Precalculus — Algebra II

Version $4 - \epsilon$

by

Carl Stitz, Ph.D.
Lakeland Community College

Jeff Zeager, Ph.D.
Lorain County Community College

$\sigma o\phi \acute{\iota}\alpha$

Sophia Publishing
CHENNAI
https://sophia-publishing.github.io/

Table of Contents

Chapter 1

Rational Functions

1.1 Introduction to Rational Functions

If we add, subtract, or multiply polynomial functions, the result is another polynomial function. When we divide polynomial functions, however, we may not get a polynomial function. The result of dividing two polynomials is a **rational function**, so named because rational functions are *ratios* of polynomials.

Definition 1.1.1. A **rational function** is a function which is the ratio of polynomial functions. Said differently, r is a rational function if it is of the form

$$r(x) = \frac{p(x)}{q(x)},$$

where p and q are polynomial functions.[a]

[a] According to this definition, all polynomial functions are also rational functions. (Take $q(x) = 1$).

1.1.1 Laurent Monomial Functions

As with polynomial functions, we begin our study of rational functions with what are, in some sense, the building blocks of rational functions, **Laurent monomial functions**.

Definition 1.1.2. A **Laurent monomial function** is either a monomial function (see Definition **??**) or a function of the form $f(x) = \frac{a}{x^n} = ax^{-n}$ for $n \in \mathbb{N}$.

Laurent monomial functions are named in honor of Pierre Alphonse Laurent[1] and generalize the notion of 'monomial function' from Chapter **??** to terms with negative

[1] https://en.wikipedia.org/wiki/Pierre_Alphonse_Laurent

exponents. Our study of these functions begins with an analysis of $r(x) = \frac{1}{x} = x^{-1}$, the reciprocal function. The first item worth noting is that $r(0)$ is not defined owing to the presence of x in the denominator. That is, the domain of r is $\{x \in \mathbb{R} \,|\, x \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. Of course excluding 0 from the domain of r serves only to pique our curiosity about the behavior of $r(x)$ when $x \approx 0$. Thinking from a number sense perspective, the closer the denominator of $\frac{1}{x}$ is to 0, the larger the value of the fraction (in absolute value.)[a] So it stands to reason that as x gets closer and closer to 0, the values for $r(x) = \frac{1}{x}$ should grow larger and larger (in absolute value.) This is borne out in Table 1.1.1 where it is apparent that for $x \approx 0$, $r(x)$ is becoming unbounded.

Technically speaking, -1×10^{117} is a a 'small' number (since it is very far to the left on the number line.) However, it's absolute value, 1×10^{117} is very large.

x	$r(x) = \frac{1}{x}$
−0.01	−100
−0.001	−1000
−0.0001	−10000
−0.00001	−100000
0	undefined
0.00001	100000
0.0001	10000
0.001	1000
0.01	100

Table 1.1.1

x	$r(x) = \frac{1}{x}$
−100000	−0.00001
−10000	−0.0001
−1000	−0.001
−100	−0.01
0	undefined
100	0.01
1000	0.001
10000	0.0001
100000	0.00001

Table 1.1.2

As we investigate the end behavior of r, we find that as $x \to \pm\infty$, $r(x) \approx 0$. Again, number sense agrees here with the data (Table 1.1.2), since as the denominator of $\frac{1}{x}$ becomes unbounded, the value of the fraction should diminish. That being said, we could ask if the graph ever reaches the x-axis. If we attempt to solve $y = r(x) = \frac{1}{x} = 0$. we arrive at the contradiction $1 = 0$ hence, 0 is not in the range of r. Every other real number besides 0 is in the range of r, however. To see this, let $c \neq 0$ be a real number. Then $\frac{1}{c}$ is defined and, moreover, $r\left(\frac{1}{c}\right) = \frac{1}{(1/c)} = c$. This shows c is in the range of r. Hence, the range of r is $\{y \in \mathbb{R} \,|\, y \neq 0\}$ or, using interval notation, $(-\infty, 0) \cup (0, \infty)$. See Figure 1.1.1.

In order to more precisely describe the behavior near 0, we say 'as x approaches 0 *from the left*,' written as $x \to 0^-$, the function values $r(x) \to -\infty$. By 'from the left' we mean we are considering x-values slightly to the *left* of 0 on the number line, such as $x = -0.001$ and $x = -0.0001$ in the table above. If we think of these numbers as all being x-values where $x =$ '0− a little bit', the the '−' in the notation '$x \to 0^-$' makes better sense. The notation to describe the $r(x)$ values, $r(x) \to -\infty$, is used here in the same manner as it was in Section **??**. That is, as $x \to 0^-$, the values $r(x)$ are becoming unbounded in the negative direction.

Similarly, we say 'as x approaches 0 *from the right*,' that is as $x \to 0^+$, $r(x) \to \infty$. Here 'from the right' means we are using x values slightly to the *right* of 0 on the number line: numbers such as $x = 0.001$ which could be described as '0 + a little bit.' For these values of x, the values of $r(x)$ become unbounded (in the positive direction) so we write $r(x) \to \infty$ here.

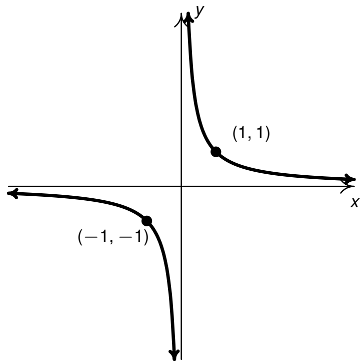


Figure 1.1.1: $y = r(x) = \frac{1}{x}$

We can also use this notation to describe the end behavior, but here the numerical roles are reversed. We see as $x \to -\infty$, $r(x) \to 0^-$ and as $x \to \infty$, $r(x) \to 0^+$.

The way we describe what is happening graphically is to say the line $x = 0$ is a **vertical asymptote** to the graph of $y = r(x)$ an the line $y = 0$ is a **horizontal asymptote** to the graph of $y = r(x)$. Roughly speaking, asymptotes are lines which approximate functions as either the inputs our outputs become unbounded. While defined more precisely using the language of Calculus, we do our best to formally define vertical and horizontal asymptotes in Definitions 1.1.3 and 1.1.4.

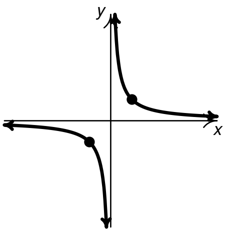
Definition 1.1.3. The line $x = c$ is called a **vertical asymptote** of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$.

Definition 1.1.4. The line $y = c$ is called a **horizontal asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to c$.

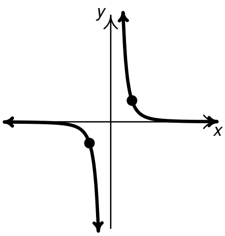
Note that in Definition 1.1.4, we write $f(x) \to c$ (not $f(x) \to c^+$ or $f(x) \to c^-$) because we are unconcerned from which direction the values $f(x)$ approach the value c, just as long as they do so. As we shall see, the graphs of rational functions may, in fact, *cross* their horizontal asymptotes. If this happens, however, it does so only a *finite* number of times (at least in this chapter), and so for each choice of $x \to -\infty$ and $x \to \infty$, $f(x)$ will approach c from either below (in the case $f(x) \to c^-$) or above (in the case $f(x) \to c^+$.) We leave $f(x) \to c$ generic in our definition, however, to allow this concept to apply to less tame specimens in the Precalculus zoo, one that cross horizontal asymptotes an infinite number of times.[b]

b See Exercise **??** in Section **??**.

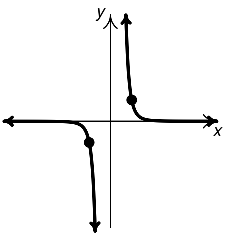
The behaviors illustrated in the graph $r(x) = \frac{1}{x}$ are typical of functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ for natural numbers, n. As with the monomial functions discussed in Section **??**, the patterns that develop primarily depend on whether n is odd or even. Having thoroughly discussed the graph of $y = \frac{1}{x} = x^{-1}$, we graph it along with $y = \frac{1}{x^3} = x^{-3}$ and $y = \frac{1}{x^5} = x^{-5}$ in Figure 1.1.2, with the data presented in Table 1.1.3. Note the points $(-1,-1)$ and $(1,1)$ are common to all three graphs as are the asymptotes $x = 0$ and $y = 0$. As the n increases, the graphs become steeper for $|x| < 1$ and flatten out more quickly for $|x| > 1$. Both the domain and range in each case appears to be $(-\infty, 0) \cup (0, \infty)$. Indeed, owing to the x in the denominator of $f(x) = \frac{1}{x^n}$, $f(0)$, and only $f(0)$, is undefined. Hence the domain is $(-\infty, 0) \cup (0, \infty)$. When thinking about the range, note the equation $f(x) = \frac{1}{x^n} = c$ has the solution $x = \sqrt[n]{\frac{1}{c}}$ as long as $c \neq 0$. Thus means $f\left(\sqrt[n]{\frac{1}{c}}\right) = c$ for every nonzero real number c. If $c = 0$, we are in the same situation as before: $\frac{1}{x^n} = 0$ has no real solution. This establishes the range is $(-\infty, 0) \cup (0, \infty)$. Finally, each of the graphs appear to be symmetric about the origin. Indeed, since n is odd, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = -x^{-n} = -f(x)$, proving every member of this function family is odd.



$y = \frac{1}{x} = x^{-1}$



$y = \frac{1}{x^3} = x^{-3}$



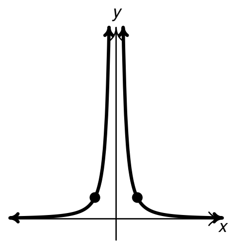
$y = \frac{1}{x^5} = x^{-5}$

Figure 1.1.2

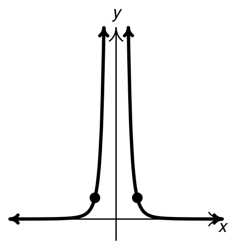
x	$\frac{1}{x} = x^{-1}$	$\frac{1}{x^3} = x^{-3}$	$\frac{1}{x^5} = x^{-5}$
-10	-0.1	-0.001	-0.00001
-1	-1	-1	-1
-0.1	-10	-1000	-100000
0	undefined	undefined	undefined
0.1	10	1000	100000
1	1	1	1
10	0.1	0.001	0.00001

Table 1.1.3

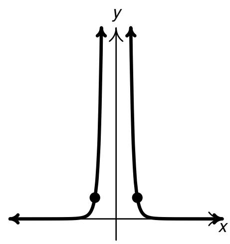
We repeat the same experiment with functions of the form $f(x) = \frac{1}{x^n} = x^{-n}$ where n is even. $y = \frac{1}{x^2} = x^2$, $y = \frac{1}{x^4} = x^{-4}$ and $y = \frac{1}{x^6} = x^{-6}$. These graphs all share the points $(-1, 1)$ and $(1, 1)$, and asymptotes $x = 0$ and $y = 0$. See Table 1.1.4 and Figure 1.1.3. The same remarks about the steepness for $|x| < 1$ and the flattening for $|x| > 1$ also apply. For the same reasons as given above, the domain of each of these functions is $(-\infty, 0) \cup (0, \infty)$. When it comes to the range, the fact n is even tells us tere are solutions to $\frac{1}{x^n} = c$ only if $c > 0$. It follows that the range is $(0, \infty)$ for each of these functions. Concerning symmetry, as n is even, $f(-x) = (-x)^{-n} = (-1)^{-n}x^{-n} = x^{-n} = f(x)$, proving each member of this function family is even. Hence, all of the graphs of these functions is symmetric about the y-axis.



$y = \frac{1}{x^2} = x^{-2}$



$y = \frac{1}{x^3} = x^{-3}$



$y = \frac{1}{x^6} = x^{-6}$

Figure 1.1.3

x	$\frac{1}{x^2} = x^{-2}$	$\frac{1}{x^4} = x^{-4}$	$\frac{1}{x^6} = x^{-6}$
-10	0.01	0.0001	1×10^{-6}
-1	1	1	1
-0.1	100	10000	1×10^{6}
0	undefined	undefined	undefined
0.1	100	10000	1×10^{6}
1	1	1	1
10	0.01	0.0001	1×10^{-6}

Table 1.1.4

Not surprisingly, we have an analog to Theorem **??** for this family of Laurent monomial functions.

Theorem 1.1.1. For real numbers a, h, and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x - h)^{-n}$.
   The vertical asymptote moves from $x = 0$ to $x = h$.
2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (x - h)^{-n}$ to $y = a(x - h)^{-n}$.
3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(x - h)^{-n}$ to $y = a(x - h)^{-n} + k$.
   The horizontal asymptote moves from $y = 0$ to $y = k$.

The proof of Theorem 1.1.1 is *identical* to the proof of Theorem **??** - just replace x^n with x^{-n}. We nevertheless encourage the reader to work through the details [c] and compare the results of this theorem with Theorems **??**, **??**, and **??**.

c We are, in fact, building to Theorem 3.4.6 in Section 3.4, so the more you see the forest for the trees, the better off you'll be when the time comes to generalize these moves to all functions.

We put Theorem 1.1.1 to good use in the following example.

Example 1.1.1. Use Theorem 1.1.1 to graph the following. Label at least two points and the asymptotes. State the domain and range using interval notation.

1. $f(x) = (2x-3)^{-2}$ 2. $g(t) = \dfrac{2t-1}{t+1}$

Solution.

1. In order to use Theorem 1.1.1, we first must put $f(x) = (2x-3)^{-2}$ into the form prescribed by the theorem. To that end, we factor:

$$f(x) = \left(2\left[x - \frac{3}{2}\right]\right)^{-2} = 2^{-2}\left(x - \frac{3}{2}\right)^{-2} = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}$$

We identify $n = 2$, $a = \frac{1}{4}$ and $h = \frac{3}{2}$ (and $k = 0$.) Per the theorem, we begin with the graph of $y = x^{-2}$ and track the two points $(-1, 1)$ and $(1, 1)$ along with the vertical and horizontal asymptotes $x = 0$ and $y = 0$, respectively through each step.

Step 1: add $\frac{3}{2}$ to each of the x-coordinates of each of the points on the graph of $y = x^{-2}$. This moves the vertical asymptote from $x = 0$ to $x = \frac{3}{2}$ (which we represent by a dashed line) in Figure 1.1.4

Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \left(x - \frac{3}{2}\right)^{-2}$ by $\frac{1}{4}$. See Figure 1.1.5.

Since we did not shift the graph vertically, the horizontal asymptote remains $y = 0$. We can determine the domain and range of f by tracking the changes to the domain and range of our progenitor function, $y = x^{-2}$. We get the domain and range of f is $\left(-\infty, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \infty\right)$ and the range of f is $(-\infty, 0) \cup (0, \infty)$.

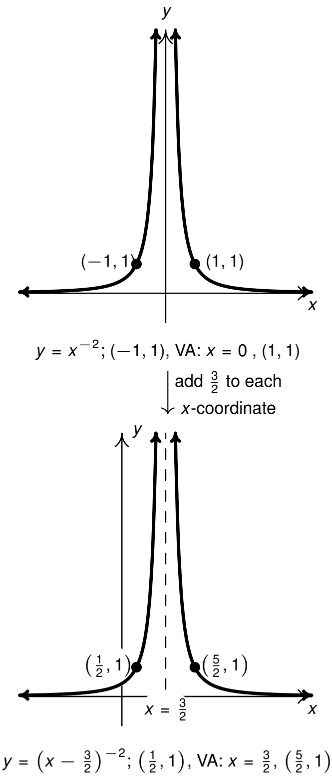


Figure 1.1.4

2. Using either long or synthetic division, we get

$$g(t) = \frac{2t-1}{t+1} = -\frac{3}{t+1} + 2 = \frac{-3}{(t-(-1))^{1}} + 2$$

so we identify $n = 1$, $a = -3$, $h = -1$, and $k = 2$. We start with the graph of $y = \frac{1}{t}$ with points $(-1, -1)$, $(1, 1)$ and asymptotes $t = 0$ and $y = 0$ and track these through each of the steps.

Step 1: Add -1 to each of the t-coordinates of each of the points on the graph of $y = \frac{1}{t}$. This moves the vertical asymptote from $t = 0$ to $t = -1$. See Figure 1.1.6.

Step 2: multiply each of the y-coordinates of each of the points on the graph of $y = \frac{1}{t+1}$ by -3. See Figure 1.1.7.

Step 3: add 2 to each of the y-coordinates of each of the points on the graph of $y = \frac{-3}{t+1}$. This moves the horizontal asymptote from $y = 0$ to $y = 2$. See Figure 1.1.8.

As above, we determine the domain and range of g by tracking the changes in the domain and range of $y = \frac{1}{t}$. We find the domain of g is $(-\infty, -1) \cup (-1, \infty)$ and the range is $(-\infty, 2) \cup (2, \infty)$. $\square$

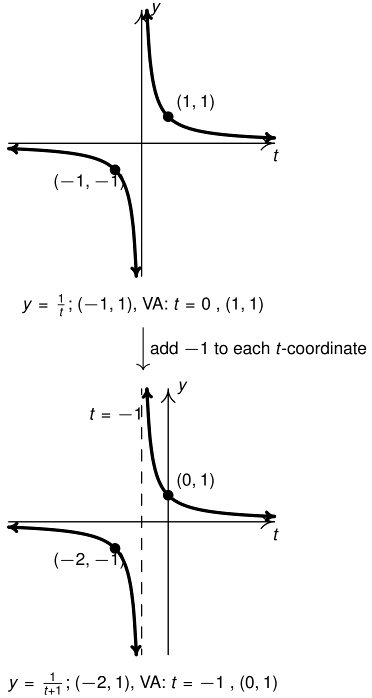


Figure 1.1.6

In Example 1.1.1, we once again see the benefit of changing the form of a function to make use of an important result. A natural question to ask is to what extent general rational functions can be rewritten to use Theorem 1.1.1. In the same way polynomial functions are sums of monomial functions, it turns out, allowing for non-real number coefficients, that every rational function can be written as a sum of (possibly shifted) Laurent monomial functions.[d]

1.1.2 Local Behavior near Excluded Values

We take time now to focus on behaviors of the graphs of rational functions near excluded values. We've already seen examples of one type of behavior: vertical asymptotes. Our next example gives us a physical interpretation of a vertical asymptote. This type of model arises

[d] i.e., Laurent 'Polynomials.' This result is a combination of Theorems **??** in Section **??** and Theorem **??** in Section **??**.

from a family of equations cheerily named 'doomsday' equations.[e]

[e] These functions arise in Differential Equations. The unfortunate name will make sense shortly.

Example 1.1.2. A mathematical model for the population $P(t)$, in thousands, of a certain species of bacteria, t days after it is introduced to an environment is given by $P(t) = \frac{100}{(5-t)^2}$, $0 \leq t < 5$.

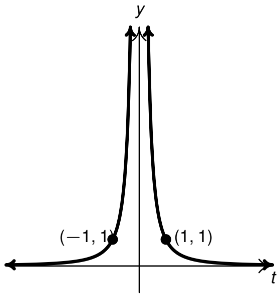
1. Find and interpret $P(0)$.
2. When will the population reach 100,000?
3. Graph $y = P(t)$.
4. Find and interpret the behavior of P as $t \to 5^-$.

Solution.

1. Substituting $t = 0$ gives $P(0) = \frac{100}{(5-0)^2} = 4$. Since t represents the number of days *after* the bacteria are introduced into the environment, $t = 0$ corresponds to the day the bacteria are introduced. Since $P(t)$ is measured in *thousands*, $P(t) = 4$ means 4000 bacteria are initially introduced into the environment.

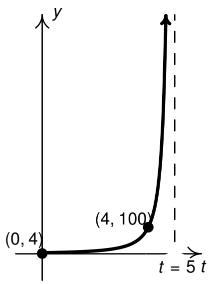
2. To find when the population reaches 100,000, we first need to remember that $P(t)$ is measured in *thousands*. In other words, 100,000 bacteria corresponds to $P(t) = 100$. Hence, we need to solve $P(t) = \frac{100}{(5-t)^2} = 100$. Clearing denominators and dividing by 100 gives $(5-t)^2 = 1$, which, after extracting square roots, produces $t = 4$ or $t = 6$. Of these two solutions, only $t = 4$ in our domain, so this is the solution we keep. Hence, it takes 4 days for the population of bacteria to reach 100,000.

3. After a slight re-write, we have $P(t) = \frac{100}{(5-t)^2} = \frac{100}{[(-1)(t-5)]^2} = \frac{100}{(t-5)^2}$. Using Theorem 1.1.1, we start with the graph of $y = \frac{1}{t^2}$ in Figure 1.1.9 on the left. After shifting the graph to the right 5 units and stretching it vertically by a factor of 100 (note, the graphs are not to scale!), we restrict the domain to



$y = \frac{1}{t^2}$; $(-1, 1)$, VA: $t = 0$, $(1, 1)$

Theorem 1.1.1



$y = P(t)$; $(4, 100)$, VA: $t = 5$

Figure 1.1.9

$0 \leq t < 5$ to arrive at the graph of $y = P(t)$ in the same figure on the right.

4. We see that as $t \to 5^-$, $P(t) \to \infty$. This means that the population of bacteria is increasing without bound as we near 5 days, which cannot actually happen. For this reason, $t = 5$ is called the 'doomsday' for this population. There is no way any environment can support infinitely many bacteria, so shortly before $t = 5$ the environment would collapse. □

Will all values excluded from the domain of a rational function produce vertical asymptotes in the graph? The short answer is 'no.' There are milder interruptions that can occur - holes in the graph - which we explore in our next example. To this end, we formalize the notion of *average velocity* - a concept we first encountered in Example **??** in Section **??**. In that example, the function $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. The function s an example of a **position function** since it provides information about *where* the rocket is at time t. In that example, we interpreted the average rate of change of s over an interval as the average velocity of the rocket over that interval. The average velocity provides two pieces of information: the average speed of the rocket along with the rocket's direction. Suppose we have a position function s defined over an interval containing some fixed time t_0. We can define the average velocity as a function of any time t other than t_0 as shown in Definition 1.1.5.

It is clear why we must exclude $t = t_0$ from the domain of $\overline{v}$ in Definition 1.1.5 since otherwise we would have a 0 in the denominator. What is interesting in this case however, is that substituting $t = t_0$ also produces 0 in the *numerator*. (Do you see why?) While '$\frac{0}{0}$' is undefined, it is more precisely called an 'indeterminate form' and is studied extensively in Calculus. We can nevertheless explore this function in the next example.

Example 1.1.3. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find and simplify an expression for the average velocity of the rocket between times t and 15.
2. Find and interpret $\overline{v}(14)$.
3. Graph $y = \overline{v}(t)$. Interpret the intercepts.
4. Interpret the behavior of $\overline{v}$ as $t \to 15$.

Definition 1.1.5. Suppose $s(t)$ gives the position of an object at time t and t_0 is a fixed time in the domain of s. The **average velocity** between time t and time t_0 is given by

$$\begin{aligned}\overline{v}(t) &= \frac{\Delta[s(t)]}{\Delta t} \\ &= \frac{s(t) - s(t_0)}{t - t_0},\end{aligned}$$

provided $t \neq t_0$.

Solution.

1. Using Definition 1.1.5 with $t_0 = 15$, we get:

$$\begin{aligned}\overline{v}(t) &= \frac{s(t) - s(15)}{t - 15} \\ &= \frac{(-5t^2 + 100t) - 375}{t - 15} \\ &= \frac{-5(t^2 - 20t + 75)}{t - 15} \\ &= \frac{-5(t-15)(t-5)}{t - 15} \\ &= \frac{-5\cancel{(t-15)}(t-5)}{\cancel{(t-15)}} \\ &= -5(t-5) = -5t + 25 \qquad t \neq 15\end{aligned}$$

   Since domain of s is $0 \leq t \leq 20$, our final answer is $\overline{v}(t) = -5t + 25$, for $t \in [0, 15) \cup (15, 20]$.

2. We find $\overline{v}(14) = -5(14) + 25 = -45$. This means between 14 and 15 seconds after launch, the rocket was traveling, on average a speed 45 feet per second *downwards*, or falling back to the Moon's surface.

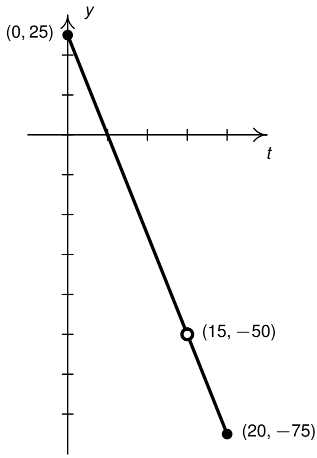
3. The graph of $\overline{v}(t)$ is a portion of the line $y = -5t + 25$. Since the domain of s is $[0, 20]$ and $\overline{v}(t)$ is not defined when $t = 15$, our graph is the line segment starting at $(0, 25)$ and ending at $(20, 75)$ with a hole at $(15, 50)$. The y-intercept is $(0, 25)$ which

f Note that the rocket has already started its descent at $t = 10$ seconds (see Example **??** in Section **??**.) However, the rocket is still at a higher altitude at when $t = 15$ than $t = 0$ which produces a positive *average* velocity.

means on average, the rocket is traveling 25 feet per second *upwards*.[f] To get the t-intercept, we set $\overline{v}(t) = -5t + 25 = 0$ and obtain $t = 5$. Hence, $\overline{v}(5) = 0$ or the average velocity between times $t = 5$ and $t = 15$ is 0. As you may recall, this is due to the rocket being at the same altitude (375 feet) at both times, hence, $\Delta[s(t)]$ and, hence $\overline{v}(t) = 0$.

4. From the graph, we see as $t \to 15$, $\overline{v}(t) \to -50$. (This is also borne out in the numerically in the tables below.) This means as we sample the average velocity between time $t_0 = 15$ and and times closer and closer to 15, the average velocity approaches -50. This value is how we define the **instantaneous velocity** - that is, *at* $t = 15$ seconds, the rocket is falling at a rate of 50 feet per second towards the surface of the Moon.

□



$y = \overline{v}(t)$

Figure 1.1.10

t	$\overline{v}(t)$
14.9	-49.5
14.99	-49.95
14.999	-49.995
15	undefined
15.001	-50.005
15.01	-50.05
15.1	-50.5

Table 1.1.5

If nothing else, Example 1.1.3 shows us that just because a value is excluded from the domain of a rational function doesn't mean there will be a vertical asymptote to the graph there. In this case, the factor $(t - 15)$ cancelled from the denominator, thereby effectively removing the threat of dividing by 0. It turns out, this situation generalizes to the Theorem 1.1.2.

In English, Theorem 1.1.2 says that if $x = c$ is not in the domain of r but, when we simplify $r(x)$, it no longer makes the denominator 0, then we have a hole at $x = c$. Otherwise, the line $x = c$ is a vertical asymptote of the graph of $y = r(x)$. Like many properties of rational functions, we owe Theorem 1.1.2 to Calculus, but that won't stop us from putting Theorem 1.1.2 to good use in the following example.

Example 1.1.4. For each function below:

- determine the values excluded from the domain.
- determine whether each excluded value corresponds to a vertical asymptote or hole in the graph.

- verify your answers using a graphing utility.
- describe the behavior of the graph near each excluded value using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{2x}{x^2 - 3}$
2. $g(t) = \dfrac{t^2 - t - 6}{t^2 - 9}$
3. $h(t) = \dfrac{t^2 - t - 6}{t^2 + 9}$
4. $r(t) = \dfrac{t^2 - t - 6}{t^2 + 6t + 9}$

Solution.

1. To use Theorem 1.1.2, we first find all of the real numbers which aren't in the domain of f. To do so, we solve $x^2 - 3 = 0$ and get $x = \pm\sqrt{3}$. Since the expression $f(x)$ is in lowest terms (can you see why?), there is no cancellation possible, and we conclude that the lines $x = -\sqrt{3}$ and $x = \sqrt{3}$ are vertical asymptotes to the graph of $y = f(x)$. The graphing utility verifies this claim, and from the graph in Figure 1.1.11, we see that as $x \to -\sqrt{3}^-$, $f(x) \to -\infty$, as $x \to -\sqrt{3}^+$, $f(x) \to \infty$, as $x \to \sqrt{3}^-$, $f(x) \to -\infty$, and finally as $x \to \sqrt{3}^+$, $f(x) \to \infty$. As a side note, the graph of f appears to be symmetric about the origin. Sure enough, we find: $f(-x) = \frac{2(-x)}{(-x)^2-3} = -\frac{2x}{x^2-3} = -f(x)$, proving f is odd.

2. As above, we find the values excluded from the domain of g by setting the denominator equal to 0. Solving $t^2 - 9 = 0$ gives $t = \pm 3$. In lowest terms $g(t) = \frac{t^2-t-6}{t^2-9} = \frac{(t-3)(t+2)}{(t-3)(t+3)} = \frac{t+2}{t+3}$. Since $t = -3$ continues to be a zero of the denominator in the reduced formula, we know the line $t = -3$ is a vertical asymptote to the graph of $y = g(t)$. Since $t = 3$ does not produce a '0' in the denominator of the reduced formula, we have a hole at $t = 3$. To find the y-coordinate of the hole, we substitute $t = 3$ into the reduced formula: $\frac{t+2}{t+3} = \frac{3+2}{3+3} = \frac{5}{6}$ so

Theorem 1.1.2. Location of Vertical Asymptotes and Holes:[a] Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros.[b] Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote to the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph.'

[b] In other words, $r(x)$ is in lowest terms.

the hole is at $\left(3, \frac{5}{6}\right)$. Graphing g (Figure 1.1.11) we can definitely see the vertical asymptote $t = -3$: as $t \to -3^-$, $g(t) \to \infty$ and as $t \to -3^+$, $g(t) \to -\infty$. Near $t = 3$, the graph seems to have no interruptions (but we know g is undefined at $t = 3$.) Since g appears to be increasing on $(-3, \infty)$, we write as $t \to 3^-$, $g(t) \to \frac{5}{6}^-$, and as $t \to 3^+$, $g(t) \to \frac{5}{6}^+$.

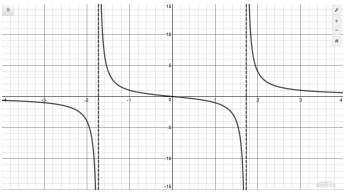


Figure 1.1.11: The graph of $y = f(x)$

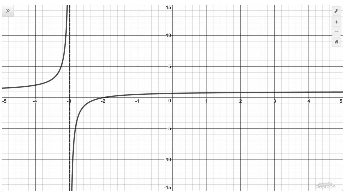


Figure 1.1.12: The graph of $y = g(t)$

3. Setting the denominator of the expression for $h(t)$ to 0 gives $t^2 + 9 = 0$, which ahs has no real solutions. Accordingly, the graph of $y = h(t)$ (at least as much as we can discern from the technology) is devoid of both vertical asymptotes and holes. See Figure 1.1.13

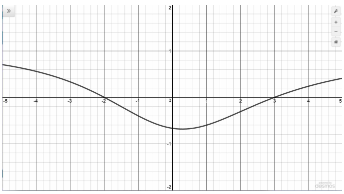


Figure 1.1.13: The graph of $y = h(t)$

4. Setting the denominator of $r(t)$ to zero gives the equation $t^2 + 6t + 4 = 0$. We get the (repeated!) solution $t = -2$. Simplifying, we see $r(t) = \frac{t^2-t-6}{t^2+4t+4} = \frac{(t-3)(t+2)}{(t+2)^2} = \frac{t-3}{t+2}$. Since $t = -2$ continues to produce a 0 in the denominator of the reduced function, we know $t = -2$ is a vertical asymptote to the graph. The calculator bears this out (Figure 1.1.14), and, moreover, we see that as $t \to -2^-$, $r(t) \to \infty$ and as $t \to -2^+$, $r(t) \to -\infty$.

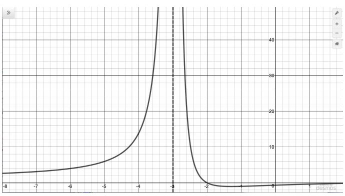


Figure 1.1.14: The graph of $y = r(t)$

□

1.1.3 End Behavior

Now that we've thoroughly discussed behavior near values excluded from the domains of rational functions, focus our attention on end behavior. We have already seen one example of this in the form of horizontal asymptotes. Our next example of the section gives us a real-world application of a horizontal asymptote.[9]

[g] Though the population below is more accurately modeled with the functions in Chapter **??**, we approximate it (using Calculus, of course!) using a rational function.

Example 1.1.5. The number of students $N(t)$ at local college who have had the flu t months after the semester begins can be modeled by:

$$N(t) = \frac{1500t + 50}{3t + 1}, \quad t \geq 0.$$

1. Find and interpret $N(0)$.
2. How long will it take until 300 students will have had the flu?
3. Use Theorem 1.1.1 to graph $y = N(t)$.
4. Find and interpret the behavior of N as $t \to \infty$.

Solution.

1. Substituting $t = 0$ gives $N(0) = \frac{1500(0)+50}{1+3(0)} = 50$. Since t represents the number of months since the beginning of the semester, $t = 0$ describes the state of the flu outbreak at the beginning of the semester. Hence, at the beginning of the semester, 50 students have had the flu.
2. We set $N(t) = \frac{1500t+50}{3t+1} = 300$ and solve. Clearing denominators gives $1500t + 50 = 300(3t + 1)$ from which we get $t = \frac{5}{12}$. This means it will take $\frac{5}{12}$ months, or about 13 days, for 300 students to have had the flu.
3. To graph $y = N(t)$, we first use long division to rewrite $N(t) = \frac{-450}{3t+1} + 500$. From there, we get

$$N(t) = -\frac{450}{3t+1} + 500 = \frac{-450}{3\left(t+\frac{1}{3}\right)} + 500 = \frac{-150}{t+\frac{1}{3}} + 500$$

   Using Theorem 1.1.1, we start with the graph of $y = \frac{1}{t}$ in **??** on the left and perform the following steps: shift the graph to the left by $\frac{1}{3}$ units, stretch the graph vertically by a factor of 150, reflect the graph across the t-axis, and finally, shift the graph up 500 units. As the domain of N is $t \geq 0$, we obtain the graph on the right of the same figure.

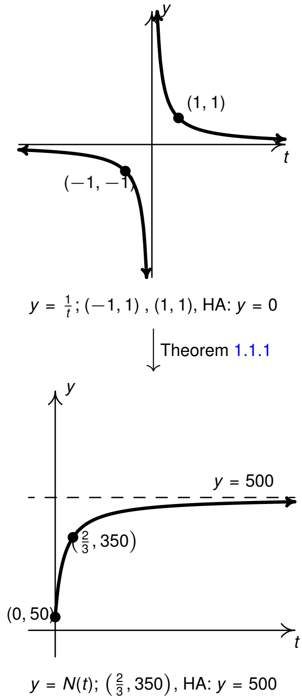


Figure 1.1.15

4. From the graph, we see as $t \to \infty$, $N(t) \to 500$. (More specifically, 500^-.) This means as time goes by, only a total of 500 students will have ever had the flu. $\Box$

We determined the horizontal asymptote to the graph of $y = N(t)$ in Example 1.1.5 by rewriting $N(t)$ into a form compatible with Theorem 1.1.1, and while there is nothing wrong with this approach, it will simply not work for general rational functions which cannot be rewritten this way. To that end, we revisit this problem using Theorem **??** from Section **??**. The end behavior of the numerator of $N(t) = \frac{1500t+50}{3t+1}$ is determined by its leading term, $1500t$, and the end behavior of the denominator is likewise determined by its leading term, $3t$. Hence, as $t \to \pm\infty$,

$$N(t) = \frac{1500t + 50}{3t + 1} \approx \frac{1500t}{3t} = 500.$$

Hence as $t \to \pm\infty$, $y = N(t) \to 500$, producing the horizontal asymptote $y = 500$. This same reasoning can be used in general to argue Theorem 1.1.3.

Theorem 1.1.3. Location of Horizontal Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.

- If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the[a] horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
- If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

[a]The use of the definite article will be justified momentarily.

So see why Theorem 1.1.3 works, suppose $r(x) = \frac{p(x)}{q(x)}$ where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$. As $x \to \pm\infty$, Theorem **??** gives $r(x) \approx \frac{ax^n}{bx^m}$, where n and m are the degrees of $p(x)$ and $q(x)$, respectively.

If the degree of $p(x)$ and the degree of $q(x)$ are the same, then $n = m$ so that $r(x) \approx \frac{ax^n}{bx^n} = \frac{a}{b}$, which means $y = \frac{a}{b}$ is the horizontal asymptote in this case.

If the degree of $p(x)$ is less than the degree of $q(x)$, then $n < m$, so $m - n$ is a positive number, and hence, $r(x) \approx \frac{ax^n}{bx^m} = \frac{a}{bx^{m-n}} \to 0$. As $x \to \pm\infty$, $r(x)$ is more or less a fraction with a constant numerator, a, but a denominator which is unbounded. Hence, $r(x) \to 0$ producing the horizontal asymptote $y = 0$.

If the degree of $p(x)$ is greater than the degree of $q(x)$, then $n > m$, and hence $n - m$ is a positive number and

$r(x) \approx \frac{ax^n}{bx^m} = \frac{ax^{n-m}}{b}$, which is a monomial function from Section **??**. As such, r becomes unbounded as $x \to \pm\infty$.

Note that in the two cases which produce horizontal asymptotes, the behavior of r is identical as $x \to \infty$ and $x \to -\infty$. Hence, if the graph of a rational function has a horizontal asymptote, there is only one.[h]

h We will (first) encounter functions with more than one horizontal asymptote in Chapter 2.1.

We put Theorem 1.1.3 to good use in the following example.

Example 1.1.6. For each function below:

- use Theorem **??** to analytically determine the horizontal asymptotes to the graph, if any.
- check your answers Theorem 1.1.3 and a graphing utility.
- describe the end behavior of the graph using proper notation.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $F(s) = \dfrac{5s}{s^2+1}$
2. $g(x) = \dfrac{x^2-4}{x+1}$
3. $h(t) = \dfrac{6t^3-3t+1}{5-2t^3}$
4. $r(x) = 2 - \dfrac{3x^2}{1-x^2}$

Solution.

1. Using Theorem **??**, we get as $s\pm\infty$, $F(s) = \frac{5s}{s^2+1} \approx \frac{5s}{s^2} = \frac{5}{s}$. Hence, as $s \to \infty$, $F(s) \to 0$, so $y = 0$ is a horizontal asymptote to the graph of $y = F(s)$. To check, we note that since the degree of the numerator of $F(s)$, 1, is less than the degree of the denominator, 2 Theorem 1.1.3 gives $y = 0$ is the horizontal asymptote. Graphically, we see as $s \to \pm\infty$, $F(s) \to 0$. More specifically, as $s \to -\infty$, $F(s) \to 0^-$ and as $s \to \infty$, $F(s) \to 0^+$. As a side note, the graph of F appears to be symmetric

about the origin. Indeed, $F(-s) = \frac{5(-s)}{(-s)^2+1} = -\frac{5s}{s^2+1}$ proving F is odd. See Figure 1.1.16.

2. As $x \to \pm\infty$, $g(x) = \frac{x^2-4}{x+1} \approx \frac{x^2}{x} = x$, and while $y = x$ is a line, it is not a horizontal line. Hence, we conclude the graph of $y = g(x)$ has no horizontal asymptotes. Sure enough, Theorem 1.1.3 supports this since the degree of the numerator of $g(x)$ is 2 which is greater than the degree of the denominator, 1. By, there is no horizontal asymptote. From the graph (Figure 1.1.17), we see that the graph of $y = g(x)$ doesn't appear to level off to a constant value, so there is no horizontal asymptote.[i]

[i] Sit tight! We'll revisit this function and its end behavior shortly.

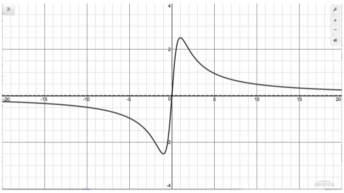


Figure 1.1.16: The graph of $y = F(s)$

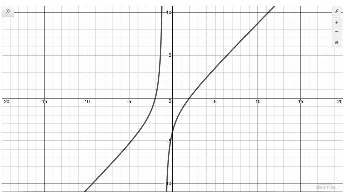


Figure 1.1.17: The graph of $y = g(x)$

3. We have $h(t) = \frac{6t^3-3t+1}{5-2t^3} \approx \frac{6t^3}{-2t^3} = -3$ as $t \to \pm\infty$, indicating a horizontal asymptote $y = -3$. Sure enough, since the degrees of the numerator and denominator of $h(t)$ are both three, Theorem 1.1.3 tells us $y = \frac{6}{-2} = -3$ is the horizontal asymptote. We see from the graph of $y = h(t)$ (Figure 1.1.18) that as $t \to -\infty$, $h(t) \to -3^+$, and as $t \to \infty$, $h(t) \to -3^-$.

4. If we apply Theorem **??** to the term $\frac{3x^2}{1-x^2}$ in the expression for $r(x)$, we find $\frac{3x^2}{1-x^2} \approx \frac{3x^2}{-x^2} = -3$ as $x \to \pm\infty$. It seems reasonable to conclude, then, that $r(x) = 2 - \frac{3x^2}{1-x^2} \approx 2-(-3) = 5$ so $y = 5$ is our horizontal asymptote. In order to use Theorem 1.1.3 as stated, however, we need to rewrite the expression $r(x)$ with a single denominator: $r(x) = 2 - \frac{3x^2}{1-x^2} = \frac{2(1-x^2)-3x^2}{1-x^2} = \frac{2-5x^2}{1-x^2}$. Now we apply Theorem 1.1.3 and note since the numerator and denominator have the same degree, we are guaranteed the horizontal asymptote is $y = \frac{-5}{-1} = 5$. These calculations are borne out graphically in Figure 1.1.19 where it appears as if as $x \to \pm\infty$, $r(x) \to 5^+$. As a final note, the graph of r appears to be symmetric about the y axis. We find $r(-x) = 2 - \frac{3(-x)^2}{1-(-x)^2} = 2 - \frac{3x^2}{1-x^2} = r(x)$, proving

r is even.

□

We close this section with a discussion of the *third* (and final!) kind of asymptote which can be associated with the graphs of rational functions. Let us return to the function $g(x) = \frac{x^2-4}{x+1}$ in Example 1.1.6. Performing long division,[j] we get $g(x) = \frac{x^2-4}{x+1} = x - 1 - \frac{3}{x+1}$. Since the term $\frac{3}{x+1} \to 0$ as $x \to \pm\infty$, it stands to reason that as x becomes unbounded, the function values $g(x) = x - 1 - \frac{3}{x+1} \approx x - 1$. Geometrically, this means that the graph of $y = g(x)$ should resemble the line $y = x - 1$ as $x \to \pm\infty$. We see this play out both numerically and graphically in Table 1.1.6 and Figure 1.1.20 respectively. (As usual, we have the asymptote $y = x - 1$ denoted by a dashed line.)

x	$g(x)$	$x-1$
-10	≈ -10.6667	-11
-100	≈ -100.9697	-101
-1000	≈ -1000.9970	-1001
-10000	≈ -10000.9997	-10001

x	$g(x)$	$x-1$
10	≈ 8.7273	9
100	≈ 98.9703	99
1000	≈ 998.9970	999
10000	≈ 9998.9997	9999

Table 1.1.6

The way we symbolize the relationship between the end behavior of $y = g(x)$ with that of the line $y = x - 1$ is to write 'as $x \to \pm\infty$, $g(x) \to x - 1$' in order to have some notational consistency with what we have done earlier in this section when it comes to end behavior.[k] In this case, we say the line $y = x - 1$ is a **slant asymptote**[l]

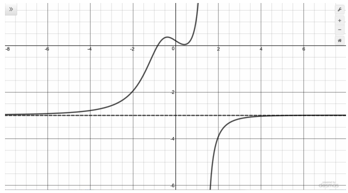


Figure 1.1.18: The graph of $y = h(t)$

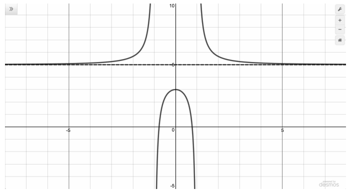


Figure 1.1.19: The graph of $y = r(x)$

j See the remarks following Theorem 1.1.3.

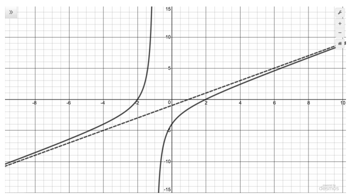


Figure 1.1.20

k Other notations include $g(x) \asymp x - 1$ or $g(x) \sim x - 1$.

l Also called an 'oblique' asymptote in some, ostensibly higher class (and more expensive), texts.

to the graph of $y = g(x)$. Informally, the graph of a rational function has a slant asymptote if, as $x \to \infty$ or as $x \to -\infty$, the graph resembles a non-horizontal, or 'slanted' line. Formally, we define a slant asymptote as in Theorem 1.1.6.

Definition 1.1.6. The line $y = mx + b$ where $m \neq 0$ is called a **slant asymptote** of the graph of a function $y = f(x)$ if as $x \to -\infty$ or as $x \to \infty$, $f(x) \to mx + b$.

A few remarks are in order. First, note that the stipulation $m \neq 0$ in Definition 1.1.6 is what makes the 'slant' asymptote 'slanted' as opposed to the case when $m = 0$ in which case we'd have a horizontal asymptote. Secondly, while we have motivated what me mean intuitively by the notation '$f(x) \to mx + b$,' like so many ideas in this section, the formal definition requires Calculus. Another way to express this sentiment, however, is to rephrase '$f(x) \to mx + b$' as '$[f(x) - (mx + b)] \to 0$.' In other words, the graph of $y = f(x)$ has the *slant* asymptote $y = mx + b$ if and only if the graph of $y = f(x) - (mx + b)$ has a *horizontal* asymptote $y = 0$. If we wanted to, we could introduce the notations $f(x) \to (mx + b)^+$ to mean $[f(x) - (mx + b)] \to 0^+$ and $f(x) \to (mx + b)^-$ to mean $[f(x) - (mx + b)] \to 0^-$, but these non-standard notations.[m]

[m] With the introduction of the symbol '?' in the next section, the authors feel we are in enough trouble already.

Our next task is to determine the conditions under which the graph of a rational function has a slant asymptote, and if it does, how to find it. In the case of $g(x) = \frac{x^2-4}{x+1}$, the degree of the numerator $x^2 - 4$ is 2, which is *exactly one more* than the degree if its denominator $x + 1$ which is 1. This results in a *linear* quotient polynomial, and it is this quotient polynomial which is the slant asymptote. Generalizing this situation gives us Theorem 1.1.4.[n]

[n] Once again, this theorem is brought to you courtesy of Theorem **??** and Calculus.

Theorem 1.1.4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is *exactly* one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In the same way that Theorem 1.1.3 gives us an easy way to see if the graph of a rational function $r(x) = \frac{p(x)}{q(x)}$ has a horizontal asymptote by comparing the degrees of the numerator and denominator, Theorem 1.1.4 gives us an easy way to check for slant asymptotes. Unlike Theorem 1.1.3, which gives us a quick way to *find* the horizontal asymptotes (if any exist), Theorem 1.1.4 gives us no such 'short-cut'. If a slant asymptote exists, we have no recourse but to use long division to find it.[o]

[o] That's OK, though. In the next section, we'll use long division to analyze end behavior and it's worth the effort!

Example 1.1.7. For each of the following functions:

- find the slant asymptote, if it exists.
- verify your answer using a graphing utility.
- investigate any apparent symmetry of the graph about the y-axis or origin.

1. $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$
2. $g(t) = \dfrac{t^2 - 4}{t - 2}$
3. $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$
4. $r(t) = 2t - 1 + \dfrac{4t^3}{1 - t^2}$

Solution.

1. The degree of the numerator is 2 and the degree of the denominator is 1, so Theorem 1.1.4 guarantees us a slant asymptote. To find it, we divide $1 - x = -x + 1$ into $x^2 - 4x + 2$ and get a quotient of $-x + 3$, so our slant asymptote is $y = -x + 3$. We confirm this graphically in Figure 1.1.21.

2. As with the previous example, the degree of the numerator $g(t) = \frac{t^2-4}{t-2}$ is 2 and the degree of the denominator is 1, so Theorem 1.1.4 applies. In this case,

$$g(t) = \frac{t^2 - 4}{t - 2} = \frac{(t+2)(t-2)}{(t-2)} = \frac{(t+2)\cancel{(t-2)}}{\cancel{(t-2)}^{\,1}} = t+2, \quad t \neq 2$$

so we have that the slant asymptote $y = t + 2$ is identical to the graph of $y = g(t)$ except at $t = 2$ (where the latter has a 'hole' at $(2, 4)$.) While the word 'asymptote' has the connotation of 'approaching but not equaling,' Definitions 1.1.6 and 1.1.4 allow for these extreme cases. See Figure 2.2.23.

3. For $h(x) = \frac{x^3+1}{x^2-4}$, the degree of the numerator is 3 and the degree of the denominator is 2 so again, we are guaranteed the existence of a slant asymptote. The long division $\left(x^3 + 1\right) \div \left(x^2 - 4\right)$ gives

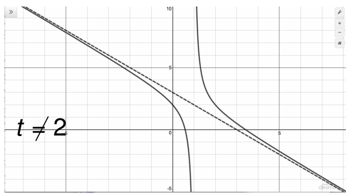


Figure 1.1.21: The graph of $y = f(x)$

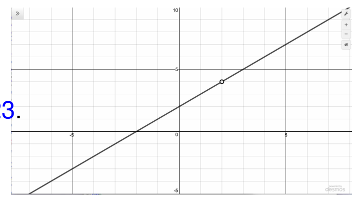


Figure 1.1.22: The graph of $y = g(t)$

a quotient of just x, so our slant asymptote is the line $y = x$. The graphing utility confirms this (Figure 1.1.23). Note the graph of h appears to be symmetric about the origin. We check $h(-x) = \frac{(-x)^3+1}{(-x)^2-4} = \frac{-x^3+1}{x^2-4} = -\frac{x^3-1}{x^2-4}$. However, $-h(x) = -\frac{x^3+1}{x^2-4}$, so it appears as if $h(-x) \neq -h(x)$ for all x. Checking $x = 1$, we find $h(1) = -\frac{2}{3}$ but $h(-1) = 0$ which shows the graph of h, is in fact, *not* symmetric about the origin.

4. For our last example, $r(t) = 2t - 1 + \frac{4t^3}{1-t^2}$, the expression $r(t)$ is not in the form to apply Theorem 1.1.4 directly. We can, nevertheless, appeal to the spirit of the theorem and use long division to rewrite the term $\frac{4t^3}{1-t^2} = -4t + \frac{4t}{1-t^2}$. We then get:

$$\begin{aligned} r(t) &= 2t - 1 + \frac{4t^3}{1-t^2} \\ &= 2t - 1 - 4t + \frac{4t}{1-t^2} \\ &= -2t - 1 + \frac{4t}{1-t^2} \end{aligned}$$

As $t \to \pm\infty$, Theorem **??** gives $\frac{4t}{1-t^2} \approx \frac{4t}{-t^2} = -\frac{4}{t} \to 0$. Hence, as $t \to \pm\infty$, $r(t) \to -2t - 1$, so $y = -2t - 1$ is the slant asymptote to the graph as confirmed by the graphing utility in Figure 1.1.24. From a distance, the graph of r appears to be symmetric about the origin. However, if we look carefully, we see the y-intercept is $(0, -1)$, as borne out by the computation $r(0) = -1$. Hence r cannot be odd. (Do you see why?)

□

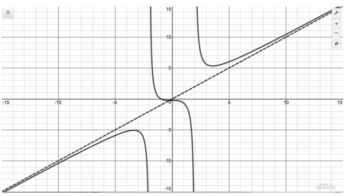


Figure 1.1.23: The graph of $y = h(x)$

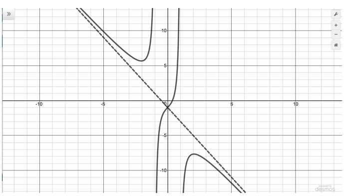


Figure 1.1.24: The graph of $y = r(t)$

Our last example gives a real-world application of a slant asymptote. The problem features the concept of **average profit**. The average profit, denoted $\overline{P}(x)$, is the total profit, $P(x)$, divided by the number of items sold, x. In English, the average profit tells us the profit made per item sold. It, along with average cost, is defined in Definition 1.1.7.

Definition 1.1.7. Let $C(x)$ and $P(x)$ represent the cost and profit to make and sell x items, respectively.

- The **average cost**, $\overline{C}(x) = \dfrac{C(x)}{x}$, $x > 0$.
  NOTE: The average cost is the cost per item produced.
- The **average profit**, $\overline{P}(x) = \dfrac{P(x)}{x}$, $x > 0$.
  NOTE: The average profit is the profit per item sold.

You'll explore average cost (and its relation to variable cost) in Exercise 37. For now, we refer the reader to to Example **??** in Section **??**.

Example 1.1.8. Recall the profit (in dollars) when x PortaBoy game systems are produced and sold is given by $P(x) = -1.5x^2 + 170x - 150$, $0 \leq x \leq 166$.

1. Find and simplify an expression for the average profit, $\overline{P}(x)$. What is the domain of $\overline{P}$?
2. Find and interpret $\overline{P}(50)$.
3. Determine the slant asymptote to the graph of $y = \overline{P}(x)$. Check your answer using a graphing utility.
4. Interpret the slope of the slant asymptote.

Solution.

1. We find $\overline{P}(x) = \frac{P(x)}{x} = \frac{-1.5x^2+170x-150}{x} = -1.5x + 170 + \frac{150}{x}$. Since the domain of P is $[0, 166]$ but $x \neq 0$, the domain of $\overline{P}$ is $(0, 166]$.
2. We find $\overline{P}(50) = -1.5(50) + 170 + \frac{150}{50} = 98$. This means that when 50 PortaBoy systems are sold, the average profit is \$98 per system.
3. Technically, the graph of $y = \overline{P}(x)$ has no slant asymptote since the domain of the function is restricted to $(0, 166]$. That being said, if we were to let $x \to \infty$, the term $\frac{150}{x} \to 0$, so we'd have $\overline{P}(x) \to -1.5x+170$. This means the slant asymptote would be $y = -1.5x+170$. We graph $y = \overline{P}(x)$ and $y = -1.5x + 170$ in Figure 1.1.25.
4. The slope of the slant asymptote $y = -1.5x + 170$ is -1.5. Since, ostensibly $\overline{P}(x) \approx -1.5x + 170$, this means that, as we sell more systems, the average profit is decreasing at about a rate of \$1.50 per system. If the number 1.5 sounds familiar to this problem situation, it should. In Example **??**

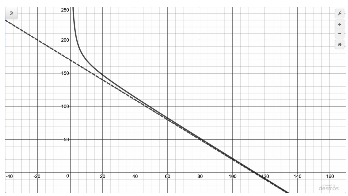
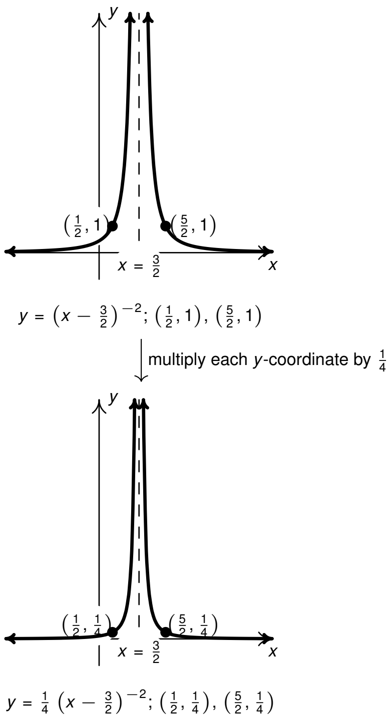


Figure 1.1.25

in Section **??**, we determined the slope of the demand function to be -1.5. In that situation, the -1.5 meant that in order to sell an additional system, the price had to drop by \$1.50. The fact the average profit is decreasing at more or less this same rate means the loss in profit per system can be attributed to the reduction in price needed to sell each additional system. [p] $\square$

p We generalize this result in Exercise 38.



y
$\left(\frac{1}{2}, 1\right)$
$\left(\frac{5}{2}, 1\right)$
$x = \frac{3}{2}$
x
$y = \left(x - \frac{3}{2}\right)^{-2}; \left(\frac{1}{2}, 1\right), \left(\frac{5}{2}, 1\right)$
multiply each y-coordinate by $\frac{1}{4}$
y
$\left(\frac{1}{2}, \frac{1}{4}\right)$
$\left(\frac{5}{2}, \frac{1}{4}\right)$
$x = \frac{3}{2}$
x
$y = \frac{1}{4}\left(x - \frac{3}{2}\right)^{-2}; \left(\frac{1}{2}, \frac{1}{4}\right), \left(\frac{5}{2}, \frac{1}{4}\right)$

Figure 1.1.5

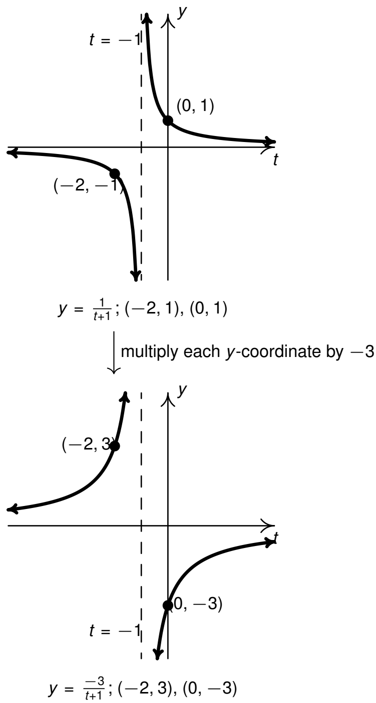
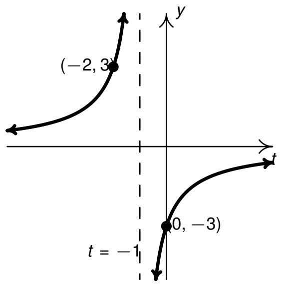
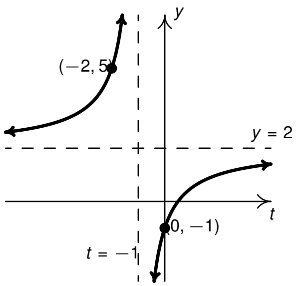

y
t = −1
(0, 1)
t
(−2, −1)
y = 1/(t+1); (−2, 1), (0, 1)
multiply each y-coordinate by −3
y
(−2, 3)
t
(0, −3)
t = −1
y = −3/(t+1); (−2, 3), (0, −3)

Figure 1.1.7



$y = \frac{-3}{t+1}$; $(-2, 3)$, $(0, -3)$, HA $y = 0$

add 2 to each y-coordinate



$y = \frac{-3}{t+1} + 2$; $(-2, 5)$, $(0, -1)$, HA $y = 2$

Figure 1.1.8

1.1.4 Exercises

(Review of Long Division):[a] In Exercises 1 - 6, use polynomial long division to perform the indicated division. Write the polynomial in the form $p(x) = d(x)q(x) + r(x)$.

[a] For more review, see Section ??.

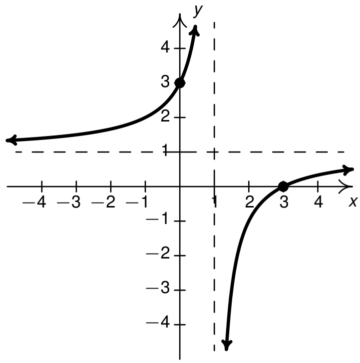
1. $\left(4x^2 + 3x - 1\right) \div (x - 3)$
2. $\left(2x^3 - x + 1\right) \div \left(x^2 + x + 1\right)$
3. $\left(5x^4 - 3x^3 + 2x^2 - 1\right) \div \left(x^2 + 4\right)$
4. $\left(-x^5 + 7x^3 - x\right) \div \left(x^3 - x^2 + 1\right)$
5. $\left(9x^3 + 5\right) \div (2x - 3)$
6. $\left(4x^2 - x - 23\right) \div \left(x^2 - 1\right)$

In Exercises 7 - 10, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 1.1.1. Track at least two points and the asymptotes. State the domain and range using interval notation.

7. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{1}{x - 2} + 1$
8. $f(x) = \dfrac{1}{x}$, $F(x) = \dfrac{2x}{x + 1}$
9. $f(x) = x^{-1}$, $F(x) = 4x(2x + 1)^{-1}$
10. $f(x) = x^{-2}$, $F(x) = -(x - 1)^{-2} + 3$

In Exercises 11 - 12, find a formula for each function below in the form $F(x) = \dfrac{a}{x - h} + k$.

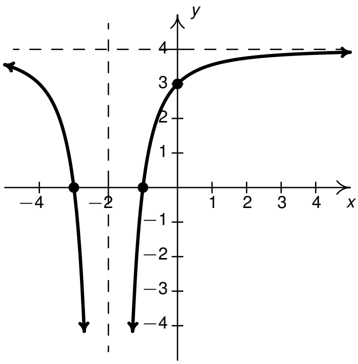
11. $y = F(x)$ in Figure 1.1.26
12. $y = F(x)$ in Figure 1.1.27



x-intercept $(3, 0)$, y-intercept $(0, 3)$

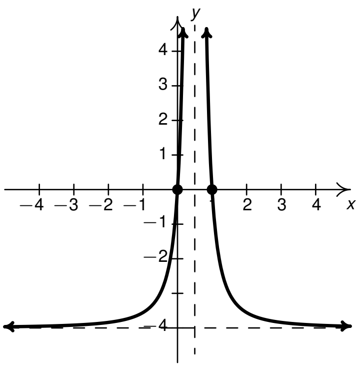
Figure 1.1.27

13. $y = F(x)$ in Figure 1.1.28
14. $y = F(x)$ in Figure 1.1.29



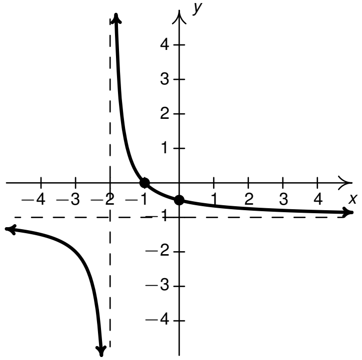
x-intercepts $(-3, 0)$, $(-1, 0)$, y-intercept $(0, 3)$

Figure 1.1.28



x-intercepts $(0, 0)$, $(1, 0)$, Vertical Asymptote: $x = \frac{1}{2}$

Figure 1.1.29



x-intercept $(-1, 0)$, y-intercept $\left(0, -\frac{1}{2}\right)$

Figure 1.1.26

In Exercises 13 - 14, find a formula for each function below in the form $F(x) = \dfrac{a}{(x-h)^2} + k$.

15. $f(x) = \dfrac{x}{3x-6}$

16. $f(x) = \dfrac{3+7x}{5-2x}$

17. $f(x) = \dfrac{x}{x^2+x-12}$

18. $g(t) = \dfrac{t}{t^2+1}$

19. $g(t) = \dfrac{t+7}{(t+3)^2}$

20. $g(t) = \dfrac{t^3+1}{t^2-1}$

21. $r(z) = \dfrac{4z}{z^2+4}$

22. $r(z) = \dfrac{4z}{z^2-4}$

23. $r(z) = \dfrac{z^2-z-12}{z^2+z-6}$

24. $f(x) = \dfrac{3x^2-5x-2}{x^2-9}$

25. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

26. $f(x) = \dfrac{x^3-3x+1}{x^2+1}$

27. $g(t) = \dfrac{2t^2+5t-3}{3t+2}$

28. $g(t) = \dfrac{-t^3+4t}{t^2-9}$

29. $g(t) = \dfrac{-5t^4-3t^3+t^2-10}{t^3-3t^2+3t-1}$

30. $r(z) = \dfrac{z^3}{1-z}$

31. $r(z) = \dfrac{18-2z^2}{z^2-9}$

32. $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1}$

33. The cost $C(p)$ in dollars to remove p% of the invasive Ippizuti fish species from Sasquatch Pond is:
$$C(p) = \frac{1770p}{100 - p}, \quad 0 \leq p < 100$$
    (a) Find and interpret $C(25)$ and $C(95)$.
    (b) What does the vertical asymptote at $x = 100$ mean within the context of the problem?
    (c) What percentage of the Ippizuti fish can you remove for \$40000?

34. In the scenario of Example 1.1.3, $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ gives the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff. For each of the times t_0 listed below, find and simplify a the formula for the average velocity $\overline{v}(t)$ between t and t_0 (see Definition 1.1.5) and use $\overline{v}(t)$ to find and interpret the instantaneous velocity of the rocket at $t = t_0$.
    (a) $t_0 = 5$
    (b) $t_0 = 9$
    (c) $t_0 = 10$
    (d) $t_0 = 11$

35. The population of Sasquatch in Portage County t years after the year 1803 is modeled by the function
$$P(t) = \frac{150t}{t + 15}.$$
Find and interpret the horizontal asymptote of the graph of $y = P(t)$ and explain what it means.

36. The cost in dollars, $C(x)$ to make x dOpi media players is $C(x) = 100x + 2000$, $x \geq 0$. You may wish to review the concepts of fixed and variable costs introduced in Example **??** in Section **??**.
    (a) Find a formula for the average cost $\overline{C}(x)$.
    (b) Find and interpret $\overline{C}(1)$ and $\overline{C}(100)$.

In Exercises 15 - 32, for the given rational function:
- State the domain.
- Identify any vertical asymptotes of the graph.
- Identify any holes in the graph.
- Find the horizontal asymptote, if it exists.
- Find the slant asymptote, if it exists.
- Graph the function using a graphing utility and describe the behavior near the asymptotes.

(c) How many dOpis need to be produced so that the average cost per dOpi is \$200?

(d) Interpret the behavior of $\overline{C}(x)$ as $x \to 0^+$.

(e) Interpret the behavior of $\overline{C}(x)$ as $x \to \infty$.

37. This exercise explores the relationships between fixed cost, variable cost, and average cost. The reader is encouraged to revisit Example **??** in Section **??** as needed. Suppose the cost in dollars $C(x)$ to make x items is given by $C(x) = mx + b$ where m and b are positive real numbers.

    (a) Show the fixed cost (the money spent even if no items are made) is b.

    (b) Show the variable cost (the increase in cost per item made) is m.

    (c) Find a formula for the average cost when making x items, $\overline{C}(x)$.

    (d) Show $\overline{C}(x) > m$ for all $x > 0$ and, moreover, $\overline{C}(x) \to m^+$ as $x \to \infty$.

    (e) Interpret $\overline{C}(x) \to m^+$ both geometrically and in terms of fixed, variable, and average costs.

38. Suppose the price-demand function for a particular product is given by $p(x) = mx + b$ where x is the number of items made and sold for $p(x)$ dollars. Here, $m < 0$ and $b > 0$. If the cost (in dollars) to make x of these products is also a linear function $C(x)$, show that the graph of the average profit function $\overline{P}(x)$ has a slant asymptote with slope m and interpret.

39. In Exercise **??** in Section **??**, we fit a few polynomial models to the following electric circuit data. The circuit was built with a variable resistor. For each of the following resistance values (measured in kilo-ohms, $k\Omega$), the corresponding power to the load (measured in milliwatts, mW) is given in Table 1.1.7. [q]

[q] The authors wish to thank Don Anthan and Ken White of Lakeland Community College for devising this problem and generating the accompanying data set.

Resistance: ($k\Omega$)	Power: (mW)
1.012	1.063
2.199	1.496
3.275	1.610
4.676	1.613
6.805	1.505
9.975	1.314

Table 1.1.7

Using some fundamental laws of circuit analysis mixed with a healthy dose of algebra, we can derive the actual formula relating power $P(x)$ to resistance x:

$$P(x) = \frac{25x}{(x+3.9)^2}, \quad x \geq 0.$$

(a) Graph the data along with the function $y = P(x)$ using a graphing utility.

(b) Use a graphing utility to approximate the maximum power that can be delivered to the load. What is the corresponding resistance value?

(c) Find and interpret the end behavior of $P(x)$ as $x \to \infty$.

40. Let $f(x) = \dfrac{ax^2 - c}{x+3}$. Find values for a and c so the graph of f has a hole at $(-3, 12)$.

41. Let $f(x) = \dfrac{ax^n - 4}{2x^2 + 1}$.

    (a) Find values for a and n so the graph of $y = f(x)$ has the horizontal asymptote $y = 3$.

    (b) Find values for a and n so the graph of $y = f(x)$ has the slant asymptote $y = 5x$.

42. Suppose p is a polynomial function and a is a real number. Define $r(x) = \dfrac{p(x) - p(a)}{x - a}$. Use the Factor Theorem, Theorem **??**, to prove the graph of $y = r(x)$ has a hole at $x = a$.

43. For each function $f(x)$ listed in Table 1.1.8, compute the average rate of change over the indicated interval. [r] What trends do you observe? How do your answers manifest themselves graphically? How do your results compare with those of Exercise **??** in Section **??**?

44. In his now famous 1919 dissertation The Learning Curve Equation, Louis Leon Thurstone presents

See Definition **??** in Section **??** for a review of this concept, as needed. [r]

$f(x)$	[0.9, 1.1]	[0.99, 1.01]	[0.999, 1.001]	[0.9999, 1.0001]
x^{-1}				
x^{-2}				
x^{-3}				
x^{-4}				

Table 1.1.8

a rational function which models the number of words a person can type in four minutes as a function of the number of pages of practice one has completed. [s] Using his original notation and original language, we have $Y = \frac{L(X+P)}{(X+P)+R}$ where L is the predicted practice limit in terms of speed units, X is pages written, Y is writing speed in terms of words in four minutes, P is equivalent previous practice in terms of pages and R is the rate of learning. In Figure 5 of the paper, he graphs a scatter plot and the curve $Y = \frac{216(X+19)}{X+148}$. Discuss this equation with your classmates. How would you update the notation? Explain what the horizontal asymptote of the graph means. You should take some time to look at the original paper. Skip over the computations you don't understand yet and try to get a sense of the time and place in which the study was conducted.

s This paper, which is now in the public domain and can be found here[2], is from a bygone era when students at business schools took typing classes on manual typewriters.

1.1.5 Answers

1. $4x^2 + 3x - 1 = (x-3)(4x+15) + 44$

2. $2x^3 - x + 1 = \left(x^2 + x + 1\right)(2x-2) + (-x+3)$

3. $5x^4 - 3x^3 + 2x^2 - 1 = \left(x^2+4\right)\left(5x^2 - 3x - 18\right) + (12x + 71)$

4. $-x^5 + 7x^3 - x = \left(x^3 - x^2 + 1\right)\left(-x^2 - x + 6\right) + \left(7x^2 - 6\right)$

5. $9x^3 + 5 = (2x-3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

6. $4x^2 - x - 23 = \left(x^2 - 1\right)(4) + (-x - 19)$

7. $F(x) = \dfrac{1}{x-2} + 1$

   Domain: $(-\infty, 2) \cup (2, \infty)$
   Range: $(-\infty, 1) \cup (1, \infty)$
   Vertical asymptote: $x = 2$
   Horizontal asymptote: $y = 1$
   See Figure 1.1.30

8. $F(x) = \dfrac{2x}{x+1} = \dfrac{-2}{x+1} + 2$

   Domain: $(-\infty, -1) \cup (-1, \infty)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -1$
   Horizontal asymptote: $y = 2$
   See Figure 1.1.31

9. $F(x) = 4x(2x+1)^{-1} = \dfrac{4x}{2x+1} = \dfrac{-1}{x+\frac{1}{2}} + 2$

   Domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, \infty\right)$
   Range: $(-\infty, 2) \cup (2, \infty)$
   Vertical asymptote: $x = -\frac{1}{2}$
   Horizontal asymptote: $y = 2$
   See Figure 1.1.32

10. $F(x) = -(x-1)^{-2} + 3 = \dfrac{-1}{(x-1)^2} + 3$

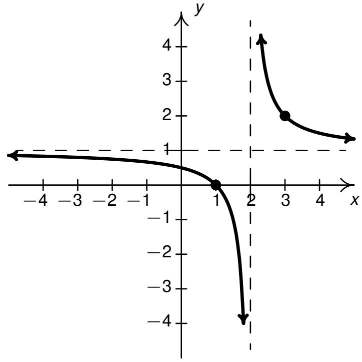


Figure 1.1.30

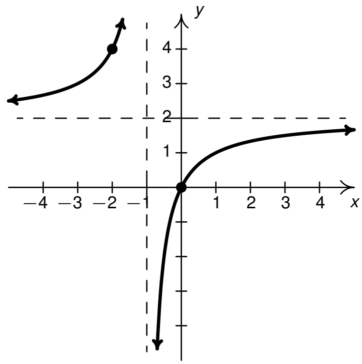


Figure 1.1.31

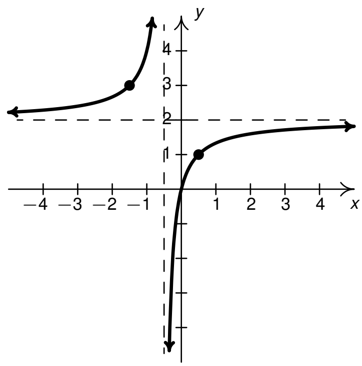


Figure 1.1.32

Domain: $(-\infty, 1) \cup (1, \infty)$
Range: $(-\infty, 3) \cup (3, \infty)$
Vertical asymptote: $x = 1$
Horizontal asymptote: $y = 3$
See Figure 1.1.33

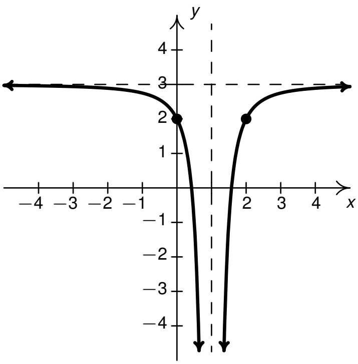


Figure 1.1.33

11. $F(x) = \dfrac{1}{x+2} - 1$

12. $F(x) = \dfrac{-2}{x-1} + 1$

13. $F(x) = \dfrac{-4}{(x+2)^2} + 4$

14. $F(x) = \dfrac{1}{\left(x - \frac{1}{2}\right)^2} - 4$

15. $f(x) = \dfrac{x}{3x-6}$
Domain: $(-\infty, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = \frac{1}{3}$
As $x \to -\infty$, $f(x) \to \frac{1}{3}^-$
As $x \to \infty$, $f(x) \to \frac{1}{3}^+$

16. $f(x) = \dfrac{3+7x}{5-2x}$
Domain: $(-\infty, \frac{5}{2}) \cup (\frac{5}{2}, \infty)$
Vertical asymptote: $x = \frac{5}{2}$
As $x \to \frac{5}{2}^-$, $f(x) \to \infty$
As $x \to \frac{5}{2}^+$, $f(x) \to -\infty$
No holes in the graph
Horizontal asymptote: $y = -\frac{7}{2}$
As $x \to -\infty$, $f(x) \to -\frac{7}{2}^+$
As $x \to \infty$, $f(x) \to -\frac{7}{2}^-$

17.
$$f(x) = \frac{x}{x^2 + x - 12} = \frac{x}{(x+4)(x-3)}$$
Domain: $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -4$, $x = 3$
As $x \to -4^-$, $f(x) \to -\infty$
As $x \to -4^+$, $f(x) \to \infty$
As $x \to 3^-$, $f(x) \to -\infty$
As $x \to 3^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$

18. $g(t) = \dfrac{t}{t^2 + 1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

19. $g(t) = \dfrac{t + 7}{(t + 3)^2}$
Domain: $(-\infty, -3) \cup (-3, \infty)$
Vertical asymptote: $t = -3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
[t]As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$

[t] This is hard to see on the calculator, but trust me, the graph is below the t-axis to the left of $t = -7$.

20. $g(t) = \dfrac{t^3 + 1}{t^2 - 1} = \dfrac{t^2 - t + 1}{t - 1}$
Domain: $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
Vertical asymptote: $t = 1$
As $t \to 1^-$, $g(t) \to -\infty$

As $t \to 1^+$, $g(t) \to \infty$
Hole at $(-1, -\frac{3}{2})$
Slant asymptote: $y = t$
As $t \to -\infty$, the graph is below $y = t$
As $t \to \infty$, the graph is above $y = t$

21. $r(z) = \dfrac{4z}{z^2 + 4}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

22. $r(z) = \dfrac{4z}{z^2 - 4} = \dfrac{4z}{(z+2)(z-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
Vertical asymptotes: $z = -2$, $z = 2$
As $z \to -2^-$, $r(z) \to -\infty$
As $z \to -2^+$, $r(z) \to \infty$
As $z \to 2^-$, $r(z) \to -\infty$
As $z \to 2^+$, $r(z) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$

23. $r(z) = \dfrac{z^2 - z - 12}{z^2 + z - 6} = \dfrac{z - 4}{z - 2}$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
Vertical asymptote: $z = 2$
As $z \to 2^-$, $r(z) \to \infty$
As $z \to 2^+$, $r(z) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$
Horizontal asymptote: $y = 1$
As $z \to -\infty$, $r(z) \to 1^+$
As $z \to \infty$, $r(z) \to 1^-$

24. $f(x) = \dfrac{3x^2 - 5x - 2}{x^2 - 9} = \dfrac{(3x+1)(x-2)}{(x+3)(x-3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $x = -3, x = 3$
As $x \to -3^-, f(x) \to \infty$
As $x \to -3^+, f(x) \to -\infty$
As $x \to 3^-, f(x) \to -\infty$
As $x \to 3^+, f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 3$
As $x \to -\infty, f(x) \to 3^+$
As $x \to \infty, f(x) \to 3^-$

25. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
Vertical asymptote: $x = 2$
As $x \to 2^-, f(x) \to -\infty$
As $x \to 2^+, f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$

26. $f(x) = \dfrac{x^3 - 3x + 1}{x^2 + 1}$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = x$
As $x \to -\infty$, the graph is above $y = x$
As $x \to \infty$, the graph is below $y = x$

27. $g(t) = \dfrac{2t^2 + 5t - 3}{3t + 2}$
Domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
Vertical asymptote: $t = -\frac{2}{3}$
As $t \to -\frac{2}{3}^-, g(t) \to \infty$
As $t \to -\frac{2}{3}^+, g(t) \to -\infty$

No holes in the graph
Slant asymptote: $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to -\infty$, the graph is above $y = \frac{2}{3}t + \frac{11}{9}$
As $t \to \infty$, the graph is below $y = \frac{2}{3}t + \frac{11}{9}$

28. $g(t) = \dfrac{-t^3 + 4t}{t^2 - 9} = \dfrac{-t^3 + 4t}{(t-3)(t+3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
Vertical asymptotes: $t = -3$, $t = 3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to -\infty$
As $t \to 3^-$, $g(t) \to \infty$
As $t \to 3^+$, $g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -t$
As $t \to -\infty$, the graph is above $y = -t$
As $t \to \infty$, the graph is below $y = -t$

29. $$\begin{aligned} g(t) &= \frac{-5t^4 - 3t^3 + t^2 - 10}{t^3 - 3t^2 + 3t - 1} \\ &= \frac{-5t^4 - 3t^3 + t^2 - 10}{(t-1)^3} \end{aligned}$$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptotes: $t = 1$
As $t \to 1^-$, $g(t) \to \infty$
As $t \to 1^+$, $g(t) \to -\infty$
No holes in the graph
Slant asymptote: $y = -5t - 18$
As $t \to -\infty$, the graph is above $y = -5t - 18$
As $t \to \infty$, the graph is below $y = -5t - 18$

30. $r(z) = \dfrac{z^3}{1 - z}$
Domain: $(-\infty, 1) \cup (1, \infty)$
Vertical asymptote: $z = 1$
As $z \to 1^-$, $r(z) \to \infty$
As $z \to 1^+$, $r(z) \to -\infty$
No holes in the graph
No horizontal or slant asymptote

As $z \to -\infty$, $r(z) \to -\infty$
As $z \to \infty$, $r(z) \to -\infty$

31. $r(z) = \dfrac{18 - 2z^2}{z^2 - 9} = -2$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
No vertical asymptotes
Holes in the graph at $(-3, -2)$ and $(3, -2)$
Horizontal asymptote $y = -2$
As $z \to \pm\infty$, $r(z) = -2$

32. $r(z) = \dfrac{z^3 - 4z^2 - 4z - 5}{z^2 + z + 1} = z - 5$
Domain: $(-\infty, \infty)$
No vertical asymptotes
No holes in the graph
Slant asymptote: $y = z - 5$
$r(z) = z - 5$ everywhere.

33. (a) $C(25) = 590$ means it costs \$590 to remove 25% of the fish and and $C(95) = 33630$ means it would cost \$33630 to remove 95% of the fish from the pond.

(b) The vertical asymptote at $x = 100$ means that as we try to remove 100% of the fish from the pond, the cost increases without bound; i.e., it's impossible to remove all of the fish.

(c) For \$40000 you could remove about 95.76% of the fish.

34. (a) $\overline{v}(t) = \frac{s(t)-s(5)}{t-5} = \frac{-5t^2+100t-375}{t-5} = -5t+75$, $t \neq 5$. The instantaneous velocity of the rocket when $t_0 = 5$ is $-5(5) + 75 = 50$ meaning it is traveling 50 feet per second upwards.

(b) $\overline{v}(t) = \frac{s(t)-s(9)}{t-9} = \frac{-5t^2+100t-495}{t-9} = -5t+55$, $t \neq 9$. The instantaneous velocity of the rocket when $t_0 = 9$ is $-5(9) + 55 = 10$, so the rocket has slowed to 10 feet per second (but still heading up.)

(c) $\overline{v}(t) = \frac{s(t)-s(10)}{t-10} = \frac{-5t^2+100t-495}{t-10} = -5t+50$, $t \neq 10$. The instantaneous velocity of the rocket when $t_0 = 10$ is $-5(10)+50 = 0$, so the rocket has momentarily stopped! In Example **??**, we learned the rocket reaches its maximum height when $t = 10$ seconds, which means the rocket must change direction from heading up to coming back down, so it makes sense that for this instant, its velocity is 0.

(d) $\overline{v}(t) = \frac{s(t)-s(11)}{t-11} = \frac{-5t^2+100t-495}{t-11} = -5t+45$, $t \neq 11$. The instantaneous velocity of the rocket when $t_0 = 11$ is $-5(11)+45 = -10$ meaning the rocket has, indeed, changed direction and is heading downwards at a rate of 10 feet per second. (Note the symmetry here between this answer and our answer when $t = 9$.)

35. The horizontal asymptote of the graph of $P(t) = \frac{150t}{t+15}$ is $y = 150$ and it means that the model predicts the population of Sasquatch in Portage County will never exceed 150.

36. (a) $\overline{C}(x) = \frac{100x+2000}{x} = 100 + \frac{2000}{x}$, $x > 0$.

(b) $\overline{C}(1) = 2100$ and $\overline{C}(100) = 120$. When just 1 dOpi is produced, the cost per dOpi is \$2100, but when 100 dOpis are produced, the cost per dOpi is \$120.

(c) $\overline{C}(x) = 200$ when $x = 20$. So to get the cost per dOpi to \$200, 20 dOpis need to be produced.

(d) As $x \to 0^+$, $\overline{C}(x) \to \infty$. This means that as fewer and fewer dOpis are produced, the cost per dOpi becomes unbounded. In this situation, there is a fixed cost of \$2000 ($C(0) = 2000$), we are trying to spread that \$2000 over fewer and fewer dOpis.

(e) As $x \to \infty$, $\overline{C}(x) \to 100^{+}$. This means that as more and more dOpis are produced, the cost per dOpi approaches \$100, but is always a little more than \$100. Since \$100 is the variable cost per dOpi ($C(x) = \underline{100}x + 2000$), it means that no matter how many dOpis are produced, the average cost per dOpi will always be a bit higher than the variable cost to produce a dOpi. As before, we can attribute this to the \$2000 fixed cost, which factors into the average cost per dOpi no matter how many dOpis are produced.

37. (a) The cost to make 0 items is $C(0) = m(0) + b = b$. Hence, so the fixed costs are b.

(b) $C(x) = mx + b$ is a linear function with slope $m > 0$. Hence, the cost increases at a rate of m dollars per item made. Hence, the variable cost is m.

(c) $\overline{C}(x) = \frac{C(x)}{x} = \frac{mx+b}{x} = m + \frac{b}{x}$ for $x > 0$.

(d) Since $b > 0$, $\overline{C}(x) = m + \frac{b}{x} > m$ for $x > 0$. As $x \to \infty$, $\frac{b}{x} \to 0$ so $\overline{C}(x) = m + \frac{b}{x} \to m$.

(e) Geometrically, the graph of $y = \overline{C}(x)$ has a horizontal asymptote $y = m$, the variable cost. In terms of costs, as more items are produced, the affect of the fixed cost on the average cost, $\frac{b}{x}$ falls away so that the average cost per item approaches the variable cost to make each item.

38. If $p(x) = mx + b$ and $C(x)$ is linear, say $C(x) = rx + s$, then we can compute the the profit function (in general) as: $P(x) = xp(x) - C(x) = x(mx + b) - (rx + s)$ which simplifies to $P(x) = mx^2 + (b -$

$r)x - s$. Hence, the average profit $\overline{P}(x) = \frac{P(x)}{x} = \frac{mx^2+(b-r)x-s}{x} = mx + (b-r) - \frac{s}{x}$. We see that as $x \to \infty$, $\frac{s}{x} \to 0$ so $\overline{P}(x) \approx mx + (b-r)$. Hence, $y = mx + (b-r)$ is the slant asymptote to $y = \overline{P}(x)$. This means that as more items are sold, the average profit is decreasing at approximately the same rate as the price function is decreasing, m dollars per item. That is, to sell one additional item, we drop the price $p(x)$ by m dollars which results in a drop in the average profit by approximately m dollars.

39. (a) See Figure 1.1.34

    (b) The maximum power is approximately 1.603 mW which corresponds to 3.9 $k\Omega$.

    (c) As $x \to \infty$, $P(x) \to 0^+$ which means as the resistance increases without bound, the power diminishes to zero.

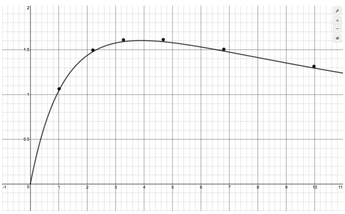


Figure 1.1.34

40. $a = -2$ and $c = -18$ so $f(x) = \dfrac{-2x^2 + 18}{x + 3}$.

41. (a) $a = 6$ and $n = 2$ so $f(x) = \dfrac{6x^2 - 4}{2x^2 + 1}$

    (b) $a = 10$ and $n = 3$ so $f(x) = \dfrac{10x^3 - 4}{2x^2 + 1}$.

42. If we define $f(x) = p(x) - p(a)$ then f is a polynomial function with $f(a) = p(a) - p(a) = 0$. The Factor Theorem guarantees $(x - a)$ is a factor of $f(x)$, that is, $f(x) = p(x) - p(a) = (x - a)q(x)$ for some polynomial $q(x)$. Hence, $r(x) = \frac{p(x)-p(a)}{x-a} = \frac{(x-a)q(x)}{x-a} = q(x)$ so the graph of $y = r(x)$ is the same as the graph of the polynomial $y = q(x)$ except for a hole when $x = a$.

43. The slope of the curves near $x = 1$ matches the exponent on x. This exactly what we saw in Exercise **??** in Section **??**. See Table 1.1.9

$f(x)$	[0.9, 1.1]	[0.99, 1.01]	[0.999, 1.001]	[0.9999, 1.0001]
x^{-1}	-1.0101	-1.0001	≈ -1	≈ -1
x^{-2}	-2.0406	-2.0004	≈ -2	≈ -2
x^{-3}	-3.1021	-3.0010	≈ -3	≈ -3
x^{-4}	-4.2057	-4.0020	≈ -4	≈ -4

Table 1.1.9

1.2 Graphs of Rational Functions

In Section 1.1, we learned about the types of behaviors to expect from graphs of rational functions: vertical asymptotes, holes in graph, horizontal and slant asymptotes. Moreover, Theorems 1.1.2, 1.1.3 and 1.1.4 tell us exactly when and where these behaviors will occur. We used graphing technology extensively in the last section to help us verify results. In this section, we delve more deeply into graphing rational functions with the goal of sketching relatively accurate graphs without the aid of a graphing utility. Your instructor will ultimately communicate the level of detail expected out of you when it comes to producing graphs of rational functions; what we provide here is an attempt to glean as much information about the graph as possible given the analytical tools at our disposal.

One of the standard tools we will use is the sign diagram which was first introduced in Section **??**, and then revisited in Section **??**. In these sections, to construct a sign diagram for a function f, we first found the zeros of f. The zeros broke the domain of f into a series of intervals. We determined the sign of $f(x)$ over the *entire* interval by finding the sign of $f(x)$ for just *one* test value per interval. The theorem that justified this approach was the Intermediate Value Theorem, Theorem **??**, which says that *continuous* functions cannot change their sign between two values unless there is a zero between those two values.

This strategy fails in general with rational functions. Indeed, the very first function we studied in Section 1.1, $r(x) = \frac{1}{x}$ changes sign between $x = -1$ and $x = 1$, but there is no zero between these two values - instead, the graph changes sign across a vertical asymptote. We could also well imagine the graph of a rational function having a hole where an x-intercept should be[a] It turns out that with Calculus we can show rational functions are continuous on their domains. What this means for us is

a Take $f(x) = \frac{x^2}{x}$, for instance.

Box 1.2.1. Steps for Constructing a Sign Diagram for a Rational Function

Suppose f is a rational function.

1. Place any values excluded from the domain of f on the number line with an '‽' above them.[a]
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

[a] '‽' is a nonstandard symbol called the interrobang[b]. We use this symbol to convey a sense of surprise, caution and wonderment - an appropriate attitude to take when approaching these points.

when we construct sign diagrams, we need to choose test values on either side of values excluded from the domain in addition to checking around zeros.[b] See Box 1.2.1.

[b] Since excluded values are zeros of the denominator, we can think of this as really just generalizing what we already do.

We now present our procedure for graphing rational functions in Box 1.2.2 and apply it to a few exhaustive examples. Please note that we decrease the amount of detail given in the explanations as we move through the examples. The reader should be able to fill in any details in those steps which we have abbreviated.

Example 1.2.1. Sketch a detailed graph of $f(x) = \dfrac{3x}{x^2 - 4}$.

Solution. We follow the six step procedure outlined above.

1. To find the domain, we first find the excluded values. To that end, we solve $x^2 - 4 = 0$ and find $x = \pm 2$. Our domain is $\{x \in \mathbb{R} \mid x \neq \pm 2\}$, or, using interval notation, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

2. We check if $f(x)$ is in lowest terms by factoring: $f(x) = \frac{3x}{(x-2)(x+2)}$. There are no common factors

Box 1.2.2. Steps for Graphing Rational Functions

Suppose r is a rational function.

1. Find the domain of r.
2. Reduce $r(x)$ to lowest terms, if applicable.
3. Determine the location of any vertical asymptotes or holes in the graph, if they exist.
4. Find the axis intercepts, if they exist.
5. Analyze the end behavior of r. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph.[a]

[a] It doesn't hurt to check for symmetry at this point, if convenient.

c The sign diagram in step 6 will also determine the behavior near the vertical asymptotes.

which means $f(x)$ is already in lowest terms.

3. Per Theorem 1.1.2, vertical asymptotes and holes in the graph come from values excluded from the domain of f. The two numbers excluded from the domain of f are $x = -2$ and $x = 2$ and since $f(x)$ didn't reduce, Theorem 1.1.2 tells us $x = -2$ and $x = 2$ are vertical asymptotes of the graph. We can actually go a step further at this point and determine exactly how the graph approaches the asymptote near each of these values. Though not absolutely necessary,[c] it is good practice for those heading off to Calculus. For the discussion that follows, we use the factored form of $f(x) = \frac{3x}{(x-2)(x+2)}$.

   - *The behavior of $y = f(x)$ as $x \to -2$:* Suppose $x \to -2^-$. If we were to build a table of values, we'd use x-values a little less than -2, say -2.1, -2.01 and -2.001. While there is no harm in actually building a table like we did in Section 1.1, we want to develop a 'number sense' here. Let's think about each factor in the formula of $f(x)$ as we imagine substituting a number like $x = -2.000001$ into $f(x)$. The quantity $3x$ would be very close to -6, the quantity $(x - 2)$ would be very close to -4, and the factor $(x + 2)$ would be very close to 0. More specifically, $(x + 2)$ would be a little less than 0, in this case, -0.000001. We will call such a number a 'very small $(-)$', 'very small' meaning close to zero in absolute value. So, mentally, as $x \to -2^-$,

$$\begin{aligned} f(x) &= \frac{3x}{(x-2)(x+2)} \\ &\approx \frac{-6}{(-4)\,(\text{very small }(-))} \\ &= \frac{3}{2\,(\text{very small }(-))} \end{aligned}$$

Now, the closer x gets to -2, the smaller $(x + 2)$ will become, so even though we are multiplying our 'very small $(-)$' by 2, the denominator will continue to get smaller and smaller, and remain negative. The result is a fraction whose numerator is positive, but whose denominator is very small and negative. Mentally,

$$f(x) \approx \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

The term 'very big $(-)$' means a number with a large absolute value which is negative.[d] What all of this means is that as $x \to -2^-$, $f(x) \to -\infty$. Now suppose we wanted to determine the behavior of $f(x)$ as $x \to -2^+$. If we imagine substituting something a little larger than -2 in for x, say -1.999999, we mentally estimate

[d] The actual retail value of $f(-2.000001)$ is approximately $-1{,}500{,}000$.

$$f(x) \approx \frac{-6}{(-4)\,(\text{very small }(+))} = \frac{3}{2\,(\text{very small }(+))} \approx \frac{3}{\text{very small }(+)} \approx \text{very big }(+)$$

We conclude that as $x \to -2^+$, $f(x) \to \infty$.

- *The behavior of $y = f(x)$ as $x \to 2$:* Consider $x \to 2^-$. We imagine substituting $x = 1.999999$. Approximating $f(x)$ as we did above, we get

$$f(x) \approx \frac{6}{(\text{very small }(-))\,(4)} = \frac{3}{2\,(\text{very small }(-))} \approx \frac{3}{\text{very small }(-)} \approx \text{very big }(-)$$

We conclude that as $x \to 2^-$, $f(x) \to -\infty$. Similarly, as $x \to 2^+$, we imagine substituting $x = 2.000001$ to get $f(x) \approx \frac{3}{\text{very small (+)}} \approx$ very big (+). So as $x \to 2^+$, $f(x) \to \infty$.

We interpret this graphically in Figure 1.2.1.

4. To find the x-intercepts of the graph, we set $y = f(x) = 0$. Solving $\frac{3x}{(x-2)(x+2)} = 0$ results in $x = 0$. Since $x = 0$ is in our domain, $(0, 0)$ is the x-intercept. This is also the y-intercept,[e] as we can quickly verify since $f(0) = \frac{3(0)}{0^2-4} = 0$.

e Per Exercise **??**, functions can have at most one y-intercept. Since $(0, 0)$ is on the graph, it is *the* y-intercept.

5. Next, we determine the end behavior of the graph of $y = f(x)$. Since the degree of the numerator is 1, and the degree of the denominator is 2, Theorem 1.1.3 tells us that $y = 0$ is the horizontal asymptote. As with the vertical asymptotes, we can glean more detailed information using 'number sense'. For the discussion below, we use the formula $f(x) = \frac{3x}{x^2-4}$.

   - *The behavior of $y = f(x)$ as $x \to -\infty$:* If we were to make a table of values to discuss the behavior of f as $x \to -\infty$, we would substitute very 'large' negative numbers in for x, say for example, $x = -1$ billion. The numerator $3x$ would then be -3 billion, whereas the denominator x^2-4 would be $(-1 \text{ billion})^2-4$, which is pretty much the same as $1(\text{billion})^2$. Hence,

$$f(-1 \text{ billion}) \approx \frac{-3 \text{ billion}}{1(\text{billion})^2} \approx -\frac{3}{\text{billion}}$$
$$\approx \text{very small } (-)$$

     Notice that if we substituted in $x = -1$ trillion, essentially the same kind of cancellation would happen, and we would be left with an even 'smaller' negative number. This not only confirms the fact that as $x \to -\infty$, $f(x) \to 0$, it tells us that $f(x) \to 0^-$. In other words, the graph of $y = f(x)$ is a little bit *below* the x-axis as we move to the far left.

- *The behavior of $y = f(x)$ as $x \to \infty$:* On the flip side, we can imagine substituting very large positive numbers in for x and looking at the behavior of $f(x)$. For example, let $x =$ 1 billion. Proceeding as before, we get

$$f\,(1 \text{ billion}) \approx \frac{3 \text{ billion}}{1(\text{billion})^2} \approx \frac{3}{\text{billion}}$$
$$\approx \text{very small } (+)$$

The larger the number we put in, the smaller the positive number we would get out. In other words, as $x \to \infty$, $f(x) \to 0^+$, so the graph of $y = f(x)$ is a little bit *above* the x-axis as we look toward the far right.

We interpret these findings graphically in Figure 1.2.2.

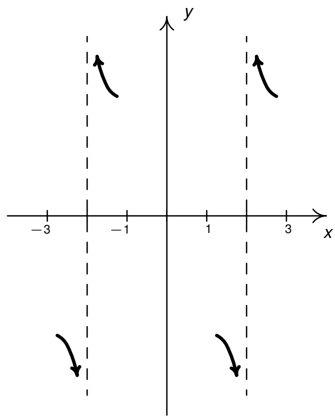


Figure 1.2.1: behavior near $x = \pm 2$

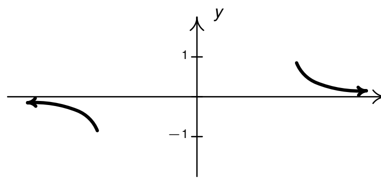


Figure 1.2.2: end behavior

6. Lastly, we construct a sign diagram for $f(x)$ (Figure 1.2.3). The x-values excluded from the domain of f are $x = \pm 2$, and the only zero of f is $x = 0$. Displaying these appropriately on the number line gives us four test intervals, and we choose the test values[f] $x = -3$, $x = -1$, $x = 1$ and $x = 3$. We find $f(-3)$ is $(-)$, $f(-1)$ is $(+)$, $f(1)$ is $(-)$ and $f(3)$ is $(+)$. As we begin our sketch, it certainly appears as if the graph could be symmetric about the origin. Taking a moment to check for symmetry, we find $f(-x) = \frac{3(-x)}{(-x)^2-4} = -\frac{3x}{x^2-4} = -f(x)$. Hence, f is odd and the graph of $y = f(x)$ is symmetric about the origin. Putting all of our work together, we get the graph in Figure 1.2.4.

□

[f] In this particular case, we don't need test values since our analysis of the behavior of f near the vertical asymptotes and our end behavior analysis have given us the signs on each of the test intervals. In general, however, this won't always be the case, so for demonstration purposes, we continue with our usual construction.

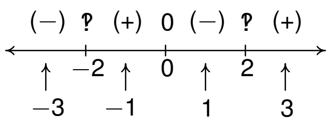


Figure 1.2.3

Something important to note about the above example is that while $y = 0$ is the horizontal asymptote, the graph of f actually crosses the x-axis at $(0, 0)$. The myth that graphs of rational functions can't cross their horizontal asymptotes is completely false,[g] as we shall see again in our next example.

Example 1.2.2. Sketch a detailed graph of

$$g(t) = \frac{2t^2 - 3t - 5}{t^2 - t - 6}$$

Solution.

1. To find the values excluded from the domain of g, we solve $t^2 - t - 6 = 0$ and find $t = -2$ and $t = 3$. Hence, our domain is $\{t \in \mathbb{R} \mid t \neq -23\}$, or using interval notation: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$.

2. To check if $g(t)$ is in lowest terms, we factor: $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$. There is no cancellation, so $g(t)$ is in lowest terms.

3. Since $g(t)$ was given to us in lowest terms, we have, once again by Theorem 1.1.2 vertical asymptotes $t = -2$ and $t = 3$. Keeping in mind $g(t) = \frac{(2t-5)(t+1)}{(t-3)(t+2)}$, we proceed to our analysis near each of these values.

   - *The behavior of $y = g(t)$ as $t \to -2$:* As $t \to -2^-$, we imagine substituting a number a little bit less than -2. We have

     $$g(t) \approx \frac{(-9)(-1)}{(-5)(\text{very small }(-))} \approx \frac{9}{\text{very small }(+)}$$
     $$\approx \text{very big }(+)$$

     so as $t \to -2^-$, $g(t) \to \infty$. On the flip side, as $t \to -2^+$, we get

     $$g(t) \approx \frac{9}{\text{very small }(-)}$$
     $$\approx \text{very big }(-)$$

     so $g(t) \to -\infty$.

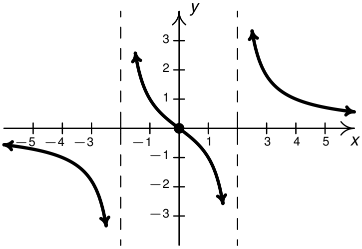


Figure 1.2.4

[g] That's why we called it a MYTH!

- *The behavior of $y = g(t)$ as $x \to 3$:* As $t \to 3^-$, we imagine plugging in a number just shy of 3. We have

$$g(t) \approx \frac{(1)(4)}{(\text{ very small }(-))(5)} \approx \frac{4}{\text{very small }(-)}$$
$$\approx \text{very big }(-)$$

Hence, as $t \to 3^-$, $g(t) \to -\infty$. As $t \to 3^+$, we get

$$g(t) \approx \frac{4}{\text{very small }(+)} \approx \text{very big }(+)$$

so $g(x) \to \infty$.

We interpret this analysis graphically in Figure 1.2.5.

4. To find the t-intercepts we set $y = g(t) = 0$. Using the factored form of $g(t)$ above, we find the zeros to be the solutions of $(2t-5)(t+1) = 0$. We obtain $t = \frac{5}{2}$ and $t = -1$. Since both of these numbers are in the domain of g, we have two t-intercepts, $\left(\frac{5}{2}, 0\right)$ and $(-1, 0)$. To find the y-intercept, we find $y = g(0) = \frac{5}{6}$, so our y-intercept is $\left(0, \frac{5}{6}\right)$.

5. Since the degrees of the numerator and denominator of $g(t)$ are the same, we know from Theorem 1.1.3 that we can find the horizontal asymptote of the graph of g by taking the ratio of the leading terms coefficients, $y = \frac{2}{1} = 2$. However, if we take the time to do a more detailed analysis, we will be able to reveal some 'hidden' behavior which would be lost otherwise. Using long division, we may rewrite $g(t)$ as $g(t) = 2 - \frac{t-7}{t^2-t-6}$. We focus our attention on the term $\frac{t-7}{t^2-t-6}$.

- *The behavior of $y = g(t)$ as $t \to -\infty$:* If imagine substituting $t = -1$ billion into $\frac{t-7}{t^2-t-6}$, we estimate $\frac{t-7}{t^2-t-6} \approx \frac{-1 \text{ billion}}{1\text{billion}^2} = \frac{-1}{\text{billion}} \approx$ very small $(-)$.[h] Hence,

[h] We are once again using the fact that for polynomials, end behavior is determined by the leading term, so in the denominator, the t^2 term dominates the t and constant terms.

$$g(t) = 2 - \frac{t-7}{t^2-t-6} \approx 2 - \text{very small }(-)$$
$$= 2 + \text{very small }(+)$$

Hence, as $t \to -\infty$, the graph is a little bit *above* the line $y = 2$.

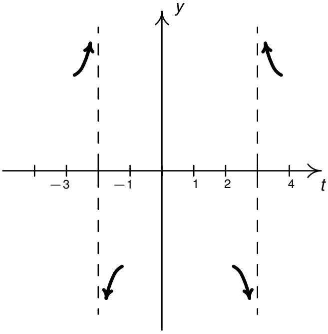


Figure 1.2.5: behavior near $t = -2$ and $t = 3$

- *The behavior of $y = g(t)$ as $t \to \infty$.* To consider $\frac{t-7}{t^2-t-6}$ as $t \to \infty$, we imagine substituting $t = 1$ billion and, going through the usual mental routine, find

$$\frac{t-7}{t^2-t-6} \approx \text{very small }(+)$$

Hence, $g(t) \approx 2 -$ very small $(+)$, so the graph is just *below* the line $y = 2$ as $t \to \infty$.

We sketch the end behavior in Figure 1.2.6.

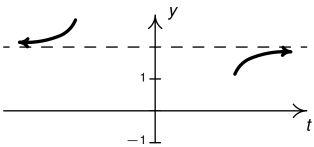


Figure 1.2.6: end behavior

6. Finally we construct our sign diagram (Figure 1.2.7). We place an '‽' above $t = -2$ and $t = 3$, and a '0' above $t = \frac{5}{2}$ and $t = -1$. Choosing test values in the test intervals gives us $g(t)$ is $(+)$ on the intervals $(-\infty, -2)$, $\left(-1, \frac{5}{2}\right)$ and $(3, \infty)$, and $(-)$ on the intervals $(-2, -1)$ and $\left(\frac{5}{2}, 3\right)$. As we piece together all of the information, it stands to reason the graph must cross the horizontal asymptote at some point after $t = 3$ in order for it to approach $y = 2$ from underneath.[i] To find where $y = g(t)$ intersects $y = 2$, we solve $g(t) = 2 - \frac{t-7}{t^2-t-6} = 2$ and get $t - 7 = 0$, or $t = 7$. Note that $t - 7$ is the remainder when $2t^2 - 3t - 5$ is divided by $t^2 - t - 6$, so it makes sense that for $g(t)$ to equal the quotient 2, the remainder from the division must be 0. Sure enough, we find $g(7) = 2$. The location of the t-intercepts alone dashes all hope of the function being even or odd (do you see why?) so we skip the symmetry check in this case. See Figure 1.2.8

i This subtlety would have been missed had we skipped the long division and subsequent end behavior analysis.

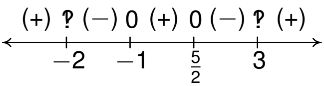


Figure 1.2.7

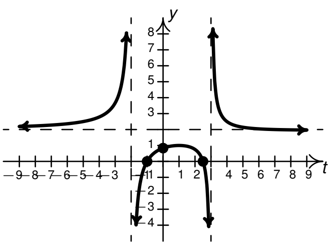


Figure 1.2.8

□

More can be said about the graph of $y = g(t)$ in Figure 1.2.8. It stands to reason that g must attain a local minimum at some point past $t = 7$ since the graph of g crosses through $y = 2$ at $(2, 7)$ but approaches $y = 2$ from below as $t \to \infty$. Calculus verifies a local minimum at $(13, 1.96)$. We invite the reader to verify this claim using a graphing utility.

Example 1.2.3. Sketch a detailed graph of

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2}$$

Solution.

1. Solving $x^2 + 3x + 2 = 0$ gives $x = -2$ and $x = -1$ as our excluded values. Hence, the domain is $\{x \in \mathbb{R} \,|\, x \neq -1, -2\}$ or, using interval notation, $(-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

2. To reduce $h(x)$, we need to factor the numerator and denominator. To factor the numerator, we use the techniques[j] set forth in Section **??** and get

$$h(x) = \frac{2x^3 + 5x^2 + 4x + 1}{x^2 + 3x + 2} = \frac{(2x+1)(x+1)^2}{(x+2)(x+1)}$$
$$= \frac{(2x+1)(x+1)^{\cancelto{1}{2}}}{(x+2)\cancel{(x+1)}} = \frac{(2x+1)(x+1)}{x+2}$$

   Note we can use this formula for $h(x)$ in our analysis of the graph of h as long as we are not substituting $x = -1$. To make this exclusion specific, we write $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$.

3. From Theorem 1.1.2, we know that since $x = -2$ is a zero of the denominator of the reduced form of $h(x)$, we have a vertical asymptote there. As for $x = -1$, the factor $(x + 1)$ was canceled from the denominator when we reduced $h(x)$, so there will be a hole when $x = -1$. To find the y-coordinate of the hole, we substitute $x = -1$ into $\frac{(2x+1)(x+1)}{x+2}$, per Theorem 1.1.2 and get 0. Hence, we have a

[j] Bet you never thought you'd never see *that* stuff again before the Final Exam!

hole on the x-axis at $(-1, 0)$. It should make you uncomfortable plugging $x = -1$ into the reduced formula for $h(x)$, especially since we've made such a big deal about the stipulation '$x \neq -1$' that goes along with that formula. What we are really doing is taking a Calculus short-cut to the more detailed kind of analysis near $x = -1$ which we will show below.

- *The behavior of $y = h(x)$ as $x \to -2$:* As $x \to -2^-$, we imagine substituting a number a little bit less than -2. We have $h(x) \approx \frac{(-3)(-1)}{(\text{very small }(-))} \approx \frac{3}{(\text{very small }(-))} \approx$ very big $(-)$ thus as $x \to -2^-$, $h(x) \to -\infty$. On the other side of -2, as $x \to -2^+$, we find that $h(x) \approx \frac{3}{\text{very small }(+)} \approx$ very big $(+)$, so $h(x) \to \infty$.
- *The behavior of $y = h(x)$ as $x \to -1$.* As $x \to -1^-$, we imagine plugging in a number a bit less than $x = -1$. We have $h(x) \approx \frac{(-1)(\text{very small }(-))}{1} =$ very small $(+)$ Hence, as $x \to -1^-$, $h(x) \to 0^+$. This means that as $x \to -1^-$, the graph is a bit above the point $(-1, 0)$. As $x \to -1^+$, we get $h(x) \approx \frac{(-1)(\text{very small }(+))}{1} =$ very small $(-)$. This gives us that as $x \to -1^+$, $h(x) \to 0^-$, so the graph is a little bit lower than $(-1, 0)$ here.

We interpret this graphically below on the left.

4. To find the x-intercepts, as usual, we set $h(x) = 0$ and solve. Solving $\frac{(2x+1)(x+1)}{x+2} = 0$ yields $x = -\frac{1}{2}$ and $x = -1$. The latter isn't in the domain of h, in fact, we know there is a hole at $(-1, 0)$, so we exclude it. Our only x-intercept is $\left(-\frac{1}{2}, 0\right)$. To find the y-intercept, we set $x = 0$. Since $0 \neq -1$, we can use the reduced formula for $h(x)$ and we get $h(0) = \frac{1}{2}$ for a y-intercept of $\left(0, \frac{1}{2}\right)$.
5. For end behavior, we note that the degree of the numerator of $h(x)$, $2x^3 + 5x^2 + 4x + 1$, is 3 and the

degree of the denominator, $x^2 + 3x + 2$, is 2 so by Theorem 1.1.4, the graph of $y = h(x)$ has a slant asymptote. For $x \to \pm\infty$, we are far enough away from $x = -1$ to use the reduced formula, $h(x) = \frac{(2x+1)(x+1)}{x+2}$, $x \neq -1$. To perform long division, we multiply out the numerator and get $h(x) = \frac{2x^2+3x+1}{x+2}$, $x \neq -1$, and rewrite $h(x) = 2x - 1 + \frac{3}{x+2}$, $x \neq -1$. By Theorem 1.1.4, the slant asymptote is $y = 2x - 1$, and to better see *how* the graph approaches the asymptote, we focus our attention on the term generated from the remainder, $\frac{3}{x+2}$.

- *The behavior of $y = h(x)$ as $x \to -\infty$:* Substituting $x = -1$ billion into $\frac{3}{x+2}$, we get the estimate $\frac{3}{-1 \text{ billion}} \approx$ very small $(-)$. Hence, $h(x) = 2x - 1 + \frac{3}{x+2} \approx 2x - 1 +$ very small $(-)$. This means the graph of $y = h(x)$ is a little bit *below* the line $y = 2x - 1$ as $x \to -\infty$.

- *The behavior of $y = h(x)$ as $x \to \infty$:* If $x \to \infty$, then $\frac{3}{x+2} \approx$ very small $(+)$. This means $h(x) \approx 2x - 1 +$ very small $(+)$, or that the graph of $y = h(x)$ is a little bit *above* the line $y = 2x - 1$ as $x \to \infty$.

We sketch the end behavior below on the right.

6. To make our sign diagram (Figure 1.2.11), we place an '?' above $x = -2$ and $x = -1$ and a '0' above $x = -\frac{1}{2}$. On our four test intervals, we find $h(x)$ is $(+)$ on $(-2, -1)$ and $\left(-\frac{1}{2}, \infty\right)$ and $h(x)$ is $(-)$ on $(-\infty, -2)$ and $\left(-1, -\frac{1}{2}\right)$. Putting all of our work together yields the graph in Figure 1.2.12.

To find if the graph of h ever crosses the slant asymptote, we solve $h(x) = 2x - 1 + \frac{3}{x+2} = 2x - 1$. This results in $\frac{3}{x+2} = 0$, which has no solution.[k] Hence, the graph of h never crosses its slant asymptote.[l] □

Our last graphing example is challenging in that our six step process provides us little information to work with.

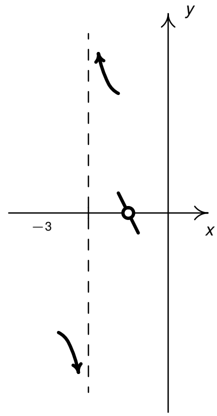


Figure 1.2.9: behavior near $x = -2$ and $x = -1$

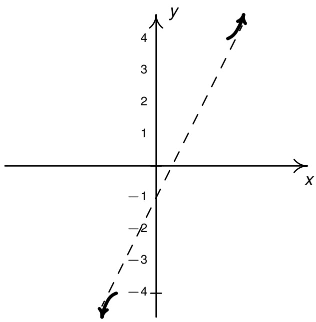


Figure 1.2.10: end behavior

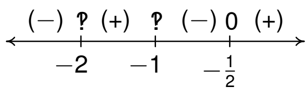


Figure 1.2.11

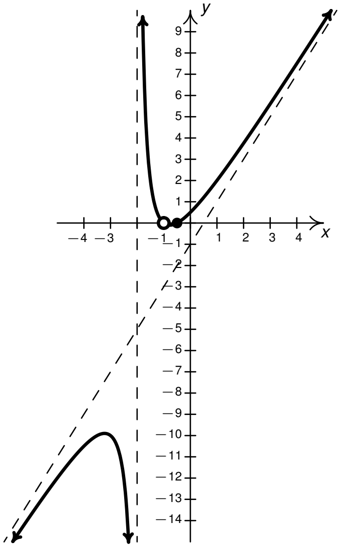


Figure 1.2.12

k Alternatively, the remainder after the long division was $r = 3$ which is never 0.

l But rest assured, some graphs do!

m This won't stop us from giving it the old community college try, however!

Example 1.2.4. Sketch the graph of $r(x) = \dfrac{x^4+1}{x^2+1}$.

Solution.

1. The denominator x^2+1 is never zero which means there are no excluded values. The domain is $\mathbb{R}$, or using interval notation, $(-\infty, \infty)$.

2. With no real zeros in the denominator, x^2+1 is an irreducible quadratic. Our only hope of reducing $r(x)$ is if x^2+1 is a factor of x^4+1. Performing long division gives us
$$\frac{x^4+1}{x^2+1} = x^2 - 1 + \frac{2}{x^2+1}$$
The remainder is not zero so $r(x)$ is already reduced.

3. Since there are no numbers excluded from the domain of r, there are no vertical asymptotes or holes in the graph of r.

4. To find the x-intercept, we'd set $r(x) = 0$. Since there are no real solutions to $x^4+1 = 0$, we have no x-intercepts. Since $r(0) = 1$, we do get $(0, 1)$ as the y-intercept.

5. For end behavior, we note that since the degree of the numerator is exactly *two* more than the degree of the denominator, neither Theorems 1.1.3 nor 1.1.4 apply.[m] We know from our attempt to reduce $r(x)$ that we can rewrite $r(x) = x^2 - 1 + \frac{2}{x^2+1}$, so we focus our attention on the term corresponding to the remainder, $\frac{2}{x^2+1}$ It should be clear that as $x \to \pm\infty$, $\frac{2}{x^2+1} \approx$ very small (+), which means $r(x) \approx x^2 - 1 +$ very small (+). So the graph $y = r(x)$ is a little bit *above* the graph of the parabola $y = x^2 - 1$ as $x \to \pm\infty$. See Figure 1.2.13.

6. There isn't much work to do for a sign diagram for $r(x)$, since its domain is all real numbers and it has no zeros. Our sole test interval is $(-\infty, \infty)$,

and since we know $r(0) = 1$, we conclude $r(x)$ is $(+)$ for all real numbers. We check for symmetry, and find $r(-x) = \frac{(-x)^4+1}{(-x)^2+1} = \frac{x^4+1}{x^2+1} = r(x)$, so r is even and, hence, the graph is symmetric about the y-axis. It may be tempting at this point to call it quits, reach for a graphing utility, or ask someone who knows Calculus.[n] It turns out, we can do a little bit better. Recall from Section **??**, that when $|x| < 1$ but $x \neq 0$, $x^4 < x^2$, hence $x^4 + 1 < x^2 + 1$. This means for $-1 < x < 0$ and $0 < x < 1$, $r(x) = \frac{x^4+1}{x^2+1} < 1$. Since we know $r(0) = 1$, this means the graph of $y = r(x)$ must fall to either side before heading off to ∞. This means $(0, 1)$ is a local maximum and, moreover, there are at least two local minimums, one on either side of $(0, 1)$. We invite the reader to confirm this using a graphing utility. See Figure 1.2.14.

n This is exactly what the authors did in the Third Edition. Special thanks go to Erik Boczko from Ohio University for showing us that, in fact, we could do more with this example algeraically.

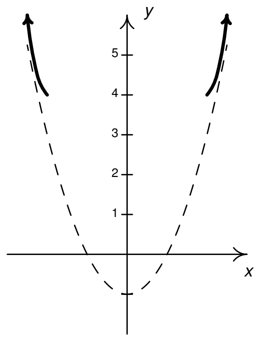


Figure 1.2.13: end behavior

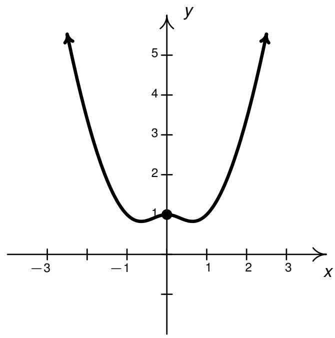


Figure 1.2.14: sketch of $y = r(x)$

□

Our last example turns the tables and invites us to write formulas for rational function given their graphs.

Example 1.2.5. Write formulas for rational functions $r(x)$ and $F(x)$ given in Figure 1.2.15 and Figure 1.2.16.

Solution. The good news is the graph of r closely resembles the graph of F, so once we know an expression for $r(x)$, we should be able to modify it to obtain $F(x)$. We are told r is a rational function, so we know there are polynomial functions p and q so that $r(x) = \frac{p(x)}{q(x)}$. We know from Theorem **??** that we can factor $p(x)$ and $q(x)$ completely in terms of their leading coefficients and their zeros. For simplicity's sake, we assume neither p nor q has any non-real zeros.

We focus our attention first on finding an expression for $p(x)$. When finding the x-intercepts, we look for the zeros of r by solving $r(x) = \frac{p(x)}{q(x)} = 0$. This equation quickly reduces to solving $p(x) = 0$. Since $\left(\frac{5}{3}, 0\right)$ is an x-intercept of the graph, we know $x = \frac{5}{3}$ is a zero of r, and, hence,

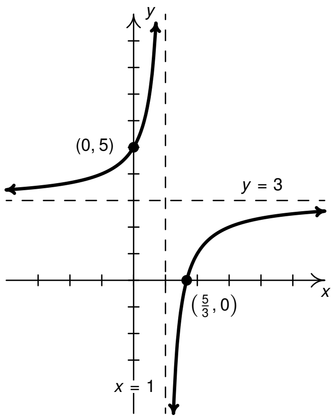


Figure 1.2.15: $y = r(x)$

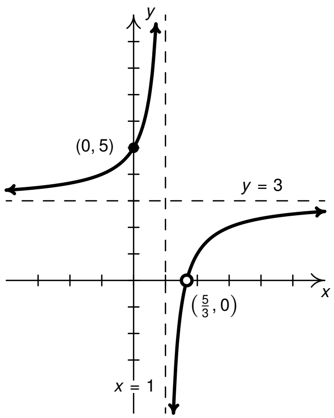


Figure 1.2.16: $y = F(x)$

a zero of p. Since we are shown no other x-intercepts, we assume r, hence p have no other real zeros (and no non-real zeros by our assumption.) Theorem **??** gives $p(x) = a\left(x - \frac{5}{3}\right)^m$ where a is the leading coefficient of $p(x)$ and m is the multiplicity of the zero $x = \frac{5}{3}$. Since the graph of $y = r(x)$ crosses through the x-axis in what appears to be a fairly linear fashion at $\left(\frac{5}{3}, 0\right)$, it seems reasonable to set $m = 1$. Hence, $p(x) = a\left(x - \frac{5}{3}\right)$.

Next, we focus our attention on finding $q(x)$. Theorem 1.1.2 $x = 1$ comes from a factor of $(x - 1)$ in the denominator of $r(x)$. This means $(x - 1)$ is a factor of $q(x)$. Since there are no other vertical asymptotes or holes in the graph, $x = 1$ is the only real zero, hence (per our assumption) only zero of q. At this point, we have $q(x) = b(x - 1)^m$ where b is the leading coefficient of $q(x)$ and m is the multiplicity of the zero $x = 1$. Since the graph of r has the *horizontal* asymptote $y = 3$,Theorem 1.1.4 tells us two things: first, degree of q must match the degree of p; second, the ratio $\frac{a}{b} = 3$. Hence, the degree of q is 1 so that:

$$
\begin{aligned}
r(x) &= \frac{a\left(x - \frac{5}{3}\right)}{b(x - 1)} \\
&= \frac{a}{b}\left(\frac{x - \frac{5}{3}}{x - 1}\right) \\
&= 3\left(\frac{x - \frac{5}{3}}{x - 1}\right) \\
&= \frac{3x - 5}{x - 1}.
\end{aligned}
$$

We have yet to use the y-intercept, $(0, 5)$. In this case, we use it as a partial check: $r(0) = \frac{3(0)-5}{0-1} = 5$, as required. We can sketch $y = r(x)$ by hand, or with a graphing utility, to give a better check of our work.

Now it is time to find a formula for $F(x)$. The graphs of r and F look identical except the graph has a hole in the graph at $\left(\frac{5}{3}, 0\right)$ instead of an x-intercept. Theorem 1.1.2 tells us this happens because a factor of $\left(x - \frac{5}{3}\right)$

cancels from the denominator when the formula for $F(x)$ is reduced. Hence, we reverse this process and multiply the numerator and denominator of our expression for $r(x)$ by $\left(x - \frac{5}{3}\right)$:

$$\begin{aligned}
F(x) &= r(x) \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)} \\
&= \frac{3x-5}{x-1} \cdot \frac{\left(x - \frac{5}{3}\right)}{\left(x - \frac{5}{3}\right)} \\
&= \frac{3x^2 - 10x + \frac{25}{3}}{x^2 - \frac{8}{3}x + \frac{5}{3}} && \text{(expand)} \\
&= \frac{9x^2 - 30x + 25}{3x^2 - 8x + 5}
\end{aligned}$$

(multiply by $1 = \frac{3}{3}$ to reduce complex fractions.)

Again, we can check our answer by applying the six step method to this function or, for a quick verification, we can use a graphing utility.[o] □

[o] Be warned, however, a graphing utility may not show the hole at $\left(\frac{5}{3}, 0\right)$.

Another way to approach Example 1.2.5 is to take a cue from Theorem 1.1.1. The graph of $y = r(x)$ certainly *appears* to be the result of moving around the graph of $f(x) = \frac{1}{x}$. To that end, suppose $r(x) = \frac{a}{x-k} + k$. Since the vertical asymptote is $x = 1$ and the horizontal asymptote is $y = 3$, we get $h = 1$ and $k = 3$. At this point, we have $r(x) = \frac{a}{x-1} + 3$. We can determine a by using the y-intercept, $(0, 5)$: $r(0) = 5$ gives us $-a + 3 = 5$ so $a = -2$. Hence, $r(x) = \frac{-2}{x-1} + 3$. At this point we could check the x-intercept $\left(\frac{5}{3}, 0\right)$ is on the graph, check our answer using a graphing utility, or even better, get common denominators and write $r(x)$ as a single rational expression to compare with our answer in the above example.

As usual, the authors offer no apologies for what may be construed as 'pedantry' in this section. We feel that the detail presented in this section is necessary to obtain a firm grasp of the concepts presented here and it also serves as an introduction to the methods employed in Calculus. In the end, your instructor will decide how much, if any, of the kinds of details presented here are 'mission critical' to your understanding of Precalculus. Without further delay, we present you with this section's Exercises.

1.2.1 Exercises

In Exercises 1 - 16, use the six-step procedure to graph the rational function. Be sure to draw any asymptotes as dashed lines.

1. $f(x) = \dfrac{4}{x+2}$

2. $f(x) = 5x(6-2x)^{-1}$

3. $g(t) = t^{-2}$

4. $g(t) = \dfrac{1}{t^2+t-12}$

5. $r(z) = \dfrac{2z-1}{-2z^2-5z+3}$

6. $r(z) = \dfrac{z}{z^2+z-12}$

7. $f(x) = 4x(x^2+4)^{-1}$

8. $f(x) = 4x(x^2-4)^{-1}$

9. $g(t) = \dfrac{t^2-t-12}{t^2+t-6}$

10. $g(t) = 3 - \dfrac{5t-25}{t^2-9}$

11. $r(z) = \dfrac{z^2-z-6}{z+1}$

12. $r(z) = -z-2+\dfrac{6}{3-z}$

13. $f(x) = \dfrac{x^3+2x^2+x}{x^2-x-2}$

14. $f(x) = \dfrac{5x}{9-x^2} - x$

15. $g(t) = \dfrac{1}{2}t - 1 + \dfrac{t+1}{t^2+1}$

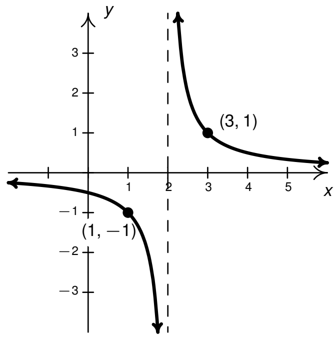
16. $g(t) = \dfrac{t^2-2t+1}{t^3+t^2-2t}$

Once you've done the six-step procedure, use a graphing utility to graph this function on the window $[0,12]\times[0,0.25]$ . . .

In Exercises 17 - 20, find a possible formula for the function whose graph is given.

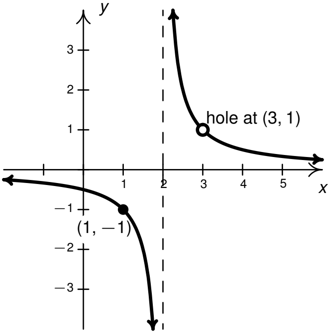
17. $y = f(x)$. See Figure 1.2.17.

18. $y = F(x)$. See Figure 1.2.18



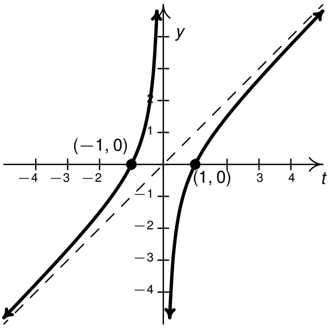
Vertical asymptote: $x = 2$

Figure 1.2.17



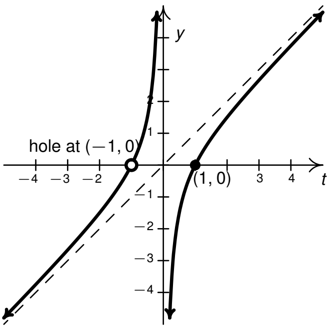
Vertical asymptote: $x = 2$

Figure 1.2.18



Slant asymptote: $y = t$

Figure 1.2.19



Slant asymptote: $y = t$

Figure 1.2.20

19. $y = g(t)$. See Figure 1.2.19

20. $y = G(t)$. See Figure 1.2.20

21. Let $g(x) = \dfrac{x^4 - 8x^3 + 24x^2 - 72x + 135}{x^3 - 9x^2 + 15x - 7}$. With the help of your classmates:

    - find the x- and y- intercepts of the graph of g.
    - find all of the asymptotes of the graph of g and any holes in the graph, if they exist.
    - find the intervals on which the function is increasing, the intervals on which it is decreasing and the local maximums and minimums, if any exist.
    - sketch the graph of g, using more than one picture if necessary to show all of the important features of the graph.

Example 1.2.4 showed us that the six-step procedure cannot tell us everything of importance about the graph of a rational function and that sometimes there are things that are easy to miss. Without Calculus, we may need to use graphing utilities to reveal the hidden behavior of rational functions. Working with your classmates, use a graphing utility to examine the graphs of the rational functions given in Exercises 22 - 25. Compare and contrast their features. Which features can the six-step process reveal and which features cannot be detected by it?

22. $f(x) = \dfrac{1}{x^2 + 1}$

23. $f(x) = \dfrac{x}{x^2 + 1}$

24. $f(x) = \dfrac{x^2}{x^2 + 1}$

25. $f(x) = \dfrac{x^3}{x^2 + 1}$

1.2.2 Answers

1. $f(x) = \dfrac{4}{x+2}$

   Domain: $(-\infty, -2) \cup (-2, \infty)$
   No x-intercepts
   y-intercept: $(0, 2)$
   Vertical asymptote: $x = -2$
   As $x \to -2^-$, $f(x) \to -\infty$
   As $x \to -2^+$, $f(x) \to \infty$
   Horizontal asymptote: $y = 0$
   As $x \to -\infty$, $f(x) \to 0^-$
   As $x \to \infty$, $f(x) \to 0^+$
   See Figure 1.2.21

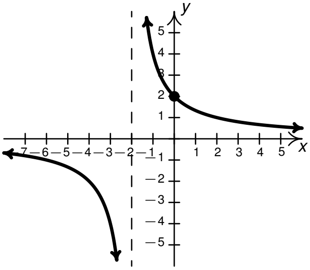


Figure 1.2.21

2. $f(x) = 5x(6-2x)^{-1} = \dfrac{5x}{6-2x}$

   Domain: $(-\infty, 3) \cup (3, \infty)$
   x-intercept: $(0, 0)$
   y-intercept: $(0, 0)$
   Vertical asymptote: $x = 3$
   As $x \to 3^-$, $f(x) \to \infty$
   As $x \to 3^+$, $f(x) \to -\infty$
   Horizontal asymptote: $y = -\frac{5}{2}$
   As $x \to -\infty$, $f(x) \to -\frac{5}{2}^+$
   As $x \to \infty$, $f(x) \to -\frac{5}{2}^-$
   See Figure 1.2.22

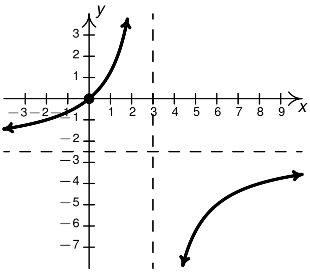


Figure 1.2.22

3. $g(t) = t^{-2} = \dfrac{1}{t^2}$

   Domain: $(-\infty, 0) \cup (0, \infty)$
   No t-intercepts
   No y-intercepts
   Vertical asymptote: $t = 0$
   As $t \to 0^-$, $g(t) \to \infty$
   As $t \to 0^+$, $g(t) \to \infty$
   Horizontal asymptote: $y = 0$
   As $t \to -\infty$, $g(t) \to 0^+$
   As $t \to \infty$, $g(t) \to 0^+$
   See Figure 1.2.23

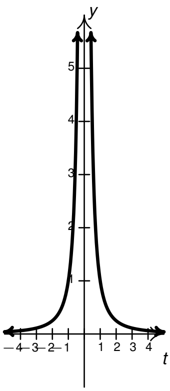


Figure 1.2.23

4. $g(t) = \dfrac{1}{t^2+t-12} = \dfrac{1}{(t-3)(t+4)}$ Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
No t-intercepts
y-intercept: $(0,-\frac{1}{12})$
Vertical asymptotes: $t=-4$ and $t=3$
As $t\to -4^-$, $g(t)\to\infty$
As $t\to -4^+$, $g(t)\to-\infty$
As $t\to 3^-$, $g(t)\to-\infty$
As $t\to 3^+$, $g(t)\to\infty$
Horizontal asymptote: $y=0$
As $t\to-\infty$, $g(t)\to 0^+$
As $t\to\infty$, $g(t)\to 0^+$
See Figure 1.2.24

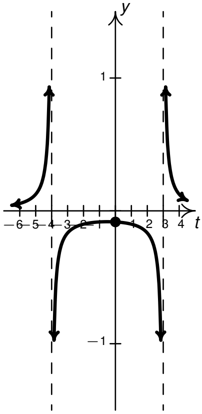


Figure 1.2.24

5. $r(z) = \dfrac{2z-1}{-2z^2-5z+3} = -\dfrac{2z-1}{(2z-1)(z+3)}$
Domain: $(-\infty,-3)\cup(-3,\frac{1}{2})\cup(\frac{1}{2},\infty)$
No z-intercepts
y-intercept: $(0,-\frac{1}{3})$
$r(z) = \dfrac{-1}{z+3},\ z\neq\frac{1}{2}$
Hole in the graph at $(\frac{1}{2},-\frac{2}{7})$
Vertical asymptote: $z=-3$
As $z\to-3^-$, $r(z)\to\infty$
As $z\to-3^+$, $r(z)\to-\infty$
Horizontal asymptote: $y=0$
As $z\to-\infty$, $r(z)\to 0^+$
As $z\to\infty$, $r(z)\to 0^-$
See Figure 1.2.25

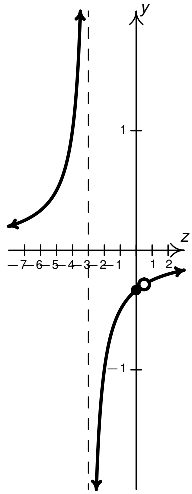


Figure 1.2.25

6. $r(z) = \dfrac{z}{z^2+z-12} = \dfrac{z}{(z-3)(z+4)}$
Domain: $(-\infty,-4)\cup(-4,3)\cup(3,\infty)$
z-intercept: $(0,0)$
y-intercept: $(0,0)$
Vertical asymptotes: $z=-4$ and $z=3$
As $z\to-4^-$, $r(z)\to-\infty$
As $z\to-4^+$, $r(z)\to\infty$
As $z\to 3^-$, $r(z)\to-\infty$
As $z\to 3^+$, $r(z)\to\infty$

Horizontal asymptote: $y = 0$
As $z \to -\infty$, $r(z) \to 0^-$
As $z \to \infty$, $r(z) \to 0^+$
See Figure 1.2.26

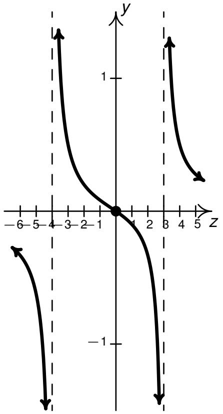


Figure 1.2.26

7. $f(x) = 4x(x^2+4)^{-1} = \dfrac{4x}{x^2+4}$
Domain: $(-\infty, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
No vertical asymptotes
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$
See Figure 1.2.27

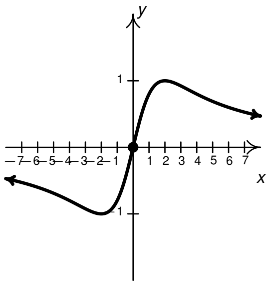


Figure 1.2.27

8. $f(x) = 4x(x^2-4)^{-1} = \dfrac{4x}{x^2-4} = \dfrac{4x}{(x+2)(x-2)}$
Domain: $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -2$, $x = 2$
As $x \to -2^-$, $f(x) \to -\infty$
As $x \to -2^+$, $f(x) \to \infty$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
No holes in the graph
Horizontal asymptote: $y = 0$
As $x \to -\infty$, $f(x) \to 0^-$
As $x \to \infty$, $f(x) \to 0^+$
See Figure 1.2.28

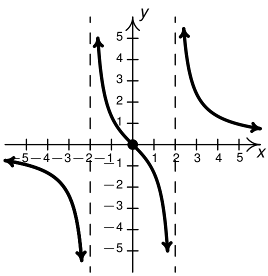


Figure 1.2.28

9. $g(t) = \dfrac{t^2-t-12}{t^2+t-6} = \dfrac{t-4}{t-2}$, $t \neq -3$
Domain: $(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$
t-intercept: $(4, 0)$
y-intercept: $(0, 2)$
Vertical asymptote: $t = 2$
As $t \to 2^-$, $g(t) \to \infty$
As $t \to 2^+$, $g(t) \to -\infty$
Hole at $\left(-3, \frac{7}{5}\right)$

Horizontal asymptote: $y = 1$
As $t \to -\infty$, $g(t) \to 1^+$
As $t \to \infty$, $g(t) \to 1^-$
See Figure 1.2.30

10. $g(t) = 3 - \dfrac{5t-25}{t^2-9} = \dfrac{3t^2-5t-2}{t^2-9} = \dfrac{(3t+1)(t-2)}{(t+3)(t-3)}$
Domain: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
t-intercepts: $\left(-\frac{1}{3},0\right)$, $(2,0)$
y-intercept: $\left(0,\frac{2}{9}\right)$
Vertical asymptotes: $t=-3$, $t=3$
As $t \to -3^-$, $g(t) \to \infty$
As $t \to -3^+$, $g(t) \to -\infty$
As $t \to 3^-$, $g(t) \to -\infty$
As $t \to 3^+$, $g(t) \to \infty$
Horizontal asymptote: $y = 3$
As $t \to -\infty$, $g(t) \to 3^+$
As $t \to \infty$, $g(t) \to 3^-$
See **??**

11. $r(z) = \dfrac{z^2-z-6}{z+1} = \dfrac{(z-3)(z+2)}{z+1}$
Domain: $(-\infty,-1)\cup(-1,\infty)$
z-intercepts: $(-2,0)$, $(3,0)$
y-intercept: $(0,-6)$
Vertical asymptote: $z=-1$
As $z \to -1^-$, $r(z) \to \infty$
As $z \to -1^+$, $r(z) \to -\infty$
Slant asymptote: $y = z-2$
As $z \to -\infty$, the graph is above $y = z-2$
As $z \to \infty$, the graph is below $y = z-2$
See Figure 1.2.31

12. $r(z) = -z-2+\dfrac{6}{3-z} = \dfrac{z^2-z}{3-z}$

Domain: $(-\infty,3)\cup(3,\infty)$
z-intercepts: $(0,0)$, $(1,0)$
y-intercept: $(0,0)$
Vertical asymptote: $z=3$

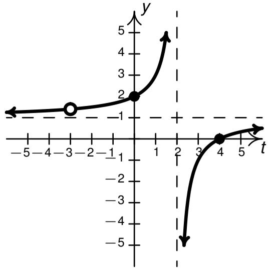


Figure 1.2.29

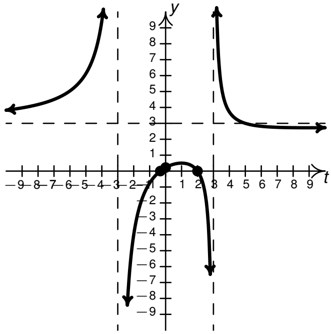


Figure 1.2.30

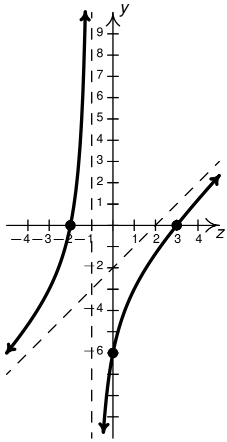


Figure 1.2.31

As $z \to 3^-$, $r(z) \to \infty$
As $z \to 3^+$, $r(z) \to -\infty$
Slant asymptote: $y = -z - 2$
As $z \to -\infty$, the graph is above $y = -z - 2$
As $z \to \infty$, the graph is below $y = -z - 2$
See Figure 1.2.32

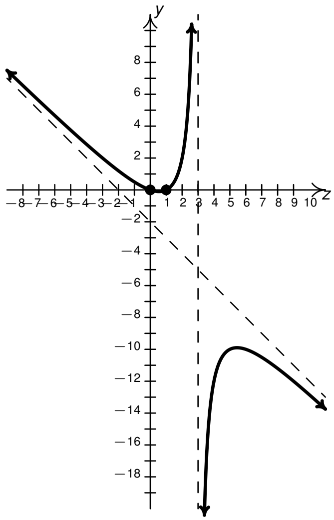


Figure 1.2.32

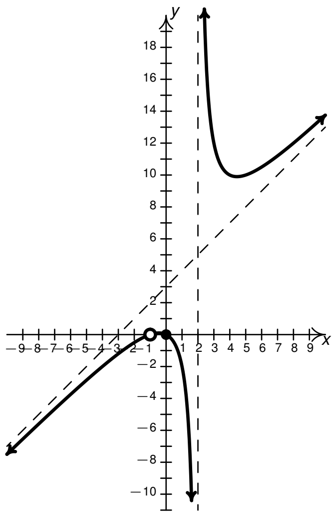


Figure 1.2.33

13. $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2} = \dfrac{x(x+1)}{x-2}$, $x \neq -1$
Domain: $(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$
x-intercept: $(0, 0)$
y-intercept: $(0, 0)$
Vertical asymptote: $x = 2$
As $x \to 2^-$, $f(x) \to -\infty$
As $x \to 2^+$, $f(x) \to \infty$
Hole at $(-1, 0)$
Slant asymptote: $y = x + 3$
As $x \to -\infty$, the graph is below $y = x + 3$
As $x \to \infty$, the graph is above $y = x + 3$
See Figure 1.2.33

14. $f(x) = \dfrac{5x}{9 - x^2} - x = \dfrac{x^3 - 4x}{9 - x^2} = \dfrac{x(x-2)(x+2)}{-(x-3)(x+3)}$
Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$
x-intercepts: $(-2, 0)$, $(0, 0)$, $(2, 0)$
y-intercept: $(0, 0)$
Vertical asymptotes: $x = -3$, $x = 3$
As $x \to -3^-$, $f(x) \to \infty$
As $x \to -3^+$, $f(x) \to -\infty$
As $x \to 3^-$, $f(x) \to \infty$
As $x \to 3^+$, $f(x) \to -\infty$
Slant asymptote: $y = -x$
As $x \to -\infty$, the graph is above $y = -x$
As $x \to \infty$, the graph is below $y = -x$
See Figure 1.2.34

15. $g(t) = \dfrac{1}{2}t - 1 + \dfrac{t+1}{t^2+1} = \dfrac{t(t^2 - 2t + 3)}{2t^2 + 2}$
Domain: $(-\infty, \infty)$
t-intercept: $(0, 0)$
y-intercept: $(0, 0)$

Slant asymptote: $y = \frac{1}{2}t - 1$
As $t \to -\infty$, the graph is below $y = \frac{1}{2}t - 1$
As $t \to \infty$, the graph is above $y = \frac{1}{2}t - 1$
See Figure 1.2.35

16. $g(t) = \dfrac{t^2 - 2t + 1}{t^3 + t^2 - 2t} = \dfrac{t-1}{t(t+2)}, \ t \neq 1$
Domain: $(-\infty, -2) \cup (-2, 0) \cup (0, 1) \cup (1, \infty)$
No t-intercepts
No y-intercepts
Vertical asymptotes: $t = -2$ and $t = 0$
As $t \to -2^-$, $g(t) \to -\infty$
As $t \to -2^+$, $g(t) \to \infty$
As $t \to 0^-$, $g(t) \to \infty$
As $t \to 0^+$, $g(t) \to -\infty$
Hole in the graph at $(1, 0)$
Horizontal asymptote: $y = 0$
As $t \to -\infty$, $g(t) \to 0^-$
As $t \to \infty$, $g(t) \to 0^+$
See Figure 1.2.36

17. $f(x) = \dfrac{1}{x-2}$

18. $F(x) = \dfrac{x-3}{(x-2)(x-3)} = \dfrac{x-3}{x^2 - 5x + 6}$

19. $g(t) = \dfrac{t^2 - 1}{t}$

20. $G(t) = \dfrac{(t^2-1)(t+1)}{t(t+1)} = \dfrac{t^3 + t^2 - t - 1}{t^2 + t}$

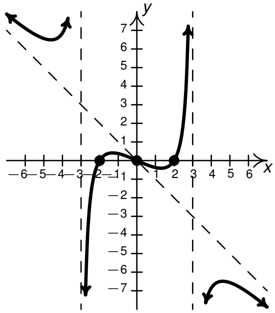


Figure 1.2.34

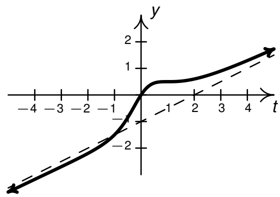


Figure 1.2.35

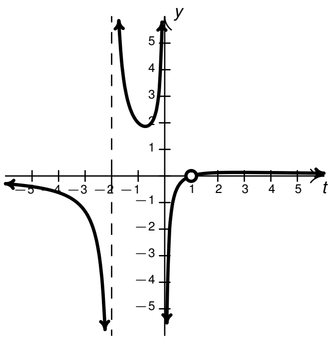


Figure 1.2.36

1.3 Inequalities involving Rational Functions and Applications

In this section, we solve equations and inequalities involving rational functions and explore associated application problems. Our first example showcases the critical difference in procedure between solving equations and inequalities.

Example 1.3.1.

1. Solve $\dfrac{x^3 - 2x + 1}{x - 1} = \dfrac{1}{2}x - 1$.
2. Solve $\dfrac{x^3 - 2x + 1}{x - 1} \geq \dfrac{1}{2}x - 1$.
3. Verify your solutions using a graphing utility.

Solution.

1. To solve the equation, we clear denominators

$$\begin{aligned}
\frac{x^3 - 2x + 1}{x - 1} &= \frac{1}{2}x - 1 \\
\left(\frac{x^3 - 2x + 1}{x - 1}\right) \cdot 2(x - 1) &= \left(\frac{1}{2}x - 1\right) \cdot 2(x - 1) \\
2x^3 - 4x + 2 &= x^2 - 3x + 2 & \text{(expand)} \\
2x^3 - x^2 - x &= 0 \\
x(2x + 1)(x - 1) &= 0 & \text{(factor)} \\
x &= -\frac{1}{2},\ 0,\ 1
\end{aligned}$$

Since we cleared denominators, we need to check for extraneous solutions. Sure enough, we see that $x = 1$ does not satisfy the original equation, so our only solutions are $x = -\frac{1}{2}$ and $x = 0$.

2. To solve the inequality, it may be tempting to begin as we did with the equation $-$ namely by multiplying both sides by the quantity $(x-1)$. The problem is that, depending on x, $(x-1)$ may be positive (which doesn't affect the inequality) or $(x-1)$ could be negative (which would reverse the inequality). Instead of working by cases, we collect all of the terms on one side of the inequality with 0 on the other and make a sign diagram using the technique given on page 46 in Section 1.2.

$$\begin{aligned}
\frac{x^3-2x+1}{x-1} &\geq \frac{1}{2}x-1 \\
\frac{x^3-2x+1}{x-1}-\frac{1}{2}x+1 &\geq 0 \\
\frac{2\left(x^3-2x+1\right)}{2(x-1)}-\frac{x(x-1)}{2(x-1)}+\frac{2(x-1)}{2(x-1)} &\geq 0 \\
&\text{(get a common denominator)} \\
\frac{2\left(x^3-2x+1\right)-x(x-1)+2(x-1)}{2(x-1)} &\geq 0 \\
\frac{2x^3-x^2-x}{2x-2} &\geq 0 \\
&\text{(expand)}
\end{aligned}$$

Viewing the left hand side as a rational function $r(x)$ we make a sign diagram. The only value excluded from the domain of r is $x=1$ which is the solution to $2x-2=0$. The zeros of r are the solutions to $2x^3-x^2-x=0$, which we have already found to be $x=0$, $x=-\frac{1}{2}$ and $x=1$, the latter was discounted as a zero because it is not in the domain. Choosing test values in each test interval, we construct the sign diagram in Equation **??**.

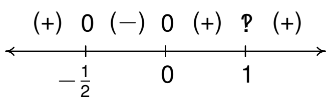


Figure 1.3.1

We are interested in where $r(x) \geq 0$. We see $r(x)>0$, or $(+)$, on the intervals $\left(-\infty,-\frac{1}{2}\right)$, $(0,1)$

and $(1, \infty)$. We know $r(x) = 0$ when $x = -\frac{1}{2}$ and $x = 0$. Hence, $r(x) \geq 0$ on $\left(-\infty, -\frac{1}{2}\right] \cup [0, 1) \cup (1, \infty)$.

3. To check our answers graphically, let $f(x) = \frac{x^3-2x+1}{x-1}$ and $g(x) = \frac{1}{2}x - 1$. The solutions to $f(x) = g(x)$ are the x-coordinates of the points where the graphs of $y = f(x)$ and $y = g(x)$ intersect. We graph both f and g in Figure 1.3.2 (the graph of g is the line and is slightly lighter in color.) We find only two intersection points, $(-0.5, -1.25)$ and $(0, -1)$ which correspond to our solutions $x = -\frac{1}{2}$ and $x = 0$. The solution to $f(x) \geq g(x)$ represents not only where the graphs meet, but the intervals over which the graph of $y = f(x)$ is above ($>$) the graph of $g(x)$. From the graph, this *appears* to happen on $\left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ which *almost* matches the answer we found analytically. We have to remember that f is not defined at $x = 1$, so it cannot be included in our solution.[a] See Figure 1.3.3.

a We invite the reader to show there is a hole in the graph of $y = f(x)$ at $(1, 1)$.

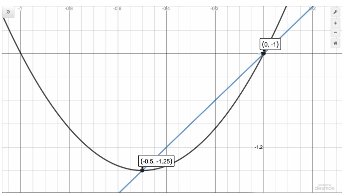


Figure 1.3.2

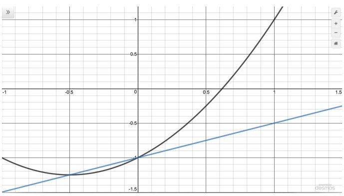


Figure 1.3.3

$\square$

The important take-away from Example 1.3.1 is not to clear fractions when working with an inequality unless you know for certain the sign of the denominators. We offer another example.

Example 1.3.2. Solve: $2t(3t-2)^{-1} \leq 3t^2(3t-2)^{-2}$. Check your answer using a graphing utility.

Solution. We begin by rewriting the terns with negative exponents as fractions and gathering all nonzero terms to one side of the inequality:

$$\begin{aligned} 2t(3t-2)^{-1} &\leq 3t^2(3t-2)^{-2} \\ \frac{2t}{3t-2} &\leq \frac{3t^2}{(3t-2)^2} \\ \frac{2t}{3t-2} - \frac{3t^2}{(3t-2)^2} &\leq 0 \end{aligned}$$

$$\frac{2t(3t-2)}{(3t-2)^2} - \frac{3t^2}{(3t-2)^2} \le 0$$
(get a common denominator)
$$\frac{2t(3t-2)-3t^2}{(3t-2)^2} \le 0$$
$$\frac{3t^2-4t}{(3t-2)^2} \le 0 \qquad \text{(expand)}$$

We define $r(t) = \frac{3t^2-4t}{(3t-2)^2}$ and set about constructing a sign diagram for r. Solving $(3t-2)^2 = 0$ gives $t = \frac{2}{3}$, our sole excluded value. To find the zeros of r, we set $r(t) = \frac{3t^2-4t}{(3t-2)^2} = 0$ and solve $3t^2 - 4t = 0$. Factoring gives $t(3t-4) = 0$ so our solutions are $t = 0$ and $t = \frac{4}{3}$. After choosing test values, we get the sign diagram in Figure 1.3.4. Since we are looking for where $r(t) \le 0$, we are looking for where $r(t)$ is $(-)$ or $r(t) = 0$. Hence, our final answer is $\left[0, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{4}{3}\right]$. In Figure 1.3.5, we graph $f(t) = 2t(3t-1)^{-1}$ (the darker curve), $g(t) = 3t^2(3t-2)^{-2}$, and vertical asymptote $x = \frac{2}{3}$, the dashed line. Sure enough, the graph of f intersects the graph of g when $t = 0$ and $t = \frac{4}{3}$. Moreover, the graph of f is below the graph of g everywhere they are defined between these values, in accordance with our algebraic solution.

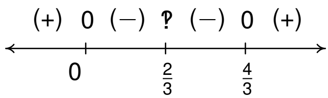


Figure 1.3.4

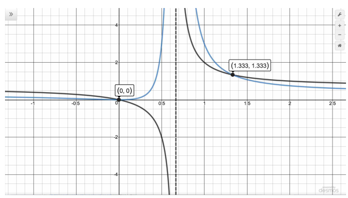


Figure 1.3.5

$\square$

One thing to note about Example 1.3.2 is that the quantity $(3t-2)^2 \ge 0$ for all values of t. Hence, as long as we remember $t = \frac{2}{3}$ is excluded from consideration, we could actually multiply both sides of the inequality in Example 1.3.2 by $(3t-2)^2$ to obtain $2t(3t-2) \le 3t^2$. We could then solve this (slightly easier) inequality using the methods of Section **??** as long as we remember to exclude $t = \frac{2}{3}$ from our solution. Once again, the more you *understand*, the less you have to *memorize*. If you know the '*why*' behind an algorithm instead of just the '*how*,' you will know when you can short-cut it.

Our next example is an application of average cost. Recall from Definition 1.1.7 if $C(x)$ represents the cost to make x items then the average cost per item is given by $\overline{C}(x) = \frac{C(x)}{x}$, for $x > 0$.

Example 1.3.3. Recall from Example **??** that the cost, $C(x)$, in dollars, to produce x PortaBoy game systems for a local retailer is $C(x) = 80x + 150$, $x \geq 0$.

1. Find an expression for the average cost function, $\overline{C}(x)$.
2. Solve $\overline{C}(x) < 100$ and interpret.
3. Determine the behavior of $\overline{C}(x)$ as $x \to \infty$ and interpret.

Solution.

1. From $\overline{C}(x) = \frac{C(x)}{x}$, we obtain $\overline{C}(x) = \frac{80x+150}{x}$. The domain of C is $x \geq 0$, but since $x = 0$ causes problems for $\overline{C}(x)$, we get our domain to be $x > 0$, or $(0, \infty)$.
2. Solving $\overline{C}(x) < 100$ means we solve $\frac{80x+150}{x} < 100$. We proceed as in the previous example.

$$\begin{aligned}
\frac{80x + 150}{x} &< 100 \\
\frac{80x + 150}{x} - 100 &< 0 \\
\frac{80x + 150 - 100x}{x} &< 0 \quad \text{(common denominator)} \\
\frac{150 - 20x}{x} &< 0
\end{aligned}$$

If we take the left hand side to be a rational function $r(x)$, we need to keep in mind that the applied domain of the problem is $x > 0$. This means we consider only the positive half of the number line for our sign diagram. On $(0, \infty)$, r is defined everywhere so we need only look for zeros of r. Setting $r(x) = 0$ gives $150 - 20x = 0$, so that $x = \frac{15}{2} = 7.5$. The test intervals on our domain are $(0, 7.5)$ and $(7.5, \infty)$. We find $r(x) < 0$ on $(7.5, \infty)$. See Figure 1.3.6.

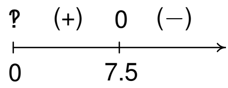


Figure 1.3.6

In the context of the problem, x represents the number of PortaBoy games systems produced and $\overline{C}(x)$ is the average cost to produce each system. Solving $\overline{C}(x) < 100$ means we are trying to find how many systems we need to produce so that the average cost is less than \$100 per system. Our solution, $(7.5, \infty)$ tells us that we need to produce more than 7.5 systems to achieve this. Since it doesn't make sense to produce half a system, our final answer is $[8, \infty)$.

3. When we apply Theorem 1.1.3 to $\overline{C}(x)$ we find that $y = 80$ is a horizontal asymptote to the graph of $y = \overline{C}(x)$. To more precisely determine the behavior of $\overline{C}(x)$ as $x \to \infty$, we first use long division[b] and rewrite $\overline{C}(x) = 80 + \frac{150}{x}$. As $x \to \infty$, $\frac{150}{x} \to 0^{+}$, which means $\overline{C}(x) \approx 80 + \text{very small } (+)$. Thus the average cost per system is getting closer to \$80 per system. If we set $\overline{C}(x) = 80$, we get $\frac{150}{x} = 0$, which is impossible, so we conclude that $\overline{C}(x) > 80$ for all $x > 0$. This means that the average cost per system is always greater than \$80 per system, but the average cost is approaching this amount as more and more systems are produced. Looking back at Example **??**, we realize \$80 is the variable cost per system – the cost per system above and beyond the fixed initial cost of \$150. Another way to interpret our answer is that 'infinitely' many systems would need to be produced to effectively 'zero out' the fixed cost. □

In this case, long division amounts to term-by-term division. b

Note that number 2 in Example 1.3.3 is another opportunity to short-cut the standard algorithm and obtain the solution more quickly if we take stock of the situation.

Since the applied domain is $x > 0$, we can multiply through the inequality $\frac{80x+150}{x} < 100$ by x without worrying about changing the sense of the inequality. This reduces the problem to $80x + 150 < 100x$, a basic linear inequality whose solution is readily seen to be $x > 7.5$. It is absolutely critical here that $x > 0$. Indeed, any time you decide to multiply an inequality by a variable expression, it is necessary to justify why the inequality is preserved. Our next example is another classic 'box with no top' problem. The reader is encouraged to compare and contrast this problem with Example **??** in Section **??**.

Example 1.3.4. A box with a square base and no top is to be constructed so that it has a volume of 1000 cubic centimeters. Let x denote the width of the box, in centimeters as seen in Figure 1.3.7.

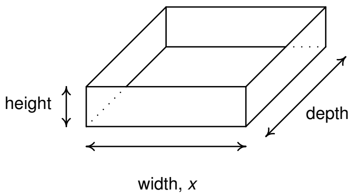


Figure 1.3.7

1. Explain why the height of the box (in centimeters) is a function of the width x. Call this function h and find an expression for $h(x)$, complete with an appropriate applied domain.

2. Solve $h(x) \geq x$ and interpret.

3. Find and interpret the behavior of $h(x)$ as $x \to 0^+$ and as $x \to \infty$.

4. Express the surface area of the box as a function of x, $S(x)$ and state the applied domain.

5. Use a graphing utility to approximate (to two decimal places) the dimensions of the box which minimize the surface area.

Solution.

1. We are told that the volume of the box is 1000 cubic centimeters and that x represents the width, in centimeters. Since x represents a physical dimension of a box, we have that $x > 0$. From geometry, we know volume = width $\times$ height $\times$ depth. Since the base of the box is a square, the width and the depth are both x centimeters. Hence, 1000 =

x^2(height). Solving for the height, we get height = $\frac{1000}{x^2}$. In other words, for each width $x > 0$, we are able to compute *the*[c] corresponding height using the formula $\frac{1000}{x^2}$. Hence, the height is a function of x. Using function notation, we write $h(x) = \frac{1000}{x^2}$. As mentioned before, our only restriction is $x > 0$ so the domain of h is $(0, \infty)$.

[c] that is, the one and only one

2. To solve $h(x) \geq x$, we proceed as before and collect all nonzero terms on one side of the inequality in order to use a sign diagram.

$$\begin{aligned} h(x) &\geq x \\ \frac{1000}{x^2} &\geq x \\ \frac{1000}{x^2} - x &\geq 0 \\ \frac{1000 - x^3}{x^2} &\geq 0 \qquad \text{(common denominator)} \end{aligned}$$

We consider the left hand side of the inequality as our rational function $r(x)$. We see immediately the only value excluded from the domain of r is 0, but since our applied domain is $x > 0$, we restrict our attention to the interval $(0, \infty)$. The sole zero of r comes when $1000 - x^3 = 0$, or when $x = 10$. Choosing test values in the intervals $(0, 10)$ and $(10, \infty)$ gives Figure 1.3.8.

We see $r(x) > 0$ on $(0, 10)$, and since $r(x) = 0$ at $x = 10$, our solution is $(0, 10]$. In the context of the problem, $h(x)$ represents the height of the box while x represents the width (and depth) of the box. Solving $h(x) \geq x$ is tantamount to finding the values of x which result in a box where the height is at least as big as the width (and, in this case, depth.) Our answer tells us the width of the box can be at most 10 centimeters for this to happen.[d]

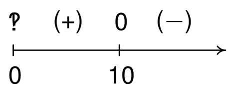


Figure 1.3.8

[d] As with the previous example, knowing $x > 0$ means $x^2 > 0$ so we can clear denominators right away and solve $x^3 \leq 1000$, or $x \leq 10$. Coupled with our applied domain, $x > 0$, we would arrive at the same solution, $(0, 10]$.

3. As $x \to 0^+$, $h(x) = \frac{1000}{x^2} \to \infty$. This means that the smaller the width x (and, in this case, depth), the larger the height h has to be in order to maintain a volume of 1000 cubic centimeters. As $x \to \infty$, we find $h(x) \to 0^+$, which means that in order to maintain a volume of 1000 cubic centimeters, the width and depth must get bigger as the height becomes smaller.

4. Since the box has no top, the surface area can be found by adding the area of each of the sides to the area of the base. The base is a square of dimensions x by x, and each side has dimensions x by $h(x)$. We get the surface area, $S(x) = x^2 + 4xh(x)$. Since $h(x) = \frac{1000}{x^2}$, we have $S(x) = x^2 + 4x\left(\frac{1000}{x^2}\right) = x^2 + \frac{4000}{x}$. The domain of S is the same as h, namely $(0, \infty)$, for the same reasons as above.

5. To graph $y = S(x)$, we create a table of values to help us define a good viewing window. Doing so, we find a local minim when $x \approx 12.60$. As far as we can tell,[e] this is the only local extremum, so it is the (absolute) minimum as well. This means that the width and depth of the box should each measure approximately 12.60 centimeters. To determine the height, we find $h(12.60) \approx 6.30$, so the height of the box should be approximately 6.30 centimeters.[f] See Figure 1.3.9

e without Calculus, that is...

f The y-coordinate here, 476.22 means the minimum surface area possible is 476.22 square centimeters. Minimizing the surface area minimizes the material required to make the box, therein helping to reduce the cost of the box.

□

Our last example uses regression to verify a very famous scientific law.

Example 1.3.5. Boyle's Law states that when temperature is held constant, the pressure of a gas is inversely proportional to the volume of the gas.[g] According to this website[1] the actual data relating the volume V of a gas

g For a review of what this means, see Section **??**.

[1] http://web.lemoyne.edu/~giunta/classicalcs/boyleverify.html

and its pressure P used by Boyle and his assistant in 1662 to formulate this law is given in Table 1.3.1. (NOTE: both pressure and volume here are given in 'arbitrary units.')

1. Assuming P and V are inversely proportional, estimate the constant of proportionality, k.
2. Use a graphing utility to fit a curve of the form $P = \dfrac{k}{V}$ to these data.

Solution.

1. Recall if P and V are inversely proportional, there is a real number k so $PV = k$ for all values of P and V. Multiplying the corresponding P and V values from the data together result in numbers which are consistently approximately 1400. This gives us confidence in the claim P and V are inversely proportional and suggests $k \approx 1400$.
2. We plot the pairs (V, P) and run a regression, the results of which are in Figure 1.3.10. To our amazement, the graphing utility reports $k \approx 1406.9$ with $R^2 \approx 1$. This means the data are a very good fit to the model $P = \frac{k}{V}$, or $PV = k$, hence verifying Boyle's Law for this set of data.

□

V	P
48	29.13
46	30.56
44	31.94
42	33.50
40	35.31
38	37.00
36	39.31
34	41.63
32	44.19
30	47.06
28	50.31
26	54.31
24	58.81

V	P
23	61.31
22	64.06
21	67.06
20	70.69
19	74.13
18	77.88
17	82.75
16	87.88
15	93.06
14	100.44
13	107.81
12	117.56

Table 1.3.1

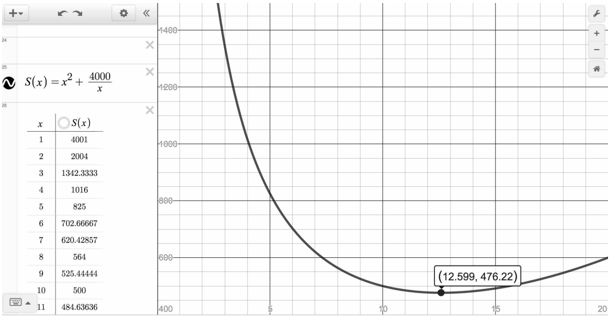


Figure 1.3.9

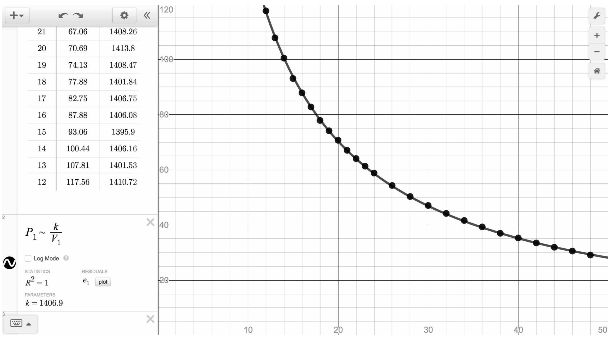


Figure 1.3.10

1.3.1 Exercises

(Review of Solving Equations):[a] In Exercises 1 - 6, solve the rational equation. Be sure to check for extraneous solutions.

[a] For more review, see Section ??.

1. $\dfrac{x}{5x+4} = 3$
2. $\dfrac{3x-1}{x^2+1} = 1$
3. $\dfrac{1}{t+3} + \dfrac{1}{t-3} = \dfrac{t^2-3}{t^2-9}$
4. $\dfrac{2t+17}{t+1} = t+5$
5. $\dfrac{z^2-2z+1}{z^3+z^2-2z} = 1$
6. $\dfrac{4z-z^3}{z^2-9} = 4z$

In Exercises 7 - 22, solve the rational inequality. Express your answer using interval notation.

7. $\dfrac{1}{x+2} \geq 0$
8. $\dfrac{5}{x+2} \geq 1$
9. $\dfrac{x}{x^2-1} < 0$
10. $\dfrac{4t}{t^2+4} \geq 0$
11. $\dfrac{2t+6}{t^2+t-6} < 1$
12. $\dfrac{5}{t-3} + 9 < \dfrac{20}{t+3}$
13. $\dfrac{6z+6}{2+z-z^2} \leq z+3$
14. $\dfrac{6}{z-1} + 1 > \dfrac{1}{z+1}$
15. $\dfrac{3z-1}{z^2+1} \leq 1$
16. $(2x+17)(x+1)^{-1} > x+5$
17. $(4x-x^3)(x^2-9)^{-1} \geq 4x$
18. $(x^2+1)^{-1} < 0$

19. $(2t - 8)(t + 1)^{-1} \leq (t^2 - 8t)(t + 1)^{-2}$

20. $(t - 3)(2t + 7)(t^2 + 7t + 6)^{-2} \geq (t^2 + 7t + 6)^{-1}$

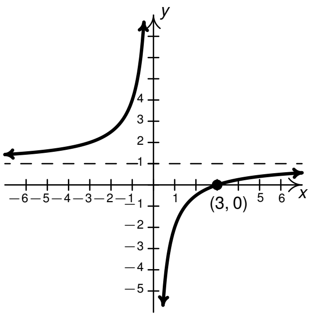
21. $60z^{-2} + 23z^{-1} \geq 7(z - 4)^{-1}$

22. $2z + 6(z - 1)^{-1} \geq 11 - 8(z + 1)^{-1}$

In Exercises 23 - 28, use the the graph of the given rational function to solve the stated inequality.

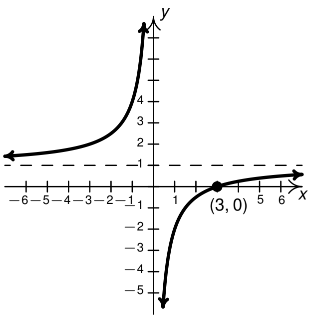
23. Solve $f(x) \geq 0$ from Figure 1.3.11.

24. Solve $f(x) < 1$ from Figure 1.3.12.



$y = f(x)$, asymptotes: $x = 0$, $y = 1$.

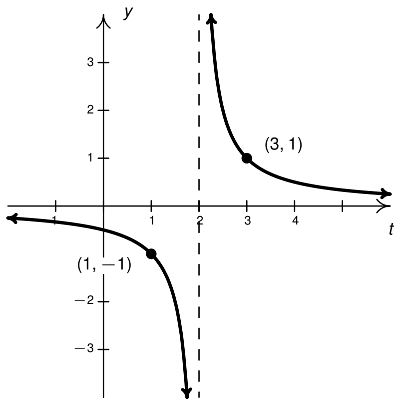
Figure 1.3.11



$y = f(x)$, asymptotes: $x = 0$, $y = 1$.

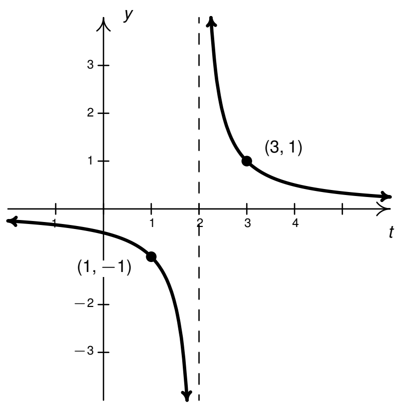
Figure 1.3.12

25. Solve $g(t) \geq -1$.



$y = g(t)$, asymptotes: $t = 2$, $y = 0$.

26. Solve $-1 \leq g(t) < 1$.



$y = g(t)$, asymptotes: $t = 2$, $y = 0$.

27. Solve $r(z) \leq 1$ from Figure 1.3.13.

28. Solve $r(z) > 0$ from Figure 1.3.14.

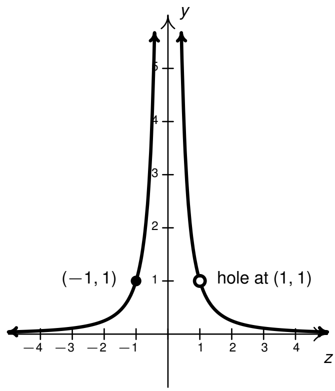
29. In Exercise **??** in Section **??**, the function $C(x) = .03x^3 - 4.5x^2 + 225x + 250$, for $x \geq 0$ was used to model the cost (in dollars) to produce x PortaBoy game systems. Using this cost function, find the number of PortaBoys which should be produced to minimize the average cost $\overline{C}$. Round your answer to the nearest number of systems.

30. Suppose we are in the same situation as Example 1.3.4. If the volume of the box is to be 500 cubic centimeters, use a graphing utility to find the dimensions of the box which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

31. The box for the new Sasquatch-themed cereal, 'Crypt-Os', is to have a volume of 140 cubic inches. For aesthetic reasons, the height of the box needs to be 1.62 times the width of the base of the box.[h] Find the dimensions of the box which will minimize the surface area of the box. What is the minimum surface area? Round your answers to two decimal places.

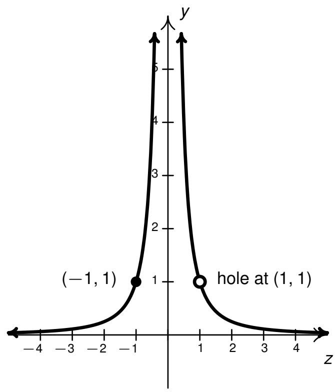
32. Sally is Skippy's neighbor from Exercise **??** in Section **??**. Sally also wants to plant a vegetable garden along the side of her home. She doesn't have any fencing, but wants to keep the size of the garden to 100 square feet. What are the dimensions of the garden which will minimize the amount of fencing she needs to buy? What is the minimum amount of fencing she needs to buy? Round your answers to the nearest foot. (Note: Since one side of the garden will border the house, Sally doesn't need fencing along that side.)

33. Another Classic Problem: A can is made in the shape of a right circular cylinder and is to hold one



$y = r(z)$, asymptotes: $z = 0$, $y = 0$.

Figure 1.3.13



$y = r(z)$, asymptotes: $z = 0$, $y = 0$.

Figure 1.3.14

[h] 1.62 is a crude approximation of the so-called 'Golden Ratio' $\phi = \frac{1+\sqrt{5}}{2}$.

pint. (For dry goods, one pint is equal to 33.6 cubic inches.)[i]

i According to www.dictionary.com[2], there are different values given for this conversion. We use 33.6in^3 for this problem.

(a) Find an expression for the volume V of the can in terms of the height h and the base radius r.

(b) Find an expression for the surface area S of the can in terms of the height h and the base radius r. (Hint: The top and bottom of the can are circles of radius r and the side of the can is really just a rectangle that has been bent into a cylinder.)

(c) Using the fact that $V = 33.6$, write S as a function of r and state its applied domain.

(d) Use your graphing calculator to find the dimensions of the can which has minimal surface area.

34. A right cylindrical drum is to hold 7.35 cubic feet of liquid. Find the dimensions (radius of the base and height) of the drum which would minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

35. In Exercise 35 in Section 1.1, the population of Sasquatch in Portage County is modeled by

$$P(t) = \frac{150t}{t+15}, \quad t \geq 0,$$

where $t = 0$ corresponds to the year 1803. According to this model, when were there fewer than 100 Sasquatch in Portage County?

1.3.2 Answers

1. $x = -\frac{6}{7}$
2. $x = 1$, $x = 2$
3. $t = -1$
4. $t = -6$, $x = 2$
5. No solution
6. $z = 0$, $z = \pm 2\sqrt{2}$
7. $(-2, \infty)$
8. $(-2, 3]$
9. $(-\infty, -1) \cup (0, 1)$
10. $[0, \infty)$
11. $(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$
12. $\left(-3, -\frac{1}{3}\right) \cup (2, 3)$
13. $(-1, 0] \cup (2, \infty)$
14. $(-\infty, -3) \cup (-2, -1) \cup (1, \infty)$
15. $(-\infty, 1] \cup [2, \infty)$
16. $(-\infty, -6) \cup (-1, 2)$
17. $(-\infty, -3) \cup \left[-2\sqrt{2}, 0\right] \cup \left[2\sqrt{2}, 3\right)$
18. No solution
19. $[-4, -1) \cup (-1, 2]$
20. $(-\infty, -6) \cup (-6, -3] \cup [9, \infty)$
21. $[-3, 0) \cup (0, 4) \cup [5, \infty)$
22. $\left(-1, -\frac{1}{2}\right] \cup (1, \infty)$
23. $f(x) \geq 0$ on $(-\infty, 0) \cup [3, \infty)$.
24. $f(x) < 1$ on $(0, \infty)$.
25. $g(t) \geq -1$ on $(-\infty, 1] \cup (2, \infty)$.

26. $-1 \leq g(t) < 1$ on $(-\infty, 1] \cup (3, \infty)$.

27. $r(z) \leq 1$ on $(-\infty, -1] \cup (1, \infty)$.

28. $r(z) > 0$ on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$.

29. The absolute minimum of $y = \overline{C}(x)$ occurs at $\approx (75.73, 59.57)$. Since x represents the number of game systems, we check $\overline{C}(75) \approx 59.58$ and $\overline{C}(76) \approx 59.57$. Hence, to minimize the average cost, 76 systems should be produced at an average cost of \$59.57 per system.

30. The width (and depth) should be 10.00 centimeters, the height should be 5.00 centimeters. The minimum surface area is 300.00 square centimeters.

31. The width of the base of the box should be approximately 4.12 inches, the height of the box should be approximately 6.67 inches, and the depth of the base of the box should be approximately 5.09 inches. The minimum surface area is approximately 164.91 square inches.

32. The dimensions are approximately 7 feet by 14 feet. Hence, the minimum amount of fencing required is approximately 28 feet.

33. (a) $V = \pi r^2 h$ (b) $S = 2\pi r^2 + 2\pi rh$
    (c) $S(r) = 2\pi r^2 + \frac{67.2}{r}$, Domain $r > 0$
    (d) $r \approx 1.749$ in. and $h \approx 3.498$ in.

34. The radius of the drum should be approximately 1.05 feet and the height of the drum should be approximately 2.12 feet. The minimum surface area of the drum is approximately 20.93 cubic feet.

35. $P(t) < 100$ on $(-15, 30)$, and the portion of this which lies in the applied domain is $[0, 30)$. Since $t = 0$ corresponds to the year 1803, from 1803 through the end of 1832, there were fewer than 100 Sasquatch in Portage County.

Chapter 2

Root, Radical and Power Functions

2.1 Root and Radical Functions

In Sections **??**, **??** and **??**, we studied constant, linear, absolute value,[a] and quadratic functions. Constant, linear and quadratic functions were specific examples of polynomial functions, which we studied in generality in Chapter **??**. Chapter **??** culminated with the Real Factorization Theorem, Theorem **??**, which says that all polynomial functions with real coefficients can be thought of as products of linear and quadratic functions. Our next step was to enlarge our field[b] of study to rational functions in Chapter 1. Being quotients of polynomials, we can ultimately view this family of functions as being built up of linear and quadratic functions as well. So in some sense, Sections **??**, **??** and **??** along with Chapters **??** and 1 can be thought of as an exhaustive study of linear and quadratic[c] functions. We now turn our attention to functions involving radicals which cannot be written in terms of linear functions. For a more detailed review of the basics of roots and radicals, we refer the reader to

a These were introduced, as you may recall, as piecewise-defined linear functions.

b This is a really bad math pun.

c If we broaden our concept of functions to allow for complex valued coefficients, the Complex Factorization Theorem, Theorem **??**, tells us every function we have studied thus far is a combination of linear functions.

Sections **??** and **??**.

2.1.1 Root Functions

As with polynomial functions and rational functions, we begin our study of functions involving radical with a special family of functions: the (principal) root functions.

Definition 2.1.1. Let $n \in \mathbb{N}$ with $n \geq 2$. The n **th (principal) root function** is the function $f(x) = \sqrt[n]{x}$.
NOTE: If n is even, the domain of f is $[0, \infty)$; if n is odd, the domain of f is $(-\infty, \infty)$.

The domain restriction for even indexed roots means that, once again, we are restricting our attention to *real* numbers.[d] We graph a few members of the root function family in Figure 2.1.1, Figure 2.1.2 and Figure 2.1.3, and quickly notice that, as with the monomial, and, more generally, the Laurent monomial functions, the behavior of the root functions depends primarily on whether the root is even or odd.

[d] Although we discussed imaginary numbers in Section **??**, we restrict our attention to real numbers in this section. See the epilogue on page **??** for more details.

In addition to having the common domain of $[0, \infty)$, the graphs of $f(x) = \sqrt[n]{x}$ for even indices n all share the points $(0, 0)$ and $(1, 1)$. As n increases, the functions become 'steeper' near the y-axis and 'flatter' as $x \to \infty$. To show $f(x) \to \infty$ as $x \to \infty$, we show, more generally, the range of f is $[0, \infty)$. Indeed, if $c \geq 0$ is a real number, then $f(c^n) = \sqrt[n]{c^n} = c$ so c is in the range of f. Note that f is increasing: that is, if $a < b$, then $f(a) = \sqrt[n]{a} < \sqrt[n]{b} = f(b)$. This property is useful in solving certain types of polynomial inequalities.[e]

[e] See Exercise 13.

Figure 2.1.1: $y = \sqrt{x}$

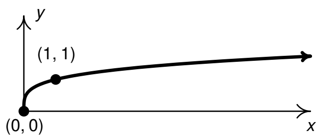


Figure 2.1.2: $y = \sqrt[4]{x}$

Figure 2.1.3: $y = \sqrt[6]{x}$

The functions $f(x) = \sqrt[n]{x}$ for odd natural numbers $n \geq 3$ also follow a predictable trend - steepening near $x = 0$ and flattening as $x \to \pm\infty$. The range for these functions is $(-\infty, \infty)$ since if c is any real number, $f(c^n) = \sqrt[n]{c^n} = c$, so c is in the range of f. Like the even indexed roots, the odd indexed roots are also increasing. Moreover, these graphs appear to be symmetric about the origin. Sure enough, when n is odd, $f(-x) = \sqrt[n]{-x} = -\sqrt[n]{x} = -f(x)$ so f is an odd function. See Figure 2.1.4, Figure 2.1.5 and Figure 2.1.6.

At this point, you're probably expecting a theorem like Theorems **??**, **??**, **??**, 1.1.1 - that is, a theorem which tells us how to obtain the graph of $F(x) = a\sqrt[n]{x-h} + k$ from the graph of $f(x) = \sqrt[n]{x}$ - and you would not be wrong. Here, however, we need to add an extra parameter 'b' to the recipe and discuss functions of the form $F(x) = a\sqrt[n]{bx-h} + k$. The reason is that, with all of the previous function families, we were always able to factor out the coefficient of x. We list some examples of this below, and invite the reader to revisit other examples in the text:

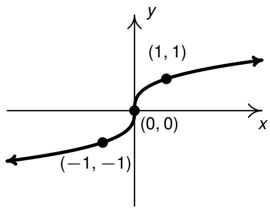


Figure 2.1.4: $y = \sqrt[3]{x}$

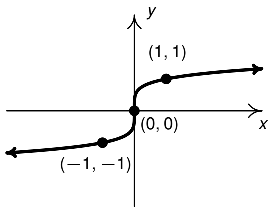


Figure 2.1.5: $y = \sqrt[5]{x}$

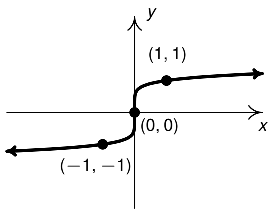


Figure 2.1.6: $y = \sqrt[7]{x}$

- $F(x) = |6-2x| = |-2x+6| = |-2(x+3)| = |-2||x+3| = 2|x+3|$.
- $F(x) = (2x-1)^2+1 = \left[2\left(x-\frac{1}{2}\right)\right]^2+1 = (2)^2\left(x-\frac{1}{2}\right)^2 + 1 = 4\left(x-\frac{1}{2}\right)^2+1$
- $F(x) = \frac{2}{(1-x)^3} - 5 = \frac{2}{[(-1)(x-1)]^3} - 5 = \frac{2}{(-1)^3(x-1)^3} - 5 = \frac{2}{-(x-1)^3} - 5 = \frac{-2}{(x-1)^3} - 5$.

For a function like $F(x) = \sqrt{4x-12}+1 = \sqrt{4(x-6)}+1 = \sqrt{4}\sqrt{x-3}+1 = 2\sqrt{x-3}+1$, this approach works fine. However, if the coefficient of x is *negative*, for example, $F(x) = \sqrt{1-x} = \sqrt{(-1)(x-1)}$ we get stuck the product rule for radicals doesn't extend to negative quantities when the index is even.[f] Hence we add an extra parameter which means we have an extra step. We state this in Theorem 2.1.1.

f Since, otherwise, $-1 = i^2 = i \cdot i = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$, a contradition.

Theorem 2.1.1. For real numbers a, b, h, and k with $a, b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx-h}+k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{x}$ to $y = \sqrt[n]{x-h}$.
2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{x-h}$ to $y = \sqrt[n]{bx-h}$.
3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}$.
4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a\sqrt[n]{bx-h}$ to $y = a\sqrt[n]{bx-h}+k$.

Proof. As usual, we 'build' the graph of $F(x) = a\sqrt[n]{bx-h}+k$ starting with the graph of $f(x) = \sqrt[n]{x}$ one step at a time. First, we consider the graph of $F_1(x) = \sqrt{x-h}$. A generic point on the graph of F_1 looks like $(x, \sqrt[n]{x-h})$. Note that if n is odd, x can be any real number whereas if n is even $x - h \geq 0$ so $x \geq h$. If we let $c = x - h$, then $x = c + h$ and we can change (dummy) variables[g] and obtain a new representation of the point: $(c+h, \sqrt[n]{c})$. Note that if n is odd, x and c vary through all real num-

[g] again this is because every real number can be represented as both $x - h$ for some value x and as $c + h$ for some value c.

bers; if n is even, $x \geq h$ and, hence, $c \geq 0$. Since a generic point on the graph of $f(x) = \sqrt[n]{x}$ can be represented as $(c, \sqrt[n]{c})$ for applicable values of c, we see that we can obtain every point on the graph of F_1 by adding h to each x-coordinate of the graph of f, establishing step 1 of the theorem.

Proceeding to (the new!) step 2, a point on the graph of $F_2(x) = \sqrt[n]{bx - h}$ has the form $(x, \sqrt[n]{bx - h})$. If n is odd, as usual, x can vary through all real numbers. If n is even, we require $bx - h \geq 0$ or $bx \geq h$. If $b > 0$, this gives $x \geq \frac{h}{b}$. If, on the other hand, $b < 0$, the we have $x \leq \frac{h}{b}$. Let $c = bx$ and since by assumption $b \neq 0$, we have $x = \frac{c}{b}$. Once again, we change dummy variables from x to c and describe a generic point on the graph of F_2 as $\left(\frac{c}{b}, \sqrt[n]{c - h}\right)$. If n is odd, x and c can vary through all real numbers. If n is even and $b > 0$, then $x \geq \frac{h}{b}$ and, hence, $c = bx \geq h$; if $b < 0$, then $x \leq \frac{h}{b}$ also gives $c = bx \geq h$. Since a generic point on the graph of F_1 can be represented as $(c, \sqrt{c - h})$ for applicable values of c, we see we can obtain every point on the graph of F_2 by dividing every x-coordinate on the graph of F_1 by b, as per step 2 of the theorem.

The proof of steps 3 and 4 of Theorem 2.1.1 are identical to the proof of Theorem **??** (just with $\sqrt[n]{\cdot}$ instead of $(\cdot)^n$) so we invite the reader to work through the details on their own. $\square$

We demonstrate Theorem 2.1.1 in the following example.

Example 2.1.1. Theorem 2.1.1 to graph the following functions. Label at least three points on the graph. State the domain and range using interval notation.

1. $f(x) = 1 - 2\sqrt[3]{x + 3}$
2. $g(t) = \dfrac{\sqrt{1 - 2t}}{4}$

Solution.

1. We begin by rewriting the expression for $f(x)$ in the form prescribed Theorem 2.1.1: $f(x) = -2\sqrt[3]{x+3}+1$. We identify $n = 3$, $a = -2$, $b = 1$, $h = -3$ and $k = 1$.

   Step 1: add -3 to each of the x-coordinates of each of the points on the graph of $y = \sqrt[3]{x}$ as in Figure 2.1.7.

   Since $b = 1$, we can proceed to Step 3 (since dividing a real by 1 just results in the same real number.)

   Step 3: multiply each of the y-coordinates of each point on the graph of $y = \sqrt[3]{x+3}$ by -2 as in Figure 2.1.8.

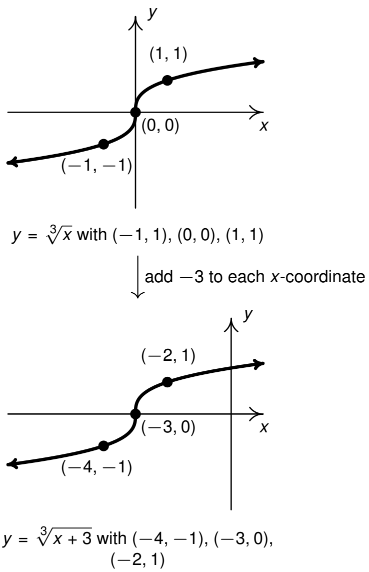


Figure 2.1.7

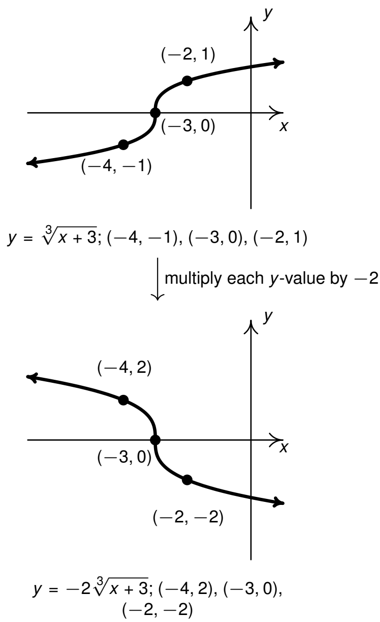


Figure 2.1.8

   Step 4: add 1 to y-coordinates of each point on the graph of $y = -2\sqrt[3]{x+3}$ as in Figure 2.1.9

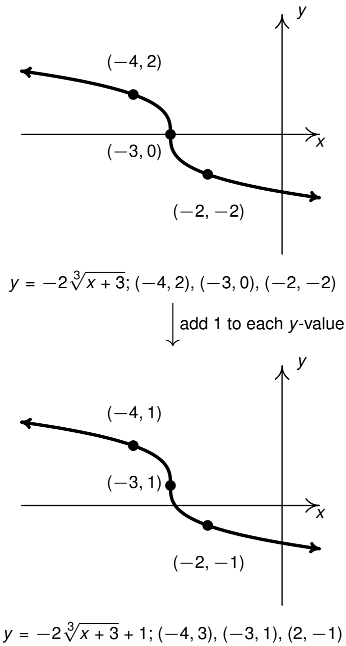


Figure 2.1.9

We get the domain and range of f are $(-\infty, \infty)$.

2. For $g(t) = \dfrac{\sqrt{1-2t}}{4} = \frac{1}{4}\sqrt{-2t+1}$, we identify $n = 2$, $a = \frac{1}{4}$, $b = -2$, $h = -1$ and $k = 0$. Since we are asked to label *three* points on the graph, we track $(4, 2)$ along with $(0, 0)$ and $(1, 1)$.[h]

   Step 1: add -1 to each of the t-coordinates of each of the points on the graph of $y = \sqrt{t}$ as in Figure 2.1.10

   Step 2: divide each of the t-coordinates of each of the points on the graph of $y = \sqrt{t+1}$ by -2 as in Figure 2.1.11.

[h] As $\sqrt{4} = 2$, we know $(4, 2)$ is on the graph of $y = \sqrt{t}$.

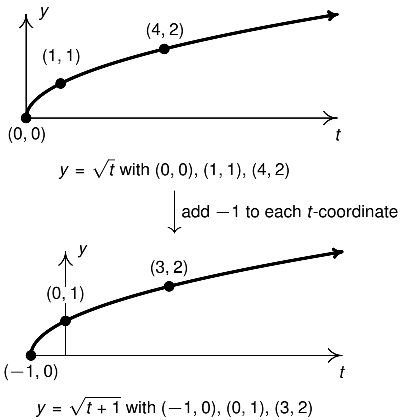


Figure 2.1.10

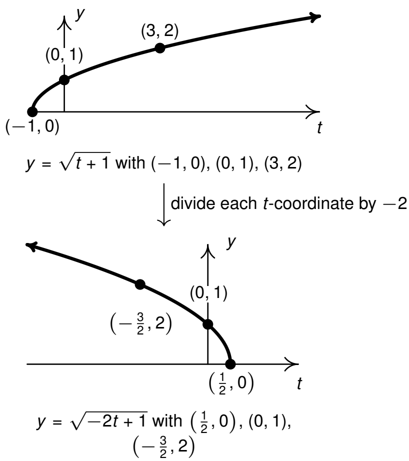


Figure 2.1.11

Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = \sqrt{-2t+1}$ by $\frac{1}{4}$ as in Figure 2.1.12.

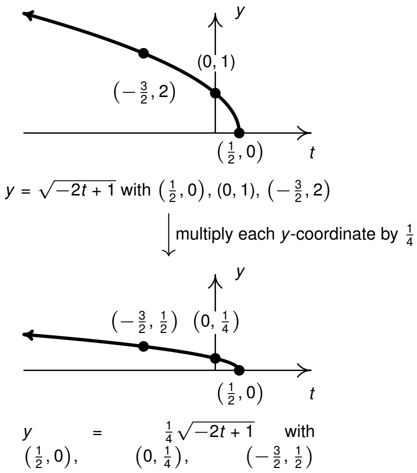


Figure 2.1.12

We get the domain is $\left(-\infty, \frac{1}{2}\right]$ and the range is $[0, \infty)$. $\square$

2.1.2 Other Functions involving Radicals

Now that we have some practice with basic root functions, we turn our attention to more general functions involving radicals. In general, Calculus is the best tool with which to study these functions. Nevertheless, we will use what algebra we know in combination with a graphing utility to help us visualize these functions and preview concepts which are studied in greater depth in later courses. In the table below, we summarize some of the properties of radicals from elsewhere in this text (and Intermediate Algebra) we will be using in the coming examples.

Theorem 2.1.2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$, and $\sqrt[n]{b}$ are real numbers.[a]
Simplifying n th powers and n th roots:[b]

- $\left(\sqrt[n]{x}\right)^n = x$.
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$.

Root Functions Preserve Inequality:[c] if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

[a] i.e., if n is odd, x, a, and b can be any real numbers; if, on the other hand n is even, $x \geq 0$, $a \geq 0$, and $b \geq 0$.
[b] a.k.a., 'Inverse Properties.' See Section 3.6.
[c] i.e., root functions are increasing.

Example 2.1.2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase.
  - intervals of decrease.
- Construct a sign diagram for each function using the intercepts and graph.[i]

i We'll revisit sign diagrams for these functions in Section 2.3 where we will use them to solve inequalities (surprised?)

1. $f(x) = 3x\sqrt[3]{2-x}$
2. $g(t) = \sqrt[3]{\dfrac{8t}{t+1}}$
3. $h(x) = \dfrac{3x}{\sqrt{x^2+1}}$
4. $r(t) = t^{-1}\sqrt{16t^4 - 1}$

Solution.

1. When looking for the domain, we have two thing to watch out for: denominators (which we must make

sure aren't 0) and even indexed radicals (whose radicands we must ensure are nonnegative.) Looking at the expression for $f(x)$, we have no denominators nor do we have an even indexed radical, so we are confident the domain is all real numbers, $(-\infty, \infty)$.

To find the x-intercepts, we find the zeros of f by solving $f(x) = 3x\sqrt[3]{2-x} = 0$. Using the zero product property, we get $3x = 0$ or $\sqrt[3]{2-x} = 0$. The former gives $x = 0$ and to solve the latter, we cube both sides and get $2 - x = 0$ or $x = 2$. Hence, the x-intercepts are $(0, 0)$ and $(2, 0)$. Since $(0, 0)$ is also on the y-axis and functions can can have at most one y-intercept, we know $(0, 0)$ is the only y-intercept.[j] That being said, we can quickly verify $f(0) = 3(0)\sqrt[3]{2-0} = 0$.

j Why is this, again?

To determine the end behavior, we consider $f(x)$ as $x \to \pm\infty$. Using 'number sense,' [k] we have

k remember this means we use the adjective 'big' here to mean large in *absolute value*

$$\begin{aligned} f(x) = 3x\sqrt[3]{2-x} = 3x\sqrt[3]{-x+2} &\approx (\text{big } (+))\sqrt[3]{\text{big } (-)} \\ &= (\text{big } (+))(\text{big } (-)) = \text{big } (-) \end{aligned}$$

So $f(x) \to -\infty$.

As $x \to -\infty$, we get

$$\begin{aligned} f(x) = 3x\sqrt[3]{-x+2} &\approx (\text{big } (-))\sqrt[3]{\text{big } (+)} \\ &= (\text{big } (-))(\text{big } (+)) = \text{big } (-) \end{aligned}$$

So $f(x) \to -\infty$ here, too.

We graph f in Figure 2.1.13. From the graph, the range appears to be $(-\infty, 3.572]$ with a local maximum (which also happens to be *the* maximum) at $(1.5, 3.572)$. We also see f appears to be increasing on $(-\infty, 1.5)$ and decreasing on $(1.5, \infty)$. It is also worth noting that there appears to be 'unusual steepness' near the x-intercept $(2, 0)$. We invite the reader to zoom in on the graph near $(2, 0)$ to see that the function is 'locally vertical.'[l]

l Of course, the Vertical Line Test prohibits the graph from actually *being* a vertical line. This behavior is more precisely defined and more closely studied in Calculus.

To create a sign diagram for $f(x)$, we note that the function has zeros $x = 0$ and $x = 2$. For $x < 0$, $f(x) < 0$ or $(-)$, for $0 < x < 2$, $f(x) > 0$ or $(+)$, and for $x > 2$, $f(x) < 0$ or $(-)$. The sign diagram for $f(x)$ is in Figure 2.1.14

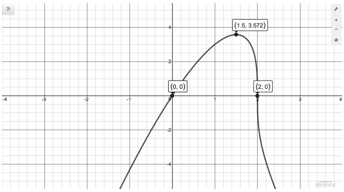


Figure 2.1.13

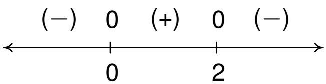


Figure 2.1.14

2. The index of the radical in the expression for $g(t)$ is odd, so our only concern is the denominator. Setting $t + 1 = 0$ gives $t = -1$, which we exclude, so our domain is $\{t \in \mathbb{R} \mid t \neq -1\}$ or using interval notation, $(-\infty, -1) \cup (-1, \infty)$. If we take the time to analyze the behavior of g near $t = -1$, we find that as $t \to -1^-$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \approx \sqrt[3]{\frac{-8}{\text{small }(-)}} \approx \sqrt[3]{\text{big}(+)} = \text{big }(+)$. That is, as $t \to -1^-$, $g(t) \to \infty$. Likewise, as $t \to -1^+$, $g(t) \approx \sqrt[3]{\frac{-8}{\text{small }(+)}} \approx \sqrt[3]{\text{big }(-)} = \text{big }(-)$. This suggests as $t \to -1^+$, $g(t) \to -\infty$. This behavior points to a vertical asymptote, $t = -1$.

   To find the t-intercepts of the graph of g, we find the zeros of g by setting $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 0$. Cubing both sides and clearing denominators gives $8t = 0$ or $t = 0$. Hence our t-, and in this case, y- intercept is $(0, 0)$.

   To determine the end behavior, we note that as $t \to \pm\infty$, $\frac{8t}{t+1} \to \frac{8}{1} = 8$. Hence, it stands to reason that as $t \to \pm\infty$, $g(t) = \sqrt[3]{\frac{8t}{t+1}} \to \sqrt[3]{8} = 2$. This suggests the graph of $y = g(t)$ has a horizontal asymptote at $y = 2$.

   We graph $y = g(t)$ in Figure 2.1.15. The graph confirms our suspicions about the asymptotes $t = -1$ and $y = 2$. Moreover, the range appears to be $(-\infty, 2) \cup (2, \infty)$. We could check if the graph ever crosses its horizontal asymptote by attempting to solve $g(t) = \sqrt[3]{\frac{8t}{t+1}} = 2$. Cubing both sides and clearing denominators gives $8t = 8(t + 1)$ which gives $0 = 8$, a contradiction. This proves 2 is not in the range, as we had suspected.

Scanning the graph, there appears to be no local extrema, and, moreover, the graph suggests g is increasing on $(-\infty, -1)$ and again on $(-1, \infty)$. As with the previous example, the graph appears locally vertical near its intercept $(0, 0)$.

To create a sign diagram for $g(t)$, we note that the function is undefined when $t = -1$ (so we place a '‽' above it) and has a zero $t = 0$. When $t < -1$, $g(t) > 0$ or $(+)$, for $-1 < t < 0$, $g(t) < 0$ or $(-)$, and for $t > 0$, $g(t) > 0$ or $(+)$. In Figure 2.1.16 is a sign diagram for $g(t)$.

3. The expression for $h(x) = \frac{3x}{\sqrt{x^2+1}}$ has both a denominator and an even-indexed radical, so we have to be extra cautious here. Fortunately for us, the quantity $x^2 + 1 > 0$ for al real numbers x. Not only does this mean $\sqrt{x^2 + 1}$ is always defined, it also tells us $\sqrt{x^2 + 1} > 0$ for all x, too. This means the domain of h is all real numbers, $(-\infty, \infty)$.

   Solving for the zeros of h gives only $x = 0$, and we find, once again, $(0, 0)$ is both our lone x- and y-intercept. Moving on to end behavior, as $x \to \pm\infty$, the term x^2 is the dominant term in the radicand in the denominator. As such, $h(x) = \frac{3x}{\sqrt{x^2+1}} \approx \frac{3x}{\sqrt{x^2}} = \frac{3x}{|x|}$. As $x \to \infty$, $|x| = x$ (since $x > 0$), so $h(x) \approx \frac{3x}{x} = 3$, so $h(x) \to 3$. Likewise, as $x \to -\infty$, $|x| = -x$ (since $x < 0$) and hence, $h(x) \approx \frac{3x}{-x} = -3$, so $h(x) \to -3$. This analysis suggests the graph of $y = h(x)$ has not one, but *two* horizontal asymptotes.[m] The graph of h in Figure 2.1.17 bears this out.

   From the graph, we see the range of h appears to be $(-3, 3)$. Attempting to solve $h(x) = \frac{3x}{\sqrt{x^2+1}} = \pm 3$ gives, in either case, $9x^2 = 9(x^2 + 1)$ which reduces to $0 = 9$, a contradiction. Hence, the graph

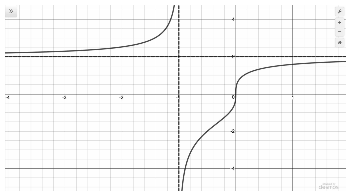


Figure 2.1.15

$(+)$ ‽ $(-)$ 0 $(+)$

-1 0

Figure 2.1.16

m We warned you this was coming . . . see the discussion following Theorem 1.1.3 in Section 1.1.

of $y = h(x)$ never reaches its horizontal asymptotes. Moreover, h appears to be always increasing, with no local extrema or 'unusual' steepness. One last remark: it appears as if the graph of h is symmetric about the origin. We check $h(-x) = \frac{3(-x)}{\sqrt{(-x)^2+1}} = -\frac{3x}{\sqrt{x^2+1}} = -h(x)$ which verifies h is odd.

Since the domain of h is all real number and the only zero of h is $x = 0$, the sign diagram for $h(x)$ is fairly straight forward. For $x < 0$, $h(x) < 0$ or $(-)$ and for $x > 0$, $h(x) > 0$ or $(+)$. The sign diagram for $h(x)$ is in Figure 2.1.18.

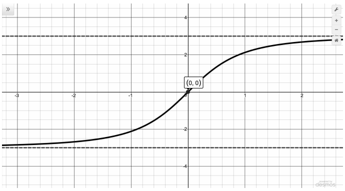


Figure 2.1.17

$(-)$ 0 $(+)$

$\longleftarrow\!\!\!+\!\!\!\longrightarrow$

0

Figure 2.1.18

4. The first thing to note about the expression $r(t) = t^{-1}\sqrt{16t^4 - 1}$ is that $t^{-1} = \frac{1}{t}$. Hence, we must exclude $t = 0$ from the domain straight away. Next, we have an even-indexed radical expression: $\sqrt{16t^4 - 1}$. In order for this to return a real number, we require $16t^4 - 1 \geq 0$. Instead of using a sign diagram to solve this,[n] we opt instead to *carefully* use properties of radicals. Isolating t^4, we have $t^4 \geq \frac{1}{16}$. Since the root functions are increasing, we can apply the fourth root to both sides and preserve the inequality: $\sqrt[4]{t^4} \geq \sqrt[4]{\frac{1}{16}}$ which gives[o] $|t| \geq \frac{1}{2}$. Note that since $t = 0$ does *not* satisfy this inequality, restricting t in this manner takes care of *both* domain issues, so the domain is $\left(-\infty, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$.

   Next, we look for zeros. Setting $r(t) = t^{-1}\sqrt{16t^4 - 1} = \frac{\sqrt{16t^4-1}}{t} = 0$ gives $\sqrt{16t^4 - 1} = 0$. After squaring both sides, we get $16t^4 - 1 = 0$ or $t^4 = \frac{1}{16}$. Extracting fourth roots, we get $t = \pm\frac{1}{2}$. Both of these are (barely!) in the domain of r, so our t intercepts are $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 0\right)$. Note, the graph of r has no y-intercept, since $r(0)$ is undefined ($t = 0$ is not in the domain of r).

   Concerning end behavior, we note the term $16t^4$ dominates the radicand $\sqrt{16t^4 - 1}$ as $t \to \pm\infty$,

See Section **??** n

o Recall: $\sqrt[n]{x^n} = |x|$, not x, if n is even.

hence, $r(t) = \frac{\sqrt{16t^4-1}}{t} \approx \frac{\sqrt{16t^4}}{t} = \frac{4t^2}{t} = 4t$. This suggests the graph of $y = r(t)$ has a slant asymptote with slope 4.[p]

p Note: this analysis suggests the slant asymptote is $y = 4t + b$, but from this analysis, we cannot determine the value of b. As with slant asymptotes in Section 1.1, we'd need to perform a more detailed analysis which we omit in this case owing to the complexity of the function.

We graph $y = r(t)$ below on the left. We see the range appears to be all real numbers, $(-\infty, \infty)$. It appears as if r is increasing on $\left(-\infty, -\frac{1}{2}\right]$ and again on $\left[\frac{1}{2}, \infty\right)$. The graph does appear to be asymptotic to $y = 4t$, and it also appears to be symmetric about the origin. Sure enough, we find $r(-t) = \frac{\sqrt{16(-t)^4-1}}{-t} = -\frac{\sqrt{16t^4-1}}{t} = -r(t)$, proving r is an odd function.

To construct the sign diagram for $r(t)$ we note r has two zeros, $t = \pm\frac{1}{2}$. For $t < \frac{1}{2}$, $r(t) < 0$ or $(-)$ and when $t > \frac{1}{2}$, $r(t) > 0$ or $(+)$. When $-\frac{1}{2} < t < \frac{1}{2}$, r is undefined so we have removed that segment from the diagram, as seen below on the right.

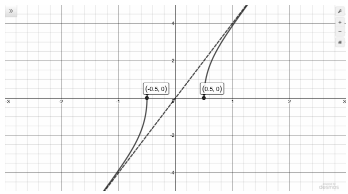


Figure 2.1.19

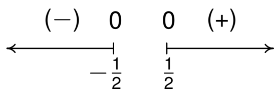


Figure 2.1.20

□

We end this section with a classic application of root functions.

Example 2.1.3. Carl wishes to get high speed internet service installed in his remote Sasquatch observation post located 30 miles from Route 117. The nearest junction box is located 50 miles down the road from the post, as indicated in the diagram below. Suppose it costs \$15 per mile to run cable along the road and \$20 per mile to run cable off of the road. (Refer Figure 2.1.21)

1. Find an expression $C(x)$ which computes the cost of connecting the Junction Box to the Outpost as a function of x, the number of miles the cable is run along Route 117 before heading off road directly towards the Outpost. Determine a reasonable applied domain for the problem.

2. Use your calculator to graph $y = C(x)$ on its domain. What is the minimum cost? How far along Route 117 should the cable be run before turning off of the road?

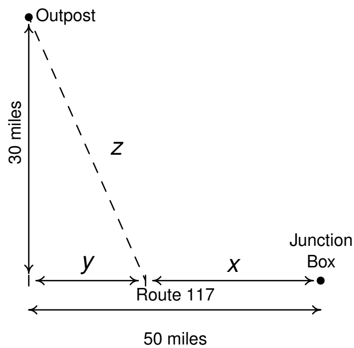


Figure 2.1.21

Solution.

1. The cost is broken into two parts: the cost to run cable along Route 117 at \$15 per mile, and the cost to run it off road at \$20 per mile. Since x represents the miles of cable run along Route 117, the cost for that portion is $15x$. From the diagram, we see that the number of miles the cable is run off road is z, so the cost of that portion is $20z$. Hence, the total cost is $15x + 20z$.

   Our next goal is to determine z in terms of x. The diagram suggests we can use the Pythagorean Theorem to get $y^2 + 30^2 = z^2$. But we also see $x + y = 50$ so that $y = 50 - x$. Substituting $(50 - x)$ in for y we obtain $z^2 = (50 - x)^2 + 900$. Solving for z, we obtain $z = \pm\sqrt{(50 - x)^2 + 900}$. Since z represents a distance, we choose $z = \sqrt{(50 - x)^2 + 900}$.

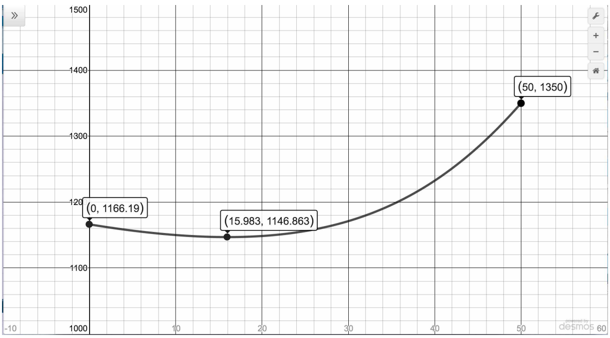
   Hence, the cost as a function of x is given by

   $$C(x) = 15x + 20\sqrt{(50 - x)^2 + 900}$$

   From the context of the problem, we have $0 \leq x \leq 50$.

2. We graph $y = C(x)$ in Figure 2.1.22 and find our (local) minimum to be at the point $(15.98, 1146.86)$. Here the x-coordinate tells us that in order to minimize cost, we should run 15.98 miles of cable along Route 117 and then turn off of the road and head towards the outpost. The y-coordinate tells us that the minimum cost, in dollars, to do so is \$1146.86. The ability to stream live Sasquatch-Casts? Priceless.

□



1500
1400
1300
120
1100
1000
-10
10
20
30
40
50
60
(0, 1166.19)
(15.983, 1146.863)
(50, 1350)
desmos

Figure 2.1.22

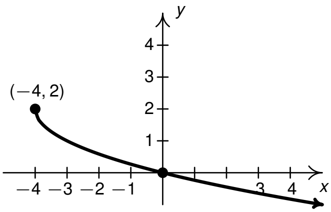
2.1.3 Exercises

In Exercises 1 - 8, given the pair of functions f and F, sketch the graph of $y = F(x)$ by starting with the graph of $y = f(x)$ and using Theorem 2.1.1. Track at least two points and state the domain and range using interval notation.

1. $f(x) = \sqrt{x}$, $F(x) = \sqrt{x+3} - 2$
2. $f(x) = \sqrt{x}$, $F(x) = \sqrt{4-x} - 1$
3. $f(x) = \sqrt[3]{x}$, $F(x) = \sqrt[3]{x-1} - 2$
4. $f(x) = \sqrt[3]{x}$, $F(x) = -\sqrt[3]{8x+8} + 4$
5. $f(x) = \sqrt[4]{x}$, $F(x) = \sqrt[4]{x-1} - 2$
6. $f(x) = \sqrt[4]{x}$, $F(x) = -3\sqrt[4]{x-7} + 1$
7. $f(x) = \sqrt[5]{x}$, $F(x) = \sqrt[5]{x+2} + 3$
8. $f(x) = \sqrt[8]{x}$, $F(x) = \sqrt[8]{-x} - 2$

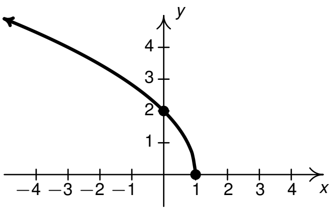
In Exercises 9 - 10, find a formula for each function below in the form $F(x) = a\sqrt{bx-h} + k$.
NOTE: There may be more than one solution!

9. $y = F(x)$. See Figure 2.1.23.
10. $y = F(x)$. See Figure 2.1.24



x, y-intercept $(0, 0)$

Figure 2.1.23



x-intercept $(1, 0)$, y-intercept $(0, 2)$

Figure 2.1.24

In Exercises 11 - 12, find a formula for each function below in the form $F(x) = a\sqrt[3]{bx-h} + k$.
NOTE: There may be more than one solution!

11. $y = F(x)$. See Figure 2.1.25.
12. $y = F(x)$. See Figure 2.1.26.
13. Use the fact that the nth root functions are increasing to solve the following polynomial inequalities:

    (a) $x^3 \leq 64$

    (b) $2 - t^5 < 34$

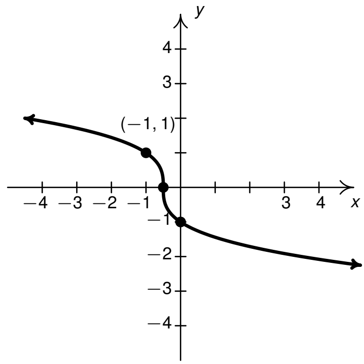
    (c) $\dfrac{(2z+1)^3}{4} \geq 2$

    For the following inequalities, remember $\sqrt[n]{x^n} = |x|$ if n is even:

    (d) $x^4 \leq 16$

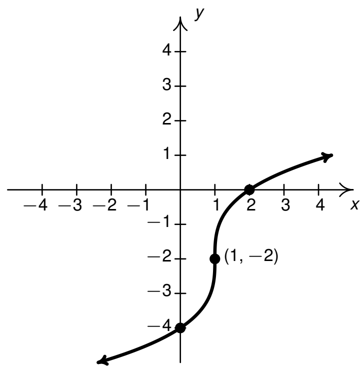
(e) $6 - t^6 < -58$

(f) $\dfrac{(2z+1)^4}{3} \geq 27$



x-intercept $\left(-\frac{1}{2}, 0\right)$, y-intercept $(0, -1)$

Figure 2.1.25



x-intercept $(2, 0)$, y-intercept $(0, -4)$

Figure 2.1.26

For each function in Exercises 14 - 21 below

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase/decrease.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Construct a sign diagram for each function using the intercepts and graph.
- Comment on any observed symmetry.

14. $f(x) = \sqrt{1 - x^2}$

15. $f(x) = \sqrt{x^2 - 1}$

16. $g(t) = t\sqrt{1 - t^2}$

17. $g(t) = t\sqrt{t^2 - 1}$

18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$

19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

20. $g(t) = \sqrt{t(t+5)(t-4)}$

21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$

22. Rework Example 2.1.3 so that the outpost is 10 miles from Route 117 and the nearest junction box is 30 miles down the road for the post.

23. The volume V of a right cylindrical cone depends on the radius of its base r and its height h and is

given by the formula $V = \frac{1}{3}\pi r^2 h$. The surface area S of a right cylindrical cone also depends on r and h according to the formula $S = \pi r\sqrt{r^2 + h^2}$. In the following problems, suppose a cone is to have a volume of 100 cubic centimeters.

(a) Use the formula for volume to find the height as a function of r, $h(r)$.

(b) Use the formula for surface area along with your answer to 23a to find the surface area as a function of r, $S(r)$.

(c) Use your calculator to find the values of r and h which minimize the surface area. What is the minimum surface area? Round your answers to two decimal places.

24. The period of a pendulum in seconds is given by

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(for small displacements) where L is the length of the pendulum in meters and $g = 9.8$ meters per second per second is the acceleration due to gravity. My Seth-Thomas antique schoolhouse clock needs $T = \frac{1}{2}$ second and I can adjust the length of the pendulum via a small dial on the bottom of the bob. At what length should I set the pendulum?

25. According to Einstein's Theory of Special Relativity, the observed mass of an object is a function of how fast the object is traveling. Specifically, if m_r is the mass of the object at rest, v is the speed of the object and c is the speed of light, then the observed mass of the object $m(v)$ is given by:

$$m(v) = \frac{m_r}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(a) Find the applied domain of the function.

(b) Compute $m(.1c)$, $m(.5c)$, $m(.9c)$ and $m(.999c)$.

(c) As $v \to c^-$, what happens to $m(x)$?

(d) How slowly must the object be traveling so that the observed mass is no greater than 100 times its mass at rest?

26. Find the inverse of $k(x) = \dfrac{2x}{\sqrt{x^2 - 1}}$.

2.1.4 Answers

1. $F(x) = \sqrt{x+3} - 2$
   Domain: $[-3, \infty)$, Range: $[-2, \infty)$
   See Figure 2.1.27

2. $F(x) = \sqrt{4-x} - 1 = \sqrt{-x+4} - 1$
   Domain: $(-\infty, 4]$, Range: $[-1, \infty)$
   See Figure 2.1.28

3. $F(x) = \sqrt[3]{x-1} - 2$
   Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$
   See Figure 2.1.29

4. $F(x) = -\sqrt[3]{8x+8} + 4$
   Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$
   See Figure 2.1.30

5. $F(x) = \sqrt[4]{x-1} - 2$
   Domain: $[1, \infty)$, Range: $[-2, \infty)$
   See Figure 2.1.31

6. $F(x) = -3\sqrt[4]{x-7} + 1$
   Domain: $[7, \infty)$, Range: $(-\infty, 1]$
   See **??**

7. $F(x) = \sqrt[5]{x+2} + 3$
   Domain: $(-\infty, \infty)$, Range: $(-\infty, \infty)$
   See Figure 2.1.33

8. $F(x) = \sqrt[8]{-x} - 2$
   Domain: $(-\infty, 0]$, Range: $[-2, \infty)$
   See Figure 2.1.34

9. One solution is: $F(x) = -\sqrt{x+4} + 2$

10. One solution is: $F(x) = 2\sqrt{-x+1}$

11. One solution is: $F(x) = -\sqrt[3]{2x+1}$

12. One solution is: $F(x) = 2\sqrt[3]{x-1} - 2$

13. (a) $(-\infty, 4]$ (b) $(-2, \infty)$
    (c) $\left[\frac{1}{2}, \infty\right)$ (d) $[-2, 2]$
    (e) $(-\infty, -2) \cup (2, \infty)$
    (f) $(-\infty, -2] \cup [1, \infty)$

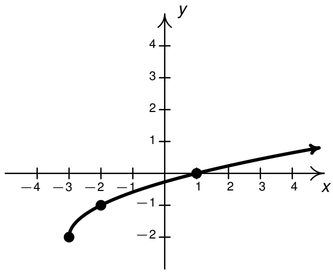


Figure 2.1.27

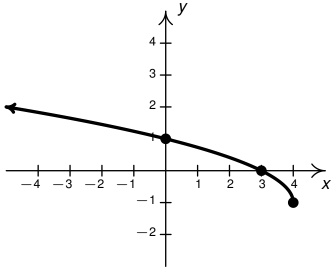


Figure 2.1.28

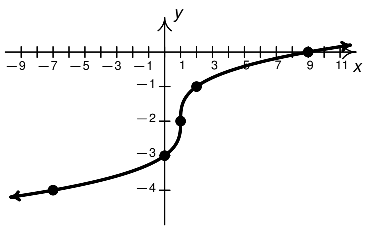


Figure 2.1.29

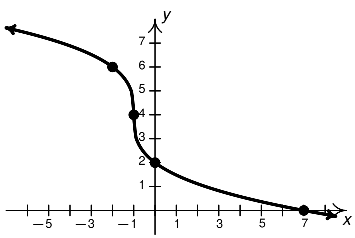


Figure 2.1.30

14. $f(x) = \sqrt{1 - x^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph: See Figure 2.1.35
Range: $[0, 1]$
Local maximum: $(0, 1)$
Increasing: $[-1, 0]$, Decreasing: $[0, 1]$
Unusual steepness[1] at $x = -1$ and $x = 1$
Sign Diagram: 0 (+) 0
-1 1
Note: f is even.

15. $f(x) = \sqrt{x^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph: See Figure 2.1.36
As $x \to \pm\infty$, $f(x) \to \infty$
Range: $[0, \infty)$
Increasing: $[1, \infty)$, Decreasing: $(-\infty, -1]$
Unusual steepness[2] at $x = -1$ and $x = 1$
Using Calculus, one can show $y = \pm x$ are slant asymptotes to the graph.
Sign Diagram: (+) 0 0 (+)
-1 1
Note: f is even.

16. $g(t) = t\sqrt{1 - t^2}$
Domain: $[-1, 1]$
Intercepts: $(-1, 0)$, $(0, 0)$, $(1, 0)$
Graph: See Figure 2.1.37
Range: $\approx [-0.5, 0.5]$
Local minimum $\approx (-0.707, -0.5)$
Local maximum: $\approx (0.707, 0.5)$
Increasing: $\approx [-0.707, 0.707]$
Decreasing: $\approx [-1, -0.707]$, $[0.707, 1]$
Unusual steepness at $t = -1$ and $t = 1$

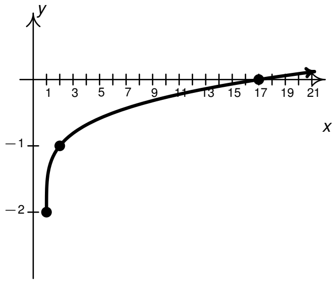


Figure 2.1.31

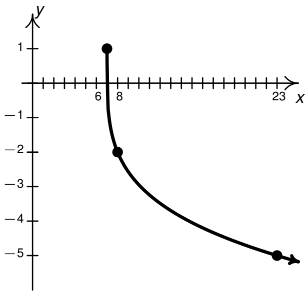


Figure 2.1.32

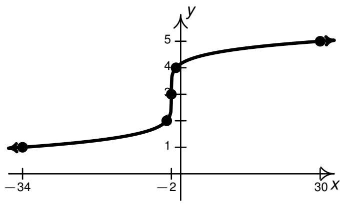


Figure 2.1.33

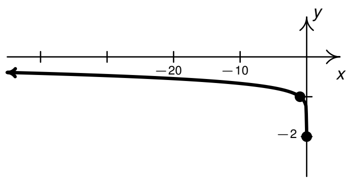


Figure 2.1.34

[1] You may need to zoom in to see this.
[2] You may need to zoom in to see this.

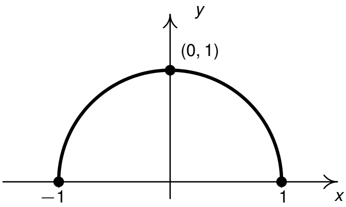


Figure 2.1.35

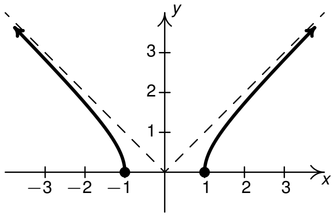


Figure 2.1.36

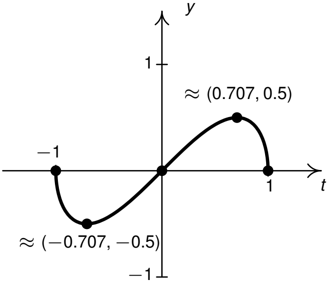


Figure 2.1.37

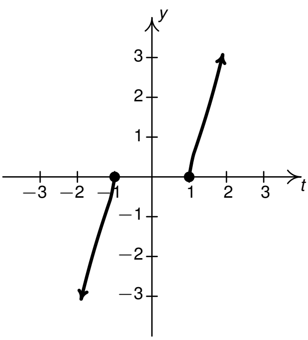


Figure 2.1.38

Sign Diagram: 0 (−) 0 (+) 0

−1 0 1

Note: g is odd.

17. $g(t) = t\sqrt{t^2 - 1}$
Domain: $(-\infty, -1] \cup [1, \infty)$
Intercepts: $(-1, 0)$, $(1, 0)$
Graph: See Figure 2.1.38
As $t \to -\infty$, $g(t) \to -\infty$
As $t \to \infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Increasing: $(-\infty, -1]$, $[1, \infty)$
Unusual steepness at $t = -1$ and $t = 1$
Sign Diagram: (−) 0 0 (+)

−1 1

Note: g is odd.

18. $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$
Domain: $(-3, 0] \cup (3, \infty)$
Graph: See Figure 2.1.39
Intercept: $(0, 0)$
As $x \to \infty$, $f(x) \to 0$
Range: $[0, \infty)$
Decreasing: $(-3, 0]$, $(3, \infty)$
Unusual steepness at $x = 0$
Vertical asymptotes: $x = -3$ and $x = 3$
Horizontal asymptote: $y = 0$
Sign Diagram: ‽ (+) 0 ‽ (+)

−3 0 3

19. $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$
Graph: See Figure 2.1.40
Domain: $(-\infty, -2) \cup (-2, \infty)$
Intercept: $(0, 0)$
As $x \to \pm\infty$, $f(x) \to 5$
Range: $(-\infty, 5) \cup (5, \infty)$
Increasing: $(-\infty, -2)$, $(-2, \infty)$
Vertical asymptote $x = -2$

Horizontal asymptote $y = 5$

Sign Diagram: $(+)$ ‽ $(-)$ 0 $(+)$ at -2, 0

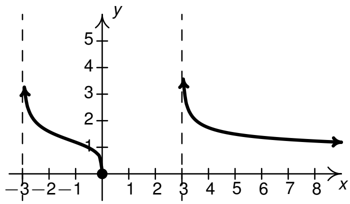


Figure 2.1.39

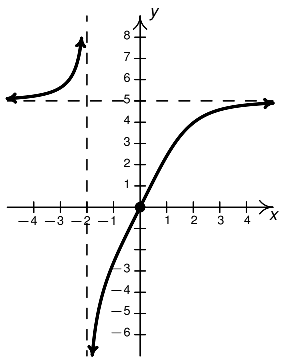


Figure 2.1.40

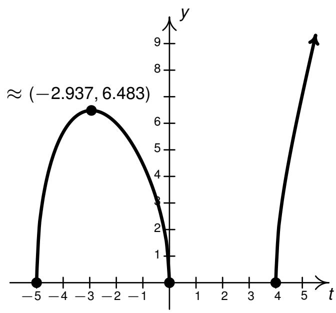


Figure 2.1.41

20. $g(t) = \sqrt{t(t+5)(t-4)}$
    Domain: $[-5, 0] \cup [4, \infty)$
    Intercepts $(-5, 0)$, $(0, 0)$, $(4, 0)$
    As $t \to \infty$, $g(t) \to \infty$
    Graph: See Figure 2.1.41
    Range: $[0, \infty)$
    Local maximum $\approx (-2.937, 6.483)$
    Increasing: $\approx [-5, -2.937]$, $[4, \infty)$
    Decreasing: $\approx [-2.937, 0]$
    Unusual steepness at $t = -5$, $t = 0$ and $t = 4$
    Sign Diagram: 0 $(+)$ 0 on $[-5, 0]$; 0 $(+)$ on $[4, \infty)$

21. $g(t) = \sqrt[3]{t^3 + 3t^2 - 6t - 8}$
    Domain: $(-\infty, \infty)$
    Intercepts: $(-4, 0)$, $(-1, 0)$, $(0, -2)$, $(2, 0)$
    Graph: See Figure 2.1.42
    as $t \to -\infty$, $g(t) \to -\infty$
    as $t \to \infty$, $g(t) \to \infty$
    Range: $(-\infty, \infty)$
    Local maximum: $\approx (-2.732, 2.182)$
    Local minimum: $\approx (0.732, -2.182)$
    Increasing: $\approx (-\infty, -2.732]$, $[0.732, \infty)$
    Decreasing: $\approx [-2.732, 0.732]$
    Unusual steepness at $t = -4$, $t = -1$ and $t = 2$
    Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.
    Sign Diagram: $(-)$ 0 $(+)$ 0 $(-)$ 0 $(+)$ at -4, -1, 2

22. $C(x) = 15x + 20\sqrt{100 + (30 - x)^2}$, $0 \leq x \leq 30$. The calculator gives the absolute minimum at approximately $(18.66, 582.29)$. This means to minimize the cost, approximately 18.66 miles of cable should be run along Route 117 before turning off

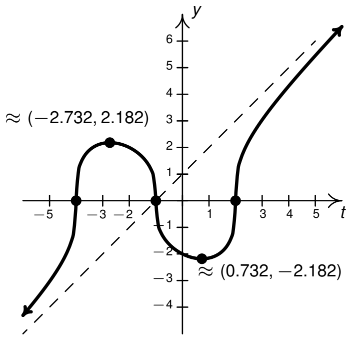


Figure 2.1.42

the road and heading towards the outpost. The minimum cost to run the cable is approximately \$582.29.

23. (a) $h(r) = \frac{300}{\pi r^2}$, $r > 0$.

 (b) $S(r) = \pi r \sqrt{r^2 + \left(\frac{300}{\pi r^2}\right)^2} = \frac{\sqrt{\pi^2 r^6 + 90000}}{r}$, $r > 0$

 (c) The calculator gives the absolute minimum at the point $\approx (4.07, 90.23)$. This means the radius should be (approximately) 4.07 centimeters and the height should be 5.76 centimeters to give a minimum surface area of 90.23 square centimeters.

24. $9.8\left(\frac{1}{4\pi}\right)^2 \approx 0.062$ meters or 6.2 centimeters

25. (a) $[0, c)$

 (b) $m(.1c) = \dfrac{m_r}{\sqrt{.99}} \approx 1.005m_r$
 $m(.5c) = \dfrac{m_r}{\sqrt{.75}} \approx 1.155m_r$
 $m(.9c) = \dfrac{m_r}{\sqrt{.19}} \approx 2.294m_r$
 $m(.999c) = \dfrac{m_r}{\sqrt{.0.001999}} \approx 22.366m_r$

 (c) As $v \to c^-$, $m(x) \to \infty$

 (d) If the object is traveling no faster than approximately 0.99995 times the speed of light, then its observed mass will be no greater than $100m_r$.

26. $k^{-1}(x) = \dfrac{x}{\sqrt{x^2 - 4}}$

2.2 Power Functions

Monomial, and, more generally, Laurent monomial functions are specific examples of a much larger class of functions called **power functions**, as defined in Definition 2.2.1.

Definition 2.2.1 broadens our scope of functions to include non-integer exponents such as $f(x) = 2x^{4/3}$, $g(t) = t^{0.4}$ and $h(w) = w^{\sqrt{2}}$. Our primary aim in this section is to ascribe meaning to these quantities.

Definition 2.2.1. Let a and p be nonzero real numbers. A **power function** is either a constant function or a function of the form $f(x) = ax^p$.

2.2.1 Rational Number Exponents

The road to real number exponents starts by defining rational number exponents. See Definition 2.2.2.

Definition 2.2.2. Let r be a rational number where in lowest terms $r = \frac{m}{n}$ where m is an integer and n is a natural number.[a] If $n = 1$, then $x^r = x^m$. If $n > 1$, then

$$x^r = x^{\frac{m}{n}} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m},$$

whenever $\left(\sqrt[n]{x}\right)^m$ is defined.[b]

[a]Recall 'lowest terms' means m and n have no common factors other than 1.

[b]That is, if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

There are quite a few items worthy of note which are consequences of Definition 2.2.2. First off, if m is an integer, then $x^{\frac{m}{1}} = x^m$ so expressions like $x^{\frac{3}{1}}$ are synonymous with x^3, as we would expect.[a] Second, the definition of $x^{\frac{m}{n}}$ can be taken as just $\left(\sqrt[n]{x}\right)^m$ and shown to be equal to $\sqrt[n]{x^m}$ (or vice-versa) courtesy of properties of radicals. We state both in Definition 2.2.2 to allow for the reader to choose whichever form is more convenient in a given situation. The critical point to remember is no matter which representation you choose, keep in mind the restrictions if n is even, $x \geq 0$ and if $m < 0$, $x \neq 0$.

Moreover, per this definition, $x^{\frac{1}{n}} = \sqrt[n]{x^1} = \sqrt[n]{x}$, so we may rewrite principal roots as exponents: $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[5]{x} = x^{\frac{1}{5}}$. This makes sense from an algebraic standpoint since per Theorem 2.1.2, $\left(\sqrt[n]{x}\right)^n = x$. Hence if we were to assign an exponent notation to $\sqrt[n]{x}$, say $\sqrt[n]{x} = x^r$, then $\left(\sqrt[n]{x}\right)^n = (x^r)^n = x$. If the properties of exponents are to hold, then, necessarily, $(x^r)^n = x^{rn} = x = x^1$, so $rn = 1$ or $r = \frac{1}{n}$. While this argument helps *motivate* the notation, as we shall see shortly, great care must be exercised in applying exponent properties in these cases. The long and short of this is that root functions as defined

a Either $n = 1$ is a special case in Definition 2.2.2 or we need to define what is meant by $\sqrt[1]{x}$. The authors chose the former.

in Section 2.1 are all members of the 'power functions' family.

Another important item worthy of note in Definition 2.2.2 is that it is absolutely essential we express the rational number r in *lowest terms* before applying the root-power definition. For example, consider $x^{0.4}$. Expressing r in lowest terms, we get: $r = 0.4 = \frac{4}{10} = \frac{2}{5}$. Hence, $x^{0.4} = x^{2/5} = (\sqrt[5]{x})^2$ or $\sqrt[5]{x^2}$, either of which is defined for all real numbers x. In contrast, consider the equivalence $r = 0.4 = \frac{4}{10}$. Here, the expression $(\sqrt[10]{x})^4$ is defined only for $x \geq 0$ owing to the presence of the even indexed root, $\sqrt[10]{x}$. Hence, $(\sqrt[10]{x})^4 \neq x^{\frac{4}{10}} = x^{\frac{2}{5}}$ unless $x \geq 0$. On the other hand, the expression $\sqrt[10]{x^4}$ *is* defined for all numbers, x, since $x^4 \geq 0$ for all x. In fact, it can be shown that $\sqrt[10]{x^4} = \sqrt[5]{x^2}$ for all real numbers. This means $\sqrt[10]{x^4} = \sqrt[5]{x^2} = x^{\frac{2}{5}} = x^{\frac{4}{10}}$. So, to review, in general we have: $x^{\frac{4}{10}} = \sqrt[10]{x^4}$, but $x^{\frac{4}{10}} \neq \left(\sqrt[10]{x}\right)^4$ unless $x \geq 0$. Once again the easiest way to avoid confusion here is to *reduce* the exponent to *lowest terms* before converting it to root-power notation.

Likewise, we have to be careful about the properties of exponents when it comes to rational exponents. Consider, for instance, the product rule for integer exponents: $x^m x^n = x^{m+n}$. Consider $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}}$ and $g(x) = x^{\frac{1}{2}+\frac{1}{2}}$. In the first case, $f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = \sqrt{x}\sqrt{x} = (\sqrt{x})^2 = x$ only for $x \geq 0$. In the second case, $g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x^1 = x$ for all real numbers x. Even though $f(x) = g(x)$ for $x \geq 0$, f and g are *different functions* since they have *different domains*.

Similarly, the power rule for integer exponents: $(x^n)^m = x^{nm}$ does not hold in general for rational exponents. To see this, consider the three functions: $f(x) = (x^{\frac{1}{2}})^2$, $g(x) = x^{\frac{2}{2}}$, and $h(x) = (x^2)^{\frac{1}{2}}$. In the first case, $f(x) = (x^{\frac{1}{2}})^2 = (\sqrt{x})^2 = x$ for $x \geq 0$ only (this is the same function f above.) In the second case, the rational number $r = \frac{2}{2} = 1$, so $g(x) = x^{\frac{2}{2}} = x^{\frac{1}{1}} = x^1 = x$ for *all* real numbers, x (this is the same function g from above.) In the last case,

$h(x) = (x^2)^{\frac{1}{2}} = \sqrt{x^2} = |x|$ for all real numbers, x. Once again, despite $f(x) = g(x) = h(x)$ for all $x \geq 0$, f, g and h and are *three different functions*. We graph f, g, and h Figure 2.2.1, Figure 2.2.2 and Figure 2.2.3 respectively.

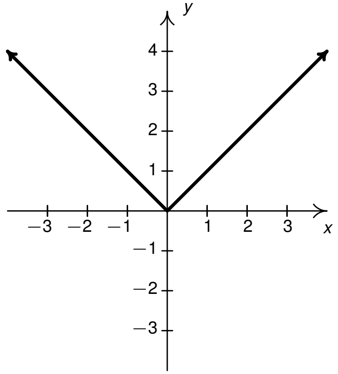


Figure 2.2.3: $h(x) = (x^2)^{\frac{1}{2}} = |x|$

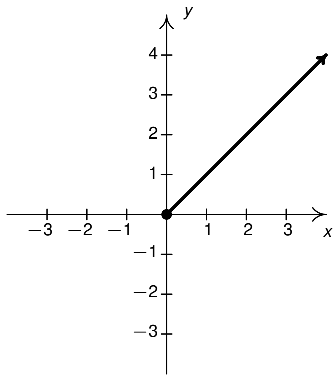


Figure 2.2.1:
$f(x) = x^{\frac{1}{2}}x^{\frac{1}{2}} = (x^{\frac{1}{2}})^2 = x$, $x \geq 0$

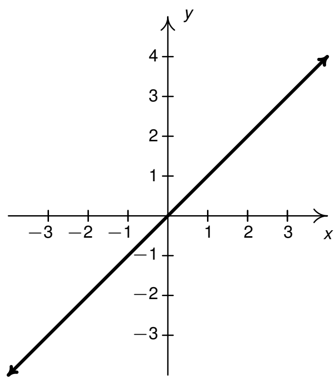


Figure 2.2.2: $g(x) = x^{\frac{1}{2}+\frac{1}{2}} = x^{\frac{2}{2}} = x$

In general, the properties of integer exponents *do not extend* to rational exponents *unless* the bases involved represent non-negative real numbers *or* the roots involved are *odd*. We have the following:

Theorem 2.2.1. Let r and s are rational numbers. The following properties hold provided none of the computations results in division by 0 and **either** r and s have odd denominators **or** $x \geq 0$ **and** $y \geq 0$:

- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$.
- **Quotient Rules:** $\dfrac{x^r}{x^s} = x^{r-s}$ and $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{y^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

Next, we turn our attention to the graphs of $f(x) = x^r = x^{\frac{m}{n}}$ for varying values of m and n. When n is even, the domain is restricted owing to the presence of the even indexed root to $[0, \infty)$. The range is likewise $[0, \infty)$, a fact leave to the reader. All of the functions below are increasing on their domains, and it turns out this is always the case provided $r > 0$. There is, however, a difference in *how* the functions are increasing—and this is the concept of *concavity*. As with many concepts we've encountered so far in the text, concavity is most precisely defined using Calculus terminology, but we can nevertheless get a sense of concavity geometrically. For us, a curve is **concave up** over an interval if it resembles a portion of a '$\smile$' shape. Similarly, a curve is called **concave down** over an interval if resembles part of a '$\frown$' shape. When $0 < r < 1$, the graphs of $f(x) = x^r$ resemble the left half of $\frown$ and so are concave down; when $r > 1$, the graphs resemble the right half of a '$\smile$' and are hence described as 'concave up.' See Figure 2.2.4 to Figure 2.2.7.

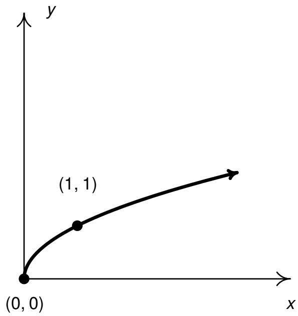


Figure 2.2.4: $f(x) = x^{\frac{1}{2}}$

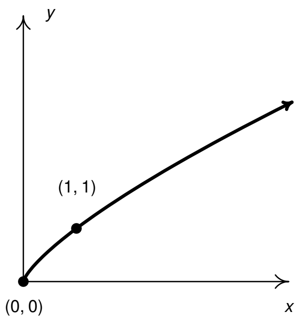


Figure 2.2.5: $f(x) = x^{\frac{3}{4}}$

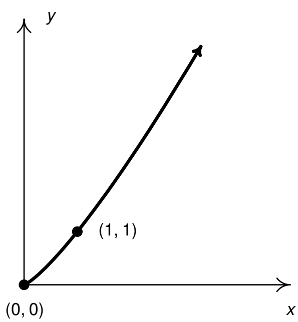


Figure 2.2.6: $f(x) = x^{\frac{5}{4}}$

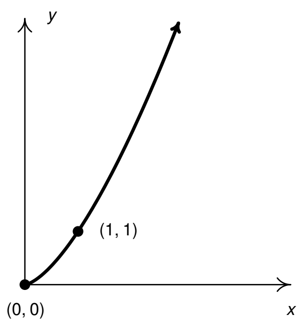


Figure 2.2.7: $f(x) = x^{\frac{3}{2}}$

In Figure 2.2.8 to Figure 2.2.11 we graph several examples of $f(x) = x^r = x^{\frac{m}{n}}$ where n is odd. Here, the domain is $(-\infty, \infty)$ since the index on the root here is odd. Note that when m is even, the graphs appear to be symmetric about the y-axis and the range looks to be $[0, \infty)$. When m is odd, the graphs appear to be symmetric about the origin with range $(-\infty, \infty)$. We leave verification of these facts to the reader. Note here also that for $x \geq 0$, the graphs are down for $0 < r < 1$ and concave up for $r > 1$.

When $r < 0$, we have variables appear in the denominator which open the opportunities for vertical and horizontal asymptotes. In Figure 2.2.12 and Figure 2.2.13 are graphed two examples.

Unsurprisingly, Theorem 2.1.1, which, as stated, applied to root functions, generalizes to all rational powers as shown in Theorem 2.2.2.

Theorem 2.2.2. For real numbers a, b, h, and k and rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx-h)^r+k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^r$ to $y = (x - h)^r$.
2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.
   NOTE: This transforms the graph of $y = (x - h)^r$ to $y = (bx - h)^r$.
3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (bx-h)^r$ to $y = a(bx - h)^r$.
4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(bx - h)^r$ to $y = a(bx - h)^r + k$.

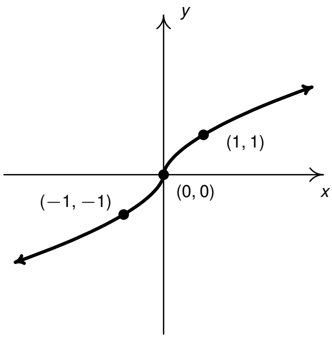


Figure 2.2.8: $f(x) = x^{\frac{3}{5}}$

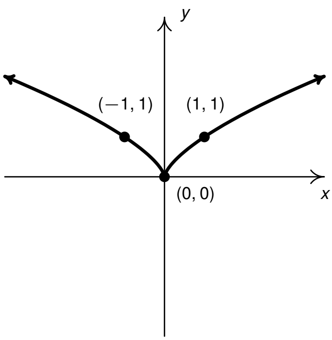


Figure 2.2.9: $f(x) = x^{\frac{2}{3}}$

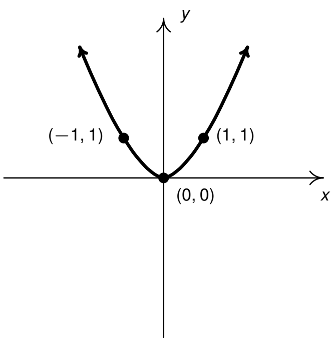


Figure 2.2.10: $f(x) = x^{\frac{8}{5}}$

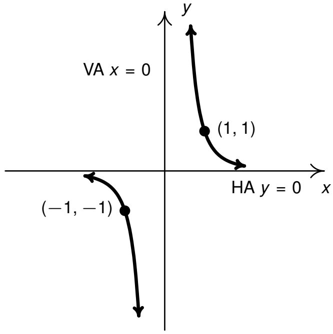


Figure 2.2.13: $f(x) = x^{-\frac{1}{3}}$

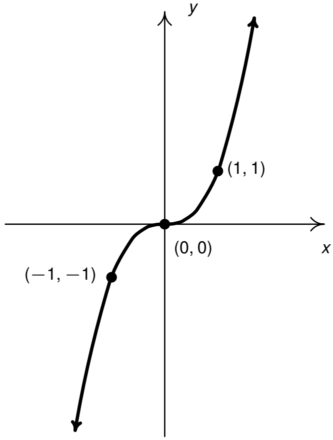


Figure 2.2.14: $f(x) = x^{2.6}$

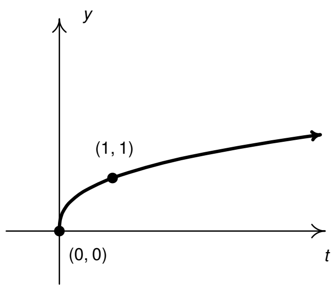


Figure 2.2.15: $g(t) = t^{\frac{3}{8}}$

The proof of Theorem 2.2.2 is identical to that of Theorem 2.1.1, and we suggest the reader work through the details. We give Theorem 2.2.2 a test run in the following example.

Example 2.2.1. Use the given graphs of f and g in Figure 2.2.14 and Figure 2.2.15, along with Theorem 2.2.2 to graph F and G. State the domain and range of F and G using interval notation.

1. Graph $F(x) = (2x - 1)^{2.6}$.
2. Graph $G(t) = 1 - 2(t + 3)^{\frac{3}{8}}$.

Solution.

1. The expression $F(x) = (2x - 1)^{2.6}$ is given to us in the form prescribed by Theorem 2.2.2, and we identify $r = 2.6$, $a = 1$, $b = 2$, $h = 1$, and $k = 0$. Even though the graph of $f(x) = x^{2.6}$ is given to us, it's worth taking a moment to reinforce some concepts. Since, in lowest terms, $2.6 = \frac{26}{10} = \frac{13}{5}$, it makes sense the domain and range of $f(x) = x^{2.6}$ are both all real numbers and the graph is symmetric about the origin.[1] Moreover, since $2.6 > 1$, the concavity matches what we would expect, too. We proceed as we have several times in the past, beginning with the horizontal shift.

   Step 1: add 1 to each of the x-coordinates of each of the points on the graph of $y = x^{2.6}$ as in Figure 2.2.16.

   Step 2: divide each of the x-coordinates of each of the points on the graph of $y = (x - 1)^{2.6}$ by 2 as in Figure 2.2.17.

   We get the domain and range here are both $(-\infty, \infty)$.

[1] The domain is all real numbers since the denominator (root) 5 is odd; the range is all real numbers since the numerator (power) 13 is odd. Since both power and root are odd, the function itself is an odd function, hence the symmetry about the origin.

2. We first need to rewrite $G(t) = 1 - 2(t+3)^{\frac{3}{8}}$ in the form required by Theorem 2.2.2: $G(t) = -2(t+3)^{\frac{3}{8}} + 1$. We identify $r = \frac{3}{8}$, $a = -2$, $b = 1$, $h = -3$, and $k = 1$. Since $\frac{3}{8}$ is in lowest terms and has an even denominator, it makes sense the domain and range of $g(t) = t^{\frac{3}{8}}$ is $[0, \infty)$, since the root here, 8 is even. Also, since $0 < \frac{3}{8} < 1$, the graph of $y = t^{\frac{3}{8}}$ is concave down, as we would expect. As usual, we start with the horizontal shift.

   Step 1: add -3 to each of the t-coordinates of each of the points on the graph of $y = t^{\frac{3}{8}}$ as in Figure 2.2.18.

   Step 2: Since $b = 1$, we proceed directly to Step 3.

   Step 3: multiply each of the y-coordinates of each of the points on the graph of $y = (t+3)^{\frac{3}{8}}$ by -2 as in Figure 2.2.19.

   Step 4: add 1 to each of the y-coordinates of each of the points on the graph of $y = -2(t+3)^{\frac{3}{8}}$ as in Figure 2.2.20.

   From the graph, we get the domain is $[-3, \infty)$ and the range is $(-\infty, 1]$.

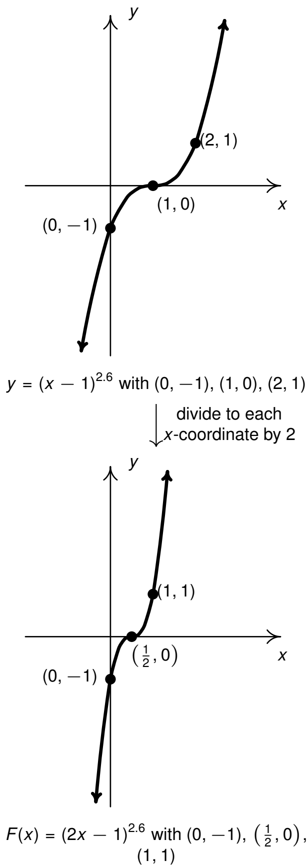


Figure 2.2.17

We now turn our attention to more complicated functions involving rational exponents.

Example 2.2.2. For the following functions:

- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.

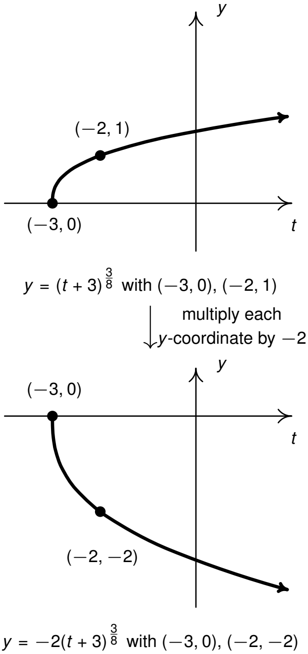


Figure 2.2.19

- the local extrema, if they exist.
- intervals of increase.
- intervals of decrease.

• Construct a sign diagram for each function using the intercepts and graph.

1. $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}}$
2. $g(t) = \dfrac{(t^2 - 4)^{\frac{3}{2}}}{t^2 - 36}$

Solution.

1. We first note that, owing to the negative exponent, the quantity $(x^3 - 8)^{\frac{2}{3}}$ is in the denominator, alerting us to a potential domain issue. Rewriting $(x^3 - 8)^{\frac{2}{3}}$ we set about solving $\sqrt[3]{(x^3 - 8)^2} = 0$. Cubing both sides and extracting square roots gives $x^3 - 8 = 0$ or $x = 2$. Hence, $x = 2$ is excluded from the domain.[b] Since the root involved here is odd (3), the only issue we have is with the denominator, hence our domain is $\{x \in \mathbb{R} \,|\, x \neq 2\}$ or $(-\infty, 2) \cup (2, \infty)$.

   While not required to do so, we analyze the behavior of f near $x = 2$. As $x \to 2^-$, $3x^2 \approx 12$ and $x^3 - 8 \approx$ small $(-)$. Hence, $(x^3 - 8)^{\frac{2}{3}} = \sqrt[3]{(x^3 - 8)^2} \approx \sqrt[3]{(\text{small }(-))^2} \approx \sqrt[3]{\text{small }(+)} \approx$ small(+). As such, $f(x) \approx \frac{12}{\text{small}(+)} \approx$ big (+). We conclude as $x \to 2^-$, $f(x) \to \infty$. As $x \to 2^+$, $3x^2 \approx 12$ and $x^3 - 8 \approx$ small (+), and we likewise get $f(x) \to \infty$. This analysis suggests $x = 2$ is a vertical asymptote to the graph.

   To find the x-intercepts, we set $f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = 0$, so that $3x^2 = 0$ or $x = 0$. We get $(0, 0)$ is our only x- (and y-)intercept.

   For end behavior, we note that in the denominator the x^3 term dominates the constant term, so as

$x \to \pm\infty$,

$$f(x) = 3x^2(x^3 - 8)^{-\frac{2}{3}} = \frac{3x^2}{(x^3 - 8)^{\frac{2}{3}}} \approx \frac{3x^2}{(x^3)^{\frac{2}{3}}}$$
$$= \frac{3x^2}{\sqrt[3]{(x^3)^2}} = \frac{3x^2}{\sqrt[3]{x^6}} = \frac{3x^2}{x^2} = 3.$$

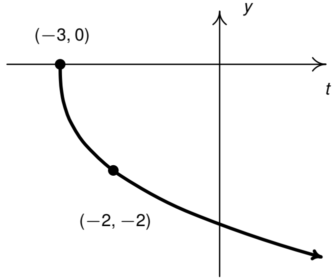
This suggests $y = 3$ is a horizontal asymptote to the graph.

Graphing $y = f(x)$ in Figure 2.2.21 bears out our analysis regarding zeros and asymptotes. The range appears to be $[0, \infty)$, with the graph of $y = f(x)$ crossing its horizontal asymptote between $x = 1$ and $x = 2$. We see we have a single local minimum at $(0, 0)$ with f is decreasing on $(-\infty, 0]$ and $(2, \infty)$ and increasing on $[0, 2)$.

For the sign diagram, we note that f has only one zero, $x = 0$ and is undefined at $x = 2$. For all x values between these two numbers, $f(x) > 0$ or $(+)$. Our sign diagram for $f(x)$ is in Figure 2.2.22.

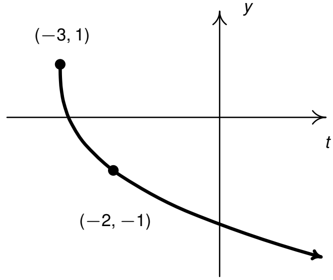
2. To find the domain of $g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36}$, we have two issues to address: the denominator and an even (square) root. Solving $t^2 - 36 = 0$ gives two excluded values, $t = \pm 6$. For the numerator, we may rewrite $(t^2 - 4)^{\frac{3}{2}} = (\sqrt{t^2 - 4})^3$, so we require $t^2 - 4 \geq 0$, or $t^2 \geq 4$. Extracting square roots, we have $\sqrt{t^2} \geq \sqrt{4}$ or $|t| \geq 2$ which means $t \leq -2$ or $t \geq 2$. Taking into account our excluded values $t = \pm 6$, we get the domain of g is $(-\infty, -6) \cup (-6, -2] \cup [2, 6) \cup (6, \infty)$.

   Looking near $t = -6$, we note that as $t \to -6$, $(t^2 - 4)^{\frac{3}{2}} \approx 32^{\frac{3}{2}} = 32^{1.5}$, a positive number. As $t \to -6^-$, $t^2 - 36 \approx$ small $(+)$, so $g(t) \approx \frac{32^{1.5}}{\text{small}(+)} \approx$ big $(+)$. This suggests as $t \to -6^-$, $g(t) \to \infty$. On the other hand, as $t \to -6^+$, $t^2 - 36 \approx$ small $(-)$, so $g(t) \approx \frac{32^{1.5}}{\text{small}(-)} \approx$ big $(-)$, suggesting $g(t) \to$



$y = -2(t+3)^{\frac{3}{8}}$ with $(-3, 0)$, $(-2, -2)$

add 1 to each y-coordinat



$y = -2(t+3)^{\frac{3}{8}} + 1$ with $(-3, 1)$, $(-2, -1)$

Figure 2.2.20

In general if $u^p = 0$ where $p > 0$, then b $u = 0$.

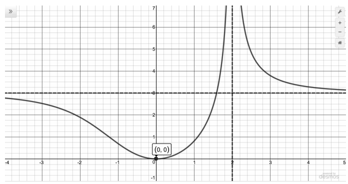


Figure 2.2.21: The graph of $y = f(x)$

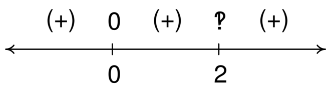


Figure 2.2.22: Sign Diagram for $f(x)$

$-\infty$. Similarly, we find as $t \to 6^-$, $g(t) \to -\infty$ and as $t \to 6^+$, $g(t) \to \infty$. This suggests we have two vertical asymptotes to the graph of $y = g(t)$: $t = -6$ and $t = 6$.

To find the t-intercepts, we set $g(t) = 0$ and solve $(t^2 - 4)^{\frac{3}{2}} = 0$. This reduces to $t^2 - 4 = 0$ or $t = \pm 2$. As these are (just barely!) in the domain of g, we have two t-intercepts, $(-2, 0)$ and $(2, 0)$. The graph of g has no y-intercepts, since 0 is not in the domain of g, so $g(0)$ is undefined.

Regarding end behavior, as $t \to \pm\infty$, the t^2 in both numerator and denominator dominate the constant terms, so we have

$$\begin{aligned} g(t) = \frac{(t^2-4)^{\frac{3}{2}}}{t^2-36} \approx \frac{\left(t^2\right)^{\frac{3}{2}}}{t^2} &= \frac{\left(\sqrt{t^2}\right)^3}{t^2} = \frac{|t|^3}{t^2} \\ &= \frac{|t||t|^2}{t^2} = \frac{|t|t^2}{t^2} = |t| \end{aligned}$$

This suggests that as $t \to \infty$, the graph of $y = g(t)$ resembles $y = |t|$. Using the piecewise definition of $|t|$, we have that as $t \to -\infty$, $g(t) \approx -t$ and as $t \to \infty$, $g(t) \approx t$. In other words, the graph of $y = g(t)$ has *two* slant asymptotes with slopes ± 1.

Graphing $y = g(t)$ in Figure 2.2.23 verifies our analysis. From the graph, the range appears to be $(-\infty, 0] \cup [14.697, \infty)$. The points $(-10, 14.697)$ and $(10, 14.697)$ are local minimums. g appears to be decreasing on $(-\infty, -10]$, $[2, 6)$, and $(6, 10]$. Likewise, g is increasing on $[-10, -6)$, $(-6, -2]$ and $[10, \infty)$. The graph of $y = g(t)$ certainly appears to be symmetric about the y-axis. We leave it to the reader to show g is, indeed, an even function.

For the sign diagram for $g(t)$, we note that g has zeros $t = \pm 2$ and is undefined at $t = \pm 6$. Moreover, there is a gap in the domain for all values in the interval $(-2, 2)$, so we excise that portion of the real number line for our discussion. We find $g(t) > 0$ or $(+)$ on the intervals $(-\infty - 6)$ and $(6, \infty)$ while $g(t) < 0$ or $(-)$ on $(-6, -2)$ and $(2, 6)$. Our sign diagram for $g(t)$ is in Figure 2.2.24.

□

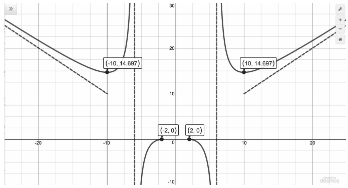


Figure 2.2.23: The graph of $y = g(t)$

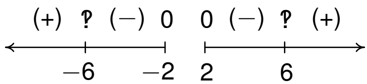


Figure 2.2.24: Sign Diagram for $g(t)$

2.2.2 Real Number Exponents

We wish now to extend the concept of 'exponent' from rational to all real numbers which means we need to discuss how to interpret an irrational exponent. Once again, the notions presented here are best discussed using the language of Calculus or Analysis, but we nevertheless do what we can with the notions we have.

Consider the wildly famous irrational number π. The number π is defined geometrically as the ratio of the circumference of a circle to that circle's diameter.[c] The reason we use the *symbol* 'π' instead of any numerical expression is that π is an irrational number, and, as such, its decimal representation neither terminates nor repeats. Hence we *approximate* π as $\pi \approx 3.14$ or $\pi \approx 3.14159265$. No matter how many digits we write, however, what we have is a *rational number* approximation of π.

c This works for each and every circle, by the way, regardless of how large or small the circle is!

The good news is we can approximate π to any desired accuracy using rational numbers by taking enough digits, so while we'll never 'reach' the *exact* value of π with rational numbers, we can get *as close as we like* to π using rational numbers. That being said, we assume π exists on the real number line, despite the fact the list of digits to pinpoint its location is, in some sense, infinite.

We take this tack when defining the value of a number raised to an irrational exponent. Consider, for instance, 2^{π}. We can compute $2^3 = 8$, $2^{3.1} = 2^{\frac{31}{10}} = \sqrt[10]{2^{31}} \approx 8.574$, $2^{3.14} = 2^{\frac{314}{100}} = 2^{\frac{157}{50}} = \sqrt[50]{2^{157}} \approx 8.8512$, and so on, so one way to *define* 2^π as the unique real number we obtain as the exponents 'approach' π.

It is with this understanding that we present the notion of a 'power function,' as described in Definition 2.2.1: $f(x) = ax^p$ where a and p are nonzero real number parameters. Here the exponent p is open to any (nonzero) real number. Because of how we define real number exponents, if p is irrational, then $x \geq 0$ to avoid having negatives under even-indexed roots as we go through the approximation process.[d]

or $x > 0$ if p is negative. d

In general, real number exponents inherit their properties from rational number exponents. For instance, Theorem 2.2.1 also holds for all real number exponents and the graphs of power functions inherit their behavior from graphs of rational exponent functions. More specifically, the graphs of functions of the form $f(x) = x^p$ where $p > 0$ all contain the points $(0, 0)$ and $(1, 1)$. Moreover, these functions are increasing and their graphs are concave down if $0 < p < 1$ and concave up if $p > 1$. See Figure 2.2.25.

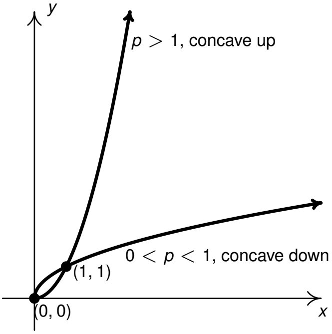


Figure 2.2.25: $f(x) = x^p$ for varying values of p

Theorem 2.2.2 generalizes to real number power functions, so, for instance to graph $F(x) = (x-2)^\pi$, one need only start with $y = x^\pi$ and shift horizontally two units to the right. (See the Exercises.)

We close this section with an application to economics. According to the US Census[2], Table 2, the share of money income (2014-2015) is given in Table 2.2.1. From these data, we can create a cumulative distribution, $y = L(x)$ called the **Lorenz Curve**.

The number $L(x)$ gives the percentage of the total national income earned by the bottom x percent of wage earners, ranked from lowest income to highest income. Since the population here is separated into 'quintiles,' each data point corresponds to 20% of the population.

[2]http://www.census.gov/library/publications/2016/demo/p60-256.html

So, for example, $L(20)$ is the percentage of money income earned by the lowest 20% of wage earners. In this case, we see $L(20) = 3.1$. The number $L(40)$ is the percentage of the money income earned by the bottom 40% of wage earners - so this includes not only the money from the Second Quintile, but also the Lowest Quintile: $L(40) = 8.2 + L(20) = 8.2 + 3.1 = 11.3$. Likewise, $L(60)$ is the total income share of the bottom 60% of wage earners which includes the income from the Middle, Second, and Lowest Quintiles: $L(60) = 14.3 + L(40) = 14.3 + (8.2 + 3.1) = 25.6$.

Continuing in this manner, we get $L(80) = 48.8$ and $L(100) = 100$, which is what we would expect: 100% of the income is earned by 100% of the population. We summarize these findings in Table 2.2.2.

Portion of Population	Percent of Money Income
Lowest Quintile	3.1
Second Quintile	8.2
Middle Quintile	14.3
Fourth Quintile	23.2
Highest Quintile	51.2

Table 2.2.1

percent wage earners, x	percent income, $L(x)$
20	3.1
40	11.3
60	25.6
80	48.8
100	100

Table 2.2.2

Example 2.2.3.

1. Find power function to model the Lorenz Curve: $L(x) = ax^p$. Comment on the goodness of fit.

2. Find and interpret $L(90)$.

Solution.

1. Using desmos[3], we get $L(x) = 0.00027901x^{2.7738}$ with $R^2 = 0.993$, indicating a pretty good fit. See Figure 2.2.26

2. We compute $L(90) = 0.00027901(90)^{2.7738} \approx 73.5$ meaning that the bottom 90% of the wage earners brought home 73.5% of the total income. Said differently, the top 10% of wage earners made over 25% of the total national income. □

[3] https://www.desmos.com/

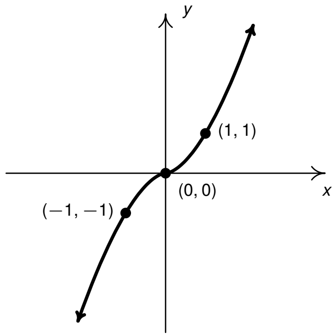


Figure 2.2.11: $f(x) = x^{\frac{5}{3}}$

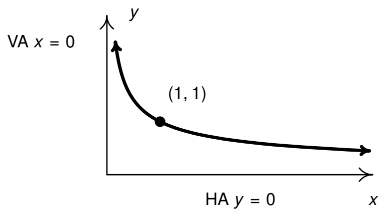


Figure 2.2.12: $f(x) = x^{-\frac{1}{2}}$

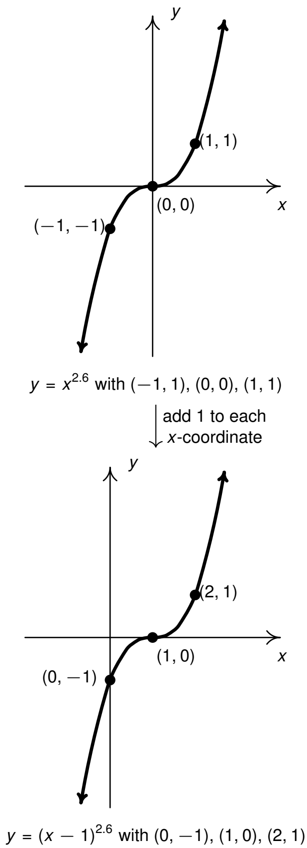

y
x
(1, 1)
(0, 0)
(−1, −1)
y = x^2.6 with (−1, 1), (0, 0), (1, 1)
add 1 to each
x-coordinate
y
x
(2, 1)
(1, 0)
(0, −1)
y = (x − 1)^2.6 with (0, −1), (1, 0), (2, 1)

Figure 2.2.16

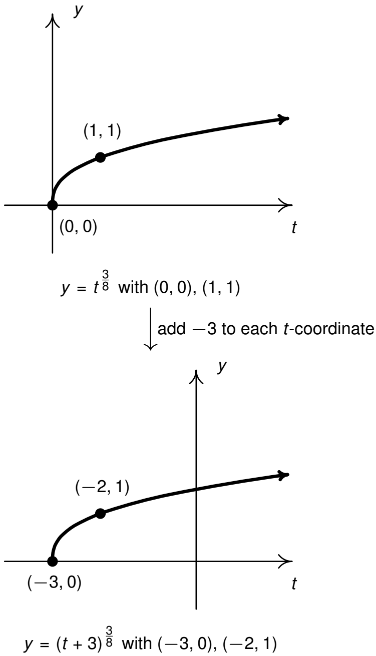


Figure 2.2.18

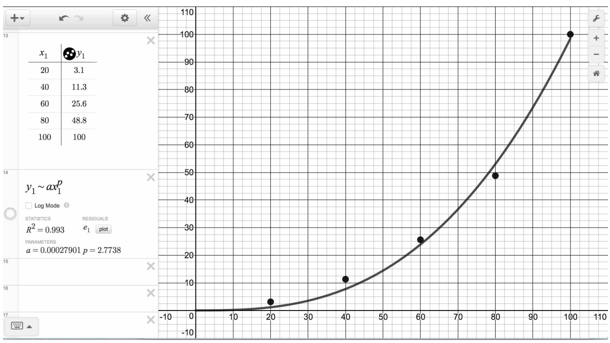


Figure 2.2.26

2.2.3 Exercises

In Exercises 1 - 6, use the graphs in Figure 2.2.27 and Figure 2.2.28 along with Theorem 2.2.2 to graph the given function. Track at least two points and state the domain and range using interval notation.

1. $F(x) = (x-2)^{\frac{2}{3}} - 1$
2. $G(t) = (t+3)^{\pi} + 1$
3. $F(x) = 3 - x^{\frac{2}{3}}$
4. $G(t) = (1-t)^{\pi} - 2$
5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$
6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1$

In Exercises 7 - 8, find a formula for each function below in the form $F(x) = a(bx-h)^{\frac{2}{3}} + k$.
NOTE: There may be more than one solution!

7. $y = F(x)$ in Figure 2.2.29
8. $y = F(x)$ in Figure 2.2.30

For each function in Exercises 9 - 16 below
- Analytically:
  - find the domain.
  - find the axis intercepts.
  - analyze the end behavior.
- Graph the function with help from a graphing utility and determine:
  - the range.
  - the local extrema, if they exist.
  - intervals of increase/decrease.
  - any 'unusual steepness' or 'local' verticality.
  - vertical asymptotes.
  - horizontal / slant asymptotes.
- Construct a sign diagram for each function using the intercepts and graph.
- Comment on any observed symmetry.

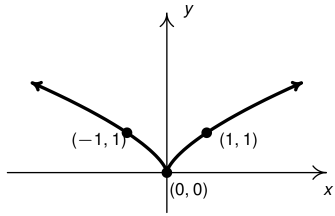


Figure 2.2.27: $f(x) = x^{\frac{2}{3}}$

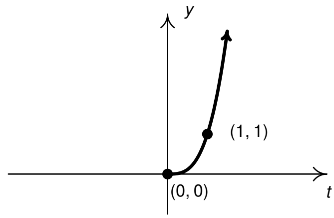


Figure 2.2.28: $g(t) = t^{\pi}$

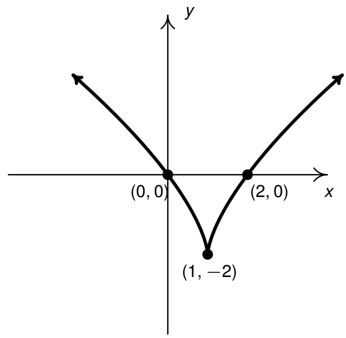


Figure 2.2.29

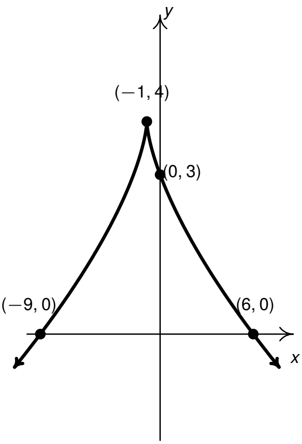


Figure 2.2.30

See Definition **??** in Section **??** for a e review of this concept, as needed.

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$
12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
13. $f(x) = x^{0.4}(3-x)^{0.6}$
14. $f(x) = x^{0.5}(3-x)^{0.5}$
15. $g(t) = 4t(9-t^2)^{-\sqrt{2}}$
16. $g(t) = 3(t^2+1)^{-\pi}$
17. For each function $f(x)$ listed below, compute the average rate of change over the indicated interval.[e] What trends do you observe? How do your answers manifest themselves graphically? Compare the results of this exercise with those of Exercise **??** in Section **??** and Exercise 43 in Section 1.1

$f(x)$	$x^{\frac{1}{2}}$	$x^{\frac{2}{3}}$	$x^{-0.23}$	x^{π}
[0.9, 1.1]				
[0.99, 1.01]				
[0.999, 1.001]				
[0.9999, 1.0001]				

18. The National Weather Service[4] uses the following formula to calculate the wind chill:

$$W = 35.74 + 0.6215\,T_a - 35.75\,V^{0.16} + 0.4275\,T_a\,V^{0.16}$$

where W is the wind chill temperature in °F, T_a is the air temperature in °F, and V is the wind speed in miles per hour. Note that W is defined only for air temperatures at or lower than 50°F and wind speeds above 3 miles per hour.

[4] http://www.nws.noaa.gov/om/windchill/windchillglossary.shtml

(a) Suppose the air temperature is 42° and the wind speed is 7 miles per hour. Find the wind chill temperature. Round your answer to two decimal places.

(b) Suppose the air temperature is 37°F and the wind chill temperature is 30°F. Find the wind speed. Round your answer to two decimal places.

19. As a follow-up to Exercise 18, suppose the air temperature is 28°F.

(a) Use the formula from Exercise 18 to find an expression for the wind chill temperature as a function of the wind speed, $W(V)$.

(b) Solve $W(V) = 0$, round your answer to two decimal places, and interpret.

(c) Graph the function W using a graphing utility and check your answer to part 19b.

20. Suppose Fritzy the Fox, positioned at a point (x, y) in the first quadrant, spots Chewbacca the Bunny at $(0, 0)$. Chewbacca begins to run along a fence (the positive y-axis) towards his warren. Fritzy, of course, takes chase and constantly adjusts his direction so that he is always running directly at Chewbacca. If Chewbacca's speed is v_1 and Fritzy's speed is v_2, the path Fritzy will take to intercept Chewbacca, provided v_2 is directly proportional to, but not equal to, v_1 is modeled by

$$y = \frac{1}{2}\left(\frac{x^{1+v_1/v_2}}{1+v_1/v_2} - \frac{x^{1-v_1/v_2}}{1-v_1/v_2}\right) + \frac{v_1 v_2}{v_2^2 - v_1^2}$$

(a) Determine the path that Fritzy will take if he runs exactly twice as fast as Chewbacca; that is, $v_2 = 2v_1$. Use your calculator to graph this path for $x \geq 0$. What is the significance of the y-intercept of the graph?

(b) Determine the path Fritzy will take if Chewbacca runs exactly twice as fast as he does; that is, $v_1 = 2v_2$. Use a graphing utility to graph this path for $x > 0$. Describe the behavior of y as $x \to 0^+$ and interpret this physically.

(c) With the help of your classmates, generalize parts (a) and (b) to two cases: $v_2 > v_1$ and $v_2 < v_1$. We will discuss the case of $v_1 = v_2$ in Exercise **??** in Section **??**.

2.2.4 Answers

1. $F(x) = (x-2)^{\frac{2}{3}} - 1$
   Graph: See Figure 2.2.31
   Domain: $(-\infty, \infty)$, Range: $[-1, \infty)$

2. $G(t) = (t+3)^{\pi} + 1$
   Graph: See Figure 2.2.32
   Domain: $[-3, \infty)$, Range: $[1, \infty)$

3. $F(x) = 3 - x^{\frac{2}{3}} = (-1)x^{\frac{2}{3}} + 3$
   Graph: See Figure 2.2.33
   Domain: $(-\infty, \infty)$, Range: $(-\infty, 3]$

4. $G(t) = (1-t)^{\pi} - 2 = ((-1)t+1)^{\pi} - 2$
   Graph: Figure 2.2.34
   Domain: $(-\infty, 1]$, Range: $[-2, \infty)$

5. $F(x) = (2x+5)^{\frac{2}{3}} + 1$

   Graph: See Figure 2.2.35
   Domain: $(-\infty, \infty)$, Range: $[1, \infty)$

6. $G(t) = \left(\dfrac{t+3}{2}\right)^{\pi} - 1 = \left(\frac{1}{2}t + \frac{3}{2}\right)^{\pi} - 1$
   Graph: See Figure 2.2.36
   Domain: $[-3, \infty)$, Range: $[-1, \infty)$

7. One solution is: $F(x) = 2(x-1)^{\frac{2}{3}} - 2$

8. One solution is: $F(x) = -(x+1)^{\frac{2}{3}} + 4$

9. $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$
   Domain: $(-\infty, \infty)$
   Intercepts: $(0,0)$, $(7,0)$
   Graph: See Figure 2.2.37

   As $x \to -\infty$, $f(x) \to -\infty$
   As $x \to \infty$, $f(x) \to \infty$
   Range: $(-\infty, \infty)$
   Local minimum: $\approx (4.667, -3.704)$
   Local maximum: $(0,0)$ (this is a cusp)
   Increasing: $(-\infty, 0]$, $\approx [4.667, \infty)$
   Decreasing: $[0, 4.667]$

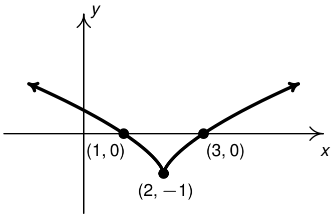


Figure 2.2.31

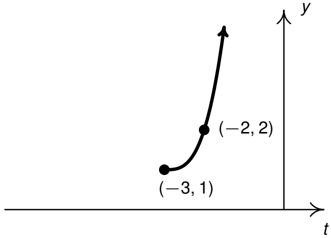


Figure 2.2.32

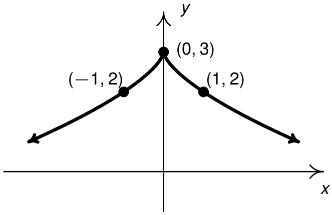


Figure 2.2.33

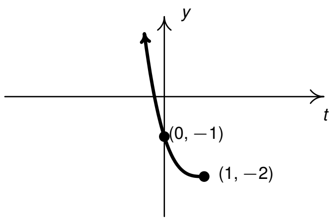


Figure 2.2.34

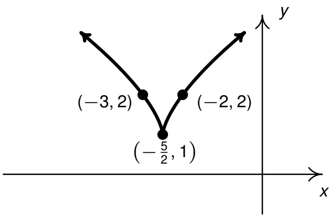


Figure 2.2.35

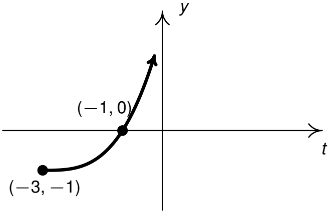


Figure 2.2.36

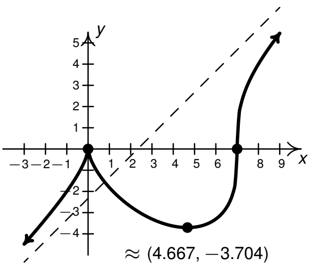


Figure 2.2.37

Unusual steepness at $x = 7$
Using Calculus it can be shown that $y = x - \frac{7}{3}$ is a slant asymptote of this graph.
Sign Diagram: $(-)$ 0 $(-)$ 0 $(+)$ on the number line with marks at 0 and 7

10. $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$
Graph: See Figure 2.2.38

Domain: $[0, \infty)$
Intercepts: $(0,0)$, $(7,0)$
As $x \to \infty$, $f(x) \to \infty$
Range: $\approx [-14.854, \infty)$
Local minimum: $\approx (5.727, -14.854)$
Increasing: $\approx [5.727, \infty)$
Decreasing: $\approx [0, 5.727]$
Unusual steepness at $x = 7$
Sign Diagram: 0 $(-)$ 0 $(+)$ on the number line with marks at 0 and 7

11. $g(t) = 2t(t+3)^{-\frac{1}{3}}$
Graph: See Figure 2.2.39

Domain: $(-\infty, -3) \cup (-3, \infty)$
Intercept: $(0,0)$
As $t \to \pm\infty$, $g(t) \to \infty$
Range: $(-\infty, \infty)$
Local minimum: $\approx (-4.5, 7.862)$
Increasing: $\approx [-4.5, -3)$, $(-3, \infty)$
Decreasing: $\approx (-\infty, -4.5]$
Vertical Asymptote: $t = -3$
Sign Diagram: $(+)$ ‽ $(-)$ 0 $(+)$ on the number line with marks at -3 and 0

12. $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$
Domain: $(2, \infty)$
As $t \to \infty$, $g(t) \to \infty$
Graph: See Figure 2.2.40

Range: $\approx [5.196, \infty)$
Local minimum: $\approx (3, 5.196)$
Increasing: $\approx [3, \infty)$
Decreasing: $\approx (2, 3]$
Vertical asymptote: $t = 2$
Using Calculus it can be shown that $y = t + 1$ is a slant asymptote of this graph.
Sign Diagram: ‽ (+)
2

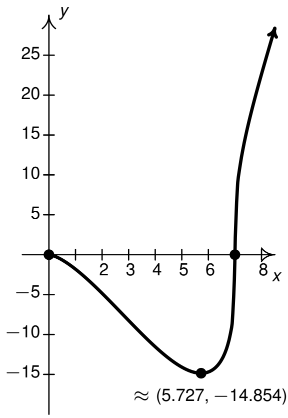


Figure 2.2.38

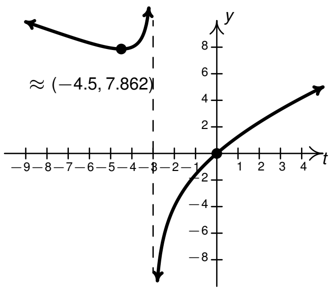


Figure 2.2.39

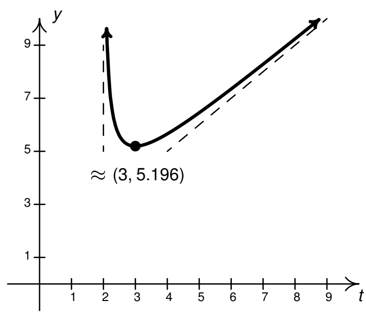


Figure 2.2.40

13. $f(x) = x^{0.4}(3-x)^{0.6}$
Domain: $(-\infty, \infty)$
Intercepts: $(0, 0)$, $(3, 0)$
Graph: See Figure 2.2.41

As $x \to -\infty$, $f(x) \to \infty$
As $x \to \infty$, $f(x) \to -\infty$
Range: $(-\infty, \infty)$
Local minimum: $(0, 0)$ (this is a cusp)
Local maximum: $\approx (1.2, 1.531)$
Increasing: $\approx [0, 1.2]$
Decreasing: $\approx (-\infty, 0], [1.2, \infty)$
Unusual Steepness: $x = 3$
Sign Diagram: (+) 0 (+) 0 (−)
0 3

14. $f(x) = x^{0.5}(3-x)^{0.5}$
Graph: See Figure 2.2.42

Domain: $[0, 3]$
Intercepts: $(0, 0)$, $(3, 0)$
Range: $\approx [0, 1.5]$
Increasing: $\approx [0, 1.5]$
Decreasing: $\approx [1.5, 3]$
Unusual Steepness:[5] $x = 0$, $x = 3$
Sign Diagram: 0 (+) 0
0 3

[5]Note you may need to zoom in to see this.

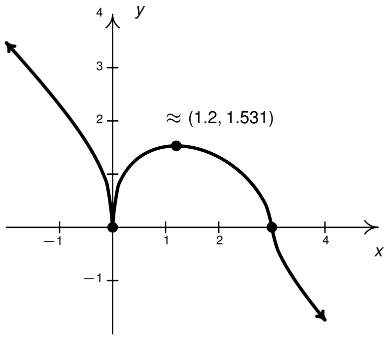


Figure 2.2.41

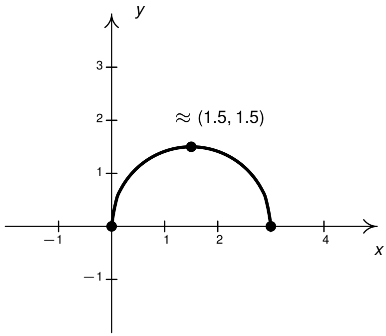


Figure 2.2.42

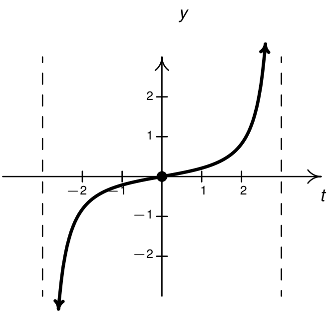


Figure 2.2.43

15. $g(t) = 4t(9 - t^2)^{-\sqrt{2}}$
    Graph: See Figure 2.2.43

    Domain: $(-3, 3)$
    Intercepts: $(0, 0)$
    Increasing: $(-3, 3)$
    Range: $(-\infty, \infty)$
    Sign Diagram: $\text{‽} \quad (-) \quad 0 \quad (+) \quad \text{‽}$ at -3, 0, 3

    Note: g is odd

16. $g(t) = 3(t^2 + 1)^{-\pi}$
    Domain: $(-\infty, \infty)$
    Graph: See Figure 2.2.44

    Intercept: $(0, 3)$
    As $t \to \pm\infty$, $g(t) \to 0$
    Range: $(0, 3]$
    Increasing: $(-\infty, 0]$
    Decreasing: $[0, \infty)$
    Horizontal asymptote: $y = 0$
    Sign Diagram: $(+)$

    Note: g is even

17. As in Exercise **??** in Section **??** and Exercise 43 in Section 1.1, the slopes of these curves near $x = 1$ approach the value of the exponent on x.

$f(x)$	$x^{\frac{1}{2}}$	$x^{\frac{2}{3}}$	$x^{-0.23}$	x^{π}
[0.9, 1.1]	0.5006	0.6672	-0.2310	3.1544
[0.99, 1.01]	$\approx \frac{1}{2}$	0.6667	≈ -0.23	3.1417
[0.999, 1.001]	$\approx \frac{1}{2}$	$\approx \frac{2}{3}$	≈ -0.23	$\approx \pi$
[0.9999, 1.0001]	$\approx \frac{1}{2}$	$\approx \frac{2}{3}$	≈ -0.23	$\approx \pi$

18. (a) $W \approx 37.55^\circ$F.

    (b) $V \approx 9.84$ miles per hour.

19. (a) $W(V) = 53.142 - 23.78V^{0.16}$. Since we are told in Exercise 18 that wind chill is only effect for wind speeds of more than 3 miles per hour, we restrict the domain to $V > 3$.

(b) $W(V) = 0$ when $V \approx 152.29$. This means, according to the model, for the wind chill temperature to be 0°F, the wind speed needs to be 152.29 miles per hour.

(c) The graph of $y = W(V)$ is given in Figure 2.2.45.

20. (a) $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{2}{3}$. The point $\left(0, \frac{2}{3}\right)$ is when Fritzy's path crosses Chewbacca's path - in other words, where Fritzy catches Chewbacca. See Figure 2.2.46.

(b) $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} - \frac{2}{3}$. We find as $x \to 0^+$, $y \to \infty$ which means, in this case, Fritzy's pursuit never ends; he never catches Chewbacca. This makes sense since Chewbacca has a head start and is running faster than Fritzy. See Figure 2.2.47.

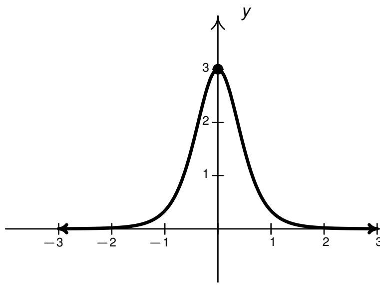


Figure 2.2.44

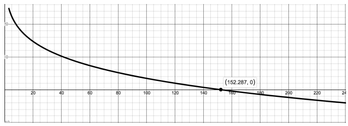


Figure 2.2.45

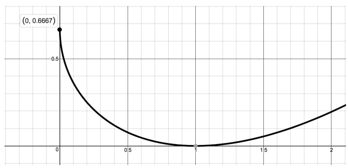


Figure 2.2.46: $y = \frac{1}{3}x^{\frac{3}{2}} - x^{\frac{1}{2}} + \frac{2}{3}$

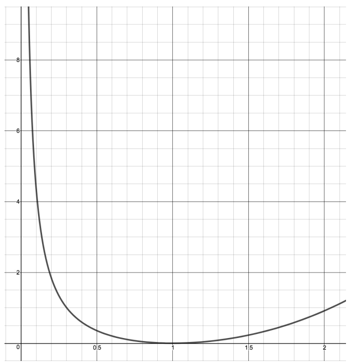


Figure 2.2.47: $y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1} - \frac{2}{3}$

2.3 Equations and Inequalities involving Power Functions

In this section, we set about solving equations and inequalities involving power functions. Our first example demonstrates the usual sorts of strategies to employ when solving equations.

Example 2.3.1. Solve the following equations analytically and verify your answers using a graphing utility.

1. $(7-x)^{\frac{3}{2}} = 8$
2. $(2t-1)^{\frac{2}{3}} - 4 = 0$
3. $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$
4. $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$
5. $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$
6. $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$

Solution.

1. One way to proceed to solve $(7-x)^{\frac{3}{2}} = 8$ is to use Definition 2.2.2 to rewrite $(7-x)^{\frac{3}{2}}$ as either $(\sqrt{7-x})^3$ or $\sqrt{(7-x)^3}$. We opt for the former since, thinking ahead, 8 is a perfect cube:

$$\begin{aligned}
(7-x)^{\frac{3}{2}} &= 8 & \\
(\sqrt{7-x})^3 &= 8 & \text{(rewrite using Definition 2.2.2)} \\
\sqrt[3]{(\sqrt{7-x})^3} &= \sqrt[3]{8} & \text{(extract cube roots)} \\
\sqrt{7-x} &= 2 & (\sqrt[3]{u^3} = u)
\end{aligned}$$

From $\sqrt{7-x} = 2$, we square both sides and obtain $7-x = 4$, so $x = 3$. We verify our answer analytically by substituting $x = 3$ into the original equation and it checks.

Geometrically, we are looking for where the graph of $f(x) = (7-x)^{\frac{3}{2}}$ intersects the graph of $g(x) = 8$. While we could sketch both curves by hand and gauge the reasonableness of the result,[a] we are instructed to use a graphing utility. In Figure 2.3.1 we see that the intersection point of both graphs is $(3, 8)$, thereby checking our solution $x = 3$.

a consider this an exercise!

2. Proceeding similarly to the above, to solve $(2t-1)^{\frac{2}{3}} - 4 = 0$, we rewrite $(2t-1)^{\frac{2}{3}}$ as $(\sqrt[3]{2t-1})^2$ and solve:

$$\begin{aligned}
(2t-1)^{\frac{2}{3}} - 4 &= 0 \\
(\sqrt[3]{2t-1})^2 - 4 &= 0 \\
&\text{(rewrite using Definition 2.2.2)} \\
(\sqrt[3]{2t-1})^2 &= 4 \quad \text{(isolate the variable term)} \\
\sqrt{(\sqrt[3]{2t-1})^2} &= \sqrt{4} \quad \text{(extract square roots)} \\
|\sqrt[3]{2t-1}| &= 2 \quad (\sqrt{u^2} = |u|) \\
\sqrt[3]{2t-1} &= \pm 2 \\
&(\text{ for } c > 0,\ |u| = c \text{ is equivalent to } u = \pm c.)
\end{aligned}$$

From $\sqrt[3]{2t-1} = 2$ we cube both sides and obtain $2t - 1 = 8$, so $t = \frac{9}{2} = 4.5$. Similarly, from $\sqrt[3]{2t-1} = -2$, we cube both sides and obtain $2t - 1 = -8$, so $t = -\frac{7}{2} = -3.5$. Both of these solutions check in the given equation.

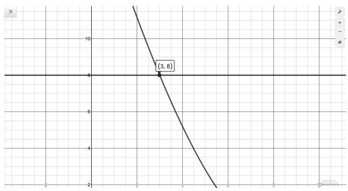


Figure 2.3.1: Checking $(7-x)^{\frac{3}{2}} = 8$

In this case we are looking for where the graph of $f(t) = (2t-1)^{\frac{2}{3}} - 4$ intersects the graph of $g(t) = 0$ - i.e., the t-intercepts of the graph of g. We find these are $(-3.5, 0)$ and $(4.5, 0)$, as predicted. See Figure 2.3.2.

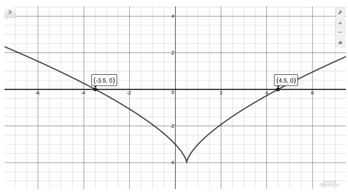


Figure 2.3.2: Checking $(2t-1)^{\frac{2}{3}} - 4 = 0$

3. Since $0.5 = \frac{1}{2}$, we may rewrite $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$ as $(x+3)^{\frac{1}{2}} = 2(7-x)^{\frac{1}{2}} + 1$. Using Definition 2.2.2, we then have $\sqrt{x+3} = 2\sqrt{7-x} + 1$. Since

one of the square roots is already isolated, we can rid ourselves of it by squaring both sides.

$$\begin{aligned}
\sqrt{x+3} &= 2\sqrt{7-x}+1 \\
(\sqrt{x+3})^2 &= (2\sqrt{7-x}+1)^2 \quad \text{(square both sides)} \\
x+3 &= (2\sqrt{7-x})^2 + 2(2\sqrt{7-x})(1) + 1 \\
&\qquad ((\sqrt{u})^2 = u \text{ and } (a+b)^2 = a^2+2ab+b^2) \\
x+3 &= 4(7-x) + 4\sqrt{7-x} + 1 \\
&\qquad ((ab)^2 = a^2b^2 \text{ and, again, } (\sqrt{u})^2 = u) \\
x+3 &= 28 - 4x + 4\sqrt{7-x} + 1 \\
5x - 26 &= 4\sqrt{7-x} \qquad \text{(isolate } \sqrt{7-x})
\end{aligned}$$

We square both sides *again* and get $(5x-26)^2 = (4\sqrt{7-x})^2$ which reduces to $25x^2 - 260x + 676 = 16(7-x)$. At last, we have a quadratic equation which we can solve by setting to zero and factoring. We get $25x^2 - 244x + 564 = 0$, so $(x-6)(25x-94) = 0$ so $x = 6$ or $x = \frac{94}{25} = 3.76$. When we go to check these answers, we find $x = 6$ does check, but $x = 3.76$ does not. Hence, $x = 3.76$ is an 'extraneous' solution.[b]

We graph both $f(x) = \sqrt{x+3}$ and $g(x) = 2\sqrt{7-x}+1$ in Figure 2.3.3 (once again, we could graph these by hand!) and confirm there is only one intersection point, $(6,3)$.

4. While we *could* approach solving $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$ as the previous example, we would encounter cubing binomials[c] which we would prefer to avoid. Instead, we take a step back and notice there are three terms here with the exponent on one term, $t^{\frac{2}{3}}$ exactly twice the exponent on another term, $t^{\frac{1}{3}}$. We have ourselves a 'quadratic in disguise.'[d] To help us see the forest for the trees, we let $u = t^{\frac{1}{3}}$ so that $u^2 = t^{\frac{2}{3}}$. (Note that since root here, 3, is odd, we can use the properties of exponents

b We invite the reader to see at which point in our machinations $x = 3.76$ *does* check. Knowing a solution is extraneous is one thing; understanding *how* it came about is another.

c that is, expanding things like $(a+b)^3$.

d See Section **??** or, more recently, Example **??** in Section **??**.

stated in Theorem 2.2.1.) Hence, in terms of u, the equation $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$ becomes the quadratic $2u^2 + 5u = 3$, or $2u^2 + 5u - 3 = 0$. Factoring gives $(2u - 1)(u + 3) = 0$ so $u = t^{\frac{1}{3}} = \frac{1}{2}$ or $u = t^{\frac{1}{3}} = -3$. Since $t^{\frac{1}{3}} = \sqrt[3]{t}$, we solve both equations by cubing both sides to get $t = \frac{1}{8} = 0.125$ and $t = -27$. Both of these solutions check in our original equation. Looking at the graphs of $f(t) = 2t^{\frac{2}{3}} + 5t^{\frac{1}{3}}$ and $g(t) = 3$, we find two intersection points, $(-27, 3)$ and $(0.125, 3)$, as required. See Figure 2.3.4.

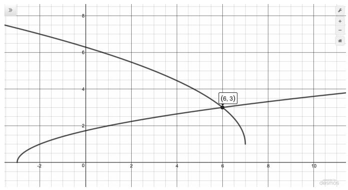


Figure 2.3.3: Checking $(x+3)^{0.5} = 2(7-x)^{0.5} + 1$

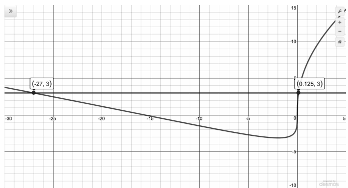


Figure 2.3.4: Checking $2t^{\frac{2}{3}} + 5t^{\frac{1}{3}} = 3$

5. Next we are to solve $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$ which, when written without negative exponents is: $\frac{2}{(3x-1)^{0.5}} = \frac{3x}{(3x-1)^{1.5}}$. Since the rational exponents here are $0.5 = \frac{1}{2}$ and $1.5 = \frac{3}{2}$, both involve an *even* indexed root (the square root in this case!) which means $3x - 1 \geq 0$. Moreover, since the $3x - 1$ resides in the *denominator* $3x - 1 \neq 0$ so our equation is really valid only for values of x where $3x - 1 > 0$ or $x > \frac{1}{3}$. Hence, we clear denominators and can apply Theorem 2.2.1:

$$\frac{2}{(3x-1)^{0.5}} = \frac{3x}{(3x-1)^{1.5}}$$

$$\left[\frac{2}{(3x-1)^{0.5}}\right] \cdot (3x-1)^{1.5} = \left[\frac{3x}{(3x-1)^{1.5}}\right] \cdot (3x-1)^{1.5}$$

$$2 \cdot \frac{(3x-1)^{1.5}}{(3x-1)^{0.5}} = 3x$$

$$2(3x-1)^{1.5-0.5} = 3x$$

(Theorem 2.2.1 applies since $3x - 1 > 0$.)

$$2(3x-1)^1 = 3x$$

We get $6x - 2 = 3x$, or $x = \frac{2}{3}$. Since $x = \frac{2}{3} > \frac{1}{3}$, we keep it and, sure enough, it checks in our original equation. Graphically we see $f(x) = 2(3x-1)^{-0.5}$ intersects $g(x) = 3x(3x-1)^{-1.5}$ at the point $(0.6667, 2)$ which is the graphing utility's way of representing $\left(\frac{2}{3}, 2\right)$. See Figure 2.3.5.

6. Our last equation to solve is $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-t^2)^{-\frac{2}{3}}$, which, when rewritten without negative exponents is: $6(9-t^2)^{\frac{1}{3}} = \frac{4t^2}{(9-t^2)^{\frac{2}{3}}}$. Again, the root here (3) is odd, so we can use the exponent properties listed in Theorem 2.2.1. We begin by clearing denominators:

$$6(9-t^2)^{\frac{1}{3}} = \frac{4t^2}{(9-t^2)^{\frac{2}{3}}}$$

$$6(9-t^2)^{\frac{1}{3}} \cdot (9-t^2)^{\frac{2}{3}} = \left[\frac{4t^2}{(9-t^2)^{\frac{2}{3}}}\right] \cdot (9-t^2)^{\frac{2}{3}}$$

$$6(9-t^2)^{\frac{1}{3}+\frac{2}{3}} = 4t^2$$

(Theorem 2.2.1 applies since the root here 3 is odd.)

$$6(9-t^2)^1 = 4t^2$$

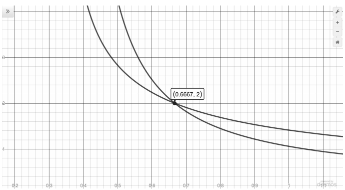


Figure 2.3.5: Checking $2(3x-1)^{-0.5} = 3x(3x-1)^{-1.5}$

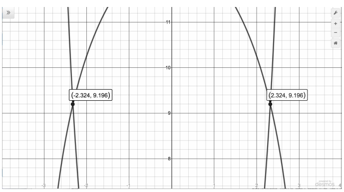


Figure 2.3.6: Checking $6(9-t^2)^{\frac{1}{3}} = 4t^2(9-x^2)^{\frac{2}{3}}$

We get $54 - 6t^2 = 4t^2$ or $10t^2 = 54$. As fraction $t^2 = \frac{54}{10} = \frac{27}{5}$ so $t = \pm\sqrt{\frac{27}{5}} = \pm 3\sqrt{155}$. While not the easiest to check analytically, both of these solutions do work in the original equation. Graphing $f(t) = 6(9-t^2)^{\frac{1}{3}}$ and $g(t) = 4t^2(9-t^2)^{-\frac{2}{3}}$ in Figure 2.3.6, we see the graphs intersect when $t \approx \pm 2.324$ which are decimal approximations of our exact answers. $\square$

Note that Example 2.3.1, there are several ways to correctly solve each equation, and we endeavored to demonstrate a variety of methods. For example, for number 1, instead of converting $(7-x)^{\frac{3}{2}}$ to a radical equation, we could use Theorem 2.2.1. Since the root here (2) is even, we know $7 - x \geq 0$ or $x \leq 7$. Hence we may apply exponent properties:

$$(7-x)^{\frac{3}{2}} = 8$$

$$\left[(7-x)^{\frac{3}{2}}\right]^{\frac{2}{3}} = 8^{\frac{2}{3}} \qquad \text{(raise both sides to the } \tfrac{2}{3} \text{ power)}$$

$$\begin{aligned} (7-x)^{\frac{3}{2}\cdot\frac{2}{3}} &= 4 && \text{(Theorem 2.2.1)} \\ (7-x)^1 &= 4 \end{aligned}$$

from which we get $x = 3$. If we try this same approach to solve number 2, however, we encounter difficulty. From $(2t-1)^{\frac{2}{3}} - 4 = 0$, we get $(2t-1)^{\frac{2}{3}} = 4$.

$$\begin{aligned} (2t-1)^{\frac{2}{3}} &= 4 \\ \left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} &= 4^{\frac{3}{2}} && \text{(raise both sides to the } \tfrac{3}{2} \text{ power)} \end{aligned}$$

Since the root here (3) is odd, we have no restriction on $2t-1$ but the exponent $\frac{3}{2}$ has an even denominator. Hence, Theorem 2.2.1 *does not apply*. That is,

$$\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} \neq (2t-1)^{\frac{2}{3}\cdot\frac{3}{2}} = (2t-1)^1 = (2t-1).$$

Note that if we weren't careful, we'd have $2t-1 = 4^{\frac{3}{2}} = 8$ which gives $t = \frac{9}{2} = 4.5$ only. We'd have missed the solution $t = -3.5$. Truth be told, you *can* simplify $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}}$ - just not using Theorem 2.2.1. We leave it as an exercise to show $\left[(2t-1)^{\frac{2}{3}}\right]^{\frac{3}{2}} = |2t-1|$ and, more generally, $\left(x^{\frac{2}{3}}\right)^{\frac{3}{2}} = |x|$.

Our next example is an application of the Cobb Douglas[1] production model of an economy. The Cobb-Douglas model states that the yearly total dollar value of the production output in an economy is a function of *two* variables: labor (the total number of hours worked in a year) and capital (the total dollar value of the physical goods required for manufacturing.) The equation relating the

[1] https://en.wikipedia.org/wiki/Cobb-Douglas_production_function

production output level P, labor L and capital K takes the form $P = aL^bK^{1-b}$ where $0 < b < 1$; that is, the production level varies jointly with some power of the labor and capital.

Example 2.3.2. In their original paper *A Theory of Production*[e] Cobb and Douglas modeled the output of the US Economy (using 1899 as a baseline) using the formula $P = 1.01L^{0.75}K^{0.25}$ where P, L, and K were percentages of the 1899 figures for total production, labor, and capital, respectively.

[e] available here[2].

1. For 1910, the recorded labor and capital figures for the US Economy are 144% and 208% of the 1899 figures, respectively. Find P using these figures and interpret your answer.
2. The recorded production value figure for 1920 is 231% of the 1899 figure. Use this to write K as a function of L, $K = f(L)$. Find and interpret $f(193)$.
3. Graph $K = f(L)$ and interpret the behavior as $L \to 0^+$ and $L \to \infty$.

Solution.

1. In this case, $P = 1.01L^{0.75}K^{0.25} = 1.01(144)^{0.75}(208)^{0.25} \approx 159$ which means the dollar value of the total US Production in 1920 was approximately 159% of what it was in 1899.[f]

[f] This answer is remarkably accurate. Note: all the dollar values here are recorded in '1880' dollars, per the source article.

2. We are given $P = 231 = 1.01L^{0.75}K^{0.25}$, so to write K as a function of L, we need to solve this equation for K. Since L and K are positive by definition, we can employ properties of exponents:

$$
\begin{aligned}
231 &= 1.01L^{0.75}K^{0.25} \\
\frac{231}{1.01L^{0.75}} &= \frac{1.01L^{0.75}K^{0.25}}{1.01L^{0.75}} && (L > 0, \text{ hence } L^{0.75} \neq 0.) \\
K^{0.25} &= 228.\overline{7128}L^{-0.75} && \text{(rewrite)}
\end{aligned}
$$

$$\left(K^{0.25}\right)^{\frac{1}{0.25}} = \left(228.\overline{7128}L^{-0.75}\right)^{\frac{1}{0.25}}$$

$$K^{\frac{0.25}{0.25}} = (228.\overline{7128})^{\frac{1}{0.25}} L^{-\frac{0.75}{0.25}} \quad \text{(Theorem 2.2.1)}$$

$$K = (228.\overline{7128})^{4} L^{-3} \quad \text{(simplify)}$$

Hence, $K = f(L) = (228.\overline{7128})^4 L^{-3}$ where $L > 0$. We find $f(193) = (228.\overline{7128})^4(193)^{-3} \approx 381$ meaning that in order to maintain a production level of 231% of 1889 with a labor level at 193% of 1889, the required capital is 381% that of 1889.[g]

g The actual recorded figure is 407.

3. The function $f(L)$ is a Laurent Monomial (see Section 1.1) with $n = 3$ and $a = (228.\overline{7128})^4$. As such, as $L \to 0^+$, $f(L) \to \infty$. This means that in order to maintain the given production level, as the available labor diminishes, the capital requirement become unbounded. As $L \to \infty$, we have $f(L) \to 0$ meaning that as the available labor increases, the need for capital diminishes. The graph of f is called an 'isoquant' - meaning 'same quantity.' In this context, the graph displays all combinations of labor and capital, (L, K) which result in the *same* production level, in this case, 231% of what was produced in 1889. See Figure 2.3.7

$\square$

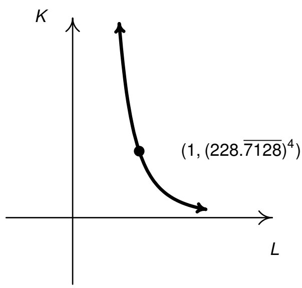


Figure 2.3.7: $K = f(L) = (228.\overline{7128})^4 L^{-3}$

Next, we move on to solving inequalities with power functions. As we've seen with other types of non-linear inequalities,[h] an invaluable tool for us is the Sign Diagram. See Box 2.3.1.

h see Sections **??**, **??**, and 1.3

As you may recall, since sign diagrams compare functions to 0, the first step in solving inequalities using a sign diagram is to gather all the nonzero terms one one side of the inequality. We demonstrate this technique in the following example.

Example 2.3.3. Solve the following inequalities. Check your answers graphically with a calculator.

Box 2.3.1. Steps for Constructing a Sign Diagram for an Algebraic Function

Suppose f is an algebraic function.

1. Place any values excluded from the domain of f on the number line with an '?' above them.
2. Find the zeros of f and place them on the number line with the number 0 above them.
3. Choose a test value in each of the intervals determined in steps 1 and 2.
4. Determine and record the sign of $f(x)$ for each test value in step 3.

1. $2 - \sqrt[4]{x+3} \geq 0$
2. $t^{2/3} < t^{4/3} - 6$
3. $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$
4. $(t-4)^{\frac{2}{3}} \geq -\dfrac{2t}{3(t-4)^{\frac{1}{3}}}$

Solution.

1. To solve $2 - \sqrt[4]{x+3} \geq 0$, it is tempting to rewrite this inequality as $2 \geq \sqrt[4]{x+3}$ and rid ourselves of the fourth root by raising both sides of this inequality to the fourth power. While this technique works *sometimes*, it doesn't work *all* the time since raising both sides of an inequality to the fourth (more generally, to an even) power does not necessarily preserve inequalities .[i] For that reason, we solve this inequality using a sign diagram since this technique will *always* produce a correct solution.

   We already have all the nonzero terms on one side of the inequality, so we let $r(x) = 2 - \sqrt[4]{x+3}$ and proceed to make a sign diagram. Owing to the presence of the fourth root, we know $x + 3 \geq 0$ or $x \geq -3$. Hence, we only concern ourselves with the portion of the number line representing $[3, \infty)$.

[i] For instance, $-2 \leq 1$ but $(-2)^4 \geq (1)^2$. We invite the reader to see what goes wrong if attempting to solve either of the following inequalities using this method: $-2 \geq \sqrt[4]{x+3}$, which has no solution, or $-2 \leq \sqrt[4]{x+3}$, whose solution is $[-3, \infty)$.

Next, we find the zeros of r by solving $r(x) = 2 - \sqrt[4]{x+3} = 0$. We get $\sqrt[4]{x+3} = 2$, so $x + 3 = 16$ and we get $x = 13$. We find this solution checks in our original equation,[j] and proceed to construct the sign diagram in Figure 2.3.8. Since we are looking for where $r(x) = 2 - \sqrt[4]{x+3} \geq 0$, we are looking for the zeros of r along with the intervals over which $r(x)$ is $(+)$. We record our answer as $[-3, 13]$. In Figure 2.3.9 is the graph of $y = 2 - \sqrt[4]{x+3}$, and we can see that, indeed, the graph is above the x-axis ($y = 0$) from $[-3, 13)$ and meets the x-axis at $x = 13$, verifying our answer.

j Recall that raising both sides to an even power could produce extraneous solutions, so it is important we check here.

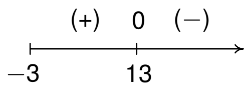


Figure 2.3.8

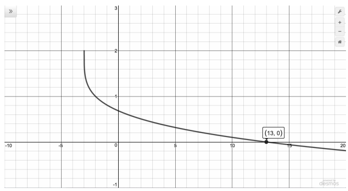


Figure 2.3.9

2. To solve $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$, we first rewrite as $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$. We set $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6$ and note that since the denominators in the exponents are 3, they correspond to cube roots, which means the domain of r is $(-\infty, \infty)$. To find the zeros for the sign diagram, we set $r(t) = 0$ and attempt to solve $t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 = 0$. Since there are three terms, and the exponent on one of the variable terms, $t^{\frac{4}{3}}$, is exactly twice that of the other, $t^{\frac{2}{3}}$, we have ourselves a 'quadratic in disguise.' If we let $u = t^{\frac{2}{3}}$, then $u^2 = t^{\frac{4}{3}}$, so in terms of u, we have $u^2 - u - 6 = 0$. Solving we get $u = -2$ or $u = 3$, hence $t^{\frac{2}{3}} = -2$ or $t^{\frac{2}{3}} = 3$. In root-power notation, these are $\sqrt[3]{t^2} = -2$ or $\sqrt[3]{t^2} = 3$. Cubing both sides of these equations results in $t^2 = -8$, which admits no real solution, or $t^2 = 27$, which gives $t = \pm 3\sqrt{3}$. Using these zeros, we construct the sign diagram in Figure 2.3.10. We find $r(t) = t^{\frac{4}{3}} - t^{\frac{2}{3}} - 6 > 0$ on $\left(-\infty, -3\sqrt{3}\right) \cup \left(3\sqrt{3}, \infty\right)$. To check our answer graphically, we set $f(t) = t^{\frac{2}{3}}$ and $g(t) = t^{\frac{4}{3}} - 6$. The solution to $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$ corresponds to the inequality $f(t) < g(t)$, which means we are looking for the t values for which the graph of f is *below* the graph of g. On the graph in Figure 2.3.11, we see the graph of f (the lighter colored curve) is below the graph of g (the darker colored curve) for

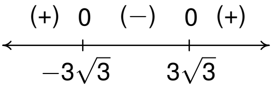


Figure 2.3.10

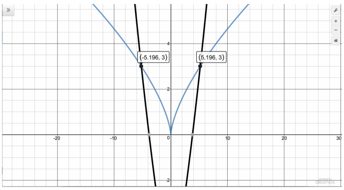


Figure 2.3.11

$t < -5.196$ and again for $t > 5.196$, which are the grapher's approximations to $\pm 3\sqrt{3}$.

3. To solve $3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$, we first gather all the nonzero terms to one side and obtain $3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \leq 0$. Setting $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note since the denominators of the rational exponents are odd, we have no domain concerns owing to even indexed roots. However, the negative exponent on the second term indicates a denominator. Rewriting $r(x)$ with positive exponents, we obtain

$$r(x) = 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}}$$

Setting the denominator equal to zero we get $(2-x)^{\frac{2}{3}} = 0$, which reduces to $2 - x = 0$, or $x = 2$. Hence, the domain of r is $(-\infty, 2) \cup (2, \infty)$.

To find the zeros of r, we set $r(x) = 0$, so we set about solving

$$3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} = 0.$$

Clearing denominators, we get $3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} - x = 0$. Since the denominators of the exponents are odd, we may use Theorem 2.2.1 to simplify this to $3(2-x)^1 - x = 0$, and obtain $6 - 4x = 0$ or $x = \frac{3}{2}$. In order for us to be able to more easily determine the sign of $r(x)$ at the test values, we rewrite $r(x)$ as a single term.[k] There are two schools of thought on how to proceed, so we demonstrate both.

- *Factoring Approach.* From $r(x) = 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}}$, we note that the quantity $(2-x)$ is common to both terms. When we factor out common factors, we factor out the quantity with the *smaller* exponent. In this case, since $-\frac{2}{3} < \frac{1}{3}$, we factor $(2-x)^{-\frac{2}{3}}$ from both

[k] This also gives us a chance to review some good intermediate algebra!

quantities. While it may seem odd to do so, we need to factor $(2-x)^{-\frac{2}{3}}$ *from* $(2-x)^{\frac{1}{3}}$, which results in subtracting the exponent $-\frac{2}{3}$ from $\frac{1}{3}$. We proceed using the usual properties of exponents.

$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{\frac{1}{3}-\left(-\frac{2}{3}\right)} - x\right] \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{\frac{3}{3}} - x\right] \\
&= (2-x)^{-\frac{2}{3}}\left[3(2-x)^{1} - x\right] \\
&= (2-x)^{-\frac{2}{3}}(6-4x) \\
&= (2-x)^{-\frac{2}{3}}(6-4x)
\end{aligned}$$

Written without negative exponents, we have $r(x) = \dfrac{6-4x}{(2-x)^{\frac{2}{3}}}$.

- *Common Denominator Approach.* We rewrite

$$\begin{aligned}
r(x) &= 3(2-x)^{\frac{1}{3}} - x(2-x)^{-\frac{2}{3}} \\
&= 3(2-x)^{\frac{1}{3}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&\qquad \text{(common denominator)} \\
&= \frac{3(2-x)^{\frac{1}{3}+\frac{2}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&\qquad \text{(Theorem 2.2.1)} \\
&= \frac{3(2-x)^{\frac{3}{3}}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}} \\
&= \frac{3(2-x)^{1}}{(2-x)^{\frac{2}{3}}} - \frac{x}{(2-x)^{\frac{2}{3}}}
\end{aligned}$$

$$= \frac{3(2-x)-x}{(2-x)^{\frac{2}{3}}}$$
$$= \frac{6-4x}{(2-x)^{\frac{2}{3}}}$$

Using either approach, we end up with the same, simpler, expression for $r(x)$ and we use that to create our sign diagram as shown in Figure 2.3.12. We find $r(x) \leq 0$ on $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. To check this graphically, we set $f(x) = 3(2-x)^{\frac{1}{3}}$ (the lighter curve) and $g(x) = x(2-x)^{-\frac{2}{3}}$ (the darker curve). We confirm that the graphs intersect at $x = \frac{3}{2}$ and the graph of f is below the graph of g for $x > \frac{3}{2}$, with the exception of $x = 2$ where it appears the graph of g has a vertical asymptote. See Figure 2.3.13.

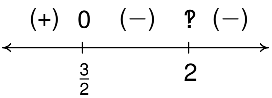


Figure 2.3.12

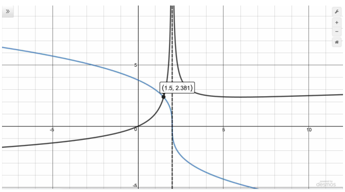


Figure 2.3.13

4. While it may be tempting to begin solving our last inequality by clearing denominators, owing to the odd root, the quantity $3(t-4)^{\frac{1}{3}}$ can be both positive and negative for different values of t. This means that if we chose to multiply both sides of our inequality by this quantity, we have no guarantee if the inequality would be preserved. Hence we proceed as usual by gathering all the nonzero terms to one side, and, with the ultimate goal of creating a sign diagram, get common denominators.

$$(t-4)^{\frac{2}{3}} \geq -\frac{2t}{3(t-4)^{\frac{1}{3}}}$$
$$(t-4)^{\frac{2}{3}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} \geq 0$$
$$\frac{(t-4)^{\frac{2}{3}} \cdot 3(t-4)^{\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} \geq 0$$
(common denominator)

$$\frac{3(t-4)^{\frac{2}{3}+\frac{1}{3}}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} \geq 0 \quad \text{(Theorem 2.2.1)}$$

$$\frac{3(t-4)^{1}}{3(t-4)^{\frac{1}{3}}} + \frac{2t}{3(t-4)^{\frac{1}{3}}} \geq 0$$

$$\frac{3(t-4)+2t}{3(t-4)^{\frac{1}{3}}} \geq 0$$

$$\frac{5t-12}{3(t-4)^{\frac{1}{3}}} \geq 0$$

We identify $r(t)$ as the left hand side of the inequality and see right away we must exclude $t = 4$ from the domain owing to the quantity $(t-4)$ in the denominator. As we have already mentioned, the root here (3) is odd, so we have no domain issues stemming from that. To find the zeros of r, we set $r(t) = 0$ which quickly reduces to solving $5t - 12 = 0$. We get $t = \frac{12}{5}$. From the sign diagram (Figure 2.3.14), we find $r(t) \geq 0$ on $\left(-\infty, \frac{5}{12}\right] \cup (4, \infty)$. Graphing $f(t) = (t-4)^{\frac{2}{3}}$ (the lighter curve) and $g(t) = -\frac{2t}{3(t-4)^{\frac{1}{3}}}$ (the darker curve), we see the graph of f is above the graph of g for $t < 2.4$ and again for $t > 4$, with an intersection point at $t = 2.4 = \frac{12}{5}$. See Figure 2.3.15. □

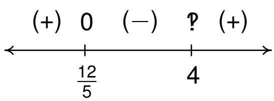


Figure 2.3.14

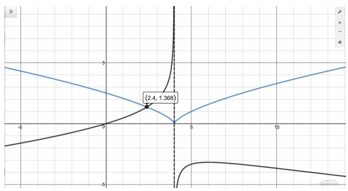


Figure 2.3.15

Note that in Example 2.3.3 number 3, since $(2-x)^{\frac{2}{3}}$ is always positive for $x \neq 2$ (owing to the squared exponent), we *could* have short-cut the sign diagram, choosing to clear denominators instead:

$$3(2-x)^{\frac{1}{3}} \leq x(2-x)^{-\frac{2}{3}}$$

$$3(2-x)^{\frac{1}{3}} \leq \frac{x}{(2-x)^{\frac{2}{3}}}$$

$$\left[3(2-x)^{\frac{1}{3}}\right]\left[(2-x)^{\frac{2}{3}}\right] \leq \frac{x}{(2-x)^{\frac{2}{3}}}\left[(2-x)^{\frac{2}{3}}\right]$$
(provided $x \neq 2$)
$$3(2-x)^{\frac{1}{3}}(2-x)^{\frac{2}{3}} \leq x$$
$$3(2-x)^{\frac{1}{3}+\frac{2}{3}} \leq x$$
$$3(2-x) \leq x$$

Hence, we get $6-3x \leq x$ or $x \geq \frac{3}{2}$, provided $x \neq 2$. This matches our solution $\left[\frac{3}{2}, 2\right) \cup (2, \infty)$. If, on the other hand, we tried this same manipulation with number 4, we would clear denominators, assuming $t \neq 4$ to obtain $3(t-4) \geq -2t$ or $t \geq \frac{12}{5}$ which is *not* the correct solution. The moral of the story is the more you understand, the less you need to rely on memorized processes and the more efficient your solution methodologies can become. The sign diagram algorithm is a fail-safe method, but, in some cases, may be far from the most efficient one. It's always best to understand the *why* of a procedure as much as the *how*.

2.3.1 Exercises

In Exercises 1 - 30, solve the equation or inequality.

1. $x + 1 = (3x + 7)^{\frac{1}{2}}$
2. $2x + 1 = (3 - 3x)^{\frac{1}{2}}$
3. $t + (3t + 10)^{0.5} = -2$
4. $3t + (6 - 9t)^{0.5} = 2$
5. $x^{-1.5} = 8$
6. $2x - 1 = (x + 1)^{-0.5}$
7. $t^{\frac{2}{3}} = 4$
8. $(t - 2)^{\frac{1}{2}} + (t - 5)^{\frac{1}{2}} = 3$
9. $(2x + 1)^{\frac{1}{2}} = 3 + (4 - x)^{\frac{1}{2}}$
10. $5 - (4 - 2x)^{\frac{2}{3}} = 1$
11. $2t^{\frac{2}{3}} = 6 - t^{\frac{1}{3}}$
12. $2t^{\frac{1}{3}} = 1 - 3t^{\frac{2}{3}}$
13. $2x^{1.5} = 15x^{0.75} + 8$
14. $35x^{-0.75} = x^{-1.5} + 216$
15. $10 - \sqrt{t - 2} \leq 11$
16. $t^{\frac{2}{3}} \leq 4$
17. $\sqrt[3]{x} \leq x$
18. $(2 - 3x)^{\frac{1}{3}} > 3x$
19. $(t^2 - 1)^{-\frac{1}{2}} \geq 2$
20. $(t^2 - 1)^{-\frac{1}{3}} \leq 2$
21. $3(x - 1)^{\frac{1}{3}} + x(x - 1)^{-\frac{2}{3}} \geq 0$
22. $3(x - 1)^{\frac{2}{3}} + 2x(x - 1)^{-\frac{1}{3}} \geq 0$
23. $2(t - 2)^{-\frac{1}{3}} - \frac{2}{3}t(t - 2)^{-\frac{4}{3}} \leq 0$
24. $-\frac{4}{3}(t - 2)^{-\frac{4}{3}} + \frac{8}{9}t(t - 2)^{-\frac{7}{3}} \geq 0$
25. $2x^{-\frac{1}{3}}(x - 3)^{\frac{1}{3}} + x^{\frac{2}{3}}(x - 3)^{-\frac{2}{3}} \geq 0$

26. $\sqrt[3]{x^3 + 3x^2 - 6x - 8} > x + 1$

27. $4(7-t)^{0.75} - 3t(7-t)^{-0.25} \leq 0$

28. $4t^{0.75}(t-3)^{-\frac{2}{3}} + 9t^{-0.25}(t-3)^{\frac{1}{3}} < 0$

29. $x^{-\frac{1}{3}}(x-3)^{-\frac{2}{3}} - x^{-\frac{4}{3}}(x-3)^{-\frac{5}{3}}(x^2 - 3x + 2) \geq 0$

30. $\frac{2}{3}(t+4)^{\frac{3}{5}}(t-2)^{-\frac{1}{3}} + \frac{3}{5}(t+4)^{-\frac{2}{5}}(t-2)^{\frac{2}{3}} \geq 0$

See Example 2.3.2 for more details on these sorts of models.

31. The Cobb-Douglas production model for the country of Sasquatchia is $P = 1.25L^{0.4}K^{0.6}$. Here, P represents the country's production (measured in thousands of Bigfoot Bullion), L represents the total labor (measured in thousands of hours) and K represents the total investment in capital (measured in Bigfoot Bullion.)

   (a) Let $P = 300$ and solve for K as a function of L. If $L = 100$, what is K? Interpret each of the quantities in this case.

   (b) Graph your answer to 31a using a graphing utility. What information does an ordered pair (L, K) on this graph represent?

2.3.2 Answers

1. $x = 3$
2. $x = \frac{1}{4}$
3. $t = -3$
4. $t = -\frac{1}{3}, \frac{2}{3}$
5. $x = \frac{1}{4}$
6. $x = \frac{\sqrt{3}}{2}$
7. $t = \pm 8$
8. $t = 6$
9. $x = 4$
10. $x = -2, 6$
11. $t = -8, \frac{27}{8}$
12. $t = -1, \frac{1}{27}$
13. $x = 16$
14. $x = \frac{1}{81}, \frac{1}{16}$
15. $[2, \infty)$
16. $[-8, 8]$
17. $[-1, 0] \cup [1, \infty)$
18. $\left(-\infty, \frac{1}{3}\right)$
19. $\left[-\frac{\sqrt{5}}{2}, -1\right) \cup \left(1, \frac{\sqrt{5}}{2}\right]$
20. $\left(-\infty, -\frac{3\sqrt{2}}{4}\right] \cup (-1, 1) \cup \left[\frac{3\sqrt{2}}{4}, \infty\right)$
21. $\left[\frac{3}{4}, 1\right) \cup (1, \infty)$
22. $\left(-\infty, \frac{3}{5}\right] \cup (1, \infty)$
23. $(-\infty, 2) \cup (2, 3]$
24. $(2, 6]$
25. $(-\infty, 0) \cup [2, 3) \cup (3, \infty)$
26. $(-\infty, -1)$
27. $[4, 7)$
28. $\left(0, \frac{27}{13}\right)$
29. $(-\infty, 0) \cup (0, 3)$
30. $(-\infty, -4) \cup \left(-4, -\frac{22}{19}\right] \cup (2, \infty)$
31. (a) $K = f(L) = (240)^{\frac{5}{3}} L^{-\frac{2}{3}}$. $f(100) \approx 430.2148$. This means in order for the production level of Sasquatchia to reach 300,000 Bigfoot Bullion with a labor investment of 100,000 hours, the country needs to invest approximately 430 Bigfoot Bullion into capital.

(b) If a point (L, K) is on the graph of this function, it means a combination of L thousand hours of labor with an investment of K Bigfoot Bullion into the Sasquatian Economy will result in a production level of 300,000 Bigfoot Bullion. See Figure 2.3.16

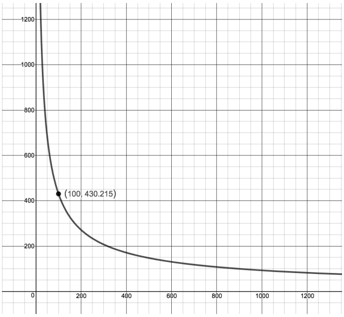


Figure 2.3.16

Chapter 3

Further Topics on Functions

3.1 Graphs of Functions

Up until this point in the text, we have primarily focused on studying particular *families* of functions. These families and their relationships to one another provide useful *examples* of more abstract function structures and relationships. The notions introduced in this chapter will not only provide us a more formal vocabulary with which to describe the connections between the function families we have already studied, but, more importantly, give us additional lenses through which to view new families of functions that we'll encounter.

In this section, we review of the concepts associated with the graphs of functions. We introduced the notion of the graph of a function in Section **??**, and the vast majority of the graphs we have encountered in this text were generated from an algebraic representation of a function. In this section, we define the functions geometrically from the outset and review the important concepts associated with the graphs of functions.

Recall the **domain** of a function is the set of inputs to the function and the **range** of a function is the set of outputs from the function. When graphing a function whose domain and range are subsets of real numbers, we plot the ordered pairs (input, output) on the Cartesian plane. Hence, the domain values are found on the horizontal axis while the range values are found on the vertical axis.

Recall from Definition **??** that the largest output from the function (if there is one) is called the **maximum** or, when there may be some confusion, the **absolute maximum** of the function. Likewise, the smallest output from the function (again, if there is one) is called the **minimum** or **absolute minimum**.

A concept related to 'absolute' maximum and minimum is the concept of 'local' maximum and minimum as described in Definition **??**. Here, a point (a, b) on the graph of a function f is a **local maximum** if b is the maximum function value for some open interval in the domain containing a. The notion of 'local' here meaning instead of surveying the entire domain, we instead restrict our attention to inputs 'local' or 'near' the input a. The concept of **local minimum** is defined similarly.

Next, we review the notions of **increasing**, **decreasing**, and **constant** as described in Definition **??**. Recall a function is increasing over an interval if, as the inputs increase, do the outputs. This means that, geometrically, the graph of the function rises as we move left to right. Similarly, a function is decreasing over an interval if the outputs decrease as the inputs increase. Geometrically, a decreasing function falls as we move left to right. Finally, a function is constant over an interval if the output is the same regardless of the input. If a function is constant over an interval, its graph remains 'flat' - a horizontal line.

Jeff a

Last, and according to some[a] least, we briefly review the notion of symmetry in the graphs of functions. Recall from Definition **??** that a function f is called **even** if

$f(-x) = f(x)$ for all x in the domain of f. The graphs of even functions are symmetric about the vertical (usually y-) axis. In a similar manner, Definition **??** tells us a function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f. Geometrically, the graphs of odd functions are symmetric about the origin.

The next example reviews all of the aforementioned concepts as well as many more.

Example 3.1.1. Given the graph of $y = f(x)$ in Figure 3.1.1, answer all of the following questions.

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the x-intercepts, if any exist.
6. List the y-intercepts, if any exist.
7. Find the zeros of f.
8. Solve $f(x) < 0$.
9. Determine $f(2)$.
10. Solve $f(x) = -3$.
11. Find the number of solutions to $f(x) = 1$.
12. Does f appear to be even, odd, or neither?
13. List the local maximums, if any exist.
14. List the local minimums, if any exist.
15. List the intervals on which f is increasing.
16. List the intervals on which f is decreasing.

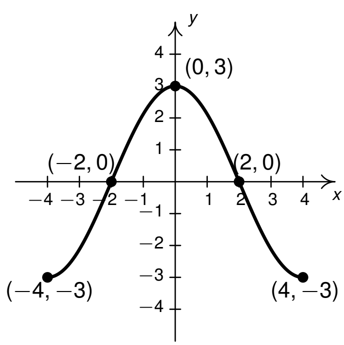


Figure 3.1.1

Solution.

1. To find the domain of f, we proceed as in Section **??**. By projecting the graph to the x-axis, we see

that the portion of the x-axis which corresponds to a point on the graph is everything from -4 to 4, inclusive. Hence, the domain is $[-4, 4]$.

2. To find the range, we project the graph to the y-axis. We see that the y values from -3 to 3, inclusive, constitute the range of f. Hence, our answer is $[-3, 3]$.

3. The maximum value of f is the largest y-coordinate which is 3.

4. The minimum value of f is the smallest y-coordinate which is -3.

5. The x-intercepts are the points on the graph with y-coordinate 0, namely $(-2, 0)$ and $(2, 0)$.

6. The y-intercept is the point on the graph with x-coordinate 0, namely $(0, 3)$.

7. The zeros of f are the x-coordinates of the x-intercepts of the graph of $y = f(x)$ which are $x = -2, 2$.

8. To solve $f(x) < 0$, we look for the x values of the points on the graph where the $y = f(x)$ is negative. Graphically, we are looking for where the graph is *below* the x-axis. This happens for the x values from -4 to -2 and again from 2 to 4. So our answer is $[-4, -2) \cup (2, 4]$.

9. Since the graph of f is the graph of the equation $y = f(x)$, $f(2)$ is the y-coordinate of the point which corresponds to $x = 2$. Since the point $(2, 0)$ is on the graph, we have $f(2) = 0$.

10. To solve $f(x) = -3$, we look where $y = f(x) = -3$. We find two points with a y-coordinate of -3, namely $(-4, -3)$ and $(4, -3)$. Hence, the solutions to $f(x) = -3$ are $x = \pm 4$.

11. As in the previous problem, to solve $f(x) = 1$, we look for points on the graph where the y-coordinate

is 1. If we imagine the horizontal like $y = 1$ superimposed over the graph of f as sketched in Figure 3.1.2, we get two intersections. Hence, even though these points aren't specified, we know there are *two* points on the graph of f whose y-coordinate is 1. Hence, there are two solutions to $f(x) = 1$.

12. The graph appears to be symmetric about the y-axis. This suggests[b] that f is even.

13. The function has its only local maximum at $(0, 3)$.

14. There are no local minimums. Why don't $(-4, -3)$ and $(4, -3)$ count? Let's consider the point $(-4, -3)$ for a moment. Recall that, in the definition of local minimum, there needs to be an open interval containing $x = -4$ which is in the domain of f. In this case, there is no open interval containing $x = -4$ which lies entirely in the domain of f, $[-4, 4]$. Because we are unable to fulfill the requirements of the definition for a local minimum, we cannot claim that f has one at $(-4, -3)$. The point $(4, -3)$ fails for the same reason – no open interval around $x = 4$ stays within the domain of f.

15. As we move from left to right, the graph rises from $(-4, -3)$ to $(0, 3)$. This means f is increasing on the interval $[-4, 0]$. (Remember, the answer here is an interval on the x-axis.)

16. As we move from left to right, the graph falls from $(0, 3)$ to $(4, -3)$. This means f is decreasing on the interval $[0, 4]$. (Again, the answer here is an interval on the x-axis.) □

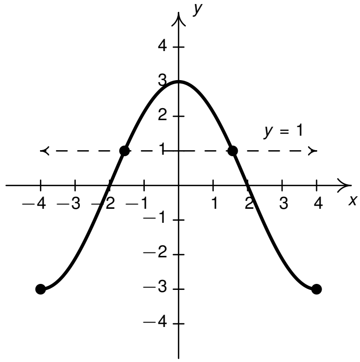
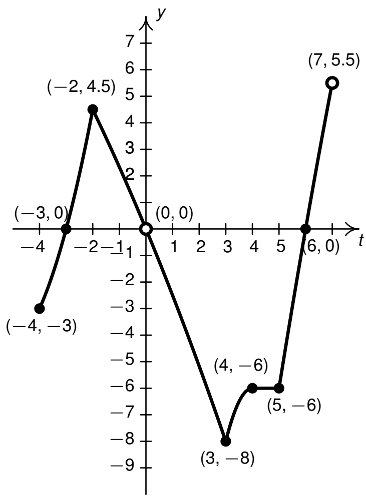


Figure 3.1.2

b but does not prove

Our next example involves a more complicated function and asks more complicated questions.

Example 3.1.2. Consider the graph of the function g in Figure 3.1.3.



The graph of $y = g(t)$

Figure 3.1.3

1. Find the domain of g.

2. Find the range of g.

3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. Solve $(t^2 - 25)g(t) = 0$.
8. Solve $\dfrac{g(t)}{t^2 + t - 30} \geq 0$.

Solution.

1. Projecting the graph of g to the t-axis, we see the domain contains values of t from -4 up to, but not including $t = 0$ and values greater than $t = 0$ up to, but not including $t = 7$. Using interval notation, we write the domain as $[-4, 0) \cup (0, 7)$.
2. Projecting the graph of g to the y-axis, we see the range of g contains all real numbers from $y = -8$ up to, but not including, $y = 5.5$. Note that even though there is a hole in the graph at $(0, 0)$, the points $(-3, 0)$ and $(6, 0)$ put $y = 0$ in the range of g. Hence, the range of g is $[-8, 5.5)$.
3. Owing to the hole in the graph at $(7, 5.5)$, g has no maximum.[c]
4. The minimum of g is -8 which occurs at the point $(3, -8)$.
5. The point $(-2, 4.5)$ is clearly a local maximum, but there are actually infinitely many more. Per Definition **??**, all points of the form $(t, -6)$ for $4 \leq t < 5$ are also local maximums. For each of these points, we can find an open interval on the t axis within which we produce no points on the graph higher than $(t, -6)$. (You may think about 'zooming in' on the point $(4.5, -6)$ to see how this works.)
6. The local minimums of the graph are $(3, -8)$ along with points of the form $(t, -6)$ for $4 < t \leq 5$. Note

[c] There is no real number 'right before' 5.5 ...

the point $(-4, -3)$ is not a local minimum since there is no open interval containing $t = -4$ which lies entirely within the domain of g.

7. To solve $(t^2 - 25)g(t) = 0$, we use the zero product property of real numbers[d] to conclude either $t^2 - 25 = 0$ or $g(t) = 0$.

   From $t^2 - 25 = 0$, we get $t = \pm 5$. However, since $t = -5$ isn't in the domain of g, it cannot be regarded as a solution to the equation $(t^2 - 25)g(t) = 0$. (If we substitute $t = -5$ into the equation, we'd get $((-5)^2 - 25)g(-5) = 0 \cdot g(-5)$. Since $g(-5)$ is undefined, so is $0 \cdot g(-5)$.)

   To solve $g(t) = 0$, we look for the zeros of g which are $t = -3$ and $t = 6$. (Again, there is a hole at $(0, 0)$, so $t = 0$ doesn't count as a zero.) Our final answer to $(t^2 - 25)g(t) = 0$ is $t = -3$, 5, or 6.

8. To solve $\frac{g(t)}{t^2+t-30} \geq 0$, we employ a sign diagram as we (most recently) have done in Section 2.3.[e] To that end, we define $F(t) = \frac{g(t)}{t^2+t-30}$ and we set about finding the domain of f.

   First, we note that since F is defined in terms of g, the domain of F is restricted to some subset of the domain of g, namely $[-4, 0) \cup (0, 7)$. Since $t^2 + t - 30$ is in the denominator of $F(t)$, we must also exclude the values where $t^2+t-30 = (t+6)(t-5) = 0$. Hence, we must exclude $t = -6$ (which isn't in the domain of g in the first place) along with $t = 5$. Hence, the domain of F is $[-4, 0) \cup (0, 5) \cup (5, 7)$.

   Next, we find the zeros of F. Setting $F(t) = \frac{g(t)}{t^2+t-30} = 0$ amounts to solving $g(t) = 0$. Graphically, we see this occurs when $t = -3$ and $t = 6$. Hence, we need to select test values in each of the following intervals: $[-4, -3)$, $(-3, 0)$, $(0, 5)$, $(5, 6)$ and $(6, 7)$.

   For the interval $[-4, -3)$, we may choose $t = -4$. $F(-4) = \frac{g(-4)}{(-4)^2+(-4)-30} = \frac{-3}{-18} > 0$ so is (+). For the interval $(-3, 0)$ we choose $t = -2$ and get $F(-2) =$

d see Section **??**, **??**

e Note that g is continuous on its domain, and hence, it follows that $\frac{g(t)}{t^2+t-30}$ is, too. (Thank Calculus!) This means the Intermediate Value Theorem applies so a Sign Diagram approach is valid.

$\frac{g(-2)}{(-2)^2+(-2)-30} = \frac{4.5}{-28} < 0$ so is $(-)$. For the interval $(0, 5)$, we choose $t = 3$ and find $F(3) = \frac{g(3)}{(3)^2+(3)-30} = \frac{-8}{-18} > 0$ which is $(+)$ again.

For the last two intervals, $(5, 6)$ and $(6, 7)$, we do not have specific function values for g. However, all we are interested in is the *sign* of the function over these intervals, and we can get that information about g graphically.

For the interval $(5, 6)$, we choose $t = 5.5$ as our test value. Since the graph of $y = g(t)$ is *below* the t-axis when $t = 5.5$. we know $g(5.5)$ is $(-)$. Hence, $F(5.5) = \frac{g(5.5)}{(5.5)^2+(5.5)-30} = \frac{(-)}{5.75} < 0$ so is $(-)$. Similarly, when $t = 6.5$, the graph of $y = g(t)$ is *above* the t-axis so $F(6.5) = \frac{g(6.5)}{(6.5)^2+(6.5)-30} = \frac{(+)}{18.75} > 0$ so is $(+)$. Putting all of this together, we get the sign diagram for $F(t) = \frac{g(t)}{t^2+t-30}$ shown in Figure 3.1.4.

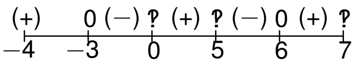


Figure 3.1.4

Hence, $F(t) \geq 0$ on $[-4, -3] \cup (0, 5) \cup [6, 7)$. □

Our last example focuses on symmetry. The reader is encouraged to review the notes about symmetry as summarized on page **??** in Section **??**.

Example 3.1.3. Figure 3.1.5 and Figure 3.1.6 show the partial graphs of functions f and g.

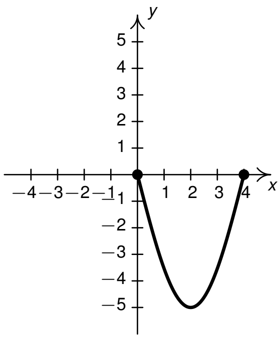


Figure 3.1.5: Partial graph of $y = f(x)$

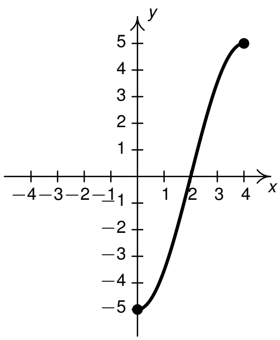


Figure 3.1.6: Partial graph of $y = g(x)$

1. If possible, complete the graphs of f and g assuming both functions are even.

2. If possible, complete the graphs of f and g assuming both functions are odd.

Solution.

1. If f and g are even then their graphs are symmetric about the y-axis. Hence, to complete each graph, we reflect each point on the graphs of f and g about the y-axis. See Figure 3.1.7 and Figure 3.1.8.

2. If f and g are odd then their graphs are symmetric about the origin. Hence, to complete each graph, we imagine reflecting each of the points on their graphs through the origin. We complete the process on the graph of f with no issues. See Figure 3.1.9.

   However, when attempting to do the same with the graph of the function g, we find the point $(0, -5)$ is reflected to the point $(0, 5)$. Hence, this new graph doesn't pass the vertical line test and hence is not a function. Therefore, g cannot be odd.[f] See Figure 3.1.10. □

f We leave it as an exercise to show that if a function f is odd and 0 is in the domain of f, then, necessarily, $f(0) = 0$.

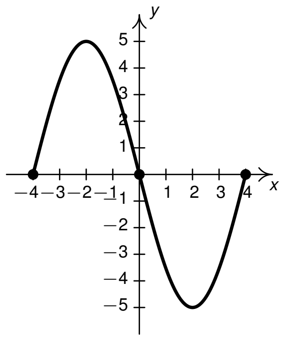


Figure 3.1.9: The graph of f assuming f is odd

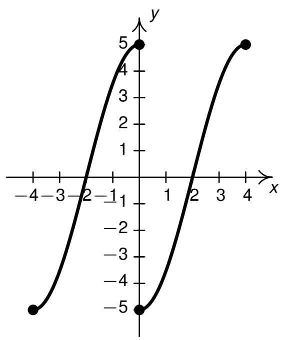


Figure 3.1.10: This graph fails the vertical line test

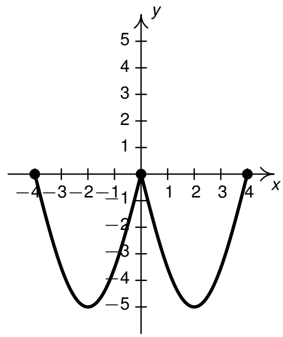


Figure 3.1.7: The graph of f assuming f is even

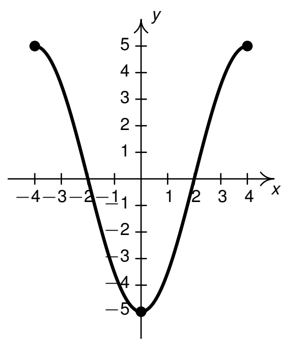


Figure 3.1.8: The graph of g assuming g is even

3.1.1 Exercises

In Exercises 1 - 20, use the graph of $y = f(x)$ given in Figure 3.1.11 to answer the question.

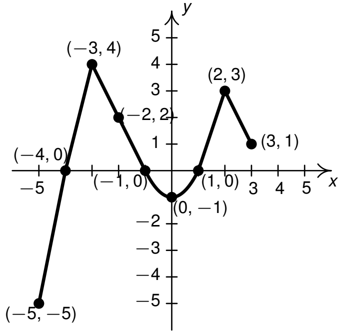


Figure 3.1.11: $y = f(x)$

1. Find the domain of f.
2. Find the range of f.
3. Find the maximum, if it exists.
4. Find the minimum, if it exists.
5. List the local maximums, if any exist.
6. List the local minimums, if any exist.
7. List the intervals where f is increasing.
8. List the intervals where f is decreasing.
9. Determine $f(-2)$.
10. Solve $f(x) = 4$.
11. List the x-intercepts, if any exist.
12. List the y-intercepts, if any exist.
13. Find the zeros of f.
14. Solve $f(x) \geq 0$.
15. Find the number of solutions to $f(x) = 1$.
16. Find the number of solutions to $|f(x)| = 1$.
17. Solve $(x^2 - x - 2)f(x) = 0$
18. Solve $(x^2 - x - 2)f(x) > 0$

With help from your classmates:

19. Find the domain of $R(x) = \dfrac{1}{f(x)}$
20. Find the range of $R(x) = \dfrac{1}{f(x)}$

In Exercises 21 - 38, use the graph of $y = g(t)$ given in Figure 3.1.12 to answer the question.

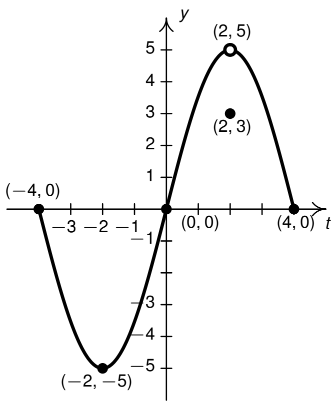


Figure 3.1.12: $y = g(t)$

21. Find the domain of g.
22. Find the range of g.
23. Find the maximum, if it exists.

24. Find the minimum, if it exists.
25. List the local maximums, if any exist.
26. List the local minimums, if any exist.
27. List the intervals where g is increasing.
28. List the intervals where g is decreasing.
29. Determine $g(2)$.
30. Solve $g(t) = -5$.
31. List the t-intercepts, if any exist.
32. List the y-intercepts, if any exist.
33. Find the zeros of g.
34. Solve $g(t) \leq 0$.
35. Find the domain of $G(t) = \dfrac{g(t)}{t+2}$.
36. Solve $\dfrac{g(t)}{t+2} \leq 0$.
37. How many solutions are there to $[g(t)]^2 = 9$?
38. Does g appear to be even, odd, or neither?
39. Prove that if f is an odd function and 0 is in the domain of f, then $f(0) = 0$.
40. Let $R(x)$ be the function defined as: $R(x) = 1$ if x is a rational number, $R(x) = 0$ if x is an irrational number. With help from your classmates, try to graph R. What difficulties do you encounter?

    NOTE: Between every pair of real numbers, there is both a rational and an irrational number . . .
41. Consider the graph of the function f given in Figure 3.1.13.
    (a) Explain why f has a local maximum but not a local minimum at the point $(-1, 1)$.
    (b) Explain why f has a local minimum but not a local maximum at the point $(1, 1)$.

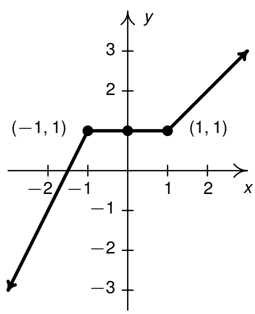


Figure 3.1.13

(c) Explain why f has a local maximum AND a local minimum at the point $(0, 1)$.

(d) Explain why f is constant on the interval $[-1, 1]$ and thus has both a local maximum AND a local minimum at every point $(x, f(x))$ where $-1 < x < 1$.

42. Explain why the function g whose graph is given in Figure 3.1.14 does not have a local maximum at $(-3, 5)$ nor does it have a local minimum at $(3, -3)$. Find its extrema, both local and absolute and find the intervals on which g is increasing and those on which g is decreasing.

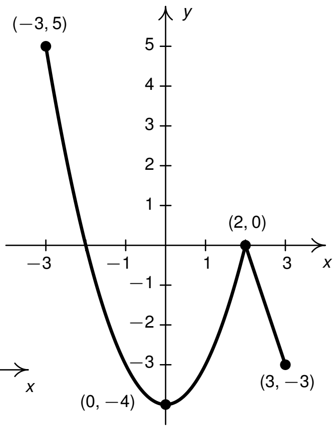
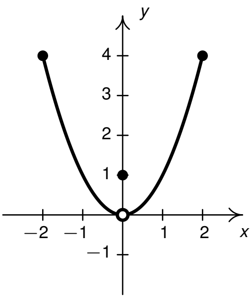
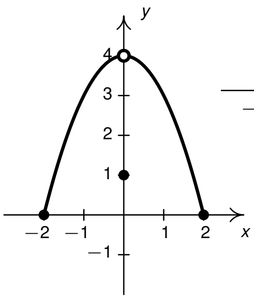


Figure 3.1.14

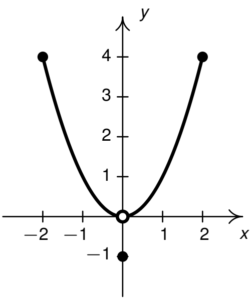
43. For each function below, find the local maximum or local minimum and list the interval over which the function is increasing and the interval over which the function is decreasing.



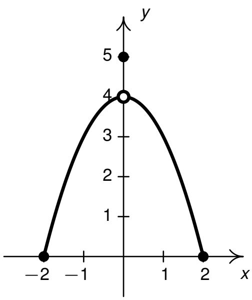
(a) Function I



(b) Function II



(c) Function III



(d) Function IV

3.1.2 Answers

1. $[-5, 3]$
2. $[-5, 4]$
3. $f(-3) = 4$
4. $f(-5) = -5$
5. $(-3, 4)$, $(2, 3)$
6. $(0, -1)$
7. $[-5, -3]$, $[0, 2]$
8. $[-3, 0]$, $[2, 3]$
9. $f(-2) = 2$
10. $x = -3$
11. $(-4, 0)$, $(-1, 0)$, $(1, 0)$
12. $(0, -1)$
13. -4, -1, 1
14. $[-4, -1]$, $[1, 3]$
15. 4
16. 6
17. $x = -4, -1, 1, 2$
18. $(-4, -1) \cup (-1, 1) \cup (2, 3)$
19. To find the domain of $R(x) = \frac{1}{f(x)}$, we start with the domain of f and exclude values where $f(x) = 0$. Hence, the domain of R is $[-5, -4) \cup (-4, -1) \cup (-1, 1) \cup (1, 3]$.
20. To find the range of $R(x) = \frac{1}{f(x)}$, we start with the range of f (excluding 0) and take reciprocals. If $-5 \leq y < 0$, then $\frac{1}{y} \leq -\frac{1}{5}$. If $0 < y \leq 4$, then $\frac{1}{y} \geq \frac{1}{4}$. Hence the range of R is $\left(-\infty, -\frac{1}{5}\right] \cup \left[\frac{1}{4}, \infty\right)$.

21. $[-4, 4]$
22. $[-5, 5)$
23. none
24. $g(-2) = -5$
25. none
26. $(-2, -5)$, $(2, 3)$
27. $[-2, 2)$
28. $[-4, -2]$, $(2, 4]$
29. $g(2) = 3$
30. $t = -2$
31. $(-4, 0)$, $(0, 0)$, $(4, 0)$
32. $(0, 0)$
33. -4, 0, 4
34. $[-4, 0] \cup \{4\}$
35. $[-4, -2) \cup (-2.4]$
36. $\{-4\} \cup (-2, 0] \cup \{4\}$
37. 5
38. Neither.
43. (a) Local maximum: $(0, 1)$, no local minimum. Increasing: $(0, 2]$, decreasing: $[-2, 0)$.
    (b) No local maximum, local minimum: $(0, 1)$. Increasing: $[-2, 0)$, decreasing: $(0, 2]$.
    (c) No local maximum, local minimum: $(0, -1)$. Increasing: $[0, 2]$, decreasing: $[-2, 0]$.
    (d) Local maximum: $(0, 5)$, no local minimum. Increasing: $[-2, 0]$, decreasing: $[0, 2]$.

3.2 Function Arithmetic

As we mentioned in Section 3.1, in this chapter, we are studying functions in a more abstract and general setting. In this section, we begin our study of what can be considered as the *algebra of functions* by defining *function arithmetic.*

Given two real numbers, we have four primary arithmetic operations available to us: addition, subtraction, multiplication, and division (provided we don't divide by 0.) Since the functions we study in this text have ranges which are sets of real numbers, it makes sense we can extend these arithmetic notions to functions.

For example, to add two functions means we add their outputs; to subtract two functions, we subtract their outputs, and so on and so forth. More formally, given two functions f and g, we *define* a new function $f + g$ whose rule is determined by adding the outputs of f and g. That is $(f + g)(x) = f(x) + g(x)$. While this looks suspiciously like some kind of distributive property, it is nothing of the sort. The '+' sign in the expression '$f + g$' is part of the *name* of the function we are defining,[a] whereas the plus sign '+' sign in the expression $f(x) + g(x)$ represents real number addition: we are adding the output from f, $f(x)$ with the output from g, $g(x)$ to determine the output from the sum function, $(f + g)(x)$.

a We could have just as easily called this new function $S(x)$ for 'sum' of f and g and defined S by $S(x) = f(x) + g(x)$.

Of course, in order to define $(f + g)(x)$ by the formula $(f + g)(x) = f(x) + g(x)$, both $f(x)$ and $g(x)$ need to be defined in the first place; that is, x must be in the domain of f *and* the domain of g. You'll recall[b] this means x must be in the *intersection* of the domains of f and g. We define Definition 3.2.1.

b see Section **??**.

We put these definitions to work for us in the next example.

Example 3.2.1. Consider the following functions:

- $f(x) = 6x^2 - 2x$

Definition 3.2.1. Suppose f and g are functions and x is in both the domain of f and the domain of g.

- The **sum** of f and g, denoted $f + g$, is the function defined by the formula

$$(f + g)(x) = f(x) + g(x)$$

- The **difference** of f and g, denoted $f - g$, is the function defined by the formula

$$(f - g)(x) = f(x) - g(x)$$

- The **product** of f and g, denoted fg, is the function defined by the formula

$$(fg)(x) = f(x)g(x)$$

- The **quotient** of f and g, denoted $\frac{f}{g}$, is the function defined by the formula

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},$$

provided $g(x) \neq 0$.

- $g(t) = 3 - \frac{1}{t}$, $t > 0$
- $h = \{(-3, 2), (-2, 0.4), (0, \sqrt{2}), (3, -6)\}$
- s whose graph is given in Figure 3.2.1.

1. Find and simplify the following function values:

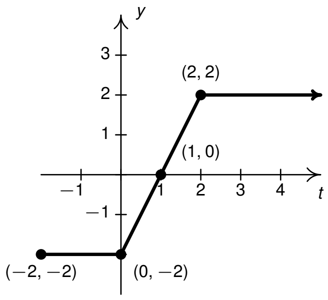
(a) $(f + g)(1)$

(b) $(s - f)(-1)$

(c) $(fg)(2)$

(d) $\left(\frac{s}{h}\right)(0)$

(e) $((s + g) + h)(3)$



$y = s(t)$

Figure 3.2.1

(f) $(s + (g + h))(3)$ (g) $\left(\frac{f + h}{s}\right)(3)$

(h) $(f(g - h))(-2)$

2. Find the domain of each of the following functions:

(a) hg (b) $\frac{f}{s}$

3. Find expressions for the functions below. State the domain for each.

(a) $(fg)(x)$ (b) $\left(\frac{g}{f}\right)(t)$

Solution.

1. (a) By definition, $(f + g)(1) = f(1) + g(1)$. We find $f(1) = 6(1)^2 - 2(1) = 4$ and $g(1) = 3 - \frac{1}{1} = 2$. So we get $(f + g)(1) = 4 + 2 = 6$.

   (b) To find $(s - f)(-1) = s(-1) - f(-1)$, we need both $s(-1)$ and $f(-1)$. To get $s(-1)$, we look to the graph of $y = s(t)$ and look for the y-coordinate of the point on the graph with the t-coordinate of -1. While not labeled directly, we infer the point $(-1, -2)$ is on the graph which means $s(-1) = -2$. For $f(-1)$, we compute: $f(-1) = 6(-1)^2 - 2(-1) = 8$. Putting it all together, we get $(s - f)(-1) = (-2) - (8) = -10$.

   (c) Since $(fg)(2) = f(2)g(2)$, we first compute $f(2)$ and $g(2)$. We find $f(2) = 6(2)^2 - 2(2) = 20$ and $g(2) = 2 + \frac{1}{2} = \frac{5}{2}$, so $(fg)(2) = f(2)g(2) = (20)\left(\frac{5}{2}\right) = 50$.

   (d) By definition, $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)}$. Since $(0, -2)$ is on the graph of $y = s(t)$, so we know $s(0) = -2$. Likewise, the ordered pair $(0, \sqrt{2}) \in h$, so $h(0) = \sqrt{2}$. We get $\left(\frac{s}{h}\right)(0) = \frac{s(0)}{h(0)} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$.

(e) The expression $((s+g)+h)(3)$ involves *three* functions. Fortunately, they are grouped so that we can apply Definition 3.2.1 by first considering the sum of the two functions $(s+g)$ and h, then to the sum of the two functions s and g: $((s+g)+h)(3) = (s+g)(3)+h(3) = (s(3)+g(3))+h(3)$. To get $s(3)$, we look to the graph of $y = s(t)$. We infer the point $(3,2)$ is on the graph of s, so $s(3) = 2$. We compute $g(3) = 3 - \frac{1}{3} = \frac{8}{3}$. To find $h(3)$, we note $(3,-6) \in h$, so $h(3) = -6$. Hence, $((s+g)+h)(3) = (s+g)(3)+h(3) = (s(3)+g(3))+h(3) = \left(2+\frac{8}{3}\right)+(-6) = -\frac{4}{3}$.

(f) The expression $(s+(g+h))(3)$ is very similar to the previous problem, $((s+g)+h)(3)$ except that the g and h are grouped together here instead of the s and g. We proceed as above applying Definition 3.2.1 twice and find $(s+(g+h))(3) = s(3)+(g+h)(3) = s(3)+(g(3)+h(3))$. Substituting the values for $s(3)$, $g(3)$ and $h(3)$, we get $(s+(g+h))(3) = 2+\left(\frac{8}{3}+(-6)\right) = -\frac{4}{3}$, which, not surprisingly, matches our answer to the previous problem.

(g) Once again, we find the expression $\left(\frac{f+h}{s}\right)(3)$ has more than two functions involved. As with all fractions, we treat '$-$' as a grouping symbol and interpret $\left(\frac{f+h}{s}\right)(3) = \frac{(f+h)(3)}{s(3)} = \frac{f(3)+h(3)}{s(3)}$. We compute $f(3) = 6(3)^2-2(3) = 48$ and have $h(3) = -6$ and $s(3) = 2$ from above. Hence, $\left(\frac{f+h}{s}\right)(3) = \frac{f(3)+h(3)}{s(3)} = \frac{48+(-6)}{2} = 21$.

(h) We need to need to exercise caution in parsing $(f(g-h))(-2)$. In this context, f, g, and h are all functions, so we interpret $(f(g-h))$ as the function and -2 as the argument. We view the function $f(g-h)$ as the product of f and the function $g-h$. Hence, $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2)-h(-2)]$. We compute $f(-2) = 6(-2)^2-2(-2) =$

28, and $g(-2) = 3 - \frac{1}{-2} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$. Since $(-2, 0.4) \in h$, $h(-2) = 0.4$. Putting this altogether, we get $(f(g-h))(-2) = f(-2)[(g-h)(-2)] = f(-2)[g(-2) - h(-2)] = 28(3.5 - 0.4) = 28(3.1) = 86.8$.

2. (a) To find the domain of hg, we need to find the real numbers in both the domain of h and the domain of g. The domain of h is $\{-3, -2, 0, 3\}$ and the domain of g is $\{t \in \mathbb{R} \mid t > 0\}$ so the only real number in common here is 3. Hence, the domain of hg is $\{3\}$, which may be small, but it's better than nothing.[c]

Since [c] $(hg)(3) = h(3)g(3) = (-6)\left(\frac{8}{3}\right) = -16$, we can write $hg = \{(3, -16)\}$.

(b) To find the domain of $\frac{f}{s}$, we first note the domain of f is all real numbers, but that the domain of s, based on the graph, is just $[-2, \infty)$. Moreover, $s(t) = 0$ when $t = 1$, so we must exclude this value from the domain of $\frac{f}{s}$. Hence, we are left with $[-2, 1) \cup (1, \infty)$.

3. (a) By definition, $(fg)(x) = f(x)g(x)$. We are given $f(x) = 6x^2 - 2x$ and $g(t) = 3 - \frac{1}{t}$ so $g(x) = 3 - \frac{1}{x}$. Hence,

$$\begin{aligned}
(fg)(x) &= f(x)g(x) \\
&= \left(6x^2 - 2x\right)\left(3 - \frac{1}{x}\right) \\
&= 18x^2 - 6x^2\left(\frac{1}{x}\right) - 2x(3) + 2x\left(\frac{1}{x}\right) \\
&\qquad \text{(distribute)} \\
&= 18x^2 - 6x - 6x + 2 \\
&= 18x^2 - 12x + 2
\end{aligned}$$

To find the domain of fg, we note the domain of f is all real numbers, $(-\infty, \infty)$ whereas the domain of g is restricted to $\{t \in \mathbb{R} \mid t >$

$0\} = (0, \infty)$. Hence, the domain of fg is likewise restricted to $(0, \infty)$. Note if we relied solely on the **simplified formula** for $(fg)(x) = 18x^2 - 12x + 2$, we would have obtained the *incorrect* answer for the domains of fg.

(b) To find an expression for $\left(\frac{g}{f}\right)(t) = \frac{f(t)}{g(t)}$ we first note $f(t) = 6t^2 - 2t$ and $g(t) = 3 - \frac{1}{t}$. Hence:

$$\begin{aligned}
\left(\frac{g}{f}\right)(t) &= \frac{g(t)}{f(t)} \\
&= \frac{3 - \frac{1}{t}}{6t^2 - 2t} \qquad = \frac{3 - \frac{1}{t}}{6t^2 - 2t} \cdot \frac{t}{t} \\
&\qquad\qquad \text{(simplify compound fractions)} \\
&= \frac{\left(3 - \frac{1}{t}\right) t}{\left(6t^2 - 2t\right) t} \qquad = \frac{3t - 1}{\left(6t^2 - 2t\right) t} \\
&= \frac{3t - 1}{2t^2(3t - 1)} \qquad = \frac{\cancel{(3t - 1)}^{1}}{2t^2\cancel{(3t - 1)}} \\
&\qquad\qquad \text{(factor and cancel)} \\
&= \frac{1}{2t^2}
\end{aligned}$$

Hence, $\left(\frac{g}{f}\right)(t) = \frac{1}{2t^2} = \frac{1}{2}t^{-2}$. To find the domain of $\frac{g}{f}$, a real number must be both in the domain of g, $(0, \infty)$, and the domain of f, $(-\infty, \infty)$ so we start with the set $(0, \infty)$. Additionally, we require $f(t) \neq 0$. Solving $f(t) = 0$ amounts to solving $6t^2 - 2t = 0$ or $2t(3t - 1) = 0$. We find $t = 0$ or $t = \frac{1}{3}$ which means we need to exclude these values from the domain. Hence, our final answer for the domain of $\frac{g}{f}$ is $\left(0, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$. Note that, once again, using the *simplified formula* for $\left(\frac{g}{f}\right)(t)$ to determine the domain of $\frac{g}{f}$, would have produced erroneous results. $\square$

A few remarks are in order. First, in number 1 parts 1e through 1h, we first encountered combinations of *three* functions despite Definition 3.2.1 only addressing combinations of *two* functions at a time. It turns out that function arithmetic inherits many of the same properties of real number arithmetic. For example, we showed above that $((s+g)+h)(3) = (s+(g+h))(3)$. In general, given any three functions f, g, and h, $(f+g)+h = f+(g+h)$ that is, function addition is *assocative*. To see this, choose an element x common to the domains of f, g, and h. Then

$$\begin{aligned}
((f+g)+h)(x) &= (f+g)(x)+h(x) \\
&\qquad \text{(definiton of } ((f+g)+h)(x)\text{)} \\
&= (f(x)+g(x))+h(x) \\
&\qquad \text{(definition of } (f+g)(x)\text{)} \\
&= f(x)+(g(x)+h(x)) \\
&\qquad \text{(associative property of real number addition)} \\
&= f(x)+(g+h)(x) \\
&\qquad \text{(definition of } (g+h)(x)\text{)} \\
&= (f+(g+h))(x) \\
&\qquad \text{(definition of } (f+(g+h))(x)\text{)}
\end{aligned}$$

The key step to the argument is that $(f(x)+g(x))+h(x) = f(x)+(g(x)+h(x))$ which is true courtesy of the associative property of real number addition. And just like with real number addition, because function addition is associative, we may write $f+g+h$ instead of $(f+g)+h$ or $f+(g+h)$ even though, when it comes down to computations, we can only add two things together at a time.[d]

d Addition is a 'binary' operation - meaning it is defined only on two objects at once. Even though we write $1+2+3=6$, mentally, we add just two of numbers together at any given time to get our answer: for example, $1+2+3=(1+2)+3=3+3=6$.

For completeness, we summarize the properties of function arithmetic in the Theorem 3.2.1. The proofs of the properties all follow along the same lines as the proof of the associative property and are left to the reader. We investigate some additional properties in the exercises.

Theorem 3.2.1. Suppose f, g and h are functions.

- **Commutative Law of Addition:** $f + g = g + f$
- **Associative Law of Addition:** $(f + g) + h = f + (g + h)$
- **Additive Identity:** The function $Z(x) = 0$ satisfies: $f + Z = Z + f = f$ for all functions f.
- **Additive Inverse:** The function $F(x) = -f(x)$ for all x in the domain of f satisfies:

$$f + F = F + f = Z.$$

- **Commutative Law of Multiplication:** $fg = gf$
- **Associative Law of Multiplication:** $(fg)h = f(gh)$
- **Multiplicative Identity:** The function $I(x) = 1$ satisfies: $fI = If = f$ for all functions f.
- **Multiplicative Inverse:** If $f(x) \neq 0$ for all x in the domain of f, then $F(x) = \dfrac{1}{f(x)}$ satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition:** $f(g + h) = fg + fh$

In the next example, we decompose given functions into sums, differences, products and/or quotients of other functions. Note that there are infinitely many different ways to do this, including some trivial ones. For example, suppose we were instructed to decompose $f(x) = x + 2$ into a sum or difference of functions. We could write $f = g + h$ where $g(x) = x$ and $h(x) = 2$ or we could choose $g(x) = 2x + 3$ and $h(x) = -x - 1$. More simply, we could write $f = g + h$ where $g(x) = x + 2$ and $h(x) = 0$. We'll call this last decomposition a 'trivial' decomposition. Likewise, if we ask for a decomposition of $f(x) = 2x$ as a product, a nontrivial solution would be $f = gh$ where $g(x) = 2$ and $h(x) = x$ whereas a trivial solution would be $g(x) = 2x$ and $h(x) = 1$. In general, non-trivial solutions to decomposition problems avoid using the additive identity, 0, for sums and differences and the multiplicative identity, 1, for products and quotients.

Example 3.2.2. 1. For $f(x) = x^2 - 2x$, find functions g, h and k to decompose f nontrivially as:

(a) $f = g - h$ (b) $f = g + h$
(c) $f = gh$ (d) $f = g(h - k)$

2. For $F(t) = \dfrac{2t+1}{\sqrt{t^2-1}}$, find functions G, H and K to decompost F nontrivially as:

(a) $F = \dfrac{G}{H}$ (b) $F = GH$
(c) $F = G + H$ (d) $F = \dfrac{G+H}{K}$

Solution.

1. (a) To decompose $f = g - h$, we need functions g and h so $f(x) = (g - h)(x) = g(x) - h(x)$. Given $f(x) = x^2 - 2x$, one option is to let $g(x) = x^2$ and $h(x) = 2x$. To check, we find $(g - h)(x) = g(x) - h(x) = x^2 - 2x = f(x)$ as required. In addition to checking the formulas match up, we also need to check domains. There isn't much work here since the domains of g and h are all real numbers which combine to give the domain of f which is all real numbers.

   (b) In order to write $f = g+h$, we need $f(x) = (g+h)(x) = g(x) + h(x)$. One way to accomplish this is to write $f(x) = x^2 - 2x = x^2 + (-2x)$ and identify $g(x) = x^2$ and $h(x) = -2x$. To check, $(g + h)(x) = g(x) + h(x) = x^2 - 2x = f(x)$. Again, the domains for both g and h are all real numbers which combine to give f its domain of all real numbers.

   (c) To write $f = gh$, we require $f(x) = (gh)(x) = g(x)h(x)$. In other words, we need to factor

$f(x)$. We find $f(x) = x^2 - 2x = x(x-2)$, so one choice is to select $g(x) = x$ and $h(x) = x - 2$. Then $(gh)(x) = g(x)h(x) = x(x-2) = x^2 - 2x = f(x)$, as required. As above, the domains of g and h are all real numbers which combine to give f the correct domain of $(-\infty, \infty)$.

(d) We need to be careful here interpreting the equation $f = g(h-k)$. What we have is an equality of *functions* so the parentheses here *do not* represent function notation here, but, rather function *multiplication*. The way to parse $g(h-k)$, then, is the function g *times* the function $h-k$. Hence, we seek functions g, h, and k so that $f(x) = [g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x))$. From the previous example, we know we can rewrite $f(x) = x(x-2)$, so one option is to set $g(x) = h(x) = x$ and $k(x) = 2$ so that $[g(h-k)](x) = g(x)[(h-k)(x)] = g(x)(h(x) - k(x)) = x(x-2) = x^2 - 2x = f(x)$, as required. As above, the domain of all constituent functions is $(-\infty, \infty)$ which matches the domain of f.

2. (a) To write $F = \frac{G}{H}$, we need $G(t)$ and $H(t)$ so $F(t) = \left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)}$. We choose $G(t) = 2t + 1$ and $H(t) = \sqrt{t^2 - 1}$. Sure enough, $\left(\frac{G}{H}\right)(t) = \frac{G(t)}{H(t)} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$ as required. When it comes to the domain of F, owing to the square root, we require $t^2 - 1 \geq 0$. Since we have a denominator as well, we require $\sqrt{t^2 - 1} \neq 0$. The former requirement is the same restriction on H, and the latter requirement comes from Definition 3.2.1. Starting with the domain of G, all real numbers, and working through the details, we arrive at the correct domain of F, $(-\infty, -1) \cup (1, \infty)$.

(b) Next, we are asked to find functions G and H so $F(t) = (GH)(t) = G(t)H(t)$. This means we need to rewrite the expression for $F(t)$ as a product. One way to do this is to convert radical notation to exponent notation:

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t+1}{\left(t^2-1\right)^{\frac{1}{2}}} = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}.$$

Choosing $G(t) = 2t+1$ and $H(t) = \left(t^2-1\right)^{-\frac{1}{2}}$, we see $(GH)(t) = G(t)H(t) = (2t+1)\left(t^2-1\right)^{-\frac{1}{2}}$ as required. The domain restrictions on F stem from the presence of the square root in the denominator - both are addressed when finding the domain of H. Hence, we obtain the correct domain of F.

(c) To express F as a sum of functions G and H, we could rewrite

$$F(t) = \frac{2t+1}{\sqrt{t^2-1}} = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}},$$

so that $G(t) = \frac{2t}{\sqrt{t^2-1}}$ and $H(t) = \frac{1}{\sqrt{t^2-1}}$. Indeed, $(G+H)(t) = G(t) + H(t) = \frac{2t}{\sqrt{t^2-1}} + \frac{1}{\sqrt{t^2-1}} = \frac{2t+1}{\sqrt{t^2-1}} = F(t)$, as required. Moreover, the domain restrictions for F are the same for both G and H, so we get agreement on the domain, as required.

(d) Last, but not least, to write $F = \frac{G+H}{K}$, we require $F(t) = \left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} = \frac{G(t)+H(t)}{K(t)}$. Identifying $G(t) = 2t$, $H(t) = 1$, and $K(t) = \sqrt{t^2-1}$, we get

$$\begin{aligned}\left(\frac{G+H}{K}\right)(t) = \frac{(G+H)(t)}{K(t)} &= \frac{G(t)+H(t)}{K(t)} \\ &= \frac{2t+1}{\sqrt{t^2-1}} = F(t).\end{aligned}$$

Concerning domains, the domain of both G and H are all real numbers, but the domain of K is restricted to $t^2 - 1 \geq 0$. Coupled with the restriction stated in Definition 3.2.1 that $K(t) \neq 0$, we recover the domain of F, $(-\infty, -1) \cup (1, \infty)$. □

3.2.1 Difference Quotients

Recall in Section **??** the concept of the average rate of change of a function over the interval $[a, b]$ is the slope between the two points $(a, f(a))$ and $(b, f(b))$ and is given by

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(b) - f(a)}{b - a}.$$

Consider a function f defined over an interval containing x and $x + h$ where $h \neq 0$. The average rate of change of f over the interval $[x, x + h]$ is thus given by the formula:[e]

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x + h) - f(x)}{h}, \quad h \neq 0.$$

e assuming $h > 0$; otherwise, we the interval is $[x + h, x]$. We get the same formula for the difference quotient either way.

The above is an example of what is traditionally called the **difference quotient** or **Newton quotient** of f, since it is the *quotient* of two *differences*, namely $\Delta[f(x)]$ and Δx. Another formula for the difference quotient sticks keeps with the notation Δx instead of h:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0.$$

It is important to understand that in this formulation of the difference quotient, the variables 'x' and 'Δx' are distinct - that is they do not combine as like terms.

In Section 1.1, the average rate of change of position function s can be interpreted as the average velocity (see Definition 1.1.5.) We can likewise re-cast this definition. After relabeling $t = t_0 + \Delta t$, we get

$$\overline{v}(\Delta t) = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}, \quad \Delta t \neq 0,$$

which measures the average velocity between time t_0 and time $t_0 + \Delta t$ as a function of Δt.

Note that, regardless of which form the difference quotient takes, when h, Δx, or Δt is 0, the difference quotient returns the indeterminant form '$\frac{0}{0}$.' As we've seen with rational functions in Section 1.1, when this happens, we can reduce the fraction to lowest terms to see if we have a vertical asymptote or hole in the graph. With this in mind, when we speak of 'simplifying the difference quotient,' we mean to manipulate the expression until the factor of 'h' or 'Δx' cancels out from the denominator.

Our next example invites us to simplify three difference quotients, each cast slightly differently. In each case, the bulk of the work involves Intermediate Algebra. We refer the reader to Sections **??** and **??** for additional review, if needed.

Example 3.2.3. Find and simplify the indicated difference quotients for the following functions:

1. For $f(x) = x^2 - x - 2$, find and simplify:

   (a) $\dfrac{f(3+h) - f(3)}{h}$ (b) $\dfrac{f(x+h) - f(x)}{h}$.

2. For $g(x) = \dfrac{3}{2x+1}$, find and simplify:

   (a) $\dfrac{g(\Delta x) - g(0)}{\Delta x}$ (b) $\dfrac{g(x+\Delta x) - g(x)}{\Delta x}$.

3. $r(t) = \sqrt{t}$, find and simplify:

   (a) $\dfrac{r(9+\Delta t) - r(9)}{\Delta t}$ (b) $\dfrac{r(t+\Delta t) - r(t)}{\Delta t}$.

Solution.

1. (a) For our first difference quotient, we find $f(3+h)$ by substituting the quantity $(3+h)$ in for x:

$$\begin{aligned} f(3+h) &= (3+h)^2 - (3+h) - 2 \\ &= 9 + 6h + h^2 - 3 - h - 2 \\ &= 4 + 5h + h^2 \end{aligned}$$

Since $f(3) = (3)^2 - 3 - 2 = 4$, we get for the difference quotient:

$$\begin{aligned} \frac{f(3+h) - f(3)}{h} &= \frac{(4 + 5h + h^2) - 4}{h} \\ &= \frac{5h + h^2}{h} \\ &= \frac{h(5+h)}{h} && \text{(factor)} \\ &= \frac{\cancel{h}(5+h)}{\cancel{h}} && \text{(cancel)} \\ &= 5 + h \end{aligned}$$

(b) For the second difference quotient, we first find $f(x+h)$, we replace every occurrence of x in the formula $f(x) = x^2 - x - 2$ with the quantity $(x+h)$ to get

$$\begin{aligned} f(x+h) &= (x+h)^2 - (x+h) - 2 \\ &= x^2 + 2xh + h^2 - x - h - 2. \end{aligned}$$

So the difference quotient is

$$\begin{aligned}
&\frac{f(x+h)-f(x)}{h} \\
&= \frac{\left(x^2+2xh+h^2-x-h-2\right)-\left(x^2-x-2\right)}{h} \\
&= \frac{x^2+2xh+h^2-x-h-2-x^2+x+2}{h} \\
&= \frac{2xh+h^2-h}{h} \\
&= \frac{h(2x+h-1)}{h} && \text{(factor)} \\
&= \frac{\cancel{h}(2x+h-1)}{\cancel{h}} && \text{(cancel)} \\
&= 2x+h-1
\end{aligned}$$

Note if we substitute $x = 3$ into this expression, we obtain $5 + h$ which agrees with our answer from the first difference quotient.

2. (a) Rewriting $\Delta x = 0 + \Delta x$, we see the first expression really is a difference quotient:

$$\frac{g(\Delta x)-g(0)}{\Delta x} = \frac{g(0+\Delta x)-g(0)}{\Delta x}.$$

Since $g(\Delta x) = \frac{3}{2\Delta x+1}$ and $g(0) = \frac{3}{2(0)+1} = 3$, our difference quotient is:

$$\begin{aligned}
\frac{g(0+\Delta x)-g(0)}{\Delta x} &= \frac{\frac{3}{2\Delta x+1}-3}{\Delta x} \\
&= \frac{\frac{3}{2\Delta x+1}-3}{\Delta x}\cdot\frac{(2\Delta x+1)}{(2\Delta x+1)} \\
&= \frac{3-3(2\Delta x+1)}{\Delta x(2\Delta x+1)}
\end{aligned}$$

$$= \frac{3 - 6\Delta x - 3}{\Delta x(2\Delta x + 1)}$$
$$= \frac{-6\Delta x}{\Delta x(2\Delta x + 1)}$$
$$= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2\Delta x + 1)}$$
$$= \frac{-6}{2\Delta x + 1}.$$

(b) For our next difference quotient, we first find $g(x+\Delta x)$ by replacing every occurrence of x in the formula for $g(x)$ with the quantity $(x + \Delta x)$:

$$\begin{aligned} g(x + \Delta x) &= \frac{3}{2(x + \Delta x) + 1} \\ &= \frac{3}{2x + 2\Delta x + 1}. \end{aligned}$$

Hence,

$$\frac{g(x + \Delta x) - g(x)}{\Delta x}$$
$$= \frac{\dfrac{3}{2x + 2\Delta x + 1} - \dfrac{3}{2x + 1}}{\Delta x}$$
$$= \frac{\dfrac{3}{2x + 2\Delta x + 1} - \dfrac{3}{2x + 1}}{\Delta x} \cdot \frac{(2x + 2\Delta x + 1)(2x + 1)}{(2x + 2\Delta x + 1)(2x + 1)}$$
$$= \frac{3(2x + 1) - 3(2x + 2\Delta x + 1)}{\Delta x(2x + 2\Delta x + 1)(2x + 1)}$$
$$= \frac{6x + 3 - 6x - 6\Delta x - 3}{\Delta x(2x + 2\Delta x + 1)(2x + 1)}$$
$$= \frac{-6\Delta x}{\Delta x(2x + 2\Delta x + 1)(2x + 1)}$$
$$= \frac{-6\cancel{\Delta x}}{\cancel{\Delta x}(2x + 2\Delta x + 1)(2x + 1)}$$

$$= \frac{-6}{(2x + 2\Delta x + 1)(2x + 1)}$$

Since we have managed to cancel the factor 'Δx' from the denominator, we are done. Substituting $x = 0$ into our final expression gives $\frac{-6}{2\Delta x+1}$ thus checking our previous answer.

3. (a) We start with $r(9 + \Delta t) = \sqrt{9 + \Delta t}$ and $r(9) = \sqrt{9} = 3$ and get:

$$\frac{r(9 + \Delta t) - r(9)}{\Delta t} = \frac{\sqrt{9 + \Delta t} - 3}{\Delta t}.$$

In order to cancel the factor 'Δt' from the *denominator*, we set about rationalizing the *numerator* by multiplying both numerator and denominator by the conjugate of the numerator, $\sqrt{9 + \Delta t} - 3$:

$$\begin{aligned}
\frac{r(9 + \Delta t) - r(9)}{\Delta t} &= \frac{\sqrt{9 + \Delta t} - 3}{\Delta t} \\
&= \frac{\left(\sqrt{9 + \Delta t} - 3\right)}{\Delta t} \cdot \frac{\left(\sqrt{9 + \Delta t} + 3\right)}{\left(\sqrt{9 + \Delta t} + 3\right)} \\
&\qquad \text{(Multiply by the conjugate.)} \\
&= \frac{\left(\sqrt{9 + \Delta t}\right)^2 - (3)^2}{\Delta t\left(\sqrt{9 + \Delta t} + 3\right)} \\
&\qquad \text{(Difference of Squares.)} \\
&= \frac{(9 + \Delta t) - 9}{\Delta t\left(\sqrt{9 + \Delta t} + 3\right)} \\
&= \frac{\Delta t}{\Delta t\left(\sqrt{9 + \Delta t} + 3\right)}
\end{aligned}$$

$$= \frac{\cancel{\Delta t}^{\,1}}{\cancel{\Delta t}\left(\sqrt{9+\Delta t}+3\right)}$$

$$= \frac{1}{\sqrt{9+\Delta t}+3}$$

(b) As one might expect, we use this same strategy to simplify our final different quotient. We have:

$$\frac{r(t+\Delta t)-r(t)}{\Delta t}$$

$$= \frac{\sqrt{t+\Delta t}-\sqrt{t}}{\Delta t}$$

$$= \frac{\left(\sqrt{t+\Delta t}-\sqrt{t}\right)}{\Delta t} \cdot \frac{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}{\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}$$

(Multiply by the conjugate.)

$$= \frac{\left(\sqrt{t+\Delta t}\right)^2-\left(\sqrt{t}\right)^2}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}$$

(Difference of Squares.)

$$= \frac{(t+\Delta t)-t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}$$

$$= \frac{\Delta t}{\Delta t\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}$$

$$= \frac{\cancel{\Delta t}^{\,1}}{\cancel{\Delta t}\left(\sqrt{t+\Delta t}+\sqrt{t}\right)}$$

$$= \frac{1}{\sqrt{t+\Delta t}+\sqrt{t}}$$

Since we have canceled the original 'Δt' factor from the denominator, we are done. Setting $t = 9$ in this expression, we get $\frac{1}{\sqrt{9+\Delta t}+3}$ which agrees with our previous answer. $\Box$

We close this section revisiting the situation in Example 1.1.3.

Example 3.2.4. Let $s(t) = -5t^2 + 100t$, $0 \leq t \leq 20$ give the height of a model rocket above the Moon's surface, in feet, t seconds after liftoff.

1. Find, and simplify: $\overline{v}(\Delta t) = \dfrac{s(15+\Delta t) - s(15)}{\Delta t}$, for $\Delta t \neq 0$.
2. Find and interpret $\overline{v}(-1)$.
3. Graph $y = \overline{v}(t)$.
4. Describe the behavior of $\overline{v}$ as $\Delta t \to 0$ and interpret.

Solution.

1. To find $\overline{v}(\Delta t)$, we first find $s(15+\Delta t)$:

$$\begin{aligned} s(15+\Delta t) &= -5(15+\Delta t)^2 + 100(15+\Delta t) \\ &= -5(225 + 30\Delta t + (\Delta t)^2) + 1500 + 100\Delta t \\ &= -5(\Delta t)^2 - 50\Delta t + 375 \end{aligned}$$

Since $s(15) = -5(15)^2 + 100(15) = 375$, we get:

$$\begin{aligned} \overline{v}(\Delta t) &= \frac{s(15+\Delta t) - s(15)}{\Delta t} \\ &= \frac{(-5(\Delta t)^2 - 50\Delta t + 375) - 375}{\Delta t} \\ &= \frac{\Delta t(-5\Delta t - 50)}{\Delta t} \\ &= \frac{\cancel{\Delta t}(-5\Delta t - 50)}{\cancel{\Delta t}} \end{aligned}$$

$$= -5\Delta t - 50 \qquad (\Delta t \neq 0)$$

In addition to the restriction $\Delta t \neq 0$, we also know the domain of s is $0 \leq t \leq 20$. Hence, we also require $0 \leq 15 + \Delta t \leq 20$ or $-15 \leq \Delta \leq 5$. Our final answer is $\overline{v}(\Delta t) = -5\Delta t - 50$, for $\Delta t \in [-15, 0) \cup (0, 5]$

2. We find $\overline{v}(-1) = -5(-1) - 50 = -45$. This means the average velocity over between time $t = 15 + (-1) = 14$ seconds and $t = 15$ seconds is -45 feet per second. This indicates the rocket is, on average, heading *downwards* at a rate of 45 feet per second.

3. The graph of $y = -5\Delta t - 50$ is a line with slope -5 and y-intercept $(0, -50)$. However, since the domain of $\overline{v}$ is $[-15, 0) \cup (0, 5]$, we the graph of $\overline{v}$ is a line *segment* from $(-15, 25)$ to $(5, -75)$ with a hole at $(0, -50)$. See Figure 3.2.2

4. As $\Delta t \to 0$, $\overline{v}(\Delta t) \to -50$ meaning as we approach $t = 15$, the velocity of the rocket approaches -50 feet per second. Recall from Example 1.1.3 that this is the so-called *instantaneous velocity* of the rocket *at* $t = 15$ seconds. That is, 15 seconds after lift-off, the rocket is heading back towards the surface of the moon at a rate of 15 feet per second. □

The reader is invited to compare Example 1.1.3 in Section 1.1 with Exercise 3.2.4 above. We obtain the exact same information because we are asking the *exact same* questions - they are just framed differently.

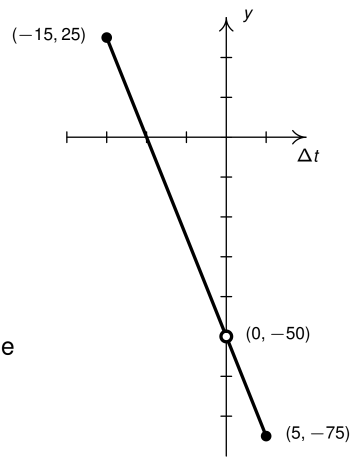


Figure 3.2.2: $y = \overline{v}(\Delta t)$

3.2.2 Exercises

In Exercises 1 - 10, use the pair of functions f and g to find the following values if they exist.

- $(f+g)(2)$
- $(f-g)(-1)$
- $(g-f)(1)$
- $(fg)\left(\frac{1}{2}\right)$
- $\left(\frac{f}{g}\right)(0)$
- $\left(\frac{g}{f}\right)(-2)$

1. $f(x) = 3x + 1$ and $g(t) = 4 - t$
2. $f(x) = x^2$ and $g(t) = -2t + 1$
3. $f(x) = x^2 - x$ and $g(t) = 12 - t^2$
4. $f(x) = 2x^3$ and $g(t) = -t^2 - 2t - 3$
5. $f(x) = \sqrt{x+3}$ and $g(t) = 2t - 1$
6. $f(x) = \sqrt{4-x}$ and $g(t) = \sqrt{t+2}$
7. $f(x) = 2x$ and $g(t) = \dfrac{1}{2t+1}$
8. $f(x) = x^2$ and $g(t) = \dfrac{3}{2t-3}$
9. $f(x) = x^2$ and $g(t) = \dfrac{1}{t^2}$
10. $f(x) = x^2 + 1$ and $g(t) = \dfrac{1}{t^2+1}$

Exercises 11 - 20 refer to the functions f and g whose graphs are given in Figure 3.2.3 and Figure 3.2.4 respectively.

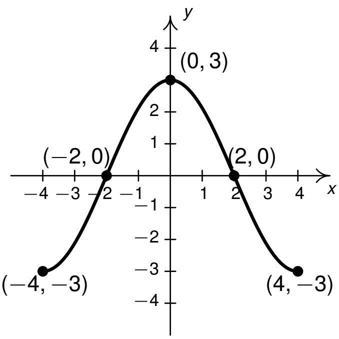


Figure 3.2.3: $y = f(x)$

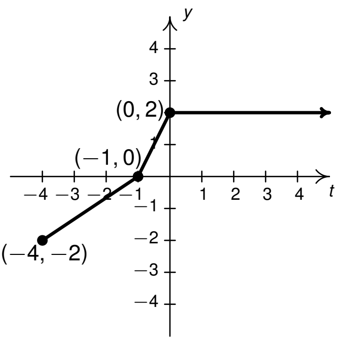


Figure 3.2.4: $y = g(t)$

11. $(f+g)(-4)$
12. $(f+g)(0)$
13. $(f-g)(4)$
14. $(fg)(-4)$
15. $(fg)(-2)$
16. $(fg)(4)$
17. $\left(\dfrac{f}{g}\right)(0)$
18. $\left(\dfrac{f}{g}\right)(2)$
19. $\left(\dfrac{g}{f}\right)(-1)$

20. Find the domains of $f+g$, $f-g$, fg, $\dfrac{f}{g}$ and $\dfrac{g}{f}$.

In Exercises 21 - 32, let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

Compute the indicated value if it exists.

21. $(f+g)(-3)$
22. $(f-g)(2)$
23. $(fg)(-1)$
24. $(g+f)(1)$
25. $(g-f)(3)$
26. $(gf)(-3)$
27. $\left(\frac{f}{g}\right)(-2)$
28. $\left(\frac{f}{g}\right)(-1)$
29. $\left(\frac{f}{g}\right)(2)$
30. $\left(\frac{g}{f}\right)(-1)$
31. $\left(\frac{g}{f}\right)(3)$
32. $\left(\frac{g}{f}\right)(-3)$

In Exercises 33 - 42, use the pair of functions f and g to find the domain of the indicated function then find and simplify an expression for it.

- $(f+g)(x)$
- $(f-g)(x)$
- $(fg)(x)$
- $\left(\frac{f}{g}\right)(x)$

33. $f(x) = 2x+1$ and $g(x) = x-2$
34. $f(x) = 1-4x$ and $g(x) = 2x-1$
35. $f(x) = x^2$ and $g(x) = 3x-1$
36. $f(x) = x^2 - x$ and $g(x) = 7x$
37. $f(x) = x^2 - 4$ and $g(x) = 3x+6$
38. $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$
39. $f(x) = \dfrac{x}{2}$ and $g(x) = \dfrac{2}{x}$
40. $f(x) = x-1$ and $g(x) = \dfrac{1}{x-1}$
41. $f(x) = x$ and $g(x) = \sqrt{x+1}$
42. $f(x) = \sqrt{x-5}$ and $g(x) = f(x) = \sqrt{x-5}$

In Exercises 43 - 47, write the given function as a nontrivial decomposition of functions as directed.

43. For $p(z) = 4z - z^3$, find functions f and g so that $p = f - g$.

44. For $p(z) = 4z - z^3$, find functions f and g so that $p = f + g$.

45. For $g(t) = 3t|2t - 1|$, find functions f and h so that $g = fh$.

46. For $r(x) = \dfrac{3 - x}{x + 1}$, find functions f and g so $r = \dfrac{f}{g}$.

47. For $r(x) = \dfrac{3 - x}{x + 1}$, find functions f and g so $r = fg$.

48. Can $f(x) = x$ be decomposed as $f = g - h$ where $g(x) = x + \dfrac{1}{x}$ and $h(x) = \dfrac{1}{x}$?

49. Discuss with your classmates how to phrase the quantities revenue and profit in Definition **??** terms of function arithmetic as defined in Definition 3.2.1.

In Exercises 50 - 59, find and simplify the difference quotients:

- $\dfrac{f(2 + h) - f(2)}{h}$
- $\dfrac{f(x + h) - f(x)}{h}$

50. $f(x) = 2x - 5$

51. $f(x) = -3x + 5$

52. $f(x) = 6$

53. $f(x) = 3x^2 - x$

54. $f(x) = -x^2 + 2x - 1$

55. $f(x) = 4x^2$

56. $f(x) = x - x^2$

57. $f(x) = x^3 + 1$

58. $f(x) = mx + b$ where $m \neq 0$

59. $f(x) = ax^2 + bx + c$ where $a \neq 0$

In Exercises 60 - 67, find and simplify the difference quotients:

- $\dfrac{f(-1 + \Delta x) - f(-1)}{\Delta x}$
- $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

60. $f(x) = \dfrac{2}{x}$

61. $f(x) = \dfrac{3}{1 - x}$

62. $f(x) = \dfrac{1}{x^2}$

63. $f(x) = \dfrac{2}{x+5}$

64. $f(x) = \dfrac{1}{4x-3}$

65. $f(x) = \dfrac{3x}{x+2}$

66. $f(x) = \dfrac{x}{x-9}$

67. $f(x) = \dfrac{x^2}{2x+1}$

68. $g(t) = \sqrt{9-t}$

69. $g(t) = \sqrt{2t+1}$

70. $g(t) = \sqrt{-4t+5}$

71. $g(t) = \sqrt{4-t}$

72. $g(t) = \sqrt{at+b}$, where $a \neq 0$.

73. $g(t) = t\sqrt{t}$

74. $g(t) = \sqrt[3]{t}$. **HINT:** $(a-b)\left(a^2+ab+b^2\right) = a^3 - b^3$

75. In this exercise, we explore decomposing a function into its positive and negative parts. Given a function f, we define the **positive part** of f, denoted f_+ and **negative part** of f, denoted f_- by:

$$f_+(x) = \frac{f(x) + |f(x)|}{2}, \qquad \text{and} \qquad f_-(x) = \frac{f(x) - |f(x)|}{2}.$$

(a) Using a graphing utility, graph each of the functions f below along with f_+ and f_-.

- $f(x) = x - 3$
- $f(x) = x^2 - x - 6$
- $f(x) = 4x - x^3$

Why is f_+ called the 'positive part' of f and f_- called the 'negative part' of f?

(b) Show that $f = f_+ + f_-$.

(c) Use Definition **??** to rewrite the expressions for $f_+(x)$ and $f_-(x)$ as piecewise defined functions.

In Exercises 68 - 74, find and simplify the difference quotients:

- $\dfrac{g(\Delta t) - g(0)}{\Delta t}$
- $\dfrac{g(t + \Delta t) - g(t)}{\Delta t}$

3.2.3 Answers

1. For $f(x) = 3x + 1$ and $g(x) = 4 - x$
   - $(f + g)(2) = 9$
   - $(f - g)(-1) = -7$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = \frac{35}{4}$
   - $\left(\frac{f}{g}\right)(0) = \frac{1}{4}$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{6}{5}$
2. For $f(x) = x^2$ and $g(x) = -2x + 1$
   - $(f + g)(2) = 1$
   - $(f - g)(-1) = -2$
   - $(g - f)(1) = -2$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{5}{4}$
3. For $f(x) = x^2 - x$ and $g(x) = 12 - x^2$
   - $(f + g)(2) = 10$
   - $(f - g)(-1) = -9$
   - $(g - f)(1) = 11$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{47}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{4}{3}$
4. For $f(x) = 2x^3$ and $g(x) = -x^2 - 2x - 3$
   - $(f + g)(2) = 5$
   - $(f - g)(-1) = 0$
   - $(g - f)(1) = -8$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{17}{16}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{3}{16}$
5. For $f(x) = \sqrt{x + 3}$ and $g(x) = 2x - 1$
   - $(f + g)(2) = 3 + \sqrt{5}$
   - $(f - g)(-1) = 3 + \sqrt{2}$
   - $(g - f)(1) = -1$
   - $(fg)\left(\frac{1}{2}\right) = 0$
   - $\left(\frac{f}{g}\right)(0) = -\sqrt{3}$

- $\left(\frac{g}{f}\right)(-2) = -5$

6. For $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$
   - $(f+g)(2) = 2 + \sqrt{2}$
   - $(f-g)(-1) = -1 + \sqrt{5}$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = \frac{\sqrt{35}}{2}$
   - $\left(\frac{f}{g}\right)(0) = \sqrt{2}$
   - $\left(\frac{g}{f}\right)(-2) = 0$

7. For $f(x) = 2x$ and $g(x) = \frac{1}{2x+1}$
   - $(f+g)(2) = \frac{21}{5}$
   - $(f-g)(-1) = -1$
   - $(g-f)(1) = -\frac{5}{3}$
   - $(fg)\left(\frac{1}{2}\right) = \frac{1}{2}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{12}$

8. For $f(x) = x^2$ and $g(x) = \frac{3}{2x-3}$
   - $(f+g)(2) = 7$
   - $(f-g)(-1) = \frac{8}{5}$
   - $(g-f)(1) = -4$
   - $(fg)\left(\frac{1}{2}\right) = -\frac{3}{8}$
   - $\left(\frac{f}{g}\right)(0) = 0$
   - $\left(\frac{g}{f}\right)(-2) = -\frac{3}{28}$

9. For $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$
   - $(f+g)(2) = \frac{17}{4}$
   - $(f-g)(-1) = 0$
   - $(g-f)(1) = 0$
   - $(fg)\left(\frac{1}{2}\right) = 1$
   - $\left(\frac{f}{g}\right)(0)$ is undefined.
   - $\left(\frac{g}{f}\right)(-2) = \frac{1}{16}$

10. For $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x^2+1}$
    - $(f+g)(2) = \frac{26}{5}$
    - $(f-g)(-1) = \frac{3}{2}$
    - $(g-f)(1) = -\frac{3}{2}$
    - $(fg)\left(\frac{1}{2}\right) = 1$
    - $\left(\frac{f}{g}\right)(0) = 1$

- $\left(\frac{g}{f}\right)(-2) = \frac{1}{25}$

11. $(f+g)(-4) = -5$
12. $(f+g)(0) = 5$
13. $(f-g)(4) = -5$
14. $(fg)(-4) = 6$
15. $(fg)(-2) = 0$
16. $(fg)(4) = -6$
17. $\left(\dfrac{f}{g}\right)(0) = \dfrac{3}{2}$
18. $\left(\dfrac{f}{g}\right)(2) = 0$
19. $\left(\dfrac{g}{f}\right)(-1) = 0$
20. The domains of $f+g$, $f-g$ and fg are all $[-4, 4]$. The domain of $\frac{f}{g}$ is $[-4,-1) \cup (-1, 4]$ and the domain of $\frac{g}{f}$ is $[-4,-2) \cup (-2, 2) \cup (2, 4]$.
21. $(f+g)(-3) = 2$
22. $(f-g)(2) = 3$
23. $(fg)(-1) = 0$
24. $(g+f)(1) = 0$
25. $(g-f)(3) = 3$
26. $(gf)(-3) = -8$
27. $\left(\frac{f}{g}\right)(-2)$ does not exist
28. $\left(\frac{f}{g}\right)(-1) = 0$
29. $\left(\frac{f}{g}\right)(2) = 4$
30. $\left(\frac{g}{f}\right)(-1)$ does not exist
31. $\left(\frac{g}{f}\right)(3) = -2$

32. $\left(\frac{g}{f}\right)(-3) = -\frac{1}{2}$

33. For $f(x) = 2x + 1$ and $g(x) = x - 2$

    - $(f + g)(x) = 3x - 1$
      Domain: $(-\infty, \infty)$
    - $(f - g)(x) = x + 3$
      Domain: $(-\infty, \infty)$
    - $(fg)(x) = 2x^2 - 3x - 2$
      Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{2x+1}{x-2}$
      Domain: $(-\infty, 2) \cup (2, \infty)$

34. For $f(x) = 1 - 4x$ and $g(x) = 2x - 1$

    - $(f + g)(x) = -2x$
      Domain: $(-\infty, \infty)$
    - $(f - g)(x) = 2 - 6x$
      Domain: $(-\infty, \infty)$
    - $(fg)(x) = -8x^2 + 6x - 1$
      Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{1-4x}{2x-1}$
      Domain: $\left(-\infty, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

35. For $f(x) = x^2$ and $g(x) = 3x - 1$

    - $(f + g)(x) = x^2 + 3x - 1$
      Domain: $(-\infty, \infty)$
    - $(f - g)(x) = x^2 - 3x + 1$
      Domain: $(-\infty, \infty)$
    - $(fg)(x) = 3x^3 - x^2$
      Domain: $(-\infty, \infty)$
    - $\left(\frac{f}{g}\right)(x) = \frac{x^2}{3x-1}$
      Domain: $\left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$

36. For $f(x) = x^2 - x$ and $g(x) = 7x$

- $(f + g)(x) = x^2 + 6x$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 8x$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 7x^3 - 7x^2$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-1}{7}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

37. For $f(x) = x^2 - 4$ and $g(x) = 3x + 6$

- $(f + g)(x) = x^2 + 3x + 2$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = x^2 - 3x - 10$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = 3x^3 + 6x^2 - 12x - 24$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = \frac{x-2}{3}$
  Domain: $(-\infty, -2) \cup (-2, \infty)$

38. For $f(x) = -x^2 + x + 6$ and $g(x) = x^2 - 9$

- $(f + g)(x) = x - 3$
  Domain: $(-\infty, \infty)$
- $(f - g)(x) = -2x^2 + x + 15$
  Domain: $(-\infty, \infty)$
- $(fg)(x) = -x^4 + x^3 + 15x^2 - 9x - 54$
  Domain: $(-\infty, \infty)$
- $\left(\frac{f}{g}\right)(x) = -\frac{x+2}{x+3}$
  Domain: $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

39. For $f(x) = \frac{x}{2}$ and $g(x) = \frac{2}{x}$

- $(f + g)(x) = \frac{x^2+4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$
- $(f - g)(x) = \frac{x^2-4}{2x}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

- $(fg)(x) = 1$
  Domain: $(-\infty, 0) \cup (0, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x^2}{4}$
  Domain: $(-\infty, 0) \cup (0, \infty)$

40. For $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$

- $(f + g)(x) = \frac{x^2-2x+2}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$

- $(f - g)(x) = \frac{x^2-2x}{x-1}$
  Domain: $(-\infty, 1) \cup (1, \infty)$

- $(fg)(x) = 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

- $\left(\frac{f}{g}\right)(x) = x^2 - 2x + 1$
  Domain: $(-\infty, 1) \cup (1, \infty)$

41. For $f(x) = x$ and $g(x) = \sqrt{x + 1}$

- $(f + g)(x) = x + \sqrt{x + 1}$
  Domain: $[-1, \infty)$

- $(f - g)(x) = x - \sqrt{x + 1}$
  Domain: $[-1, \infty)$

- $(fg)(x) = x\sqrt{x + 1}$
  Domain: $[-1, \infty)$

- $\left(\frac{f}{g}\right)(x) = \frac{x}{\sqrt{x+1}}$
  Domain: $(-1, \infty)$

42. For $f(x) = \sqrt{x - 5}$ and $g(x) = f(x) = \sqrt{x - 5}$

- $(f + g)(x) = 2\sqrt{x - 5}$
  Domain: $[5, \infty)$

- $(f - g)(x) = 0$
  Domain: $[5, \infty)$

- $(fg)(x) = x - 5$
  Domain: $[5, \infty)$

- $\left(\frac{f}{g}\right)(x) = 1$
  Domain: $(5, \infty)$

43. One solution is $f(z) = 4z$ and $g(z) = z^3$.
44. One solution is $f(z) = 4z$ and $g(z) = -z^3$.
45. One solution is $f(t) = 3t$ and $h(t) = |2t - 1|$
46. One solution is $f(x) = 3 - x$ and $g(x) = x + 1$.
47. One solution is $f(x) = 3 - x$ and $g(x) = (x + 1)^{-1}$.
48. No. The equivalence does not hold when $x = 0$.
50. 2, 2.
51. -3, -3.
52. 0, 0
53. $3h + 11$, $6x + 3h - 1$
54. $-h - 2$, $-2x - h + 2$
55. $4h + 16$, $8x + 4h$
56. $-h - 3$, $-2x - h + 1$
57. $h^2 + 6h + 12$, $3x^2 + 3xh + h^2$
58. m, m
59. $ah + 4a + b$, $2ax + ah + b$
60. $\dfrac{2}{\Delta x - 1}, \dfrac{-2}{x(x + \Delta x)}$
61. $\dfrac{-3}{2(\Delta x - 2)}, \dfrac{3}{(x + \Delta x - 1)(x - 1)}$
62. $\dfrac{2 - \Delta x}{(\Delta x - 1)^2}, \dfrac{-(2x + \Delta x)}{x^2(x + \Delta x)^2}$
63. $\dfrac{-1}{2(\Delta x + 4)}, \dfrac{-2}{(x + 5)(x + \Delta x + 5)}$
64. $\dfrac{4}{7(4\Delta x - 7)}, \dfrac{-4}{(4x - 3)(4x + 4\Delta x - 3)}$

65. $\dfrac{6}{\Delta x+1}$, $\dfrac{6}{(x+2)(x+\Delta x+2)}$

66. $\dfrac{9}{10(\Delta x-10)}$, $\dfrac{-9}{(x-9)(x+\Delta x-9)}$

67. $\dfrac{\Delta x}{2\Delta x-1}$, $\dfrac{2x^2+2x\Delta x+2x+\Delta x}{(2x+1)(2x+2\Delta x+1)}$

68. $\dfrac{-1}{\sqrt{9-\Delta t}+3}$, $\dfrac{-1}{\sqrt{9-t-\Delta t}+\sqrt{9-t}}$

69. $\dfrac{2}{\sqrt{2\Delta t+1}+1}$, $\dfrac{2}{\sqrt{2t+2\Delta t+1}+\sqrt{2t+1}}$

70. $\dfrac{-4}{\sqrt{5-4\Delta t}+\sqrt{5}}$, $\dfrac{-4}{\sqrt{-4t-4\Delta t+5}+\sqrt{-4t+5}}$

71. $\dfrac{-1}{\sqrt{4-\Delta t}+2}$, $\dfrac{-1}{\sqrt{4-t-\Delta t}+\sqrt{4-t}}$

72. $\dfrac{a}{\sqrt{a\Delta t+b}+\sqrt{b}}$, $\dfrac{a}{\sqrt{at+a\Delta t+b}+\sqrt{at+b}}$

73. $(\Delta t)^{\frac{1}{2}}$, $\dfrac{3t^2+3t\Delta t+(\Delta t)^2}{(t+\Delta t)^{3/2}+t^{3/2}}$

74. $\dfrac{1}{(\Delta t)^{2/3}}$, $\dfrac{1}{(t+\Delta t)^{2/3}+(t+\Delta t)^{1/3}t^{1/3}+t^{2/3}}$

75. (b) $(f_+ + f_-)(x) = f_+(x) + f_-(x) = \dfrac{f(x)+|f(x)|}{2} + \dfrac{f(x)-|f(x)|}{2} = \dfrac{2f(x)}{2} = f(x)$.

(c)

$$f_+(x) = \begin{cases} 0 & \text{if } f(x) < 0 \\ f(x) & \text{if } f(x) \geq 0 \end{cases},$$

$$f_-(x) = \begin{cases} f(x) & \text{if } f(x) < 0 \\ 0 & \text{if } f(x) \geq 0 \end{cases}$$

3.3 Function Composition

In Section 3.2, we saw how the arithmetic of real numbers carried over into an arithmetic of functions. In this section, we discuss another way to combine functions which is unique to functions and isn't shared with real numbers - function **composition**.

Definition 3.3.1. Let f and g be functions where the real number x is in the domain of f and the real number $f(x)$ is in the domain of g. The **composite** of g with f, denoted $g \circ f$, and read 'g composed with f' is defined by the formula: $(g \circ f)(x) = g(f(x))$.

To compute $(g \circ f)(x)$, we use the formula given in Defintion 3.3.1: $(g \circ f)(x) = g(f(x))$. However, from a procedural viewpoint, Defintion 3.3.1 tells us the output from $g \circ f$ is found by taking the output from f, $f(x)$, and then making that the input to g. From this perspective, we see $g \circ f$ as a two step process taking an input x and first applying the procedure f then applying the procedure g. Abstractly, we have as in Figure 3.3.1.

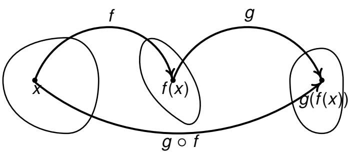


Figure 3.3.1

In the expression $g(f(x))$, the function f is often called the 'inside' function while g is often called the 'outside' function. When evaluating composite function values we present two methods in the example below: the 'inside out' and 'outside in' methods.

Example 3.3.1. Let $f(x) = x^2 - 4x$, $g(t) = 2 - \sqrt{t+3}$, and $h(s) = \dfrac{2s}{s+1}$.

In numbers 1 - 3, find the indicated function value.

1. $(g \circ f)(1)$
2. $(f \circ g)(1)$
3. $(g \circ g)(6)$

In numbers 4 - 10, find and simplify the indicated composite functions. State the domain of each.

4. $(g \circ f)(x)$
5. $(f \circ g)(t)$
6. $(g \circ h)(s)$
7. $(h \circ g)(t)$
8. $(h \circ h)(x)$

9. $(h \circ (g \circ f))(x)$
10. $((h \circ g) \circ f)(x)$

Solution.

1. Using Definition 3.3.1, $(g \circ f)(1) = g(f(1))$. Since $f(1) = (1)^2 - 4(1) = -3$ and $g(-3) = 2 - \sqrt{(-3)+3} = 2$, we have $(g \circ f)(1) = g(f(1)) = g(-3) = 2$.

2. By definition, $(f \circ g)(1) = f(g(1))$. We find $g(1) = 2 - \sqrt{1+3} = 0$, and $f(0) = (0)^2 - 4(0) = 0$, so $(f \circ g)(1) = f(g(1)) = f(0) = 0$. Comparing this with our answer to the last problem, we see that $(g \circ f)(1) \neq (f \circ g)(1)$ which tells us function composition is not commutative,[a]

3. Since $(g \circ g)(6) = g(g(6))$, we 'iterate' the process g: that is, we apply the process g to 6, then apply the process g again. We find $g(6) = 2 - \sqrt{6+3} = -1$, and $g(-1) = 2 - \sqrt{(-1)+3} = 2 - \sqrt{2}$, so $(g \circ g)(6) = g(g(6)) = g(-1) = 2 - \sqrt{2}$.

4. By definition, $(g \circ f)(x) = g(f(x))$. We now illustrate *two* ways to approach this problem.

   - *inside out*: We substitute $f(x) = x^2 - 4x$ in for t in the expression $g(t)$ and get

     $$\begin{aligned} (g \circ f)(x) = g(f(x)) = g\left(x^2 - 4x\right) &= 2 - \sqrt{\left(x^2 - 4x\right) + 3} \\ &= 2 - \sqrt{x^2 - 4x + 3} \end{aligned}$$

     Hence, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

   - *outside in*: We use the formula for g first to get

     $$\begin{aligned} (g \circ f)(x) = g(f(x)) &= 2 - \sqrt{f(x) + 3} \\ &= 2 - \sqrt{\left(x^2 - 4x\right) + 3} \\ &= 2 - \sqrt{x^2 - 4x + 3} \end{aligned}$$

[a] That is, in general, $g \circ f \neq f \circ g$. This shouldn't be too surprising, since, in general, the order of processes matters: adding eggs to a cake batter then baking the cake batter has a much different outcome than baking the cake batter then adding eggs.

We get the same answer as before, $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$.

To find the domain of $g \circ f$, we need to find the elements in the domain of f whose outputs $f(x)$ are in the domain of g. Since the domain of f is all real numbers, we focus on finding the range elements compatible with g. Owing to the presence of the square root in the formula $g(t) = 2 - \sqrt{t+3}$ we require $t \geq -3$. Hence, we need $f(x) \geq -3$ or $x^2 - 4x \geq -3$. To solve this inequality we rewrite as $x^2 - 4x + 3 \geq 0$ and use a sign diagram. Letting $r(x) = x^2 - 4x + 3$, we find the zeros of r to be $x = 1$ and $x = 3$ and obtain

$$\begin{array}{ccccc} (+) & 0 & (-) & 0 & (+) \\ \longleftarrow & \mid & \longrightarrow & \mid & \longrightarrow \\ & 1 & & 3 & \end{array}$$

Our solution to $x^2 - 4x + 3 \geq 0$, and hence the domain of $g \circ f$, is $(-\infty, 1] \cup [3, \infty)$.

5. To find $(f \circ g)(t)$, we find $f(g(t))$.

   - *inside out*: We substitute the expression $g(t) = 2 - \sqrt{t+3}$ in for x in the formula $f(x)$ and get

$$\begin{aligned} (f \circ g)(t) &= f(g(t)) \\ &= f\left(2 - \sqrt{t+3}\right) \\ &= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\ &= 4 - 4\sqrt{t+3} + \left(\sqrt{t+3}\right)^2 - 8 + 4\sqrt{t+3} \\ &= 4 + t + 3 - 8 \\ &= t - 1 \end{aligned}$$

   - *outside in*: We use the formula for $f(x)$ first to get

$$
\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= (g(t))^2 - 4\,(g(t)) \\
&= \left(2 - \sqrt{t+3}\right)^2 - 4\left(2 - \sqrt{t+3}\right) \\
&= t - 1 \quad \text{(same algebra as before)}
\end{aligned}
$$

Thus we get $(f \circ g)(t) = t - 1$. To find the domain of $f \circ g$, we look for the elements t in the domain of g whose outputs, $g(t)$ are in the domain of f. As mentioned previously, the domain of g is limited by the presence of the square root to $\{t \in \mathbb{R} \,|\, t \geq -3\}$ while the domain of f is all real numbers. Hence, the domain of $f \circ g$ is restricted only by the domain of g and is $\{t \in \mathbb{R} \,|\, t \geq -3\}$ or, using interval notation, $[-3, \infty)$. Note that as with Example 3.2.1 in Section 3.2, had we used the simplified formula for $(f \circ g)(t) = t - 1$ to determine domain, we would have arrived at the incorrect answer.

6. To find $(g \circ h)(s)$, we compute $g(h(s))$.

   - *inside out*: We substitute $h(s)$ in for t in the expression $g(t)$ to get

$$
\begin{aligned}
(g \circ h)(s) &= g(h(s)) \\
&= g\left(\frac{2s}{s+1}\right) \\
&= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\
&= 2 - \sqrt{\frac{2s}{s+1} + \frac{3(s+1)}{s+1}} \\
&\quad \text{(get common denominators)} \\
&= 2 - \sqrt{\frac{5s+3}{s+1}}
\end{aligned}
$$

- *outside in*: We use the formula for $g(t)$ first to get

$$\begin{aligned}(g \circ h)(s) &= g(h(s)) \\ &= 2 - \sqrt{h(s) + 3} \\ &= 2 - \sqrt{\left(\frac{2s}{s+1}\right) + 3} \\ &= 2 - \sqrt{\frac{5s+3}{s+1}}\end{aligned}$$

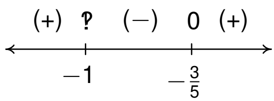
(get common denominators as before)

To find the domain of $g \circ h$, we need the elements in the domain of h so that $h(s)$ is in the domain of g. Owing to the $s + 1$ in the denominator of the expression $h(s)$, we require $s \neq -1$. Once again, because of the square root in $g(t) = 2 - \sqrt{t+3}$, we need $t \geq -3$ or, in this case $h(s) \geq -3$. To use a sign diagram to solve, we rearrange this inequality:

$$\begin{aligned}\frac{2s}{s+1} &\geq -3 \\ \frac{2s}{s+1} + 3 &\geq 0 \\ \frac{5s+3}{s+1} &\geq 0\end{aligned}$$

(get common denominators as before)

Defining $r(s) = \frac{5s+3}{s+1}$, we see r is undefined at $s = -1$ (a carry over from the domain restriction of h) and $r(s) = 0$ at $s = -\frac{3}{5}$. Our sign diagram is



hence our domain is $(-\infty, -1) \cup \left[-\frac{3}{5}, \infty\right)$.

7. We find $(h \circ g)(t)$ by finding $h(g(t))$.

   - *inside out*: We substitute the expression $g(t)$ for s in the formula $h(s)$

$$\begin{aligned}(h \circ g)(t) &= h(g(t)) = h\left(2 - \sqrt{t+3}\right) \\ &= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\ &= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}\end{aligned}$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$\begin{aligned}(h \circ g)(t) &= h(g(t)) = \frac{2\,(g(t))}{(g(t)) + 1} \\ &= \frac{2\left(2 - \sqrt{t+3}\right)}{\left(2 - \sqrt{t+3}\right) + 1} \\ &= \frac{4 - 2\sqrt{t+3}}{3 - \sqrt{t+3}}\end{aligned}$$

   To find the domain of $h \circ g$, we need the elements of the domain of g so that $g(t)$ is in the domain of h. As we've seen already, for t to be in the domain of g, $t \geq -3$. For s to be in the domain of h, $s \neq -1$, so we require $g(t) \neq -1$. Hence, we solve $g(t) = 2 - \sqrt{t+3} = -1$ with the intent of excluding the solutions. Isolating the radical expression gives $\sqrt{t+3} = 3$ or $t = 6$. Sure enough, we check $g(6) = -1$ so we exclude $t = 6$ from the domain of $h \circ g$. Our final answer is $[-3, 6) \cup (6, \infty)$.

8. To find $(h \circ h)(s)$ we find $h(h(s))$:

   - *inside out*: We substitute the expression $h(s)$ for s in the expression $h(s)$ into h to get

$$
\begin{aligned}
(h \circ h)(s) &= h(h(s)) = h\left(\frac{2s}{s+1}\right) \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{\dfrac{4s}{s+1}}{\dfrac{2s}{s+1}+1} \cdot \frac{(s+1)}{(s+1)} \\
&= \frac{\dfrac{4s}{s+1}\cdot(s+1)}{\left(\dfrac{2s}{s+1}\right)\cdot(s+1)+1\cdot(s+1)} \\
&= \frac{\dfrac{4s}{\cancel{(s+1)}}\cdot\cancel{(s+1)}}{\dfrac{2s}{\cancel{(s+1)}}\cdot\cancel{(s+1)}+s+1} \\
&= \frac{4s}{3s+1}
\end{aligned}
$$

- *outside in*: This approach yields

$$
\begin{aligned}
(h \circ h)(s) = h(h(s)) &= \frac{2(h(s))}{h(s)+1} \\
&= \frac{2\left(\dfrac{2s}{s+1}\right)}{\left(\dfrac{2s}{s+1}\right)+1} \\
&= \frac{4s}{3s+1}
\end{aligned}
$$

(same algebra as before)

To find the domain of $h \circ h$, we need to find the elements in the domain of h so that the outputs, $h(s)$ are also in the domain of h. The only domain restriction for h comes from the denominator: $s \neq -1$, so in addition to this, we also need $h(s) \neq -1$. To this end, we solve $h(s) = -1$ and exclude the answers. Solving $\frac{2s}{s+1} = -1$ gives $s = -\frac{1}{3}$. The domain of $h \circ h$ is $(-\infty, -1) \cup \left(-1, -\frac{1}{3}\right) \cup \left(-\frac{1}{3}, \infty\right)$.

9. The expression $(h \circ (g \circ f))(x)$ indicates that we first find the composite, $g \circ f$ and compose the function h with the result. We know from number 4 that $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3}$ with domain $(-\infty, 1] \cup [3, \infty)$. We now proceed as usual.

   - *inside out*: We substitute the expression $(g \circ f)(x)$ for s in the expression $h(s)$ first to get

$$
\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h\left(2 - \sqrt{x^2 - 4x + 3}\right) \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}
$$

   - *outside in*: We use the formula for $h(s)$ first to get

$$
\begin{aligned}
(h \circ (g \circ f))(x) &= h((g \circ f)(x)) = \frac{2\left((g \circ f)(x)\right)}{((g \circ f)(x)) + 1} \\
&= \frac{2\left(2 - \sqrt{x^2 - 4x + 3}\right)}{\left(2 - \sqrt{x^2 - 4x + 3}\right) + 1} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}
$$

   To find the domain of $h \circ (g \circ f)$, we need the domain elements of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, so that $(g \circ f)(x)$ is in the domain of h. As we've seen several times already, the only domain restriction for h is $s \neq -1$, so we set $(g \circ f)(x) = 2 - \sqrt{x^2 - 4x + 3} = -1$ and exclude the solutions. We get $\sqrt{x^2 - 4x + 3} = 3$, and, after squaring both sides, we have $x^2 - 4x + 3 = 9$. We solve $x^2 - 4x - 6 = 0$ using the quadratic formula and obtain $x = 2 \pm \sqrt{10}$. The reader is encouraged to check that both of these numbers satisfy the original equation, $2 - \sqrt{x^2 - 4x + 3} = -1$ and also belong to the domain of $g \circ f$, $(-\infty, 1] \cup [3, \infty)$, and so must be excluded from our final answer.[b] Our final domain for $h \circ (f \circ g)$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$.

We can approximate $\sqrt{10} \approx 3$ so [b] $2 - \sqrt{10} \approx -1$ and $2 + \sqrt{10} \approx 5$.

10. The expression $((h \circ g) \circ f)(x)$ indicates that we first find the composite $h \circ g$ and then compose that with f. From number 7, we have

$$(h \circ g)(t) = \frac{4 - 2\sqrt{t + 3}}{3 - \sqrt{t + 3}}$$

with domain $[-3, 6) \cup (6, \infty)$.

- *inside out*: We substitute the expression $f(x)$ for t in the expression $(h \circ g)(t)$ to get

$$\begin{aligned}
((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = (h \circ g)\left(x^2 - 4x\right) \\
&= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}} \\
&= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}
\end{aligned}$$

- *outside in*: We use the formula for $(h \circ g)(t)$ first to get

$$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = \frac{4 - 2\sqrt{(f(x)) + 3}}{3 - \sqrt{f(x)) + 3}}$$

$$= \frac{4 - 2\sqrt{\left(x^2 - 4x\right) + 3}}{3 - \sqrt{\left(x^2 - 4x\right) + 3}}$$

$$= \frac{4 - 2\sqrt{x^2 - 4x + 3}}{3 - \sqrt{x^2 - 4x + 3}}$$

Since the domain of f is all real numbers, the challenge here to find the domain of $(h \circ g) \circ f$ is to determine the values $f(x)$ which are in the domain of $h \circ g$, $[-3, 6) \cup (6, \infty)$. At first glance, it appears as if we have two (or three!) inequalities to solve: $-3 \leq f(x) < 6$ and $f(x) > 6$. Alternatively, we could solve $f(x) = x^2 - 4x \geq -3$ and exclude the solutions to $f(x) = x^2 - 4x = 6$ which is not only easier from a procedural point of view, but also easier since we've already done both calculations. In number 4, we solved $x^2 - 4x \geq -3$ and obtained the solution $(-\infty, 1] \cup [3, \infty)$ and in number 9, we solved $x^2 - 4x - 6 = 0$ and obtained $x = 2 \pm \sqrt{10}$. Hence, the domain of $(h \circ g) \circ f$ is $(-\infty, 2 - \sqrt{10}) \cup (2 - \sqrt{10}, 1] \cup \left[3, 2 + \sqrt{10}\right) \cup \left(2 + \sqrt{10}, \infty\right)$. □

As previously mentioned, it should be clear from Example 3.3.1 that, in general, $g \circ f \neq f \circ g$, in other words, function composition is not *commutative*. However, numbers 9 and 10 demonstrate the **associative** property of function composition. That is, when composing three (or more) functions, as long as we keep the order the same, it doesn't matter which two functions we compose first. We summarize the important properties of function composition in **??**.

Theorem 3.3.1. Properties of Function Composition: Suppose f, g, and h are functions.

- **Associative Law of Composition:** $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x) = x$ satisfies: $I \circ f = f \circ I = f$ for all functions, f.

By repeated applications of Definition 3.3.1, we find $(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$. Similarly, $((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$. This establishes that the formulas for the two functions are the same. We leave it to the reader to think about why the domains of these two functions are identical, too. These two facts establish the equality $h \circ (g \circ f) = (h \circ g) \circ f$. A consequence of the associativity of function composition is that there is no need for parentheses when we write $h \circ g \circ f$. The second property can also be verified using Definition 3.3.1. Recall that the function $I(x) = x$ is called the *identity function* and was introduced in Exercise **??** in Section **??**. If we compose the function I with a function f, then we have $(I \circ f)(x) = I(f(x)) = f(x)$, and a similar computation shows $(f \circ I)(x) = f(I(x)) = f(x)$. This establishes that we have an identity for function composition much in the same way the function $I(x) = 1$ is an identity for function multiplication.

As we know, not all functions are described by formulas, and, moreover, not all functions are described by just *one* formula. The next example applies the concept of function composition to functions represented in various and sundry ways.

Example 3.3.2. Consider the following functions:

- $f(x) = 6x - x^2$
- $g(t) \begin{cases} 2t - 1 & \text{if } -1 \leq t < 3, \\ t^2 & \text{if } t \geq 3. \end{cases}$
- $h = \{(-3,1), (-2,6), (0,-2), (1,5), (3,-1)\}$
- s whose graph is given in Figure 3.3.2:

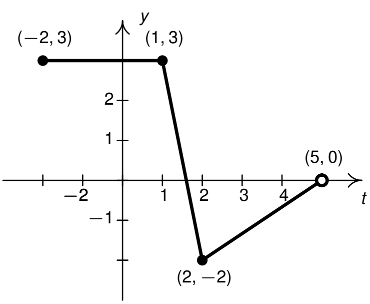


Figure 3.3.2: $y = s(t)$

1. Find and simplify the following function values:

   (a) $(g \circ f)(2)$ (b) $(h \circ g)(-1)$
   (c) $(h \circ s)(-2)$ (d) $(f \circ s)(0)$

2. Find and simplify a formula for $(g \circ f)(x)$.

3. Write $s \circ h$ as a set of ordered pairs.

Solution.

1. (a) To find $(g \circ f)(2) = g(f(2))$ we first find $f(2) = 6(2) - (2)^2 = 8$. Since $8 \geq 3$, we use the rule $g(t) = t^2$ so $g(8) = (8)^2 = 64$. Hence, $(g \circ f)(3) = g(f(3)) = g(8) = 64$.

   (b) Since $(h \circ g)(-1) = h(g(-1))$ we first need $g(-1)$. Since $-1 \leq -1 < 3$, we use the rule $g(t) = 2t - 1$ and find $g(-1) = 2(-1) - 1 = -3$. Next, we need $h(-3)$. Since $(-3, 1) \in h$, we have that $h(-3) = 1$. Putting it all together, we find $(h \circ g)(-1) = h(g(-1)) = h(-3) = 1$.

   (c) To find $(h \circ s)(-2) = h(s(-2))$, we first need $s(-2)$. We see the point $(-2, 3)$ is on the graph of s, so $s(-2) = 3$. Next, we see $(3, -1) \in h$, so $h(3) = -1$. Hence, $(h \circ s)(-2) = h(s(-2)) = h(3) = -1$.

   (d) To find $(f \circ s)(0) = f(s(0))$ we infer from the graph of s that it contains the point $(0, 3)$, so $s(0) = 3$. Sine $f(3) = 6(3) - (3)^2 = 9$, we have $(f \circ s)(0) = f(s(0)) = f(3) = 9$.

2. To find a formula for $(g \circ f)(x) = g(f(x))$, we substitute $f(x) = 6x - x^2$ in for t in the formula for $g(t)$:

$$(g \circ f)(x) = g(f(x)) = g(6x - x^2)$$
$$= \begin{cases} 2(6x - x^2) - 1 & \text{if } -1 \leq 6x - x^2 < 3, \\ (6x - x^2)^2 & \text{if } 6x - x^2 \geq 3. \end{cases}$$

   Simplifying each expression, we get $2(6x - x^2) - 1 = -2x^2 + 12x - 1$ for the first piece and $(6x - x^2)^2 = x^4 - 12x^3 + 36x^2$ for the second piece. The real challenge comes in solving the inequalities $-1 \leq 6x - x^2 < 3$ and $6x - x^2 \geq 3$. While we could solve each individually using a sign diagram, a graphical approach works best here. We graph the parabola $y = 6x - x^2$, finding the vertex is $(3, 9)$ with intercepts $(0, 0)$ and $(6, 0)$ along with the horizontal lines $y = -1$ and $y = 3$ below. We determine the intersection points by solving $6x - x^2 = -1$ and $6x - x^2 = 3$. Using the quadratic formula, we find the solutions to each equation are $x = 3 \pm \sqrt{10}$ and $x = 3 \pm \sqrt{6}$, respectively.

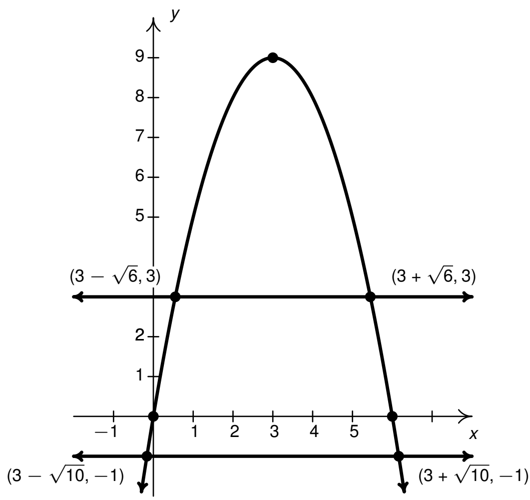


Figure 3.3.3: $y = 6x - x^2$, $y = -1$, $y = 3$

From the graph in Figure 3.3.3, we see the parabola $y = 6x - x^2$ is between the lines $y = -1$ and $y = 3$ from $x = 3 - \sqrt{10}$ to $x = 3 - \sqrt{6}$ and again from $x = 3 + \sqrt{6}$ to $x = 3 + \sqrt{10}$. Hence the solution to $-1 \leq 6x - x^2 < 3$ is $[3 - \sqrt{10}, 3 - \sqrt{6}) \cup (3 + \sqrt{6}, 3 + \sqrt{10}]$. We also note $y = 6x - x^2$ is above the line $y = 3$ for all x between $x = 3 - \sqrt{6}$ and $3 + \sqrt{6}$. Hence, the solution to $6x - x^2 \geq 3$ is $[3 - \sqrt{6}, 3 + \sqrt{6}]$. Hence,

$$(g \circ f)(x) = \begin{cases} -2x^2 + 12x - 1 \\ \qquad \text{if } x \in [3 - \sqrt{10}, 3 - \sqrt{6}) \\ \qquad\qquad \cup (3 + \sqrt{6}, 3 + \sqrt{10}], \\ x^4 - 12x^3 + +36x^2 \\ \qquad \text{if } x \in [3 - \sqrt{6}, 3 + \sqrt{6}]. \end{cases}$$

3. Last but not least, we are tasked with representing $s \circ h$ as a set of ordered pairs. Since h is described by the discrete set of points

$$h = \{(-3, 1), (-2, 6), (0, -2), (1, 5), (3, -1)\}$$

, we find $s \circ h$ point by point. From the graph of s in Figure 3.3.2 we construct Table 3.3.7 to help us organize our work.

Since neither 6 nor 5 are in the domain of s, -2 and 1 are not in the domain of $s \circ h$. Hence, we get $s \circ h = \{(-3, 3), (0, 3), (3, 3)\}$. □

x	$h(x)$	$s(h(x))$
-3	1	3
-2	6	undefined
0	-2	3
1	5	undefined
3	-1	3

Table 3.3.7

A useful skill in Calculus is to be able to take a complicated function and break it down into a composition of easier functions which our last example illustrates. As with Example 3.2.2, we want to avoid trivial decompositions, which, when it comes to function composition, are those involving the identity function $I(x) = x$ as described in Theorem 3.3.1.

Example 3.3.3.

1. Write each of the following functions as a composition of two or more (non-identity) functions. Check your answer by performing the function composition.

   (a) $F(x) = |3x - 1|$

   (b) $G(t) = \dfrac{2}{t^2 + 1}$

(c) $H(s) = \dfrac{\sqrt{s}+1}{\sqrt{s}-1}$

2. For $F(x) = \sqrt{\dfrac{2x-1}{x^2+4}}$, find functions f, g, and h to decompose F nontrivially as $F = f \circ \left(\dfrac{g}{h}\right)$.

Solution. There are many approaches to this kind of problem, and we showcase a different methodology in each of the solutions below.

1. (a) Our goal is to express the function F as $F = g \circ f$ for functions g and f. From Definition 3.3.1, we know $F(x) = g(f(x))$, and we can think of $f(x)$ as being the 'inside' function and g as being the 'outside' function. Looking at $F(x) = |3x-1|$ from an 'inside versus outside' perspective, we can think of $3x-1$ being inside the absolute value symbols. Taking this cue, we define $f(x) = 3x-1$. At this point, we have $F(x) = |f(x)|$. What is the outside function? The function which takes the absolute value of its input, $g(x) = |x|$. Sure enough, this checks: $(g \circ f)(x) = g(f(x)) = |f(x)| = |3x-1| = F(x)$.

   (b) We attack deconstructing G from an operational approach. Given an input t, the first step is to square t, then add 1, then divide the result into 2. We will assign each of these steps a function so as to write G as a composite of *three* functions: f, g and h. Our first function, f, is the function that squares its input, $f(t) = t^2$. The next function is the function that adds 1 to its input, $g(t) = t+1$. Our last function takes its input and divides it into 2, $h(t) = \frac{2}{t}$. The claim is that $G = h \circ g \circ f$ which checks:

$$(h \circ g \circ f)(t) = h(g(f(t))) = h(g\left(t^2\right))$$

$$= h\left(t^2+1\right) = \frac{2}{t^2+1} = G(x).$$

(c) If we look $H(s) = \frac{\sqrt{s}+1}{\sqrt{s}-1}$ with an eye towards building a complicated function from simpler functions, we see the expression $\sqrt{s}$ is a simple piece of the larger function. If we define $f(s) = \sqrt{s}$, we have $H(s) = \frac{f(s)+1}{f(s)-1}$. If we want to decompose $H = g \circ f$, then we can glean the formula for $g(s)$ by looking at what is being done to $f(s)$. We take $g(s) = \frac{s+1}{s-1}$, and check below:

$$(g \circ f)(s) = g(f(s)) = \frac{f(s)+1}{f(s)-1} = \frac{\sqrt{s}+1}{\sqrt{s}-1} = H(s).$$

□

2. To write $F = f \circ \left(\frac{g}{h}\right)$ means

$$F(x) = \sqrt{\frac{2x-1}{x^2+4}} = \left(f \circ \left(\frac{g}{h}\right)\right)(x)$$
$$= f\left(\left(\frac{g}{h}\right)(x)\right) = f\left(\frac{g(x)}{h(x)}\right).$$

Working from the inside out, we have a rational expression with numerator $g(x)$ and denominator $h(x)$. Looking at the formula for $F(x)$, one choice is $g(x) = 2x - 1$ and $h(x) = x^2 + 4$. Making these identifications, we have

$$F(x) = \sqrt{\frac{2x-1}{x^2+4}} = \sqrt{\frac{g(x)}{h(x)}}.$$

Since F takes the square root of $\frac{g(x)}{h(x)}$, the our last function f is the function that takes the square root of its input, i.e., $f(x) = \sqrt{x}$. We leave it to the reader to check that, indeed, $F = f \circ \left(\frac{g}{h}\right)$. □

We close this section of a real-world application of function composition.

Example 3.3.4. The surface area of a sphere is a function of its radius r and is given by the formula $S(r) = 4\pi r^2$. Suppose the sphere is being inflated so that the radius of the sphere is increasing according to the formula $r(t) = 3t^2$, where t is measured in seconds, $t \geq 0$, and r is measured in inches. Find and interpret $(S \circ r)(t)$.

Solution. If we look at the functions $S(r)$ and $r(t)$ individually, we see the former gives the surface area of a sphere of a given radius while the latter gives the radius at a given time. So, given a specific time, t, we could find the radius at that time, $r(t)$ and feed that into $S(r)$ to find the surface area at that time. From this we see that the surface area S is ultimately a function of time t and we find $(S \circ r)(t) = S(r(t)) = 4\pi(r(t))^2 = 4\pi \left(3t^2\right)^2 = 36\pi t^4$. This formula allows us to compute the surface area directly given the time without going through the 'intermediary variable' r. $\square$

3.3.1 Exercises

In Exercises 1 - 12, use the given pair of functions to find the following values if they exist.

- $(g \circ f)(0)$
- $(f \circ g)(-1)$
- $(f \circ f)(2)$
- $(g \circ f)(-3)$
- $(f \circ g)\left(\frac{1}{2}\right)$
- $(f \circ f)(-2)$

1. $f(x) = x^2$, $g(t) = 2t + 1$
2. $f(x) = 4 - x$, $g(t) = 1 - t^2$
3. $f(x) = 4 - 3x$, $g(t) = |t|$
4. $f(x) = |x - 1|$, $g(t) = t^2 - 5$
5. $f(x) = 4x + 5$, $g(t) = \sqrt{t}$
6. $f(x) = \sqrt{3 - x}$, $g(t) = t^2 + 1$
7. $f(x) = 6 - x - x^2$, $g(t) = t\sqrt{t + 10}$
8. $f(x) = \sqrt[3]{x + 1}$, $g(t) = 4t^2 - t$
9. $f(x) = \dfrac{3}{1 - x}$, $g(t) = \dfrac{4t}{t^2 + 1}$
10. $f(x) = \dfrac{x}{x + 5}$, $g(t) = \dfrac{2}{7 - t^2}$
11. $f(x) = \dfrac{2x}{5 - x^2}$, $g(t) = \sqrt{4t + 1}$
12. $f(x) = \sqrt{2x + 5}$, $g(t) = \dfrac{10t}{t^2 + 1}$

In Exercises 13 - 24, use the given pair of functions to find and simplify expressions for the following functions and state the domain of each using interval notation.

- $(g \circ f)(x)$
- $(f \circ g)(t)$
- $(f \circ f)(x)$

13. $f(x) = 2x + 3$, $g(t) = t^2 - 9$
14. $f(x) = x^2 - x + 1$, $g(t) = 3t - 5$
15. $f(x) = x^2 - 4$, $g(t) = |t|$
16. $f(x) = 3x - 5$, $g(t) = \sqrt{t}$
17. $f(x) = |x + 1|$, $g(t) = \sqrt{t}$
18. $f(x) = 3 - x^2$, $g(t) = \sqrt{t + 1}$
19. $f(x) = |x|$, $g(t) = \sqrt{4 - t}$
20. $f(x) = x^2 - x - 1$, $g(t) = \sqrt{t - 5}$
21. $f(x) = 3x - 1$, $g(t) = \dfrac{1}{t + 3}$
22. $f(x) = \dfrac{3x}{x - 1}$, $g(t) = \dfrac{t}{t - 3}$

23. $f(x) = \dfrac{x}{2x+1}$, $g(t) = \dfrac{2t+1}{t}$

24. $f(x) = \dfrac{2x}{x^2-4}$, $g(t) = \sqrt{1-t}$

In Exercises 25 - 30, use $f(x) = -2x$, $g(t) = \sqrt{t}$ and $h(s) = |s|$ to find and simplify expressions for the following functions and state the domain of each using interval notation.

25. $(h \circ g \circ f)(x)$

26. $(h \circ f \circ g)(t)$

27. $(g \circ f \circ h)(s)$

28. $(g \circ h \circ f)(x)$

29. $(f \circ h \circ g)(t)$

30. $(f \circ g \circ h)(s)$

In Exercises 31 - 43, let f be the function defined by

$$f = \{(-3,4), (-2,2), (-1,0), (0,1), (1,3), (2,4), (3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2), (-2,0), (-1,-4), (0,0), (1,-3), (2,1), (3,2)\}.$$

Find the following, if it exists.

31. $(f \circ g)(3)$

32. $f(g(-1))$

33. $(f \circ f)(0)$

34. $(f \circ g)(-3)$

35. $(g \circ f)(3)$

36. $g(f(-3))$

37. $(g \circ g)(-2)$

38. $(g \circ f)(-2)$

39. $g(f(g(0)))$

40. $f(f(f(-1)))$

41. $f(f(f(f(f(1)))))$

42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0)$

43. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 44 - 50, use the graphs of $y = f(x)$ and $y = g(x)$ in **??** and Figure 3.3.5 to find the following if it exists.

44. $(g \circ f)(1)$

45. $(f \circ g)(3)$

46. $(g \circ f)(2)$

47. $(f \circ g)(0)$ 48. $(f \circ f)(4)$ 49. $(g \circ g)(1)$

50. Find the domain and range of $f \circ g$ and $g \circ f$.

In Exercises 51 - 60, write the given function as a composition of two or more non-identity functions. (There are several correct answers, so check your answer using function composition.)

51. $p(x) = (2x+3)^3$

52. $P(x) = \left(x^2 - x + 1\right)^5$

53. $h(t) = \sqrt{2t-1}$

54. $H(t) = |7 - 3t|$

55. $r(s) = \dfrac{2}{5s+1}$

56. $R(s) = \dfrac{7}{s^2-1}$

57. $q(z) = \dfrac{|z|+1}{|z|-1}$

58. $Q(z) = \dfrac{2z^3+1}{z^3-1}$

59. $v(x) = \dfrac{2x+1}{3-4x}$

60. $w(x) = \dfrac{x^2}{x^4+1}$

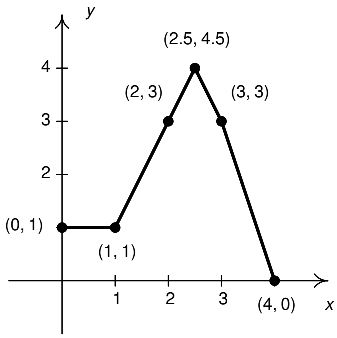
61. Write the function $F(x) = \sqrt{\dfrac{x^3+6}{x^3-9}}$ as a composition of three or more non-identity functions.

62. Let $g(x) = -x$, $h(x) = x+2$, $j(x) = 3x$ and $k(x) = x-4$. In what order must these functions be composed with $f(x) = \sqrt{x}$ to create $F(x) = 3\sqrt{-x+2}-4$?

63. What linear functions could be used to transform $f(x) = x^3$ into $F(x) = -\frac{1}{2}(2x-7)^3+1$? What is the proper order of composition?

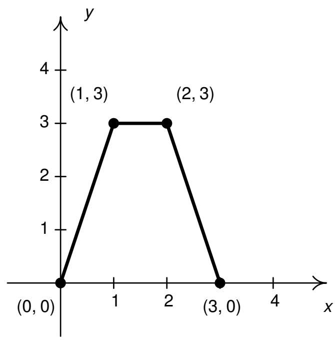
64. Let $f(x) = 3x+1$ and let $g(x) = \begin{cases} 2x-1 & \text{if } x \leq 3 \\ 4-x & \text{if } x > 3 \end{cases}$.
Find expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$.

65. The volume V of a cube is a function of its side length x. Let's assume that $x = t+1$ is also a



$y = f(x)$

Figure 3.3.4: fig:exfcompyeqfx



$y = g(x)$

Figure 3.3.5

function of time t, where x is measured in inches and t is measured in minutes. Find a formula for V as a function of t.

66. Suppose a local vendor charges \$2 per hot dog and that the number of hot dogs sold per hour x is given by $x(t) = -4t^2 + 20t + 92$, where t is the number of hours since 10 AM, $0 \leq t \leq 4$.

    (a) Find an expression for the revenue per hour R as a function of x.

    (b) Find and simplify $(R \circ x)(t)$. What does this represent?

    (c) What is the revenue per hour at noon?

67. Discuss with your classmates how 'real-world' processes such as filling out federal income tax forms or computing your final course grade could be viewed as a use of function composition. Find a process for which composition with itself (iteration) makes sense.

3.3.2 Answers

1. For $f(x) = x^2$ and $g(t) = 2t + 1$,
   - $(g \circ f)(0) = 1$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 16$
   - $(g \circ f)(-3) = 19$
   - $(f \circ g)\left(\frac{1}{2}\right) = 4$
   - $(f \circ f)(-2) = 16$
2. For $f(x) = 4 - x$ and $g(t) = 1 - t^2$,
   - $(g \circ f)(0) = -15$
   - $(f \circ g)(-1) = 4$
   - $(f \circ f)(2) = 2$
   - $(g \circ f)(-3) = -48$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{13}{4}$
   - $(f \circ f)(-2) = -2$
3. For $f(x) = 4 - 3x$ and $g(t) = |t|$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = 10$
   - $(g \circ f)(-3) = 13$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{5}{2}$
   - $(f \circ f)(-2) = -26$
4. For $f(x) = |x - 1|$ and $g(t) = t^2 - 5$,
   - $(g \circ f)(0) = -4$
   - $(f \circ g)(-1) = 5$
   - $(f \circ f)(2) = 0$
   - $(g \circ f)(-3) = 11$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{23}{4}$
   - $(f \circ f)(-2) = 2$
5. For $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$,
   - $(g \circ f)(0) = \sqrt{5}$
   - $(f \circ g)(-1)$ is not real
   - $(f \circ f)(2) = 57$
   - $(g \circ f)(-3)$ is not real
   - $(f \circ g)\left(\frac{1}{2}\right) = 5 + 2\sqrt{2}$
   - $(f \circ f)(-2) = -7$
6. For $f(x) = \sqrt{3 - x}$ and $g(t) = t^2 + 1$,
   - $(g \circ f)(0) = 4$
   - $(f \circ g)(-1) = 1$
   - $(f \circ f)(2) = \sqrt{2}$
   - $(g \circ f)(-3) = 7$
   - $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt{7}}{2}$
   - $(f \circ f)(-2) = \sqrt{3 - \sqrt{5}}$
7. For $f(x) = 6 - x - x^2$ and $g(t) = t\sqrt{t + 10}$,

- $(g \circ f)(0) = 24$
- $(f \circ g)(-1) = 0$
- $(f \circ f)(2) = 6$
- $(g \circ f)(-3) = 0$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{27-2\sqrt{42}}{8}$
- $(f \circ f)(-2) = -14$

8. For $f(x) = \sqrt[3]{x+1}$ and $g(t) = 4t^2 - t$,

- $(g \circ f)(0) = 3$
- $(f \circ g)(-1) = \sqrt[3]{6}$
- $(f \circ f)(2) = \sqrt[3]{\sqrt[3]{3}+1}$
- $(g \circ f)(-3) = 4\sqrt[3]{4} + \sqrt[3]{2}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{\sqrt[3]{12}}{2}$
- $(f \circ f)(-2) = 0$

9. For $f(x) = \frac{3}{1-x}$ and $g(t) = \frac{4t}{t^2+1}$,

- $(g \circ f)(0) = \frac{6}{5}$
- $(f \circ g)(-1) = 1$
- $(f \circ f)(2) = \frac{3}{4}$
- $(g \circ f)(-3) = \frac{48}{25}$
- $(f \circ g)\left(\frac{1}{2}\right) = -5$
- $(f \circ f)(-2)$ is undefined

10. For $f(x) = \frac{x}{x+5}$ and $g(t) = \frac{2}{7-t^2}$,

- $(g \circ f)(0) = \frac{2}{7}$
- $(f \circ g)(-1) = \frac{1}{16}$
- $(f \circ f)(2) = \frac{2}{37}$
- $(g \circ f)(-3) = \frac{8}{19}$
- $(f \circ g)\left(\frac{1}{2}\right) = \frac{8}{143}$
- $(f \circ f)(-2) = -\frac{2}{13}$

11. For $f(x) = \frac{2x}{5-x^2}$ and $g(t) = \sqrt{4t+1}$,

- $(g \circ f)(0) = 1$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = -\frac{8}{11}$
- $(g \circ f)(-3) = \sqrt{7}$
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{3}$
- $(f \circ f)(-2) = \frac{8}{11}$

12. For $f(x) = \sqrt{2x+5}$ and $g(t) = \frac{10t}{t^2+1}$,

- $(g \circ f)(0) = \frac{5\sqrt{5}}{3}$
- $(f \circ g)(-1)$ is not real
- $(f \circ f)(2) = \sqrt{11}$
- $(g \circ f)(-3)$ is not real
- $(f \circ g)\left(\frac{1}{2}\right) = \sqrt{13}$
- $(f \circ f)(-2) = \sqrt{7}$

13. For $f(x) = 2x + 3$ and $g(t) = t^2 - 9$

- $(g \circ f)(x) = 4x^2 + 12x$, domain: $(-\infty, \infty)$

- $(f \circ g)(t) = 2t^2 - 15$, domain: $(-\infty, \infty)$
- $(f \circ f)(x) = 4x + 9$, domain: $(-\infty, \infty)$

14. For $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$
    - $(g \circ f)(x) = 3x^2 - 3x - 2$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = 9t^2 - 33t + 31$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 + 2x^2 - x + 1$, domain: $(-\infty, \infty)$
15. For $f(x) = x^2 - 4$ and $g(t) = |t|$
    - $(g \circ f)(x) = |x^2 - 4|$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |t|^2 - 4 = t^2 - 4$, domain: $(-\infty, \infty)$
    - $(f \circ f)(x) = x^4 - 8x^2 + 12$, domain: $(-\infty, \infty)$
16. For $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$
    - $(g \circ f)(x) = \sqrt{3x - 5}$, domain: $\left[\frac{5}{3}, \infty\right)$
    - $(f \circ g)(t) = 3\sqrt{t} - 5$, domain: $[0, \infty)$
    - $(f \circ f)(x) = 9x - 20$, domain: $(-\infty, \infty)$
17. For $f(x) = |x + 1|$ and $g(t) = \sqrt{t}$
    - $(g \circ f)(x) = \sqrt{|x + 1|}$, domain: $(-\infty, \infty)$
    - $(f \circ g)(t) = |\sqrt{t} + 1| = \sqrt{t} + 1$, domain: $[0, \infty)$
    - $(f \circ f)(x) = ||x + 1| + 1| = |x + 1| + 1$, domain: $(-\infty, \infty)$
18. For $f(x) = 3 - x^2$ and $g(t) = \sqrt{t + 1}$
    - $(g \circ f)(x) = \sqrt{4 - x^2}$, domain: $[-2, 2]$
    - $(f \circ g)(t) = 2 - t$, domain: $[-1, \infty)$
    - $(f \circ f)(x) = -x^4 + 6x^2 - 6$, domain: $(-\infty, \infty)$
19. For $f(x) = |x|$ and $g(t) = \sqrt{4 - t}$
    - $(g \circ f)(x) = \sqrt{4 - |x|}$, domain: $[-4, 4]$
    - $(f \circ g)(t) = |\sqrt{4 - t}| = \sqrt{4 - t}$, domain: $(-\infty, 4]$
    - $(f \circ f)(x) = ||x|| = |x|$, domain: $(-\infty, \infty)$

20. For $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t-5}$
    - $(g \circ f)(x) = \sqrt{x^2 - x - 6}$, domain: $(-\infty, -2] \cup [3, \infty)$
    - $(f \circ g)(t) = t - 6 - \sqrt{t-5}$, domain: $[5, \infty)$
    - $(f \circ f)(x) = x^4 - 2x^3 - 2x^2 + 3x + 1$, domain: $(-\infty, \infty)$
21. For $f(x) = 3x - 1$ and $g(t) = \frac{1}{t+3}$
    - $(g \circ f)(x) = \frac{1}{3x+2}$, domain: $\left(-\infty, -\frac{2}{3}\right) \cup \left(-\frac{2}{3}, \infty\right)$
    - $(f \circ g)(t) = -\frac{t}{t+3}$, domain: $(-\infty, -3) \cup (-3, \infty)$
    - $(f \circ f)(x) = 9x - 4$, domain: $(-\infty, \infty)$
22. For $f(x) = \frac{3x}{x-1}$ and $g(t) = \frac{t}{t-3}$
    - $(g \circ f)(x) = x$, domain: $(-\infty, 1) \cup (1, \infty)$
    - $(f \circ g)(t) = t$, domain: $(-\infty, 3) \cup (3, \infty)$
    - $(f \circ f)(x) = \frac{9x}{2x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 1\right) \cup (1, \infty)$
23. For $f(x) = \frac{x}{2x+1}$ and $g(t) = \frac{2t+1}{t}$
    - $(g \circ f)(x) = \frac{4x+1}{x}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, 0\right), \cup (0, \infty)$
    - $(f \circ g)(t) = \frac{2t+1}{5t+2}$, domain: $\left(-\infty, -\frac{2}{5}\right) \cup \left(-\frac{2}{5}, 0\right) \cup (0, \infty)$
    - $(f \circ f)(x) = \frac{x}{4x+1}$, domain: $\left(-\infty, -\frac{1}{2}\right) \cup \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(-\frac{1}{4}, \infty\right)$
24. For $f(x) = \frac{2x}{x^2-4}$ and $g(t) = \sqrt{1-t}$
    - $(g \circ f)(x) = \sqrt{\frac{x^2-2x-4}{x^2-4}}$, domain: $(-\infty, -2) \cup \left[1 - \sqrt{5}, 2\right) \cup \left[1 + \sqrt{5}, \infty\right)$
    - $(f \circ g)(t) = -\frac{2\sqrt{1-t}}{t+3}$, domain: $(-\infty, -3) \cup (-3, 1]$
    - $(f \circ f)(x) = \frac{4x - x^3}{x^4 - 9x^2 + 16}$, domain: $\left(-\infty, -\frac{1+\sqrt{17}}{2}\right) \cup \left(-\frac{1+\sqrt{17}}{2}, -2\right) \cup \left(-2, \frac{1-\sqrt{17}}{2}\right) \cup \left(\frac{1-\sqrt{17}}{2}, \frac{-1+\sqrt{17}}{2}\right) \cup \left(\frac{-1+\sqrt{17}}{2}, 2\right) \cup \left(2, \frac{1+\sqrt{17}}{2}\right) \cup \left(\frac{1+\sqrt{17}}{2}, \infty\right)$

25. $(h \circ g \circ f)(x) = |\sqrt{-2x}| = \sqrt{-2x}$, domain: $(-\infty, 0]$

26. $(h \circ f \circ g)(t) = |-2\sqrt{t}| = 2\sqrt{t}$, domain: $[0, \infty)$

27. $(g \circ f \circ h)(s) = \sqrt{-2|s|}$, domain: $\{0\}$

28. $(g \circ h \circ f)(x) = \sqrt{|-2x|} = \sqrt{2|x|}$, domain: $(-\infty, \infty)$

29. $(f \circ h \circ g)(t) = -2|\sqrt{t}| = -2\sqrt{t}$, domain: $[0, \infty)$

30. $(f \circ g \circ h)(s) = -2\sqrt{|s|}$, , domain: $(-\infty, \infty)$

31. $(f \circ g)(3) = f(g(3)) = f(2) = 4$

32. $f(g(-1)) = f(-4)$ which is undefined

33. $(f \circ f)(0) = f(f(0)) = f(1) = 3$

34. $(f \circ g)(-3) = f(g(-3)) = f(-2) = 2$

35. $(g \circ f)(3) = g(f(3)) = g(-1) = -4$

36. $g(f(-3)) = g(4)$ which is undefined

37. $(g \circ g)(-2) = g(g(-2)) = g(0) = 0$

38. $(g \circ f)(-2) = g(f(-2)) = g(2) = 1$

39. $g(f(g(0))) = g(f(0)) = g(1) = -3$

40. $f(f(f(-1))) = f(f(0)) = f(1) = 3$

41. $f(f(f(f(f(1))))) = f(f(f(f(3)))) = f(f(f(-1))) = f(f(0)) = f(1) = 3$

42. $\underbrace{(g \circ g \circ \cdots \circ g)}_{n \text{ times}}(0) = 0$

43.
   - The domain of $f \circ g$ is $\{-3, -2, 0, 1, 2, 3\}$ and the range of $f \circ g$ is $\{1, 2, 3, 4\}$.
   - The domain of $g \circ f$ is $\{-2, -1, 0, 1, 3\}$ and the range of $g \circ f$ is $\{-4, -3, 0, 1, 2\}$.

44. $(g \circ f)(1) = 3$

45. $(f \circ g)(3) = 1$

46. $(g \circ f)(2) = 0$

47. $(f \circ g)(0) = 1$

48. $(f \circ f)(4) = 1$

49. $(g \circ g)(1) = 0$

50.
    - The domain of $f \circ g$ is $[0, 3]$ and the range of $f \circ g$ is $[1, 4.5]$.
    - The domain of $g \circ f$ is $[0, 2] \cup [3, 4]$ and the range is $[0, 3]$.

51. Let $f(x) = 2x + 3$ and $g(x) = x^3$, then $p(x) = (g \circ f)(x)$.

52. Let $f(x) = x^2 - x + 1$ and $g(x) = x^5$, $P(x) = (g \circ f)(x)$.

53. Let $f(t) = 2t - 1$ and $g(t) = \sqrt{t}$, then $h(t) = (g \circ f)(t)$.

54. Let $f(t) = 7 - 3t$ and $g(t) = |t|$, then $H(t) = (g \circ f)(t)$.

55. Let $f(s) = 5s + 1$ and $g(s) = \frac{2}{s}$, then $r(s) = (g \circ f)(s)$.

56. Let $f(s) = s^2 - 1$ and $g(s) = \frac{7}{s}$, then $R(s) = (g \circ f)(s)$.

57. Let $f(z) = |z|$ and $g(z) = \frac{z+1}{z-1}$, then $q(z) = (g \circ f)(z)$.

58. Let $f(z) = z^3$ and $g(z) = \frac{2z+1}{z-1}$, then $Q(z) = (g \circ f)(z)$.

59. Let $f(x) = 2x$ and $g(x) = \frac{x+1}{3-2x}$, then $v(x) = (g \circ f)(x)$.

60. Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then $w(x) = (g \circ f)(x)$.

61. $F(x) = \sqrt{\frac{x^3+6}{x^3-9}} = (h(g(f(x)))$ where $f(x) = x^3$, $g(x) = \frac{x+6}{x-9}$ and $h(x) = \sqrt{x}$.

62. $F(x) = 3\sqrt{-x+2} - 4 = k(j(f(h(g(x)))))$

63. One solution is $F(x) = -\frac{1}{2}(2x-7)^3 + 1 = k(j(f(h(g(x)))))$ where $g(x) = 2x$, $h(x) = x - 7$, $j(x) = -\frac{1}{2}x$ and $k(x) = x + 1$. You could also have $F(x) = H(f(G(x)))$ where $G(x) = 2x - 7$ and $H(x) = -\frac{1}{2}x + 1$.

64. $(f \circ g)(x) = \begin{cases} 6x - 2 & \text{if } x \leq 3 \\ 13 - 3x & \text{if } x > 3 \end{cases}$ and

$$(g \circ f)(x) = \begin{cases} 6x + 1 & \text{if } x \leq \dfrac{2}{3} \\ 3 - 3x & \text{if } x > \dfrac{2}{3} \end{cases}$$

65. $V(x) = x^3$ so $V(x(t)) = (t + 1)^3$

66. (a) $R(x) = 2x$

    (b) $(R \circ x)(t) = -8t^2 + 40t + 184$, $0 \leq t \leq 4$. This gives the revenue per hour as a function of time.

    (c) Noon corresponds to $t = 2$, so $(R \circ x)(2) = 232$. The hourly revenue at noon is \$232 per hour.

3.4 Transformations of Graphs

Theorems **??**, **??**, **??**, 1.1.1, 2.1.1 and 2.2.2 all describe ways in which the graph of a function can changed, or 'transformed' to obtain the graph of a related function. The results and proofs of each of these theorems are virtually identical, and with the language of function composition, we can see better why.

Consider, for instance, Theorem 2.2.2, in which we describe how to transform the graph of $f(x) = x^r$ to $F(x) = a(bx - h)^r + k$. We may think of F as being build up from f by composing f with linear functions. Specifically, if we let $i(x) = bx - h$, then $(f \circ i)(x) = f(i(x)) = f(bx - h) = (bx - h)^r$. If, additionally, we let and $j(x) = ax + k$, then $(j \circ (f \circ i))(x) = j((f \circ i)(x)) = j((bx - h)^r) = a(bx - h)^r + k = F(x)$. Hence, we can view $F = j \circ f \circ i$. In this section, our goal is to generalize the aforementioned theorems to the graphs of *all* functions. Along the way, you'll see some very familiar arguments, but, additionally, we hope this section affords the reader an opportunity to not only see *how* these transformations work they way they do, but *why*.

Our motivational example for the results in this section is the graph of $y = f(x)$ in Figure 3.4.1. While we could formulate an expression for $f(x)$ as a piecewise-defined function consisting of linear and constant parts, we wish to focus more on the geometry here. That being said, we do record some of the function values—the 'key points' if you will—to track through each transformation (Table 3.4.1).

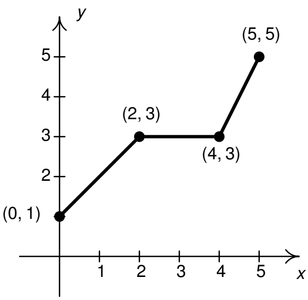


Figure 3.4.1: $y = f(x)$

x	$(x, f(x))$	$f(x)$
0	$(0, 1)$	1
2	$(2, 3)$	3
4	$(4, 3)$	3
5	$(5, 5)$	5

Table 3.4.1

3.4.1 Vertical and Horizontal Shifts

Suppose we wished to graph $g(x) = f(x) + 2$. From a procedural point of view, we start with an input x to the function f and we obtain the output $f(x)$. The function g takes the output $f(x)$ and adds 2 to it. Using the sample values for f from Table 3.4.1 we can create a table of values for g as shown in Table 3.4.2, hence generating points on the graph of g.

x	$(x, f(x))$	$f(x)$	$g(x) = f(x) + 2$	$(x, g(x))$
0	$(0, 1)$	1	$1 + 2 = 3$	$(0, 3)$
2	$(2, 3)$	3	$3 + 2 = 5$	$(2, 5)$
4	$(4, 3)$	3	$3 + 2 = 5$	$(4, 5)$
5	$(5, 5)$	5	$5 + 2 = 7$	$(5, 7)$

Table 3.4.2

In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Hence, $g(a) = f(a) + 2 = b + 2$, so the point $(a, b + 2)$ is on the graph of g. In other words, to obtain the graph of g, we add 2 to the y-coordinate of each point on the graph of f.

Geometrically, adding 2 to the y-coordinate of a point moves the point 2 units above its previous location. Adding 2 to every y-coordinate on a graph *en masse* is moves or 'shifts' the entire graph of f up 2 units. Notice that the graph in Figure 3.4.2 retains the same basic shape as before, it is just 2 units above its original location. In other words, we connect the four 'key points' we moved in the same manner in which they were connected before.

You'll note that the domain of f and the domain of g are the same, namely $[0, 5]$, but that the range of f is $[1, 5]$ while the range of g is $[3, 7]$. In general, shifting a function vertically like this will leave the domain unchanged, but could very well affect the range.

You can easily imagine what would happen if we wanted to graph the function $j(x) = f(x) - 2$. Instead of adding 2 to each of the y-coordinates on the graph of f, we'd be subtracting 2. Geometrically, we would be moving the graph down 2 units. We leave it to the reader to verify

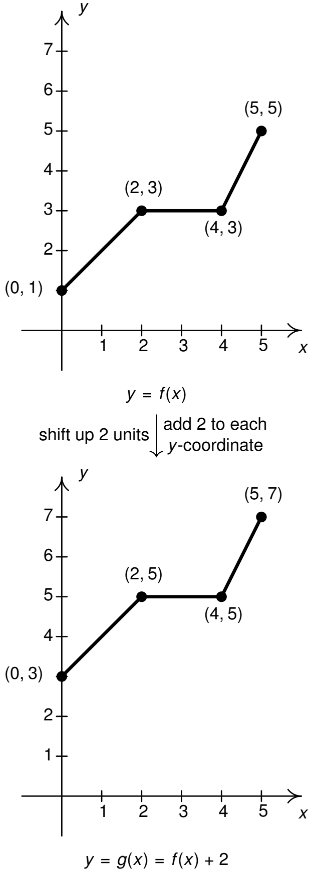


Figure 3.4.2

x	$(x, f(x))$	$f(x)$	$g(x) = f(x+2)$	$(x, g(x))$
0	$(0, 1)$	1	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	$(2, 3)$	3	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$g(4) = f(4+2) = f(6) = ?$	
5	$(5, 5)$	5	$g(5) = f(5+2) = f(7) = ?$	

Table 3.4.3

that the domain of j is the same as f, but the range of j is $[-1, 3]$. In general, we have Theorem 3.4.1.

Theorem 3.4.1. Vertical Shifts. Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

To prove Theorem 3.4.1, we first note that f and F have the same domain (why?) Let c be an element in the domain of F and, hence, the domain of f. The fact that f and F are *functions* guarantees there is *exactly one* point on each of their graphs corresponding to $x = c$. On $y = f(x)$, this point is $(c, f(c))$; on $y = F(x)$, this point is $(c, F(c)) = (c, f(c) + k)$. This sets up a nice correspondence between the two graphs and shows that each of the points on the graph of F can be obtained to by adding k to each of the y-coordinates of the corresponding point on the graph of f. This proves Theorem 3.4.1. In the language of 'inputs' and 'outputs', Theorem 3.4.1 says adding to the *output* of a function causes the graph to shift *vertically*.

Keeping with the graph of $y = f(x)$ above, suppose we wanted to graph $g(x) = f(x + 2)$. In other words, we are looking to see what happens when we add 2 to the input of the function. Let's try to generate a table of values of g based on those we know for f. We quickly find that we run into some difficulties. For instance, when we substitute $x = 4$ into the formula $g(x) = f(x + 2)$, we are asked to find $f(4 + 2) = f(6)$ which doesn't exist because the domain of f is only $[0, 5]$. The same thing happens when we attempt to find $g(5)$. See Table 3.4.3.

What we need here is a new strategy. We know, for in-

x	$x+2$	$g(x) = f(x+2)$	$(x, g(x))$
-2	0	$g(-2) = f(-2+2) = f(0) = 1$	$(-2, 1)$
0	2	$g(0) = f(0+2) = f(2) = 3$	$(0, 3)$
2	4	$g(2) = f(2+2) = f(4) = 3$	$(2, 3)$
3	5	$g(3) = f(3+2) = f(5) = 5$	$(3, 5)$

Table 3.4.4

stance, $f(0) = 1$. To determine the corresponding point on the graph of g, we need to figure out what value of x we must substitute into $g(x) = f(x+2)$ so that the quantity $x+2$, works out to be 0. Solving $x+2 = 0$ gives $x = -2$, and $g(-2) = f((-2)+2) = f(0) = 1$ so $(-2, 1)$ on the graph of g. To use the fact $f(2) = 3$, we set $x+2 = 2$ to get $x = 0$. Substituting gives $g(0) = f(0+2) = f(2) = 3$. Continuing in this fashion, we produce Table 3.4.4.

In summary, the points $(0, 1)$, $(2, 3)$, $(4, 3)$ and $(5, 5)$ on the graph of $y = f(x)$ give rise to the points $(-2, 1)$, $(0, 3)$, $(2, 3)$ and $(3, 5)$ on the graph of $y = g(x)$, respectively. In general, if (a, b) is on the graph of $y = f(x)$, then $f(a) = b$. Solving $x+2 = a$ gives $x = a-2$ so that $g(a-2) = f((a-2)+2) = f(a) = b$. As such, $(a-2, b)$ is on the graph of $y = g(x)$. The point $(a-2, b)$ is exactly 2 units to the *left* of the point (a, b) so the graph of $y = g(x)$ is obtained by shifting the graph $y = f(x)$ to the left 2 units, as pictured in Figure 3.4.3.

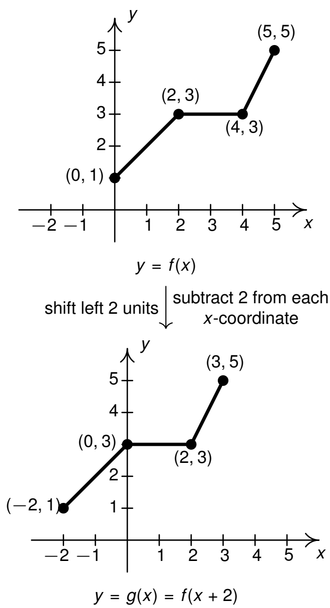


Figure 3.4.3

Note that while the ranges of f and g are the same, the domain of g is $[-2, 3]$ whereas the domain of f is $[0, 5]$. In general, when we shift the graph horizontally, the range will remain the same, but the domain could change. If we set out to graph $j(x) = f(x-2)$, we would find ourselves *adding* 2 to all of the x values of the points on the graph of $y = f(x)$ to effect a shift to the *right* 2 units. Generalizing these notions produces the result in Theorem 3.4.2.

To prove Theorem 3.4.2, we first note the domains of f

Theorem 3.4.2. Horizontal Shifts. Suppose f is a function and h is a real number.
To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left h units if $h < 0$.

and F may be different. If c is in the domain of f, then the only number we know for sure is in the domain of F is $c + h$, since $F(c + h) = f((c + h) - h) = f(c)$. This sets up a nice correspondence between the domain of f and the domain of F which spills over to a correspondence between their graphs, The point $(c, f(c))$ is the one and only point on the graph of $y = f(x)$ corresponding to $x = c$ just as the point $(c+h, F(c+h)) = (c+h, f(c))$ is the one and only point on the graph of $y = F(x)$ corresponding to $x = c + h$. This correspondence shows we may obtain the graph of F by adding h to each x-coordinate of each point on the graph of f, which establishes the theorem. In words, Theorem 3.4.2 says that subtracting from the *input* to a function amounts to shifting the graph *horizontally*.

Theorems 3.4.1 and 3.4.2 present a theme which will run common throughout the section: changes to the *outputs* from a function result in some kind of *vertical change*; changes to the *inputs* to a function result in some kind of *horizontal* change. We demonstrate Theorems 3.4.1 and 3.4.2 in the example below.

Example 3.4.1. Use Theorems 3.4.1 and 3.4.2 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 3)$ is on the graph of $y = f(x)$. Find a point on the graph of:
   (a) $y = f(x) + 5$
   (b) $y = f(x + 5)$
   (c) $f(x - 7) + 4$
2. Find a formula for a function $g(t)$ whose graph is the same as $f(t) = |t| - 2t$ but is shifted:
   (a) to the right 4 units. (b) down 2 units.
3. Predict how the graph of $F(x) = \dfrac{(x-2)^{\frac{2}{3}}}{x}$ relates

to the graph of $f(x) = \dfrac{x^{\frac{2}{3}}}{x+2}$.

4. In Figure 3.4.4 is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(x-2)$

   (b) $F(x) = f(x) + 1$

   (c) $F(x) = f(x+1) - 2$

5. In Figure 3.4.5 is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.

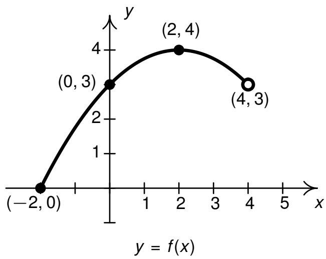


Figure 3.4.4

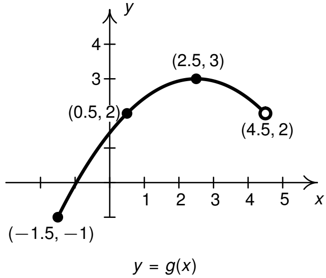


Figure 3.4.5

Solution.

1. (a) To apply Theorem 3.4.1, we identify $f(x) + 5 = f(x) + k$ so $k = 5$. Hence, we add 5 to the y-coordinate of $(-1, 3)$ and get $(-1, 3+5) = (-1, 8)$. To check our answer note since $(-1, 3)$ is on the graph of f this means $f(-1) = 3$. Substituting $x = -1$ into the formula $y = f(x) + 5$, we get $y = f(-1) + 5 = 3 + 5 = 8$. Hence, $(-1, 8)$ is on the graph of $f(x) + 5$.

   (b) We note that $f(x+5)$ can be written as $f(x - (-5)) = f(x - h)$ so we apply Theorem 3.4.2 with $h = -5$. Adding -5 to (subtracting 5 from) the x-coordinate of $(-1, 3)$ gives $(-1 + (-5), 3) = (-6, 3)$. To check our answer, since $(-1, 3)$ is on the graph of f, $f(-1) = 3$. Substituting $x = -6$ into $y = f(x+5)$ gives $y = f(-6+5) = f(-1) = 3$, proving $(-6, 3)$ is on the graph of $y = f(x+5)$.

   (c) Note that the expression $f(x-7) + 4$ differs from $f(x)$ in *two* ways indicating two different transformations. In situations like this, its best if we handle each transformation in turn, starting with the graph of $y = f(x)$ and 'building up' to the graph of $y = f(x-7) + 4$.

   We choose to work from the 'inside' (argument) out and use Theorem 3.4.2 to first get a point on the graph of $y = f(x-7) = f(x-h)$. Identifying $h = 7$, we add 7 to the x-coordinate of $(-1, 3)$ to get $(-1+7, 3) = (6, 3)$. Hence, $(6, 3)$ is a point on the graph of $y = f(x-7)$.

Next, we apply Theorem 3.4.1 to graph $y = f(x-7)+4$ starting with $y = f(x-7)$. Viewing $f(x-7)+4 = f(x-7)+k$, we identify $k = 4$ and add 4 to the y-coordinate of $(6, 3)$ to get $(6, 3+4) = (6, 7)$. To check, we note that if we substitute $x = 6$ into $y = f(x-7)+4$, we get $y = f(6-7)+4 = f(-1)+4 = 3+4 = 7$.

2. Here the independent variable is t instead of x which doesn't affect the geometry in any way since our convention is the independent variable is used to label the horizontal axis and the dependent variable is used to label the vertical axis.

   (a) Per Theorem 3.4.2, the graph of $g(t) = f(t-4) = |t-4| - 2(t-4) = |t-4| - 2t + 8$ should be the graph of $f(t) = |t| - 2t$ shifted to the right 4 units. Our check is in Figure 3.4.6.

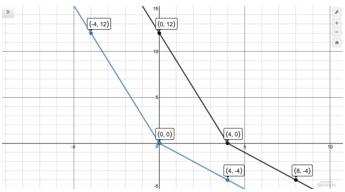


Figure 3.4.6: $y = |t| - 2t$ (lighter color) and $y = |t-4| - 2t + 8$

   (b) Per Theorem 3.4.1, the graph of $g(t) = f(t) + (-2) = |t| - 2t + (-2) = |t| - 2t - 2$ should be the graph of $f(t) = |t| - 2t$ shifted down 2 units. Our check is in Figure 3.4.7.

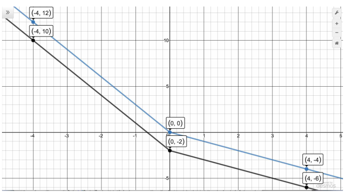


Figure 3.4.7: $y = |t| - 2t$ (lighter color) and $y = |t| - 2t - 2$

3. Comparing *formulas*, it appears as if $F(x) = f(x-2)$. We check:

$$f(x-2) = \frac{(x-2)^{\frac{2}{3}}}{(x-2)+2} = \frac{(x-2)^{\frac{2}{3}}}{x} = F(x),$$

so, per Theorem 3.4.2, the graph of $y = F(x)$ should be the graph of $y = f(x)$ but shifted to the right 2 units. We graph both functions in Figure 3.4.8 to confirm our answer.

4. (a) We recognize $F(x) = f(x-2) = f(x-h)$. With $h = 2$, Theorem 3.4.2 tells us to add 2

to each of the x-coordinates of the points on the graph of f, moving the graph of f to the *right* two units. See Figure 3.4.9.

We can check our answer by showing each ordered pair (x, y) listed on our final graph satisfies the equation $y = f(x-2)$. Starting with $(0,0)$, we substitute $x = 0$ into $y = f(x-2)$ and get $y = f(0-2) = f(-2)$. Since $(-2,0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(0-2) = f(-2) = 0$, showing the point $(0,0)$ is on the graph of $y = f(x-2)$. We invite the reader to check the remaining points.

(b) We have $F(x) = f(x) + 1 = f(x) + k$ where $k = 1$, so Theorem 3.4.1 tells us to move the graph of f *up* 1 unit by adding 1 to each of the y-coordinates of the points on the graph of f. See Figure 3.4.10.

To check our answer, we proceed as above. Starting with the point $(-2,1)$, we substitute $x = -2$ into $y = f(-2)+1$ to get $y = f(-2)+1$. Since $(-2,0)$ is on the graph of f, we know $f(-2) = 0$. Hence, $y = f(-2) + 1 = 0 + 1 = 1$. This proves $(-2,1)$ is on the graph of $y = f(x) + 1$. We encourage the reader to check the remaining points in kind.

(c) We are asked to graph $F(x) = f(x+1) - 2$. As above, when we have more than one modification to do, we work from the inside out and build up to $F(x) = f(x+1) - 2$ from $f(x)$ in stages. First, we apply Theorem 3.4.2 to graph $y = f(x+1)$ from $y = f(x)$. Rewriting $f(x+1) = f(x-(-1))$, we identify $h = -1$, so we add -1 to (subtract 1 from) each of the x-coordinates on the graph of f, shifting it to the *left* 1 unit. See Figure 3.4.11.

Next, we apply Theorem 3.4.1 to graph $y =$

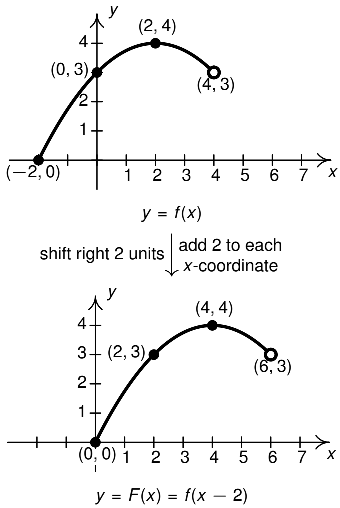


Figure 3.4.9

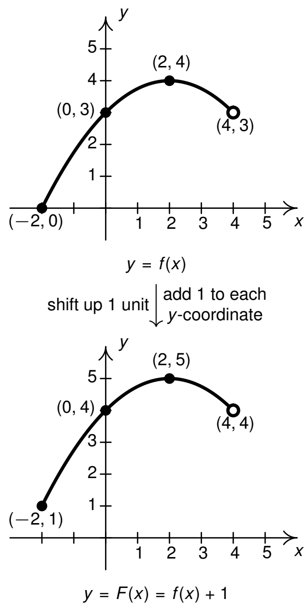


Figure 3.4.10

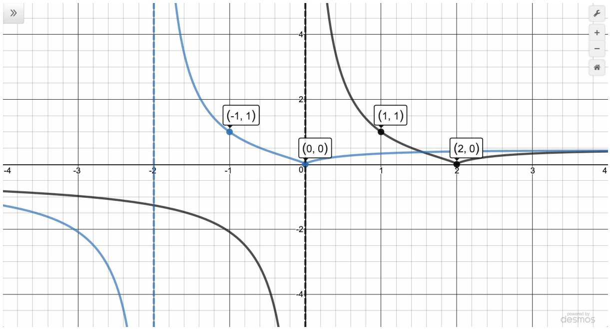


Figure 3.4.8: $y = \dfrac{x^{\frac{2}{3}}}{x+2}$ (lighter color) and $y = \dfrac{(x-2)^{\frac{2}{3}}}{x}$

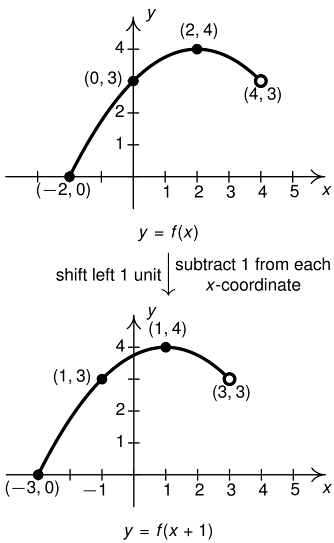


Figure 3.4.11

$f(x+1)-2$ starting with the graph of $y = f(x+1)$. Writing $f(x+1)-2 = f(x+1)+(-2) = f(x+1)+k$, we identify $k = -2$ so Theorem 3.4.1 instructs us to add -2 to (subtract 2 from) each of the y-coordinates on the graph of $y = f(x+1)$, thereby shifting the graph *down* two units. See Figure 3.4.12.

To check, we start with the point $(-3,-2)$. We find when we substitute $x = -3$ into the equation $y = f(x+1)-2$ we get $y = f(-3+1)-2 = f(-2)-2$. Since $(-2,0)$ is on the graph of f, we know $f(-2) = 0$, so $y = f(-3+1)-2 = f(-2)-2 = 0-2 = -2$. This proves $(-3,-2)$ is on the graph of $y = f(x+1)-2$. We leave the checks of the remaining points to the reader.

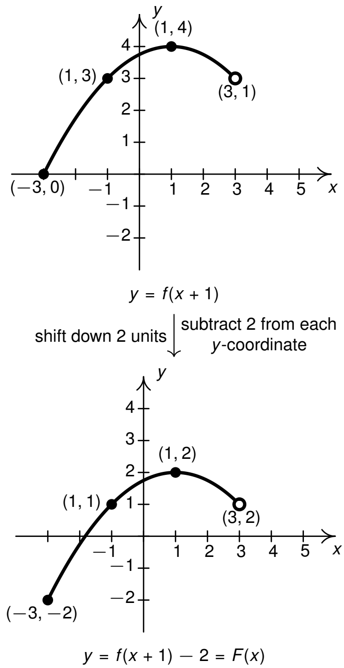


Figure 3.4.12

5. To write $g(x)$ in terms of $f(x)$, we note that based on points which are labeled, it appears as if the graph of g can be obtained from the graph of f by shifting the graph of f to the right 0.5 units and down 1 unit.

   Per Theorems 3.4.2 and 3.4.1, $g(x)$ must take the form $g(x) = f(x-h)+k$. Since the horizontal shift is to the *right* 0.5 units, $h = 0.5$ and since the vertical shift is *down* 1 unit, $k = -1$. Hence, we get $g(x) = f(x-0.5)-1$.

   We can check our answer by working through both transformations, in sequence, as in the previous example. To write $f(x)$ in terms of $g(x)$, we need to reverse the process - that is, we need to shift the graph of g *left* one half of a unit and *up* one unit. Theorems 3.4.2 and 3.4.1 suggest the formula $f(x) = g(x+0.5)+1$. We leave it to the reader to check. □

3.4.2 Reflections about the Coordinate Axes

We now turn our attention to reflections. We know from Section **??** that to reflect a point (x, y) across the x-axis, we replace y with $-y$. If (x, y) is on the graph of f, then $y = f(x)$, so replacing y with $-y$ is the same as replacing $f(x)$ with $-f(x)$. Hence, the graph of $y = -f(x)$ is the graph of f reflected across the x-axis. Similarly, the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis.[a]

[a] The expressions $-f(x)$ and $f(-x)$ should look familiar - they are the quantities we used in Section **??** to determine if a function was even, odd or neither. We explore impact of symmetry on reflections in Exercise 74.

Theorem 3.4.3. Reflections. Suppose f is a function.
To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the x-axis.
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the y-axis.

The proof of Theorem 3.4.3 follows in much the same way as the proofs of Theorems 3.4.1 and 3.4.2. If c is an element of the domain of f and $F(x) = -f(x)$, then the point $(c, f(c))$ corresponds to the point $(c, F(c)) = (c, -f(c))$. Comparing the corresponding points $(c, f(c))$ and $(c, -f(c))$, we see that the only difference is the y-coordinates are the exact opposite — indicating they are mirror-images across the x-axis. Similarly, if c is an element in the domain of f, then c corresponds to the element $-c$ in the domain of $F(x) = f(-x)$ since $F(-c) = f(-(-c)) = f(c)$. Hence, the corresponding points here are $(c, f(c))$ and $(-c, F(-c)) = (-c, f(c))$. Comparing $(c, f(c))$ with $(-c, f(c))$, we see they are reflections about the y-axis.

Using the language of inputs and outputs, Theorem 3.4.3 says that multiplying the *outputs* from a function by -1 reflects its graph across the *horizontal* axis, while multiplying the *inputs* to a function by -1 reflects the graph across the *vertical* axis.

Applying Theorem 3.4.3 to the graph of $y = f(x)$ given at the beginning of the section, we can graph $y = -f(x)$ by reflecting the graph of f about the x-axis. See Table 3.4.5 and Figure 3.4.13.

By reflecting the graph of f across the y-axis, we obtain the graph of $y = f(-x)$. See Table 3.4.6 and Figure 3.4.14.

Example 3.4.2. Use Theorems 3.4.1, 3.4.2 and 3.4.3 to

x	$(x, f(x))$	$f(x)$	$g(x) = -f(x)$	$(x, g(x))$
0	$(0, 1)$	1	-1	$(0, -1)$
2	$(2, 3)$	3	-3	$(2, -3)$
4	$(4, 3)$	3	-3	$(4, -3)$
5	$(5, 5)$	5	-5	$(5, -5)$

Table 3.4.5

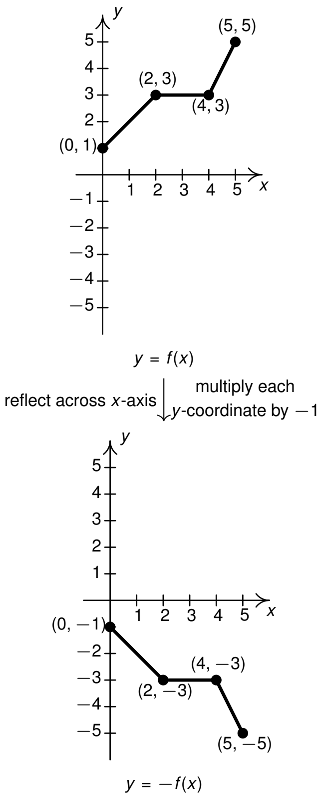


Figure 3.4.13

x	$-x$	$g(x) = f(-x)$	$(x, g(x))$
0	0	$g(0) = f(-(-0)) = f(0) = 1$	$(0, 1)$
-2	2	$g(-2) = f(-(-2)) = f(2) = 3$	$(-2, 3)$
-4	4	$g(-4) = f(-(-4)) = f(4) = 3$	$(-4, 3)$
-5	5	$g(-5) = f(-(-5)) = f(5) = 5$	$(-5, 5)$

Table 3.4.6

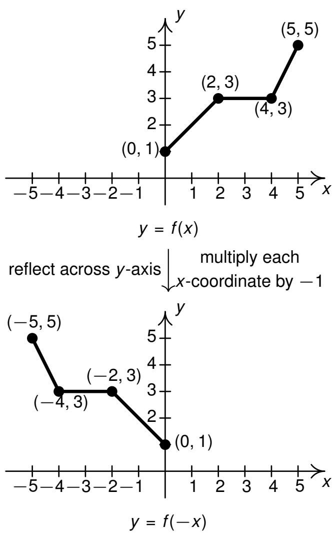


Figure 3.4.14

answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(2, -5)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = f(-x)$

   (b) $y = -f(x+2)$

   (c) $f(8-x)$

2. Find a formula for a function $H(s)$ whose graph is the same as $t = h(s) = s^3 - s^2$ but is reflected across the t-axis.

3. Predict how the graph of $G(t) = \dfrac{t+4}{t-3}$ relates to the graph of $g(t) = \dfrac{t+4}{3-t}$.

4. In Figure 3.4.15 is the graph of $y = f(x)$. Use it to sketch the graph of

   (a) $F(x) = f(-x) + 1$ (b) $F(x) = 1 - f(2-x)$

5. Figure 3.4.16 is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.

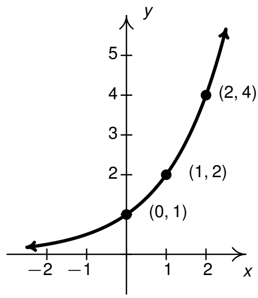


Figure 3.4.15: NOTE: The x-axis, $y = 0$, is a horizontal asymptote to the graph of $y = f(x)$

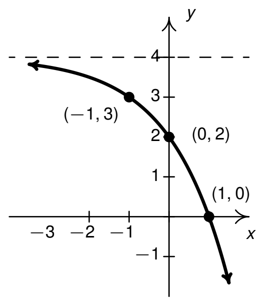


Figure 3.4.16: NOTE: The line $y = 4$ is a horizontal asymptote to the graph of $y = g(x)$

Solution.

1. (a) To find a point on the graph of $y = f(-x)$, Theorem 3.4.3 tells us to multiply the x-coordinate of the point on the graph of $y = f(x)$ by -1: $((-1)2, -5) = (-2, -5)$.

   To check, since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, when we substitute $x = -2$ into $y = f(-x)$, we get $y = f(-(-2)) = f(2) = -5$, proving $(-2, -5)$ is on the graph of $y = f(-x)$.

   (b) To find a point on the graph of $y = -f(x+2)$, we first note we have two transformations at work here, so we work our way from the inside out and build $f(x)$ to $-f(x+2)$.

First, we find a point on the graph of $y = f(x+2)$. Writing $f(x+2) = f(x-(-2))$, we apply Theorem 3.4.2 with $h = -2$ and add -2 to (or subtract 2 from) the x-coordinate of the point we know is on $y = f(x)$: $(2-2, -5) = (0, -5)$.

Next we apply Theorem 3.4.3 to the graph of $y = f(x+2)$ to get a point on the graph of $y = -f(x+2)$ by multiplying the y-coordinate of $(0, -5)$ by -1: $(0, (-1)(-5)) = (0, 5)$.

To check, recall $f(2) = -5$ so that when we substitute $x = 0$ into the equation $y = -f(x+2)$, we get $y = -f(0+2) = -f(2) = -(-5) = 5$, as required.

(c) Rewriting $f(8-x) = f(-x+8)$ we see we have two transformations at play here: a reflection across the y-axis and a horizontal shift. Since both of these transformations affect the x-coordinates of the graph, the question becomes which transformation to address first. To help us with this decision, we attack the problem algebraically.

Recall that since $(2, -5)$ is on the graph of f, we know $f(2) = -5$. Hence, to get a point on the graph of $y = f(-x+8)$, we need to match up the arguments of $f(-x+8)$ and $f(2)$: $-x+8 = 2$.

To solve this equation, we first subtract 8 from both sides to get $-x = -6$. Geometrically, subtracting 8 from the x-coordinate of $(2, -5)$, shifts the point $(2, -5)$ left 8 units to get the point $(-6, -5)$.

Next, we multiply both sides of the equation $-x = -6$ by -1 to get $x = 6$. Geometrically, multiplying the x-coordinate of $(-6,-5)$ by -1 reflects the point $(-6,-5)$ across the y-axis to $(6,-5)$.

To check we substitute $x = 6$ into $y = f(-x+8)$, and obtain $y = f(-6+8) = f(2) = -5$.

Even though we have found our answer, we re-examine this process from a 'build' perspective. We began with a point on the graph of $y = f(x)$ and first shifted the graph to the left 8 units. Per Theorem 3.4.2, this point is on the graph of $y = f(x+8)$.

Next we took a point on the graph of $y = f(x+8)$ and reflected it about the y-axis. Per Theorem 3.4.3, this put the point on the graph of $y = f(-x+8)$.

In general, when faced with graphing functions in which there is both a horizontal shift and a reflection about the y-axis, we'll deal with the shift first.

2. In this example, the independent variable is s and the dependent variable is t. We are asked to reflect the graph of h about the t-axis, which in this case is the *vertical* axis. Hence, $H(s) = h(-s) = (-s)^3 - (-s)^2 = -s^3 - s^2$. Our confirmation is in Figure 3.4.17.

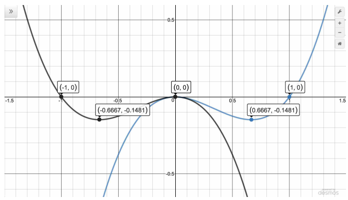


Figure 3.4.17: $t = s^3 - s^2$ (lighter color) and $t = -s^3 - s^2$

3. Comparing the formulas for $G(t) = \frac{t+4}{t-3}$ and $g(t) = \frac{t+4}{3-t}$, we have the same numerators, but in the denominator, we have $(t-3) = -(3-t)$:

$$G(t) = \frac{t+4}{t-3} = \frac{t+4}{-(3-t)} = -\frac{t+4}{3-t} = -g(t).$$

Hence, the graph of $y = G(t)$ should be the graph of $y = g(t)$ reflected across the t-axis. We check our answer in Figure 3.4.18.

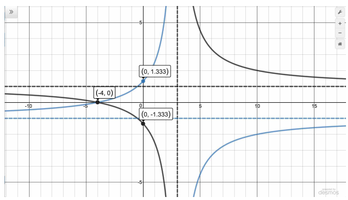


Figure 3.4.18: $y = \dfrac{t+4}{3-t}$ (lighter color) and $y = \dfrac{t+4}{t-3}$

4. (a) We have two transformations indicated with the formula $F(x) = f(-x) + 1$: a reflection across the y-axis and a vertical shift. Working from the inside out, we first tackle the reflection. Per Theorem 3.4.3, to obtain the graph of $y = f(-x)$ from $y = f(x)$, we multiply each of the x-coordinates of each of the points on the graph of $y = f(x)$ by (-1). See Figure 3.4.19.

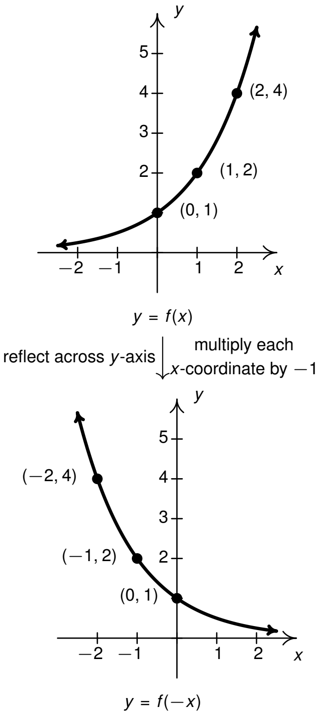


Figure 3.4.19

Next, we use Theorem 3.4.1 to obtain the graph of $y = f(-x) + 1$ from the graph of $y = f(-x)$ by adding 1 to each of the y-coordinates of each of the points on the graph of $y = f(-x)$. This shifts the graph of $y = f(-x)$ up one unit. Note, the horizontal asymptote $y = 0$ is also shifted up 1 unit to $y = 1$. See Figure 3.4.20.

To check our answer, we begin with the point $(0, 2)$. Substituting $x = 0$ into $y = f(-x) + 1$, we get $y = f(-0)+1 = f(0)+1$. Since the point $(0, 1)$ is on the graph of f, we know $f(0) = 1$. Hence, $y = f(0) + 1 = 1 + 1 = 2$, so $(0, 2)$ is, indeed, on the graph of $y = f(-x) + 1$. We leave it to the reader to check the remaining points.

(b) In order to graph $F(x) = 1 - f(2 - x)$, we first rewrite as $F(x) = -f(-x + 2) + 1$ and note there are *four* modifications to the formula $f(x)$ indicated here.

Working from the inside out, we see we have a reflection about the y-axis indicated as well as a horizontal shift. From our work above, we know we first handle the shift: that is, we apply Theorem 3.4.2 to graph $y = f(x + 2) = f(x - (-2))$ by adding -2 to (subtracting 2 from) the x-coordinates of the points on the graph of $y = f(x)$. See Figure 3.4.21.

Next, we use Theorem 3.4.3 to graph $y = f(-x+2)$ starting with the graph of $y = f(x+2)$ by multiplying each of the x-coordinates of the points of the graph of $y = f(x+2)$ by -1. This reflects the graph of $f(x+2)$ about the y-axis. See Figure 3.4.22.

We have the graph of $y = f(-x+2)$ and need to build towards the graph of $y = -f(-x+2)+1$. The transformations that remain are a reflection about the x-axis and a vertical shift. The question is which to do first.

Once again, we can think algebraically about the problem. We know the point $(0,1)$ is on the graph of f which means $f(0) = 1$. This point corresponds to the point $(2,1)$ on the graph of $f(-x+2)$. Indeed, when we substitute $x = 2$ into $y = f(-x+2)$, we get $y = f(-2+2) = f(0) = 1$.

If we substitute $x = 2$ into the formula $y = -f(-x+2)+1$, we get $y = -f(-2+2)+1 = -f(0)+1 = -1(1)+1 = 0$. That is, we first multiply the y-coordinate of $(2,1)$ by -1 then add 1. This suggests we take care of the reflection about the x-axis first, then the vertical shift.

We proceed as shown in Figure 3.4.23 to obtain the graph of $y = -f(-x+2)$ from $y = f(-x+2)$ by multiplying each of the y-coordinates on the graph of $y = f(-x+2)$ by -1. Note the horizontal asymptote remains unchanged: $y = (-1)(0) = 0$.

Finally, we take care of the vertical shift. Per Theorem 3.4.1, we graph $y = -f(-x+2)+1$ by adding 1 to the y-coordinates of each of the points on the graph of $y = -f(-x+2)$. This moves the graph up one unit, including

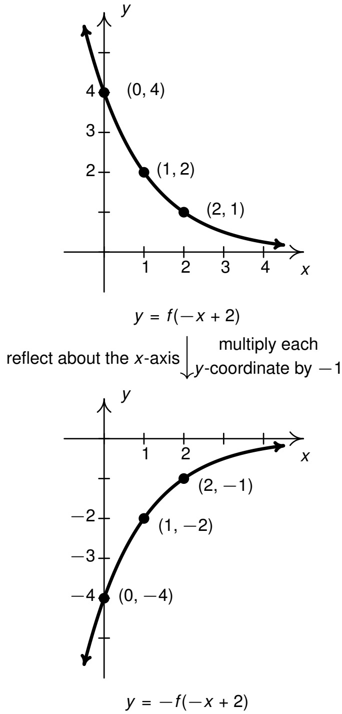


Figure 3.4.23

the horizontal asymptote: $y = 0 + 1 = 1$. See Figure 3.4.24.

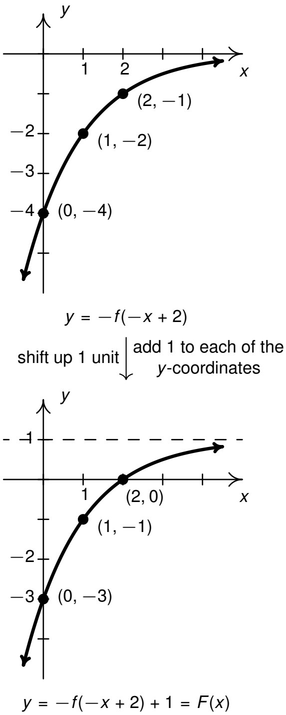


Figure 3.4.24

To check, we begin with the point $(2, 0)$. Substituting $x = 2$ into $y = 1 - f(2 - x)$, we obtain $y = 1 - f(2 - 2) = 1 - f(0)$. Since $(0, 1)$ is on the graph of f, we know $f(0) = 1$. This means $y = 1 - f(2 - 2) = 1 - f(0) = 1 - 1 = 0$. This proves $(2, 0)$ is on the graph of $y = 1 - f(2 - x)$, and we recommend the reader check the remaining points.

5. With the transformations at our disposal, our task amounts to finding values of h and k and choosing between signs $\pm$ so that $g(x) = \pm f(\pm x - h) + k$.

   Based on the horizontal asymptote, $y = 4$, we choose $k = 4$. Note, however, in the graph of $y = f(x)+4$, the entire graph is *above* the line $y = 4$. Since the graph of g approaches the asymptote from below, we know $y = -f(\pm x - h) + 4$.

   Hence, two of transformations applied to the graph of f are a reflection across the x-axis followed by a shift up 4 units. This means the point $(0, 1)$ on the graph of f must correspond to the point $(-1, 3)$ on the graph of g, since these are the points closest to the asymptote on each graph.

   Likewise, the points $(1, 2)$ and $(2, 4)$ on the graph of f must correspond to $(0, 2)$ and $(1, 0)$, respectively, on the graph of g. Looking at the x-coordinates only, we have $x = 0$ moves to $x = -1$, $x = 1$ moves to $x = 0$, and $x = 2$ moves to $x = 1$. Hence, the net effect on the x-values is a shift left 1 unit. Hence, we guess the formula for $g(x)$ to be $g(x) = -f(x + 1) + 4$.

   We can readily check by going through the transformations: first, shift left 1 unit; next, reflect across the x-axis; finally, shift up 4. We leave it to the reader to verify that tracking each of the points on the graph of f along with the horizontal asymptote through this sequence of transformations results in the graph of g.

One way to recover the graph of f from the graph of g is to reverse the process by which we obtained g from f. The challenge here comes from the fact that two different operations were done which affected the y-values: reflection and shifting - and the order in which these are done matters.

To motivate our methodology, let's consider a more down-to-earth example like putting on socks and then putting on shoes. Unless we're very talented, to reverse this process, we take off the shoes first, then the socks - that is, we undo each step in the reverse order.[b] In the same way, when we think about reversing the steps transforming the graph of f to the graph of g, we need to undo each transformation in the opposite order.

b We'll have more to say about this sort of thing in Section 3.6.

To review, we obtained the graph of g from the graph of f by first shifting the graph to the left 1 unit, then reflecting the graph about the x-axis, then, finally, shifting the graph up 4 units. Hence, we first undo the vertical shift. Instead of shifting the graph *up* four units, we shift the graph *down* four units. This takes the graph of $y = g(x)$ to $y = g(x) - 4$.

Next, we have to undo the refection across the x-axis. Thinking at the level of points, to recover the point (a, b) from its reflection across the x-axis, $(a, -b)$, we simply reflect across the x-axis again: $(a, -(-b)) = (a, b)$. Per Theorem 3.4.3, this takes the graph the graph of $y = g(x) - 4$ to the graph of $y = -[g(x) - 4] = -g(x) + 4$.[c]

c To see this better, let us temporarily write $F(x) = g(x) - 4$. Theorem 3.4.3 tells us to reflect the graph of F about the x-axis, graph $y = -F(x) = -[g(x) - 4] = -g(x) + 4$.

Last, to undo moving the graph to the *left* 1 unit, we move the graph of $y = -g(x) + 4$ to the *right* 1 unit. Per Theorem 3.4.2, we accomplish this by graphing $y = -g(x - 1) + 4$. We leave it to the reader to start with the graph of $y = g(x)$ and graph $y = -g(x - 1) + 4$ and show it matches the graph of $y = f(x)$. $\square$

Some remarks about Example 3.4.2 are in order. In number 1c above, to find a point on the graph of $y = f(-x+8)$, we took the given x-coordinate on our starting graph, 2, and subtracted 8 first then multiplied by -1. If this seems somehow 'backwards' it should.

When *evaluating* the expression $-x+8$, the order of operations mandates we multiply by -1 first then add 8. Here, however, we weren't *evaluating* an expression - we were *solving* an equation: $-x + 8 = 2$, which meant we did the exact opposite steps in the opposite order.[d] This exemplifies a larger theme with transformations: when adjusting inputs, the resulting points on the graph are obtained by applying the opposite operations indicated by the formula in the opposite order of operations.

[d] Note that dividing by -1 is the same as multiplying by -1, so to keep with the 'opposite steps in opposite order' theme, we could more precisely say we subtracted 8 and *divided* by -1.

On the other hand, when it came to multiple transformations involving the y-coordinates, we followed the order of operations. As in 4b above, when it came to applying a reflection about the x-axis and a vertical shift, we applied the reflection first, then the shift. This is because instead of *solving* an *equation* to find the new y-coordinates, we were *simplifying* an *expression*. Again, this is an example of a much larger theme: when adjusting outputs, the resulting points on the graph are obtained by applying the stated operations in the usual order.

Last but not least, in number 5, to find f in terms of g, we reversed the steps used to transform f into g. Another tact is to approach the problem in the same way we approached transforming f into g: namely, starting with the graph of g, determine values h and k and signs $\pm$ so that $f(x) = \pm g(\pm x - h) + k$. We leave this to the reader.

3.4.3 Scalings

We now turn our attention to our last class of transformations: **scalings**. A thorough discussion of scalings can get complicated because they are not as straight-forward as the previous transformations. A quick review of what we've covered so far, namely vertical shifts, horizontal shifts and reflections, will show you why those transformations are known as **rigid transformations**.

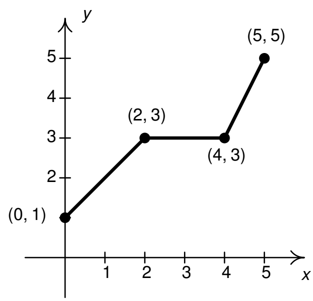
Simply put, rigid transformations preserve the distances between points on the graph - only their position and orientation in the plane change.[e] If, however, we wanted to make a new graph twice as tall as a given graph, or one-third as wide, we would be affecting the distance between points. These sorts of transformations are hence called **non-rigid**. As always, we motivate the general theory with an example.

e Another word that can be used here instead of 'rigid transformation' is 'isometry' - meaning 'same distance.'

Suppose we wish to graph the function $g(x) = 2f(x)$ where $f(x)$ is the function whose graph is given at the beginning of the section. From its graph in Figure 3.4.25, we can build a table of values for g as before. See Table 3.4.7.

x	$(x, f(x))$	$f(x)$	$g(x) = 2f(x)$	$(x, g(x))$
0	$(0, 1)$	1	2	$(0, 2)$
2	$(2, 3)$	3	6	$(2, 6)$
4	$(4, 3)$	3	6	$(4, 6)$
5	$(5, 5)$	5	10	$(5, 10)$

Table 3.4.7



$y = f(x)$

Figure 3.4.25

Graphing, we get Figure 3.4.26.

In general, if (a, b) is on the graph of f, then $f(a) = b$ so that $g(a) = 2f(a) = 2b$ puts $(a, 2b)$ on the graph of g. In other words, to obtain the graph of g, we multiply all of the y-coordinates of the points on the graph of f by 2.

Multiplying all of the y-coordinates of all of the points on the graph of f by 2 causes what is known as a 'vertical scaling[f] by a factor of 2.'

If we wish to graph $y = \frac{1}{2}f(x)$, we multiply the all of the y-coordinates of the points on the graph of f by $\frac{1}{2}$. This creates a 'vertical scaling[g] by a factor of $\frac{1}{2}$' as seen in Figure 3.4.27.

These results are generalized in Theorem 3.4.4.

Theorem 3.4.4. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number.
To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph of $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a] expansion, dilation
[b] compression, contraction

The proof of Theorem 3.4.4 mimics the proofs of Theorems 3.4.1 and 3.4.3. If c is in the domain of f, then $(c, f(c))$ is on the graph of f and the corresponding point on the graph of $F(x) = af(x)$ is $(c, F(c)) = (c, af(c))$. Comparing the points $(c, f(c))$ and $(c, af(c))$ proves the theorem.

A few remarks about Theorem 3.4.4 are in order. First, a note about the verbiage. To the authors, the words 'stretch', 'expansion', and 'dilation' all indicate something getting bigger. Hence, 'stretched by a factor of 2' makes sense if we are scaling something by multiplying it by 2. Similarly, we believe words like 'shrink', 'compression' and 'contraction' all indicate something getting smaller, so if we scale something by a factor of $\frac{1}{2}$, we would say it 'shrinks by a factor of 2' - not 'shrinks by a factor of $\frac{1}{2}$'. This is why we have written the descriptions 'stretch by a

f Also called a 'vertical stretch,' 'vertical expansion' or 'vertical dilation' by a factor of 2.

g Also called 'vertical shrink,' 'vertical compression' or 'vertical contraction' by a factor of 2.

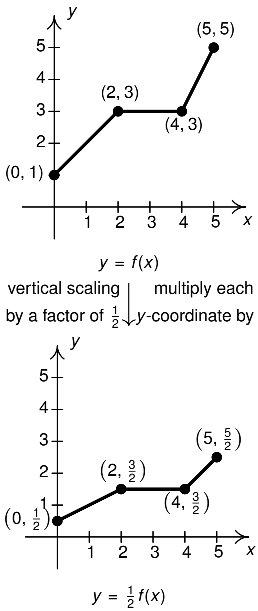


Figure 3.4.27

x	$(x, f(x))$	$f(x)$	$g(x) = f(2x)$	$(x, g(x))$
0	$(0, 1)$	1	$f(2 \cdot 0) = f(0) = 1$	$(0, 1)$
2	$(2, 3)$	3	$f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
4	$(4, 3)$	3	$f(2 \cdot 4) = f(8) = ?$	
5	$(5, 5)$	5	$f(2 \cdot 5) = f(10) = ?$	

Table 3.4.8

factor of a' and 'shrink by a factor of $\frac{1}{a}$' in the statement of the theorem.

Second, in terms of inputs and outputs, Theorem 3.4.4 says multiplying the *outputs* from a function by positive number a causes the graph to be vertically scaled by a factor of a. It is natural to ask what would happen if we multiply the *inputs* of a function by a positive number. This leads us to our last transformation of the section.

Referring to the graph of f given at the beginning of this section, suppose we want to graph $g(x) = f(2x)$. In other words, we are looking to see what effect multiplying the inputs to f by 2 has on its graph. If we attempt to build a table directly, we quickly run into the same problem we had in our discussion leading up to Theorem 3.4.2, as seen in Table 3.4.8.

We solve this problem in the same way we solved this problem before. For example, if we want to determine the point on g which corresponds to the point $(2, 3)$ on the graph of f, we set $2x = 2$ so that $x = 1$. Substituting $x = 1$ into $g(x)$, we obtain $g(1) = f(2 \cdot 1) = f(2) = 3$, so that $(1, 3)$ is on the graph of g. Continuing in this fashion, we obtain Table 3.4.9.

In general, if (a, b) is on the graph of f, then $f(a) = b$. Hence $g\left(\frac{a}{2}\right) = f\left(2 \cdot \frac{a}{2}\right) = f(a) = b$ so that $\left(\frac{a}{2}, b\right)$ is on the graph of g. In other words, to graph g we divide the x-coordinates of the points on the graph of f by 2. This results in a horizontal scaling[h] by a factor of $\frac{1}{2}$. See Figure 3.4.28.

h Also called 'horizontal shrink,' 'horizontal compression' or 'horizontal contraction' by a factor of 2.

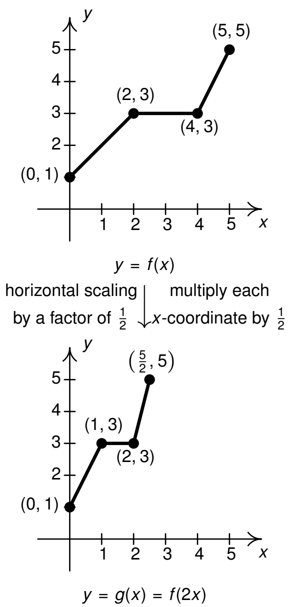


Figure 3.4.28

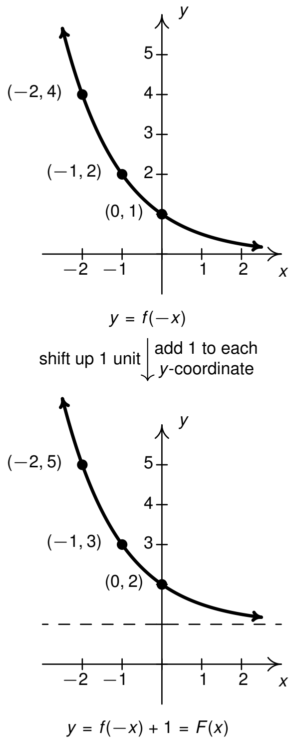


Figure 3.4.20

x	$2x$	$g(x) = f(2x)$	$(x, g(x))$
0	0	$g(0) = f(2 \cdot 0) = f(0) = 1$	$(0, 0)$
1	2	$g(1) = f(2 \cdot 1) = f(2) = 3$	$(1, 3)$
2	4	$g(2) = f(2 \cdot 2) = f(4) = 3$	$(2, 3)$
$\frac{5}{2}$	5	$g\left(\frac{5}{2}\right) = f\left(2 \cdot \frac{5}{2}\right) = f(5) = 5$	$\left(\frac{5}{2}, 5\right)$

Table 3.4.9

If, on the other hand, we wish to graph $y = f\left(\frac{1}{2}x\right)$, we end up multiplying the x-coordinates of the points on the graph of f by 2 which results in a horizontal scaling[i] by a factor of 2, as demonstrated in Figure 3.4.29.

i Also called 'horizontal stretch,' 'horizontal expansion' or 'horizontal dilation' by a factor of 2.

We have Theorem 3.4.5.

Theorem 3.4.5. Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number.
To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f has undergone a horizontal stretch[a] by a factor of $\frac{1}{b}$.
- If $b > 1$, we say the graph of f has undergone a horizontal shrink[b] by a factor of b.

[a] expansion, dilation
[b] compression, contraction

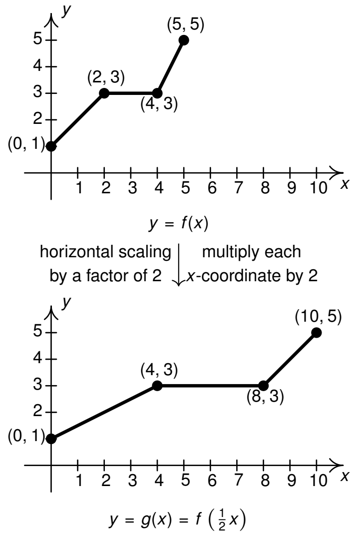


Figure 3.4.29

The proof of Theorem 3.4.5 follows closely the spirit of the proof of Theorems 3.4.2 and 3.4.3. If c is an element of the domain of f, them the number $\frac{c}{b}$ corresponds to a domain element of $F(x) = f(bx)$ since $F\left(\frac{c}{b}\right) = f\left(b \cdot \frac{c}{b}\right) = f(c)$. Hence, there is a correspondence between the point $(c, f(c))$ on the graph of f and the point $\left(\frac{c}{b}, F\left(\frac{c}{b}\right)\right) = \left(\frac{c}{b}, f(c)\right)$ on the graph of F. We can obtain $\left(\frac{c}{b}, f(c)\right)$ by dividing the x-coordinate of $(c, f(c))$ by b and the result follows.

Theorem 3.4.5 tells us that if we multiply the input to a function by b, the resulting graph is scaled horizontally by a factor of $\frac{1}{b}$. The next example explores how vertical and horizontal scalings sometimes interact with each other and with the other transformations introduced in this section.

Example 3.4.3. Use Theorems 3.4.1, 3.4.2, 3.4.3, 3.4.4 and 3.4.5 to answer the questions below. Check your answers using a graphing utility where appropriate.

1. Suppose $(-1, 4)$ is on the graph of $y = f(x)$. Find a point on the graph of:

   (a) $y = 3f(x-2)$

   (b) $y = f\left(-\frac{1}{2}x\right)$

   (c) $f(2x-3)+1$

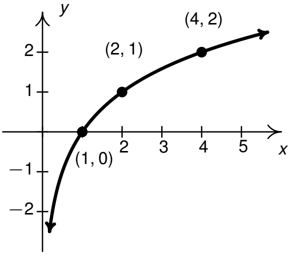
2. Find a formula for a function $G(t)$ whose graph is the same as $y = g(t) = \frac{2t+1}{t-1}$ but is vertically stretched by a factor of 4.

3. Predict how the graph of $H(s) = 8s^3 - 12s^2$ relates to the graph of $h(s) = s^3 - 3s^2$.

4. In Figure 3.4.30 is the graph of $y = f(x)$. Use it to sketch the graph of

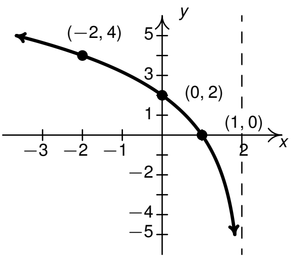
   (a) $F(x) = \dfrac{1-f(x)}{2}$ (b) $F(x) = f\left(\dfrac{1-x}{2}\right)$

5. In Figure 3.4.31 is the graph of $y = g(x)$. Write $g(x)$ in terms of $f(x)$ and vice-versa.



$y = f(x)$

Figure 3.4.30: NOTE: The y-axis, $x = 0$, is a vertical asymptote to the graph of $y = f(x)$



$y = g(x)$

Figure 3.4.31: NOTE: The line $x = 2$ is a vertical asymptote to the graph of $y = g(x)$

Solution.

1. (a) As we examine the formula $y = 3f(x-2)$, we note two modifications from $y = f(x)$. Building from the inside out, we start with obtaining a point on the graph of $y = f(x-2)$.

   Per Theorem 3.4.2, this shifts all of the points on the graph of $y = f(x)$ 2 units to the right. Hence, the point $(-1, 4)$ on the graph of $y = f(x)$ moves to the point $(-1+2, 4) = (1, 4)$ on the graph of $y = f(x-2)$.

   To get a point on the graph of $y = 3f(x-2) = af(x-3)$, we apply Theorem 3.4.4 with $a = 3$ to the point $(1, 4)$ on the graph of $y = f(x-2)$ to get the point $(1, 3(4)) = (1, 12)$ on the graph of $y = 3f(x-2)$.

To check, we note that since $(-1, 4)$ is on the graph of $y = f(x)$, we know $f(-1) = 4$. Hence, when we substitute $x = 1$ into the $y = 3f(x-2)$, we get $y = 3f(1-2) = 3f(-1) = 3(4) = 12$.

(b) The formula $y = f\left(-\frac{1}{2}x\right)$ also indicates two transformations: a horizontal scaling, indicated by $\frac{1}{2}$ factor, as well as a reflection across the y-axis. The question before us is which to do first.

If we return to algebra for inspiration, we know $f(-1) = 4$, so we match up the arguments of $f\left(-\frac{1}{2}x\right)$ and $f(-1)$ and get the equation $-\frac{1}{2}x = -1$. We solve this equation by multiplying both sides by -2: $x = (-2)(-1) = 2$. That is, we take the original x-value on the graph of $y = f(x)$ and multiply it by -2.

If we think of $-2 = (-1)(2)$ then multiplying by the '2' in '$(-1)(2)$' produces a horizontal stretch by a factor of 2 while multiplying by the '-1' reflects the point across the y-axis.

Applying the horizontal stretch first, we use Theorem 3.4.5 and start with the point $(-1, 4)$ on the graph of $y = f(x)$ and multiply the x-coordinate by 2 to obtain a point on the graph of $y = f\left(\frac{1}{2}x\right)$: $(-1(2), 4) = (-2, 4)$.

Next, we take care of the reflection about the y-axis using Theorem 3.4.3 . Starting with $(-2, 4)$ on the graph of $y = f\left(\frac{1}{2}x\right)$, we multiply the x-coordinate by -1 to obtain a point on the graph of $y = f\left(\frac{1}{2}(-x)\right) = f\left(-\frac{1}{2}x\right)$: $((-1)(-2), 4) = (2, 4)$.

To check, note when $x = 2$ is substituted into $y = f\left(-\frac{1}{2}x\right)$, we get $y = f\left(-\frac{1}{2}(2)\right) = f(-1) = 4$.

Of course, we could have equally written the multiple $-2 = (2)(-1)$ and reversed these steps: doing the reflection first, then the horizontal scaling.

Proceeding this way, we start with the point $(-1,4)$ on the graph of $y = f(x)$ and reflect across the y-axis to obtain the point $((-1)(-1),4) = (1,4)$ on the graph of $y = f(-x)$.

Next, we stretch the graph of $y = f(-x)$ by a factor of 2 by multiplying the x-coordinates of the points on the graph by 2 and obtain $(2(1),4) = (2,4)$ on the graph of $y = f\left(-\frac{1}{2}x\right)$. In general when it comes to reflections and scalings, whether horizontal or, as we'll see soon, vertical, either order will produce the same results.

(c) The formula $f(2x-3)+1$ indicates *three* transformations: a horizontal shift, a horizontal scaling, and a vertical shift. As usual, we appeal to algebra to give us guidance on which horizontal transformation to apply first.

Since we know $f(-1) = 4$, we set $2x - 3 = -1$ and solve. Our first step is to add 3 to both sides: $2x = (-1) + 3 = 2$. Since we are adding 3 to the given x-value -1, this corresponds to a shift to the right 3 units, so the point $(-1,4)$ is moved to the point $(2,4)$.

Next, to solve $2x = 2$, we divide this new x-coordinate 2 by 2 and get $x = \frac{2}{2} = 1$ which corresponds to a horizontal compression by a factor of 2. This moves the point $(2,4)$ to $(1,4)$.

Hence, the algebra suggests we use Theorem 3.4.2 first and follow it up with Theorem 3.4.5. Starting with $(-1,4)$ on the graph of $y = f(x)$, we shift to the right 3 units to obtain the point $(-1+3,4) = (2,4)$ on the graph of $y = f(x-3)$.

Next, we start with the point $(2, 4)$ on the graph of $y = f(x - 3)$ and horizontally shrink the x-axis by a factor of 2 to get the point $\left(\frac{2}{2}, 4\right) = (1, 4)$ on the graph of $y = f(2x - 3)$.

Last, but not least, we take care of the vertical shift using Theorem 3.4.1. Starting with the point $(1, 4)$ on the graph of $y = f(2x - 3)$, we add 1 to the y-coordinate to get the point $(1, 4 + 1) = (1, 5)$ on the graph of $y = f(2x - 3) + 1$.

To check, we substitute $x = 1$ into the formula $y = f(2x - 3) + 1$ and get $y = f(2(1) - 3) + 1 = f(-1) + 1 = 4 + 1 = 5$, as required.

2. To vertically stretch the graph of $y = g(t)$ by 4, we use Theorem 3.4.4 with $a = 4$ to get

$$G(t) = 4g(t) = 4\frac{2t + 1}{t - 1} = \frac{4(2t + 1)}{t - 1} = \frac{8t + 4}{t - 1}.$$

We check our answer in Figure 3.4.32.

3. When comparing the formulas for $H(s) = 8s^3 - 12s^2$ and $h(s) = s^3 - 3s^2$, it doesn't appear as if any shifting or reflecting is going on (why not?)

We also note that since the coefficient of s^3 in the expression of $H(s)$ is 8 times that of the coefficient of s^3 in $h(s)$, but the coefficient of s^2 in $H(s)$ is only 4 times the coefficient of s^2 in $h(s)$, the change is not the result of a vertical scaling (again, why not?)

Hence, if anything, we are looking for a horizontal scaling. In other words, we are looking for a real number $b > 0$ so $h(bs) = H(s)$, that is, $(bs)^3 - 3(bs)^2 = b^3s^3 - 3b^2s^2 = 8s^3 - 12s^2$.

Matching up coefficients of s^3 gives $b^3 = 8$ so $b = 2$ which checks with the coefficients of $s^2 : 3b^2 = 3(2)^2 = 12$.

Hence, we predict the graph of $y = H(s)$ to be the same as $y = h(s)$ except horizontally compressed by a factor of 2. Our check is in Figure 3.4.33.

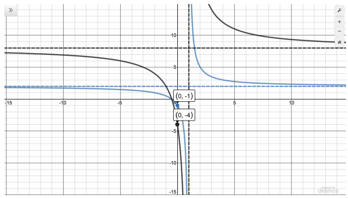


Figure 3.4.32: $y = g(t) = \frac{2t+1}{t-1}$ (lighter color) and $y = 4g(t) = \dfrac{8t + 4}{t - 1}$

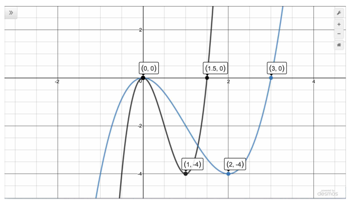


Figure 3.4.33: $y = h(s) = s^3 - 3s^2$ (lighter color) and $y = H(s) = 8s^3 - 12s^2$

4. (a) We first rewrite the expression for $F(x) = \frac{1-f(x)}{2} = -\frac{1}{2}f(x)+\frac{1}{2}$ in order to use the theorems available to us. Note we have two modifications to the formula of $f(x)$ which correspond to three transformations.

Multiplying $f(x)$ by $-\frac{1}{2}$ indicates a vertical compression by a factor of 2 along with a reflection about the x-axis. Adding $\frac{1}{2}$ indicates a vertical shift up $\frac{1}{2}$ units.

As always the question is which to do first. Once again, we look to algebra for the answer. Picking the point $(1, 0)$ on the graph of $f(x)$, we know $f(1) = 0$. To see which point this corresponds to on the graph of $y = F(x)$, we find $F(1) = -\frac{1}{2}f(1) + \frac{1}{2} = -\frac{1}{2}(0) + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$.

Hence, we first multiplied the y-value 0 by $-\frac{1}{2}$. As above, we can think of $-\frac{1}{2} = (-1)\frac{1}{2}$ so that multiplying by $-\frac{1}{2}$ amounts to a vertical compression by a factor of 2 first, then the refection about the x-axis second. Lastly, adding the $\frac{1}{2}$ is the vertical shift up $\frac{1}{2}$ unit.

Beginning with the vertical scaling by a factor of $\frac{1}{2}$, we use Theorem 3.4.4 to graph $y = \frac{1}{2}f(x)$ starting from $y = f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = f(x)$ by $\frac{1}{2}$. See Figure 3.4.34.

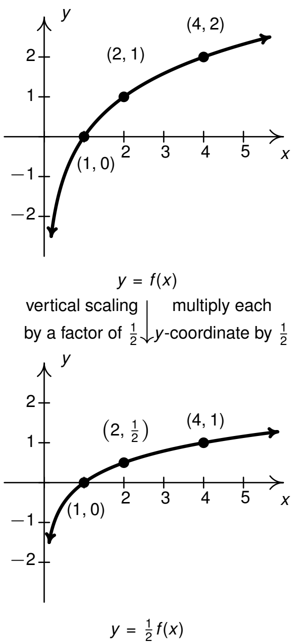


Figure 3.4.34

Next, we reflect the graph of $y = \frac{1}{2}f(x)$ across the x-axis to produce the graph of $y = -\frac{1}{2}f(x)$ by multiplying each of the y-coordinates of the points on the graph of $y = \frac{1}{2}f(x)$ by -1. See Figure 3.4.35.

Finally, we shift the graph of $y = -\frac{1}{2}f(x)$ vertically up $\frac{1}{2}$ unit by adding $\frac{1}{2}$ to each of the y-coordinates of each of the points to obtain the graph of $y = -\frac{1}{2}f(x) + \frac{1}{2} = F(x)$. See Figure 3.4.36.

Note that as with horizontal scalings and reflections about the y-axis, the order of vertical scalings and reflections across the x-axis is interchangeable. Had we decided to think of the factor $-\frac{1}{2} = \frac{1}{2} \cdot (-1)$, we could have just as well started with the graph of $y = f(x)$ and produced the graph of $y = -f(x)$ first. See Figure 3.4.37.

Next, we vertically scale the graph of $y = -f(x)$ by multiplying each of the y-coordinates of each of the points on the graph of $y = -f(x)$ by $\frac{1}{2}$ to obtain the graph of $y = -\frac{1}{2}f(x)$. See Figure 3.4.38.

Notice we've reached the same graph of $y = -\frac{1}{2}f(x)$ that we had before, and, hence we arrive at the same final answer as before. See Figure 3.4.39.

We check our answer as we have so many times before. We start with the point $\left(1, \frac{1}{2}\right)$ and substitute $x = 1$ into $y = \frac{1-f(x)}{2}$ to get $y = \frac{1-f(1)}{2}$. From the graph of f, we know $f(1) = 0$, so we get $y = \frac{1-f(1)}{2} = \frac{1-0}{2} = \frac{1}{2}$. This proves $\left(1, \frac{1}{2}\right)$ is on the graph of $y = \frac{1-f(x)}{2}$. We invite the reader to check the remaining points.

Note that in the preceding example, since none of the transformations included adjusting the x-coordinates of points, the vertical asymptote, $x = 0$ remained in place.

(b) As with the previous example, we first rewrite $F(x) = f\left(\frac{1-x}{2}\right) = F\left(-\frac{1}{2}x + \frac{1}{2}\right)$. Here again, we have two modifications to the formula $f(x)$, the $-\frac{1}{2}$ multiple indicating a horizontal scaling and a reflection across the y-axis and a horizontal shift.

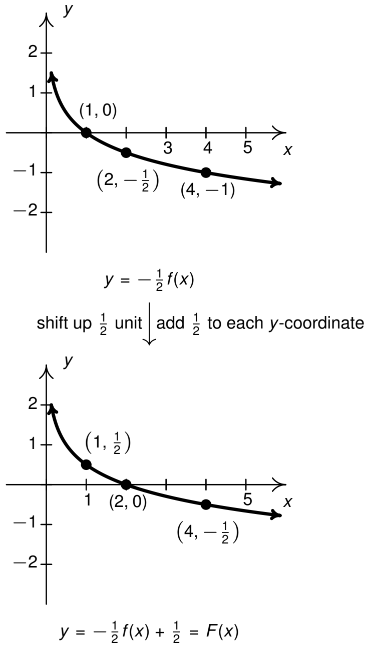


Figure 3.4.36

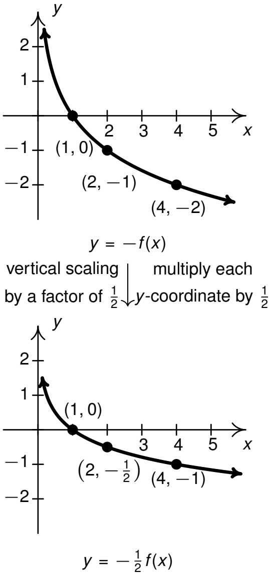


Figure 3.4.38

Based on our experience from previous examples, we do the horizontal shift first, with the order of the scaling and reflection more or less irrelevant.

To produce the graph of $y = f\left(x + \frac{1}{2}\right)$ we subtract $\frac{1}{2}$ from each of the x-coordinates of each of the points on the graph of $y = f(x)$. This moves the graph to the left $\frac{1}{2}$ unit, including the vertical asymptote $x = 0$ which moves to $x = -\frac{1}{2}$. See Figure 3.4.40.

Next, we graph $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ starting with $y = f\left(x + \frac{1}{2}\right)$ by horizontally expanding the graph by a factor of 2. That is, we multiply each x-coordinates on the graph of $y = f\left(x + \frac{1}{2}\right)$ by 2, including the vertical asymptote, $x = -\frac{1}{2}$ which moves to $x = 2\left(-\frac{1}{2}\right) = -1$. See Figure 3.4.41.

Finally, we reflect the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ about the y-axis to graph $y = f\left(-\frac{1}{2}x + \frac{1}{2}\right)$. We accomplish this by multiplying each of the x-coordinates of each of the points on the graph of $y = f\left(\frac{1}{2}x + \frac{1}{2}\right)$ by -1. This includes the vertical asymptote which is moved to $x = (-1)(-1) = 1$. See Figure 3.4.42.

To check our answer, we begin with the point $(-1, 0)$ and substitute $x = -1$ into $y = f\left(\frac{1-x}{2}\right)$. We get $y = f\left(\frac{1-(-1)}{2}\right) = f\left(\frac{2}{2}\right) = f(1)$. From the graph of f, we know $f(1) = 0$, hence we have $y = f(1) = 0$, proving $(-1, 0)$ is on the graph of $y = f\left(\frac{1-x}{2}\right)$. The reader is encouraged to check the remaining points.

As mentioned previously, instead of doing the horizontal scaling first, then the reflection, we could have done the reflection first, then the scaling. We leave this to the reader to check.

5. To write $g(x)$ in terms of $f(x)$, we assume we can

find real numbers a, b, h, and k and choose signs $\pm$ so that $g(x) = \pm af(\pm bx - h) + k$.

The most notable change we see is the vertical asymptote $x = 0$ has moved to $x = 2$. Moreover, instead of the graph increasing off to the right, it is decreasing coming in from the left. This suggests a horizontal shift of 2 units as well as a reflection across the y-axis.

Since we always shift first then reflect, we have a shift *left* of 2 units followed by a reflection about the y-axis. In other words, $g(x) = \pm af(-x+2)+k$.

Comparing y-values, the y-values on the graph of g appear to be exactly twice the corresponding values on the graph of f, indicating a vertical stretch by a factor of 2. Hence, we get $g(x) = 2f(-x + 2)$. We leave it to the reader to check the graph of $y = 2f(-x + 2)$ matches the graph of $y = g(x)$.

To write $f(x)$ in terms of $g(x)$, we reverse the steps done in obtaining the graph of $g(x)$ from $f(x)$ in the reverse order.

Since to get from the graph of f to the graph of g, we: first, shifted left 2 units; second reflected across the y-axis; third, vertically stretched by a factor of 2, our first step in taking g back to f is to implement a vertical compression by a factor of 2. Hence, starting with the graph of $y = g(x)$, our first step results in the formula $y = \frac{1}{2}g(x)$.

Next, we need to undo the reflection abut the y-axis. If the point (a, b) is reflected about the y-axis, we obtain the point $(-a, b)$. To return to the point (a, b), we reflect $(-a, b)$ across the y-axis again: $(-(-a), b) = (a, b)$. Hence, we take the graph of $y = \frac{1}{2}g(x)$ and reflect it across the y-axis to obtain $y = \frac{1}{2}g(-x)$.

Our last step is to undo a horizontal shift to the

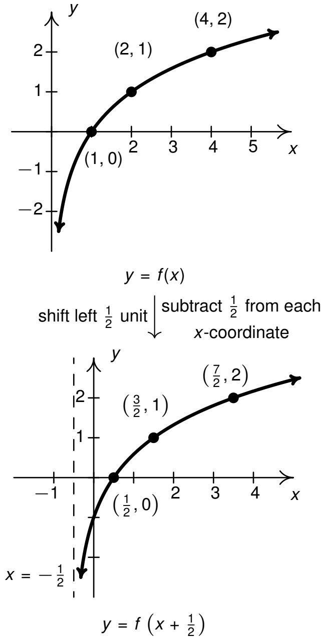


Figure 3.4.40

left 2 units. The reverse of this process is shifting the graph to the *right* two units, so we get $y = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$.[j]

We leave it to the reader to start with the graph of $y = g(x)$ and check the graph of $y = \frac{1}{2}g(-x+2)$ matches the graph of $y = f(x)$. □

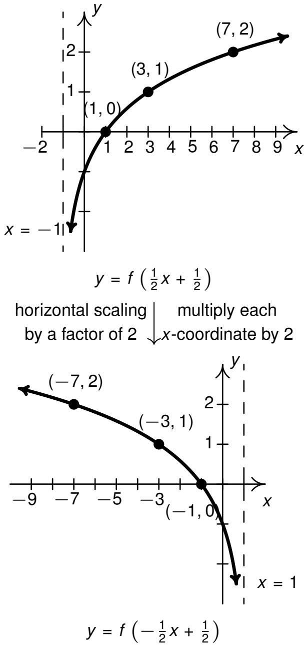


Figure 3.4.42

[j] To see this better, let $F(x) = \frac{1}{2}g(-x)$. Per Theorem 3.4.2, the graph of $F(x-2) = \frac{1}{2}g(-(x-2)) = \frac{1}{2}g(-x+2)$ is the same as the graph of F but shifted 2 units to the right.

[k] See the remarks at the beginning of the section.

[l] See Exercise 72.

3.4.4 Transformations in Sequence

Now that we have studied three basic classes of transformations: shifts, reflections, and scalings, we present a result below which provides one algorithm to follow to transform the graph of $y = f(x)$ into the graph of $y = af(bx-h)+k$ without the need of using Theorems 3.4.1, 3.4.2, 3.4.3, 3.4.4 and 3.4.5 individually.

Theorem 3.4.6 is the ultimate generalization of Theorems **??**, **??**, **??**, 1.1.1, 2.1.1 and 2.2.2. We note the underlying assumption here is that regardless of the order or number of shifts, reflections and scalings applied to the graph of a function f, we can always represent the final result in the form $g(x) = af(bx-h)+k$. Since each of these transformations can ultimately be traced back to composing f with linear functions,[k] this fact is verified by showing compositions of linear functions results in a linear function.[l]

Theorem 3.4.6. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.
   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.
   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.
   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.
4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.
   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

Theorem 3.4.6 can be established by generalizing the techniques developed in this section. Suppose $(c, f(c))$ is on the graph of f. To match up the inputs of $f(bx - h)$ and $f(c)$, we solve $bx - h = c$ and solve.

We first add the h (causing the horizontal shift) and then divide by b. If b is a positive number, this induces only a horizontal scaling by a factor of $\frac{1}{b}$. If $b < 0$, then we have a factor of -1 in play, and dividing by it induces a reflection about the y-axis. So we have $x = \frac{c+h}{b}$ as the input to g which corresponds to the input $x = c$ to f.

We now evaluate $g\left(\frac{c+h}{b}\right) = af\left(b \cdot \frac{c+h}{b} - h\right) + K = af(c + h - h) = af(c) + k$. We notice that the output from f is first multiplied by a. As with the constant b, if $a > 0$, this induces only a vertical scaling. If $a < 0$, then the -1 induces a reflection across the x-axis. Finally, we add k to the result, which is our vertical shift.

A less precise, but more intuitive way to paraphrase Theorem 3.4.6 is to think of the quantity $bx - h$ is the 'inside' of the function f. What's happening inside f affects the inputs or x-coordinates of the points on the graph of f. To find the x-coordinates of the corresponding points on g, we undo what has been done to x in the same way we would solve an equation.

What's happening to the output can be thought of as things happening 'outside' the function, f. Things happening outside affect the outputs or y-coordinates of the points on the graph of f. Here, we follow the usual order of operations to simplify the new y-value: we first multiply by a then add k to find the corresponding y-coordinates on the graph of g.

It needs to be stressed that our approach to handling multiple transformations, as summarized in Theorem 3.4.6 is only one approach. Your instructor may have a different algorithm. As always, the more you understand, the less you'll ultimately need to memorize, so whatever algorithm you choose to follow, it is worth thinking through each step both algebraically and geometrically.

We make good use of Theorem 3.4.6 in the following example.

Example 3.4.4. Figure 3.4.43 shows the complete graph of $y = f(x)$. Use Theorem 3.4.6 to graph $g(x) = \dfrac{4 - 3f(1 - 2x)}{2}$.

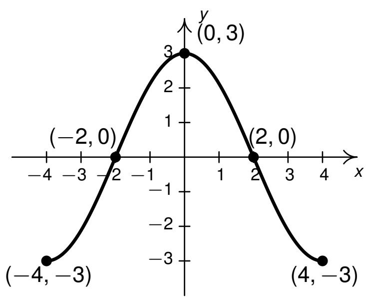


Figure 3.4.43

Solution. We use Theorem 3.4.6 to track the five 'key points' $(-4, -3)$, $(-2, 0)$, $(0, 3)$, $(2, 0)$ and $(4, -3)$ indicated on the graph of f to their new locations.

We first rewrite $g(x)$ in the form presented in Theorem 3.4.6, $g(x) = -\frac{3}{2}f(-2x + 1) + 2$. We set $-2x + 1$ equal to the x-coordinates of the key points and solve.

For example, solving $-2x + 1 = -4$, we first subtract 1 to get $-2x = -5$ then divide by -2 to get $x = \frac{5}{2}$. Subtracting the 1 is a horizontal shift to the left 1 unit. Dividing by -2 can be thought of as a two step process: dividing by

$(c, f(c))$	c	$-2x+1=c$	x
$(-4,-3)$	-4	$-2x+1=-4$	$x=\frac{5}{2}$
$(-2,0)$	-2	$-2x+1=-2$	$x=\frac{3}{2}$
$(0,3)$	0	$-2x+1=0$	$x=\frac{1}{2}$
$(2,0)$	2	$-2x+1=2$	$x=-\frac{1}{2}$
$(4,-3)$	4	$-2x+1=4$	$x=-\frac{3}{2}$

Table 3.4.10

2 which compresses the graph horizontally by a factor of 2 followed by dividing (multiplying) by -1 which causes a reflection across the y-axis. We summarize the results in Table 3.4.10.

Next, we take each of the x values and substitute them into $g(x) = -\frac{3}{2}f(-2x+1)+2$ to get the corresponding y-values. Substituting $x = \frac{5}{2}$, and using the fact that $f(-4) = -3$, we get

$$g\left(\frac{5}{2}\right) = -\frac{3}{2}f\left(-2\left(\frac{5}{2}\right)+1\right)+2 = -\frac{3}{2}f(-4)+2$$
$$= -\frac{3}{2}(-3)+2 = \frac{9}{2}+2 = \frac{13}{2}$$

We see that the output from f is first multiplied by $-\frac{3}{2}$. Thinking of this as a two step process, multiplying by $\frac{3}{2}$ then by -1, we have a vertical stretching by a factor of $\frac{3}{2}$ followed by a reflection across the x-axis. Adding 2 results in a vertical shift up 2 units. Continuing in this manner, we get Table 3.4.11.

x	$g(x)$	$(x, g(x))$
$\frac{5}{2}$	$\frac{13}{2}$	$\left(\frac{5}{2}, \frac{13}{2}\right)$
$\frac{3}{2}$	2	$\left(\frac{3}{2}, 2\right)$
$\frac{1}{2}$	$-\frac{5}{2}$	$\left(\frac{1}{2}, -\frac{5}{2}\right)$
$-\frac{1}{2}$	2	$\left(-\frac{1}{2}, 2\right)$
$-\frac{3}{2}$	$\frac{13}{2}$	$\left(-\frac{3}{2}, \frac{13}{2}\right)$

Table 3.4.11

To graph g, we plot each of the points in Table 3.4.11 and connect them in the same order and fashion as the points to which they correspond. Plotting f and g gives Figure 3.4.44 and Figure 3.4.45. □

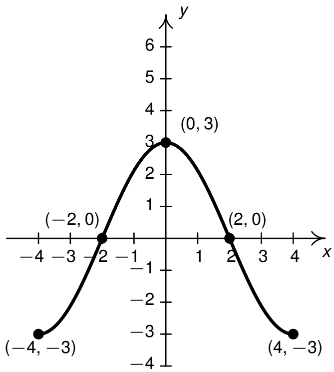


Figure 3.4.44

The reader is strongly encouraged to graph the series of functions which shows the gradual transformation of the graph of f into the graph of g in Example 3.4.4. We have

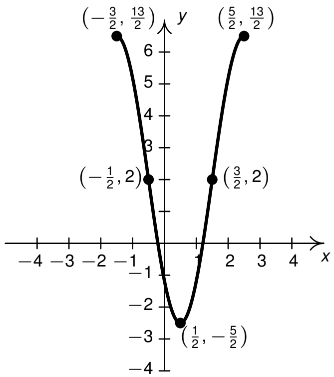


Figure 3.4.45

outlined the sequence of transformations in the above exposition; all that remains is to plot the five intermediate stages. Our next example turns the tables and asks for the formula of a function given a desired sequence of transformations.

Example 3.4.5. Let $f(x) = x^2 - |x|$. Find and simplify the formula of the function $g(x)$ whose graph is the result of the graph of $y = f(x)$ undergoing the following sequence of transformations. Check your answer to each step using a graphing utility.

1. Vertical shift up 2 units.
2. Reflection across the x-axis.
3. Horizontal shift right 1 unit.
4. Horizontal compression by a factor of 2.
5. Vertical shift up 3 units.
6. Reflection across the y-axis.

Solution. To help keep us organized we will label each intermediary function. The function g_1 will be the result of applying the first transformation to f. The function g_2 will be the result of applying the first two transformations to f—which is also the result of applying the second transformation to g_1, and so on.[m]

So, we can think of $g_0 = f$ and $g_6 = g$. m

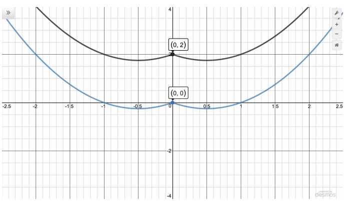


Figure 3.4.46: $y = f(x)$ (lighter color) and $y = g_1(x) = f(x) + 2$

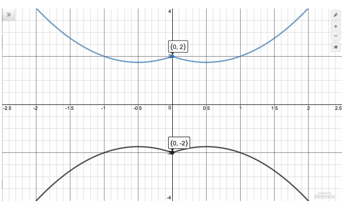


Figure 3.4.47: $y = g_1(x)$ (lighter color) and $y = g_2(x) = -g_1(x)$

1. Per Theorem 3.4.1, $g_1(x) = f(x) + 2 = x^2 - |x| + 2$. See Figure 3.4.46.
2. Per Theorem 3.4.3, $g_2(x) = -g_1(x) = -[x^2 - |x| + 2] = -x^2 + |x| - 2$. See Figure 3.4.47.
3. Per Theorem 3.4.2, $g_3(x) = g_2(x-1) = -(x-1)^2 + |x-1| - 2$. See Figure 3.4.48.
4. Per Theorem 3.4.5, $g_4(x) = g_3(2x) = -(2x-1)^2 + |2x-1| - 2$. See Figure 3.4.49.
5. Per Theorem 3.4.1, $g_5(x) = g_4(x) + 3 = -(2x-1)^2 + |2x-1| - 2 + 3 = -(2x-1)^2 + |2x-1| + 1$. See Figure 3.4.50.

6. Per Theorem 3.4.3, $g_6(x) = g_5(-x)$. See Figure 3.4.51.

$$\begin{aligned}
g_6(x) &= g_5(-x) \\
&= -(2(-x)-1)^2 + |2(-x)-1| + 1 \\
&= -(-2x-1)^2 + |-2x-1| + 1 \\
&= -[(-1)(2x+1)]^2 + |[(-1)(2x+1)| + 1 \\
&= -(-1)^2(2x+1)^2 + |-1||2x+1| + 1 \\
&= -(2x+1)^2 + |2x+1| + 1
\end{aligned}$$

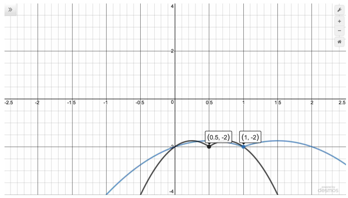


Figure 3.4.49: $y = g_3(x)$ (lighter color) and $y = g_4(x) = g_3(2x)$

Hence, $g(x) = g_6(x) = -(2x+1)^2 + |2x+1| + 1$. □

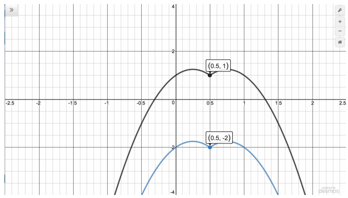


Figure 3.4.50: $y = g_4(x)$ (lighter color) and $y = g_5(x) = g_4(x) + 3$

It is instructive to show that the expression $g(x)$ in Example 3.4.4 can be written as $g(x) = af(bx - h) + k$.

One way is to compare the graphs of f and g and work backwards. A more methodical way is to repeat the work of Example 3.4.4, but never substitute the formula for $f(x)$ as follows:

1. Per Theorem 3.4.1, $g_1(x) = f(x) + 2$.
2. Per Theorem 3.4.3, $g_2(x) = -g_1(x) = -[f(x)+2] = -f(x) - 2$.
3. Per Theorem 3.4.2, $g_3(x) = g_2(x-1) = -f(x-1) - 2$.
4. Per Theorem 3.4.5, $g_4(x) = g_3(2x) = -f(2x-1) - 2$.
5. Per Theorem 3.4.1, $g_5(x) = g_4(x) + 3 = -f(2x-1) - 2 + 3 = -f(2x-1) + 1$.
6. Per Theorem 3.4.3, $g_6(x) = g_5(-x) = -f(2(-x)-1) + 1 = -f(-2x-1) + 1$.

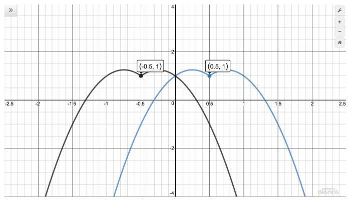


Figure 3.4.51: $y = g_5(x)$ (lighter color) and $y = g_6(x) = g_5(-x)$

Hence $g(x) = -f(-2x-1) + 1$. Note we can show f is even,[n] so $f(-2x-1) = f(-(2x+1)) = f(2x+1)$ and obtain $g(x) = -f(2x+1) + 1$.

At the beginning of this section, we discussed how all of the transformations we'd be discussing are the result of

n Recall this means $f(-x) = f(x)$.

composing given functions with linear functions. Not all transformations, not even all rigid transformations,[o] fall into these categories.

o See Section **??**.

For example, consider the graphs of $y = f(x)$ and $y = g(x)$ in Figure 3.4.52.

In Exercise 76, we explore a non-linear transformation and revisit the pair of functions f and g then.

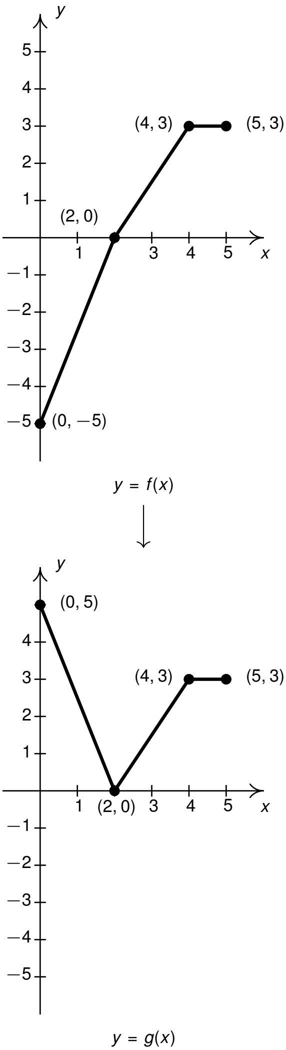


Figure 3.4.52

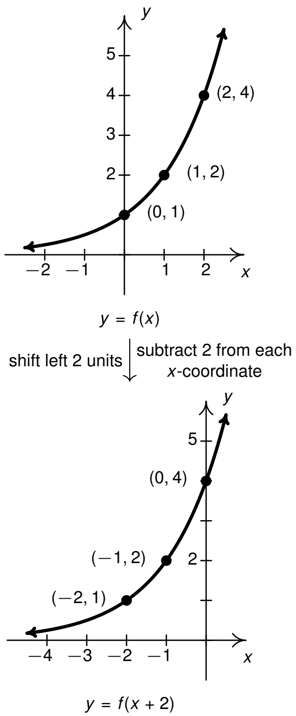

y
5
4
3
2
(2, 4)
(1, 2)
(0, 1)
−2
−1
1
2
x
$y = f(x)$
shift left 2 units
subtract 2 from each
x-coordinate
y
5
(0, 4)
(−1, 2)
2
(−2, 1)
−4
−3
−2
−1
x
$y = f(x + 2)$

Figure 3.4.21

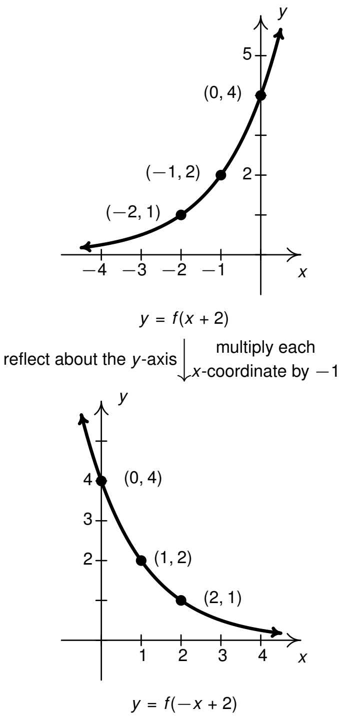

y
5
(0, 4)
(−1, 2)
2
(−2, 1)
−4 −3 −2 −1
x
y = f(x + 2)
reflect about the y-axis
multiply each
x-coordinate by −1
y
4
(0, 4)
3
2
(1, 2)
(2, 1)
1 2 3 4
x
y = f(−x + 2)

Figure 3.4.22

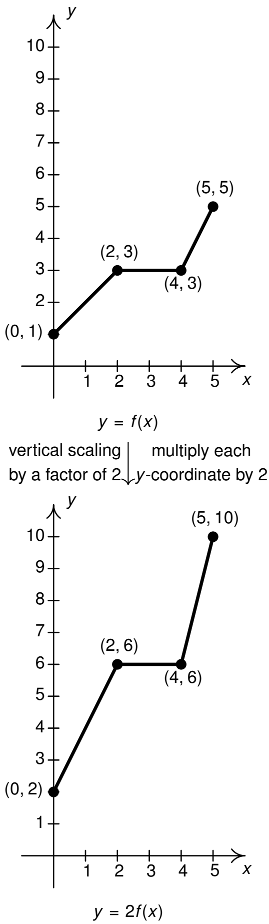

y
10
9
8
7
6
5
4
3
2
(5, 5)
(2, 3)
(4, 3)
(0, 1)
1
2
3
4
5
x
y = f(x)
vertical scaling
by a factor of 2
multiply each
y-coordinate by 2
y
(5, 10)
10
9
8
7
6
5
4
3
1
(2, 6)
(4, 6)
(0, 2)
1
2
3
4
5
x
y = 2f(x)

Figure 3.4.26

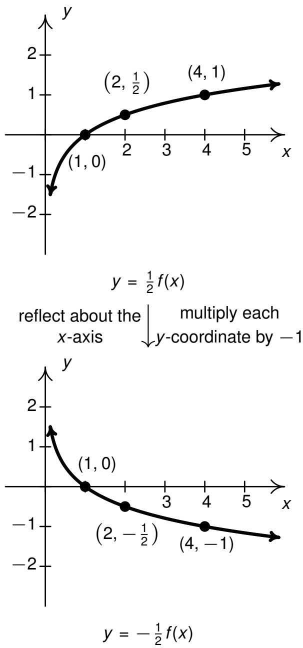

y
2
1
(2, 1/2)
(4, 1)
(1, 0)
2 3 4 5
x
−1
−2
y = 1/2 f(x)
reflect about the x-axis
multiply each y-coordinate by −1
y
2
1
(1, 0)
3 4 5
x
−1
(2, −1/2)
(4, −1)
−2
y = −1/2 f(x)

Figure 3.4.35

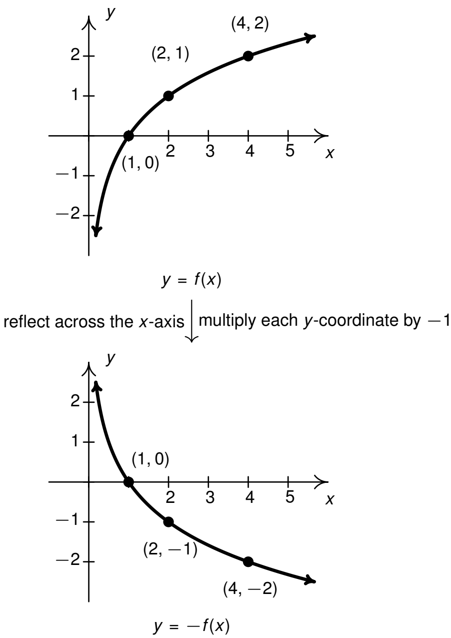

y
(4, 2)
(2, 1)
2
1
(1, 0)
2
3
4
5
x
−1
−2
y = f(x)
reflect across the x-axis
multiply each y-coordinate by −1
y
2
1
(1, 0)
2
3
4
5
x
−1
−2
(2, −1)
(4, −2)
y = −f(x)

Figure 3.4.37

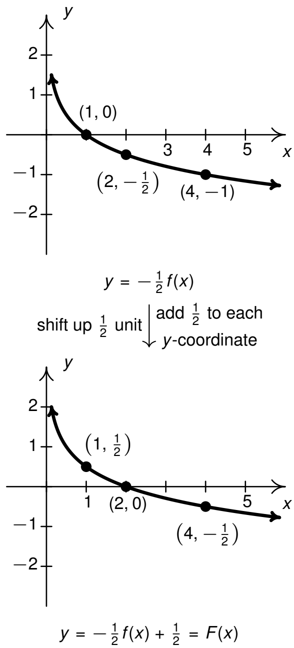

y
2
1
(1, 0)
3
4
5
x
−1
−2
(2, −1/2)
(4, −1)
y = −1/2 f(x)
shift up 1/2 unit
add 1/2 to each
y-coordinate
y
2
1
(1, 1/2)
1
(2, 0)
5
x
−1
−2
(4, −1/2)
y = −1/2 f(x) + 1/2 = F(x)

Figure 3.4.39

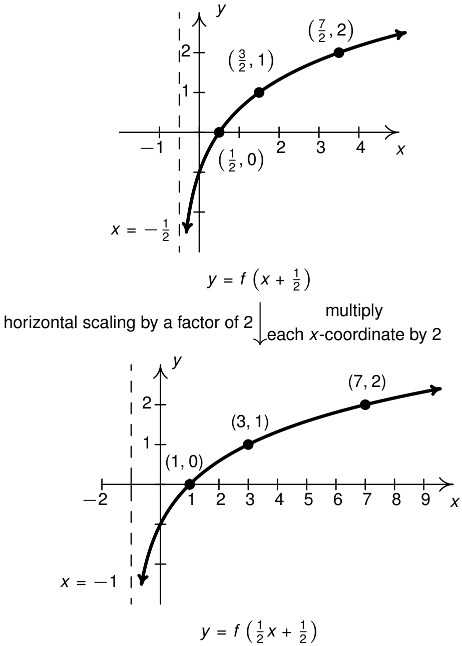


Figure 3.4.41

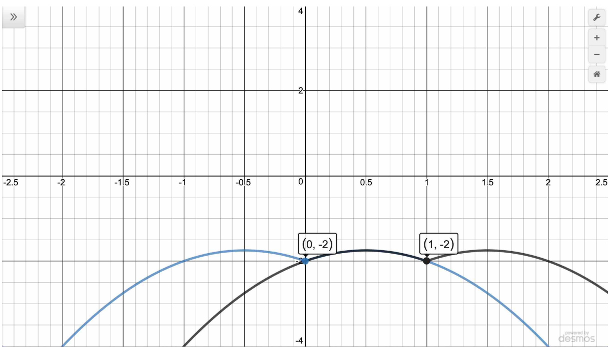


Figure 3.4.48: $y = g_2(x)$ (lighter color) and $y = g_3(x) = g_2(x-1)$

3.4.5 Exercises

Suppose $(2, -3)$ is on the graph of $y = f(x)$. In Exercises 1 - 18, use Theorem 3.4.6 to find a point on the graph of the given transformed function.

1. $y = f(x) + 3$
2. $y = f(x + 3)$
3. $y = f(x) - 1$
4. $y = f(x - 1)$
5. $y = 3f(x)$
6. $y = f(3x)$
7. $y = -f(x)$
8. $y = f(-x)$
9. $y = f(x - 3) + 1$
10. $y = 2f(x + 1)$
11. $y = 10 - f(x)$
12. $y = 3f(2x) - 1$
13. $y = \frac{1}{2}f(4 - x)$
14. $y = 5f(2x + 1) + 3$
15. $y = 2f(1 - x) - 1$
16. $y = f\left(\dfrac{7 - 2x}{4}\right)$
17. $y = \dfrac{f(3x) - 1}{2}$
18. $y = \dfrac{4 - f(3x - 1)}{7}$

The complete graph of $y = f(x)$ is given in Figure 3.4.53. In Exercises 19 - 27, use it and Theorem 3.4.6 to graph the given transformed function.

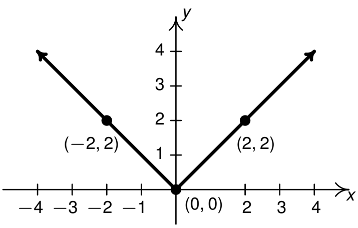


Figure 3.4.53

19. $y = f(x) + 1$
20. $y = f(x) - 2$
21. $y = f(x + 1)$
22. $y = f(x - 2)$
23. $y = 2f(x)$
24. $y = f(2x)$

25. $y = 2 - f(x)$

26. $y = f(2 - x)$

27. $y = 2 - f(2 - x)$

28. Some of the answers to Exercises 19 - 27 above should be the same. Which ones match up? What properties of the graph of $y = f(x)$ contribute to the duplication?

29. The function f used in Exercises 19 - 27 should look familiar. What is $f(x)$? How does this this explain some of the duplication in the answers to Exercises 19 - 27 mentioned in Exercise 28?

The complete graph of $y = g(t)$ is given in Figure 3.4.54. In Exercises 30 - 38, use it and Theorem 3.4.6 to graph the given transformed function.

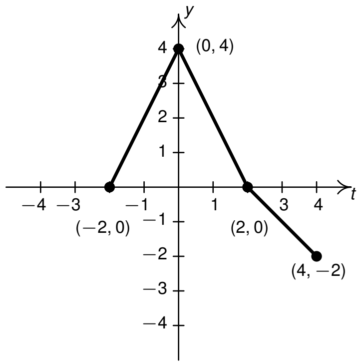


Figure 3.4.54

30. $y = g(t) - 1$

31. $y = g(t + 1)$

32. $y = \frac{1}{2}g(t)$

33. $y = g(2t)$

34. $y = -g(t)$

35. $y = g(-t)$

36. $y = g(t + 1) - 1$

37. $y = 1 - g(t)$

38. $y = \frac{1}{2}g(t + 1) - 1$

The complete graph of $y = f(x)$ is given in Figure 3.4.55. In Exercises 39 - 50, use it and Theorem 3.4.6 to graph the given transformed function.

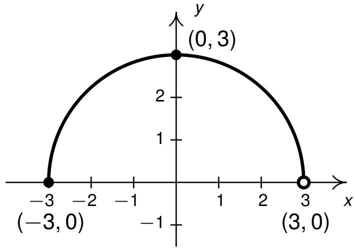


Figure 3.4.55

39. $g(x) = f(x) + 3$

40. $h(x) = f(x) - \frac{1}{2}$

41. $j(x) = f\left(x - \frac{2}{3}\right)$

42. $a(x) = f(x + 4)$

43. $b(x) = f(x + 1) - 1$

44. $c(x) = \frac{3}{5}f(x)$

45. $d(x) = -2f(x)$

46. $k(x) = f\left(\frac{2}{3}x\right)$

47. $m(x) = -\frac{1}{4}f(3x)$

48. $n(x) = 4f(x-3) - 6$

49. $p(x) = 4 + f(1-2x)$

50. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3$

The complete graph of $y = S(t)$ is given in Figure 3.4.56.
The purpose of Exercises 51 - 54 is to build up to the graph of $y = \frac{1}{2}S(-t+1)+1$ one step at a time.

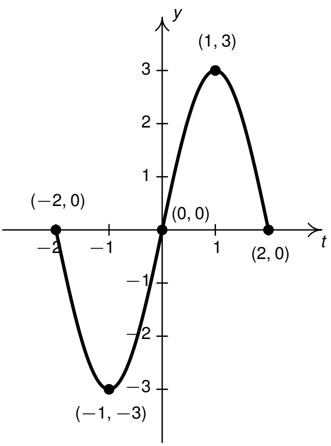


Figure 3.4.56: The graph of $y = S(t)$

51. $y = S_1(t) = S(t+1)$

52. $y = S_2(t) = S_1(-t) = S(-t+1)$

53. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$

54. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$

Let $f(x) = \sqrt{x}$. For Exercise 55 to 64 Find a formula for a function g whose graph is obtained from f from the given sequence of transformations.

55. (1) shift right 2 units; (2) shift down 3 units

56. (1) shift down 3 units; (2) shift right 2 units

57. (1) reflect across the x-axis; (2) shift up 1 unit

58. (1) shift up 1 unit; (2) reflect across the x-axis

59. (1) shift left 1 unit; (2) reflect across the y-axis; (3) shift up 2 units

60. (1) reflect across the y-axis; (2) shift left 1 unit; (3) shift up 2 units

61. (1) shift left 3 units; (2) vertical stretch by a factor of 2; (3) shift down 4 units

62. (1) shift left 3 units; (2) shift down 4 units; (3) vertical stretch by a factor of 2

63. (1) shift right 3 units; (2) horizontal shrink by a factor of 2; (3) shift up 1 unit

64. (1) horizontal shrink by a factor of 2; (2) shift right 3 units; (3) shift up 1 unit

65. $y = g(x)$ See Figure 3.4.58.

66. $y = h(x)$. See Figure 3.4.59

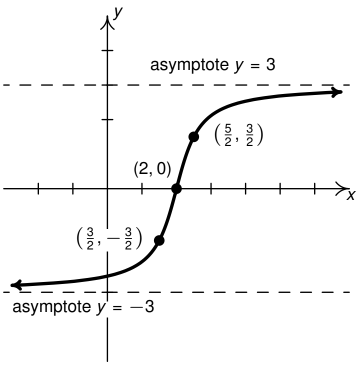


Figure 3.4.59

67. $y = p(x)$. See Figure 3.4.60.

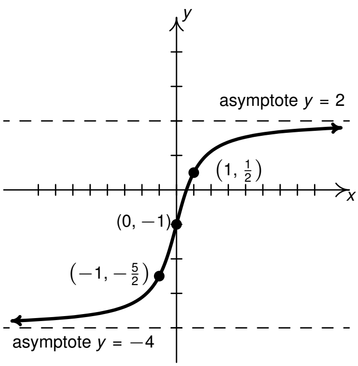


Figure 3.4.60

68. $y = q(x)$. See Figure 3.4.61.

For Exercises 65 - 70, use the graph of $y = f(x)$ given in Figure 3.4.57 to write each function in terms of $f(x)$.

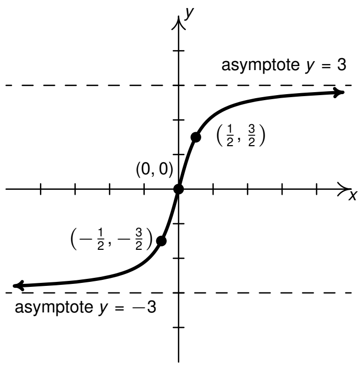


Figure 3.4.57: The graph of $y = f(x)$ for Ex. 65 - 70

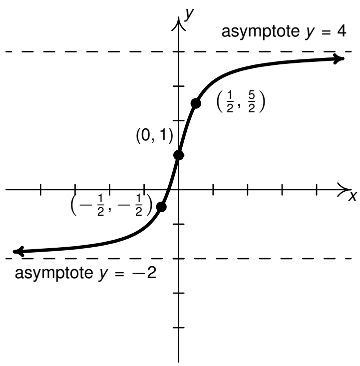


Figure 3.4.58

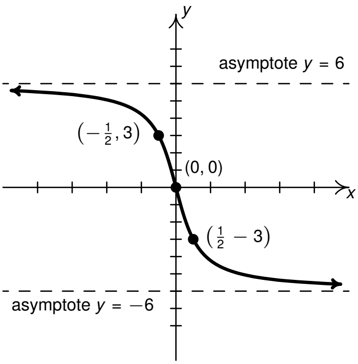


Figure 3.4.61

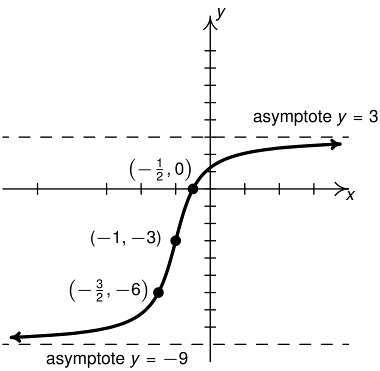


Figure 3.4.62

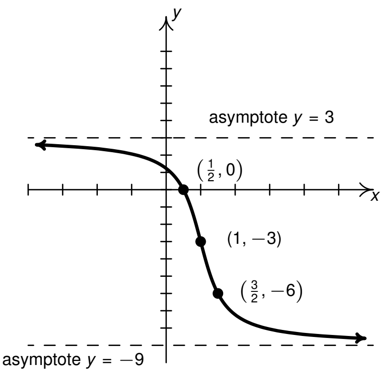


Figure 3.4.63

69. $y = r(x)$. See Figure 3.4.62.

70. $y = s(x)$. See Figure 3.4.63.

71. The graph of $y = f(x) = \sqrt[3]{x}$ is given in Figure 3.4.64 and the graph of $y = g(x)$ is given in Figure 3.4.65. Find a formula for g based on transformations of the graph of f. Check your answer by confirming that the points shown on the graph of g satisfy the equation $y = g(x)$.

72. Show that the composition of two linear functions is a linear function. Hence any (finite) sequence of transformations discussed in this section can be combined into the form given in Theorem 3.4.6.

    (HINT: Let $f(x) = ax + b$ and $g(x) = cx + d$. Find $(f \circ g)(x)$.)

73. For many common functions, the properties of Algebra make a horizontal scaling the same as a vertical scaling by (possibly) a different factor. For example, $\sqrt{9x} = 3\sqrt{x}$, so a horizontal compression of $y = \sqrt{x}$ by a factor of 9 results in the same graph as a vertical stretch of $y = \sqrt{x}$ by a factor of 3.

    With the help of your classmates, find the equivalent vertical scaling produced by the horizontal scalings $y = (2x)^3$, $y = |5x|$, $y = \sqrt[3]{27x}$ and $y = \left(\frac{1}{2}x\right)^2$.

    What about $y = (-2x)^3$, $y = |-5x|$, $y = \sqrt[3]{-27x}$ and $y = \left(-\frac{1}{2}x\right)^2$?

74. Discuss the following questions with your classmates.

    - If f is even, what happens when you reflect the graph of $y = f(x)$ across the y-axis?
    - If f is odd, what happens when you reflect the graph of $y = f(x)$ across the y-axis?

- If f is even, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
- If f is odd, what happens when you reflect the graph of $y = f(x)$ across the x-axis?
- How would you describe symmetry about the origin in terms of reflections?

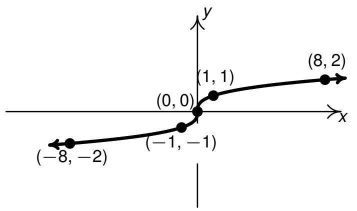


Figure 3.4.64: $y = \sqrt[3]{x}$

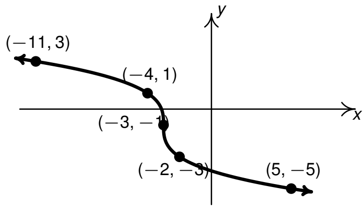


Figure 3.4.65: $y = g(x)$

75. We mentioned earlier in the section that, in general, the order in which transformations are applied matters, yet in our first example with two transformations the order did not matter. (You could perform the shift to the left followed by the shift down or you could shift down and then left to achieve the same result.) With the help of your classmates, determine the situations in which order does matter and those in which it does not.

76. This Exercise is a follow-up to Exercise **??** in Section **??**.

    (a) For each of the following functions, use a graphing utility to compare the graph of $y = f(x)$ with the graphs of $y = |f(x)|$ and $y = f(|x|)$.

    - $f(x) = 3 - x$
    - $f(x) = x^2 - x - 6$
    - $f(x) = \sqrt{x+3} - 1$

    (b) In general, how does the graph of $y = |f(x)|$ compare with that of $y = f(x)$? What about the graph of $y = f(|x|)$ and $y = f(x)$?

    (c) Referring to the functions f and g graphed on page 272, write g in terms of f.

3.4.6 Answers

1. $(2, 0)$
2. $(-1, -3)$
3. $(2, -4)$
4. $(3, -3)$
5. $(2, -9)$
6. $\left(\frac{2}{3}, -3\right)$
7. $(2, 3)$
8. $(-2, -3)$
9. $(5, -2)$
10. $(1, -6)$
11. $(2, 13)$
12. $y = (1, -10)$
13. $\left(2, -\frac{3}{2}\right)$
14. $\left(\frac{1}{2}, -12\right)$
15. $(-1, -7)$
16. $\left(-\frac{1}{2}, -3\right)$
17. $\left(\frac{2}{3}, -2\right)$
18. $(1, 1)$
19. $y = f(x) + 1$. See Figure 3.4.66.
20. $y = f(x) - 2$. See Figure 3.4.67.
21. $y = f(x + 1)$. See Figure 3.4.68.
22. $y = f(x - 2)$. See Figure 3.4.69
23. $y = 2f(x)$. See Figure 3.4.70.
24. $y = f(2x)$. See Figure 3.4.71
25. $y = 2 - f(x)$. See Figure 3.4.72.

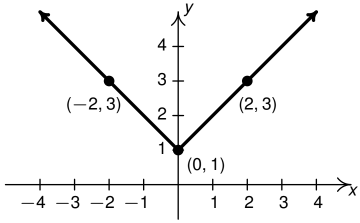


Figure 3.4.66

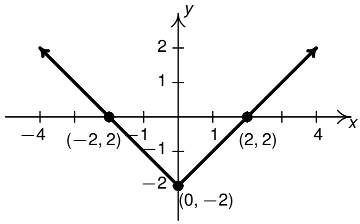


Figure 3.4.67

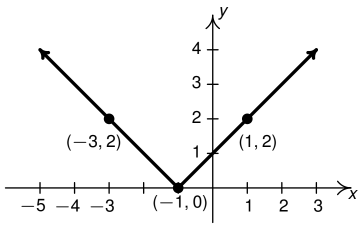


Figure 3.4.68

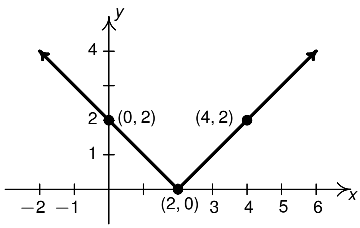


Figure 3.4.69

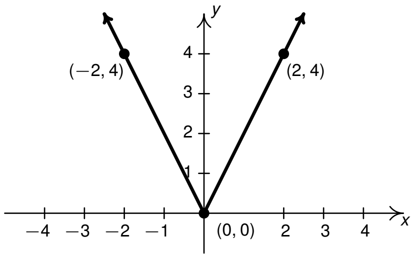


Figure 3.4.70

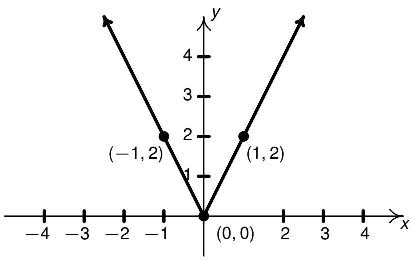


Figure 3.4.71

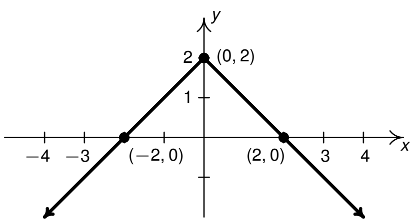


Figure 3.4.72

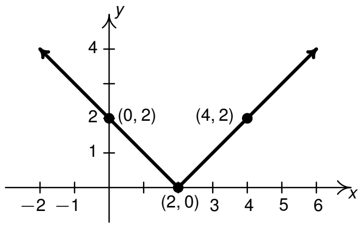


Figure 3.4.73

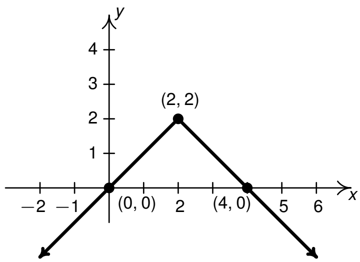


Figure 3.4.74

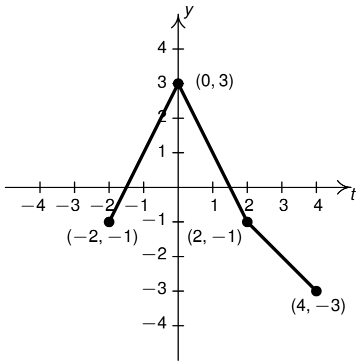


Figure 3.4.75

26. $y = f(2 - x)$. See Figure 3.4.73.

27. $y = 2 - f(2 - x)$. See Figure 3.4.74.

28. $y = g(t) - 1$. See Figure 3.4.75.

29. $y = g(t + 1)$. See Figure 3.4.76.

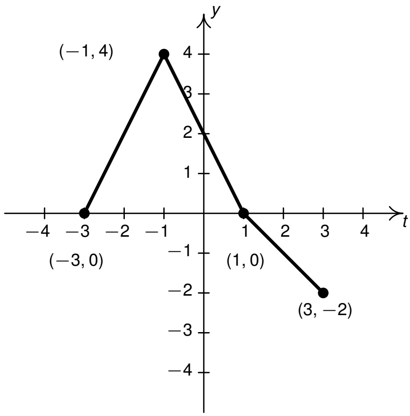


Figure 3.4.76

30. $y = \frac{1}{2}g(t)$. See Figure 3.4.77.

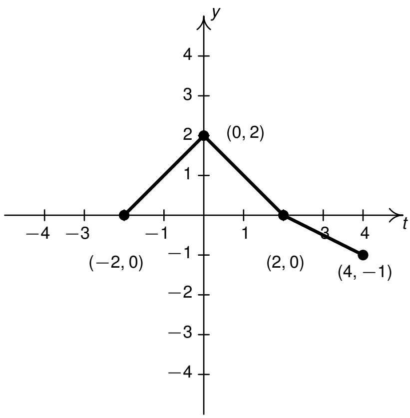


Figure 3.4.77

31. $y = g(2t)$. See Figure 3.4.78.

32. $y = -g(t)$. See Figure 3.4.79.

33. $y = g(-t)$. See Figure 3.4.80.

34. $y = g(t + 1) - 1$. See Figure 3.4.81.

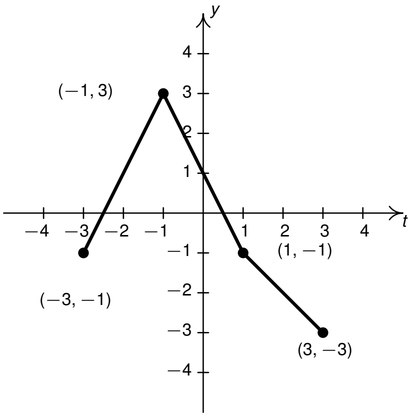


Figure 3.4.81

35. $y = 1 - g(t)$. See Figure 3.4.82.

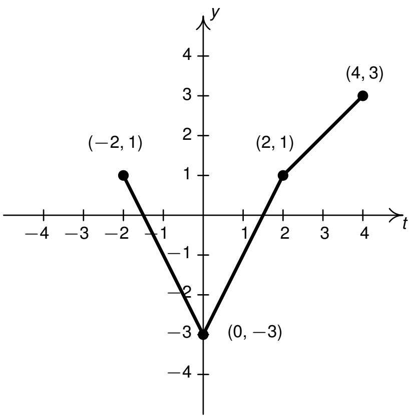


Figure 3.4.82

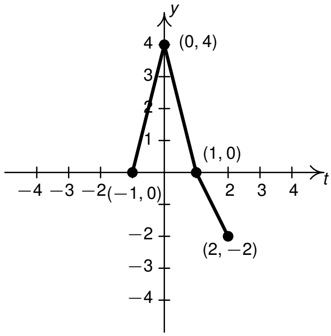


Figure 3.4.78

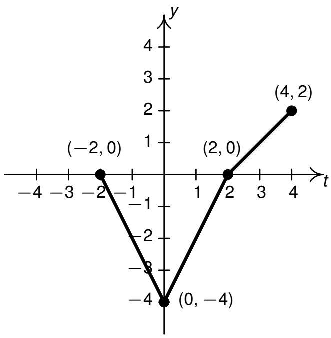


Figure 3.4.79

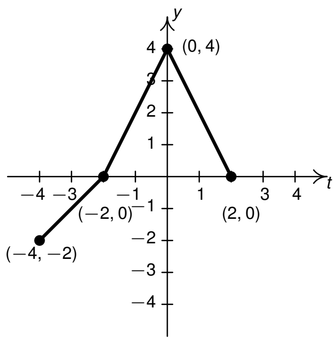


Figure 3.4.80

36. $y = \frac{1}{2}g(t+1) - 1$. See Figure 3.4.83.

37. $g(x) = f(x) + 3$. See Figure 3.4.84.

38. $h(x) = f(x) - \frac{1}{2}$. See Figure 3.4.85.

39. $j(x) = f\left(x - \frac{2}{3}\right)$. See Figure 3.4.86.

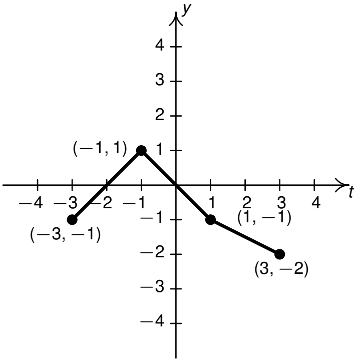


Figure 3.4.83

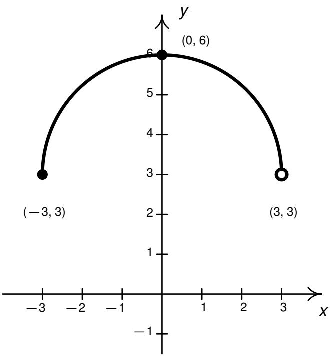


Figure 3.4.84

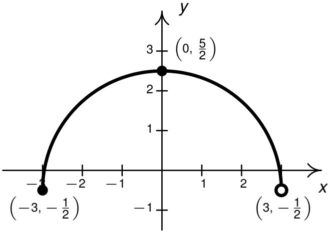


Figure 3.4.85

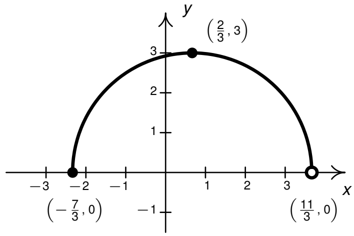


Figure 3.4.86

40. $a(x) = f(x+4)$. See Figure 3.4.87.

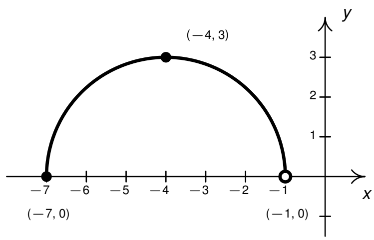


Figure 3.4.87

41. $b(x) = f(x+1) - 1$. See Figure 3.4.88

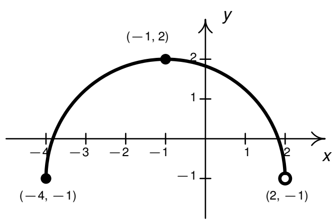


Figure 3.4.88

42. $c(x) = \frac{3}{5}f(x)$. See Figure 3.4.89

43. $d(x) = -2f(x)$. See Figure 3.4.90

44. $k(x) = f\left(\frac{2}{3}x\right)$. See Figure 3.4.91.

45. $m(x) = -\frac{1}{4}f(3x)$. See Figure 3.4.92

46. $n(x) = 4f(x-3) - 6$. See **??**.

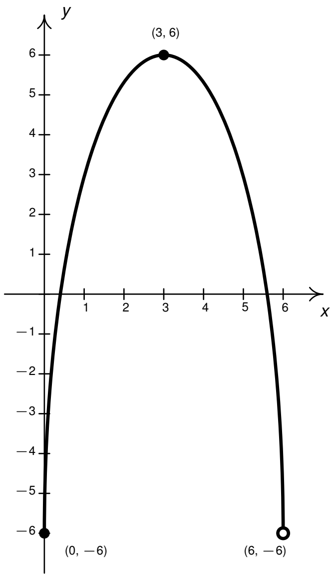


Figure 3.4.93

47. $p(x) = 4 + f(1-2x) = f(-2x+1) + 4$. See Figure 3.4.94

48. $q(x) = -\frac{1}{2}f\left(\frac{x+4}{2}\right) - 3 = -\frac{1}{2}f\left(\frac{1}{2}x + 2\right) - 3$. See Figure 3.4.95

49. $y = S_1(t) = S(t+1)$. See Figure 3.4.96.

50. $y = S_2(t) = S_1(-t) = S(-t+1)$. See Figure 3.4.97.

51. $y = S_3(t) = \frac{1}{2}S_2(t) = \frac{1}{2}S(-t+1)$. See Figure 3.4.98.

52. $y = S_4(t) = S_3(t) + 1 = \frac{1}{2}S(-t+1) + 1$. See Figure 3.4.99.

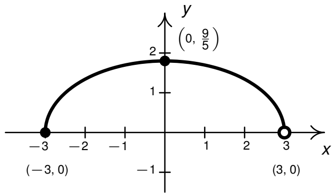


Figure 3.4.89

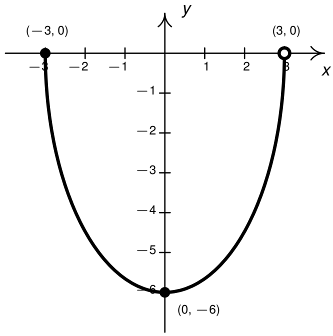


Figure 3.4.90

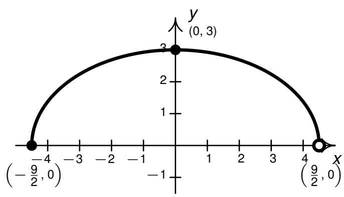


Figure 3.4.91

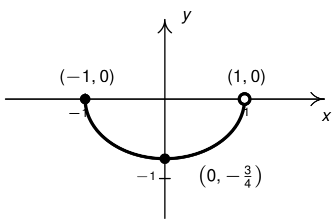


Figure 3.4.92

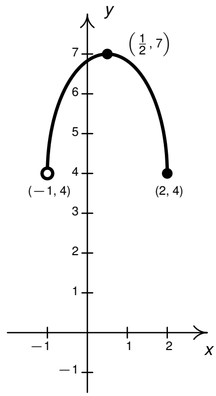


Figure 3.4.94

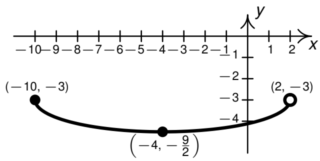


Figure 3.4.95

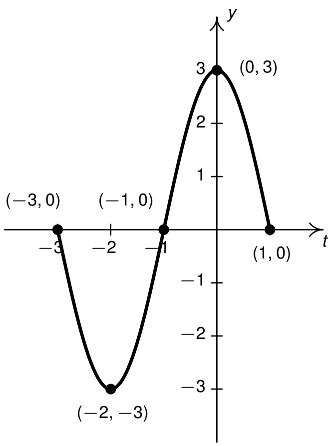


Figure 3.4.96

53. $g(x) = \sqrt{x-2} - 3$

54. $g(x) = \sqrt{x-2} - 3$

55. $g(x) = -\sqrt{x} + 1$

56. $g(x) = -(\sqrt{x} + 1) = -\sqrt{x} - 1$

57. $g(x) = \sqrt{-x+1} + 2$

58. $g(x) = \sqrt{-(x+1)} + 2 = \sqrt{-x-1} + 2$

59. $g(x) = 2\sqrt{x+3} - 4$

60. $g(x) = 2\left(\sqrt{x+3} - 4\right) = 2\sqrt{x+3} - 8$

61. $g(x) = \sqrt{2x-3} + 1$

62. $g(x) = \sqrt{2(x-3)} + 1 = \sqrt{2x-6} + 1$

63. $g(x) = f(x) + 1$

64. $h(x) = f(x-2)$

65. $p(x) = f\left(\frac{x}{2}\right) - 1$

66. $q(x) = -2f(x) = 2f(-x)$

67. $r(x) = 2f(x+1) - 3$

68. $s(x) = 2f(-x+1) - 3 = -2f(x-1) + 3$

69. $g(x) = -2\sqrt[3]{x+3} - 1$ or $g(x) = 2\sqrt[3]{-x-3} - 1$

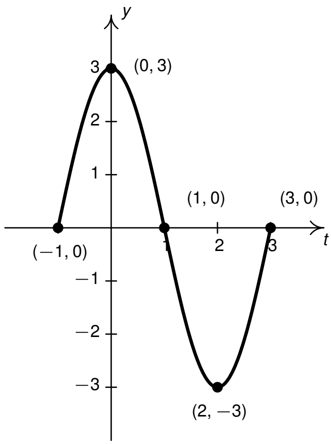


Figure 3.4.97

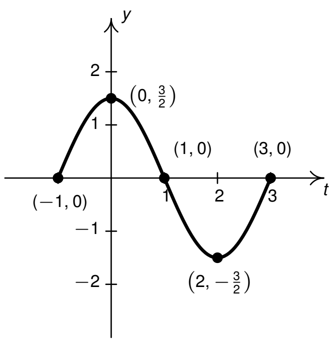


Figure 3.4.98

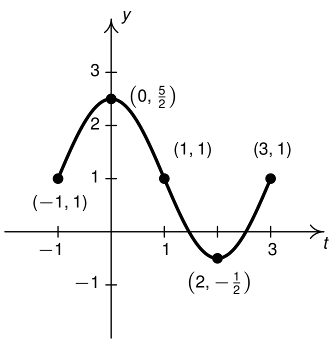


Figure 3.4.99

3.5 Relations and Implicit Functions

Up until now in this text, we have seen exclusively special kinds of mappings called *functions*. In this section, we broaden our horizons to study more general mappings called *relations*. The reader is encouraged to revisit Definition **??** in Section **??** before proceeding with **??** for *relation*.

Definition 3.5.1. Given two sets A and B, a **relation** from A to B is a process by which elements of A are matched with (or 'mapped to') elements of B.

Unlike Definition **??**, Definition 3.5.1 puts no conditions on the process which maps elements of A to elements of B. This means that while all functions are relations, not all relations need be functions. For example, consider the mappings f and g in Figure 3.5.1 and Figure 3.5.2 from Section **??**.

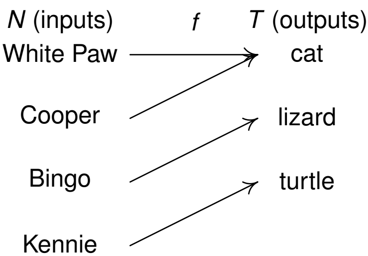


Figure 3.5.1

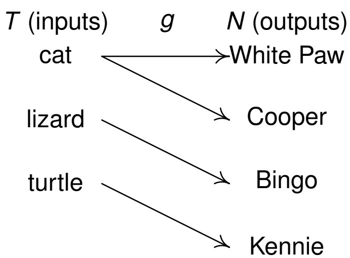


Figure 3.5.2

Both f and g are relations. More specifically, f is a *function* from N to T while g is merely *relation* from T to N. As with functions, we may describe general relations in a variety of different ways: verbally, as mapping diagrams, or a set of ordered pairs. For example, just as we may describe the function f above as

$$f = \{(\text{White Paw}, \text{cat}), (\text{Cooper}, \text{cat}), (\text{Bingo}, \text{lizard}), (\text{Kennie}, \text{turtle})\},$$

we may represent g as

$$g = \{(\text{cat}, \text{White Paw}), (\text{cat}, \text{Cooper}), (\text{lizard}, \text{Bingo}), (\text{turtle}, \text{Kennie})\}.$$

Note here the grammar 'g is a relation *from T to N*' is evidenced by the elements of T being listed first in the ordered pairs (i.e., the abscissae) and the elements of N being listed second (i.e., the ordinates.)

Unlike functions, we do not use function notation when describing the input/output relationship for general relations. For example, we may write 'f(White Paw) = cat' since f maps the input 'White Paw' to only one output, 'cat.' However, g(cat) is ambiguous since it could mean 'White Paw' or 'Cooper.'[a]

a In more advanced texts, we would write 'cat g White Paw' and 'cat g Cooper' to indicate g maps 'cat' to both 'White Paw' and 'Cooper.' Our study of relations, however, isn't deep enough to necessitate introducing and using this notation. Similarly, we won't introduce the notions of 'domain,' 'codomain,' and 'range' for relations, either.

As with functions, our focus in this course will rest with relations of real numbers. Consider the relation R described as follows:

$$R = \{(-1,3),(0,-3),(4,-2),(4,1)\}$$

Figure 3.5.3 shows a mapping diagram of R. However, since R relates real numbers, we can also create the graph of R in the same way we graphed functions—by interpreting the ordered pairs which comprise R as points in the plane. Since we have no context, we use the default labels 'x' for the horizontal axis and 'y' for the vertical axis. See Figure 3.5.12.

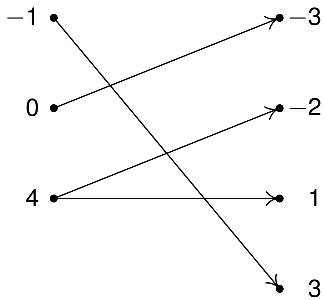


Figure 3.5.3: A Mapping Diagram of R

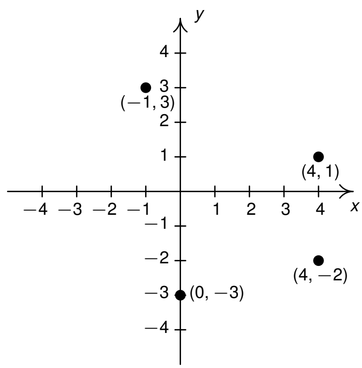


Figure 3.5.4: The graph of R

Our next example focuses on using relations to describe sets of points in the plane and vice-versa.

Example 3.5.1.

1. Graph the following relations.

   (a) $S = \{(k, 2^k) \mid k = 0, \pm1, \pm2\}$
   (b) $P = \{(j, j^2) \mid j \text{ is an integer}\}$
   (c) $V = \{(3, y) \mid y \text{ is a real number}\}$
   (d) $R = \{(x, y) \mid x \text{ is a real number}, 1 < y \leq 3\}$

2. Find a roster or set-builder description for each of the relations below.

   (a) See Figure 3.5.5.

   (b) See Figure 3.5.6.

   (c) See Figure 3.5.7.

   (d) See Figure 3.5.8.

Solution.

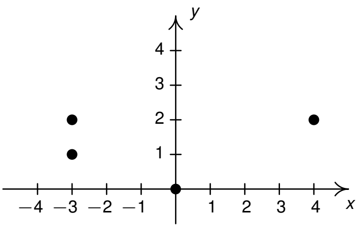


Figure 3.5.5: The graph of A

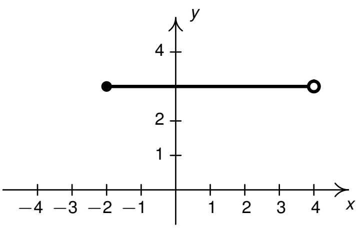


Figure 3.5.6: The graph of H

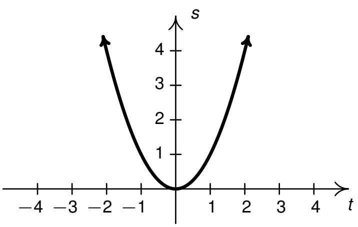


Figure 3.5.7: The graph of Q

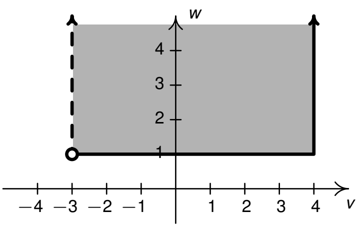


Figure 3.5.8: The graph of T

1. (a) The relation S is described using *set-builder notation.*[1] To generate the ordered pairs which belong to S, we substitute the given values of k, $k = 0, \pm 1, \pm 2$, into the formula $\left(k, 2^k\right)$.

   Starting with $k = 0$, we get $\left(0, 2^0\right) = (0,1)$. For $k = 1$, we get $\left(1, 2^1\right) = (1,2)$, and for $k = -1$, we get $\left(-1, 2^{-1}\right) = \left(-1, \frac{1}{2}\right)$. Continuing, we get $\left(2, 2^2\right) = (2,4)$ for $k = 2$ and, finally $\left(-2, 2^{-2}\right) = \left(-2, \frac{1}{4}\right)$ for $k = -2$. Hence, a roster description of S is $S = \left\{\left(-2, \frac{1}{4}\right), \left(-1,\right.\right.$

   When we graph S, we label the horizontal axis as the k-axis, since 'k' was the variable chosen used to generate the ordered pairs and keep the default label 'y' for the vertical axis. The graph of S is given in Figure 3.5.9.

   (b) To graph the relation $P = \left\{\left(j, j^2\right) \mid j \text{ is an integer}\right\}$, we proceed as above when we graphed the relation S. Here, j is restricted to being an integer, which means $j = 0, \pm 1, \pm 2$, etc.

   Plugging in these sample values for j, we obtain the ordered pairs $(0,0)$, $(1,1)$, $(-1,1)$, $(2,4)$, $(-2,4)$, etc. Since the variable j takes on only integer values, we could write P using the roster notation: $P = \{(0,0), (\pm 1, 1), (\pm 2, 4), \ldots\}$

   We plot a few of these points and use some periods of ellipsis to indicate the complete graph contains additional points not in the current field of view. The graph of P is given in Figure 3.5.10.

   (c) Next, we come to the relation V, described, once again, using set-builder notation. In this case, V consists of all ordered pairs of the form $(3, y)$ where y is free to be whatever real number we like, without any restriction.[2] For

[1] See Section **??** to review this, if needed.

[2] We'll revisit the concept of a 'free variable' in Section **??**.

example, $(3,0)$, $(3,-1)$, and $(3,117)$ all belong to V as do $\left(3, \frac{1}{2}\right)$, $(3,-1.0342)$, $(3,\sqrt{2})$, etc.

After plotting some sample points, becomes apparent that the ordered pairs which belong to V correspond to points which lie on the vertical line $x = 3$, and vice-versa. That is, every point on the line $x = 3$ has coordinates which correspond to an ordered pair belonging to V. The graph of V is in Figure 3.5.11.

(d) In the relation $R = \{(x,y) \mid 1 < y \leq 3\}$, we see y is restricted by the inequality $1 < y \leq 3$, but x is free to be whatever it likes.

Since x is unrestricted, this means whatever the graph of R is, it will extend indefinitely off to the right and left. The restriction $y > 1$ means all points on the graph of R have a y-coordinate larger than one, so they are *above* the horizontal line $y = 1$. The restriction $y \leq 3$, on the other hand, means all the points on the graph of R have a y-coordinate less than or equal to 3, meaning they are either *on* or *below* the horizontal line $y = 3$.

In other words, the graph of R is the region in the plane between $y = 1$ and $y = 3$, including $y = 3$ but not $y = 1$. We signify this by *shading* the region between these two horizontal lines.

How do we communicate $y = 1$ is not part of the graph? One way is to visualize putting 'holes' all along the line $y = 1$ to indicate this is not part of the graph. In practice, however, this looks cluttered and could be confusing. Instead, we 'dash' the line $y = 1$ as seen in Figure 3.5.12.

2. (a) Since A consists of finitely many points, we

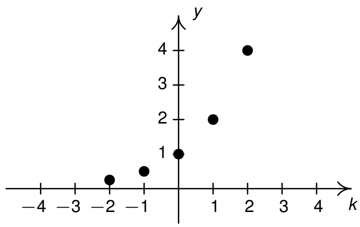


Figure 3.5.9: The graph of S

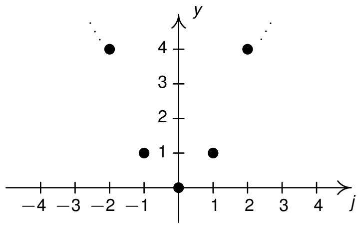


Figure 3.5.10: The graph of P

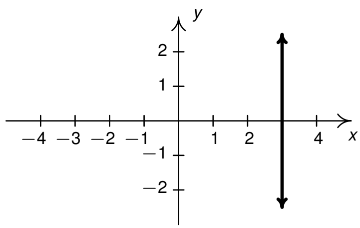


Figure 3.5.11: The graph of V

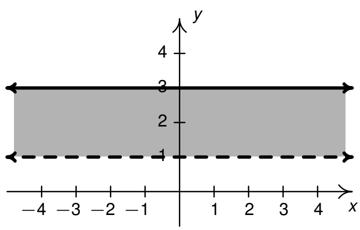


Figure 3.5.12: The graph of R

can describe A using the roster method:

$$A = \{(-3,2),(-3,1),(0,0),(4,2)\}.$$

(b) The graph of H appears to be a portion of the horizontal line $y = 3$ from $x = -2$ (including $x = -2$) up to, but not including $x = 4$. Since it is impossible[b] to *list* each and every one of these points, we'll opt to describe H using set-builder as opposed to the roster method. Taking a cue from the description of the relations V and R above, we write $H = \{(x,3) \,|\, -2 \leq x < 4\}$.

[b] Really impossible. The interested reader is encouraged to research **countable**[3] versus **uncountable**[4] sets.

(c) The graph of Q appears to be the graph of the function $s = f(t) = t^2$. Again, as the graph consists of infinitely many points, we will use set-builder notation to describe Q out of necessity.

There are a couple of different ways to do this. Taking a cue from the relation P above, we could write $Q = \{(t,t^2) \,|\, t \text{ is a real number}\}$. Alternatively, we could introduce the dependent variable, s into the description by writing $Q = \{(t,s) \,|\, s = t^2\}$ where here the assumption is x takes in all real number values.

(d) As with the relation R above, the relation T describes a region in the plane, The v-values appear to range between -3 (not including -3) and up to, and including, $v = 4$. The only restriction on the w-values is that $w \geq 1$, so we have $T = \{(v,w) \,|\, -3 < v \leq 4,\, w \geq 1\}$.
□

As with functions, we can describe relations algebraically using equations. For example, the equation $v^2 + w^3 = 1$ relates two variables v and w each of which represent real numbers. More formally, we can express this sentiment by defining the relation $R = \{(v,w) \,|\, v^2 + w^3 = 1\}$.

An ordered pair $(v, w) \in R$ means v and w are *related* by the equation $v^2 + w^3 = 1$; that is, the pair (v, w) *satisfy* the equation.

For example, to show $(3, -2) \in R$, we check that when we substitute $v = 3$ and $w = -2$, the equation $v^2 + w^3 = 1$ is true. Sure enough, $(3)^2 + (-2)^3 = 9 - 8 = 1$. Hence, R maps 3 to -2. Note, however, that $(-2, 3) \notin R$ since $(-2)^2 + (3)^3 = -8 + 27 \neq 1$ which means R does not map -2 to 3.

When asked to 'graph the equation' $v^2 + w^3 = 1$, we really have two options. We could graph the relation R above. In this case, we would be graphing $v^2 + w^3 = 1$ on the vw-plane.[c] Alternatively, we could define $S = \{(w, v) \mid v^2 + w^3 = 1\}$ and graph S. This is equivalent to graphing $v^2 + w^3 = 1$ on the wv-plane. We do both in our next example.

c Recall this means the horizontal axis is labeled 'v' and the vertical axis is labeled 'w.'

Example 3.5.2. Graph the equation $v^2 + w^3 = 1$ in the vw- and wv-planes. Include the axis-intercepts.

Solution.

- *graphing in the vw-plane:* We begin by finding the axis intercepts of the graph. To obtain a point on the v-axis, we require $w = 0$. To see if we have any v-intercepts on the graph of the equation $v^2 + w^3 = 1$, we substitute $w = 0$ into the equation and solve for v: $v^2 + (0)^3 = 1$. We get $v^2 = 1$ or $v = \pm 1$ so our two v-intercepts, as described in the vw-plane, are $(1, 0)$ and $(-1, 0)$.

  Likewise, to find w-intercepts of the graph, we substitute $v = 0$ into the equation $v^2 + w^3 = 1$ and get $w^3 = 1$ or $w = 1$. Hence, he have only one w-intercept, $(0, 1)$.

  One way to efficiently produce additional points is to solve the equation $v^2 + w^3 = 1$ for one of the variables, say w, in terms of the other, v. In this way, we are treating w as the dependent variable and v as the independent variable. From $v^2+w^3=1$, we get $w^3=1-v^2$ or $w=\sqrt[3]{1-v^2}$.

We now substitute a value in for v, determine the corresponding value w, and plot the resulting point (v,w). We summarize our results in Table 3.5.1. By plotting additional points (or getting help from a graphing utility), we produce the graph in Figure 3.5.13.

v	w	(v,w)
-3	-2	$(-3,-2)$
-2	$-\sqrt[3]{3}$	$(-2,-\sqrt[3]{3})$
-1	0	$(-1,0)$
0	1	$(0,1)$
1	0	$(1,0)$
2	$-\sqrt[3]{3}$	$(2,-\sqrt[3]{3})$
3	-2	$(3,-2)$

Table 3.5.1

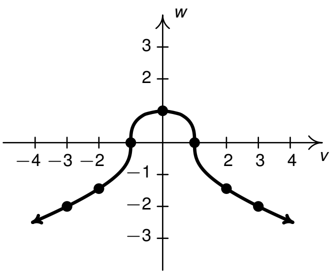


Figure 3.5.13: $v^2+w^3=1$

v	w	(w,v)
-3	-2	$(-2,-3)$
-2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3},-2)$
-1	0	$(0,-1)$
0	1	$(1,0)$
1	0	$(0,1)$
2	$-\sqrt[3]{3}$	$(-\sqrt[3]{3},2)$
3	-2	$(-2,3)$

Table 3.5.2

- *graphing in the wv-plane:* To graph $v^2+w^3=1$ in the wv-plane, all we need to do is reverse the coordinates of the ordered pairs we obtained for our graph in the vw-plane. In particular, the v-intercepts are written $(0,1)$ and $(0,-1)$ and the w-intercept is written $(1,0)$. Using Table 3.5.2 we produce the graph in Figure 3.5.14. $\square$

Note that regardless of which geometric depiction we choose for $v^2+w^3=1$, the graph appears to be symmetric about the w-axis. To prove this is the case, consider a generic point (v,w) on the graph of $v^2+w^3=1$ in the vw-plane.

To show the point symmetric about the w-axis, $(-v,w)$ is also on the graph of $v^2+w^3=1$, we need to show that the coordinates of the point $(-v,w)$ satisfy the equation $v^2+w^3=1$. That is, we need to show $(-v)^2+w^3=1$. Since $(-v)^2+w^3=v^2+w^3$, and we know by assumption $v^2+w^3=1$, we get $(-v)^2+w^3=v^2+w^3=1$, proving $(-v,w)$ is also on the graph of the equation.

The key reason our proof above is successful is that algebraically, the equation $v^2+w^3=1$ is unchanged if v is replaced with $-v$. Geometrically, this means the graph is the same if it undergoes a reflection across the w-axis. We generalize this reasoning in the following result. Note that, as usual, we default to the more common x and y-axis labels.

Theorem 3.5.1. Testing the Graph of an Equation for Symmetry:
To test the graph of an equation in the xy-plane for symmetry:

- about the x-axis: substitute $(x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the x-axis.
- about the y-axis: substitute $(-x, y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the y-axis.
- about the origin: substitute $(-x, -y)$ into the equation and simplify. If the result is equivalent to the original equation, the graph is symmetric about the origin.

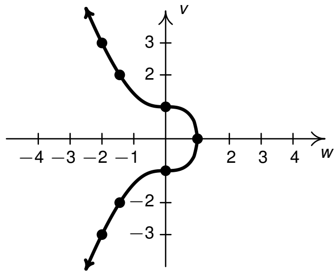


Figure 3.5.14: $v^2 + w^3 = 1$

Parts of Theorem 3.5.1 should look familiar from our work with even and odd functions. Indeed if a function f is even, $f(-x) = f(x)$. Hence, the equation $y = f(-x)$ reduces to the equation $y = f(x)$, so the graph of f is symmetric about the y-axis.

Likewise if f is odd, then $f(-x) = -f(x)$. In this case, the equation $-y = f(-x)$ reduces to $-y = -f(x)$, or $y = f(x)$, proving the graph is symmetric about the origin.

When it comes to symmetry about the x-axis, most of the time this indicates a violation of the Vertical Line Test, which is why we haven't discussed that particular kind of symmetry until now.

We put Theorem 3.5.1 to good use in the following example.

Example 3.5.3. Graph each of the equations below in the xy-plane. Find the axis intercepts, if any, and prove any symmetry suggested by the graphs.

1. $x^2 - y^2 = 4$
2. $(x-1)^2 + 4y^2 = 16$

Solution.

1. We begin graphing $x^2 - y^2 = 4$ by checking for axis intercepts. To check for x-intercepts, we set $y = 0$ and solve $x^2 - (0)^2 = 4$. We get $x = \pm 2$ and obtain two x-intercepts $(-2, 0)$ and $(2, 0)$.

   When looking for y-intercepts, we set $x = 0$ and get $(0)^2 - y^2 = 4$ or $y^2 = -4$. Since this equation has no real number solutions, we have no y-intercepts.

   In order to produce more points on the graph, we solve $x^2 - y^2 = 4$ for y and obtain $y = \pm\sqrt{x^2 - 4}$. Since we know $x^2 - 4 \geq 0$ in order to produce real number results for y, we restrict our attention to $x \leq -2$ and $x \geq 2$. Doing so produces Table 3.5.3. Using these, we construct the graph in Figure 3.5.17.

x	y	(x, y)
-4	$\pm 2\sqrt{3}$	$(-4, \pm 2\sqrt{3})$
-3	$\pm\sqrt{5}$	$(-3, \pm\sqrt{5})$
-2	0	$(-2, 0)$
2	0	$(2, 0)$
3	$\pm\sqrt{5}$	$(3, \pm\sqrt{5})$
4	$\pm 2\sqrt{3}$	$(4, \pm 2\sqrt{3})$

Table 3.5.3

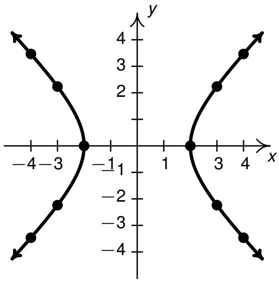


Figure 3.5.15: $x^2 - y^2 = 4$

   The graph certainly appears to be symmetric about both axes and the origin. To prove this, we note that the equation $x^2 - (-y)^2 = 4$ quickly reduces to $x^2 - y^2 = 4$, proving the graph is symmetric about the x-axis.

   Likewise, the equations $(-x)^2 - y^2 = 4$ and $(-x)^2 - (-y)^2 = 4$ also reduce to $x^2 - y^2 = 4$, proving the graph is, indeed, symmetric about the y-axis and origin, respectively.

2. To determine if there are any x-intercepts on the graph of $(x-1)^2 + 4y^2 = 16$, we set $y = 0$ and solve $(x-1)^2 + 4(0)^2 = 16$. This reduces to $(x-1)^2 = 16$ which gives $x = -3$ and $x = 5$. Hence, we have two x-intercepts, $(-3, 0)$ and $(5, 0)$.

   Looking for y-intercepts, we set $x = 0$ and solve $(0-1)^2 + 4y^2 = 16$ or $1 + 4y^2 = 16$. This gives

$y^2 = \frac{15}{4}$ so $y = \pm\frac{\sqrt{15}}{2}$. Hence, we have two y-intercepts: $\left(0, \pm\frac{\sqrt{15}}{2}\right)$.

In this case, it is slightly easier[d] to solve for x in terms of y. From $(x-1)^2 + 4y^2 = 16$ we get $(x-1)^2 = 16 - 4y^2$ which gives $x = 1 \pm \sqrt{16-4y^2}$.

d Read this as we're avoiding fractions.

Since we know $16 - 4y^2 \geq 0$ to produce real number results for x, we require $-2 \leq y \leq 2$. Selecting values in that range produces Table 3.5.4. Plotting these points, along with the y-intercepts produces the graph in Figure 3.5.16.

y	x	(x, y)
-2	1	$(1, -2)$
-1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, -1)$
0	$1 \pm 4 = -3, 5$	$(-3, 0), (5, 0)$
1	$1 \pm 2\sqrt{3}$	$(1 \pm 2\sqrt{3}, 1)$
2	1	$(1, 2)$

Table 3.5.4

The graph certainly appears to be symmetric about the x-axis. To check, we substitute $(-y)$ in for y and get $(x-1)^2 + 4(-y)^2 = 16$ which reduces to $(x-1)^2 + 4y^2 = 16$.

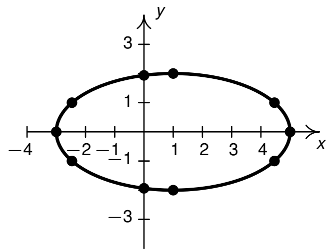


Figure 3.5.16: $(x-1)^2 + 4y^2 = 16$

Owing to the placement of the x-intercepts, $(-3, 0)$ and $(5, 0)$, the graph is most certainly not symmetric about the y-axis nor about the origin. □

Looking at the graphs of the equations $x^2 - y^2 = 4$ and $(x-1)^2 + 4y^2 = 16$ in Example 3.5.3, it is evident neither of these equations represents y as a function of x nor x as a function of y. (Do you see why?)

With the concept of 'function' being touted in the opening remarks of Section **??** as being one of the 'universal tools' with which scientists and engineers solve a wide variety of problems, you may well wonder if we can't somehow apply what we know about functions to these sorts of relations. It turns out that while, taken all at once, these equations do not describe functions, taken in parts, they do.

For example, consider the equation $x^2 - y^2 = 4$ (Figure 3.5.17). Solving for y, we obtained $y = \pm\sqrt{x^2 - 4}$. Defining $f_1(x) = \sqrt{x^2 - 4}$ and $f_2(x) = -\sqrt{x^2 - 4}$, we get a functional description for the upper and lower halves, or *branches* of the curve, respectively (Figure 3.5.18 and Figure 3.5.19).[e]

[e] There are many more ways to break this relation into functional parts. We could, for instance, go piecewise and take portions of the graph which lie in Quadrants I and III as one function and leave the parts in Quadrants II and IV as the other; we could look at this as being comprised of *four* functions, and so on.

If, for instance, we wanted to analyze this curve near $(3, -\sqrt{5})$, we could use the *function* f_2 and all the associated function tools[f] to do just that.

[f] including, when the time comes, Calculus

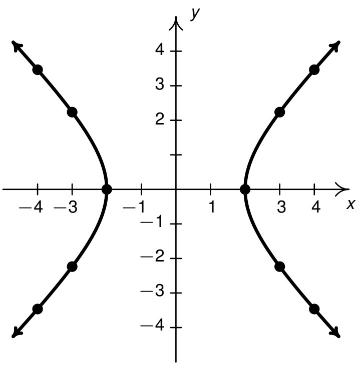


Figure 3.5.17: $x^2 - y^2 = 4$

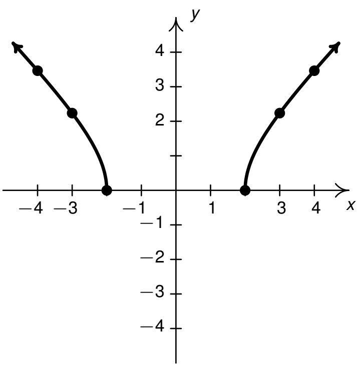


Figure 3.5.18: $f_1(x) = \sqrt{x^2 - 4}$

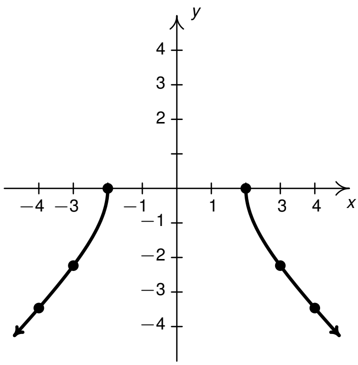


Figure 3.5.19: $f_2(x) = -\sqrt{x^2 - 4}$

In this way we say the equation $x^2 - y^2 = 4$ *implicitly* describes y as a function of x meaning that given any point (x_0, y_0) on $x^2 - y^2 = 4$, we can find a function f defined (on an interval) containing x_0 so that $f(x_0) = y_0$ and whose graph lies on the curve $x^2 - y^2 = 4$.

Note that in this case, we are fortunate to have two *explicit* formulas for functions that cover the entire curve, namely $f_1(x) = \sqrt{x^2 - 4}$ and $f_2(x) = -\sqrt{x^2 - 4}$. We explore this concept further in the next example.

Example 3.5.4. Consider the graph of the relation R in Figure 3.5.20.

1. Explain why this curve does not represent y as a function of x.
2. Resolve the graph of R into two or more graphs of implicitly defined functions.
3. Explain why this curve represents x as a function of y and find a formula for $x = g(y)$.

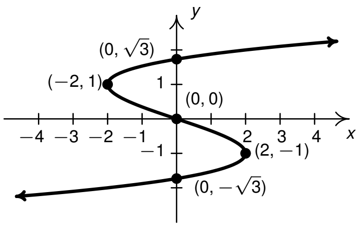


Figure 3.5.20: The graph of R

Solution.

1. Using the Vertical Line Test, Theorem **??**, we find several instances where vertical lines intersect the graph of R more than once. The y-axis, $x = 0$ is one such line. We have $x = 0$ matched with *three* different y-values: $-\sqrt{3}$, 0, and $\sqrt{3}$.

2. Since the maximum number of times a vertical line intersects the graph of R is three, it stands to reason we need to resolve the graph of R into at least three pieces.

   One strategy is to begin at the far left and begin tracing the graph until it begins to 'double back' and repeat y-coordinates. Doing so we get three functions (represented by the bold solid lines) in Figure 3.5.21, Figure 3.5.22 and Figure 3.5.23.

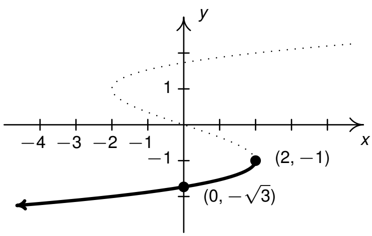


Figure 3.5.21: $y = f_1(x)$

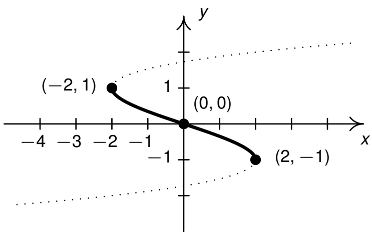


Figure 3.5.22: $y = f_2(x)$

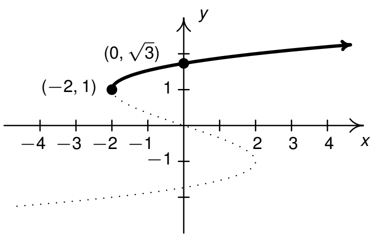


Figure 3.5.23: $y = f_3(x)$

3. To verify that R represents x as a function of y, we check to see if any y-value has more than one x associated with it. One way to do this is to employ the the Horizontal Line Test (Exercise **??** in Section **??**.) Since every horizontal line intersects the graph at most once, x is a function of y.

   Using Theorem **??** from Chapter **??**, we get $x = (1)y(y - \sqrt{3})(y + \sqrt{3}) = y^3 - 3y$, a fact we can readily check using a graphing utility. □

Not all equations implicitly define y as a function of x. For a quick example, take $x = 117$ or any other vertical line.

Even if an equation implicitly describes y as a function of x near one point, there's no guarantee we can find an explicit algebraic representation for that function.[g]

g An example of this is $y^5 - y - x = 1$ near $(-1, 0)$.

While the theory of implicit functions is well beyond the scope of this text, we will nevertheless see this concept come into play in Section 3.6. For our purposes, it suffices to know that just because a relation is not a function doesn't mean we cannot find a way to apply what we know about functions to analyze the relation locally through a functional lens.

3.5.1 Exercises

In Exercises 1 - 20, graph the given relation in the xy-plane.

1. $\{(-3,9),\ (-2,4),\ (-1,1),\ (0,0),\ (1,1),\ (2,4),\ (3,9)\}$
2. $\{(-2,0),\ (-1,1),\ (-1,-1),\ (0,2),\ (0,-2),\ (1,3),\ (1,-3)\}$
3. $\{(m,2m)\mid m=0,\pm1,\pm2\}$
4. $\left\{\left(\frac{6}{k},k\right)\mid k=\pm1,\pm2,\pm3,\pm4,\pm5,\pm6\right\}$
5. $\left\{\left(n,4-n^2\right)\mid n=0,\pm1,\pm2\right\}$
6. $\left\{\left(\sqrt{j},j\right)\mid j=0,1,4,9\right\}$
7. $\{(x,-2)\mid x>-4\}$
8. $\{(x,3)\mid x\leq 4\}$
9. $\{(-1,y)\mid y>1\}$
10. $\{(2,y)\mid y\leq 5\}$
11. $\{(-2,y)\mid -3<y\leq 4\}$
12. $\{(3,y)\mid -4\leq y<3\}$
13. $\{(x,2)\mid -2\leq x<3\}$
14. $\{(x,-3)\mid -4<x\leq 4\}$
15. $\{(x,y)\mid x>-2\}$
16. $\{(x,y)\mid x\leq 3\}$
17. $\{(x,y)\mid y<4\}$
18. $\{(x,y)\mid x\leq 3,\ y<2\}$
19. $\{(x,y)\mid x>0,\ y<4\}$
20. $\left\{(x,y)\mid -\sqrt{2}\leq x\leq \frac{2}{3},\ \pi<y\leq\frac{9}{2}\right\}$

In Exercises 21 - 30, describe the given relation using either the roster or set-builder method.

21. See Figure 3.5.24.
22. See Figure 3.5.25.
23. See Figure 3.5.26.
24. See Figure 3.5.27.

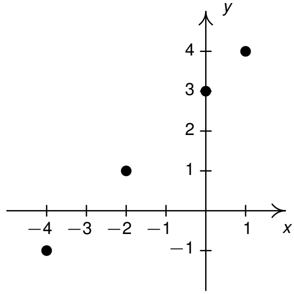


Figure 3.5.24: Relation A

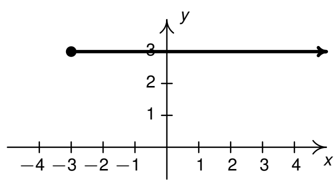


Figure 3.5.25: Relation B

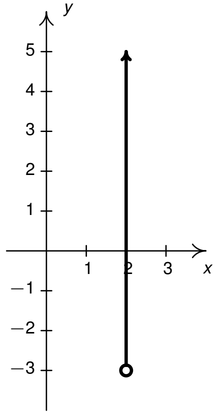


Figure 3.5.26: Relation C

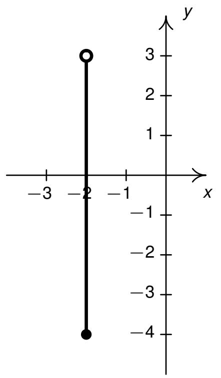


Figure 3.5.27: Relation D

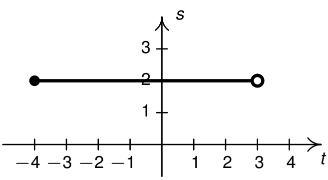


Figure 3.5.28: Relation E

25. See Figure 3.5.28.

26. See Figure 3.5.29.

27. See Figure 3.5.30.

28. See Figure 3.5.31

29. See Figure 3.5.32

30. See Figure 3.5.33

Some relations are fairly easy to describe in words or with the roster method but are rather difficult, if not impossible, to graph. Discuss with your classmates how you might graph the relations given in Exercises 31 - 34. Note that in the notation below we are using the ellipsis, '. . . ,' to denote that the list does not end, but rather, continues to follow the established pattern indefinitely.
For the relations in Exercises 31 and 32, give two examples of points which belong to the relation and two points which do not belong to the relation.

31. $\{(x, y) \mid x$ is an odd integer, and y is an even integer.$\}$

32. $\{(x, 1) \mid x$ is an irrational number $\}$

33. $\{(1, 0), (2, 1), (4, 2), (8, 3), (16, 4), (32, 5), ...\}$

34. $\{... , (-3, 9), (-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4), (3, 9), ...\}$

For each equation given in Exercises 35 - 38:

- Graph the equation in the xy-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes y as a function of x. If not, describe the graph of the equation using two or more explicit functions of x. Check your answers using a graphing utility.

35. $(x+2)^2 + y^2 = 16$

36. $x^2 - y^2 = 1$

37. $4y^2 - 9x^2 = 36$

38. $x^3 y = -4$

For each equation given in Exercises 39 - 8:

- Graph the equation in the vw-plane by creating a table of points.
- Find the axis intercepts, if they exist.
- Test the equation for symmetry. If the equation fails a symmetry test, find a point on the graph of the equation whose symmetric point is not on the graph of the equation.
- Determine if the equation describes w as a function of v. If not, describe the graph of the equation using two or more explicit functions of v. Check your answers using a graphing utility.

39. $v + w^2 = 4$

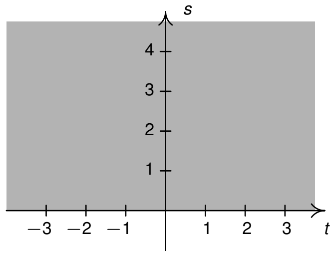


Figure 3.5.29: Relation F

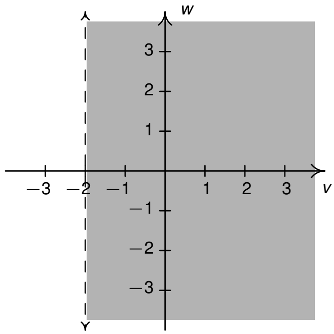


Figure 3.5.30: Relation G

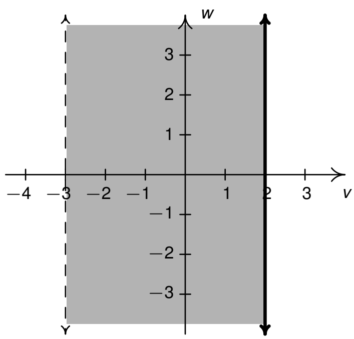


Figure 3.5.31: Relation H

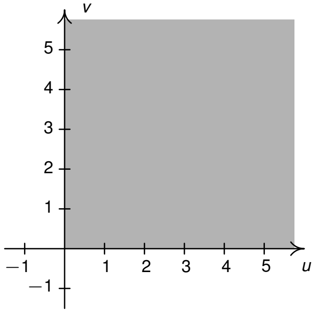


Figure 3.5.32: Relation I

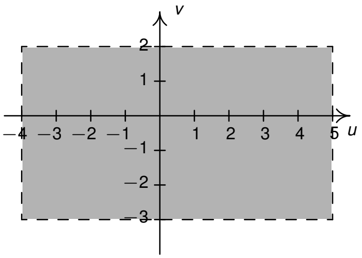


Figure 3.5.33: Relation J

HINT: $v^4 - 2v^2w + w^2 = \left(v^2 - w\right)^2$ h

...

40. $v^3 + w^3 = 8$

41. $v^2w^3 = 8$

42. $v^4 - 2v^2w + w^2 = 16$

> The procedures which we have outlined in the Examples of this section and used in Exercises 35 - 38 all rely on the fact that the equations were "well-behaved". Not everything in Mathematics is quite so tame, as the following equations will show you. Discuss with your classmates how you might approach graphing the equations given in Exercises 43 - 46. What difficulties arise when trying to apply the various tests and procedures given in this section? For more information, including pictures of the curves, each curve name is a link to its page at www.wikipedia.org. For a much longer list of fascinating curves, click here[a].
>
> [a] http://en.wikipedia.org/wiki/List_of_curves

43. $x^3 + y^3 - 3xy = 0$ Folium of Descartes[7]

44. $x^4 = x^2 + y^2$ Kampyle of Eudoxus[8]

45. $y^2 = x^3 + 3x^2$ Tschirnhausen cubic[9]

46. $(x^2 + y^2)^2 = x^3 + y^3$ Crooked egg[10]

47. With the help of your classmates, find examples of equations whose graphs possess
    - symmetry about the x-axis only
    - symmetry about the y-axis only

[7] http://en.wikipedia.org/wiki/Folium_of_descartes
[8] http://en.wikipedia.org/wiki/Kampyle_of_Eudoxus
[9] http://en.wikipedia.org/wiki/Tschirnhausen_cubic
[10] https://en.wikipedia.org/wiki/File:Crooked_egg_curve.svg

- symmetry about the origin only
- symmetry about the x-axis, y-axis, and origin

Can you find an example of an equation whose graph possesses exactly *two* of the symmetries listed above? Why or why not?

3.5.2 Answers

1. See Figure 3.5.34.
2. See Figure 3.5.35.
3. See Figure 3.5.36.
4. See Figure 3.5.37

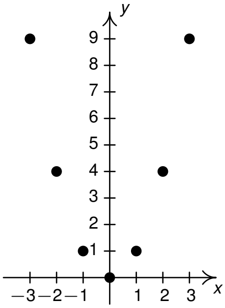


Figure 3.5.34

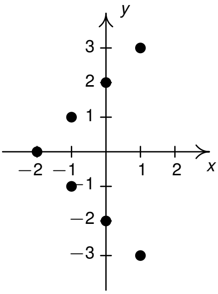


Figure 3.5.35

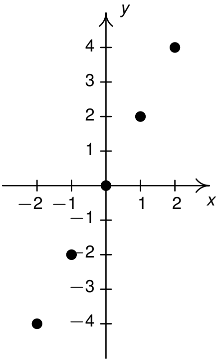


Figure 3.5.36

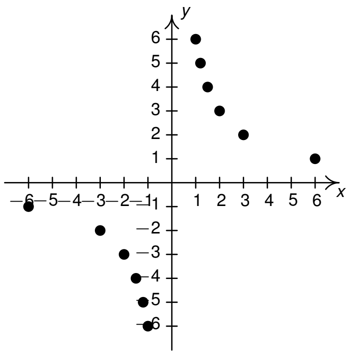


Figure 3.5.37

5. See Figure 3.5.38

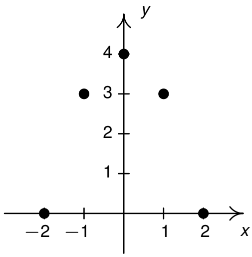


Figure 3.5.38

6. See Figure 3.5.39
7. See Figure 3.5.40
8. See Figure 3.5.41
9. See Figure 3.5.42

10. See Figure 3.5.43

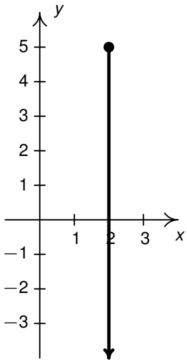


Figure 3.5.43

11. See Figure 3.5.44

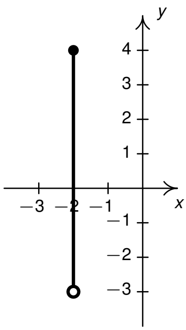


Figure 3.5.44

12. See Figure 3.5.45
13. See Figure 3.5.46
14. See Figure 3.5.47
15. See Figure 3.5.48
16. See Figure 3.5.49
17. See Figure 3.5.50

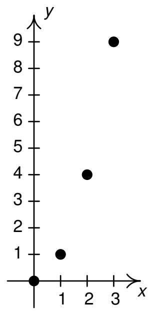


Figure 3.5.39

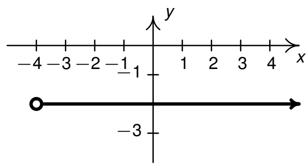


Figure 3.5.40

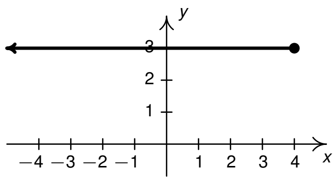


Figure 3.5.41

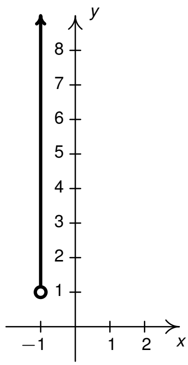


Figure 3.5.42

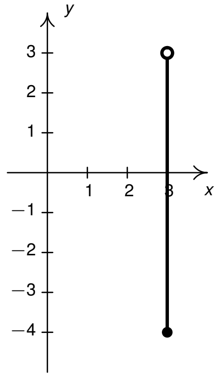


Figure 3.5.45

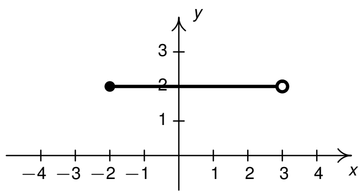


Figure 3.5.46

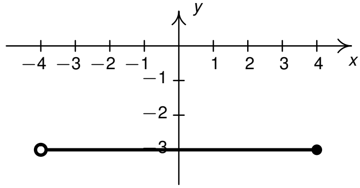


Figure 3.5.47

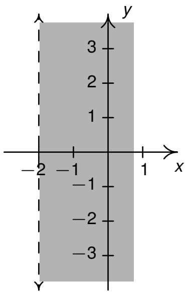


Figure 3.5.48

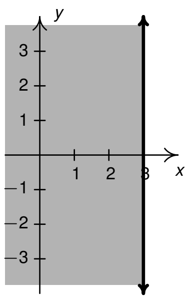


Figure 3.5.49

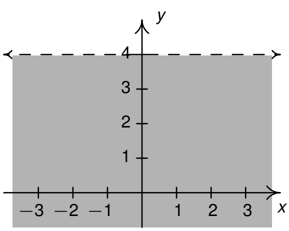


Figure 3.5.50

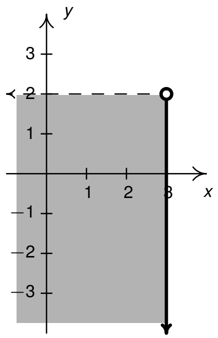


Figure 3.5.51

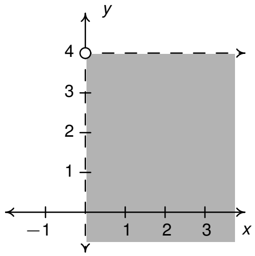


Figure 3.5.52

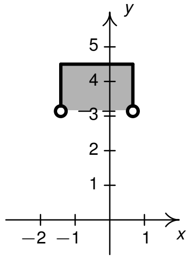


Figure 3.5.53

x	y	(x, y)
-6	0	$(-6, 0)$
-4	$\pm 2\sqrt{3}$	$\left(-4, \pm 2\sqrt{3}\right)$
-2	± 4	$(-2, \pm 4)$
0	$\pm 2\sqrt{3}$	$\left(0, \pm 2\sqrt{3}\right)$
2	0	$(2, 0)$

Table 3.5.5

18. See Figure 3.5.51.
19. See Figure 3.5.52
20. See Figure 3.5.53
21. $A = \{(-4, -1), (-2, 1), (0, 3), (1, 4)\}$
22. $B = \{(x, 3) \mid x \geq -3\}$
23. $C = \{\left(2, y\right) \mid y > -3\}$
24. $D = \{\left(-2, y\right) \mid -4 \leq y < 3\}$
25. $E = \{(t, 2) \mid -4 < t \leq 3\}$
26. $F = \{\left(t, s\right) \mid s \geq 0\}$
27. $G = \{(v, w) \mid v > -2\}$
28. $H = \{(v, w) \mid -3 < v \leq 2\}$
29. $I = \{\left(u, v\right) \mid u \geq 0, v \geq 0\}$
30. $J = \{(u, v) \mid -4 < u < 5,\ -3 < v < 2\}$

35. $(x + 2)^2 + y^2 = 16$
Re-write as $y = \pm\sqrt{16 - (x + 2)^2}$.
x-intercepts: $(-6, 0)$, $(2, 0)$
y-intercepts: $\left(0, \pm 2\sqrt{3}\right)$
See Table 3.5.5.
See Figure 3.5.54.
The graph is symmetric about the x-axis.
The graph is not symmetric about the y-axis: $(-6, 0)$ is on the graph but $(6, 0)$ is not.
The graph is not symmetric about the origin: $(-6, 0)$ is on the graph but $(6, 0)$ is not.
The equation does not describe y as a function of x.
The graph of the equation is the graphs of $f_1(x) = \sqrt{16 - (x + 2)^2}$ together with $f_2(x) = -\sqrt{16 - (x + 2)^2}$.

36. $x^2 - y^2 = 1$
Re-write as: $y = \pm\sqrt{x^2 - 1}$.
x-intercepts: $(-1, 0)$, $(1, 0)$
The graph has no y-intercepts

See Table 3.5.6.
See Figure 3.5.55.
The graph is symmetric about the x-axis.
The graph is symmetric about the y-axis.
The graph is symmetric about the origin.
The equation does not describe y as a function of x.
The graph of the equation is the graphs of $f_1(x) = \sqrt{x^2-1}$ together with $f_2(x) = -\sqrt{x^2-1}$.

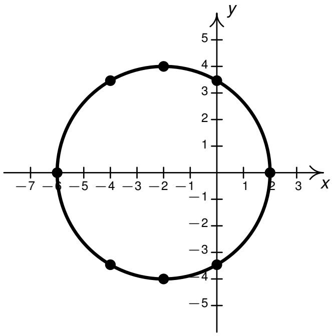


Figure 3.5.54

x	y	(x, y)
-3	$\pm\sqrt{8}$	$(-3, \pm\sqrt{8})$
-2	$\pm\sqrt{3}$	$(-2, \pm\sqrt{3})$
-1	0	$(-1, 0)$
1	0	$(1, 0)$
2	$\pm\sqrt{3}$	$(2, \pm\sqrt{3})$
3	$\pm\sqrt{8}$	$(3, \pm\sqrt{8})$

Table 3.5.6

37. $4y^2 - 9x^2 = 36$

Re-write as: $y = \pm\dfrac{\sqrt{9x^2+36}}{2}$.
The graph has no x-intercepts
y-intercepts: $(0, \pm 3)$
See Table 3.5.7
See Figure 3.5.56
The graph is symmetric about the x-axis.
The graph is symmetric about the y-axis.
The graph is symmetric about the origin.
The equation does not describe y as a function of x.
The graph of the equation is the graphs of $f_1(x) = \dfrac{\sqrt{9x^2+36}}{2}$ together with $f_2(x) = -\dfrac{\sqrt{9x^2+36}}{2}$.

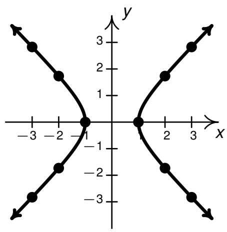


Figure 3.5.55

x	y	(x, y)
-4	$\pm 3\sqrt{5}$	$\left(-4, \pm 3\sqrt{5}\right)$
-2	$\pm 3\sqrt{2}$	$\left(-2, \pm 3\sqrt{2}\right)$
0	± 3	$(0, \pm 3)$
2	$\pm 3\sqrt{2}$	$\left(2, \pm 3\sqrt{2}\right)$
4	$\pm 3\sqrt{5}$	$\left(4, \pm 3\sqrt{5}\right)$

Table 3.5.7

38. $x^3 y = -4$

Re-write as: $y = -\dfrac{4}{x^3} = -4x^{-3}$.
The graph has no x-intercepts
The graph has no y-intercepts
See Table 3.5.8
See Figure 3.5.57
The graph is not symmetric about the x-axis: $(1, -4)$ is on the graph but $(1, 4)$ is not.
The graph is not symmetric about the y-axis: $(1, -4)$ is on the graph but $(-1, -4)$ is not.
The graph is symmetric about the origin.
The equation does describe y as a function of x,

namely $y = f(x) = -4x^{-3}$.

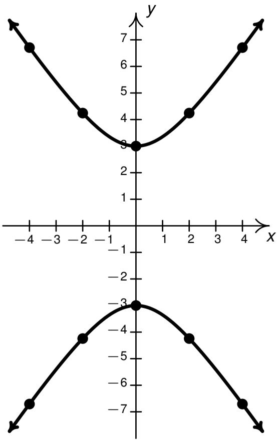


Figure 3.5.56

x	y	(x, y)
-2	$\frac{1}{2}$	$(-2, \frac{1}{2})$
-1	4	$(-1, 4)$
$-\frac{1}{2}$	32	$(-\frac{1}{2}, 32)$
$\frac{1}{2}$	-32	$(\frac{1}{2}, -32)$
1	-4	$(1, -4)$
2	$-\frac{1}{2}$	$(2, -\frac{1}{2})$

Table 3.5.8

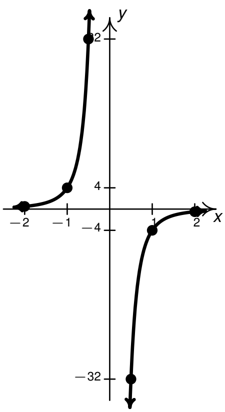


Figure 3.5.57

39. $v + w^2 = 4$
Re-write as $w = \pm\sqrt{4 - v}$.
v-intercept: $(4, 0)$
w-intercepts: $(0, \pm 2)$
See Table 3.5.9
See Figure 3.5.58
The graph is symmetric about the v-axis
The graph is not symmetric about the w-axis: $(4, 0)$ is on the graph but $(-4, 0)$ is not.
The graph is not symmetric about the origin: $(4, 0)$ is on the graph but $(-4, 0)$ is not.
The equation does not describe w as a function of v.
The graph of the equation is the graphs of $f_1(v) = \sqrt{4 - v}$ together with $f_2(v) = -\sqrt{4 - v}$.
The graph is not symmetric about the v-axis: $(0, 2)$ is on the graph but $(0, -2)$ is not.
The graph is not symmetric about the w-axis: $(2, 0)$ is on the graph but $(-2, 0)$ is not.
The graph is not symmetric about the origin: $(0, 2)$ is on the graph but $(0, -2)$ is not.
The equation does describe w as a function of v, namely $w = f(v) = \sqrt[3]{8 - v^3}$.

40. $v^3 + w^3 = 8$
Re-write as: $w = \sqrt[3]{8 - v^3}$.
v-intercept: $(2, 0)$
w-intercept: $(0, 2)$
See Table 3.5.10
See Figure 3.5.59

41. $v^2 w^3 = 8$
Re-write as $w = \dfrac{2}{\sqrt[3]{v^2}} = 2v^{-\frac{2}{3}}$.
The graph has no v-intercepts.
The graph has no w-intercepts.
See Table 3.5.11

See Figure 3.5.60
The graph is not symmetric about the v-axis: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.
The graph is symmetric about the w-axis.
The graph is not symmetric about the origin: $(-1, 2)$ is on the graph but $(-1, -2)$ is not.
The equation does describe w as a function of v, namely $w = f(v) = 2v^{-\frac{2}{3}}$.

v	w	(x, y)
-8	$\frac{1}{2}$	$\left(-8, \frac{1}{2}\right)$
-1	2	$(-1, 2)$
$-\frac{1}{8}$	8	$\left(-\frac{1}{8}, 8\right)$
$\frac{1}{8}$	8	$\left(\frac{1}{8}, 8\right)$
1	2	$(1, 2)$
8	$\frac{1}{2}$	$\left(8, \frac{1}{2}\right)$

Table 3.5.11

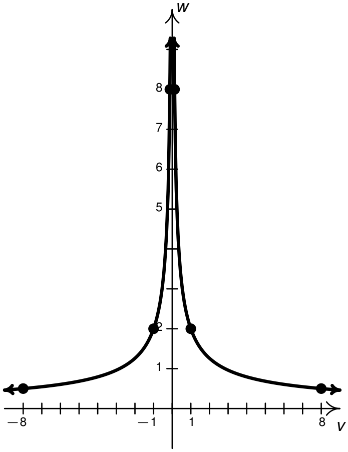


Figure 3.5.60

v	w	(x, y)
-5	± 3	$(-5, \pm 3)$
-2	$\pm\sqrt{6}$	$\left(-2, \pm\sqrt{6}\right)$
0	± 2	$(0, \pm 2)$
2	$\pm\sqrt{2}$	$\left(1, \pm\sqrt{3}\right)$
4	0	$(4, 0)$

Table 3.5.9

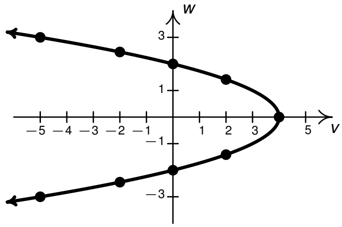


Figure 3.5.58

v	w	(v, w)
-3	$\sqrt[3]{35}$	$(-3, \sqrt[3]{35})$
-1	$\sqrt[3]{9}$	$(-1, \sqrt[3]{9})$
0	2	$(0, 2)$
1	$\sqrt[3]{7}$	$(1, \sqrt[3]{7})$
2	0	$(2, 0)$
3	$-\sqrt[3]{19}$	$(3, -\sqrt[3]{19})$

Table 3.5.10

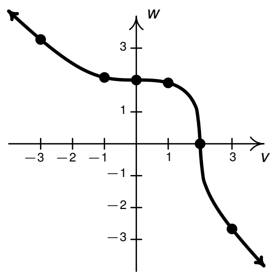


Figure 3.5.59

42. $v^4 - 2v^2w + w^2 = 16$

Re-write as: $\left(v^2 - w\right)^2 = 16$
Extracting square roots gives:
$w = v^2 + 4$ and $w = v^2 - 4$
v-intercepts: $(-2, 0), (2, 0)$.
w-intercepts: $(0, -4), (0, 4)$
See Table 3.5.12
See Figure 3.5.61
The graph is not symmetric about the v-axis: $(1, 5)$ is on the graph but $(1, -5)$ is not.
The graph is symmetric about the w-axis.
The graph is not symmetric about the origin: $(1, 5)$ is on the graph but $(-1, -5)$ is not.
The equation does not describe w as a function of v.
The graph of the equation is the graphs of $f_1(v) = v^2 + 4$ together with $f_2(v) = v^2 - 4$.

v	w	(v, w)
-2	8	$(-2, 8)$
-2	0	$(-2, 0)$
-1	5	$(-1, 5)$
-1	-3	$(-1, -3)$
0	± 4	$(0, \pm 4)$
1	5	$(1, 5)$
1	-3	$(1, -3)$
2	8	$(2, 8)$
2	0	$(2, 0)$

Table 3.5.12

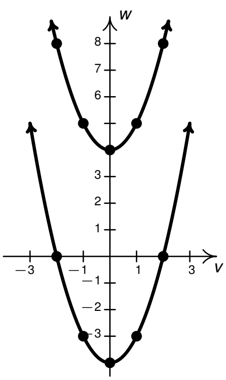


Figure 3.5.61

3.6 Inverse Functions

In Section **??**, we defined functions as processes. In this section, we seek to reverse, or 'undo' those processes. As in real life, we will find that some processes (like putting on socks and shoes) are reversible while some (like baking a cake) are not.

Consider the function $f(x) = 3x + 4$. Starting with a real number input x, we apply two steps in the following sequence: first we multiply the input by 3 and, second, we add 4 to the result.

To reverse this process, we seek a function g which will undo each of these steps and take the output from f, $3x + 4$, and return the input x. If we think of the two-step process of first putting on socks then putting on shoes, to reverse the process, we first take off the shoes and then we take off the socks. In much the same way, the function g should undo each step of f but in the opposite order. That is, the function g should first *subtract* 4 from the input x then *divide* the result by 3. This leads us to the formula $g(x) = \frac{x-4}{3}$.

Let's check to see if the function g does the job. If $x = 5$, then $f(5) = 3(5) + 4 = 15 + 4 = 19$. Taking the output 19 from f, we substitute it into g to get $g(19) = \frac{19-4}{3} = \frac{15}{3} = 5$, which is our original input to f. To check that g does the job for all x in the domain of f, we take the generic output from f, $f(x) = 3x + 4$, and substitute that into g. That is, we simplify $g(f(x)) = g(3x + 4) = \frac{(3x+4)-4}{3} = \frac{3x}{3} = x$, which is our original input to f. If we carefully examine the arithmetic as we simplify $g(f(x))$, we actually see g first 'undoing' the addition of 4, and then 'undoing' the multiplication by 3.

Not only does g undo f, but f also undoes g. That is, if we take the output from g, $g(x) = \frac{x-4}{3}$, and substitute that into f, we get $f(g(x)) = f\left(\frac{x-4}{3}\right) = 3\left(\frac{x-4}{3}\right) + 4 = (x - 4) + 4 = x$. Using the language of function composition developed in Section 3.3, the statements $g(f(x)) = x$ and

$f(g(x)) = x$ can be written as $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$, respectively.[a] Abstractly, we can visualize the relationship between f and g in Figure 3.6.1.

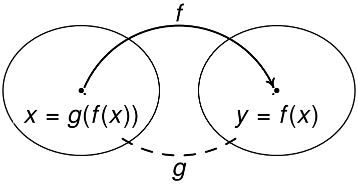


Figure 3.6.1

The main idea to get from the diagram is that g takes the outputs from f and returns them to their respective inputs, and conversely, f takes outputs from g and returns them to their respective inputs. We now have enough background to state the central definition of the section specified in Definition 3.6.1.

Definition 3.6.1. Suppose f and g are two functions such that

1. $(g \circ f)(x) = x$ for all x in the domain of f
   and
2. $(f \circ g)(x) = x$ for all x in the domain of g

then f and g are **inverses** of each other and the functions f and g are said to be **invertible**.

If we abstract one step further, we can express the sentiment in Definition 3.6.1 by saying that f and g are inverses if and only if $g \circ f = I_1$ and $f \circ g = I_2$ where I_1 is the identity function restricted[b] to the domain of f and I_2 is the identity function restricted to the domain of g.

In other words, $I_1(x) = x$ for all x in the domain of f and $I_2(x) = x$ for all x in the domain of g. Using this description of inverses along with the properties of function composition listed in Theorem 3.3.1, we can show that function inverses are unique.[c]

Suppose g and h are both inverses of a function f. By Theorem 3.6.1, the domain of g is equal to the domain of h, since both are the range of f. This means the identity function I_2 applies both to the domain of h and the domain of g. Thus $h = h \circ I_2 = h \circ (f \circ g) = (h \circ f) \circ g = I_1 \circ g = g$, as required.

We summarize the important properties of invertible functions in the following theorem.[d] Apart from introducing

[a] At the level of functions, $g \circ f = f \circ g = I$, where I is the identity function as defined as $I(x) = x$ for all real numbers, x.

[b] The identity function I, first introduced in Exercise **??** in Section **??** and mentioned in Theorem 3.3.1, has a domain of all real numbers. Since the domains of f and g may not be all real numbers, we need the restrictions listed here.

[c] In other words, invertible functions have exactly one inverse.

[d] In the interests of full disclosure, the authors would like to admit that much of the discussion in the previous paragraphs could have easily been avoided had we appealed to the description of a function as a set of ordered pairs. We make no apology for our discussion from a function composition standpoint, however, since it exposes the reader to more abstract ways of thinking of functions and inverses. We will revisit this concept again in Chapter **??**.

notation, each of the results below are immediate consequences of the idea that inverse functions map the outputs from a function f back to their corresponding inputs.

Theorem 3.6.1. Properties of Inverse Functions: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$
  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a, c) is on the graph of f if and only if (c, a) is on the graph of f^{-1}
  NOTE: This means graph of $y = f^{-1}(x)$ is the reflection of the graph of $y = f(x)$ across $y = x$.[a]
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

[a] See Example **??** in Section **??** and Example **??** in Section **??**.

The notation f^{-1} is an unfortunate choice since you've been programmed since Elementary Algebra to think of this as $\frac{1}{f}$. This is most definitely *not* the case since, for instance, $f(x) = 3x + 4$ has as its inverse $f^{-1}(x) = \frac{x-4}{3}$, which is certainly different than $\frac{1}{f(x)} = \frac{1}{3x+4}$.

Why does this confusing notation persist? As we mentioned in Section 3.3, the identity function I is to function composition what the real number 1 is to real number multiplication. The choice of notation f^{-1} alludes to the property that $f^{-1} \circ f = I_1$ and $f \circ f^{-1} = I_2$, in much the same way as $3^{-1} \cdot 3 = 1$ and $3 \cdot 3^{-1} = 1$.

Before we embark on an example, we demonstrate the pertinent parts of Theorem 3.6.1 to the inverse pair $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$. Suppose we wanted to solve $3x + 4 = 7$. Going through the usual machinations, we obtain $x = 1$.

If we view this equation as $f(x) = 7$, however, then we are looking for the input x corresponding to the output $f(x) = 7$. This is exactly the question f^{-1} was built to answer. In other words, the solution to $f(x) = 7$ is $x = f^{-1}(7) = 1$. In other words, the formula $f^{-1}(x)$ encodes all of the algebra required to 'undo' what the formula $f(x)$ does to x. More generally, any time you have ever solved an equation, you have really been working through an inverse problem.

We also note the graphs of $f(x) = 3x + 4$ and $g(x) = f^{-1}(x) = \frac{x-4}{3}$ are easily seen to be reflections across the line $y = x$ as seen below. In particular, note that the y-intercept $(0, 4)$ on the graph of $y = f(x)$ corresponds to the x-intercept on the graph of $y = f^{-1}(x)$. Indeed, the point $(0, 4)$ on the graph of $y = f(x)$ can be interpreted as $(0, 4) = (0, f(0)) = (f^{-1}(4), 4)$ just as the point $(4, 0)$ on the graph of $y = f^{-1}(x)$ can be interpreted as $(4, 0) = (4, f^{-1}(4)) = (f(0), 0)$. See Figure 3.6.2.

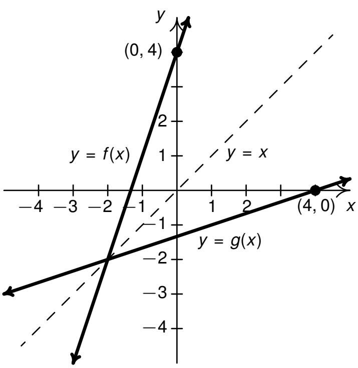


Figure 3.6.2: Graphs of inverse functions $y = f(x) = 3x + 4$ and $y = f^{-1}(x) = \dfrac{x-4}{3}$

Example 3.6.1. For each pair of functions f and g below:

1. Verify each pair of functions f and g are inverses: (a) algebraically and (b) graphically.

2. Use the fact f and g are inverses to solve $f(x) = 5$ and $g(x) = -3$

- $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$
- $f(t) = \dfrac{2t}{t+1}$ and $g(t) = \dfrac{t}{2-t}$

Solution.

Solution for $f(x) = \sqrt[3]{x-1} + 2$ *and* $g(x) = (x-2)^3 + 1$.

1. (a) To verify $f(x) = \sqrt[3]{x-1} + 2$ and $g(x) = (x-2)^3 + 1$ are inverses, we appeal to Definition 3.6.1 and show $(g \circ f)(x) = x$ and $(f \circ g)(x) = x$ for all real numbers, x.

$$\begin{aligned}
(g \circ f)(x) &= g(f(x)) \\
&= g(\sqrt[3]{x-1}+2) \\
&= [(\sqrt[3]{x-1}+2)-2]^3+1 \\
&= (\sqrt[3]{x-1})^3+1 \\
&= x-1+1 \\
&= x \checkmark
\end{aligned}$$

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f((x-2)^3+1) \\
&= \sqrt[3]{[(x-2)^3+1]-1}+2 \\
&= \sqrt[3]{(x-2)^3}+2 \\
&= x-4+4 \\
&= x \checkmark
\end{aligned}$$

Since the root here, 3, is odd, Theorem 2.1.2 gives $(\sqrt[3]{x-1})^3 = x-1$ and $\sqrt[3]{(x-2)^3} = x-2$.

(b) To show f and g are inverses graphically, we graph $y = f(x)$ and $y = g(x)$ on the same set of axes and check to see if they are reflections about the line $y = x$.

The graph of $y = f(x) = \sqrt[3]{x-1}+2$ appears in Figure 3.6.14 courtesy of Theorem 2.1.1 in Section 2.1. The graph of $y = g(x) = (x-2)^3+1$ appears in Figure 3.6.4 thanks to Theorem **??** in Section **??**.

We can immediately see three pairs of corresponding points: $(0,1)$ and $(1,0)$, $(1,2)$ and $(2,1)$, $(2,3)$ and $(3,2)$. When graphed on the same pair of axes, the two graphs certainly appear to be symmetric about the line $y = x$, as required. See Figure 3.6.5.

2. Since f and g are inverses, the solution to $f(x) = 5$ is $x = f^{-1}(5) = g(5) = (5-2)^3+1 = 28$. To check, we find $f(28) = \sqrt[3]{28-1}+2 = \sqrt[3]{27}+2 = 3+2 = 5$, as required.

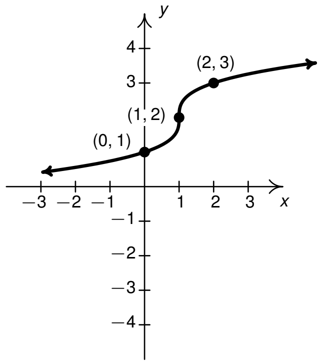


Figure 3.6.3: $y = f(x)$

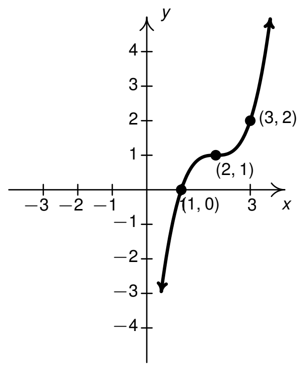


Figure 3.6.4: $y = g(x)$

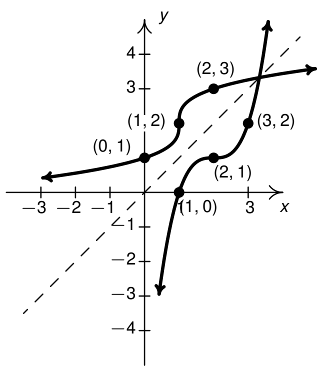


Figure 3.6.5: $y = f(x)$, $y = g(x)$, $y = x$

Likewise, the solution to $g(x) = -3$ is $x = g^{-1}(-3) = f(-3) = \sqrt[3]{(-3) - 1} + 2 = 2 - \sqrt[3]{4}$. Once again, to check, we find $g(2 - \sqrt[3]{4}) = (2 - \sqrt[3]{4} - 2)^3 + 1 = (-\sqrt[3]{4})^3 + 1 = -4 + 1 = -3$.

Solution for $f(t) = \dfrac{2t}{t+1}$ *and* $g(t) = \dfrac{t}{2-t}$.

1. (a) Note the domain of f excludes $t = -1$ and the domain of g excludes $t = 2$. Hence, when simplifying $(g \circ f)(t)$ and $(f \circ g)(t)$, we tacitly assume $t \neq -1$ and $t \neq 2$, respectively.

$$\begin{aligned}
(g \circ f)(t) &= g(f(t)) \\
&= g\left(\frac{2t}{t+1}\right) \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \\
&= \frac{\frac{2t}{t+1}}{2 - \frac{2t}{t+1}} \cdot \frac{(t+1)}{(t+1)} \\
&= \frac{2t}{2(t+1) - 2t} \\
&= \frac{2t}{2t + 2 - 2t} \\
&= \frac{2t}{2} \\
&= t \ \checkmark
\end{aligned}$$

$$\begin{aligned}
(f \circ g)(t) &= f(g(t)) \\
&= f\left(\frac{t}{2-t}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right)+1} \\
&= \frac{2\left(\frac{t}{2-t}\right)}{\left(\frac{t}{2-t}\right)+1} \cdot \frac{(2-t)}{(2-t)} \\
&= \frac{2t}{t+(1)(2-t)} \\
&= \frac{2t}{t+2-t)} \\
&= \frac{2t}{2} \\
&= t \ \checkmark
\end{aligned}$$

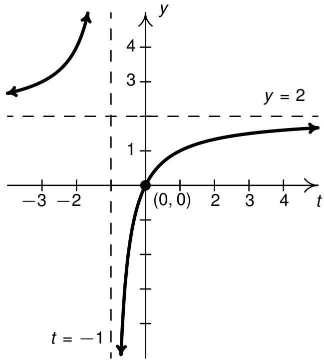


Figure 3.6.6: $y = f(t)$

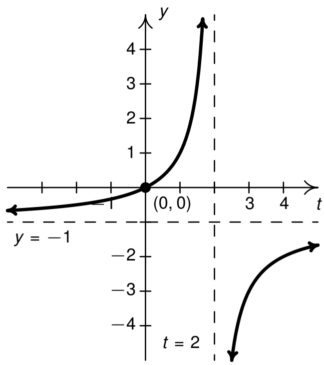


Figure 3.6.7: $y = g(t)$

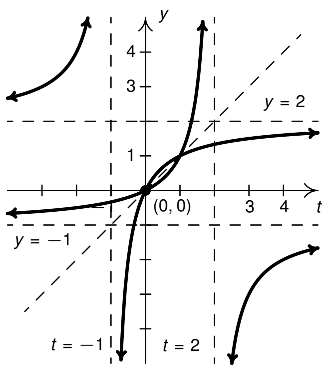


Figure 3.6.8: $y = f(t)$, $y = g(t)$, and $y = t$

(b) We graph $y = f(t)$ and $y = g(t)$ using the techniques discussed in Sections 1.1 and 1.2. See Figure 3.6.6 and Figure 3.6.7.

We find the graph of f has a vertical asymptote $t = -1$ and a horizontal asymptote $y = 2$. Corresponding to the *vertical* asymptote $t = -1$ on the graph of f, we find the graph of g has a *horizontal* asymptote $y = -1$.

Likewise, the *horizontal* asymptote $y = 2$ on the graph of f corresponds to the *vertical* asymptote $t = 2$ on the graph of g. Both graphs share the intercept $(0, 0)$. When graphed together on the same set of axes, the graphs of f and g do appear to be symmetric about the line $y = t$. See Figure 3.6.8.

2. Don't let the fact that f and g in this case were defined using the independent variable, 't' instead of 'x' deter you in your efforts to solve $f(x) = 5$. Remember that, ultimately, the function f here is

the *process* represented by the formula $f(t)$, and is the same process (with the same inverse!) regardless of the letter used as the independent variable. Hence, the solution to $f(x) = 5$ is $x = f^{-1}(1) = g(5)$. We get $g(5) = \frac{5}{2-5} = -\frac{5}{3}$.

To check, we find $f\left(-\frac{5}{3}\right) = \left(-\frac{10}{3}\right) / \left(-\frac{2}{3}\right) = 5$. Similarly, we solve $g(x) = -3$ by finding $x = g^{-1}(-3) = f(-3) = \frac{-6}{-2} = 3$. Sure enough, we find $g(3) = \frac{3}{2-3} = -3$. □

We now investigate under what circumstances a function is invertible. As a way to motivate the discussion, we consider $f(x) = x^2$. A likely candidate for the inverse is the function $g(x) = \sqrt{x}$. However, $(g \circ f)(x) = g(f(x)) = \sqrt{x^2} = |x|$, which is not equal to x unless $x \geq 0$.

For example, when $x = -2$, $f(-2) = (-2)^2 = 4$, but $g(4) = \sqrt{4} = 2$. That is, g failed to return the input -2 from its output 4. Instead, g matches the output 4 to a *different* input, namely 2, which satisfies $f(2) = 4$. This is shown schematically in Figure 3.6.9.

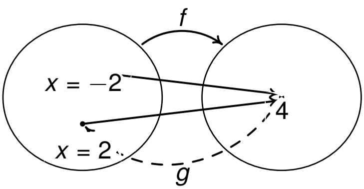


Figure 3.6.9

We see from the diagram that since both $f(-2)$ and $f(2)$ are 4, it is impossible to construct a *function* which takes 4 back to *both* $x = 2$ and $x = -2$ since, by definition, a function can match 4 with only *one* number.

In general, in order for a function to be invertible, each output can come from only *one* input. Since, by definition, a function matches up each input to only *one* output, invertible functions have the property that they match one input to one output and vice-versa. We formalize this concept in Definition 3.6.2.

Definition 3.6.2. A function f is said to be **one-to-one** if whenever $f(a) = f(b)$, then $a = b$.

Note that an equivalent way to state Definition 3.6.2 is that a function is one-to-one if *different* inputs go to *different* outputs. That is, if $a \neq b$, then $f(a) \neq f(b)$.

Before we solidify the connection between invertible functions and one-to-one functions, we take a moment to see what goes wrong graphically when trying to find the inverse of $f(x) = x^2$.

Per Theorem 3.6.1, the graph of $y = f^{-1}(x)$, if it exists, is obtained from the graph of $y = x^2$ by reflecting $y = x^2$ about the line $y = x$. Procedurally, this is accomplished by interchanging the x and y coordinates of each point on the graph of $y = x^2$. Algebraically, we are swapping the variables 'x' and 'y' which results in the equation $x = y^2$ whose graph is shown in Figure 3.6.10.

We see immediately the graph of $x = y^2$ fails the Vertical Line Test, Theorem **??**. In particular, the vertical line $x = 4$ intersects the graph at two points, $(4, -2)$ and $(4, 2)$ meaning the relation described by $x = y^2$ matches the x-value 4 with two different y-values, -2 and 2.

Note that the *vertical* line $x = 4$ and the points $(4, \pm 2)$ on the graph of $x = y^2$ correspond to the *horizontal* line $y = 4$ and the points $(\pm 2, 4)$ on the graph of $y = x^2$ which brings us right back to the concept of one-to-one. The fact that both $(-2, 4)$ and $(2, 4)$ are on the graph of f means $f(-2) = f(2) = 4$. Hence, f takes different inputs, -2 and 2, to the same output, 4, so f is not one-to-one.

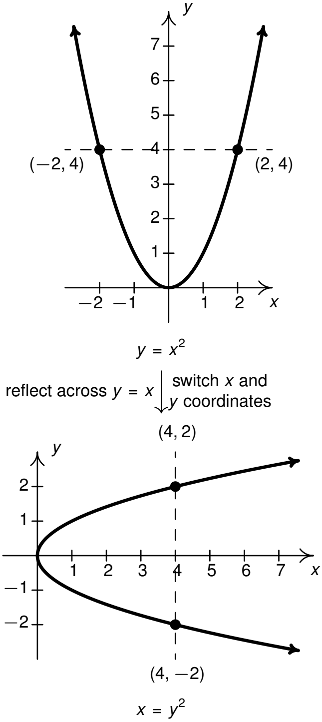


Figure 3.6.10

Recall the Horizontal Line Test from Exercise **??** in Section **??**. Applying that result to the graph of f we say the graph of f 'fails' the Horizontal Line Test since the horizontal line $y = 4$ intersects the graph of $y = x^2$ more than once. This means that the equation $y = x^2$ does not represent x is not a function of y.

Said differently, the Horizontal Line Test detects when there is at least one y-value (4) which is matched to more than one x-value (± 2). In other words, the Horizontal Line Test can be used to detect whether or not a function is one-to-one.

So, to review, $f(x) = x^2$ is not invertible, not one-to-one, and its graph fails the Horizontal Line Test. It turns out that these three attributes: being invertible, one-to-one, and having a graph that passes the Horizontal Line Test are mathematically equivalent. That is to say if one if these things is true about a function, then they all are; it also means that, as in this case, if one of these things *isn't* true about a function, then *none* of them are. We summarize this result in Theorem 3.6.2.

Theorem 3.6.2. Equivalent Conditions for Invertibility:
For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a] i.e., no horizontal line intersects the graph more than once.

To prove Theorem 3.6.2, we first suppose f is invertible. Then there is a function g so that $g(f(x)) = x$ for all x in the domain of f. If $f(a) = f(b)$, then $g(f(a)) = g(f(b))$. Since $g(f(x)) = x$, the equation $g(f(a)) = g(f(b))$ reduces to $a = b$. We've shown that if $f(a) = f(b)$, then $a = b$, proving f is one-to-one.

Next, assume f is one-to-one. Suppose a horizontal line $y = c$ intersects the graph of $y = f(x)$ at the points (a, c) and (b, c). This means $f(a) = c$ and $f(b) = c$ so $f(a) = f(b)$. Since f is one-to-one, this means $a = b$ so the points (a, c) and (b, c) are actually one in the same. This establishes that each horizontal line can intersect the graph of f at most once, so the graph of f passes the Horizontal Line Test.

Last, but not least, suppose the graph of f passes the Horizontal Line Test. Let c be a real number in the range of f. Then the horizontal line $y = c$ intersects the graph of $y = f(x)$ just *once*, say at the point $(a, c) = (a, f(a))$. Define the mapping g so that $g(c) = g(f(a)) = a$. The mapping g is a *function* since each horizontal line $y = c$ where c is in the range of f intersects the graph of f only *once*. By construction, we have the domain of g is the range of f and that for all x in the domain of f,$g(f(x)) = x$. We leave it to the reader to show that for all x in the domain of g, $f(g(x)) = x$, too.

Hence, we've shown: first, if f invertible, then f is one-to-one; second, if f is one-to-one, then the graph of f passes the Horizontal Line Test; and third, if f passes the Horizontal Line Test, then f is invertible. Hence if f is satisfies any one of these three conditions, we can show f must satisfy the other two.[e]

[e] For example, if we know f is one-to-one, we showed the graph of f passes the HLT which, in turn, guarantees f is invertible.

We put this result to work in the next example.

Example 3.6.2. Determine if the following functions are one-to-one: (a) analytically using Definition 3.6.2 and (b) graphically using the Horizontal Line Test. For the functions that are one-to-one, graph the inverse.

1. $f(x) = x^2 - 2x + 4$
2. $g(t) = \dfrac{2t}{1-t}$
3. $F = \{(-1,1),(0,2),(1,-3),(2,1)\}$
4. $G = \{(t^3+1, 2t) \mid t \text{ is a real number.}\}$

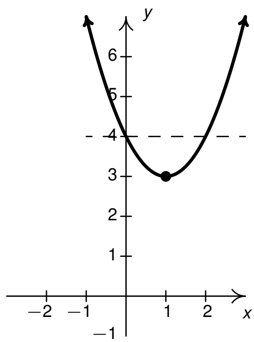
Solution.

1. (a) To determine whether or not f is one-to-one analytically, we assume $f(a) = f(b)$ and work to see if we can deduce $a = b$.

$$\begin{array}{rcl} f(a) & = & f(b) \\ a^2 - 2a + 4 & = & b^2 - 2b + 4 \\ a^2 - 2a & = & b^2 - 2b \\ a^2 - b^2 - 2a + 2b & = & 0 \\ (a+b)(a-b) - 2(a-b) & = & 0 \\ (a-b)((a+b)-2) & = & 0 \\ a - b = 0 & \text{or} & a + b - 2 = 0 \\ a = b & \text{or} & a = 2 - b \end{array}$$

As we work our way through the problem, we encounter a quadratic equation. We rewrite the equation so it equals 0 and factor by grouping. We get $a = b$ as one possibility, but we also get the possibility that $a = 2 - b$. This suggests that f may not be one-to-one. Taking $b = 0$, we get $a = 0$ or $a = 2$. Since $f(0) = 4$ and $f(2) = 4$, we have two different inputs with the same output, proving f is neither one-to-one nor invertible.

(b) We note that f is a quadratic function and we graph $y = f(x)$ using the techniques presented in Section **??** in Figure 3.6.11. We see the graph fails the Horizontal Line Test quite often - in particular, crossing the line $y = 4$ at the points $(0,4)$ and $(2,4)$.



$y = f(x)$

Figure 3.6.11

2. (a) We begin with the assumption that $g(a) = g(b)$ for a, b in the domain of g (That is, we

assume $a \neq 1$ and $b \neq 1$.) Through our work below, we deduce $a = b$, proving g is one-to-one.

$$\begin{aligned} g(a) &= g(b) \\ \frac{2a}{1-a} &= \frac{2b}{1-b} \\ 2a(1-b) &= 2b(1-a) \\ 2a - 2ab &= 2b - 2ba \\ 2a &= 2b \\ a &= b \,\checkmark \end{aligned}$$

(b) We graph $y = g(t)$ in Figure 3.6.12 using the procedure outlined in Section 1.2. We find the sole intercept is $(0, 0)$ with asymptotes $t = 1$ and $y = -2$. Based on our graph, the graph of g appears to pass the Horizontal Line Test, verifying g is one-to-one.

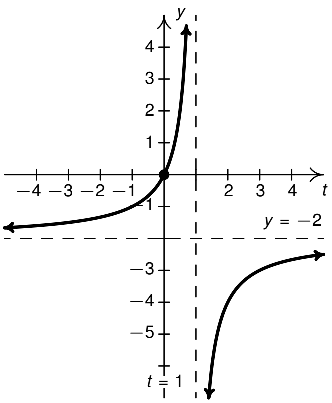


Figure 3.6.12: $y = g(t)$

Since g is one-to-one, g is invertible. Even though we do not have a *formula* for $g^{-1}(t)$, we can nevertheless sketch the graph of $y = g^{-1}(t)$ by reflecting the graph of $y = g(t)$ across $y = t$. See Figure 3.6.13.

Corresponding to the *vertical* asymptote $t = 1$ on the graph of g, the graph of $y = g^{-1}(t)$ will have a *horizontal* asymptote $y = 1$. Similarly, the *horizontal* asymptote $y = -2$ on the graph of g corresponds to a *vertical* asymptote $t = -2$ on the graph of g^{-1}. The point $(0, 0)$ remains unchanged when we switch the t and y coordinates, so it is on both the graph of g and g^{-1}.

3. (a) The function F is given to us as a set of ordered pairs. Recall each ordered pair is of the form $(a, F(a))$. Since $(-1, 1)$ and $(2, 1)$ are both elements of F, this means $F(-1) = 1$ and $F(2) = 1$. Hence, we have two distinct inputs, -1 an 2 with the same output, 1, so F

is not one-to-one and, hence, not invertible.

(b) To graph F, we plot the points in F as shown in Figure 3.6.14. We see the horizontal line $y = 1$ crosses the graph more than once. Hence, the graph of F fails the Horizontal Line Test.

4. Like the function F above, the function G is described as a set of ordered pairs. Before we set about determining whether or not G is one-to-one, we take a moment to show G is, in fact, a function. That is, we must show that each real number input to G is matched to only one output.

We are given $G = \{(t^3+1, 2t) \mid t \text{ is a real number.}\}$. and we know that when represented in this way, each ordered pair is of the form (input, output). Hence, the inputs to G are of the form $t^3 + 1$ and the outputs from G are of the form $2t$. To establish G is a function, we must show that each input produces only one output. If it should happen that $a^3 + 1 = b^3 + 1$, then we must show $2a = 2b$. The equation $a^3 + 1 = b^3 + 1$ gives $a^3 = b^3$, or $a = b$. From this it follows that $2a = 2b$ so G is a function.

(a) To show G is one-to-one, we must show that if two outputs from G are the same, the corresponding inputs must also be the same. That is, we must show that if $2a = 2b$, then $a^3 + 1 = b^3 + 1$. We see almost immediately that if $2a = 2b$ then $a = b$ so $a^3 + 1 = b^3 + 1$ as required. This shows G is one-to-one and, hence, invertible.

(b) We graph G in Figure 3.6.15 by plotting points in the default xy-plane by choosing different values for t. For instance, $t = 0$ corresponds to the point $(0^3 + 1, 2(0)) = (1, 0)$, $t = 1$ corresponds to the point $(1^3 + 1, 2(1)) = (2, 2)$, $t = -1$ corresponds to the point $((-1)^3 + 1, 2(-1)) = (0, -2)$, etc.[f] Our graph appears

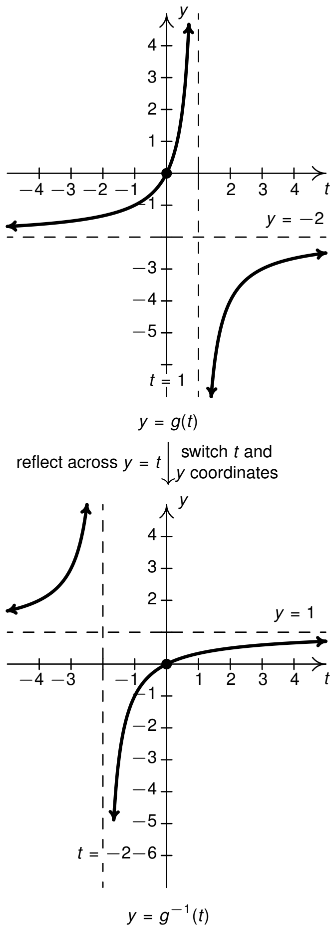


Figure 3.6.13

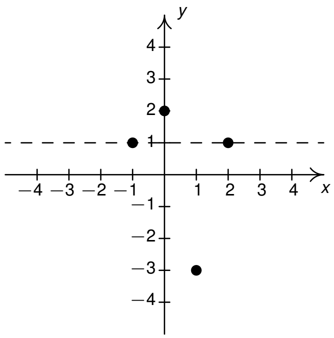


Figure 3.6.14: $y = F(x)$

f Foreshadowing Section **??**, we could let $x = t^3 + 1$ so that $t = \sqrt[3]{x-1}$. Hence, $y = 2t = 2\sqrt[3]{x-1}$.

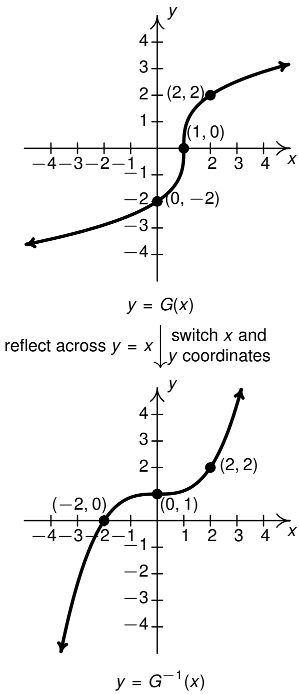


Figure 3.6.15

to pass the Horizontal Line Test, confirming G is one-to-one. We obtain the graph of G^{-1} below on the right by reflecting the graph of G about the line $y = x$.

□

In Example 3.6.2, we showed the functions G and g are invertible and graphed their inverses. While graphs are perfectly fine representations of functions, we have seen where they aren't the most accurate. Ideally, we would like to represent G^{-1} and g^{-1} in the same manner in which G and g are presented to us. The key to doing this is to recall that inverse functions take outputs back to their associated inputs.

Consider $G = \left\{(t^3+1, 2t) \,\middle|\, t \text{ is a real number.}\right\}$. As mentioned in Example 3.6.2, the ordered pairs which comprise G are in the form (input, output). Hence to find a compatible description for G^{-1}, we simply interchange the expressions in each of the coordinates to obtain $G^{-1} = \left\{(2t, t^3+1) \,\middle|\, t \text{ is a real number.}\right\}$.

Since the function g was defined in terms of a formula we would like to find a formula representation for g^{-1}. We apply the same logic as above. Here, the input, represented by the independent variable t, and the output, represented by the dependent variable y, are related by the equation $y = g(t)$. Hence, to exchange inputs and outputs, we interchange the 't' and 'y' variables. Doing so, we obtain the equation $t = g(y)$ which is an *implicit* description for g^{-1}. Solving for y gives an explicit formula for g^{-1}, namely $y = g^{-1}(t)$. We demonstrate this technique below.

$$
\begin{aligned}
y &= g(t) \\
y &= \frac{2t}{1-t} \\
t &= \frac{2y}{1-y} && \text{(interchange variables: } t \text{ and } y\text{)}
\end{aligned}
$$

$$\begin{aligned} t(1-y) &= 2y \\ t - ty &= 2y \\ t &= ty + 2y \\ t &= y(t+2) && \text{(factor)} \\ y &= \frac{t}{t+2} \end{aligned}$$

We claim $g^{-1}(t) = \frac{t}{t+2}$, and leave the algebraic verification of this to the reader.

We generalize this approach in Box 3.6.1. As always, we resort to the default 'x' and 'y' labels for the independent and dependent variables, respectively.

Box 3.6.1. Steps for finding a formula for the Inverse of a one-to-one function

1. Write $y = f(x)$
2. Interchange x and y
3. Solve $x = f(y)$ for y to obtain $y = f^{-1}(x)$

We now return to $f(x) = x^2$. We know that f is not one-to-one, and thus, is not invertible, but our goal here is to see what way to see what goes wrong algebraically.

If we attempt to follow the algorithm above to find a formula for $f^{-1}(x)$, we start with the equation $y = x^2$ and interchange the variables 'x' and 'y' to produce the equation $x = y^2$. Solving for y gives $y = \pm\sqrt{x}$. It's this '$\pm$' which is causing the problem for us since this produces *two* y-values for any $x > 0$. See Figure 3.6.16.

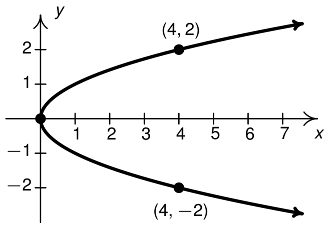


Figure 3.6.16: $x = y^2$

Using the language of Section 3.5, the equation $x = y^2$ implicitly defines *two* functions, $g_1(x) = \sqrt{x}$ and $g_2(x) = -\sqrt{x}$, each of which represents the top and bottom halves, respectively, of the graph of $x = y^2$. See Figure 3.6.17 and Figure 3.6.18.

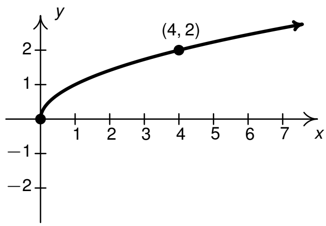


Figure 3.6.17: $y = g_1(x) = \sqrt{x}$

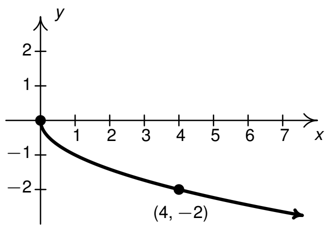


Figure 3.6.18: $y = g_2(x) = -\sqrt{x}$

Hence, in some sense, we have two *partial* inverses for $f(x) = x^2$ (shown in Figure 3.6.19): $g_1(x) = \sqrt{x}$ (shown in Figure 3.6.20) returns the *positive* inputs from f and $g_2(x) = -\sqrt{x}$ (shown in Figure 3.6.21) returns the *negative* inputs to f. In order to view each of these functions as strict inverses, however, we need to split f into two parts: $f_1(x) = x^2$ for $x \geq 0$ and $f_2(x) = x^2$ for $x \leq 0$.

We claim that f_1 and g_1 are an inverse function pair as are f_2 and g_2.

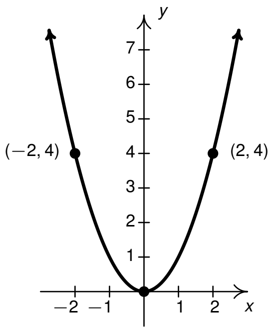


Figure 3.6.19: $y = f(x) = x^2$

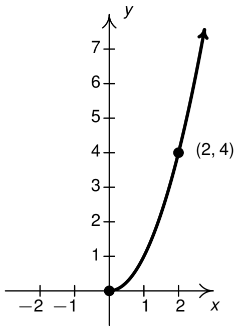


Figure 3.6.20: $y = f_1(x) = x^2$, $x \geq 0$

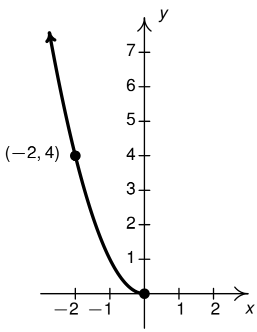


Figure 3.6.21: $y = f_2(x) = x^2$, $x \leq 0$

$$\begin{aligned}(g_1 \circ f_1)(x) &= g_1(f_1(x))\\ &= g_1(x^2)\\ &= \sqrt{x^2}\\ &= |x| = x, && \text{(as } x \geq 0\text{)}\end{aligned}$$

$$\begin{aligned}(f_1 \circ g_1)(x) &= f_1(g_1(x))\\ &= f_1(\sqrt{x})\\ &= (\sqrt{x})^2\\ &= x\end{aligned}$$

$$\begin{aligned}(g_2 \circ f_2)(x) &= g_2(f_2(x))\\ &= g_2(x^2)\\ &= -\sqrt{x^2}\\ &= -|x|\\ &= -(-x) = x && \text{(as } x \leq 0\text{)}\end{aligned}$$

$$\begin{aligned}(f_2 \circ g_2)(x) &= f_2(g_2(x))\\ &= f_2(-\sqrt{x})\\ &= (-\sqrt{x})^2\\ &= (\sqrt{x})^2\\ &= x\end{aligned}$$

Indeed, we find in Figure 3.6.22 and Figure 3.6.23.

Hence, by restricting the domain of f we are able to produce invertible functions. Said differently, in much the same way the equation $x = y^2$ implicitly describes a pair of *functions*, the equation $y = x^2$ implicitly describes a pair of *invertible* functions.

Our next example continues the theme of restricting the domain of a function to find inverse functions.

Example 3.6.3. Graph the following functions to show they are one-to-one and find their inverses. Check your answers analytically using function composition and graphically.

1. $j(x) = x^2 - 2x + 4$, $x \leq 1$.
2. $k(t) = \sqrt{t+2} - 1$

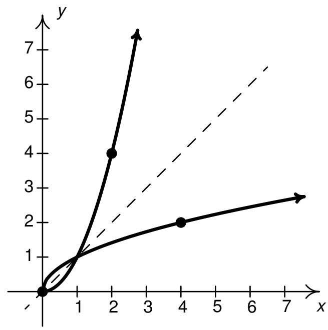


Figure 3.6.22: $y = f_1(x) = x^2$, $x \geq 0$ and $y = g_1(x) = \sqrt{x}$

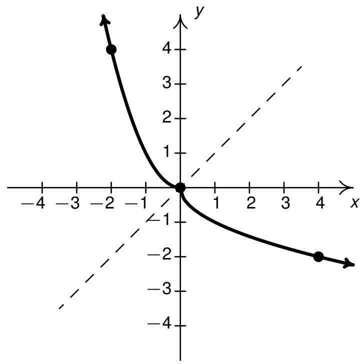


Figure 3.6.23: $y = f_2(x) = x^2$, $x \leq 0$ and $y = g_2(x) = -\sqrt{x}$

Solution.

1. The function j is a restriction of the function f from Example 3.6.2. Since the domain of j is restricted to $x \leq 1$, we are selecting only the 'left half' of the parabola. Hence, the graph of j, seen in Figure 3.6.24, passes the Horizontal Line Test and thus j is invertible. Below, we find an explicit formula for $j^{-1}(x)$ using our standard algorithm.[g]

$$y = j(x)$$
$$y = x^2 - 2x + 4, \quad x \leq 1$$
$$x = y^2 - 2y + 4, \quad y \leq 1 \qquad \text{(switch } x \text{ and } y\text{)}$$
$$0 = y^2 - 2y + 4 - x$$
$$y = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4-x)}}{2(1)} \qquad \text{(quadratic formula, } c = 4 - x\text{)}$$
$$y = \frac{2 \pm \sqrt{4x - 12}}{2}$$
$$y = \frac{2 \pm \sqrt{4(x-3)}}{2}$$
$$y = \frac{2 \pm 2\sqrt{x-3}}{2}$$
$$y = \frac{2\left(1 \pm \sqrt{x-3}\right)}{2}$$

g Here, we use the Quadratic Formula to solve for y. For 'completeness,' we note you can (and should!) also consider solving for y by 'completing' the square.

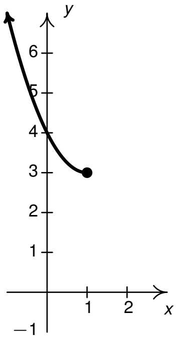


Figure 3.6.24: $y = j(x)$

$$y = 1 \pm \sqrt{x-3}$$
$$y = 1 - \sqrt{x-3} \qquad \text{(since } y \leq 1\text{)}$$

Hence, $j^{-1}(x) = 1 - \sqrt{x-3}$.

To check our answer algebraically, we simplify $(j^{-1} \circ j)(x)$ and $(j \circ j^{-1})(x)$ Note the importance of the domain restriction $x \leq 1$ when simplifying $(j^{-1} \circ j)(x)$.

$$\begin{aligned}
\left(j^{-1} \circ j\right)(x) &= j^{-1}(j(x)) \\
&= j^{-1}\left(x^2 - 2x + 4\right), \quad x \leq 1 \\
&= 1 - \sqrt{\left(x^2 - 2x + 4\right) - 3} \\
&= 1 - \sqrt{x^2 - 2x + 1} \\
&= 1 - \sqrt{(x-1)^2} \\
&= 1 - |x - 1| \\
&= 1 - (-(x-1)) \qquad \text{(since } x \leq 1\text{)} \\
&= x \ \checkmark
\end{aligned}$$

$$\begin{aligned}
\left(j \circ j^{-1}\right)(x) &= j\left(j^{-1}(x)\right) \\
&= j\left(1 - \sqrt{x-3}\right) \\
&= \left(1 - \sqrt{x-3}\right)^2 - 2\left(1 - \sqrt{x-3}\right) + 4 \\
&= 1 - 2\sqrt{x-3} + \left(\sqrt{x-3}\right)^2 - 2 \\
&\quad + 2\sqrt{x-3} + 4 \\
&= 1 + x - 3 - 2 + 4 \\
&= x \ \checkmark
\end{aligned}$$

We graph both j and j^{-1} on the axes in Figure 3.6.25. They appear to be symmetric about the line $y = x$.

2. Graphing $y = k(t) = \sqrt{t+2} - 1$ (Figure 3.6.26), we see k is one-to-one, so we proceed to find an formula for k^{-1}.

$$\begin{aligned} y &= k(t) \\ y &= \sqrt{t+2} - 1 \\ t &= \sqrt{y+2} - 1 && \text{(switch } t \text{ and } y\text{)} \\ t + 1 &= \sqrt{y+2} \\ (t+1)^2 &= \left(\sqrt{y+2}\right)^2 \\ t^2 + 2t + 1 &= y + 2 \\ y &= t^2 + 2t - 1 \end{aligned}$$

We have $k^{-1}(t) = t^2 + 2t - 1$. Based on our experience, we know something isn't quite right. We determined k^{-1} is a quadratic function, and we have seen several times in this section that these are not one-to-one unless their domains are suitably restricted.

Theorem 3.6.1 tells us that the domain of k^{-1} is the range of k. From the graph of k, we see that the range is $[-1, \infty)$, which means we restrict the domain of k^{-1} to $t \geq -1$.

We now check that this works in our compositions. Note the importance of the domain restriction, $t \geq -1$ when simplifying $(k \circ k^{-1})(t)$.

$$\begin{aligned} \left(k^{-1} \circ k\right)(t) &= k^{-1}(k(t)) \\ &= k^{-1}\left(\sqrt{t+2} - 1\right) \end{aligned}$$

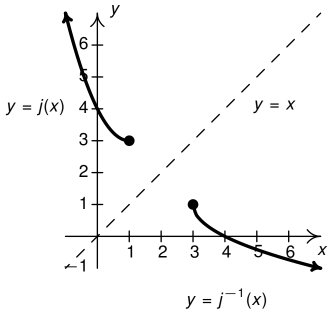


Figure 3.6.25

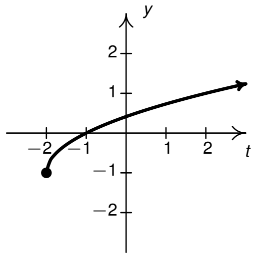


Figure 3.6.26: $y = k(t)$

$$
\begin{aligned}
&= \left(\sqrt{t+2}-1\right)^2 + 2\left(\sqrt{t+2}-1\right) - 1 \\
&= \left(\sqrt{t+2}\right)^2 - 2\sqrt{t+2} + 1 + 2\sqrt{t+2} - 2 - 1 \\
&= t + 2 - 2 \\
&= t \ \checkmark
\end{aligned}
$$

$$
\begin{aligned}
\left(k \circ k^{-1}\right)(t) &= k\left(t^2 + 2t - 1\right), \quad t \geq -1 \\
&= \sqrt{\left(t^2 + 2t - 1\right) + 2} - 1 \\
&= \sqrt{t^2 + 2t + 1} - 1 \\
&= \sqrt{(t+1)^2} - 1 \\
&= |t+1| - 1 \\
&= t + 1 - 1, \ \text{since } t \geq -1 \\
&= t \ \checkmark
\end{aligned}
$$

Graphically, everything checks out, provided that we remember the domain restriction on k^{-1} means we take the right half of the parabola. See Figure 3.6.27.

□

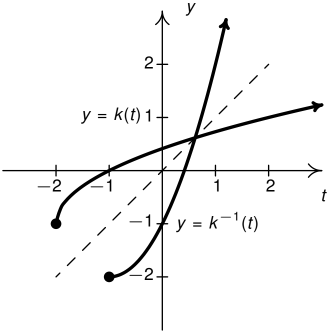


Figure 3.6.27

Our last example of the section gives an application of inverse functions. Recall in Example **??** in Section **??**, we modeled the demand for PortaBoy game systems as the price per system, $p(x)$ as a function of the number of systems sold, x. In the following example, we find $p^{-1}(x)$ and interpret what it means.

Example 3.6.4. Recall the price-demand function for PortaBoy game systems is modeled by the formula $p(x) = -1.5x + 250$ for $0 \leq x \leq 166$ where x represents the number of systems sold (the demand) and $p(x)$ is the price per system, in dollars.

1. Explain why p is one-to-one and find a formula for $p^{-1}(x)$. State the restricted domain.
2. Find and interpret $p^{-1}(220)$.
3. Recall from Section **??** that the profit P, in dollars, as a result of selling x systems is given by $P(x) = -1.5x^2 + 170x - 150$. Find and interpret $\left(P \circ p^{-1}\right)(x)$.
4. Use your answer to part 3 to determine the price per PortaBoy which would yield the maximum profit. Compare with Example **??**.

Solution.

1. Recall the graph of $p(x) = -1.5x + 250$, $0 \leq x \leq 166$, is a line segment from $(0, 250)$ to $(166, 1)$, and as such passes the Horizontal Line Test. Hence, p is one-to-one. We find the expression for $p^{-1}(x)$ as usual and get $p^{-1}(x) = \frac{500-2x}{3}$. The domain of p^{-1} should match the range of p, which is $[1, 250]$, and as such, we restrict the domain of p^{-1} to $1 \leq x \leq 250$.
2. We find $p^{-1}(220) = \frac{500-2(220)}{3} = 20$. Since the function p took as inputs the number of systems sold and returned the price per system as the output, p^{-1} takes the price per system as its input and returns the number of systems sold as its output. Hence, $p^{-1}(220) = 20$ means 20 systems will be sold in if the price is set at \$220 per system.
3. We compute

$$\left(P \circ p^{-1}\right)(x) = P\left(p^{-1}(x)\right) = P\left(\frac{500-2x}{3}\right)$$
$$= -1.5\left(\frac{500-2x}{3}\right)^2 + 170\left(\frac{500-2x}{3}\right) - 150$$

After a hefty amount of Elementary Algebra,[h] we obtain $\left(P \circ p^{-1}\right)(x) = -\frac{2}{3}x^2 + 220x - \frac{40450}{3}$.

h It is good review to actually do this!

To understand what this means, recall that the original profit function P gave us the profit as a function of the number of systems sold. The function p^{-1} gives us the number of systems sold as a function of the price. Hence, when we compute $(P \circ p^{-1})(x) = P(p^{-1}(x))$, we input a price per system, x into the function p^{-1}.

The number $p^{-1}(x)$ is the number of systems sold at that price. This number is then fed into P to return the profit obtained by selling $p^{-1}(x)$ systems. Hence, $\left(P \circ p^{-1}\right)(x)$ gives us the profit (in dollars) as a function of the price per system, x.

4. We know from Section **??** that the graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards. The maximum profit is realized at the vertex. Since we are concerned only with the price per system, we need only find the x-coordinate of the vertex. Identifying $a = -\frac{2}{3}$ and $b = 220$, we get, by the Vertex Formula, Equation **??**, $x = -\frac{b}{2a} = 165$.

   Hence, weekly profit is maximized if we set the price at \$165 per system. Comparing this with our answer from Example **??**, there is a slight discrepancy to the tune of \$0.50. We leave it to the reader to balance the books appropriately. □

3.6.1 Exercises

In Exercises 1 - 8, verify the given pairs of functions are inverses algebraically and graphically.

1. $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$
2. $f(x) = \dfrac{5-3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$.
3. $f(t) = \dfrac{5}{t-1}$ and $g(t) = \dfrac{t+5}{t}$
4. $f(t) = \dfrac{t}{t-1}$ and $g(t) = f(t) = \dfrac{t}{t-1}$
5. $f(x) = \sqrt{4-x}$ and $g(x) = -x^2 + 4$, $x \geq 0$
6. $f(x) = 1 - \sqrt{x+1}$ and $g(x) = x^2 - 2x$, $x \leq 1$.
7. $f(t) = (t-1)^3 + 5$ and $g(t) = \sqrt[3]{t-5} + 1$
8. $f(t) = -\sqrt[4]{t-2}$ and $g(t) = t^4 + 2$, $t \leq 0$.

In Exercises 9 - 28, show that the given function is one-to-one and find its inverse. Check your answers algebraically and graphically. Verify the range of the function is the domain of its inverse and vice-versa.

9. $f(x) = 6x - 2$
10. $f(x) = 42 - x$
11. $g(t) = \dfrac{t-2}{3} + 4$
12. $g(t) = 1 - \dfrac{4+3t}{5}$
13. $f(x) = \sqrt{3x-1} + 5$
14. $f(x) = 2 - \sqrt{x-5}$
15. $g(t) = 3\sqrt{t-1} - 4$
16. $g(t) = 1 - 2\sqrt{2t+5}$
17. $f(x) = \sqrt[5]{3x-1}$
18. $f(x) = 3 - \sqrt[3]{x-2}$
19. $g(t) = t^2 - 10t$, $t \geq 5$
20. $g(t) = 3(t+4)^2 - 5$, $t \leq -4$
21. $f(x) = x^2 - 6x + 5$, $x \leq 3$
22. $f(x) = 4x^2 + 4x + 1$, $x < -1$

23. $g(t) = \dfrac{3}{4-t}$

24. $g(t) = \dfrac{t}{1-3t}$

25. $f(x) = \dfrac{2x-1}{3x+4}$

26. $f(x) = \dfrac{4x+2}{3x-6}$

27. $g(t) = \dfrac{-3t-2}{t+3}$

28. $g(t) = \dfrac{t-2}{2t-1}$

29. Explain why each set of ordered pairs below represents a one-to-one function and find the inverse.

    (a) $F = \{(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3)\}$

    (b) $G = \{(0,0),(1,1),(2,-1),(3,2),(4,-2),(5,3),(6,-3),...\}$

    NOTE: The difference between F and G is the '....'

    (c) $P = \{(2t^5, 3t-1) \mid t \text{ is a real number.}\}$

    (d) $Q = \{(n, n^2) \mid n \text{ is a } \textit{natural} \text{ number.}\}$[i]

Recall this means $n = 0, 1, 2, ...$ [i]

In Exercises 30 - 33, explain why each graph represents[a] a one-to-one function and graph its inverse.

[a] or, more precisely, *appears* to represent . . .

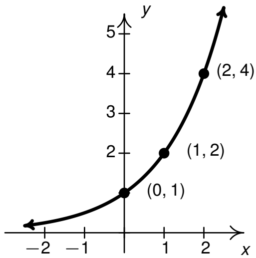


Figure 3.6.28: Asymptote: $y = 0$.

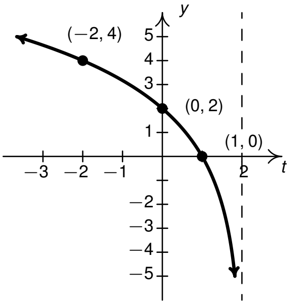


Figure 3.6.29: Asymptote: $t = 2$.

30. $y = f(x)$. See Figure 3.6.28.

31. $y = g(t)$. See Figure 3.6.29.

32. $y = S(t)$. See Figure 3.6.30.

33. $y = R(s)$. See Figure 3.6.31.

34. The price of a dOpi media player, in dollars per dOpi, is given as a function of the weekly sales x according to the formula $p(x) = 450 - 15x$ for $0 \leq x \leq 30$.

    (a) Find $p^{-1}(x)$ and state its domain.

    (b) Find and interpret $p^{-1}(105)$.

(c) The profit (in dollars) made from producing and selling x dOpis per week is given by the formula $P(x) = -15x^2+350x-2000$, for $0 \leq x \leq 30$. Find $\left(P \circ p^{-1}\right)(x)$ and determine what price per dOpi would yield the maximum profit. What is the maximum profit? How many dOpis need to be produced and sold to achieve the maximum profit?

35. Show that the Fahrenheit to Celsius conversion function found in Exercise **??** in Section **??** is invertible and that its inverse is the Celsius to Fahrenheit conversion function.

36. Analytically show that the function $f(x) = x^3+3x+1$ is one-to-one. Use Theorem 3.6.1 to help you compute $f^{-1}(1)$, $f^{-1}(5)$, and $f^{-1}(-3)$. What happens when you attempt to find a formula for $f^{-1}(x)$?

37. Let $f(x) = \dfrac{2x}{x^2-1}$.

(a) Graph $y = f(x)$ using the techniques in Section 1.2. Check your answer using a graphing utility.

(b) Verify that f is one-to-one on the interval $(-1, 1)$.

(c) Use the procedure outlined on Page 337 to find the formula for $f^{-1}(x)$ for $-1 < x < 1$.

(d) Since $f(0) = 0$, it should be the case that $f^{-1}(0) = 0$. What goes wrong when you attempt to substitute $x = 0$ into $f^{-1}(x)$? Discuss with your classmates how this problem arose and possible remedies.

38. With the help of your classmates, explain why a function which is either strictly increasing or strictly decreasing on its entire domain would have to be one-to-one, hence invertible.

39. If f is odd and invertible, prove that f^{-1} is also odd.

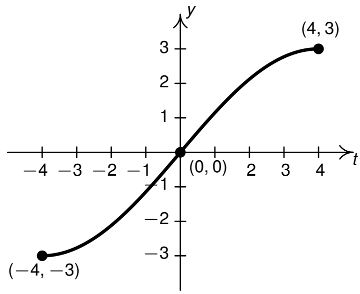


Figure 3.6.30: Domain: $[-4, 4]$.

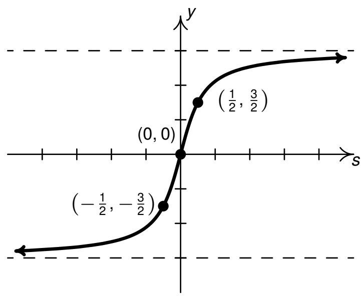


Figure 3.6.31: Asymptotes: $y = \pm 3$.

With help from your classmates, find the inverses of the functions in Exercises 41 - 44.

40. Let f and g be invertible functions. With the help of your classmates show that $(f \circ g)$ is one-to-one, hence invertible, and that $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

41. $f(x) = ax + b, \;\; a \neq 0$

42. $f(x) = a\sqrt{x - h} + k, \;\; a \neq 0, x \geq h$

43. $f(x) = ax^2 + bx + c$ where $a \neq 0$, $x \geq -\dfrac{b}{2a}$.

44. $f(x) = \dfrac{ax + b}{cx + d}$, (See Exercise 45 below.)

45. What conditions must you place on the values of a, b, c and d in Exercise 44 in order to guarantee that the function is invertible?

46. The function given in number 4 is an example of a function which is its own inverse.

    (a) Algebraically verify every function of the form: $f(x) = \dfrac{ax + b}{cx - a}$ is its own inverse.

    What assumptions do you need to make about the values of a, b, and c?

    (b) Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

3.6.2 Answers

9. $f^{-1}(x) = \dfrac{x+2}{6}$
10. $f^{-1}(x) = 42 - x$
11. $g^{-1}(t) = 3t - 10$
12. $g^{-1}(t) = -\frac{5}{3}t + \frac{1}{3}$
13. $f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}$, $x \geq 5$
14. $f^{-1}(x) = (x-2)^2 + 5$, $x \leq 2$
15. $g^{-1}(t) = \frac{1}{9}(t+4)^2 + 1$, $t \geq -4$
16. $g^{-1}(t) = \frac{1}{8}(t-1)^2 - \frac{5}{2}$, $t \leq 1$
17. $f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$
18. $f^{-1}(x) = -(x-3)^3 + 2$
19. $g^{-1}(t) = 5 + \sqrt{t+25}$
20. $g^{-1}(t) = -\sqrt{\frac{t+5}{3}} - 4$
21. $f^{-1}(x) = 3 - \sqrt{x+4}$
22. $f^{-1}(x) = -\frac{\sqrt{x}+1}{2}$, $x > 1$
23. $g^{-1}(t) = \dfrac{4t-3}{t}$
24. $g^{-1}(t) = \dfrac{t}{3t+1}$
25. $f^{-1}(x) = \dfrac{4x+1}{2-3x}$
26. $f^{-1}(x) = \dfrac{6x+2}{3x-4}$
27. $g^{-1}(t) = \dfrac{-3t-2}{t+3}$
28. $g^{-1}(t) = \dfrac{t-2}{2t-1}$

29. (a) None of the first coordinates of the ordered pairs in F are repeated, so F is a function and none of the second coordinates of the ordered pairs of F are repeated, so F is one-to-one.

$$F^{-1} = \{(0,0),(1,1),(-1,2),(2,3),\\ (-2,4),(3,5),(-3,6)\}$$

(b) Because of the '...' it is helpful to determine a formula for the matching. For the even numbers n, $n = 0, 2, 4, ...$, the ordered pair $\left(n, -\frac{n}{2}\right)$ is in G. For the odd numbers $n = 1, 3, 5, ...$, the ordered pair $\left(n, \frac{n+1}{2}\right)$ is in G. Hence, given any input to G, n, whether it be even or odd, there is only one output from G, either $-\frac{n}{2}$ or $\frac{n+1}{2}$, both of which are functions of n. To show G is one to one, we note that if the output from G is 0 or less, then it must be of the form $-\frac{n}{2}$ for an even number n. Moreover, if $-\frac{n}{2} = -\frac{m}{2}$, then $n = m$. In the case we are looking at outputs from G which are greater than 0, then it must be of the form $\frac{n+1}{2}$ for an odd number n. In this, too, if $\frac{n+1}{2} = \frac{m+1}{2}$, then $n = m$. Hence, in any case, if the outputs from G are the same, then the inputs to G had to be the same so G is one-to-one and

$$G^{-1} = \{(0,0),(1,1),(-1,2),(2,3),(-2,4),\\ (3,5),(-3,6),...\}$$

(c) To show P is a function we note that if we have the same inputs to P, say $2t^5 = 2u^5$, then $t = u$. Hence the corresponding outputs, $2t - 1$ and $3u - 1$, are equal, too. To show P is one-to-one, we note that if we have the same outputs from P, $3t - 1 = 3u - 1$, then $t = u$. Hence, the corresponding inputs

$2t^5$ and $2u^5$ are equal, too. Hence P is one-to-one and

$$P^{-1} = \left\{(3t-1, 2t^5) \,\middle|\, t \text{ is a real number.}\right\}$$

(d) To show Q is a function, we note that if we have the same inputs to Q, say $n = m$, then the outputs from Q, namely n^2 and m^2 are equal. To show Q is one-to-one, we note that if we get the same output from Q, namely $n^2 = m^2$, then $n = \pm m$. However since n and m are *natural* numbers, both n and m are positive so $n = m$. Hence Q is one-to-one and

$$Q^{-1} = \left\{(n^2, n) \,\middle|\, \right\} \quad (n \text{ is a } \textit{natural} \text{ number.})$$

30. $y = f^{-1}(x)$. Asymptote: $x = 0$. See Figure 3.6.32.

31. $y = g^{-1}(t)$. Asymptote: $y = 2$. See **??**.

32. $y = S^{-1}(t)$. Domain $[-3, 3]$. See Figure 3.6.35.

33. $y = R^{-1}(s)$. Asymptotes: $s = \pm 3$. See Figure 3.6.35

34. (a) $p^{-1}(x) = \frac{450-x}{15}$. The domain of p^{-1} is the range of p which is $[0, 450]$

(b) $p^{-1}(105) = 23$. This means that if the price is set to \$105 then 23 dOpis will be sold.

The graph of $y = \left(P \circ p^{-1}\right)(x)$ is a parabola opening downwards with vertex $\left(275, \frac{125}{3}\right) \approx (275, 41.67)$. This means that the maximum profit is a whopping \$41.67 when the price per dOpi is set to \$275. At this price, we can produce and sell $p^{-1}(275) = 11.\overline{6}$ dOpis. Since we cannot sell part of a system, we need to adjust the price to sell either 11 dOpis or 12 dOpis. We find $p(11) = 285$ and $p(12) = 270$, which means we set the price per dOpi

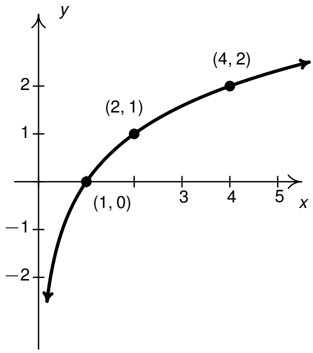


Figure 3.6.32

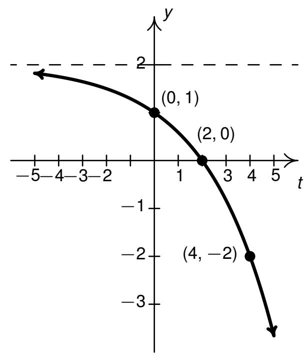


Figure 3.6.33

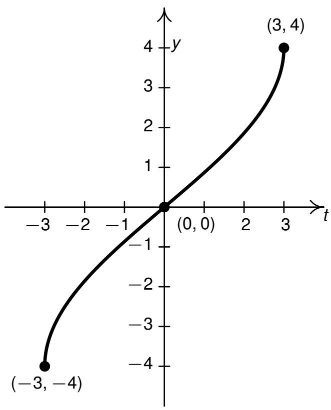


Figure 3.6.34

at either \$285 or \$270, respectively. The profits at these prices are $\left(P \circ p^{-1}\right)(285) = 35$ and $\left(P \circ p^{-1}\right)(270) = 40$, so it looks as if the maximum profit is \$40 and it is made by producing and selling 12 dOpis a week at a price of \$270 per dOpi.

36. Given that $f(0) = 1$, we have $f^{-1}(1) = 0$. Similarly $f^{-1}(5) = 1$ and $f^{-1}(-3) = -1$

46. (b) If $b = 0$, then $m = \pm 1$. If $b \neq 0$, then $m = -1$ and b can be any real number.

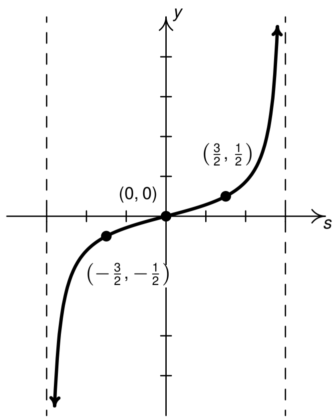


Figure 3.6.35