Measuring The Viscosity Of Water Using Poiseuille's Equation

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The viscosity of water, η, at 287.15 K can be found, using Poiseuille's law, by measuring water's flow rate through a capillary tube for various pressures. From these data, two values, η_i and η_{ii}, are calculated using two approaches, both involving Poiseuille's law. The second of these, 2.14 $\pm$ 0.04 mPa.s, is taken to be the final value due to its higher precision. η_{ii} is then compared to the value of η given by the 2009 IAPWS reference equation for liquid water. After χ^2 minimisation and analysis of normalised residuals, which suggest that Poiseuille's law is an unsuitable model, flaws in the experimental method and model are investigated, including the possibility of turbulent flow (investigated by calculation of Reynold's number) and the potential to include additional terms in the equation to account for kinetic energy and pressure changes.

I Introduction

The viscosity of a fluid is a measure of its internal friction, or the forces acting against the movement of one part of a fluid relative to another. For liquids, η is inversely proportional to temperature. In this work, we find η for water, using Poiseuille's law, which gives the rate of fluid flow through a tube as a function of pressure change, tube dimensions and η. For the streamlined motion of an incompressible Newtonian fluid through a tube of radius a and length L, this is:

$$\frac{dV}{dt} = \frac{\pi}{8}\frac{\rho g h}{\eta}\frac{a^4}{L}, \tag{1}$$

where $\rho g h = \Delta p$, the pressure change, and $\frac{dV}{dt}$ is volume flow rate. ρ is the density of water, and g is acceleration due to gravity. It should also be noted that η is absolute viscosity, with units mPa.s, as opposed to kinematic viscosity, and that in a Newtonian fluid, shear stress is proportional to rate of deformation [1].

The experimentally determined value is then compared to the true value of η, which is calculated, in μPa.s, for different temperatures T using the 2009 IAPWS reference equation for liquid water:

$$\begin{aligned}\eta = 280.68T_r^{-1.9} + 511.45T_r^{-7.7}\\ +61.131T_r^{-19.6} + 0.45903T_r^{-40.0},\end{aligned} \tag{2}$$

where $T_r = \frac{T}{300}$, with T in Kelvin [3].

Poiseuille's law does not apply to turbulent flow and therefore would not give an accurate value of η for such. It can be verified whether flow is turbulent by calculating Reynold's number, which has no dimensions and describes whether viscous forces are negligible. For pipe flow:

$$Re = \frac{\rho v D}{\eta} \tag{3}$$

where v is velocity and D is diameter. If Re is large, flow is turbulent; if small ($Re \leq 2300$), the flow is laminar [1].

II Methods

As shown in figure 1, a tank with a capillary tube screwed into an outlet on one side was mounted on a raised platform. The radius of the capillary tube, a, was measured to be 0.0009 $\pm$ 0.0001 m. This was done using a travelling microscope to find the position of one side of the tube, then the other. The tube's diameter is the difference between these values. Repeat measurements were taken at different orientations to account for irregularities in the tube and resulting error. Tube length, L, was measured using a ruler to be 0.149 $\pm$ 0.001 m. Below the tube was a mass balance with a waterproof cover, onto which was placed a 100 ml beaker. The balance was connected to a PC, through which measurements of mass, m, and time, t, could be gathered, using a Python program provided in the lab.

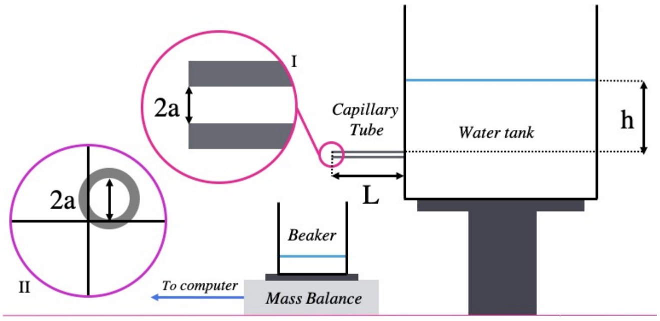

Figure 1: Diagram of experimental set-up, with a, L and h as defined above. I is a close-up of the capillary tube to show a. II is the end of the tube as viewed through the travelling microscope; the crosshairs were aligned with the edges of the hollow inside of the tube for each measurement.

With the beaker empty, the balance was tared. The end of the capillary tube was covered and the tank was filled to a depth h above the tube with water. The temperature of the water was measured to be 287.15 K using an electronic thermometer, hence ρ = 999.33 kg m^{-3} [2]. h was measured using a ruler. The capillary tube was then uncovered as the data collection program was started, and water dripped into the beaker. Measurements of m and corresponding t were taken once per second for 60 s in order to find $\frac{dm}{dt}$. The experiment was repeated over different values of h, allowing for a plot of $\frac{dm}{dt}$ against h. Assuming a linear fit, η can be determined from the graph's gradient. η for each $\frac{dm}{dt}$ and h pair can also be found and a mean of these ηs taken. $\frac{dV}{dt}$ becomes $\frac{dm}{dt}$ using the definition of ρ as mass per unit volume:

$$\frac{dm}{dt} = \frac{\pi}{8}\frac{\rho^2 g h}{\eta}\frac{a^4}{L}. \tag{4}$$

III Results

Two different approaches were used to find η to see which yielded the value with the lowest error. For both, the first step was to find $\frac{dm}{dt}$ for each h. As m and t are homoscedastic data sets, method of least squares was used to find $\frac{dm}{dt}$. An attempt was also made using χ^2 minimisation. Both methods gave equal values of $\frac{dm}{dt}$ for each h to a tolerance of 0.005%, but the χ^2 analysis was discarded due to the implausibility of the errors it produced (described further in the errors appendix).

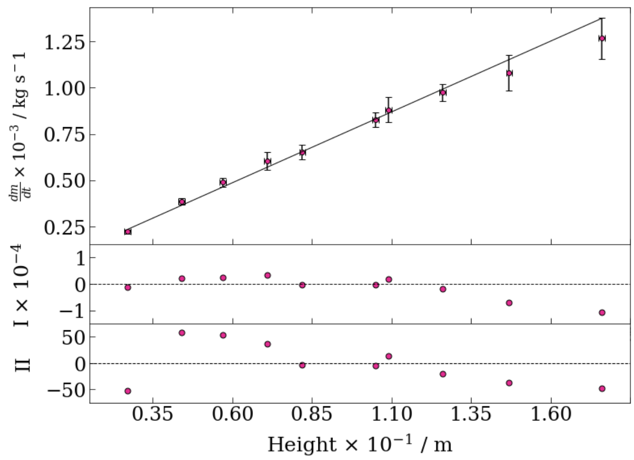

Figure 2: Graph to show $\frac{dm}{dt}$ against h. The y error bars are scaled up by a factor of 25 to be made visible; in reality, they are of the order of 10^{-6}. Error bars in the x-direction are of the order of 10^{-3}. I is the residual plot and II shows the normalised residuals. Neither residual shows a clear structure, implying that no additional terms should necessarily be incorporated into the model.

III.1 Gradient Method

The first approach obtained η by plotting $\frac{dm}{dt}$ against h, as seen in figure 2, and then calculating a value from Poiseuille's law and the gradient of the plot, as found by method of χ^2 minimisation:

$$\eta = \frac{\pi}{8}\frac{\rho^2 g}{gradient}\frac{a^4}{L}, \tag{5}$$

The χ^2 analysis gave χ^2_{min} = 14400 and reduced χ^2, χ^2_ν =1800 (these values are further explored in the discussion section). This method yielded $\eta_{ii} = 2.2 \pm 0.1$ mPa.s.

III.2 Averages Method

The second approach calculated η from each specific pair of $\frac{dm}{dt}$ and h:

$$\eta = \frac{\pi}{8}\frac{\rho^2 gh}{\frac{dm}{dt}}\frac{a^4}{L}, \tag{6}$$

The mean was then taken of all the η values, giving $\eta_{ii} =$ 2.12 $\pm$ 0.04 mPa.s.

IV Discussion

From equation 2, at T = 287.15 K, η_{ref} = 1.1683 mPa.s. η_{ref} is not within the range of the error in either η_i or η_{ii}. The percentage differences between η_{ref} and the experimentally calculated η values are as follows: η_i: 47.3%; η_{ii} 44.9%. As η_{ii} is only 4.40% different to η_i, and their errors overlap, η_i and η_{ii} are in agreement. However, η_{ii}'s error is 4% of that of η_i. Due to its higher precision, the final value of η from this experiment was determined to be η_{ii}, 2.12 $\pm$ 0.04 mPa.s. However, this value was not found through a weighted fit, which may be why different values were given by the different approaches .

A factor that could reduce flow rate and therefore increase η is air bubbles in the capillary tube. Another potential reason for discrepancy in η from the literature value is that h was taken to be constant, whereas in reality, as water left the tank, h decreased. This was not anticipated to be a problem. Due to the relatively large area of the tank, the change in h as volume decreased was relatively small. The contribution of friction between the water and the capillary tube itself could also account for some difference between the experimental and literature values.

a was the dominant source of error in this experiment. a was calculated from the mean of 8 measurements of the tube diameter at different orientations. By taking more repeat measurements, overall error could be reduced, and a more accurate value of a obtained. No repeat measurements of m and t were taken for each height. The error in mass flow rate could have been reduced by doing so. Additionally, the experiment could have been repeated for multiple tubes of different a and L, rather than just using the one. η could then be found for each a, and an average of these values taken.

Despite their lack of structure, and mean of -1.12$\times 10^{-5}$ ($\approx$ 0), the normalised residuals in figure 1 are not as expected for a good fit. 65% should be within ± 1 and 96% within ± 2; however, the normalised residuals are in the range ± 60 due to small errors in $\frac{dm}{dt}$. Additionally, for a reasonable fit, $\chi^2_\nu \approx$ 1, which is not the case in this work [4].

The combination of these statistical analyses tells us that the chosen model, Poiseuille's law, is a not a good fit, and that method 1 does not give a reasonable η. To improve this model, the Hagenbach and Couette corrections to pressure change could be incorporated. These respectively model pressure loss from the change in kinetic energy and the parabolic pressure velocity profile of the fluid as it enters the capillary tube, whereas the Poiseuille equation only accounts for viscous contribution to pressure change [5].

Another reason for the model's failure could be turbulence. Using Reynold's law, with $v = \frac{dV}{dt}\frac{1}{\pi a^2}$ and $D = 2a$, Re was calculated to be 6.9$\times 10^{-2}$. This, being 5 orders of magnitude below the threshold for turbulence, implies that flow was laminar, and so disproves this hypothesis.

V Conclusions

The final value found for η at 287.15 K was 2.14 $\pm$ 0.04 mPa.s. This is 44.9% different to the literature value of viscosity at this temperature, η_{ref}. This discrepancy may be from the application of an unsuitable model, Poiseuille's law; to replicate η_{ref}, the Hagenbach and Couette corrections should have been applied.

References

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[5] "Visco handbook: Theory and Application of Viscometry with Glass Capillary Viscometers", http://www.si-analytics.com/fileadmin/upload/Informationen/Kapillarviskosimetrie/INT/Visco-Handbook_2015_2.7-MB_PDF-English.pdf (accessed 04-12-18)

VI Error Appendix

All the following error propagation uses techniques from [4].

VI.1 Errors in Single Measurements

α_m, α_t , α_L and α_h were determined from the precision of equipment:

$$\alpha_m = 0.00001, \tag{7}$$

$$\alpha_t = 0.01, \tag{8}$$

$$\alpha_h = 0.001, \tag{9}$$

and

$$\alpha_L = 0.001. \tag{10}$$

$\alpha_h = \alpha_L$, as L and h were measured with the same ruler.

The tube's diameter, D, was measured rather than the radius. α_D is standard error in D:

$$\alpha_D = \frac{\sigma}{\sqrt{N}}, \tag{11}$$

where σ is the standard deviation in the mean and N is the number of measurements of D. Consequently α_a was calculated:

$$\alpha_a = \frac{\alpha_d}{2}. \tag{12}$$

α_a is a large contributor to the overall error in η. From the calculus method, as seen in [4]:

$$\alpha_{a^4} = \left|4a^3\right| \alpha_a. \tag{13}$$

Units of α_D, α_a, α_h and α_L are m. Units of α_m and α_s are kg and s respectively.

The method of least squares was used to find the values of the mass rate, $\frac{dm}{dt}$ and its error, $\alpha_{\frac{dm}{dt}}$. This gave $\alpha_{\frac{dm}{dt}}$ of the order of 10^{-7}. χ^2 was used in parallel with this to find another set of values of $\frac{dm}{dt}$ and $\alpha_{\frac{dm}{dt}}$. However, the χ^2 analysis gave $\alpha_{\frac{dm}{dt}}$ of around the order of 10^{-20}, which was deemed implausibly small, so this method was discarded. Overall, the method used here would not have made a significant difference to the final value of η, as the dominant error is in a^4.

χ^2 analysis was carried out on $\frac{dm}{dt}$ and h to give:

$$\alpha_{gradient} = 5.40 \times 10^{-8}. \tag{14}$$

$\alpha_{gradient}$ is needed to calculate the error in η_i, α_{η_i}.

VI.2 The Source of Dominant Error, a

By comparison of percentage errors (in the smallest value for each measurement with multiple values; m and t are orders of magnitude smaller and so not considered), dominant error was determined to be in a:

Error	Percentage Error
α_a	11.1
α_{a^4}	44.9
α_L	0.67
α_h	3.7

As well as having the largest percentage error from precision of instrument relative to size of measurement, the measurement of a offered many more opportunities for uncertainty. For one, alignment of the tube's edge with the travelling microscope's crosshairs changed fractionally when the instrument was screwed tight, and as each set of measurements was taken by a different team member in an attempt to eliminate systematic error, how the edge of the tube was defined may have varied.

VI.3 Combining Errors

α_{η_i} was found using the calculus method:

$$\alpha_{\eta_i} = \eta\sqrt{(\frac{\alpha_{a^4}}{a^4})^2 + (\frac{\alpha_L}{L})^2 + (\frac{\alpha_{gradient}}{gradient})^2} \tag{15}$$

For the second value's error, $\alpha_{\eta_{ii}}$, standard error was calculated, as η_{ii} is the mean of a set of values:

$$\alpha_{\eta_{ii}} = \frac{\sigma}{\sqrt{N}} \tag{16}$$

where N is the number of measurements.

VI.4 χ^2 Analysis

χ^2 minimisation was one of the methods of analysis used in this work, as it gives a fit for non-uniform error bars. A Python program provided by the Physics department was used to compute χ^2_{min}, χ^2_ν and fit parameters for the data in this work.

[5] describes χ^2 as a 'goodness of fit parameter' and defines it as the sum of the square of the normalised residuals:

$$\chi^2 = \sum_i \frac{(y_i - y(x_i))^2}{\alpha_i^2}, \tag{17}$$

Where y_i, $y(x_i)$ and α_i are the i^{th} measurement, fit value and error respectively. In this work, the fit was chosen to be linear, i.e.:

$$y(x_i) = Ax_i + B \tag{18}$$

The best fit (A, B, and their errors) can be determined by χ^2 minimisation.

For each point, χ^2 is calculated. A contour plot of equal χ^2 values for different values of parameters A and B can be produced. The values of the best fit parameters correspond to χ^2_{min}, the χ^2 value at the midpoint of the contour plot. 1-σ error for each parameter is found from the difference between that parameter at χ^2_{min} and that parameter at the $\chi^2_{min} + 1$ contour line.

χ^2_ν, reduced χ^2, is calculated as follows:

$$\chi^2_\nu = \frac{\chi^2_{min}}{\nu}. \tag{19}$$

ν, degrees of freedom, is defined by:

$$\nu = n - N, \tag{20}$$

where n is number of measurements and N is the number of parameters (in the case of $\frac{dm}{dt}$ vs h, ν = 8).