32AH CHALLENGE PROBLEM REPORT 4

FALL 2022

1. The Can problem

Consider the following situation: we are working for a canned beverage manufacturer and are tasked with designing the optimal can. The manufacturer would like us to find such a can in two ways:

(1) The more material a can requires, the more expensive it is to manufacture. Given a certain volume, what are the dimensions of the can with the smallest possible surface area?
(2) The manufacturer is also interested in the most amount of volume a certain amount of material can hold. Given a certain surface area, what is the most amount of beverage a can could hold?

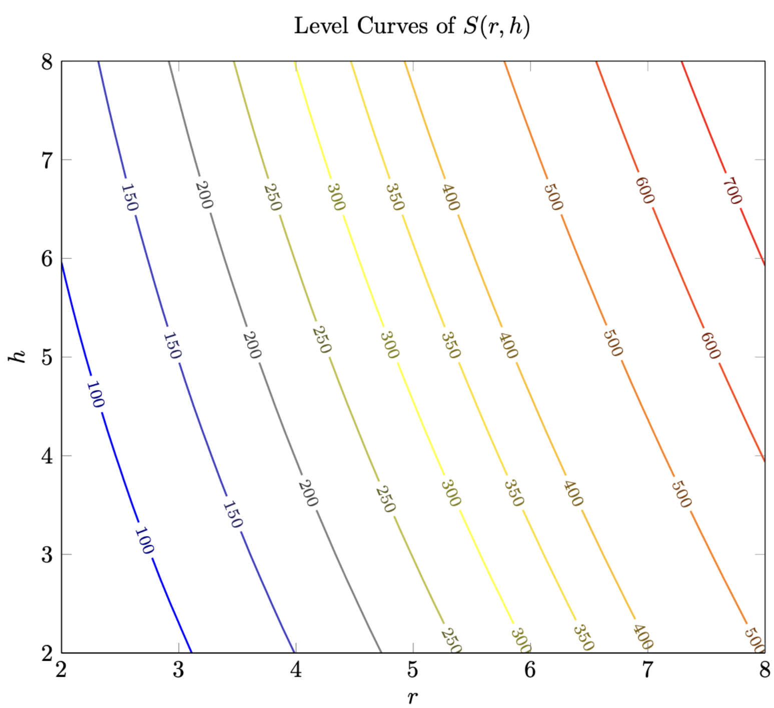
For the purposes of our analysis, we will assume that the can is a perfect cylinder. This means that our surface area and volume functions in terms of the radius r and the height h of the can are the following:

$$S(r,h) = 2\pi rh + 2\pi r^2$$
$$V(r,h) = \pi r^2 h$$

For all analyses, we will use the domain $D = \{(r,h)\,|\,r > 0, h > 0\}$. Since a can is a three-dimensional object, it would not make sense for us to consider $r = 0$ and $h = 0$.

1.1. Minimizing Surface Area with Constrained Volume

Let's begin by considering problem (1). Given a fixed volume V_0, we wish to minimize $f_1(r,h) = S(r,h)$ subject to the constraint $g_1(r,h) = V(r,h) - V_0 = 0$. Here are the level curves of $f_1(r,h)$:



Taking the gradient of f and g, we get:

$$\nabla f = \langle 2\pi h + 4\pi r, 2\pi r\rangle$$
$$\nabla g = \langle 2\pi rh, \pi r^2\rangle$$

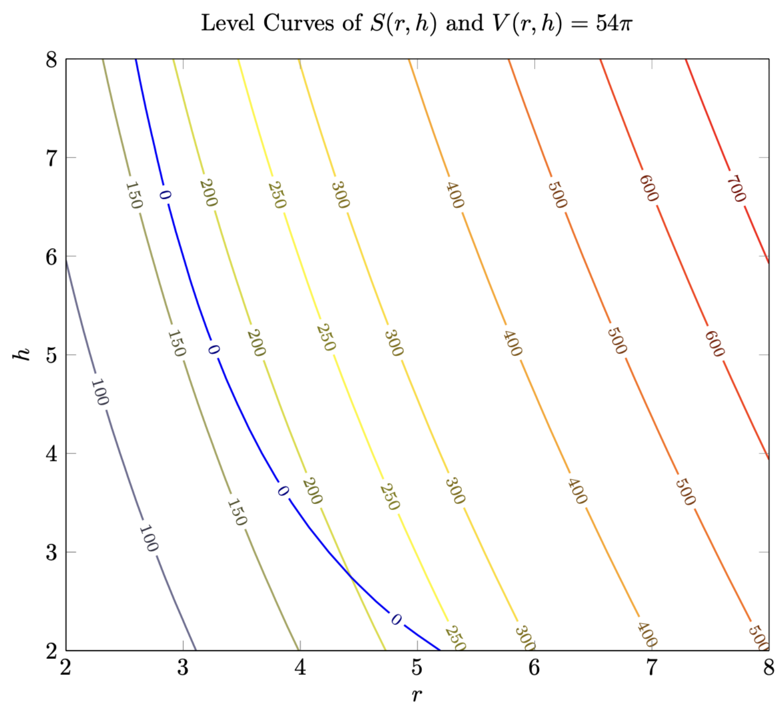
Setting $\nabla f = \lambda \nabla g$, we obtain the following system of equations:

$$\begin{aligned} f_r &= \lambda g_r & f_h &= \lambda g_h \\ 2\pi h + 4\pi r &= \lambda 2\pi r h & 2\pi r &= \lambda \pi r^2 \\ h + 2r &= \lambda r h & 2r - \lambda r^2 &= 0 \end{aligned}$$

The right equation factors to $r(2 - \lambda r) = 0$, so we get that either $r = 0$ or $2 = \lambda r$. Our domain D does not include $r = 0$, so we do not need to consider it further. Plugging $r\lambda = 2$ into the left equation, we get $h + 2r = (2)h$, or $2r = h$. Substituting this relationship back into the original constraint, we get that surface area is minimized when $\pi r^2(2r) = V_0$, which means that

$$r = \left(\frac{V_0}{2\pi}\right)^{1/3} \text{ and } h = 2r = \left(\frac{4V_0}{\pi}\right)^{1/3}.$$

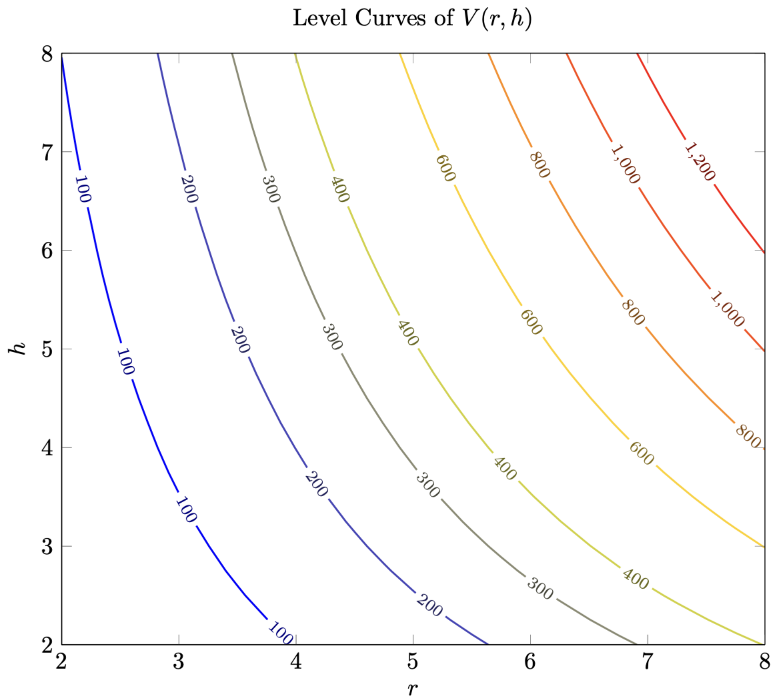
Although our math tells us this (r, h) pair is a critical point, we still have not determined whether it is a global minimum or maximum. To determine this, we can inspect the level curves of f_1 and g_1. Let's say our manufacturer gives us a constrained volume V_0 of 54π. Using the relationships above, we get the critical point $(r, h) = (3, 6)$. Graphing the level curves of $f_1(r, h)$ against the curve $g_1(r, h) - 54\pi = 0$, we get:



Indeed, it appears that $g_1(r, h) - 54\pi = 0$ is parallel to the closest level curve of $f(r, h)$ around $(3, 6)$. Further, notice that $(3, 6)$ is where the constraint gets the closest to the level curve 150; when $r < 3$ or $r > 3$, the constraint gets further from 150. This observation tells us that $(3, 6)$ is a constrained minimum, as desired.

1.2. Maximizing Volume with Constrained Surface Area

Let's proceed to question (2). Here, we are interested in finding the maximum volume of a can that has a fixed surface area S_0. This means that we are optimizing $f_2(r, h) = V(r, h)$ and our constraint is $g_2(r, h) = S(r, h) - S_0 = 0$. The level curves of f_2 are the following:



Taking the gradient of $f_2(r,h)$ and $g_2(r,h)$, we get

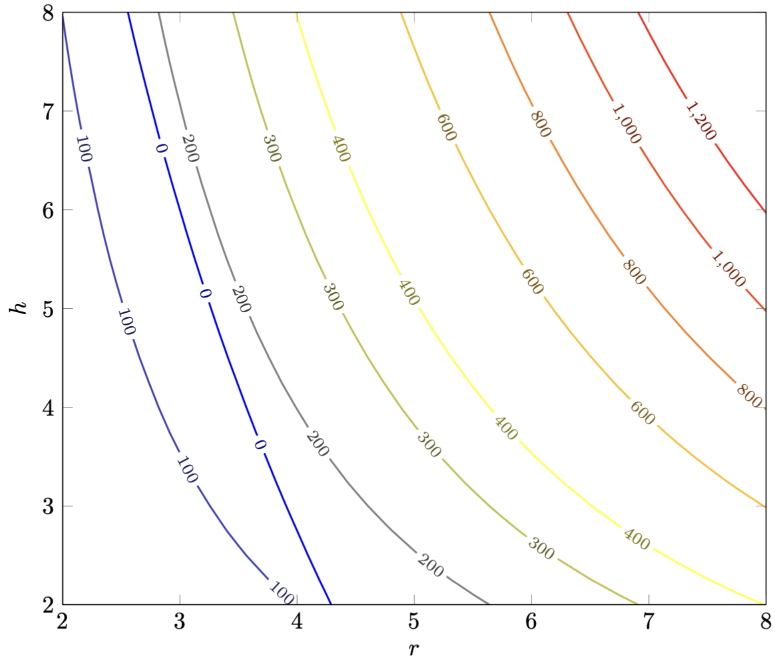
$$\nabla f = \langle 2\pi rh, \pi r^2 \rangle$$
$$\nabla g = \langle 2\pi h + 4\pi r, 2\pi r \rangle.$$

Notice that these are the exact same gradients we were working with in the previous problem – the only difference is that ∇f and ∇g are flipped. Thus, we already know the two gradients are parallel when $2r = h$. Substituting this relationship back into our original constraint, we get $2\pi r(2r) + 2\pi r^2 = S_0$, which means that volume is maximized when

$$r = \sqrt{\frac{S_0}{6\pi}} \text{ and } h = 2\sqrt{\frac{S_0}{6\pi}}.$$

For $S_0 = 54\pi$, we get that $r = 3$ and $h = 6$. To check that this is a maximum, we can look at the level curves of $f_2(r,h)$ and compare it to the curve $g_2(r,h) = 0$:

Level Curves of $V(r,h)$ and $S(r,h) = 54\pi$



It appears that $g_2(r,h) = 0$ gets closest to $f_2(r,h) = 200$ at around $(3,6)$, and is further from that level curve everywhere else. This confirms that $(r,h) = (3,6)$ is a constrained maximum.

1.3. **Converse Questions**

We might wonder whether the converses of the questions above are meaningful. That is, given a fixed volume, is it possible to find a can with maximal surface area? Similarly, given a fixed surface area, is it possible to find a can with minimal volume?

Let's start by thinking about the first question. To maximize surface area, we could make r as large as possible. To preserve the fixed volume, we'll decrease h accordingly. For small h, the surface area is effectively the top and bottom of the can, or $S \approx 2\pi r^2$. However, notice that r can be arbitrarily large – using the volume constraint, we can always find a sufficiently small h. This means there is no maximum surface area, as no matter how big we choose an r, a larger r always exists that will increase the surface area. In other words, since our domain is not *bounded*, S does not have a maximum.

We can apply similar reasoning to the question of whether it makes sense to minimize volume with fixed surface area. One way we could try to minimize volume is seeing how small we can get h. Since $V_h = 2\pi r$, we know that $V_h > 0$ for all $r > 0$, which means that volume is a strictly increasing function along the h-axis on D. This means that the greater h is, the greater V must be; similarly, the smaller h is, the smaller V is. Thus, we could approach the problem by trying to get h as small as possible, so long as $h > 0$. However, therein lies the problem: no matter how small of an h we choose, there always exists a smaller h that makes the volume smaller. In other words, because the domain of V is not *closed*, there is no smallest h in the domain, which means there is no smallest S in the range.

We have deduced above that since our domain D is not closed or bounded, the existence of global extrema is not guaranteed. A more rigorous statement of this intuition is that we do not satisfy the conditions of the multivariable analog of the Extreme Value Theorem:

Theorem 1.1 (Multivariable global minima and maxima)**.** *If D is a closed and bounded subset of $\mathbb{R}^n$, and f is a continuous function on D, then f has a global maximum and a global minimum in D.*

Since our D is neither closed nor bounded, the existence of global extrema is not guaranteed.

1.4. **Summary**

We began by finding the dimensions of a can that has minimal surface area subject to a volume constraint. There, we found that given a fixed volume V_0, the ideal dimensions are

$$r = \left(\frac{V_0}{2\pi}\right)^{1/3} \text{ and } h = \left(\frac{4V_0}{\pi}\right)^{1/3}.$$

Next, we found the dimensions of a can that has maximal volume subject to a surface area constraint. For a can with fixed surface area S_0, we found the ideal dimensions to be

$$r = \sqrt{\frac{S_0}{6\pi}} \text{ and } h = 2\sqrt{\frac{S_0}{6\pi}}.$$

These relationships will be helpful in the next section.

2. Tolerance intervals

Manufacturing machines in the real world are not perfect: they might output a bit too much or too little beverage, or manufacturing defects might make the surface area of the can larger or smaller than desired. Because of this, we might want to optimize again to adjust for the following situations:

(1) In situation (1) in the previous section, the manufacturing machines will not always produce a can with the exact surface area, radius, and height we specified. As a result, some cans might hold a bit more or less volume than the ideal can. To avoid wasting beverage, we'd like to find the minimum volume of all acceptable produced cans.
(2) Similarly, in situation (2) in the section above, the manufacturing machines will not always output the exact amount of volume desired. Supposing our manufacturing machines are already made and we cannot significantly change the radius and height of the can we designed in problem 2, we must find the maximum surface area needed per can, which might help us accurately estimate the cost per can.

We'll use a .1% level of tolerance for each of these quantities.

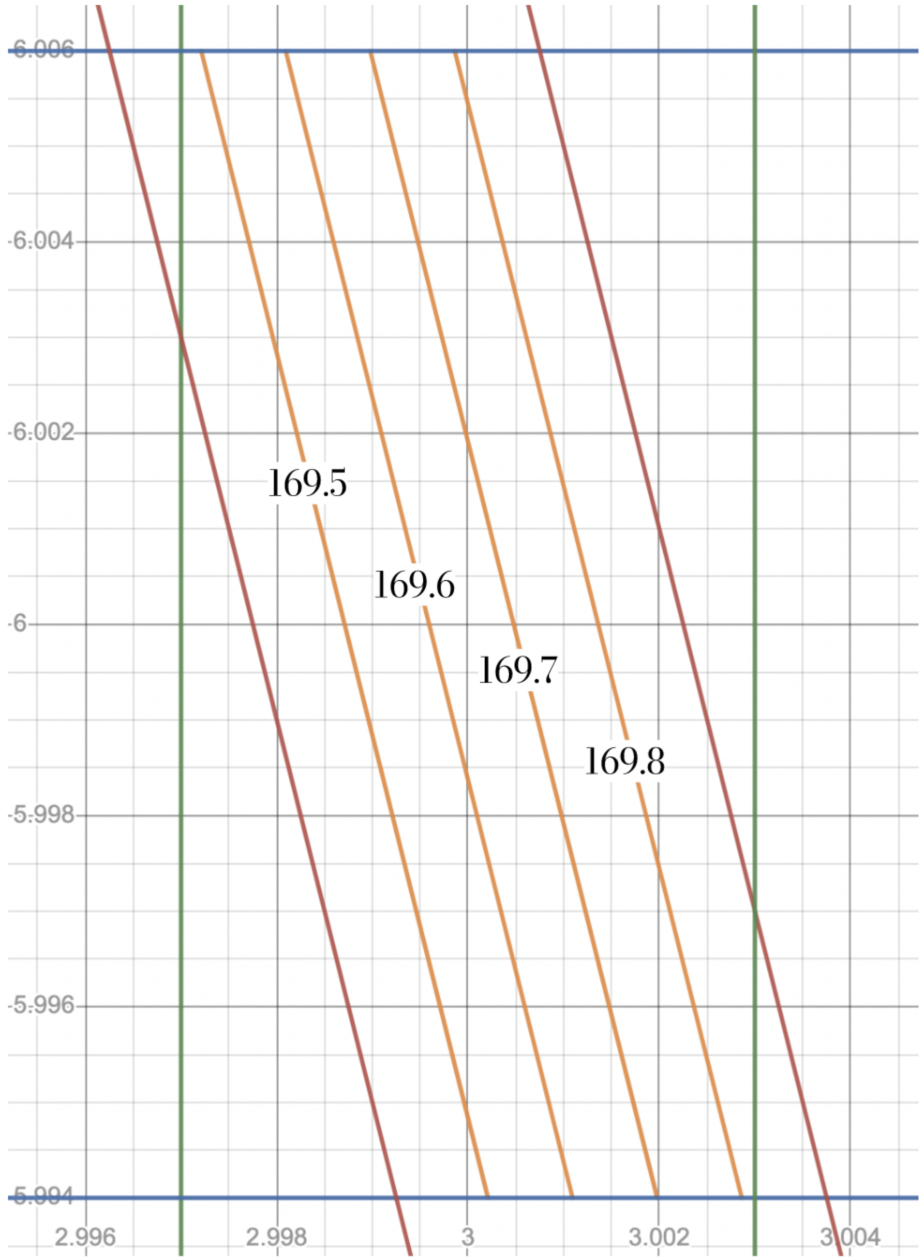
2.1. **Minimizing Volume Subject to Tolerance Intervals**

In the previous section, we found that for a fixed volume, surface area is minimized when the relationship $2r = h$ holds. For the volume of 54π we are interested in, this yielded $r = 3$ and $h = 6$. The can-producing machine at our manufacturing plant produces cans that have a surface area of $54\pi \pm 0.1\%$, and we establish a level of tolerance for the radius and height of $r = 3 \pm 0.1\%$ and $h = 6 \pm 0.1\%$. We would like to find the minimum amount of beverage that will find all cans that meet these constraints.

More explicitly, we wish to find the global minimum of $V(r, h) = \pi r^2 h$ of the region within the following constraints, where r_0, h_0, and S_0 represent the ideal values we found previously:

$$\begin{aligned} r &\geq (0.999)r_0 & r &\leq (1.001)r_0 \\ h &\geq (0.999)h_0 & h &\leq (1.001)h_0 \\ S(r,h) &\geq (0.999)S_0 & S(r,h) &\leq (1.001)S_0 \end{aligned}$$

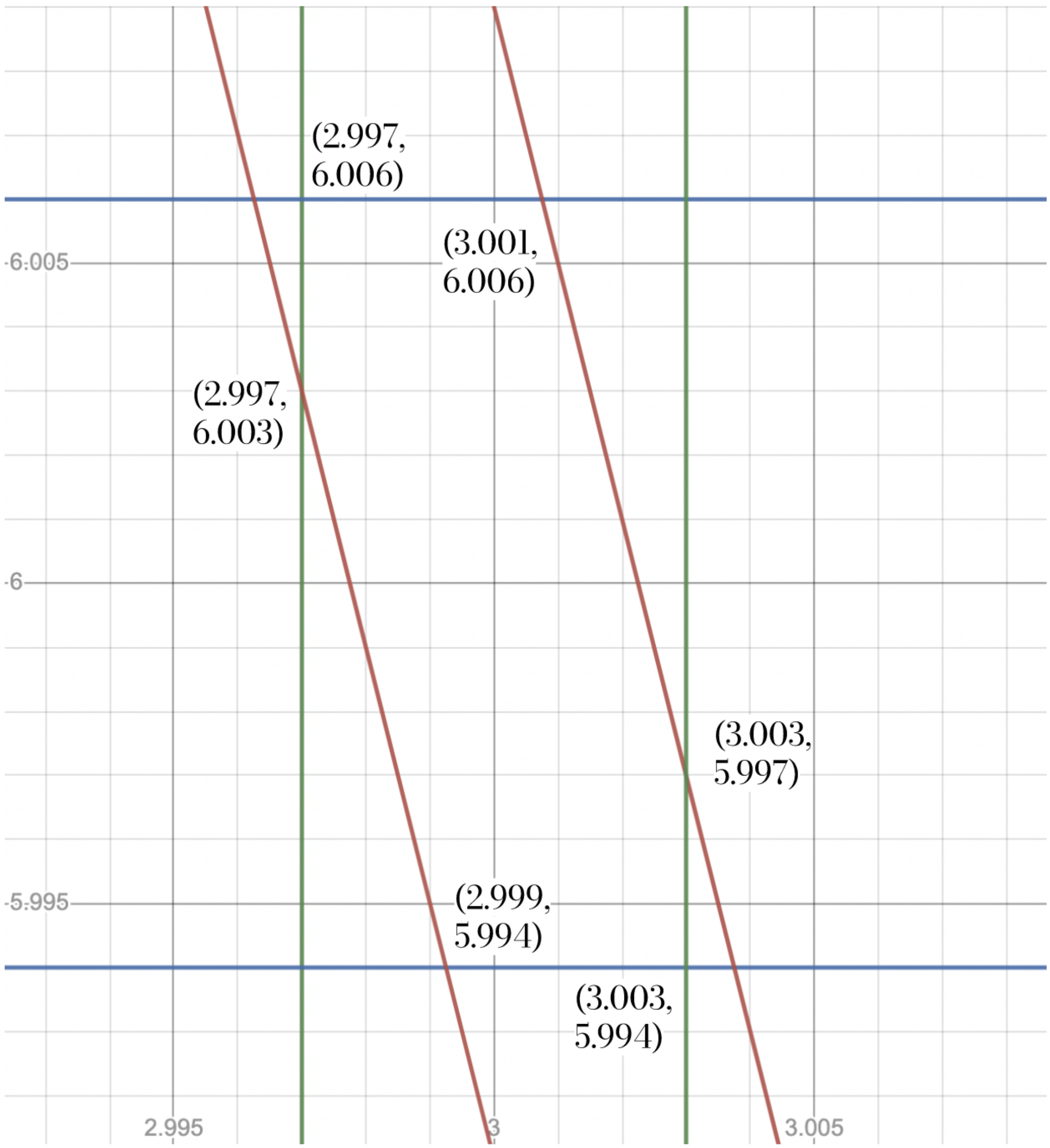
When graphed, these constraints define the region below. Green represents the r constraints, blue the h constraints, red the S constraints, and orange the level curves of V.



First, we must precisely define the region by finding the points of intersection of the constraints. The intersection of the r and h constraints are just the values of the constraints, $(0.999r_0, 1.001h_0)$ and $(1.001r_0, 0.999h_0)$. To find the points of intersection with the S constraints, we can use the relationship $54\pi k = S(r,h) = 2\pi rh + 2\pi r^2$, or

$$2rh + 2r^2 = 27k,$$

where k is either 0.999 or 1.001. Using this equation, we find that the points of intersection are the following:



Let's call this region R. To begin, let's check if there are any critical points we could find using local optimization. Setting the first partial derivatives of V equal to 0, we find that:

$$V_r = 2\pi rh, \text{ so } V_r = 0 \text{ when } r = 0 \text{ or } h = 0.$$

$$V_h = \pi r^2, \text{ so } V_h = 0 \text{ when } r = 0.$$

Thus, the only critical points occur when $r = 0$, which means there are no local extrema over R. This makes sense, as volume only increases as r and h increase, so it follows that V_r and V_h are never 0 for $r > 0$ and $h > 0$.

Further, notice that this means that we cannot have any critical points on the blue and green boundaries, where either r or h is held constant. For instance, by finding all ordered pairs for which $\frac{\partial V}{\partial r} = 0$, we have already confirmed that $\frac{\partial V}{\partial r} \neq 0$ on the line $h = kh_0$; by checking when $V_r = 0$, we are implicitly checking if

$V'(kr_0, h) = 0$ along the h constraints. The diagram of the level curves above supports this conclusion, as the level curves are never parallel to the r and h constraints within R.

This leaves us with only the red boundaries to check, which represent $S = kS_0$, where k is either 0.999 or 1.001. We can use Lagrange multipliers here. We have that $f = V$, since V is the function we are optimizing. The constraining function is $g(r, h) = S(r, h) - kS_0$. Notice that we are maximizing volume subject to a surface area constraint. We have already solved this problem in the previous section and derived the equations

$$r = \sqrt{\frac{S_0}{6\pi}} \text{ and } h = 2\sqrt{\frac{S_0}{6\pi}}.$$

When $S_0 = 0.999(54\pi)$, we get $r = 2.998$ and 5.997; when $S_0 = 1.001(54\pi)$, we have that $r = 3.001$ and $h = 6.003$. Both of these (r, h) pairs fall within our levels of tolerance, they are critical points we must also consider.

Thus, we have 8 critical points in total: 6 on the intersections of boundaries, and 2 along the surface area constraints. Evaluating V at each of these (r, h), we get that:

$$\begin{aligned} f(2.997, 6.003) &= 169.392 & \qquad f(3.001, 6.006) &= 169.929 \\ f(2.997, 6.006) &= 169.476 & \qquad f(3.003, 5.994) &= 169.815 \\ f(2.999, 5.994) &= 169.363 & \qquad f(3.003, 5.997) &= 169.900 \\ f(2.998, 6.003) &= 169.392 & \qquad f(3.001, 6.003) &= 169.901 \end{aligned}$$

The smallest value out of these points is 169.363, so that is the volume that will fit into all cans under this set of constraints.

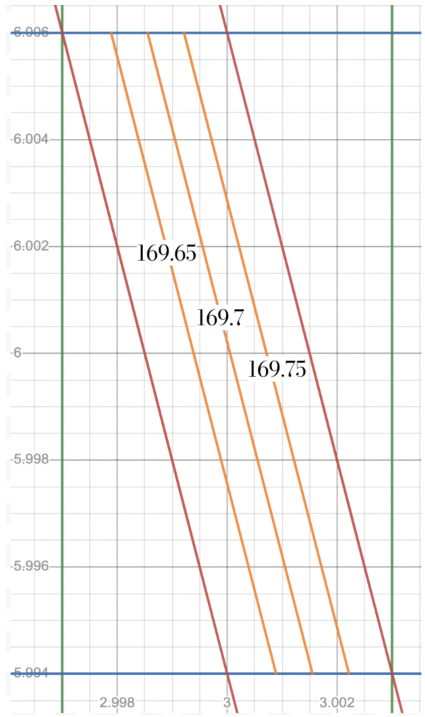
2.2. **Maximizing Surface Area Subject to Tolerance Intervals**

Now, let's place a similar set of constraints on the complement of this problem: we might have machines that do not quite output the same amount of beverage each time. Let's suppose that machine outputs $54\pi \pm .1\%$ of beverage, and since the machines are already made, we cannot change r and h by more than $\pm.1\%$ as well. To ensure that we do not waste beverage, we wish to maximize surface area subject to these constraints.

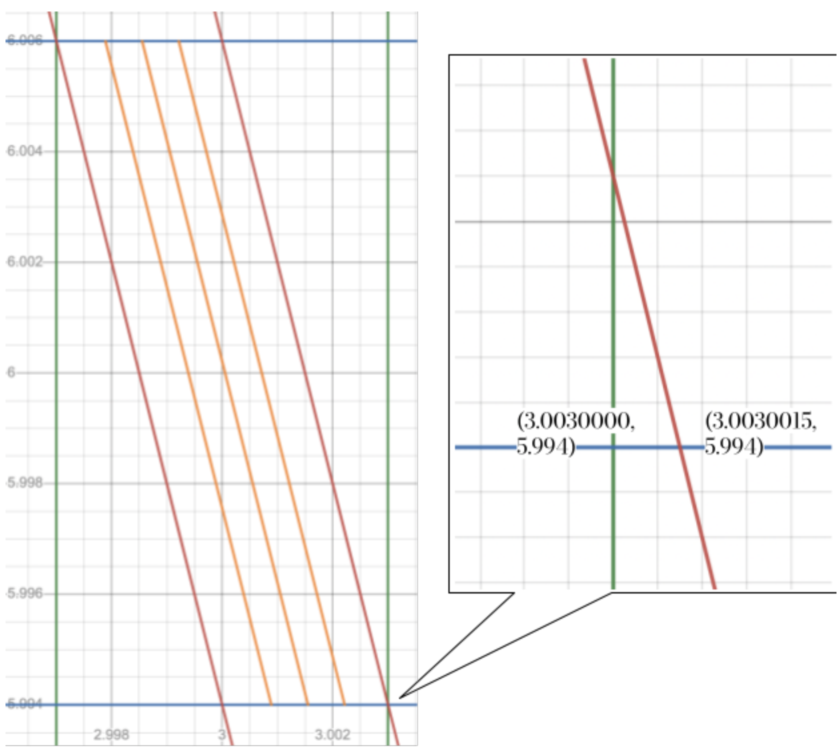
Since we are optimizing surface area, $f_4(r, h) = S(r, h) = 2\pi rh + 2\pi r^2$. Our constraints are the following:

$$\begin{aligned} r &\geq (0.999)r_0 & \qquad r &\leq (1.001)r_0 \\ h &\geq (0.999)h_0 & \qquad h &\leq (1.001)h_0 \\ V(r, h) &\geq (0.999)V_0 & \qquad V(r, h) &\leq (1.001)V_0 \end{aligned}$$

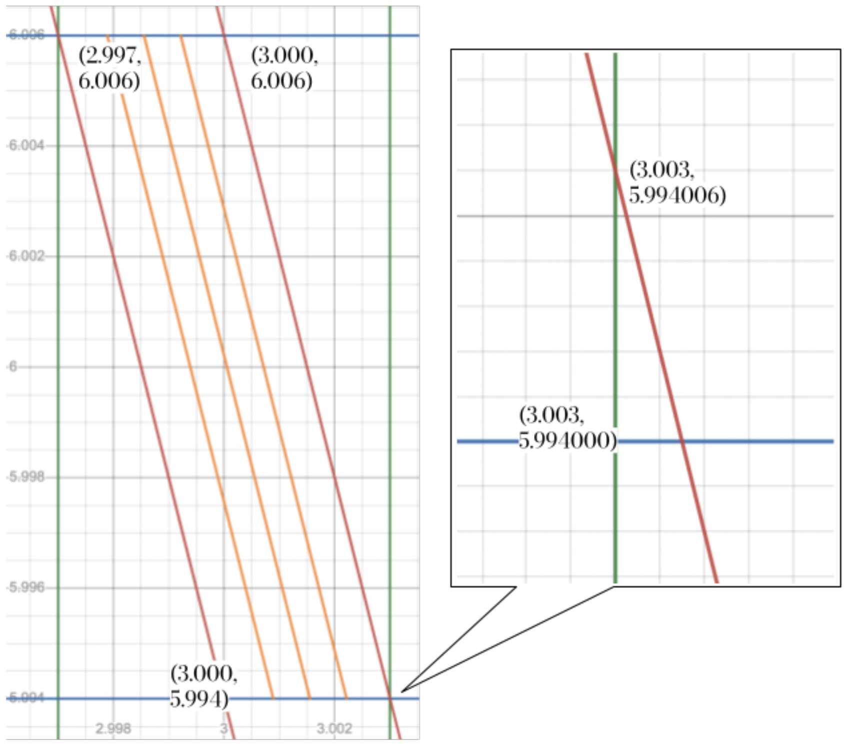
Let's graph these constraints along with the level curves of $f_4(r, h)$ to better understand the region we are optimizing over. In the following diagram, green represents the r constraints, blue the h constraints, red the V constraints, and orange the level curves of S:



From the picture, we might think that we do not even have to worry about the r constraints, as the region seems to be defined by only the h and V constraints. However, zooming in on the lower right corner of the constraint, we see that the V constraint barely misses the intersection of the h and r constraints:



This means that we have 5 constraint intersection points to consider. Finding each of the intersections algebraically, we get that the (r, h) points are the following:



Using similar reasoning to the previous problem, we know that S_h and S_r are always positive (and therefore cannot be 0) on the region we are interested in, so there are no critical points to find there. Further, this means that there will be no critical points along any of the r or h constraints.

Lastly, we must check the V constraints for critical points. Again, we notice this is a minimum surface area subject to a volume constraint problem, which we already solved in the previous section and got the equations

$$r = \left(\frac{V_0}{2\pi}\right)^{1/3} \text{ and } h = \left(\frac{4V_0}{\pi}\right)^{1/3}.$$

Substituting in $V_0 = 0.999(54\pi)$ and $V_0 = 1.001(54\pi)$, we find that our two critical points are $(2.999, 5.998)$ and $(3.001, 6.002)$, both of which fall into the region of interest.

In total we have 7 critical points: 5 from the intersections of the constraints and 2 from the V constraints. Evaluating f_4 at these points, we get:

$$f(2.999, 5.998) = 169.533$$
$$f(3.001, 6.002) = 169.7591$$
$$f(2.997, 6.006) = 169.533$$
$$f(3.000, 6.006) = 169.7591$$
$$f(3.000, 5.994) = 169.533$$
$$f(3.003, 5.994000) = 169.7590$$
$$f(3.003, 5.994006) = 169.7592$$

Thus, the global maximum for surface area subject to these constraints is 169.7592 at $r = 3.003$ and $h = 5.994006$.

2.3. **Summary**

We expanded upon our work in the first section by setting levels of tolerance of $\pm.1\%$ for all constant values. In the first problem, we found that 169.363 is the smallest volume that fits into all cans within our levels of tolerance. In the second problem, we found that 169.7592 is the largest surface area within our levels of tolerance.

Department of Mathematics, UCLA, Los Angeles, CA 90095
Current address: Department of Mathematics, UCLA, Los Angeles, CA 90095
Email address: `richardwong@math.ucla.edu`