A Lagrangian Approach to Scalable Mixed Constraint Portfolio Optimization

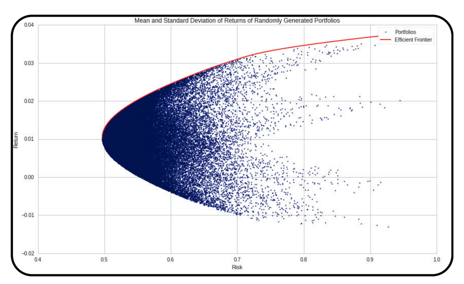
Introduction

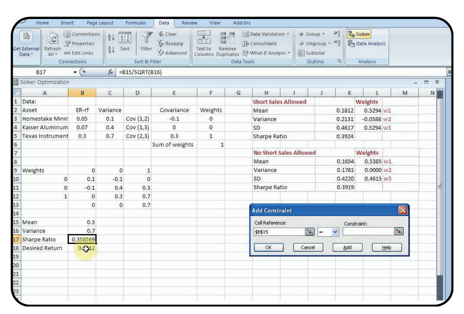
Introduction: The Lagrangian method (LM) is used to **mathematically** solve an objective function subject to constraints; SLSQP is an **algorithmic** iterative method for constrained nonlinear optimization. These methods hold significant ability for wide-range problem-solving in financial portfolio optimization.

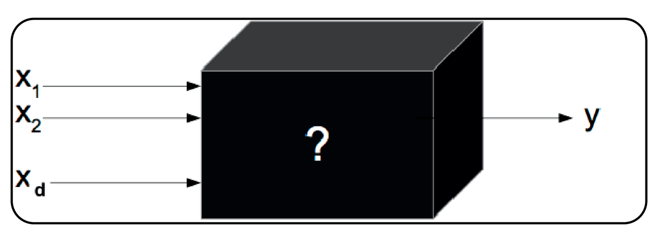
- **Computationally taxing** portfolio optimization tools dominate 82% of modern solutions
- ~95% of investors **lack the specialized capacity** needed^^
- ~65% **lack informed investment decisions** ---> (financial exclusion & lack of well-being)

Research Objectives

Current Portfolio Optimization Solutions
Inaccessibly computer-resource intensive
No constraints/excessive distortive constraints
Complex output interpretation/non-user friendly
Blackbox/derivative-free optimization
No insight into input/correlation impact







Engineering Objectives
Incorporate essential user-defined constraints
Numerical/analytical solutions cross validation
Sensitivity analysis insight; graphical output
Accessibility via simplified web based interface
Derive scalable solutions

Project Goal: Build a novel **simplified and accessible** web-based constrained **portfolio optimization tool** by developing a generalized **SLSQP algorithm** and mathematically prove its convergence with a **Lagrangian method.**

Abstract

This research aims to enhance constrained portfolio optimization by building an accessible web framework, employing a Lagrangian technique and generalized Sequential Least Squares Programming (SLSQP) algorithm to compute optimal allocation outputs. This development offers a groundbreaking advancement in Markowitz Mean-Variance optimization by providing a simplified white-box constraint-efficient model alternative to prohibitively complex derivative-free optimization tools.

By integrating multivariable calculus and linear programming methods, we develop an application of the Lagrange Multipliers approach, facilitating optimization subject to an essential combination of constraints. We mathematically derive an analytical solution equation for optimal allocation proportions using arbitrary variables for expected return, standard deviation, and portfolio risk limit. All analytical solutions are cross-validated using numerical solutions derived from the implementation of a quasi-Newton SLSQP algorithm, run from the Streamlit backend. Statistical analysis of coded graphical outputs is conducted to investigate the impact of inputs on portfolio performance, and a model comparison is simulated to analyze performance against alternative strategies.

The web framework streamlines the optimization process, instantly outputting optimal asset allocation and expected return with graphical insight of the portfolio against user-defined variables and constraints— removing the friction from inaccessible methods and their complex output data into tailored, intuitive visualizations any investor can understand and act upon. We prove the accuracy of convergence results of this model software with the Lagrangian method and find that the Lagrangian proportions consistently outperform alternative strategies. This superior model presents significant democratizing access to informed investment decision-making through a novel, user-efficient approach to advanced portfolio optimization.

Software Methodology Framework

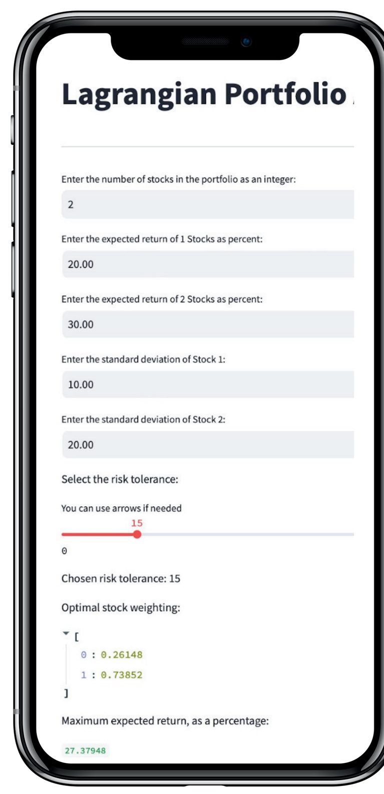
Software Design → Case Study → Problem Formulation → Cross-Validation

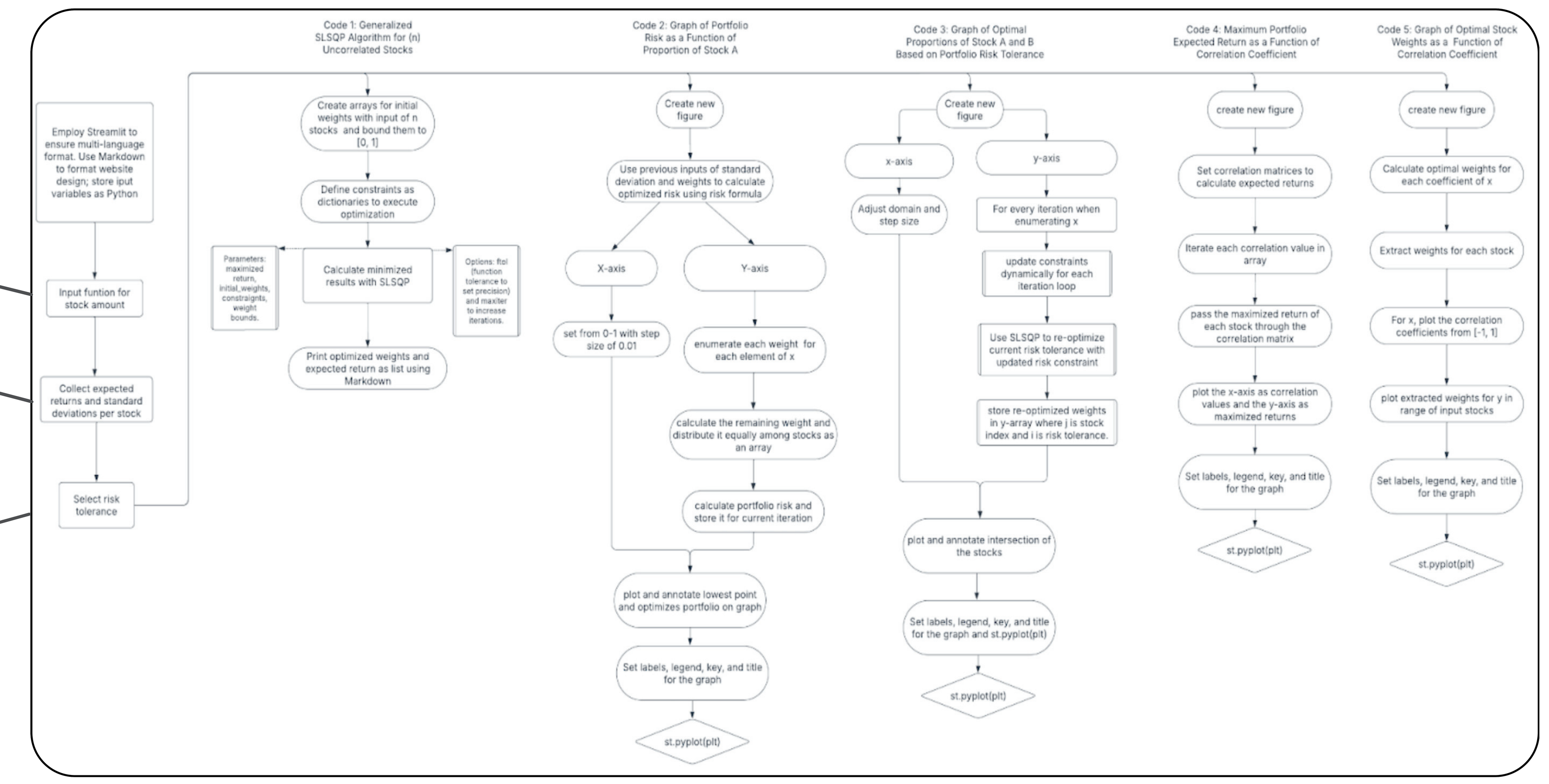
Code 1 is used to implement the generalized SLSQP alogorithm, whihc accurately determines the optimal allocation of funds. **Code 2** is used to generate a graph that visually represents the impact of varying the allocation of Stock A on the resulting portfotlio risk. **Code 3** provides valuable insights into the allocation strategy based on risk tolerance. **Code 4** generates a graph that offers valuable insights into the influence of the correlation coefficient on the portfolio's performance. **Code 5** generates a graph that showcases the dynamic allocation strategies that evolve as the correlation coefficient changes. (See examples in **Statistical** Analysis)

Case Study 1:
Stock A: Expected return of 20% and standard deviation of 10%
Stock B: Expected return of 30% and standard deviation of 20%

The web framework outputs are coded with a generalized **Sequential Least Squares Programming (SLSQP) algorithm**. Requiring only essential and straightforward user-defined constraints, optimal stock weighting and portfolio maximized expected return is given by the algorithm. Solutions generated here are **numerical solutions**, to be cross-validated with analytical solutions.

User Frontend





Lagrangian Methodology Framework

Lagrangian Method - Proof and Solution Derivation

Define Optimization Problem → Slack Variables → Lagrangian Function → Derive the Quadratic → Compute Portfolio Weights

Define Optimization Problem

$$\text{Let } f(\omega_A,\omega_B) = E(r_p)$$

and

$$g(\omega_A,\omega_B) = \omega_A + \omega_B - 1 = 0$$

Denotes the expected return of the portfolio to be maximized and the budget constraint.

The general optimization problem can be formualted as follows:

$$\textbf{max} \quad E(r_p) = E(r_A)\omega_A + E(r_B)\omega_B$$

subject to the following constraints:

$$\omega_A + \omega_B = 1$$
$$\omega_A, \omega_B \geq 0$$
$$\sigma^2_A\omega^2_A + \sigma^2_B\omega^2_B \leq R^2$$

Slack Variables

$$h(\omega_A, s_1) = \omega_A - s_1^2 = 0$$
$$j(\omega_B, s_2) = \omega_B - s_2^2 = 0$$
$$k(\omega_A,\omega_B,s_3) = \sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 + s_3^2 = 0.$$

Introduce 3 non-negative slack variables to convert the inequality constraints to equality constraints.

These transformed constraints, along with the equality constraint

$$g(\omega_A,\omega_B) = \omega_A + \omega_B - 1 = 0$$

Forms a unified set of 4 equality constraints. To incorporate the constraints, introduce 4 Lagrangian Multipliers.

Lagrangian Function

$$E(r_A) = \lambda_1 + \lambda_2 + 2\sigma_A^2\omega_A\lambda_4$$
$$E(r_B) = \lambda_1 + \lambda_3 + 2\sigma_B^2\omega_B\lambda_4$$
$$0 = -2s_1\lambda_2$$
$$0 = -2s_2\lambda_3$$
$$0 = 2s_3\lambda_4$$
$$\omega_A + \omega_B - 1 = 0$$
$$\omega_A - s_1^2 = 0$$
$$\omega_B - s_2^2 = 0$$
$$\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 + s_3^2 = 0.$$

Formulate the gradient equation with Lagrange Multipliers; derive a 9 system equation by evaluating the partial derivatives & performing the necessary substitutions.

Derive the Quadratic

$$\omega_A + \omega_B - 1 = 0$$
$$\sigma_A^2\omega_A^2 + \sigma_B^2\omega_B^2 - R^2 = 0.$$
$$(\sigma_A^2 + \sigma_B^2)\omega_A^2 - (2\sigma_B^2)\omega_A + (\sigma_B^2 - R^2) = 0.$$

Establish a system of 2 equations; solve 1st equation for Stock A weight and substitute into 2nd; derive equation equation

Compute Portfolio Weights

Using the derived arbitrary analytical solution equation, we can solve for the optimal proportions of any unique input case, given expected return and standard deviation of assets, and risk limit.

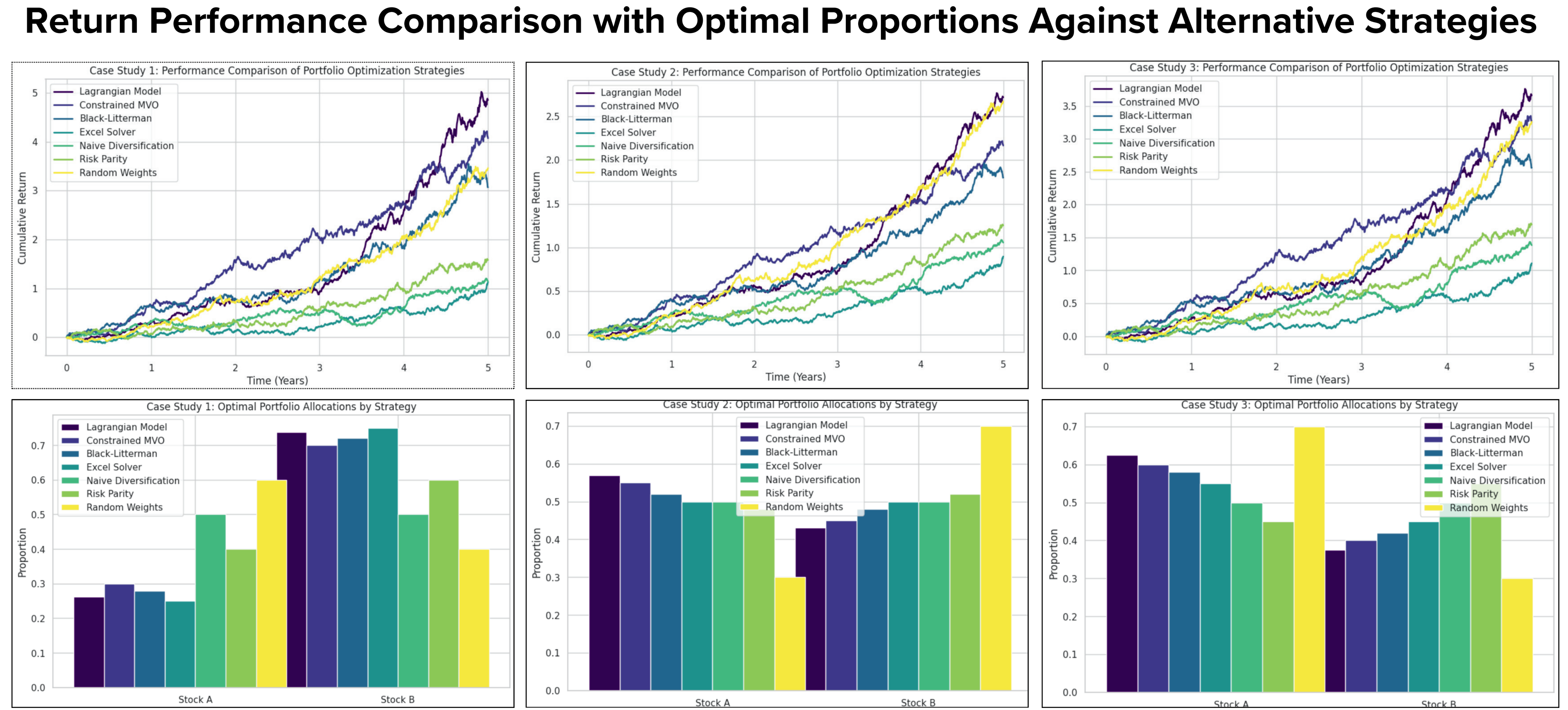
Arbitrary Solution

Assuming positive proportions for both Stocks A and B, the optimal allocation proportions that maximize the expected return of the portfolio are

$$\omega_A = \frac{\sigma_B^2 \pm \sqrt{\sigma_B^4 - (\sigma_B^2 - R^2)(\sigma_A^2 + \sigma_B^2)}}{\sigma_A^2 + \sigma_B^2} \quad \text{and} \quad \omega_B = 1 - \omega_A.$$

Model Performance Analysis

Return Performance Comparison with Optimal Proportions Against Alternative Strategies



By simulating cumulative returns forward, we prove the superiority of the outputs generated by our model over outputs generated by the comparable alternative strategies (all strategies follow the same input constraints):

- **Constrained Mean Variance Optimization (MVO)**
- **Constrained Black-Litterman**
- **Excel-Solver Based Optimization**
- **Naive diversification (Equal Weight)**
- **Risk Parity Model**
- **Random Weights**

Simulation Method:
Returns are simulated across a 5-year period (to ensure enough granularity for realistic performance trajectory and market cycles) using a geometric Brownian motion (GBM) lognormal model aligning with given expected return, standard deviation, and volatility.

Optimal Allocation Weights:
Computed for each strategy based on their standard models. Each strategy generates different optimal proportions, influencing performance analyzed in the cumulative return graphs.

Case Studies Used (Lagrangian):
Case 1: Stock A has an expected return of 20% and a standard deviation of 10%, while Stock B has an expected return of 30% and a standard deviation of 20%. The predefined risk limit is set at 15%.
Case 2: Stock A has an expected return of 15% and a standard deviation of 8%, while Stock B has an expected return of 25% and a standard deviation of 18%. The predefined risk limit is set at 9%.
Case 3: Stock A has an expected return of 22% and a standard deviation of 12%, while Stock B has an expected return of 28% and a standard deviation of 25%. The predefined risk limit is set at 12%.

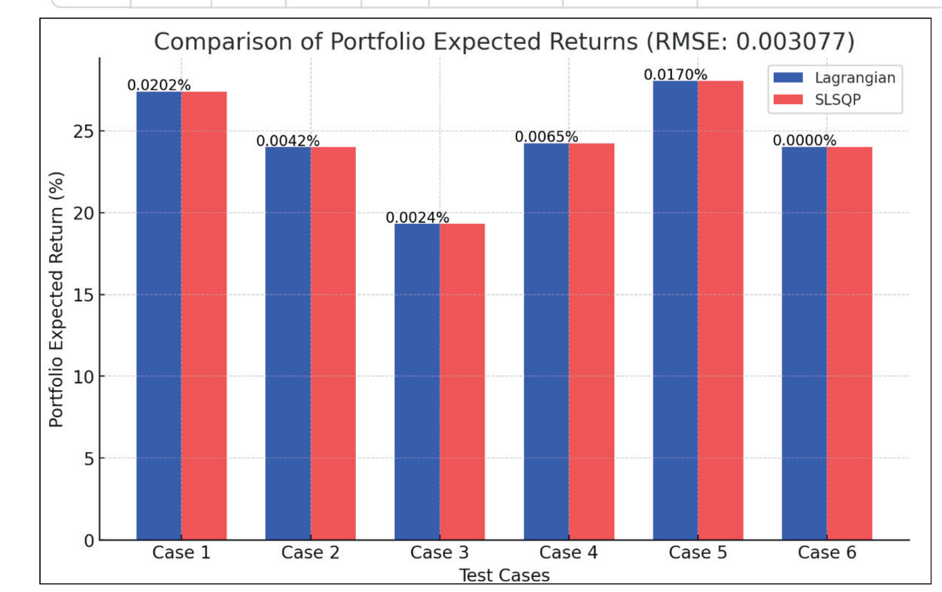
Cross-Validation Analysis

Table 1: Portfolio Performance from Lagrangian Method

E_A	E_B	σ_A	σ_B	R	ω_A	ω_B	Portfolio Expected Return (%)	Portfolio Risk (%)
20	30	10	20	15	0.26148	0.73852	27.385	15
20	30	10	20	10	0.6	0.4	24	10
15	25	8	18	9	0.56855	0.43145	19.31449	9
22	28	12	25	12	0.62549	0.37451	24.24707	12
18	32	9	22	16	0.28193	0.71807	28.05297	16
21	27	11	19	11	0.49793	0.50207	24.01245	11

Table 2: Portfolio Performance from SLSQP Algorithm

E_A	E_B	σ_A	σ_B	R	ω_A	ω_B	Portfolio Expected Return (%)	Portfolio Risk (%)
20	30	10	20	15	0.26148	0.73852	27.37948	15
20	30	10	20	10	0.60000	0.40000	23.99900	10
15	25	8	18	9	0.56855	0.43145	19.31402	9
22	28	12	25	12	0.62549	0.37451	24.24550	12
18	32	9	22	16	0.28193	0.71807	28.04821	16
21	27	11	19	11	0.49793	0.50201	24.01245	11

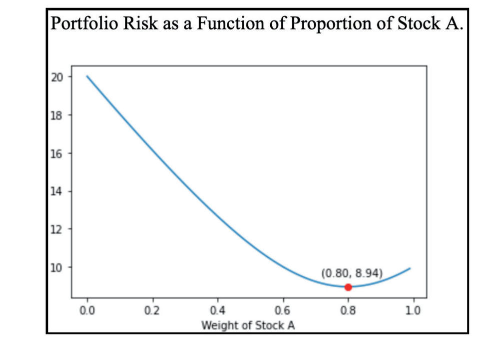


To evaluate the overall deviation between methods, we compute the Absolute Percentage Error (APE) and the Root Mean Squared Error (RMSE). **The APE values are extremely low, ranging from 0.0000 to 0.0202**, indicating a nearly perfect match between the two methods. **The RMSE is 0.003077**, suggesting negligible deviation. The bar chart shown in Figure 1 visualizes the expected returns from both methods. Differences are nearly negligible, further validated by the annotated APE values on each bar. This cross-validation confirms the validity of the SLSQP algorithm as an accurate approximation of the Lagrangian method, providing rigorous mathematical backing to our model software. The minuscule errors likely stem from numerical precision differences rather than fundamental algorithmic issues.

Statistical/Sensitivity Analysis

In addition to computing optimal allocation weights and portfolio expected return, our model software generates graphical outputs of portfolio performance against user-defined variables and constraints, providing sensitivity analysis insight beyond black-box optimization to clearly provide the user insight in the impact of inputs and correlation, allowing for more informed investment decision-making. We conduct a statistical analysis of the graphical outputs resulted from Case 1 to investigate these implications.

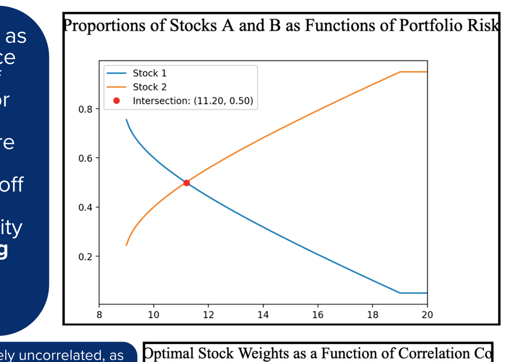
Portfolio Risk as a Function of Proportion of Stock A.



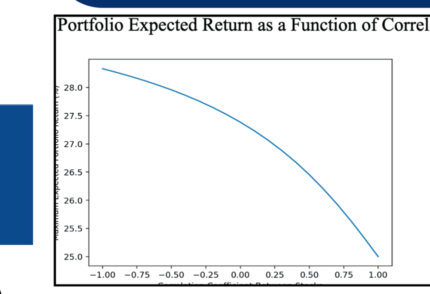
When considering the standard deviations (σA and σB) of Stocks A (10%) and B (20%), Stock A exhibits lower investment risk. This graph highlights that **minimizing portfolio risk does not require investing solely in the less risky stock, or the stock with lower variance**. By reducing the weight of Stock A (ωA) from 1 to 0.8, the expected return of the portfolio increases while the portfolio risk falls below Stock A's risk level (10%), reaching 8.94%, highlighting the importance of diversification. Allocating 20% of the initially Stock A-dominated portfolio to Stock B leads to a risk reduction of over 1% (Figure 2), while enhancing the expected portfolio return by 2%.

This graphical output demonstrates the optimal weighting of ωA and ωB as a function of portfolio risk tolerance. As the level of portfolio risk tolerance increases, the optimal investment in Stock B rises. At a tolerance level of 20%, Stock B becomes the sole constituent of the portfolio. This behavior arises from the inclusion of stocks with higher standard deviation when there is a greater risk tolerance. Allocating a larger proportion to the more volatile Stock B results in an increased expected return due to its higher expected return compared to Stock A. These findings reaffirm the trade-off between risk and return in portfolio management. The optimal allocation proportions is dependent on investor objectives, highlighting the necessity for user-defined variable constraints in portfolio management. **Evaluating this sensitivity graph will enable the investor to make an informed decision about their trade-off as it is relative to their objectives.**

Proportions of Stocks A and B as Functions of Portfolio Risk

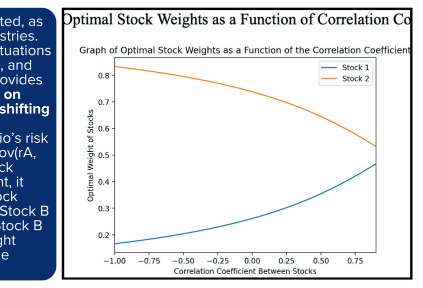


Portfolio Expected Return as a Function of Correlation



In real-world financial markets, stocks are rarely uncorrelated, as there are inter-dependencies among companies and industries. Stocks can become more or less correlated based on fluctuations in market conditions or broader macroeconomic variables, and portfolio adjustments may be necessary. The left figure provides valuable insights into the **effect of changes in correlation on portfolio performance**, and the right figure highlights the **shifting allocation strategies with varying correlations**. As the correlation between Stocks A and B increases, the portfolio's risk tends to rise. This can be attributed to the term 2ωAωB Cov(rA, rB) in the portfolio risk calculation, assuming constant stock weights. To maintain the risk within the specified constraint, it becomes necessary to reduce the weight of the riskier stock (Stock B) in the portfolio. Consequently, the proportion of Stock B decreases while the proportion of Stock A increases. As Stock B typically offers higher profitability, the reduction in its weight leads to a decrease in the maximum expected return of the portfolio.

Optimal Stock Weights as a Function of Correlation Co



Conclusion

- **Cross-validation** of the Lagrangian Method (LM) mathematical derivation and Sequential Least Squares Programming (SLSQP) algorithm method demonstrated an impressive co-accuracy, with both methods yielding **nearly identical results**. This ensures software was correctly implemented and **not subject to iterative computational error** or local optima issues.
- Developed a **novel solver optimization model** allows investors to dynamically customize risk tolerance; tailored **optimization to individual needs** without excessive constraints disputing the model's fundamental efficiency. Proved as the **Lagrangian proportions consistently outperform alternative strategies** under analyzed simulated testing.
- Derivation of an arbitrary solution equation that allows for **scalable solutions** mathematically, & grounded mathematical cross-validation understanding; s**traightforward user experience** while r**etaining theoretical rigor**.
- Transparent and rigorous mathematical backing justifies portfolio decisions **beyond derivative-free** numerical outputs; sensitivity analysis with graphical outputs clearly provides the user **insight into the impact of inputs** and correlation, allowing for **informed investment decision-making**.
- An intuitive and simplified web-based and device-agnostic interface was developed **fully operational** and **streamlining the constrained optimization process**, presenting significant **democratizing access** to rigorous portfolio optimization tools to retail investors and financial practitioners alike.