Inductive NFA proof - Sample Solution

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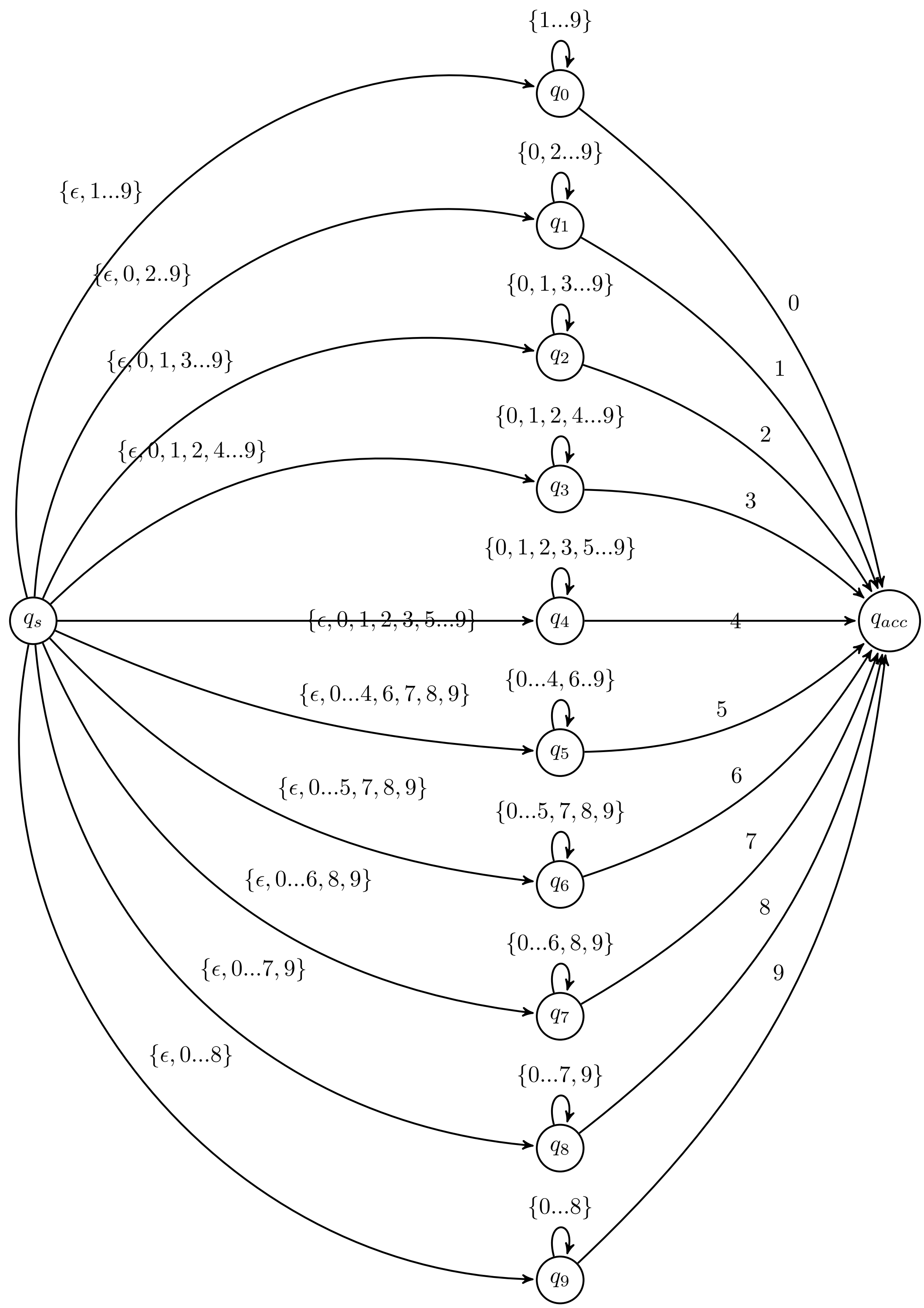
Problem: Let L be the language of all strings over the alphabet $\Sigma = 0, ..., 9$ such that $w \neq \epsilon$ and the last (rightmost) digit in w has not appeared in the string before. For instance 3261, 12334, and 0 are elements of L, but 1221, 7898, and 55 are not. Show that L is regular by giving a DFA or NFA that accepts precisely this language.

Sample Solution: (NFA on the next page)
Base Case: For our base case, we will consider all strings w over the alphabet s.t. $|w| = 1$.
$\hat{\delta}(q_s, \epsilon) = \{q_0, ...q_9\}$, meaning there exists a transition from q_s to all states q_0 through q_9 on the empty string. From a given state q_n, $n \in L$, there is a transition from q_n to q_{acc} on n. For instance, $\hat{\delta}(q_0, 0) = q_{acc}$, $\hat{\delta}(q_1, 1) = q_{acc}$, $\hat{\delta}(q_2, 2) = q_{acc}$ and so forth for all $|w| = 1, w \in L$. Thus, all strings w of length 1 s.t. $w \in L$ are accepted by the NFA.
Inductive Hypothesis: If we are in state q_n after processing k elements of w, then there are no instances of the element n in our string of length k.
Inductive Step:
We assume the inductive hypothesis holds, and extend it the $k + 1_{th}$ element. We call the $k + 1_{th}$ element of the string x.
Suppose $x \neq n$. Given that we are in state q_n, we cannot guarantee that there have been no prior instances of x, so our NFA defines a loop on all $x \neq n$ and does not accept *see note.
Suppose on the other hand that $x = n$. Then by inductive hypothesis, we guarantee that there are no instances of n in our string up to this point, so we accept the string on the transition $\hat{\delta}(q_n, n) = q_{acc}$.
We have now shown that this NFA accepts the language of all strings whose rightmost digit has not appeared in the string before.

> ***Note for students:*** because the NFA is trying all possible paths, any accepted string will be rejected by the other paths. For instance, 123 cannot go from $q_s \to q_4 \to q_{acc}$, but because all paths are being tried, the NFA will simultaneously choose the correct path $q_s \to q_3 \to q_{acc}$. As long as *there exists* some computation path for a given string, the NFA accepts the string.


$\{1...9\}$
q_0
$\{0, 2...9\}$
q_1
$\{0, 1, 3...9\}$
q_2
$\{0, 1, 2, 4...9\}$
q_3
$\{0, 1, 2, 3, 5...9\}$
q_4
$\{0...4, 6..9\}$
q_5
$\{0...5, 7, 8, 9\}$
q_6
$\{0...6, 8, 9\}$
q_7
$\{0...7, 9\}$
q_8
$\{0...8\}$
q_9
q_s
q_{acc}
$\{\epsilon, 1...9\}$
$\{\epsilon, 0, 2..9\}$
$\{\epsilon, 0, 1, 3...9\}$
$\{\epsilon, 0, 1, 2, 4...9\}$
$\{\epsilon, 0, 1, 2, 3, 5...9\}$
$\{\epsilon, 0...4, 6, 7, 8, 9\}$
$\{\epsilon, 0...5, 7, 8, 9\}$
$\{\epsilon, 0...6, 8, 9\}$
$\{\epsilon, 0...7, 9\}$
$\{\epsilon, 0...8\}$
0
1
2
3
4
5
6
7
8
9