CHALLENGE PROBLEM REPORT 2 SOLUTIONS

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1. Polar Coordinates

Let us begin by determining whether the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(r, \theta) \mapsto (r\cos(\theta), r\sin(\theta))$ is linear. If f is linear, it must satisfy

$$f(r_1, \theta_1) + f(r_2, \theta_2) = f(r_1 + r_2, \theta_1 + \theta_2)$$
$$\lambda f(r, \theta) = f(\lambda r, \lambda\theta)$$

Neither condition is satisfied. If f were a linear map, we would find that $f(2, \pi/2) + f(3, \pi) = f(5, 3\pi/2)$. However, $f(2, \pi/2) + f(3, \pi) = \langle -3, 2\rangle$ and $f(5, 3\pi/2) = \langle 0, 5\rangle$, so vector addition is not preserved. Further, we must have that $2f(1, \pi) = f(2, 2\pi)$. But $2f(1, \pi) = \langle -2, 0\rangle$ and $f(2, 2\pi) = \langle 2, 0\rangle$, so scalar multiplication is not preserved either. Thus, f is not a linear map.

The reason f is not linear is because $\sin\theta$ and $\cos\theta$ are not linear. For example, $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$; if vector addition were preserved, it would have to be $\sin(a + b) = \sin(a) + \sin(b)$. It follows from this that scalar multiplication is not preserved either:

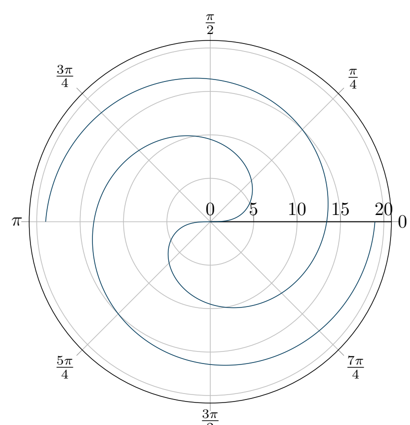
$$\sin(2\theta) = \sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta = 2\sin\theta\cos\theta \neq 2\sin\theta.$$

Because $\sin\theta$ and $\cos\theta$ are not linear, f is not linear either. **(1)**

We continue by graphing the polar equation $\theta = r^2$. We can start by creating an r-θ table:

r	-3	-2	-1	0	1	2	3
θ	9	4	1	0	1	4	9

We can see that θ is growing at a faster rate than r. Geometrically, this means that as the graph gets further from the origin, $\Delta\theta$ increases as Δr is held constant. Combining this information with the data in the r-θ table, we get the graph:



(2)

2. Limits in Polar coordinates

The following theorem states how we can use polar coordinates to find Cartesian limits:

Theorem 2.1 (Limits using polar coordinates)**.** *Let $f(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be a function of two variables, which we can express in polar coordinates as $g(r, \theta) := f(r\cos(\theta), r\sin(\theta))$. Then*

$$\lim_{(x,y)\to(0,0)} f(x, y) = L$$

if and only if there exists $\delta > 0$ and a function $h : \mathbb{R} \to \mathbb{R}$ such that

- *If $0 < r < \delta$, then $|g(r, \theta) - L| \leq h(r)$ for all θ, AND*
- $\lim_{r\to 0} h(r) = 0$

Corollary 2.2. If $\lim_{r\to 0} g(r,\theta)$ depends on θ, then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

This theorem applies to limits approaching $(0,0)$. To generalize this theorem[1] to limits approaching any point $(a,b)\in\mathbb{R}^2$, we can translate the function and treat (a,b) as the origin.

We begin by defining a new function $f'(x,y) := f(x+a, y+b)$. By definition,

$$\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{(x,y)\to(0,0)} f(x+a,y+b) = \lim_{(x,y)\to(0,0)} f'(x,y).$$

The polar form of $f'(x,y)$ is $g'(x,y) = f'(r\cos\theta, r\sin\theta)$. By Theorem 2.1, we have that

$$\lim_{(x,y)\to(0,0)} f'(x,y) = L$$

if and only if there exists $\delta > 0$ and a function $h : \mathbb{R}\to\mathbb{R}$ such that

- If $0 < r < \delta$, then $|g'(r,\theta) - L| \le h(r)$ for all θ, AND
- $\lim_{r\to 0} h(r) = 0$

We can show that this version of the theorem works using the ϵ-δ definition of limits. To prove the limit exists, we want to show that for every $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1 \quad\Rightarrow\quad |f(x,y) - L| < \epsilon_1.$$

Substituting in $x = a + r\cos\theta$ and $y = b + r\sin\theta$, we get that

$$0 < \sqrt{(a + r\cos\theta - a)^2 + (b + r\sin\theta - b)^2} < \delta_1$$

$$0 < \sqrt{r^2(\cos^2\theta + \sin^2\theta)} < \delta_1$$

$$0 < |r| < \delta_1$$

Now substituting in $f(x,y) = g(r,\theta)$, we get that we are trying to prove that

$$0 < |r| < \delta_1 \quad\Rightarrow\quad |g(r,\theta) - L| < \epsilon_1.$$

We'll put this aside for now. From the given conditions, we know that $\lim_{r\to 0} h(r) = 0$. This means that for every $\epsilon_2 > 0$, there exists a $\delta_2 > 0$ such that

$$0 < |r - 0| < \delta_2 \quad\Rightarrow\quad |h(r) - 0| < \epsilon_2.$$

We also know that $|g'(r,\theta) - L| \le |h(r)|$, so

$$0 < |r| < \delta_2 \quad\Rightarrow\quad |g(r,\theta) - L| \le |h(r)| < \epsilon_2.$$

Choose $\epsilon_2 = \epsilon_1$. This gives us that there exists a $\delta_2 > 0$ such that

$$0 < |r| < \delta_2 \quad\Rightarrow\quad |g(r,\theta) - L| < \epsilon_1.$$

Looping back to the beginning, choose $\delta_1 = \delta_2$. Then,

$$\begin{aligned}
0 < |r| < \delta_1 &\Rightarrow 0 < |r| < \delta_2 \\
&\Rightarrow |h(r)| < \epsilon_2 && \text{(By definition of limits)} \\
&\Rightarrow |h(r)| < \epsilon_1 && \text{(Because we chose } \epsilon_2 = \epsilon_1\text{)} \\
&\Rightarrow |g(r,\theta)| < \epsilon_1 && \text{(Because } g(r,\theta) \le h(r)\text{)}
\end{aligned}$$

Thus, we have show that for every $\epsilon_1 > 0$, there exists a $\delta_1 > 0$ such that

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta_1 \quad\Rightarrow\quad |f(x,y) - L| < \epsilon_1. \quad \textbf{(3)}$$

[1] Thanks to Julia for this new definition of Theorem 2.1!

We proceed to use polar coordinates to show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist. Substituting in $x = r\cos\theta$ and $r^2 = x^2 + y^2$, we get

$$\lim_{r\to 0} \frac{r^2\cos^2\theta}{r^2} = \lim_{r\to 0} \cos^2\theta$$

However, our limit depends on θ. Thus, by Corollary 2.1, $\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$ does not exist. **(4)**

An limit that pushes the boundaries of Theorem 2.1 is

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$$

Using the polar method, we get that this limit evaluates to 0. To see this, we substitute in $x = r\cos\theta$ and $y = r\sin\theta$:

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2} = \lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$$

.

Observe that both $r^4\cos^4\theta$ and $r^2\sin^2\theta$ are always positive, so $r^4\cos^2\theta + r^2\sin^2\theta > r^2\sin^2\theta$ for $r \neq 0$. Thus,

$$\lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^4\cos^2\theta + r^2\sin^2\theta} < \left|\lim_{r\to 0} \frac{r^3\cos^2\theta\sin\theta}{r^2\sin^2\theta}\right| = \left|\lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{\sin^2\theta}\right|$$

Because of the squeeze theorem, we can evaluate the limit of this new expression to find the limit we're interested in. By the limit quotient law,

$$\left|\lim_{r\to 0} \frac{r\cos^2\theta\sin\theta}{\sin^2\theta}\right| = \frac{\lim_{r\to 0} r\cos^2\theta\sin\theta}{\lim_{r\to 0}\sin^2\theta} = \frac{0}{\lim_{r\to 0}\sin^2\theta}.$$

This limit evaluates to 0 for any fixed θ such that $\theta \neq 0 + \pi n$. When $\theta = 0 + \pi n$ we still find that the limit evaluates to 0:

$$\lim_{r\to 0} \frac{r^3\cos^2(0+\pi n)\sin(0+\pi n)}{r^4\cos^4(0+\pi n) + r^2\sin^2(0+\pi n)} = \lim_{r\to 0} \frac{r^3(1)(0)}{r^4(1) + r^2(0)} = \lim_{r\to 0} \frac{0}{r^4(1)} = 0.$$

Thus, $\lim_{r\to 0} g(r,\theta) = 0$ for all fixed θ. **(5)**

In Cartesian, we find that the limit does not exist. Consider the two paths $r_1(t) = \langle t, 0\rangle$ and $r_2(t) = \langle t, t^2\rangle$. If the limit exists, we should find that $\lim_{t\to 0} f(r_1(t)) = 0 = \lim_{t\to 0} f(r_2(t))$. However, evaluating the two limits, we see that

$$\lim_{t\to 0} f(r_1(t)) = \lim_{t\to 0} \frac{t^2(0)}{t^4+0} = \lim_{t\to 0} \frac{0}{t^4} = 0.$$
$$\lim_{t\to 0} f(r_2(t)) = \lim_{t\to 0} \frac{t^2(t^2)}{t^4+t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = \frac{1}{2}.$$

Because there are two paths on $f(x,y)$ through $(0,0)$ that do not agree on the value of the limit, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. **(6)**

At first glance, it might seem like these results contradict Theorem 2.1. However, in question 5, we only evaluated the limit for fixed θ, which correspond to straight paths through $(0,0)$ in the polar plane. Finding the limit is independent of fixed θ does not necessarily mean it is independent of all θ. For example, consider the path $r(t) = \langle t, \arcsin(t)\rangle$:[2]

$$\lim_{t\to 0} g(r(t)) = \lim_{t\to 0} \frac{t\cos^2(\arcsin(t))\sin(\arcsin(t))}{t^2\cos^4(\arcsin(t)) + \sin^2(\arcsin(t))} = \lim_{t^2\to 0} \frac{t^2\cos^2(\arcsin(t))}{t^2\cos^4(\arcsin(t)) + t^2}$$

[2]Thanks for Charlotte for this example!

We know that both $\arcsin(x)$ and $\cos(x)$ are continuous at $x = 0$. Since the compositions of continuous functions are continuous, we can simply evaluate these for $t = 0$. Substituting in $\cos(\arcsin(0)) = 1$, we get

$$\lim_{t^2 \to 0} \frac{t^2}{t^2 + t^2} = \frac{1}{2}.$$

Thus, the value of this limit does depends on θ. By Corollary 2.2, this means that $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

Coming back to Theorem 2.1, we find that $g(r, \theta)$ does not actually meet its conditions. For example, consider $\theta = \frac{\pi}{2}$. Using direct substitution into $g(r, \theta)$, we get that

$$g(r, \frac{\pi}{2}) = \frac{r^4(\cos(\frac{\pi}{2}))^2(\sin(\frac{\pi}{2}))}{r^2(\cos(\frac{\pi}{2}))^4 + (\sin(\frac{\pi}{2}))^2} = \frac{0}{1} = 0$$

Per the example above, we can claim that $\lim_{r\to 0} g(r, \theta) = 1/2$. So, to satisfy Theorem 2.1, we must have that there exist a δ and $h(r)$ such that

- If $0 < r < \delta$, then $|g(r, \theta) - L| = |0 - 1/2| \leq |h(r)|$, AND
- $\lim_{r\to 0} h(r) = 0$

However, we cannot have both that $|h(r)| \geq 1/2$ and $\lim_{r\to 0} h(r) = 0$. To see this, note that if $\lim_{r\to 0} h(r) = 0$, for all $\epsilon > 0$, there must exist a $\delta > 0$ such that

$$0 < |r| < \delta \Rightarrow |h(r)| < \epsilon.$$

Choose $\epsilon = 1/4$. If the limit is 0, there must exist a delta such that

$$0 < |r| < \delta \Rightarrow |h(r)| < \frac{1}{4}.$$

This is impossible because we know that $|h(r)| \geq 1/2$ close to the origin. Thus, $\lim_{r\to 0} h(r) \neq 0$, so we do not satisfy both conditions. By Theorem 2.1, the limit does not exist. **(7)**

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