CHALLENGE PROBLEM SET 1 SOLUTIONS

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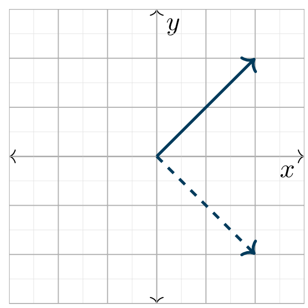
In this Challenge Problem set, we explore the geometry of two-dimensional linear maps. In the first section, we study how to understand linear maps as transformations of e_1 and e_2. In the second section, we explore how to calculate the eigenvalues and eigenvectors of linear maps $\mathbb{R}^2 \mapsto \mathbb{R}^2$.

1. Linear Maps and Basis Vectors

To better understand how a matrix represents a linear transformation, we begin by finding a 2×2 matrix R such that the associated linear map $T_R : \mathbb{R}^2 \to \mathbb{R}^2$ reflects vectors across the x-axis. Generally, we reflect vectors across the x-axis by making their y-component negative. For example, the reflection of $\langle 2, 2\rangle$ is $\langle 2, -2\rangle$:



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To find T_R, we consider the basis vectors e_1 and e_2 are transformed. Since e_1 does not have a y-component, it does not change. However, e_2 becomes $\langle 0, -1\rangle$. Thus, our matrix is

$$\begin{bmatrix} | & | \\ T(e_1) & T(e_2) \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We can test it by seeing how it transforms an arbitrary vector $\langle x, y\rangle$:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

As desired, the x-component of the transformed matrix is unchanged and the y-component becomes negative.

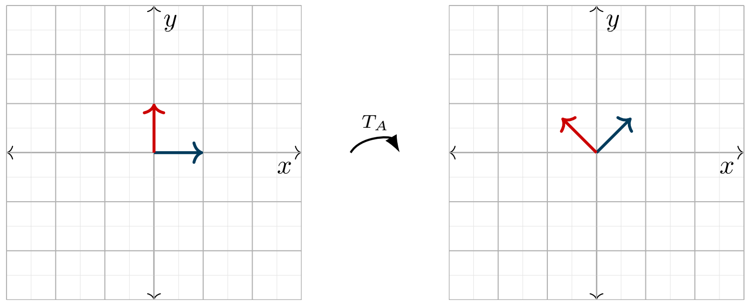
Now, let us study the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

by fixing an angle θ and then describing and sketching the linear map T_θ associated with R_θ. We begin with the concrete example $\theta = \frac{\pi}{4}$:

$$R_{\frac{\pi}{4}} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

This matrix is associated with the following linear transformation of e_1 and e_2:



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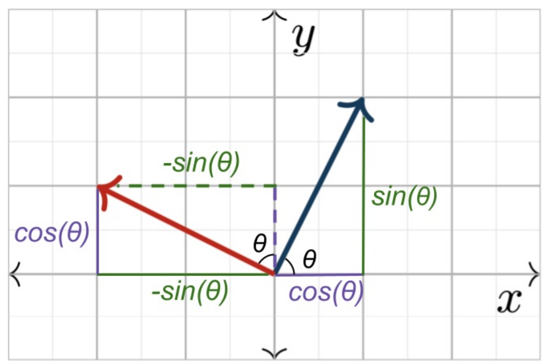
Observe that e_1 and e_2 have both rotated counterclockwise by θ but have maintained a magnitude of 1. The basis vector e_1 is sent to $\langle \cos\theta, \sin\theta \rangle$, a vector of magnitude 1 that intersects the x-axis at an angle of θ. Because we must preserve $\theta = \arctan(\sin\theta/\cos\theta)$, the θ of e_2 is instead relative to the positive y-axis. The magnitude of e_1 and e_2 stay constant at 1 because $||e|| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$. Visually, the transformation looks like this (assume the magnitude of both vectors is 1):



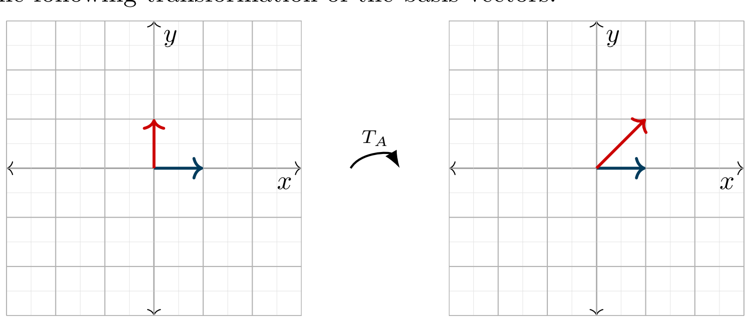
Thus, R_θ transforms vectors by rotating them counterclockwise by θ but maintaining their original magnitude.

2. Eigenvectors and Eigenvalues

Now, we study how to calculate and interpret the eigenvectors and eigenvalues of a linear matrix. Let us begin by sketching, describing, and finding the eigenvectors of the linear map T_S associated to the matrix

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

S corresponds to the following transformation of the basis vectors:



In this case, it appears that S does not change e_1 but slides the tip of e_2 right by a unit. To understand S more generally, we need to understand how exactly it transforms input vectors. We could express the transformation of an arbitrary input vector $\langle x, y \rangle$ like so:

$$\begin{aligned} S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} y \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + e_1 y \end{aligned}$$

.

The identity matrix does not transform the input vector, so the meaningful change here is the $+e_1y$, which adds e_1 scaled by y to the original input vector. Thus, S horizontally translates the tip of the input vector by y.

Now that we understand S, we proceed to find its eigenvectors. Notice that in the equation above, the output vector will be parallel to the input vector if and only if $e_1y = 0$. If $e_1y \neq 0$, the output vector cannot be a scalar multiple of the input vector, so it will not be an eigenvector. Because the transformation causes the tip of the input vector to translate right horizontally by the magnitude of its y-component, any input vector with a non-zero y-component cannot be an eigenvector. Thus, it seems that only input vectors with $y = 0$ will be eigenvectors.

Let's confirm this intuition algebraically. We begin by finding the eigenvalues using the S's characteristic polynomial equation

$$p(\lambda) = (1 - \lambda)^2.$$

We see that $p(\lambda) = 0$ when $\lambda = 1$, so the only eigenvalue of S is 1. Now, to find the eigenvectors, we must solve the system of equations

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1x \\ 1y \end{bmatrix}.$$

After matrix multiplication, we get the equations (1) $x + y = x$ and (2) $y = y$. Subtracting x from both sides in equation (1), we have that $y = 0$. Substitution $y = 0$ back in, we get $x = x$. Equation (2) does not place any additional constraints on the system, the solutions are $y = 0$ and $x = \{a | a \in \mathbb{R}\}$. Thus, the eigenvectors of S are $\{\langle x, 0\rangle | x \in \mathbb{R}\}$.

Now, consider the matrix

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Suppose we are interested in the values of θ for which R_θ has real eigenvalues and eigenvectors. In this case, we can use characteristic polynomial equation to determine which values of θ lead R_θ to have real eigenvalues. Substituting the appropriate values into $p(\lambda)$, we get that

$$p(\lambda) = \lambda^2 - 2\cos\theta\lambda + \cos^2\theta + \sin^2\theta.$$

Using the Pythagorean identity to simplify $\cos^2\theta + \sin^2\theta$,

$$p(\lambda) = \lambda^2 - 2\cos\theta\lambda + 1.$$

A polynomial has real solutions when its discriminant is greater than or equal to 0. Thus, we are interested in the θ that satisfy the inequality

$$\begin{gathered} 4\cos^2\theta - 4 \geq 0 \\ \cos^2\theta \geq 1 \end{gathered}$$

The inequality above is satisfied only when $\cos\theta = 1$ or $\cos\theta = -1$, so R_θ has real eigenvalues only when $\theta = 0 + \pi n$. In retrospect, this makes sense; the only time transformed vectors will be parallel to their original input vectors is when they are either not rotated at all ($\theta = 0 + 2\pi n$) or they are flipped ($\theta = \pi + 2\pi n$). To find the eigenvectors, we can substitute $\cos\theta = 1$ and $\cos\theta = -1$ into $p(\lambda)$ and find the roots:

$$cos(\theta) = 1 \rightarrow p(\lambda) = (\lambda + 1)^2 \rightarrow \lambda = -1$$
$$cos(\theta) = -1 \rightarrow p(\lambda) = (\lambda - 1)^2 \rightarrow \lambda = 1$$

Thus, the eigenvalues are $\lambda = 1$ and $\lambda = -1$. Solving the system of equations $R_\theta \vec{v} = \lambda \vec{v}$, we get

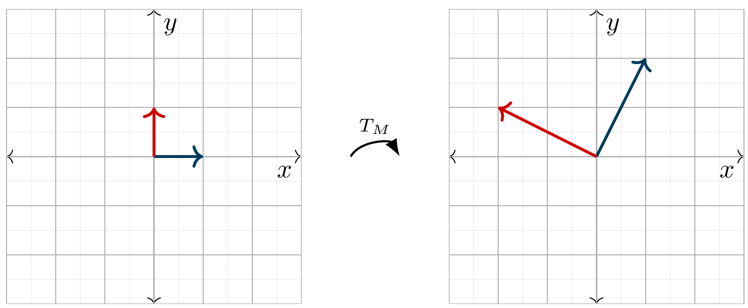
$$\lambda = 1 : R_0 \vec{v} = \vec{v} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow x = x, y = y$$
$$\lambda = -1 : R_\pi \vec{v} = -\vec{v} \Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix} \Rightarrow x = x, y = y.$$

Because all real numbers satisfy the equations above, the eigenvectors for $\theta = \pi n$ are $\{\langle x, y \rangle | x, y \in \mathbb{R}\}$. This makes sense considering the transformation matrices for $\theta = \pi n$ are either the identity matrix or the negative of the identity matrix, which both output a scalar multiple of the input vector.

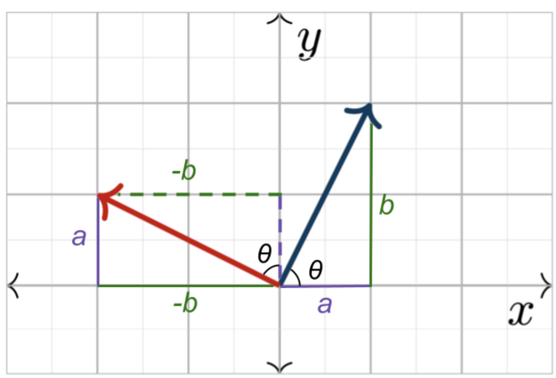
Now, suppose we are interested in finding the real eigenvectors of the matrix

$$M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $a, b \in \mathbb{R}$ such that a and b are not both zero. We can try to gain an intuitive understanding of this problem by sketching the transformation of e_1 and e_2 for if $a = 1$ and $b = 2$:



With this matrix, e_1 is sent to $\langle a, b \rangle$ and e_2 is sent to $\langle b, -a \rangle$. Let θ be the angle between the original input vector and the output vector. Geometrically, the transformation looks like this:



Thus, we can express θ as $\arctan(\frac{b}{a})$. (Note that in the second quadrant θ is still $\arctan(\frac{b}{a})$ and not $\arctan(-\frac{b}{a})$ because we are interested in the acute angle between the transformed vector and the y-axis, not the obtuse angle between the transformed vector and the x-axis.) So, it appears that T_M rotates the basis vectors counterclockwise by $\theta = \arctan(\frac{b}{a})$.

From the diagram, it appears that the transformation also scales input vectors by $\sqrt{a^2 + b^2}$. We can confirm this algebraically[1] by calculating the magnitude of the transformation of an arbitrary vector $\langle x, y \rangle$. After matrix multiplication, we see that $\langle x, y \rangle$ is sent to $\langle ax - by, bx + ay \rangle$:

[1]Thank you to Cassie for this algebraic interpretation of the scaling of input vectors!

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}.$$

Calculating the magnitude of this generalized output vector, we get

$$\begin{aligned} \sqrt{(ax - by)^2 + (bx + ay)^2} &= \sqrt{a^2x^2 + b^2y^2 + b^2x^2 + a^2y^2} \\ &= \sqrt{(a^2 + b^2)(x^2 + y^2)} \\ &= \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}. \end{aligned}$$

We see the magnitude of the output vector is the magnitude of the input vector times $\sqrt{a^2 + b^2}$. Thus, the transformation M results in input vectors being rotated by $\arctan(\frac{b}{a})$ and scaled by $\sqrt{a^2 + b^2}$.

Now we proceed to find the eigenvalues and eigenvectors. By definition, a transformed vector is an eigenvector only if it is parallel to its original input vector. Thus, we would expect there to be real eigenvalues and eigenvectors when $\theta = 0$ or $\theta = \pi$. Substituting both into $\theta = \arctan(\frac{b}{a})$, we get

$$0 = \arctan\left(\frac{b}{a}\right) \Rightarrow tan(0) = \frac{b}{a} \Rightarrow 0 = \frac{b}{a}$$

$$\pi = \arctan\left(\frac{b}{a}\right) \Rightarrow tan(\pi) = \frac{b}{a} \Rightarrow 0 = \frac{b}{a}$$

In both cases, the equation is satisfied when $b = 0$ and a is a non-zero real number. Thus, we might guess that M only has real eigenvalue and eigenvectors when $b = 0$ and a is a non-zero real number.

We can confirm this algebraically by using the characteristic polynomial equation. Using the formula, we have that $p(\lambda) = \lambda^2 - 2a\lambda + (a^2 + b^2)$. We again use the fact that a polynomial has real solutions only when its discriminant is greater than or equal to 0:

$$\begin{aligned} 4a^2 - 4(a^2 + b^2) \geq 0 \\ b^2 \leq 0. \end{aligned}$$

This inequality is only satisfied when b=0, confirming our intuition above. Substituting b=0 into $p(\lambda)$, we get

$$p(\lambda) = (\lambda - a)^2.$$

The only root of the equation is a, which means that our only eigenvalue is $\lambda = a$. To find the eigenvectors, we must solve for a in

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix},$$

which translates to the equations

$$ax + 0y = ax$$

$$0x + ay = ay.$$

The first equation simplifies to $x = x$ and the second to $y = y$. Thus, all real vectors in $\mathbb{R}^2$ are eigenvectors when $b = 0$. Similar to the first matrix we examined, this makes sense because the transformation matrix

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$$

is just a scalar multiple of the identity matrix. So, while it will scale input vectors by a, it will not rotate them at all. Thus, when $b = 0$ and $a \neq 0$, eigenvectors take the form $\{\langle x, y\rangle | x, y\ in \mathbb{R}\}$ When $b \neq 0$, there are no real eigenvectors.

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