of asteroids to align with Newtonian physics. [**Ret20**]

After 1915, by utilizing the Schwarzschild solution, astronomers were able to calculate the precession of Mercury to be about 0.001 degrees per century, which aligned with physical evidence. This result was one of the most important confirmations of general relativity as a correct physical model.

One intuition for the disparity between the precession of Mercury and Earth is because of Mercury's proximity to the Sun. We have seen that the general relativity term in the effective potential equation involves a term r^{-3}. Because

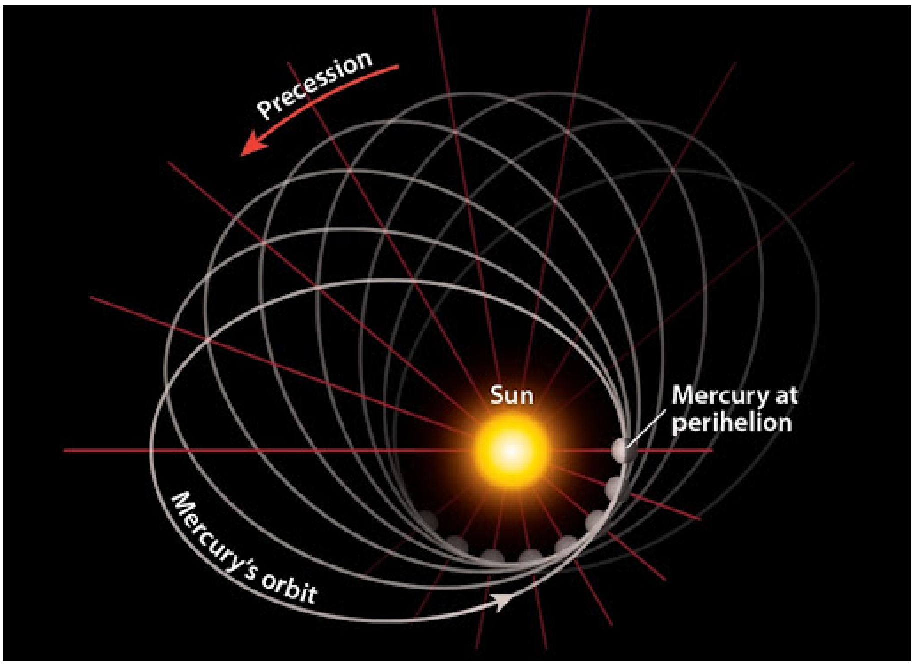

Figure 4. Diagram of Mercury's precession around the Sun.

of Earth's larger distance, the general relativity term is very small in-regard to Earth, whereas it is much larger for Mercury.

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Appendix A: Mathematica Code

```
Manipulate[
 Module[{Veff, topPlot, bottomPlot, hValues, colors},
  Veff[r_] := -1/r + L^2/(2 r^2) - (L^2)/(r^3);
  hValues = Range[-0.2, 0.2, 0.01];
  colors = ColorData["Rainbow"] /@ Rescale[Range[Length[hValues]]];
  topPlot =
   LogLinearPlot[
    Evaluate[Join[{List[Veff[x]]}, hValues]], {x, 1, 100},
    PlotRange -> {{2, 100}, {-0.2, 0.2}},
    PlotStyle ->
     Table[{Directive[AbsoluteThickness[1], color]}, {color, colors}],
     Frame -> True, Axes -> False, FrameTicks -> None,
    FrameStyle -> Directive[Black, 18],
    FrameLabel -> {"r", "Veff(r)"}, RotateLabel -> False];
  bottomPlot =
   LogLinearPlot[
    Evaluate[
     Flatten@
      Table[{Sqrt[2 h - Veff[x]], -Sqrt[2 h - Veff[x]]}, {h,
        hValues}]], {x, 1, 100}, PlotRange -> {{2, 100}, {-1, 1}},
    PlotStyle ->
     Table[{Directive[AbsoluteThickness[1], color]}, {color, colors}],
     Frame -> True, Axes -> True, FrameTicks -> None,
    FrameStyle -> Directive[Black, 18],
    FrameLabel -> {"r", "radial speed"}, RotateLabel -> False,
    Epilog -> {Text[Style["L = " <> ToString[L], Black, Bold],
       Scaled[{0.8, 0.9}]]}];
  GraphicsColumn[{topPlot, bottomPlot}, ImageSize -> Large]], {{L, 5},
   3, 5, 0.5}, ControlPlacement -> Right]
```

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Li Zhaoyang, Jesse Campbell