32AH CHALLENGE PROBLEM REPORT 3

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1. Calculus of vector-valued functions

Let us begin by showing that the Frenet frame is an orthonormal basis. By the definition of the cross product, we know that $\boldsymbol{B}$ is perpendicular to both $\boldsymbol{T}$ and $\boldsymbol{N}$. Thus, we only have to show that $\boldsymbol{T}$ is perpendicular to $\boldsymbol{N}$, or that $\boldsymbol{T} \cdot \boldsymbol{N} = 0$. Since $||T|| = 1$, we have that

$$T(t) \cdot T(t) = ||T||^2 = 1.$$

Taking the derivative of both sides, we obtain

$$T'(t) \cdot T(t) + T(t) \cdot T'(t) = 2(T(t) \cdot T'(t)) = 0.$$

Substituting in $T'(t) = N(t)||T'(t)||$, we get

$$2(T(t) \cdot N(t)||T'(t)||) = 0.$$

Because $||T'(t)||$ is a scalar, we can factor it out of the dot product. Dividing out the scalars, we are left with

$$T(t) \cdot N(t) = 0.$$

This that proves that the Frenet frame is an orthonormal basis. (1)

An interesting property of Frenet Frames is that they remain identical for all possible parametrizations of a curve. In other words, Frenet Frames solely describe the geometry of the curve – they are independent of speed. To explore this concept, we can calculate and compare the Frenet Frames and speeds of two different parametrizations of a curve.

First, let's inspect the vector-valued function $\boldsymbol{c}(t) = \langle -\sin(t), \cos(t), t\rangle$. Taking the derivative of $c(t)$, we get

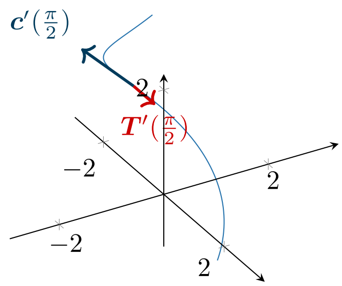
$$c'(t) = \langle -\cos(t), -\sin(t), 1\rangle$$
$$c'(\frac{\pi}{2}) = \langle 0, -1, 1\rangle.$$
$$||c'(t)|| = \sqrt{(-\cos t)^2 + (\sin t)^2 + 1} = \sqrt{2}$$

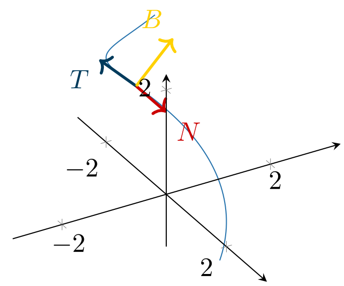
.

So, for this parametrization, the speed at $\langle 0, -1, 1\rangle$ is $\sqrt{2}$. Now, let's find the Frenet Frame:

$$\begin{aligned}
\boldsymbol{T}(t) &= \frac{1}{||\boldsymbol{c}'(t)||}\boldsymbol{c}'(t) = \frac{1}{\sqrt{2}}\langle -\cos(t), -\sin(t), 1\rangle \\
\boldsymbol{T}(\frac{\pi}{2}) &= \frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle \\
\boldsymbol{T}'(t) &= \frac{1}{\sqrt{2}}\langle \sin(t), -\cos(t), 0\rangle \\
\boldsymbol{T}'(\frac{\pi}{2}) &= \frac{1}{\sqrt{2}}\langle 1, 0, 0\rangle \\
\boldsymbol{N}(\frac{\pi}{2}) &= \frac{1}{||\boldsymbol{T}'(\frac{\pi}{2})||}\boldsymbol{T}'(\frac{\pi}{2}) = \langle 1, 0, 0\rangle \\
\boldsymbol{B}(\frac{\pi}{2}) &= \boldsymbol{T}(\frac{\pi}{2}) \times \boldsymbol{N}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle \times \langle 1, 0, 0\rangle \\
&= \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle
\end{aligned}$$

Hence, the Frenet Frame at $\langle 0, -1, 1\rangle$ is $\{\frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle, \langle 1, 0, 0\rangle, \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle\}$. Let's sketch $c'(t)$, $T'(t)$, and the Frenet Frame at $\langle -1, 0, \frac{\pi}{2}\rangle$ – this will give us a basis for comparison later. (2)

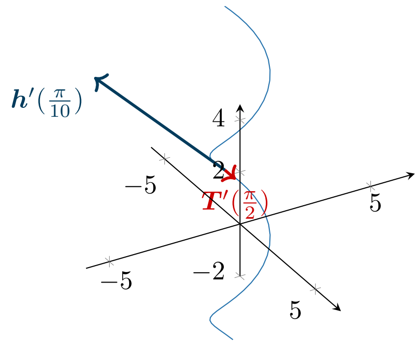


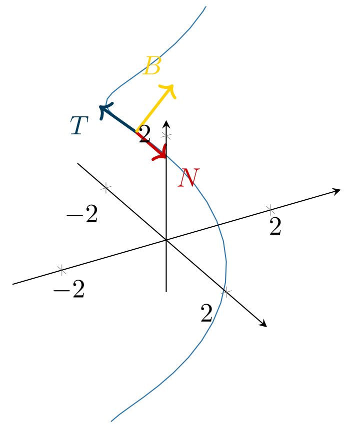


Now, consider the vector-valued function $h(s) = \langle -\sin(5s), \cos(5s), 5s\rangle$, which describes the same curve as $c(t)$. Let's again compute $h'(s)$, $T'(s)$, and the Frenet Frame, but this time for $s = \frac{\pi}{10}$ so that we are inspecting the same position. Going through the same process, we find that

$$
\begin{aligned}
h'(s) &= \langle -5\cos(5s), -5\sin(5s), 5\rangle. \\
h'(\frac{\pi}{10}) &= \langle 0, -5, 5\rangle \\
||h'(s)|| &= \sqrt{(-5\cos s)^2 + (5\sin s)^2 + 5^2} = 5\sqrt{2} \\
\boldsymbol{T}(s) &= \frac{1}{||\boldsymbol{h'}(s)||}\boldsymbol{h}'(s) \\
&= \frac{1}{5\sqrt{2}}\langle -5\cos(5s), -5\sin(5s), 5\rangle \\
&= \frac{1}{\sqrt{2}}\langle -\cos(5s), -\sin(5s), 1\rangle \\
\boldsymbol{T}(\frac{\pi}{2}) &= \frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle \\
\boldsymbol{T}'(s) &= \frac{5}{\sqrt{2}}\langle \sin(5s), -\cos(5s), 0\rangle \\
\boldsymbol{T}'(\frac{\pi}{2}) &= \frac{5}{\sqrt{2}}\langle 1, 0, 0\rangle \\
||\boldsymbol{T}'(\frac{\pi}{2})|| &= \frac{5}{\sqrt{2}} \\
\boldsymbol{N}(\frac{\pi}{2}) &= \frac{1}{||\boldsymbol{T}'(\frac{\pi}{2})||}\boldsymbol{T}'(\frac{\pi}{2}) = \langle 1, 0, 0\rangle \\
\boldsymbol{B}(\frac{\pi}{2}) &= \boldsymbol{T}(\frac{\pi}{2}) \times \boldsymbol{N}(\frac{\pi}{2}) = \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle
\end{aligned}
$$

Thus, the Frenet Frame for $h(s)$ at $\langle -1, 0, \frac{\pi}{2}\rangle$ is $\{\frac{1}{\sqrt{2}}\langle 0, -1, 1\rangle, \langle 1, 0, 0\rangle, \frac{1}{\sqrt{2}}\langle 0, 1, 1\rangle\}$. These are the sketches of $h'(s)$, $T'(s)$, and the Frenet Frame at $s = \frac{\pi}{10}$: (3)





As one might expect, $h'(s)$ and $T'_h(s)$ are 5 times greater than $c'(t)$ and $T_c(t)$ respectively. However, $c(\frac{\pi}{2})$ and $h(\frac{\pi}{10})$ have identical Frenet Frames. In our calculations for T and N, we divided by the $||r'(t)||$ and $||T'(t)||$, respectively. This makes T and N independent of the speed at which their functions trace the curve. This reinforces the idea that Frenet Frames describe the geometry of a curve without regard for how it is parameterized. (4)

2. Curvature

Let's begin by exploring the relationship between the curvature and radius of a circle. If we think of "low curvature" as "relatively flat," we might guess that the curvature of a circle is inversely related to radius; the larger the radius of a circle, the "flatter" its curve.

We can confirm this intuition using the formula for curvature. One parametrization of a circle in $\mathbb{R}^3$ is $r(t) = \langle -R\cos(t), R\sin(t), 0\rangle$. To calculate the curvature of this function, we much first compute $||r'(t)||$ and $||T'(t)||$. Taking the derivative of $r(t)$, we get that

$$r'(t) = \langle R\sin(t), R\cos(t), 0\rangle,$$

which means $||r'(t)|| = R$. Now that we have $r'(t)$ and $||r'(t)||$, we know that

$$T(t) = \langle \sin(t), \cos(t), 0\rangle.$$

Taking the derivative of $T(t)$, we find that

$$T'(t) = \langle \cos(t), -\sin(t), 0\rangle.$$

Thus, $||T'(t)|| = 1$. Substituting $||T'(t)||$ and $||r'(t)||$ into the formula for κ, we discover

$$\kappa = \frac{1}{R}.$$

As expected, the radius and curvature of a circle are inversely proportional. More generally, this equation reveals that the radius of the osculating circle on any point of a curve $r(t)$ defines the curvature of that point. The smaller the radius of the oscullating circle, the greater the curvature. (5)

Let's try to apply what we have learned by calculating the curvature of $\boldsymbol{r}(t) = \langle \ln(t), 1, t\rangle$. To start, we compute $||r'(t)||$:

$$\begin{aligned} r'(t) &= \langle \frac{1}{t}, 0, 1\rangle \\ ||r'(t)|| &= \sqrt{\frac{1}{t^2} + 1} \\ &= \sqrt{\frac{1+t^2}{t^2}} \end{aligned}$$

Now we can compute $||T'(t)||$:

$$\begin{aligned} T(t) &= \frac{1}{||r'(t)||} r(t) \\ &= \sqrt{\frac{t^2}{1+t^2}} \left\langle \frac{1}{t}, 0, 1 \right\rangle \\ &= \left\langle \frac{1}{\sqrt{1+t^2}}, 0, \frac{|t|}{\sqrt{1+t^2}} \right\rangle \end{aligned}$$

It seems like this will be a tedious process. Conveniently, there is another formula for curvature for situations like these:

$$\kappa(t) = \frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3}$$

Before we apply it to this problem, we'd like to prove that this is a valid alternative formula for curvature. In other words, we wish to show that

$$\frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3} = \frac{1}{||\boldsymbol{r}'(t)||} ||\boldsymbol{T}'(t)||$$

We can start with the numerator. Because $T(t) = \frac{r'(t)}{||r'(t)||}$, we know that $r'(t) = T(t)||r'(t)||$. Taking the derivative of both sides, we obtain

$$r''(t) = T'(t)||r'(t)|| + T(t)||r'(t)||'.$$

Thus,

$$\begin{aligned} ||r'(t) \times r''(t)|| &= T(t)||r'(t)|| \times (T'(t)||r'(t)|| + T(t)||r'(t)||') \\ &= \Big| \Big| \, ||r'(t)||^2 \, (T(t) \times T'(t)) + ||r'(t)|| ||r'(t)||' \, (T(t) \times T(t)) \, \Big| \Big| \, . \end{aligned}$$

Because a vector crossed itself always evaluates to 0, we are left with

$$||r'(t) \times r''(t)|| = ||r'(t)||^2 ||T(t) \times T'(t)||.$$

If we knew the angle between $T(t)$ and $T'(t)$, we could substitute $||T(t)|| ||T'(t)|| \sin\theta$ for $||T(t) \times T'(t)||$. To find θ, notice that since $T(t)$ is defined to have a constant magnitude, its derivative should only indicate a direction change. So, $T'(t)$ should be perpendicular to $T(t)$. More rigorously, we wish to show that $T(t) \cdot T'(t) = 0$. Since $||T(t)|| = 1$, we know that $T(t) \cdot T(t) = ||T(t)||^2 = 1$. Taking the derivative of both sides, we get that $T'(t) \cdot T(t) + T(t) \cdot T'(t) = 0$, or $2T(t) \cdot T'(t) = 0$. Thus, we find that $T(t) \cdot T'(t) = 0$, so the angle between $T(t)$ and $T'(t)$ must be $\pi/2$. This gives us that $||T(t) \times T'(t)|| = (1)||T'(t)|| \sin(\pi/2) = ||T'(t)||$.

Substituting this back into $||r'(t) \times r''(t)||$, we get that

$$\begin{aligned} ||r'(t) \times r''(t)|| &= ||r'(t)||^2 ||T(t) \times T'(t)|| \\ &= ||r'(t)||^2 ||T'(t)|| \end{aligned}$$

Finally, substituting this back into the proposed new equation for curvature, we find that

$$\frac{||r'(t) \times r''(t)||}{||r'(t)||^3} = \frac{||r'(t)||^2 ||T'(t)||}{||r'(t)||^3} = \frac{1}{||r'(t)||} ||T'(t)||.$$

Hence, we have shown that $\kappa(t) = \frac{||\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)||}{||\boldsymbol{r}'(t)||^3}$ is a valid formula for curvature. (7)

Circling back to the original problem, let's try using this formula to calculate the curvature of $r(t) = \langle \ln t, 1, t \rangle$. Taking the first and second derivatives of $r(t)$, we get that

$$r'(t) = \langle t^{-1}, 0, 1\rangle$$
$$r''(t) = \langle -t^{-2}, 0, 0\rangle$$
$$r'(t) \times r''(t) = \langle 0, -t^{-2}, 0\rangle$$
$$||r'(t) \times r''(t)|| = \sqrt{(-t^{-2})^2} = t^{-2}.$$

Finally, $||r'(t)|| = \sqrt{t^{-2}+1}$, so

$$||r'(t)||^3 = (t^{-2}+1)^{\frac{3}{2}} = \left(\frac{1+t^2}{t^2}\right)^{3/2}.$$

Substituting these values back into our new formula for κ, we get that

$$\kappa = \left(\frac{1}{t^2}\right)\left(\frac{t^2}{1+t^2}\right)^{3/2} = \frac{|t|}{(1+t^2)^{3/2}}. (6)$$

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