32BH CHALLENGE PROBLEM REPORT 3

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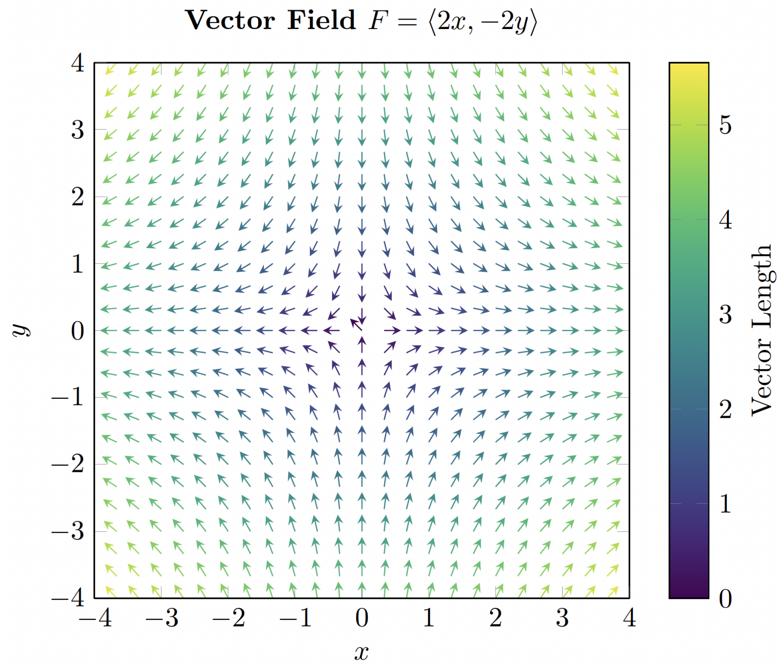
In this report, we study vector fields and their properties. The first section explores conservative vector fields and their relationship to their potential functions. We will also prove that any two potential functions for a conservative vector field must only differ by a constant. The second section studies the divergence and curl of vector fields. Here, we will understand how these operators interact and their physical implications.

1. Conservative Vector Fields

In this report, we will assume for convenience that our functions $f : \mathbb{R}^n \to \mathbb{R}$ are smooth. We will begin with a concrete example to understand how to compute potential functions of conservative vector fields and understand the relationship between conservative vector fields and their potential functions. Consider the vector field $\boldsymbol{F} = \langle 2x, -2y \rangle$. Here is a sketch of $\boldsymbol{F}$:



To find a potential function, we start by integrating the x-component of F:

$$\int F_x dx = \int 2x dx$$
$$f(x, y) = x^2 + g(y)$$

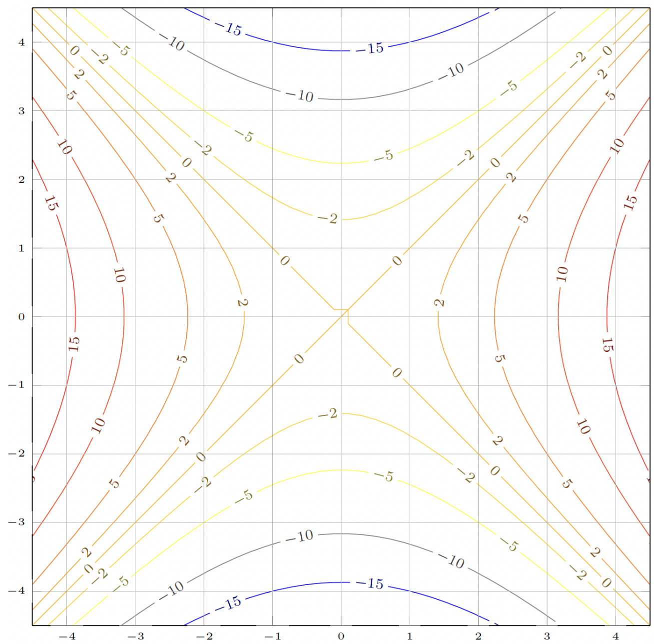
Next, we take the derivative of f with respect to to y, and substitute $F_y = -2y$ for $g'(y)$:

$$\begin{aligned} f_y &= \frac{\partial}{\partial y}\Big(x^2 + g(y)\Big) \\ &= 2x + g'(y) \\ &= 2x - 2y \end{aligned}$$

Lastly, we need to integrate f_y with respect to y to find f:

$$\begin{aligned} f &= \int f_y dy \\ &= \int (2x - 2y) dy \\ &= x^2 - y^2 + C \end{aligned}$$

Thus, potential functions of $\boldsymbol{F}$ will take the form $f(x, y) = x^2 - y^2 + C$, so one possible potential function is $f_1(x, y) = x^2 - y^2$. These are the level curves of f_1:



Notice that at each point (x, y), $\boldsymbol{F}$ is perpendicular to the level curve of its potential function f. This is the case because $\boldsymbol{F} = \nabla f$, and a function's gradient perpendicular to the function's level curves. **(1)**

Now that we have acquired an understanding of conservative vector fields and how they relate to their potential functions, we'd like to prove that any two potential functions of a vector field will only differ by a constant. To assist us, we'll first prove that if $\nabla f(x, y) = \mathbf{0}$ for all $(x, y) \in D$, then $f(x, y) : D \to \mathbb{R}$ is a constant function.

To intuitively understand why this must be true, consider three points $P, Q, R \in D$, where $P = (a, b)$, $Q = (c, b)$, $R = (c, d)$. Notice that points P and Q both lie on the line $y = b$ and that Q and R both lie on $x = c$. We can find the values of f along the line $y = b$ with the function $g(x) = f(x, b)$. Since this is a single-variable function, we can apply the Fundamental Theorem of Calculus to find the difference between $f(P)$ and $f(Q)$. Since we are assuming that functions are smooth, we know that g is real-valued and continuous on $[a, c]$, which means that

$$\int_a^c g'(x)dx = g(c) - g(a).$$

However, we know that $g'(x) = 0$ because $\frac{\partial f}{\partial x} = 0$, which tells us that $0 = g(c) - g(a)$, or that $g(c) = g(a)$ and $f(P) = f(Q)$. Since points P and Q are arbitrary, this tells us that f is constant along the line segment PQ.

Similarly, we can find the values of f along the line $x = c$ with the single-variable function $h(y) = f(c, y)$. Because we're assuming functions are smooth, we know that h is real-valued and continuous on $[b, d]$, so we can apply the Fundamental Theorem of Calculus to get that

$$\int_b^d h'(y)dy = h(d) - h(b).$$

Since $\frac{\partial f}{\partial y} = 0$, we know that $h'(y) = 0$, which means that $h(d) = h(b)$ and $f(R) = f(Q)$. Thus, f is also constant along the line segment QR.

From this information, we can understand intuitively why f must be a constant function if $\nabla f = 0$. It is possible to travel from any point X_1 to X_2 along a series of line segments perpendicular to the x and y

axes, and f is constant along each of these perpendicular line segments. Thus, it makes sense that f must be constant on D.

To prove this more rigorously, note that since D is a path-connected region, there must exist a path from any point X_1 to any point X_2 within D. Let $r(t)$ be a parametrization of such a path. Recall that by the chain rule,

$$\frac{d}{dt}f(r(t)) = \nabla f(r(t)) \cdot r'(t).$$

We know that $\nabla f = 0$ for all points, so $\frac{d}{dt}f(r(t)) = 0 \cdot r'(t) = 0$, which means that f is constant over $r(t)$. Thus, we must have that $f(X_1) = f(X_2)$.

This fact is useful because it allows us to understand why potential functions for conservative vector fields must only differ by a constant. To see this, let f_1 and f_2 be two different potential functions of a conservative vector field $\boldsymbol{F}$. Let their difference be the function $G(x, y) = f_1(x, y) - f_2(x, y)$. If we take the gradient of both sides, we see that

$$\begin{aligned}\nabla G(x, y) &= \nabla f_1(x, y) - \nabla f_2(x, y) \\ &= F(x, y) - F(x, y) \\ &= 0.\end{aligned}$$

From our work above, we know that $\nabla G = 0$ implies that G is a constant function. Let $G(x, y) = c$. Substituting this into our original equation, we get that

$$f_1(x, y) - f_2(x, y) = c,$$

which means that any two potential functions of F only differ by a constant. **(2)**

2. Divergence and Curl

Divergence and curl allow us to understand how vector fields behave. Interestingly, although the component of a vector that divergence measures is orthogonal to the component of a vector that curl is concerned with, it is possible for a non-constant vector field to have both a divergence and curl of 0. As an example, take $F = \langle y, x, 0\rangle$. For this field,

$$\begin{aligned}\text{div}\boldsymbol{F} &= \frac{\partial}{\partial x}y + \frac{\partial}{\partial y}x + \frac{\partial}{\partial z}0 = 0 \\ \text{curl}\boldsymbol{F} &= \left\langle \frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}x, \frac{\partial}{\partial z}y - \frac{\partial}{\partial x}0, \frac{\partial}{\partial x}x - \frac{\partial}{\partial y}y \right\rangle = \vec{0}. \quad \textbf{(3)}\end{aligned}$$

That aside, we can discover some interesting properties when we start applying divergence, curl, and the gradient in different orders. One fundamental fact is that for any smooth function f, $\text{curl}(\nabla f) = \mathbf{0}$. To see this, let $\nabla f = \langle f_x, f_y, f_z\rangle$. Then by the definition of curl,

$$\text{curl}(\nabla f) = \left\langle f_{xy} - f_{yz}, f_{xz} - f_{zx}, f_{yx} - f_{xy} \right\rangle.$$

Since f is smooth, we know that f has defined and continuous second partial derivatives. This means that by Clairaut's Theorem, $f_{zy} = f_{yz}$, $f_{xz} = f_{zx}$, and $f_{yx} = f_{xy}$. Subsituting these relationships into our equation for $\text{curl}(\nabla f)$, we get that

$$\text{curl}(\nabla f) = \vec{0}.$$

Notice that this implies that for a conservative vector field $\boldsymbol{F}$, $\text{curl}(\boldsymbol{F}) = \vec{0}$, as there must exist a function f such that $\nabla f = \boldsymbol{F}$. The physical implication of this is that bodies placed conservative vector fields will not naturally rotate, as the curl at any point must be 0. **(4)**

A fundamental property that generalizes to all vector fields is the fact that $\text{div}(\text{curl}\boldsymbol{F}) = 0$. For the sake of legibility, let superscripts refer to components of $\boldsymbol{F}$ and subscripts denote partial derivatives of $\boldsymbol{F}$. Let $F = \langle F^1, F^2, F^3\rangle$. By the definition of curl,

$$\text{curl}(\boldsymbol{F}) = \left\langle F^3_y - F^2_z, F^1_z - F^3_x, F^2_x - F^1_y \right\rangle.$$

Taking the divergence, we get

$$\begin{aligned}\operatorname{div}(\operatorname{curl}\boldsymbol{F}) &= (F^3_{yx} - F^2_{zx}) + (F^1_{zy} - F^3_{xy}) + (F^2_{xz} - F^1_{yz}) \\ &= (F^1_{zy} - F^1_{yz}) + (F^2_{xz} - F^2_{zx}) + (F^3_{yx} - F^3_{xy})\end{aligned}$$

Since F is smooth, it has continuous and defined partial derivatives, so we can again apply Clairaut's Theorem to get the relationships $F^1_{zy} = F^1_{yz}$, $F^2_{xz} = F^2_{zx}$, and $F^3_{yx} = F^3_{xy}$. Substituting these into the equation for $\operatorname{div}(\operatorname{curl}\boldsymbol{F})$, we get that

$$\operatorname{div}(\operatorname{curl}\boldsymbol{F}) = 0.$$

This tells us that it is impossible for the curl of a vector field to be a source or a sink; it will always be incompressible. **(5)**

3. Summary

This report studied several fundamental properties of vector fields. First, we prove that any two potential functions of a conservative vector field must only differ by a constant. We then found that for any vector field f, $\operatorname{curl}(\nabla f) = \mathbf{0}$, which implies that the curl of a conservative vector field is 0 at all points. Finally, we found that for any vector field $\boldsymbol{F}$, $\operatorname{div}(\operatorname{curl}\boldsymbol{F}) = 0$.

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