Precalculus — Algebra I

Version $4 - \epsilon$

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Chapter 1

Introduction to Functions

1.1 Functions and their Representations

1.1.1 Functions as Mappings

Mathematics can be thought of as the study of patterns. In most disciplines, Mathematics is used as a language to express, or codify, relationships between quantities - both algebraically and geometrically - with the ultimate goal of solving real-world problems. The fact that the same algebraic equation which models the growth of bacteria in a petri dish is also used to compute the account balance of a savings account or the potency of radioactive material used in medical treatments speaks to the universal nature of Mathematics. Indeed, Mathematics is more than just about solving a specific problem in a specific situation, it's about abstracting problems and creating universal tools which can be used by a variety of scientists and engineers to solve a variety of problems.

This power of abstraction has a tendency to create a language that is initially intimidating to students. Mathematical definitions are precise and adherence to that precision is often a source of confusion and frustration. It doesn't help matters that more often than not very common words are used in Mathematics with slightly different definitions than is commonly expected. The first 'universal tool' we wish to highlight - the concept of a 'function' - is a perfect example of this phenomenon in that we redefine a word that already has multiple meanings in English.

Definition 1.1.1. Given two sets[a] A and B, a **function** from A to B is a process by which each element of A is matched with (or 'mapped to') one and only one element of B.

[a]Please refer to Section **??** for a review of this terminology.

The grammar here '*from A to B*' is important. Thinking of a function as a process, we can view the elements of the set A as our starting materials, or *inputs* to the process. The function processes these inputs according to some specified rule and the result is a set of *outputs* - elements of the set B. In terms of inputs and outputs, Definition 1.1.1 says that a function is a process in which each *input* is matched to one and only one *output*.

For example, let's take a look at some of the pets in the Stitz household. Taylor's pets include White Paw and Cooper (both cats), Bingo (a lizard) and Kennie (a turtle). Let N be the set of pet names: $N = \{\text{White Paw, Cooper, Bingo, Kennie}\}$, and let T be the set of pet types: $T = \{\text{cat, lizard, turtle}\}$. Let f be the process that takes each pet's name as the input and returns that pet's type as the output. Let g be the reverse of f: that is, g takes each pet type as the input and returns the names of the pets of that type as the output. Note that both f and g are codifying the *same* given information about Taylor's pets, but one of them is a function and the other is not.

To help identify which process f or g is a function and why the other is not, we create **mapping diagrams** for f and g as shown in Figure 1.1.1. In each case, we organize the inputs in a column on the left and the outputs on a column on the right. We draw an arrow connecting each input to its corresponding output(s). Note that the arrows communicate the grammatical bias: the arrow originates at the input and points to the output.

The process f is a function since f matches each of its inputs (each pet name) to just one output (the pet's type). The fact that different inputs (White Paw and Cooper) are matched to the same output (cat) is fine. On the other hand, g matches the input 'cat' to the two different outputs 'White Paw' and 'Cooper', so g is not a function. Functions are favored in mathematical circles because they are processes which produce only one answer (output) for any given query (input). In this scenario, for instance, there is only one answer to the question: 'What type of pet is White Paw?' but there is more than one answer to the question 'Which of Taylor's pets are cats?'

As you might expect, with functions being such an important concept in Mathe-

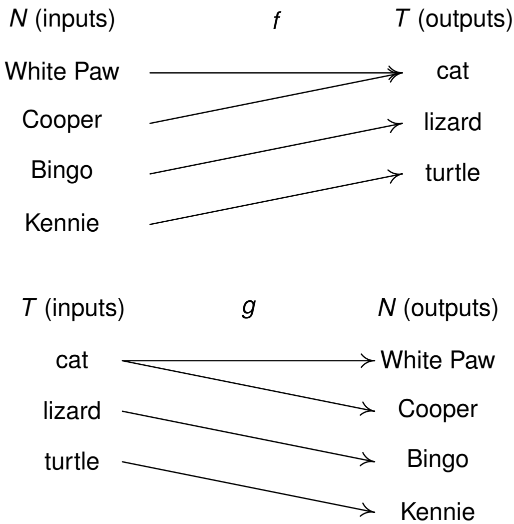

Figure 1.1.1: Mapping diagram for functins f and g

matics, we need to build a vocabulary to assist us when discussing them. To that end, we have the following definitions.[1]

Definition 1.1.2. Suppose f is a function from A to B.

- If $a \in A$, we write $f(a)$ (read 'f of a') to denote the unique element of B to which f matches a.
  That is, if we view 'a' as the input to f, then '$f(a)$' is the output from f.
- The set A is called the **domain**.
  Said differently, the domain of a function is the set of inputs to the function.
- The set $\{f(a) \mid a \in A\}$ is called the **range** of f.
  Said differently, the range of a function is the set of outputs from the function.

Some remarks about Definition 1.1.2 are in order. First, and most importantly, the notation '$f(a)$' in Definition 1.1.2 introduces yet another mathematical use for parentheses. Parentheses are used in some cases as grouping symbols,

[1] Please refer to Section **??** for a review of the terminology used in these definitions.

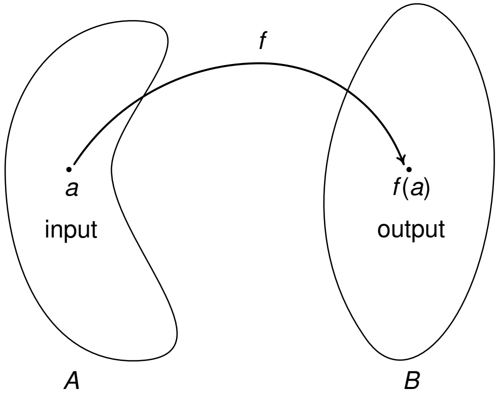

Figure 1.1.2: Function as a mapping process

to represent ordered pairs, and to delineate intervals of real numbers. More often than not, the use of parentheses in expressions like '$f(a)$' is confused with multiplication. As always, paying attention to the context is key. If f is a function and 'a' is in the domain of f, then '$f(a)$' is the output from f when you input a. The diagram in Figure 1.1.2 provides a nice generic picture to keep in mind when thinking of a function as a mapping process with input 'a' and output '$f(a)$'.

In the preceding pet example, the symbol f(Bingo), read 'f of Bingo', is asking what type of pet Bingo is, so f(Bingo) = lizard. The fact that f is a function means f(Bingo) is unambiguous because f matches the name 'Bingo' to only one pet type, namely 'lizard'. In contrast, if we tried to use the notation 'g(cat)' to indicate what pet name g matched to 'cat', we have *two* possibilities, White Paw and Cooper, with no way to determine which one (or both) is indicated.

Continuing to apply Definition 1.1.2 to our pet example, we find that the domain of the function f is N, the set of pet names. Finding the range takes a little more work, mostly because it's easy to be caught off guard by the notation used in the definition of 'range'. The description of the range as '$\{f(a) \mid a \in A\}$' is an example of 'set-builder' notation. In English, '$\{f(a) \mid a \in A\}$' reads as 'the set of $f(a)$ such that a is in A'. In other words, the range consists of all of the outputs from f - all of the $f(a)$ values - as a varies through each of the elements in the domain A. Note that while every element of the set A is, by definition, an element of the domain of f, not every element of the set B is necessarily part of the range of f.[2]

[2]For purposes of completeness, the set B is called the **codomain** of f. For us, the concepts of

In our pet example, we can obtain the range of f by looking at the mapping diagram or by constructing the set $\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\}$ which lists all of the outputs from f as we run through all of the inputs to f. Keep in mind that we list each element of a set only once so the range of f is:[3]

$$\{f(\text{White Paw}), f(\text{Cooper}), f(\text{Bingo}), f(\text{Kennie})\} = \{\text{cat}, \text{lizard}, \text{turtle}\} = T.$$

If we let n denote a generic element of N then $f(n)$ is some element t in T, so we write $t = f(n)$. In this equation, n is called the **independent variable** and t is called the **dependent variable**.[4] Moreover, we say 't is a function of n', or, more specifically, 'the type of pet is a function of the pet name' meaning that every pet name n corresponds to one, and only one, pet type t. Even though f and t are different things,[5] it is very common for the function and its outputs to become more-or-less synonymous, even in what are otherwise precise mathematical definitions.[6] We will endeavor to point out such ambiguities as we move through the text.

While the concept of a function is very general in scope, we will be focusing primarily on functions of real numbers because most disciplines use real numbers to quantify data. Our next example explores a function defined using a table of numerical values.

Example 1.1.1. Suppose Skippy records the outdoor temperature every two hours starting at 6 a.m. and ending at 6 p.m. and summarizes the data in the table below:

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domain and range suffice since our codomain will most always be the set of real numbers, $\mathbb{R}$.

[3]If instead of mapping N into T, we could have mapped N into $U = \{\text{cat}, \text{lizard}, \text{turtle}, \text{dog}\}$ in which case the range of f would not have been the entire codomain U.

[4]These adjectives stem from the fact that the value of t *depends* entirely on our (independent) choice of n.

[5]Specifically, f is a function so it requires and domain, a range and a rule of assignment whereas t is simply the output from f.

[6]In fact, it is not uncommon to see the name of the function as the same as the dependent variable. For example, writing '$y = y(x)$' would be a way to communicate the idea that 'y is a function of x'.

time (hours after 6 a.m.)	outdoor temperature in degrees Fahrenheit
0	64
2	67
4	75
6	80
8	83
10	83
12	82

1. Explain why the recorded outdoor temperature is a function of the corresponding time.
2. Is time a function of the outdoor temperature? Explain.
3. Let f be the function which matches time to the corresponding recorded outdoor temperature.
   (a) Find and interpret the following:
   - $f(2)$
   - $f(4)$
   - $f(2+4)$
   - $f(2)+f(4)$
   - $f(2)+4$

   (b) Solve and interpret $f(t) = 83$.
   (c) State the range of f. What is lowest recorded temperature of the day? The highest?

Solution.

1. The outdoor temperature is a function of time because each time value is associated with only one recorded temperature.
2. Time is not a function of the outdoor temperature because there are instances when different times are associated with a given temperature. For example, the temperature 83 corresponds to both of the times 8 and 10.
3. (a)
   - To find $f(2)$, we look in the table to find the recorded outdoor temperature that corresponds to when the time is 2. We get $f(2) = 67$ which means that 2 hours after 6 a.m. (i.e., at 8 a.m.), the temperature is 67°F.

- Per the table, $f(4) = 75$, so the recorded outdoor temperature at 10 a.m. (4 hours after 6 a.m.) is 75°F.

- From the table, we find $f(2 + 4) = f(6) = 80$, which means that at noon (6 hours after 6 a.m.), the recorded outdoor temperature is 80°F.

- Using results from above we see that $f(2) + f(4) = 67 + 75 = 142$. When adding $f(2) + f(4)$, we are adding the recorded outdoor temperatures at 8 a.m. (2 hours after 6 a.m.) and 10 a.m. (4 hours after 6 AM), respectively, to get 142°F.

- We compute $f(2) + 4 = 67 + 4 = 71$. Here, we are adding 4°F to the outdoor temperature recorded at 8 a.m..

(b) Solving $f(t) = 83$ means finding all of the input (time) values t which produce an output value of 83. From the data, we see that the temperature is 83 when the time is 8 or 10, so the solution to $f(t) = 83$ is $t = 8$ or $t = 10$. This means the outdoor temperature is 83°F at 2 p.m. (8 hours after 6 a.m.) and at 4 p.m. (10 hours after 6 a.m.).

(c) The range of f is the set of all of the outputs from f, or in this case, the outside recorded temperatures. Based on the data, we get $\{64, 67, 75, 80, 82, 83\}$. (Here again, we list elements of a set only once.) The lowest recorded temperature of the day is 64°F and the highest recorded temperature of the day is 83°F. □

A few remarks about Example 1.1.1 are in order. First, note that $f(2+4)$, $f(2)+f(4)$ and $f(2) + 4$ all work out to be numerically different, and more importantly, all represent different things.[7] One of the common mistakes students make is to misinterpret expressions like these, so it's important to pay close attention to the syntax here.

Next, when solving $f(t) = 83$, the variable 't' is being used as a convenient 'dummy' variable or placeholder in the sense that solving $f(t) = 83$ produces the same solutions as solving $f(x) = 83$, $f(w) = 83$, or even $f(?) = 83$. All of these equations are asking for the same thing: what inputs to f produce an output of 83. The choice of the letter 't' here makes sense since the inputs are **t**ime values. Throughout the text, we will endeavor to use meaningful labels when working in

[7] You may be wondering why one would ever compute these quantities. Rest assured that we will use expressions like these in examples throughout the text. For now, it suffices just to know that they are different.

applied situations, but the fact remains that the choice of letters (or symbols) is completely arbitrary.

Finally, given that the range in this example was a finite set of real numbers, we could find the smallest and largest elements of it. Here, they correspond to the coolest and warmest temperatures of the day, respectively, but the meaning would change if the function related different quantities. In many applications involving functions, the end goal is to find the minimum or maximum values of the outputs of those functions (called **optimizing** the function) so for that reason, we have the following definition.

Definition 1.1.3. Suppose f is a function whose range is a set of real numbers containing m and M.

- The value m is called the **minimum**[a] of f if $m \leq f(x)$ for all x in the domain of f.
  That is, the minimum of f is the smallest output from f, if it exists.
- The value M is called the **maximum**[b] of f if $f(x) \leq M$ for all x in the domain of f.
  That is, the maximum of f is the largest output from f, if it exists.
- Taken together, the values m and M (if they exist) are called the **extrema**[c] of f.

[a] also called 'absolute' or 'global' minimum
[b] also called 'absolute' or 'global' maximum
[c] also called the 'absolute' or 'global' extrema or the 'extreme values'

Definition 1.1.3 is an example where the name of the function, f, is being used almost synonymously with its outputs in that when we speak of 'the minimum and maximum of the *function* f' we are really talking about the minimum and maximum values of the *outputs* $f(x)$ as x varies through the domain of f. Thus we say that the maximum of f is 83 and the minimum of f is 64 when referring to the highest and lowest recorded temperatures in the previous example.

1.1.2 Algebraic Representations of Functions

By focusing our attention to functions that involve real numbers, we gain access to all of the structures and tools from prior courses in Algebra. In this subsection, we discuss how to represent functions algebraically using formulas and begin with the following example.

Example 1.1.2.

1. Let f be the function which takes a real number and performs the following sequence of operations:
   - Step 1: add 2
   - Step 2: multiply the result of Step 1 by 3
   - Step 3: subtract 1 from the result of Step 2.

   (a) Find and simplify $f(-5)$.

   (b) Find and simplify a formula for $f(x)$.

2. Let $h(t) = -t^2 + 3t + 4$.

   (a) Find and simplify the following:

      i. $h(-1)$, $h(0)$ and $h(2)$.

      ii. $h(2x)$ and $2h(x)$.

      iii. $h(t + 2)$, $h(t) + 2$ and $h(t) + h(2)$.

   (b) Solve $h(t) = 0$.

Solution.

1. (a) We take -5 and follow it through each step:
      - Step 1: adding 2 gives us $-5 + 2 = -3$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(-3)(3) = -9$.
      - Step 3: subtracting 1 from the result of Step 2 produces $-9 - 1 = -10$.

      Hence, $f(-5) = -10$.

   (b) To find a formula for $f(x)$, we repeat the above process but use the variable 'x' in place of the number -5:
      - Step 1: adding 2 gives us the quantity $x + 2$.
      - Step 2: multiplying the result of Step 1 by 3 yields $(x + 2)(3) = 3x + 6$.
      - Step 3: subtracting 1 from the result of Step 2 produces $(3x + 6) - 1 = 3x + 5$.

Hence, we have codified f using the formula $f(x) = 3x + 5$. In other words, the function f matches each real number 'x' with the value of the expression '$3x + 5$'. As a partial check of our answer, we use this formula to find $f(-5)$. We compute $f(-5)$ by substituting $x = -5$ into the formula $f(x)$ and find $f(-5) = 3(-5) + 5 = -10$ as before.

2. As before, representing the function h as $h(t) = -t^2 + 3t + 4$ means that h matches the real number t with the value of the expression $-t^2 + 3t + 4$.

   (a) To find $h(-1)$, we substitute -1 for t in the expression $-t^2 + 3t + 4$. It is highly recommended that you be generous with parentheses here in order to avoid common mistakes:

$$\begin{aligned} h(-1) &= -(-1)^2 + 3(-1) + 4 \\ &= -(1) + (-3) + 4 \\ &= 0. \end{aligned}$$

   Similarly, $h(0) = -(0)^2 + 3(0) + 4 = 4$, and $h(2) = -(2)^2 + 3(2) + 4 = -4 + 6 + 4 = 6$.

   (b) To find $h(2x)$, we substitute $2x$ for t:

$$\begin{aligned} h(2x) &= -(2x)^2 + 3(2x) + 4 \\ &= -(4x^2) + (6x) + 4 \\ &= -4x^2 + 6x + 4. \end{aligned}$$

   The expression $2h(x)$ means that we multiply the expression $h(x)$ by 2. We first get $h(x)$ by substituting x for t: $h(x) = -x^2 + 3x + 4$. Hence,

$$\begin{aligned} 2h(x) &= 2\left(-x^2 + 3x + 4\right) \\ &= -2x^2 + 6x + 8. \end{aligned}$$

   (c) To find $h(t + 2)$, we substitute the quantity $t + 2$ in place of t:

$$\begin{aligned} h(t+2) &= -(t+2)^2 + 3(t+2) + 4 \\ &= -\left(t^2 + 4t + 4\right) + (3t + 6) + 4 \\ &= -t^2 - 4t - 4 + 3t + 6 + 4 \\ &= -t^2 - t + 6. \end{aligned}$$

To find $h(t)+2$, we add 2 to the expression for $h(t)$

$$\begin{aligned} h(t)+2 &= \left(-t^2+3t+4\right)+2 \\ &= -t^2+3t+6. \end{aligned}$$

From our work above, we see that $h(2)=6$ so

$$\begin{aligned} h(t)+h(2) &= \left(-t^2+3t+4\right)+6 \\ &= -t^2+3t+10. \end{aligned}$$

3. We know $h(-1)=0$ from above, so $t=-1$ should be one of the answers to $h(t)=0$. In order to see if there are any more, we set $h(t)=-t^2+3t+4=0$. Factoring[8] gives $-(t+1)(t-4)=0$, so we get $t=-1$ (as expected) along with $t=4$. □

A few remarks about Example 1.1.2 are in order. First, note that $h(2x)$ and $2h(x)$ are different expressions. In the former, we are multiplying the *input* by 2; in the latter, we are multiplying the *output* by 2. The same goes for $h(t+2)$, $h(t)+2$ and $h(t)+h(2)$. The expression $h(t+2)$ calls for adding 2 to the input t and then performing the function h. The expression $h(t)+2$ has us performing the process h first, then adding 2 to the output $h(t)$. Finally, $h(t)+h(2)$ directs us to first find the outputs $h(t)$ and $h(2)$ and then add the results. As we saw in Example 1.1.1, we see here again the importance paying close attention to syntax.[9]

Let us return for a moment to the function f in Example 1.1.2 which we ultimately represented using the formula $f(x)=3x+5$. If we introduce the dependent variable y, we get the equation $y=f(x)=3x+5$, or, more simply $y=3x+5$. To say that the equation $y=3x+5$ describes y as a function of x means that for each choice of x, the formula $3x+5$ determines only one associated y-value.

We could turn the tables and ask if the equation $y=3x+5$ describes x as a function of y. That is, for each value we pick for y, does the equation $y=3x+5$ produce only one associated x value? One way to proceed is to solve $y=3x+5$ for x and get $x=\frac{1}{3}(y-5)$. We see that for each choice of y, the expression $\frac{1}{3}(y-5)$ evaluates to just one number, hence, x is a function of y. If we give this function a name, say g, we have $x=g(y)=\frac{1}{3}(y-5)$, where in this equation, y is the independent variable and x is the dependent variable. We explore this idea in the next example.

[8]You may need to review Section **??**.

[9]As was mentioned before, we will give meanings to the these quantities in other examples throughout the text.

Example 1.1.3.

1. Consider the equation $x^3 + y^2 = 25$.
   (a) Does this equation represent y as a function of x? Explain.
   (b) Does this equation represent x as a function of y? Explain.
2. Consider the equation $u^4 + t^3 u = 16$.
   (a) Does this equation represent t as a function of u? Explain.
   (b) Does this equation represent u as a function of t? Explain.

Solution.

1. (a) To say that $x^3 + y^2 = 25$ represents y as a function of x, we need to show that for each x we choose, the equation produces only one associated y-value. To help with this analysis, we solve the equation for y in terms of x.

$$\begin{aligned} x^3 + y^2 &= 25 \\ y^2 &= 25 - x^3 \\ y &= \pm\sqrt{25 - x^3} \end{aligned}$$

(extract square roots. (See Section **??** for a review, if needed.))

The presence of the '$\pm$' indicates that there is a good chance that for some x-value, the equation will produce *two* corresponding y-values. Indeed, $x = 0$ produces $y = \pm\sqrt{25 - 0^3} = \pm 5$. Hence, $x^3 + y^2 = 25$ equation does *not* represent y as a function of x because $x = 0$ is matched with more than one y-value.

   (b) To see if $x^3 + y^2 = 25$ represents x as a function of y, we solve the equation for x in terms of y:

$$\begin{aligned} x^3 + y^2 &= 25 \\ x^3 &= 25 - y^2 \\ x &= \sqrt[3]{25 - y^2} \end{aligned}$$

(extract cube roots. (See Section **??** for a review, if needed.))

In this case, each choice of y produces only *one* corresponding value for x, so $x^3 + y^2 = 25$ represents x as a function of y.

2. (a) To see if $u^4 + t^3u = 16$ represents t as a function of u, we proceed as above and solve for t in terms of u:

$$\begin{aligned} u^4 + t^3u &= 16 \\ t^3u &= 16 - u^4 \\ t^3 &= \frac{16 - u^4}{u} && \text{(assumes } u \neq 0\text{)} \\ t &= \sqrt[3]{\frac{16 - u^4}{u}} && \text{(extract cube roots.)} \end{aligned}$$

Although it's a bit cumbersome, as long as $u \neq 0$ the expression $\sqrt[3]{\frac{16-u^4}{u}}$ will produce just one value of t for each value of u. What if $u = 0$? In that case, the equation $u^4 + t^3u = 16$ reduces to $0 = 16$ - which is never true - so we don't need to worry about that case.[10] Hence, $u^4 + t^3u = 16$ represents t as a function of u.

(b) In order to determine if $u^4+t^3u = 16$ represents u as a function of t, we could attempt to solve $u^4+t^3u = 16$ for u in terms of t, but we won't get very far.[11] Instead, we take a different approach and experiment with looking for solutions for u for specific values of t. If we let $t = 0$, we get $u^4 = 16$ which gives $u = \pm\sqrt[4]{16} = \pm 2$. Hence, $t = 0$ corresponds to more than one u-value which means $u^4 + t^3u = 16$ does not represent u as a function of t. □

We'll have more to say about using equations to describe functions in Section **??**. For now, we turn our attention to a geometric way to represent functions.

1.1.3 Geometric Representations of Functions

In this section, we introduce how to graph functions. As we'll see in this and later sections, visualizing functions geometrically can assist us in both analyzing

[10] Said differently, $u = 0$ is not in the domain of the function represented by the equation $u^4 + t^3u = 16$.

[11] Try it for yourself!

them and using them to solve associated application problems. Our playground, if you will, for the Geometry in this course is the Cartesian Coordinate Plane. The reader would do well to review Section **??** as needed.

Our path to the Cartesian Plane requires ordered pairs. In general, we can represent every function as a set of ordered pairs. Indeed, given a function f with domain A, we can represent $f = \{(a, f(a)) \mid a \in A\}$. That is, we represent f as a set of ordered pairs $(a, f(a))$, or, more generally, (input, output). For example, the function f which matches Taylor's pet's names to their associated pet type can be represented as:

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\}$$

Moving on, we next consider the function f from Example 1.1.1 which relates time to temperature. In this case,

$$f = \{(0, 64), (2, 67), (4, 75), (6, 80), (8, 83), (10, 83), (12, 82)\}$$

This function has numerical values for both the domain and range so we can identify these ordered pairs with points in the Cartesian Plane. The first coordinates of these points (the abscissae) represent time values so we'll use t to label the horizontal axis. Likewise, we'll use T to label the vertical axis since the second coordinates of these points (the ordinates) represent temperature values. Note that labeling these axes in this way determines our independent and dependent variable names, t and T, respectively.

The plot of these points is called 'the **graph** of f'. More specifically, we could describe this plot as 'the graph of $f(t)$', because we have decided to name the independent variable t. Most specifically, we could describe the plot as 'the graph of $T = f(t)$', given that we have named the independent variable t and the dependent variable T.

In Figure 1.1.3 and Figure 1.1.4, we present two plots, both of which are graphs of the function f. In both cases, the vertical axis has been scaled in order to save space. In Figure 1.1.3, the same increment on the horizontal axis to measure 1 unit measures 10 units on the vertical axis whereas in Figure 1.1.4, this ratio is 1 : 2. The '$\asymp$' symbol on the vertical axis in the graph on the right is used to indicate a jump in the vertical labeling. Both are perfectly accurate data plots, but they have different visual impacts. Note here that the extrema of f, 64 and 83, correspond to the lowest and highest points on the graph, respectively: $(0, 64)$,

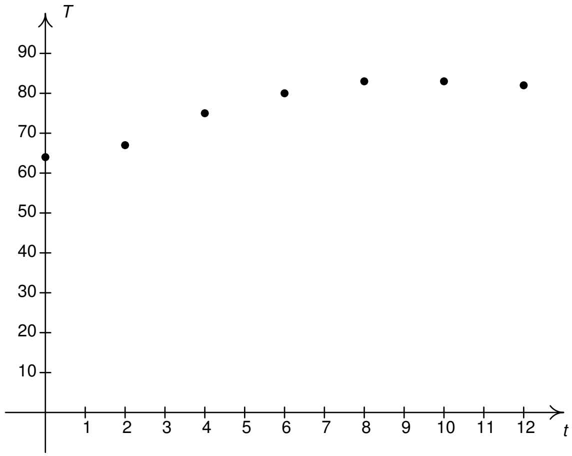

Figure 1.1.3: The graph of $T = f(t)$.

$(8, 83)$ and $(10, 83)$. More often than not, we will use the graph of a function to help us optimize that function.[12]

If you found yourself wanting to connect the dots in the graphs above, you're not alone. As it stands, however, the function f matches only seven inputs to seven outputs, so those seven points - and just those seven points - comprise the graph of f. That being said, common everyday experience tells us that while the data Skippy collected in his table gives some good information about the relationship between time and temperature on a given day, it is by no means a complete description of the relationship.

For example, Skippy's data cannot tell us what the temperature was at 7 a.m. or 12:13 p.m, although we are pretty sure there were outdoor temperatures at those times. Also, given that at some point it was 64°F and later on it was 83°F, it seems reasonable to assume that at some point it was 70°F or even 79.923°F.

Skippy's temperature function f is an example of a **discrete** function in the sense that each of the data points are 'isolated' with measurable gaps in between. The idea of 'filling in' those gaps is a quest to find a **continuous** function to model this same phenomenon.[13] We'll return to this example in Sections 1.2 and 1.4 in an

[12] One major use of Calculus is to optimize functions analytically - that is, without a graph.

[13] Roughly speaking, a *continuous variable* is a variable which takes on values over an *interval* of real numbers as opposed to values in a discrete list. In this case we would think of time as a

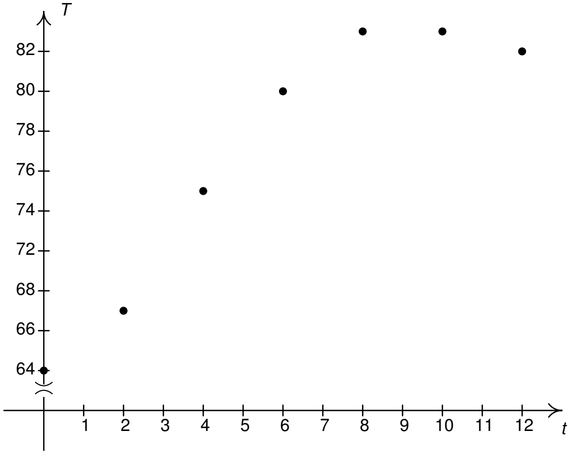

Figure 1.1.4: The graph of $T = f(t)$.

attempt to do just that.

In the meantime, our next example involves a function whose domain is (almost) an *interval* of real numbers and whose graph consists of a (mostly) *connected* arc.

Example 1.1.4. Consider the graph in Figure 1.1.5.

1. (a) Explain why this graph suggests that w is a function of v, $w = F(v)$.

   (b) Find $F(0)$ and solve $F(v) = 0$.

   (c) Find the domain and range of F using interval notation.[14] Find the extrema of F, if any exist.

2. Does this graph suggest v is a function of w? Explain.

Solution. The challenge in working with only a graph is that unless points are specifically labeled (as some are in this case), we are forced to approximate values. In addition to the labeled points, there are other interesting features of

---

'continuum' - an interval of real numbers as opposed to 7 or so isolated times. A *continuous function* is a function which takes an interval of real numbers and maps it in such a way that its graph is a connected curve with no holes or gaps. This is technically a Calculus idea, but we'll need to discuss the notion of continuity a few times in the text.

[14] Please consult Section **??** for a review of interval notation if need be.

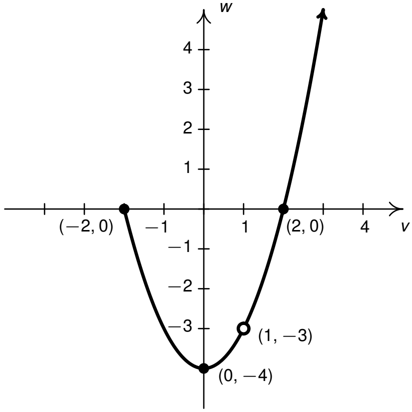

Figure 1.1.5

the graph; a gap or 'hole' labeled $(1, -3)$ and an arrow on the upper right hand part of the curve. We'll have more to say about these two features shortly.

1. (a) In order for w to be a function of v, each v-value on the graph must be paired with only one w-value. What if this weren't the case? We'd have at least two points with the *same* v-coordinate with *different* w-coordinates. Graphically, we'd have two points on graph on the same vertical line, one above the other. This never happens so we may conclude that w is a function of v.

   (b) The value $F(0)$ is the output from F when $v = 0$. The points on the graph of F are of the form $(v, F(v))$ thus we are looking for the w-coordinate of the point on the graph where $v = 0$. Given that the point $(0, -4)$ is labeled on the graph, we can be sure $F(0) = -4$.

   To solve $F(v) = 0$, we are looking for the v-values where the output, or associated w value, is 0. Hence, we are looking for points on the graph with a w-coordinate of 0. We find two such points, $(-2, 0)$ and $(2, 0)$, so our solutions to $F(v) = 0$ are $v = \pm 2$. Pictures highlighting the relevant graphical features are given in Figure 1.1.6 and Figure 1.1.7.

   (c) The domain of F is the set of inputs to F. With v as the input here, we need to describe the set of v-values on the graph. We can accomplish this by **projecting** the graph to the v-axis and seeing what part of the

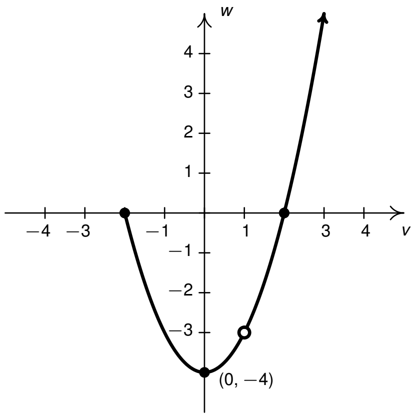

Figure 1.1.6: Finding $F(0) = -4$.

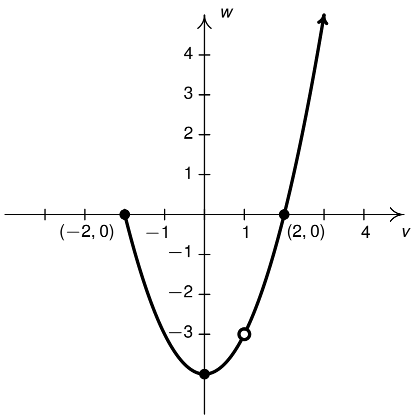

Figure 1.1.7: Solving $F(v) = 0$.

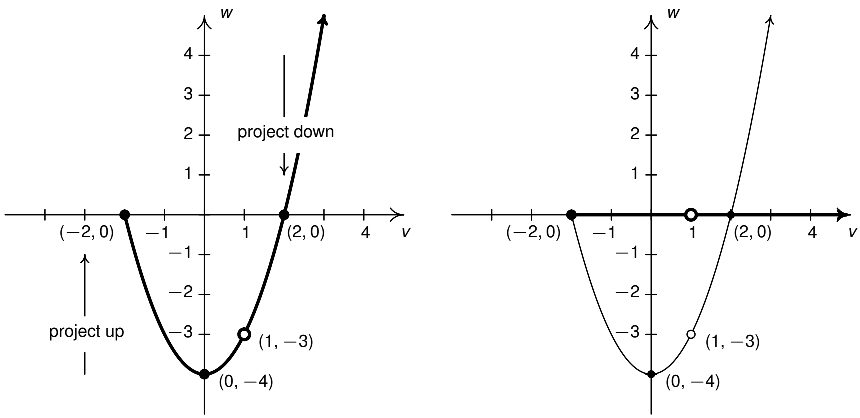

Figure 1.1.8: Finding the domain

v-axis is covered. The leftmost point on the graph is $(-2, 0)$, so we know that the domain starts at $v = -2$. The graph continues to the right until we encounter the 'hole' labeled at $(1, -3)$. This indicates one and only one point, namely $(1, -3)$ is missing from the curve which for us means $v = 1$ is not in the domain of F. The graph continues to the right and the arrow on the graph indicates that the graph goes upwards to the right indefinitely. Hence, our domain is $\{v \,|\, v \geq -2,\ v \neq 1\}$ which, in interval notation, is $[-2, 1) \cup (1, \infty)$. Pictures demonstrating the process of projecting the graph to the v-axis are shown in Figure 1.1.8.

To find the range of F, we need to describe the set of outputs - in this case, the w-values on the graph. Here, we project the graph to the w-axis. Vertically, the graph starts at $(0, -4)$ so our range starts at $w = -4$. Note that even though there is a hole at $(1, -3)$, the w-value -3 is covered by what *appears* to be the point $(-1, -3)$ on the graph.[15] The arrow indicates that the graph extends upwards indefinitely so the range of F is $\{w \,|\, w \geq -4\}$ or, in interval notation, $[-4, \infty)$. Regarding extrema, F has a minimum of -4 when $v = 0$, but given that the graph extends upwards indefinitely, F has no maximum.

Pictures showing the projection of the graph onto the w-axis are given

[15] For all we know, it could be $(-0.992, -3)$.

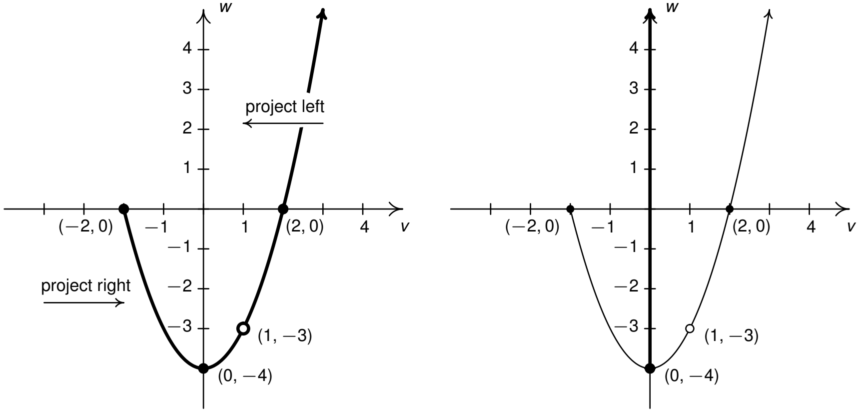

Figure 1.1.9: Finding the range

in Figure 1.1.9.

2. Finally, to determine if v is a function of w, we look to see if each w-value is paired with only one v-value on the graph. We have points on the graph, namely $(-2, 0)$ and $(2, 0)$, that clearly show us that $w = 0$ is matched with the *two* v-values $v = 2$ and $v = -2$. Hence, v is not a function of w. □

It cannot be stressed enough that when given a graphical representation of a function, certain assumptions must be made. In the previous example, for all we know, the minimum of the graph is at $(0.001, -4.0001)$ instead of $(0, -4)$. If we aren't given an equation or table of data, or if specific points aren't labeled, we really have no way to tell. We also are assuming that the graph depicted in the example, while ultimately made of infinitely many points, has no gaps or holes other than those noted. This allows us to make such bold claims as the existence of a point on the graph with a w-coordinate of -3.

Before moving on to our next example, it is worth noting that the geometric argument made in Example 1.1.4 to establish that w is a function of v can be generalized to any graph. This result is the celebrated Vertical Line Test and it enables us to detect functions geometrically. Note that the statement of the theorem resorts to the 'default' x and y labels on the horizontal and vertical axes, respectively.

Theorem 1.1.1. The Vertical Line Test: A graph in the xy-plane[a] represents y as a function of x if and only if no vertical line intersects the graph more than once.

[a]That is, the horizontal axis is labeled with 'x' and the vertical axis is labeled with 'y'.

Let's take a minute to discuss the phrase 'if and only if' used in Theorem 1.1.1. The statement 'the graph represents y as a function of x *if and only if* no vertical line intersects the graph more than once' is actually saying two things. First, it's saying 'the graph represents y as a function of x *if* no vertical line intersects the graph more than once' and, second, 'the graph represents y as a function of x *only if* no vertical line intersects the graph more than once'.

Logically, these statements are saying two different things. The first says that if no vertical line crosses the graph more than once, then the graph represents y as a function of x. But the question remains: could a graph represent y as a function of x and yet there be a vertical line that intersects the graph more than once? The answer to this is 'no' because the second statement says that the *only* way the graph represents y as a function of x is the case when no vertical line intersects the graph more than once.

Applying the Vertical Line Test to the graph given in Example 1.1.4, we see in Figure 1.1.10 that all of the vertical lines meet the graph at most once (several are shown for illustration) showing w is a function of v. Notice that some of the lines ($x = -3$ and $x = 1$, for example) don't hit the graph at all. This is fine because the Vertical Line Test is looking for lines that hit the graph more than once. It does not say *exactly* once so missing the graph altogether is permitted.

There is also a geometric test to determine if the graph above represents v as a function of w. We introduce this aptly-named **Horizontal Line Test** in Exercise 57 and revisit it in Sections **??** and **??**.

Our next example revisits the function h from Example 1.1.2 from a graphical perspective.

Example 1.1.5. With the help of a graphing utility graph $h(t) = -t^2 + 3t + 4$. From your graph, state the domain, range and extrema, if any exist.

Solution. The dependent variable wasn't specified so we use the default 'y' label for the vertical axis and set about graphing $y = h(t)$. From our work in Example 1.1.2, we already know $h(-1) = 0$, $h(0) = 4$, $h(2) = 6$ and $h(4) = 0$. These give us

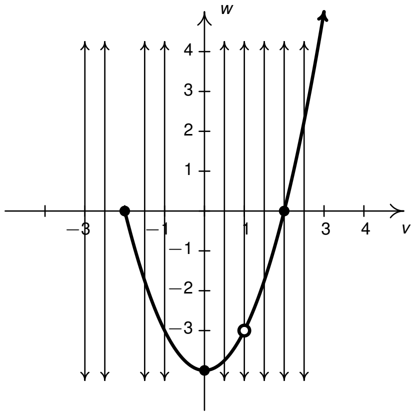

Figure 1.1.10: The Vertical Line Test

the points $(-1,0)$, $(0,4)$, $(2,6)$ and $(4,0)$, respectively. Using these as a guide, we can use desmos[16] to produce the graph in Figure 1.1.11.[17]

As nice as the graph is, it is still technically incomplete. There is no restriction stated on the independent variable t so the domain of h is all real numbers. However, the graph as presented shows only the behavior of h between roughly $t = -2.5$ and $t = 4.25$. By zooming out, we see that the graph extends downwards indefinitely which we indicate by adding the arrows you see in the graph in Figure 1.1.12. We find that the domain is $(-\infty, \infty)$ and the range is $(-\infty, 6.25]$. There is no minimum, but the maximum of h is 6.25 and it occurs at $t = 1.5$. The point $(1.5, 6.25)$ is shown on both graphs.

□

Our last example of the section uses the interplay between algebraic and graphical representations of a function to solve a real-world problem.

Example 1.1.6. The United States Postal Service mandates that when shipping parcels using 'Parcel Select' service, the sum of the length (the longest dimension) and the girth (the distance around the thickest part of the parcel perpendicular to the length) must not exceed 130 inches.[18] Suppose we wish to ship

[16] https://www.desmos.com/

[17] The curve in this example is called a 'parabola'. In Section 1.4, we'll learn how to graph these accurately *by hand*.

[18] See here[19].

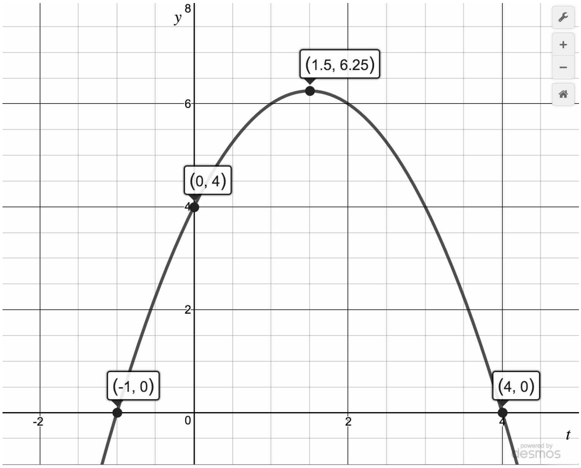

Figure 1.1.11: $h(t) = -t^2 + 3t + 4$ graphed using desmos

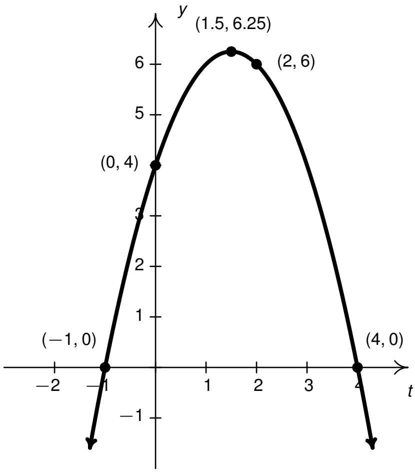

Figure 1.1.12: Graph of $h(t) = -t^2 + 3t + 4$

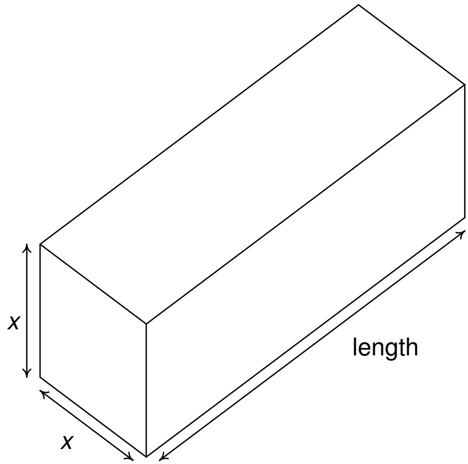

Figure 1.1.13: Dimensions of shipment

a rectangular box whose girth forms a square measuring x inches per side as shown in Figure 1.1.13.

It turns out[20] that the volume of a box, $V(x)$, measured in cubic inches, whose length plus girth is exactly 130 inches is given by the formula: $V(x) = x^2(130-4x)$ for $0 < x \leq 26$.

1. Find and interpret $V(5)$.
2. Make a table of values and use these along with a graphing utility to graph $y = V(x)$.
3. What is the largest volume box that can be shipped? What value of x maximizes the volume? Round your answers to two decimal places.

Solution.

1. To find $V(5)$, we substitute $x = 5$ into the expression $V(x)$: $V(5) = (5)^2(130-4(5)) = 25(110) = 2750$. Our result means that when the length and width of the square measure 5 inches, the volume of the resulting box is 2750 cubic inches.[21]

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[20] We'll skip the explanation for now because we want to focus on just the different representations of the function. Rest assured, you'll be asked to construct this very model in Exercise 56a in Section 2.1.

[21] Note that we have $V(5)$ and $25(110)$ in the same string of equality. The first set of parentheses is

2. The domain of V is specified by the inequality $0 < x \leq 26$, so we can begin graphing V by sampling V at finitely many x-values in this interval to help us get a sense of the range of V. This, in turn, will help us determine an adequate viewing window on our graphing utility when the time comes.

   It seems natural to start with what's happening near $x = 0$. Even though the expression $x^2(130 - 4x)$ is defined when we substitute $x = 0$ (it reduces very quickly to 0), it would be incorrect to state $V(0) = 0$ because $x = 0$ is not in the domain of V. However, there is nothing stopping us from evaluating $V(x)$ at values x 'very close' to $x = 0$. A table of such values is given below.

x	$V(x)$
0.1	1.296
0.01	0.012996
0.001	0.000129996
10^{-23}	$\approx 1.3 \times 10^{-44}$

   There is no such thing as a 'smallest' positive number,[22] so we will have points on the graph of V to the right of $x = 0$ leading to the point $(0, 0)$. We indicate this behavior by putting a hole at $(0, 0)$.[23]

   Moving forward, we start with $x = 5$ and sample V at steps of 5 in its domain. Our goal is to graph $y = V(x)$, so we plot our points $(x, V(x))$ using the domain as a guide to help us set the horizontal bounds (i.e., the bounds on x) and the sample values from the range to help us set the vertical bounds (i.e., the bounds on y). The right endpoint, $x = 26$, is included in the domain $0 < x \leq 26$ so we finish the graph by plotting the point $(26, V(26)) = (26, 17576)$. Table 1.1.1 shows the data and Figure 1.1.14 shows a graph produced with some help from desmos[24].

3. The largest volume in this case refers to the maximum of V. The biggest y-value in our table of data is 20,000 cubic inches which occurs at $x = 20$ inches, but the graph produced by the graphing utility shown in Figure 1.1.15 indicates that there are points on the graph of V with y-values

function notation and directs us to substitute 5 for x in the expression $V(x)$ while the second indicates multiplying 25 by 110. Context is key!

[22]If p is any positive real number, $0 < 0.5p < p$, so we can always find a smaller positive real number.

[23]What's really needed here is the precise definition of 'closeness' discussed in Calculus. This hand-waving will do for now.

[24]https://www.desmos.com/

x	$V(x)$	$(x, V(x))$
≈ 0	≈ 0	hole at $(0,0)$
5	2750	$(5, 2750)$
10	9000	$(10, 9000)$
15	15,750	$(15, 15{,}750)$
20	20,000	$(20, 20{,}000)$
25	18,750	$(25, 18{,}750)$
26	17,576	$(26, 17{,}576)$

Table 1.1.1: Sampling V

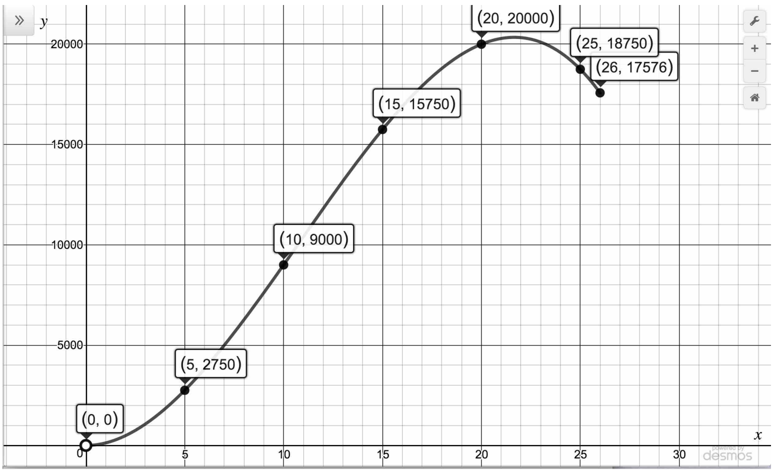


Figure 1.1.14: The graph of $y = V(x)$

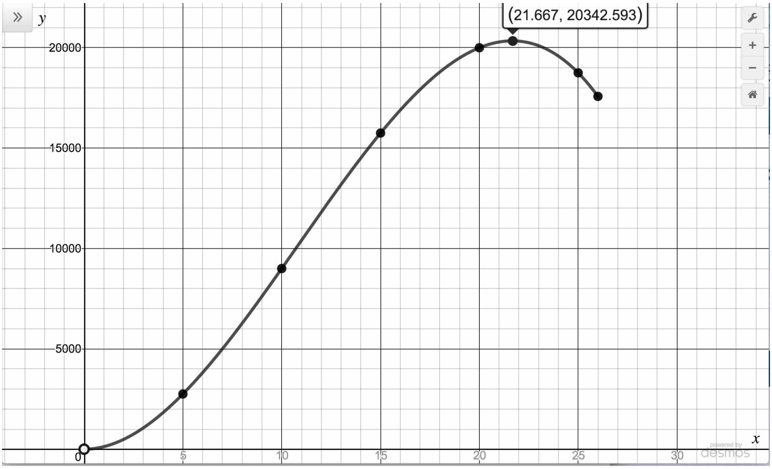


Figure 1.1.15: Finding the maximum volume using the graph of $y = V(x)$

(hence $V(x)$ values) greater than 20,000. Indeed, the graph continues to rise to the right of $x = 20$ and the graphing utility reports the maximum y-value to be $y \approx 20{,}342.593$ when $x \approx 21.667$. Rounding to two decimal places, we find the maximum volume obtainable under these conditions is about 20,342.59 cubic inches which occurs when the length and width of the square side of the box are approximately 21.67 inches.[25]

It is worth noting that while the function V has a maximum, it did not have a minimum. Even though $V(x) > 0$ for all x in its domain,[26] the presence of the hole at $(0, 0)$ means that 0 is not in the range of V. Hence, based on our model, we can never make a box with a 'smallest' volume.[27] □

Example 1.1.6 typifies the interplay between Algebra and Geometry which lies ahead. Both the algebraic description of V: $V(x) = x^2(130 - 4x)$ for $0 < x \leq 26$, and the graph of $y = V(x)$ were useful in describing aspects of the physical situation at hand. Wherever possible, we'll use the algebraic representations of functions to *analytically* produce *exact* answers to certain problems and use the graphical descriptions to check the reasonableness of our answers.

[25] We could also find the length of the box in this case as well. The sum of length and girth is 130 inches so the length is 130 minus the girth, or $130 - 4x \approx 130 - 4(21.67) = 43.32$ inches.

[26] said differently, the values of $V(x)$ are **bounded below** by 0.

[27] How realistic is this?

That being said, we'll also encounter problems which we simply *cannot* answer analytically (such as determining the maximum volume in the previous example), so we will be forced to resort to using technology (specifically graphing technology) in order to find *approximate* solutions. The most important thing to keep in mind is that while technology may *suggest* a result, it is ultimately Mathematics that *proves* it.

We close this section with a summary of the different ways to represent functions as shown in Box 1.1.1

Box 1.1.1: Ways to Represent a Function

Suppose f is a function with domain A. Then f can be represented:

- verbally; that is, by describing how the inputs are matched with their outputs.
- using a mapping diagram.
- as a set of ordered pairs of the form (input, output): $\{(a, f(a)) \mid a \in A\}$.

If f is a function whose domain and range are subsets of real numbers, then f can be represented:

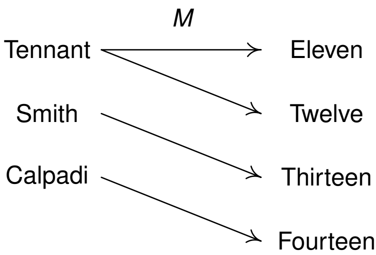
- algebraically as a formula for $f(a)$.
- graphically by plotting the points $\{(a, f(a)) \mid a \in A\}$ in the plane.

Note: An important consequence of the last bulleted item is that the point (a, b) is on the graph of $y = f(x)$ if and only if $f(a) = b$.

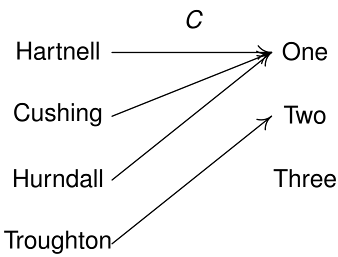
1.1.4 Exercises

In Exercises 1 - 2, determine whether or not the mapping diagram represents a function. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

1.



2.



In Exercises 3. - 4., determine whether or not the data in the given table represents y as a function of x. Explain your reasoning. If the mapping does represent a function, state the domain, range, and represent the function as a set of ordered pairs.

3.

x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

4.

x	y
0	0
1	1
1	-1
2	2
2	-2
3	3
3	-3

5. Suppose W is the set of words in the English language and we set up a mapping from W into the set of natural numbers $\mathbb{N}$ as follows: word $\rightarrow$

number of letters in the word. Explain why this mapping is a function. What would you need to know to determine the range of the function?

6. Suppose L is the set of last names of all the people who have served or are currently serving as the President of the United States. Consider the mapping from L into $\mathbb{N}$ as follows: last name $\rightarrow$ number of their presidency. For example, Washington $\rightarrow$ 1 and Obama $\rightarrow$ 44. Is this mapping a function? What if we use full names instead of just last names? (**HINT:** Research Grover Cleveland.)

7. Under what conditions would the time of day be a function of the outdoor temperature?

For the functions f described in Exercises 8 - 13, find $f(2)$ and find and simplify an expression for $f(x)$ that takes a real number x and performs the following three steps in the order given:

8. (1) multiply by 2; (2) add 3; (3) divide by 4.
9. (1) add 3; (2) multiply by 2; (3) divide by 4.
10. (1) divide by 4; (2) add 3; (3) multiply by 2.
11. (1) multiply by 2; (2) add 3; (3) take the square root.
12. (1) add 3; (2) multiply by 2; (3) take the square root.
13. (1) add 3; (2) take the square root; (3) multiply by 2.

In Exercises 14 - 19, use the given function f to find and simplify the following:

- $f(3)$
- $f(4x)$
- $f(x-4)$
- $f(-1)$
- $4f(x)$
- $f(x)-4$
- $f\left(\frac{3}{2}\right)$
- $f(-x)$
- $f\left(x^2\right)$

14. $f(x) = 2x+1$
15. $f(x) = 3-4x$
16. $f(x) = 2-x^2$
17. $f(x) = x^2-3x+2$
18. $f(x) = 6$
19. $f(x) = 0$

In Exercises 20 - 25, use the given function f to find and simplify the following:

- $f(2)$
- $2f(a)$
- $f\left(\dfrac{2}{a}\right)$
- $f(-2)$
- $f(a+2)$
- $\dfrac{f(a)}{2}$
- $f(2a)$
- $f(a)+f(2)$
- $f(a+h)$

20. $f(x) = 2x-5$
21. $f(t) = 5-2t$
22. $f(w) = 2w^2-1$
23. $f(q) = 3q^2+3q-2$
24. $f(r) = 117$
25. $f(z) = \dfrac{z}{2}$

In Exercises 26 - 29, use the given function f to find $f(0)$ and solve $f(x) = 0$

26. $f(x) = 2x - 1$
27. $f(x) = 3 - \frac{2}{5}x$
28. $f(x) = 2x^2 - 6$
29. $f(x) = x^2 - x - 12$

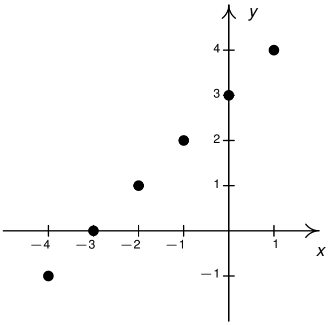
In Exercises 30 - 44, determine whether or not the equation represents y as a function of x.

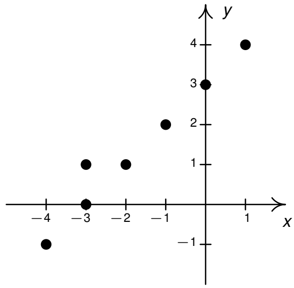
30. $y = x^3 - x$
31. $y = \sqrt{x - 2}$
32. $x^3 y = -4$
33. $x^2 - y^2 = 1$
34. $y = \dfrac{x}{x^2 - 9}$
35. $x = -6$
36. $x = y^2 + 4$
37. $y = x^2 + 4$
38. $x^2 + y^2 = 4$
39. $y = \sqrt{4 - x^2}$
40. $x^2 - y^2 = 4$
41. $x^3 + y^3 = 4$
42. $2x + 3y = 4$
43. $2xy = 4$
44. $x^2 = y^2$

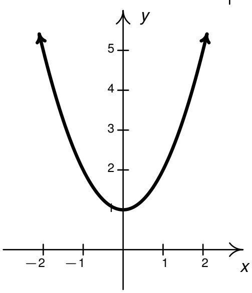
Exercises 45 - 56 give a set of points in the xy-plane. Determine if y is a function of x. If so, state the domain and range.

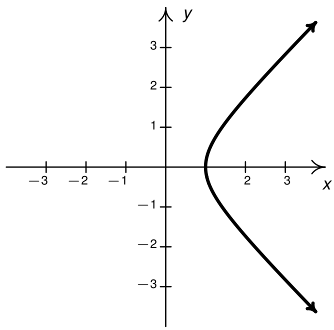
45. $\{(-3, 9),\ (-2, 4),\ (-1, 1),\ (0, 0),\ (1, 1),\ (2, 4),\ (3, 9)\}$
46. $\{(-3, 0), (1, 6), (2, -3), (4, 2), (-5, 6), (4, -9), (6, 2)\}$
47. $\{(-3, 0), (-7, 6), (5, 5), (6, 4), (4, 9), (3, 0)\}$
48. $\{(1, 2), (4, 4), (9, 6), (16, 8), (25, 10), (36, 12), ...\}$
49. $\{(x, y) \,|\, x$ is an odd integer, and y is an even integer$\}$
50. $\{(x, 1) \,|\, x$ is an irrational number$\}$
51. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$
52. $\{... (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$
53. $\{(-2, y) \,|\, -3 < y < 4\}$
54. $\{(x, 3) \,|\, -2 \leq x < 4\}$
55. $\{\left(x, x^2\right) \,|\, x$ is a real number$\}$
56. $\{\left(x^2, x\right) \,|\, x$ is a real number$\}$
57. The Vertical Line Test is a quick way to determine from a graph if the vertical axis variable is a function of the horizontal axis variable. If we are given a graph and asked to determine if the horizontal axis variable is a function of the vertical axis variable, we can use horizontal lines instead of vertical lines to check. Using Theorem 1.1.1 as a guide, formulate a 'Horizontal Line Test.' (We'll refer back to this exercise in Section **??**.)

In Exercises 58. - 61., determine whether or not the graph suggests y is a function of x. For the ones which do, state the domain and range.

58. 

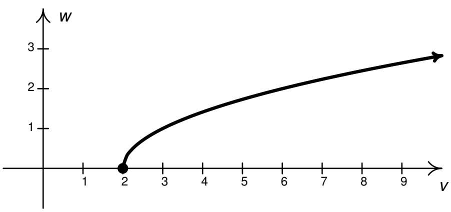
59. 

60. 

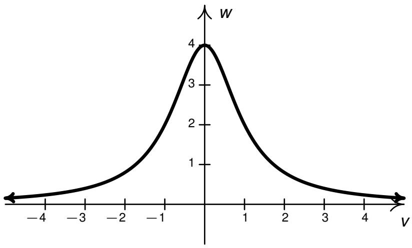
61. 

62. Determine which, if any, of the graphs in numbers 58. - 61. represent x as a function of y. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

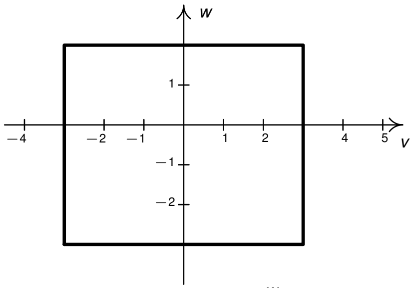
In Exercises 63. - 66., determine whether or not the graph suggests w is a function of v. For the ones which do, state the domain and range.

63. 

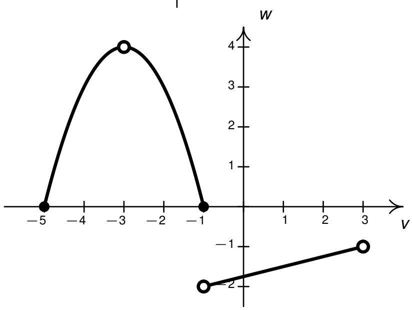
64.



65.



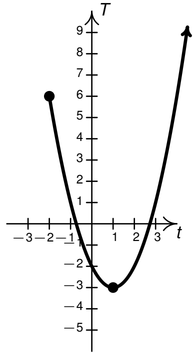
66.



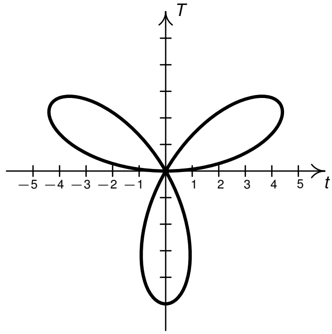
67. Determine which, if any, of the graphs in numbers 63. - 66. represent v as a function of w. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 68. - 71., determine whether or not the graph suggests T is a function of t. For the ones which do, state the domain and range.

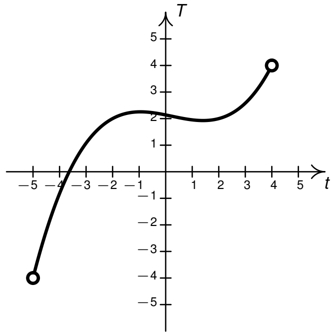
68.



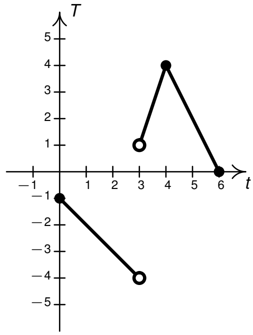
69.



70.



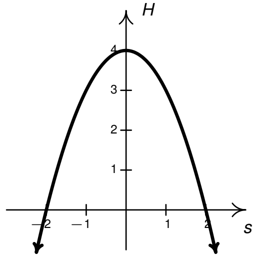
71.



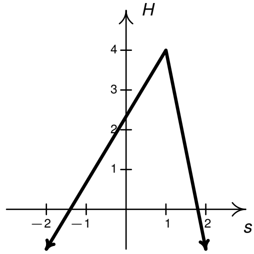
72. Determine which, if any, of the graphs in numbers 68. - 71. represent t as a function of T. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 73. - 76., determine whether or not the graph suggests H is a function of s. For the ones which do, state the domain and range.

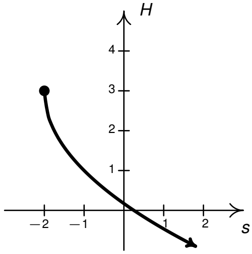
73.



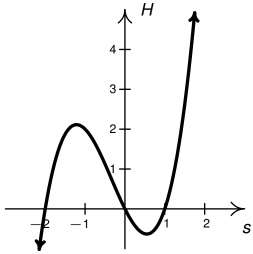
74.



75.



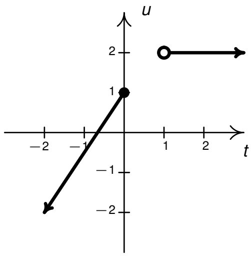
76.



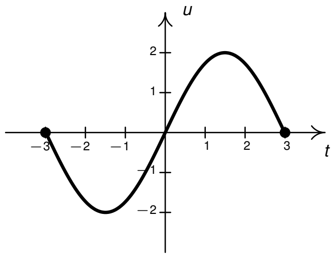
77. Determine which, if any, of the graphs in numbers 73. - 76. represent s as a function of H. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 78. - 81., determine whether or not the graph suggests u is a function of t. For the ones which do, state the domain and range.

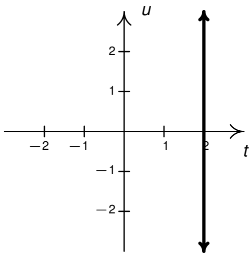
78.



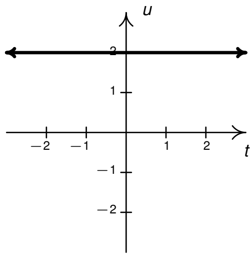
79.



80.



81.



82. Determine which, if any, of the graphs in numbers 78. - 81. represent t as a function of u. For the ones which do, state the domain and range. (Feel free to use Exercise 57.)

In Exercises 83 - 92, use the graphs of f and g shown in Figure 1.1.16 and Figure 1.1.17 to find the indicated values.

83. $f(-2)$

84. $g(-2)$

85. $f(2)$

86. $g(2)$

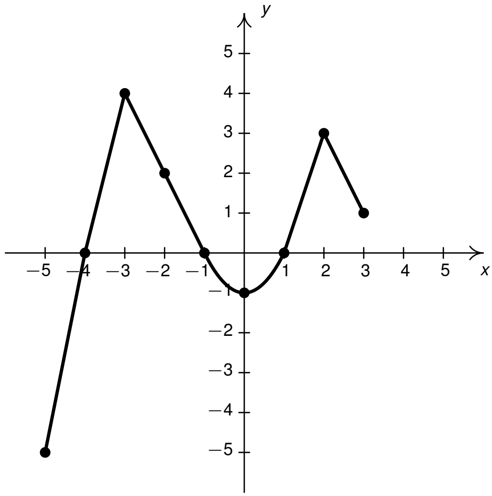


Figure 1.1.16: The graph of $y = f(x)$.

87. $f(0)$
88. $g(0)$
89. Solve $f(x) = 0$.
90. Solve $g(t) = 0$.
91. State the domain and range of f.
92. State the domain and range of g .

In Exercises 93 - 104, graph each function by making a table, plotting points, and using a graphing utility (if needed.) Use the independent variable as the horizontal axis label and the default 'y' label for the vertical axis label. State the domain and range of each function.

93. $f(x) = 2 - x$
94. $g(t) = \dfrac{t-2}{3}$
95. $h(s) = s^2 + 1$
96. $f(x) = 4 - x^2$
97. $g(t) = 2$
98. $h(s) = s^3$
99. $f(x) = x(x-1)(x+2)$
100. $g(t) = \sqrt{t-2}$
101. $h(s) = \sqrt{5-s}$
102. $f(x) = 3 - 2\sqrt{x+2}$
103. $g(t) = \sqrt[3]{t}$
104. $h(s) = \dfrac{1}{s^2+1}$

105. Consider the function f described below:

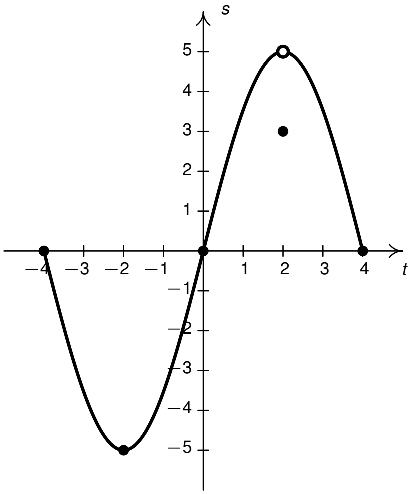
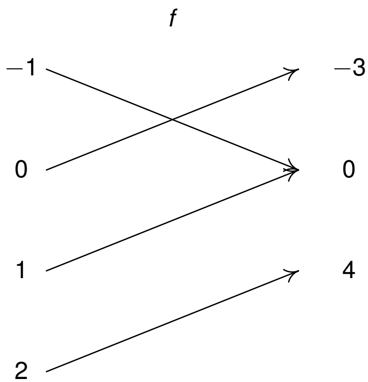


Figure 1.1.17: The graph of $s = g(t)$.



(a) State the domain and range of f.

(b) Find $f(0)$ and solve $f(x) = 0$.

(c) Write f as a set of ordered pairs.

(d) Graph f.

106. Let $g = \{(-1, 4), (0, 2), (2, 3), (3, 4)\}$

(a) State the domain and range of g.

(b) Create a mapping diagram for g.

(c) Find $g(0)$ and solve $g(x) = 0$.

(d) Graph g.

107. Let $F = \{(t, t^2) \mid t \text{ is a real number}\}$. Find $F(4)$ and solve $F(x) = 4$.

HINT: Elements of F are of the form $(x, F(x))$.

108. Let $G = \{(2t, t + 5) \mid t \text{ is a real number}\}$. Find $G(4)$ and solve $G(x) = 4$.

HINT: Elements of G are of the form $(x, G(x))$.

109. The area enclosed by a square, in square inches, is a function of the length of one of its sides ℓ, when measured in inches. This function is represented by the formula $A(\ell) = \ell^2$ for $\ell > 0$. Find $A(3)$ and solve $A(\ell) = 36$. Interpret your answers to each. Why is ℓ restricted to $\ell > 0$?

110. The area enclosed by a circle, in square meters, is a function of its radius r, when measured in meters. This function is represented by the formula $A(r) = \pi r^2$ for $r > 0$. Find $A(2)$ and solve $A(r) = 16\pi$. Interpret your answers to each. Why is r restricted to $r > 0$?

111. The volume enclosed by a cube, in cubic centimeters, is a function of the length of one of its sides s, when measured in centimeters. This function is represented by the formula $V(s) = s^3$ for $s > 0$. Find $V(5)$ and solve $V(s) = 27$. Interpret your answers to each. Why is s restricted to $s > 0$?

112. The volume enclosed by a sphere, in cubic feet, is a function of the radius of the sphere r, when measured in feet. This function is represented by the formula $V(r) = \frac{4\pi}{3}r^3$ for $r > 0$. Find $V(3)$ and solve $V(r) = \frac{32\pi}{3}$. Interpret your answers to each. Why is r restricted to $r > 0$?

113. The height of an object dropped from the roof of an eight story building is modeled by the function: $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. Find $h(0)$ and solve $h(t) = 0$. Interpret your answers to each. Why is t restricted to $0 \leq t \leq 2$?

114. The temperature in degrees Fahrenheit t hours after 6 AM is given by $T(t) = -\frac{1}{2}t^2 + 8t + 3$ for $0 \leq t \leq 12$. Find and interpret $T(0)$, $T(6)$ and $T(12)$.

115. The function $C(x) = x^2 - 10x + 27$ models the cost, in *hundreds* of dollars, to produce x *thousand* pens. Find and interpret $C(0)$, $C(2)$ and $C(5)$.

116. Using data from the Bureau of Transportation Statistics[28], the average fuel economy in miles per gallon for passenger cars in the US can be modeled by $E(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Use a calculator to find $E(0)$, $E(14)$ and $E(28)$. Round your answers to two decimal places and interpret your answers to each.

117. The perimeter of a square, in centimeters, is four times the length of one if its sides, also measured in centimeters. Represent the function P which takes as its input the length of the side of a square in centimeters, s and returns the perimeter of the square in inches, $P(s)$ using a formula.

118. The circumference of a circle, in feet, is π times the diameter of the circle, also measured in feet. Represent the function C which takes as its input the length of the diameter of a circle in feet, D and returns the circumference of a circle in inches, $C(D)$ using a formula.

119. Suppose $A(P)$ gives the amount of money in a retirement account (in dollars) after 30 years as a function of the amount of the monthly payment (in dollars), P.

    (a) What does $A(50)$ mean?

    (b) What is the significance of the solution to the equation $A(P) = 250000$?
    .

    (c) Explain what each of the following expressions mean: $A(P + 50)$, $A(P) + 50$, and $A(P) + A(50)$.

120. Suppose $P(t)$ gives the chance of precipitation (in percent) t hours after 8 AM.

    (a) Write an expression which gives the chance of precipitation at noon.

    (b) Write an inequality which determines when the chance of precipitation is more than 50%.

121. Explain why the graph in Exercise 63. suggests that not only is v as a function of w but also w is a function of v. Suppose $w = f(v)$ and $v = g(w)$. That is, f is the name of the function which takes v values as inputs and returns w values as outputs and g is the name of the function which does vice-versa. Find the domain and range of g and compare these to the domain and range of f.

[28] http://www.bts.gov/publications/national_transportation_statistics/html/table_04_23.html

122. Sketch the graph of a function with domain $(-\infty, 3) \cup [4, 5)$ with range $\{2\} \cup (5, \infty)$.

1.1.5 Answers

1. The mapping M is not a function since 'Tennant' is matched with both 'Eleven' and 'Twelve.'

2. The mapping C is a function since each input is matched with only one output. The domain of C is $\{\text{Hartnell}, \text{Cushing}, \text{Hurndall}, \text{Troughton}\}$ and the range is $\{\text{One}, \text{Two}\}$. We can represent C as the following set of ordered pairs:

$$\{(\text{Hartnell}, \text{One}), (\text{Cushing}, \text{One}), (\text{Hurndall}, \text{One}), (\text{Troughton}, \text{Two})\}$$

3. In this case, y is a function of x since each x is matched with only one y.

   The domain is $\{-3, -2, -1, 0, 1, 2, 3\}$ and the range is $\{0, 1, 2, 3\}$.

   As ordered pairs, this function is

$$\{(-3, 3), (-2, 2), (-1, 1), (0, 0), (1, 1), (2, 2), (3, 3)\}$$

4. In this case, y is not a function of x since there are x values matched with more than one y value. For instance, 1 is matched both to 1 and -1.

5. The mapping is a function since given any word, there is only one answer to 'how many letters are in the word?' For the range, we would need to know what the length of the longest word is and whether or not we could find words of all the lengths between 1 (the length of the word 'a') and it. See here[29].

6. Since Grover Cleveland was both the 22nd and 24th POTUS, neither mapping described in this exercise is a function.

7. The outdoor temperature could never be the same for more than two different times - so, for example, it could always be getting warmer or it could always be getting colder.

8. $f(2) = \frac{7}{4}$, $f(x) = \frac{2x+3}{4}$

9. $f(2) = \frac{5}{2}$, $f(x) = \frac{2(x+3)}{4} = \frac{x+3}{2}$

10. $f(2) = 7$, $f(x) = 2\left(\frac{x}{4} + 3\right) = \frac{1}{2}x + 6$

11. $f(2) = \sqrt{7}$, $f(x) = \sqrt{2x + 3}$

[29] https://en.wikipedia.org/wiki/Longest_word_in_English

12. $f(2) = \sqrt{10}$, $f(x) = \sqrt{2(x+3)} = \sqrt{2x+6}$

13. $f(2) = 2\sqrt{5}$, $f(x) = 2\sqrt{x+3}$

14. For $f(x) = 2x + 1$

- $f(3) = 7$
- $f(-1) = -1$
- $f\left(\frac{3}{2}\right) = 4$
- $f(4x) = 8x + 1$
- $4f(x) = 8x + 4$
- $f(-x) = -2x + 1$
- $f(x-4) = 2x - 7$
- $f(x) - 4 = 2x - 3$
- $f\left(x^2\right) = 2x^2 + 1$

15. For $f(x) = 3 - 4x$

- $f(3) = -9$
- $f(-1) = 7$
- $f\left(\frac{3}{2}\right) = -3$
- $f(4x) = 3 - 16x$
- $4f(x) = 12 - 16x$
- $f(-x) = 4x + 3$
- $f(x-4) = 19 - 4x$
- $f(x) - 4 = -4x - 1$
- $f\left(x^2\right) = 3 - 4x^2$

16. For $f(x) = 2 - x^2$

- $f(3) = -7$
- $f(-1) = 1$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 2 - 16x^2$
- $4f(x) = 8 - 4x^2$
- $f(-x) = 2 - x^2$
- $f(x-4) = -x^2 + 8x - 14$
- $f(x) - 4 = -x^2 - 2$
- $f\left(x^2\right) = 2 - x^4$

17. For $f(x) = x^2 - 3x + 2$

- $f(3) = 2$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = -\frac{1}{4}$
- $f(4x) = 16x^2 - 12x + 2$
- $4f(x) = 4x^2 - 12x + 8$
- $f(-x) = x^2 + 3x + 2$
- $f(x-4) = x^2 - 11x + 30$
- $f(x) - 4 = x^2 - 3x - 2$
- $f\left(x^2\right) = x^4 - 3x^2 + 2$

18. For $f(x) = 6$

- $f(3) = 6$
- $f(-1) = 6$
- $f\left(\frac{3}{2}\right) = 6$
- $f(4x) = 6$
- $4f(x) = 24$
- $f(-x) = 6$
- $f(x-4) = 6$
- $f(x) - 4 = 2$
- $f\left(x^2\right) = 6$

19. For $f(x) = 0$

- $f(3) = 0$
- $f(-1) = 0$
- $f\left(\frac{3}{2}\right) = 0$
- $f(4x) = 0$
- $4f(x) = 0$
- $f(-x) = 0$
- $f(x-4) = 0$
- $f(x) - 4 = -4$
- $f\left(x^2\right) = 0$

20. For $f(x) = 2x - 5$

- $f(2) = -1$
- $f(-2) = -9$
- $f(2a) = 4a - 5$
- $2f(a) = 4a - 10$
- $f(a+2) = 2a - 1$
- $f(a) + f(2) = 2a - 6$
- $f\left(\frac{2}{a}\right) = \frac{4}{a} - 5 = \frac{4-5a}{a}$
- $\frac{f(a)}{2} = \frac{2a-5}{2}$
- $f(a+h) = 2a + 2h - 5$

21. For $f(x) = 5 - 2x$
    - $f(2) = 1$
    - $f(-2) = 9$
    - $f(2a) = 5 - 4a$
    - $2f(a) = 10 - 4a$
    - $f(a+2) = 1 - 2a$
    - $f(a) + f(2) = 6 - 2a$
    - $f\left(\frac{2}{a}\right) = 5 - \frac{4}{a} = \frac{5a-4}{a}$
    - $\frac{f(a)}{2} = \frac{5-2a}{2}$
    - $f(a+h) = 5 - 2a - 2h$

22. For $f(x) = 2x^2 - 1$
    - $f(2) = 7$
    - $f(-2) = 7$
    - $f(2a) = 8a^2 - 1$
    - $2f(a) = 4a^2 - 2$
    - $f(a+2) = 2a^2 + 8a + 7$
    - $f(a) + f(2) = 2a^2 + 6$
    - $f\left(\frac{2}{a}\right) = \frac{8}{a^2} - 1 = \frac{8-a^2}{a^2}$
    - $\frac{f(a)}{2} = \frac{2a^2-1}{2}$
    - $f(a+h) = 2a^2 + 4ah + 2h^2 - 1$

23. For $f(x) = 3x^2 + 3x - 2$
    - $f(2) = 16$
    - $f(-2) = 4$
    - $f(2a) = 12a^2 + 6a - 2$
    - $2f(a) = 6a^2 + 6a - 4$
    - $f(a+2) = 3a^2 + 15a + 16$
    - $f(a) + f(2) = 3a^2 + 3a + 14$
    - $f\left(\frac{2}{a}\right) = \frac{12}{a^2} + \frac{6}{a} - 2 = \frac{12+6a-2a^2}{a^2}$
    - $\frac{f(a)}{2} = \frac{3a^2+3a-2}{2}$
    - $f(a+h) = 3a^2 + 6ah + 3h^2 + 3a + 3h - 2$

24. For $f(x) = 117$
    - $f(2) = 117$
    - $f(-2) = 117$
    - $f(2a) = 117$
    - $2f(a) = 234$
    - $f(a+2) = 117$
    - $f(a) + f(2) = 234$
    - $f\left(\frac{2}{a}\right) = 117$
    - $\frac{f(a)}{2} = \frac{117}{2}$
    - $f(a+h) = 117$

25. For $f(x) = \frac{x}{2}$
    - $f(2) = 1$
    - $f(-2) = -1$
    - $f(2a) = a$
    - $2f(a) = a$
    - $f(a+2) = \frac{a+2}{2}$
    - $f(a) + f(2) = \frac{a}{2} + 1 = \frac{a+2}{2}$
    - $f\left(\frac{2}{a}\right) = \frac{1}{a}$
    - $\frac{f(a)}{2} = \frac{a}{4}$
    - $f(a+h) = \frac{a+h}{2}$

26. For $f(x) = 2x - 1$, $f(0) = -1$ and $f(x) = 0$ when $x = \frac{1}{2}$
27. For $f(x) = 3 - \frac{2}{5}x$, $f(0) = 3$ and $f(x) = 0$ when $x = \frac{15}{2}$
28. For $f(x) = 2x^2 - 6$, $f(0) = -6$ and $f(x) = 0$ when $x = \pm\sqrt{3}$

29. For $f(x) = x^2 - x - 12$, $f(0) = -12$ and $f(x) = 0$ when $x = -3$ or $x = 4$

30. Function
31. Function
32. Function
33. Not a function
34. Function
35. Not a function
36. Not a function
37. Function
38. Not a function
39. Function
40. Not a function
41. Function
42. Function
43. Function
44. Not a function

45. Function
    domain = $\{-3, -2, -1, 0, 1, 2, 3\}$
    range = $\{0, 1, 4, 9\}$

46. Not a function

47. Function
    domain = $\{-7, -3, 3, 4, 5, 6\}$
    range = $\{0, 4, 5, 6, 9\}$

48. Function
    domain = $\{1, 4, 9, 16, 25, 36, ...\} = \{x \mid x \text{ is a perfect square}\}$
    range = $\{2, 4, 6, 8, 10, 12, ...\} = \{y \mid y \text{ is a positive even integer}\}$

49. Not a function

50. Function
    domain = $\{x \mid x \text{ is irrational}\}$
    range = $\{1\}$

51. Function
    domain = $\{x \mid 1, 2, 4, 8, ...\} = \{x \mid x = 2^n \text{ for some whole number } n\}$
    range = $\{0, 1, 2, 3, ...\} = \{y \mid y \text{ is any whole number}\}$

52. Function
    domain = $\{x \mid x \text{ is any integer}\}$
    range = $\{y \mid y \text{ is the square of an integer}\}$

53. Not a function

54. Function
    domain = $\{x \mid -2 \le x < 4\} = [-2, 4)$,
    range = $\{3\}$

55. Function
    domain = $\{x \mid x \text{ is a real number}\} = (-\infty, \infty)$
    range = $\{y \mid y \geq 0\} = [0, \infty)$

56. Not a function

57. **Horizontal Line Test:** A graph on the xy-plane represents x as a function of y if and only if no **horizontal** line intersects the graph more than once.

58. Function
domain = $\{-4, -3, -2, -1, 0, 1\}$
range = $\{-1, 0, 1, 2, 3, 4\}$

59. Not a function

60. Function
domain = $(-\infty, \infty)$
range = $[1, \infty)$

61. Not a function

62.
    - Number 58. represents x as a function of y. domain = $\{-1, 0, 1, 2, 3, 4\}$ and range = $\{-4, -3, -2, -1, 0, 1\}$
    - Number 61. represents x as a function of y.

      domain = $(-\infty, \infty)$ and range = $[1, \infty)$

63. Function
domain = $[2, \infty)$
range = $[0, \infty)$

64. Function
domain = $(-\infty, \infty)$
range = $(0, 4]$

65. Not a function

66. Function
domain = $[-5, -3) \cup (-3, 3)$
range = $(-2, -1) \cup [0, 4)$

67. Only number 63. represents v as a function of w; domain = $[0, \infty)$ and range = $[2, \infty)$

68. Function
domain = $[-2, \infty)$
range = $[-3, \infty)$

69. Not a function

70. Function
domain = $(-5, 4)$
range = $(-4, 4)$

71. Function
domain = $[0, 3) \cup (3, 6]$
range = $(-4, -1] \cup [0, 4]$

72. None of numbers 68. - 71. represent t as a function of T.

73. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

74. Function
domain = $(-\infty, \infty)$
range = $(-\infty, 4]$

75. Function
domain = $[-2, \infty)$
range = $(-\infty, 3]$

76. Function
domain = $(-\infty, \infty)$
range = $(-\infty, \infty)$

77. Only number 75. represents s as a function of H; domain = $(-\infty, 3]$ and range = $[-2, \infty)$

78. Function
domain = $(-\infty, 0] \cup (1, \infty)$
range = $(-\infty, 1] \cup \{2\}$

79. Function
domain = $[-3, 3]$
range = $[-2, 2]$

80. Not a function

81. Function
domain = $(-\infty, \infty)$
range = $\{2\}$

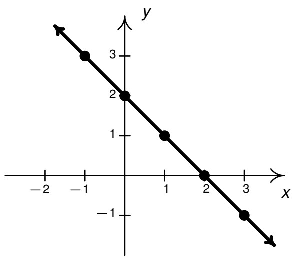
82. Only number 80. represents t as a function of u; domain = $(-\infty, \infty)$ and range=$\{2\}$.

83. $f(-2) = 2$ 84. $g(-2) = -5$ 85. $f(2) = 3$
86. $g(2) = 3$ 87. $f(0) = -1$ 88. $g(0) = 0$
89. $x = -4, -1, 1$ 90. $t = -4, 0, 4$

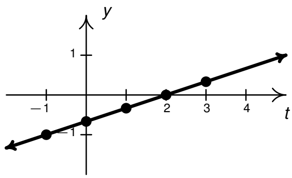
91. Domain: $[-5, 3]$, Range: $[-5, 4]$.

92. Domain: $[-4, 4]$, Range: $[-5, 5)$.

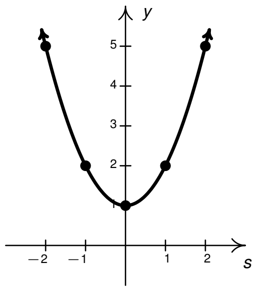
93. $f(x) = 2 - x$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



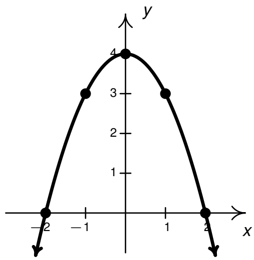
94. $g(t) = \dfrac{t-2}{3}$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$



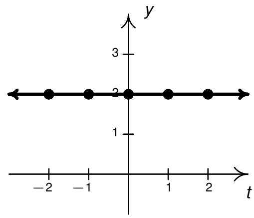
95. $h(s) = s^2 + 1$
Domain: $(-\infty, \infty)$
Range: $[1, \infty)$



96. $f(x) = 4 - x^2$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$



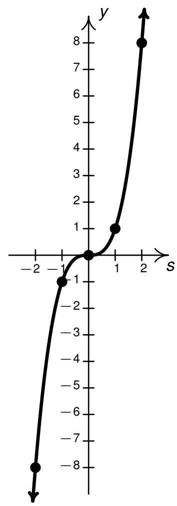
97. $g(t) = 2$
Domain: $(-\infty, \infty)$
Range: $\{2\}$



98. $h(s) = s^3$

Domain: $(-\infty, \infty)$

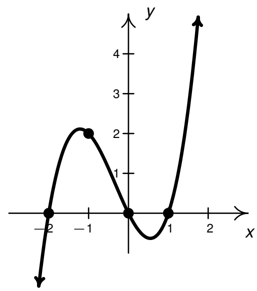
Range: $(-\infty, \infty)$



99. $f(x) = x(x-1)(x+2)$

Domain: $(-\infty, \infty)$

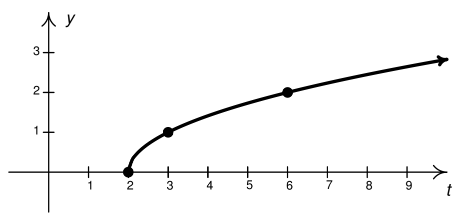
Range: $(-\infty, \infty)$



100. $g(t) = \sqrt{t-2}$

Domain: $[2, \infty)$

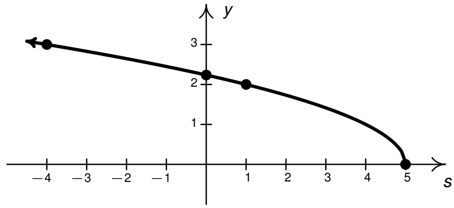
Domain: $[0, \infty)$



101. $h(s) = \sqrt{5 - s}$

Domain: $(-\infty, 5]$

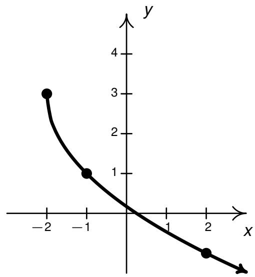
Range: $[0, \infty)$



102. $f(x) = 3 - 2\sqrt{x + 2}$

Domain: $[-2, \infty)$

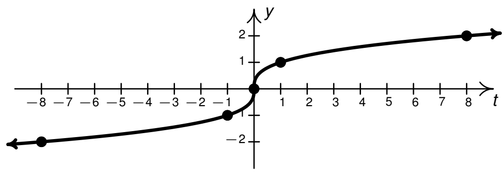
Range: $(-\infty, 3]$



103. $g(t) = \sqrt[3]{t}$

Domain: $(-\infty, \infty)$

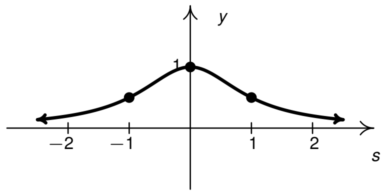
Range: $(-\infty, \infty)$



104. $h(s) = \dfrac{1}{s^2 + 1}$

Domain: $(-\infty, \infty)$

Range: $(0, 1]$

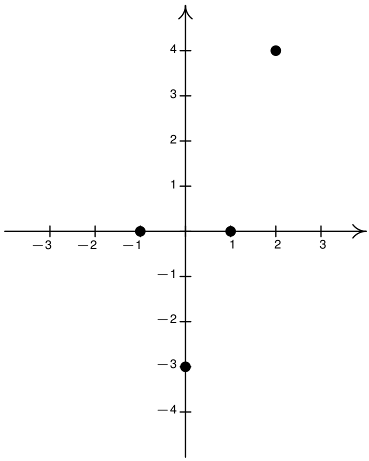


105. (a) domain = $\{-1, 0, 1, 2\}$, range = $\{-3, 0, 4\}$

(b) $f(0) = -3$, $f(x) = 0$ for $x = -1, 1$.

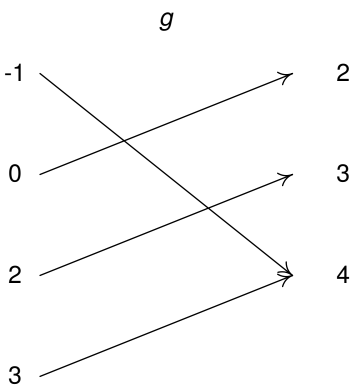
(c) $f = \{(-1, 0), (0, -3), (1, 0), (2, 4)\}$

(d)



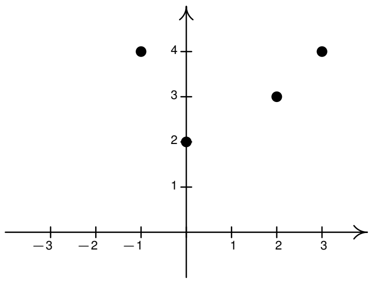
106. (a) domain = $\{-1, 0, 2, 3\}$, range = $\{2, 3, 4\}$

(b)



(c) Find $g(0) = 2$ and $g(x) = 0$ has no solutions.

(d)



107. $F(4) = 4^2 = 16$ (when $t = 4$), the solutions to $F(x) = 4$ are $x = \pm 2$ (when $t = \pm 2$).

108. $G(4) = 7$ (when $t = 2$), the solution to $G(t) = 4$ is $x = -2$ (when $t = -1$)

109. $A(3) = 9$, so the area enclosed by a square with a side of length 3 inches is 9 square inches. The solutions to $A(\ell) = 36$ are $\ell = \pm 6$. Since ℓ is restricted to $\ell > 0$, we only keep $\ell = 6$. This means for the area enclosed

by the square to be 36 square inches, the length of the side needs to be 6 inches. Since ℓ represents a length, $\ell > 0$.

110. $A(2) = 4\pi$, so the area enclosed by a circle with radius 2 meters is 4π square meters. The solutions to $A(r) = 16\pi$ are $r = \pm 4$. Since r is restricted to $r > 0$, we only keep $r = 4$. This means for the area enclosed by the circle to be 16π square meters, the radius needs to be 4 meters. Since r represents a radius (length), $r > 0$.

111. $V(5) = 125$, so the volume enclosed by a cube with a side of length 5 centimeters is 125 cubic centimeters. The solution to $V(s) = 27$ is $s = 3$. This means for the volume enclosed by the cube to be 27 cubic centimeters, the length of the side needs to 3 centimeters. Since x represents a length, $x > 0$.

112. $V(3) = 36\pi$, so the volume enclosed by a sphere with radius 3 feet is 36π cubic feet. The solution to $V(r) = \frac{32\pi}{3}$ is $r = 2$. This means for the volume enclosed by the sphere to be $\frac{32\pi}{3}$ cubic feet, the radius needs to 2 feet. Since r represents a radius (length), $r > 0$.

113. $h(0) = 64$, so at the moment the object is dropped off the building, the object is 64 feet off of the ground. The solutions to $h(t) = 0$ are $t = \pm 2$. Since we restrict $0 \leq t \leq 2$, we only keep $t = 2$. This means 2 seconds after the object is dropped off the building, it is 0 feet off the ground. Said differently, the object hits the ground after 2 seconds. The restriction $0 \leq t \leq 2$ restricts the time to be between the moment the object is released and the moment it hits the ground.

114. $T(0) = 3$, so at 6 AM (0 hours after 6 AM), it is 3° Fahrenheit. $T(6) = 33$, so at noon (6 hours after 6 AM), the temperature is 33° Fahrenheit. $T(12) = 27$, so at 6 PM (12 hours after 6 AM), it is 27° Fahrenheit.

115. $C(0) = 27$, so to make 0 pens, it costs[30] \$2700. $C(2) = 11$, so to make 2000 pens, it costs \$1100. $C(5) = 2$, so to make 5000 pens, it costs \$2000.

116. $E(0) = 16.00$, so in 1980 (0 years after 1980), the average fuel economy of passenger cars in the US was 16.00 miles per gallon. $E(14) = 20.81$, so in 1994 (14 years after 1980), the average fuel economy of passenger cars in the US was 20.81 miles per gallon. $E(28) = 22.64$, so in 2008 (28 years after 1980), the average fuel economy of passenger cars in the US was 22.64 miles per gallon.

[30] This is called the 'fixed' or 'start-up' cost. We'll revisit this concept in Example 1.2.3 in Section 1.2.

117. $P(s) = 4s$, $s > 0$.

118. $C(D) = \pi D$, $D > 0$.

119. (a) The amount in the retirement account after 30 years if the monthly payment is \$50.

 (b) The solution to $A(P) = 250000$ is what the monthly payment needs to be in order to have \$250,000 in the retirement account after 30 years.

 (c) $A(P+50)$ is how much is in the retirement account in 30 years if \$50 is added to the monthly payment P. $A(P)+50$ represents the amount of money in the retirement account after 30 years if \$$P$ is invested each month plus an additional \$50. $A(P) + A(50)$ is the sum of money from two retirement accounts after 30 years: one with monthly payment \$$P$ and one with monthly payment \$50.

120. (a) Since noon is 4 hours after 8 AM, $P(4)$ gives the chance of precipitation at noon.

 (b) We would need to solve $P(t) \geq 50\%$ or $P(t) \geq 0.5$.

121. The graph in question passes the horizontal line test meaning for each w there is only one v. The domain of g is $[0, \infty)$ (which is the range of f) and the range of g is $[2, \infty)$ which is the domain of f.

122. Answers vary.

1.2 Constant and Linear Functions

1.2.1 Constant Functions

Now that we have defined the concept of a function, we'll spend the rest of Chapter 1 revisiting families of curves from prior courses in Algebra by viewing them through a 'function lens'. We start with lines and refer the reader to Section **??** for a review of the basic properties of lines. The simplest lines are vertical and horizontal lines. We leave it to the reader (see Exercise 58) to think about why we eschew vertical lines in our discussion here, and begin with a functional description of horizontal lines.

Consider the horizontal lines graphed in the xy-plane as shown in Figure 1.2.1, Figure 1.2.2 and Figure 1.2.3. The Vertical Line Test, Theorem 1.1.1, tells us that each describes y as a function of x so the question becomes how to represent these functions algebraically. The key here is to remember that the equation relating the independent variable x, the dependent variable y, and the function f is given by $y = f(x)$.

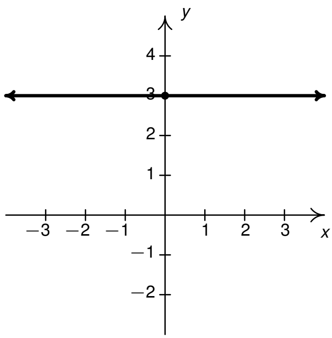


Figure 1.2.1: $y = 3$

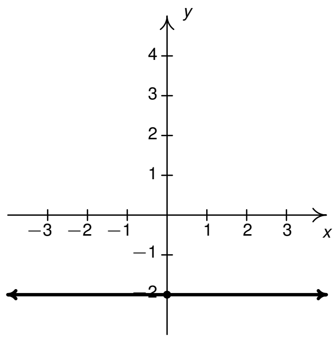


Figure 1.2.2: $y = -2$

In the graph on the left, y always equals 3 so we have $f(x) = 3$. Procedurally, '$f(x) = 3$' says that the rule f takes the input x, and, regardless of that input, gives the output 3. This is an example of what is called a **constant** function - a function which returns the *same* value regardless of the input. Likewise, the function represented by the graph in the middle is $f(x) = -2$, and the graph on the right (the x-axis) is the graph of $f(x) = 0$. In general, we have the following definition:

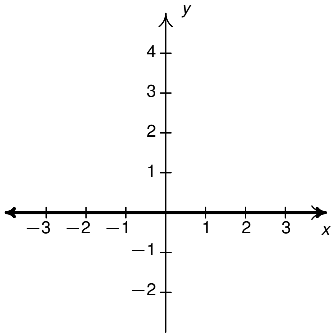


Figure 1.2.3: $y = 0$

Definition 1.2.1. A **constant function** is a function of the form

$$f(x) = b$$

where b is real number. The domain of a constant function is $(-\infty, \infty)$.

Some remarks about Definition 1.2.1 are in order. First, note that we are using 'x' as the independent variable, 'f' as the function name, and the letter 'b' as a **parameter**. In this context, a parameter is a fixed, but arbitrary, constant used to describe a *family* of functions. Different values of b determine different constant functions. For example, $b = 3$ gives $f(x) = 3$, $b = -2$ gives $f(x) = -2$, and so on. Once b is chosen, however, it does not change as the independent variable, x, changes.

Also note that we are using the generic defaults for function names and independent variables, namely f and x, respectively. The functions $G(t) = \sqrt{\pi}$ and $Z(\rho) = 0$ are also fine examples of constant functions. Recall that inherent in the definition of a function is the notion of domain, so we record (as part of the definition) that a constant function has domain $(-\infty, \infty)$. The range of a constant function is the set $\{b\}$. The value b in this case is both the maximum and minimum of f, attained at each value in its domain.[1]

The next example showcases an application of constant functions and introduces the notion of a **piecewise-defined** function.

[1] It gets much weirder than that as we explore other more complicated functions. The key is to pay attention to the precision in the definitions of the terms involved in the discussion. Stay tuned!

Example 1.2.1. The price of admission to see a matinee showing at a local movie theater is a function of the age of the ticket holder. If a person is aged A years, the price per ticket is $p(A)$ dollars and is given by:

$$p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$$

1. Find and interpret $p(3)$, $p(6)$ and $p(62)$.
2. Explain the pricing structure verbally.
3. Graph p.

Solution. The function p described above is an example of a **piecewise-defined** function because the *rule* to determine outputs, not just the value of the output, changes depending on the inputs.

1. To find $p(3)$, we note that the value $A = 3$ satisfies the inequality $0 \leq A < 6$ so we use the rule $p(A) = 5.75$. Hence, $p(3) = 5.75$ which means a ticket for a 3 year old is \$5.75. The next age, $A = 6$, just barely satisfies the inequality $6 \leq A < 50$ so we use the rule $p(A) = 7.25$, This yields $p(6) = 7.25$ which means a ticket for a 6 year old is \$7.25. Lastly, $A = 62$ satisfies the inequality $A \geq 50$, so we are back to the rule $p(A) = 5.75$. Thus $p(62) = 5.75$ which means someone 62 years young gets in for \$5.75.

2. Now that we've had some practice interpreting function values, we can begin to verbalize what the function is really saying. In the first 'piece' of the function, the inequality $0 \leq A < 6$ describes ticket holders under the age of 6 years and the inequality $A \geq 50$ describes ticket holders fifty years old or or older. For folks in these two age demographics, $p(A) = 5.75$ so the price per ticket is \$5.75. For everyone else, that is for folks at least 6 but younger than 50, the price is \$7.25 per ticket.

3. The independent variable here is specified as A, so we'll label our horizontal axis that way as shown in Figure 1.2.4. The dependent variable remains unspecified so we can use the default y. The graph of $y = p(A)$ consists of three horizontal line pieces: the first is $y = 5.75$ for $0 \leq A < 6$, the second piece is $y = 7.25$ for $6 \leq A < 50$, and the last piece is $y = 5.75$ for $A \geq 50$.

   For the first piece, note that $A = 0$ is included in the inequality $0 \leq A < 6$ but $A = 6$ is not. For this reason, we have a point indicated at $(0, 5.75)$ but leave a hole[2] at $(6, 5.75)$. Similarly, to graph the second piece, we

[2]See our discussion about holes in graphs in Example 1.1.6 in Section 1.1.

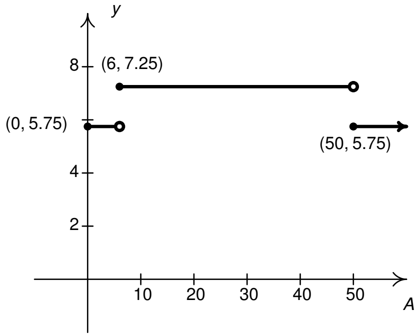


Figure 1.2.4: $y = p(A) = \begin{cases} 5.75 & \text{if } 0 \leq A < 6 \text{ or } A \geq 50 \\ 7.25 & \text{if } 6 \leq A < 50 \end{cases}$

begin with a point at $(6, 7.25)$ and continue the horizontal line to a hole at $(50, 7.25)$. Lastly, we finish the graph with a point at $(50, 5.75)$ and continue to the right indefinitely.[3] Note the scaling on the horizontal axis compared to the vertical axis. □

One of the favorite piecewise-defined functions in mathematical circles is the **greatest integer of *x***, denoted by $\lfloor x \rfloor$. In **??** we defined the set of **integers** as $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.[4] The value $\lfloor x \rfloor$ is defined to be the largest integer k with $k \leq x$. That is, $\lfloor x \rfloor$ is the unique integer k such that $k \leq x < k+1$. Said differently, given any real number x, if x is an integer, then $\lfloor x \rfloor = x$. If not, then x lies in an interval between two integers, k and $k + 1$ and we choose $\lfloor x \rfloor = k$, the left endpoint.

Example 1.2.2. Let $\lfloor x \rfloor$ denote the greatest integer function.

1. Find $\lfloor 0.785 \rfloor$, $\lfloor 117 \rfloor$, $\lfloor -2.001 \rfloor$ and $\lfloor \pi + 6 \rfloor$
2. Explain how we can view $\lfloor x \rfloor$ as a piecewise-defined function and use this to graph $y = \lfloor x \rfloor$.

Solution.

1. To find $\lfloor 0.785 \rfloor$, we note that $0 \leq 0.785 < 1$ so $\lfloor 0.785 \rfloor = 0$. Given that 117 is an integer, we have $\lfloor 117 \rfloor = 117$. To find $\lfloor -2.001 \rfloor$, we note that

[3] The domain of p is $[0, \infty)$ by definition, even though few 327 year olds are out and about these days.

[4] The use of the letter $\mathbb{Z}$ for the integers is ostensibly because the German word *zahlen* means 'to count'.

$$\lfloor x \rfloor = \begin{cases} \vdots & \\ -5 & \text{if } -5 \le x < -4 \\ -4 & \text{if } -4 \le x < -3 \\ -3 & \text{if } -3 \le x < -2 \\ -2 & \text{if } -2 \le x < -1 \\ -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2 \\ 2 & \text{if } 2 \le x < 3 \\ 3 & \text{if } 3 \le x < 4 \\ 4 & \text{if } 4 \le x < 5 \\ 5 & \text{if } 5 \le x < 6 \\ \vdots & \end{cases}$$

Figure 1.2.5: Partial description of $\lfloor x \rfloor$

$-3 \le -2.001 < -2$, so $\lfloor -2.001 \rfloor = -3$. Finally, with $\pi \approx 3.14$, we get $\pi + 6 \approx 9.14$ and $9 \le \pi + 6 < 10$ so $\lfloor \pi + 6 \rfloor = 9$.

2. The first step in evaluating $\lfloor x \rfloor$ is to determine the interval $[k, k+1)$ containing x so it seems reasonable that these are the intervals which produce the 'pieces'. In this case, there happen to be infinitely many pieces. The inequality '$k \le x < k + 1$' includes the left endpoint but excludes the right endpoint, so we have points at the left endpoints of our horizontal line segments while we have holes at the right endpoints.

   A partial description of $\lfloor x \rfloor$ is given in Figure 1.2.5 and a partial graph in Figure 1.2.6. (A full description or a complete graph would require infinitely large paper!) We use the vertical dots $\vdots$ to indicate that both the rule and the graph continue indefinitely following the established pattern.[5]

   □

[5] It is always dangerous to leave the rest of the pattern to the reader. See, for instance, this paper.[6]

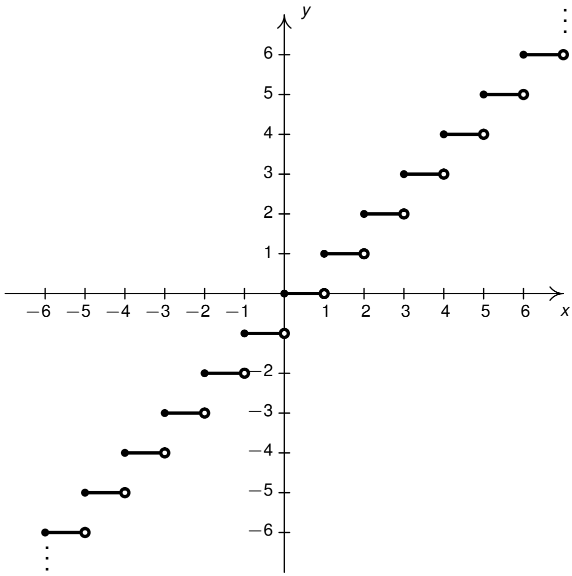


Figure 1.2.6: The graph of $y = \lfloor x \rfloor$

1.2.2 Linear Functions

Now that we've discussed the functions which correspond to horizontal lines, $y = b$, we move to discussing the functions which can be represented by lines of the form $y = mx + b$ where $m \neq 0$. These functions are called **linear** functions and are described below.

Definition 1.2.2. A **linear function** is a function of the form

$$f(x) = mx + b,$$

where m and b are real numbers with $m \neq 0$. The domain of a linear function is $(-\infty, \infty)$.

As with Definition 1.2.1, in Definition 1.2.2, x is the independent variable, f is the function name, and both m and b are parameters. Notice that m is restricted by $m \neq 0$ for if $m = 0$ then the function $f(x) = mx + b$ would reduce to the constant function $f(x) = b$. The domain of linear functions, like that of constant functions, is specified as $(-\infty, \infty)$

Recall[7] that the form of the line $y = mx + b$ is called the slope-intercept form of the line and the slope, m, and the y-intercept $(0, b)$, are easily determined when the line is written this way. Likewise, the form of the function in Definition 1.2.2, $f(x) = mx + b$, is often called the **slope-intercept form** of a linear function.

The graph of a linear function is the graph of the line $y = mx + b$. Lines are uniquely determined by two points, and two points of geometric interest are the axis intercepts. We've already reminded you of the y-intercept, $(0, b)$, which is obtained by setting $x = 0$. Similarly, to find the x-intercept, we set $y = 0$ and solve $mx + b = 0$ for x. We leave this to the reader in Exercise 38. In addition to having special graphical significance, axis intercepts quite often play important roles in applications involving both linear and non-linear functions. For that reason, we take the time to define them here using function notation.

[7] or see Section **??**

Definition 1.2.3. Suppose f is a function represented by the graph of $y = f(x)$.

- If 0 is in the domain of f then the point $(0, f(0))$ is the ***y*-intercept** of the graph of $y = f(x)$.
  That is, $(0, f(0))$ is where the graph meets the y-axis.
- If 0 is in the range of f then the solutions to $f(x) = 0$ are called the **zeros** of f. If c is a zero of f then the point $(c, 0)$ is an ***x*-intercept** of the graph of $y = f(x)$.
  That is, $(c, 0)$ is where the graph meets the x-axis.

As is customary in this text, Definition 1.2.3 uses the default independent variable x, function name f, and dependent variable y, so these letters will change depending on the context. Also note that the 'zeros' of a function are the solutions to $f(x) = 0$ - so they are *real numbers*. The x-intercepts are, on the other hand, *points* on the graph. As a quick example, consider $f(x) = x - 3$. The zeros of f are found by solving $f(x) = 0$, or $x - 3 = 0$. We get one solution, $x = 3$. Therefore, $x = 3$ is the *zero* of f that corresponds graphically to the x-*intercept* $(3, 0)$.

We now turn our attention to slope. The role of slope, or more generally a 'rate of change', in Science and Mathematics cannot be overstated.[8] As you may recall, or quickly read about on page **??**, the slope of a line that has been graphed in the xy-plane is defined geometrically as follows:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x},$$

where the capital Greek letter 'Δ' denotes 'change in.'[9] In this course, it is vital that we regard the slope of a linear function as a rate of change of *function outputs* to *function inputs*. That is, given the graph of a linear function $y = f(x) = mx + b$:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\Delta\text{outputs}}{\Delta\text{inputs}}.$$

What is important to note here is that for linear functions, the rate of change m is constant for all values in the domain.[10] We'll see the importance of this statement in the upcoming examples.

[8] The first half of any introductory Calculus course is about slope.

[9] More specifically, if (x_0, y_0) and (x_1, y_1) are two distinct points in the plane, then $\Delta x = x_1 - x_0$ and $\Delta y = y_1 - y_0$.

[10] See Exercise 57 for more details.

Geometrically, the sign of the slope has a profound impact on the graph of the line. Recall that if the slope $m > 0$, the line rises as we read from left to right; if $m < 0$, the line falls as we read from left to right; if $m = 0$, we have a horizontal line and the graph plateaus. We define these notions more precisely for general functions in the following definition.

Definition 1.2.4. Let f be a function defined on an interval I. Then f is said to be:

- **increasing** on I if, whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of an increasing function **rises** as one moves from left to right.
- **decreasing** on I if, whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a decreasing function **falls** as one moves from left to right.
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.

Again, as with Definition 1.2.3, Definition 1.2.4 applies to any function, not just linear and constant functions. Also, note that, like Definition 1.1.3, Definition 1.2.4 blurs the line between the function, f, and its outputs, $f(x)$, because the verbiage 'f is increasing' is really a statement about the outputs, $f(x)$. Finally, when we ask 'where' a function is increasing, decreasing or constant, we are looking for an interval of *inputs*. We'll have more to say about this in later sections, but for now, we summarize these ideas graphically in Figure 1.2.7, Figure 1.2.8 and Figure 1.2.9.

From the graphs, we see that regardless if $m > 0$ or $m < 0$, the range of linear functions is $(-\infty, \infty)$. Therefore, linear functions have no maximum or minimum.[11]

Example 1.2.3. The cost, in dollars, to produce x PortaBoy[12] game systems for a local retailer is given by $C(x) = 80x + 150$ for $x \geq 0$.

1. Find and interpret $C(0)$ and $C(5)$ and use these to graph $y = C(x)$.

---

[11] This is one of the more pedantic reasons why we distinguish between constant and linear functions. See the discussion concerning the range of a constant function on page 54.

[12] The similarity of this name to PortaJohn[13] is deliberate.

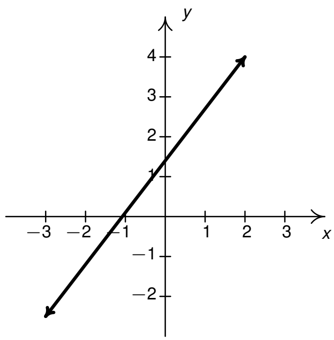


Figure 1.2.7: ‘increasing’, $m > 0$

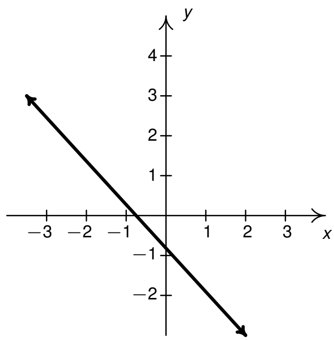


Figure 1.2.8: ‘decreasing’, $m < 0$

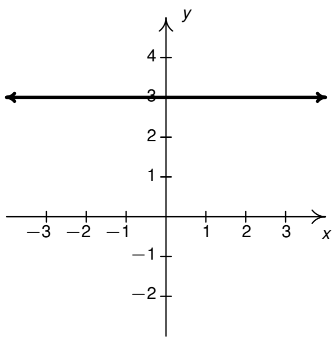


Figure 1.2.9: ‘constant’, $m = 0$

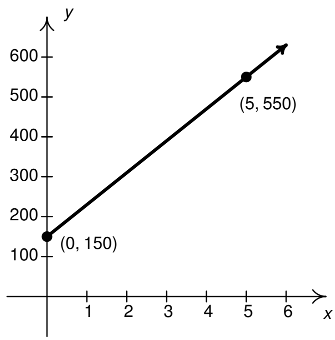


Figure 1.2.10: $y = C(x)$

2. Explain the significance of the restriction on the domain, $x \geq 0$.
3. Interpret the slope of $y = C(x)$ geometrically and as a rate of change.
4. How many PortaBoys can be produced for \$15,000?

Solution.

1. To find $C(0)$, we substitute 0 for x in the formula $C(x)$ and obtain: $C(0) = 80(0) + 150 = 150$. Given that x represents the number of PortaBoys produced and $C(x)$ represents the cost to produce said PortaBoys, $C(0) = 150$ means it costs \$150 even if we don't produce any PortaBoys at all. At first, this may not seem realistic, but that \$150 is often called the **fixed** or **start-up** cost of the venture. Things like re-tooling equipment, leasing space, or any other 'up front' costs get lumped into the fixed cost. To find $C(5)$, we substitute 5 for x in the formula $C(x)$: $C(5) = 80(5) + 150 = 550$. This means it costs \$550 to produce 5 PortaBoys for the local retailer. These two computations give us two points on the graph: $(0, C(0))$ and $(5, C(5))$. Along with the domain restriction $x \geq 0$, we get the graph shown in Figure 1.2.10

2. In this context, x represents the number of PortaBoys produced. It makes no sense to produce a negative quantity of game systems,[14] so $x \geq 0$.

3. The cost function $C(x) = 80x + 150$ is in slope-intercept form so we recognize the slope as the coefficient of x, $m = 80$. With $m > 0$, the function C

[14] Actually, it makes no sense to produce a fractional part of a game system, either, which we'll discuss later in this example.

is always increasing. This means that it costs more money to make more game systems. To interpret the slope as a rate of change, we note that the output, $C(x)$, is the cost in dollars, while the input, x, is the number of PortaBoys produced:

$$m = 80 = \frac{80}{1} = \frac{\Delta[C(x)]}{\Delta x} = \frac{\$80}{1 \text{ PortaBoy produced}}.$$

Hence, the cost to produce PortaBoys is increasing at a rate of \$80 per PortaBoy produced. This is often called the **variable cost** for the venture.

4. To find how many PortaBoys can be produced for \$15,000, we solve $C(x) = 15000$, which means $80x + 150 = 15000$. This yields $x = 185.625$. We can produce only a whole number amount of PortaBoys so we are left with two options: produce 185 or 186 PortaBoys. Given that $C(185) = 14950$ and $C(186) = 15030$, we would be over budget if we produced 186 PortaBoys. Hence, we can produce 185 PortaBoys for \$15,000 (with \$50 to spare). □

A couple of remarks about Example 1.2.3 are in order. First, if x represents the number of PortaBoy game systems being produced, then x can really only take on whole number values. We will revisit this scenario in Section 1.4 where we will see how the approach presented here allows us to use more elegant techniques when analyzing the situation than a discrete data set would allow.[15]

Second, once we know that the variable cost is \$80 per PortaBoy, we can revisit a computation we did earlier in the example. We computed $C(185) = 14950$ and needed to compute $C(186)$. With 186 being just one more PortaBoy than 185, we can use the variable cost to get

$$C(186) = C(185) + 80(1) = 14950 + 80 = 15030,$$

which agrees with our earlier computation.[16] If we wanted to find $C(300)$, we could do something similar. Using $300 - 185 = 115$, we can find $C(300)$ as follows:

$$C(300) = C(185) + 80(115) = 14950 + 9200 = 24150.$$

[15] This is an example of using a 'continuous' variable to model a 'discrete' scenario. Contrast this with the discussion following Example 1.1.1 in Section 1.1.

[16] The cost to produce 'just one more item' is called the **marginal cost**. The difference between variable and marginal costs in this case are the units used: the variable cost is \$80 per Portaboy whereas the marginal cost is simply \$80.

In general, we could rewrite $C(x) = C(185) + 80(x - 115)$. This same reasoning shows that for any x_0 in the domain of C, we have $C(x) = C(x_0) + 80(x - x_0)$ - a fact we invite the reader to verify.[17]

Indeed, the computations above are at the heart of what it means to be a linear function: linear functions change at a constant rate known as the slope. To better see this algebraically, recall that given a point (x_0, y_0) on a line along with the slope, m, the **point-slope form of the line** is: $y - y_0 = m(x - x_0)$.[18] Rewriting, we get $y = y_0 + m(x - x_0)$ and setting $y = f(x)$ and $y_0 = f(x_0)$ yields:

Equation 1.2.1. The **point-slope form** of a linear function is

$$f(x) = f(x_0) + m(x - x_0)$$

A few remarks are in order. First note that if the point $(x_0, f(x_0))$ is the y-intercept $(0, b)$, Equation 1.2.1 immediately reduces to the slope-intercept form of the line: $f(x) = f(x_0) + m(x - x_0) = b + m(x - 0) = mx + b$, so you can use Equation 1.2.1 exclusively from this point forward.[19]

Second, if we write $\Delta x = x - x_0$, then $x = x_0 + \Delta x$ so we can rewrite Equation 1.2.1 as follows:

$$\begin{array}{ccccc} f(x_0 + \Delta x) & = & f(x_0) & + & m\Delta x \\ \text{(new output)} & = & \text{(known output)} & + & \text{(change in outputs)} \end{array}$$

In other words, changing the *input* by Δx results in changing the *output* by $m\Delta x$. This tracks since

$$m\Delta x = \frac{\Delta[f(x)]}{\Delta x}\Delta x = \Delta[f(x)] = \Delta\text{outputs}.$$

The fact that we can write $\Delta\text{outputs} = m\Delta x$ for any choice of x_0 is another way to see that for linear functions, the rate of change is constant. That is, the rate of change, m, is the same for all values x_0 in the domain. We'll put Equation 1.2.1 to good use in the next example.

Example 1.2.4. The local retailer in Example 1.2.3 is trying to mathematically model the relationship between the number of PortaBoy systems sold and the

[17] In the case $x_0 = 0$, this formula reduces to $C(x) = C(0) + 80(x - 0) = 150 + 80x = 80x + 150$. To show the formula in general, consider $C(x_0) = 80x_0 + 150$ . . .

[18] See Section **??** for a review of this form.

[19] In other words, the slope intercept form of a line is just a special case of the point-slope form.

price per system. Suppose 20 systems were sold when the price was \$220 per system but when the systems went on sale for \$190 each, sales doubled.

1. Find a formula for a linear function p which represents the price $p(x)$ as a function of the number of systems sold, x. Graph $y = p(x)$, find and interpret the intercepts, and determine a reasonable domain for p.

2. Interpret the slope of $p(x)$ in terms of price and game system sales.

3. If the retailer wants to sell 150 PortaBoys next week, what should the price be?

4. How many systems would sell if the price per system were set at \$150?

Solution.

1. We are asked to find a linear function $p(x)$ ostensibly because the retailer has only two data points and two points are all that is needed to determine a unique line. We know that 20 PortaBoys were sold when the price was 220 dollars and double that, so 40 units, were sold when the price was 190 dollars. Using the language of function notation, these statements translate to $p(20) = 220$ and $p(40) = 190$, respectively. We first find the slope

$$m = \frac{\Delta[p(x)]}{\Delta x} = \frac{190 - 220}{40 - 20} = \frac{-30}{20} = -1.5$$

and then substitute it and a pair $(x_0, p(x_0))$ into the point-slope formula. We have two choices: $x_0 = 20$ and $p(x_0) = 220$ or $x_0 = 40$ and $p(x_0) = 190$. We'll choose the former and invite the reader to use the latter - both will result in the same simplified expression. The point-slope formula yields

$$p(x) = p(x_0) + m(x - x_0) = 220 + (-1.5)(x - 40)$$

which simplifies to $p(x) = -1.5x + 250$. (To check this algebraically, we can verify that $p(20) = 220$ and $p(40) = 190$.) To find the y-intercept of the graph, we substitute $x = 0$ and find $p(0) = 250$. Hence our y-intercept is $(0, 250)$. To find the x-intercept, we set $p(x) = 0$. Solving $-1.5x + 250 = 0$ gives $x = 166.\overline{6}$, so our x-intercept is $(166.\overline{6}, 0)$.[20] The graph in Figure 1.2.11 is that of the line $y = -1.5x + 250$.

To determine a reasonable domain for p, we certainly require $x \geq 0$, because we can't sell a negative number of game systems.[21] Next, we require

[20] The exact value is $x = \frac{500}{3}$. Recall that the bar over the 6 indicates that the decimal repeats. See page **??** for details.

[21] ignoring returns, that is.

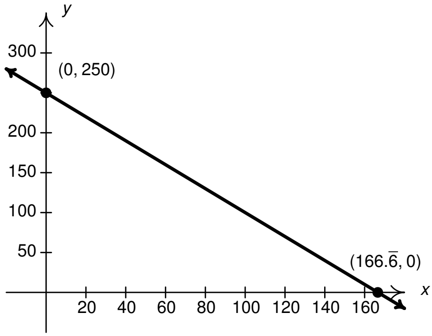


Figure 1.2.11: $y = -1.5x + 250$

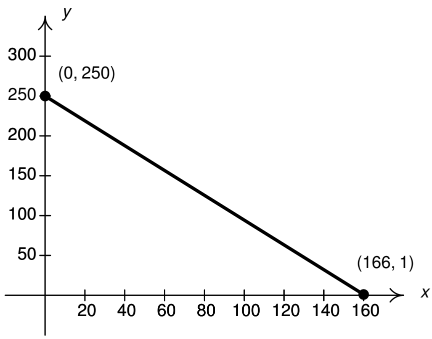


Figure 1.2.12: $y = p(x)$

$p(x) \geq 0$, otherwise we'd be *paying* customers to 'buy' PortaBoys. Solving $-1.5x + 250 \geq 0$ results in $x \leq 166.\overline{6}$. This shouldn't be too surprising since our graph passes through the x-axis at $(166.\overline{6}, 0)$, going from positive y-values (hence, positive $p(x)$ values) to negative y (hence negative $p(x)$ values).[22]

Given that x represents the number of PortaBoys sold, we need to choose to end the domain at either $x = 166$ or $x = 167$. We have that $p(166) = 1 > 0$ but $p(167) = -0.5 < 0$ so we settle on the domain $[0, 166]$. Our final answer is $p(x) = -1.5x + 250$ restricted to $0 \leq x \leq 166$ which is graphed in Figure 1.4.10.

2. The slope $m = -1.5$ represents the rate of change of the price of a system with respect to sales of PortaBoys. The slope is negative so we have that the price is *decreasing* at a rate of \$1.50 per PortaBoy sold. (Said differently, you can sell one more PortaBoy for every \$1.50 drop in price.)

3. To determine the price which will move 150 PortaBoys, we find $p(150) = -1.5(150) + 250 = 25$. That is, the price would have to be \$25 per system.

4. If the price of a PortaBoy were set at \$150, we'd have $p(x) = 150$, or $-1.5x + 250 = 150$. This yields $-1.5x = -100$ or $x = 66.\overline{6}$. Again our algebraic solution lies between two whole numbers, so we find $p(66) = 151$ and $p(67) = 149.5$. If the price were set at \$150, we'd sell 66 systems, since to sell 67 systems, we'd have to drop the price just under \$150. □

The function p in Example 1.2.4 is called the **price-demand** function (or, sometimes called more simply a 'demand function') because it returns the price $p(x)$ associated with a certain demand x - that is, how many products will sell.[23] These functions, along with cost functions like the one in Example 1.2.3, will be revisited in Example 1.4.3.

Our next two examples focus on writing formulas for piecewise-defined functions, the second of which models a real-world situation.

Example 1.2.5. Find a formula for the function L graphed in Figure 1.2.13.

Solution. From the graph of $W = L(t)$ we see that there are two distinct pieces. Taking note of the point at $(1, 4)$, we get $L(t) = 4$ for $t \leq 1$. To represent L for $t > 1$, we use the point-slope form of a linear function: $L(t) = L(t_0) + m(t - t_0)$.

[22] We'll discuss these sorts of connections in greater depth in Section 1.3.

[23] It may seem counter-intuitive to express price as a function of demand. Shouldn't the price determine how many systems people will buy? We will address this issue later.

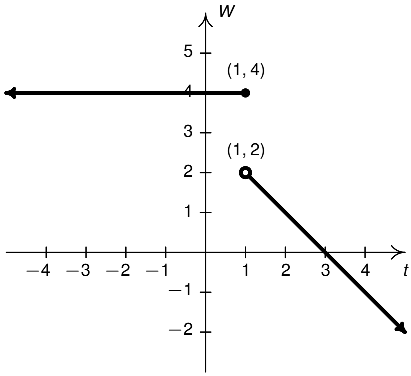


Figure 1.2.13: $W = L(t)$

The only 'point' labeled with this part of the graph is the hole at $(1, 2)$ and it isn't technically part of the graph, so we will avoid using it.[24] Instead, we infer from the graph two other points: $(2, 1)$ and $(3, 0)$. We get the slope to be

$$m = \frac{\Delta W}{\Delta t} = \frac{\Delta[L(t)]}{\Delta t} = \frac{3-2}{0-1} = -1.$$

Next, we choose a point to plug into $L(t) = L(t_0) + m(t - t_0)$. We have two options: $t_0 = 2$ and $L(t_0) = 1$ or $t_0 = 3$ and $L(t_0) = 0$. Using the latter, we get $L(t) = 0 + (-1)(t - 3)$, or $L(t) = -t + 3$. Putting this together with the first part, we get:

$$L(t) = \begin{cases} 4 & \text{if } t \leq 1 \\ -t+3 & \text{if } t > 1 \end{cases}$$

Note that when $t = 1$ is substituted into the expression $-t + 3$, we get 2, so the hole at $(1, 2)$ checks.[25] □

Example 1.2.6. A popular Fōn-i smartphone carrier offers the following smartphone data plan: use any amount of data up to and including 4 gigabytes for \$60 per month with an 'overage' charge of \$5 per gigabyte. Determine a formula that computes the cost in dollars as a function of using g gigabytes of data per month. Graph your answer.

[24]We actually could use the point $(1, 2)$ to find the equation of the *line* containing $(1, 2)$ and, say $(3, 0)$, which is $y = -t + 3$. It's just that the graph of $L(t)$ and the line $y = -t + 3$ only agree for $t > 1$, so it would be incorrect to write $L(1) = 2$.

[25]Alternatively, for t values larger than 1 but getting closer and closer to 1, $L(t) \approx 2$.

Solution. It is clear from context that we are to use the variable g (for 'g'igabytes) as the independent variable. We are asked to compute the cost so it seems natural to name the function C. Hence, we are after a formula for $C(g)$. Knowing that g represents the amount of data used each month, we must have $g \geq 0$. In order to get a feel for the formula for $C(g)$, we can choose some specific values for g and determine the cost, $C(g)$. For example, if we use no data at all, 1 gigabyte of data, or 3.796 gigabytes of data, the cost is the same: \$60. Indeed, per the plan, for any amount of data up to and including 4 gigabytes, the cost is \$60.

Translating this to function notation means $C(0) = 60$, $C(1) = 60$, $C(3.796) = 60$, and, in general, $C(g) = 60$ for $0 \leq g \leq 4$. What happens if we use more than 4 gigabytes? Let's say we use 6 gigabytes. Per the plan, we are charged \$60 for the first 4 and then \$5 for each gigabyte over 4. Using 6 gigabytes means that we are 2 gigabytes over and our overage charge is (\$5)(2) = \$10. The total cost is the base plus the overages or \$60 + \$10 = \$70. In general, if $g > 4$, the expression $(g-4)$ computes the amount of data used over 4 gigabytes. Our base plus overage then comes to: $60 + 5(g - 4) = 5g + 40$. Putting this together with our previous work, we get

$$C(g) = \begin{cases} 60 & \text{if } 0 \leq g \leq 4 \\ 5g + 40 & \text{if } g > 4 \end{cases}$$

To graph C, we graph $y = C(g)$. For $0 \leq g \leq 4$, we have the horizontal line $y = 60$ from $(0, 60)$ to $(4, 60)$. For $g > 4$, we have the line $y = 5g + 40$. Even though the inequality $g > 4$ is strict, we nevertheless substitute $g = 4$ into the formula $y = 5g+40$ and get $y = 60$. Normally, this would produce a hole at $(4, 60)$, but in this case, the point $(4, 60)$ is already on the graph from the first piece of the function. Essentially, the point $(4, 60)$ from $C(g) = 60$ for $0 \leq g \leq 4$ 'plugs' the hole from $C(g) = 5g + 40$ when $g > 4$.

We are graphing a line so we need to plot just one more point to determine the graph. From our work above, we know $C(6) = 70$, so we use $(6, 70)$ as our second point. Our graph is shown in Figure 1.2.14. As with the graphs shown on page **??** from Example 1.1.1, we use '$\asymp$' to denote a break in the vertical axis in order to better display the graph.

□

1.2.3 Linear Regression

We have demonstrated in this section that constant, linear, and piecewise combinations of these two function types can be used to model a variety of phenomena

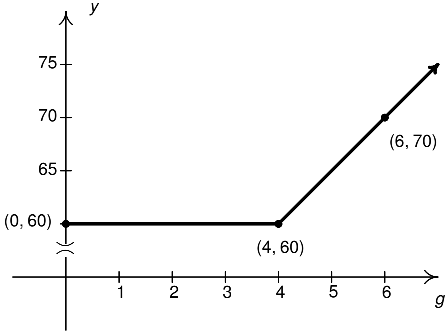


Figure 1.2.14: $y = C(g)$

inspired by real-world situations. What happens, as if often the case in real-world situations, when we are given data sets that are not precisely linear, but still have a definite linear trend? An example of this is Skippy's time and temperature data from Example 1.1.1 in Section 1.1 and shown in Table 1.2.1.

In that example, t represented the time (number of hours after 6 a.m.) and T represented the outdoor temperature in degrees Fahrenheit. The data Skippy collected along with a plot of the function $T = f(t)$ are given below. Even though the data points as t varies from $t = 0$ to $t = 8$ do not all lie on the same line - a fact we could prove analytically by checking slopes - there does appear to be a linear *trend* evident. The same can be said for the data as t varies from $t = 8$ to $t = 12$. As we'll see, there are statistical methods which can produce linear functions that are in some sense 'closest' to all of the data, and they are represented below by the dashed lines in Figure 1.2.15.

How do we measure how 'close' a set of points is to a given line? Let's leave Skippy's data for the moment and focus on a smaller data set. Suppose we collected three data points: $\{(1, 0.5), (3, 2), (4, 3)\}$. At the top of the next page (on the left) we plot these points along with the line $y = 0.5x + 0.5$. The way we measure how close the line is to these points is by computing the **total squared (vertical) error** between the data points and the line as follows. For each of our data points, we find the vertical distance between the point and the line. To accomplish this, we need to find a point on the line directly above or below each data point. In other words, we need a point on the line with the same x-coordinate as our data point.

For example, to find the point on the line directly above $(1, 0.5)$, we plug $x = 1$

t: hours after 6 a.m.	T: temperature $^\circ$F
0	64
2	67
4	75
6	80
8	83
10	83
12	82

Table 1.2.1: Skippy's time and temperature data

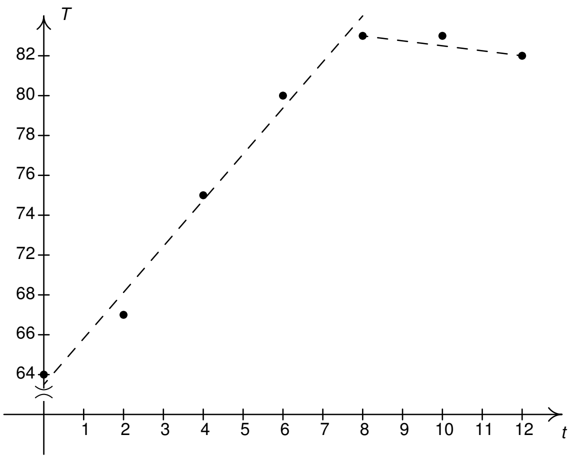


Figure 1.2.15: The graph of $T = f(t)$

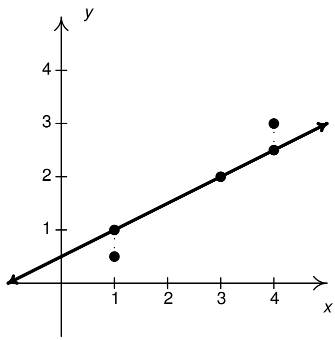


Figure 1.2.16: Least squares regression line

into $y = 0.5x + 0.5$ and we get the point $(1, 1)$. Similarly, we find $(3, 2)$ is on the line already and $(4, 2.5)$ is the point on the line directly beneath $(4, 3)$. We find the total squared error E by taking the sum of the squares of the differences of the y-coordinates of each data point and its corresponding point on the line. For the data and line in this discussion $E = (0.5 - 1)^2 + (2 - 2)^2 + (3 - 2.5)^2 = 0.5$.

Using advanced mathematical machinery,[26] it is possible to find the line which results in the lowest value of E. This line is called the **least squares regression line**, or sometimes the 'line of best fit'. This is shown in Figure 1.2.16. The formula for the line of best fit requires notation we won't present until Chapter **??**, so we will revisit it then. Most graphing utilities have a built-in regression feature, so at this point we turn the computations over to the technology. A screenshot from desmos[27] is given in Figure 1.2.17 the right at the top of the next page.

Our graphing utility produces the model[28] $y = mx + b$ where the slope is $m \approx 0.821$ and the y-coordinate of the y-intercept is $b \approx -0.357$. The value r is the **correlation coefficient** and is a measure of how close the data is to being on the same line. The closer $|r|$ is to 1, the better the linear fit.[29] Having $r \approx 0.997$ tells us that the points have a strong, positive correlation - that is, they are very close to being on a line with a positive slope, namely $y = 0.821x - 0.357$. Indeed, the total

---

[26] like Calculus or Linear Algebra ...

[27] www.desmos.com

[28] We chose to use three decimal places for the approximations in this demonstration. How many you get to use in reality varies from one application to another.

[29] The value r^2 is called the **coefficient of determination** and is also a measure of the goodness of fit. We refer the interested reader to a course in Statistics to explore the significance of r and r^2.

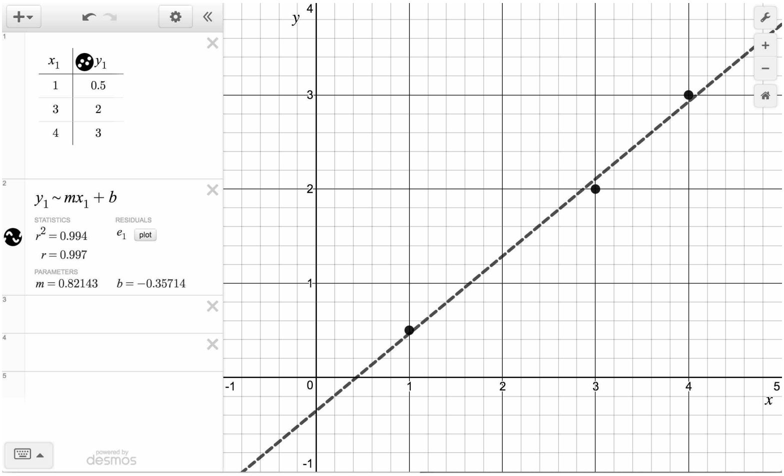


Figure 1.2.17: Screenshot from desmos

t: hours after 6 a.m.	T: temperature °F
0	64
2	67
4	75
6	80
8	83

Table 1.2.2: Dataset 1

squared error between our data set and this line is $E \approx 0.018$. The mathematics tells us that this is the smallest we can get E by modifying the parameters m and b, even though none of the data points actually lie on the line.

Now that we have this new mathematical machinery, let's revisit Skippy's time and temperature data.

Example 1.2.7.

1. Use a graphing utility to find best fit linear models for each of the data sets shown in Table 1.2.2 and Table 1.2.3. Comment on the fit and interpret the slope of each.

t: hours after 6 a.m.	T: temperature °F
8	83
10	83
12	82

Table 1.2.3: Dataset 2

2. Use your models to predict the temperature at 7 a.m. and 3 p.m., rounded to one decimal place.

Solution.

1. For our first set of data, we get the line $T = F(t) = 2.55t + 63.6$. The value $r = 0.987$ tells us that it is a fairly good fit and we see this graphically, too.[30] Thus we can be confident in using this model to predict the temperature during between the hours of 6 a.m. and 2 p.m. with reasonable accuracy.

   To interpret the slope, we recognize t as the independent variable (input) and T as the dependent variable (output), so the slope $m = \frac{\Delta T}{\Delta t}$ is the rate of change of temperature with respect to time. In this case, $m = 2.55$ means that the temperature is increasing (getting warmer) at a rate of 2.55°F per hour. A screenshot from Desmos is given in Figure 1.2.18.

   For the second set of data, we get $T = G(t) = -0.25t + 85.167$ and we have $r = -0.866$. Here, the negative sign on r indicates a negative correlation which means our line has a negative slope.[31] While the fit looks OK, it certainly isn't as strong as with the first data set, so using this model to predict the temperature between 2 p.m. and 6 p.m. (let alone beyond) is a bit risky.

   The slope in this case is $m = -0.25$ which corresponds to the temperature decreasing (getting cooler) at a rate of 0.25°F per hour. That's what a negative correlation means - an increase in input (more time passes) yields a decrease in output (cooler temperatures). As shown in screenshot from Desmos, given in Figure 1.2.19.

2. The time 7 a.m. corresponds to $t = 1$. This falls between $t = 0$ and $t = 8$ so we use our first model. Substituting $t = 1$ gives $T = F(1) = 2.55(1) + 63.6 \approx$

[30]We use F as the name of the function here to distinguish it from f - the function determined solely by the given set of data.

[31]We use G as the function name here to distinguish it from the given function f and the regression for the first data set, F.

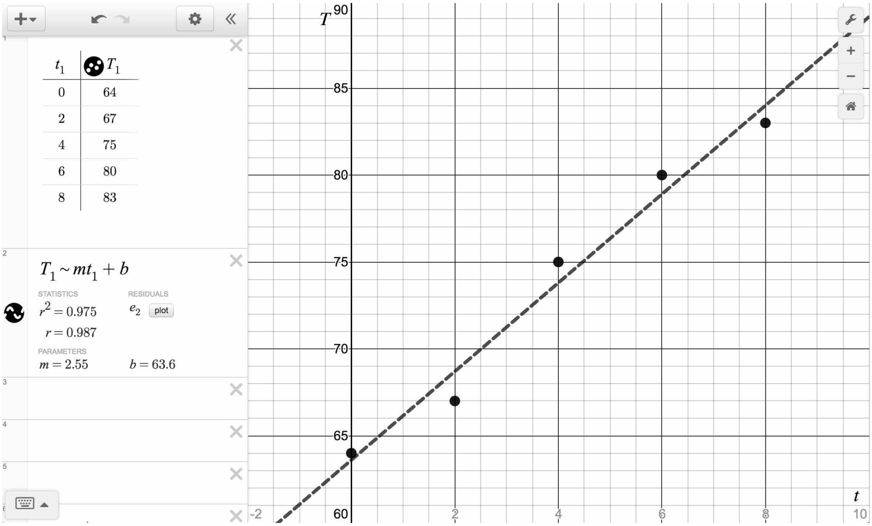


Figure 1.2.18: Screenshot from desmos

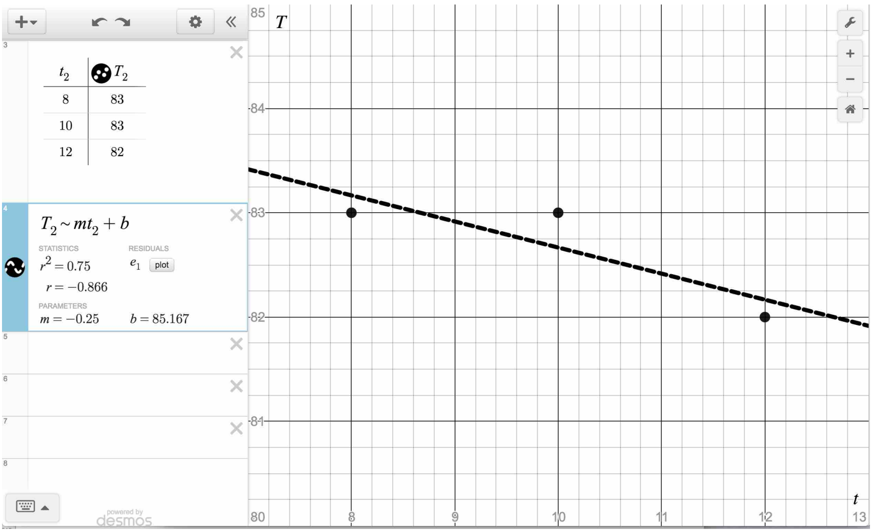


Figure 1.2.19: Screenshot from desmos

66.2. Therefore, the model predicts the temperature to be 66.2°F at 7 a.m.. Likewise, 3 p.m. corresponds to $t = 9$. This is greater than 8, so we use the second model: $T = G(9) = -0.25(9) + 85.167 \approx 82.9$. The model predicts the temperature at 3 p.m. to be 82.9°F. Based on the goodness of fit of each model, we have more confidence in the former prediction than in the latter. □

Examples 1.2.3, 1.2.4 and 1.2.7 (among others) represent three different levels of mathematical modeling. In Example 1.2.3, the mathematical model (the cost function) was provided and our task was to use the model to *interpret* the mathematics in that context. In Example 1.2.4, we were given a minimal amount of information, namely, two data points, and then asked to *construct* a model which fit those data exactly. Lastly, in Example 1.2.7, we were given several data points and we used statistical methods to construct a *best fit* model to the data.

The validity of the models rests on the validity of the underlying assumptions used to create the models. For instance, is there any reason to assume a price-demand function would be linear? Is it reasonable to assume that the temperature changes at a constant rate? These are questions for economists and scientists. Mathematicians often take on a role of equal parts translator and prophet: they codify ideas into formulas and then use them to make predictions about yet-to-be observed phenomena.

1.2.4 The Average Rate of Change of a Function

As mentioned earlier in the section, the concepts of slope and the more general rates of change are important concepts not just in Mathematics, but also in other fields. Many important phenomena are modeled using non-linear functions, and while the rates of change of these functions are not constant, we can sample the function at two points and compute what is known as an **average rate of change** between them to give some sense as to the function's behavior over that interval.[32]

[32] We are basically pretending that the function is linear on a short interval to see what we can say about its behavior.

Definition 1.2.5. Let f be a function defined on the interval $[a, b]$. The **average rate of change** of f over $[a, b]$ is defined as:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Geometrically, the average rate of change is the slope of the line[a] containing $(a, f(a))$ and $(b, f(b))$.

[a]This line is called a *secant* line.

As with Definitions 1.1.3 and 1.2.4, the wording in Definition 1.2.5, while referring to the function f, is really making a statement about its outputs $f(x)$.

If f is increasing over $[a, b]$, then the average rate of change will be positive. Likewise, if f is decreasing or constant, the average rate of change will be negative or 0, respectively. (Think about this for a moment.) However, as the next example demonstrates, the converses of these statements aren't always true.[33]

Example 1.2.8. The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time t, in seconds after lift-off.

1. Find $s(0)$, $s(5)$, $s(10)$, $s(15)$ and $s(20)$ and use these along with a graphing utility to graph $y = s(t)$.
2. State the range of s and interpret the extrema, if any exist.
3. Find and interpret the t- and y-intercepts.
4. Find and interpret the interval(s) over which s is increasing, decreasing or constant.
5. Find and interpret the average rate of change of s over the intervals $[0, 5]$, $[5, 10]$, $[10, 20]$ and $[5, 15]$.

Solution.

1. To find $s(0)$, we substitute $t = 0$ into the formula for $s(t)$: $s(0) = -5(0)^2 + 100(0) = 0$. Similarly, $s(5) = -5(5)^2 + 100(5) = -5(25) + 500 = -125 + 500 = 375$. Continuing, we obtain: $s(10) = 500$, $s(15) = 375$ and $s(20) = 0$. Using these, we construct a table of values and with the help of a graphing utility we obtain the graph shown in Figure 1.2.20:

[33]For example, the average rate of change over an interval could be positive yet the function could decrease over part of that interval and then increase on a different part.

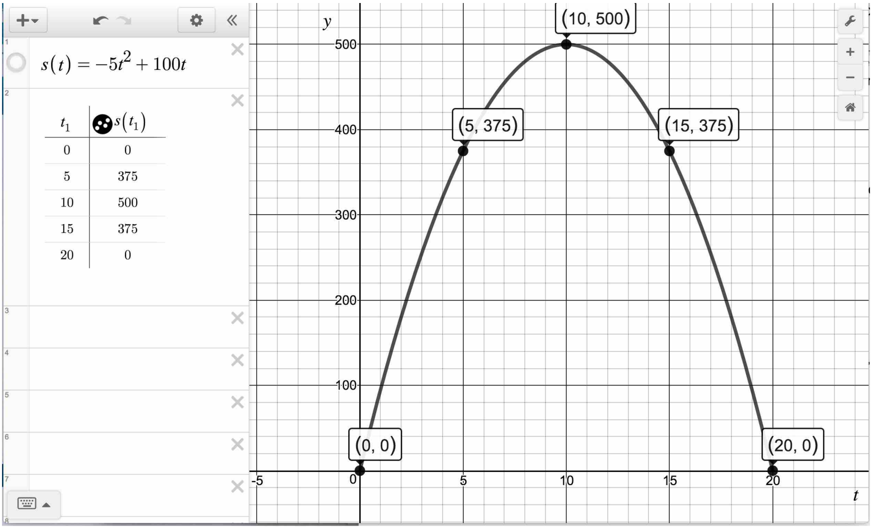


Figure 1.2.20: Screenshot from desmos

2. Projecting the graph to the y-axis, we see that the range of s is $[0, 500]$ so the minimum of s is 0 and the maximum is 500. This means that the rocket at some point is on the surface of the Moon and reaches its highest altitude of 500 feet above the lunar surface.

3. The first intercept we see is $(0, 0)$ which is both a t- and a y-intercept. Given that t represents the time after lift-off and $y = s(t)$ represents the height above the Moon's surface, the point $(0, 0)$ means that the model rocket was launched ($t = 0$) from the Moon's surface ($s(t) = 0$). The remaining intercept, $(20, 0)$, is another t-intercept. This means that 20 seconds after lift-off ($t = 20$), the model rocket returns to the Moon's surface ($s(t) = 0$). Said differently, 20 seconds is the 'time of flight' of the model rocket.

4. Referring to Definition 1.2.4, s increases over the interval $[0, 10]$, since for those values of t, as we read from left to right, the graph of the function is rising meaning the y values (hence $s(t)$ values) are getting larger. Thus the model rocket is heading upwards for the first 10 seconds of its flight. We find that s decreases over the interval $[10, 20]$, indicating once it has reached its highest altitude of 500 feet 10 seconds into the flight, the rocket begins to fall back to the surface of the Moon, landing 20 seconds after

lift-off.

5. To find the average rate of change of s over the interval $[0,5]$ we compute

$$\frac{\Delta[s(t)]}{\Delta t} = \frac{s(5)-s(0)}{5-0} = \frac{375 \text{ feet}}{5 \text{ seconds}} = 75 \text{ feet per second.}$$

In other words, the height is *increasing* at an *average rate* of 75 feet per second during the first 5 seconds of flight. The rate here is called the **average velocity** of the rocket over this interval. Velocity differs from speed in that velocity comes with a direction. In this case, a positive velocity indicates that the rocket is traveling *upwards*, since when s is increasing, the model rocket is climbing higher.

Similarly, the average rate of change of s over the interval $[5,10]$ works out to be 25. This means that the average velocity over the next 5 seconds of the flight has slowed to 25 feet per second. The model rocket is still, on average, traveling upwards, albeit more slowly than before.

Over the interval $[10,20]$, the average rate of change of s works out to be -50. This means that, on average, the rocket is *falling* at a rate of 50 feet per second. The rocket has managed to fall from its highest point 500 feet above the surface of the Moon back to the Moon's surface in 10 seconds so this makes sense. Last, but not least, the average rate of change of s over $[5,15]$ turns out to be 0. This means that the model is the same height above the ground after 5 seconds (375 feet) as it is after 15 seconds.

Geometrically, the average rate of change of a function over an interval can be interpreted as the slope of a secant line. Figure 1.2.21 shows a dotted line containing $(0,0)$ and $(5,375)$ (which has slope 75) along with a dotted line containing the points $(5,375)$ and $(10,500)$ (which has slope 25). Visually, the lines help demonstrate that, while s is increasing over $[0,10]$, the rate of increase is slowing down as t nears 10.

The graph in Figure 1.2.22 depicts a dotted line through $(10,500)$ and $(20,0)$ indicating a net decrease over that interval. We also have a horizontal line (0 slope) containing the points $(5,375)$ and $(15,375)$, which shows no net change between those two points, despite the fact that the rocket rose to its maximum height then began its descent during the interval $[5,15]$.

□

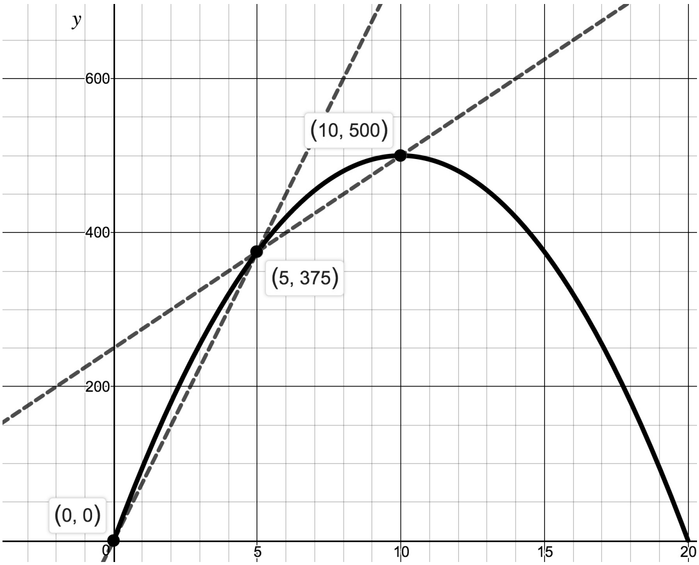


Figure 1.2.21: Rate of increase of s

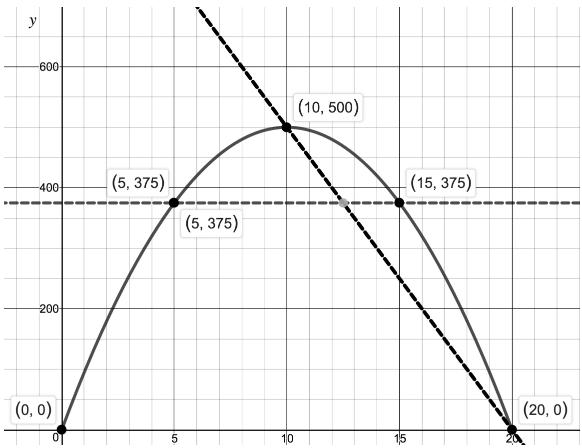


Figure 1.2.22: Rate of decrease of s

An important lesson from the last example is that average rates of change give us a snapshot of what is happening *at the endpoints* of an interval, but not necessarily what happens *over the course* of the interval. Calculus gives us tools to compute slopes *at* points which correspond to *instantaneous* rates of changes. While we don't quite have the machinery to properly express these ideas, we can hint at them in the Exercises. Speaking of exercises . . .

1.2.5 Exercises

In Exercises 1 - 6, graph the function. Find the slope and axis intercepts, if any.

1. $f(x) = 2x - 1$
2. $g(t) = 3 - t$
3. $F(w) = 3$
4. $G(s) = 0$
5. $h(t) = \frac{2}{3}t + \frac{1}{3}$
6. $j(w) = \dfrac{1 - w}{2}$

In Exercises 7 - 10, graph the function. Find the domain, range, and axis intercepts, if any.

7. $f(x) = \begin{cases} 4 - x & \text{if} \quad x \leq 3 \\ 2 & \text{if} \quad x > 3 \end{cases}$
8. $g(x) = \begin{cases} 2 - x & \text{if} \quad x < 2 \\ x - 2 & \text{if} \quad x \geq 2 \end{cases}$
9. $F(t) = \begin{cases} -2t - 4 & \text{if} \quad t < 0 \\ 3t & \text{if} \quad t \geq 0 \end{cases}$
10. $G(t) = \begin{cases} -3 & \text{if} \quad t < 0 \\ 2t - 3 & \text{if} \quad 0 < t < 3 \\ 3 & \text{if} \quad t > 3 \end{cases}$

11. The **unit step function** is defined as $U(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$

    (a) Graph $y = U(t)$.

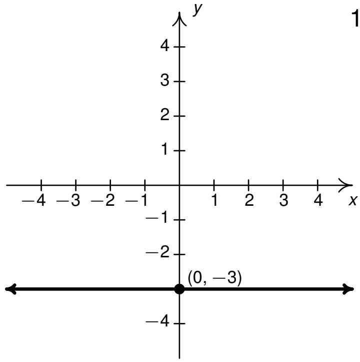
    (b) State the domain and range of U.

    (c) List the interval(s) over which U is increasing, decreasing, and/or constant.

    (d) Write $U(t - 2)$ as a piecewise defined function and graph.

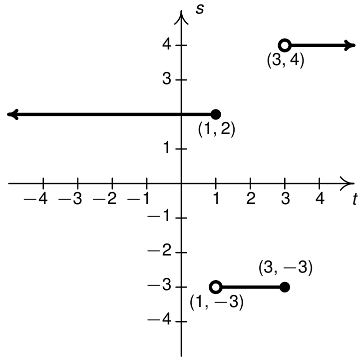
In Exercises 12. - 15., find a formula for the function.

12.



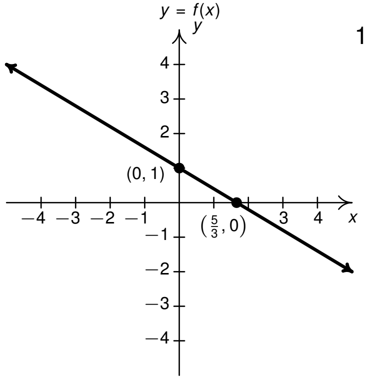
$y = f(x)$

13.



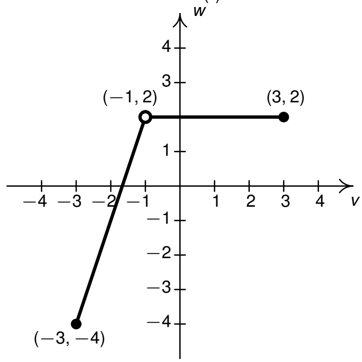
$s = F(t)$

14.



$y = L(x)$

15.



$w = g(v)$

16. For n copies of the book *Me and my Sasquatch*, a print on-demand company charges $C(n)$ dollars, where $C(n)$ is determined by the formula

$$C(n) = \begin{cases} 15n & \text{if } \quad 1 \leq n \leq 25 \\ 13.50n & \text{if } \quad 25 < n \leq 50 \\ 12n & \text{if } \quad n > 50 \end{cases}$$

(a) Find and interpret $C(20)$.

(b) How much does it cost to order 50 copies of the book? What about 51 copies?

(c) Your answer to 16b should get you thinking. Suppose a bookstore estimates it will sell 50 copies of the book. How many books can, in fact, be ordered for the same price as those 50 copies? (Round your answer to a whole number of books.)

17. An on-line comic book retailer charges shipping costs according to the following formula

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if} \quad 1 \leq n \leq 14 \\ 0 & \text{if} \quad n \geq 15 \end{cases}$$

where n is the number of comic books purchased and $S(n)$ is the shipping cost in dollars.

(a) What is the cost to ship 10 comic books?

(b) What is the significance of the formula $S(n) = 0$ for $n \geq 15$?

18. The cost in dollars $C(m)$ to talk m minutes a month on a mobile phone plan is modeled by

$$C(m) = \begin{cases} 25 & \text{if} \quad 0 \leq m \leq 1000 \\ 25 + 0.1(m - 1000) & \text{if} \quad m > 1000 \end{cases}$$

(a) How much does it cost to talk 750 minutes per month with this plan?

(b) How much does it cost to talk 20 hours a month with this plan?

(c) Explain the terms of the plan verbally.

19. Jeff can walk comfortably at 3 miles per hour. Find an expression for a linear function $d(t)$ that represents the total distance Jeff can walk in t hours, assuming he doesn't take any breaks.

20. Carl can stuff 6 envelopes per *minute*. Find an expression for a linear function $E(t)$ that represents the total number of envelopes Carl can stuff after t *hours*, assuming he doesn't take any breaks.

21. A landscaping company charges \$45 per cubic yard of mulch plus a delivery charge of \$20. Find an expression for a linear function $C(x)$ which computes the total cost in dollars to deliver x cubic yards of mulch.

22. A plumber charges \$50 for a service call plus \$80 per hour. If she spends no longer than 8 hours a day at any one site, find an expression for a linear function $C(t)$ that computes her total daily charges in dollars as a function of the amount of time spent in hours, t at any one given location.

23. A salesperson is paid \$200 per week plus 5% commission on her weekly sales of x dollars. Find an expression for a linear function $W(x)$ which computes her total weekly pay in dollars as a function of x. What must her weekly sales be in order for her to earn \$475.00 for the week?

24. An on-demand publisher charges \$22.50 to print a 600 page book and \$15.50 to print a 400 page book. Find an expression for a linear function which models the cost of a book in dollars $C(p)$ as a function of the number of pages p. Find and interpret both the slope of the linear function and $C(0)$.

25. The Topology Taxi Company charges \$2.50 for the first fifth of a mile and \$0.45 for each additional fifth of a mile. Find an expression for a linear function which models the taxi fare $F(m)$ as a function of the number of miles driven, m. Find and interpret both the slope of the linear function and $F(0)$.

26. Water freezes at 0° Celsius and 32° Fahrenheit and it boils at 100°C and 212°F.

    (a) Find an expression for a linear function $F(T)$ that computes temperature in the Fahrenheit scale as a function of the temperature T given in degrees Celsius. Use this function to convert 20°C into Fahrenheit.

    (b) Find an expression for a linear function $C(T)$ that computes temperature in the Celsius scale as a function of the temperature T given in degrees Fahrenheit. Use this function to convert 110°F into Celsius.

    (c) Is there a temperature T such that $F(T) = C(T)$?

27. Legend has it that a bull Sasquatch in rut will howl approximately 9 times per hour when it is $40^\circ F$ outside and only 5 times per hour if it's $70^\circ F$. Assuming that the number of howls per hour, N, can be represented by a linear function of temperature Fahrenheit, find the number of howls per hour he'll make when it's only $20^\circ F$ outside. What troubles do you encounter when trying to determine a reasonable applied domain?

28. Economic forces have changed the cost function for PortaBoys to $C(x) = 105x + 175$. Rework Example 1.2.3 with this new cost function.

29. In response to the economic forces in Exercise 28 above, the local retailer sets the selling price of a PortaBoy at \$250. Remarkably, 30 units were sold each week. When the systems went on sale for \$220, 40 units per week were sold. Rework Example 1.2.4 with this new data.

30. A local pizza store offers medium two-topping pizzas delivered for \$6.00 per pizza plus a \$1.50 delivery charge per order. On weekends, the store runs a 'game day' special: if six or more medium two-topping pizzas are

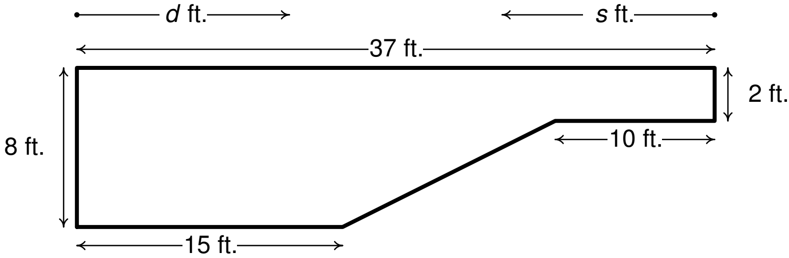


Figure 1.2.23: Cross-secion of swimming pool

ordered, they are \$5.50 each with no delivery charge. Write a piecewise-defined linear function which calculates the cost in dollars $C(p)$ of p medium two-topping pizzas delivered during a weekend.

31. A restaurant offers a buffet which costs \$15 per person. For parties of 10 or more people, a group discount applies, and the cost is \$12.50 per person. Write a piecewise-defined linear function which calculates the total bill $T(n)$ of a party of n people who all choose the buffet.

32. A mobile plan charges a base monthly rate of \$10 for the first 500 minutes of air time plus a charge of 15¢ for each additional minute. Write a piecewise-defined linear function which calculates the monthly cost in dollars $C(m)$ for using m minutes of air time.

    HINT: You may wish to refer to number 18 for inspiration.

33. The local pet shop charges 12¢ per cricket up to 100 crickets, and 10¢ per cricket thereafter. Write a piecewise-defined linear function which calculates the price in dollars $P(c)$ of purchasing c crickets.

34. The cross-section of a swimming pool is shown in Figure 1.2.23. Write a piecewise-defined linear function which describes the depth of the pool, D (in feet) as a function of:

    (a) the distance (in feet) from the edge of the shallow end of the pool, d.

    (b) the distance (in feet) from the edge of the deep end of the pool, s.

    (c) Graph each of the functions in (a) and (b). Discuss with your classmates how to transform one into the other and how they relate to the diagram of the pool.

35. The function defined by $I(x) = x$ is called the Identity Function. Thinking from a procedural perspective, explain a possible origin of this name.

36. Why must the graph of a function $y = f(x)$ have at most one y-intercept?

    HINT: Consider what would happen graphically if there were more than one ...

37. Why is a discussion of vertical lines omitted when discussing functions?

38. Find a formula for the x-intercept of the graph of $f(x) = mx + b$. Assume $m \neq 0$.

39. Suppose $(c, 0)$ is the x-intercept of a linear function f. Use the point-slope form of a liner function, Equation 1.2.1 to show $f(x) = m(x - c)$. This is the 'slope x-intercept' form of the linear function.

40. Prove that for all linear functions L with with slope 3, $L(120) = L(100) + 60$.

41. Find the slopes between the following points from the data set given in Example 1.2.7 and compare them with the slope of the corresponding regression line:

    (a) $(0, 64)$, $(4, 75)$ (b) $(4, 75)$, $(8, 83)$ (c) $(8, 83)$, $(10, 83)$
    (d) $(10, 83)$, $(12, 82)$

42. According to this website[34][35], the census data for Lake County, Ohio is:

Year	1970	1980	1990	2000
Population	197200	212801	215499	227511

    (a) Find the least squares regression line for these data and comment on the goodness of fit.[37] Interpret the slope of the line of best fit.

    (b) Use the regression line to predict the population of Lake County in 2010. (The recorded figure from the 2010 census is 230,041)

    (c) Use the regression line to predict when the population of Lake County will reach 250,000.

[34] http://www.ohiobiz.com/census/Lake.pdf
[35] http://www.ohiobiz.com/census/Lake.pdf[36]
[37] We'll develop more sophisticated models for the growth of populations in Chapter **??**. For the moment, we use a theorem from Calculus to approximate those functions with lines.

43. According to this website[38][39], the census data for Lorain County, Ohio is:

Year	1970	1980	1990	2000
Population	256843	274909	271126	284664

(a) Find the least squares regression line for these data and comment on the goodness of fit. Interpret the slope of the line of best fit.

(b) Use the regression line to predict the population of Lorain County in 2010. (The recorded figure from the 2010 census is 301,356)

(c) Use the regression line to predict when the population of Lake County will reach 325,000.

44. Table 1.2.4 contains a portion of the fuel consumption information for a 2002 Toyota Echo that Jeffrey used to own. The first row is the cumulative number of gallons of gasoline that I had used and the second row is the odometer reading when I refilled the gas tank. So, for example, the fourth entry is the point (28.25, 1051) which says that I had used a total of 28.25 gallons of gasoline when the odometer read 1051 miles.

Find the least squares line for this data. Is it a good fit? What does the slope of the line represent? Do you and your classmates believe this model would have held for ten years had I not crashed the car on the Turnpike a few years ago?

45. Using the energy production data given below

Year	1950	1960	1970	1980	1990	2000
Production (in Quads)	35.6	42.8	63.5	67.2	70.7	71.2

(a) Plot the data using a graphing utility and explain why it does not appear to be linear.

(b) Discuss with your classmates why ignoring the first two data points may be justified from a historical perspective.

(c) Find the least squares regression line for the last four data points and comment on the goodness of fit. Interpret the slope of the line of best fit.

[38] `http://www.ohiobiz.com/census/Lorain.pdf`
[39] http://www.ohiobiz.com/census/Lorain.pdf[40]

Gasoline Used (Gallons)	Odometer (Miles)
0	41
9.26	356
19.03	731
28.25	1051
36.45	1347
44.64	1631
53.57	1966
62.62	2310
71.93	2670
81.69	3030
90.43	3371

Table 1.2.4: Fuel consumption of 2002 Toyota Echo

(d) Use the regression line to predict the annual US energy production in the year 2010.

(e) Use the regression line to predict when the annual US energy production will reach 100 Quads.

In Exercises 46 - 51, compute the average rate of change of the function over the specified interval.

46. $f(x) = x^3$, $[-1, 2]$

47. $g(x) = \dfrac{1}{x}$, $[1, 5]$

48. $f(t) = \sqrt{t}$, $[0, 16]$

49. $g(t) = x^2$, $[-3, 3]$

50. $F(s) = \dfrac{s+4}{s-3}$, $[5, 7]$

51. $G(s) = 3s^2 + 2s - 7$, $[-4, 2]$

52. The height of an object dropped from the roof of a building is modeled by: $h(t) = -16t^2 + 64$, for $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground in feet t seconds after the object is dropped. Find and interpret the average rate of change of h over the interval $[0, 2]$.

53. Using data from Bureau of Transportation Statistics[41], the average fuel econ-

[41] http://www.bts.gov/publications/national_transportation_statistics/html/table_04_23.html

omy $F(t)$ in miles per gallon for passenger cars in the US can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$, where t is the number of years since 1980. Find and interpret the average rate of change of F over the interval $[0, 28]$.

54. The temperature $T(t)$ in degrees Fahrenheit t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

   (a) Find and interpret $T(4)$, $T(8)$ and $T(12)$.

   (b) Find and interpret the average rate of change of T over the interval $[4, 8]$.

   (c) Find and interpret the average rate of change of T from $t = 8$ to $t = 12$.

   (d) Find and interpret the average rate of temperature change between 10 AM and 6 PM.

55. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. Find and interpret the average rate of change as production is increased from making 3000 to 5000 pens.

56. Recall from Example 1.2.8 The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds.

   (a) Find and interpret the average rate of change of s over the following intervals:

      i. $[14.9, 15]$ ii. $[15, 15.1]$ iii. $[14.99, 15]$
      iv. $[15, 15.01]$

   (b) What value does the average rate of change appear to be approaching as the interval shrinks closer to the value $t = 15$?

   (c) Find the equation of the line containing $(15, 375)$ with slope $m = -50$ and graph it along with s on the same set of axes using a graphing utility. What happens as you zoom in near $(15, 375)$?

57. Show the average rate of change of a function of the form $f(x) = mx + b$ over *any* interval is m.

58. Why doesn't the graph of the vertical line $x = b$ in the xy-plane represent y as a function of x?

59. With help from a graphing utility, graph the following pairs of functions on the same set of axes:[42]

   - $f(x) = 2 - x$ and $g(x) = \lfloor 2 - x \rfloor$
   - $f(x) = x^2 - 4$ and $g(x) = \lfloor x^2 - 4 \rfloor$
   - $f(x) = x^3$ and $g(x) = \lfloor x^3 \rfloor$
   - $f(x) = \sqrt{x} - 4$ and $g(x) = \lfloor \sqrt{x} - 4 \rfloor$

   Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = \lfloor f(x) \rfloor$ until you can explain how, in general, one would obtain the graph of $y = \lfloor f(x) \rfloor$ given the graph of $y = f(x)$.

60. The Lagrange Interpolate[43] function L for two points (x_0, y_0) and (x_1, y_1) where $x_0 \neq x_1$ is given by:

$$L(x) = y_0 \frac{x - x_1}{x_0 - x_1} + y_1 \frac{x - x_0}{x_1 - x_0}$$

   (a) For each of the following pairs of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

      i. $(-1, 3)$, $(2, 3)$
      ii. $(-3, -2)$, $(5, -2)$
      iii. $(-3, -2)$, $(0, 1)$
      iv. $(-1, 5)$, $(2, -1)$

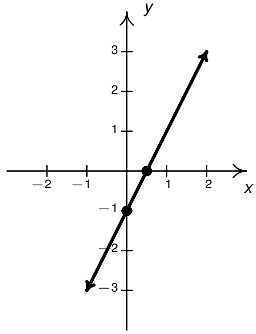
   (b) Verify that, in general, $L(x_0) = y_0$ and $L(x_1) = y_1$.

   (c) Show the point-slope form of a linear function, Equation 1.2.1 is equivalent to the formula given for $L(x)$ after making the identifications: $f(x_0) = y_0$ and $m = \dfrac{y_1 - y_0}{x_1 - x_0}$.

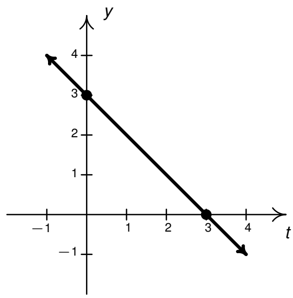
[42] See Example 1.2.2 for the definition of $\lfloor x \rfloor$.
[43] https://en.wikipedia.org/wiki/Lagrange_polynomial

1.2.6 Answers

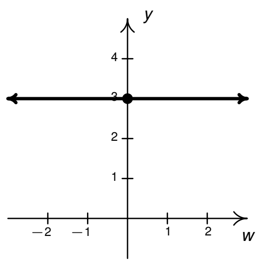
1. $f(x) = 2x - 1$
   slope: $m = 2$
   y-intercept: $(0, -1)$
   x-intercept: $\left(\frac{1}{2}, 0\right)$



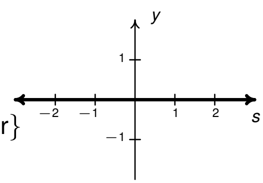
2. $g(t) = 3 - t$
   slope: $m = -1$
   y-intercept: $(0, 3)$
   t-intercept: $(3, 0)$



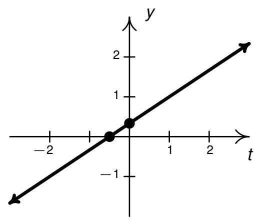
3. $F(w) = 3$
   slope: $m = 0$
   y-intercept: $(0, 3)$
   w-intercept: none



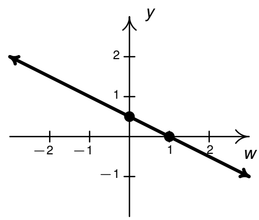
4. $G(s) = 0$
   slope: $m = 0$
   y-intercept: $(0, 0)$
   s-intercept: $\{(s, 0) \mid s \text{ is a real number}\}$



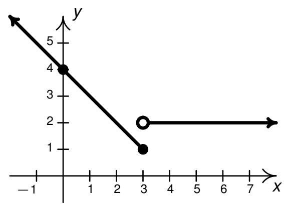
5. $h(t) = \frac{2}{3}x + \frac{1}{3}$
   slope: $m = \frac{2}{3}$
   y-intercept: $\left(0, \frac{1}{3}\right)$
   t-intercept: $\left(-\frac{1}{2}, 0\right)$



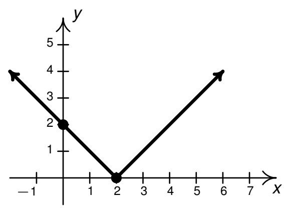
6. $j(w) = \dfrac{1 - w}{2}$
   slope: $m = -\frac{1}{2}$
   y-intercept: $\left(0, \frac{1}{2}\right)$
   w-intercept: $(1, 0)$



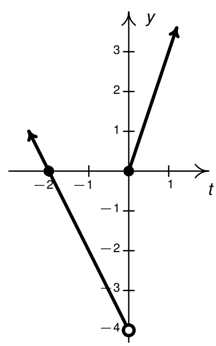
7. domain: $(-\infty, \infty)$
   range: $[1, \infty)$
   y-intercept: $(0, 4)$
   x-intercept: none



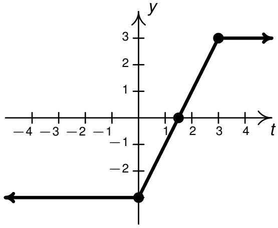
8. domain: $(-\infty, \infty)$
   range: $[0, \infty)$
   y-intercept: $(0, 2)$
   x-intercept: $(2, 0)$



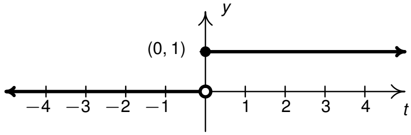
9. domain: $(-\infty, \infty)$
   range: $(-4, \infty)$
   y-intercept: $(0, 0)$
   t-intercepts: $(-2, 0)$, $(0, 0)$



10. domain: $(-\infty, \infty)$
range: $[-3, 3]$
y-intercept: $(0, -3)$
t-intercept: $\left(\frac{3}{2}, 0\right) = (1.5, 0)$



11. (a)

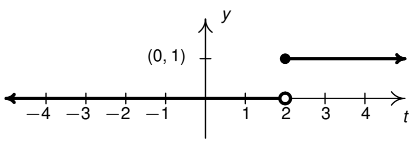


$y = U(t)$

(b) domain: $(-\infty, \infty)$, range: $\{0, 1\}$

(c) U is constant on $(-\infty, 0)$ and $[0, \infty)$.

(d) $U(t-2) = \begin{cases} 0 & \text{if } t < 2, \\ 1 & \text{if } t \geq 2. \end{cases}$



$y = U(t-2)$

12. $f(x) = -3$

13. $F(t) = \begin{cases} 2 & \text{if } t \leq 1, \\ -3 & \text{if } 1 < t \leq 3, \\ 4 & \text{if } t > 3. \end{cases}$

14. $L(x) = -\frac{3}{5}x + 1$

15. $g(v) = \begin{cases} 3v + 5 & \text{if } -3 \leq v < -1, \\ 2 & \text{if } -1 < v \leq 3, \end{cases}$

16. (a) $C(20) = 300$. It costs \$300 for 20 copies of the book.

(b) $C(50) = 675$, \$675. $C(51) = 612$, \$612.

(c) 56 books.

17. (a) $S(10) = 17.5$, \$17.50.

(b) There is free shipping on orders of 15 or more comic books.

18. (a) $C(750) = 25$, \$25.

(b) $C(1200) = 45$, \$45.

(c) It costs \$25 for up to 1000 minutes and 10 cents per minute for each minute over 1000 minutes.

19. $d(t) = 3t$, $t \geq 0$.
20. $E(t) = 360t$, $t \geq 0$.
21. $C(x) = 45x + 20$, $x \geq 0$.
22. $C(t) = 80t + 50$, $0 \leq t \leq 8$.

23. $W(x) = 200 + .05x$, $x \geq 0$ She must make \$5500 in weekly sales.

24. $C(p) = 0.035p + 1.5$ The slope 0.035 means it costs 3.5¢ per page. $C(0) = 1.5$ means there is a fixed, or start-up, cost of \$1.50 to make each book.

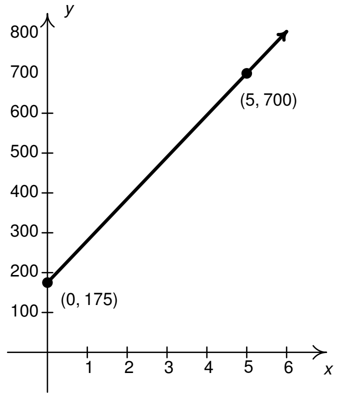
25. $F(m) = 2.25m + 2.05$ The slope 2.25 means it costs an additional \$2.25 for each mile beyond the first 0.2 miles. $F(0) = 2.05$, so according to the model, it would cost \$2.05 for a trip of 0 miles. Would this ever really happen? Depends on the driver and the passenger, we suppose.

26. (a) $F(T) = \frac{9}{5}T + 32$ (b) $C(T) = \frac{5}{9}(T - 32) = \frac{5}{9}T - \frac{160}{9}$
(c) $F(-40) = -40 = C(-40)$.

27. $N(T) = -\frac{2}{15}T + \frac{43}{3}$ and $N(20) = \frac{35}{3} \approx 12$ howls per hour.

Having a negative number of howls makes no sense and since $N(107.5) = 0$ we can put an upper bound of $107.5^\circ F$ on the domain. The lower bound is trickier because there's nothing other than common sense to go on. As it gets colder, he howls more often. At some point it will either be so cold that he freezes to death or he's howling non-stop. So we're going to say that he can withstand temperatures no lower than $-42^\circ F$ so that the applied domain is $[-42, 107.5]$.

28. (a) $C(0) = 175$, so our start-up costs are \$175. $C(5) = 700$, so to produce 5 systems, it costs \$700.



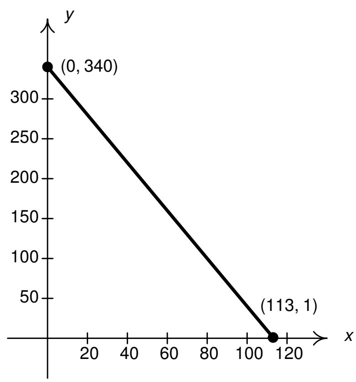
$y = C(x)$

(b) Since we can't make a negative number of game systems, $x \geq 0$.

(c) The slope is $m = 105$ so for each additional system produced, it costs an additional \$105.

(d) Solving $C(x) = 15000$ gives $x \approx 141.19$ so 141 can be produced for \$15,000.

29. (a) $p(x) = -3x + 340$, $0 \leq x \leq 113$.



$y = p(x)$

(b) The slope is $m = -3$ so for each \$3 drop in price, we sell one additional game system.

(c) Since $x = 150$ is not in the domain of p, $p(150)$ is not defined. (In other words, under these conditions, it is impossible to sell 150 game systems.)

(d) Solving $p(x) = 150$ gives $x \approx 63.33$ so if the price \$150 per system, we would sell 63 systems.

30. $C(p) = \begin{cases} 6p + 1.5 & \text{if} \quad 1 \leq p \leq 5 \\ 5.5p & \text{if} \quad p \geq 6 \end{cases}$

31. $T(n) = \begin{cases} 15n & \text{if} \quad 1 \leq n \leq 9 \\ 12.5n & \text{if} \quad n \geq 10 \end{cases}$

32. $C(m) = \begin{cases} 10 & \text{if} \quad 0 \leq m \leq 500 \\ 10 + 0.15(m - 500) & \text{if} \quad m > 500 \end{cases}$

33. $P(c) = \begin{cases} 0.12c & \text{if} \quad 1 \leq c \leq 100 \\ 12 + 0.1(c - 100) & \text{if} \quad c > 100 \end{cases}$

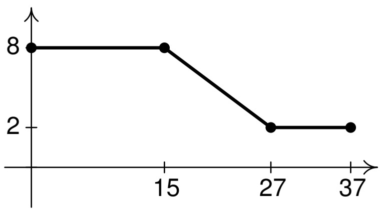
34. (a)

$$D(d) = \begin{cases} 8 & \text{if} \quad 0 \leq d \leq 15 \\ -\frac{1}{2}\,d + \frac{31}{2} & \text{if} \quad 15 \leq d \leq 27 \\ 2 & \text{if} \quad 27 \leq d \leq 37 \end{cases}$$

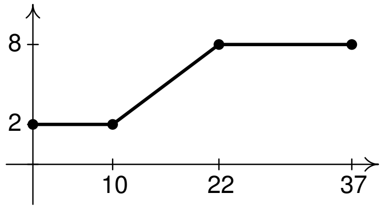
(b)

$$D(s) = \begin{cases} 2 & \text{if} \quad 0 \leq s \leq 10 \\ \frac{1}{2}\,s - 3 & \text{if} \quad 10 \leq s \leq 22 \\ 8 & \text{if} \quad 22 \leq s \leq 37 \end{cases}$$

(c)



$y = D(d)$



$y = D(s)$

35. Since $I(x) = x$ for all real numbers x, the function I doesn't change the 'identity' of the input at all.

36. If a graph contains more than one y-intercept, it would violate the Vertical Line Test since $x = 0$ would be matched with (at least) two different y-values.

37. Vertical Lines fail the Vertical Line Test.

38. $\left(-\frac{b}{m}, 0\right)$. (Note the importance here of $m \neq 0$.)

39. Plugging in $(c, 0)$ for $(x_0, f(x_0))$, we get $f(x) = f(x_0) + m(x - x_0) = 0 + m(x - c)$ or $f(x) = m(x - c)$.

40. Since L is linear with slope 3, $L(x) = L(x_0) + m\Delta x = L(100) + (3)(120 - 100) = L(100) + 60$.

41. (a) $m = \frac{75-64}{4-0} = 2.75$ (b) $m = \frac{83-75}{8-4} = 2$
(c) $m = \frac{83-83}{10-8} = 0$ (d) $m = \frac{82-83}{12-10} = -0.5$

The first two points contributed to a regression line slope of $m = 2.55$; the last two points contributed to a regression line slope of $m = -0.25$.

42. (a) $y = 936.31x - 1645322.6$ with $r = 0.9696$ which indicates a good fit. The slope 936.31 indicates Lake County's population is increasing at a rate of (approximately) 936 people per year.

(b) According to the model, the population in 2010 will be 236,660.

(c) According to the model, the population of Lake County will reach 250,000 sometime between 2024 and 2025.

43. (a) $y = 796.8x - 1309762.5$ with $r = 0.8916$ which indicates a reasonable fit. The slope 796.8 indicates Lorain County's population is increasing at a rate of (approximately) 797 people per year.

(b) According to the model, the population in 2010 will be 291,805.

(c) According to the model, the population of Lake County will reach 325,000 sometime between 2051 and 2052.

44. The regression line is $y = 36.8x + 16.39$ with $r = .99987$, so this is an excellent fit. The slope 36.8 represents mileage in miles per gallon.

45. (c) $y = 0.266x - 459.86$ with $r = 0.9607$ which indicates a good fit. The slope 0.266 indicates the country's energy production is increasing at a rate of 0.266 Quad per year.

(d) According to the model, the production in 2010 will be 74.8 Quad.

(e) According to the model, the production will reach 100 Quad in the year 2105.

46. $\frac{2^3-(-1)^3}{2-(-1)} = 3$ 47. $\frac{\frac{1}{5}-\frac{1}{1}}{5-1} = -\frac{1}{5}$ 48. $\frac{\sqrt{16}-\sqrt{0}}{16-0} = \frac{1}{4}$

49. $\frac{3^2-(-3)^2}{3-(-3)} = 0$ 50. $\dfrac{\frac{7+4}{7-3} - \frac{5+4}{5-3}}{7-5} = -\frac{7}{8}$

51. $\frac{(3(2)^2+2(2)-7)-(3(-4)^2+2(-4)-7)}{2-(-4)} = -4$

52. The average rate of change is $\frac{h(2)-h(0)}{2-0} = -32$. During the first two seconds after it is dropped, the object has fallen at an average rate of 32 feet per second.

53. The average rate of change is $\frac{F(28)-F(0)}{28-0} = 0.2372$. From 1980 to 2008, the average fuel economy of passenger cars in the US increased, on average, at a rate of 0.2372 miles per gallon per year.

54. (a) $T(4) = 56$, so at 10 AM (4 hours after 6 AM), it is 56°F. $T(8) = 64$, so at 2 PM (8 hours after 6 AM), it is 64°F. $T(12) = 56$, so at 6 PM (12 hours after 6 AM), it is 56°F.

(b) The average rate of change is $\frac{T(8)-T(4)}{8-4} = 2$. Between 10 AM and 2 PM, the temperature increases, on average, at a rate of 2°F per hour.

(c) The average rate of change is $\frac{T(12)-T(8)}{12-8} = -2$. Between 2 PM and 6 PM, the temperature decreases, on average, at a rate of 2°F per hour.

(d) The average rate of change is $\frac{T(12)-T(4)}{12-4} = 0$. Between 10 AM and 6 PM, the temperature, on average, remains constant.

55. The average rate of change is $\frac{C(5)-C(3)}{5-3} = -2$. As production is increased from 3000 to 5000 pens, the cost decreases at an average rate of \$200 per 1000 pens produced (20¢ per pen.)

56. (a) i. -49.5 so the average velocity of the rocket between 14.9 and 15 seconds after lift off is -49.5 feet per second (49.5 feet per second directed *downwards*.)

ii. -50.5 so the average velocity of the rocket between 14 and 15.1 seconds after lift off is -50.5 feet per second. (50.5 feet per second directed *downwards*.)

iii. -49.95 so the average velocity of the rocket between 14.99 and 15 seconds after lift off is -49.95 feet per second. (49.95 feet per second directed *downwards*.)

iv. -50.05 so the average velocity of the rocket between 15.01 and 15 seconds after lift off is -50.05 feet per second. (50.05 feet per second directed *downwards*.)

(b) The average rate of change seem to be approaching -50.

(c) Line: $y = -50(t - 15) + 375$ or $y = -50t + 1125$. Graphing this line along with the s on a graphing utility we find the two graphs become indistinguishable as we zoom in near $(15, 375)$.

60. (a) i. $L(x) = 3$ ii. $L(x) = -2$ iii. $L(x) = x + 1$
iv. $L(x) = -2x + 3$

1.3 Absolute Value Functions

1.3.1 Graphs of Absolute Value Functions

In Section 1.2, we revisited lines in a function context. In this section, we revisit the absolute value in a similar manner, so it may be useful to refresh yourself with the basics in Section **??**. Recall that the absolute value of a real number x, denoted $|x|$, can be defined as the distance from x to 0 on the real number line.[1] This definition is very useful for several applications, and lends itself well to solving equations and inequalities such as $|x-2|+1=5$ or $2|t+1|>4$.

We now wish to explore solving more complicated equations and inequalities, such as $|x-2|+1=x$ and $2|t+1|\geq t+4$. We'll approach these types of problems from a function standpoint and use the interplay between the graphical and analytical representations of these functions to obtain solutions. The key to this section is understanding the absolute value from that function (or procedural) standpoint.

Consider a real number $x \geq 0$ such as $x=0$, $x=\pi$ or $x=117.42$. When computing absolute values, we find $|0|=0$, $|\pi|=\pi$ and $|117.42|=117.42$. In general, if $x \geq 0$, the absolute value function does nothing to change the input, so $|x|=x$. On the other hand, if $x<0$, say $x=-1$, $x=-\sqrt{42}$ or $x=-117.42$, we get $|-1|=1$, $|-\sqrt{42}|=\sqrt{42}$ and $|-117.42|=117.42$. That is, if $x<0$, $|x|$ returns the exact *opposite* of the input x, so $|x|=-x$.

Putting these two observations together, we have the following.

Definition 1.3.1. The **absolute value** of a real number x, denoted $|x|$, is given by

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

In Definition 1.3.1, it is *absolutely* essential to read '$-x$' as 'the *opposite* of x' as *opposed* to 'negative x' in order to avoid serious errors later. To see that this description agrees with our previous experience, consider $|117.42|$. Given that $117.42 \geq 0$, we use the rule $|x|=x$. Hence, $|117.42|=117.42$. Likewise, $|0|=0$. To compute $|-\sqrt{42}|$, we note that $-\sqrt{42}<0$ we use the rule $|x|=-x$ in this case. We get $|-\sqrt{42}| = -(-\sqrt{42})$ (the opposite of $-\sqrt{42}$), so $|-\sqrt{42}| = -(-\sqrt{42}) = \sqrt{42}$.

[1] More generally, $|x-c|$ is the distance from x to c on the number line.

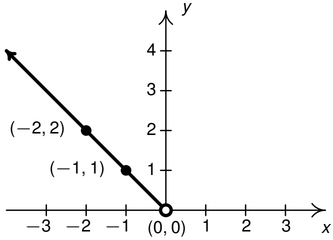


Figure 1.3.1: $f(x) = |x|$, $x < 0$

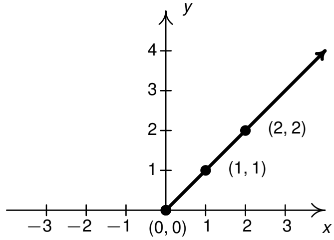


Figure 1.3.2: $f(x) = |x|$, $x \geq 0$

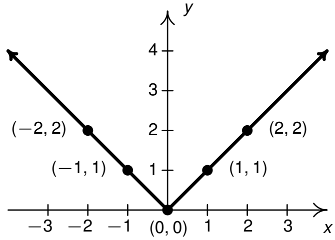


Figure 1.3.3: $f(x) = |x|$

Another way to view Definition 1.3.1 is to think of $-x = (-1)x$ and $x = (1)x$. That is, $|x|$ multiplies negative inputs by -1 and non-negative inputs by 1. This viewpoint is especially useful in graphing $f(x) = |x|$. For $x < 0$, $|x| = (-1)x$, so the graph of $y = |x|$ is the graph of $y = -x = (-1)x$: a line with slope -1 and y-intercept $(0, 0)$. Likewise, for $x \geq 0$, $|x| = x$, so the graph of $y = |x|$ is the graph of $y = x = (1)x$: a line with slope 1 and y-intercept $(0, 0)$.

We graph each piece in Figure 1.3.1 and Figure 1.3.2 and then put them together in Figure 1.3.3. Note that when graphing $f(x) = |x|$ for $x < 0$, we have a hole at $(0, 0)$ because the inequality $x < 0$ is strict. However, the point $(0, 0)$ is included in the graph of $f(x) = |x|$ for $x \geq 0$, so there is no hole in our final graph.

The graph of $f(x) = |x|$ is a very distinctive '$\vee$' shape and is worth remembering. The point $(0, 0)$ on the graph is called the **vertex**. This terminology makes sense from a geometric viewpoint because $(0, 0)$ is the point where two lines meet to form an angle. We will also see this term used in Section 1.4 where, more generally, it corresponds to the graphical location of the sole maximum or minimum of a quadratic function.

We put Definition 1.3.1 to good use in the next example and review the basics of graphing along the way.

Example 1.3.1. For each of the functions below, analytically find the zeros of the function and the axis intercepts of the graph, if any exist. Rewrite the function using Definition 1.3.1 as a piecewise-defined function and sketch its graph. From the graph, determine the vertex, find the range of the function and any extrema, and then list the intervals over which the function is increasing, decreasing or constant.

1.	$f(x) = \lvert x-3 \rvert$	2.	$g(t) = \lvert t \rvert - 3$
3.	$h(u) = \lvert 2u-1 \rvert - 3$	4.	$i(w) = 4 - 2\lvert 3w-1 \rvert$

Solution. In what follows below, we will be doing quite a bit of substitution. As we have mentioned before, when substituting one expression in for another, the use of parentheses or other grouping symbols is highly recommended. Also, the dependent variable wasn't specified so we use the default y in each case.

1. To find the zeros of f, we solve $f(x) = 0$ or $\lvert x-3 \rvert = 0$. We get $x = 3$ so the sole x-intercept of the graph of f is $(3,0)$. To find the y-intercept, we compute $f(0) = \lvert 0-3 \rvert = 3$ and obtain $(0,3)$. Using Definition 1.3.1 to rewrite the expression for $f(x)$ means that we substitute the expression $x-3$ in for x and simplify. Note that when substituting the $x-3$ in for x, we do so for *every* instance of x – both in the formula (output) as well as the inequality (input).

$$f(x) = \lvert x-3 \rvert = \begin{cases} -(x-3) & \text{if } (x-3) < 0 \\ (x-3) & \text{if } (x-3) \geq 0 \end{cases}$$

$$\longrightarrow f(x) = \begin{cases} -x+3 & \text{if } x < 3 \\ x-3 & \text{if } x \geq 3 \end{cases}$$

As both pieces of the graph of f are lines, we need just two points for each piece. We already have two points for the graph: $(0,3)$ and $(3,0)$. These two points both lie on the line $y = -x+3$ but the strictness of the inequality means $f(x) = -x+3$ only for $x < 3$, not $x = 3$, so we would have a hole at $(3,0)$ instead of a point there. For $x \geq 3$, $f(x) = x-3$, so the hole we thought we had at $(3,0)$ gets plugged because $f(3) = 3-3 = 0$. We need just one more point for $f(x)$ where $x \geq 3$ and choose somewhat arbitrarily $x = 6$. We find $f(6) = \lvert 6-3 \rvert = 3$ so our final point on the graph is $(6,3)$. Now that we have a complete graph,[2] we see that the vertex is

[2] We know it's complete because we did the Math - no trusting technology on this example!

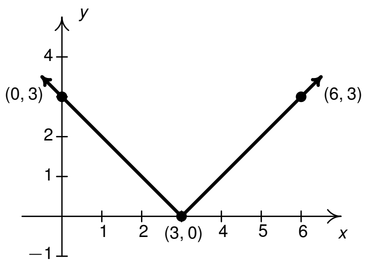


Figure 1.3.4: $f(x) = |x - 3|$

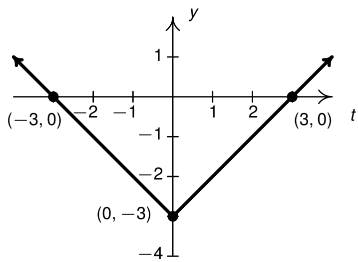


Figure 1.3.5: $g(t) = |t| - 3$

$(3, 0)$ and the range is $[0, \infty)$. The minimum of f is 0 when $x = 3$ and f has no maximum. Also, f is decreasing over $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is given in Figure 1.3.4.

2. To find the zeros of g, we solve $g(t) = |t| - 3 = 0$ and get $|t| = 3$ or $t = \pm 3$. Hence, the t-intercepts of the graph of g are $(-3, 0)$ and $(3, 0)$. To find the y-intercept, we compute $g(0) = |0| - 3 = -3$ and get $(0, -3)$. To rewrite $g(t)$ has a piecewise defined function, we first substitute t in for x in Definition 1.3.1 to get a piecewise definition of $|t|$. This breaks the domain into two pieces: $t < 0$ and $t \geq 0$. For $t < 0$, $|t| = -t$, so $g(t) = |t| - 3 = (-t) - 3 = -t - 3$. Likewise, for $t \geq 0$, $|t| = t$ so $g(t) = |t| - 3 = t - 3$.

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases}$$

$$\longrightarrow g(t) = |t| - 3 = \begin{cases} -t - 3 & \text{if } t < 0 \\ t - 3 & \text{if } t \geq 0 \end{cases}$$

Once again, we have two lines to graph, but in this case we have three points: $(-3, 0)$, $(0, -3)$ and $(3, 0)$. Both $(-3, 0)$ and $(0, -3)$ lie on $y = -t - 3$, but $g(t) = -t - 3$ only for $t < 0$. This would yield a hole at $(0, -3)$, but, just like in the previous example, the hole is plugged thanks to the second piece of the function because $g(0) = 0 - 3 = -3$. We also pick up the second t-intercept, $(3, 0)$ and this helps us complete our graph. We see that the vertex is $(0, -3)$ and the range is $[-3, \infty)$. The minimum of g is -3 at $t = 0$ and there is no maximum. Also, g is decreasing on $(-\infty, 0]$ and increasing on $[0, \infty)$. The graph of g is shown in Figure 1.3.5.

3. Solving $h(u) = |2u-1|-3 = 0$ gives $|2u-1| = 3$ or $2u-1 = \pm 3$. We get two zeros: $u = -1$ and $u = 2$ which correspond to two u-intercepts: $(-1, 0)$ and $(2, 0)$. We find $h(0) = |2(0)-1|-3 = -2$ so our y-intercept is $(0, -2)$. To rewrite $h(u)$ as a piecewise defined function, we first rewrite $|2u-1|$ as a piecewise function. Substituting the expression $2u-1$ in for x in Definition 1.3.1 gives:

$$|2u-1| = \begin{cases} -(2u-1) & \text{if } 2u-1 < 0 \\ 2u-1 & \text{if } 2u-1 \geq 0 \end{cases}$$

$$\longrightarrow |2u-1| = \begin{cases} -2u+1 & \text{if } u < \frac{1}{2} \\ 2u-1 & \text{if } u \geq \frac{1}{2} \end{cases}$$

Hence, for $u < \frac{1}{2}$, $|2u-1| = -2u+1$ so $h(u) = |2u-1|-3 = (-2u+1)-3 = -2u-2$. Likewise, for $u \geq \frac{1}{2}$, $|2u-1| = 2u-1$ so $h(u) = |2u-1|-3 = (2u-1)-3 = 2u-4$.

$$h(u) = |2u-1|-3 = \begin{cases} (-2u+1)-3 & \text{if } u < \frac{1}{2} \\ (2u-1)-3 & \text{if } u \geq \frac{1}{2} \end{cases}$$

$$\longrightarrow h(u) = \begin{cases} -2u-2 & \text{if } u < \frac{1}{2} \\ 2u-4 & \text{if } u \geq \frac{1}{2} \end{cases}$$

We have three points to help us graph $y = h(u)$: $(-1, 0)$, $(0, -2)$ and $(2, 0)$. Unlike in the last two examples, these points do not give us information at the value $u = \frac{1}{2}$ where the rule for $h(u)$ changes. Substituting $u = \frac{1}{2}$ into the expression $-2u-2$ gives -3, so from $h(u) = -2u-2$, $u < \frac{1}{2}$, we get a hole at $\left(\frac{1}{2}, -3\right)$. However, this hole is filled because $h\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right) - 4 = -3$ and this produces the vertex at $\left(\frac{1}{2}, -3\right)$. The range of h is $[-3, \infty)$, with the minimum of h being -3 at $t = \frac{1}{2}$. Moreover, h is decreasing on $\left(-\infty, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, \infty\right)$. The graph of h is given in Figure 1.3.6.

4. Solving $i(w) = 4-2|3w-1| = 0$ yields $|3w-1| = 2$ or $3w-1 = \pm 2$. This gives two zeros, $w = -\frac{1}{3}$ and $w = 1$, which correspond to two w-intercepts, $\left(-\frac{1}{3}, 0\right)$ and $(1, 0)$. Also, $i(0) = 4-2|3(0)-1| = 2$, so the y-intercept of the graph is $(0, 2)$. As in the previous example, the first step in rewriting $i(w)$ as a piecewise defined function is to rewrite $|3w-1|$ as a piecewise function. Once again, we substitute the expression $3w-1$ in for every occurrence

of x in Definition 1.3.1:

$$|3w-1| = \begin{cases} -(3w-1) & \text{if } 3w-1<0 \\ 3w-1 & \text{if } 3w-1\geq 0 \end{cases}$$

$$\longrightarrow |3w-1| = \begin{cases} -3w+1 & \text{if } w<\frac{1}{3} \\ 3w-1 & \text{if } w\geq \frac{1}{3} \end{cases}$$

Thus for $w < \frac{1}{3}$, $|3w-1| = -3w+1$, so $i(w) = 4-2|3w-1| = 4-2(-3w+1) = 6w+2$. Likewise, for $w \geq \frac{1}{3}$, $|3w-1| = 3w-1$ so $i(w) = 4-2|3w-1| = 4-2(3w-1) = -6w+6$.

$$i(w) = 4-2|3w-1| = \begin{cases} 4-2(-3w+1) & \text{if } w<\frac{1}{3} \\ 4-2(3w-1) & \text{if } w\geq \frac{1}{3} \end{cases}$$

$$\longrightarrow i(w) = \begin{cases} 6w+2 & \text{if } w<\frac{1}{3} \\ -6w+6 & \text{if } w\geq \frac{1}{3} \end{cases}$$

As with the previous example, we have three points on the graph of i: $\left(-\frac{1}{3},0\right)$, $(0,2)$ and $(1,0)$, but no information about happens at $w = \frac{1}{3}$. Substituting this value of w into the formula $6w+2$ would produce a hole at $\left(\frac{1}{3},4\right)$. As we've seen several times already, however, $i\left(\frac{1}{3}\right) = 4$ so we don't have a hole at $\left(\frac{1}{3},4\right)$ but, rather, the vertex. From the graph we see that the range of i is $(-\infty,4]$ with the maximum of i being 4 when $w = \frac{1}{3}$. Also, i is increasing over $\left(-\infty,\frac{1}{3}\right]$ and decreasing on $\left[\frac{1}{3},\infty\right)$. Its graph is given in Figure 1.3.7.

□

As we take a step back and look at the graphs produced in Example 1.3.1, some patterns begin to emerge. Indeed, each of the graphs has the common '$\vee$' shape (in the case of the function i it's a '$\wedge$') with the vertex located at the x-value where the rule for each function changes from one formula to the other. It turns out that, independent variable labels aside, each and every function in Example 1.3.1 can be rewritten in the form $F(x) = a|x-h|+k$ for real number parameters a, h and k.

Each of the functions from Example 1.3.1 is rewritten in this form below and we record the vertex along with the slopes of the lines in the graph.

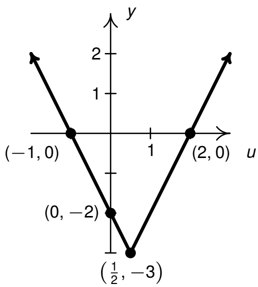


Figure 1.3.6: $h(u) = |2u - 1| - 3$

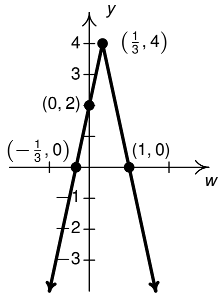


Figure 1.3.7: $i(w) = 4 - 2|3w - 1|$

- $f(x) = |x - 3| = (1)|x - 3| + 0$: $a = 1$, $h = 3$, $k = 0$; vertex $(3, 0)$; slopes ± 1
- $g(t) = |t| - 3 = (1)|t - 0| + (-3)$: $a = 1$, $h = 0$, $k = -3$; vertex $(0, -3)$; slopes ± 1
- $h(u) = |2u - 1| - 3 = 2\left|u - \frac{1}{2}\right| + (-3)$: $a = 2$, $h = \frac{1}{2}$, $k = -3$; vertex $\left(\frac{1}{2}, -3\right)$; slopes ± 2
- $i(w) = 4 - 2|3w - 1| = -6\left|w - \frac{1}{3}\right| + 4$: $a = -6$, $h = \frac{1}{3}$, $k = 4$; vertex $\left(\frac{1}{3}, 4\right)$; slopes ± 6

These specific examples suggest the following theorem.

Theorem 1.3.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a|x - h| + k$ consists of parts of two lines with slopes $\pm a$ which meet at a vertex (h, k). If $a > 0$, the shape resembles '$\vee$'. If $a < 0$, the shape resembles '$\wedge$'. Moreover, the graph is symmetric about the line $x = h$.

Proof. What separates Mathematics from the other sciences is its ability to actually *prove* patterns like the one stated in the theorem above as opposed to just *verifying* it by working more examples. The proof of Theorem 1.3.1 uses the exact same concepts as were used in Example 1.3.1, just in a more general context by which we mean using letters as parameters instead of numbers.

The first step is to rewrite $|x - h|$ as a piecewise function.

$$|x-h| = \begin{cases} -(x-h) & \text{if } x-h < 0 \\ x-h & \text{if } x-h \geq 0 \end{cases} \quad \longrightarrow \quad |x-h| = \begin{cases} -x+h & \text{if } x < h \\ x-h & \text{if } x \geq h \end{cases}$$

We plug that work into $F(x)$ to rewrite it as a piecewise function. For $x < h$, we have $|x - h| = -x + h$, so

$$F(x) = a|x-h| + k = a(-x+h) + k = -ax + ah + k = -ax + (ah+k)$$

Similarly, for $x \geq h$, we have $|x - h| = x - h$, so

$$F(x) = a|x-h| + k = a(x-h) + k = ax - ah + k = ax + (-ah+k)$$

Hence,

$$F(x) = a|x-h| + k = \begin{cases} a(-x+h)+k & \text{if } x < h \\ a(x-h)+k & \text{if } x \geq h \end{cases}$$

$$\longrightarrow F(x) = \begin{cases} -ax + (ah+k) & \text{if } x < h \\ ax + (-ah+k) & \text{if } x \geq h \end{cases}$$

All three parameters, a, h and k, are fixed (but arbitrary) real numbers. Thus, for any given choice of a, h and k the numbers $ah + k$ and $-ah + k$ are also just numbers as opposed to variables. This shows that the graph of F is comprised of pieces of two lines, $y = -ax+(ah+k)$ and $y = ax+(-ah+k)$, the former with slope $-a$ and the latter with slope a. Note that substituting $x = h$ into $y = -ax+(ah+k)$ produces $y = -ah + (ah + k) = k$ and substituting $x = h$ into $y = ax + (-ah + k)$ also produces $y = ah + (-ah + k) = k$. This tells us that the two linear pieces meet at the point (h, k).

If $a > 0$ then $-a < 0$ so the line $y = -ax + (ah + k)$, hence F, is decreasing on $(-\infty, h]$. Similarly, the line $y = ax + (-ah + k)$, hence F, is increasing on $[h, \infty)$. This produces a '$\vee$' shape. On the other hand, if $a < 0$ then $-a > 0$ which produces a '$\wedge$' shape because F is increasing on $(-\infty, h]$ followed by decreasing on $[h, \infty)$. (Said another way, $-a > 0$ means that the first linear piece has a positive slope and $a < 0$ means that the second piece has a negative slope.)

To show that the graph is symmetric about the line $x = h$, we need to show that if we move left or right the same distance away from $x = h$, then we get the same

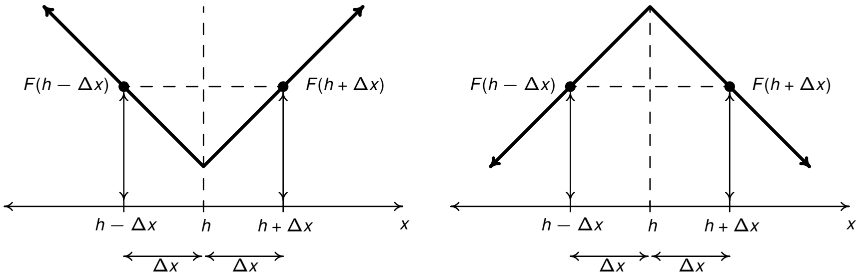


Figure 1.3.8: Axis of symmetry of $y = F(x)$

y-value on the graph. Suppose we move Δx to the right or left of h. The y-values are the function values so we need to show that $F(a + \Delta x) = F(a - \Delta x)$. Given that

$$F(a + \Delta x) = a|a + \Delta x - a| + k = a|\Delta x| + k$$

and

$$F(a - \Delta x) = a|a - \Delta x - a| + k = a| - \Delta x| + k = a|\Delta x| + k$$

we see that $F(a + \Delta x) = F(a - \Delta x)$. Thus we have shown that the y-values on the graph on either side of $x = h$ are equal provided we move the same distance away from $x = a$. This completes the proof. □

The line $x = a$ in Theorem 1.3.1 is called the **axis of symmetry** of the graph of $y = F(x)$. This language is consistent with the basics of symmetry discussed in Section **??** and we will build upon our work here in several upcoming sections. For now, we simply present two graphs illustrating the concept of the axis of symmetry in Figure 1.3.8.

While Theorem 1.3.1 and its proof are specific to the particular family of absolute value functions, there are ideas here that apply to all functions. Thus we wish to take a slight detour away from the main narrative to argue this result again from an even more generalized viewpoint. Our goal is to 'build' the formula $F(x) = a|x - h| + k$ from $f(x) = |x|$ in three stages, each corresponding to the role of one of the parameters a, h and k, and track the geometric changes that go along with each stage. We will revisit all of the ideas described below in complete generality in Section **??**.

The graph of $f(x) = |x|$ consists of the points $\{(c, |c|) \mid c \in \mathbb{R}\}$.[3] Consider $F_1(x) = |x - h|$. The graph of F_1 is the set of points $\{(x, |x - h|) \mid x \in \mathbb{R}\}$. If we relabel $x - h = c$, then $x = c + h$, and as x varies through all of the real numbers, so does c and vice-versa.[4]

Hence, we can write $\{(x, |x - h|) \mid x \in \mathbb{R}\} = \{(c + h, |c|) \mid c \in \mathbb{R}\}$. If we fix a y-coordinate, $|c|$, we see that the corresponding points on the graph of f and F_1, $(c, |c|)$ and $(c + h, |c|)$, respectively, differ only in that one is horizontally shifted by h. In other words, to get the graph of F_1, we simply take the graph of f and shift each point horizontally by adding h to the x-coordinate. Translating the graph in this manner preserves the '$\vee$' shape and symmetry, but moves the vertex from $(0, 0)$ to $(h, 0)$.

Next, we examine $F_2(x) = a|x - h|$ and compare its graph to that of $F_1(x) = |x - h|$. The graph of F_2 consists of the points $\{(x, a|x - h|) \mid x \in \mathbb{R}\}$ whereas the graph of F_1 consists of the points $\{(x, |x - h|) \mid x \in \mathbb{R}\}$. The only difference between the points $(x, |x - h|)$ and $(x, a|x - h|)$ is that the y-coordinate of one is a times the y-coordinate of the other. If $a > 0$, all we are doing is scaling the y-axis by a factor of a. As we've seen when plotting points and graphing functions, the scaling of the y-axis affects only the relative vertical displacement of points[5] and not the overall shape.

If $a < 0$, then in addition to scaling the vertical axis, we are reflecting the points across the x-axis.[6] Such a transformation doesn't change the '$\vee$' shape except for flipping it upside-down to make it a '$\wedge$'. In either case, the vertex $(h, 0)$ stays put at $(h, 0)$ because the y-value of the vertex is 0 and $a \cdot 0 = 0$ regardless if $a > 0$ or $a < 0$.

Last, we examine the graph of $F(x) = a|x - h| + k$ to see how it relates to the graph of $F_2(x) = a|x - h|$. The graph of F consists of the points $\{(x, a|x - h| + k) \mid x \in \mathbb{R}\}$ whereas the graph of F_2 consists of the points $\{(x, a|x - h|) \mid x \in \mathbb{R}\}$. The difference between the corresponding points $(x, a|x - h|)$ and $(x, a|x - h| + k)$ is the addition of k in the y-coordinate of the latter. Adding k to each of the y-values translates the graph of F_2 vertically by k units. The basic shape doesn't change but the vertex goes from $(h, 0)$ to (h, k).

[3] See the box on page **??**. Also, we use 'c' as our dummy variable to avoid the confusion that would arise by over-using 'x'.

[4] That is, every real number c can be written as $x - h$ for some x, and every real number x can be written as $c + h$ for some c.

[5] See the discussion following Example 1.1.1 regarding the plot of Skippy's data.

[6] See the box on page **??** in Section **??**.

In summary, the graph of $F(x) = a|x - h| + k$ can be obtained from the graph of $f(x) = |x|$ in three steps: first, add h to each of the x-coordinates; second, multiply each y-coordinate by a; and third, add k to each y-coordinate. Geometrically, these steps mean that we first move the graph left or right, then scale the y-axis by a factor of a (and reflect across the x-axis if $a < 0$), and then move the graph up or down. Throughout all of these *transformations*, the graph maintains its '$\vee$' or '$\wedge$' shape.

Of course, not every function involving absolute values can be written in the form given in Theorem 1.3.1. A good example of this is $G(x) = |x - 2| - x$. However recognizing the ones that can be rewritten will greatly simplify the graphing process. In the next example, we graph four more absolute value functions, two using Theorem 1.3.1 and two using Definition 1.3.1.

Example 1.3.2.

1. Graph each of the functions below using Theorem 1.3.1 or by rewriting it as a piecewise defined function using Definition 1.3.1. Find the zeros, axis-intercepts and the extrema (if any exist) and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $F(x) = |x + 3| + 2$

   (b) $f(t) = \dfrac{4 - |5 - 3t|}{2}$

   (c) $G(x) = |x - 2| - x$

   (d) $g(t) = |t - 2| - |t|$

2. Use Theorem 1.3.1 to write a possible formula for $H(x)$ whose graph is given in Figure 1.3.9.

Solution.

1. (a) Rewriting $F(x) = |x+3|+2 = (1)|x-(-3)|+2$, we have $F(x)$ in the form stated in Theorem 1.3.1 with $a = 1$, $h = -3$ and $k = 2$. The vertex is $(-3, 2)$ and the graph will be a '$\vee$' shape. Seeing as the vertex is already above the x-axis and the graph opens upwards, there are no x-intercepts on the graph of F, hence there are no zeros.[7] With $F(0) = 5$, the y-intercept is $(0, 5)$. To get a third point, we can pick an arbitrary x-value to the left of the vertex or we could use symmetry: three units to the *right* of the vertex the y-value is 5, so the same must be true three units to the *left* of the vertex, at $x = -6$. Sure enough, $F(-6) = |-6+3|+2 = |-3|+2 = 5$. The range of F is $[2, \infty)$ with

[7] Alternatively, setting $|x + 3| + 2 = 0$ gives $|x + 3| = -2$. Absolute values are never negative thus we have no solution.

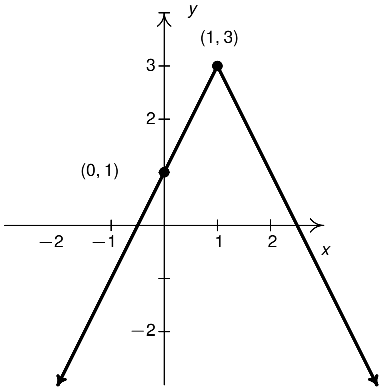


Figure 1.3.9: $y = H(x)$

its minimum of 2 when $x = -3$ and F decreasing on $(-\infty, -3]$ then increasing on $[-3, \infty)$. The graph is in Figure 1.3.10.

(b) We see in the formula for $f(t)$ that t appears only once to the first power inside the absolute values, so we proceed to rewrite it in the form $a|t - h| + k$:

$$\begin{aligned} f(x) &= \frac{4 - |5 - 3t|}{2} \\ &= -\frac{|5 - 3t|}{2} + \frac{4}{2} \\ &= \left(-\frac{1}{2}\right)\left|(-3)\left(t - \frac{5}{3}\right)\right| + 2 \\ &= \left(-\frac{1}{2}\right)|-3|\left|t - \frac{5}{3}\right| + 2 \\ &= -\frac{3}{2}\left|t - \frac{5}{3}\right| + 2. \end{aligned}$$

Matching up the constants in the formula $f(t)$ to the parameters of $F(x)$ in Theorem 1.3.1, we identify $a = -\frac{3}{2}$, $h = \frac{5}{3}$ and $k = 2$. Hence the vertex is $\left(\frac{5}{3}, 2\right)$, and the graph is shaped like '$\wedge$' comprised of pieces of lines with slopes $\pm\frac{3}{2}$. To find the zeros of f, we set $f(t) = 0$. (We can use either expression here.) Solving $-\frac{3}{2}\left|t - \frac{5}{3}\right| + 2 = 0$, we

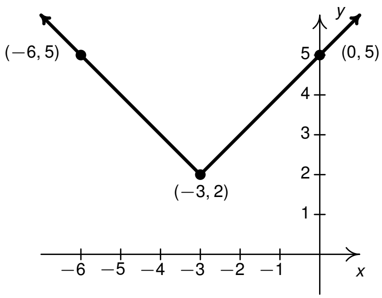


Figure 1.3.10: $F(x) = |x+3| + 2$

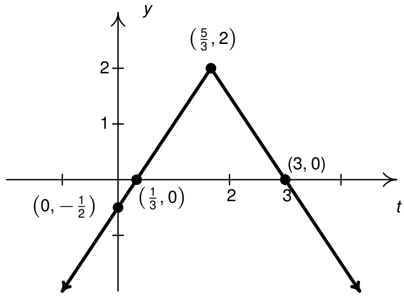


Figure 1.3.11: $f(t) = \dfrac{4 - |5-3t|}{2}$

get $\left|t - \frac{5}{3}\right| = \frac{4}{3}$, so $t - \frac{5}{3} = \pm\frac{4}{3}$. Hence our zeros are $t = \frac{1}{3}$ and $t = 3$, producing the t-intercepts $\left(\frac{1}{3}, 0\right)$ and $(3, 0)$. Using either formula gives $f(0) = -\frac{1}{2}$, so our y-intercept is $\left(0, -\frac{1}{2}\right)$. Plotting the vertex, along with the intercepts, gives us enough information to produce the graph in Figure 1.3.11. The range is $(-\infty, 2]$ with a maximum of 2 at $t = \frac{5}{3}$ and f is increasing on $\left(-\infty, \frac{5}{3}\right]$ then decreasing on $\left[\frac{5}{3}, \infty\right)$.

(c) We are unable to apply Theorem 1.3.1 to $G(x) = |x-2| - x$ because there is an x both inside and outside of the absolute value. We can, however, rewrite the function as a piecewise function using Definition 1.3.1. Our first step is to rewrite $|x-2|$ as a piecewise function:

$$|x-2| = \begin{cases} -(x-2) & \text{if } x-2 < 0 \\ x-2 & \text{if } x-2 \geq 0 \end{cases}$$

$$\longrightarrow |x-2| = \begin{cases} -x+2 & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2 \end{cases}$$

Hence, for $x < 2$, $|x-2| = -x+2$ so $G(x) = |x-2| - x = (-x+2) - x = -2x+2$. Likewise, for $x \geq 2$, $|x-2| = x-2$ so $G(x) = |x-2| - x = x - 2 - x = -2$.

$$G(x) = |x-2| - x = \begin{cases} (-x+2) - x & \text{if } x < 2 \\ (x-2) - x & \text{if } x \geq 2 \end{cases}$$

$$\longrightarrow G(x) = \begin{cases} -2x+2 & \text{if } x < 2 \\ -2 & \text{if } x \geq 2 \end{cases}$$

To find the zeros of G, we set $G(x) = 0$. Solving $|x - 2| - x = 0$ can be problematic, given that x is both inside and outside of the absolute values.[8] We can, however, use the piecewise description of $G(x)$. With $G(x) = -2x + 2$ for $x < 2$, we solve $-2x + 2 = 0$ to get $x = 1$. This works because $1 < 2$, so we have $x = 1$ as the zero of G corresponding to the x-intercept $(1, 0)$. The other piece of $G(x)$ is $G(x) = -2$ which is never 0. For the y-intercept, we find $G(0) = 2$, and get $(0, 2)$.

To graph $y = G(x)$, we have the line $y = -2x+2$ which contains $(0, 2)$ and $(1, 0)$ and continues to a hole at $(2, -2)$. At this point, $G(x) = -2$ takes over and we have a horizontal line containing $(2, -2)$ extending indefinitely to the right. The range of G is $[-2, \infty)$ with a minimum value of -2 attained for all $x \geq 2$. Moreover, G is decreasing on $(-\infty, 2]$ and then constant on $[2, \infty)$. The graph is in Figure 1.3.12.

(d) Once again we are unable to use Theorem 1.3.1 because $g(t) = |t-2| - |t|$ has two absolute values with no apparent way to combine them. Thus we proceed by re-writing the function g with two separate applications of Definition 1.3.1 to remove each instance of the absolute values. To start with we have:

$$|t| = \begin{cases} -t & \text{if } t < 0 \\ t & \text{if } t \geq 0 \end{cases} \quad \text{and} \quad |t-2| = \begin{cases} -t+2 & \text{if } t < 2 \\ t-2 & \text{if } t \geq 2 \end{cases}$$

Taken together, these break the domain into *three* pieces: $t < 0$, $0 \leq t < 2$ and $t \geq 2$. For $t < 0$, $|t| = -t$ and $|t-2| = -t+2$. Therefore $g(t) = |t-2| - |t| = (-t+2) - (-t) = 2$ for $t < 0$. For $0 \leq t < 2$, $|t| = t$ and $|t-2| = -t+2$, so $g(t) = |t-2| - |t| = (-t+2) - t) = -2t+2$.

Last, for $t \geq 2$, $|t| = t$ and $|t-2| = t-2$, so $g(t) = |t-2| - |t| = (t-2) - (t) = -2$. Putting all three parts together yields:

$$\begin{aligned} g(t) = |t-2| - |t| &= \begin{cases} (-t+2) - (-t) & \text{if } t < 0 \\ (-t+2) - (t) & \text{if } 0 \leq t < 2 \\ (t-2) - (t) & \text{if } t \geq 2 \end{cases} \\ &= \begin{cases} 2 & \text{if } t < 0 \\ -2t+2 & \text{if } 0 \leq t < 2 \\ -2 & \text{if } t \geq 2 \end{cases} \end{aligned}$$

[8]We'll return to this momentarily.

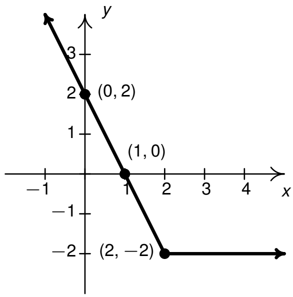


Figure 1.3.12: $G(x) = |x - 2| - x$

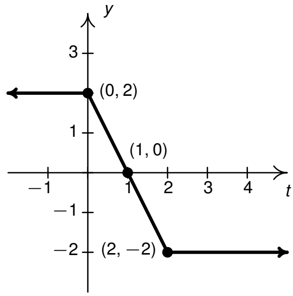


Figure 1.3.13: $g(t) = |t - 2| - |t|$

As with the previous example, we'll delay discussing the absolute value algebra needed to find the zeros of g and use the piecewise description instead. To graph g, we have the horizontal line $y = 2$ up to, but not including, the point $(0, 2)$. For $0 \leq t < 2$, we have the line $y = -2t + 2$ which has a y-intercept at $(0, 2)$ (thus picking up where the first part left off) and a t-intercept at $(1, 0)$. This piece ends with a hole at $(2, -2)$ which is promptly plugged by the horizontal line $y = -2$ for $t \geq 2$. Hence the only zero of t is $t = 1$.

The range of g is $[-2, 2]$ with a minimum of -2 achieved for all $t \geq 2$, and a maximum of 2 for $t \leq 0$. We note that g is constant on $(-\infty, 0]$ and $[2, \infty)$, but with different values, and g is decreasing on $[0, 2]$. The graph is given in Figure 1.3.13.

2. We are told to use Theorem 1.3.1 to find a formula for $H(x)$ so we start with $H(x) = a|x - h| + k$ and look for real numbers a, h and k that make sense. The vertex is labeled as $(1, 3)$, meaning $h = 1$ and $k = 3$. Hence we know $H(x) = a|x - 1| + 3$, so all that is left for us to find is the value of a. The only other point labeled for us is $(0, 1)$, meaning $H(0) = 1$. Substituting $x = 0$ into our formula for $H(x)$ gives: $H(0) = a|0 - 1| + 3 = a + 3$. Given that $H(0) = 1$, we have $a + 3 = 1$, so $a = -2$. Our final answer is $H(x) = -2|x - 1| + 3$. $\square$

If nothing else, Example 1.3.2 demonstrates the value of *changing forms* of functions and the utility of the interplay between algebraic and graphical descriptions of functions. These themes resonate time and time again in this and later courses in Mathematics.

1.3.2 Graphical Solution Techniques for Equations and Inequalities

Consider the basic equation and related inequalities: $|x| = 3$, $|x| < 3$ and $|x| > 3$. At some point you learned how to solve these using properties of the absolute value inspired by the distance definition. (If not, see Section **??**.) While there is nothing wrong with this understanding, we wish to use these problems to motivate powerful graphical techniques which we'll use to solve more complicated equations and inequalities in this section, and in many other sections of the textbook.

To that end, let's call $f(x) = |x|$ and $g(x) = 3$. If we graph $y = f(x)$ and $y = g(x)$ on the same set of axes then, by looking for x values where $f(x) = g(x)$, we are looking for x-values which have the same y-value on both graphs. That is, the solutions to $f(x) = g(x)$ are the x-coordinates of the *intersection points* of the two graphs. We graph $y = f(x) = |x|$ (the characteristic '$\vee$') along with $y = g(x) = 3$ (the horizontal line) in Figure 1.3.14. Indeed, the two graphs intersect at $(-3, 3)$ and $(3, 3)$ so our solutions to $f(x) = g(x)$ are the x-values of these points, $x = \pm 3$.

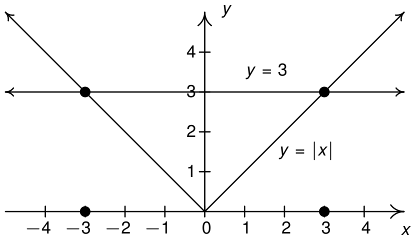


Figure 1.3.14: Graphically solving $|x| = 3$

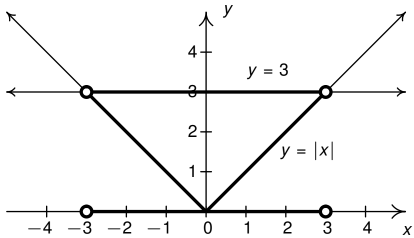


Figure 1.3.15: Graphically solving $|x| < 3$

Likewise, if we wish to solve $|x| < 3$, we can view this as a functional inequality $f(x) < g(x)$ which means we are looking for the x-values where the $f(x)$ values are less than the corresponding $g(x)$ values. On the graphs, this means we'd be looking for the x-values where the y-values of $y = f(x)$ are less than, hence *below*, those on the graph of $y = g(x)$.

In Figure 1.3.15 we see that the graph of f is below the graph of g between $x = -3$ and $x = 3$, so our solution is $-3 < x < 3$, or in interval notation, $(-3, 3)$. Finally, the inequality $|x| > 3$ is equivalent to $f(x) > g(x)$ so we are looking for

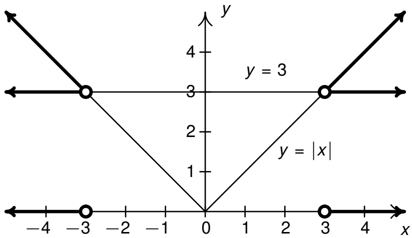


Figure 1.3.16: Graphically solving $|x| > 3$

the x-values where the graph of f is *above* the graph of g.[9] Figure 1.3.16 shows that this is true for all $x < -3$ or for all $x > 3$. In interval notation, the solution set is $(-\infty, -3) \cup (3, \infty)$.

The methodology and reasoning behind solving the above equation and inequalities extend to any pair of functions f and g, since when graphed on the same set of axes, function outputs are always the dependent variable or the ordinate (second coordinate) of the ordered pairs which comprise the graph. In general:

Graphical Interpretation of Equations and Inequalities

Suppose f and g are functions whose domains and ranges are sets of real numbers.

- The solutions to $f(x) = g(x)$ are the x-values where the graphs of f and g intersect.
- The solution to $f(x) < g(x)$ is the set of x-values where the graph of f is *below* the graph of g.
- The solution to $f(x) > g(x)$ is the set of x-values where the graph of f *above* the graph of g.

Let's return to Example 1.3.2 where were asked to find the zeros of the functions $G(x) = |x - 2| - x$ and $g(t) = |t - 2| - |t|$. In that Example, instead of tackling the algebra involving the absolute values head on we rewrote each function as a piecewise-defined function and obtained our solutions that way.

Let's see what this looks like graphically. Note that solving $|x - 2| - x = 0$ is equivalent to solving $|x - 2| = x$. We graphed $y = |x - 2|$ and $y = x$ on the

[9] Solving $f(x) > g(x)$ is equivalent to solving $g(x) < f(x)$ - that is, finding where the graph of g is below the graph of f.

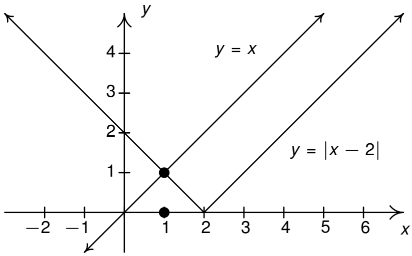


Figure 1.3.17: Graphically solving $|x-2|=x$.

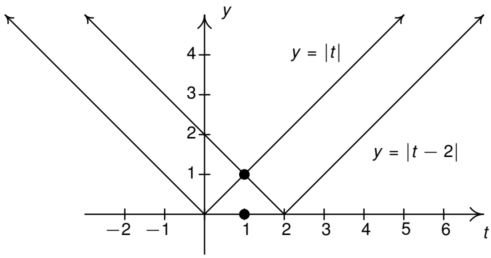


Figure 1.3.18: Graphically solving $|t-2|=|t|$

same set of axes in Figure 1.3.17 and it appears as if we have just one point of intersection, corresponding to just one solution.

Indeed, we can *show* that there is just one point of intersection. The graph of $y=|x-2|$ is comprised of parts of two lines, $y=-(x-2)$ and $y=x-2$. The first line has a slope of -1 and the second has slope 1. The line $y=x$ also has a slope 1 meaning it and the 'right half' of $y=|x-2|$ are parallel, so they never intersect. If our graphs are accurate enough, we may even be able to guess that the solution is $x=1$, which we can verify by substituting $x=1$ into $|x-2|=x$ and seeing that it checks.

Likewise, solving $|t-2|-|t|=0$ is equivalent to solving $|t-2|=|t|$. We graphed $y=|t-2|$ and $y=|t|$ in Figure 1.3.18 and used the same arguments to get the solution $t=1$ here as well.

There is more to see here. Consider solving $|x-2|-x=0$ algebraically using the techniques from a previous Algebra course (or Section **??**). Our first step would be to isolate the absolute value quantity: $|x-2|=x$. We then 'drop' the absolute value by paying the price of a '$\pm$': $x - 2 = \pm x$. This gives us two equations: $x - 2 = x$ and $x - 2 = -x$. The first equation, $x - 2 = x$ reduces to $-2 = 0$ which has no solution. The second equation, $x - 2 = -x$, does have a solution, namely $x = 1$.

How does the algebra tie into the graphs above? Instead of 'dropping' the absolute value and tagging the right hand side with a $\pm$, we can think about the piecewise definition of $|x - 2|$ and write $|x - 2| = \pm(x - 2)$ depending on if $x < 2$ or if $x \geq 2$. That is, $|x - 2| = x$ is more precisely equivalent to the two equations: $-(x - 2) = x$ which is valid for $x < 2$ or $x - 2 = x$ which is valid for $x \geq 2$.

Graphically, the first equation is looking for intersection points between the 'left half' of the '$\vee$' of $y = |x - 2|$ and the line $y = x$. Indeed, $-(x - 2) = x$ is equivalent to $x - 2 = -x$ from which we obtain our solution $x = 1$. Likewise, the second equation, $x - 2 = x$ is looking for intersection points of the 'right half' of the '$\vee$' and the line $y = x$, but there is none. The equation $-2 = 0$ is telling us that for us to have any solutions, the lines $y = x - 2$ and $y = x$, which have the same slope, must also have the same y-intercepts: that is, -2 would have to equal 0 and that's just silly.

Similarly, when solving $|t - 2| - |t| = 0$ or $|t - 2| = |t|$, we can use our graphs to prove that the only intersection point is when the 'left half' of $y = |t - 2|$ intersects the 'right half' of $y = |t|$ - that is, when $-(t - 2) = t$. The moral of the story is this: careful graphs can help us simplify the algebra, because we can narrow down the cases. This is especially useful in solving inequalities, as we'll see in our next example.

Example 1.3.3. Solve the following equations and inequalities.

1. $4 - |x| = 0.9x - 3.6$
2. $|t - 3| - |t| = 3$
3. $|x + 1| \geq \dfrac{x + 4}{2}$
4. $2 < |t - 1| \leq 5$

Solution.

1. We begin by graphing $y = 4 - |x|$ and $y = 0.9x - 3.6$ to look for intersection points. Using Theorem 1.3.1, we know that the graph of $y = 4 - |x| = -|x| + 4$ has a vertex at $(0, 4)$ and is a '$\wedge$' shape, so there are x-intercepts to find. Solving $4 - |x| = 0$, we get $|x| = 4$, or $x = \pm 4$. Hence, we have two x-intercepts: $(-4, 0)$ and $(4, 0)$.

   We know from Section **??** that the graph of $y = 0.9x - 3.6$ is a line with slope 0.9 and y-intercept $(0, -3.6)$. To find the x-intercept here we solve

$0.9x - 3.6 = 0$ and get $x = 4$. Hence, $(4, 0)$ is an x-intercept here as well, and we have stumbled upon one solution to $4 - |x| = 0.9x - 3.6$, namely $x = 4$. The question is if there are any other solutions. Our graph (Figure 1.3.19) certainly looks as if there is just one intersection point, but we know from Theorem 1.3.1 that the slopes of the linear parts of $y = 4 - |x|$ are ± 1. The slope of $y = 0.9x - 3.6$ is 0.9 and $0.9 \neq 1$ so we know that the left hand side of the'$\wedge$' must meet up with the graph of the line because they are not parallel.[10]

Definition 1.3.1 tells us that when $x < 0$, $|x| = -x$, so $4 - |x| = 4 - (-x) = 4 + x$. Hence we set about solving $4 + x = 0.9x - 3.6$ and get $x = -76$. Both $x = -76$ and $x = 4$ check in our original equation, $4 - |x| = 0.9x - 3.6$, so we have found our two solutions.[11]

2. While we could graph $y = |t - 3| - |t|$ and $y = 3$ to help us find solutions, we choose to rewrite the equation as $|t - 3| = |t| + 3$. This way, we have somewhat easier graphs to deal with, namely $y = |t - 3|$ and $y = |t| + 3$. The first graph, $y = |t - 3|$, has a vertex at $(3, 0)$ and is shaped like a '$\vee$' with slopes ± 1 and a y-intercept of $(0, 3)$. The second graph, $y = |t| + 3$, has a vertex at $(0, 3)$ and is also shaped like a '$\vee$,' with slopes ± 1, and has no t-intercepts.

   To our surprise and delight, the graphs (Figure 1.3.20) appear to overlap for $t \leq 0$. Indeed, for $t \leq 0$, $|t - 3| = -(t - 3) = -t + 3$ and $|t| + 3 = -t + 3$. Since the formulas are *identical* for these values of t, our solutions are all values of t with $t \leq 0$. Using interval notation, we state our solution as $(-\infty, 0]$. (The other parts of the graphs are non-intersecting parallel lines so we ignored them.)

3. To solve $|x + 1| \geq \frac{x+4}{2}$, we first graph $y = |x + 1|$ and $y = \frac{x+4}{2} = \frac{1}{2}x + 2$. The former is '$\vee$' shaped with a vertex at $(-1, 0)$ and a y-intercept of $(0, 1)$. The latter is a line with y-intercept $(0, 2)$, slope $m = \frac{1}{2}$ and x-intercept $(-4, 0)$. The picture in Figure 1.3.21 shows two intersection points. To find these, we solve the equations: $-(x + 1) = \frac{x+4}{2}$, obtaining $x = -2$, and $x + 1 = \frac{x+4}{2}$ obtaining $x = 2$.

   Graphically, the inequality $|x + 1| \geq \frac{x+4}{2}$ is looking for where the graph of $y = |x + 1|$, the '$\vee$,' intersects (=) or is above (>) the line $y = \frac{x+4}{2}$. The graph shows this happening whenever $x \leq -2$ or $x \geq 2$. Using interval notation,

[10] See Theorem **??**.

[11] Our picture shows only one of the solutions. We encourage you to take the time with a graphing utility to get the picture to show both points of intersection.

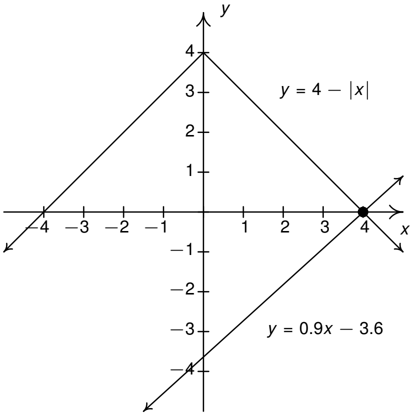


Figure 1.3.19: Solving $4 - |x| = 0.9x - 3.6$

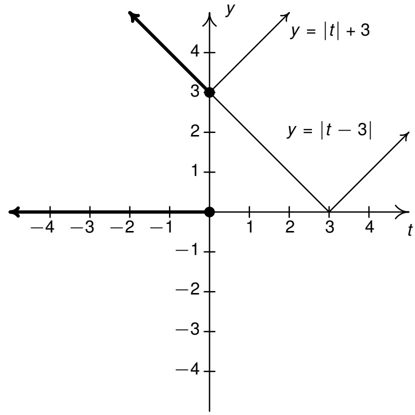


Figure 1.3.20: Solving $|t - 3| - |t| = 3$

our solution is $(-\infty, -2] \cup [2, \infty)$. While we cannot check every single x value individually, choosing test values $x < -2$, $x = 2$, $-2 < x < 2$, $x = 2$, and $x > 2$ to see if the original inequality $|x + 1| \geq \frac{x+4}{2}$ holds would help us verify our solution.

4. Recall that the inequality $2 < |t - 1| \leq 5$ is an example of a 'compound' inequality in that is two inequalities in one.[12] The values of t in the solution set need to satisfy $2 < |t - 1|$ *and* $|t - 1| \leq 5$. To help us sort through the cases, we graph the horizontal lines $y = 2$ and $y = 5$ along with the '$\vee$' shaped $y = |t - 1|$ with vertex $(1, 0)$ and y-intercept $(0, 1)$.

   Geometrically, we are looking for where $y = |t - 1|$ is strictly *above* the line $y = 2$ but *below* (or meets) the line $y = 5$. Solving $|t - 1| = 2$ gives $t = -1$ and $t = 3$ whereas solving $|t - 1| = 5$ gives $t = -4$ or $t = 6$. Per the graph (Figure 1.3.22), we see that $y = |t - 1|$ lies between $y = 2$ and $y = 5$ when $-4 \leq t < -1$ and again when $3 < t \leq 6$.

   In interval notation, our solution is $[-4, -1) \cup (3, 6]$. As with the previous example, it is impossible to check each and every one of these solutions, but choosing t values both in and around the solution intervals would give us some numerical confidence we have the correct and complete solution. $\square$

[12]See Example **??** for examples of linear compound inequalities.

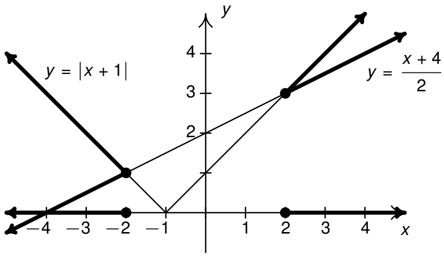


Figure 1.3.21: Solving $|x+1| \geq \dfrac{x+4}{2}$

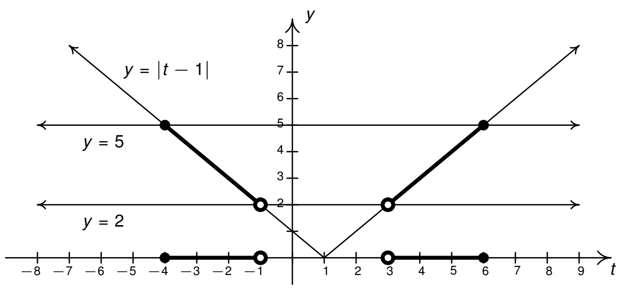


Figure 1.3.22: Solving $2 < |t-1| \leq 5$

We will see the interplay of Algebra and Geometry throughout the rest of this course. In the Exercises, do not hesitate to use whatever mix of algebraic and graphical methods you deem necessary to solve the given equation or inequality. Indeed, there is great value in checking your algebraic answers graphically and vice-versa.

One of the classic applications of inequalities involving absolute values is the notion of tolerances.[13] Recall that for real numbers x and c, the quantity $|x - c|$ may be interpreted as the distance from x to c. Solving inequalities of the form $|x - c| \leq d$ for $d > 0$ can then be interpreted as finding all numbers x which lie within d units of c. We can think of the number d as a 'tolerance' and our solutions x as being within an accepted tolerance of c. We use this principle in the next example.

Example 1.3.4. Suppose a manufacturer needs to produce a 24 inch by 24 inch square piece of particle board as part of a home office desk kit. How close does the side of the piece of particle board need to be cut to 24 inches to guarantee that the area of the piece is within a tolerance of 0.25 square inches of the target area of 576 square inches?

Solution. Let x denote the length of the side of the square piece of particle board so that the area of the board is x^2 square inches. Our tolerance specifies that the area of the board, x^2, needs to be within 0.25 square inches of 576. Mathematically, this translates to $|x^2 - 576| \leq 0.25$. Rewriting, we get $-0.25 \leq x^2 - 576 \leq 0.25$, or $575.75 \leq x^2 \leq 576.25$. At this point, we take advantage of the fact that the square root is increasing.[14] Therefore, taking square roots preserves the inequality. When simplifying, we keep in mind that since x represents a length, $x > 0$.

$$\begin{array}{rcll}
575.75 \leq & x^2 & \leq 576.25 & \\
\sqrt{575.75} \leq & \sqrt{x^2} & \leq \sqrt{576.25} & \text{(take square roots.)} \\
\sqrt{575.75} \leq & |x| & \leq \sqrt{576.25} & (\sqrt{x^2} = |x|) \\
\sqrt{575.75} \leq & x & \leq \sqrt{576.25} & (|x| = x \text{ since } x > 0)
\end{array}$$

The side of the piece of particle board must be between $\sqrt{575.75} \approx 23.995$ and $\sqrt{576.25} \approx 24.005$ inches, a tolerance of (approximately) 0.005 inches of the

[13] The underlying concept of Calculus can be phrased in terms of tolerances, so this is well worth your attention.

[14] This means that for $a, b \geq 0$, if $a \leq b$, then $\sqrt{a} \leq \sqrt{b}$.

target length of 24 inches, to ensure that the area is within 0.25 square inches of 576. □

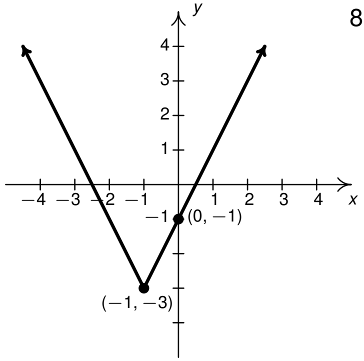
1.3.3 Exercises

In Exercises 1 - 6, graph the function using Theorem 1.3.1. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

1. $f(x) = |x + 4|$
2. $f(x) = |x| + 4$
3. $f(x) = |4x|$
4. $g(t) = -3|t|$
5. $g(t) = 3|t + 4| - 4$
6. $g(t) = \frac{1}{3}|2t - 1|$

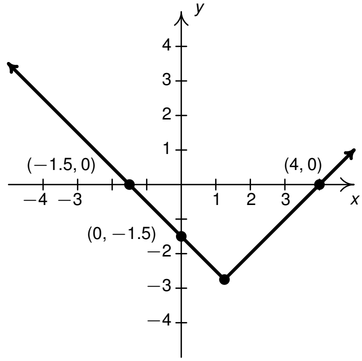
In Exercises 7. - 10., find a formula for each function below in the form $F(x) = a|x - h| + k$.

7.



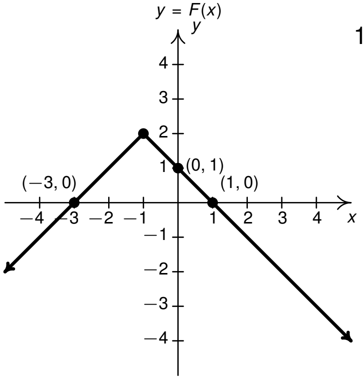
$y = F(x)$

8.



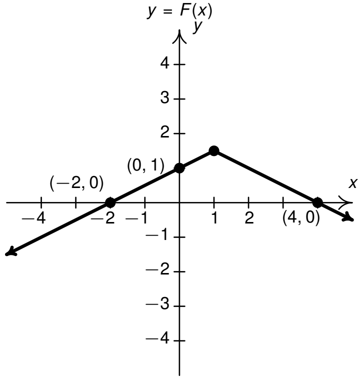
$y = F(x)$

9.



$y = F(x)$

10.



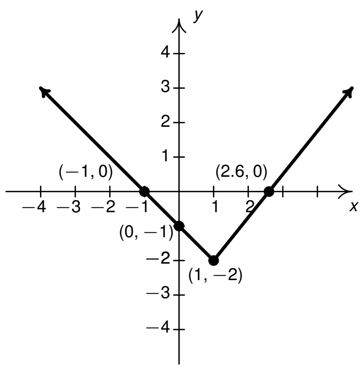
$y = F(x)$

11. With help from a graphing utility, graph the following pairs of functions on the same set of axes:

- $f(x) = 2 - x$ and $g(x) = |2 - x|$
- $f(x) = x^2 - 4$ and $g(x) = |x^2 - 4|$
- $f(x) = x^3$ and $g(x) = |x^3|$
- $f(x) = \sqrt{x} - 4$ and $g(x) = |\sqrt{x} - 4|$

Choose more functions $f(x)$ and graph $y = f(x)$ alongside $y = |f(x)|$ until you can explain how, in general, one would obtain the graph of $y = |f(x)|$ given the graph of $y = f(x)$. How does your explanation tie in with with Definition 1.3.1?

12. Explain why the function below cannot be written in the form $F(x) = a|x - h| + k$. Write $F(x)$ as a piecewise-defined linear function.

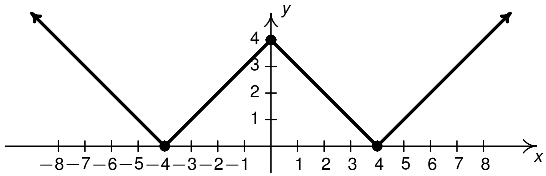


$y = F(x)$

In Exercises 13 - 18, graph the function by rewriting each function as a piecewise defined function using Definition 1.3.1. Find the axis intercepts of each graph, if any exist. From the graph, determine the domain and range of each function, the maximum and minimum of each function, if they exist, and list the intervals on which the function is increasing, decreasing or constant.

13. $f(x) = x + |x| - 3$
14. $f(x) = |x + 2| - x$
15. $f(x) = |x + 2| - |x|$
16. $g(t) = |t + 4| + |t - 2|$
17. $g(t) = \dfrac{|t + 4|}{t + 4}$
18. $g(t) = \dfrac{|2 - t|}{2 - t}$

19. With the help of your classmates, find an absolute value function whose graph is given below.



In Exercises 20 - 31, solve the equation.

20. $|x| = 6$
21. $|3x - 1| = 10$
22. $|4 - x| = 7$
23. $4 - |t| = 3$
24. $2|5t + 1| - 3 = 0$
25. $|7t - 1| + 2 = 0$
26. $\dfrac{5 - |w|}{2} = 1$
27. $\frac{2}{3}|5 - 2w| - \frac{1}{2} = 5$
28. $|w| = w + 3$
29. $|2x - 1| = x + 1$
30. $4 - |x| = 2x + 1$
31. $|x - 4| = x - 5$

Solve the equations in Exercises 32 - 37 using the property that if $|a| = |b|$ then $a = \pm b$.

32. $|3x - 2| = |2x + 7|$
33. $|3x + 1| = |4x|$
34. $|1 - 2x| = |x + 1|$
35. $|4 - t| - |t + 2| = 0$
36. $|2 - 5t| = 5|t + 1|$
37. $3|t - 1| = 2|t + 1|$

In Exercises 38 - 53, solve the inequality. Write your answer using interval notation.

38. $|3x - 5| \leq 4$
39. $|7x + 2| > 10$
40. $|2t + 1| - 5 < 0$
41. $|2 - t| - 4 \geq -3$
42. $|3w + 5| + 2 < 1$
43. $2|7 - w| + 4 > 1$
44. $2 \leq |4 - x| < 7$
45. $1 < |2x - 9| \leq 3$
46. $|t + 3| \geq |6t + 9|$
47. $|t - 3| - |2t + 1| < 0$
48. $|1 - 2x| \geq x + 5$
49. $x + 5 < |x + 5|$
50. $x \geq |x + 1|$
51. $|2x + 1| \leq 6 - x$
52. $t + |2t - 3| < 2$
53. $|3 - t| \geq t - 5$

54. Show that if δ is a real number with $\delta > 0$, the solution to $|x - a| < \delta$ is the interval: $(a - \delta, a + \delta)$. That is, an interval centered at a with 'radius' δ.

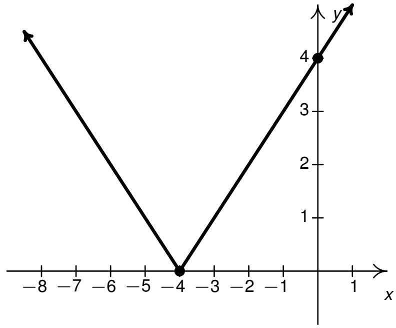
55. The Triangle Inequality[15] for real numbers states that for all real numbers x and a, $|x + a| \leq |x| + |a|$ and, moreover, $|x + a| = |x| + |a|$ if and only if x and a are both positive, both negative, or one or the other is 0. Graph each pair of functions below on the same pair of axes and use the graphs to verify the triangle inequality in each instance.

[15] http://en.wikipedia.org/wiki/Triangle_inequality

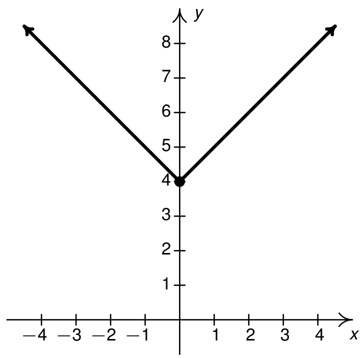
- $f(x) = |x+2|$ and $g(x) = |x| + 2$.
- $f(x) = |x+4|$ and $g(x) = |x| + 4$.

1.3.4 Answers

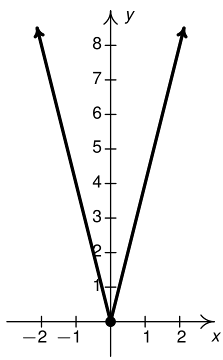
1. $f(x) = |x+4|$
   x-intercept $(-4, 0)$
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, -4]$
   Increasing on $[-4, \infty)$
   Minimum is 0 at $(-4, 0)$
   No maximum



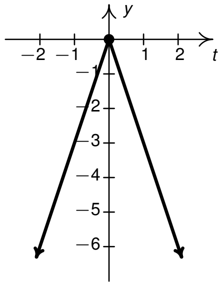
2. $f(x) = |x| + 4$
   No x-intercepts
   y-intercept $(0, 4)$
   Domain $(-\infty, \infty)$
   Range $[4, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 4 at $(0, 4)$
   No maximum



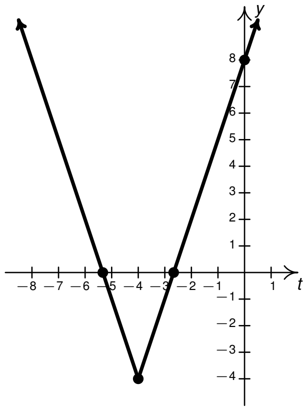
3. $f(x) = |4x|$
   x-intercept $(0, 0)$
   y-intercept $(0, 0)$
   Domain $(-\infty, \infty)$
   Range $[0, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Minimum is 0 at $(0, 0)$
   No maximum



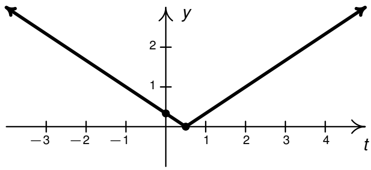
4. $g(t) = -3|t|$
t-intercept $(0,0)$
y-intercept $(0,0)$
Domain $(-\infty,\infty)$
Range $(-\infty,0]$
Increasing on $(-\infty,0]$
Decreasing on $[0,\infty)$
Maximum is 0 at $(0,0)$
No minimum



5. $g(t) = 3|t+4| - 4$
t-intercepts $\left(-\frac{16}{3},0\right)$, $\left(-\frac{8}{3},0\right)$
y-intercept $(0,8)$
Domain $(-\infty,\infty)$
Range $[-4,\infty)$
Decreasing on $(-\infty,-4]$
Increasing on $[-4,\infty)$
Minimum is -4 at $(-4,-4)$
No maximum



6. $g(t) = \frac{1}{3}|2t-1|$
t-intercepts $\left(\frac{1}{2},0\right)$
y-intercept $\left(0,\frac{1}{3}\right)$
Domain $(-\infty,\infty)$
Range $[0,\infty)$
Decreasing on $\left(-\infty,\frac{1}{2}\right]$
Increasing on $\left[\frac{1}{2},\infty\right)$
Minimum is 0 at $\left(\frac{1}{2},0\right)$
No maximum



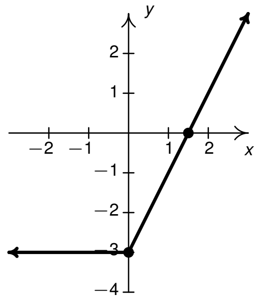
7. $F(x) = 2|x+1| - 3$
8. $F(x) = |x-1.25| - 2.75$
9. $F(x) = -|x+1| + 2$
10. $F(x) = -\frac{1}{2}|x+1| + \frac{3}{2}$

11. In each case, the graph of g can be obtained from the graph of f by reflecting the portion of the graph of f which lies below the x-axis about the x-axis. This meshes with Definition 1.3.1 since what we are doing algebraically is making the negative y-values positive.

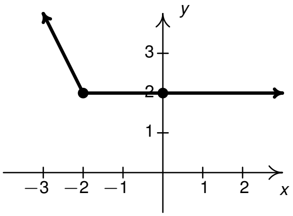
12. If $F(x) = a|x - h| + k$, then for the vertex to be at $(1, -2)$, $h = 1$ and $k = -2$ so $F(x) = a|x - 1| - 2$. Since $(0, -1)$ is on the graph, $F(0) = -1$ so $-1 = a|0 - 1| - 2$ which means $a = 1$. This means $F(x) = |x - 1| - 2$. However, $(2.6, 0)$ is also on the graph, so it should work out that $F(2.6) = 0$. However, we find $F(2.6) = |2.6 - 1| - 2 = -0.4 \neq 0$.

$$F(x) = \begin{cases} -x - 1 & \text{if } x \leq 1, \\ \frac{5}{4}x - \frac{13}{4} & \text{if } x \geq 1, \end{cases}$$

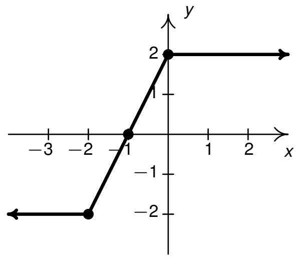
13. Re-write $f(x) = x + |x| - 3$ as
$f(x) = \begin{cases} -3 & \text{if} \quad x < 0 \\ 2x - 3 & \text{if} \quad x \geq 0 \end{cases}$
x-intercept $\left(\frac{3}{2}, 0\right)$
y-intercept $(0, -3)$
Domain $(-\infty, \infty)$
Range $[-3, \infty)$
Increasing on $[0, \infty)$
Constant on $(-\infty, 0]$
Minimum is -3 at $(x, -3)$ where $x \leq 0$
No maximum



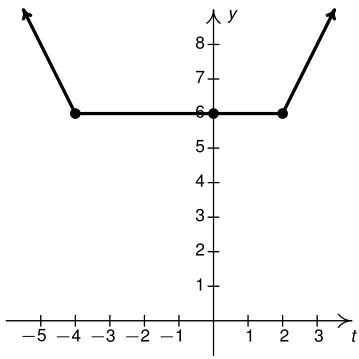
14. Re-write $f(x) = |x + 2| - x$ as
$f(x) = \begin{cases} -2x - 2 & \text{if} \quad x < -2 \\ 2 & \text{if} \quad x \geq -2 \end{cases}$
No x-intercepts
y-intercept $(0, 2)$
Domain $(-\infty, \infty)$
Range $[2, \infty)$
Decreasing on $(-\infty, -2]$
Constant on $[-2, \infty)$
Minimum is 2 at every point $(x, 2)$ where $x \geq -2$
No maximum



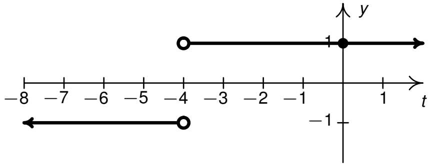
15. Re-write $f(x) = |x+2| - |x|$ as
$$f(x) = \begin{cases} -2 & \text{if} \quad x < -2 \\ 2x+2 & \text{if} \quad -2 \le x < 0 \\ 2 & \text{if} \quad x \ge 0 \end{cases}$$
x-intercept $(-1, 0)$
y-intercept $(0, 2)$
Domain $(-\infty, \infty)$
Range $[-2, 2]$
Increasing on $[-2, 0]$
Constant on $(-\infty, -2]$
Constant on $[0, \infty)$
Minimum is -2 at $(x, -2)$ where $x \le -2$
Maximum is 2 at $(x, 2)$ where $x \ge 0$



16. Re-write $g(t) = |t+4| + |t-2|$ as
$$g(t) = \begin{cases} -2t-2 & \text{if} \quad t < -4 \\ 6 & \text{if} \quad -4 \le t < 2 \\ 2t+2 & \text{if} \quad t \ge 2 \end{cases}$$
No t-intercept
y-intercept $(0, 6)$
Domain $(-\infty, \infty)$
Range $[6, \infty)$
Decreasing on $(-\infty, -4]$
Constant on $[-4, 2]$
Increasing on $[2, \infty)$
Minimum is 6 at $(t, 6)$ where $-4 \le t \le 2$
No maximum



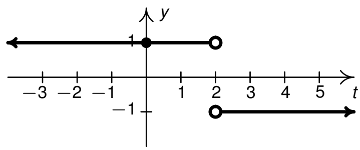
17. Re-write $g(t) = \dfrac{|t+4|}{t+4}$ as



$$g(t) = \begin{cases} -1 & \text{if} \quad t < -4 \\ 1 & \text{if} \quad t > -4 \end{cases}$$

No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, -4) \cup (-4, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, -4)$
Constant on $(-4, \infty)$
Minimum is -1 at every point $(t, -1)$ where $t < -4$
Maximum is 1 at $(t, 1)$ where $t > -4$

18. Re-write $g(t) = \dfrac{|2-t|}{2-t}$ as



$$g(t) = \begin{cases} 1 & \text{if} \quad t < 2 \\ -1 & \text{if} \quad t > 2 \end{cases}$$

No t-intercept
y-intercept $(0, 1)$
Domain $(-\infty, 2) \cup (2, \infty)$
Range $\{-1, 1\}$
Constant on $(-\infty, 2)$
Constant on $(2, \infty)$
Minimum is -1 at $(t, -1)$ where $t > 2$
Maximum is 1 at every point $(t, 1)$ where $t < 2$

19. $f(x) = \big||x| - 4\big|$
20. $x = -6$ or $x = 6$
21. $x = -3$ or $x = \frac{11}{3}$
22. $x = -3$ or $x = 11$
23. $t = -1$ or $t = 1$
24. $t = -\frac{1}{2}$ or $t = \frac{1}{10}$
25. no solution
26. $w = -3$ or $w = 3$
27. $w = -\frac{13}{8}$ or $w = \frac{53}{8}$
28. $w = -\frac{3}{2}$
29. $x = 0$ or $x = 2$
30. $x = 1$
31. no solution
32. $x = -1$ or $x = 9$
33. $x = -\frac{1}{7}$ or $x = 1$
34. $x = 0$ or $x = 2$
35. $t = 1$
36. $t = -\frac{3}{10}$
37. $t = \frac{1}{5}$ or $t = 5$
38. $\left[\frac{1}{3}, 3\right]$
39. $\left(-\infty, -\frac{12}{7}\right) \cup \left(\frac{8}{7}, \infty\right)$
40. $(-3, 2)$
41. $(-\infty, 1] \cup [3, \infty)$

42. No solution
43. $(-\infty, \infty)$
44. $(-3, 2] \cup [6, 11)$
45. $[3, 4) \cup (5, 6]$
46. $\left[-\frac{12}{7}, -\frac{6}{5}\right]$
47. $(-\infty, -4) \cup \left(\frac{2}{3}, \infty\right)$
48. $\left(-\infty, -\frac{4}{3}\right] \cup [6, \infty)$
49. $(-\infty, -5)$
50. No Solution.
51. $\left[-7, \frac{5}{3}\right]$
52. $\left(1, \frac{5}{3}\right)$
53. $(-\infty, \infty)$

1.4 Quadratic Functions

1.4.1 Graphs of Quadratic Functions

You may recall studying quadratic equations in a previous Algebra course. If not, you may wish to refer to Section **??** to revisit this topic. In this section, we review those equations in the context of our next family of functions: the quadratic functions.

Definition 1.4.1. A **quadratic function** is a function of the form

$$f(x) = ax^2 + bx + c,$$

where a, b and c are real numbers with $a \neq 0$. The domain of a quadratic function is $(-\infty, \infty)$.

As in Definitions 1.2.1 and 1.2.2, the independent variable in Definition 1.4.1 is x while the values a, b and c are parameters. Note that $a \neq 0$ - otherwise we would have a linear function (see Definition 1.2.2).

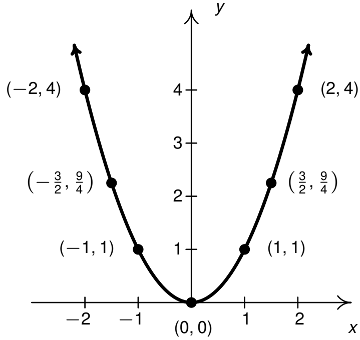
The most basic quadratic function is $f(x) = x^2$, the squaring function, whose graph appears in Figure 1.4.1 along with a corresponding table of values. Its shape may look familiar from your previous studies in Algebra – it is called a **parabola**. The point $(0, 0)$ is called the **vertex** of the parabola because it is the sole point where the function obtains its extreme value, in this case, a minimum of 0 when $x = 0$.

Indeed, the range of $f(x) = x^2$ appears to be $[0, \infty)$ from the graph. We can substantiate this algebraically since for all x, $f(x) = x^2 \geq 0$. This tells us that the range of f is a subset of $[0, \infty)$. To show that the range of f actually equals $[0, \infty)$, we need to show that every real number c in $[0, \infty)$ is in the range of f. That is, for every $c \geq 0$, we have to show c is an output from f. In other words, we have to show there is a real number x so that $f(x) = x^2 = c$. Choosing $x = \sqrt{c}$, we find $f(x) = f(\sqrt{c}) = (\sqrt{c})^2 = c$, as required.[1]

The techniques we used to graph many of the absolute value functions in Section 1.3 can be applied to quadratic functions, too. In fact, knowing the graph of $f(x) = x^2$ enables us to graph *every* quadratic function, but there's some extra work involved. We start with the following theorem:

[1] This assumes, of course, $\sqrt{c}$ is a real number for all real numbers $c \geq 0$ . . .

x	$f(x) = x^2$
-2	4
$-\frac{3}{2}$	$\frac{9}{4}$
-1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



$f(x) = x^2$

Figure 1.4.1: Graphing $f(x) = x^2$

> **Theorem 1.4.1.** For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x - h)^2 + k$ is a parabola with vertex (h, k). If $a > 0$, the graph resembles '$\smile$.' If $a < 0$, the graph resembles '$\frown$.' Moreover, the vertical line $x = h$ is the **axis of symmetry** of the graph of $y = F(x)$.

To prove Theorem 1.4.1 the reader is encouraged to revisit the discussion following the proof of Theorem 1.3.1, replacing every occurrence of absolute value notation with the squared exponent.[2] Alternatively, the reader can skip ahead and read the statement and proof of Theorem 2.1.1 in Section 2.1. In the meantime we put Theorem 1.4.1 to good use in the next example.

Example 1.4.1.

1. Graph the following functions using Theorem 1.4.1. Find the vertex, zeros and axis-intercepts (if any exist). Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

   (a) $f(x) = \dfrac{(x-3)^2}{2}$ (b) $g(x) = (x+2)^2 - 3$

[2] i.e., replace $|x|$ with x^2, $|c|$ with c^2, $|x - h|$ with $(x - h)^2$.

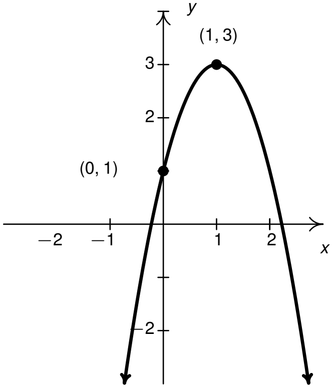


Figure 1.4.2: $y = H(x)$

(c) $h(t) = -2(t-3)^2 + 1$

(d) $i(t) = \dfrac{(3-2t)^2+1}{2}$

2. Use Theorem 1.4.1 to write a possible formula for $H(x)$ whose graph is given in Figure 1.4.2.

Solution.

1. (a) For $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}(x-3)^2 + 0$, we identify $a = \frac{1}{2}$, $h = 3$ and $k = 0$. Thus the vertex is $(3, 0)$ and the parabola opens upwards. The only x-intercept is $(3, 0)$. Since $f(0) = \frac{1}{2}(0-3)^2 = \frac{9}{2}$, our y-intercept is $\left(0, \frac{9}{2}\right)$. To help us graph the function, it would be nice to have a third point and we'll use symmetry to find it. The y-value three units to the *left* of the vertex is 4.5, so the y-value must be 4.5 three units to the *right* of the vertex as well. Hence, we have our third point: $\left(6, \frac{9}{2}\right)$. From the graph, we get that the range is $[0, \infty)$ and see that f has the minimum value of 0 at $x = 3$ and no maximum. Also, f is decreasing on $(-\infty, 3]$ and increasing on $[3, \infty)$. The graph is in Figure 1.4.3.

   (b) For $g(x) = (x+2)^2-3 = (1)(x-(-2))^2+(-3)$, we identify $a = 1$, $h = -2$ and $k = -3$. This means that the vertex is $(-2, -3)$ and the parabola opens upwards. Thus we have two x-intercepts. To find them, we set $y = g(x) = 0$ and solve. Doing so yields the equation $(x+2)^2 - 3 = 0$, or $(x+2)^2 = 3$. Extracting square roots gives us the two zeros of g: $x + 2 = \pm\sqrt{3}$, or $x = -2 \pm \sqrt{3}$. Our x-intercepts are $(-2-\sqrt{3}, 0) \approx$

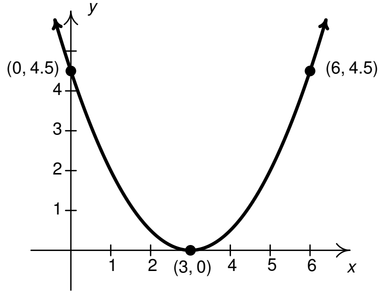


Figure 1.4.3: $f(x) = \frac{(x-3)^2}{2}$

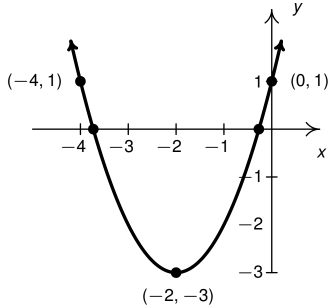


Figure 1.4.4: $g(x) = (x+2)^2 - 3$

$(-3.73, 0)$ and $(-2+\sqrt{3}, 0) \approx (-0.27, 0)$. We find $g(0) = (0+2)^2 - 3 = 1$ so our y-intercept is $(0, 1)$. Using symmetry, we get $(-4, 1)$ as another point to help us graph. The range of g is $[-3, \infty)$. The minimum of g is -3 at $x = -2$, and g has no maximum. Moreover, g is decreasing on $(-\infty, -2]$ and g is increasing on $[-2, \infty)$. The graph is in Figure 1.4.4

(c) Given $h(t) = -2(t-3)^2 + 1$, we identify $a = -2$, $h = 3$ and $k = 1$. Hence the vertex of the graph is $(3, 1)$ and the parabola opens downwards. Solving $h(t) = -2(t-3)^2 + 1 = 0$ gives $(t-3)^2 = \frac{1}{2}$. Extracting square roots[3] gives $t - 3 = \pm\frac{\sqrt{2}}{2}$, so that when we add 3 to each side,[4] we get $t = \frac{6\pm\sqrt{2}}{2}$. Hence, our t-intercepts are $\left(\frac{6-\sqrt{2}}{2}, 0\right) \approx (2.29, 0)$ and $\left(\frac{6+\sqrt{2}}{2}, 0\right) \approx (3.71, 0)$. To find the y-intercept, we compute $h(0) = -2(0-3)^2 + 1 = -17$. Thus the y-intercept is $(0, -17)$. Using symmetry, we also have that $(6, -17)$ is on the graph in Figure 1.4.5.

(d) We have some work ahead of us to put $i(t)$ into a form we can use to exploit Theorem 1.4.1:

$$\begin{aligned} i(t) = \frac{(3-2t)^2+1}{2} &= \frac{1}{2}(-2t+3)^2 + \frac{1}{2} \\ &= \frac{1}{2}\left[-2\left(t - \frac{3}{2}\right)\right]^2 + \frac{1}{2} \end{aligned}$$

[3] and rationalizing denominators!
[4] and get common denominators!

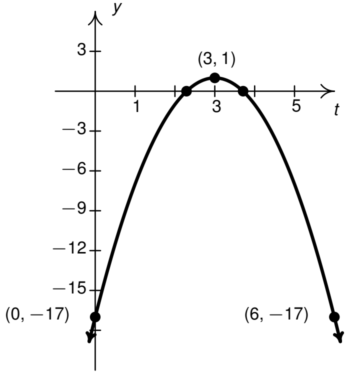


Figure 1.4.5: $h(t) = -2(t-3)^2 + 1$

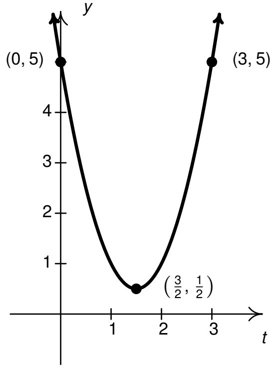


Figure 1.4.6: $i(t) = \frac{(3-2t)^2+1}{2}$

$$= \frac{1}{2}(-2)^2\left(t - \frac{3}{2}\right)^2 + \frac{1}{2}$$

$$= 2\left(t - \frac{3}{2}\right)^2 + \frac{1}{2}$$

We identify $a = 2$, $h = \frac{3}{2}$ and $k = \frac{1}{2}$. Hence our vertex is $\left(\frac{3}{2}, \frac{1}{2}\right)$ and the parabola opens upwards, meaning there are no t-intercepts. Since $i(0) = \frac{(3-2(0))^2+1}{2} = 5$, we get $(0, 5)$ as the y-intercept. Using symmetry, this means we also have $(3, 5)$ on the graph. The range is $\left[\frac{1}{2}, \infty\right)$ with the minimum of i, $\frac{1}{2}$, occurring when $t = \frac{3}{2}$. Also, i is decreasing on $\left(-\infty, \frac{3}{2}\right]$ and increasing on $\left[\frac{3}{2}, \infty\right)$. The graph is given in Figure 1.4.6.

2. We are instructed to use Theorem 1.4.1, so we know $H(x) = a(x-h)^2 + k$ for some choice of parameters a, h and k. The vertex is $(1, 3)$ so we know $h = 1$ and $k = 3$, and hence $H(x) = a(x-1)^2 + 3$. To find the value of a, we use the fact that the y-intercept, as labeled, is $(0, 1)$. This means $H(0) = 1$, or $a(0-1)^2 + 3 = 1$. This reduces to $a + 3 = 1$ or $a = -2$. Our final answer[5] is $H(x) = -2(x-1)^2 + 3$. □

A few remarks about Example 1.4.1 are in order. First note that none of the functions are in the form of Definition 1.4.1. However, if we took the time to perform the indicated operations and simplify, we'd find:

[5]The reader is encouraged to compare this example with number 2 of Example 1.3.2.

- $f(x) = \frac{(x-3)^2}{2} = \frac{1}{2}x^2 - 3x + \frac{9}{2}$
- $h(t) = -2(t-3)^2 + 1 = -2t^2 + 12t - 17$
- $g(x) = (x+2)^2 - 3 = x^2 + 4x + 1$
- $i(t) = \frac{(3-2t)^2+1}{2} = 2t^2 - 6t + 5$

While the y-intercepts of the graphs of the each of the functions are easier to see when the formulas for the functions are written in the form of Definition 1.4.1, the vertex is not. For this reason, the form of the functions presented in Theorem 1.4.1 are given a special name.

Definition 1.4.2. Standard and General Form of Quadratic Functions:
- The **general form** of the quadratic function f is $f(x) = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.
- The **standard form** of the quadratic function f is $f(x) = a(x-h)^2+k$, where a, h and k are real numbers with $a \neq 0$.

If we proceed as in the remarks following Example 1.4.1, we can convert any quadratic function given to us in standard form and convert to general form by performing the indicated operation and simplifying:

$$\begin{aligned} f(x) &= a(x-h)^2 + k \\ &= a\left(x^2 - 2hx + h^2\right) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k). \end{aligned}$$

With the identifications $b = -2ah$ and $c = ah^2 + k$, we have written $f(x)$ in the form $f(x) = ax^2 + bx + c$. Likewise, through a process known as 'completing the square', we can take any quadratic function written in general form and rewrite it in standard form. We briefly review this technique in the following example – for a more thorough review the reader should see Section **??**.

Example 1.4.2. Graph the following functions. Find the vertex, zeros and axis-intercepts, if any exist. Find the extrema and then list the intervals over which the function is increasing, decreasing or constant.

1. $f(x) = x^2 - 4x + 3$.
2. $g(t) = 6 - 4t - 2t^2$

Solution.

1. We follow the procedure for completing the square in Section **??**. The only difference here is instead of the quadratic equation being set to 0, it is equal

to $f(x)$. This means when we are finished completing the square, we need to solve for $f(x)$.

$$\begin{aligned} f(x) &= x^2 - 4x + 3 & \\ f(x) - 3 &= x^2 - 4x & \text{(Subtract 3 from both sides)} \\ f(x) - 3 + (-2)^2 &= x^2 - 4x + (-2)^2 & \text{(Add } \left(\tfrac{1}{2}(-4)\right)^2 \text{ to both sides.)} \\ f(x) + 1 &= (x-2)^2 & \text{(Factor the perfect square trinomial.)} \\ f(x) &= (x-2)^2 - 1 & \text{(Solve for } f(x)\text{.)} \end{aligned}$$

The reader is encouraged to start with $f(x) = (x-2)^2 - 1$, perform the indicated operations and simplify the result to $f(x) = x^2 - 4x + 3$. From the standard form, $f(x) = (x-2)^2 - 1$, we see that the vertex is $(2, 1)$ and that the parabola opens upwards. To find the zeros of f, we set $f(x) = 0$.

We have two equivalent expressions for $f(x)$ so we could use either the general form or standard form. We solve the former and leave it to the reader to solve the latter to see that we get the same results either way. To solve $x^2 - 4x + 3 = 0$, we factor: $(x-3)(x-1) = 0$ and obtain $x = 1$ and $x = 3$. We get two x-intercepts, $(1, 0)$ and $(3, 0)$.

To find the y-intercept, we need $f(0)$. Again, we could use either form of $f(x)$ for this and we choose the general form and find that the y-intercept is $(0, 3)$. From symmetry, we know the point $(4, 3)$ is also on the graph. We see that the range of f is $[-1, \infty)$ with the minimum -1 at $x = 2$. Finally, f is decreasing on $(-\infty, 2]$ and increasing from $[2, \infty)$. The graph is given in Figure 1.4.7

2. We first rewrite $g(t) = 6 - 4t - 2t^2$ as $g(t) = -2t^2 - 4t + 6$. As with the previous example, once we complete the square, we solve for $g(t)$:

$$\begin{aligned} g(t) &= -2t^2 - 4t + 6 & \\ g(t) - 6 &= -2t^2 - 4t & \text{(Subtract 6 from both sides.)} \\ \frac{g(t) - 6}{-2} &= \frac{-2t^2 - 4t}{-2} & \text{(Divide both sides by } -2\text{.)} \\ \frac{g(t) - 6}{-2} + (1)^2 &= t^2 + 2t + (1)^2 & \text{(Add } \left(\tfrac{1}{2}(2)\right)^2 \text{ to both sides.)} \end{aligned}$$

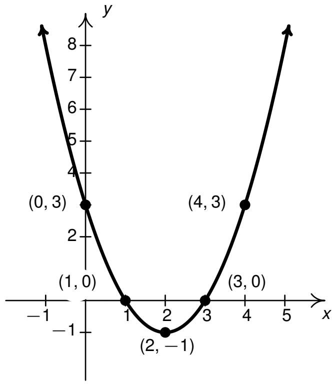


Figure 1.4.7: $f(x) = x^2 - 4x + 3$

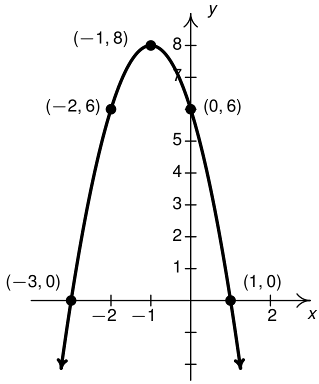


Figure 1.4.8: $g(x) = 6 - 4t - 2t^2$

$$\frac{g(t)-6}{-2} + 1 = (t+1)^2 \qquad \text{(Factor the prefect square trinomial.)}$$

$$\frac{g(t)-6}{-2} = (t+1)^2 - 1$$

$$g(t) - 6 = -2\left[(t+1)^2 - 1\right]$$

$$g(t) = -2(t+1)^2 + 2 + 6$$

$$g(t) = -2(t+1)^2 + 8$$

We can check our answer by expanding $-2(t+1)^2 + 8$ and show that it simplifies to $-2t^2 - 4t + 6$. From the standard form, we find that the vertex is $(-1, 8)$ and that the parabola opens downwards. Setting $g(t) = -2t^2 - 4t + 6 = 0$, we factor to get $-2(t-1)(t+3) = 0$ so $t = -3$ and $t = 1$. Hence, our two t-intercepts are $(-3, 0)$ and $(1, 0)$.

Since $g(0) = 6$, we get the y-intercept to be $(0, 6)$. Using symmetry, we also have the point $(-2, 6)$ on the graph. The range is $(-\infty, 8]$ with a maximum of 8 when $t = -1$. Finally we note that g is increasing on $(-\infty, -1]$ and decreasing on $[-1, \infty)$. The graph is in Figure 1.4.8.

□

We now generalize the procedure demonstrated in Example 1.4.2. Let $f(x) = ax^2 + bx + c$ for $a \neq 0$:

$$\begin{aligned}
f(x) &= ax^2 + bx + c \\
f(x) - c &= ax^2 + bx && \text{(Subtract } c \text{ from both sides)} \\
\frac{f(x) - c}{a} &= \frac{ax^2 + bx}{a} && \text{(Divide both sides by } a \neq 0\text{)} \\
\frac{f(x) - c}{a} &= x^2 + \frac{b}{a}x \\
\frac{f(x) - c}{a} + \left(\frac{b}{2a}\right)^2 &= x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 && \text{(Add } \left(\frac{b}{2a}\right)^2 \text{ to both sides)} \\
\frac{f(x) - c}{a} + \frac{b^2}{4a^2} &= \left(x + \frac{b}{2a}\right)^2 && \text{(Factor the perfect square trinomial)} \\
\frac{f(x) - c}{a} &= \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} && \text{(Solve for } f(x)\text{)} \\
f(x) - c &= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] \\
f(x) - c &= a\left(x + \frac{b}{2a}\right)^2 - a\frac{b^2}{4a^2} \\
f(x) &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \\
f(x) &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} && \text{(Get a common denominator)}
\end{aligned}$$

By setting $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$, we have written the function in the form $f(x) = a(x-h)^2 + k$. This establishes the fact that every quadratic function can be written in standard form.[6] Moreover, writing a quadratic function in standard form allows us to identify the vertex rather quickly, and so our work also shows us that the vertex of $f(x) = ax^2 + bx + c$ is $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$. It is not worth memorizing the expression $\frac{4ac-b^2}{4a}$ especially since we can write this as $f\left(-\frac{b}{2a}\right)$. (This about this last statement for a moment.)

[6]To avoid completing the square, we could solve the equations $b = -2ah$ and $c = ah^2 + k$ for h and k. See Exercise 54.

We summarize the information detailed above in the following box:

Equation 1.4.1. Vertex Formulas for Quadratic Functions: Suppose a, b, c, h and k are real numbers where $a \neq 0$.

- If $f(x) = a(x-h)^2 + k$ then the vertex of the graph of $y = f(x)$ is the point (h, k).
- If $f(x) = ax^2 + bx + c$ then the vertex of the graph of $y = f(x)$ is the point $\left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

Completing the square is also the means by which we may derive the celebrated Quadratic Formula, a formula which returns the solutions to $ax^2+bx+c = 0$ for $a \neq 0$. Before we state it here for reference, we wish to encourage the reader to pause a moment and read the derivation if the Quadratic Formula found in Section **??**. The work presented in this section transforms the general form of a quadratic *function* into the standard form whereas the work in Section **??** finds a formula to solve an *equation*. There is great value in understanding the similarities and differences between the two approaches.

Equation 1.4.2. The Quadratic Formula: The zeros of the quadratic function $f(x) = ax^2 + bx + c$ are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It is worth pointing out the symmetry inherent in Equation 1.4.2. We may rewrite the zeros as:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

so that, if there are real zeros, they (like the rest of the parabola) are symmetric about the line $x = -\frac{b}{2a}$. Another way to view this symmetry is that the x-coordinate of the vertex is the average of the zeros. We encourage the reader to verify this fact in all of the preceding examples, where applicable.

Next, recall that if the quantity $b^2 - 4ac$ is strictly negative then we do not have any real zeros. This quantity is called the *discriminant* and is useful in determining the number and nature of solutions to a quadratic equation. We remind the reader of this below.

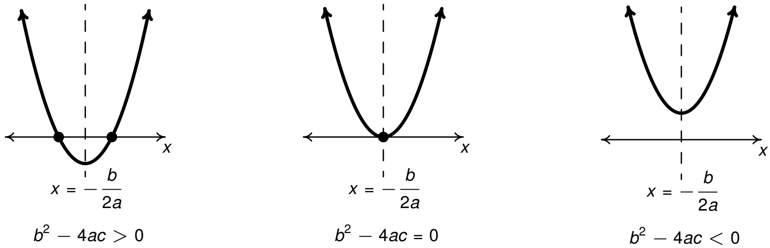


Figure 1.4.9: Zeroes of $ax^2 + bx + c$

Equation 1.4.3. The Discriminant of a Quadratic Function: Given a quadratic function in general form $f(x) = ax^2 + bx + c$, the **discriminant** is the quantity $b^2 - 4ac$.

- If $b^2 - 4ac > 0$ then f has two unequal (distinct) real zeros.
- If $b^2 - 4ac = 0$ then f has one (repeated) real zero.
- If If $b^2 - 4ac < 0$ then f has two unequal (distinct) non-real zeros.

We'll talk more about what we mean by a 'repeated' zero and how to compute 'non-real' zeros in Chapter 2. For us, the discriminant has the graphical implication that if $b^2 - 4ac > 0$ then we have two x-intercepts; if $b^2 - 4ac = 0$ then we have just one x-intercept, namely, the vertex; and if $b^2 - 4ac < 0$ then we have no x-intercepts because the parabola lies entirely above or below the x-axis. We sketch each of these scenarios in Figure 1.4.9 assuming $a > 0$. (The sketches for $a < 0$ are similar - see Exercise 49.)

We now revisit the economic scenario first described in Examples 1.2.3 and 1.2.4 where we were producing and selling PortaBoy game systems. Recall that the cost to produce x PortaBoys is denoted by $C(x)$ and the price-demand function, that is, the price to charge in order to sell x systems is denoted by $p(x)$. We introduce two more related functions below: the **revenue** and **profit** functions.

Definition 1.4.3. Revenue and Profit: Suppose $C(x)$ represents the cost to produce x units and $p(x)$ is the associated price-demand function. Under the assumption that we are producing the same number of units as are being sold:

- The **revenue** obtained by selling x units is $R(x) = x\,p(x)$.
  That is, revenue = (number of items sold) · (price per item).
- The **profit** made by selling x units is $P(x) = R(x) - C(x)$.
  That is, profit = (revenue) − (cost).

Said differently, the *revenue* is the amount of money *collected* by selling x items whereas the *profit* is how much money is *left over* after the costs are paid.

Example 1.4.3. In Example 1.2.3 the cost to produce x PortaBoy game systems for a local retailer was given by $C(x) = 80x + 150$ for $x \geq 0$ and in Example 1.2.4 the price-demand function was found to be $p(x) = -1.5x + 250$, for $0 \leq x \leq 166$.

1. Find formulas for the associated revenue and profit functions; include the domain of each.
2. Find and interpret $P(0)$.
3. Find and interpret the zeros of P.
4. Graph $y = P(x)$. Find the vertex and axis intercepts.
5. Interpret the vertex of the graph of $y = P(x)$.
6. What should the price per system be in order to maximize profit?
7. Find and interpret the average rate of change of P over the interval $[0, 57]$.

Solution.

1. The formula for the revenue function is $R(x) = x\,p(x) = x(-1.5x + 250) = -1.5x^2 + 250x$. Since the domain of p is restricted to $0 \leq x \leq 166$, so is the domain of R. To find the profit function $P(x)$, we subtract $P(x) = R(x) - C(x) = \left(-1.5x^2 + 250x\right) - (80x + 150) = -1.5x^2 + 170x - 150$. The cost function formula is valid for $x \geq 0$, but the revenue function is valid when $0 \leq x \leq 166$. Hence, the domain of P is likewise restricted to $[0, 166]$.
2. We find $P(0) = -1.5(0)^2 + 170(0) - 150 = -150$. This means that if we produce and sell 0 PortaBoy game systems, we have a profit of $-\$150$. Since profit = (revenue) − (cost), this means our costs exceed our revenue

by \$150. This makes perfect sense, since if we don't sell any systems, our revenue is \$0 but our fixed costs (see Example 1.2.3) are \$150.

3. To find the zeros of P, we set $P(x) = 0$ and solve $-1.5x^2 + 170x - 150 = 0$. Factoring here would be challenging to say the least, so we use the Quadratic Formula, Equation 1.4.2. Identifying $a = -1.5$, $b = 170$ and $c = -150$, we obtain

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-170 \pm \sqrt{170^2 - 4(-1.5)(-150)}}{2(-1.5)} \\ &= \frac{-170 \pm \sqrt{28000}}{-3} \\ &= \frac{170 \pm 20\sqrt{70}}{3} \\ &\approx 0.89,\ 112.44. \end{aligned}$$

Given that profit = (revenue) − (cost), if profit = 0, then revenue = cost. Hence, the zeros of P are called the 'break-even' points - where just enough product is sold to recover the cost spent to make the product. Also, x represents a number of game systems, which is a whole number, so instead of using the exact values of the zeros, or even their approximations, we consider $x = 0$ and $x = 1$ along with $x = 112$ and $x = 113$. We find $P(0) = -150$, $P(1) = 18.5$, $P(112) = 74$ and $P(113) = -93.5$. These data suggest that, in order to be profitable, at least 1 but not more than 112 systems should be produced and sold, as borne out in the graph below.

4. Knowing the zeros of P, we have two x-intercepts: $\left(\frac{170-20\sqrt{70}}{3}, 0\right) \approx (0.89, 0)$ and $\left(\frac{170+20\sqrt{70}}{3}, 0\right) \approx (112.44, 0)$. Since $P(0) = -150$, we get the y-intercept is $(0, -150)$. To find the vertex, we appeal to Equation 1.4.1. Substituting $a = -1.5$ and $b = 170$, we get $x = -\frac{170}{2(-1.5)} = \frac{170}{3} = 56.\overline{6}$. To find the y-coordinate of the vertex, we compute $P\left(\frac{170}{3}\right) = \frac{14000}{3} = 4666.\overline{6}$. Hence, the vertex is $(56.\overline{6}, 4666.\overline{6})$. The domain is restricted $0 \leq x \leq 166$ and we find $P(166) = -13264$. Attempting to plot all of these points on the same graph to any sort of scale is challenging. Instead, we present a portion of the graph for $0 \leq x \leq 113$. Even with this, the intercepts near the origin are crowded.

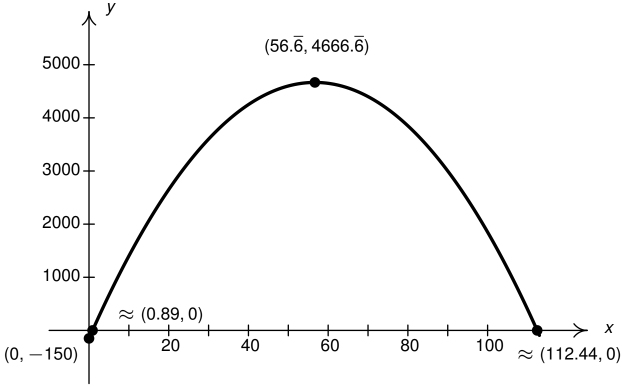


Figure 1.4.10: $y = P(x)$

5. From the vertex, we see that the maximum of P is $4666.\overline{6}$ when $x = 56.\overline{6}$. As before, x represents the number of PortaBoy systems produced and sold, so we cannot produce and sell $56.\overline{6}$ systems. Hence, by comparing $P(56) = 4666$ and $P(57) = 4666.5$, we conclude that we will make a maximum profit of \$4666.50 if we sell 57 game systems.

6. We've determined that we need to sell 57 PortaBoys to maximize profit, so we substitute $x = 57$ into the price-demand function to get $p(57) = -1.5(57) + 250 = 164.5$. In other words, to sell 57 systems, and thereby maximize the profit, we should set the price at \$164.50 per system.

7. To find the average rate of change of P over $[0, 57]$, we compute

$$\frac{\Delta[P(x)]}{\Delta x} = \frac{P(57) - P(0)}{57 - 0} = \frac{4666.5 - (-150)}{57} = 84.5.$$

This means that as the number of systems produced and sold ranges from 0 to 57, the average profit per system is increasing at a rate of \$84.50. In other words, for each additional system produced and sold, the profit increased by \$84.50 on average. □

We hope Example 1.4.3 shows the value of using a continuous model to describe a discrete situation. True, we could have 'run the numbers' and computed $P(1)$, $P(2)$, ..., $P(166)$ to eventually determine the maximum profit, but the vertex formula made much quicker work of the problem.

Along these same lines, in our next example we revisit Skippy's temperature data from Example 1.1.1 in Section 1.1. We found a piecewise-linear model in Section 1.2 to model the temperature over the course the day and now we seek a quadratic function to do the job. The methodology used here is similar to that of the least squares regression line discussed in Section 1.2.3 but instead of finding the line closest to the data points, we want the *parabola* closest to them that comes from a function of the form $f(x) = ax^2 + bx + c$. The Mathematics required to find the desired quadratic function is beyond the scope of this text, but most graphing utilities can do these quickly. In the quadratic case, the machine will return a value of R^2 such that $0 \leq R^2 \leq 1$. The closer R^2 is to 1, the better the fit. (Again, how R^2 is computed is beyond this text.)

Example 1.4.4.

1. Use a graphing utility to fit a quadratic model to the time and temperature data in Example 1.1.1. Comment on the goodness of fit.
2. Use your model to predict the temperature at 7 AM and 3 PM. Round your answers to one decimal place. How do your results compare with those from Example 1.2.7?
3. According to the model, what was the warmest temperature of the day? When did that occur? Round your answers to one decimal place.

Solution.

1. Entering the data in Desmos we find $T = F(t) = -0.1905t^2 + 3.9643t + 62.405$ with an R^2 value of 0.97, indicating a pretty strong fit. See Figure 1.4.11.
2. Since 7 AM corresponds to $t = 1$, we find $T = F(1) \approx 66.18$. Hence our quadratic model predicts a temperature of 66.2° F at 7 AM - identical (when rounded) to the 66.2° F predicted in Example 1.2.7. Similarly, 3 PM corresponds to $t = 9$, so we find $T = F(9) \approx 82.65$. Thus the model predicts an outdoor temperature of 82.6° F which is very close to the 82.9°F prediction from Example 1.2.7.
3. The model is quadratic with $a < 0$ so the maximum (warmest) temperature can be determined by finding the vertex. We get

$$t = -\frac{b}{2a} = -\frac{3.9643}{2(-0.1905)} \approx -10.40, \quad T = F(-10.40) \approx 83.03,$$

or, in other words, the warmest temperature is 83.0° F at 4:24 PM (10.40 hours after 6 AM.)

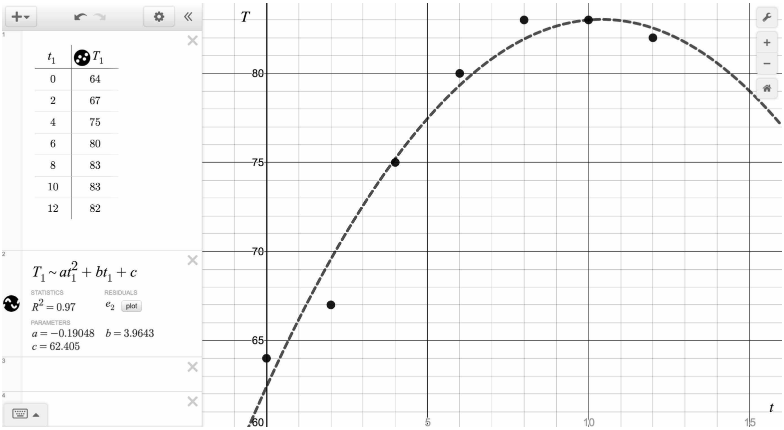


Figure 1.4.11: Graphing using desmos

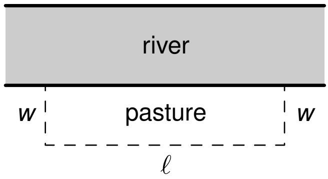
□

It is interesting how close the predictions from Examples 1.2.7 and 1.4.4 despite one using linear models and one using a quadratic model. Which model is the 'better' model? We leave that discussion to the reader and their classmates.

Our next example is classic application of optimizing a quadratic function.

Example 1.4.5. Much to Donnie's surprise and delight, he inherits a large parcel of land in Ashtabula County from one of his (e)strange(d) relatives so the time is right for him to pursue his dream of raising alpaca. He wishes to build a rectangular pasture and estimates that he has enough money for 200 linear feet of fencing material. If he makes the pasture adjacent to a river (so that no fencing is required on that side), what are the dimensions of the pasture which maximize the area? What is the maximum area? If an average alpaca needs 25 square feet of grazing area, how many alpaca can Donnie keep in his pasture?

Solution. We are asked to find the dimensions of the pasture which would give a maximum area, so we begin by sketching the diagram seen in Figure 1.4.12 on the left. We let w denote the width of the pasture and we let ℓ denote the length of the pasture. The units given to us in the statement of the problem are feet, so we assume that w and ℓ are measured in feet. The area of the pasture, which we'll call A, is related to w and ℓ by the equation $A = w\ell$. Since w and ℓ are both measured in feet, A has units of feet^2, or square feet.



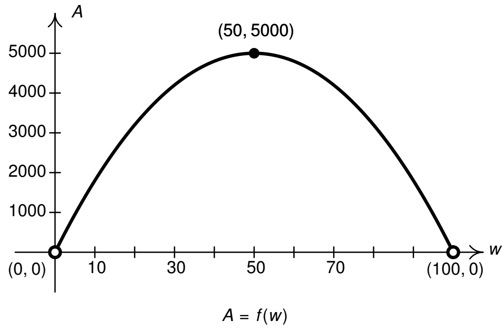


Figure 1.4.12: Fencing the pasture

We are also told that the total amount of fencing available is 200 feet, which means $w + \ell + w = 200$, or, $\ell + 2w = 200$. We now have two equations, $A = w\ell$ and $\ell + 2w = 200$. In order to use the tools given to us in this section to *maximize* A, we need to use the information given to write A as a function of just *one* variable, either w or ℓ. This is where we use the equation $\ell + 2w = 200$. Solving for ℓ, we find $\ell = 200 - 2w$, and we substitute this into our equation for A. We get $A = w\ell = w(200 - 2w) = 200w - 2w^2$. We now have A as a function of w, $A = f(w) = 200w - 2w^2 = -2w^2 + 200w$.

Before we go any further, we need to find the applied domain of f so that we know what values of w make sense in this situation.[7] Given that w represents the width of the pasture we need $w > 0$. Likewise, ℓ represents the length of the pasture, so $\ell = 200 - 2w > 0$. Solving this latter inequality yields $w < 100$. Hence, the function we wish to maximize is $f(w) = -2w^2 + 200w$ for $0 < w < 100$. We know two things about the quadratic function f: the graph of $A = f(w)$ is a parabola and (since the coefficient of w^2 is -2) the parabola opens downwards.

This means that there is a maximum value to be found, and we know it occurs at the vertex. Using the vertex formula, we find $w = -\frac{200}{2(-2)} = 50$, and $A = f(50) = -2(50)^2 + 200(50) = 5000$. Since $w = 50$ lies in the applied domain, $0 < w < 100$, we have that the area of the pasture is maximized when the width is 50 feet. To find the length, we use $\ell = 200 - 2w$ and find $\ell = 200 - 2(50) = 100$, so the length of the pasture is 100 feet. The maximum area is $A = f(50) = 5000$,

[7] Donnie would be very upset if, for example, we told him the width of the pasture needs to be -50 feet.

or 5000 square feet. If an average alpaca requires 25 square feet of pasture, Donnie can raise $\frac{5000}{25} = 200$ average alpaca. $\square$

The function f in Example 1.4.5 is called the **objective function** for this problem - it's the function we're trying to optimize. In the case above, we were trying to maximize f. The equation $\ell + 2w = 200$ along with the inequalities $w > 0$ and $\ell > 0$ are called the **constraints**. As we saw in this example, and as we'll see again and again, the constraint equation is used to rewrite the objective function in terms of just one of the variables where constraint inequalities, if any, help determine the applied domain.

1.4.2 Inequalities involving Quadratic Functions

We now turn our attention to solving inequalities involving quadratic functions. Consider the inequality $x^2 \leq 6$. We could use the fact that the square root is increasing[8] to get: $\sqrt{x^2} \leq \sqrt{6}$, or $|x| \leq \sqrt{6}$. This reduces to $-\sqrt{6} \leq x \leq \sqrt{6}$ or, using interval notation, $[-\sqrt{6}, \sqrt{6}]$. If, however, we had to solve $x^2 \leq x + 6$, things are more complicated. One approach is to complete the square:

$$\begin{aligned}
x^2 &\leq x + 6 \\
x^2 - x &\leq 6 \\
x^2 - x + \frac{1}{4} &\leq 6 + \frac{1}{4} \\
\left(x - \frac{1}{2}\right)^2 &\leq \frac{25}{4} \\
\sqrt{\left(x - \frac{1}{2}\right)^2} &\leq \sqrt{\frac{25}{4}} \\
\left|x - \frac{1}{2}\right| &\leq \frac{5}{2} \\
-\frac{5}{2} \leq x - \frac{1}{2} &\leq \frac{5}{2} \\
-2 \quad \leq x &\leq 3
\end{aligned}$$

We get the solution $[-2, 3]$. While there is nothing wrong with this approach, we seek methods here that will generalize to higher degree polynomials such as those we'll see in Chapter 2.

[8] That is, if $a < b$, then $\sqrt{a} < \sqrt{b}$.

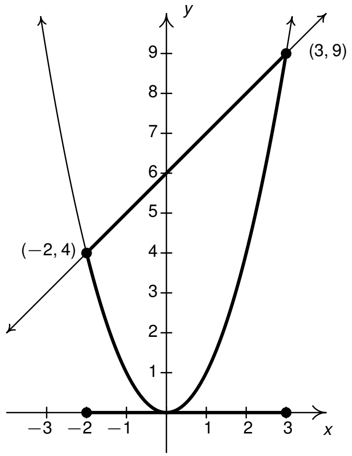


Figure 1.4.13: Solving $x^2 \leq x + 6$

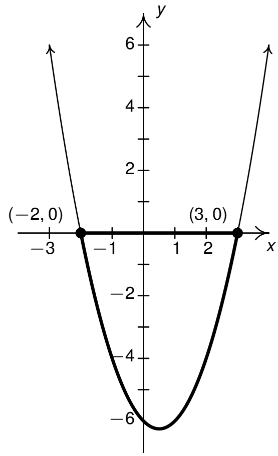


Figure 1.4.14: Solving $x^2 - x - 6 \leq 0$

To that end, we look at the inequality $x^2 \leq x + 6$ graphically. Identifying $f(x) = x^2$ and $g(x) = x + 6$, we graph f and g on the same set of axes in Figure 1.4.13 and look for where the graph of f (the parabola) meets or is below the graph of g (the line). There are two points of intersection which we determine by solving $f(x) = g(x)$ or $x^2 = x + 6$. As usual, we rewrite this equation as $x^2 - x - 6 = 0$ in order to use the primary tools we've developed to handle these types[9] of quadratic equations: factoring, or failing that, the Quadratic Formula. We find $x^2 - x - 6 = (x + 2)(x - 3)$ so we get two solutions to $(x + 2)(x - 3) = 0$, namely $x = -2$ and $x = 3$. Putting these together with the graph, we obtain the same solution: $[-2, 3]$.

Yet a third way to attack $x^2 \leq x + 6$ is to rewrite the inequality as $x^2 - x - 6 \leq 0$. Here, we graph $f(x) = x^2 - x - 6$ to look for where the graph meets or is below the graph of $g(x) = 0$, a.k.a. the x-axis. Doing so requires us to find the zeros of f, that is, solve $f(x) = x^2 - x - 6 = 0$ from which we obtain $x = -2$ and $x = 3$ as before. We find the same solution, $[-2, 3]$ as is showcased in the graph in Figure 1.4.14.

One advantage to using this last approach is that we are essentially concerned with one function and its *zeros*. This approach can be generalized to all functions—not just quadratics, so we take the time to develop this method more thoroughly

[9]Namely ones with a nonzero coefficient of 'x'.

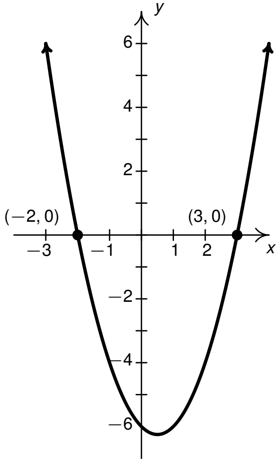


Figure 1.4.15: $f(x) = x^2 - x - 6$.

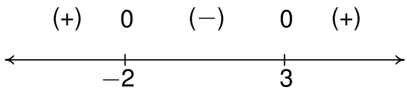


Figure 1.4.16: A sign diagram for $f(x) = x^2 - x - 6$

now.

Consider the graph of $f(x) = x^2 - x - 6$ in Figure 1.4.15. The zeros of f are $x = -2$ and $x = 3$ and they divide the domain (the x-axis) into three intervals: $(-\infty, -2)$, $(-2, 3)$ and $(3, \infty)$. For every number in $(-\infty, -2)$, the graph of f is above the x-axis; in other words, $f(x) > 0$ for all x in $(-\infty, -2)$. Similarly, $f(x) < 0$ for all x in $(-2, 3)$, and $f(x) > 0$ for all x in $(3, \infty)$. We represent this schematically with the **sign diagram** shown in Figure 1.4.16.

The (+) above a portion of the number line indicates $f(x) > 0$ for those values of x and the $(-)$ indicates $f(x) < 0$ there. The numbers labeled on the number line are the zeros of f, so we place 0 above them. For the inequality $f(x) = x^2 - x - 6 \leq 0$, we read from the sign diagram that the solution is $[-2, 3]$.

Our next goal is to establish a procedure by which we can generate the sign diagram without graphing the function. While parabolas aren't that bad to graph knowing what we know, our sights are set on more general functions whose graphs are more complicated.

An important property of parabolas is that a parabola can't be above the x-axis at one point and below the x-axis at another point without crossing the x-axis at some point in between. Said differently, if the function is positive at one point and negative at another, the function must have at least one zero in between. This property is a consequence of quadratic functions being **continuous**. A precise definition of 'continuous' requires the language of Calculus, but it suffices for us to know that the graph of a continuous function has no gaps or holes. This allows us to determine the sign of *all* of the function values on a given interval by testing the function at just *one* value in the interval.

The result in Box 1.4.1 applies to all continuous functions defined on an interval of real numbers, but we restrict our attention to quadratic functions for the time being,

Box 1.4.1: Steps for Creating A Sign Diagram for A Quadratic Function

Suppose f is a quadratic function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

To use a sign diagram to solve an inequality, we must always remember to compare the function to 0 as explained in Box 2.3.2.

Box 1.4.2: Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so some function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

We practice this approach in the following example.

Example 1.4.6. Solve the following inequalities analytically[10] and check your so-

[10] By 'solve analytically' we mean 'algebraically' using a sign diagram.

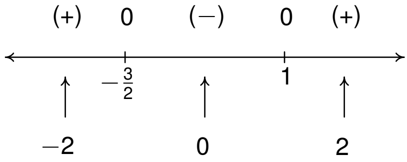


Figure 1.4.17: Sign chart of g and h

lutions graphically.

1. $2x^2 \leq 3 - x$ 2. $t^2 - 2t > 1$ 3. $x^2 + 1 \leq 2x$
4. $2t - t^2 \geq |t - 1| - 1$

Solution.

1. To solve $2x^2 \leq 3 - x$, we rewrite it as $2x^2 + x - 3 \leq 0$. We find the zeros of $f(x) = 2x^2 + x - 3$ by solving $2x^2 + x - 3 = 0$. Factoring gives $(2x + 3)(x - 1) = 0$, so $x = -\frac{3}{2}$ or $x = 1$. We place these values on the number line with 0 above them and choose test values in the intervals $\left(-\infty, -\frac{3}{2}\right)$, $\left(-\frac{3}{2}, 1\right)$ and $(1, \infty)$. For the interval $\left(-\infty, -\frac{3}{2}\right)$, we choose[11] $x = -2$; for $\left(-\frac{3}{2}, 1\right)$, we pick $x = 0$; and for $(1, \infty)$, $x = 2$. Evaluating the function at the three test values gives us $f(-2) = 3 > 0$, so we place (+) above $\left(-\infty, -\frac{3}{2}\right)$; $f(0) = -3 < 0$, so $(-)$ goes above the interval $\left(-\frac{3}{2}, 1\right)$; and, $f(2) = 7$, which means (+) is placed above $(1, \infty)$.

   We are solving $2x^2 + x - 3 \leq 0$ so we need solutions to $2x^2 + x - 3 < 0$ as well as solutions for $2x^2 + x - 3 = 0$. For $2x^2 + x - 3 < 0$, we need the intervals which we have a $(-)$ above them. The sign diagram shows only one: $\left(-\frac{3}{2}, 1\right)$. Also, we know $2x^2 + x - 3 = 0$ when $x = -\frac{3}{2}$ and $x = 1$, so our final answer is $\left[-\frac{3}{2}, 1\right]$.

   To verify our solution graphically, we refer to the original inequality, $2x^2 \leq 3 - x$. We let $g(x) = 2x^2$ and $h(x) = 3 - x$. We are looking for the x values where the graph of g is below that of h (the solution to $g(x) < h(x)$) as well as the points of intersection (the solutions to $g(x) = h(x)$). The graphs of g and h are given in Figure 1.4.18 with the sign chart in Figure 1.4.17.

2. Once again, we re-write $t^2 - 2t > 1$ as $t^2 - 2t - 1 > 0$ and we identify $f(t) = t^2 - 2t - 1$. When we go to find the zeros of f, we find, to our chagrin, that the quadratic $t^2 - 2t - 1$ doesn't factor nicely. Hence, we resort to

[11] We have to choose *something* in each interval. If you don't like our choices, please feel free to choose different numbers. You'll get the same sign chart.

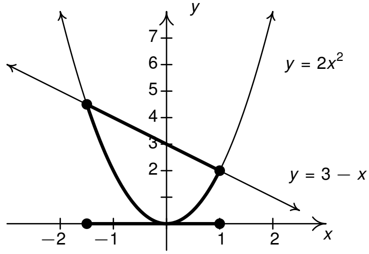


Figure 1.4.18: Graph of g and h

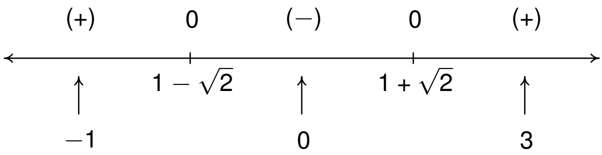


Figure 1.4.19: Sign chart of g and h

the Quadratic Formula and find $t = 1 \pm \sqrt{2}$. As before, these zeros divide the number line into three pieces. To help us decide on test values, we approximate $1 - \sqrt{2} \approx -0.4$ and $1 + \sqrt{2} \approx 2.4$. We choose $t = -1$, $t = 0$ and $t = 3$ as our test values and find $f(-1) = 2$, which is (+); $f(0) = -1$ which is (−); and $f(3) = 2$ which is (+) again. Our solution to $t^2 - 2t - 1 > 0$ is where we have (+), so, in interval notation $\left(-\infty, 1 - \sqrt{2}\right) \cup \left(1 + \sqrt{2}, \infty\right)$.

To check the inequality $t^2 - 2t > 1$ graphically, we set $g(t) = t^2 - 2t$ and $h(t) = 1$. We are looking for the t values where the graph of g is above the graph of h. As before we present the graphs in Figure 1.4.20 and the sign chart in Figure 1.4.19.

3. To solve $x^2 + 1 \leq 2x$, as before, we solve $x^2 - 2x + 1 \leq 0$. Setting $f(x) = x^2 - 2x + 1 = 0$, we find only one zero of f: $x = 1$. This one x value divides the number line into two intervals, from which we choose $x = 0$ and $x = 2$ as test values. We find $f(0) = 1 > 0$ and $f(2) = 1 > 0$. Since we are looking for solutions to $x^2 - 2x + 1 \leq 0$, we are looking for x values where $x^2 - 2x + 1 < 0$ as well as where $x^2 - 2x + 1 = 0$. Looking at our sign diagram, there are no places where $x^2 - 2x + 1 < 0$ (there are no (−)), so our solution is only $x = 1$ (where $x^2 - 2x + 1 = 0$). We write this as $\{1\}$.

   Graphically, we solve $x^2 + 1 \leq 2x$ by graphing $g(x) = x^2 + 1$ and $h(x) = 2x$.

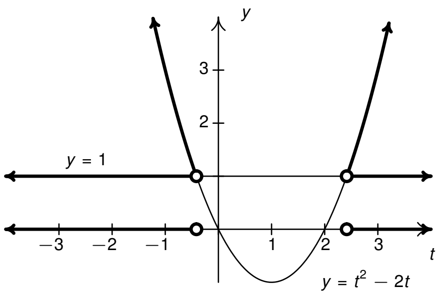


Figure 1.4.20: Graph of g and h

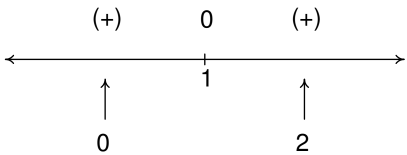


Figure 1.4.21: Sign chart of g and h

We are looking for the x values where the graph of g is below the graph of h (for $x^2+1 < 2x$) and where the two graphs intersect ($x^2+1 = 2x$). Notice that the line and the parabola touch at $(1, 2)$, but the parabola is always above the line otherwise.[12] See Figure 1.4.21 and Figure 1.4.22.

4. To solve $2t - t^2 \geq |t-1| - 1$ analytically we first rewrite the absolute value using cases. For $t < 1$, $|t-1| = -(t-1) = -t+1$, so we get $2t - t^2 \geq (-t+1) - 1$ which simplifies to $t^2 - 3t \leq 0$. Finding the zeros of $f(t) = t^2 - 3t$, we get $t = 0$ and $t = 3$. However, we are concerned only with the portion of the number line where $t < 1$, so the only zero that we deal with is $t = 0$. This divides the interval $t < 1$ into two intervals: $(-\infty, 0)$ and $(0, 1)$. We choose $t = -1$ and $t = \frac{1}{2}$ as our test values. We find $f(-1) = 4$ and $f\left(\frac{1}{2}\right) = -\frac{5}{4}$. Hence, our solution to $t^2 - 3t \leq 0$ for $t < 1$ is $[0, 1)$. See Figure 1.4.23.

   Next, we turn our attention to the case $t \geq 1$. Here, $|t-1| = t-1$, so our original inequality becomes $2t - t^2 \geq (t-1) - 1$, or $t^2 - t - 2 \leq 0$. Setting $g(t) = t^2 - t - 2$, we find the zeros of g to be $t = -1$ and $t = 2$. Of these,

[12] In this case, we say the line $y = 2x$ is **tangent** to $y = x^2 + 1$ at $(1, 2)$. Finding tangent lines to arbitrary functions is a fundamental problem solved, in general, with Calculus.

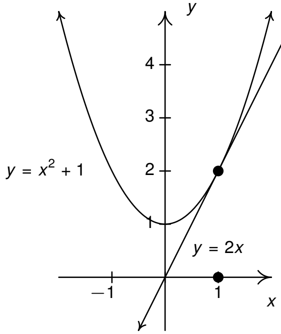


Figure 1.4.22: Graph of g and h

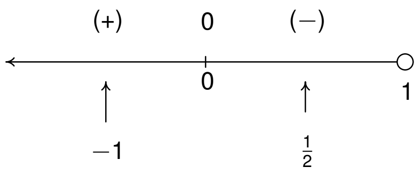


Figure 1.4.23: Solving $2t - t^2 \geq |t - 1| - 1$ for $t < 1$

only $t = 2$ lies in the region $t \geq 1$, so we ignore $t = -1$. Our test intervals are now $[1, 2)$ and $(2, \infty)$. We choose $t = 1$ and $t = 3$ as our test values and find $g(1) = -2$ and $g(3) = 4$. Hence, our solution to $g(t) = t^2 - t - 2 \leq 0$, in this region is $[1, 2)$. See Figure 1.4.24

Combining these into one sign diagram, we have that our solution is $[0, 2]$. Graphically, to check $2t - t^2 \geq |t - 1| - 1$, we set $h(t) = 2t - t^2$ and $i(t) = |t - 1| - 1$ and look for the t values where the graph of h intersects or is above the the graph of i The combined sign chart is given in Figure 1.4.25 and the graphs are given in Figure 1.4.26.

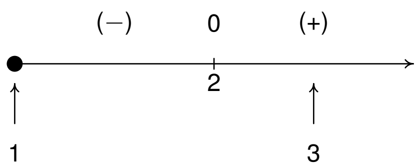


Figure 1.4.24: Solving $2t - t^2 \geq |t - 1| - 1$ for $t \geq 1$

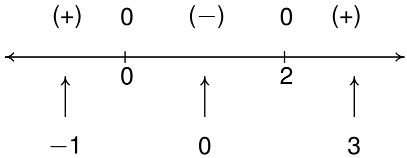


Figure 1.4.25: Sign chart of h and i

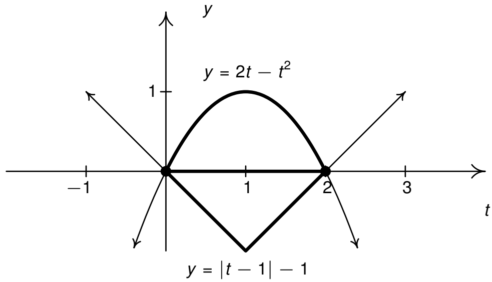


Figure 1.4.26: Graph of h and i

$\square$

We end this section with an example that combines quadratic inequalities with piecewise functions.

Example 1.4.7. Rewrite $g(x) = |x^2 - x - 6|$ as a piecewise function and graph.

Solution. Using the definition of absolute value, Definition 1.3.1 and the sign diagram we constructed for $f(x) = x^2 - x - 6$ near the beginning of the subsection, we get:

$$g(x) = |x^2 - x - 6| = \begin{cases} -(x^2 - x - 6) & \text{if } (x^2 - x - 6) < 0, \\ (x^2 - x - 6) & \text{if } (x^2 - x - 6) \geq 0. \end{cases}$$

$$\longrightarrow g(x) = \begin{cases} -x^2 + x + 6 & \text{if } -2 < x < 3, \\ x^2 - x - 6 & \text{if } x \leq -2 \text{ or } x \geq 3. \end{cases}$$

Going through the usual machinations results on the graph in Figure 1.4.28. Compare it to the graph in Figure 1.4.27. Notice anything?

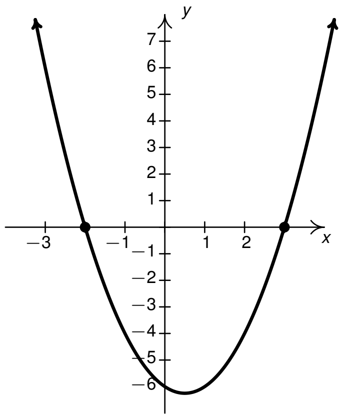


Figure 1.4.27: $y = f(x) = x^2 - x - 6$

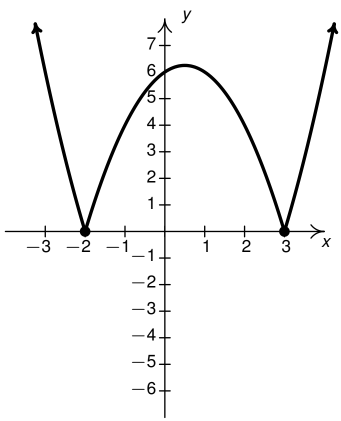


Figure 1.4.28: $y = g(x) = |x^2 - x - 6|$

If we take a step back and look at the graphs of f and g, we notice that to obtain the graph of g from the graph of f, we reflect a *portion* of the graph of f about the x-axis. In general, if $g(x) = |f(x)|$, then:

$$g(x) = |f(x)| = \begin{cases} -f(x) & \text{if } f(x) < 0, \\ f(x) & \text{if } f(x) \geq 0. \end{cases}$$

The function g is defined so that when $f(x)$ is negative (i.e., when its graph is below the x-axis), the graph of g is the refection of the graph of f across the x-axis. This is a general method to graph functions of the form $g(x) = |f(x)|$. Indeed, the graph of $g(x) = |x|$ can be obtained by reflection the portion of the line $f(x) = x$ which is below the x-axis back above the x-axis creating the characteristic '$\vee$' shape.[13] $\square$

[13] See Exercise 11 in Section 1.3.

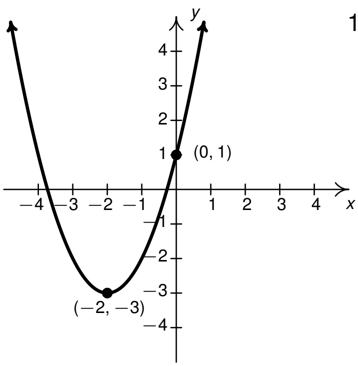
1.4.3 Exercises

In Exercises 1 - 9, graph the quadratic function. Find the vertex and axis intercepts of each graph, if they exist. State the domain and range, identify the maximum or minimum, and list the intervals over which the function is increasing or decreasing. If the function is given in general form, convert it into standard form; if it is given in standard form, convert it into general form.

1. $f(x) = x^2 + 2$
2. $f(x) = -(x+2)^2$
3. $f(x) = x^2 - 2x - 8$
4. $g(t) = -2(t+1)^2 + 4$
5. $g(t) = 2t^2 - 4t - 1$
6. $g(t) = -3t^2 + 4t - 7$
7. $h(s) = s^2 + s + 1$
8. $h(s) = -3s^2 + 5s + 4$
9. $h(s) = s^2 - \frac{1}{100}s - 1$

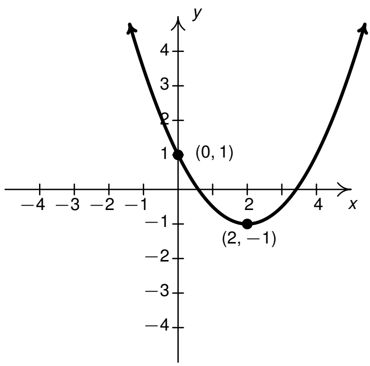
In Exercises 10. - 13., find a formula for each function below in the form $F(x) = a(x-h)^2 + k$.

10.



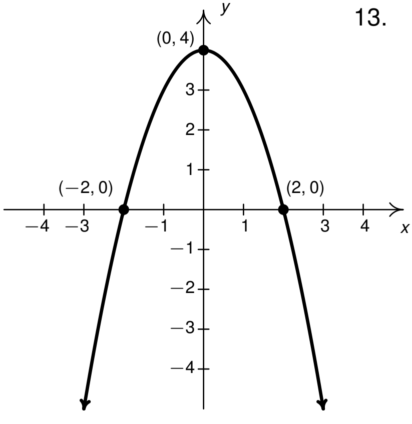
$y = F(x)$

11.

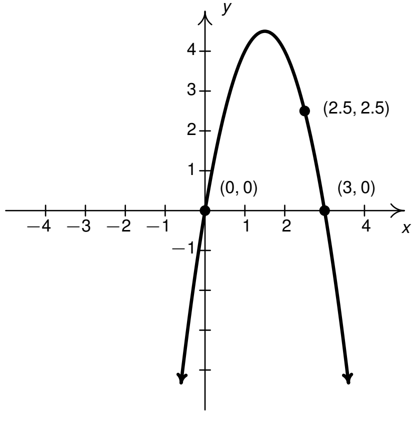


$y = F(x)$

12.



13.



In Exercises 14 - 29, solve the inequality. Write your answer using interval notation.

14. $x^2 + 2x - 3 \geq 0$
15. $16x^2 + 8x + 1 > 0$
16. $t^2 + 9 < 6t$
17. $9t^2 + 16 \geq 24t$
18. $u^2 + 4 \leq 4u$
19. $u^2 + 1 < 0$
20. $3x^2 \leq 11x + 4$
21. $x > x^2$
22. $2t^2 - 4t - 1 > 0$
23. $5t + 4 \leq 3t^2$
24. $2 \leq |x^2 - 9| < 9$
25. $x^2 \leq |4x - 3|$
26. $t^2 + t + 1 \geq 0$
27. $t^2 \geq |t|$
28. $x|x + 5| \geq -6$
29. $x|x - 3| < 2$

In Exercises 30 - 34, cost and price-demand functions are given. For each scenario,

- Find the profit function $P(x)$.
- Find the number of items which need to be sold in order to maximize profit.
- Find the maximum profit.
- Find the price to charge per item in order to maximize profit.
- Find and interpret break-even points.

30. The cost, in dollars, to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$ and the price-demand function, in dollars per shirt, is $p(x) = 30 - 2x$, for $0 \leq x \leq 15$.

31. The cost, in dollars, to produce x bottles of 100% All-Natural Certified Free-Trade Organic Sasquatch Tonic is $C(x) = 10x + 100$, $x \geq 0$ and the price-demand function, in dollars per bottle, is $p(x) = 35 - x$, for $0 \leq x \leq 35$.

32. The cost, in cents, to produce x cups of Mountain Thunder Lemonade at Junior's Lemonade Stand is $C(x) = 18x+240$, $x \geq 0$ and the price-demand function, in cents per cup, is $p(x) = 90 - 3x$, for $0 \leq x \leq 30$.

33. The daily cost, in dollars, to produce x Sasquatch Berry Pies is $C(x) = 3x + 36$, $x \geq 0$ and the price-demand function, in dollars per pie, is $p(x) = 12 - 0.5x$, for $0 \leq x \leq 24$.

34. The monthly cost, in *hundreds* of dollars, to produce x custom built electric scooters is $C(x) = 20x + 1000$, $x \geq 0$ and the price-demand function, in *hundreds* of dollars per scooter, is $p(x) = 140 - 2x$, for $0 \leq x \leq 70$.

35. The International Silver Strings Submarine Band holds a bake sale each year to fund their trip to the National Sasquatch Convention. It has been determined that the cost in dollars of baking x cookies is $C(x) = 0.1x + 25$ and that the demand function for their cookies is $p = 10 - .01x$ for $0 \leq x \leq 1000$. How many cookies should they bake in order to maximize their profit?

36. Using data from Bureau of Transportation Statistics[14], the average fuel economy $F(t)$ in miles per gallon for passenger cars in the US t years after 1980 can be modeled by $F(t) = -0.0076t^2 + 0.45t + 16$, $0 \leq t \leq 28$. Find and interpret the coordinates of the vertex of the graph of $y = F(t)$.

37. The temperature T, in degrees Fahrenheit, t hours after 6 AM is given by:

$$T(t) = -\frac{1}{2}t^2 + 8t + 32, \quad 0 \leq t \leq 12$$

What is the warmest temperature of the day? When does this happen?

38. Suppose $C(x) = x^2 - 10x + 27$ represents the costs, in *hundreds*, to produce x *thousand* pens. How many pens should be produced to minimize the cost? What is this minimum cost?

39. Skippy wishes to plant a vegetable garden along one side of his house. In his garage, he found 32 linear feet of fencing. Since one side of the garden will border the house, Skippy doesn't need fencing along that side. What are the dimensions of the garden which will maximize the area of the garden? What is the maximum area of the garden?

[14] http://www.bts.gov/publications/national_transportation_statistics/html/table_04_23.html

40. In the situation of Example 1.4.5, Donnie has a nightmare that one of his alpaca fell into the river. To avoid this, he wants to move his rectangular pasture *away* from the river so that all four sides of the pasture require fencing. If the total amount of fencing available is still 200 linear feet, what dimensions maximize the area of the pasture now? What is the maximum area? Assuming an average alpaca requires 25 square feet of pasture, how many alpaca can he raise now?

41. What is the largest rectangular area one can enclose with 14 inches of string?

42. The height of an object dropped from the roof of an eight story building is modeled by by the function $h(t) = -16t^2 + 64$, $0 \leq t \leq 2$. Here, $h(t)$ is the height of the object off the ground, in feet, t seconds after the object is dropped. How long before the object hits the ground?

43. The height $h(t)$ in feet of a model rocket above the ground t seconds after lift-off is given by the function $h(t) = -5t^2 + 100t$, for $0 \leq t \leq 20$. When does the rocket reach its maximum height above the ground? What is its maximum height?

44. Carl's friend Jason participates in the Highland Games. In one event, the hammer throw, the height $h(t)$ in feet of the hammer above the ground t seconds after Jason lets it go is modeled by the function $h(t) = -16t^2 + 22.08t + 6$. What is the hammer's maximum height? What is the hammer's total time in the air? Round your answers to two decimal places.

45. Assuming no air resistance or forces other than the Earth's gravity, the height above the ground at time t of a falling object is given by $s(t) = -4.9t^2 + v_0 t + s_0$ where s is in meters, t is in seconds, v_0 is the object's initial velocity in meters per second and s_0 is its initial position in meters.

    (a) What is the applied domain of this function?

    (b) Discuss with your classmates what each of $v_0 > 0$, $v_0 = 0$ and $v_0 < 0$ would mean.

    (c) Come up with a scenario in which $s_0 < 0$.

    (d) Let's say a slingshot is used to shoot a marble straight up from the ground ($s_0 = 0$) with an initial velocity of 15 meters per second. What is the marble's maximum height above the ground? At what time will it hit the ground?

(e) If the marble is shot from the top of a 25 meter tall tower, when does it hit the ground?

(f) What would the height function be if instead of shooting the marble up off of the tower, you were to shoot it straight DOWN from the top of the tower?

46. The two towers of a suspension bridge are 400 feet apart. The parabolic cable[15] attached to the tops of the towers is 10 feet above the point on the bridge deck that is midway between the towers. If the towers are 100 feet tall, find the height of the cable directly above a point of the bridge deck that is 50 feet to the right of the left-hand tower.

47. On New Year's Day, Jeff started weighing himself every morning in order to have an interesting data set for this section of the book. (Discuss with your classmates if that makes him a nerd or a geek. Also, the professionals in the field of weight management strongly discourage weighing yourself every day. When you focus on the number and not your overall health, you tend to lose sight of your objectives. Jeff was making a noble sacrifice for science, but you should not try this at home.) The whole chart would be too big to put into the book neatly, so we've decided to give only a small portion of the data to you. This then becomes a Civics lesson in honesty, as you shall soon see. There are two charts given here. Table 1.4.1 has Jeff's weight for the first eight Thursdays of the year (January 1, 2009 was a Thursday and we'll count it as Day 1.) and Table 1.4.2 has Jeff's weight for the first 10 Saturdays of the year.

(a) Find the least squares line for the Thursday data and comment on its goodness of fit.

(b) Find the least squares line for the Saturday data and comment on its goodness of fit.

(c) Use Quadratic Regression to find a parabola which models the Saturday data and comment on its goodness of fit.

(d) Compare and contrast the predictions the three models make for Jeff's weight on January 1, 2010 (Day #366). Can any of these models be used to make a prediction of Jeff's weight 20 years from now? Explain your answer.

---

[15] The weight of the bridge deck forces the bridge cable into a parabola and a free hanging cable such as a power line does not form a parabola. We shall see in Exercise **??** in Section **??** what shape a free hanging cable makes.

Day # (Thursday)	My weight in pounds
1	238.2
8	237.0
15	235.6
22	234.4
29	233.0
36	233.8
43	232.8
50	232.0

Table 1.4.1: Jeff's weight on Thursdays

Day # (Saturday)	My weight in pounds
3	238.4
10	235.8
17	235.0
24	234.2
31	236.2
38	236.2
45	235.2
52	233.2
59	236.8
66	238.2

Table 1.4.2: Jeff's weight on Saturdays

(e) Why is this a Civics lesson in honesty? Well, compare the two linear models you obtained above. One was a good fit and the other was not, yet both came from careful selections of real data. In presenting the tables to you, we've not lied about Jeff's weight, nor have you used any bad math to falsify the predictions. The word we're looking for here is 'disingenuous'. Look it up and then discuss the implications this type of data manipulation could have in a larger, more complex, politically motivated setting.

48. (Data that is neither linear nor quadratic.) We'll close this exercise set with two data sets that, for reasons presented later in the book, cannot be modeled correctly by lines or parabolas. It is a good exercise, though, to see what happens when you attempt to use a linear or quadratic model when it's not appropriate.

    (a) This first data set in Table 1.4.3 came from a Summer 2003 publication of the Portage County Animal Protective League called "Tattle Tails". They make the following statement and then have a chart of data that supports it. "It doesn't take long for two cats to turn into 80 million. If two cats and their surviving offspring reproduced for ten years, you'd end up with 80,399,780 cats." We assume $N(0) = 2$.

Year x	Number of Cats $N(x)$
1	12
2	66
3	382
4	2201
5	12680
6	73041
7	420715
8	2423316
9	13968290
10	80399780

Table 1.4.3: Cat growth

Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

(b) This next data set shown in Table 1.4.4 comes from the U.S. Naval Observatory[16]. That site has loads of awesome stuff on it, but for this exercise I used the sunrise/sunset times in Fairbanks, Alaska for 2009 to give you a chart of the number of hours of daylight they get on the 21$^{\text{st}}$ of each month. We'll let $x = 1$ represent January 21, 2009, $x = 2$ represent February 21, 2009, and so on.

Use Quadratic Regression to find a parabola which models this data and comment on its goodness of fit. (Spoiler Alert: Does anyone know what type of function we need here?)

49. Redraw the three scenarios discussed in the discriminant box for $a < 0$.

50. Graph $f(x) = |1 - x^2|$

51. Find all of the points on the line $y = 1 - x$ which are 2 units from $(1, -1)$.

52. Let L be the line $y = 2x + 1$. Find a function $D(x)$ which measures the distance *squared* from a point on L to $(0, 0)$. Use this to find the point on L closest to $(0, 0)$.

[16] http://aa.usno.navy.mil/data/docs/RS_OneYear.php

Month Number	Hours of Daylight
1	5.8
2	9.3
3	12.4
4	15.9
5	19.4
6	21.8
7	19.4
8	15.6
9	12.4
10	9.1
11	5.6
12	3.3

Table 1.4.4: Daylight in Fairbanks

53. With the help of your classmates, show that if a quadratic function $f(x) = ax^2 + bx + c$ has two real zeros then the x-coordinate of the vertex is the midpoint of the zeros.

54. On page 139, we argued that any quadratic function in standard form $f(x) = a(x - h)^2 + k$ can be converted to a quadratic function in general form $f(x) = ax^2 + bx + c$ by making the identifications $b = -2ah$ and $c = ah^2 + k$. In this exercise, we use same identifications to show every parabola given in general form can be converted to standard form without completing the square.

    Solve $b = -2ah$ for h and substitute the result into the equation $c = ah^2 + k$ and then solve for k. Show $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$ so that

$$f(x) = ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}.$$

In Exercises 55 - 60, solve the quadratic equation for the indicated variable.

55. $x^2 - 10y^2 = 0$ for x
56. $y^2 - 4y = x^2 - 4$ for x
57. $x^2 - mx = 1$ for x
58. $y^2 - 3y = 4x$ for y

59. $y^2 - 4y = x^2 - 4$ for y
60. $-gt^2 + v_0 t + s_0 = 0$ for t (Assume $g \neq 0$.)
61. (This is a follow-up to Exercise 60 in Section 1.2.) The Lagrange Interpolate[17] function L for three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) where x_0, x_1, and x_2 are three distinct real numbers is given by:

$$L(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

(a) For each of the following sets of points, find $L(x)$ using the formula above and verify each of the points lies on the graph of $y = L(x)$.

i. $(-1, 1)$, $(1, 1)$, $(2, 4)$
ii. $(1, 3)$, $(2, 10)$, $(3, 21)$
iii. $(0, 1)$, $(1, 5)$, $(2, 7)$

(b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, and $L(x_2) = y_2$.

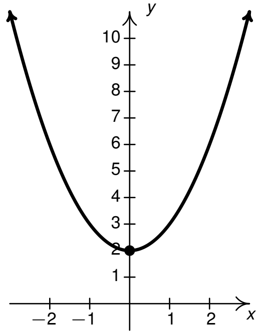
(c) Find $L(x)$ for the points $(-1, 6)$, $(1, 4)$ and $(3, 2)$. What happens?

(d) Under what conditions will $L(x)$ produce a quadratic function? Make a conjecture, test some cases, and prove your answer.

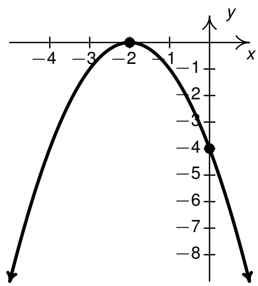
[17] https://en.wikipedia.org/wiki/Lagrange_polynomial

1.4.4 Answers

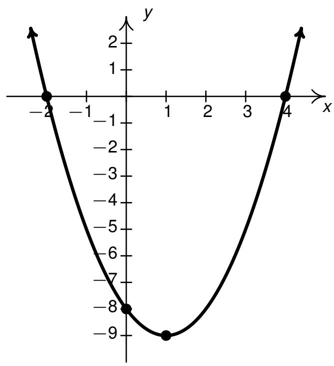
1. $f(x) = x^2 + 2$ (this is both forms!)
   No x-intercepts
   y-intercept $(0, 2)$
   Domain: $(-\infty, \infty)$
   Range: $[2, \infty)$
   Decreasing on $(-\infty, 0]$
   Increasing on $[0, \infty)$
   Vertex $(0, 2)$ is a minimum
   Axis of symmetry $x = 0$



2. $f(x) = -(x+2)^2 = -x^2 - 4x - 4$
   x-intercept $(-2, 0)$
   y-intercept $(0, -4)$
   Domain: $(-\infty, \infty)$
   Range: $(-\infty, 0]$
   Increasing on $(-\infty, -2]$
   Decreasing on $[-2, \infty)$
   Vertex $(-2, 0)$ is a maximum
   Axis of symmetry $x = -2$

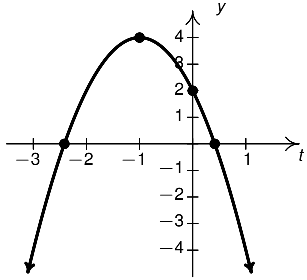


3. $f(x) = x^2 - 2x - 8 = (x-1)^2 - 9$
   x-intercepts $(-2, 0)$ and $(4, 0)$
   y-intercept $(0, -8)$
   Domain: $(-\infty, \infty)$
   Range: $[-9, \infty)$
   Decreasing on $(-\infty, 1]$
   Increasing on $[1, \infty)$
   Vertex $(1, -9)$ is a minimum
   Axis of symmetry $x = 1$

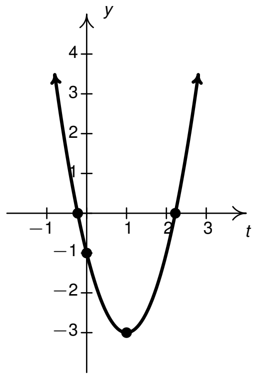


4.

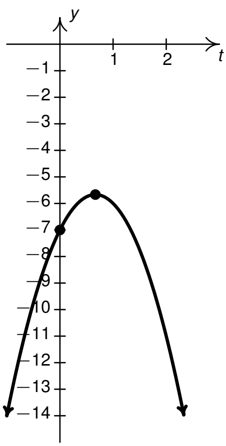
$g(t) = -2(t+1)^2 + 4 = -2t^2 - 4t + 2$
t-intercepts $(-1-\sqrt{2}, 0)$ and $(-1+\sqrt{2}, 0)$
y-intercept $(0, 2)$
Domain: $(-\infty, \infty)$
Range: $(-\infty, 4]$
Increasing on $(-\infty, -1]$
Decreasing on $[-1, \infty)$
Vertex $(-1, 4)$ is a maximum
Axis of symmetry $t = -1$



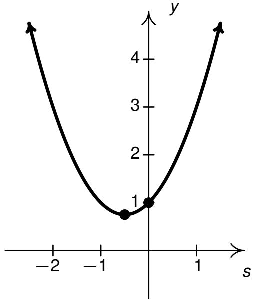
5. $g(t) = 2t^2 - tx - 1 = 2(t-1)^2 - 3$
t-intercepts $\left(\frac{2-\sqrt{6}}{2}, 0\right)$ and $\left(\frac{2+\sqrt{6}}{2}, 0\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $[-3, \infty)$
Increasing on $[1, \infty)$
Decreasing on $(-\infty, 1]$
Vertex $(1, -3)$ is a minimum
Axis of symmetry $t = 1$



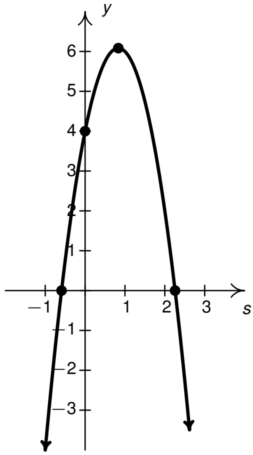
6. $g(t) = -3t^2 + 4t - 7 = -3\left(t - \frac{2}{3}\right)^2 - \frac{17}{3}$
No t-intercepts
y-intercept $(0, -7)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, -\frac{17}{3}\right]$
Increasing on $\left(-\infty, \frac{2}{3}\right]$
Decreasing on $\left[\frac{2}{3}, \infty\right)$
Vertex $\left(\frac{2}{3}, -\frac{17}{3}\right)$ is a maximum
Axis of symmetry $t = \frac{2}{3}$



7. $h(s) = s^2 + s + 1 = \left(s + \frac{1}{2}\right)^2 + \frac{3}{4}$
No s-intercepts
y-intercept $(0, 1)$
Domain: $(-\infty, \infty)$
Range: $\left[\frac{3}{4}, \infty\right)$
Increasing on $\left[-\frac{1}{2}, \infty\right)$
Decreasing on $\left(-\infty, -\frac{1}{2}\right]$
Vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ is a minimum
Axis of symmetry $s = -\frac{1}{2}$

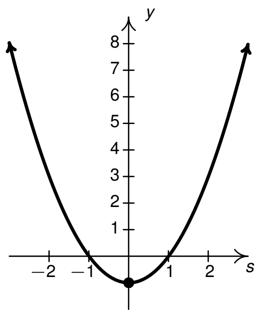


8. $h(s) = -3s^2+5s+4 = -3\left(s - \frac{5}{6}\right)^2 + \frac{73}{12}$
s-intercepts $\left(\frac{5-\sqrt{73}}{6}, 0\right)$ and $\left(\frac{5+\sqrt{73}}{6}, 0\right)$
y-intercept $(0, 4)$
Domain: $(-\infty, \infty)$
Range: $\left(-\infty, \frac{73}{12}\right]$
Increasing on $\left(-\infty, \frac{5}{6}\right]$
Decreasing on $\left[\frac{5}{6}, \infty\right)$
Vertex $\left(\frac{5}{6}, \frac{73}{12}\right)$ is a maximum
Axis of symmetry $s = \frac{5}{6}$



9. $h(s) = s^2 - \frac{1}{100}s - 1 = \left(s - \frac{1}{200}\right)^2 - \frac{40001}{40000}$
s-intercepts $\left(\frac{1+\sqrt{40001}}{200}\right)$ and $\left(\frac{1-\sqrt{40001}}{200}\right)$
y-intercept $(0, -1)$
Domain: $(-\infty, \infty)$
Range: $\left[-\frac{40001}{40000}, \infty\right)$
Decreasing on $\left(-\infty, \frac{1}{200}\right]$
Increasing on $\left[\frac{1}{200}, \infty\right)$
Vertex $\left(\frac{1}{200}, -\frac{40001}{40000}\right)$ is a minimum[18]
Axis of symmetry $s = \frac{1}{200}$

[18] You'll need to use your calculator to zoom in far enough to see that the vertex is not the y-intercept.



10. $F(x) = (x+2)^2 - 3$
11. $F(x) = \frac{1}{2}(x-2)^2 - 1$
12. $F(x) = -x^2 + 4$
13. $F(x) = -2(x-1.5)^2 + 4.5$
14. $(-\infty, -3] \cup [1, \infty)$
15. $\left(-\infty, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
16. No solution
17. $(-\infty, \infty)$
18. $\{2\}$
19. No solution
20. $\left[-\frac{1}{3}, 4\right]$
21. $(0, 1)$
22. $\left(-\infty, 1 - \frac{\sqrt{6}}{2}\right) \cup \left(1 + \frac{\sqrt{6}}{2}, \infty\right)$
23. $\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$
24. $\left(-3\sqrt{2}, -\sqrt{11}\right] \cup \left[-\sqrt{7}, 0\right) \cup \left(0, \sqrt{7}\right] \cup \left[\sqrt{11}, 3\sqrt{2}\right)$
25. $\left[-2-\sqrt{7}, -2+\sqrt{7}\right] \cup [1, 3]$
26. $(-\infty, \infty)$
27. $(-\infty, -1] \cup \{0\} \cup [1, \infty)$
28. $[-6, -3] \cup [-2, \infty)$
29. $(-\infty, 1) \cup \left(2, \frac{3+\sqrt{17}}{2}\right)$

30.
- $P(x) = -2x^2 + 28x - 26$, for $0 \leq x \leq 15$.
- 7 T-shirts should be made and sold to maximize profit.
- The maximum profit is \$72.
- The price per T-shirt should be set at \$16 to maximize profit.
- The break even points are $x = 1$ and $x = 13$, so to make a profit, between 1 and 13 T-shirts need to be made and sold.

31.
- $P(x) = -x^2 + 25x - 100$, for $0 \leq x \leq 35$
- Since the vertex occurs at $x = 12.5$, and it is impossible to make or sell 12.5 bottles of tonic, maximum profit occurs when either 12 or 13 bottles of tonic are made and sold.
- The maximum profit is \$56.
- The price per bottle can be either \$23 (to sell 12 bottles) or \$22 (to sell 13 bottles.) Both will result in the maximum profit.

- The break even points are $x = 5$ and $x = 20$, so to make a profit, between 5 and 20 bottles of tonic need to be made and sold.

32.
- $P(x) = -3x^2 + 72x - 240$, for $0 \leq x \leq 30$
- 12 cups of lemonade need to be made and sold to maximize profit.
- The maximum profit is 192¢ or \$1.92.
- The price per cup should be set at 54¢ per cup to maximize profit.
- The break even points are $x = 4$ and $x = 20$, so to make a profit, between 4 and 20 cups of lemonade need to be made and sold.

33.
- $P(x) = -0.5x^2 + 9x - 36$, for $0 \leq x \leq 24$
- 9 pies should be made and sold to maximize the daily profit.
- The maximum daily profit is \$4.50.
- The price per pie should be set at \$7.50 to maximize profit.
- The break even points are $x = 6$ and $x = 12$, so to make a profit, between 6 and 12 pies need to be made and sold daily.

34.
- $P(x) = -2x^2 + 120x - 1000$, for $0 \leq x \leq 70$
- 30 scooters need to be made and sold to maximize profit.
- The maximum monthly profit is 800 hundred dollars, or \$80,000.
- The price per scooter should be set at 80 hundred dollars, or \$8000 per scooter.
- The break even points are $x = 10$ and $x = 50$, so to make a profit, between 10 and 50 scooters need to be made and sold monthly.

35. 495 cookies
36. The vertex is (approximately) $(29.60, 22.66)$, which corresponds to a maximum fuel economy of 22.66 miles per gallon, reached sometime between 2009 and 2010 (29 – 30 years after 1980.) Unfortunately, the model is only valid up until 2008 (28 years after 1908.) So, at this point, we are using the model to *predict* the maximum fuel economy.
37. 64° at 2 PM (8 hours after 6 AM.)
38. 5000 pens should be produced for a cost of \$200.
39. 8 feet by 16 feet; maximum area is 128 square feet.

40. 50 feet by 50 feet; maximum area is 2500 feet; he can raise 100 average alpacas.

41. The largest rectangle has area 12.25 square inches.

42. 2 seconds.

43. The rocket reaches its maximum height of 500 feet 10 seconds after lift-off.

44. The hammer reaches a maximum height of approximately 13.62 feet. The hammer is in the air approximately 1.61 seconds.

45. (a) The applied domain is $[0, \infty)$.

    (d) The height function is this case is $s(t) = -4.9t^2 + 15t$. The vertex of this parabola is approximately $(1.53, 11.48)$ so the maximum height reached by the marble is 11.48 meters. It hits the ground again when $t \approx 3.06$ seconds.

    (e) The revised height function is $s(t) = -4.9t^2 + 15t + 25$ which has zeros at $t \approx -1.20$ and $t \approx 4.26$. We ignore the negative value and claim that the marble will hit the ground after 4.26 seconds.

    (f) Shooting down means the initial velocity is negative so the height functions becomes $s(t) = -4.9t^2 - 15t + 25$.

46. Make the vertex of the parabola $(0, 10)$ so that the point on the top of the left-hand tower where the cable connects is $(-200, 100)$ and the point on the top of the right-hand tower is $(200, 100)$. Then the parabola is given by $p(x) = \frac{9}{4000}x^2 + 10$. Standing 50 feet to the right of the left-hand tower means you're standing at $x = -150$ and $p(-150) = 60.625$. So the cable is 60.625 feet above the bridge deck there.

47. (a) The line for the Thursday data is $y = -.12x + 237.69$. We have $r = -.9568$ and $r^2 = .9155$ so this is a really good fit.

    (b) The line for the Saturday data is $y = -0.000693x + 235.94$. We have $r = -0.008986$ and $r^2 = 0.0000807$ which is horrible. This data is not even close to linear.

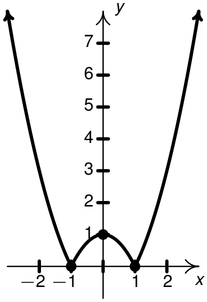
    (c) The parabola for the Saturday data is $y = 0.003x^2 - 0.21x + 238.30$. We have $R^2 = .47497$ which isn't good. Thus the data isn't modeled well by a quadratic function, either.

    (d) The Thursday linear model had my weight on January 1, 2010 at 193.77 pounds. The Saturday models give 235.69 and 563.31 pounds, respectively. The Thursday line has my weight going below 0 pounds in about five and a half years, so that's no good. The quadratic has a positive leading coefficient which would mean unbounded weight gain for the rest of my life. The Saturday line, which mathematically does not fit the data at all, yields a plausible weight prediction in the end. I think this is why grown-ups talk about "Lies, Damned Lies and Statistics."

48. (a) The quadratic model for the cats in Portage county is $y = 1917803.54x^2 - 16036408.29x + 24094857.7$. Although $R^2 = .70888$ this is not a good model because it's so far off for small values of x. The model gives us 24,094,858 cats when $x = 0$ but we know $N(0) = 2$.

(b) The quadratic model for the hours of daylight in Fairbanks, Alaska is $y = .51x^2 + 6.23x - .36$. Even with $R^2 = .92295$ we should be wary of making predictions beyond the data. Case in point, the model gives -4.84 hours of daylight when $x = 13$. So January 21, 2010 will be "extra dark"? Obviously a parabola pointing down isn't telling us the whole story.

50. $y = \left|1 - x^2\right|$



51. $\left(\frac{3-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}\right), \left(\frac{3+\sqrt{7}}{2}, \frac{-1-\sqrt{\ }}{2}\right.$

52. $D(x) = x^2 + (2x+1)^2 = 5x^2 + 4x + 1$ is minimized when $x = -\frac{2}{5}$. Hence to find the point on $y = 2x + 1$ closest to $(0, 0)$ we substitute $x = -\frac{2}{5}$ into $y = 2x + 1$ to get $\left(-\frac{2}{5}, \frac{1}{5}\right)$.

55. $x = \pm y\sqrt{10}$ 56. $x = \pm(y - 2)$ 57. $x = \dfrac{m \pm \sqrt{m^2+4}}{2}$

58. $y = \dfrac{3 \pm \sqrt{16x+9}}{2}$ 59. $y = 2 \pm x$ 60. $t = \dfrac{v_0 \pm \sqrt{v_0^2 + 4gs_0}}{2g}$

61. (a) i. $L(x) = x^2$ ii. $L(x) = 2x^2 + x$ iii. $L(x) = -x^2 + 5x + 1$

(c) The three points lie on the same line and we get $L(x) = -x + 5$.

(d) To obtain a quadratic function, we require that the points are not collinear (i.e., they do not all lie on the same line.)

Chapter 2

Polynomial Functions

2.1 Graphs of Polynomial Functions

In Chapter 1, we studied functions of the form $f(x) = b$ (constant functions), $f(x) = mx+b$, $m \neq 0$ (linear functions), and $f(x) = ax^2+bx+c$, $a \neq 0$ (quadratic functions). In each case, we learned how to construct graphs, find zeros, describe behavior, and use the functions in each family to model real-world phenomena. One might wonder about functions of the form $f(x) = ax^3 + bx^2 + cx + d$, $a \neq 0$, or functions containing even higher powers of x. These are the **polynomial functions** and are the subject of study in this chapter.[1] As you may recall, *poly*nomials are the result of adding *mon*omials, so we begin our study of polynomial functions with monomial functions.

[1] You've seen polynomials before - see Section **??**, for instance. Here, we restrict our attention to polynomial *functions* which for us means *one* independent variable instead of expressions with more than one variable.

2.1.1 Monomial Functions

Definition 2.1.1. A **monomial function** is a function of the form

$$f(x) = b \qquad \text{or} \qquad f(x) = ax^n,$$

where a and b are real numbers, $a \neq 0$ and $n \in \mathbb{N}$. The domain of a monomial function is $(-\infty, \infty)$.

Monomial functions, by definition, contain the constant functions along with a two parameter family of functions, $f(x) = ax^n$. We use x as the default independent variable here with a and n as parameters. From Section **??**, we recall that the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers, so examples of monomial functions include $f(x) = 2x = 2x^1$, $g(t) = -0.1t^2$, and $H(s) = \sqrt{2}\, s^{117}$. Note that the function $f(x) = x^0$ is *not* a monomial function. Even though $x^0 = 1$ for all *nonzero* values of x, 0^0 is undefined,[2] and hence $f(x) = x^0$ does *not* have a domain of $(-\infty, \infty)$.[3]

We begin our study of the graphs of polynomial functions by studying graphs of monomial functions. Starting with $f(x) = x^n$ where n is even, we investigate the cases $n = 2$, 4 and 6 in Table 2.1.1 and Figure 2.1.1. Numerically, we see that if $-1 < x < 1$, x^n becomes much smaller as n increases whereas if $x < -1$ or $x > 1$, x^n becomes much larger as n increases. These trends manifest themselves geometrically as the graph 'flattening' for $|x| < 1$ and 'narrowing' for $|x| > 1$ as n increases.[4]

From the graphs, it appears as if the range of each of these functions is $[0, \infty)$. When n is even, $x^n \geq 0$ for all x so the range of $f(x) = x^n$ is contained in $[0, \infty)$. To show that the range of f is all of $[0, \infty)$, we note that the equation $x^n = c$ for $c \geq 0$ has (at least) one solution for every even integer n, namely $x = \sqrt[n]{c}$. (See Section **??** for a review of this notation.) Hence, $f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$ which shows that every non-negative real number is in the range of f.[5]

Another item worthy of note is the symmetry about the line $x = 0$ a.k.a the y-axis. (See Definition **??** for a review of this concept.) With n being even, $f(-x) =$

[2] More specifically, 0^0 is an *indeterminate form.* These are studied extensively in Calculus.

[3] This is why we do not describe monomial functions as having the form $f(x) = ax^n$ for any *whole* number n. See Section **??**

[4] Recall that $|x| < 1$ is equivalent to $-1 < x < 1$ and $|x| > 1$ is equivalent to $x < -1$ or $x > 1$. Using absolute values allow us to describe these sets of real numbers more succinctly.

[5] This should sound familiar - see the comments regarding the range of $f(x) = x^2$ in Section 1.4.

x	x^2	x^4	x^6
-2	4	16	64
-1	1	1	1
-0.5	0.25	0.0625	0.015625
0	0	0	0
0.5	0.25	0.0625	0.015625
1	1	1	1
2	4	16	64

Table 2.1.1: Values of $f(x) = x^n$ for $n = 2$, 4 and 6

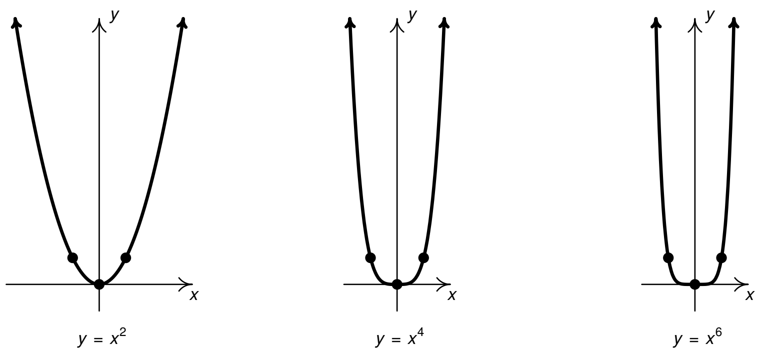


Figure 2.1.1: Graphs of $f(x) = x^n$ for $n = 2$, 4 and 6

x	x^3	x^5	x^7
-2	-8	-32	-128
-1	-1	-1	-1
-0.5	0.125	-0.03125	-0.0078125
0	0	0	0
0.5	0.125	0.03125	0.0078125
1	1	1	1
2	8	32	128

Table 2.1.2: Values of $f(x) = x^n$ for $x = 3,5$ and 7

$(-x)^n = x^n = f(x)$. At the level of points, we have that for all x, $(-x, f(-x)) = (-x, f(x))$. Hence for every point $(x, f(x))$ on the graph of f, the point symmetric about the y-axis, $(-x, f(x))$ is on the graph, too. We give this sort of symmetry a name honoring its roots here with even-powered monomial functions:

Definition 2.1.2. A function f is said to be **even** if $f(-x) = f(x)$ for all x in the domain of f.
NOTE: A function f is even if and only if the graph of $y = f(x)$ is symmetric about the y-axis.

An investigation of the odd powered monomial functions ($n \geq 3$) yields similar results with the major difference being that when a negative number is raised to an odd natural number power the result is still negative. See Table 2.1.2 and Figure 2.1.2. Numerically we see that for $|x| > 1$ the values of $|x^n|$ increase as n increases and the values of $|x^n|$ get closer to 0 as n increases. This translates graphically into a flattening behavior on the interval $(-1, 1)$ and a narrowing elsewhere. The graphs are shown on the top of the next page.

The range of these functions appear to be all real numbers, $(-\infty, \infty)$ which is algebraically sound as the equation $x^n = c$ has a solution for every real number,[6] namely $x = \sqrt[n]{c}$. Hence, for every real number c, choose $x = \sqrt[n]{c}$ so that $f(x) = f(\sqrt[n]{c}) = (\sqrt[n]{c})^n = c$. This shows that every real number is in the range of f.

Here, since n is odd, $f(-x) = (-x)^n = -x^n = -f(x)$. This means that whenever $(x, f(x))$ is on the graph, so is the point symmetric about the origin, $(-x, -f(x))$.

[6] Do you see the importance of n being odd here?

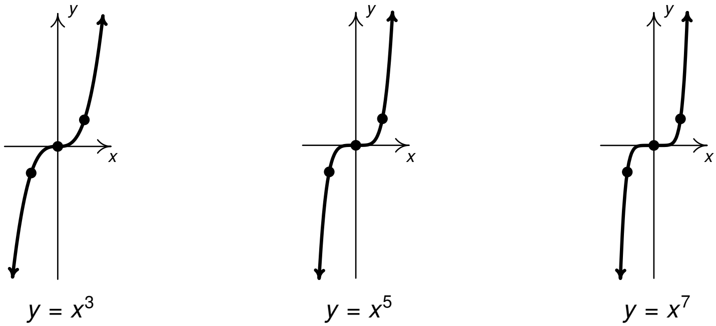


Figure 2.1.2: Graphs of $f(x) = x^n$ for $x = 3,5$ and 7

(Again, see Definition **??**.) We generalize this property below. Not surprisingly, we name it in honor of its odd powered heritage:

> **Definition 2.1.3.** A function f is said to be **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
> **NOTE:** A function f is odd if and only if the graph of $y = f(x)$ is symmetric about the origin.

The most important thing to take from the discussion above is the basic shape and common points on the graphs of $y = x^n$ for each of the families when n even and n is odd. While symmetry is nice and should be noted when present, even and odd symmetry are comparatively rare. The point of Definitions 2.1.2 and 2.1.3 is to give us the vocabulary to point out the symmetry when appropriate.

Moving on, we take a cue from Theorem 1.3.1 and prove the following.

Theorem 2.1.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^n + k$ can be obtained from the graph of $f(x) = x^n$ by performing the following operations, in sequence:

1. add h to the x-coordinates of each of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^n$ to $y = (x-h)^n$.
2. multiply the y-coordinates of each of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (x-h)^n$ to $y = a(x-h)^n$.
3. add k to the y-coordinates of each of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(x-h)^n$ to $y = a(x-h)^n + k$

Proof. Our goal is to start with the graph of $f(x) = x^n$ and build it up to the graph of $F(x) = a(x-h)^n + k$. We begin by examining $F_1(x) = (x-h)^n$. The graph of $f(x) = x^n$ can be described as the set of points $\{(c, c^n) \,|\, c \in \mathbb{R}\}$.[7] Likewise, the graph of F_1 can be described as the set of points $\{(x, (x-h)^n) \,|\, x \in \mathbb{R}\}$. If we relabel $c = x - h$ so that $x = c + h$, then as x varies through all real numbers so does c.[8] Hence, we can describe the graph of F_1 as $\{(c+h, c^n) \,|\, c \in \mathbb{R}\}$. This means that we can obtain the graph of F_1 from the graph of f by adding h to each of the x-coordinates of the points on the graph of f and that establishes the first step of the theorem.

Next, we consider the graph of $F_2(x) = a(x-h)^n$ as compared to the graph of $F_1(x) = (x-h)^n$. The graph of F_1 is the set of points $\{(x, (x-h)^n \,|\, x \in \mathbb{R}\}$ while the graph of F_2 is the set of points $\{(x, a(x-h)^n) \,|\, x \in \mathbb{R}\}$. The only difference between the points $(x, (x-h)^n)$ and $(x, a(x-h)^n)$ is that the y-coordinate in the latter is a times the y-coordinate of the former.

In other words, to produce the graph of F_2 from the graph of F_1, we take the y-coordinate of each point on the graph of F_1 and multiply it by a to get the corresponding point on the graph of F_2. If $a > 0$, all we are doing is scaling the y-axis by a. If $a < 0$, then, in addition to scaling the y-axis, we are also reflecting

[7]We are using the dummy variable c here instead of x for reasons that will become apparent shortly.

[8]That is, for a fixed number h every real number c can be written as $x - h$ for some real number x, and every real number x can be written as $c + h$ for some real number c.

each point across the x-axis. In either case, we have established the second step of the theorem.

Last, we compare the graph of $F(x) = a(x-h)^n + k$ to that of $F_2(x) = a(x-h)^n$. Once again, we view the graphs as sets of points in the plane. The graph of F_2 is $\{(x, a(x-h)^n) \mid x \in \mathbb{R}\}$ and the graph of F is $\{(x, a(x-h)^n + k) \mid x \in \mathbb{R}\}$. Looking at the corresponding points, $(x, a(x-h)^n)$ and $(x, a(x-h)^n + k)$, we see that we can obtain all of the points on the graph of F by adding k to each of the y-coordinates to points on the graph of F_2. This is equivalent to shifting every point vertically by k units which establishes the third and final step in the theorem. $\square$

This argument should sound familiar. The proof we presented above is more-or-less the same argument we presented after the proof of Theorem 1.3.1 in Section 1.3 but with '$|\cdot|$' replaced by '$(\cdot)^n$.' Also note that using $n = 2$ in Theorem 2.1.1 establishes Theorem 1.4.1 in Section 1.4.

We now use Theorem 2.1.1 to graph two different "transformed" monomial functions. To provide the reader an opportunity to compare and contrast the graphical behaviors exhibited in the case when n is even versus when n is odd, we graph one of each case.

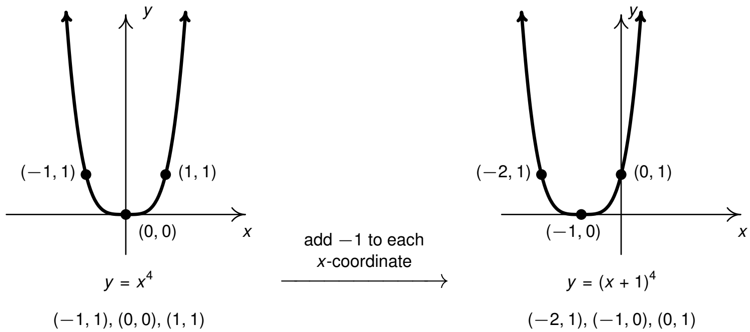
Example 2.1.1. Use Theorem 2.1.1 to graph the following functions. Label at least three points on each graph. State the domain and range using interval notation.

1. $f(x) = -2(x+1)^4 + 3$
2. $g(t) = \dfrac{(2t-1)^3}{5}$

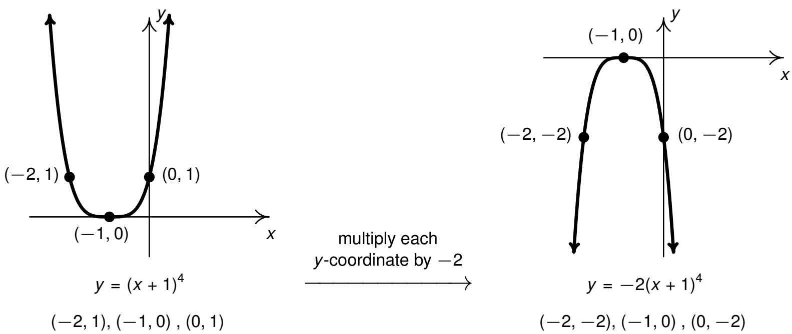
Solution.

1. For $f(x) = -2(x+1)^4 + 3 = -2(x-(-1))^4 + 3$, we identify $n = 4$, $a = -2$, $h = -1$, and $k = 3$. Thus to graph f, we start with $y = x^4$ and perform the following steps, in sequence, tracking the points $(-1, 1)$, $(0, 0)$ and $(1, 1)$ through each step:

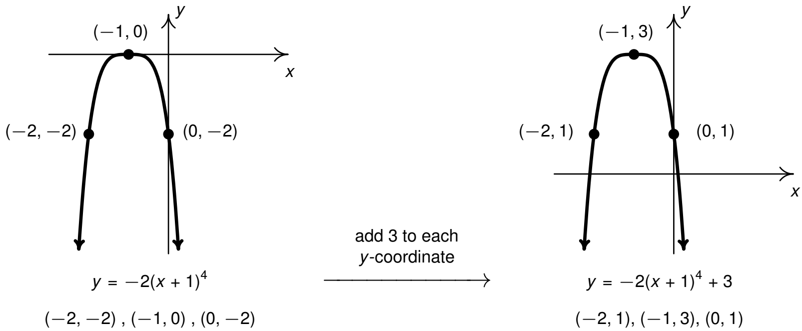
   Step 1: add -1 to the x-coordinates of each of the points on the graph of $y = x^4$:



Step 2: multiply the y-coordinates of each of the points on the graph of $y = (x + 1)^4$ by -2:



Step 3: add 3 to the y-coordinates of each of the points on the graph of $y = -2(x + 1)^4$:



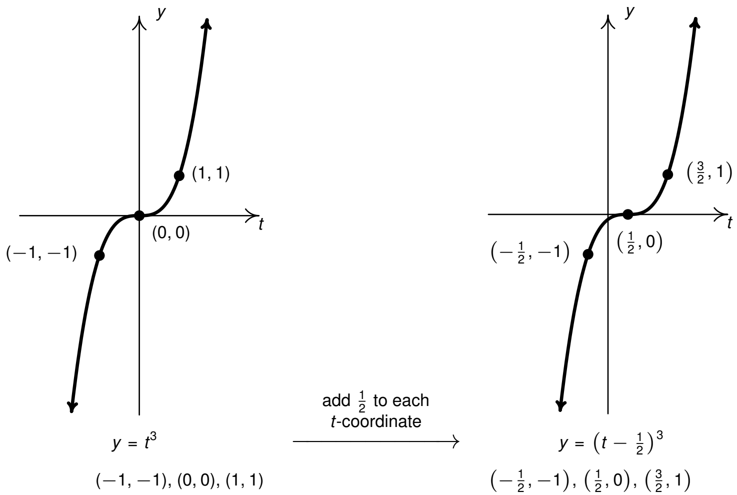
The domain here is $(-\infty, \infty)$ while the range is $(-\infty, 3]$.

2. To use Theorem 2.1.1 to graph $g(t) = \dfrac{(2t-1)^3}{5}$, we must rewrite the expression for $g(t)$:

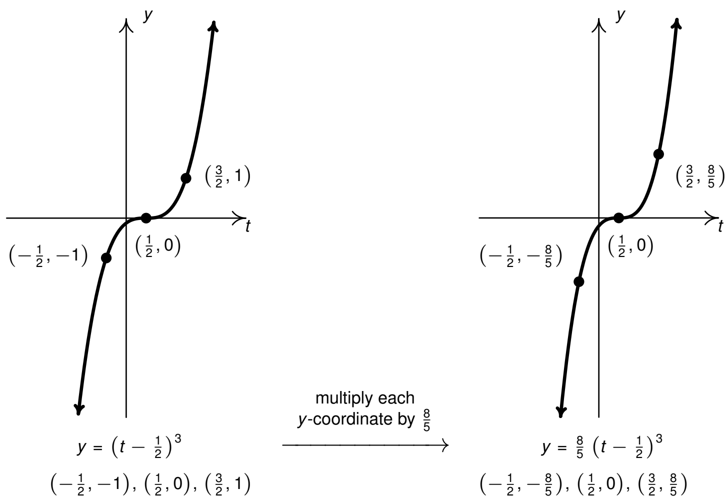
$$g(t) = \frac{(2t-1)^3}{5} = \frac{1}{5}\left(2\left(t-\frac{1}{2}\right)\right)^3 = \frac{1}{5}(2)^3\left(t-\frac{1}{2}\right)^3 = \frac{8}{5}\left(t-\frac{1}{2}\right)^3$$

We identify $n = 3$, $h = \frac{1}{2}$ and $a = \frac{8}{5}$. Hence, we start with the graph of $y = t^3$ and perform the following steps, in sequence, tracking the points $(-1, -1)$, $(0, 0)$ and $(1, 1)$ through each step:

Step 1: add $\frac{1}{2}$ to each of the t-coordinates of each of the points on the graph of $y = t^3$:



Step 2: multiply each of the y-coordinates of the graph of $y = \left(t - \frac{1}{2}\right)^3$ by $\frac{8}{5}$.



Both the domain and range of g is $(-\infty, \infty)$. $\square$

x	$f(x) = x^2$
-1000	1000000
-100	10000
-10	100
0	0
10	100
100	10000
1000	1000000

Table 2.1.3: $f(x) = x^2$

Example 2.1.1 demonstrates two big ideas in mathematics: first, resolving a complex problem into smaller, simpler steps, and, second, the value of changing form.[9]

Next we wish to focus on the so-called **end behavior** presented in each case.[10] The end behavior of a function is a way to describe what is happening to the outputs from a function as the inputs approach the 'ends' of the domain. Since domain of monomial functions is $(-\infty, \infty)$, we are looking to see what these functions do as their inputs 'approach' $\pm\infty$. The best we can do is sample inputs and outputs and infer general behavior from these observations.[11] The good news is we've wrestled with this concept before. Indeed, every time we add 'arrows' to the graph of a function, we've indicated its end behavior. Let's revisit the graph of $f(x) = x^2$ using Table 2.1.3.

As x takes on smaller and smaller values,[12], we see $f(x)$ takes on larger and larger positive values. The smaller x we use, the larger the $f(x)$ becomes, seemingly without bound.[13] We codify this behavior by writing as $x \to -\infty$, $f(x) \to \infty$. Graphically, the farther to the left we travel on the x-axis, the farther up the y-axis the function values travel as seen in Figure 2.1.3. This is why we use an 'arrow' on the graph in Quadrant II heading upwards to the left. Similarly, we write as $x \to \infty$, $f(x) \to \infty$ since as the x values increase, so do the $f(x)$ values - seemingly without bound. Graphically we indicate this by an arrow on the graph

[9]We've seen the importance of changing form several times already, but it never hurts to point it out.

[10]Sometimes called the 'long run' behavior.

[11]and let Calculus students prove our claims.

[12]said differently, negative values that are larger in absolute value

[13]That is, the $f(x)$ values grow larger than any positive number. They are '*unbounded*.'

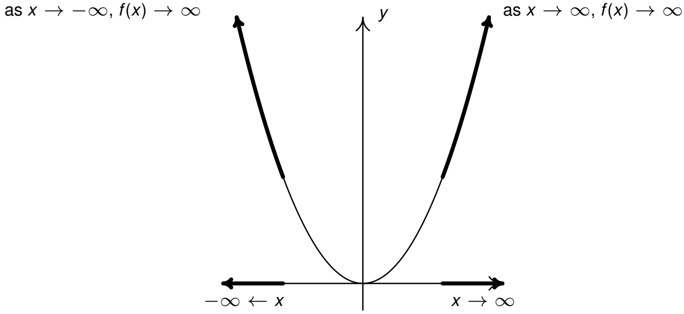


Figure 2.1.3: $f(x) = x^2$

x	$f(x) = x^3$
-1000	-1000000000
-100	-1000000
-10	-1000
0	0
10	1000
100	1000000
1000	1000000000

Table 2.1.4

in Quadrant I heading upwards to the right. This behavior holds for all functions $f(x) = x^n$ where $n \geq 2$ is even.

Repeating this investigation for for $f(x) = x^3$, we find as $x \to -\infty$, $f(x)$ becomes unbounded in the negative direction, so we write $f(x) \to -\infty$. As $x \to \infty$, $f(x)$ becomes unbounded in the positive direction, so we write $f(x) \to \infty$. This trend holds for all functions $f(x) = x^n$ where n is odd.

Theorem 2.1.2 summarizes the end behavior of monomial functions. The results are a consequence of Theorem 2.1.1 in that the end behavior of a function of the form $y = ax^n$ only differs from that of $y = x^n$ if there is a reflection, that is, if $a < 0$.

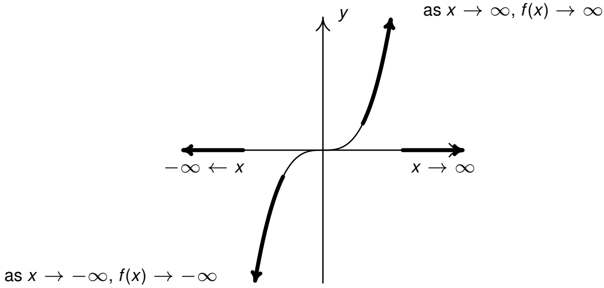
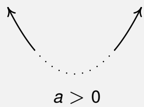


Figure 2.1.4: $f(x) = x^3$

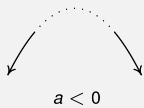
Theorem 2.1.2. End Behavior of Monomial Functions:
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \in \mathbb{N}$.

- If n is even:

  if $a > 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to \infty$:

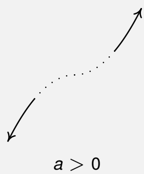
  

  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to -\infty$:

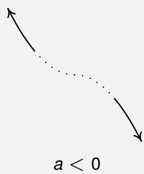
  

- If n is odd:

  for $a < 0$, as $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$:

  for $a < 0$, as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$:

2.1.2 Polynomial Functions

We are now in the position to discuss **polynomial** functions. Simply stated, *poly*nomial functions are sums of *mono*mial functions. The challenge becomes how to describe one of these beasts in general. Up until now, we have used distinct letters to indicate different parameters in our definitions of function families. In other words, we define constant functions as $f(x) = b$, linear functions as $f(x) = mx + b$, and quadratic functions as $f(x) = ax^2 + bx + c$. We even hinted at a function of the form $f(x) = ax^3 + bx^2 + cx + d$. What happens if we wanted to describe a generic polynomial that required, say, 117 different parameters? Our work around is to use subscripted parameters, a_k, that denote the coefficient of x^k. For example, instead of writing a quadratic as $f(x) = ax^2+bx+c$, we describe it as $f(x) = a_2x^2 + a_1x + a_0$, where a_2, a_1, and a_0 are real numbers and $a_2 \neq 0$. As an added example, consider $f(x) = 4x^5 - 3x^2 + 2x - 5$. We can re-write the formula for f as $f(x) = 4x^5 + 0x^4 + 0x^3 + (-3)x^2 + 2x + (-5)$. and identify $a_5 = 4$, $a_4 = 0$, $a_3 = 0$, $a_2 = -3$, $a_1 = 2$ and $a_0 = -5$. This is the notation we use in the following definition.

Definition 2.1.4. A **polynomial function** is a function of the form

$$f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0,$$

where a_0, a_1, $\ldots$, a_n are real numbers and $n \in \mathbb{N}$. The domain of a polynomial function is $(-\infty, \infty)$.

As usual, x is used in Definition 2.1.4 as the independent variable with the a_k each being a parameter. Even though we specify $n \in \mathbb{N}$ so $n \geq 1$, the value of the a_k are unrestricted. Hence, any constant function $f(x) = b$ can be written as $f(x) = 0x + a_0$, and so they are polynomials. Polynomials have an associated vocabulary,[14] and hence, so do polynomial functions.

[14] See Section **??**.

Definition 2.1.5.

- Given $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ with $n \in \mathbb{N}$ and $a_n \neq 0$, we say
  - The natural number n is called the **degree** of the polynomial f.
  - The term $a_n x^n$ is called the **leading term** of the polynomial f.
  - The real number a_n is called the **leading coefficient** of the polynomial f.
  - The real number a_0 is called the **constant term** of the polynomial f.
- If $f(x) = a_0$, and $a_0 \neq 0$, we say f has degree 0.
- If $f(x) = 0$, we say f has no degree.[a]

[a] Some authors say $f(x) = 0$ has degree $-\infty$ for reasons not even we will go into.

Again, constant functions are split off in their own separate case Definition 2.1.5 because of the ambiguity of 0^0. (See the remarks following Definition 2.1.1.) A consequence of Definition 2.1.5 is that we can now think of nonzero constant functions as 'zeroth' degree polynomial functions, linear functions as 'first' degree polynomial functions, and quadratic functions as 'second' degree polynomial functions.

Example 2.1.2. Find the degree, leading term, leading coefficient and constant term of the following polynomial functions.

1. $f(x) = 4x^5 - 3x^2 + 2x - 5$
2. $g(t) = 12t - t^3$
3. $H(w) = \dfrac{4 - w}{5}$
4. $p(z) = (2z - 1)^3(z - 2)(3z + 2)$

Solution.

1. There are no surprises with $f(x) = 4x^5 - 3x^2 + 2x - 5$. It is written in the form of Definition 2.1.5, and we see that the degree is 5, the leading term is $4x^5$, the leading coefficient is 4 and the constant term is -5.

2. Two changes here: first, the independent variable is t, not x. Second, the form given in Definition 2.1.5 specifies the function be written in descending order of the powers of x, or in this case, t. To that end, we re-write $g(t) = 12t - t^3 = -t^3 + 12t$, and see that the degree of g is 3, the leading term is $-t^3$, the leading coefficient is -1 and the constant term is 0.

3. We need to rewrite the formula for $H(w)$ so that it resembles the form given in Definition 2.1.5: $H(w) = \frac{4-w}{5} = \frac{4}{5} - \frac{w}{5} = -\frac{1}{5}w + \frac{4}{5}$. We see the degree

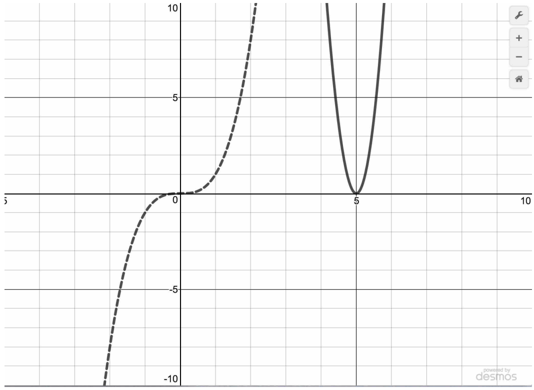


Figure 2.1.5: $f(x)$ and $y = x^3$ near the origin

of H is 1, the leading term is $-\frac{1}{5}w$, the leading coefficient is $-\frac{1}{5}$ and the constant term is $\frac{4}{5}$.

4. It may seem that we have some work ahead of us to get p in the form of Definition 2.1.5. However, it is possible to glean the information requested about p without multiplying out the entire expression $(2z-1)^3(z-2)(3z+2)$. The leading term of p will be the term which has the highest power of z. The way to get this term is to multiply the terms with the highest power of z from each factor together - in other words, the leading term of $p(z)$ is the product of the leading terms of the *factors* of $p(z)$. Hence, the leading term of p is $(2z)^3(z)(3z) = 24z^5$. This means that the degree of p is 5 and the leading coefficient is 24. As for the constant term, we can perform a similar operation. The constant term of p is obtained by multiplying the constant terms from each of the *factors*: $(-1)^3(-2)(2) = 4$. □

We now turn our attention to graphs of polynomial functions. Since polynomial functions are sums of monomial functions, it stands to reason that some of of the properties of those graphs carry over to more general polynomials. We first discuss end behavior. Consider $f(x) = x^3 - 75x + 250$. Below is the graph of $f(x)$ (solid line) along with the graph of its leading term, $y = x^3$ (dashed line.) Figure 2.1.5 shows a view 'near' the origin while Figure 2.1.6shows a 'zoomed out' view. Near the origin, the graphs have little in common, but as we look farther out, it becomes that the functions begin to look quite similar.

This observation is borne out numerically as well. Based on Table 2.1.5, as

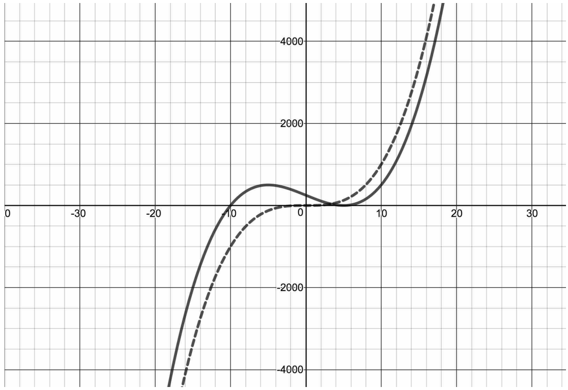


Figure 2.1.6: $f(x)$ and $y = x^3$ zoomed out

$x \to \pm\infty$, it certainly appears as if $f(x) \approx g(x)$. One way to think about what is happening numerically is that the leading term x^3 *dominates* the lower order terms $-75x$ and 250 as $x \to \pm\infty$. In other words, x^3 grows so much faster than $-75x$ and 250 that these 'lower order terms' don't contribute anything of significance to the x^3 so $f(x) \approx x^3$. Another way to see this is to rewrite $f(x)$ as[15]

$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right).$$

As $x \to \pm\infty$, both $\frac{75}{x^2}$ and $\frac{250}{x^3}$ have constant numerators but denominators that are becoming unbounded. As such, both $\frac{75}{x^2}$ and $\frac{250}{x^3} \to 0$. Therefore, as $x \to \pm\infty$,

$$f(x) = x^3 - 75x + 250 = x^3\left(1 - \frac{75}{x^2} + \frac{250}{x^3}\right) \approx x^3(1 + 0 + 0) = x^3.$$

Next, consider $g(x) = -0.01x^4 + 5x^2$. Following the logic of the above example, we would expect the end behavior of $y = g(x)$ to mimic that of $y = -0.01x^4$. When we graph $y = g(x)$ (solid line) on the same set of axes as $y = -0.01x^4$ (dashed line), a view near the origin (Figure 2.1.7) seems to suggest the exact opposite. However, zooming out Figure 2.1.8) reveals that the two graphs do share the same end behavior.[16]

[15] Since we are considering $x \to \pm\infty$, we are not concerned with x even being close to 0, so these fractions will all be defined.

[16] Or at least they appear to within the limits of the technology.

x	$f(x) = x^3 - 75x + 250$	x^3	$-75x$	250	$\frac{75}{x^2}$	$\frac{250}{x^3}$
-1000	$\approx -1 \times 10^9$	-1×10^9	75000	250	7.5×10^{-5}	-2.5×10^{-7}
-100	$\approx -9.9 \times 10^5$	-1×10^6	7500	250	0.0075	-2.5×10^{-4}
-10	0	-1000	750	250	0.75	-0.25
10	500	1000	-750	250	0.75	0.25
100	$\approx 9.9 \times 10^5$	1×10^6	-7500	250	0.0075	2.5×10^{-4}
1000	$\approx 1 \times 10^9$	1×10^9	-75000	250	7.5×10^{-5}	2.5×10^{-7}

Table 2.1.5: Numerical analysis of $f(x) = x^3 - 75x + 250$

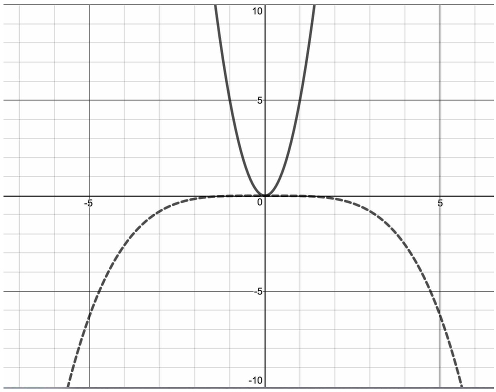


Figure 2.1.7: $g(x) = -0.01x^4 + 5x^2$ near origin

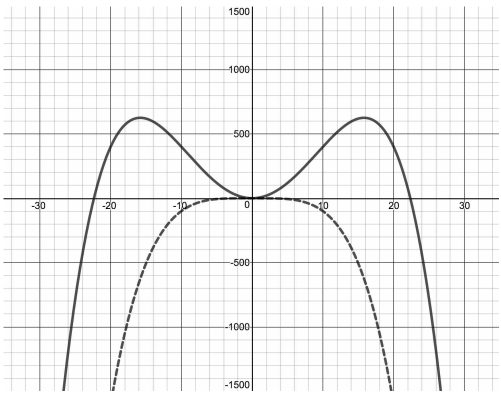


Figure 2.1.8: $g(x) = -0.01x^4 + 5x^2$ zoomed out

Algebraically, for $x \to \pm\infty$, even with the small coefficient of -0.01, $-0.01x^4$ dominates the $5x^2$ term so $g(x) \approx -0.01x^4$. More precisely,

$$g(x) = -0.01x^4 + 5x^2 = x^4\left(-0.01 + \frac{5}{x^2}\right) \approx x^4(-0.01 + 0) = -0.01x^4.$$

The results of these last two examples generalize below in Theorem 2.1.3.

Theorem 2.1.3. End Behavior for Polynomial Functions:
The end behavior of polynomial function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_n x^n$.
That is, the end behavior of a polynomial function is determined by its leading term.

We argue Theorem 2.1.3 using an argument similar to ones used above. As $x \to \pm\infty$,

$$f(x) = x^n\left(a_n + \frac{a_{n-1}}{x} + \ldots + \frac{a_2}{x^{n-2}} + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n}\right) \approx x^n(a_n + 0 + \ldots 0) = a_n x^n$$

If this argument looks a little fuzzy, it should. In Calculus, we have the tools necessary to more explicitly state what we mean by ≈ 0. For now, we'll rely on number sense and algebraic intuition.[17]

Now that we know how to determine the end behavior of polynomial functions, it's time to investigate what happens 'in between' the ends. First and foremost, polynomial functions are **continuous**. Recall from Section 1.4 that, informally, graphs of continuous functions have no 'breaks' or 'holes' in them.[18] Since monomial functions are continuous (as far as we can tell) and polynomials are sums of monomial functions, it turns out that polynomial functions are continuous as well. Moreover, the graphs of monomial functions, hence polynomial functions, are **smooth**. Once again, 'smoothness' is a concept defined precisely in Calculus, but for us, functions have no 'corners' or 'sharp turns'. In Figure 2.1.9 we find the graph of a function which is neither smooth nor continuous, and in Figure 2.1.10 we have a graph of a polynomial, for comparison. The function

[17] Both of which, by the way, can lead one astray, so we must proceed cautiously.

[18] Again, the formal definition of 'continuity' and properties of continuous functions are discussed in Calculus.

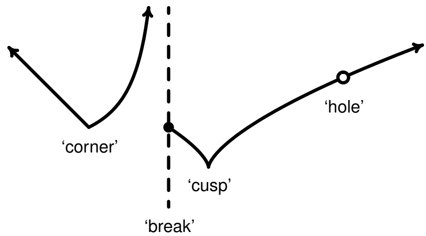


Figure 2.1.9: Pathologies not found on graphs of polynomials functions

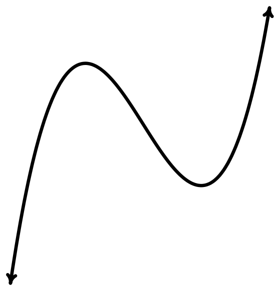


Figure 2.1.10: The graph of a polynomial function

whose graph appears on the left fails to be continuous where it has a 'break' or 'hole' in the graph; everywhere else, the function is continuous. The function is continuous at the 'corner' and the 'cusp', but we consider these 'sharp turns', so these are places where the function fails to be smooth. Apart from these four places, the function is smooth and continuous. Polynomial functions are smooth and continuous everywhere, as exhibited in the graph on the right. The notion of smoothness is what tells us graphically that, for example, $f(x) = |x|$, whose graph is the characteristic '$\vee$' shape, cannot be a polynomial function, even though it is a piecewise-defined function comprised of polynomial functions. Knowing polynomial functions are continuous and smooth gives us an idea of how to 'connect the dots' when sketching the graph from points that we're able to find analytically such as intercepts.

Speaking of intercepts, we next focus our attention on the behavior of the graphs of polynomial functions near their zeros. Recall a zero c of a function f is a solution to $f(x) = 0$. Geometrically, the zeros of a function are the x-coordinates of the x-intercepts of the graph of $y = f(x)$. Consider the polynomial function

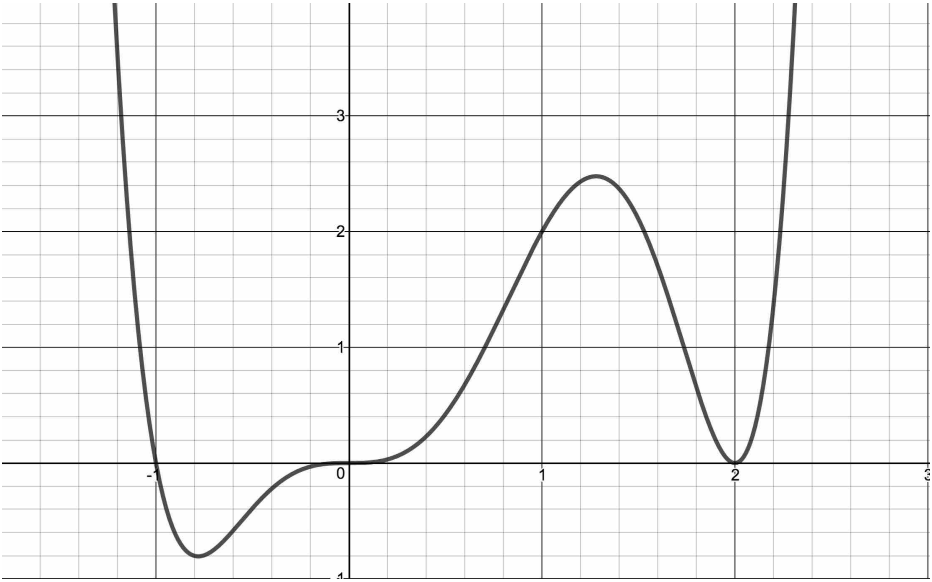


Figure 2.1.11: $f(x) = x^3(x-2)^2(x+1)$

$f(x) = x^3(x-2)^2(x+1)$. To find the zeros of f, we set $f(x) = x^3(x-2)^2(x+1) = 0$. Since the expression $f(x)$ is already factored, we set each factor equal to zero.[19] Solving $x^3 = 0$ gives $x = 0$, $(x-2)^2 = 0$ gives $x = 2$, and $x+1 = 0$ gives $x = -1$. Hence, our zeros are $x = -1$, $x = 0$, and $x = 2$. In Figure 2.1.11, we graph $y = f(x)$ and observe the x-intercepts $(-1,0)$, $(0,0)$ and $(2,0)$. We first note that the graph *crosses* through the x-axis at $(-1,0)$ and $(0,0)$, but the graph *touches* and *rebounds* at $(2,0)$. Moreover, at $(-1,0)$, the graph crosses through the axis is a fairly 'linear' fashion whereas there is a substantial amount of 'flattening' going on near $(0,0)$. Or aim is to explain these observations and generalize them.

First, let's look at what's happening with the formula $f(x) = x^3(x-2)^2(x+1)$ when $x \approx -1$. We know the x-intercept at $(-1,0)$ is due to the presence of the $(x+1)$ factor in the expression for $f(x)$. So, in this sense, the factor $(x+1)$ is determining a major piece of the behavior of the graph near $x = -1$. For that reason, we focus instead on the other two factors to see what contribution they make. We

[19] in accordance with the Zero Product Property of the Real Numbers - see Section **??**.

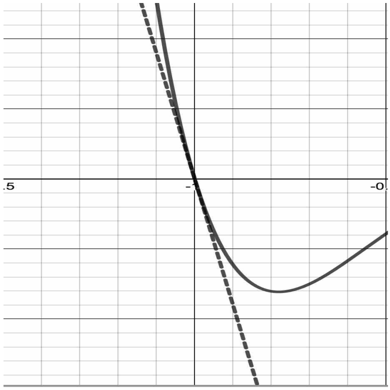


Figure 2.1.12: $y = f(x)$ and $y = -9(x+1)$

find when $x \approx -1$, $x^3 \approx (-1)^3 = -1$ and $(x-2)^2 \approx (-1-2)^2 = 9$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (-1)^3(-1-2)^2(x+1) = -9(x+1)$. In Figure 2.1.12 is a graph of $y = f(x)$ (the solid line) and the graph of $y = -9(x+1)$ (the dashed line.) Sure enough, these graphs approximate one another near $x = -1$.

Likewise, let's look near $x = 0$. The x-intercept $(0,0)$ is due to the x^3 term. For $x \approx 0$, $(x-2)^2 \approx (0-2)^2 = 4$ and $(x+1) \approx (0+1) = 1$, so $f(x) = x^3(x-3)^2(x+1) \approx x^3(-2)^2(1) = 4x^3$. In Figure 2.1.13, we have the graph of $y = f(x)$ (again, the solid line) and $y = 4x^3$ (the dashed line) near $x = 0$. Once again, the graphs verify our analysis.

Last, but not least, we analyze f near $x = 2$. Here, the intercept $(2,0)$ is due to the $(x-2)^2$ factor, so we look at the x^3 and $(x+1)$ factors. If $x \approx 2$, $x^3 \approx (2)^3 = 8$ and $(x+1) \approx (2+1) = 3$. Hence, $f(x) = x^3(x-3)^2(x+1) \approx (2)^3(x-2)^2(2+1) = 24(x-2)^2$. Sure enough, as evidenced in Figure 2.1.14, the graphs of $y = f(x)$ and $y = 24(x-2)^2$.

We generalize our observations in Theorem 2.1.4 below. Like many things we've seen in this text, a more precise statement and proof can be found in a course on Calculus.

Theorem 2.1.4. Suppose f is a polynomial function and $f(x) = (x-c)^m q(x)$ where $m \in \mathbb{N}$ and $q(c) \neq 0$. Then the the graph of $y = f(x)$ near $(c, 0)$ resembles that of $y = q(c)(x-c)^m$.

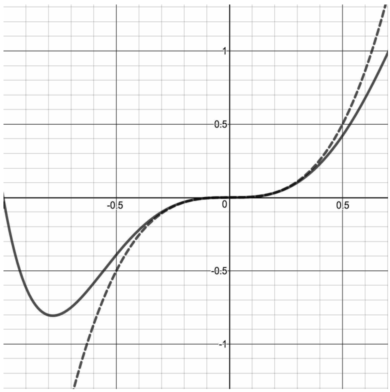


Figure 2.1.13: $y = f(x)$ and $y = 4x^3$

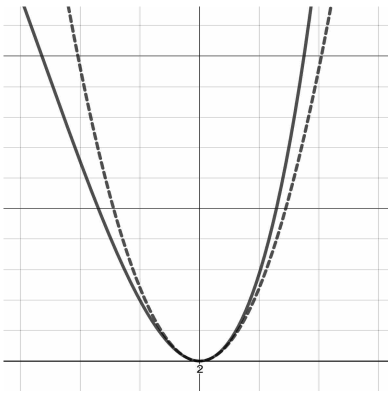


Figure 2.1.14: $y = f(x)$ and $y = 24(x - 2)^2$

Let's see how Theorem 2.1.4 applies to our findings regarding $f(x) = x^3(x-2)^2(x+1)$. For $c = -1$, $(x-c) = (x-(-1)) = (x+1)$. We rewrite $f(x) = x^3(x-2)^2(x+1) = (x-(-1))^1\left[x^3(x-2)^2\right]$ and identify $m = 1$ and $q(x) = x^3(x-2)^2$. We find $q(c) = q(-1) = (-1)^3(-1-2)^2 = -9$ so Theorem 2.1.4 says that near $(-1, 0)$, the graph of $y = f(x)$ resembles $y = q(-1)(x-(-1))^1 = -9(x+1)$. For $c = 0$, $(x-c) = (x-0) = x$ and we can rewrite $f(x) = x^3(x-2)^2(x+1) = (x-0)^3\left[(x-2)^2(x+1)\right]$. We identify $m = 3$ and $q(x) = (x-2)^2(x+1)$. In this case $q(c) = q(0) = (0-2)^2(0+1) = 4$, so Theorem 2.1.4 guarantees the graph of $y = f(x)$ near $x = 0$ resembles $y = q(0)(x-0)^3 = 4x^3$. Lastly, for $c = 2$, we see $f(x) = (x-2)^2\left[x^3(x+1)\right]$ and we identify $m = 2$ and $q(x) = x^3(x+1)$. We find $q(2) = 2^3(2+1) = 24$, so Theorem 2.1.4 guarantees the graph of $y = f(x)$ resembles $y = 24(x-2)^2$ near $x = 2$.

As we already mentioned, the formal statement and proof of Theorem 2.1.4 require Calculus. For now, we can understand the theorem as follows. If we factor a polynomial function as $f(x) = (x-c)^m q(x)$ where $m \geq 1$, then $x = c$ is a zero of f, since $f(c) = (c-c)^m q(c) = 0 \cdot q(c) = 0$. The stipulation that $q(c) \neq 0$ means that we have essentially factored the expression $f(x) = (x-c)^m q(x) = (\text{going to } 0) \cdot (\text{not going to } 0)$. Thinking back to Theorem 2.1.1, the graph $y = q(c)(x-c)^m$ has an x-intercept at $(c, 0)$, a basic overall shape determined by the exponent m, and end behavior determined by the sign of $q(c)$. The fact that *if* $x = c$ is a zero *then* we are guaranteed we can factor $f(x) = (x-c)^m q(x)$ were $q(c) \neq 0$ and, moreover, such a factorization is unique (so that there's only one value of m possible for each zero) is a consequence of two theorems, Theorem 2.2.1 and The Factor Theorem, Theorem 2.2.3 which we'll review in Section 2.2. For now, we assume such a factorization is unique in order to define the following.

Definition 2.1.6. Suppose f is a polynomial function and $m \in \mathbb{N}$. If $f(x) = (x-c)^m q(x)$ where $q(c) \neq 0$, we say $x = c$ is a zero of **multiplicity** m.

So, for $f(x) = x^3(x-2)^2(x+1) = (x-0)^3(x-2)^2(x-(-1))^1$, $x = 0$ is a zero of multiplicity 3, $x = 2$ is a zero of multiplicity 2, and $x = -1$ is a zero of multiplicity 1. Theorems 2.1.3 and 2.1.4 give us the following:

> **Theorem 2.1.5. The Role of Multiplicity:** Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.
> - If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
> - If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

Our next example showcases how all of the above theory can assist in sketching relatively good graphs of polynomial functions without the assistance of technology.

Example 2.1.3. Let $p(x) = (2x - 1)(x + 1)(1 - x^4)$.

1. Find all real zeros of p and state their multiplicities.
2. Describe the behavior of the graph of $y = p(x)$ near each of the x-intercepts.
3. Determine the end behavior and y-intercept of the graph of $y = p(x)$.
4. Sketch $y = p(x)$ and check your answer using a graphing utility.

Solution.

1. To find the zeros of p, we set $p(x) = (2x - 1)(x + 1)(1 - x^4) = 0$. Since the expression $p(x)$ is already (partially) factored, we set each factor equal to 0 and solve. From $(2x - 1) = 0$, we get $x = \frac{1}{2}$; from $(x + 1) = 0$ we get $x = -1$; and from solving $1 - x^4 = 0$ we get $x = \pm 1$. Hence, the zeros are $x = -1$, $x = \frac{1}{2}$, and $x = 1$. In order to determine the multiplicities, we need to factor $p(x)$ as so we can identify the m and $q(x)$ as described in Definition 2.1.6. The zero $x = -1$ corresponds to the factor $(x + 1)$. Notice, however, that writing $p(x) = (x + 1)^1 \left[(2x - 1)(1 - x^4)\right]$ with $m = 1$ and $q(x) = (2x - 1)(1 - x^4)$ does *not* satisfy Definition 2.1.6 since here, $q(-1) = (2(-1) - 1)(1 - (-1)^4) = 0$. Indeed, we can factor $(1 - x^4) = (1 - x^2)(1 + x^2) = (1 - x)(1 + x)(x^2 + 1)$ so that

   $$p(x) = (2x-1)(x+1)(1-x^4) = (2x-1)(x+1)(1-x)(1+x)(x^2+1) = (x+1)^2 \big[(2x - 1)(1 - x)(x$$

   Identifying $q(x) = (2x - 1)(1 - x)(x^2 + 1)$, we find $q(-1) = (2(-1) - 1)(1 - (-1))((-1)^2 + 1) = -12 \neq 0$, which means the multiplicity of $x = -1$ is $m = 2$.

   The zero $x = \frac{1}{2}$ came from the factor $(2x - 1) = 2(x - \frac{1}{2})$, so we have

   $$p(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1) = (x - \frac{1}{2})^1 \left[2(x + 1)^2(1 - x)(x^2 + 1)\right].$$

If we identify $q(x) = 2(x+1)^2(1-x)(x^2+1)$, we find $q(\frac{1}{2}) = \frac{45}{16} \neq 0$ so multiplicity here is $m = 1$.

Last but not least, we turn our attention to our last zero, $x = 1$, which we obtained from solving $1 - x^4 = 0$. However, from $p(x) = (2x-1)(x+1)^2(1-x)(x^2+1)$, we see the zero $x = 1$ corresponds to the factor $(1-x) = -(x-1)$. We have $p(x) = (x-1)^1\left[-(2x-1)(x+1)^2(x^2+1)\right]$. Identifying $q(x) = -(2x-1)(x+1)^2(x^2+1)$, we see $q(1) = -8$, so the multiplicity $m = 1$ here as well.

2. From Theorem 2.1.5, since the multiplicities of $x = \frac{1}{2}$ and $x = 1$ are both *odd*, we know the graph of $y = p(x)$ *crosses* through the x-axis at $(\frac{1}{2}, 0)$ and $(1, 0)$. More specifically, since the multiplicity for both of these zeros is 1, the graph will look locally linear at these points. More specifically, based on our calculations above, near $x = \frac{1}{2}$, the graph will resemble the increasing line $y = \frac{45}{16}(x - \frac{1}{2})$, and near $x = 1$, the graph will resemble the decreasing line $y = -8(x-1)$. Since the multiplicity of $x = -1$ is *even*, we know the graph of $y = p(x)$ *touches* and *rebounds* at $(-1, 0)$. Since the multiplicity of $x = -1$ is 2, it will look locally like a parabola. More specifically, the graph near $x = -1$ will resemble $y = -12(x+1)^2$.

3. Per Theorem 2.1.3, the end behavior of $y = p(x)$, matches the end behavior of its leading term. As in Example 2.1.2, we multiply the leading terms from each factor together to obtain the leading term for $p(x)$: $p(x) = (2x-1)(x+1)(1-x^4) = (2x)(x)(-x^4)+\ldots = -2x^6+\ldots$. Since the degree here, 6, is even and the leading coefficient $-2 < 0$, we know as $x \to \pm\infty$, $p(x) \to -\infty$. To find the y-intercept, we find $p(0) = (2(0)-1)(0+1)(1-0^4) = -1$, hence, the y-intercept is $(0, -1)$.

4. From the end behavior, $x \to -\infty$, $p(x) \to -\infty$, we start the graph in Quadrant III and head towards $(-1, 0)$. At $(-1, 0)$, we 'bounce' off of the x-axis and head towards the y-intercept, $(0, -1)$. We then head towards $\left(\frac{1}{2}, 0\right)$ and cross through the x-axis there. Finally, we head back to the x-axis and cross through at $(1, 0)$. Owing to the end behavior $x \to \infty$, $p(x) \to -\infty$, we exit the picture in Quadrant IV. Since polynomial functions are continuous and smooth, we have no holes or gaps in the graph, and all the 'turns' are rounded (no abrupt turns or corners.) We produce something resembling the graph in Figure 2.1.15.

□

A couple of remarks about Example 2.1.3 are in order. First, notice that the factor (x^2+1) was more of a spectator in our discussion of the zeros of p. Indeed, if

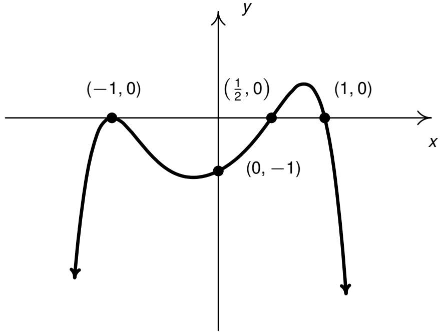


Figure 2.1.15: $y = p(x)$

we set $x^2 + 1 = 0$, we have $x^2 = -1$ which provides no *real* solutions.[20] That being said, the factor $x^2 + 1$ *does* affect the shape of the graph (See Exercise 60.) Next, when connecting up the graph from $(-1, 0)$ to $(0, -1)$ to $\left(\frac{1}{2}, 0\right)$, there really is no way for us to know how low the graph goes, or where the lowest point is between $x = -1$ and $x = \frac{1}{2}$ unless we plot more points. Likewise, we have no idea how high the graph gets between $x = \frac{1}{2}$ and $x = 1$. While there are ways to determine these points analytically, more often than not, finding them requires Calculus. Since these points do play an important role in many applications, we'll need to discuss them in this course and, when required, we'll use technology to find them. For that reason, we have the following definition:

[20]The solutions are $x = \pm i$ - see Section **??**.

Definition 2.1.7. Suppose f is a function with $f(a) = b$.

- We say f has a **local minimum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \leq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local minimum value of f.'
  That is, b is the minimum $f(x)$ value over an *open interval* containing a.
  Graphically, no points 'near' a local minimum are lower than (a, b).
- We say f has a **local maximum** at the point (a, b) if and only if there is an open interval I containing a for which $f(a) \geq f(x)$ for all x in I. The value $f(a) = b$ is called 'a local maximum value of f.'
  That is, b is the maximum $f(x)$ value over an *open interval* containing a.
  Graphically, no points 'near' a local maximum are higher than (a, b).

Taken together, the local maximums and local minimums of a function, if they exist, are called the **local extrema** of the function.

Once again, the terminology used in Definition 2.1.7 blurs the line between the function f and its outputs, $f(x)$. Also, some textbooks use the terms 'relative' minimum and 'relative' maximum instead of the adjective 'local.' Lastly, note the definition of local extrema requires an *open* interval exist in the domain containing a in order for $(a, f(a))$ to be a candidate for a local maximum or local minimum. We'll have more to say about this in later chapters. If our open interval happens to be $(-\infty, \infty)$, then our local extrema are the extrema of f - we'll see an example of this momentarily.

In Figure 2.1.16 we use a graphing utility to graph $y = p(x) = (2x-1)(x+1)(1-x^4)$. We first consider the point $(-1, 0)$. Even though there are points on the graph of $y = p(x)$ that are higher than $(-1, 0)$, locally, $(-1, 0)$ is the top of a hill. To satisfy Definition 2.1.7, we need to provide an open interval on which $p(-1) = 0$ is the largest, or maximum function value. Note the definition requires us to provide *just one* open interval. One that works is the interval $(-1.5, -0.5)$. We could use any smaller interval or go as large as $\left(-\infty, \frac{1}{2}\right)$ (can you see why?) Next we encounter a 'low' point at approximately $(-0.2353, -1.1211)$. More specifically, for all x in the interval, say, $(-0.5, 0)$, $p(x) \geq 1.1211$, Hence, we have a local minimum at $(-0.2353, -1.1211)$. Lastly, at $(0.811, 0.639)$, we are back to a high point. In fact, 0.639 isn't just a local maximum value, based on the graph, it is *the* maximum of p. Here, we may choose the open interval $(-\infty, \infty)$ as the open interval required by Definition 2.1.7, since for all x, $p(x) \leq 0.639$. It is important to note that there is no minimum value of p despite there being a local minimum

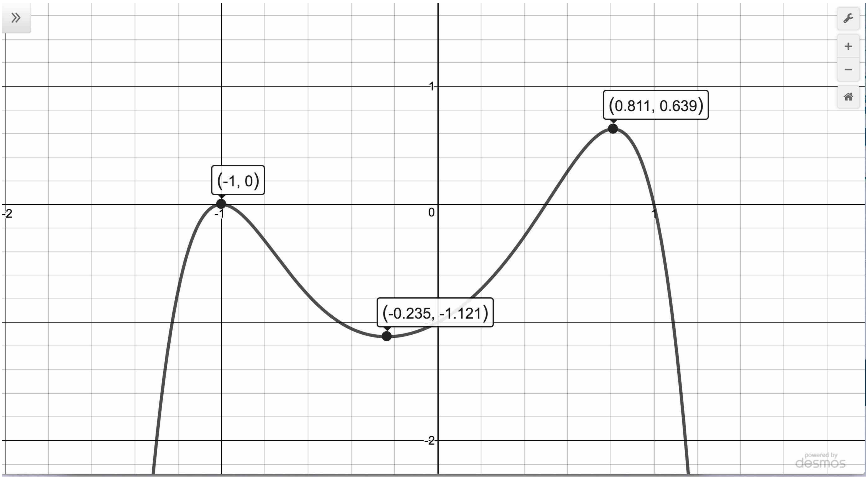


Figure 2.1.16: $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$

value.[21]

We close this section with a classic application of a third degree polynomial function.

Example 2.1.4. A box with no top is to be fashioned from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs (Figure 2.1.17). Let x denote the length of the side of the square which is removed from each corner.

1. Find an expression for $V(x)$, the volume of the box produced by removing squares of edge length x. Include an appropriate domain.

2. Use a graphing utility to help you determine the value of x which produces the box with the largest volume. What is the largest volume? Round your answers to two decimal places.

Solution.

1. From Geometry, we know that Volume = width $\times$ height $\times$ depth. The key is to find each of these quantities in terms of x. From the figure, we see that the height of the box is x itself. The cardboard piece is initially

[21] Some books use the adjectives 'global' or 'absolute' when describing the extreme values of a function to distinguish them from their local counterparts.

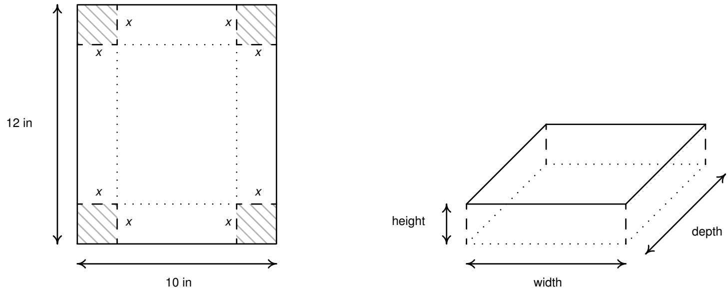


Figure 2.1.17

10 inches wide. Removing squares with a side length of x inches from each corner leaves $10 - 2x$ inches for the width.[22] As for the depth, the cardboard is initially 12 inches long, so after cutting out x inches from each side, we would have $12 - 2x$ inches remaining. Hence, we get $V(x) = x(10 - 2x)(12 - 2x)$. To find a suitable applied domain, we note that to make a box at all we need $x > 0$. Also the shorter of the two dimensions of the cardboard is 10 inches, and since we are removing $2x$ inches from this dimension, we also require $10 - 2x > 0$ or $x < 5$. Hence, our applied domain is $0 < x < 5$.

2. Using a graphing utility, we find a local maximum at approximately $(1.811, 96.771)$ (Figure 2.1.18). Because the domain of V is restricted to the interval $(0, 5)$, the maximum of V is here as well.

   This means the maximum volume attainable is approximately 96.77 cubic inches when we remove squares approximately 1.81 inches per side. □

Notice that there is a very slight, but important, difference between the function $V(x) = x(10 - 2x)(12 - 2x)$, $0 < x < 5$ from Example 2.1.4 and the function $p(x) = x(10 - 2x)(12 - 2x)$: their domains. The domain of V is restricted to the interval $(0, 5)$ while the domain of p is $(-\infty, \infty)$. Indeed, the function V has a maximum of (approximately) 96.771 at (approximately) $x = 1.811$ whereas for the function p, 96.771 is a local maximum value only. We leave it to the reader to verify that V has neither a minimum nor a local minimum.

[22] There's no harm in taking an extra step here and making sure this makes sense. If we chopped out a 1 inch square from each side, then the width would be 8 inches, so chopping out x inches would leave $10 - 2x$ inches.

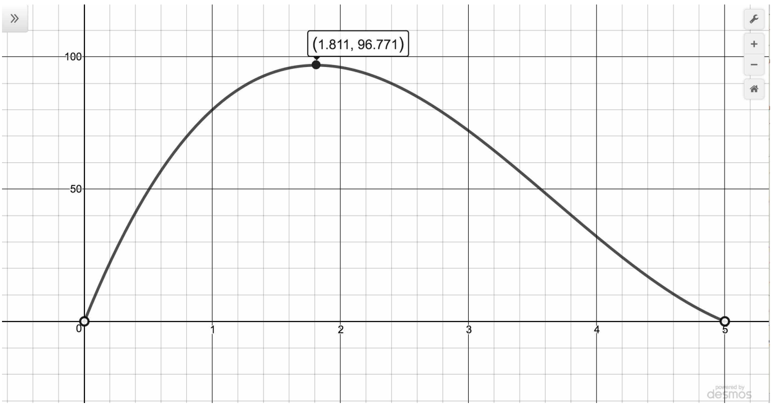


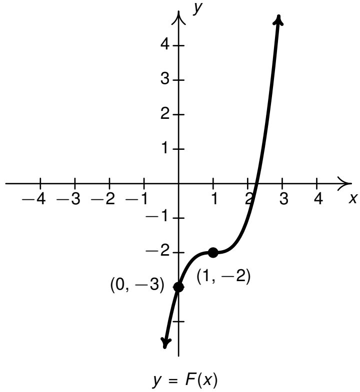
Figure 2.1.18

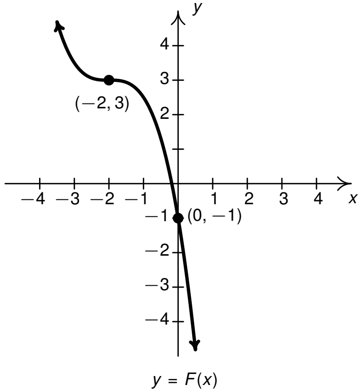
2.1.3 Exercises

In Exercises 1 - 6, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 2.1.1. Track at least three points of your choice through the transformations. State the domain and range of g.

1. $f(x) = x^3$, $F(x) = (x+2)^3 + 1$
2. $f(x) = x^4$, $F(x) = (x+2)^4 + 1$
3. $f(x) = x^4$, $F(x) = 2 - 3(x-1)^4$
4. $f(x) = x^5$, $F(x) = -x^5 - 3$
5. $f(x) = x^5$, $F(x) = (x+1)^5 + 10$
6. $f(x) = x^6$, $F(x) = 8 - x^6$

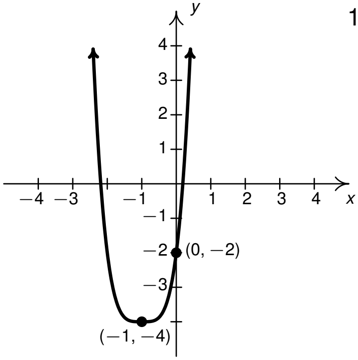
In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(x-h)^3 + k$.

7. 

8. 

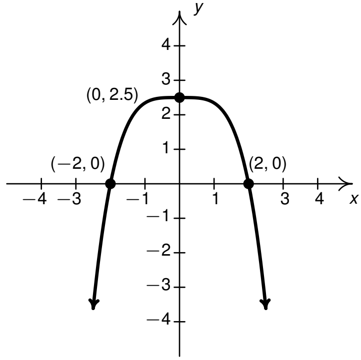
In Exercises 9. - 10., find a formula for each function below in the form $F(x) = a(x-h)^4 + k$.

9.



$y = F(x)$

10.



$y = F(x)$

In Exercises 11 - 20, find the degree, the leading term, the leading coefficient, the constant term and the end behavior of the given polynomial function.

11. $f(x) = 4 - x - 3x^2$
12. $g(x) = 3x^5 - 2x^2 + x + 1$
13. $q(r) = 1 - 16r^4$
14. $Z(b) = 42b - b^3$
15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
16. $s(t) = -4.9t^2 + v_0 t + s_0$
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
18. $p(t) = -t^2(3-5t)(t^2+t+4)$
19. $f(x) = -2x^3(x+1)(x+2)^2$
20. $G(t) = 4(t-2)^2\left(t + \frac{1}{2}\right)$

In Exercises 21 - 30, find the real zeros of the given polynomial and their corresponding multiplicities. Use this information along with end behavior to provide a rough sketch of the graph of the polynomial function. Compare your answer with the result from a graphing utility.

21. $a(x) = x(x+2)^2$
22. $g(t) = t(t+2)^3$
23. $f(z) = -2(z-2)^2(z+1)$
24. $g(x) = (2x+1)^2(x-3)$
25. $F(t) = t^3(t+2)^2$
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
27. $Q(x) = (x+5)^2(x-3)^4$
28. $h(t) = t^2(t-2)^2(t+2)^2$
29. $H(z) = (3-z)(z^2+1)$
30. $Z(x) = x(42 - x^2)$

In Exercises 31 - 45, determine analytically if the following functions are even, odd or neither. Confirm your answer using a graphing utility.

31. $f(x) = 7x$
32. $g(t) = 7t + 2$
33. $p(z) = 7$
34. $F(s) = 3s^2 - 4$
35. $h(t) = 4 - t^2$
36. $g(x) = x^2 - x - 6$
37. $f(x) = 2x^3 - x$
38. $p(z) = -z^5 + 2z^3 - z$
39. $G(t) = t^6 - t^4 + t^2 + 9$
40. $G(s) = s(s^2 - 1)$
41. $f(x) = (x^2+1)(x-1)$
42. $H(t) = (t^2-1)(t^4+t^2+3)$
43. $g(t) = t(t-2)(t+2)$
44. $P(z) = (2z^5 - 3z)(5z^3 + z)$
45. $f(x) = 0$

46. Suppose $p(x)$ is a polynomial function written in the form of Definition 2.1.4.

    (a) If the nonzero terms of $p(x)$ consist of even powers of x (or a constant), explain why p is even.

    (b) If the nonzero terms of $p(x)$ consist of odd powers of x, explain why p is odd.

    (c) If $p(x)$ the nonzero terms of $p(x)$ contain at least one odd power of x and one even power of x (or a constant term), then p is neither even nor odd.

47. Use the results of Exercise 46 to determine whether the following functions are even, odd, or neither.

    (a) $p(x) = 3x^4 + x^2 - 1$
    (b) $F(s) = s^3 - 14s$
    (c) $f(t) = 2t^5 - t^2 + 1$
    (d) $g(x) = x^3(x^2 + 1)$

48. Show $f(x) = |x|$ is an even function.

49. Rework Example 2.1.4 assuming the box is to be made from an 8.5 inch by 11 inch sheet of paper. Using scissors and tape, construct the box. Are you surprised?[23]

50. For each function $f(x)$ listed in Table 2.1.6, compute the average rate of change over the indicated interval.[24] What trends do you observe? How do your answers manifest themselves graphically?

51. For each function $f(x)$ listed in Table 2.1.7, compute the average rate of change over the indicated interval.[25] What trends do you observe? How do your answers manifest themselves graphically?

In Exercises 52 - 54, suppose the revenue R, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $R(x) = -5x^3 + 35x^2 + 155x$ for $0 \leq x \leq 10.07$.

52. Use a graphing utility to graph $y = R(x)$ and determine the number of TVs which should be sold to maximize revenue. What is the maximum revenue?

[23] Consider decorating the box and presenting it to your instructor. If done well enough, maybe your instructor will issue you some bonus points. Or maybe not.

[24] See Definition 1.2.5 in Section 1.2.4 for a review of this concept, as needed.

[25] See Definition 1.2.5 in Section 1.2.4 for a review of this concept, as needed.

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1						
x						
x^2						
x^3						
x^4						
x^5						

Table 2.1.6

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1				
x				
x^2				
x^3				
x^4				
x^5				

Table 2.1.7

53. Assume the cost, in *thousands* of dollars, to produce x *hundred* LCD TVs is given by the function $C(x) = 200x + 25$ for $x \geq 0$. Find and simplify an expression for the profit function $P(x)$.

    (Remember: Profit = Revenue - Cost.)

54. Use a graphing utility to graph $y = P(x)$ and determine the number of TVs which should be sold to maximize profit. What is the maximum profit?

55. While developing their newest game, Sasquatch Attack!, the makers of the PortaBoy (from Example 1.2.3) revised their cost function and now use $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$. As before, $C(x)$ is the cost to make x PortaBoy Game Systems. Market research indicates that the demand function $p(x) = -1.5x + 250$ remains unchanged. Use a graphing utility to find the production level x that maximizes the *profit* made by producing and selling x PortaBoy game systems.

56. According to US Postal regulations, a rectangular shipping box must satisfy the following inequality: "Length + Girth $\leq$ 130 inches" for Parcel Post and "Length + Girth $\leq$ 108 inches" for other services.

    Let's assume we have a closed rectangular box with a square face of side length x as drawn in Figure 2.1.19. The length is the longest side and is clearly labeled. The girth is the distance around the box in the other two dimensions so in our case it is the sum of the four sides of the square, $4x$.

    (a) Assuming that we'll be mailing a box via Parcel Post where Length + Girth = 130 inches, express the length of the box in terms of x and then express the volume V of the box in terms of x.

    (b) Find the dimensions of the box of maximum volume that can be shipped via Parcel Post.

    (c) Repeat parts 56a and 56b if the box is shipped using "other services".

57. This exercise revisits the data set from Exercise 48b in Section 1.4. In that exercise, you were given a chart of the number of hours of daylight they get on the 21^{st} of each month in Fairbanks, Alaska based on the 2009 sunrise and sunset data found on the U.S. Naval Observatory[26] website. Here $x = 1$ represents January 21, 2009, $x = 2$ represents February 21, 2009, and so on. See Table 2.1.8

[26] http://aa.usno.navy.mil/data/docs/RS_OneYear.php

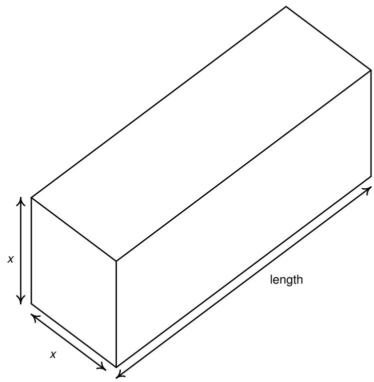


Figure 2.1.19

Month Number	Hours of Daylight
1	5.8
2	9.3
3	12.4
4	15.9
5	19.4
6	21.8
7	19.4
8	15.6
9	12.4
10	9.1
11	5.6
12	3.3

Table 2.1.8

- Find cubic (third degree) and quartic (fourth degree) polynomials which model this data and comment on the goodness of fit for each. What can we say about using either model to make predictions about the year 2020? (Hint: Think about the end behavior of polynomials.)
- Use the models to see how many hours of daylight they got on your birthday and then check the website to see how accurate the models are.
- Sasquatch are largely nocturnal, so what days of the year according to your models allow for at least 14 hours of darkness for field research on the elusive creatures?

58. An electric circuit is built with a variable resistor installed. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in the table below. [27]

Resistance: ($k\Omega$)	1.012	2.199	3.275	4.676	6.805	9.975
Power: (mW)	1.063	1.496	1.610	1.613	1.505	1.314

   (a) Make a scatter diagram of the data using the Resistance as the independent variable and Power as the dependent variable.

   (b) Use your calculator to find quadratic (2nd degree), cubic (3rd degree) and quartic (4th degree) regression models for the data and judge the reasonableness of each.

   (c) For each of the models found above, find the predicted maximum power that can be delivered to the load. What is the corresponding resistance value?

   (d) Discuss with your classmates the limitations of these models - in particular, discuss the end behavior of each.

59. In Figure 2.1.20 is a graph of a polynomial function $y = p(x)$ as generated by a graphing utility. Answer the following questions about p based on the graph provided.

   (a) Describe the end behavior of $y = p(x)$.

   (b) List the real zeros of p along with their respective multiplicities.

[27] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

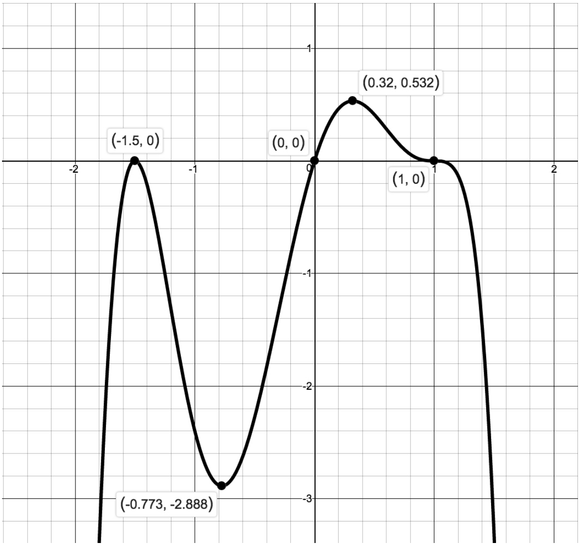


Figure 2.1.20

(c) List the local minimums and local maximums of the graph of $y = p(x)$.

(d) What can be said about the degree of and leading coefficient $p(x)$?

(e) It turns out that $p(x)$ is a seventh degree polynomial.[28] How can this be?

60. (This Exercise is a follow up to Example 2.1.3.) Use a graphing utility to compare and contrast the graphs of $f(x) = (2x - 1)(x + 1)^2(1 - x)(x^2 + 1)$ and $g(x) = (2x - 1)(x + 1)^2(1 - x)$.

61. Use the graph of $y = p(x) = (2x - 1)(x + 1)(1 - x^4)$ on page 206 to estimate the largest open interval containing $x = -0.235$ which satisfies the the criteria for 'local minimum' in Definition 2.1.7.

62. In light of Definition 2.1.7, explain why *every* point on the graph of a constant function is both a local maximum and a local minimum.

63. This exercise involves the greatest integer function, $f(x) = \lfloor x \rfloor$, introduced in Example 1.2.2. Explain why the points (k, k) for integers k are local maximums but not local minimums.

[28] to be exact, $p(x) = -0.1\,(x + 1.5)^2\,(3x)\,(x - 1)^3\,(x + 5)$.

64. Use Theorems 2.1.3 and 2.1.4 prove Theorem 2.1.5.

65. Here are a few other questions for you to discuss with your classmates.

    (a) How many and how few local extrema could a polynomial of degree n have?

    (b) Could a polynomial have two local maxima but no local minima?

    (c) If a polynomial has two local maxima and two local minima, can it be of odd degree? Can it be of even degree?

    (d) Can a polynomial have local extrema without having any real zeros?

    (e) Why must every polynomial of odd degree have at least one real zero?

    (f) Can a polynomial have two distinct real zeros and no local extrema?

    (g) Can an x-intercept yield a local extrema? Can it yield an absolute extrema?

    (h) If the y-intercept yields an absolute minimum, what can we say about the degree of the polynomial and the sign of the leading coefficient?

66. (This is a follow-up to Exercises 60 in Section 1.2 and 61 in Section 1.4.) The Lagrange Interpolate[29] function L for four points: (x_0, y_0), (x_1, y_1), (x_2, y_2), (x_3, y_3) where x_0, x_1, x_2, and x_3 are four distinct real numbers is given by the formula:

$$\begin{aligned} L(x) \quad &= \quad y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\ &\quad + y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \end{aligned}$$

    (a) Choose four points with different x-values and construct the Lagrange Interpolate for those points. Verify each of the points lies on the polynomial.

    (b) Verify that, in general, $L(x_0) = y_0$, $L(x_1) = y_1$, $L(x_2) = y_2$, and $L(x_3) = y_3$.

[29] https://en.wikipedia.org/wiki/Lagrange_polynomial

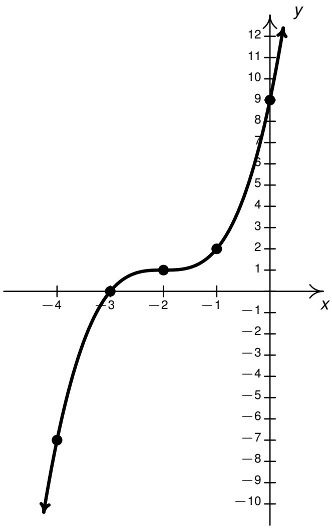
(c) Find $L(x)$ for the points $(-1, 1)$, $(0, 0)$, $(1, 1)$ and $(2, 4)$. What happens?

(d) Find $L(x)$ for the points $(-1, 0)$, $(0, 1)$, $(1, 2)$ and $(2, 3)$. What happens?

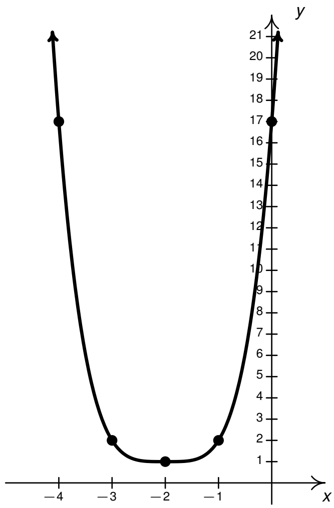
(e) Generalize the formula for $L(x)$ to five points. What's the pattern?

2.1.4 Answers

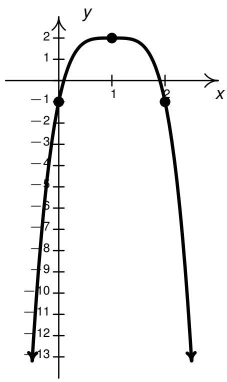
1. $F(x) = (x+2)^3 + 1$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



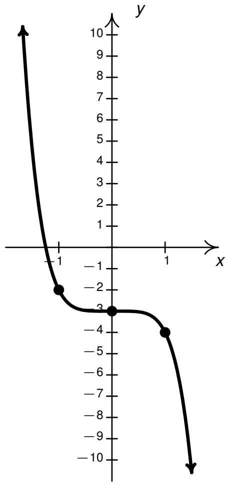
2. $F(x) = (x+2)^4 + 1$
   domain: $(-\infty, \infty)$
   range: $[1, \infty)$



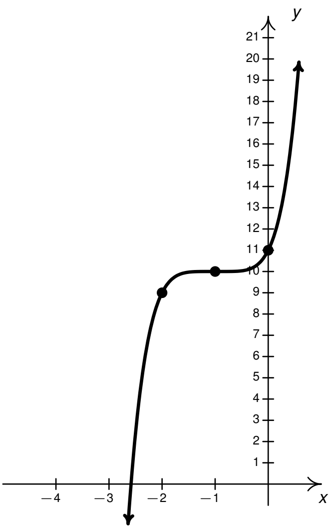
3. $F(x) = 2 - 3(x-1)^4$
   domain: $(-\infty, \infty)$
   range: $(-\infty, 2]$



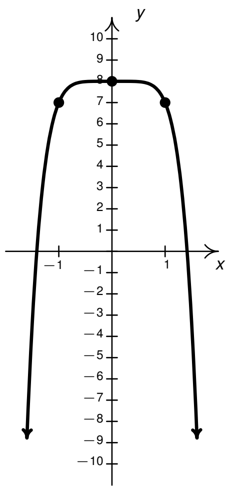
4. $F(x) = -x^5 - 3$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



5. $F(x) = (x + 1)^5 + 10$
   domain: $(-\infty, \infty)$
   range: $(-\infty, \infty)$



6. $F(x) = 8 - x^6$
   domain: $(-\infty, \infty)$
   range: $(-\infty, 8]$



7. $F(x) = (x-1)^3 - 2$

8. $F(x) = -\frac{1}{2}(x+2)^3 + 3$

9. $F(x) = 2(x+1)^4 - 4$

10. $F(x) = -0.15625x^4 + 2.5$

11. $f(x) = 4 - x - 3x^2$
    Degree 2
    Leading term $-3x^2$
    Leading coefficient -3
    Constant term 4
    As $x \to -\infty,\ f(x) \to -\infty$
    As $x \to \infty,\ f(x) \to -\infty$

12. $g(x) = 3x^5 - 2x^2 + x + 1$
    Degree 5
    Leading term $3x^5$
    Leading coefficient 3
    Constant term 1
    As $x \to -\infty,\ g(x) \to -\infty$
    As $x \to \infty,\ g(x) \to \infty$

13. $q(r) = 1 - 16r^4$
    Degree 4
    Leading term $-16r^4$
    Leading coefficient -16
    Constant term 1
    As $r \to -\infty,\ q(r) \to -\infty$
    As $r \to \infty,\ q(r) \to -\infty$

14. $Z(b) = 42b - b^3$
    Degree 3
    Leading term $-b^3$
    Leading coefficient -1
    Constant term 0
    As $b \to -\infty,\ Z(b) \to \infty$
    As $b \to \infty,\ Z(b) \to -\infty$

15. $f(x) = \sqrt{3}x^{17} + 22.5x^{10} - \pi x^7 + \frac{1}{3}$
    Degree 17
    Leading term $\sqrt{3}x^{17}$
    Leading coefficient $\sqrt{3}$
    Constant term $\frac{1}{3}$
    As $x \to -\infty,\ f(x) \to -\infty$
    As $x \to \infty,\ f(x) \to \infty$

16. $s(t) = -4.9t^2 + v_0t + s_0$
    Degree 2
    Leading term $-4.9t^2$
    Leading coefficient -4.9
    Constant term s_0
    As $t \to -\infty,\ s(t) \to -\infty$
    As $t \to \infty,\ s(t) \to -\infty$

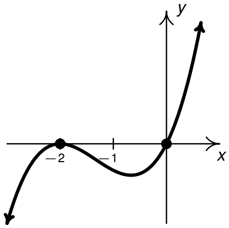
17. $P(x) = (x-1)(x-2)(x-3)(x-4)$
    Degree 4
    Leading term x^4
    Leading coefficient 1
    Constant term 24
    As $x \to -\infty,\ P(x) \to \infty$
    As $x \to \infty,\ P(x) \to \infty$

18. $p(t) = -t^2(3-5t)(t^2+t+4)$
    Degree 5
    Leading term $5t^5$
    Leading coefficient 5
    Constant term 0
    As $t \to -\infty,\ p(t) \to -\infty$
    As $t \to \infty,\ p(t) \to \infty$

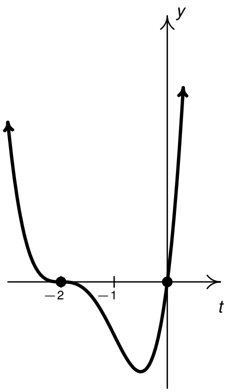
19. $f(x) = -2x^3(x+1)(x+2)^2$
Degree 6
Leading term $-2x^6$
Leading coefficient -2
Constant term 0
As $x \to -\infty$, $f(x) \to -\infty$
As $x \to \infty$, $f(x) \to -\infty$

20. $G(t) = 4(t-2)^2\left(t+\frac{1}{2}\right)$
Degree 3
Leading term $4t^3$
Leading coefficient 4
Constant term 8
As $t \to -\infty$, $G(t) \to -\infty$
As $t \to \infty$, $G(t) \to \infty$

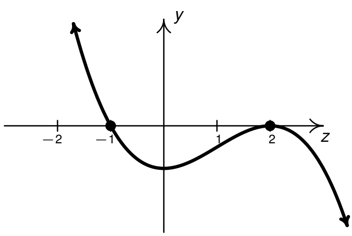
21. $a(x) = x(x+2)^2$
$x = 0$ multiplicity 1
$x = -2$ multiplicity 2



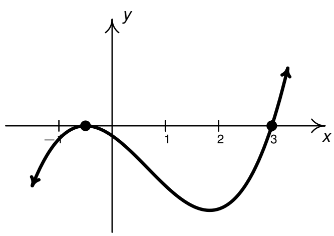
22. $g(t) = t(t+2)^3$
$t = 0$ multiplicity 1
$t = -2$ multiplicity 3



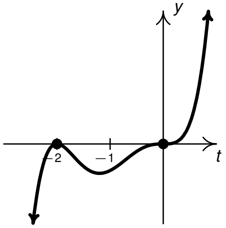
23. $f(z) = -2(z-2)^2(z+1)$
$z = 2$ multiplicity 2
$z = -1$ multiplicity 1



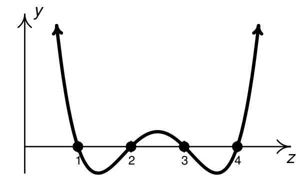
24. $g(x) = (2x+1)^2(x-3)$
$x = -\frac{1}{2}$ multiplicity 2
$x = 3$ multiplicity 1



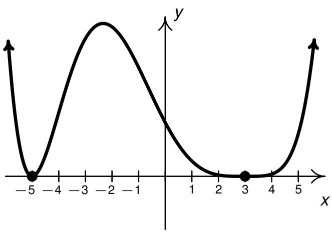
25. $F(t) = t^3(t+2)^2$
$t = 0$ multiplicity 3
$t = -2$ multiplicity 2



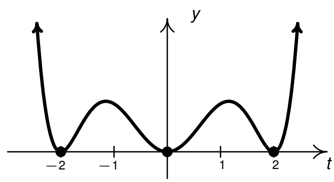
26. $P(z) = (z-1)(z-2)(z-3)(z-4)$
$z = 1$ multiplicity 1
$z = 2$ multiplicity 1
$z = 3$ multiplicity 1
$z = 4$ multiplicity 1



27. $Q(x) = (x+5)^2(x-3)^4$
$x = -5$ multiplicity 2
$x = 3$ multiplicity 4

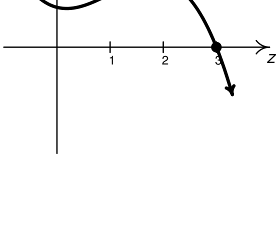


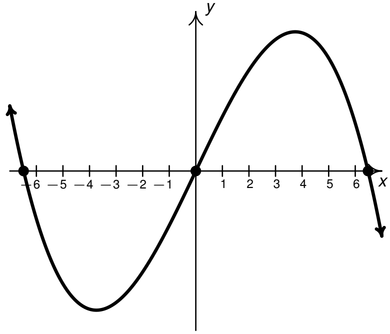
28. $f(t) = t^2(t-2)^2(t+2)^2$
$t = -2$ multiplicity 2
$t = 0$ multiplicity 2
$t = 2$ multiplicity 2



29. $H(z) = (3-z)\left(z^2+1\right)$
$z = 3$ multiplicity 1

30. $Z(x) = x(42-x^2)$
$x = -\sqrt{42}$ multiplicity 1
$x = 0$ multiplicity 1
$x = \sqrt{42}$ multiplicity 1





31. odd
32. neither
33. even
34. even
35. even
36. neither
37. odd
38. odd
39. even
40. odd
41. neither
42. even
43. odd
44. even
45. even **and** odd

47. (a) even (b) odd (c) neither (d) odd[30]

48. For $f(x) = |x|$, $f(-x) = |-x| = |(-1)x| = |-1||x| = (1)|x| = |x|$. Hence, $f(-x) = f(x)$.

49. $V(x) = x(8.5 - 2x)(11 - 2x) = 4x^3 - 39x^2 + 93.5x$, $0 < x < 4.25$. Volume is maximized when $x \approx 1.58$, so we get the dimensions of the box with maximum volume are: height ≈ 1.58 inches, width ≈ 5.34 inches, and depth ≈ 7.84 inches. The maximum volume is ≈ 66.15 cubic inches.

50. Each of these average rates of change indicate slope of the curve over the given interval. Smaller slopes correspond to 'flatter' curves and higher slopes correspond to 'steeper' curves.

[30]You need to first multiply out the expression for $g(x)$ so it is in the form prescribed by Definition 2.1.4.

$f(x)$	$[-0.1, 0]$	$[0, 0.1]$	$[0.9, 1]$	$[1, 1.1]$	$[1.9, 2]$	$[2, 2.1]$
1	0	0	0	0	0	0
x	1	1	1	1	1	1
x^2	-0.1	0.1	1.9	2.1	3.9	4.1
x^3	0.01	0.01	2.71	3.31	11.41	12.61
x^4	-0.001	0.001	3.439	4.641	29.679	34.481
x^5	0.0001	0.0001	4.0951	6.1051	72.3901	88.4101

51. As we sample points closer to $x = 1$, the slope of the curve approaches the exponent on x.

$f(x)$	$[0.9, 1.1]$	$[0.99, 1.01]$	$[0.999, 1.001]$	$[0.9999, 1.0001]$
1	0	0	0	0
x	1	1	1	1
x^2	2	2	2	2
x^3	3.01	3.0001	≈ 3	≈ 3
x^4	4.04	4.0004	≈ 4	≈ 4
x^5	5.1001	≈ 5.001	≈ 5	≈ 5

52. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 6.305$ and $y \approx 1115.417$. Since x represents the number of TVs sold in hundreds, $x = 6.305$ corresponds to 630.5 TVs. Since we can't sell half of a TV, we compare $R(6.30) \approx 1115.415$ and $R(6.31) \approx 1115.416$, so selling 631 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$1,115,416.

53. $P(x) = R(x) - C(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$.

54. The calculator gives the location of the absolute maximum (rounded to three decimal places) as $x \approx 3.897$ and $y \approx 35.255$. Since x represents the number of TVs sold in hundreds, $x = 3.897$ corresponds to 389.7 TVs. Since we can't sell 0.7 of a TV, we compare $P(3.89) \approx 35.254$ and $P(3.90) \approx 35.255$, so selling 390 TVs results in a (slightly) higher revenue. Since y represents the revenue in *thousands* of dollars, the maximum revenue is \$35,255.

55. Making and selling 71 PortaBoys yields a maximized profit of \$5910.67.

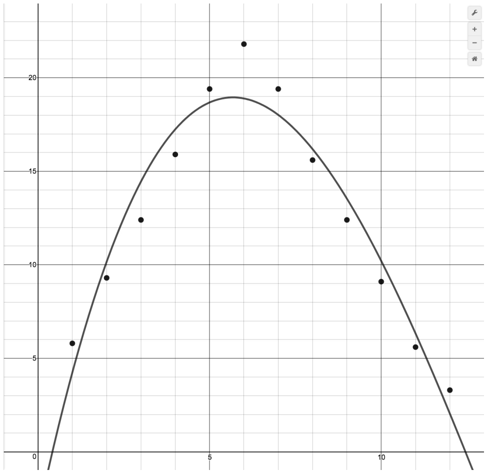
56. (a) To maximize the volume, we assume we start with the maximum Length + Girth of 130, so the length is $130 - 4x$. The volume of a rectangular box is 'length $\times$ width $\times$ height' so we get $V(x) = x^2(130 - 4x) = -4x^3 + 130x^2$.

    (b) Using a graphing utility, we get a (local) maximum of $y = V(x)$ at $(21.67, 20342.59)$. Hence, the maximum volume is 20342.59in.^{3} using a box with dimensions 21.67in. $\times$ 21.67in. $\times$ 43.32in..

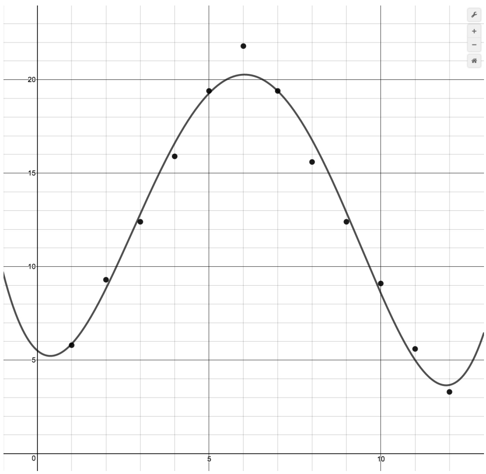
    (c) If we start with Length + Girth = 108 then the length is $108 - 4x$ so $V(x) = -4x^3 + 108x^2$. Graphing $y = V(x)$ shows a (local) maximum at $(18.00, 11664.00)$ so the dimensions of the box with maximum volume are 18.00in. $\times$ 18.00in. $\times$ 36in. for a volume of 11664.00in.^{3}. (Calculus will confirm that the measurements which maximize the volume are exactly 18in. by 18in. by 36in., however, as I'm sure you are aware by now, we treat all numerical results as approximations and list them as such.)

57.
- The cubic regression model is $p_3(x) = 0.0226x^3 - 0.9508x^2 + 8.615x - 3.446$. It has $R^2 = 0.9377$ which isn't bad. The graph of $y = p_3(x)$ along with the data is shown below on the left. Note p_3 hits the x-axis at about $x = 12.45$ making this a bad model for future predictions.
- To use the model to approximate the number of hours of sunlight on your birthday, you'll have to figure out what decimal value of x is close enough to your birthday and then plug it into the model. Jeff's birthday is July 31 which is 10 days after July 21 ($x = 7$). Assuming 30 days in a month, I think $x = 7.33$ should work for my birthday and $p_3(7.33) \approx 17.5$. The website says there will be about 18.25 hours of daylight that day.
- To have 14 hours of darkness we need 10 hours of daylight. We see that $p_3(1.96) \approx 10$ and $p_3(10.05) \approx 10$ so it seems reasonable to say that we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to February 21, 2009 ($x = 2$) and then again from October 21,2009 ($x = 10$) to December 21, 2009 ($x = 12$).
- The quartic regression model is $p_4(x) = 0.0144x^4 - 0.3507x^3 + 2.259x^2 - 1.571x + 5.513$. It has $R^2 = 0.9859$ which is good. The graph of $y = p_4(x)$ along with data is shown below on the right. Note $p_4(15)$ is above 24 making this a bad model as well for future predictions.

- Here, $p_4(7.33) \approx 18.71$ so this model more accurately predicts the number of hours of daylight on Jeff's birthday.
- This model says we'll have at least 14 hours of darkness from December 21, 2008 ($x = 0$) to about March 1, 2009 ($x = 2.30$) and then again from October 10, 2009 ($x = 9.667$) to December 21, 2009 ($x = 12$).



$y = p_3(x)$

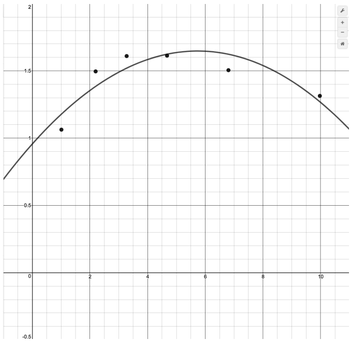


$y = p_4(x)$

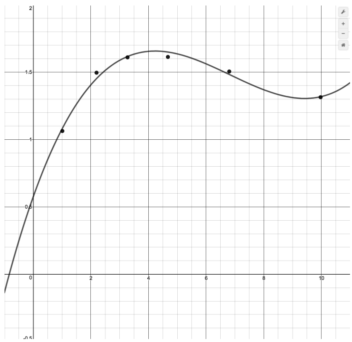
58. (a) The scatter plot is shown below with each of the three regression models.

(b) The quadratic model is $P_2(x) = -0.021x^2 + 0.241x + 0.956$, $R^2 = 0.7771$.
The cubic model is $P_3(x) = 0.005x^3 - 0.103x^2 + 0.602x + 0.573$, $R^2 = 0.9815$.
The quartic model is $P_4(x) = -0.000969x^4 + 0.0253x^3 - 0.240x^2 + 0.944x + 0.330$, $R^2 = 0.9993$.

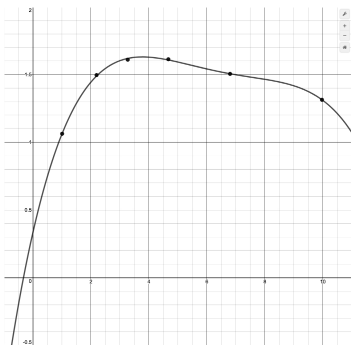
(c) The models give maximums: $P_2(5.737) \approx 1.648$, $P_3(4.232) \approx 1.657$ and $P_4(3.784) \approx 1.630$.



$y = P_2(x)$



$y = P_3(x)$



$y = P_4(x)$

59. (a) as $x \to -\infty$, $p(x) \to -\infty$ and as $x \to \infty$, $p(x) \to -\infty$

(b) The zeros appear to be: $x = -1.5$, even multiplicity - probably 2 since it doesn't 'look like' the graph is very flat near $x = 2$; $x = 0$, odd multiplicity - probably 1 since the graph seems fairly linear as it passes through the origin; $x = 1$ odd multiplicity - probably 3 or higher since the graph seems fairly 'flat' near $x = 1$.

(c) local minimum: approximately $(-0.773, -2.888)$; local maximums: approximately $(-1.5, 0)$, and $(0.32, 0.532)$

(d) Based on the graph, even degree (at least 6 based on multiplicities) with a negative leading coefficient based on the end behavior.

(e) We only have a *portion* of the graph represented here.

61. We are looking for the largest open interval containing $x = -0.235$ for which the graph of $y = p(x)$ is at or above $y = -1.121$. Since each of the gridlines on the x-axis correspond to 0.2 units, we approximate this interval as $(-1.25\,\text{ish}, 1.1\,\text{ish})$.

66. (c) $L(x) = x^2$ (d) $L(x) = x + 1$

2.2 The Remainder and Factor Theorems

In Section 2.1 we saw how much of the 'local' behavior of the graph of a polynomial function is determined by the zeros of the polynomial function. In that section, the polynomial functions we were given were mostly, if not completely, factored which greatly simplified the process for determining zeros. In this section, we revisit the relationship between zeros and factors with the ultimate aim of taking a polynomial function given to us in the form stated in Definition 2.1.4 and determining its zeros.

We start by way of example: suppose we wish to determine the zeros of $f(x) = x^3 + 4x^2 - 5x - 14$. Setting $f(x) = 0$ results in the polynomial equation $x^3 + 4x^2 - 5x - 14 = 0$. Despite all of the factoring techniques we learned (and forgot!) in Intermediate Algebra, this equation foils[1] us at every turn. Knowing that the zeros of f correspond to x-intercepts on the graph of $y = f(x)$, we use a graphing utility to produce the graph in Figure 2.2.1. The graph suggests that the function has three zeros, one of which appears to be $x = 2$ and two others for whom we are provided what we assume to be decimal approximations: $x \approx -4.414$ and $x \approx -1.586$. We can verify if these are zeros easily enough. We find $f(2) = (2)^2 + 4(2)^2 - 5(2) - 14 = 0$, but $f(-4.414) \approx 0.0039$ and $f(-1.586) \approx 0.0022$, While these last two values are probably by some measures, 'close' to 0, they are not *exactly* equal to 0. The question becomes: is there a way to use the fact that $x = 2$ is a zero to obtain the other two zeros? Based on our experience, if $x = 2$ is a zero, it seems that there should be a factor of $(x - 2)$ lurking around in the factorization of $f(x)$. In other words, we should expect that $x^3 + 4x^2 - 5x - 14 = (x - 2)\,q(x)$, where $q(x)$ is some other polynomial. How could we find such a $q(x)$, if it even exists? The answer comes from our old friend, polynomial division. (See Section **??**.) Below, we perform the long division: $(x^3 + 4x^2 - 5x - 14) \div (x - 2)$ and obtain $x^2 + 6x + 7$.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & +\,6x & +\,7 & \\
\cline{2-5}
x-2 & x^3 & +\,4x^2 & -\,5x & -\,14 \\
 & -\underline{\left(x^3 - 2x^2\right)} & & & \\
 & & 6x^2 & -\,5x & \\
 & & -\underline{\left(6x^2 - 12x\right)} & & \\
 & & & 7x & -\,14 \\
 & & & -\underline{(7x} & \underline{-\,14)} \\
 & & & & 0
\end{array}
$$

[1] pun intended

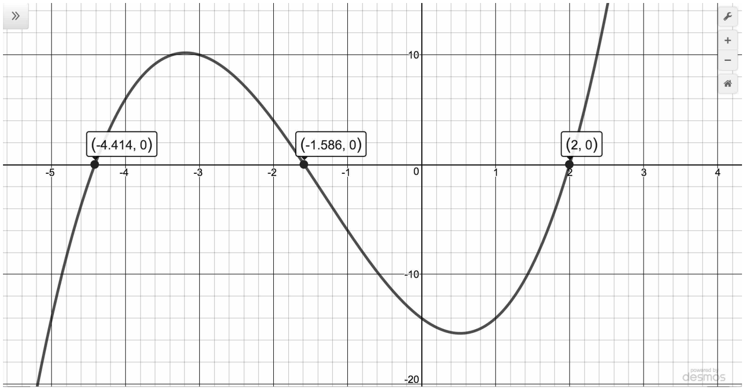


Figure 2.2.1

Said differently, $f(x) = x^3 + 4x^2 - 5x - 14 = (x-2)\left(x^2+6x+7\right)$. Using this form of $f(x)$, we find the zeros by solving $(x-2)\left(x^2+6x+7\right) = 0$. Setting each factor equal to 0, we get $x - 2 = 0$ (which gives us our known zero, $x = 2$) as well as $x^2+6x+7 = 0$. The latter doesn't factor nicely, so we apply the Quadratic Formula to get $x = -3 \pm \sqrt{2}$. Sure enough, $-3-\sqrt{2} \approx -4.414$ and $-3+\sqrt{2} \approx -1.586$. We leave it to the reader to show $f(-3-\sqrt{2}) = 0$ and $f(-3+\sqrt{2}) = 0$. (See Exercise 36.)

The point of this section is to generalize the technique applied here. First up is a friendly reminder of what we can expect when we divide polynomials.

Theorem 2.2.1. Polynomial Division:
Suppose $d(x)$ and $p(x)$ are nonzero polynomial functions where the degree of p is greater than or equal to the degree of d. There exist two unique polynomial functions, $q(x)$ and $r(x)$, such that $p(x) = d(x)\,q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

As you may recall, all of the polynomials in Theorem 2.2.1 have special names. The polynomial p is called the **dividend**; d is the **divisor**; q is the **quotient**; r is the **remainder**. If $r(x) = 0$ then d is called a **factor** of p. The word 'unique' here is critical in that it guarantees there is only *one* quotient and remainder for each division problem.[2] The proof of Theorem 2.2.1 is usually relegated to a course in Abstract Algebra, but we can still use the result to establish two important facts which are the basis of the rest of the chapter.

Theorem 2.2.2. The Remainder Theorem: Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$. Said differently, there is a polynomial function $q(x)$ such that:

$$p(x) = (x - c)q(x) + p(c)$$

The proof of Theorem 2.2.2 is a direct consequence of Theorem 2.2.1. Since $x - c$ has degree 1, when a polynomial function is divided by $x - c$, the remainder is either 0 or degree 0 (i.e., a nonzero constant.) In either case, $p(x) = (x - c)\,q(x) + r$, where r, the remainder, is a real number, possibly 0. It follows that $p(c) = (c - c)\,q(c) + r = 0 \cdot q(c) + r = r$, so we get $r = p(c)$ as required. There is one last 'low hanging fruit'[3] to collect which we present below.

Theorem 2.2.3. The Factor Theorem:
Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

Once again, we see the phrase 'if and only if' which means there are really two things being said in The Factor Theorem: if $(x - c)$ is a factor of $p(x)$, then c is a zero of p and the *only* way c is a zero of p is if $(x - c)$ is a factor of $p(x)$. We argue the Factor Theorem as follows: if $(x - c)$ is a factor of $p(x)$, then $p(x) = (x - c)\,q(x)$ for some polynomial q. Hence, $p(c) = (c - c)\,q(c) = 0$, so c is a zero of p. Conversely, suppose c is a zero of p, so $p(c) = 0$. The Remainder Theorem tells us $p(x) = (x - c)q(x) + p(c) = (x - c)q(x) + 0 = (x - c)q(x)$. Hence, $(x - c)$ is a factor of $p(x)$.

We have enough theory to explain why the concept of multiplicity (Definition 2.1.6) is well-defined. If c is a zero of p, then The Factor Theorem tells us there is a polynomial function q_1 so that $p(x) = (x - c)q_1(x)$. If $q_1(c) = 0$, then we apply the Factor Theorem to q_1 and find a polynomial q_2 so that $q_1(x) = (x - c)q_2(x)$.

[2] Hence the use of the definite article 'the' when speaking of *the* quotient and *the* remainder.
[3] Jeff hates this expression and Carl included it just to annoy him.

Hence, we have

$$p(x) = (x - c)q_1(x) = (x - c)(x - c)q_2(x) = (x - c)^2 q_2(x).$$

We now 'rinse and repeat' this process. Since the degree of p is a finite number, this process has to end at some point. That is we arrive at a factorization $p(x) = (x - c)^m q(x)$ where $q(c) \neq 0$. Suppose we arrive at a different factorization of p using other methods. That is, we find $p(x) = (x - c)^k Q(x)$, where Q is a polynomial function with $Q(c) \neq 0$. Then we have $(x - c)^m q(x) = (x - c)^k Q(x)$. If $m \neq k$, then either $m < k$ or $m > k$. Assuming the former, then we may divide both sides by $(x-c)^m$ to get: $q(x) = (x-c)^{k-m}Q(x)$. Since $k > m$, $k-m > 0$ and we would have $q(c) = (c-c)^{k-m}Q(c) = 0$, a contradiction since we are assuming $q(c) \neq 0$. The assumption that $m > k$ likewise ends in a contradiction. Therefore, we have $m = k$, so $p(x) = (x - c)^m q(x) = (x - c)^m Q(x)$. By the uniqueness guaranteed in Theorem 2.2.1, we must have that $q(x) = Q(x)$. Hence, we have shown the number m, as well as the quotient polynomial $q(x)$ are unique. The process outlined above, in which we coax out factors of $p(x)$ one at a time until we have all of them serves as a template for our work to come.

Of the things The Factor Theorem tells us, the most pragmatic is that we had better find a more efficient way to divide polynomial functions by quantities of the form $x - c$. Fortunately, people like Ruffini[4] and Horner[5] have already blazed this trail. Let's take a closer look at the long division we performed at the beginning of the section and try to streamline it. First off, let's change all of the subtractions into additions by distributing through the -1s.

$$\begin{array}{rrrrr}
 & & x^2 & +\ 6x & +\ 7 \\ \cline{2-5}
x-2 & \multicolumn{1}{|r}{x^3} & +\ 4x^2 & -\ 5x & -14 \\
 & -x^3 & +\ 2x^2 & & \\ \cline{2-3}
 & & 6x^2 & -\ 5x & \\
 & & -6x^2 & +\ 12x & \\ \cline{3-4}
 & & & 7x & -14 \\
 & & & -7x & +14 \\ \cline{4-5}
 & & & & 0
\end{array}$$

Next, observe that the terms $-x^3$, $-6x^2$ and $-7x$ are the exact opposite of the terms above them. The algorithm we use ensures this is always the case, so we

[4] http://en.wikipedia.org/wiki/Synthetic_division
[5] http://en.wikipedia.org/wiki/Horner_scheme

can omit them without losing any information. Also note that the terms we 'bring down' (namely the $-5x$ and -14) aren't really necessary to recopy, so we omit them, too.

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +\,6x & +\,7 \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
\multicolumn{1}{r}{} & & 2x^2 & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & 6x^2 & & \\
\multicolumn{1}{r}{} & & & 12x & \\ \cline{3-4}
\multicolumn{1}{r}{} & & & 7x & \\
\multicolumn{1}{r}{} & & & & 14 \\ \cline{4-5}
\multicolumn{1}{r}{} & & & & 0
\end{array}$$

Let's move terms up a bit and copy the x^3 into the last row.

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & +\,6x & +\,7 \\ \cline{2-5}
x-2 & x^3 & +4x^2 & -\,5x & -14 \\
\multicolumn{1}{r}{} & & 2x^2 & 12x & 14 \\ \cline{2-5}
\multicolumn{1}{r}{} & x^3 & 6x^2 & 7x & 0
\end{array}$$

Note that by arranging things in this manner, each term in the last row is obtained by adding the two terms above it. Notice also that the quotient polynomial can be obtained by dividing each of the first three terms in the last row by x and adding the results. If you take the time to work back through the original division problem, you will find that this is exactly the way we determined the quotient polynomial. This means that we no longer need to write the quotient polynomial down, nor the x in the divisor, to determine our answer.

$$\begin{array}{r|rrrr}
-2 & x^3 & +4x^2 & -\,5x & -14 \\
\multicolumn{1}{r}{} & & 2x^2 & 12x & 14 \\ \cline{2-5}
\multicolumn{1}{r}{} & x^3 & 6x^2 & 7x & 0
\end{array}$$

We've streamlined things quite a bit so far, but we can still do more. Let's take a moment to remind ourselves where the $2x^2$, $12x$ and 14 came from in the second row. Each of these terms was obtained by multiplying the terms in the quotient, x^2, $6x$ and 7, respectively, by the -2 in $x-2$, then by -1 when we changed the subtraction to addition. Multiplying by -2 then by -1 is the same as multiplying by 2, so we replace the -2 in the divisor by 2. Furthermore, the coefficients of the quotient polynomial match the coefficients of the first three terms in the last row, so we now take the plunge and write only the coefficients of the terms to get

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & 0 \end{array}$$

We have constructed a **synthetic division tableau** for this polynomial division problem. Let's re-work our division problem using this tableau to see how it greatly streamlines the division process. To divide $x^3 + 4x^2 - 5x - 14$ by $x - 2$, we write 2 in the place of the divisor and the coefficients of $x^3 + 4x^2 - 5x - 14$ in for the dividend. Then 'bring down' the first coefficient of the dividend.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & & & & \\ \hline & & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & & & \\ \hline & 1 & & & \end{array}$$

Next, take the 2 from the divisor and multiply by the 1 that was 'brought down' to get 2. Write this underneath the 4, then add to get 6.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

Now take the 2 from the divisor times the 6 to get 12, and add it to the -5 to get 7.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & \\ \hline & 1 & 6 & 7 & \end{array}$$

Finally, take the 2 in the divisor times the 7 to get 14, and add it to the -14 to get 0.

$$\begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & 4 & -5 & -14 \\ & \downarrow & 2 & 12 & 14 \\ \hline & 1 & 6 & 7 & \boxed{0} \end{array}$$

The first three numbers in the last row of our tableau are the coefficients of the quotient polynomial. Remember, we started with a third degree polynomial and divided by a first degree polynomial, so the quotient is a second degree polynomial. Hence the quotient is $x^2 + 6x + 7$. The number in the box is the remainder. Synthetic division is our tool of choice for dividing polynomials by divisors of the form $x - c$. It is important to note that it works *only* for these kinds of divisors.[6] Also take note that when a polynomial (of degree at least 1) is divided by $x - c$, the result will be a polynomial of exactly one less degree. Finally, it is worth the time to trace each step in synthetic division back to its corresponding step in long division. While the authors have done their best to indicate where the algorithm comes from, there is no substitute for working through it yourself.

Example 2.2.1. Use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the dividend, quotient and remainder in the form given in Theorem 2.2.1.

1. $\left(5x^3 - 2x^2 + 1\right) \div (x - 3)$
2. $\left(t^3 + 8\right) \div (t + 2)$
3. $\dfrac{4 - 8z - 12z^2}{2z - 3}$

Solution.

1. When setting up the synthetic division tableau, the coefficients of even 'missing' terms need to be accounted for, so we enter 0 for the coefficient of x in the dividend.

$$\begin{array}{r|rrrr} 3 & 5 & -2 & 0 & 1 \\ & \downarrow & 15 & 39 & 117 \\ \hline & 5 & 13 & 39 & \boxed{118} \end{array}$$

Since the dividend was a third degree polynomial function, the quotient is a second degree (quadratic) polynomial function with coefficients 5, 13 and 39: $q(x) = 5x^2 + 13x + 39$. The remainder is $r(x) = 118$. According to Theorem 2.2.1, we have $5x^3 - 2x^2 + 1 = (x - 3)\left(5x^2 + 13x + 39\right) + 118$, which we leave to the reader to check.

[6] You'll need to use good old-fashioned polynomial long division for divisors of degree larger than 1.

2. To use synthetic division here, we rewrite $t+2$ as $t-(-2)$ and proceed as before

$$\begin{array}{r|rrrr} -2 & 1 & 0 & 0 & 8 \\ & \downarrow & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

We get the quotient $q(t) = t^2 - 2t + 4$ and the remainder $r(t) = 0$. Relating the dividend, quotient and remainder gives: $t^3 + 8 = (t+2)\left(t^2 - 2t + 4\right)$, which is a specific instance of the 'sum of cubes' formula some of you may recall from Intermediate Algebra.

3. To divide $4 - 8z - 12z^2$ by $2z - 3$, two things must be done. First, we write the dividend in descending powers of z as $-12z^2 - 8z + 4$. Second, since synthetic division works only for factors of the form $z - c$, we factor $2z - 3$ as $2\left(z - \frac{3}{2}\right)$. Hence, we are dividing $-12z^2 - 8z + 4$ by two factors: 2 and $\left(z - \frac{3}{2}\right)$. Dividing first by 2, we obtain $-6z^2 - 4z + 2$. Next, we divide $-6z^2 - 4z + 2$ by $\left(z - \frac{3}{2}\right)$:

$$\begin{array}{r|rrr} \frac{3}{2} & -6 & -4 & 2 \\ & \downarrow & -9 & -\frac{39}{2} \\ \hline & -6 & -13 & \boxed{-\frac{35}{2}} \end{array}$$

Hence, $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$. However when it comes to writing the dividend, quotient and remainder in the form given in Theorem 2.2.1, we need to find $q(z)$ and $r(z)$ so that $-12z^2 - 8z + 4 = (2z-3)q(z) + r(z)$. Hence, starting with $-6z^2 - 4z + 2 = \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}$, we multiply 2 back on both sides:

$$\begin{array}{rcl} -6z^2 - 4z + 2 & = & \left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2} \\ 2\left(-6z^2 - 4z + 2\right) & = & 2\left[\left(z - \frac{3}{2}\right)(-6z - 13) - \frac{35}{2}\right] \\ -12z^2 - 8z + 4 & = & 2\left(z - \frac{3}{2}\right)(-6z - 13) - 2\left(\frac{35}{2}\right) \\ -12z^2 - 8z + 4 & = & (2z - 3)(-6z - 13) - 35 \end{array}$$

At this stage, we have written $-12z^2 - 8z + 4$ in the **form** $(2z - 3)q(z) + r(z)$, so we identify the quotient as $q(z) = -6z - 13$ and the remainder is $r(z) = -35$. But how can we be sure these are the same quotient and remainder polynomial functions we would have obtained if we had taken the time to do the long division in the first place? Because of the word 'unique' in Theorem 2.2.1. The theorem states that there is only *one* way to decompose $-12z^2 - 8z + 4$ as $(2z - 3)q(z) + r(z)$. Since we have found such a way, we can be sure it is the only way.[7] □

The next example pulls together all of the concepts discussed in this section.

Example 2.2.2. Let $p(x) = 2x^3 - 5x + 3$.

1. Find $p(-2)$ using The Remainder Theorem. Check your answer by substitution.
2. Verify $x = 1$ is a zero of p and use this information to all the real zeros of p.

Solution.

1. The Remainder Theorem states $p(-2)$ is the remainder when $p(x)$ is divided by $x - (-2)$. We set up our synthetic division tableau below. We are careful to record the coefficient of x^2 as 0:

$$\begin{array}{r|rrrr} -2 & 2 & 0 & -5 & 3 \\ & \downarrow & -4 & 8 & -6 \\ \hline & 2 & -4 & 3 & \boxed{-3} \end{array}$$

According to the Remainder Theorem, $p(-2) = -3$. We can check this by direct substitution into the formula for $p(x)$: $p(-2) = 2(-2)^3 - 5(-2) + 3 = -16 + 10 + 3 = -3$.

2. We verify $x = 1$ is a zero of p by evaluating $p(1) = 2(1)^3 - 5(1) + 3 = 0$. To see if there are any more real zeros, we need to solve $p(x) = 2x^3 - 5x + 3 = 0$. From the Factor Theorem, we know since $p(1) = 0$, we can factor $p(x)$ as $(x - 1)q(x)$. To find $q(x)$, we use synthetic division:

$$\begin{array}{r|rrrr} 1 & 2 & 0 & -5 & 3 \\ & \downarrow & 2 & 2 & -3 \\ \hline & 2 & 2 & -3 & \boxed{0} \end{array}$$

As promised, our remainder is 0, and we get $p(x) = (x - 1)\left(2x^2 + 2x - 3\right)$. Setting this form of $p(x)$ equal to 0 we get $(x - 1)\left(2x^2 + 2x - 3\right) = 0$. We

[7] But it wouldn't hurt to check, just this once.

recover $x = 1$ from setting $x - 1 = 0$ but we also obtain $x = \frac{-1 \pm \sqrt{7}}{2}$ from $2x^2 + 2x - 3 = 0$, courtesy of the Quadratic Formula. $\Box$

Our next example demonstrates how we can extend the synthetic division tableau to accommodate zeros of multiplicity greater than 1.

Example 2.2.3. Let $p(x) = 4x^4 - 4x^3 - 11x^2 + 12x - 3$. Show $x = \frac{1}{2}$ is a zero of multiplicity 2 and find all of the remaining real zeros of p.

Solution. While computing $p\left(\frac{1}{2}\right) = 0$ shows $x = \frac{1}{2}$ is a zero of p, to prove it has multiplicity 2, we need to factor $p(x) = \left(x - \frac{1}{2}\right)^2 q(x)$ with $q\left(\frac{1}{2}\right) \neq 0$,. We set up for synthetic division, but instead of stopping after the first division, we continue the tableau downwards and divide $\left(x - \frac{1}{2}\right)$ directly into the quotient we obtained from the first division as follows:

$$
\begin{array}{c|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
 & 4 & 0 & -12 & \boxed{0} &
\end{array}
$$

We get:[8] $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)^2 \left(4x^2 - 12\right)$. Note if we let $q(x) = 4x^2 - 12$, then $q\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^2 - 12 = -11 \neq 0$ which proves $x = \frac{1}{2}$ is a zero of p of multiplicity 2. To find the remaining zeros of p, we set the quotient $4x^2 - 12 = 0$, so $x^2 = 3$ and extract square roots to get $x = \pm\sqrt{3}$. $\Box$

A couple of things about the last example are worth mentioning. First, the extension of the synthetic division tableau for repeated divisions will be a common site in the sections to come. Typically, we will start with a higher order polynomial and peel off one zero at a time until we are left with a quadratic, whose roots can always be found using the Quadratic Formula. Secondly, we found $x = \pm\sqrt{3}$ are zeros of p. The Factor Theorem guarantees $\left(x - \sqrt{3}\right)$ and $\left(x - \left(-\sqrt{3}\right)\right)$ are both factors of p. We can certainly put the Factor Theorem to the test and continue the synthetic division tableau from above to see what happens.

[8] For those wanting more detail: the first division gives: $4x^4 - 4x^3 - 11x^2 + 12x - 3 = \left(x - \frac{1}{2}\right)\left(4x^3 - 2x^2 - 12x + 6\right)$. The second division gives: $4x^3 - 2x^2 - 12x + 6 = \left(x - \frac{1}{2}\right)\left(4x^2 - 12\right)$.

$$
\begin{array}{r|rrrrr}
\frac{1}{2} & 4 & -4 & -11 & 12 & -3 \\
 & \downarrow & 2 & -1 & -6 & 3 \\
\hline
\frac{1}{2} & 4 & -2 & -12 & 6 & \boxed{0} \\
 & \downarrow & 2 & 0 & -6 & \\
\hline
\sqrt{3} & 4 & 0 & -12 & \boxed{0} & \\
 & \downarrow & 4\sqrt{3} & 12 & & \\
\hline
-\sqrt{3} & 4 & 4\sqrt{3} & \boxed{0} & & \\
 & \downarrow & -4\sqrt{3} & & & \\
\hline
 & 4 & \boxed{0} & & &
\end{array}
$$

This gives us

$$
\begin{array}{rcl}
p(x) & = & 4x^4 - 4x^3 - 11x^2 + 12x - 3 \\
 & = & \left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right) \left(x - \left(-\sqrt{3}\right)\right) (4) \\
 & = & 4 \left(x - \frac{1}{2}\right)^2 \left(x - \sqrt{3}\right) \left(x - \left(-\sqrt{3}\right)\right)
\end{array}
$$

We have shown that p is a product of its leading coefficient times linear factors of the form $(x - c)$ where c are zeros of p. It may surprise and delight the reader that, in theory, all polynomials can be reduced to this kind of factorization. We leave that discussion to Section 2.4, because the zeros may not be real numbers. Our final theorem in the section gives us an upper bound on the number of real zeros.

> **Theorem 2.2.4.** Suppose f is a polynomial of degree $n \geq 1$. Then f has at most n real zeros, counting multiplicities.

Theorem 2.2.4 is a consequence of the Factor Theorem and polynomial multiplication. Every zero c of f gives us a factor of the form $(x - c)$ for $f(x)$. Since f has degree n, there can be at most n of these factors. The next section provides us some tools which not only help us determine where the real zeros are to be found, but which real numbers they may be.

We close this section with Box 2.2.1, containing a summary of several concepts previously presented. You should take the time to look back through the text to see where each concept was first introduced and where each connection to the other concepts was made.

Box 2.2.1: Connections Between Zeros, Factors and Graphs of Polynomial Functions

Suppose p is a polynomial function of degree $n \geq 1$. The following statements are equivalent:

- The real number c is a zero of p
- $p(c) = 0$
- $x = c$ is a solution to the polynomial equation $p(x) = 0$
- $(x - c)$ is a factor of $p(x)$
- The point $(c, 0)$ is an x-intercept of the graph of $y = p(x)$

2.2.1 Exercises

In Exercises 1 - 14, use synthetic division to perform the following polynomial divisions. Identify the quotient and remainder. Write the divisor, quotient and remainder in the form given in Theorem 2.2.1.

1. $\left(3x^2 - 2x + 1\right) \div (x - 1)$
2. $\left(x^2 - 5\right) \div (x - 5)$
3. $\left(3 - 4t - 2t^2\right) \div (t + 1)$
4. $\left(4t^2 - 5t + 3\right) \div (t + 3)$
5. $\left(z^3 + 8\right) \div (z + 2)$
6. $\left(4z^3 + 2z - 3\right) \div (z - 3)$
7. $\left(18x^2 - 15x - 25\right) \div \left(x - \frac{5}{3}\right)$
8. $\left(4x^2 - 1\right) \div \left(x - \frac{1}{2}\right)$
9. $\left(2t^3 + t^2 + 2t + 1\right) \div \left(t + \frac{1}{2}\right)$
10. $\left(3t^3 - t + 4\right) \div \left(t - \frac{2}{3}\right)$
11. $\left(2z^3 - 3z + 1\right) \div \left(z - \frac{1}{2}\right)$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) \div \left(z - \frac{3}{2}\right)$
13. $\left(x^4 - 6x^2 + 9\right) \div \left(x - \sqrt{3}\right)$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) \div \left(x + \sqrt{2}\right)$

In Exercises 15 - 24, find $p(c)$ using the Remainder Theorem. If $p(c) = 0$, use the Factor Theorem to partially factor the polynomial function.

15. $p(x) = 2x^2 - x + 1$, $c = 4$
16. $p(x) = 4x^2 - 33x - 180$, $c = 12$
17. $p(t) = 2t^3 - t + 6$, $c = -3$
18. $p(t) = t^3 + 2t^2 + 3t + 4$, $c = -1$
19. $p(z) = 3z^3 - 6z^2 + 4z - 8$, $c = 2$
20. $p(z) = 8z^3 + 12z^2 + 6z + 1$, $c = -\frac{1}{2}$
21. $p(x) = x^4 - 2x^2 + 4$, $c = \frac{3}{2}$
22. $p(x) = 6x^4 - x^2 + 2$, $c = -\frac{2}{3}$
23. $p(t) = t^4 + t^3 - 6t^2 - 7t - 7$, $c = -\sqrt{7}$
24. $p(t) = t^2 - 4t + 1$, $c = 2 - \sqrt{3}$

In Exercises 25 - 34, you are given a polynomial function and one of its zeros. Find the remaining real zeros and factor the polynomial.

25. $x^3 - 6x^2 + 11x - 6$, $c = 1$
26. $x^3 - 24x^2 + 192x - 512$, $c = 8$
27. $3t^3 + 4t^2 - t - 2$, $c = \frac{2}{3}$
28. $2t^3 - 3t^2 - 11t + 6$, $c = \frac{1}{2}$
29. $z^3 + 2z^2 - 3z - 6$, $c = -2$
30. $2z^3 - z^2 - 10z + 5$, $c = \frac{1}{2}$
31. $4x^4 - 28x^3 + 61x^2 - 42x + 9$, $c = \frac{1}{2}$ is a zero of multiplicity 2
32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12$, $c = -1$ is a zero of multiplicity 3
33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324$, $c = -\frac{3}{5}$ is a zero of multiplicity 3
34. $x^2 - 2x - 2$, $c = 1 - \sqrt{3}$

35. Find a quadratic polynomial with <u>integer</u> coefficients which has $x = \frac{3}{5} \pm \frac{\sqrt{29}}{5}$ as its real zeros.

36. For $f(x) = x^3 + 4x^2 - 5x - 14$, show $f(-3 - \sqrt{2}) = 0$ and $f(-3 + \sqrt{2}) = 0$ two ways:

    (a) By direct substitution.

    (b) Using synthetic division and the Factor Theorem

37. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ be a polynomial function with the property that $a_n + a_{n-1} + ... + a_1 + a_0 = 0$. (That is, the sum of the coefficients and the constant term is 0.)

    Prove that $(x - 1)$ is a factor of $f(x)$.

    HINT: Show $f(1) = 0$ and invoke the Factor Theorem

38. Verify the result in number 37 with the functions: $f(x) = x^3 - 2x + 1$ and $f(x) = 3x^4 - x - 2$.

39. Suppose a is a nonzero real number. Find the quotients below, using synthetic division as required.

    - $\frac{x - a}{x - a}$
    - $\frac{x^2 - a^2}{x - a}$
    - $\frac{x^3 - a^3}{x - a}$
    - $\frac{x^4 - a^4}{x - a}$
    - $\frac{x^5 - a^5}{x - a}$

    Based on the pattern that evolves, find the quotient: $\frac{x^{10} - a^{10}}{x - a}$. What about $\frac{x^n - a^n}{x - a}$?

40. Use your result from number 39 to rewrite the sum: $1+r+r^2+\cdots+r^{n-2}+r^{n-1}$ as a quotient. What assumptions need to be made about r?

2.2.2 Answers

1. $\left(3x^2 - 2x + 1\right) = (x - 1)(3x + 1) + 2$
2. $\left(x^2 - 5\right) = (x - 5)(x + 5) + 20$
3. $\left(3 - 4t - 2t^2\right) = (t + 1)(-2t - 2) + 5$
4. $\left(4t^2 - 5t + 3\right) = (t + 3)(4t - 17) + 54$
5. $\left(z^3 + 8\right) = (z + 2)\left(z^2 - 2z + 4\right) + 0$
6. $\left(4z^3 + 2z - 3\right) = (z - 3)\left(4z^2 + 12z + 38\right) + 111$
7. $\left(18x^2 - 15x - 25\right) = \left(x - \frac{5}{3}\right)(18x + 15) + 0$
8. $\left(4x^2 - 1\right) = \left(x - \frac{1}{2}\right)(4x + 2) + 0$
9. $\left(2t^3 + t^2 + 2t + 1\right) = \left(t + \frac{1}{2}\right)\left(2t^2 + 2\right) + 0$
10. $\left(3t^3 - t + 4\right) = \left(t - \frac{2}{3}\right)\left(3t^2 + 2t + \frac{1}{3}\right) + \frac{38}{9}$
11. $\left(2z^3 - 3z + 1\right) = \left(z - \frac{1}{2}\right)\left(2z^2 + z - \frac{5}{2}\right) - \frac{1}{4}$
12. $\left(4z^4 - 12z^3 + 13z^2 - 12z + 9\right) = \left(z - \frac{3}{2}\right)\left(4z^3 - 6z^2 + 4z - 6\right) + 0$
13. $\left(x^4 - 6x^2 + 9\right) = \left(x - \sqrt{3}\right)\left(x^3 + \sqrt{3}\,x^2 - 3x - 3\sqrt{3}\right) + 0$
14. $\left(x^6 - 6x^4 + 12x^2 - 8\right) = \left(x + \sqrt{2}\right)\left(x^5 - \sqrt{2}\,x^4 - 4x^3 + 4\sqrt{2}\,x^2 + 4x - 4\sqrt{2}\right) + 0$

15. $p(4) = 29$
16. $p(12) = 0$, $p(x) = (x - 12)(4x + 15)$
17. $p(-3) = -45$
18. $p(-1) = 2$
19. $p(2) = 0$, $p(z) = (z - 2)\left(3z^2 + 4\right)$
20. $p\left(-\frac{1}{2}\right) = 0$, $p(z) = \left(z + \frac{1}{2}\right)\left(8z^2 + 8z + 2\right)$
21. $p\left(\frac{3}{2}\right) = \frac{73}{16}$
22. $p\left(-\frac{2}{3}\right) = \frac{74}{27}$
23. $p(-\sqrt{7}) = 0$, $p(t) = (t + \sqrt{7})\left(t^3 + (1 - \sqrt{7})t^2 + (1 - \sqrt{7})t - \sqrt{7}\right)$
24. $p(2 - \sqrt{3}) = 0$, $p(t) = (t - (2 - \sqrt{3}))(t - (2 + \sqrt{3}))$
25. $x^3 - 6x^2 + 11x - 6 = (x - 1)(x - 2)(x - 3)$
26. $x^3 - 24x^2 + 192x - 512 = (x - 8)^3$
27. $3t^3 + 4t^2 - t - 2 = 3\left(t - \frac{2}{3}\right)(t + 1)^2$
28. $2t^3 - 3t^2 - 11t + 6 = 2\left(t - \frac{1}{2}\right)(t + 2)(t - 3)$
29. $z^3 + 2z^2 - 3z - 6 = (z + 2)(z + \sqrt{3})(z - \sqrt{3})$

30. $2z^3 - z^2 - 10z + 5 = 2\left(z - \frac{1}{2}\right)(z + \sqrt{5})(z - \sqrt{5})$

31. $4x^4 - 28x^3 + 61x^2 - 42x + 9 = 4\left(x - \frac{1}{2}\right)^2 (x - 3)^2$

32. $t^5 + 2t^4 - 12t^3 - 38t^2 - 37t - 12 = (t + 1)^3(t + 3)(t - 4)$

33. $125z^5 - 275z^4 - 2265z^3 - 3213z^2 - 1728z - 324 = 125\left(z + \frac{3}{5}\right)^3 (z+2)(z-6)$

34. $x^2 - 2x - 2 = (x - (1 - \sqrt{3}))(x - (1 + \sqrt{3}))$

35. $p(x) = 5x^2 - 6x - 4$

38.
- For $f(x) = x^3 - 2x + 1$, the coefficients $1 + (-2) + 1 = 0$ and $f(x) = (x - 1)(x^2 + x - 1)$.
- For $f(x) = 3x^4 - x - 2$ the coefficients $3 + (-1) + (-2) = 0$ and $f(x) = (x - 1)(3x^3 + 3x^2 + 3x + 2)$.

39.
- $\dfrac{x - a}{x - a} = 1$
- $\dfrac{x^2 - a^2}{x - a} = x + a$
- $\dfrac{x^3 - a^3}{x - a} = x^2 + ax + a^2$
- $\dfrac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3$
- $\dfrac{x^5 - a^5}{x - a} = x^4 + ax^3 + a^2x^2 + a^3x + a^4$

Following the pattern:

- $\dfrac{x^{10} - a^{10}}{x - a} = x^9 + ax^8 + a^2x^7 + a^3x^6 + a^4x^5 + a^5x^4 + a^6x^3 + a^7x^2 + a^8x + a^9$
- $\dfrac{x^n - a^n}{x - a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + ... + a^{n-2}x + a^{n-1}$

40. Put $x = 1$ and $a = r$ so that $1 + r + r^2 + \cdots + r^{n-2} + r^{n-1} = \dfrac{1 - r^n}{1 - r}$. Here, $r \neq 1$ as otherwise we'd be dividing by 0.

2.3 Real Zeros of Polynomials

In Section 2.2, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division. There are two approaches to the topic of finding the real zeros of a polynomial. The first approach is to use a little bit of Mathematics followed by a good use of technology like graphing utilities. The second approach makes good use of mathematical machinery (theorems) only. For completeness, we include the two approaches but in separate subsections. Both approaches benefit from the following two theorems, the first of which is due to the famous mathematician Augustin Cauchy[1]. It gives us an interval on which *all* of the real zeros of a polynomial can be found.

> **Theorem 2.3.1. Cauchy's Bound:** Suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all the real zeros of f lie in in the interval $[-(M+1), M+1]$.

There's a lot going on in the statement of Cauchy's Bound, so we'll get right to an example and show how it is used. For those wanting a proof of Cauchy's Bound, see Exercise **??** in Section **??**.

Example 2.3.1. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Determine an interval which contains all of the real zeros of f.

Solution. To find the M stated in Cauchy's Bound, we take the absolute value of the leading coefficient, in this case $|2| = 2$ and divide it into the largest (in absolute value) of the remaining coefficients, in this case $|-6| = 6$. This yields $M = 3$ so it is guaranteed that all of the real zeros of f lie in the interval $[-4, 4]$. □

Whereas the previous result tells us *where* we can find the real zeros of a polynomial, the next theorem gives us a list of *possible* real zeros.

> **Theorem 2.3.2. Rational Zeros Theorem:** Suppose $f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$ is a polynomial of degree n with $n \geq 1$, and $a_0, a_1, \ldots a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$, where p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

[1] http://en.wikipedia.org/wiki/Cauchy

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose c is a zero of f and $c = \frac{p}{q}$ in lowest terms. This means p and q have no common factors. Since $f(c) = 0$, we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \ldots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

Multiplying both sides of this equation by q^n, we clear the denominators to get

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0$$

Rearranging this equation, we get

$$a_n p^n = -a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1} - a_0 q^n$$

Now, the left hand side is an integer multiple of p, and the right hand side is an integer multiple of q. (Can you see why?) This means $a_n p^n$ is both a multiple of p and a multiple of q. Since p and q have no common factors, a_n must be a multiple of q. If we rearrange the equation

$$a_n p^n + a_{n-1} p^{n-1} q + \ldots + a_1 p q^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1} p^{n-1} q - \ldots - a_1 p q^{n-1}$$

we can play the same game and conclude a_0 is a multiple of p, and we have the result.

Example 2.3.2. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use the Rational Zeros Theorem to list all of the possible rational zeros of f.

Solution. To generate a complete list of rational zeros, we need to take each of the factors of constant term, $a_0 = -3$, and divide them by each of the factors of the leading coefficient $a_4 = 2$. The factors of -3 are ± 1 and ± 3. Since the Rational Zeros Theorem tacks on a $\pm$ anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list $\left\{\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}\right\}$ or $\left\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\right\}$. $\square$

Our discussion now diverges between those who wish to use technology and those who do not.

2.3.1 For Those Wishing to use a Graphing Utility

At this stage, we know not only the interval in which all of the zeros of $f(x) = 2x^4+4x^3-x^2-6x-3$ are located, but we also know some potential candidates. We can now use our calculator to help us determine all of the real zeros of f, as illustrated in the next example.

Example 2.3.3. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Graph $y = f(x)$ using a graphing utility over the interval obtained in Example 2.3.1.
2. Use the graph to shorten the list of possible rational zeros obtained in Example 2.3.2.
3. Use synthetic division to find the real zeros of f, and state their multiplicities.

Solution.

1. In Example 2.3.1, we determined all of the real zeros of f lie in the interval $[-4, 4]$, so we restrict our attention to that portion of the x-axis.
2. In Example 2.3.2, we learned that any rational zero of f must be in the list $\left\{\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3\right\}$. From the graph (Figure 2.3.1), it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the x-axis at $x \approx 1.225$. On the negative side, $x = -1$ looks good. Indeed, the shape of the graph near $(-1, 0)$ suggests that if $x = -1$ is a zero, it is of multiplicity at least three. We set about synthetically dividing:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline & 2 & 2 & -3 & -3 & \boxed{0} \end{array}$$

   Since f is a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have reduced the quotient to a quadratic, and we can use the quadratic formula, if needed, to find the two remaining zeros. Continuing with $x = -1$:

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

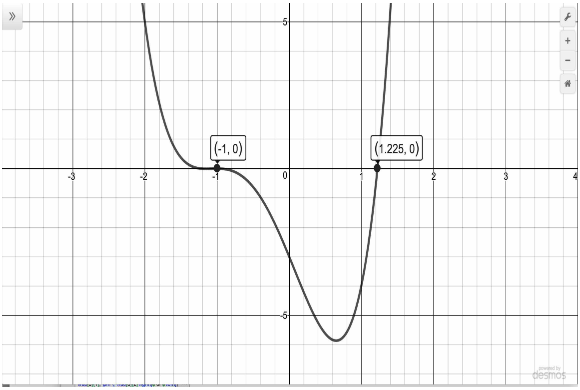


Figure 2.3.1

Our quotient polynomial is now $2x^2-3$. Setting this to zero gives $2x^2-3=0$, or $x^2=\frac{3}{2}$, which gives us $x=\pm\frac{\sqrt{6}}{2}$. Based on our division work, we know that -1 has a multiplicity of *at least* 2. The Factor Theorem tells us our remaining zeros, $\pm\frac{\sqrt{6}}{2}$, each have multiplicity at least 1. However, Theorem 2.2.4 tells us f can have at most 4 real zeros, counting multiplicity, and so we conclude that -1 is of multiplicity *exactly* 2 and $\pm\frac{\sqrt{6}}{2}\approx\pm1.225$ each has multiplicity 1. Thus, we were incorrect in thinking -1 was a zero of multiplicity 3. Sure enough, if we adjust zoom in near $(-1,0)$ using graphing utility (Figure 2.3.2), we find the graph of $y=f(x)$ touches and rebounds from the x-axis at $(-1,0)$, typical behavior near a zero of even multiplicity. The lesson here is, once again, technology may *suggest* a result, but it is only the mathematics which can *prove* (or in this case, *disprove*) it.

□

Our next example shows how even a mild-mannered polynomial can cause problems.

Example 2.3.4. Let $f(x)=x^4+x^2-12$.

1. Use Cauchy's Bound to determine an interval in which all of the real zeros of f lie.

2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of f.

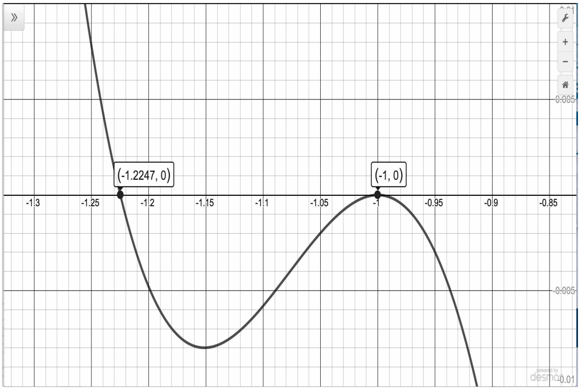


Figure 2.3.2

3. Graph $y = f(x)$ using a graphing utility.
4. Find all of the real zeros of f and their multiplicities.

Solution.

1. Applying Cauchy's Bound, we find $M = 12$, so all of the real zeros lie in the interval $[-13, 13]$.
2. Applying the Rational Zeros Theorem with constant term $a_0 = -12$ and leading coefficient $a_4 = 1$, we get the list $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$.
3. Graphing $y = f(x)$ on the interval $[-13, 13]$ produces the graph below in Figure 2.3.3. Zooming in a bit gives the graph in Figure 2.3.4. Based on the graph, none of our rational zeros will work. (Do you see why not?)
4. From the graph, we know f has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of f by setting $f(x) = x^4 + x^2 - 12 = 0$ and solving. If we stare at this equation long enough, we may recognize it as a 'quadratic in disguise' or 'quadratic in form'. (See Section **??**.) In other words, we have three terms: x^4, x^2 and 12, and the exponent on the first term, x^4, is exactly twice that of the second term, x^2. We may rewrite this as $\left(x^2\right)^2 + \left(x^2\right) - 12 = 0$. To better see the forest for the trees, we momentarily replace x^2 with the variable u. In terms of u, our equation becomes $u^2 + u - 12 = 0$, which we can readily factor as $(u + 4)(u - 3) = 0$. In terms of x, this means $x^4 + x^2 - 12 = \left(x^2 - 3\right)\left(x^2 + 4\right) = 0$. We get $x^2 = 3$, which gives us

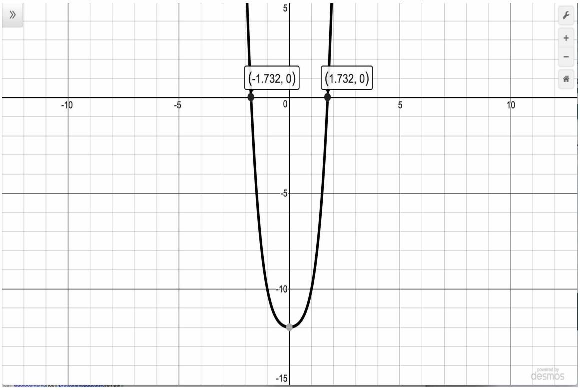


Figure 2.3.3

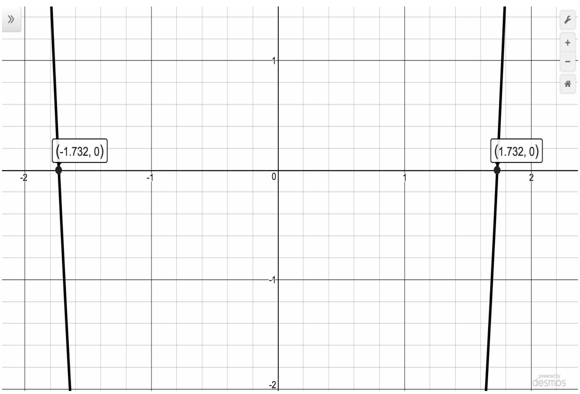


Figure 2.3.4

$x = \pm\sqrt{3}$, or $x^2 = -4$, which admits no real solutions. Since $\sqrt{3} \approx 1.73$, the two zeros match what we expected from the graph. Turning our attention now to multiplicities, the Factor Theorem guarantees that since $x = \pm\sqrt{3}$ are zeros, $(x - \sqrt{3})$ and $(x + \sqrt{3})$ are factors of $f(x)$. We've already partially factored $f(x)$ as $f(x) = \left(x^2 - 3\right)\left(x^2 + 4\right)$. Since $x^2 + 4$ as no real zeros, we know both $(x - \sqrt{3})$ and $(x + \sqrt{3})$ must divide $x^2 - 3$. By Theorem 2.2.4, $x^2 - 3$ can only have a total of two zeros, including multiplicities, so we are forced to conclude $x = \pm\sqrt{3}$ are each zeros of multiplicity 1 of $x^2 - 3$, and hence, $f(x)$.[2] □

A couple of remarks are in order. First, the graph of $f(x) = x^4 + x^2 - 12$ appears to be symmetric about the y-axis. Sure enough, we find $f(-x) = (-x)^4 + (-x)^2 - 12 = x^4 + x^2 = 12 = f(x)$ proving f is, indeed, an even function, thus *proving* the symmetry *suggested* by the graph. Second, the technique used to factor $f(x)$ in Example 2.3.4 is called ***u*-substitution**. We shall this technique now and then in the sections to come, so it is worth taking the time to let this idea sink in. In general, substitution can help us identify a 'quadratic in disguise' - in essence, it helps us 'see the forest for the trees.' Last, but not least, it is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate using technology.

2.3.2 For Those Wishing NOT to use a Graphing Calculator

Suppose we wish to find the zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ *without* using the calculator. In this subsection, we present some more advanced mathematical tools (theorems) to help us. Our first result is due to René Descartes[3].

[2] Alternatively, we could recognize $x^2 - 3 = x^2 - (\sqrt{3})^2 = (x - \sqrt{3})(x + \sqrt{3})$, but the above argument works for all quadratic functions, even those which aren't as easy to factor.

[3] http://en.wikipedia.org/wiki/Descartes

Theorem 2.3.3. Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P, P-2, P-4, \ldots\}$.
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N, N-2, N-4, \ldots\}$.

A few remarks are in order. First, to use Descartes' Rule of Signs, we need to understand what is meant by a '**variation in sign**' of a polynomial function. Consider $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. If we focus on only the *signs* of the coefficients, we start with a (+), followed by another (+), then switch to ($-$), and stay ($-$) for the remaining two coefficients. Since the signs of the coefficients switched *once* as we read from left to right, we say that $f(x)$ has *one* variation in sign. When we speak of the variations in sign of a polynomial function f we assume the formula for $f(x)$ is written with descending powers of x, as in Definition 2.1.4, and concern ourselves only with the nonzero coefficients. Second, unlike the Rational Zeros Theorem, Descartes' Rule of Signs gives us an estimate to the *number* of positive and negative real zeros, not the actual *value* of the zeros. Lastly, Descartes' Rule of Signs counts multiplicities. This means that, for example, if one of the zeros has multiplicity 2, Descscartes' Rule of Signs would count this as *two* zeros. Lastly, note that the number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if $f(x)$ has 7 sign changes, then, counting multplicities, f has either 7, 5, 3 or 1 positive real zero. This implies that the graph of $y = f(x)$ crosses the positive x-axis at least once. If $f(-x)$ results in 4 sign changes, then, counting multiplicities, f has 4, 2 or 0 negative real zeros; hence, the graph of $y = f(x)$ may not cross the negative x-axis at all. The proof of Descartes' Rule of Signs is a bit technical, and can be found here[4].

Example 2.3.5. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$. Use Descartes' Rule of Signs to determine the possible number and location of the real zeros of f.

Solution. As noted above, the variations of sign of $f(x)$ is 1. This means, counting multiplicities, f has exactly 1 positive real zero. Since $f(-x) = 2(-x)^4 + 4(-x)^3 - (-x)^2 - 6(-x) - 3 = 2x^4 - 4x^3 - x^2 + 6x - 3$ has 3 variations in sign, f has either 3 negative real zeros or 1 negative real zero, counting multiplicities. □

[4] http://www.cut-the-knot.org/fta/ROS2.shtml

Cauchy's Bound gives us a general bound on the zeros of a polynomial function. Our next result helps us determine bounds on the real zeros of a polynomial as we synthetically divide which are often sharper[5] bounds than Cauchy's Bound.

Theorem 2.3.4. Upper and Lower Bounds: Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.
  NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as (+) or $(-)$ as needed.

The Upper and Lower Bounds Theorem works because of Theorem 2.2.1. For the upper bound part of the theorem, suppose $c > 0$ is divided into f and the resulting line in the division tableau contains, for example, all nonnegative numbers. This means $f(x) = (x-c)q(x)+r$, where the coefficients of the quotient polynomial and the remainder are nonnegative. (Note that the leading coefficient of q is the same as f so $q(x)$ is not the zero polynomial.) If $b > c$, then $f(b) = (b-c)q(b)+r$, where $(b - c)$ and $q(b)$ are both positive and $r \geq 0$. Hence $f(b) > 0$ which shows b cannot be a zero of f. Thus no real number $b > c$ can be a zero of f, as required. A similar argument proves $f(b) < 0$ if all of the numbers in the final line of the synthetic division tableau are non-positive. To prove the lower bound part of the theorem, we note that a lower bound for the negative real zeros of $f(x)$ is an upper bound for the positive real zeros of $f(-x)$, since all we are doing is reflecting the numbers across the $x = 0$. Applying the upper bound portion to $f(-x)$ gives the result. (Do you see where the alternating signs come in?) With the additional mathematical machinery of Descartes' Rule of Signs and the Upper and Lower Bounds Theorem, we can find the real zeros of $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ without the use of a graphing utility.

Example 2.3.6. Let $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$.

1. Find all of the real zeros of f and their multiplicities.

[5] That is, better, or more accurate.

2. Sketch the graph of $y = f(x)$.

Solution.

1. We know from Cauchy's Bound that all of the real zeros lie in the interval $[-4, 4]$ and that our possible rational zeros are $\pm\frac{1}{2}$, ± 1, $\pm\frac{3}{2}$ and ± 3. Descartes' Rule of Signs guarantees us at least one negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational zeros, starting with the smallest, $\frac{1}{2}$. Since the remainder isn't zero, we know $\frac{1}{2}$ isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so $\frac{1}{2}$ is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us $f\left(\frac{1}{2}\right) = -\frac{45}{8}$ so the point $\left(\frac{1}{2}, -\frac{45}{8}\right)$ is on the graph of f. We continue to our next possible zero, 1. As before, the only information we can glean from this is that $(1, -4)$ is on the graph of f. When we try our next possible zero, $\frac{3}{2}$, we get that it is not a zero, and we also see that it is an upper bound on the zeros of f, since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational.[6]

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array} \qquad \begin{array}{r|rrrrr} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array}$$

$$\begin{array}{r|rrrrr} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

We now turn our attention to negative real zeros. We try the largest possible zero, $-\frac{1}{2}$. Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to -1. This division shows -1 is a zero. Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try -1 again, and it works once more. At this point, we have taken f, a fourth degree polynomial, and performed two

[6]Since polynomials are continuous, we know the zero lies between 1 and $\frac{3}{2}$, since $f(1) < 0$ and $f\left(\frac{3}{2}\right) > 0$.

successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\ \hline & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}} \end{array}$$

$$\begin{array}{r|rrrrr} -1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & -2 & -2 & 3 & 3 \\ \hline -1 & 2 & 2 & -3 & -3 & \boxed{0} \\ & \downarrow & -2 & 0 & 3 & \\ \hline & 2 & 0 & -3 & \boxed{0} & \end{array}$$

Setting the quotient polynomial equal to zero yields $2x^2 - 3 = 0$, so that $x^2 = \frac{3}{2}$, or $x = \pm\frac{\sqrt{6}}{2}$. Descartes' Rule of Signs tells us that the positive real zero we found, $\frac{\sqrt{6}}{2}$, has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces -1 to be a zero of multiplicity 2 and $-\frac{\sqrt{6}}{2}$ to have multiplicity 1.

2. We know the end behavior of $y = f(x)$ resembles that of its leading term $y = 2x^4$. This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since $\pm\frac{\sqrt{6}}{2}$ are zeros of multiplicity 1, we have that the graph crosses through the x-axis at the points $\left(-\frac{\sqrt{6}}{2}, 0\right)$ and $\left(\frac{\sqrt{6}}{2}, 0\right)$ in a fairly linear fashion. Since -1 is a zero of multiplicity 2, the graph of $y = f(x)$ touches and rebounds off the x-axis at $(-1, 0)$ in a parabolic manner. Last, but not least, since $f(0) = -3$, we get the y-intercept is $(0, -3)$. Putting all of this together results in the graph in Figure 2.3.5.

□

2.3.3 The Intermediate Value Theorem and Inequalities

As we mentioned in Section 2.1, polynomial functions are continuous. An important property of continuous functions is that they cannot change sign between two values unless there is a zero in between. We used this property of quadratic functions when constructing sign diagrams to help us solve inequalities (see Section 1.4.2.) This property is a version of the celebrated **Intermediate Value Theorem**.

> **Theorem 2.3.5. The Intermediate Value Theorem (Zero Version):** If f is continuous over an interval containing a and b and $f(a)$ and $f(b)$ have different signs, then f has a zero between a and b. That is, for at least one value c between a and b, $f(c) = 0$.

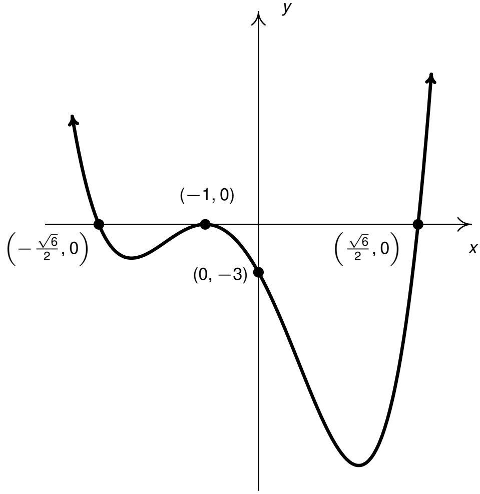


Figure 2.3.5: $y = f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$

The Intermediate Value Theorem is discussed in greater detail in Calculus, and its proof is usually delayed until a formal analysis course. It is an example of an 'existence' theorem - it tells us that, under suitable conditions, a zero exists - but offers us no algorithm to find it.[7] Its use to us in this section is that it provides the justification needed to create sign diagrams for general polynomial functions in the same manner in which we constructed them for quadratic functions. See Box 2.3.1.

Box 2.3.1: Steps for Constructing a Sign Diagram for a Polynomial Function

Suppose f is a polynomial function.

1. Find the zeros of f and place them on the number line with the number 0 above them.
2. Choose a real number, called a **test value**, in each of the intervals determined in step 1.
3. Determine and record the sign of $f(x)$ for each test value in step 2.

[7] See the notes on the 'Bisection Method' at the end of this section.

The Intermediate Value Theorem justifies the use of just one 'test' value in the algorithm above, since a continuous function cannot change signs on an interval without there being a zero on that interval. Since we have found the zeros in Step 1 of the algorithm and used these to create the intervals for Step 2, there cannot be any sign changes on any of the intervals in Step 2.

Not surprisingly, we use sign diagrams to solve inequalities involving higher order polynomial functions in the same way we used them to solve inequalities involving quadratic functions. We reproduce our algorithm from section 1.4.2 for reference in Box 2.3.2.

Box 2.3.2: Solving Inequalities using Sign Diagrams

To solve an inequality using a sign diagram:

1. Rewrite the inequality so a function $f(x)$ is being compared to '0.'
2. Make a sign diagram for f.
3. Record the solution.

Example 2.3.7.

1. Find all of the real solutions to the equation $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$.
2. Solve the inequality $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$.
3. Interpret your answer to part 2 graphically, and verify using a graphing utility.

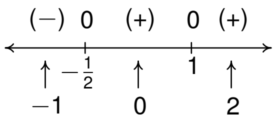
Solution.

1. Finding the real solutions to $2x^5+6x^3+3 = 3x^4+8x^2$ is the same as finding the real solutions to $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$. In other words, we are looking for the real zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$. Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr}
1 & 2 & -3 & 6 & -8 & 0 & 3 \\
 & \downarrow & 2 & -1 & 5 & -3 & -3 \\ \hline
1 & 2 & -1 & 5 & -3 & -3 & \boxed{0} \\
 & \downarrow & 2 & 1 & 6 & 3 & \\ \hline
-\frac{1}{2} & 2 & 1 & 6 & 3 & \boxed{0} & \\
 & \downarrow & -1 & 0 & -3 & & \\ \hline
 & 2 & 0 & 6 & \boxed{0} & &
\end{array}$$

The quotient polynomial is $2x^2+6$ which has no real zeros so we get $x = -\frac{1}{2}$ and $x = 1$.

2. Our first step is to rewrite this inequality so as to compare a function $f(x)$ to 0. We have two options, but choose $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \leq 0$, since we found the zeros of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ to be $x = -\frac{1}{2}$ and $x = 1$. We construct our sign diagram below using the test values -1, 0, and 2.



The solution to $p(x) < 0$ is $\left(-\infty, -\frac{1}{2}\right)$, and we know $p(x) = 0$ at $x = -\frac{1}{2}$ and $x = 1$. Hence, the solution to $p(x) \leq 0$ is $\left(-\infty, -\frac{1}{2}\right] \cup \{1\}$.

3. To interpret this solution graphically, we set $f(x) = 2x^5 + 6x^3 + 3$ and $g(x) = 3x^4 + 8x^2$. Recall from Section 1.3 the solution to $f(x) \leq g(x)$ is the set of x values for which the graph of f is below the graph of g (where $f(x) < g(x)$) along with the x values where the two graphs intersect ($f(x) = g(x)$). Graphing f and g using a graphing utility produces the graph in **??**. (The end behavior should tell you which is which.) We see that the graph of f is below the graph of g on $\left(-\infty, -\frac{1}{2}\right)$. However, it is difficult to see what is happening near $x = 1$. Zooming in (and making the graph of g lighter), we see that the graphs of f and g do intersect at $x = 1$, but the graph of g remains below the graph of f on either side of $x = 1$. See Figure 2.3.7

□

Note that we could have used end behavior and the concept of multiplicity to create the sign diagram used in Example 2.3.7 as follows. We know the end behavior of $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ matches that of $y = 2x^5$ which means as $x \to -\infty$, $p(x) \to -\infty$. This means for the interval $\left(-\infty, -\frac{1}{2}\right)$, $p(x) < 0$ or $(-)$. From our work finding the zeros of p, we can deduce the multiplicity of the zero $x = -\frac{1}{2}$ is 1 which means the graph of $y = p(x)$ crosses through the x-axis at $\left(-\frac{1}{2}, 0\right)$, hence, changing sign from $(-)$ to $(+)$. Finally, we can deduce the multiplicity of the zero $x = 1$ is 2 which means the graph of $y = p(x)$ rebounds here, meaning the sign of $p(x)$ for $x > 1$ is $(+)$. This matches the end behavior, since as $x \to \infty$, $p(x) \to \infty$. The reader is encouraged to tackle any given problem using whatever tools are comfortable and convenient, but it also never hurts to think outside the box and revisit a problem from a variety of perspectives.

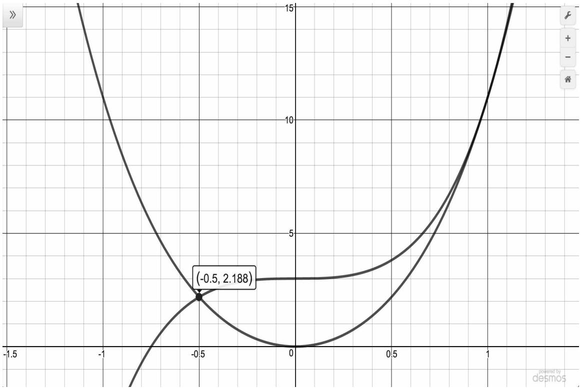


Figure 2.3.6

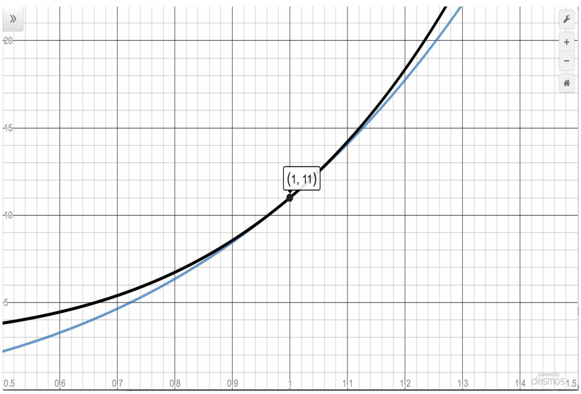


Figure 2.3.7

Next up is an application problem torn from page 212 in the Exercises of Section 2.1.

Example 2.3.8. Suppose the profit P, in *thousands* of dollars, from producing and selling x *hundred* LCD TVs is given by $P(x) = -5x^3 + 35x^2 - 45x - 25$, $0 \leq x \leq 10.07$. How many TVs should be produced to make a profit? Check your answer using a graphing utility.

Solution. To 'make a profit' means to solve $P(x) = -5x^3 + 35x^2 - 45x - 25 > 0$, which we do analytically using a sign diagram. To simplify things, we first factor out the -5 common to all the coefficients to get $-5\left(x^3 - 7x^2 + 9x - 5\right) > 0$, so we can just focus on finding the zeros of $f(x) = x^3 - 7x^2 + 9x + 5$. The possible rational zeros of f are ± 1 and ± 5, and going through the usual computations, we find $x = 5$ is the only rational zero. Using this, we factor $f(x) = x^3 - 7x^2 + 9x + 5 = (x - 5)\left(x^2 - 2x - 1\right)$, and we find the remaining zeros by applying the Quadratic Formula to $x^2 - 2x - 1 = 0$. We find three real zeros, $x = 1 - \sqrt{2} = -0.414\ldots$, $x = 1 + \sqrt{2} = 2.414\ldots$, and $x = 5$, of which only the last two fall in the applied domain of $[0, 10.07]$. We choose $x = 0$, $x = 3$ and $x = 10.07$ as our test values and plug them into the function $P(x) = -5x^3 + 35x^2 - 45x - 25$ (not $f(x) = x^3 - 7x^2 + 9x - 5$) to get the sign diagram below.

$(-)$	0	$(+)$	0	$(-)$
$\uparrow$ 0	$1+\sqrt{2}$	$\uparrow$ 3	5	$\uparrow$ 10.07

We see immediately that $P(x) > 0$ on $(1 + \sqrt{2}, 5)$. Since x measures the number of TVs in *hundreds*, $x = 1 + \sqrt{2}$ corresponds to $241.4\ldots$ TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that $P(2.41) < 0$ but $P(2.42) > 0$ so, in this case we take the next *larger* integer value and set the minimum production to 242 TVs. At the other end of the interval, we have $x = 5$ which corresponds to 500 TVs. Here, we take the next *smaller* integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. We graph $y = P(x)$ in Figure 2.3.8 using a graphing utility and see $P(x) > 0$ between $x \approx 2.414$ and $x = 5$, as predicted.

□

It would be a sin of omission if the authors left the reader with the impression that the theory in this section is complete in that given *any* polynomial function, provided here are the tools to find all of its real zeros exactly. The reality is this couldn't be further from the truth. In general, no matter how many theorems you

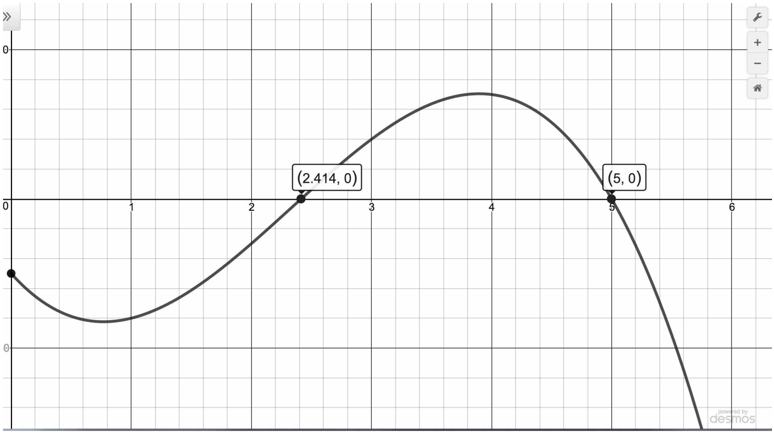


Figure 2.3.8

throw at a polynomial, it may well be impossible to express its zeros exactly. The polynomial $f(x) = x^5 - x - 1$ is one such beast.[8] According to Descartes' Rule of Signs, f has exactly one positive real zero, and it could have two negative real zeros, or none at all. The Rational Zeros Test gives us ± 1 as rational zeros to try but neither of these work since $f(1) = f(-1) = -1$. If we try the substitution technique we used in Example 2.3.4, we find $f(x)$ has three terms, but the exponent on the x^5 isn't exactly twice the exponent on x. How could we go about approximating the positive zero? We use the **Bisection Method**.

The first step in the Bisection Method is to find an interval on which f changes sign. We know $f(1) = -1$ and we find $f(2) = 29$. By the Intermediate Value Theorem, we know that the zero of f lies in the interval $[1, 2]$. Next, we 'bisect' this interval by finding the midpoint, 1.5. We compute $f(1.5) \approx 5.09$. Once again, the Intermediate Value Theorem guarantees our zero is between 1 and 1.5, since f changes sign on this interval. Now, we 'bisect' the interval $[1, 1.5]$ and find $f(1.25) \approx 0.80$, so now we have the zero between 1 and 1.25. Bisecting $[1, 1.25]$, we find $f(1.125) \approx -0.32$, which means the zero of f is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers whose digits agree to a desired amount.[10] You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and

[8] See this page[9].

[10] We ask you to approximate this zero to three decimal places using the Bisection Method in Exercise 64.

checking the sign of f using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for *approximating* a real zero when the other analytical methods fail us.

2.3.4 Exercises

In Exercises 1 - 10, for the given polynomial:

- Use Cauchy's Bound to find an interval containing all of the real zeros.
- Use the Rational Zeros Theorem to make a list of possible rational zeros.
- Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros, counting multiplicities.

1. $f(x) = x^3 - 2x^2 - 5x + 6$
2. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
3. $p(z) = z^4 - 9z^2 - 4z + 12$
4. $p(z) = z^3 + 4z^2 - 11z + 6$
5. $g(t) = t^3 - 7t^2 + t - 7$
6. $g(t) = -2t^3 + 19t^2 - 49t + 20$
7. $f(x) = -17x^3 + 5x^2 + 34x - 10$
8. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
9. $p(z) = 3z^3 + 3z^2 - 11z - 10$
10. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$

In Exercises 11 - 30, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11. $f(x) = x^3 - 2x^2 - 5x + 6$
12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
13. $p(z) = z^4 - 9z^2 - 4z + 12$
14. $p(z) = z^3 + 4z^2 - 11z + 6$
15. $g(t) = t^3 - 7t^2 + t - 7$
16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
21. $g(t) = 9t^3 - 5t^2 - t$
22. $g(t) = 6t^4 - 5t^3 - 9t^2$
23. $f(x) = x^4 + 2x^2 - 15$
24. $f(x) = x^4 - 9x^2 + 14$
25. $p(z) = 3z^4 - 14z^2 - 5$
26. $p(z) = 2z^4 - 7z^2 + 6$
27. $g(t) = t^6 - 3t^3 - 10$
28. $g(t) = 2t^6 - 9t^3 + 10$
29. $f(x) = x^5 - 2x^4 - 4x + 8$
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 - 33, use your calculator,[11] to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$

[11] You *can* do these without your calculator, but it may test your mettle!

34. Find the real zeros of $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$ by first finding a polynomial $q(x)$ with integer coefficients such that $q(x) = N \cdot f(x)$ for some integer N. (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that f and q have the same real zeros.

In Exercises 35 - 44, find the real solutions of the polynomial equation. (See Example 2.3.7.)

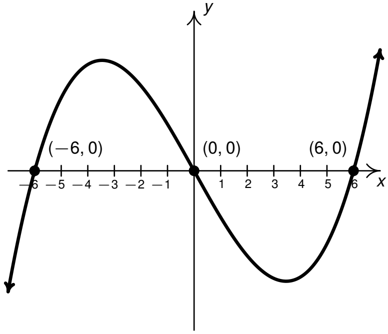
35. $9x^3 = 5x^2 + x$
36. $9x^2 + 5x^3 = 6x^4$
37. $z^3 + 6 = 2z^2 + 5z$
38. $z^4 + 2z^3 = 12z^2 + 40z + 32$
39. $t^3 - 7t^2 = 7 - t$
40. $2t^3 = 19t^2 - 49t + 20$
41. $x^3 + x^2 = \dfrac{11x + 10}{3}$
42. $x^4 + 2x^2 = 15$
43. $14z^2 + 5 = 3z^4$
44. $2z^5 + 3z^4 = 18z + 27$

In Exercises 45 - 54, solve the polynomial inequality and state your answer using interval notation.

45. $-2x^3 + 19x^2 - 49x + 20 > 0$
46. $x^4 - 9x^2 \leq 4x - 12$
47. $(z - 1)^2 \geq 4$
48. $4z^3 \geq 3z + 1$
49. $t^4 \leq 16 + 4t - t^3$
50. $3t^2 + 2t < t^4$
51. $\dfrac{x^3 + 2x^2}{2} < x + 2$
52. $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$
53. $2z^4 > 5z^2 + 3$
54. $z^6 + z^3 \geq 6$

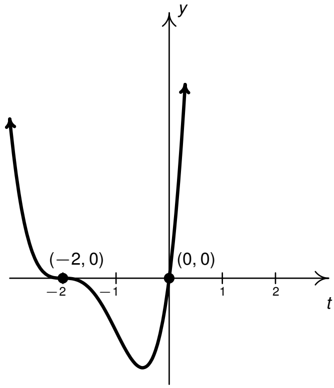
In Exercises 55. - 60., use the the graph of the given polynomial function to solve the stated inequality.

55. Solve $f(x) < 0$.



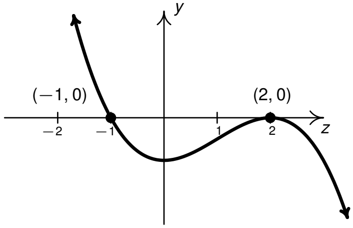
$y = f(x)$

56. Solve $g(t) > 0$.



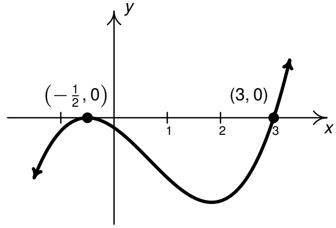
$y = g(t)$

57. Solve $p(z) \geq 0$



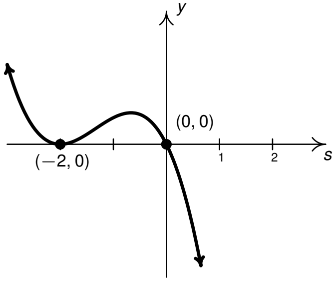
$y = p(z)$

58. Solve $f(x) < 0$.



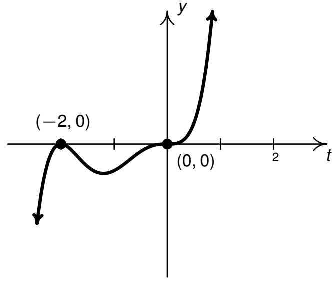
$y = f(x)$

59. Solve $F(s) \leq 0$.



$y = F(s)$

60. Solve $G(t) \geq 0$.



$y = G(t)$

61. Use the Intermediate Value Theorem, Theorem 2.3.5 to prove that $f(x) = x^3 - 9x + 5$ has a real zero in each of the following intervals: $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

62. Use the concepts of End Behavior and the Intermediate Value Theorem

to prove any odd-degree polynomial function with real number coefficients has at least one real zero.

63. Find an even-degree polynomial function with real number coefficients which has no real zeros.

64. Continue the Bisection Method as introduced on 262 to approximate the real zero of $f(x) = x^5 - x - 1$ to three decimal places.

65. In this exercise, we prove $\sqrt{2}$ is an irrational number and approximate its value. Let $f(x) = x^2 - 2$.

    (a) Use Decartes' Rule of Signs to prove f has exactly one positive real zero.

    (b) Use the Intermediate Value Theorem to prove f has a zero in $[1, 2]$.

    (c) Use the Rational Zeros Theorem to prove f has no rational zeros.

    (d) Use the Bisection Method (see 262) to approximate the zero of f on $[1, 2]$ to three decimal places.

66. Generalize the argument given in Exercise 65c to prove:

    (a) If N is not the perfect square of an integer, then $\sqrt{N}$ is irrational. (HINT: Consider $f(x) = x^2 - N$.)

    (b) For natural numbers $n \geq 2$, if N is not the perfect n^{th} power of an integer, then $\sqrt[n]{N}$ is irrational. (HINT: Consider $f(x) = x^n - N$.)

67. In Example 2.1.4 in Section 2.1, a box with no top is constructed from a 10 inch $\times$ 12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by the function $V(x) = 4x^3 - 44x^2 + 120x$, where x denotes the length of the side of the square which is removed from each corner (in inches), $0 < x < 5$. Solve the inequality $V(x) \geq 80$ analytically and interpret your answer in the context of that example.

68. From Exercise 55 in Section 2.1, $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ models the cost, in dollars, to produce x PortaBoy game systems. If the production budget is \$5000, find the number of game systems which can be produced and still remain under budget.

69. Let $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$. With the help of your classmates, find the x- and y- intercepts of the graph of

f. Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of f, using more than one picture if necessary to show all of the important features of the graph.

70. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

2.3.5 Answers

1. For $f(x) = x^3 - 2x^2 - 5x + 6$
   - All of the real zeros lie in the interval $[-7, 7]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
2. For $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
   - All of the real zeros lie in the interval $[-41, 41]$
   - Possible rational zeros are ± 1, ± 2, ± 4, ± 8, ± 16, ± 32
   - There is 1 positive real zero; there are 3 or 1 negative real zeros
3. For $p(z) = z^4 - 9z^2 - 4z + 12$
   - All of the real zeros lie in the interval $[-13, 13]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 4, ± 6, ± 12
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
4. For $p(z) = z^3 + 4z^2 - 11z + 6$
   - All of the real zeros lie in the interval $[-12, 12]$
   - Possible rational zeros are ± 1, ± 2, ± 3, ± 6
   - There are 2 or 0 positive real zeros; there is 1 negative real zero
5. For $g(t) = t^3 - 7t^2 + t - 7$
   - All of the real zeros lie in the interval $[-8, 8]$
   - Possible rational zeros are ± 1, ± 7
   - There are 3 or 1 positive real zeros; there are no negative real zeros
6. For $g(t) = -2t^3 + 19t^2 - 49t + 20$
   - All of the real zeros lie in the interval $\left[-\frac{51}{2}, \frac{51}{2}\right]$
   - Possible rational zeros are $\pm\frac{1}{2}$, ± 1, ± 2, $\pm\frac{5}{2}$, ± 4, ± 5, ± 10, ± 20
   - There are 3 or 1 positive real zeros; there are no negative real zeros
7. For $f(x) = -17x^3 + 5x^2 + 34x - 10$

- All of the real zeros lie in the interval $[-3, 3]$
- Possible rational zeros are $\pm\frac{1}{17}$, $\pm\frac{2}{17}$, $\pm\frac{5}{17}$, $\pm\frac{10}{17}$, ±1, ±2, ±5, ±10
- There are 2 or 0 positive real zeros; there is 1 negative real zero

8. For $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
   - All of the real zeros lie in the interval $\left[-\frac{4}{3}, \frac{4}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{36}$, $\pm\frac{1}{18}$, $\pm\frac{1}{12}$, $\pm\frac{1}{9}$, $\pm\frac{1}{6}$, $\pm\frac{1}{4}$, $\pm\frac{1}{3}$, $\pm\frac{1}{2}$, ±1
   - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros

9. For $p(z) = 3z^3 + 3z^2 - 11z - 10$
   - All of the real zeros lie in the interval $\left[-\frac{14}{3}, \frac{14}{3}\right]$
   - Possible rational zeros are $\pm\frac{1}{3}$, $\pm\frac{2}{3}$, $\pm\frac{5}{3}$, $\pm\frac{10}{3}$, ±1, ±2, ±5, ±10
   - There is 1 positive real zero; there are 2 or 0 negative real zeros

10. For $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
    - All of the real zeros lie in the interval $\left[-\frac{9}{2}, \frac{9}{2}\right]$
    - Possible rational zeros are $\pm\frac{1}{2}$, ±1, $\pm\frac{3}{2}$, ±3
    - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros

11. $f(x) = x^3 - 2x^2 - 5x + 6$
    $x = -2$, $x = 1$, $x = 3$ (each has mult. 1)

12. $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$
    $x = -2$ (mult. 3), $x = 4$ (mult. 1)

13. $p(z) = z^4 - 9z^2 - 4z + 12$
    $z = -2$ (mult. 2), $z = 1$ (mult. 1), $z = 3$ (mult. 1)

14. $p(z) = z^3 + 4z^2 - 11z + 6$
    $z = -6$ (mult. 1), $z = 1$ (mult. 2)

15. $g(t) = t^3 - 7t^2 + t - 7$
    $t = 7$ (mult. 1)

16. $g(t) = -2t^3 + 19t^2 - 49t + 20$
    $t = \frac{1}{2}$, $t = 4$, $t = 5$ (each has mult. 1)

17. $f(x) = -17x^3 + 5x^2 + 34x - 10$
$x = \frac{5}{17}$, $x = \pm\sqrt{2}$ (each has mult. 1)
18. $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
$x = \frac{1}{2}$ (mult. 2), $x = -\frac{1}{3}$ (mult. 2)
19. $p(z) = 3z^3 + 3z^2 - 11z - 10$
$z = -2$, $z = \frac{3\pm\sqrt{69}}{6}$ (each has mult. 1)
20. $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$
$z = -1$, $z = \frac{1}{2}$, $z = \pm\sqrt{3}$ (each mult. 1)
21. $g(t) = 9t^3 - 5t^2 - t$
$t = 0$, $t = \frac{5\pm\sqrt{61}}{18}$ (each has mult. 1)
22. $g(t) = 6t^4 - 5t^3 - 9t^2$
$t = 0$ (mult. 2), $t = \frac{5\pm\sqrt{241}}{12}$ (each has mult. 1)
23. $f(x) = x^4 + 2x^2 - 15$
$x = \pm\sqrt{3}$ (each has mult. 1)
24. $f(x) = x^4 - 9x^2 + 14$
$x = \pm\sqrt{2}$, $x = \pm\sqrt{7}$ (each has mult. 1)
25. $p(z) = 3z^4 - 14z^2 - 5$
$z = \pm\sqrt{5}$ (each has mult. 1)
26. $p(z) = 2z^4 - 7z^2 + 6$
$z = \pm\frac{\sqrt{6}}{2}$, $z = \pm\sqrt{2}$ (each has mult. 1)
27. $g(t) = t^6 - 3t^3 - 10$
$t = \sqrt[3]{-2} = -\sqrt[3]{2}$, $t = \sqrt[3]{5}$ (each has mult. 1)
28. $g(t) = 2t^6 - 9t^3 + 10$
$t = \frac{\sqrt[3]{20}}{2}$, $t = \sqrt[3]{2}$ (each has mult. 1)
29. $f(x) = x^5 - 2x^4 - 4x + 8$
$x = 2$, $x = \pm\sqrt{2}$ (each has mult. 1)
30. $f(x) = 2x^5 + 3x^4 - 18x - 27$
$x = -\frac{3}{2}$, $x = \pm\sqrt{3}$ (each has mult. 1)
31. $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$
$x = -4$ (mult. 3), $x = 6$ (mult. 2)
32. $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$
$x = \frac{3}{5}$ (mult. 2), $x = 1$ (mult. 3)

33. $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$
$x = \frac{2}{3}$, $x = \frac{3}{2}$, $x = \frac{5}{3}$, $x = \frac{3}{5}$ (each has mult. 1)

34. We choose $q(x) = 72x^3 - 6x^2 - 7x + 1 = 72 \cdot f(x)$. Clearly $f(x) = 0$ if and only if $q(x) = 0$ so they have the same real zeros. In this case, $x = -\frac{1}{3}$, $x = \frac{1}{6}$ and $x = \frac{1}{4}$ are the real zeros of both f and q.

35. $x = 0, \frac{5 \pm \sqrt{61}}{18}$
36. $x = 0, \frac{5 \pm \sqrt{241}}{12}$
37. $z = -2, 1, 3$
38. $z = -2, 4$
39. $t = 7$
40. $t = \frac{1}{2}, 4, 5$
41. $x = -2, \frac{3 \pm \sqrt{69}}{6}$
42. $x = \pm\sqrt{3}$
43. $z = \pm\sqrt{5}$
44. $z = -\frac{3}{2}, \pm\sqrt{3}$
45. $(-\infty, \frac{1}{2}) \cup (4, 5)$
46. $\{-2\} \cup [1, 3]$
47. $(-\infty, -1] \cup [3, \infty)$
48. $\left\{-\dfrac{1}{2}\right\} \cup [1, \infty)$
49. $[-2, 2]$
50. $(-\infty, -1) \cup (-1, 0) \cup (2, \infty)$
51. $(-\infty, -2) \cup \left(-\sqrt{2}, \sqrt{2}\right)$
52. $\{2\} \cup [4, \infty)$
53. $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$
54. $(-\infty, -\sqrt[3]{3}) \cup (\sqrt[3]{2}, \infty)$
55. $f(x) < 0$ on $(-\infty, -6) \cup (0, 6)$
56. $g(t) > 0$ on $(-\infty, -2) \cup (0, \infty)$
57. $p(z) \geq 0$ on $(-\infty, -1] \cup \{2\}$
58. $f(x) < 0$ on $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 3\right)$
59. $F(s) \leq 0$ on $\{-2\} \cup [0, \infty)$
60. $G(t) \geq 0$ on $\{-2\} \cup [0, \infty)$

61. Since $f(-4) = -23$, $f(-3) = 5$, $f(0) = 5$, $f(1) = -3$, $f(2) = -5$ and $f(3) = 5$ the Intermediate Value Theorem gives that $f(x) = x^3 - 9x + 5$ has real zeros in the intervals $[-4, -3]$, $[0, 1]$ and $[2, 3]$.

62. An odd degree polynomial function f has 'mismatched' end behavior. That is, the end behavior of $f(x)$ is either: $x \to -\infty$, $f(x) \to -\infty$ and as $x \to \infty$, $f(x) \to \infty$ or as $x \to -\infty$, $f(x) \to \infty$ and as $x \to \infty$, $f(x) \to -\infty$. This means at some point, $f(x) > 0$ and at some other point $f(x) < 0$. The Intermediate Value Theorem guarantees at least one place where $f(x) = 0$.

63. The function $f(x) = x^2 + 1$ has no real zeros.

64. $x \approx 1.167$.

65. (a) $f(x)$ has only one variation in sign, so the result follows from Descartes' Rule of Signs.

(b) $f(1) = -1 < 0$ and $f(2) = 2 > 0$ so the Intermediate Value Theorem promises a zero in $[1, 2]$.

(c) The Rational Zeros Theorem gives the only possible rational zeros of f are ± 1 and ± 2. Since $f(\pm 1) = -1$ and $f(\pm 2) = 2$, f has no rational zeros.

(d) The zero of f is $\sqrt{2} \approx 1.414$.

66. $V(x) \geq 80$ on $[1, 5 - \sqrt{5}] \cup [5 + \sqrt{5}, \infty)$. Only the portion $[1, 5 - \sqrt{5}]$ lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than $5 - \sqrt{5} \approx 2.76$ inches.

67. $C(x) \leq 5000$ on (approximately) $(-\infty, 82.18]$. The portion of this which lies in the applied domain is $(0, 82.18]$. Since x represents the number of game systems, we check $C(82) = 4983.04$ and $C(83) = 5078.11$, so to remain within the production budget, anywhere between 1 and 82 game systems can be produced.

2.4 Complex Zeros and the Fundamental Theorem of Algebra

In Section 2.3, we were focused on finding the real zeros of a polynomial function. In this section, we expand our horizons and look for the non-real zeros as well. By 'non-real' here we mean we will be discussing 'imaginary,' and, more generally, 'complex' numbers. Even though the monikers 'non-real' and 'imaginary' suggests these numbers play no role in 'real' world applications, we assure you that electrical engineers live a 'complex' life and these numbers are invaluable to them.[1] That being said, our main use of complex numbers in this section is to present some powerful structure theorems for polynomial functions (this is, after all, a math book!) For a detailed review of the Complex Number system, we refer the reader to Section **??**. For us, it suffices to review the basic vocabulary.

- The imaginary unit $i = \sqrt{-1}$ satisfies the two following properties
  1. $i^2 = -1$
  2. If c is a real number with $c \geq 0$ then $\sqrt{-c} = i\sqrt{c}$
- The **complex numbers** are the set of numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$
- Given a complex number $z = a + bi$, the **complex conjugate** of z, $\overline{z} = \overline{a + bi} = a - bi$.

Note that every real number is a complex number, that is $\mathbb{R} \subseteq \mathbb{C}$. To see this, take your favorite real number, say 117. We may write $117 = 117 + 0i$ which puts in the form $a + bi$. Hence, we we speak of the 'complex zeros' of a polynomial function, we are talking about not just the non-real, but also the real zeros.

Complex numbers, by their very definition, are two dimensional creatures. To see this, we may identify a complex number $z = a + bi$ with the point in the Cartesian plane (a, b). The horizontal axis is called the 'real' axis since points here have the form $(a, 0)$ which corresponds to numbers of the form $z = a + 0i = a$ which are the real numbers. The vertical axis is called the 'imaginary' axis since points here are of the form $(0, b)$ which correspond to numbers of the form $z = 0 + bi = bi$, the so-called 'purely imaginary' numbers. In Figure 2.4.1 we

[1] Even a cursory web search for 'use of imaginary numbers in electrical engineering' provides a wealth of source material - enough to convince anyone of their importance to the field (pun intended.) Most of it, however, requires more electrical background than the authors feel comfortable including in the text. Be aware, however, that in electrical applications, the letter j is used to represent $\sqrt{-1}$ since the letter i is reserved for current.

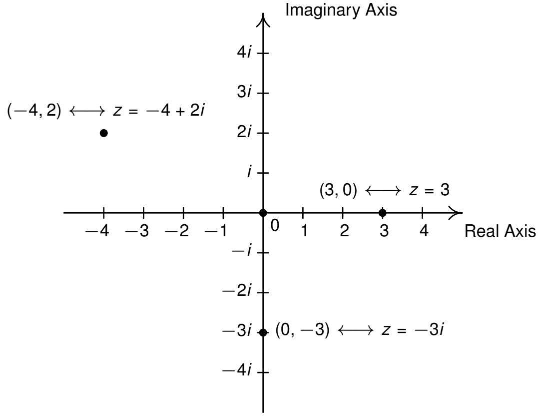


Figure 2.4.1: The Complex Plane

plot some complex numbers on this so-called 'Complex Plane.' Plotting a set of complex numbers this way is called an Argand Diagram[2], and opens up a wealth of opportunities to explore many algebraic properties of complex numbers geometrically. For example, complex conjugation amounts to a reflection about the real axis, and multiplication by i amounts to a 90° rotation.[3] While we won't have much use for the Complex Plane in this section, it is worth introducing this concept now, if, for no other reason, it gives the reader a sense of the vastness of the complex number system and the role of the real numbers in it.

Returning to zeros of polynomials, suppose we wish to find the zeros of $f(x) = x^2 - 2x + 5$. To solve the equation $x^2 - 2x + 5 = 0$, we note that the quadratic doesn't factor nicely, so we resort to the Quadratic Formula, Equation 1.4.2 and obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(5)}}{2(1)} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

Two things are important to note. First, the zeros $1 + 2i$ and $1 - 2i$ are complex conjugates. If ever we obtain non-real zeros to a quadratic function with *real number* coefficients, the zeros will be a complex conjugate pair. (Do you see why?)

[2] https://en.wikipedia.org/wiki/Complex_plane

[3] See Exercises 39 - 42.

We could ask if all of the theory from Section2.2 holds for non-real zeros, in particular the division algorithm and the Remainder and Factor Theorems. The answer is 'yes.'

$$\begin{array}{r|rrr} 1+2i & 1 & -2 & 5 \\ & \downarrow & 1+2i & -5 \\ \hline & 1 & -1+2i & \boxed{0} \end{array}$$

Indeed, the above shows $x^2 - 2x + 5 = (x - [1+2i])(x - 1 + 2i) = (x - [1+2i])(x - [1 - 2i])$ which demonstrates both $(x - [1+2i])$ and $(x - [1 - 2i])$ are factors of $x^2 - 2x + 5$.[4]

But how do we know if a general polynomial has any complex zeros at all? We have many examples of polynomials with no real zeros. Can there be polynomials with no zeros whatsoever? The answer to that last question is "No." and the theorem which provides that answer is The Fundamental Theorem of Algebra.

Theorem 2.4.1. The Fundamental Theorem of Algebra: Suppose f is a polynomial function with complex number coefficients of degree $n \geq 1$, then f has at least one complex zero.

The Fundamental Theorem of Algebra is an example of an 'existence' theorem in Mathematics. Like the Intermediate Value Theorem, Theorem 2.3.5, the Fundamental Theorem of Algebra guarantees the existence of at least one zero, but gives us no algorithm to use in finding it. In fact, as we mentioned in Section 2.3, there are polynomials whose real zeros, though they exist, cannot be expressed using the 'usual' combinations of arithmetic symbols, and must be approximated. It took mathematicians literally hundreds of years to prove the theorem in its full generality,[5] and some of that history is recorded here[6]. Note that the Fundamental Theorem of Algebra applies to not only polynomial functions with real coefficients, but to those with complex number coefficients as well.

Suppose f is a polynomial function of degree $n \geq 1$. The Fundamental Theorem of Algebra guarantees us at least one complex zero, z_1. The Factor Theorem guarantees that $f(x)$ factors as $f(x) = (x - z_1)\, q_1(x)$ for a polynomial function q_1,

[4] It is a good review of the algebra of complex numbers to start with $(x - [1+2i])(x - [1 - 2i])$, perform the indicated operations, and simplify the result to $x^2 - 2x + 5$. See part 6 of Example **??** in Section **??**.

[5] So if its profound nature and beautiful subtlety escape you, no worries!

[6] http://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

which has degree $n-1$. If $n-1 \geq 1$, then the Fundamental Theorem of Algebra guarantees a complex zero of q_1 as well, say z_2, so then the Factor Theorem gives us $q_1(x) = (x - z_2)\, q_2(x)$, and hence $f(x) = (x - z_1)\,(x - z_2)\, q_2(x)$. We can continue this process exactly n times, at which point our quotient polynomial q_n has degree 0 so it's a constant. This constant is none-other than the leading coefficient of f which is carried down line by line each time we divide by factors of the form $x - c$.

> **Theorem 2.4.2. Complex Factorization Theorem:** Suppose f is a polynomial function with complex number coefficients. If the degree of f is n and $n \geq 1$, then f has exactly n complex zeros, counting multiplicity. If $z_1, z_2, \ldots, z_k$ are the distinct zeros of f, with multiplicities $m_1, m_2, \ldots, m_k$, respectively, then $f(x) = a\,(x - z_1)^{m_1}\,(x - z_2)^{m_2} \cdots (x - z_k)^{m_k}$.

Theorem 2.4.2 says two important things: first, every polynomial is a product of linear factors; second, every polynomial function is completely determined by its zeros, their multiplicities, and its leading coefficient. We put this theorem to good use in the next example.

Example 2.4.1. Let $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$.

1. Find all of the complex zeros of f and state their multiplicities.

2. Factor $f(x)$ using Theorem 2.4.2

Solution.

1. Since f is a fifth degree polynomial, we know that we need to perform at least three successful divisions to get the quotient down to a quadratic function. At that point, we can find the remaining zeros using the Quadratic Formula, if necessary. Using the techniques developed in Section 2.3:

$$
\begin{array}{r|rrrrrr}
\frac{1}{2} & 12 & -20 & 19 & -6 & -2 & 1 \\
 & \downarrow & 6 & -7 & 6 & 0 & -1 \\
\hline
\frac{1}{2} & 12 & -14 & 12 & 0 & -2 & \boxed{0} \\
 & \downarrow & 6 & -4 & 4 & 2 & \\
\hline
-\frac{1}{3} & 12 & -8 & 8 & 4 & \boxed{0} & \\
 & \downarrow & -4 & 4 & -4 & & \\
\hline
 & 12 & -12 & 12 & \boxed{0} & &
\end{array}
$$

Our quotient is $12x^2 - 12x + 12$, whose zeros we find to be $\frac{1 \pm i\sqrt{3}}{2}$. From Theorem 2.4.2, we know f has exactly 5 zeros, counting multiplicities, and as such we have the zero $\frac{1}{2}$ with multiplicity 2, and the zeros $-\frac{1}{3}$, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$, each of multiplicity 1.

2. Applying Theorem 2.4.2, we are guaranteed that f factors as

$$f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(x - \left[\frac{1+i\sqrt{3}}{2}\right]\right)\left(x - \left[\frac{1-i\sqrt{3}}{2}\right]\right)$$

$\square$

A true test of Theorem 2.4.2 would be to take the factored form of $f(x)$ in the previous example and multiply it out[7] to see that it really does reduce to $f(x) = 12x^5 - 20x^4 + 19x^3 - 6x^2 - 2x + 1$. When factoring a polynomial using Theorem 2.4.2, we say that it is **factored completely over the complex numbers**, meaning that it is impossible to factor the polynomial any further using complex numbers. If we wanted to completely factor $f(x)$ over the **real numbers** then we would have stopped short of finding the nonreal zeros of f and factored f using our work from the synthetic division to write $f(x) = \left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(12x^2 - 12x + 12\right)$, or $f(x) = 12\left(x - \frac{1}{2}\right)^2\left(x + \frac{1}{3}\right)\left(x^2 - x + 1\right)$. Since the zeros of $x^2 - x + 1$ are nonreal, we call $x^2 - x + 1$ an **irreducible quadratic** meaning it is impossible to break it down any further using *real* numbers.

The last two results of the section show us that, theoretically, the non-real zeros of polynomial functions with real number coefficients come exclusively from irreducible quadratics.

Theorem 2.4.3. Conjugate Pairs Theorem: If f is a polynomial function with real number coefficients and z is a complex zero of f, then so is $\overline{z}$.

To prove the theorem, let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ be a polynomial function with real number coefficients. If z is a zero of f, then $f(z) = 0$, which means $a_nz^n + a_{n-1}z^{n-1} + \ldots + a_2z^2 + a_1z + a_0 = 0$. Next, we consider $f(\overline{z})$ and apply Theorem **??** below.

[7]This is a good chance to test your algebraic mettle and see that all of this does actually work.

$$
\begin{aligned}
f(\overline{z}) &= a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \ldots + a_2(\overline{z})^2 + a_1\overline{z} + a_0 \\
&= a_n\overline{z^n} + a_{n-1}\overline{z^{n-1}} + \ldots + a_2\overline{z^2} + a_1\overline{z} + a_0 && (\text{ since } (\overline{z})^n = \overline{z^n}) \\
&= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \ldots + \overline{a_2}\,\overline{z^2} + \overline{a_1}\,\overline{z} + \overline{a_0} && (\text{since the coefficients are real}) \\
&= \overline{a_n z^n} + \overline{a_{n-1} z^{n-1}} + \ldots + \overline{a_2 z^2} + \overline{a_1 z} + \overline{a_0} && (\text{ since } \overline{z}\,\overline{w} = \overline{zw}) \\
&= \overline{a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0} && (\text{ since } \overline{z} + \overline{w} = \overline{z+w}) \\
&= \overline{f(z)} \\
&= \overline{0} \\
&= 0
\end{aligned}
$$

This shows that $\overline{z}$ is a zero of f. So, if f is a polynomial function with real number coefficients, Theorem 2.4.3 tells us that if $a+bi$ is a nonreal zero of f, then so is $a-bi$. In other words, nonreal zeros of f come in conjugate pairs. The Factor Theorem kicks in to give us both $(x-[a+bi])$ and $(x-[a-bi])$ as factors of $f(x)$ which means $(x-[a+bi])(x-[a-bi]) = x^2 + 2ax + \left(a^2+b^2\right)$ is an irreducible quadratic factor of f. As a result, we have our last theorem of the section.

Theorem 2.4.4. Real Factorization Theorem: Suppose f is a polynomial function with real number coefficients. Then $f(x)$ can be factored into a product of linear factors corresponding to the real zeros of f and irreducible quadratic factors which give the nonreal zeros of f.

We now present an example which pulls together all of the major ideas of this section.

Example 2.4.2. Let $f(x) = x^4 + 64$.

1. Use synthetic division to show that $x = 2 + 2i$ is a zero of f.
2. Find the remaining complex zeros of f.
3. Completely factor $f(x)$ over the complex numbers.
4. Completely factor $f(x)$ over the real numbers.

Solution.

1. Remembering to insert the 0's in the synthetic division tableau we have

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline & 1 & 2+2i & 8i & -16+16i & \boxed{0} \end{array}$$

2. Since f is a fourth degree polynomial, we need to make two successful divisions to get a quadratic quotient. Since $2+2i$ is a zero, we know from Theorem 2.4.3 that $2-2i$ is also a zero. We continue our synthetic division tableau.

$$\begin{array}{r|rrrrr} 2+2i & 1 & 0 & 0 & 0 & 64 \\ & \downarrow & 2+2i & 8i & -16+16i & -64 \\ \hline 2-2i & 1 & 2+2i & 8i & -16+16i & \boxed{0} \\ & \downarrow & 2-2i & 8-8i & 16-16i & \\ \hline & 1 & 4 & 8 & \boxed{0} & \end{array}$$

Our quotient polynomial is x^2+4x+8. Using the quadratic formula, we solve $x^2+4x+8=0$ and find the remaining zeros are $-2+2i$ and $-2-2i$.

3. Using Theorem 2.4.2, we get $f(x) = (x-[2-2i])(x-[2+2i])(x-[-2+2i])(x-[-2-2i])$.

4. To find the irreducible quadratic factors of $f(x)$, we multiply the factors together which correspond to the conjugate pairs. We find $(x-[2-2i])(x-[2+2i]) = x^2-4x+8$, and $(x-[-2+2i])(x-[-2-2i]) = x^2+4x+8$, so $f(x) = \left(x^2-4x+8\right)\left(x^2+4x+8\right)$. □

We close this section with an example where we are asked to manufacture a polynomial function with certain characteristics.

Example 2.4.3. 1. Find a polynomial function p of lowest degree that has integer coefficients and satisfies all of the following criteria:

- the graph of $y=p(x)$ touches and rebounds from the x-axis at $\left(\frac{1}{3},0\right)$
- $x=3i$ is a zero of p.
- as $x\to-\infty$, $p(x)\to-\infty$
- as $x\to\infty$, $p(x)\to-\infty$

2. Find a possible formula for the polynomial function p graphed in Figure 2.4.2. You may leave your answer in factored form.

Solution.

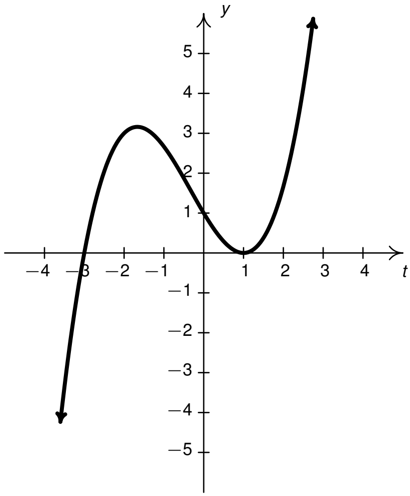


Figure 2.4.2

1. To solve this problem, we will need a good understanding of the relationship between the x-intercepts of the graph of a function and the zeros of a function, the Factor Theorem, the role of multiplicity, complex conjugates, the Complex Factorization Theorem, and end behavior of polynomial functions. (In short, you'll need most of the major concepts of this chapter.) Since the graph of p touches the x-axis at $\left(\frac{1}{3}, 0\right)$, we know $x = \frac{1}{3}$ is a zero of even multiplicity. Since we are after a polynomial of lowest degree, we need $x = \frac{1}{3}$ to have multiplicity exactly 2. The Factor Theorem now tells us $\left(x - \frac{1}{3}\right)^2$ is a factor of $p(x)$. Since $x = 3i$ is a zero and our final answer is to have integer (hence, real) coefficients, $x = -3i$ is also a zero. The Factor Theorem kicks in again to give us $(x - 3i)$ and $(x + 3i)$ as factors of $p(x)$. We are given no further information about zeros or intercepts so we conclude, by the Complex Factorization Theorem that $p(x) = a\left(x - \frac{1}{3}\right)^2 (x - 3i)(x + 3i)$ for some real number a. Expanding this, we get $p(x) = ax^4 - \frac{2a}{3}x^3 + \frac{82a}{9}x^2 - 6ax + a$. In order to obtain integer coefficients, we know a must be an integer multiple of 9. Our last concern is end behavior. Since the leading term of $p(x)$ is ax^4, we need $a < 0$ to get $p(x) \to -\infty$ as $x \to \pm\infty$. Hence, if we choose $x = -9$, we get $p(x) = -9x^4 + 6x^3 - 82x^2 + 54x - 9$. We can verify our handiwork using the techniques developed in this chapter.

2. The first thing to note is the independent variable here is t, not x as evidenced by the labeling on the horizontal axis. Next, the graph appears to cross through the t-axis at $(-3, 0)$ in a fairly linear fashion, so $t = -3$ is likely a zero of multiplicity 1. Also, the graph touches and rebounds at $(1, 0)$, indicating $t = 1$ is a zero of even multiplicity. Since the graph doesn't appear too 'flat,' we'll go with multiplicity 2 (though there is really no way of telling.) Using the Complex Factorization Theorem and assuming we have no non-real zeros, we now have $p(t) = a(t-(-3))(t-1)^2 = a(t+3)(t-1)^2$. To determine the leading coefficient, a, we note the graph appears to have a y-intercept at $(0, 1)$. Solving $p(0) = 1$ gives $a(3)(-1)^2 = 1$ or $3a = 1$. Hence, $a = \frac{1}{3}$ so $p(t) = \frac{1}{3}(t+3)(t-1)^2$. Since we may leave our answer in factored form, we are done. $\square$

This example concludes our study of polynomial functions.[8] The last few sections have contained what is considered by many to be 'heavy' Mathematics. Like a heavy meal, heavy Mathematics takes time to digest. Don't be overly concerned if it doesn't seem to sink in all at once, and pace yourself in the Exercises or you're liable to get mental cramps. But before we get to the Exercises, we'd like to offer a bit of an epilogue.

Our main goal in presenting the material on the complex zeros of a polynomial was to give the chapter a sense of completeness. Given that it can be shown that some polynomials have real zeros which cannot be expressed using the usual algebraic operations, and still others have no real zeros at all, it was nice to discover that every polynomial of degree $n \geq 1$ has n complex zeros. So like we said, it gives us a sense of closure.[9] As mentioned at the top of the section, complex numbers are very useful in many applied fields such as electrical engineering, but most of the applications require science and mathematics well beyond precalculus material to fully understand them. That does not mean you'll never be be able to understand them; in fact, it is the authors' sincere hope that all of you will reach a point in your studies when the glory, awe and splendor of complex numbers are revealed to you. For now, however, the really good stuff is beyond the scope of this text. We invite you and your classmates to find a few examples of complex number applications and see what you can make of them.

For the remainder of the text, with the exception of Section **??** and a few exploratory exercises scattered about, we will restrict our attention to real numbers. We do this primarily because the first Calculus sequence you will take, ostensibly

[8] With the exception of the Exercises on the next page, of course.

[9] This is a very deep math pun.

the one that this text is preparing you for, studies only functions of real variables. Also, lots of really cool scientific things don't require any deep understanding of complex numbers to study them, but they do need more Mathematics like exponential, logarithmic and trigonometric functions. We believe it makes more sense pedagogically for you to learn about those functions now then take a course in Complex Function Theory in your junior or senior year once you've completed the Calculus sequence. It is in that course that the true power of the complex numbers is released. But for now, in order to fully prepare you for life immediately after Precalculus, we will say that functions like $f(x) = \frac{1}{x^2+1}$, which we'll study in the very next chapter, have a domain of all real numbers, even though we know $x^2 + 1 = 0$ has two complex solutions, namely $x = \pm i$ which produce a '0' in the denominator. Since $x^2 + 1 > 0$ for all *real* numbers x, the fraction $\frac{1}{x^2+1}$ is never undefined in the real variable setting.

2.4.1 Exercises

In Exercises 1 - 22, find all of the zeros of the polynomial then completely factor it over the real numbers and completely factor it over the complex numbers.

1. $f(x) = x^2 - 4x + 13$
2. $f(x) = x^2 - 2x + 5$
3. $p(z) = 3z^2 + 2z + 10$
4. $p(z) = z^3 - 2z^2 + 9z - 18$
5. $g(t) = t^3 + 6t^2 + 6t + 5$
6. $g(t) = 3t^3 - 13t^2 + 43t - 13$
7. $f(x) = x^3 + 3x^2 + 4x + 12$
8. $f(x) = 4x^3 - 6x^2 - 8x + 15$
9. $p(z) = z^3 + 7z^2 + 9z - 2$
10. $p(z) = 9z^3 + 2z + 1$
11. $g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3$
12. $g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6$
13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18$
14. $f(x) = 6x^4 + 17x^3 - 55x^2 + 16x + 12$
15. $p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5$
16. $p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4$
17. $g(t) = t^4 + 9t^2 + 20$
18. $g(t) = t^4 + 5t^2 - 24$
19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12$
20. $f(x) = x^6 - 64$
21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26$ (Hint: $x = i$ is one of the zeros.)
22. $p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5$ (Hint: $z = -1 + 2i$ is a zero.)

In Exercises 23 - 32, use Theorem 2.4.2 to create a polynomial function with real number coefficients which has all of the desired characteristics. You may leave the polynomial in factored form.

23.
   - The zeros of f are $c = \pm 2$ and $c = \pm 1$.
   - The leading term of $f(x)$ is $117x^4$.

24.
   - The zeros of p are $c = 1$ and $c = 3$.
   - $c = 3$ is a zero of multiplicity 2.
   - The leading term of $p(z)$ is $-5z^3$.

25.
   - The solutions to $g(t) = 0$ are $t = \pm 3$ and $t = 6$.
   - The leading term of $g(t)$ is $7t^4$.
   - The point $(-3, 0)$ is a local minimum on the graph of $y = g(t)$.

26.
   - The solutions to $f(x) = 0$ are $x = \pm 3$, $x = -2$, and $x = 4$.
   - The leading term of $f(x)$ is $-x^5$.

- The point $(-2, 0)$ is a local maximum on the graph of $y = f(x)$.

27.
- p is degree 4.
- as $z \to \infty$, $p(z) \to -\infty$.
- p has exactly three z-intercepts: $(-6, 0)$, $(1, 0)$ and $(117, 0)$.
- The graph of $y = p(z)$ crosses through the z-axis at $(1, 0)$.

28.
- The zeros of g are $c = \pm 1$ and $c = \pm i$.
- The leading term of $g(t)$ is $42t^4$.

29.
- $c = 2i$ is a zero.
- the point $(-1, 0)$ is a local minimum on the graph of $y = f(x)$.
- the leading term of $f(x)$ is $117x^4$.

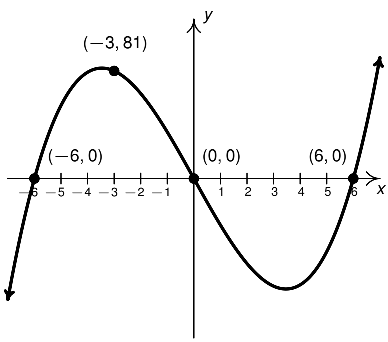
30.
- The solutions to $p(z) = 0$ are $z = \pm 2$ and $z = \pm 7i$.
- The leading term of $p(z)$ is $-3z^5$.
- The point $(2, 0)$ is a local maximum on the graph of $y = p(z)$.

31.
- g is degree 5.
- $t = 6$, $t = i$ and $t = 1 - 3i$ are zeros of g.
- as $t \to -\infty$, $g(t) \to \infty$.

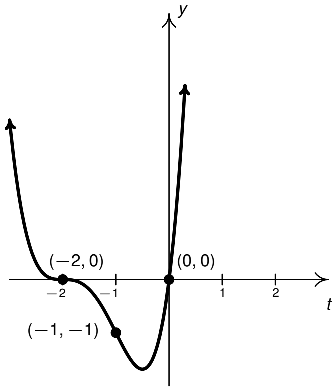
32.
- The leading term of $f(x)$ is $-2x^3$.
- $c = 2i$ is a zero.
- $f(0) = -16$.

In Exercises 33 - 38, find a possible formula for the polynomial function given its graph. You may leave the polynomial in factored form.

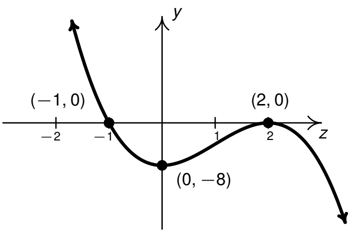
33. $y = f(x)$.



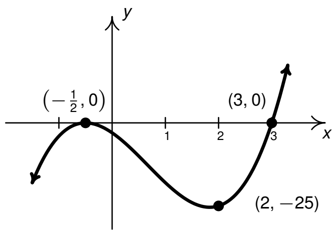
34. $y = g(t)$



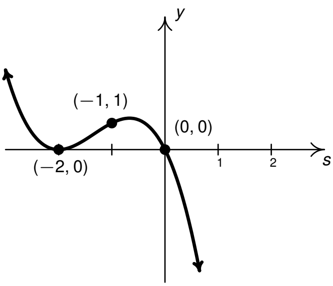
35. $y = p(z)$



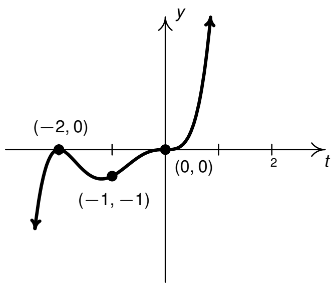
36. $y = f(x)$



37. $y = F(s)$



38. $y = G(t)$



39. With help from your classmates, choose several nonzero complex numbers z, find their complex conjugates $\overline{z}$. Plot each pair z and $\overline{z}$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.

40. With help from your classmates, choose several nonzero complex numbers z and find $-z$. Plot each pair z and $-z$ in the Complex Plane. What appears to be the relationship between these numbers geometrically? State and prove a general result.

41. With help from your classmates, choose several different complex numbers z and find the product of i and z, iz. Plot each pair of z and iz in the Complex Plane. In each case, show the line containing the origin and the point corresponding to z is perpendicular[10] to the line containing the origin and the point corresponding to iz. Show this result holds in general for every nonzero complex number.

42. Given a complex number $z = a + bi$, we define the **modulus** of z, $|z|$, by $|z| = \sqrt{a^2 + b^2}$. With help from your classmates, calculate $|z|$ for several different complex numbers, z. What does z measure geometrically? Show that if x is a real number, then the modulus of x is the same as the absolute value of x, and comment how all this relates to Definition **??** in Section **??**.

43. Let z and w be arbitrary complex numbers. Show that $\overline{z}\,\overline{w} = \overline{zw}$ and $\overline{\overline{z}} = z$.

[10] See Theorem **??** in Section **??** of you need a refresher on how to do this.

2.4.2 Answers

1. $f(x) = x^2 - 4x + 13 = (x - (2 + 3i))(x - (2 - 3i))$
   Zeros: $x = 2 \pm 3i$

2. $f(x) = x^2 - 2x + 5 = (x - (1 + 2i))(x - (1 - 2i))$
   Zeros: $x = 1 \pm 2i$

3. $p(z) = 3z^2 + 2z + 10 = 3\left(z - \left(-\frac{1}{3} + \frac{\sqrt{29}}{3}i\right)\right)\left(z - \left(-\frac{1}{3} - \frac{\sqrt{29}}{3}i\right)\right)$ Zeros: $z = -\frac{1}{3} \pm \frac{\sqrt{29}}{3}i$

4. $p(z) = z^3 - 2z^2 + 9z - 18 = (z - 2)\left(z^2 + 9\right) = (z - 2)(z - 3i)(z + 3i)$
   Zeros: $z = 2, \pm 3i$

5. $g(t) = t^3+6t^2+6t+5 = (t+5)(t^2+t+1) = (t+5)\left(t - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(t - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right)$
   Zeros: $t = -5,\ t = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

6. $g(t) = 3t^3 - 13t^2 + 43t - 13 = (3t - 1)(t^2 - 4t + 13) = (3t - 1)(t - (2 + 3i))(t - (2 - 3i))$
   Zeros: $t = \frac{1}{3},\ t = 2 \pm 3i$

7. $f(x) = x^3 + 3x^2 + 4x + 12 = (x + 3)\left(x^2 + 4\right) = (x + 3)(x + 2i)(x - 2i)$
   Zeros: $x = -3,\ \pm 2i$

8. $f(x) = 4x^3 - 6x^2 - 8x + 15 = \left(x + \frac{3}{2}\right)\left(4x^2 - 12x + 10\right)$
   $= 4\left(x + \frac{3}{2}\right)\left(x - \left(\frac{3}{2} + \frac{1}{2}i\right)\right)\left(x - \left(\frac{3}{2} - \frac{1}{2}i\right)\right)$
   Zeros: $x = -\frac{3}{2},\ x = \frac{3}{2} \pm \frac{1}{2}i$

9. $p(z) = z^3 + 7z^2 + 9z - 2 = (z + 2)\left(z - \left(-\frac{5}{2} + \frac{\sqrt{29}}{2}\right)\right)\left(z - \left(-\frac{5}{2} - \frac{\sqrt{29}}{2}\right)\right)$
   Zeros: $z = -2,\ z = -\frac{5}{2} \pm \frac{\sqrt{29}}{2}$

10. $p(z) = 9z^3 + 2z + 1 = \left(z + \frac{1}{3}\right)\left(9z^2 - 3z + 3\right)$
    $= 9\left(z + \frac{1}{3}\right)\left(z - \left(\frac{1}{6} + \frac{\sqrt{11}}{6}i\right)\right)\left(z - \left(\frac{1}{6} - \frac{\sqrt{11}}{6}i\right)\right)$
    Zeros: $z = -\frac{1}{3},\ z = \frac{1}{6} \pm \frac{\sqrt{11}}{6}i$

11. $g(t) = 4t^4 - 4t^3 + 13t^2 - 12t + 3 = \left(t - \frac{1}{2}\right)^2\left(4t^2 + 12\right) = 4\left(t - \frac{1}{2}\right)^2(t + i\sqrt{3})(t - i\sqrt{3})$
    Zeros: $t = \frac{1}{2},\ t = \pm\sqrt{3}i$

12. $g(t) = 2t^4 - 7t^3 + 14t^2 - 15t + 6 = (t - 1)^2\left(2t^2 - 3t + 6\right)$
    $= 2(t - 1)^2\left(t - \left(\frac{3}{4} + \frac{\sqrt{39}}{4}i\right)\right)\left(t - \left(\frac{3}{4} - \frac{\sqrt{39}}{4}i\right)\right)$
    Zeros: $t = 1,\ t = \frac{3}{4} \pm \frac{\sqrt{39}}{4}i$

13. $f(x) = x^4 + x^3 + 7x^2 + 9x - 18 = (x+2)(x-1)\left(x^2+9\right) = (x+2)(x-1)(x+3i)(x-3i)$
Zeros: $x = -2,\ 1,\ \pm 3i$

14. $f(x) = 6x^4+17x^3-55x^2+16x+12 = 6\left(x+\frac{1}{3}\right)\left(x-\frac{3}{2}\right)\left(x-\left(-2+2\sqrt{2}\right)\right)\left(x-\left(-2\right.\right.$
Zeros: $x = -\frac{1}{3},\ x = \frac{3}{2},\ x = -2 \pm 2\sqrt{2}$

15. $p(z) = -3z^4 - 8z^3 - 12z^2 - 12z - 5 = (z+1)^2\left(-3z^2-2z-5\right)$
$= -3(z+1)^2\left(z-\left(-\frac{1}{3}+\frac{\sqrt{14}}{3}i\right)\right)\left(z-\left(-\frac{1}{3}-\frac{\sqrt{14}}{3}i\right)\right)$
Zeros: $z = -1,\ z = -\frac{1}{3} \pm \frac{\sqrt{14}}{3}i$

16. $p(z) = 8z^4 + 50z^3 + 43z^2 + 2z - 4 = 8\left(z+\frac{1}{2}\right)\left(z-\frac{1}{4}\right)(z-(-3+\sqrt{5}))(z-(-3-\sqrt{5}))$
Zeros: $z = -\frac{1}{2},\ \frac{1}{4},\ z = -3 \pm \sqrt{5}$

17. $g(t) = t^4 + 9t^2 + 20 = \left(t^2+4\right)\left(t^2+5\right) = (t-2i)(t+2i)\left(t-i\sqrt{5}\right)\left(t+i\sqrt{5}\right)$
Zeros: $t = \pm 2i, \pm i\sqrt{5}$

18. $g(t) = t^4+5t^2-24 = \left(t^2-3\right)\left(t^2+8\right) = (t-\sqrt{3})(t+\sqrt{3})\left(t-2i\sqrt{2}\right)\left(t+2i\sqrt{2}\right)$
Zeros: $t = \pm\sqrt{3}, \pm 2i\sqrt{2}$

19. $f(x) = x^5 - x^4 + 7x^3 - 7x^2 + 12x - 12 = (x-1)\left(x^2+3\right)\left(x^2+4\right)$
$= (x-1)(x-i\sqrt{3})(x+i\sqrt{3})(x-2i)(x+2i)$
Zeros: $x = 1,\ \pm\sqrt{3}i,\ \pm 2i$

20. $f(x) = x^6 - 64 = (x-2)(x+2)\left(x^2+2x+4\right)\left(x^2-2x+4\right)$
$= (x-2)(x+2)\left(x-\left(-1+i\sqrt{3}\right)\right)\left(x-\left(-1-i\sqrt{3}\right)\right)\left(x-\left(1+i\sqrt{3}\right)\right)\left(x-\left(1-\right.\right.$
Zeros: $x = \pm 2,\ x = -1 \pm i\sqrt{3},\ x = 1 \pm i\sqrt{3}$

21. $f(x) = x^4 - 2x^3 + 27x^2 - 2x + 26 = (x^2-2x+26)(x^2+1) = (x-(1+5i))(x-(1-5i))(x+i)(x-i)$
Zeros: $x = 1 \pm 5i,\ x = \pm i$

22. $p(z) = 2z^4 + 5z^3 + 13z^2 + 7z + 5 = \left(z^2+2z+5\right)\left(2z^2+z+1\right)$
$= 2(z-(-1+2i))(z-(-1-2i))\left(z-\left(-\frac{1}{4}+i\frac{\sqrt{7}}{4}\right)\right)\left(z-\left(-\frac{1}{4}-i\frac{\sqrt{7}}{4}\right)\right)$
Zeros: $z = -1 \pm 2i, -\frac{1}{4} \pm i\frac{\sqrt{7}}{4}$

23. $f(x) = 117(x+2)(x-2)(x+1)(x-1)$

24. $p(z) = -5(z-1)(z-3)^2$

25. $g(t) = 7(t+3)^2(t-3)(t-6)$

26. $f(x) = -(x+2)^2(x-3)(x+3)(x-4)$

27. $p(z) = a(z+6)^2(z-1)(z-117)$ where a can be any real number as long as $a < 0$

28. $g(t) = 42(t-1)(t+1)(t-i)(t+i)$

29. $f(x) = 117(x+1)^2(x-2i)(x+2i)$

30. $p(z) = -3(z-2)^2(z+2)(z-7i)(z+7i)$

31. $g(t) = a(t-6)(t-i)(t+i)(t-(1-3i))(t-(1+3i))$ where a is any real number, $a < 0$

32. $f(x) = -2(x-2i)(x+2i)(x+2)$

33. $f(x) = x(x+6)(x-6)$
34. $g(t) = t(t+2)^3$
35. $p(z) = -2(z+1)(z-2)^2$
36. $f(x) = 4\left(x+\frac{1}{2}\right)^2(x-3)$
37. $F(s) = -s(s+2)^2$
38. $G(t) = t^3(t+2)^2$

39. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $\overline{z} = \overline{a+bi} = a-bi$ corresponds to the point $(a, -b)$. Hence, the points corresponding to z and $\overline{z}$ are reflections about the x-axis.

40. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Hence, $-z = -(a+bi) = -a-bi$ corresponds to the point $(-a, -b)$. Hence, the points corresponding to z and $-z$ are reflections through the origin.

41. If $z = a+bi$, then z corresponds to the point (a, b) in the xy-plane. Writing out the product iz, we get: $iz = i(a+bi) = ia+bi^2 = ia-b = -b+ia$. Hence, iz corresponds to the point $(-b, a)$. If $z \neq 0$, then neither a nor b is 0 (do you see why?) Hence, the slope of the line containing $(0, 0)$ and (a, b) is $\frac{b}{a}$ and the slope of the line containing $(0, 0)$ and $(-b, a)$ is $-\frac{a}{b}$. Per Theorem **??**, since the slopes of these lines are negative reciprocals, the lines themselves are perpendicular.[11]

42. $|z| = \sqrt{a^2+b^2}$ measures the distance from the origin to the point (a, b). Hence, $|z|$ measures the distance from z to 0 in the Complex Plane. This is exactly how $|x|$ is defined in Definition **??** in Section **??**. In that section, however, the only part of the Complex Plane under discussion is the real number line.

[11] We'll be able to show in Section **??** that, more precisely, multiplication by i rotates the complex number counter-clockwise by 90°.