32BH CHALLENGE PROBLEM REPORT 2

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Abstract. In this Challenge Problem Report, we learn how to recognize and identify manifolds using the implicit function theorem.

1. Identifying manifolds

Manifolds generalize the notion of curves and surfaces and allows us to model surfaces that do not act like graphs globally. Here is one definition of a manifold:

Definition 1.1. A subset $M \subset \mathbb{R}^n$ is a **differentiable k-dimensional manifold embedded in** $\mathbb{R}^n$ if for all $\boldsymbol{x} \in M$, there exists an open neighborhood U such that $M \cap U$ is the graph of a C^1 mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$.

However, it can be difficult to prove that a particular set X is a manifold using this definition, as we must show that every single point in X locally acts like a graph. For example, it takes 4 open neighborhoods just to prove that the unit circle $S^1 := \{(x, y) \mid x^2 + y^2 = 1\}$ is a 1-manifold embedded in $\mathbb{R}^2$. However, a more powerful theorem allows us to prove that the unit circle is a manifold with just a single open neighborhood:

Theorem 1.2. *Let M be a subset of $\mathbb{R}^n$. If for every $\boldsymbol{z} \in M$, there exists an open set $U \subset \mathbb{R}^n$ containing $\boldsymbol{z}$, and a C^1-mapping $F : U \subset \mathbb{R}^n \to \mathbb{R}^{n-k}$ such that*

(1) $M \cap U = \{\boldsymbol{z} \in U \mid F(\boldsymbol{z}) = \boldsymbol{0}\}$, and
(2) $[J_F(\boldsymbol{z})]$ is surjective for every $\boldsymbol{z} \in M$,

then M is a differentiable k-dimensional manifold.

To apply this theorem, we would have to prove that $[J_F(\boldsymbol{z})]$ is surjective for all z in a set, which can be a tedious process. A very useful fact when applying Theorem 1.2 is the equivalence of the following two statements:

(a) The derivative of F, $[J_F(\boldsymbol{z})]$, is surjective.
(b) At least one of the partial derivatives $\frac{\partial}{\partial x_i} F$ is non-zero, or equivalently, that $[J_F(\boldsymbol{z})] \neq 0$.

To prove that statements (a) and (b) are interchangeable, we will first show that $(a) \Rightarrow (b)$, and then that $(b) \Rightarrow (a)$.

First, we wish to show that if $[J_F(\boldsymbol{z})]$ is surjective, $[J_F(\boldsymbol{z})] \neq 0$. Since $[J_F(\boldsymbol{z})]$ surjects onto $\mathbb{R}$, there must exist a $\vec{v} \in \mathbb{R}$ such that $[J_F(\boldsymbol{z})]\vec{v} = 1$. However, if $[J_F(\boldsymbol{z})] = 0$, then $[J_F(\boldsymbol{z})]\vec{v} = 0$ for all $v \in \mathbb{R}^n$, so we cannot have that $[J_F(\boldsymbol{z})]\vec{v} = 1$. Thus, if $[J_F(\boldsymbol{z})]$ is surjective, then $[J_F(\boldsymbol{z})]$ must have at least one non-zero partial derivitive.

Next, we wish to show that if $[J_F(\boldsymbol{z})] \neq 0$, $[J_F(\boldsymbol{z})]$ must be surjective. That is, if $[J_F(\boldsymbol{z})]$ has at least one non-zero partial derivative, there exists $\vec{v} \in \mathbb{R}^n$ such that $[J_F(\boldsymbol{z})]\vec{v} = x$ for all $x \in \mathbb{R}$. Because we defined $F : \mathbb{R}^n \to \mathbb{R}$, $[J_F(\boldsymbol{z})]$ will be a 1 by n matrix, and $\vec{v}$ will be an n-dimensional vector. Let $\frac{\partial F}{\partial x_j}\Big|_z$ be a non-zero entry of $[J_F(\boldsymbol{z})]$. Choose the jth entry of $\vec{v}$ to be

$$v_j = x \left(\frac{\partial F}{\partial x_j}\Big|_z \right)^{-1}$$

and all the other entries to be 0. Then

$$[J_F(\boldsymbol{z})]\vec{v} = \frac{\partial F}{\partial x_j} \left(\frac{\partial F}{\partial x_j}\Big|_z \right)^{-1} x = x,$$

so $[J_F(\boldsymbol{z})]$ surjects onto $\mathbb{R}$.

We've now shown both that $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$, so statements (a) and (b) are equivalent **(1)**. We can use this fact in conjunction with Theorem 1.2 to show that the the unit circle $S_1 = \{(x, y) \mid x^2 + y^2 = 1\}$ is a 1-manifold with just one open neighborhood. Let $F(x, y) = x^2 + y^2 - 1$. The partial derivatives of F

exist and are continuous, so F is a C^1 mapping. To satisfy the first criterion of Theorem 1.2, choose $U = \mathbb{R}^2$ and notice that the sets

$$S_1 \cap \mathbb{R}^2 = \{(x, y) \mid x^2 + y^2 = 1\}$$
$$\{z \in \mathbb{R}^2 \mid F(x, y) = 0\} = \{(x, y) \mid 0 = x^2 + y^2 - 1\}$$

are equivalent. To verify the second criterion, note that $[J_{S^1}(x, y)] = [2x, 2y]$ is 0 only at $(0, 0)$, which is not a point on the unit circle. This means that $[J_{S^1}(x, y)] \neq 0$ on S_1, or equivalently, that $[J_{S^1}]$ is surjective for all $(x, y) \in S_1$. Thus, by Theorem 1.2, the unit circle is a manifold. **(2)**

However, it is important to note that the Jacobian matrix does not necessarily have to be surjective for a manifold to exist. To demonstrate this point, consider the vanishing locus of the function $F(x, y) = (x^2 + y^2 - 1)^2$, which also describes the unit circle. From our work above, we know that the unit circle is a differentiable 1-manifold. However, the Jacobian of F is

$$[J_F(x, y)] = \begin{bmatrix} 4x(x^2 + y^2 - 1) & 4y(x^2 + y^2 - 1) \end{bmatrix},$$

so $[J_F(x, y)] = 0$ whenever $x^2 + y^2 - 1 = 0$. This means that for every point on the manifold, $[J_F(x, y)]$ is not surjective – yet we know that the vanishing locus of F is a differentiable 1-manifold! This example demonstrates a set M might be a manifold even if $[J_M(z)]$ is not surjective. **(3)**

2. Vanishing Loci and Critical Points

We can apply what we learned above to understand when vanishing loci fail to be manifolds. To explore this idea concretely, we'll begin with identifying the values of c for which the equation $x^4 + y^4 + x^2 - y^2 = c$ is a differentiable manifold. This set of points is equivalently described by the vanishing locus of $G_c(x, y) = x^4 + y^4 + x^2 - y^2 - c$. We will again apply Theorem 1.2 to find the c for which G_c is a differentiable manifold.

We should begin by verifying that that $G_c = 0$ is a C^1 mapping $\mathbb{R}^2 \to \mathbb{R}$ that describes $x^4 + y^4 + x^2 - y^2 = c$ when intersected with an open set U. The Jacobian of G_c is

$$\begin{aligned} [J_G(x, y)] &= \begin{bmatrix} 4x^3 + 2x & 4y^3 - 2y \end{bmatrix} \\ &= \begin{bmatrix} 2x(2x^2 + 1) & 2y(2y^2 - 1) \end{bmatrix}. \end{aligned}$$

These parital derivatives exist and are continuous, so G_c is indeed a C^1 mapping. If we choose $U = \mathbb{R}^2$, then we see that

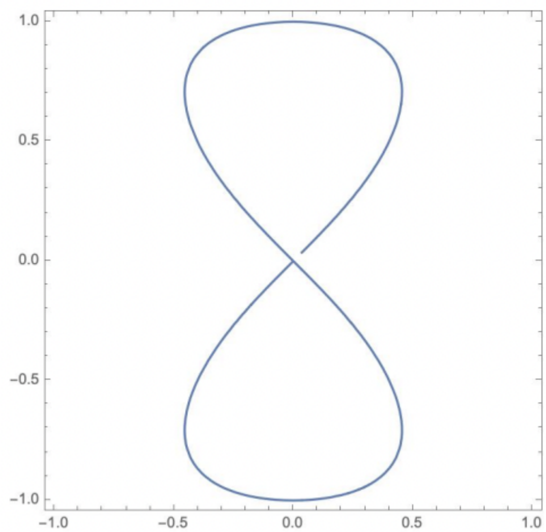
$$\{(x, y) \mid x^4 + y^4 + x^2 - y^2 = c\} \cap \mathbb{R}^2 = \{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 + x^2 - y^2 - c = 0\},$$

so we are describing the correct curve. This means that G_c satisfies the first criterion of Theorem 1.2 for all c.

Now, let's find the values of c for which $[J_G]$ is surjective and thus guaranteed to be a manifold by computing when $[J_G(x, y)] \neq 0$. Since $2x^2 + 1$ is always positive, $\frac{\partial G}{\partial x} = 0$ only when $x = 0$. On the other hand, $\frac{\partial G}{\partial y} = 0$ when $y = \pm\frac{\sqrt{2}}{2}$ and $y = 0$. Solving for c at the points $(0, 0), (0, \frac{\sqrt{2}}{2})$, and $(0, -\frac{\sqrt{2}}{2})$, we get that the contentious values are $c = 0$ or $c = -\frac{1}{4}$. So, for $c \neq 0$ and $c \neq -\frac{1}{4}$, $G(x, y) = 0$ is guaranteed to be a differentiable manifold because $[J_G(x, y)]$ is surjective. However, as we discovered earlier, $[J_G(x, y)]$ not being surjective does not imply that $G_c(x, y) = 0$ is not a manifold – we will have to check those points through other means.

The only points on the vanishing locus of $G_{-1/4}$ are $(0, \frac{\sqrt{2}}{2})$, and $(0, -\frac{\sqrt{2}}{2})$. Since all points are differentiable 0-manifolds and $G_{-1/4} = 0$ is simply two points, $G_{-1/4} = 0$ is a differentiable manifold.

For $c = 0$, we need a bit more analysis. Here is a graph of G_0:



This does not appear to be a differentiable manifold since the curve crosses itself at the origin. To make this intuition rigorous, we wish to show that there does not exist an open neighborhood U such that $\{(x, y) \,|\, G_0 = 0\} \cap U$ is a graph of a function. One way we could accomplish this is by showing that no matter how small we restrict an open neighborhood around the origin, G_0 will not produce a unique output for each input.

Consider the open neighborhood $B_R(0)$, and let $x = \alpha$ such that $0 < \alpha < R$. We can find the outputs corresponding to α by substituting it into $G_0 = 0$ and solving for y:

$$y^4 - y^2 = \alpha^4 + \alpha^2$$

Notice that if $y = \beta$ is a solution, then $y = -\beta$ must also be a solution, as $(\beta)^2 - (\beta)^4 = (-\beta)^2 - (-\beta)^4$. And since $\alpha > 0$, we must also have the $\beta > 0$, which means that β and $-\beta$ must be distinct values. Thus, no matter how much we restrict the open neighborhood around the origin, $G_0 = 0$ does not behave like a graph, so it is not a manifold. This means that the vanishing locus of $G(x, y)$ is a differentiable manifold for all c except $c = 0$. **(4)**

Note that the only places that $G_c = 0$ was not guaranteed to be a manifold were the points at which $J_{G_c}(x, y) = 0$. In other words, the critical points of G_c identify the c for which the vanishing locus of G_c might not be a differentiable manifold. However, we can be even more specific; the class of a critical point can determine whether the vanishing locus it lives in is a differentiable manifold.

To see this, notice that for each c, the vanishing locus of $G_c(x, y) = x^2 + y^2 - c$ represents the same set as the level curve of $g(x, y) = x^2 + y^2$ at $z = c$. A level curve that contains a maximum or minumum will be a set of discrete points, which means that the level sets at extrema are always 0-manifolds. Level curves containing saddle points, on the other hand, are never be manifolds, as a level curve containing a saddle point must cross itself. This is exactly what we saw in the diagram of the vanishing locus of G_0.

We can test this intuition by classifying the critical points of G_c and checking if they match up with our analysis. The Hessian matrix of G_c is the following:

$$H_{G_c} = \begin{bmatrix} 12x^2 + 2 & 0 \\ 0 & 12y^2 - 2 \end{bmatrix}$$

Since the second derivatives exist and are continuous, we can apply the second derivative test to classify the critical points of G_c. Taking the determinant of H_{G_c}, we get

$$D = (12x^2 + 2)(12y^2 - 2).$$

Evaluating D and G_{xx} at our critical points, we get the following classifications:

$$\left(0, \frac{\sqrt{2}}{2}\right) \quad \Rightarrow \quad D > 0, G_{xx} > 0 \qquad \text{local minimum; 0-manifold}$$
$$\left(0, -\frac{\sqrt{2}}{2}\right) \quad \Rightarrow \quad D > 0, G_{xx} > 0 \qquad \text{local minimum; 0-manifold}$$
$$\left(0, 0\right) \quad \Rightarrow \quad D < 0 \qquad \text{saddle point; not a manifold}$$

These results match up with what we found previously. **(5)**

3. Summary

In this report, we explored how to identify k-dimensional manifolds embedded in $\mathbb{R}^n$ as a C^1 mapping of $F : U \to \mathbb{R}^{n-k}$, and how the implicit function theorem generalizes our notion of manifolds. Additionally, using the fact that $[J_F(\mathbf{z})]$ is surjective if and only if $[J_F(\mathbf{z})] \neq 0$, we discovered that the vanishing locus of a function $G : \mathbb{R}^2 \to \mathbb{R}$ is a manifold unless it contains a saddle point of G.

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