Case Study

Modelling the Steady State Temperature Distribution in a Ceramic Plate

Mathematical model

- The law of conservation of energy, along with Fourier's law of heat conduction, leads to the following **partial differential equation**:

$$\rho c \frac{\partial T}{\partial t} = k\nabla^2 T + g$$

- For the **steady state** problem, we obtain the **Poisson equation:**

$$\nabla^2 T + \frac{g}{k} = 0$$

- If the source is zero everywhere, we obtain the **Laplace equation:**

$$\nabla^2 T = 0$$

Two dimensions

- Two-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

- Poisson equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{g}{k} = 0$$

- Laplace equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Boundary Conditions

Three types of boundaries in heat conduction problems:

Prescribed Temperature (Dirichlet condition):

$$T\ \ x,b\ \ = f\ \ x$$

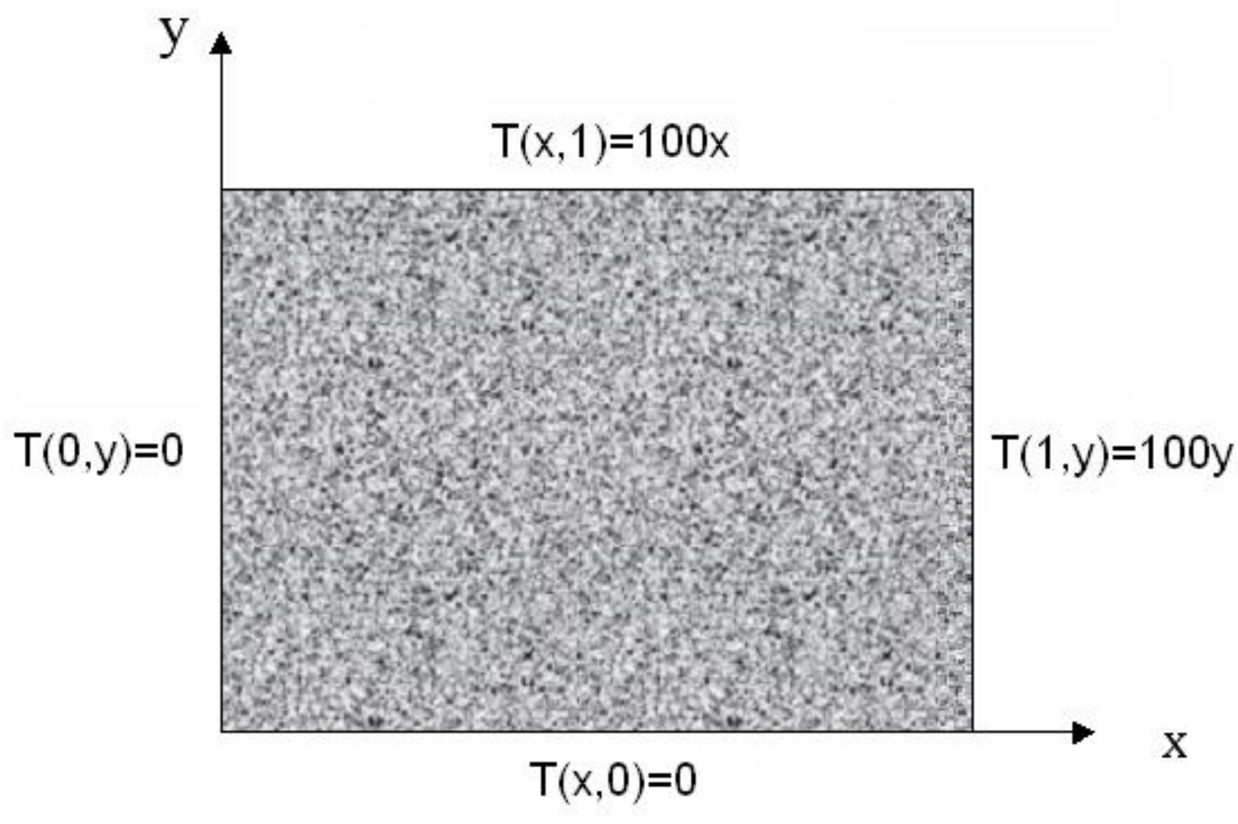
Insulated (Neumann condition):

$$\nabla T \cdot \hat{\mathbf{n}} = 0$$

Convective (Robin condition):

$$k\nabla T \cdot \hat{\mathbf{n}} = h\ \ T_{\infty} - T$$

First example: Laplace equation



Ceramic plate with Dirichlet boundary conditions

Summary of the Mathematical Model

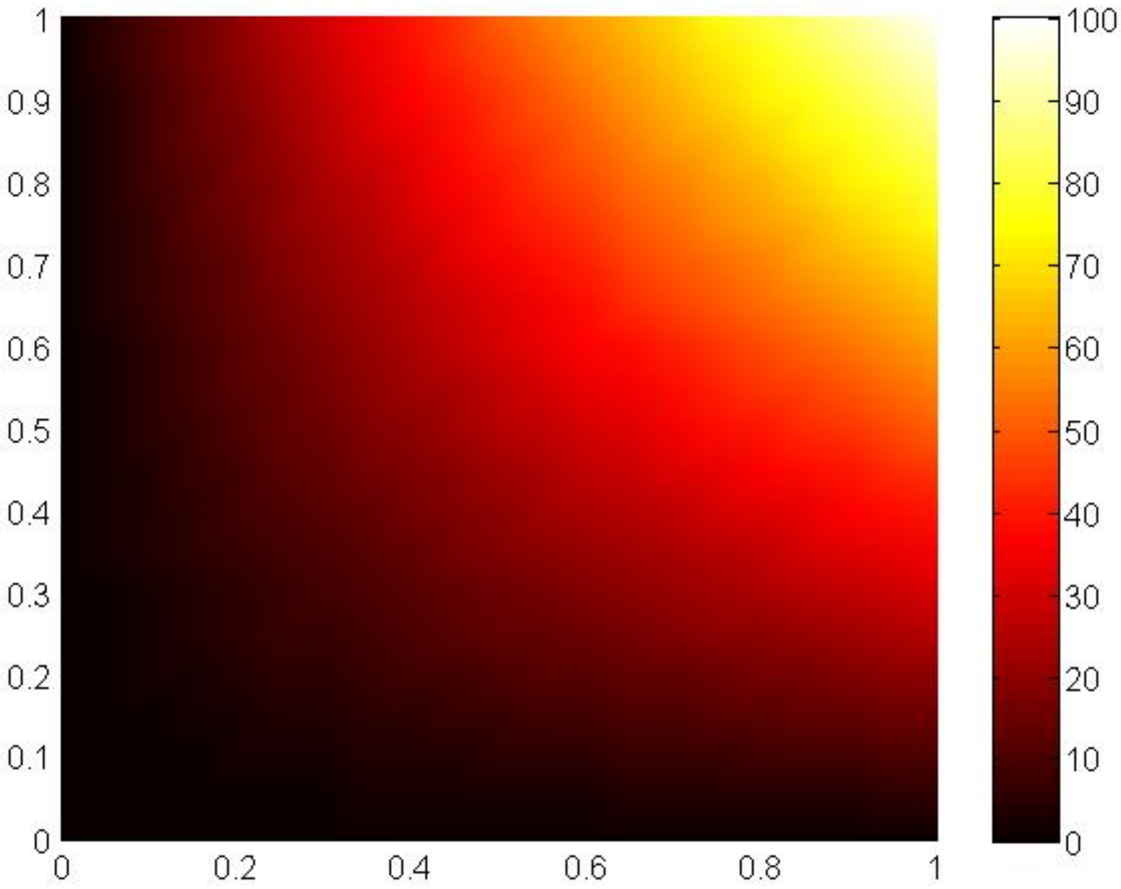
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad 0 < x < 1 \quad , \quad 0 < y < 1$$

$$subject\ \ to$$

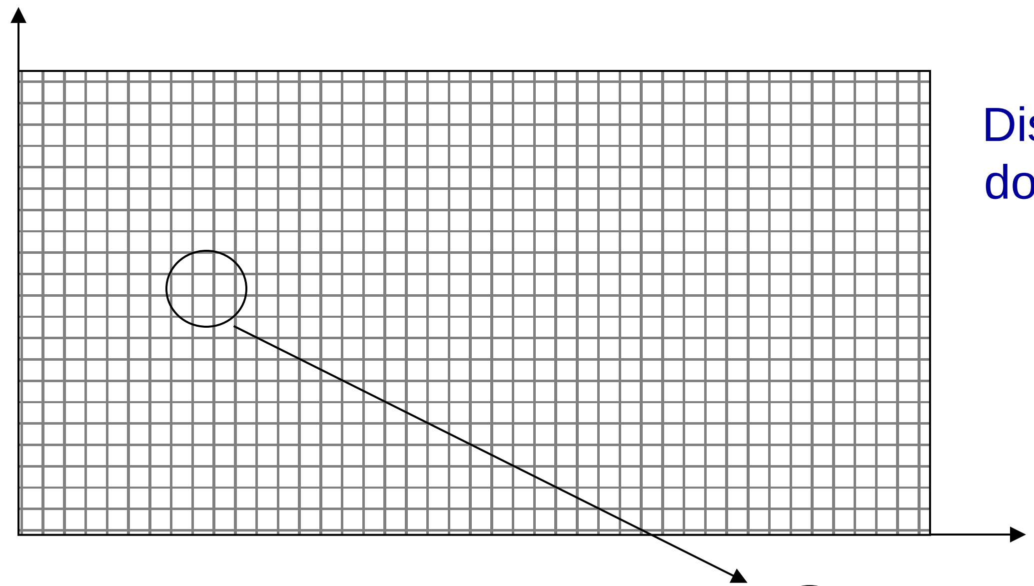
$$\begin{aligned}
at\ \ x = 0 \quad &: \quad T = 0 \\
at\ \ x = 1 \quad &: \quad T = 100y \\
at\ \ y = 0 \quad &: \quad T = 0 \\
at\ \ y = 1 \quad &: \quad T = 100x
\end{aligned}$$

Analytical Solution

$$T(x, y) = 100xy$$



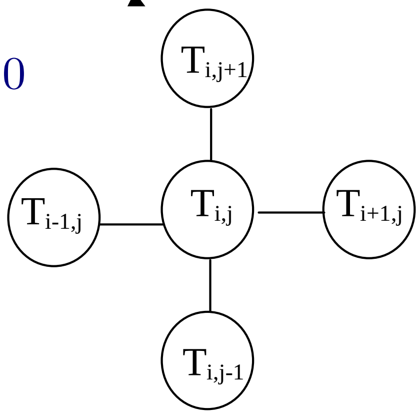
Conversion to a Matrix Problem



Discretise the domain using a mesh

The model is converted to a matrix problem by discretising the domain and forming *finite difference equations*.

$$-T_{i,j-1} - T_{i-1,j} + 4T_{i,j} - T_{i+1,j} - T_{i,j+1} = 0$$

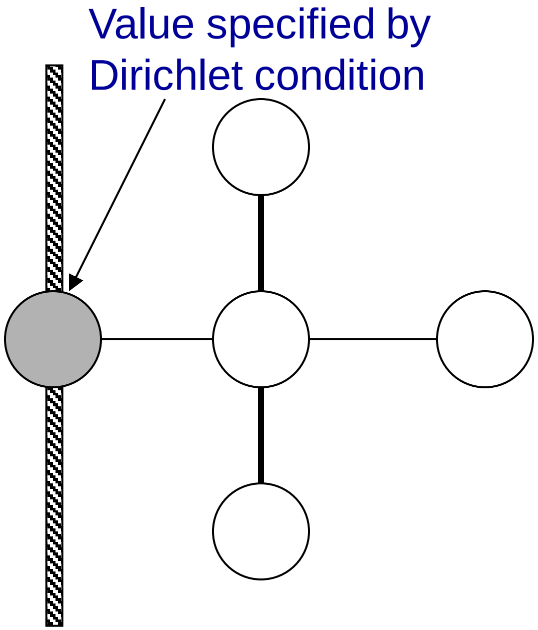


Finite Difference Stencil used for each internal node on the mesh

Treatment of Boundary Nodes

Prescribed boundary:

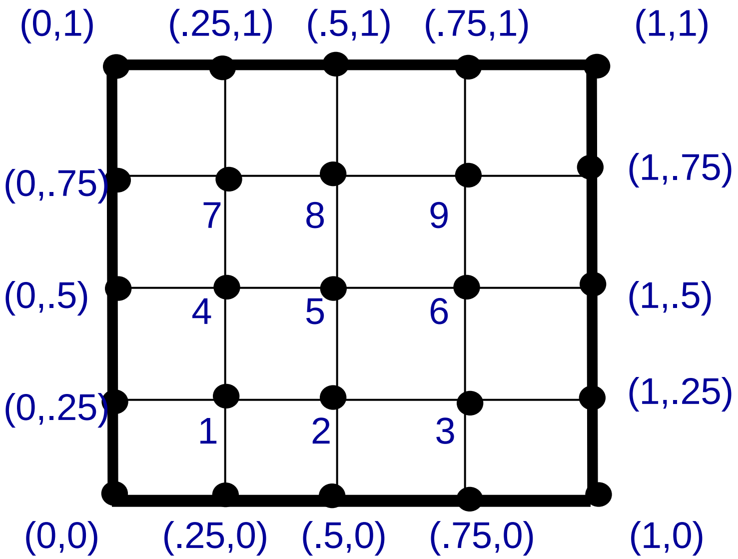
Dirichlet condition gives required value for use in FDE



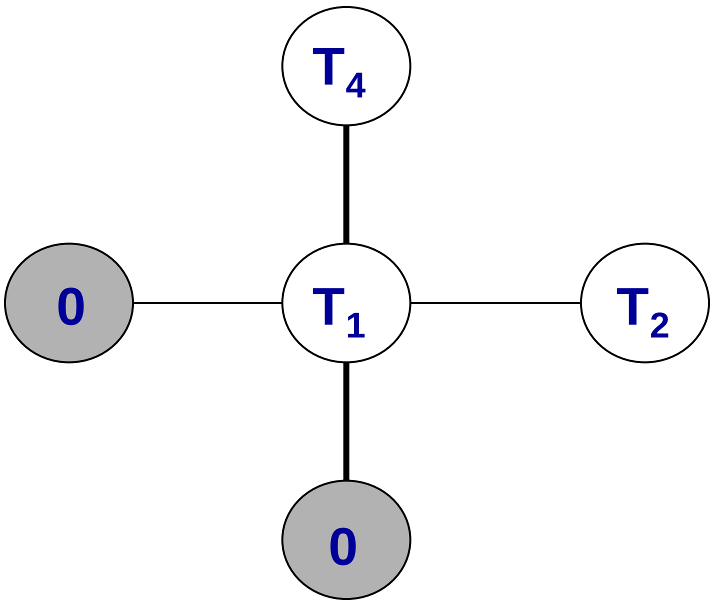
Prescribed Boundary

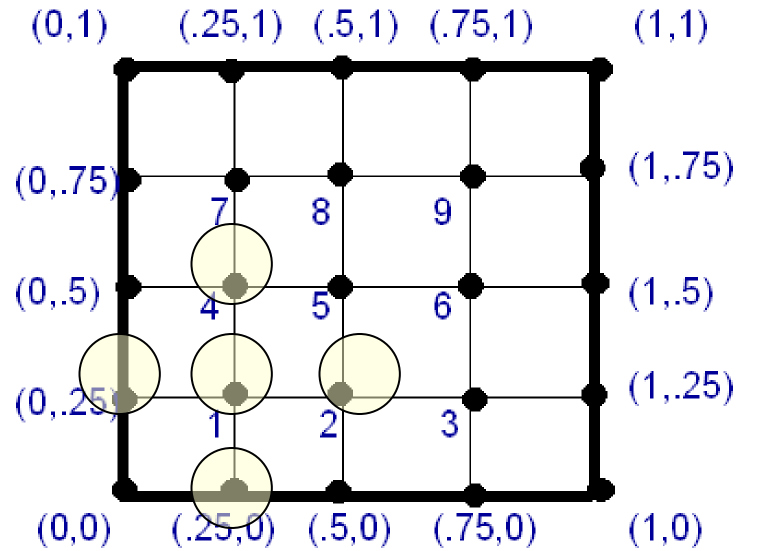
How does this process generate a matrix?

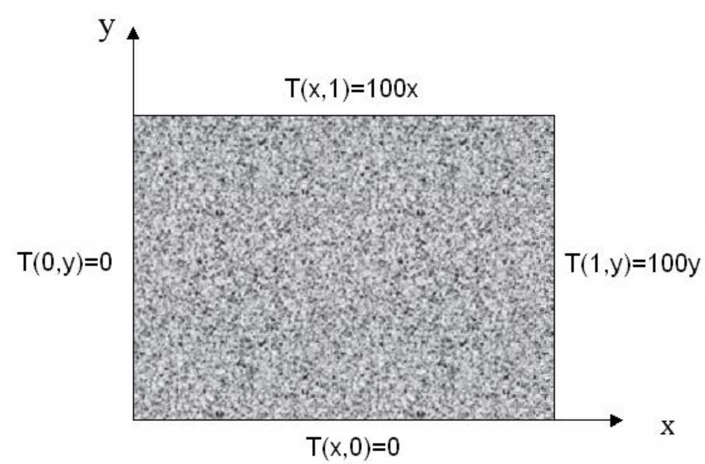
- Introduce a node numbering scheme and map (i,j) coordinates to these nodes:



Node 1

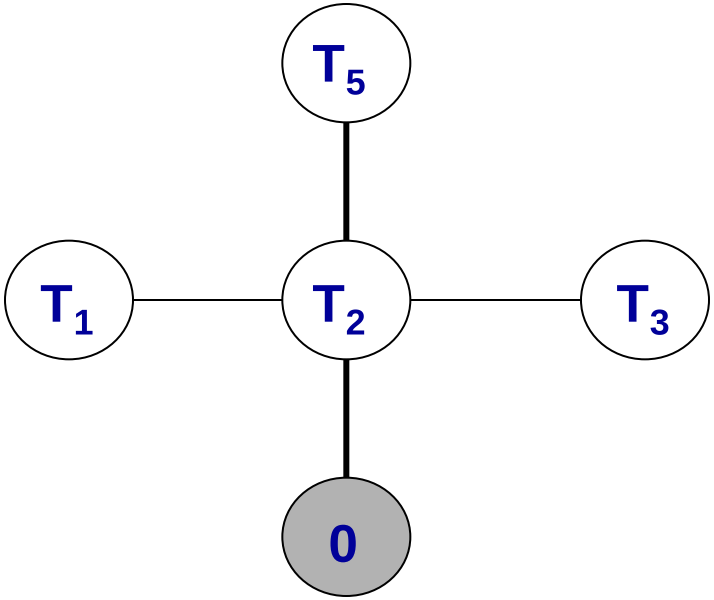


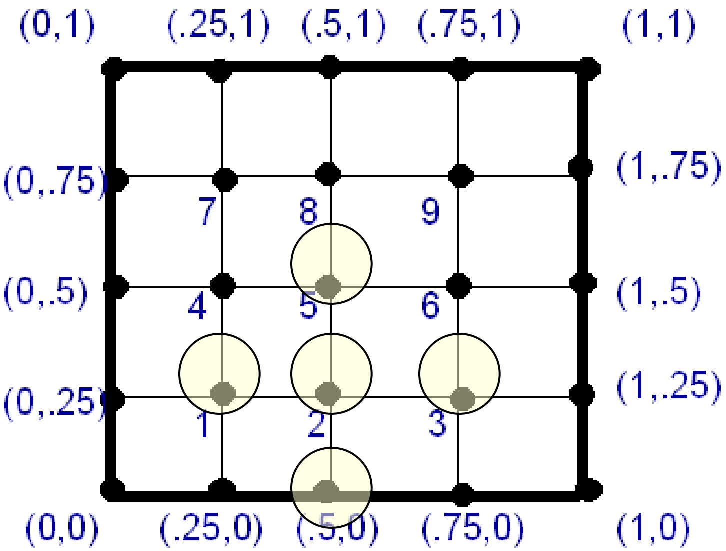




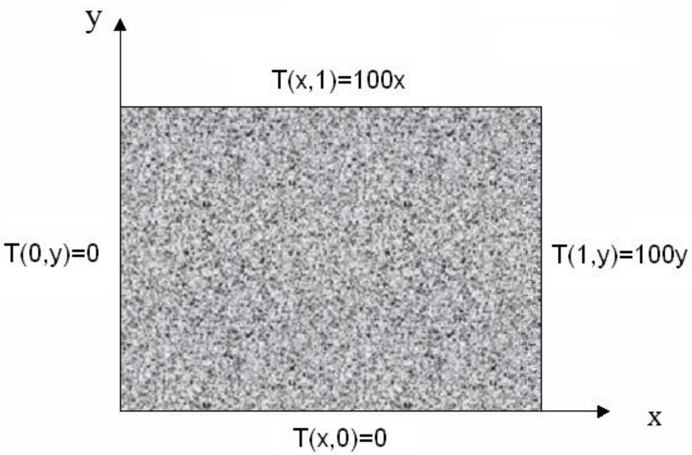
$$4T_1 - T_2 - T_4 = 0$$

Node 2

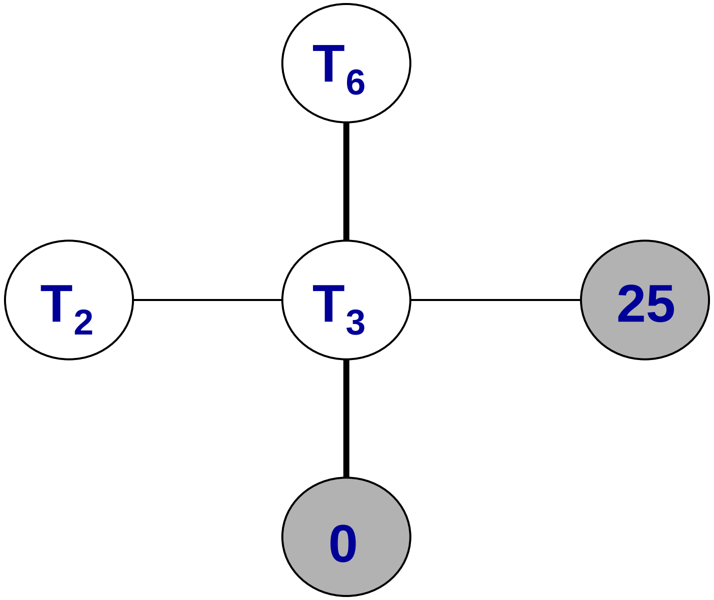


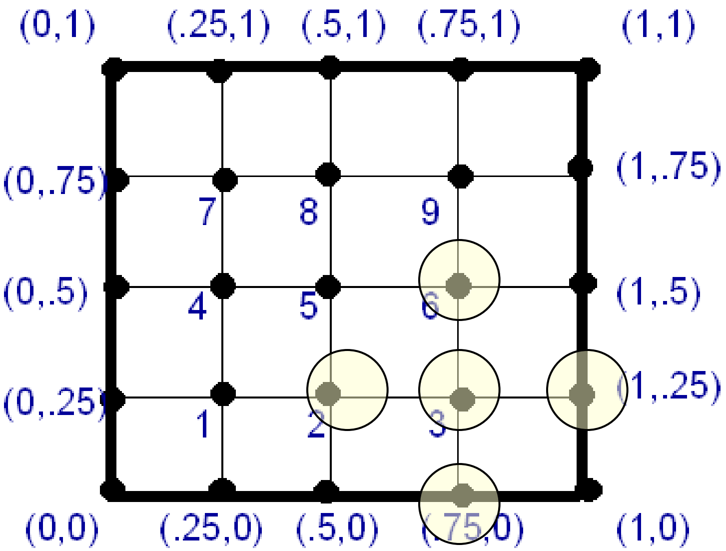


$$-T_1 + 4T_2 - T_3 - T_5 = 0$$

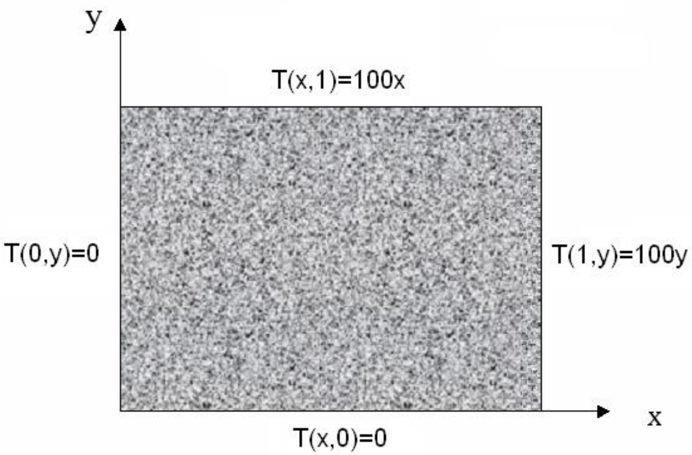


Node 3

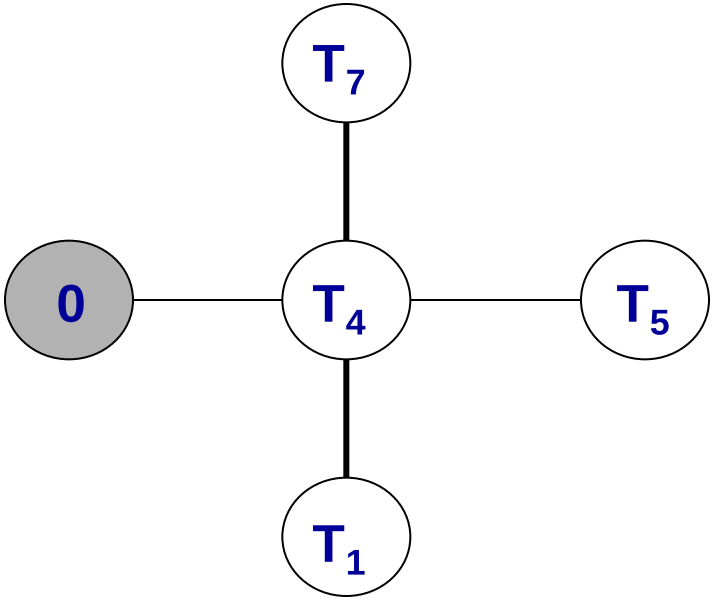


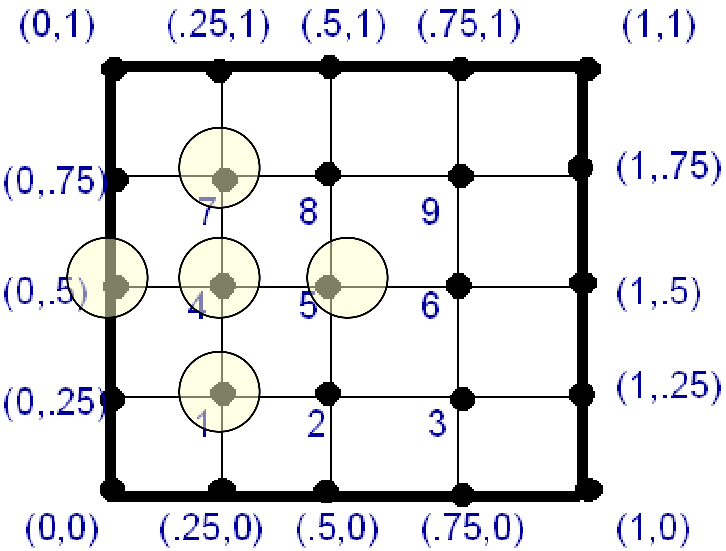


$$-T_2 + 4T_3 - T_6 = 25$$

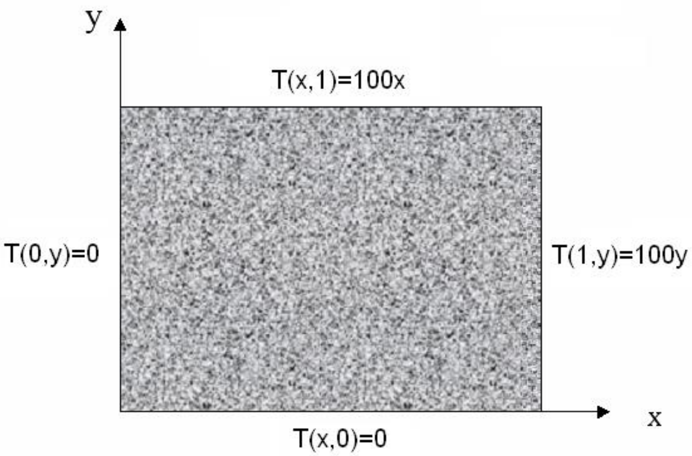


Node 4

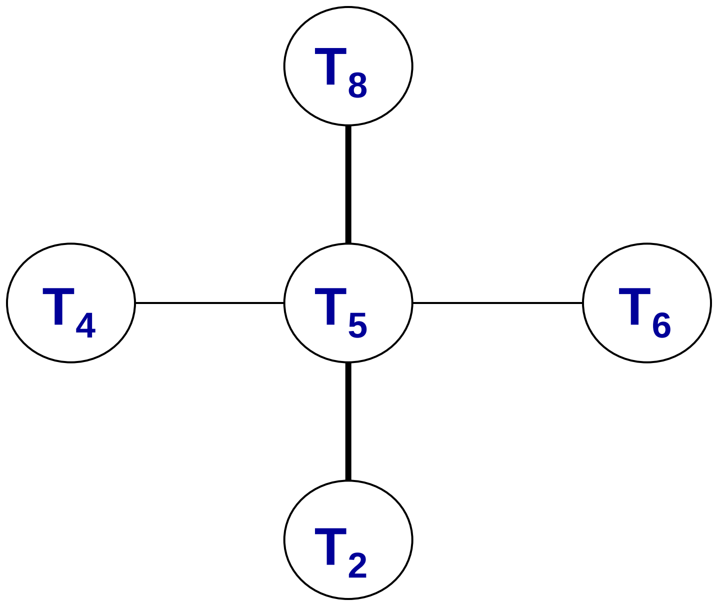


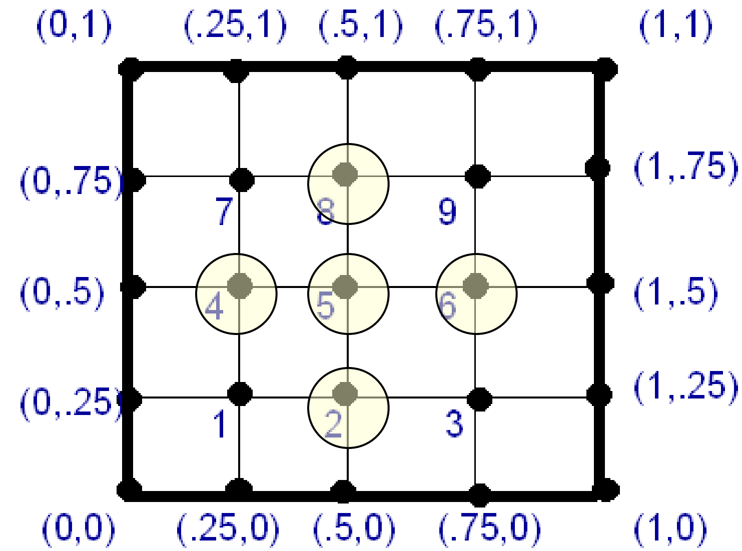


$$-T_1 + 4T_4 - T_5 - T_7 = 0$$

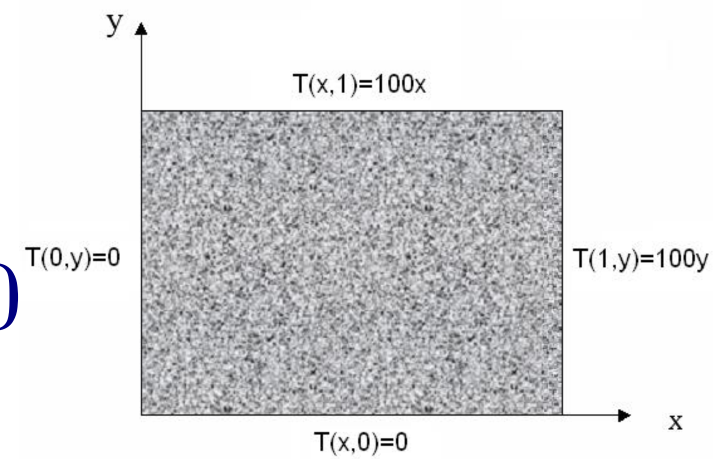


Node 5

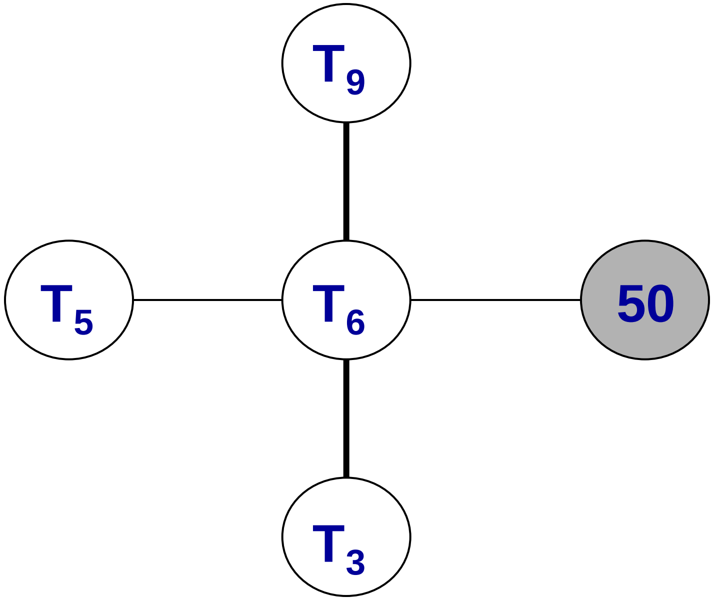


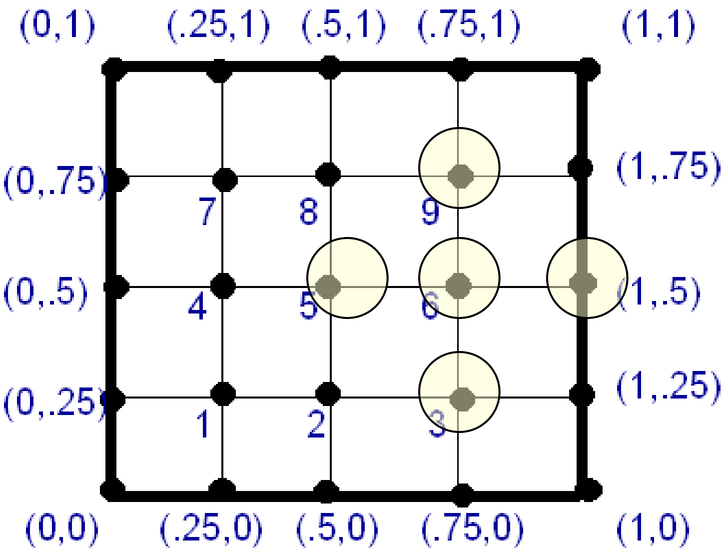


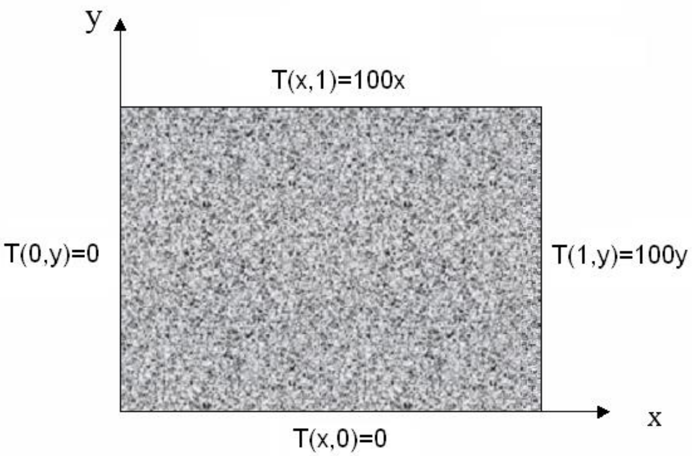
$$-T_2 - T_4 + 4T_5 - T_6 - T_8 = 0$$



Node 6

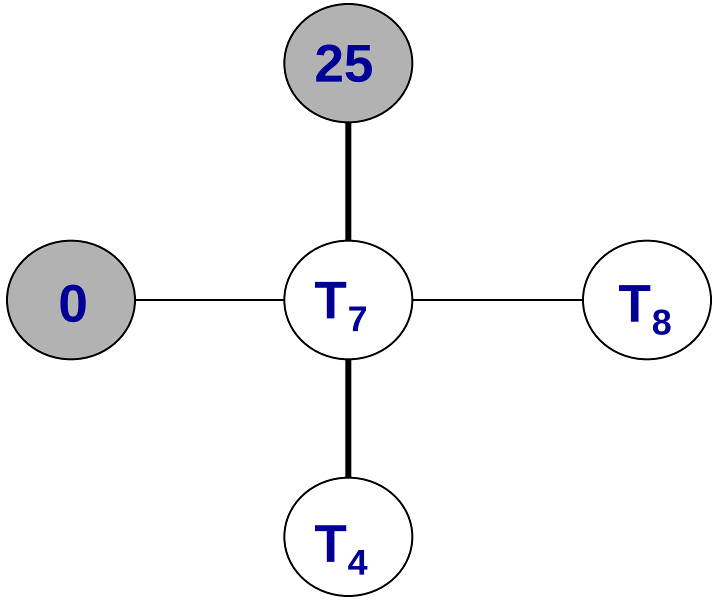


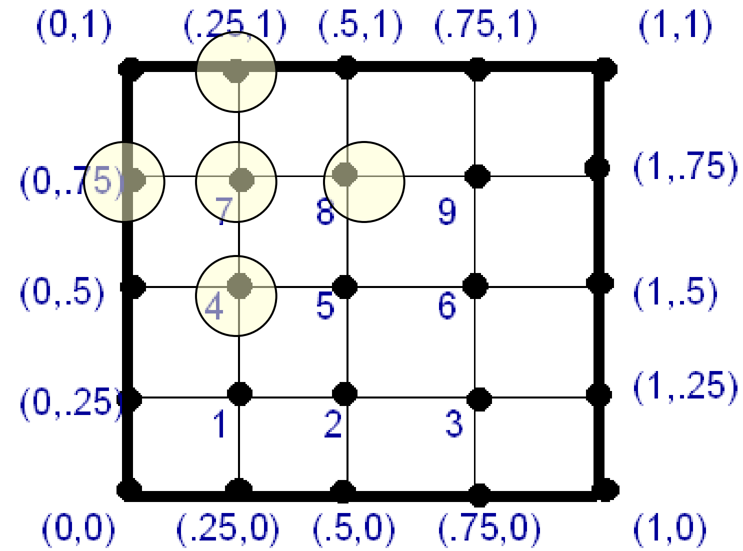




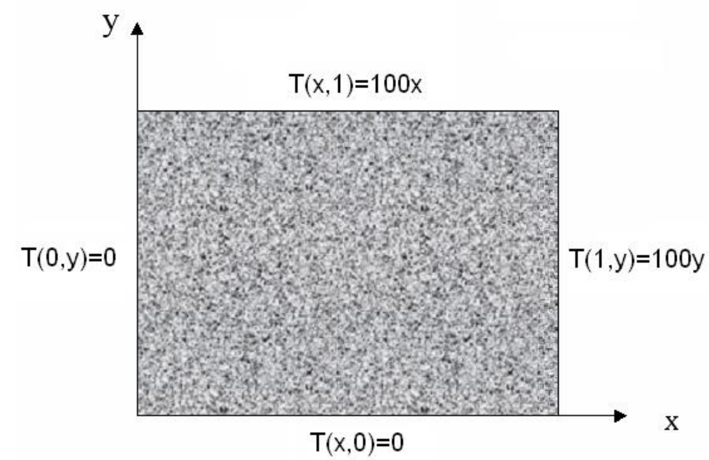
$$-T_3 - T_5 + 4T_6 - T_9 = 50$$

Node 7

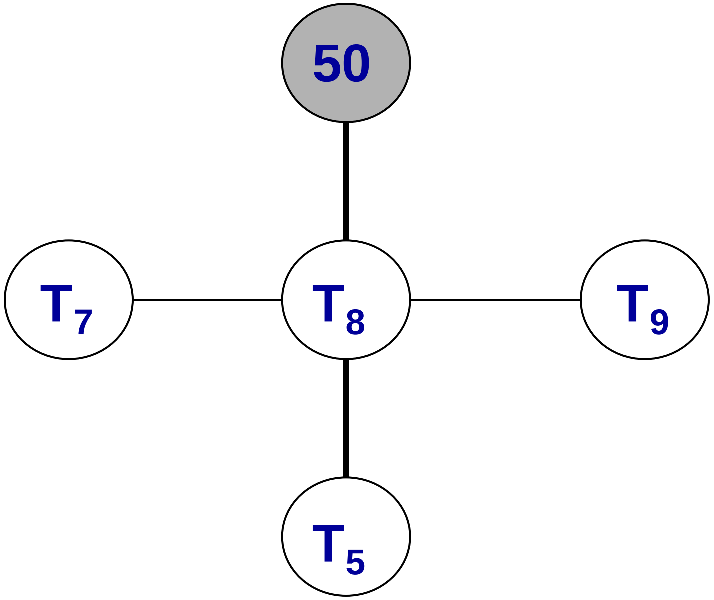


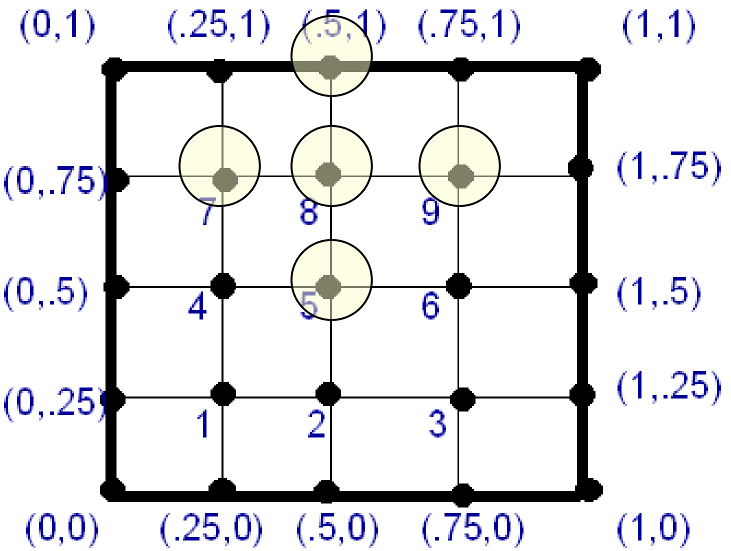


$$-T_4 + 4T_7 - T_8 = 25$$

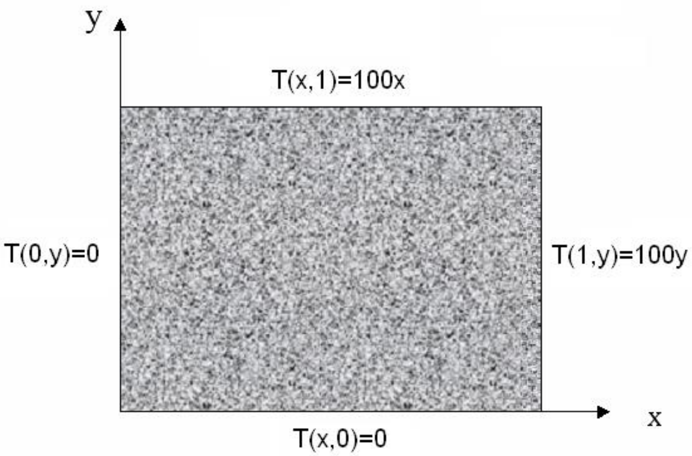


Node 8

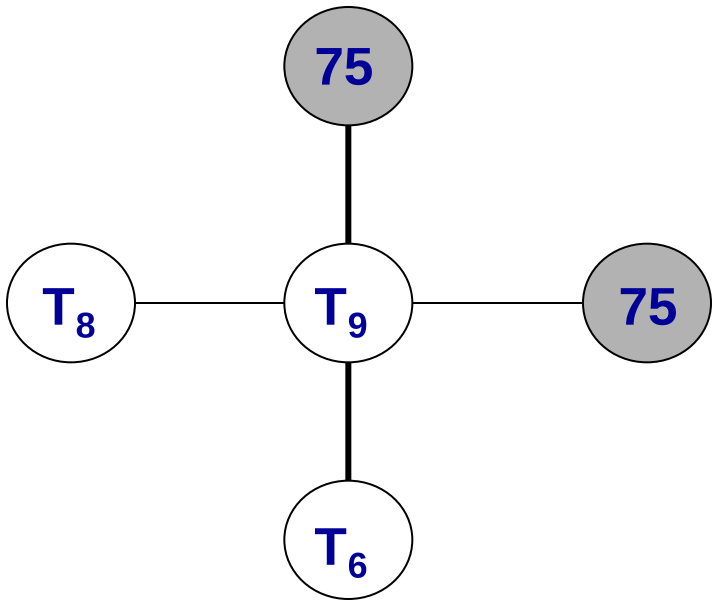


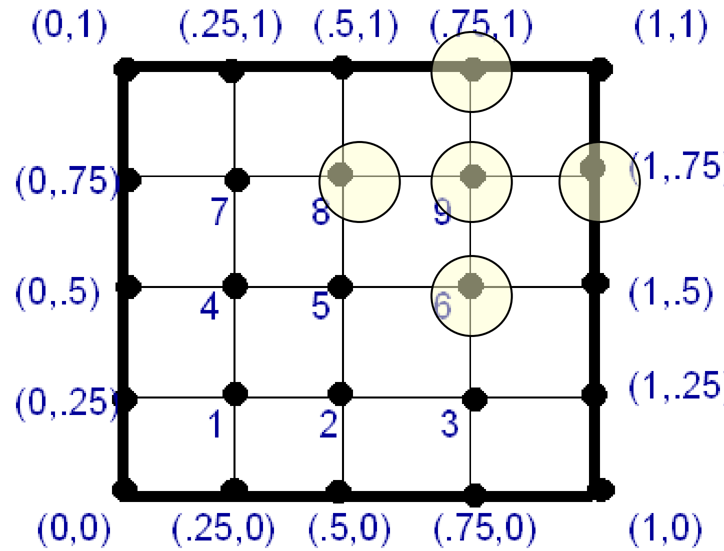


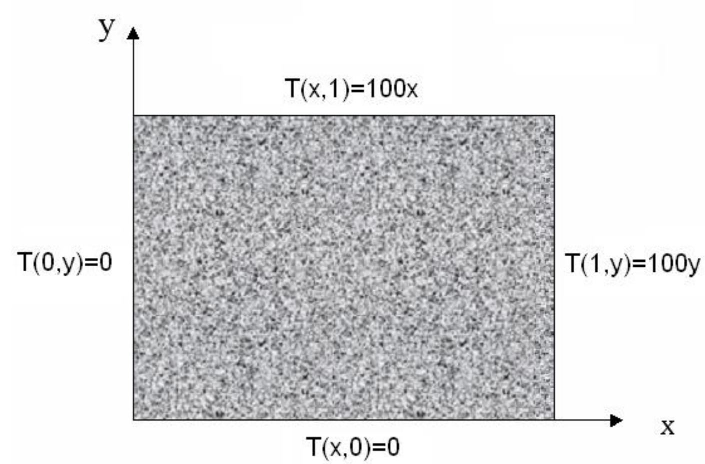
$$-T_5 - T_7 + 4T_8 - T_9 = 50$$



Node 9







$$-T_6 - T_8 + 4T_9 = 150$$

Discrete Laplace equation: $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 25 \\ 0 \\ 0 \\ 50 \\ 25 \\ 50 \\ 150 \end{bmatrix}$$

Discrete Laplace equation: $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 25 \\ 0 \\ 0 \\ 50 \\ 25 \\ 50 \\ 150 \end{bmatrix}$$

Banded structure

Discrete Laplace equation: $\mathbf{A}\mathbf{x} = \mathbf{b}$

$$
\mathbf{A} = \left[\begin{array}{ccc|ccc|ccc}
4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
\hline
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{array}\right]
\qquad
\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 25 \\ 0 \\ 0 \\ 50 \\ 25 \\ 50 \\ 150 \end{bmatrix}
$$

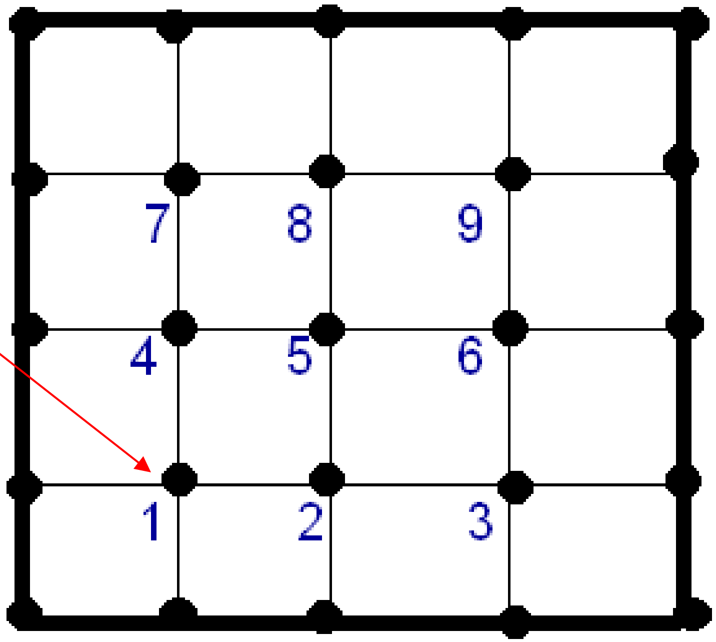
Block structure

Solution Strategy

- Coefficient matrix is:
  - large: for n x n mesh, $\mathbf{A} \in \mathbb{R}^{N \times N},\ N = O(n^2)$
  - sparse: most elements are zero
  - structured: banded / block pattern
- How do we handle this matrix efficiently?
  - efficient storage
    - naïve implementation requires $O(N^2)$ storage
  - efficient algorithms
    - naïve implementation requires $O(N^3)$ flops
- Chapter 2!

Numerical solution

$$\mathbf{x} = \begin{bmatrix} 6.25 \\ 12.50 \\ 18.75 \\ 12.50 \\ 25.00 \\ 37.50 \\ 18.75 \\ 37.50 \\ 56.25 \end{bmatrix}$$



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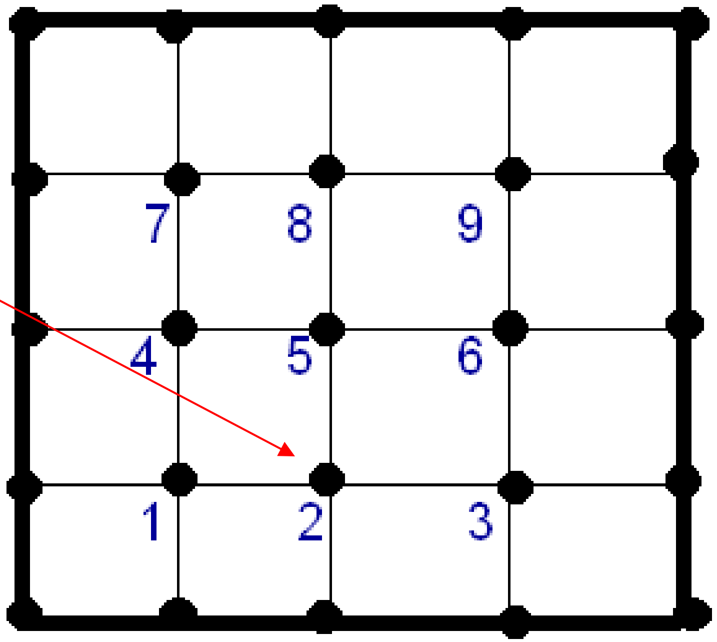

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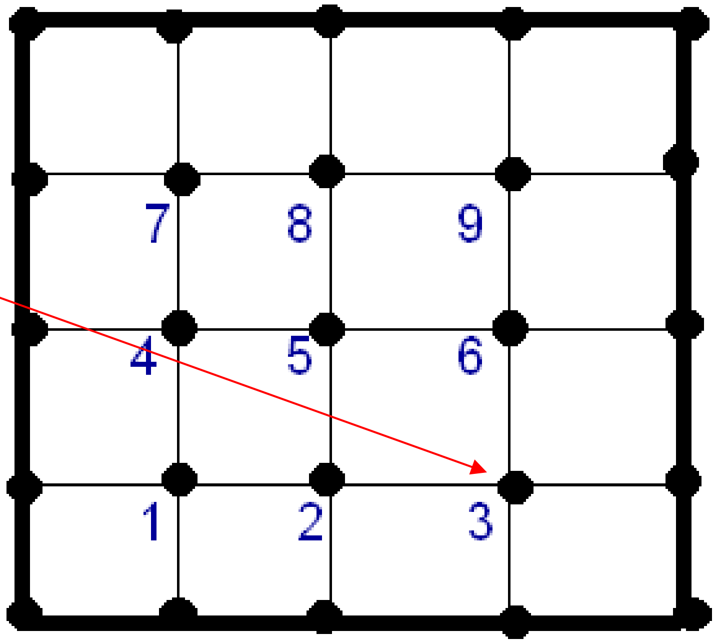

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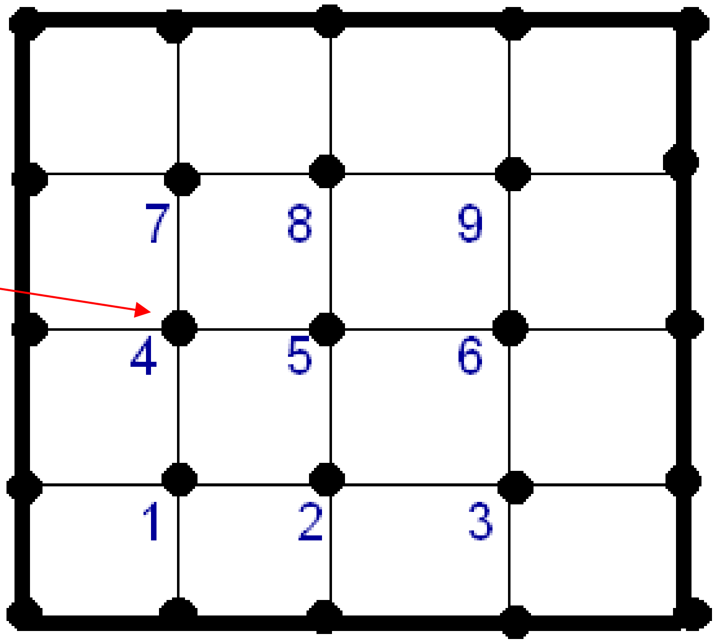

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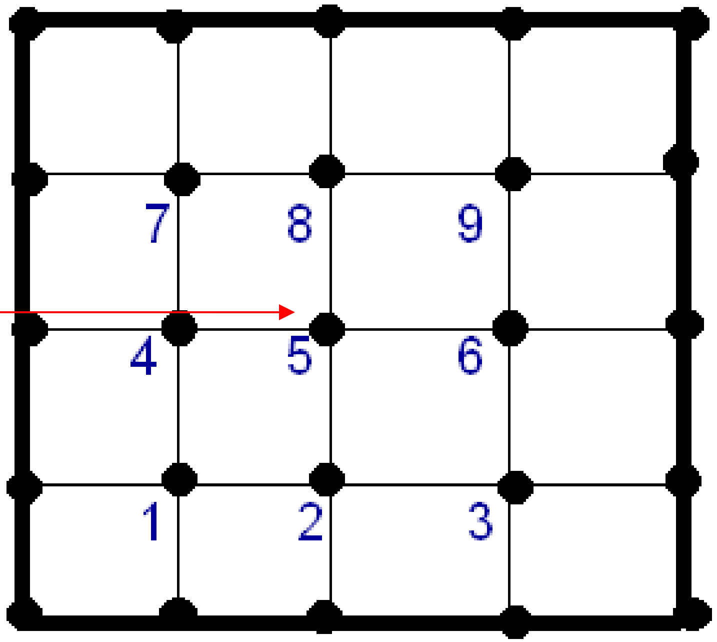

Numerical solution

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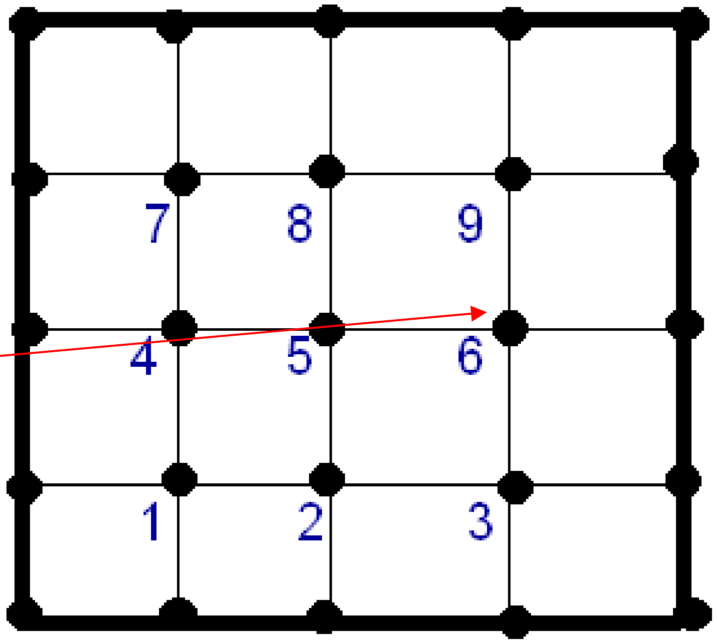

Numerical solution

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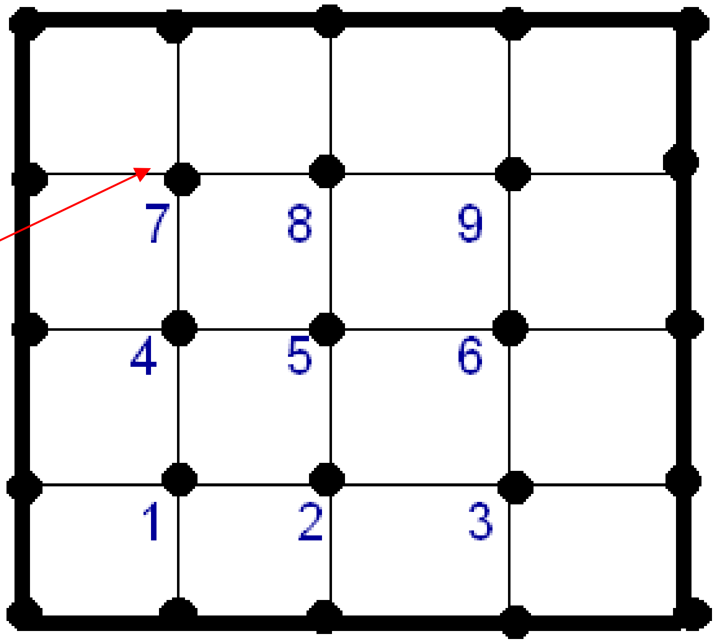
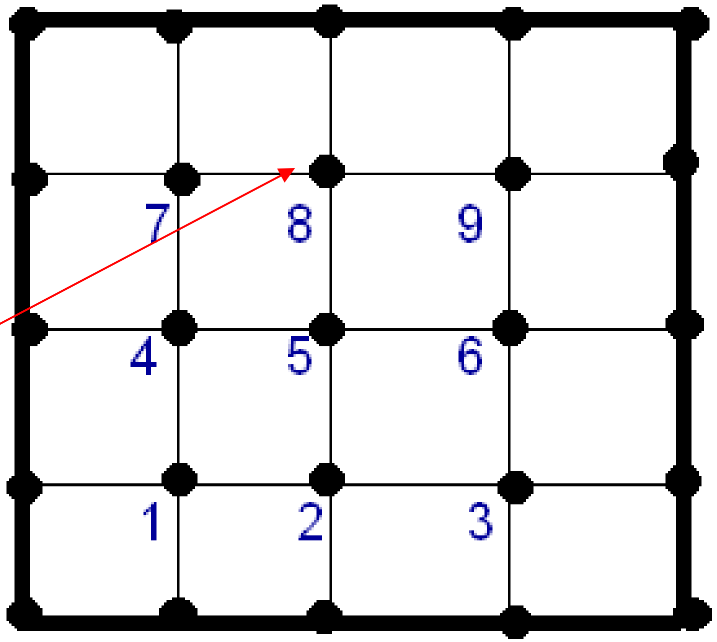


Numerical solution

$$\mathbf{x} = \begin{bmatrix} 6.25 \\ 12.50 \\ 18.75 \\ 12.50 \\ 25.00 \\ 37.50 \\ 18.75 \\ 37.50 \\ 56.25 \end{bmatrix}$$



Numerical solution

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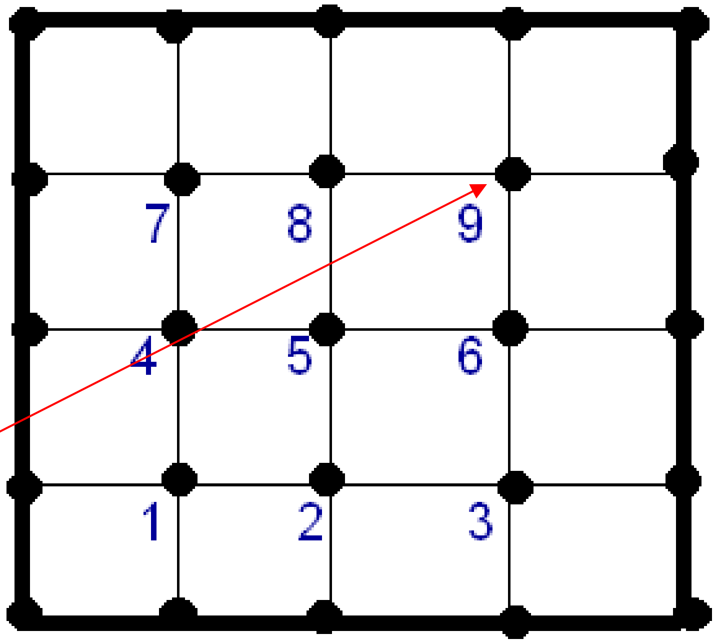

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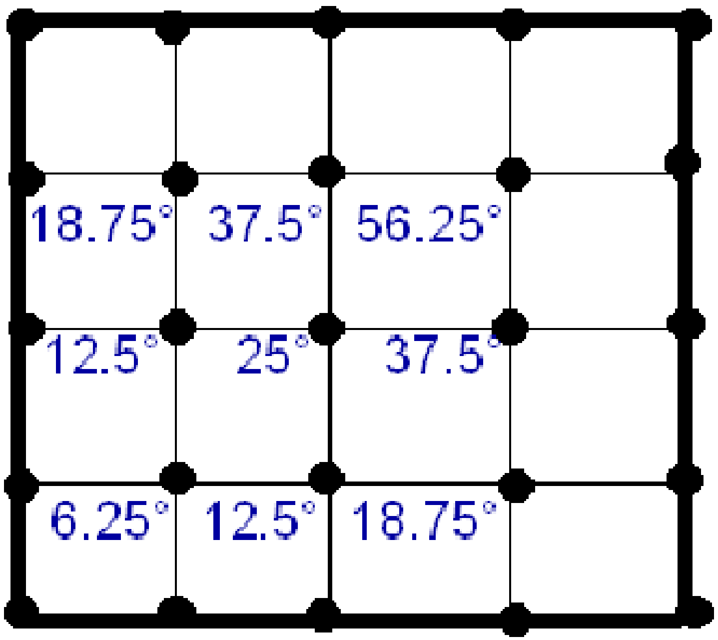
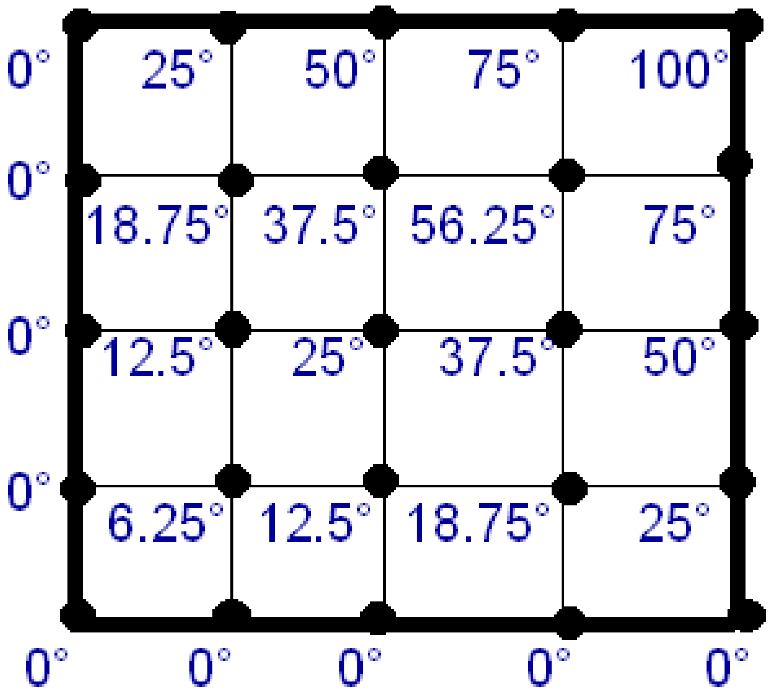


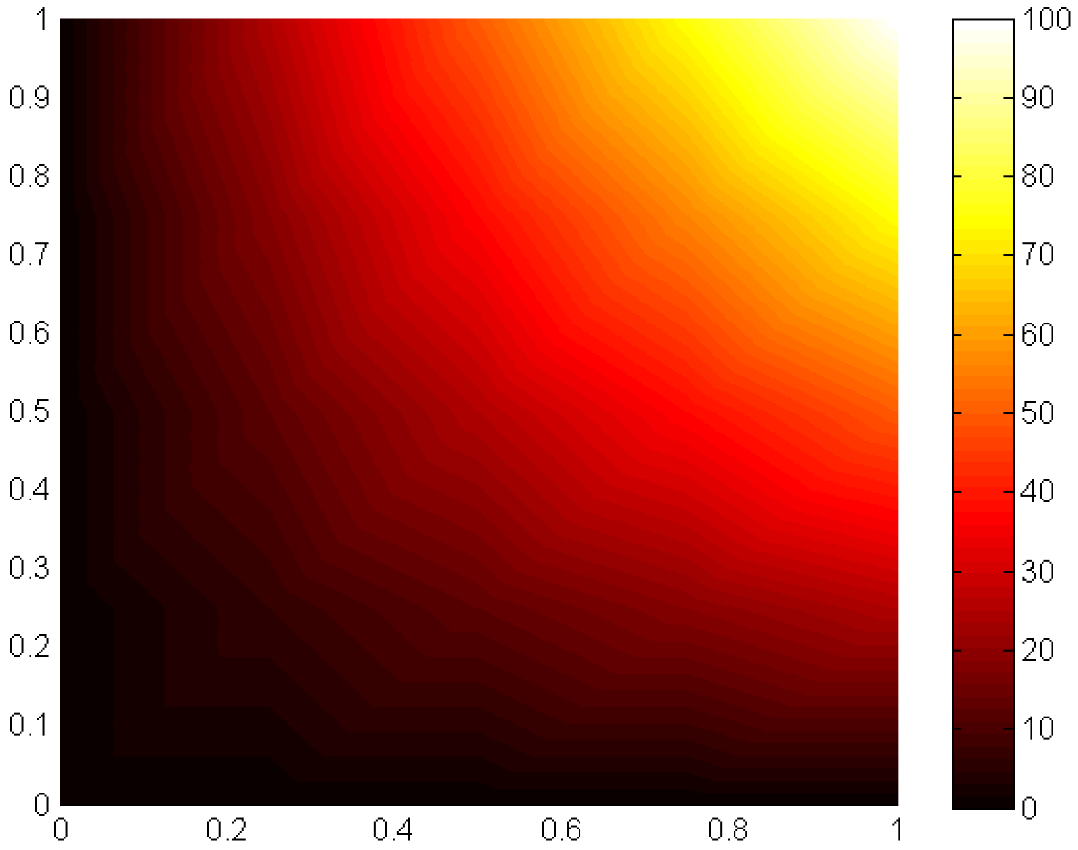
Numerical solution

$$\mathbf{x} = \begin{bmatrix} 6.25 \\ 12.50 \\ 18.75 \\ 12.50 \\ 25.00 \\ 37.50 \\ 18.75 \\ 37.50 \\ 56.25 \end{bmatrix}$$



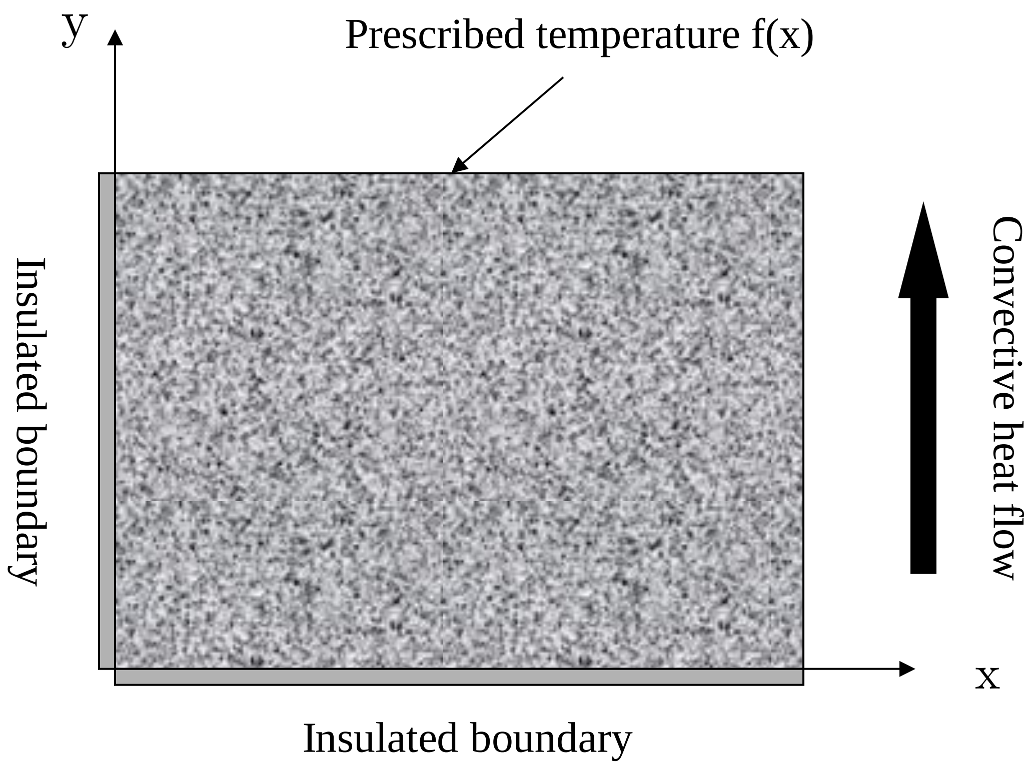
Don’t forget the boundary conditions!

Numerical solution



For this problem, the numerical solution is <u>exact</u> (do you know why?)

Second example: Poisson equation



Ceramic plate with Dirichlet, Neumann and Robin boundary conditions

Second example: Poisson equation

- Prescribed boundary:

$$f(x) = 200 - 150x^2$$

- Insulated boundaries:

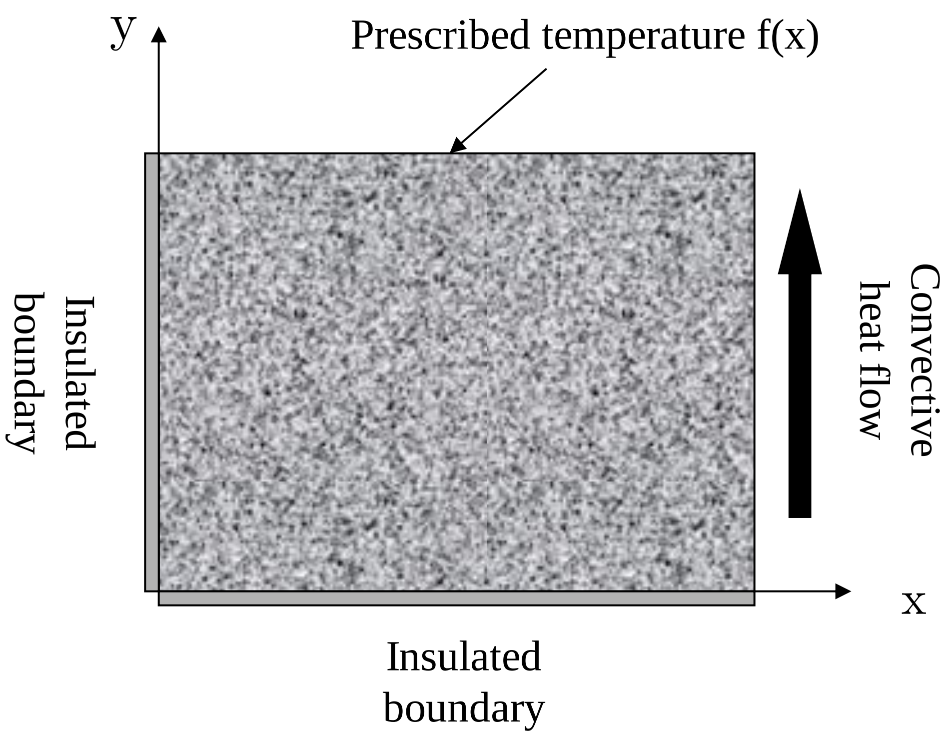
$$\nabla T \cdot \hat{\mathbf{n}} = 0$$

- Convective boundary:

$$k\nabla T \cdot \hat{\mathbf{n}} = h \;\; T_\infty - T$$

- Uniform heat source:

$$g(x, y) = g_0\,(\text{constant})$$



Summary of the Mathematical Model

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{g_0}{k} = 0, \qquad 0 < x < a \quad , \quad 0 < y < b$$

$$subject\ \ to$$

$$at\ \ x = 0 \qquad : \qquad \frac{\partial T}{\partial x} = 0$$

$$at\ \ x = a \qquad : \qquad k\frac{\partial T}{\partial x} + hT = hT_\infty$$

$$at\ \ y = 0 \qquad : \qquad \frac{\partial T}{\partial y} = 0$$

$$at\ \ y = b \qquad : \qquad T \;\; x, b \;\; = f \;\; x$$

Analytical Solution

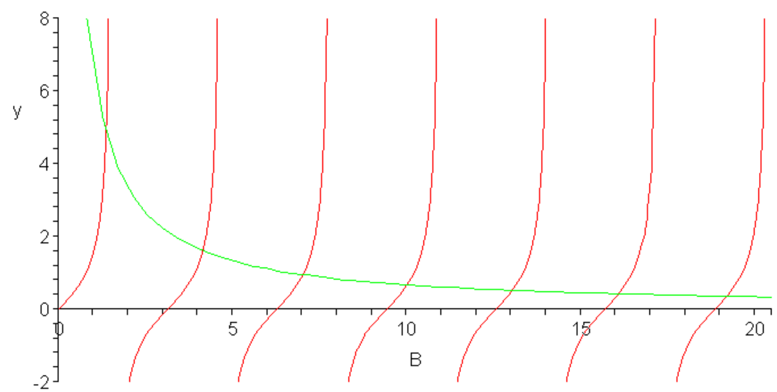
(Not examinable! You'll learn about this in MXB322.)

$$T(x,y) = \varphi(x,y) + \frac{g_0}{2k}\left(a^2 - x^2\right) + \frac{g_0 a}{h} + T_\infty$$

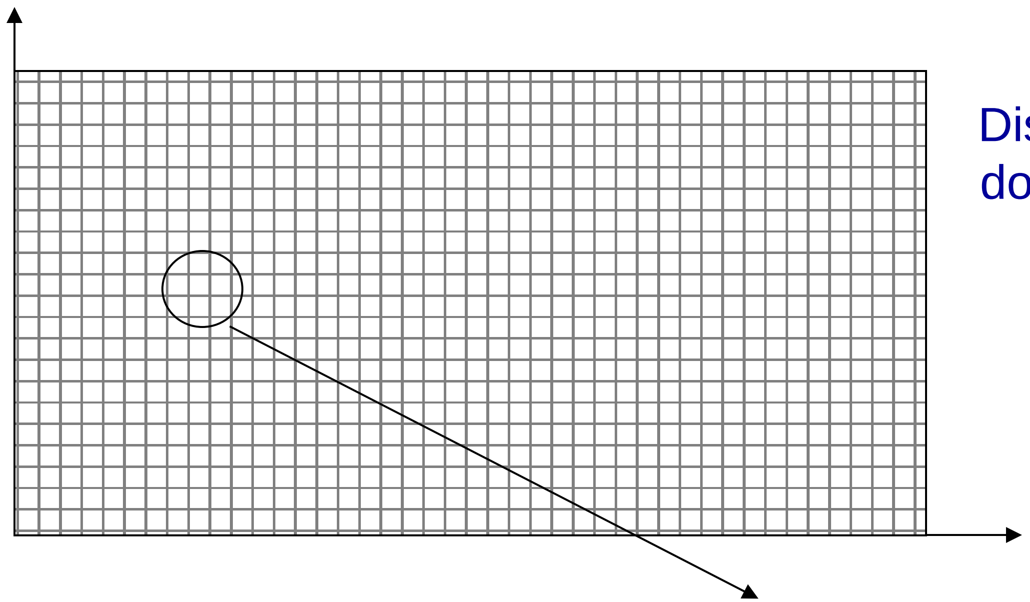
$$\varphi(x,y) = 2\sum_{m=1}^{\infty} \frac{\beta_m^2 + H^2}{a\left(\beta_m^2 + H^2\right) + H} \frac{\cosh\left(\beta_m y\right)}{\cosh\left(\beta_m b\right)} \cos\left(\beta_m x\right) \int_0^a \tilde{\varphi}(\xi) \cos\left(\beta_m \xi\right) d\xi$$

$$where\;\; \beta_m \;\; are\; the\; roots\; of\;\; \tan\left(\beta_m a\right) = H/\beta_m$$

$$and\;\; \tilde{\varphi}(x) = f(x) - \frac{g_0}{2k}\left(a^2 - x^2\right) - \frac{g_0 a}{h} - T_\infty, \quad H = \frac{h}{k}$$

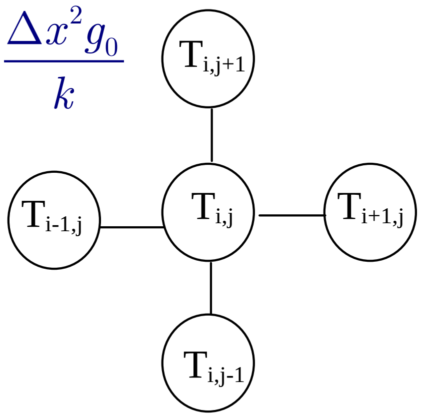


Conversion to a Matrix Problem



Discretise the domain using a mesh

$$-T_{i,j-1} - T_{i-1,j} + 4T_{i,j} - T_{i+1,j} - T_{i,j+1} = \frac{\Delta x^2 g_0}{k}$$



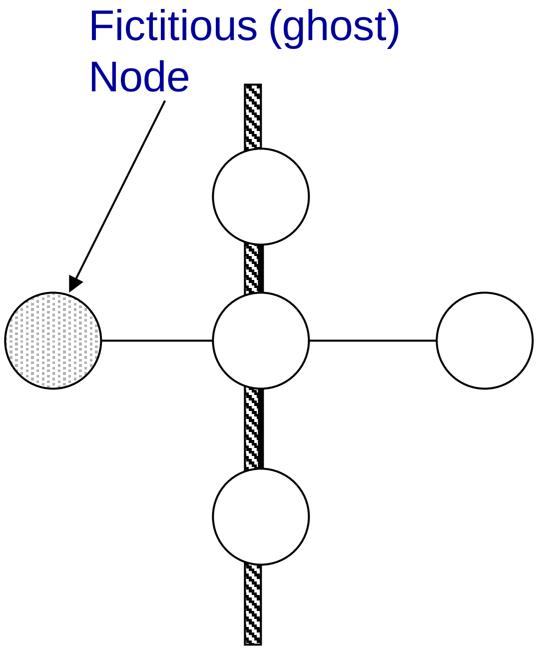
Finite Difference Stencil used for each internal node on the mesh

The model is converted to a matrix problem by discretising the domain and forming *finite difference equations*.

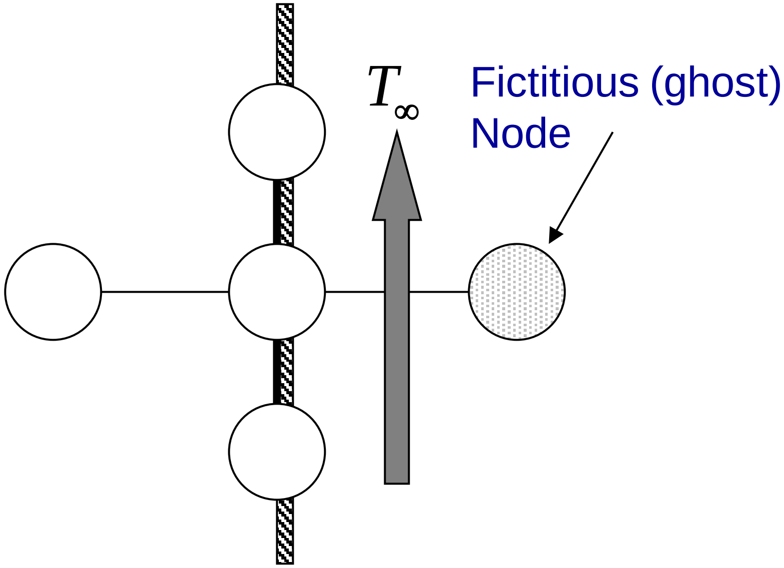
Treatment of Boundary Nodes

Insulated / Convection boundary:

Use Neumann / Robin condition to construct fictitious node



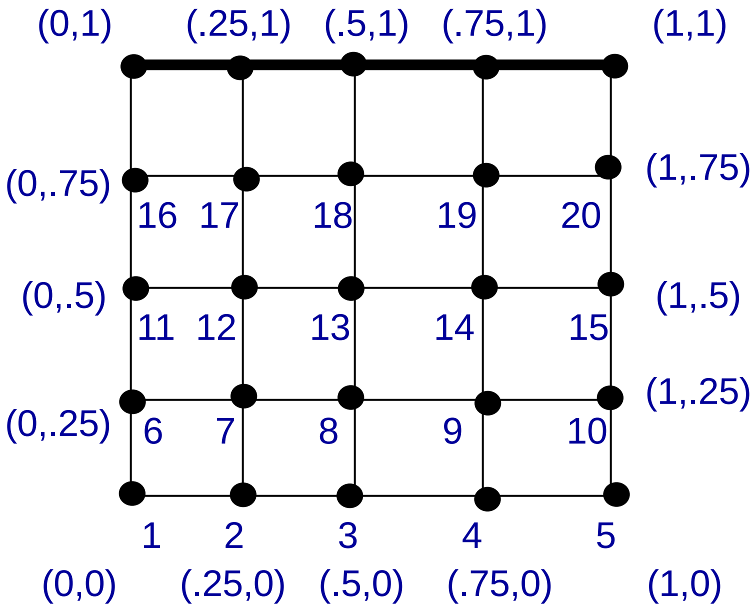
Insulated Boundary



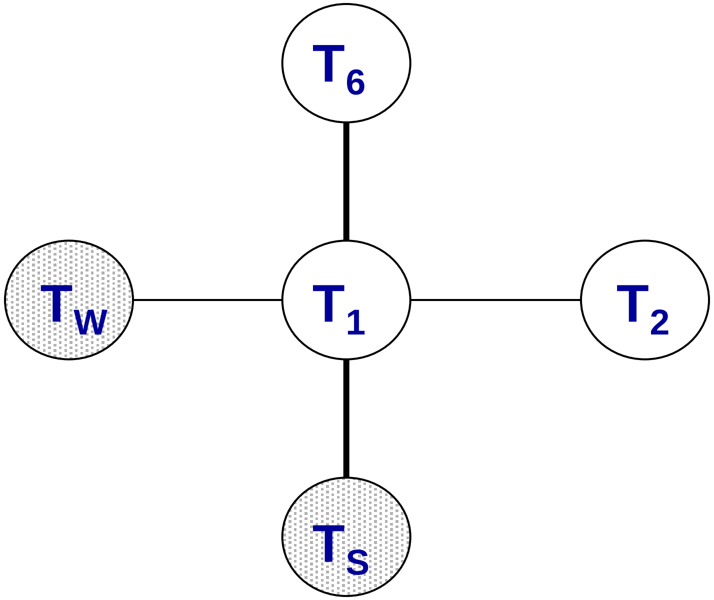
Convection Boundary

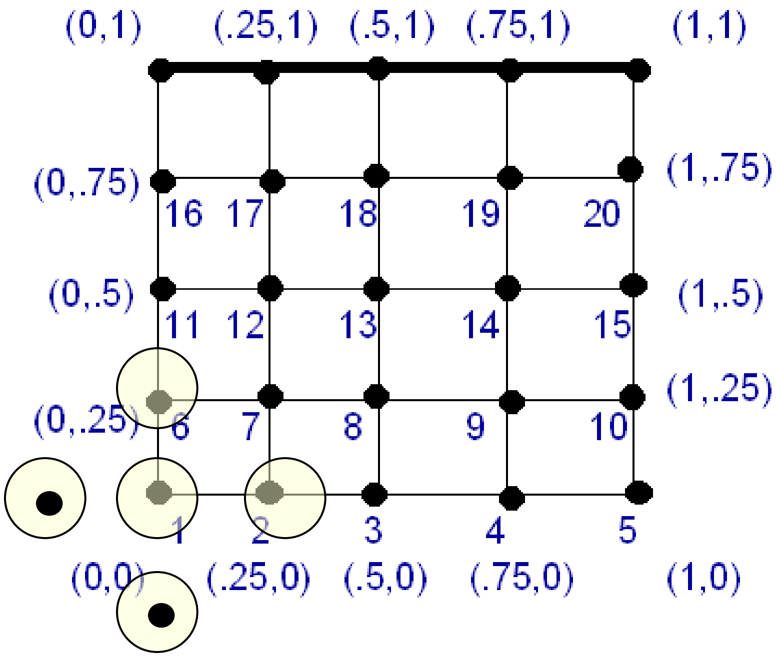
How does this process generate a matrix?

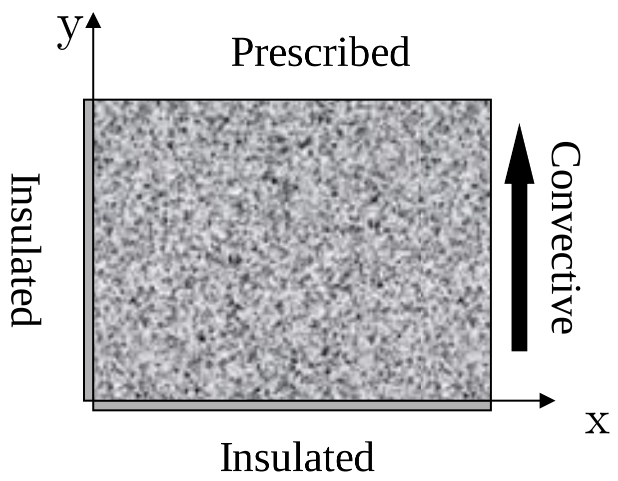
- Introduce a node numbering scheme and map (i,j) coordinates to these nodes:



Node 1

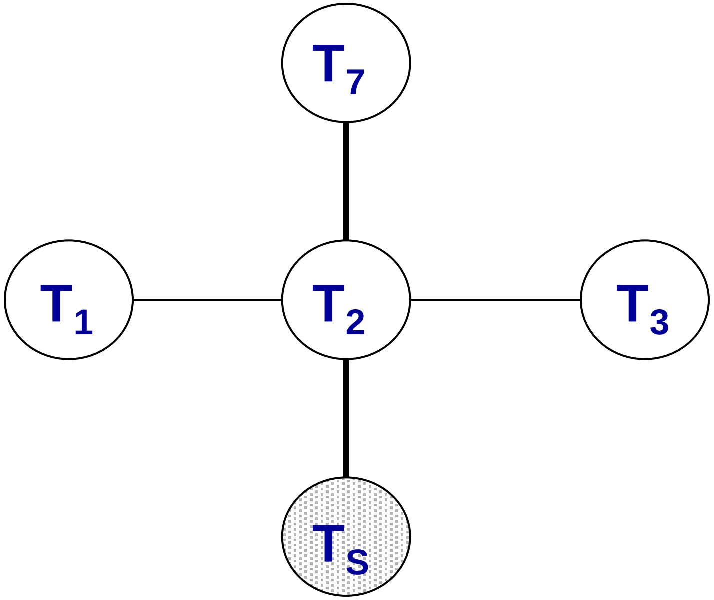


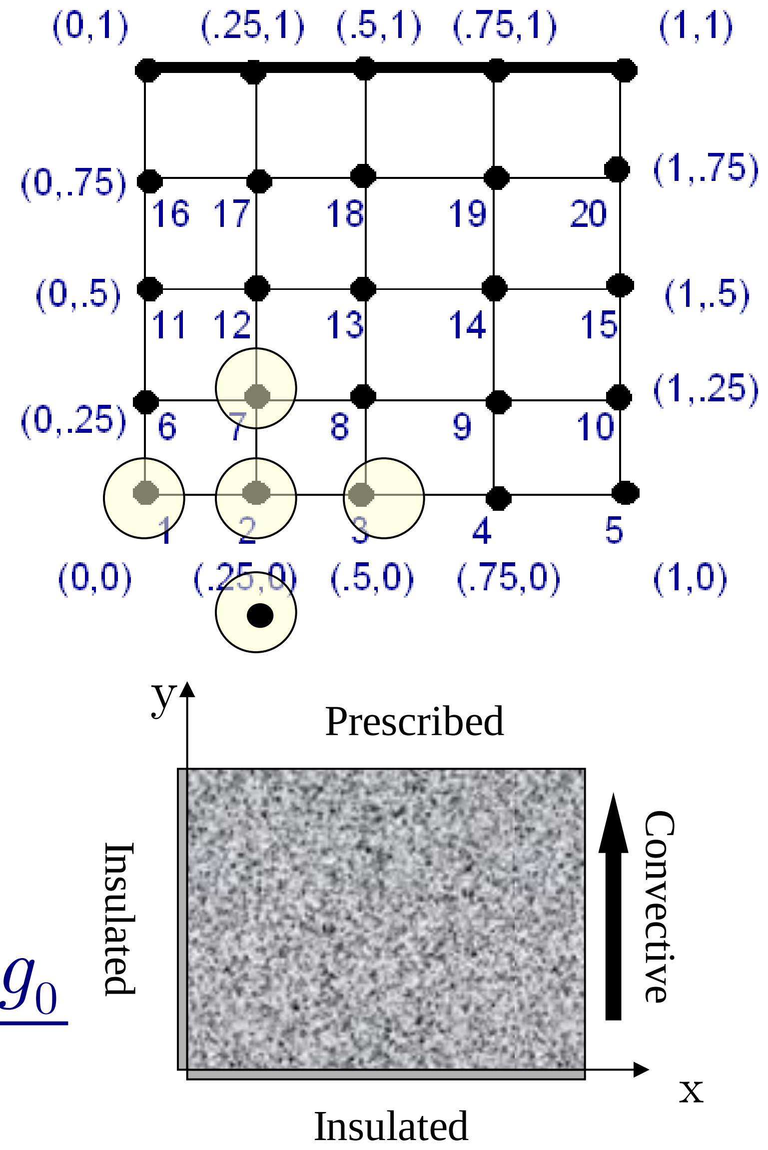




$$-T_S - T_W + 4T_1 - T_2 - T_6 = \frac{\Delta x^2 g_0}{k}$$

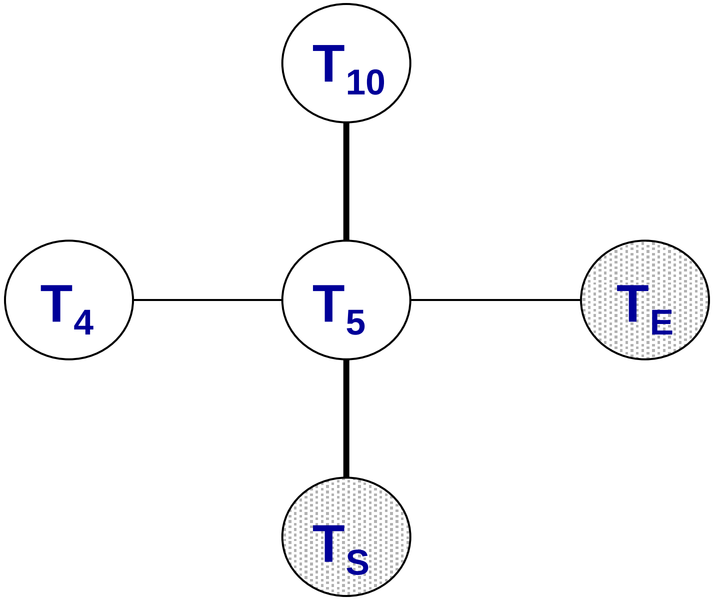
Node 2 (3,4 similar)

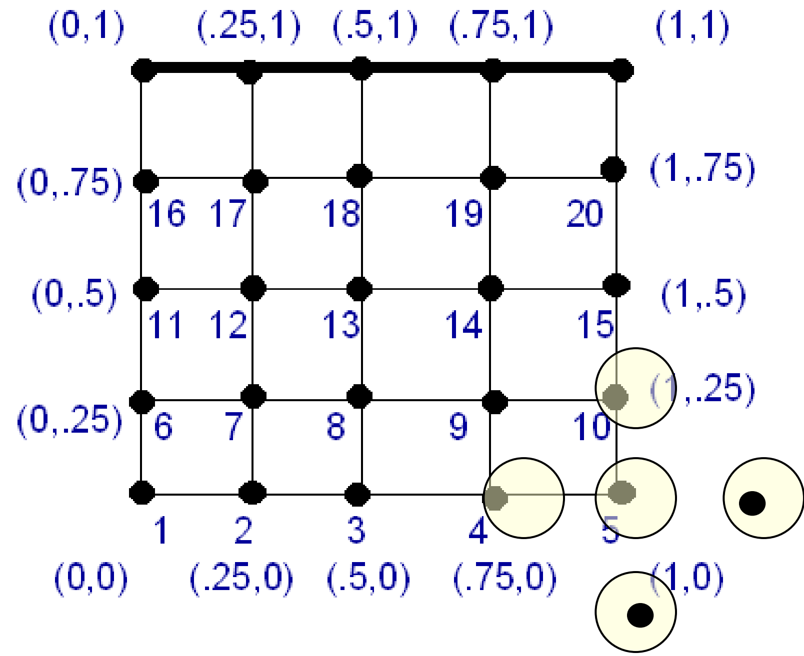


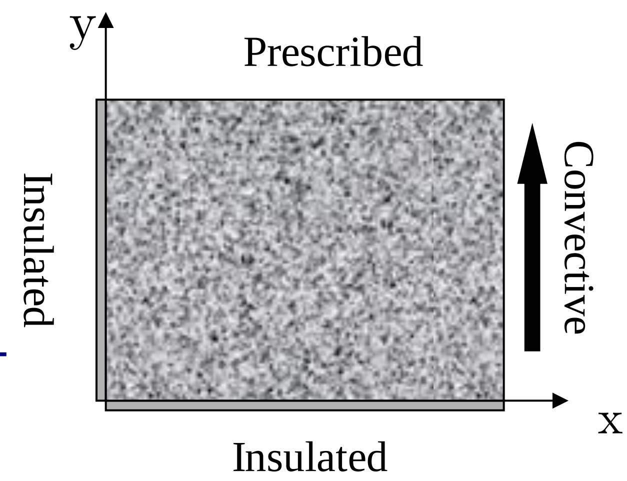


$$-T_S - T_1 + 4T_2 - T_3 - T_7 = \frac{\Delta x^2 g_0}{k}$$

Node 5

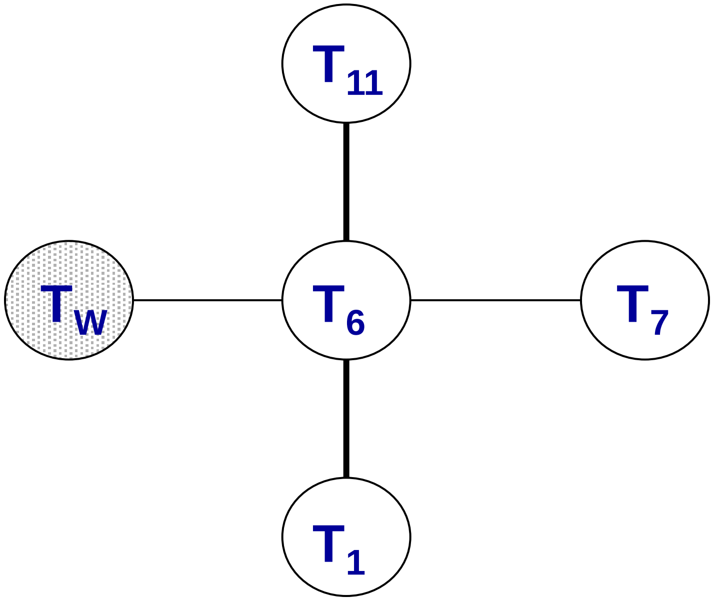


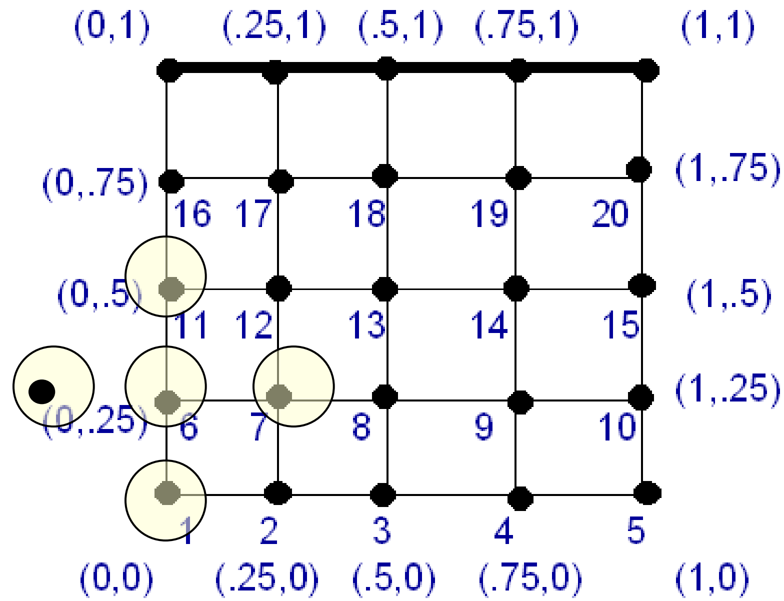


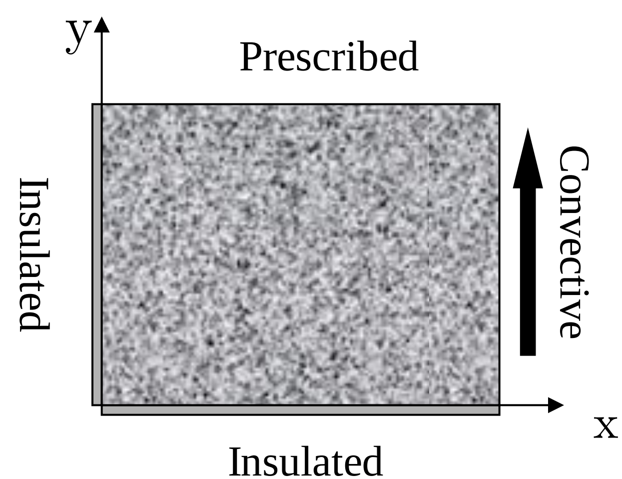


$$-T_S - T_4 + 4T_5 - T_E - T_{10} = \frac{\Delta x^2 g_0}{k}$$

Node 6 (11 similar)

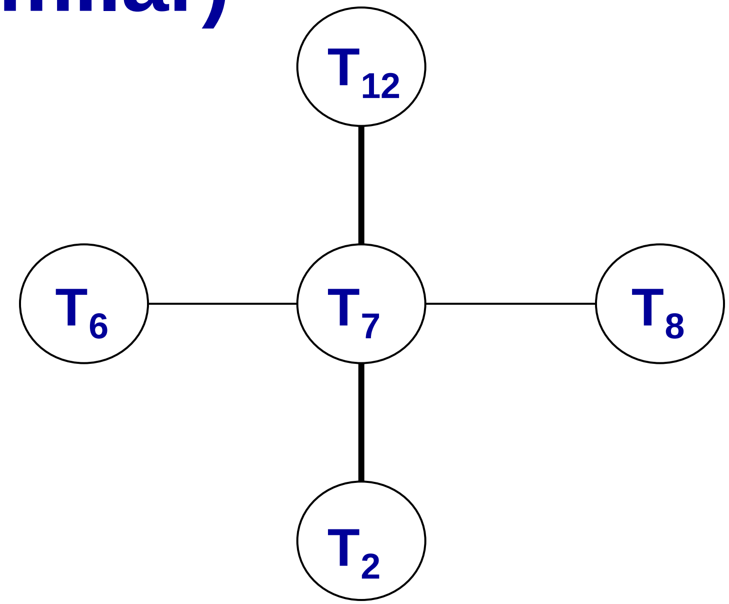


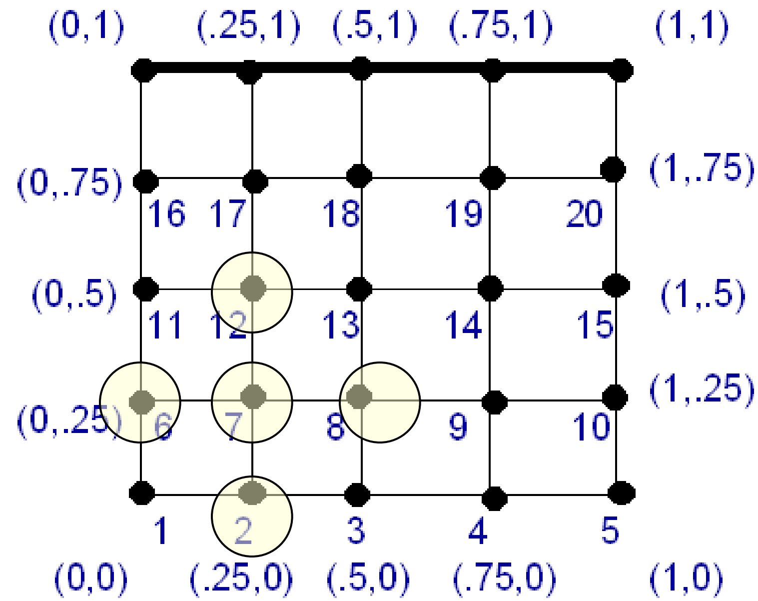


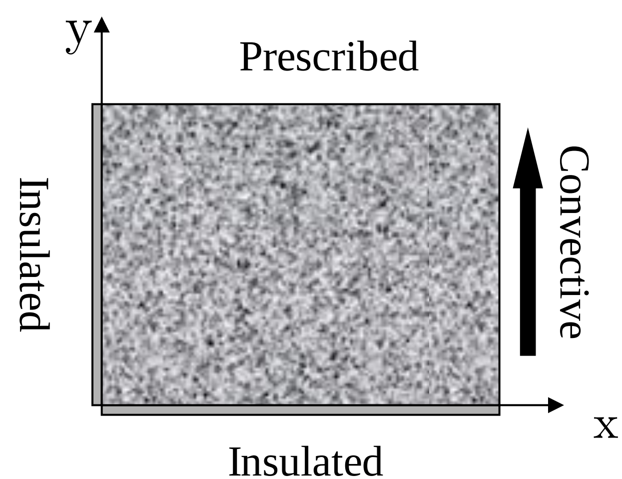


$$-T_1 - T_W + 4T_6 - T_7 - T_{11} = \frac{\Delta x^2 g_0}{k}$$

Node 7 (8,9,12,13,14 similar)

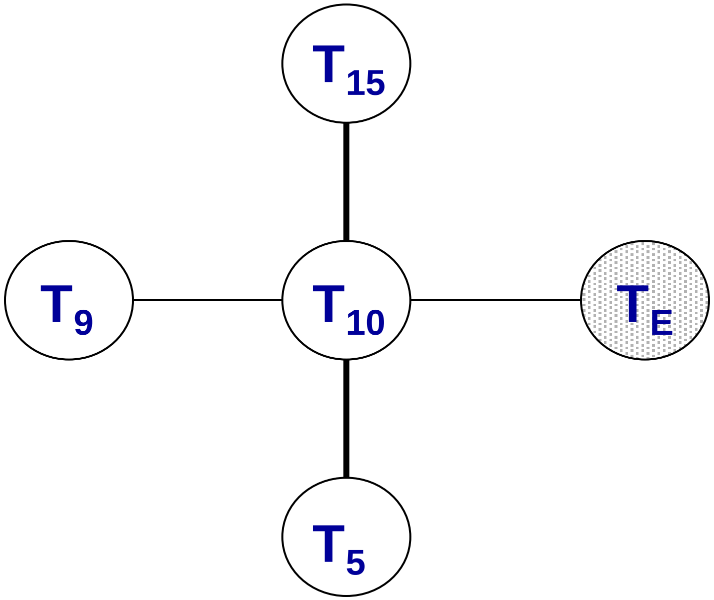


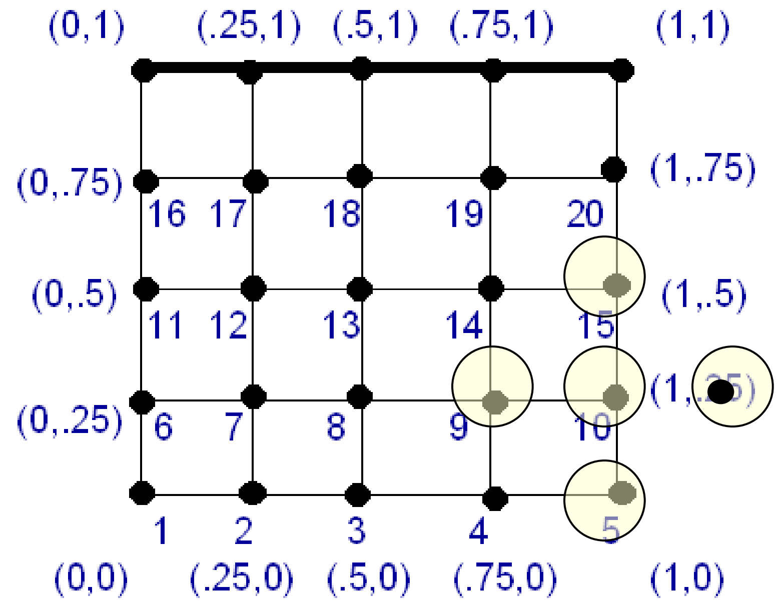


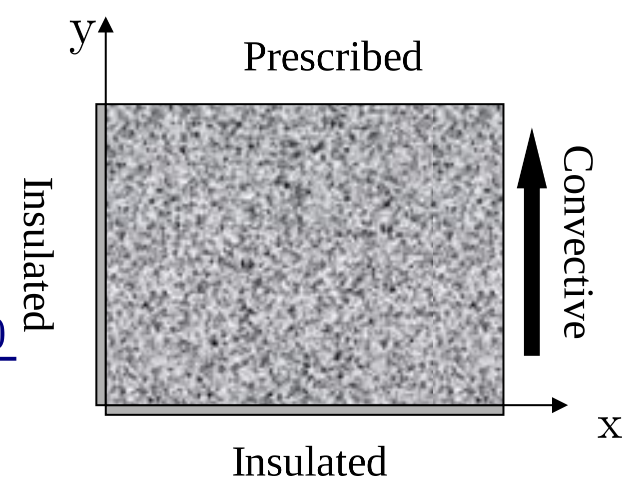


$$-T_2 - T_6 + 4T_7 - T_8 - T_{12} = \frac{\Delta x^2 g_0}{k}$$

Node 10 (15 similar)

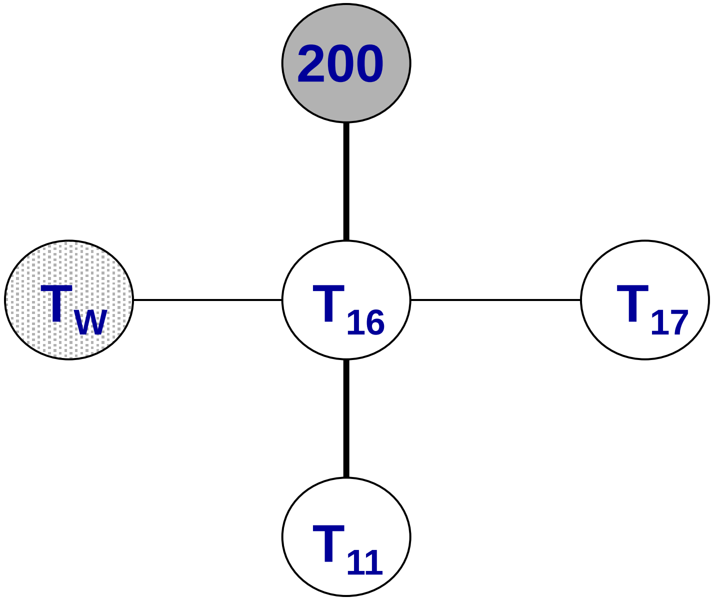


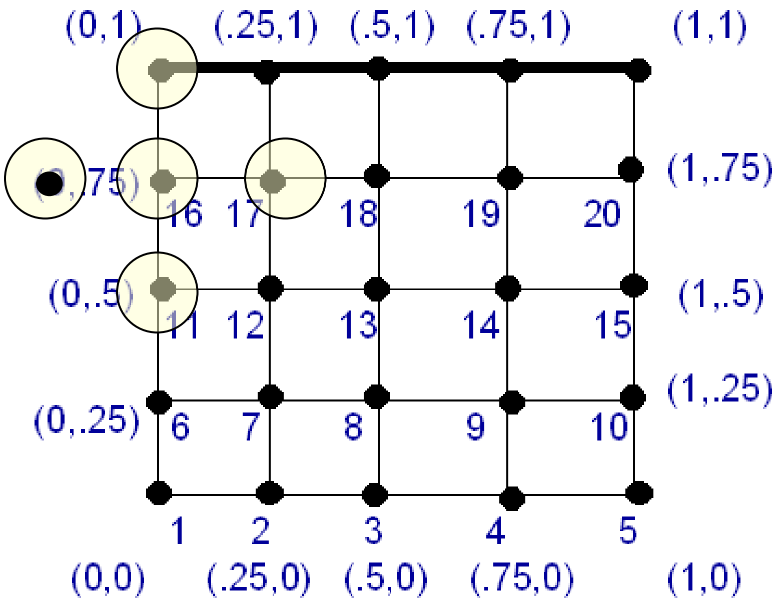


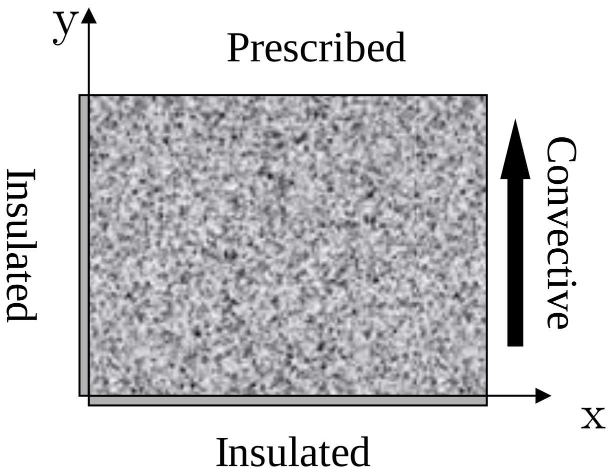


$$-T_5 - T_9 + 4T_{10} - T_E - T_{15} = \frac{\Delta x^2 g_0}{k}$$

Node 16

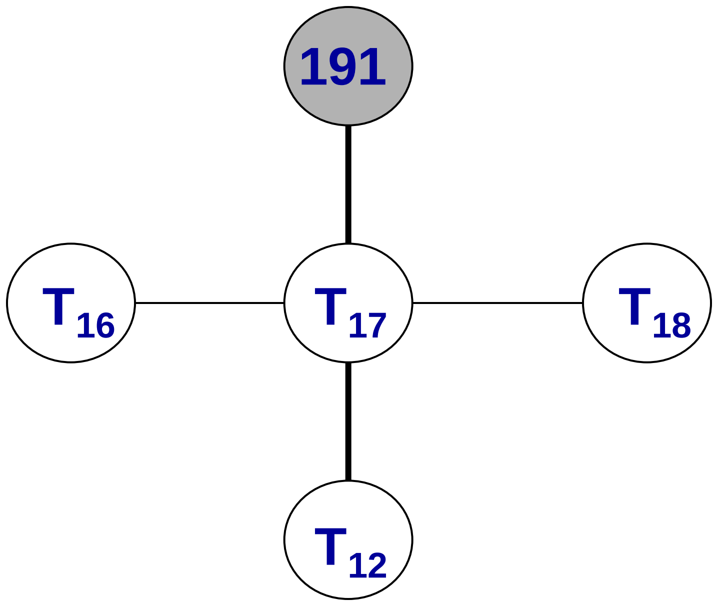


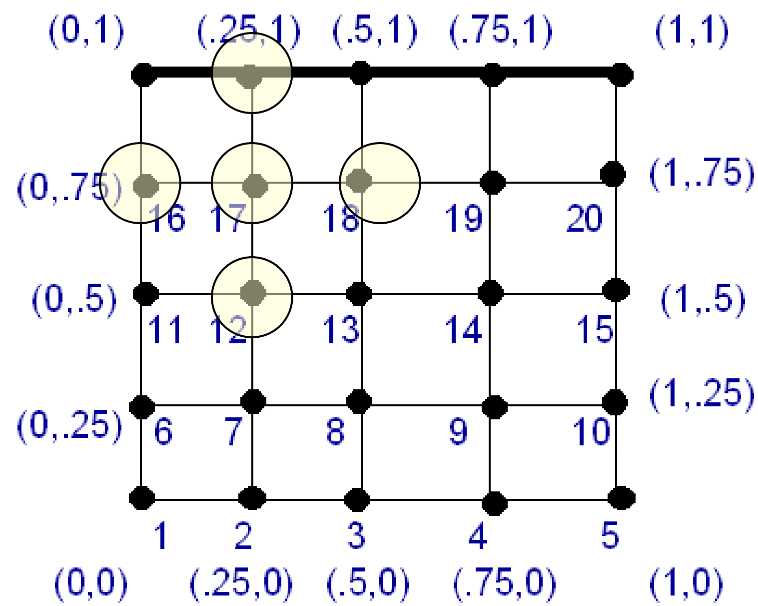


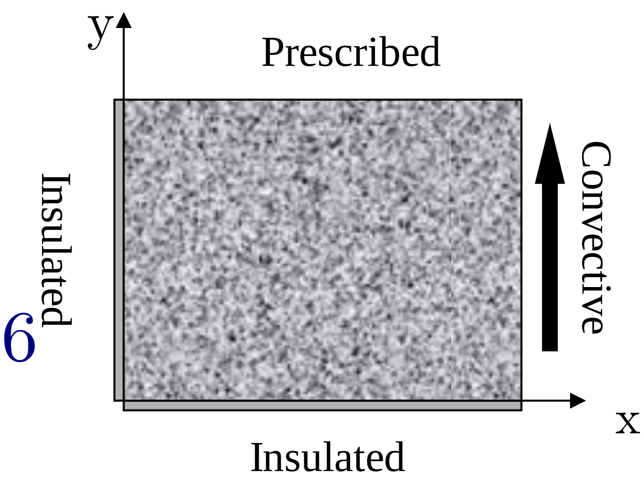


$$-T_{11} - T_W + 4T_{16} - T_{17} = \frac{\Delta x^2 g_0}{k} + 200$$

Node 17

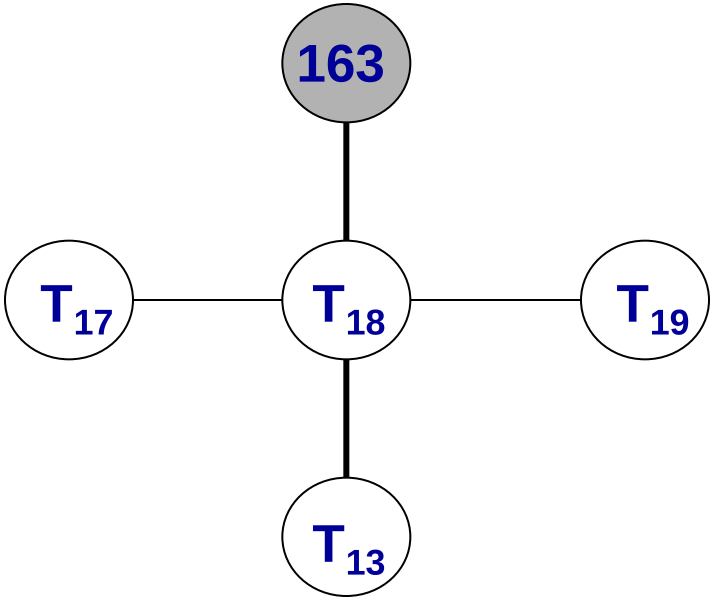


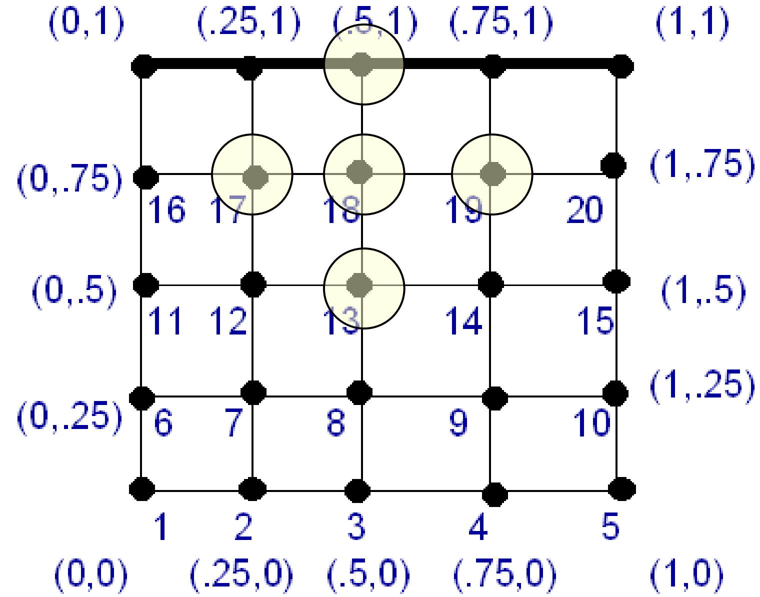


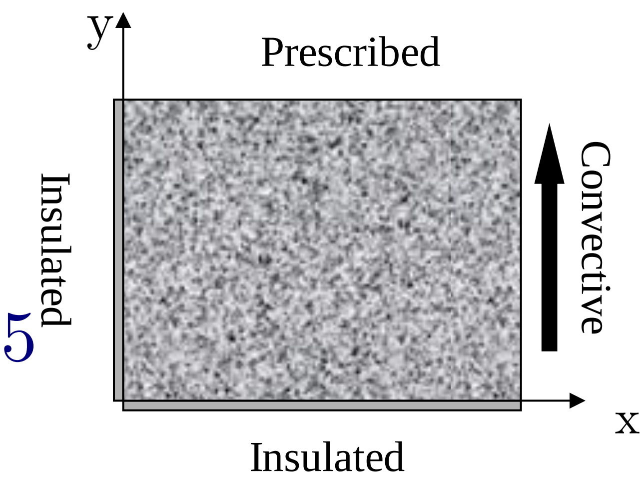


$$-T_{12} - T_{16} + 4T_{17} - T_{18} = \frac{\Delta x^2 g_0}{k} + 190.6$$

Node 18

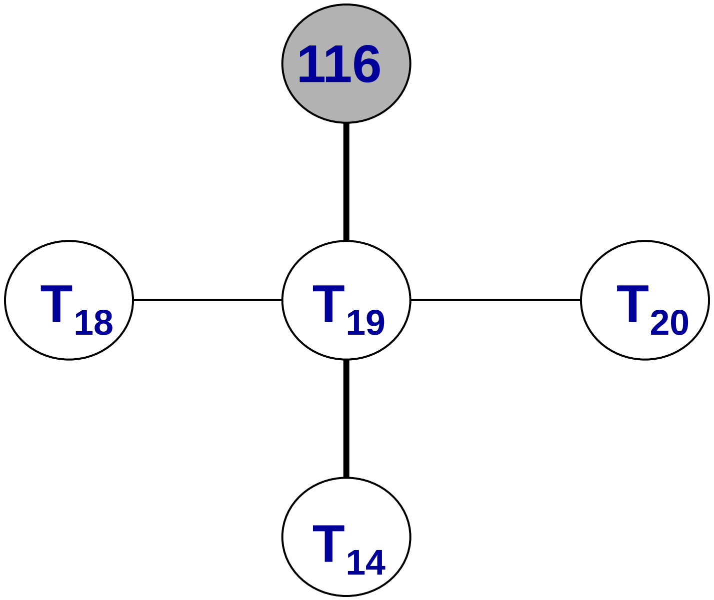


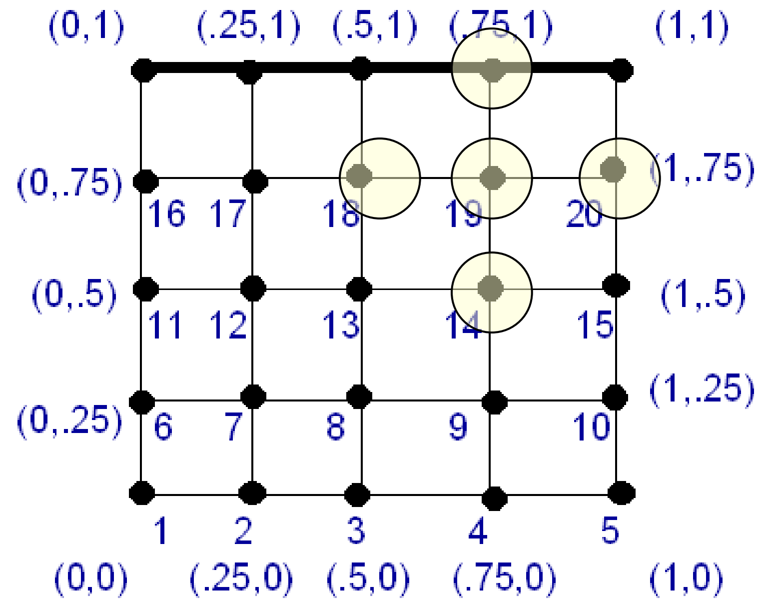




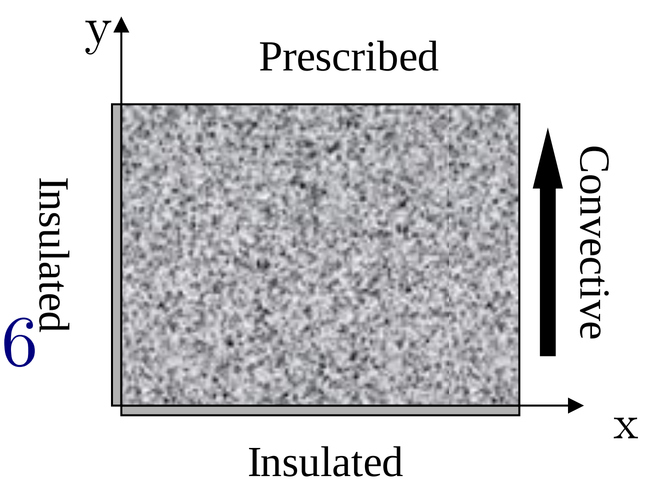
$$-T_{13} - T_{17} + 4T_{18} - T_{19} = \frac{\Delta x^2 g_0}{k} + 162.5$$

Node 19

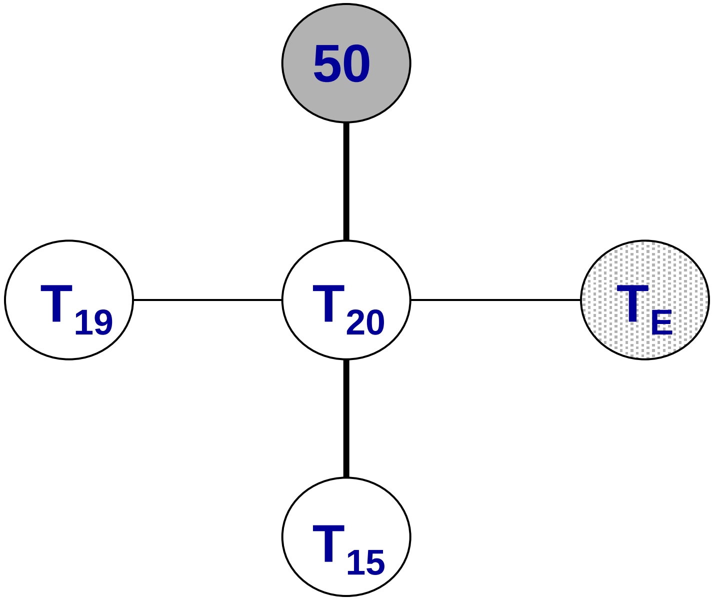


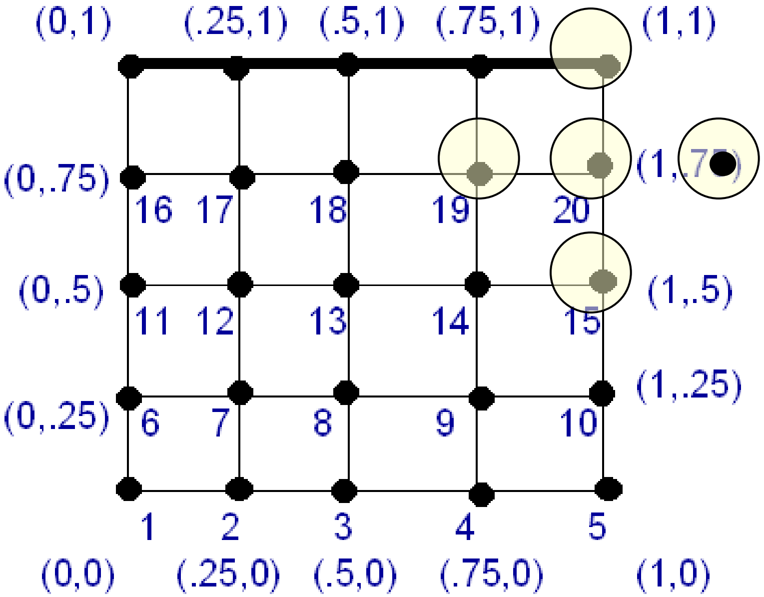


$$-T_{14} - T_{18} + 4T_{19} - T_{20} = \frac{\Delta x^2 g_0}{k} + 115.6$$

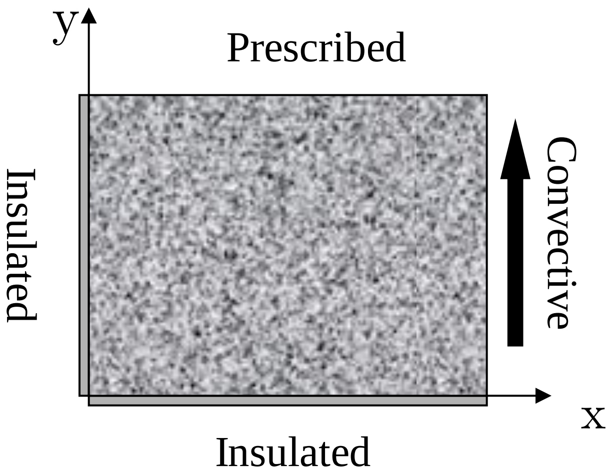


Node 20





$$-T_{15} - T_{19} + 4T_{20} - T_E = \frac{\Delta x^2 g_0}{k} + 50$$



Fictitious (or “ghost”) Nodes

- Use second order central differences
  - Maintains second order local accuracy
  - Destroys symmetry of the matrix

- Use first order forward/backward differences
  - Maintains symmetry of the matrix
  - Only first order local accuracy (but globally still a second order method)

- For this example, we’ll use second order boundary FDEs

Summary of Finite Difference Equations (FDEs)

$$(0,0)\ :\ 4T_{0,0} - 2T_{1,0} - 2T_{0,1} = \frac{\Delta x^2 g_0}{k}$$

$$(i,0)\ :\ -T_{i-1,0} + 4T_{i,0} - T_{i+1,0} - 2T_{i,1} = \frac{\Delta x^2 g_0}{k}$$

$$(N,0)\ :\ -2T_{N-1,0} + (4 + 2\Delta x H)T_{N,0} - 2T_{N,1} = \frac{\Delta x^2 g_0}{k} + 2\Delta x H T_\infty$$

$$(0,j)\ :\ -T_{0,j-1} + 4T_{0,j} - 2T_{1,j} - T_{0,j+1} = \frac{\Delta x^2 g_0}{k}$$

$$(i,j)\ :\ -T_{i,j-1} - T_{i-1,j} + 4T_{i,j} - T_{i+1,j} - T_{i,j+1} = \frac{\Delta x^2 g_0}{k}$$

$$(N,j)\ :\ -T_{N,j-1} - 2T_{N-1,j} + (4 + 2\Delta x H)T_{N,j} - T_{N,j+1} = \frac{\Delta x^2 g_0}{k} + 2\Delta x H T_\infty$$

$$(0,N-1):\ -T_{0,N-2} + 4T_{0,N-1} - 2T_{1,N-1} = \frac{\Delta x^2 g_0}{k} + f(0)$$

$$(i,N-1):\ -T_{i,N-2} - T_{i-1,N-1} + 4T_{i,N-1} - T_{i+1,N-1} = \frac{\Delta x^2 g_0}{k} + f(i\Delta x)$$

$$(N,N-1):\ -T_{N,N-2} - 2T_{N-1,N-1} + (4 + 2\Delta x H)T_{N,N-1} = \frac{\Delta x^2 g_0}{k} + 2\Delta x H T_\infty + f(a)$$

$$H = \frac{h}{k}$$

Note: These finite difference equations (FDEs) have been generated using second order central differences for fictitious nodes. A different system would be generated if first order forward or backward differences were used.

Discrete Poisson equation: $\mathbf{A}$

$$
\left[\begin{array}{cccccccccccccccccccc}
4.00 & -2.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33
\end{array}\right]
$$

Discrete Poisson equation: $\mathbf{A}$

$$
\left[\begin{array}{cccccccccccccccccccc}
4.00 & -2.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33
\end{array}\right]
$$

Discrete Poisson equation: $\mathbf{A}$

$$
\begin{bmatrix}
4.00 & -2.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -2.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
-1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. & 0. & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33 & 0. & 0. & 0. & 0. & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & 0. & 4.00 & -2.00 & 0. & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 & 0. \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -1.00 & 4.00 & -1.00 \\
0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & -1.00 & 0. & 0. & 0. & -2.00 & 7.33
\end{bmatrix}
$$

Discrete Poisson equation: $\mathbf{b}$

$$\begin{bmatrix} 0.4167 \\ 0.4167 \\ 0.4167 \\ 0.4167 \\ 83.7500 \\ 0.4167 \\ 0.4167 \\ 0.4167 \\ 0.4167 \\ 83.7500 \\ 0.4167 \\ 0.4167 \\ 0.4167 \\ 0.4167 \\ 83.7500 \\ 200.4167 \\ 191.0417 \\ 162.9167 \\ 116.0417 \\ 133.7500 \end{bmatrix}$$

Analytical Solution

$$\begin{bmatrix} 200.000 & 190.625 & 162.500 & 115.625 & 50.0000 \\ 158.149 & 150.619 & 128.493 & 93.6310 & 51.1706 \\ 131.081 & 124.983 & 107.293 & 79.9501 & 46.3005 \\ 115.982 & 110.743 & 95.6002 & 72.2595 & 43.3732 \\ 111.138 & 106.179 & 91.8575 & 69.7816 & 42.4199 \end{bmatrix}$$

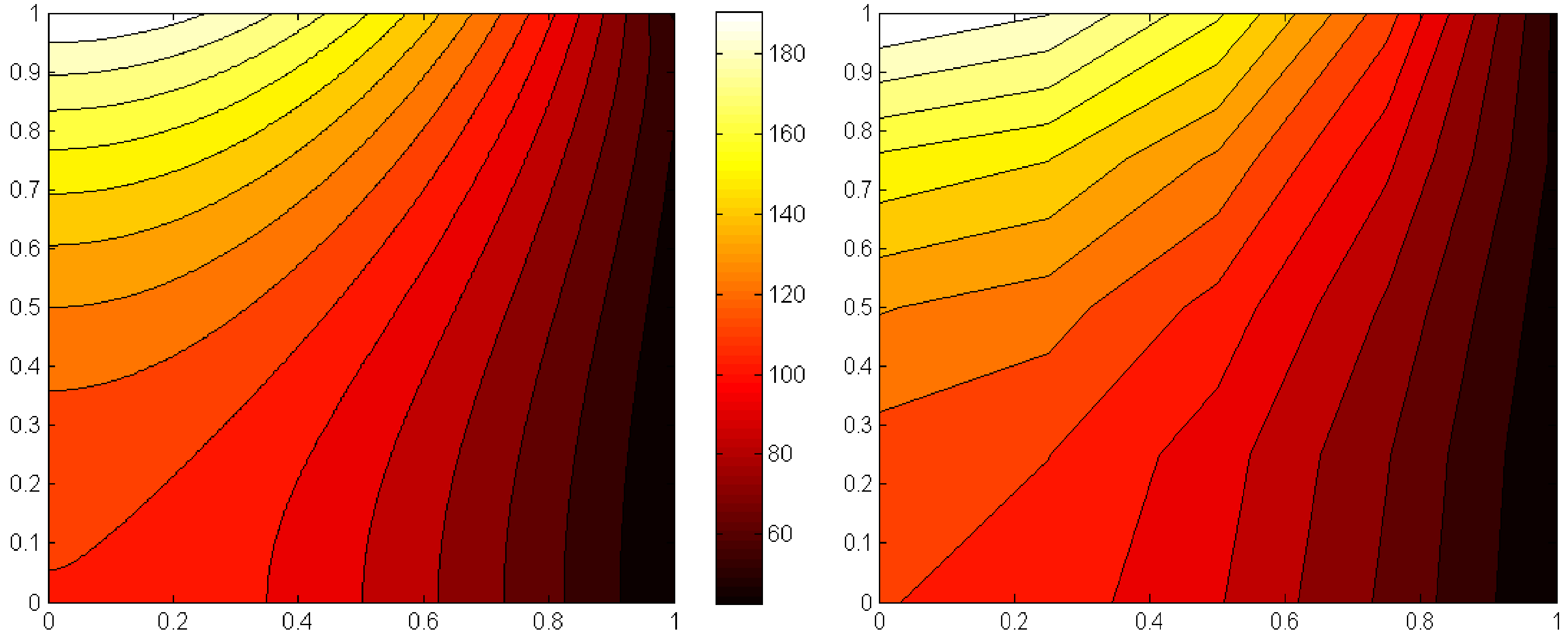
Numerical Solution

$$\begin{bmatrix} 200.000 & 190.625 & 162.500 & 115.625 & 50.0000 \\ 158.542 & 151.014 & 128.873 & 93.7785 & 50.0865 \\ 131.725 & 125.598 & 107.784 & 80.1128 & 45.9941 \\ 116.745 & 111.452 & 96.1339 & 72.4782 & 43.2283 \\ 111.934 & 106.915 & 92.4052 & 70.0212 & 42.3067 \end{bmatrix}$$

The physical parameters used for the computations are:

$$a = b = 1m,\ k = 3\ W\,m^{-1}\,{}^{\circ}C^{-1},\ h = 20\,W\,m^{-2}\,{}^{\circ}C^{-1},\ g_0 = 20\ W\,m^{-3},\ T_{\infty} = 25\,{}^{\circ}C, f(x) = 200 - 150x^2$$

Analytical Solution | Numerical Solution



Higher accuracy could be obtained in the numerical solution by using a finer mesh (more nodes) .