32BH CHALLENGE PROBLEM REPORT 1

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1. Improper Integrals: Unbounded Functions

Integration is well-defined for bounded functions with bounded support. However, although integrals with unbounded functions or unbounded support are not guaranteed, they still may exist. An interesting case study of this idea are rational functions of the form $f(x) = \frac{1}{x^p}$ with $p > 0$, which are unbounded over the region $[0, 1]$. Because $f(0) = \frac{1}{0^p}$ is undefined, the fundamental theorem of calculus does not apply to any $f(x)$ on this region, so $\int_0^1 f(x)$ is not guaranteed to exist. However, there are some values of p for which the improper integral does exist. To find them, we can apply the definition of the improper integral:

Definition 1.1. Let $f(x)$ be a function that is continuous but unbounded on the open interval (a, b). Let $\alpha, \beta \in (a, b)$ such that $a < \alpha < \beta < b$. Then we can compute the integral

$$I(\alpha, \beta) := \int_\alpha^\beta f(x)\ dx$$

If the limit $\lim_{(\alpha,\beta)\to(a,b)} I(\alpha, \beta) = L$ exists, then we can define the **improper integral** $\int_a^b f(x)\ dx$ to be L. That is,

$$\int_a^b f(x)\ dx := \lim_{\substack{\alpha\to a\\ \beta\to b}} \int_\alpha^\beta f(x)\ dx$$

Notice that $f(x)$ is continuous and bounded on the region $[a, 1]$ where $0 < a < 1$. Our new integral then becomes $\int_a^1 \frac{1}{x^p}$, on which we *can* now integrate. For $p \neq 1$, the integral is as follows:

$$\int_a^1 x^{-p} dx = \frac{x^{1-p}}{1-p}\Big|_a^1 = \frac{1^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}$$

All quantities within this expression are defined for $p > 0$ except for a^{1-p}. We are ultimately trying to find in whether $\lim_{a\to 0} \int_a^1 x^{-p}$ is finite, so we are interested in p that make $\lim_{a\to 0} a^{1-p}$ exist. When $1 - p < 0$, a has a negative exponent, so it takes the form $\frac{1}{a^{p-1}}$, which makes the limit undefined. Thus, we must have that $1 - p > 0$ or $p < 1$ for the limit to exist.

In the case $p = 1$, we instead get the following integral:

$$\int_a^1 x^{-1} dx = \ln|x|\Big|_a^1 = \ln|1| - \ln|a| = -\ln|a|$$

Since $\lim_{a\to 0^+} \ln|a| = -\infty$, the integral is not finite and does not exist for $p = 1$. Thus, $\int_0^1 \frac{1}{x^p} dx$ exists for $p < 1$ **(1)**.

2. Improper Integrals: Unbounded Regions

The fundamental theorem of calculus also does not guarantee the existence of integrals of functions with unbounded support. However, this does not necessarily mean that the integrals of such functions do not exist. As an example, let's explore the integral $\int_0^\infty f(x)dx$, where $f(x) = \frac{1}{1+x^2}$. As in the example above, we can calculate this integral using limits:

Definition 2.1. Let $a \in \mathbb{R}$, and suppose that $f(x)$ is bounded and integrable on $[a, b]$ for every $b > a$. Then we can compute the integral

$$\int_a^b f(x)\ dx$$

If the limit $\lim_{b\to\infty} \int_a^b f(x)\ dx = L$ exists, then we can define the **improper integral** $\int_a^\infty f(x)\ dx$ to be L. That is,

$$\int_a^\infty f(x)\ dx := \lim_{b\to\infty} \int_a^b f(x)\ dx$$

By this definition, we *can* integrate from 0 to a positive constant B, and then take the limit as $B \to \infty$:

$$\begin{aligned}
\lim_{B\to\infty} \int_0^B \frac{1}{1+x^2} dx &= \lim_{B\to\infty} \arctan(x)\Big|_0^B \\
&= \lim_{B\to\infty} \arctan(B) - \arctan(0) \\
&= \lim_{B\to\infty} \arctan(B) \\
&= \frac{\pi}{2} \quad \textbf{(2)}
\end{aligned}$$

However, we may also be interested in calculating integrals that take the form $\int_{-\infty}^b f(x)$. Applying the same idea, we can generalize Definition 2.1 to the following:

Definition 2.2. Let $a \in \mathbb{R}$, and suppose that $f(x)$ is bounded and integrable on $[a, b]$ for every $b > a$. Then we can compute the integral

$$\int_a^b f(x)\ dx$$

If the limit $\lim_{a\to-\infty} \int_a^b f(x)\ dx = L$ exists, then we can define the **improper integral** $\int_{-\infty}^b f(x)\ dx$ to be L. That is,

$$\int_{-\infty}^b f(x)\ dx := \lim_{a\to-\infty} \int_a^b f(x)\ dx$$

The definitions above allow us to integrate a region with unbounded support. **(3)**

3. Cauchy Principal Values

We've now defined how to integrate unbounded functions and functions with unbounded support. Combining definitions 2.1 and 2.2 from the previous section, we get the following definition:

Definition 3.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $\mathbb{R}$. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$. Then we can compute the integral

$$I(\alpha, \beta) := \int_\alpha^\beta f(x)\ dx$$

If the limit $\lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} I(\alpha, \beta) = L$ exists, then we can define the **improper integral** $\int_{-\infty}^{\infty} f(x)\ dx$ as the limit. That is,

$$\int_{-\infty}^{\infty} f(x)\ dx := \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} \int_\alpha^\beta f(x)\ dx = L$$

However, in some situations it is convenient to define the two limits of integration with the same variable:

Definition 3.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that is continuous on $\mathbb{R}$. Then we can compute the integral

$$I(R) := \int_{-R}^{R} f(x)\ dx$$

If the limit $\lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx = C$ exists, then we can define the **Cauchy principal value** to be C. That is,

$$P.V. \int_{-\infty}^{\infty} f(x)\ dx := \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx$$

Intuitively, it may seem like it would not matter whether we take the limits independently of each other or not. However, as we work through some examples, we will find that is not the case.

We will begin with an example where the Cauchy Principle Value does agree with the improper integral. Consider the function $f(x) = xe^{-x^2}$. We begin by calculating the Cauchy Principle Value:

$$\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{R} xe^{-x^2}\,dx &= \lim_{R\to\infty}\Big[-\frac{1}{2}e^{-x^2}\Big]_{-R}^{R} \\
&= \lim_{R\to\infty} -\frac{1}{2}\Big[\frac{1}{e^{R^2}} - \frac{1}{e^{R^2}}\Big] \\
&= 0
\end{aligned}$$

Now, let's calculate the improper integral $\int_{-\infty}^{\infty} xe^{-x^2}\ dx$:

$$\begin{aligned}
\lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}}\int_{\alpha}^{\beta} xe^{-x^2}\,dx &= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}}\Big[-\frac{1}{2}e^{-x^2}\Big]_{\alpha}^{\beta} \\
&= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}}\Big[-\frac{1}{2}e^{-x^2}\Big]_{\alpha}^{\beta} \\
&= \lim_{\substack{\beta\to\infty \\ \alpha\to-\infty}} -\frac{1}{2}\Big[\frac{1}{e^{\beta^2}} - \frac{1}{e^{\alpha^2}}\Big] \\
&= 0
\end{aligned}$$

For this integral, the Cauchy Principle value happens to agree with the value of the improper integral **(4)**. However, this is not always the case. As an example, let's study the function $g(x) = \frac{2x}{x^2+1}$. First, we calculate the Cauchy Principle Value:

$$\begin{aligned}
P.V.\int_{-\infty}^{\infty} g(x)\ dx &= \lim_{R\to\infty}\int_{-R}^{R}\frac{2x}{x^2+1} \\
&= \lim_{R\to\infty}\Big[\ln(x^2+1)\Big]_{-R}^{R} \\
&= \lim_{R\to\infty}\Big[\ln(R^2+1) - \ln((-R)^2+1)\Big] \\
&= \lim_{R\to\infty}\Big[\ln\Big(\frac{R^2+1}{R^2+1}\Big)\Big] \\
&= 0
\end{aligned}$$

Now, we calculate the same integral again, but approaching ∞ at a different rate; instead of using $\alpha = -R, \beta = R$ as $R \to \infty$, we will instead use $\alpha = -R, \beta = 2R$, which we can do because we still have that $a \to -\infty$ and $\beta \to \infty$. If the improper integral exists, we should find that the value of this limit agrees with the value of the Cauchy Principle Value.

$$
\begin{aligned}
\lim_{R\to\infty}\int_{-R}^{2R}\frac{2x}{x^2+1}dx &= \lim_{R\to\infty}\Big[\ln(x^2+1)\Big]_{-R}^{2R} \\
&= \lim_{R\to\infty}\Big[\ln((2R)^2+1)-\ln((-R)^2+1)\Big] \\
&= \lim_{R\to\infty}\Big[\ln\Big(\frac{4R^2+1}{R^2+1}\Big)\Big] \\
&= \ln(4)
\end{aligned}
$$

The value of the integral is not equal to what we found for the Cauchy Principle value, so this improper integral does not exist. **(5)**

Our current definition of the improper integral is still limiting[1] - there still exist functions we are not able to integrate. For example, we cannot currently the function $f(x) = \frac{1}{x}$ over the interval $[-1, 1]$ because $f(x)$ is undefined at $x = 0$.

We can define integrals of this form by applying our current definitions and the sum rule of integrals. Applying Definition 1.1, we can calculate $\int_{-1}^{0} f(x)dx$ and $\int_{0}^{1} f(x)dx$. Then, we can and add the two resulting values together to find $\int_{-1}^{1} \frac{1}{x}dx$. Note that $\lim_{x\to 0} f(x)$ does not exist, so we will have to approach 0 from the left side for $\int_{-1}^{0} f(x)dx$ and approach from the right side for $\int_{0}^{1} f(x)dx$.

$$
\begin{aligned}
\int_{-1}^{1} f(x)dx &= \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx \\
&= \lim_{a\to 0^-}\int_{-1}^{a}\frac{dx}{x} + \lim_{b\to 0^+}\int_{b}^{1}\frac{dx}{x}
\end{aligned}
$$

We determined earlier that integrals of functions that take the form $\frac{1}{x^p}$ do not exist when $p \geq 1$. Thus, neither of these sub-integrals exist, so $\int_{-1}^{1} f(x)dx$ does not exist either.

We can generalize this idea to integrals of any functions $h(x)$ that are unbounded at finitely many points $c_1 < \cdots < c_k \subset [a, b]$. We split up $\int_a^b f(x)$ into sub-integrals to which we can apply our notion of the improper integral:

Definition 3.3. Let $[a, b]$ be a closed interval, and let $h(x)$ be a function that is unbounded at finitely many points $c_1 < \cdots < c_k \subset [a, b]$, but is otherwise continuous on the closed interval $[a, b]$. Then we can compute the integral $I(a, b) := \int_a^b f(x)\ dx$ by applying the sum rule of integration and Definition 3.1.

$$
\int_a^b f(x)\ dx := \lim_{x_1\to c_1^+}\int_a^{x_1} f(x)dx + ... + \lim_{\substack{x_i\to c_i^+ \\ x_{i+1}\to c_{i+1}^-}}\int_{x_i}^{x_{i+1}} f(x)dx + ... + \lim_{x_k\to c_k^-}\int_{x_k}^{b} f(x)dx
$$

If any of the sub-integrals above are not finite, then $\int_a^b f(x)\ dx$ does not exist. **(6)**

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[1]pun intended

4. Summary

In this challenge Problem Report, we explored how to calculate improper integrals with unbounded functions, unbounded support, or both. We also found that we can use the Cauchy Principle Value to assign an integral a value even if it does not exist.

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