Contents

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Statement:
e.g. "Sky is blue" – True; "1/0 = 7" – False
x: variable f(x)
P/Q: Statemtent variables
Operations:
-And:
if P,Q are statements, then "P and Q" is true if both of them are individually true
-Not
"Not P" is true exactly when P is false
-Or
"P or Q" is true when at lease one of P or Q is true
can mean one is true or both are true
-Implies:
"P implies Q" means that when ever P is true, then Q is true.
$P \to Q$:"If P, then Q"
Either P is false or Q is true.(Another definition of implication when it fails.)
$P \to Q$ just similar to notQ $\to$ notP from using true tables

$$\begin{cases} \text{If P is false, Q can be anything} \\ \text{If Q is true, P can be anything} \end{cases}$$

P	Q	not P	P and Q	P or Q	P → Q	Q → P	P if and only if Q
T	T	F	T	T	T	T	T
T	F	F	F	T	F	T	F
F	T	T	F	T	T	F	F
F	F	T	F	F	T	T	T

not(P or Q) is logically equivalent (not P and not Q)
If the assumption P is false, whatever Q is true or false, then P → Q is always True.

P	Q	not P	not Q	not P and not Q	P or Q	not (P or Q)
T	T	F	F	F	T	F
F	F	T	T	T	F	T

Definition:
e.g.f(x)= 3x+12
Theorem:
Some follow up properties that follows a definition, a interesting relationship
Proof:
A combination of **true** statemetns that combinates in the theorem

Example:
let x be an integer, if x is odd, then $x^2 + 1$ is even.
Proof(Pf): Since x is odd, there is an integar z
such that

$$x = 2z + 1$$

$$x^2 + 1 = (2z + 1)^2 + 1 = 4z^2 + 4z + 2$$

$$x^2 + 1 = 2(2z^2 + 2z + 1)$$

So that a number is even exactly if it can be written as 2w for some integar w
So $x^2 + 1$ is even
Basic steps:
Proof usually include 2 steps:
1. one direction eg x is odd, then there is an integar
2. opposit direction...

2 Jun 18

Activity 1:
For eery integer x, x is even(P) if and only if $\frac{x}{2}$ is an integer(Q)
* meaning that P $\rightarrow$ Q and Q $\rightarrow$ P
Proof:
$\Rightarrow$] Assuming x is an even integer, want to to show $\frac{x}{2}$ is an integer.
Since x is even, there is an integer y such that x = 2y
So, $\frac{x}{2} = \frac{2y}{2} = y$, and we know that y is an integer.
$\Leftarrow$] Suppose $\frac{x}{2}$ is an integer, we want x is even.
$x = 2 * (\frac{x}{2})$, so x is even by definition.

Activity 2:
For every integer x, x is even iff(if and only if) x+1 is odd
Proof:
$\Rightarrow$]: Assuming x is even, want to show x+1 is odd.
There is an integer y such that

$$x = 2y$$

this means that

$$x + 1 = 2y + 1$$

So x+1 is odd by definition.
$\Leftarrow$]: Suppose x+1 is odd, want to show x is even.
There is an integer k such that

$$x + 1 = 2k + 1$$

then

$$x = 2k$$

By definition, x is even.

Proof Techniques: (For statements at the form P $\rightarrow$ Q)
- Direct Proof: Start from assumption then use definitions/tricks/previous theorems/etc to get the conlcusion.
- Proof by cases: Break P into several cases (Must cover all cases)
- Contrapositive: Instead of proving P $\rightarrow$ Q, we prove not Q $\rightarrow$ not P
- Contradiction: Assume the theorem is false, and reach a contradiction; means that P is true and Q is false and you prove something nonsensical(e.g. 0=1)

example 1:
Suppose x and y are integers, if $x + y \geq 19$, then $x \geq 10$ or $y \geq 10$
Pf (by cases):
Suppose $x + y \geq 19$, want to show $x \geq 10$ or $y \geq 10$
Case 1: Assume $x \geq 10$. Done
Case 2: Assume $x < 10$. Since x is an integer, $x \leq 9$

$$x + y \geq 19$$

then

$$y \geq 19 - x$$

since $x \leq 9$, we have that

$$y \geq 19 - 9 = 10$$

Pf (By contrapositive): Assume x<10 and y<10, want to show that x+y<19
since x and y are integers, $x \leq 9$ and $y \leq 9$,

$$x + y \leq 9 + 9 = 18$$

so

$$x + y < 19$$

Pf (by contradiction):
Assume towards a contradiction that $x + y \geq 19$ and x<10 and y<10
As before, $x \leq 9$ and $y \leq 9$

$$19 \leq x + y \leq 18$$

So $19 \leq 18$, a contradiction

Take home activity:
Suppose a,b are positive real numbers, if $ab \geq 9$, then $a \geq 3$ or $b \geq 3$
1. What is not P?

$$ab < 9$$

2. What is not Q?

$$a < 3 \text{ and } b < 3$$

3. What is the contrapositive?

Assuming a<3 and b<3, want to show ab<9

4. Proof the theorem by cases contrapositive and contradiction.
- Contrapositive: If a<3 and b<3, then ab<9

Assume $a<3$ and $b<3$, so

$$ab < 3b < 3 \cdot 3 < 9$$

- Contradiction: Assume towards a contradiction that $ab \geq 9$, $a<3$ and $b<3$

  We know $ab \geq 9$ but $a \cdot b < 3 \cdot 3 < 9$

  So $9<9$, contradiction.

- Case:

  Case 1: $a \geq 3$, done

  Case 2: $a<3$, we want to show $b \geq 3$

$$b \geq \frac{9}{a} \geq 3$$

  So $b \geq 3$

General Tips:

If trying to prove an "or" statement, suppose one of the options is false and prove the other.

3 Jun 19

Axioms or Postulates: State properties of mathematical objects without a proof.
Definitions: Describe mathematical objects
Theorems: State properties of mathematical objects.

The integers are the elements of the set $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$
A set is a collection of objects.
If S is a set and x is in the set S, then we write $x \in S$.
If x is not in S, we write $x \notin S$

$\mathbb{Z}$ is the set of all integers
$\mathbb{R}$ is the set of all real numbers
Interval notation:

(a,b) is the set of all real numbers strictly greater than a and strictly lesser than b
* a $\notin$ (a,b) and b $\notin$ (a,b)
[a,b] is the set of all real numbers greater or equal to a and lesser or equal to b
* a $\in$ [a,b] and b $\in$ [a,b]
* there are no elements in (b,a)

4 Jun 20

4.1 Set(continue)

Def: The empty set is the set with no elements, we denote it by $\emptyset$
Note: Two sets A, B are equal iff they have the same elements.
Def: Let A, B be sets. If every elements of A is an element of B, we say that A is a subset of B and write A $\subseteq$ B
Example: [0,1] $\subseteq$ [-1,1] $\subseteq$ (-2,2) $\subseteq \mathbb{R}$; A $\subseteq$ A
Note:
A $\subseteq$ B means set A can be smaller or equal to set B
and A $\subsetneq$ B (or A $\subset$ B) means A is a subset of B but A $\neq$ B
Distinguish between (a,b) in coordinate and (a,b) as the interval

Building Sets:
$\rightarrow$ Builder notation:
Let $A \coloneqq \{1, 0, -1\}$ (we use $\coloneqq$ to introduce new notation)
Infinity Set: $\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$
$\{2, 4, 6, 8, 10\} = \{2, 4, 4, 6, 10, 8\}$

Subset Notation:

$$[a, b] = \{x \in \mathbb{R} | a \leq x \text{ and } x \leq b\}$$

$$\text{e.g. Even integer } = \{m \in \mathbb{Z} | m = 2k \text{ for some } k \in \mathbb{Z}\}$$

Operations on Sets:
-Union:
Let A,B be sets, Then

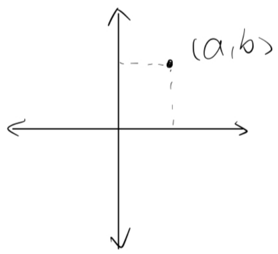
$$A \cup B \coloneqq \{x | x \in A \text{ or } x \in B\}$$

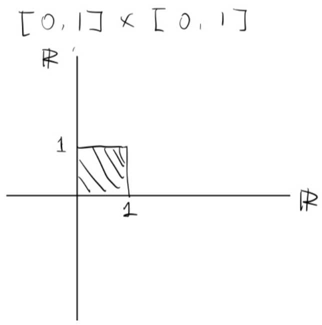
-Intersection:

$$A \cap B \coloneqq \{x | x \in A \text{ and } x \in B\}$$

e.g. $(0, 2) \cap (1, 3) = (1, 2)$; $(0, 2) \cup (1, 3) = (0, 3)$
-(Cartesian) Product:
* $\mathbb{R} \times \mathbb{R}$ is the cartesian plane

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$





The upright figure can be expressed by :

$$[0,1] \times [0,1] = \{(a,b) \in \mathbb{R} \times \mathbb{R} | 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1\}$$

4.2 Natural Numbers

We call $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ the natural numebrs or positive integers.
Note: $0 \notin \mathbb{N}$
Peano Axioms:

(N1) $1 \in \mathbb{N}$

(N2) If n $\in \mathbb{N}$, then its succession n+1 $\in \mathbb{N}$

(N3) 1 is not the successor of any natural number

(N4) if n+1 = m+1. then n = m for any n,m $\in \mathbb{N}$

(N5) if $S \subseteq \mathbb{N}$ and satisfies the following conditons:

i. 1$\in \mathbb{S}$

ii. For all n $\in \mathbb{N}$, if n $\in$ S, then n+1 $\in$ S

then $\mathbb{N}$ = S

5 Jun 24

Mathematical Induciton:

If want to prove every natural number has some property, it is enough to show two things:

(i) 1 has the property (**Base Case**)

(ii) if n has the property, they n+1 has the property (**Inductive Step**)

Example: (Gauss) For any natural number n,

$$\sum_{i=1}^{n} i \coloneqq 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Proof: By induction

Base case: n=1

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{2} = 1$$

so we are done

Inductive Step: Let n $\in \mathbb{N}$

Assume that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

(Inductive Hypothesis)

We want to show

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$$

$$\begin{aligned}\sum_{i=1}^{n+1} i &= \sum_{i=1}^{n} i + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

$\square$

Well-Ordering Principle(WOP):

If S $\subseteq \mathbb{N}$ and is nonempty, then it has a least element

(WO) $\to$ (N5) by contrapositive:

not(N5): there is S $\in \mathbb{N}$ such that (i) 1 $\in$ S and (ii) if n $\in$ S, then n+1 $\in$ S. But S $\neq \mathbb{N}$

not(WO): there is R $\subseteq \mathbb{N}$ nonempty that does not have a least element *Proof*: Let S $\subseteq \mathbb{N}$ be such that 1 $\in$ S, if n $\in$ S, then n+1 $\in$ S and S $\neq \mathbb{N}$.

We want to find a nonempty set R $\subseteq \mathbb{N}$ with no least element

$$R \coloneqq \{n \in \mathbb{N} | n \notin S\} = S^c$$

Assume towards a contradiction that R has a least element, call it m.
Since m is the least element in $\mathbb{R}$, $m - 1 \notin \mathbb{R}$
So, $m = 1 \in S$
But by hypothesis, $m = (m-1) + 1 \in S$.
Thus, $m \in S$ and $m \notin S$, a contradiction
So, R is nonempty but doesn't have a least element. $\square$

6 Jun 25

Now we want to show that (N5) implies (WO)

Proof: Let S $\subseteq \mathbb{N}$ nonempty

For any natural number n, if S contains a natural number k with k $\leq$ n, then S has a least element.

By induction:

Proof of claims

Base Case: We want to prove that if S contains a natural number k $\leq$ 1, then S has a least element.

Assume there is k $\in$ S with k $\leq$ 1,

So k = 1 $\in$ S.

Since 1 is the least natural number, S has a least element.

Inductive Step: Let n $\in \mathbb{N}$,

IH(Inductive Hypothesis): If there is k $\leq$ n such that k $\in$ S, then S has a least element

Suppose that there is a number j $\leq$ n+1, with j $\in$ S, we want to show that S has a least element.

Case 1: There is some k $\leq$ n with k $\in$ S.

By inductive hypothesis, S has a least element.

Case 2: Every number k $\leq$ n is such that k $\notin$ S

By assumption, j = n+1 $\in$ S and it is the least element of S

End of proof of claim

Since S is nonempty, it has an element called m.

By the claim, we know that if S contains an element k $\leq$ m , then S has a least element.

Since m $\leq$ m, and m $\in$ S, then S has a least element. $\square$

6.1 Rational numbers

$$\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$$

The set of rational numbers is

$$Q = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \text{ and } n \neq 0\}$$

Not every real number is a rational number, e.g. $\sqrt{2}, \pi, e$

Lemma 6.1. *Every fraction $\frac{m}{n}$ with m,n $\in \mathbb{Z}$ has a simplified form. That is, we can write it as a fraction where m and n are as small as possible and only m is allowed to be negative.*

Proof: (By cases)

Case 1: $\frac{m}{n} = 0$

Then $\frac{0}{1}$ is the simplified form

Case 2: $\frac{m}{n}$ is positive
By the WOP(Well-Ordering-Principle),

$$A := \{j \in \mathbb{N} | \text{ There is some } k \in \mathbb{N} \text{ with } \frac{j}{k} = \frac{m}{n}\}$$

has a least element called it m_0
Now, again by the WOP,

$$\{k \in \mathbb{N} | \frac{m_0}{k} = \frac{m}{n}\}$$

has a least element. (Notice: it is nonempty because $m_0 \in A$) call the least element n_0
Then $\frac{m_0}{n_0} = \frac{m}{n}$ and m_0, n_0 are as small as possible.
Case 3: $\frac{m}{n}$ is negative
We have that $\frac{-m}{n}$ is positive.
By Case 2, there are m_0 and $n_0 \in \mathbb{N}$ such that $\frac{-m}{n} = \frac{m_0}{n_0}$ with m_0 and n_0 as small as possible.
Now, $\frac{-m_0}{n_0} = \frac{m}{n}$ and it is the simplied form

$\square$

7 Jun 26

Theorem: $\sqrt{2}$ is not rational
Proof: Assume towards a contradiction that $\sqrt{2} \in \mathbb{Q}$. That is there are $m, n \in \mathbb{Z}$ with $n \neq 0$ so that $\sqrt{2} = \frac{m}{n}$
Without loss of generality, assume $\frac{m}{n}$ is the simplified form. (because of Lemma)
Squaring both sides we get

$$2 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

then

$$2n^2 = m^2$$

By definition m^2 is even. By homework, m is even.
That is there is some integer k such that $m = 2k$
So

$$2n^2 = m^2 = (2k)^2 = 4k^2$$

Dividing both side by 2,

$$n^2 = 2k^2$$

So n^2 is even. Again, n is even by homework.
By definition, there is an integer j such that $n = 2j$
Notice that $j \neq 0$ becasue $n \neq 0$, then

$$\frac{m}{n} = \frac{2k}{2j} = \frac{k}{j}$$

If we show that $\frac{k}{j}$ is more simplified than $\frac{m}{n}$, we get a contradiction.
But we know that $k = \frac{m}{2}, j = \frac{j}{2}$ and dividing by 2 gives a number closer to 0.
Since $\frac{k}{j}$ is more simplified, so there is a contradiction. $\square$

Activity: $\sqrt{5}$ is not rational but $\sqrt{4}$ is rational
Proof: ($\sqrt{5}$) Assume towards a contradiction that $\sqrt{5} \in \mathbb{Q}$. That is there are $a, b \in \mathbb{Z}$ with $b \neq 0$ so that $\sqrt{5} = \frac{a}{b}$
Without loss of generality, assume $\frac{a}{b}$ is the simplified form. Squaring both side we get

$$5 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

then

$$m^2 = 5n^2$$

So that there is some integer c such that $m = 5c$

So

$$5n^2 = m^2 = (5c)^2 = 25c^2$$

$$n^2 = 5c^2$$

Which means that there is an integer d such that $n = 5d$
With knowing $d \neq 0$ since $n \neq 0$, then

$$\frac{m}{n} = \frac{5a}{5b} = \frac{a}{b}$$

Since $a = \frac{m}{5}$ and $b = \frac{n}{5}$, it means that $\frac{a}{b}$ is more simplified than $\frac{m}{n}$ so a contradiction □
Proof:($\sqrt{4}$) Assume $m, n \in \mathbb{Z}$ and $\sqrt{4} = \frac{m}{n}$. Squaring both side we get

$$4 = \frac{m^2}{n^2}$$

then

$$m^2 = 4n^2 = (2n)^2$$

This means that there is an integer a such that $m = a$
So

$$(2n)^2 = m^2 = a^2$$

Which means that

$$2n = a$$

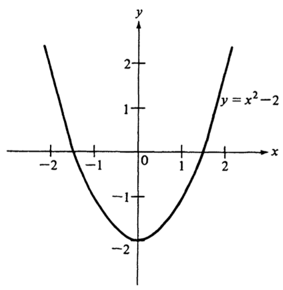
$$n = \frac{a}{2}$$

So $\sqrt{4}$ can be expressed in the form

$$\sqrt{4} = \frac{m}{n} = \frac{a}{\frac{a}{2}} = 2$$

Which indicates that $\sqrt{4}$ is a rational number. □

7.1 The Real Numbers



Axioms of an ordered field:Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields

- Addition Axioms: For all a, b, c,

(A1) $a + (b + c) = (a + b) + c$ (Associativity)

(A2) $a + b = b + a$ (Commutativity)

(A3) $a + 0 = a$ (Identity)

(A4) There is a number $-a$ such that $a + (-a) = 0$ (Inverse)

- Multiplication Axioms: For all a, b, c,

(M1) $a(bc) = (ab)c$

(M2) $ab = ba$

(M3) $a \cdot 1 = a$

(M4) If $a \neq 0$, there is a number a^{-1} such that $a \cdot a^{-1} = 1$

(DL) $a(b + c) = ab + ac$ (Distributivity)

- Order Axioms: For all a, b, c

(O1) either $a \leq b$ or $b \leq a$ (Dichotomy)

(O2) if $a \leq b$ and $b \leq a$, then $a = b$ (Antisymmytry)

(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$ (transitivity)

(O4) if $a \leq b$, then $a + c \leq b + c$

(O5) if $a \leq b$ and $c \geq 0$, then $ac \leq bc$

- Completeness Axiom:

(CA) Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound

8 Jun 27

Activity: prove as many as possible

For all $a, b, c \in \mathbb{R}$

(i) $a + c = b + c$ implies $a = b$

(ii) $a \cdot 0 = 0$

(iii) $(-a)b = -(ab)$

(iv) $(-a)(-b) = ab$

(v) if $c \neq 0$ and $ac = bc$, then $a = b$

(vi) if $ab = 0$, then $a = 0$ or $b = 0$

(vii) if $a \leq b$, then $-b \leq -a$

(viii) if $a \leq b$ and $c \leq 0$, then $ac \geq bc$

(ix) if $a \geq 0$ and $b \geq 0$, then $ab \geq 0$

(x) $a^2 \geq 0$

(xi) 0<1

(xii) if $0 < a$ then $0 < a^{-1}$

(xiii) if $0 < a < b$, then $0 < b^{-1} < a^{-1}$

Proof:
(i): Adding $-c$ on both side, we can get

$$a + c + (-c) = b + c + (-c)$$

Using A1, there is

$$a + (c + (-c)) = b + (c + (-c))$$

Using A4, we can get

$$a + 0 = b + 0$$

which indicates $a = b$
(ii): Based on DL, there is equation

$$a \cdot 0 + a \cdot 0 = a(0 + 0)$$

then

$$a \cdot 0 + a \cdot 0 = a(0 + 0) = a \cdot 0$$

Applying theory (i) to this equation, assuming $c = a \cdot 0$, we can get

$$a \cdot 0 = 0$$

(iii): From A4 and theory (ii), we can write the equation

$$(-a + a)b = 0$$

Using DL, there is

$$(-a)b + ab = 0$$

Adding $-ab$ on both side, we can get

$$(-a)b + ab + (-ab) = 0 + (-ab)$$

Based on associative property of addition (A1),

$$(-a)b + (ab + (-ab)) = 0 + (-ab)$$

Using commutative property of addition (A4),

$$(-a)b + 0 = 0 + (-ab)$$

Using A3 and M1,

$$(-a)b = (-ab) = -(ab)$$

(iv): Using equation from theory (iii) and adding $(-a)(-b)$ on both side, we can get

$$-ab + (-a)(-b) = (-a)b + (-a)(-b)$$

With Distributive axiom, there is

$$-ab + (-a)(-b) = (-a)(b + (-b))$$

Using inverse for addition and theory (ii),

$$-ab + (-a)(-b) = (-a) \cdot 0 = 0$$

Adding ab on both side, we can get

$$ab + (-ab) + (-a)(-b) = ab + 0$$

Using A4 and Identity for addition (A3),

$$0 + (-a)(-b) = ab + 0$$

$$(-a)(-b) = ab$$

(v): Since $c \neq 0$, c^{-1} exist.

Multiplying c^{-1} on both side, we can get

$$ac \cdot c^{-1} = bc \cdot c^{-1}$$

Using inverse for multiplication, we can get

$$a \cdot 1 = b \cdot 1$$

Using identity for multiplication,

$$a = b$$

(vi):
Case 1: $a = 0$, Done
Case 2: $a \neq 0$

$$(a \cdot a^{-1})b = a^{-1} \cdot 0$$
$$b = 0$$

(vii):

$$a + c < b + c$$

Assume $c = -a - b$

$$a + c < b + c$$
$$a + (-a - b) < b + (-a - b)$$
$$(a - a) - b < -a + (b - b)$$
$$-b < -a$$

(viii):
Since $c \leq 0$, it means $-c \geq 0$.
Using (vii),there is when $a \leq b$, $-b \leq -a$. So

$$-b \cdot (-c) \leq -a \cdot (-c)$$

Which indicates

$$ac \geq bc$$

(ix): multiply both side by b
(x): Using result from (ix), assume $b = a$ in this case.
Then there is

$$a \cdot a \geq 0$$

So

$$a^2 \geq 0$$

(xi):

$$a < a + 1$$
$$a - a < a + 1 - a$$
$$0 < 1$$

(xii):

$$0 < 1$$
$$0 < a \cdot a^{-1}$$
$$0 \cdot a^{-1} < a \cdot a^{-1} \cdot a^{-1}$$
$$0 < (a \cdot a^{-1}) \cdot a^{-1}$$
$$0 < 1 \cdot a^{-1}$$
$$0 < a^{-1}$$

(xiii): let $c = a^{-1}b^{-1}$ from (xii) we can know $c > 0$

$$0 \cdot c < a \cdot c < b \cdot c$$
$$0 < a \cdot a^{-1}b^{-1} < b \cdot a^{-1}b^{-1} \text{ (ii)}$$
$$0 < 1 \cdot b^{-1} < a^{-1} \cdot 1$$
$$0 < b^{-1} < a^{-1}$$

4/5/11/13 choose practice proof

9 Jul 1

9.1 Absolute Value

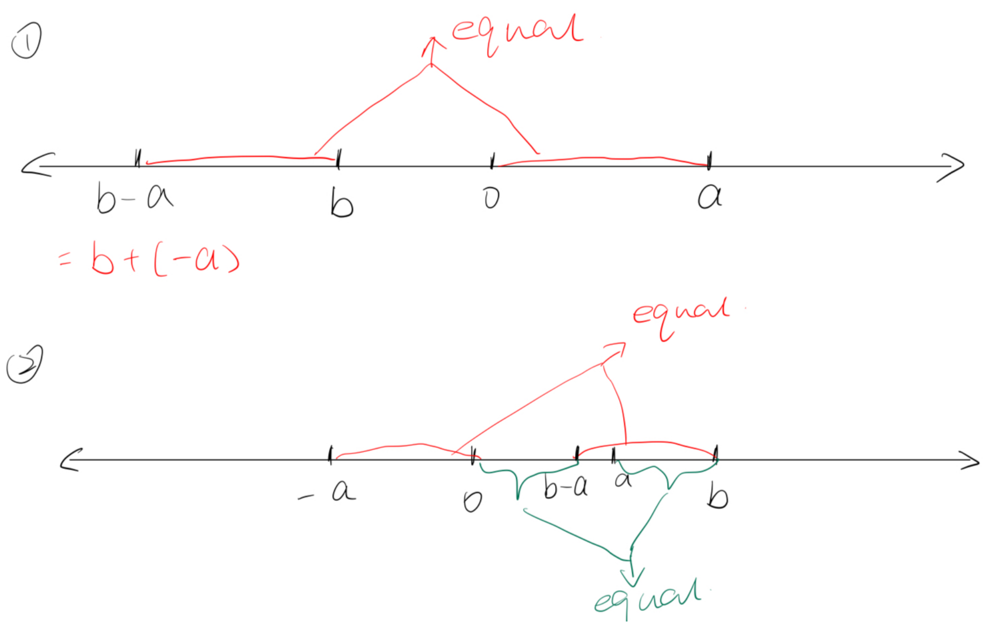
Def: Let $a \in \mathbb{R}$, the absolute value of a, denoted by $|a|$ is

$$\begin{cases} \text{If } a \geq 0: \text{ then } |a| = a \\ \text{If } a < 0: \text{ then } |a| = -a \end{cases}$$

$|a|$ can be considered as the distance from a to 0 in the number line.

Def: For all $a, b \in \mathbb{R}$, the distance between a and b is

$$dist(a, b) := |b - a| = |a - b|$$



Theorem: let $a, b \in \mathbb{R}$

(i) $|a| \geq 0$

Proof: Case 1: $a \geq 0$, By definition, $|a| = a \geq 0$

Case 2: $a < 0$, By definition, $|a| = -a$

Also since $a < 0$, then adding $-a$ to both sides, under O4,

$$a + (-a) < 0 + (-a)$$

By A3 and A4,

$$0 < -a$$

Thus,

$$0 < |a|$$

$\square$

(ii) $|ab| = |a| \cdot |b|$
Proof: Case 1: Assuming $a \geq 0$ and $b \geq 0$:
By definition, $|a| = a, |b| = b$. So

$$|a| \cdot |b| = ab$$

Also since $a \geq 0$ and $b \geq 0$, $ab \geq 0$
So

$$|ab| = ab$$

by definition. Done
Case 2:Assuming $a \geq 0$ and $b < 0$:
By definition, $|a| = a, |b| = -b$, By theory (iii) on Thursday's activity

$$|a| \cdot |b| = a \cdot (-b) = -(ab)$$

For $|ab|$ to be equal to $-(ab)$, it would have to be the case that $ab \leq 0$,
Since $b < 0$, then $b \leq 0$, also $a \geq 0$. By O5, we have

$$ab \leq a \cdot 0 = 0$$

So

$$a \cdot b \leq 0$$

Then $|ab| = -ab$ by definition and we are done
Case 3:Assuming $a < 0$ and $b \geq 0$:
Similar to the proof in Case 2, just switching a, b.
Case 4:Assuming $a < 0$ and $b < 0$:
By definition, $|a| = -a, |b| = -b$. So by theory (iv), there is

$$|a| \cdot |b| = (-a)(-b) = ab$$

Also since $a < 0$ and $b < 0$, by theory (iv) and (vi), $ab = (-a)(-b) > 0$
So

$$|ab| = ab$$

$\square$

(iii) The triangle of inequality $|a + b| \leq |a| + |b|$

Proof: Since $\begin{cases} |a| = a \text{ when } a \geq 0 \\ |a| = -a \text{ when } a < 0 \end{cases}$ and $\begin{cases} |b| = b \text{ when } b \geq 0 \\ |b| = -b \text{ when } b < 0 \end{cases}$

There is $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$

Based on O4, there is

$$a + b \leq |a| + b \leq |a| + |b|$$

$$-(|a| + |b|) \leq -|a| + b \leq a + b$$

So

$$-(|a| + |b|) \leq a + b \leq |a| + |b|$$

Since $|a + b|$ is equal to either $a + b$ or $-(a + b)$, we can condluce that

$$|a + b| \leq |a| + |b|$$

Case 1: $a + b \geq 0$

$$|a + b| = a + b \leq |a| + |b|$$

Case 2: $a + b \leq 0$

$$|a + b| = -(a + b)$$

Since $-(|a| + |b|) \leq a + b$,by act and $-(-c) = c$, we have that

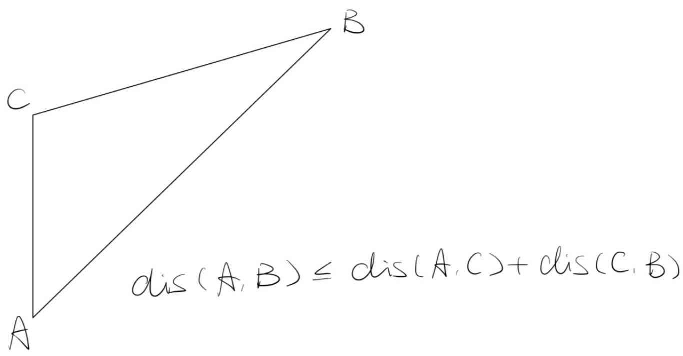
$$|a + b| = -(a + b) \leq |a| + |b|$$

$\square$

10 Jul 2

Or prooving the theory that $a \leq b$ and $c \leq d$, then $a + c \leq b + d$

Proof: Exercise $\square$



Theorem of triangle: let $a, b, c \in \mathbb{R}$, then

$$dist(a, b) \leq dist(a, c) + dist(c, b)$$

equivalent

$$|b - a| \leq |c - a| + |b - c|$$

Proof: Assuming $x = b - c, y = c - a$, then

$$x + y = b - c + (c - a) = b - a$$

So by triangle inequality, $|x + y| \leq |x| + |y|$, in other words,

$$|b - a| \leq |c - a| + |b - c|$$

$\square$

The Completness Axiom:

Every nonempty subset for $\mathbb{R}$ that is bounded above has a least upper bound.

Def: Let $S \subseteq \mathbb{R}$ nonempty

(i) If $s \in S$ is such that for any $r \in S$ and $s \leq r$, we say that s is the minimum of S, denoted by

$$s = \text{min}S$$

(ii) If $s \in S$ is such that for any $r \in S$ and $r \leq s$, we say that s is the maximum of S, denoted by

$$s = \text{max}S$$

Examples:

(i) Nonempty finite Sets have both max and min

(ii) The open interval $(a, b) = \{x \in \mathbb{R} | a < x < b\}$ does not have min or max

Note: min and max **Must** be in the set.
(iii) The closed interval $[a, b]$ has

$$\begin{cases} b = \max[a, b] \\ a = \min[a, b] \end{cases}$$

(iv) $\mathbb{N}$:

$$\min\mathbb{N} = 1, \text{ but has no max}$$

Def: Let $S \subseteq \mathbb{R}$ nonempty,
(i) we say that S is bounded above if there is $r \in \mathbb{R}$ suc htaht for all $s \in S$,

$$s \leq r \text{ and we say r is an upper bound of S}$$

(ii) we say that S is bounded below if there is $r \in \mathbb{R}$ such taht for all $s \in S$,

$$r \leq s \text{ and we say r is an lower bound of S}$$

Prop: If S has a maximum, then it is bounded above.
If S has a min, then it is bounded below.
Exercise of proof: Examples:
(i) $(0, 1)$ does not have a max or a min but is is bounded above and below.
Def: A set is bounded if it is bounded above and bounded below
(ii): $A := \{q \in \mathbb{Q} | 0 \leq q\}$
$\min A = 0$ it is bounded below
has no max, it is not bounded above.
Def: Let $S \subseteq \mathbb{R}$ nonempty
(i) consider the set $R = \{x \in \mathbb{R} \mid \text{x is an upper bound of } S\}$
if R has a min, we call that element the supremmm of S (least upper bound) and denote it $\sup S$
(ii): $Q = \{x \in \mathbb{R} \mid \text{x is a lower bound of S}\}$
if Q has a max, we call it the infimum of S $\inf S$

11 Jul 8

Def: Let $S \subseteq \mathbb{R}$ be nonempty,
(a): If S is bounded above and has a Least Upper Bound (LUB), then we call the LUB of S the supermum of S and denote it by supS.
(b): If S is bounded below and has a Greatest Lower Bound (GLB), then we call the GLB of S the infimum of S and denote it by infS.

Exercise:

$$\cdot A = \{\frac{1}{n^2} : n \in \mathbb{N} \text{ and } n \geq 3\}$$

$\sup A = \frac{1}{9}$, $\inf A = 0$.
$\dfrac{1}{n^2}$ is a decreasing funciton.

$$\cdot B = \{r \in \mathbb{Q} : r^3 \leq 7\} = \{r \in \mathbb{Q} : r \leq \sqrt[3]{7}\}$$

$\sup B = \sqrt[3]{7}$, no $\inf S$

$$\cdot C = \{m + n\sqrt{2} : m, n \in \mathbb{Z}\}$$

no $\sup C$, no $\inf C$, since it set is not bounded

Note: $S \subseteq T \subseteq \mathbb{R}$ not empty, then

$$\sup S \leq \sup T$$

$$\cdot D = \{x \in \mathbb{R} : x^2 < 10\} = (-\sqrt{10}, \sqrt{10})$$

$\sup D = \sqrt{10}$, $\inf D = -\sqrt{10}$

$$\cdot E = \{x \in \mathbb{R} : x^2 + 1 = 0\}$$

no $\sup E$ nor $\inf E$, since it is a empty set

$$\cdot F = \{x \in \mathbb{R} : \forall n \in \mathbb{N}, 0 < x < \frac{1}{n}\} = \emptyset$$

Empty.

$$\cdot G = \{r \in \mathbb{Q} : r^2 - 2 = 0\} = \emptyset$$

Empty.

Completness Axiom: Every $S \subseteq \mathbb{R}$ which is nonempty and bound above has a least upper bound, in other word, supremum exists in $\mathbb{R}$
- Remark: Let $S \subseteq \mathbb{R}$ which nonempty and bounded above. Then $M = \sup S$ iff the following two conditios are met:
(i): For all $s \in S$, $s \leq M$. $\longleftarrow$ M is an upper bound of S

(ii): If $m < M$, then there is $s \in S$ sit $m < s$. $\longleftarrow$ Nothing less than M is an upper bound for S

12 July 9

Corollary 4.5: Every subset S of R which is nonempty and bounded below has an infimum in R (but not necessarily in S)

$$L = \{r \in \mathrm{R} : r \text{ is a lower bound for } S\}$$

(one proof is in textbook, below is another proof)
we know
(1): L is nonempty (since S is bounded below and S has a lower bound)
(2): Claim: s is an upper bound for L. Take $r \in L$, so r is less than any element of S. Inparticular $r < s$. So s is an upper bound for L.
So L has a supremum which we call M.

Claim: $M = \sup L = \inf S$, if we want to prove this, we need to show:
Proof: **(1)** M is a lower bound for S.
Towards a contradiction, suppose that M is not a lower bound for S, then there is $m \in S$ such that $m < M$.
since $m < M$, m is not an upper bound for L.
So there is an element $r \in L$ such that $m < r$. Note that r is a lower bound for S. So it should be the case that $r \leq m$. Contradiction.
(2) Nothing greater than M is a lower bound for S.
TAC: suppose there is $r > M$ which is also a lower bound for S.
Since $r \in L$ and $M = \sup L$, so $r \leq M$. Contradiction.
So M is a greatest lower bound for S, that is $M = \inf S$ □

(i): If $a > 0$, then $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < a$
(ii): If $b > 0$, then $\exists m \in \mathbb{N}$ s.t. $b < m$
The Archimedean Property 4.6: If $a, b > 0$, then $\exists n \in \mathbb{N}$ s.t. $na > b$.
Proof: TAC suppose that for all $n \in \mathbb{N}$, $na \leq b$, we will study the set

$$S = \{na : n \in \mathbb{N}\}$$

(1) S is bounded above by b.
(2) $a \in S$ so S is not empty.
So by the completeness axiom, S has a supremum we call M.
By $0 < a < 2a \leq M \Rightarrow 0 < a < M$.

$$M - a < M$$

So $M - a$ si not an upper bound for S. There is $m \in S$ s.t. $M - a < m$.
There is $k \in \mathbb{N}$ s.t. $M - a < ka$.

$$M < ka + a = (k+1)a \in S$$

This contradicts the fact that M is the supremum of S. □

If I have F an ordered field NA, then there is $b \in F$ so that $na < b$ for all $n \in \mathbb{N}$.
The rationals are dense in the reals (4.7),
If $a < b$ for $a, b \in \mathbb{R}$, then $\exists r \in \mathbb{Q}$ s.t. $a < r < b$.
By AP, $\exists n \in \mathbb{N}$ s.t. $n(b-a) > 1$.

$$
\begin{aligned}
&b - a > 0 \\
&na < nb \\
&na < m < nb \\
&a < \frac{m}{n} < b
\end{aligned}
$$

13 July 10

The density of the rationals in the reals: For any $a, b \in \mathbb{R}$ such that $a < b$, $\exists r \in \mathbb{Q}$ such that $a < r < b$.
(Between any two distinct real numbers, there is a rational number).

Showing that there are integers $m, n (n \neq 0)$ such that $na < m < nb$. If we succed in showing this, then $a < \frac{m}{n} < b$ and $\frac{m}{n} \in \mathbb{Q}$.
Proof: Because $a < b, 0 < b - a$, By AP, $\exists n \in \mathbb{N}$ s.t. $1 < n(b-a)$
So

$$na + 1 < nb$$

Simplifying assumption $na \notin \mathbb{Z}$, since if $na \in \mathbb{Z}$, then $na < na + 1 < nb$, DNE.
Condiser $S = \{N \in \mathbb{Z} : N < na\}$
Claim: S has a maximum.
Case 1. $0 < na$: By AP(ii), $\exists m \in \mathbb{N}$ s.t. $na < m$
 -$na \neq 0$ since na is not a integer but 0 is a integer.
$M = \{m \in \mathbb{N} : na < m\}$ is nonempty, by WOP, M has a least element we call m.
So $m - 1 < na < m$ (since m is the least element in M, $m - 1$ does not belong to M so it is less than na.)
 -no integer less than m is greater than na.
$m - 1$ is the largest integer less than na, $m - 1$ is the maximum of S.
Case 2. $na < 0$.
$0 < -na$. Now essentially in the first case, thus $\exists m \in \mathbb{N}$ s.t. $m - 1 < -na < m$

$$-m < na < -m + 1$$

$-m$ is the maximum of S.
So S has a maximum N

$$N < na < N + 1$$

Since $N + 1 < na + 1 < nb$ (in the second line)
So $N < na + 1 < N + 1 < nb$ □

Let $A, B \subseteq \mathbb{R}$. Define

$$A + B = \{a + b \in \mathbb{R} : a \in A, b \in B\}$$

Claim: Let $A, B \subseteq \mathbb{R}$ nonempty and bounded above. The $\sup(A + B)$ exists and is euqla to $\sup A + \sup B$
Proof: (1): $A + B$ is nonempty: $\exists a \in A, \exists b \in B, a + b \in A + B$
$A + B$ is bounded above: let $M_A = \sup A, M_B = \sup B$
(2): Take $c \in A + B$, $\exists a \in A, \exists b \in B$ s.t. $c = a + b \leq M_A + M_B$

$M_A + M_B$ is an upper bound for A_B,
$\sup A + \sup B$ is an upper bound for $A + B$
(3): $\sup A + \sup B$ is the least such upper bound for $A + B$ (any thing less than $\sup A + \sup B$ is not an upper bound for $A + B$)
Suppose $d < \sup A + \sup B$.

$$d - \sup B < \sup A$$

$\exists a \in A$ s.t.

$$d - \sup B < a$$

$$d - a < \sup B$$

$\exists b \in B$ s.t. $d - a < b \iff d < a + b \in A + B$
So $\sup A + \sup B = \sup(A + B)$
Note:
$M = \sup A$iff
(1): M is an upper bound for A
(2): Whenever $m < M$, $\exists a \in A$ s.t. $m < a$ $\square$

Assignment: Let A, B be nonempty subsets of $\mathbb{R}$ bounded below. Show that $\inf(A \cup B)$ exists and is euqal to $\min\{\inf A, \inf B\}$

note: use greates lower bound not least upper bounds

14 July 11

Another proof for "Ratinals are dense in he reals"
Proof:

$$n.a_1a_2\dots a_m\bar{o} \text{ is a decimal and equals to } \frac{n.a_1a_2\dots a_m \times 10^m}{10^m}$$

$a < b$

$$\begin{aligned} a &= a_1a_2\cdots a_{i-1}a_ia_{i+1}\dots\bar{0} \\ c &= a_1a_2\cdots a_{i-1}a_i(a_{i+1}+1)\dots\bar{0} \\ b &= b_1b_2\cdots b_{i-1}b_ib_{i+1}\dots\bar{0} \end{aligned}$$

means that

$$a < c < b$$

Digits are always naturals between 0 and 9.
What happens if $a_{i+1} = 9$?
Then move right by one (or until you find a digit of a which is less than 9)
What if I never find such as digit?

$$\begin{aligned} 1 &= 0.\bar{9} = 0.9 + 0.09 + 0.009 \\ &= \end{aligned}$$

$$1 = 0.\bar{9} = 0.9 + 0.09 + 0.009$$

□

14.1 Unbounded sets of R

$$\mathbb{R} \cup \{-\infty, \infty\}$$

$$-\infty \le a \le \infty \text{ for all } a \text{ from this new set.}$$

-For $a \neq -\infty$,

$$a + \infty = \infty + a = \infty$$

-For $a \neq \infty$,

$$a + (-\infty) = -\infty + a = -\infty$$

-Not able to define $\infty + (-\infty)$ and $-\infty + \infty$

Closed unbounded above,

$$[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$$

Closed unbounded below,

$$(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$$

Open unbounded above,

$$(a, \infty) = \{x \in \mathbb{R} : a < x\}$$

open unbounded below,

$$(-\infty, a) = \{x \in \mathbb{R} : x < a\}$$

Def: Let $S \subseteq \mathbb{R}$ be nonempty.
(1) If S is unbounded above, $\sup S = \infty$
(2) If S is unbounded below, $\inf S = -\infty$
Rare to say: For $\{\}, \sup\{\} = -\infty, \inf\{\} = \infty$

Proposition: Let $A, B \subseteq \mathbb{R}$ be nonempty, then

$$\sup(A + B) = \sup(A) + \sup(B)$$

Proof: If A and B are both bounded above, then we are done by the work we did yesterday.
So suppose that one of A or B is unbounded above, WLOG (without loss of generality), let it be A.

$$\sup A = \infty$$

Claim: $A + B$ is unbounded above
Since B is nonempty, fix $b \in B$. let $r \in \mathbb{R}$. Then $r - b \in \mathbb{R}$.
Since A is not boundd above, there is $a \in A$ s.t.

$$r - b < a \iff r < a + b \in A + B$$

So r is not an upper bound for $A + B$ and this argument holds for every $r \in \mathbb{R}$

$$\sup(A + B) = \infty = \sup A + \sup B$$

The only possibility of $\sup B$ is ∞ or a real. $\square$

15 July 15

Sequences:

Def: A sequence is a function $s : \mathbb{N} \to \mathbb{R}$

$$s(n) = s_n$$

We write this sequence as $(S_n)_{n\in\mathbb{N}}, (S_n)_{n=1}^{\infty}$
example: $S_n = \frac{1}{n^2}$,

$$\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}} = \left(1, \frac{1}{2}, \frac{1}{9}, \frac{1}{16}, \cdots\right)$$

Note: Sequences are not sets, order matters.

$$\begin{cases} (a_n) = (-1)^n = (-1, 1, -1, 1, -1, \cdots) \\ (b_n)_{n\in\mathbb{N}} = (-1, 1, 1, 1, \cdots) \end{cases} \text{are not the same}$$

We say that $(S_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ are equal when

$$s_n = r_n \text{ for all } n \in \mathbb{N}$$

Def: A sequence $(s_n)_{n\in\mathbb{N}}$ converges to s if all $\epsilon > 0$ there is $N \in \mathbb{R}$ such that for all $n > N$, we have

$$|s_n - s| < \epsilon$$

we can say that

$$\lim_{n\to\infty} s_n = s$$

Property:

$$\lim_{n\to\infty} = \frac{1}{n^2} = 0$$

Proof: Let $\epsilon > 0$, take

$$N = \frac{1}{\sqrt{\epsilon}} \text{ for } n > N$$

Let $n > N$, then

$$n > \frac{1}{\sqrt{\epsilon}}$$

So

$$n^2 > \frac{1}{\epsilon}$$

Hence

$$\epsilon > \frac{1}{n^2} = |\frac{1}{n^2} - 0|$$

which is what we wanted to prove □

Prop:

$$\lim_{n\to\infty}\frac{3n+1}{7n-4}=\frac{3}{7}$$

$$\frac{21n+7-21n+12}{7(7n-4)}<\epsilon$$

$$\frac{19}{7(7n-4)}<\epsilon$$

$$n>\frac{\frac{19}{7\epsilon}+4}{7}=N$$

Proof: Let $\epsilon>0$, take

$$N=\frac{\frac{19}{7\epsilon}+4}{7}$$

Let $n>N$, then

$$n>\frac{\frac{19}{7\epsilon}+4}{7}$$

So

$$7n>\frac{19}{7\epsilon}+4$$

Then

$$7n-4>\frac{19}{7\epsilon}$$

Then

$$7\epsilon>\frac{19}{7n-4}$$

Hence

$$\epsilon>\frac{19}{7(7n-4)}$$

Hence

$$\epsilon>|\frac{21n+7-21n+12}{7(7n-4)}|$$

Hence

$$\epsilon>|\frac{3n+1}{7n-4}-\frac{3}{7}|$$

□

16 July 16

Theorem: Let $(s_N)_{n\in\mathbb{R}}$ be a convergent sequence with limit $s \neq 0$ and such that $\mathbf{s_n \neq 0}$ for all $n \in \mathbb{N}$. Then

$$\inf\{|s_n| \mid n \in \mathbb{N}\} > 0$$

example:
(1): $s_n = \frac{1}{n^2}, \lim s_n = 0$

$$A \coloneqq \{|s_n| \mid n \in \mathbb{N}\} = \{\frac{1}{n^2} \mid n \in \mathbb{N}\}$$

$$\inf A = 0$$

(2): $r_n = 1 - \frac{1}{n+1}, \lim r_n = 1$

$$B \coloneqq \{|s_n| \mid n \in \mathbb{N}\} = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$$

$$\inf B = \min B = \frac{1}{2}$$

*This satisty the theorem that infimum can be close to 0 but not actually equal to 0.
Proof: Let $\epsilon = \frac{|s|}{10} > 0$
There is $N \in \mathbb{N}$ such taht for all $n > N$,

$$|s_n - S| < \frac{|s|}{10}$$

we want to show that

$$|s_n| \geq \frac{9|s|}{10} > 0$$

*Trick:

$$|s| = |s - s_n + s_n| \leq |s - s_n| + |s_n| < \frac{|s|}{10} + |s_n|$$

$$|s| < \frac{|s|}{10} + |s_n|$$

$$|s| - \frac{|s|}{10} < |s_n|$$

then

$$\frac{9|s|}{10} < |s_n|$$

Let

$$m = \min\{|s_1|, |s_2|, \cdots, |s_{_N}|, \frac{9|s|}{10}\}$$

$m > 0$ because each $s_n \neq 0$ and $\frac{9|s|}{10}$.
m is a lower bound for $\{|s_n| \mid n \neq \mathbb{N}\}$
Thus, by definition,

$$0 < m \leq \inf\{|s_n| \mid n \in \mathbb{N}\}$$

□

<u>Def</u>: A sequence $(s_n)_{n\in\mathbb{N}}$ is bounded if $\{S_n|n \in \mathbb{N}\}$ is bounded. That is, if there is $M \in \mathbb{R}$ such that

$$|s_n| \leq M \text{ for all } n \in \mathbb{N}$$

<u>Theorem</u>: Convergent Sequences are bounded. Let $(s_n)_{n\in\mathbb{N}}$ and $s \in \mathbb{R}$ such that $\lim s_n = s$
Proof: Let $\epsilon = 5$(any positive number will work)
Since $s_n \to S$, there is $N \in \mathbb{N}$ such that for all $n > N$,

$$|s_n - s| < \epsilon = 5$$

$$|s_n| = |s_n - s + s| \leq |s_n - s| + |s| = |s| + 5$$

Let

$$M = \max\{|s_1|, |s_2|, \cdots, |s_N|, |s| + 5\}$$

By definition of maximum, for every $m \in \mathbb{N}$ we have

$$|s_m| \leq M$$

*Note: finite site always have maximum □

17 July 17

Theorem: (i): Let $(s_n)_{n\in\mathbb{N}}$ be a convergent sequence, $s \in \mathbb{R}$ such that $\lim s_n = s$ and $r \in \mathbb{R}$
Prove that

$$\lim rs_n = rs$$

Proof: Case 1: $r \neq 0$
Since $s_n \to s$ and $\frac{\epsilon}{|r|} > 0$m there is some $N \in \mathbb{N}$ such that for all $n > N$,

$$|s_n - s| < \frac{\epsilon}{|r|}$$

Multiplying both side by $|r|$, we have that

$$|r| \cdot |s_n - s| < \epsilon$$

$$|rs_n - rs| < \epsilon$$

Thus $rs_n \to rs$ Case 2: $r = 0$
Let $\epsilon > 0$ and take $N = 1$. For $n > N$,

$$|rs_n - rs| = |0 - 0| = 0 < \epsilon$$

□

Theorem: (ii): Let $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$sequences that coverge to s and r, respectively. Then

$$\lim(r_n + s_n) = (\lim r_n) + (\lim s_n) = r + s$$

Proof: Let $\epsilon > 0$, since $s_n \to s$, there is $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$|s_n - s| < \frac{\epsilon}{2}$$

Similarly, there is $N_2 \in \mathbb{N}$ such that for all $m > N_2$

$$|r_m - r| < \frac{\epsilon}{2}$$

Take $N = \max\{N_1, N_2\}$, let $n > N$, then

$$|(r_n + s_n) - (r + s)| = |(r_n - r) + (s_n - s)| \leq |r_n - r| + |s_n - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Because $n > N \geq N_1$ and $n > N \geq N_2$ □

Theorem: (iii) Let $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ sequences tht converge to s and r, respectively Then

$$\lim(s_n \cdot r_n) = (\lim s_n) \cdot (\lim r_n) = r \cdot s$$

$$\begin{aligned}|r_n s_n - rs| &= |r_n s_n + (r_n s - r_n s) - rs| \\ &\leq |r_n s_n - r_n s| + |r_n s - rs| \\ &= |r_n||s_n - s| + |s||r_n - r|\end{aligned}$$

Since $|r_n|$ is bounded, there is

$$|r_n s_n - rs| \leq M|s_n - s| + |s||r_n - r|$$

which means we can make the first part $M|s_n - s|$ small enough.
So using

$$|r_n - r| < \frac{\epsilon}{|s|}$$

$$|s||r_n - r| < \frac{\epsilon}{|s|} \cdot |s|$$

Theorem: (iv) Let $(s_n)_{n\in\mathbb{N}}$ be a sequence that converges to s with $s \neq 0$ and $s_n \neq 0$ for all n.
Then

$$\lim \frac{1}{s_n} = \frac{1}{\lim s_n} = \frac{1}{s}$$

$$|\frac{1}{s_n} - \frac{1}{s}| < \epsilon$$

$$|\frac{s - s_n}{s_n s}| = |s - s_n| \cdot \frac{1}{|s_n s|}$$

Since $|s_n| < M$

$$\frac{1}{|s_n|} > \frac{1}{M}$$

Means that |s| is not too close to 0

Recall: If (s_n) is a sequence with $\lim s_n = s \neq 0$ and $s_n \neq 0$ for all $n \in \mathbb{N}$ then

$$0 < \inf\{|s_n| \big| n \in \mathbb{N}\}$$

Means that there always have a gap with 0 though may be close to 0.
Proof: Let $\epsilon > 0$, let $m = \inf\{|s_n| \big| n \in \mathbb{N}\}$, we wan to show $0 < m \leq |s_n|$ for all n.
There is $N \in \mathbb{N}$ such taht for all $n > N, |s_n - s| < \epsilon$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s - s_n}{s_n s}\right| = \frac{|s - s_n|}{|s_n s|}| \leq \frac{|s - s_n|}{m} < \epsilon$$

using property $|a - b| = |b - a|$ □

Trickes we can use

- Triangle inequality

- Basic factoring and properties from calc 2
- Adding 0

18 July 18

Theorem: (v) If $s_n \neq 0$ for all n and $s \neq 0$,

$$\lim \frac{r_n}{s_n} = \frac{r}{s}$$

Activity: prove the following
(i):

$$\lim_{n\to\infty} \left(\frac{1}{n^p}\right) = 0 \text{ if } p > 0$$

Proof: Let $\epsilon > 0$, take $N = \frac{1}{\sqrt[p]{\epsilon}}$ and let $n > N$, then

$$n > \frac{1}{\sqrt[p]{\epsilon}}$$

So

$$n^p > \frac{1}{\epsilon}$$

Hence

$$\epsilon > \frac{1}{n^p} = |\frac{1}{n^p} - 0|$$

□

(ii):

$$\lim_{n\to\infty} a^n = 0 \text{ if } |a| < 1$$

(iii):

$$\lim_{n\to\infty} \left(n^{\frac{1}{n}}\right) = 1$$

(iv):

$$\lim_{n\to\infty} \left(a^{\frac{1}{n}}\right) = 1 \text{ for } a > 0$$

Proof: If $n > a$ then $n^{1/n} > a^{1/n}$
Case 1: $a > 1$
First we show that $a^{1/n} > 1$ for all $n \in \mathbb{N}$
Suppose $a^{1/n} \leq 1$ then $a^{1/n} \cdot a^{1/n} \leq a^{1/n}$ so $a^{2/n} \leq 1$.
Repeating that artument n-times,

$$a = a^{n/n} \leq 1$$

□

19 July 22

Proving $\lim a^{1/n} = 1$ if $a > 0$.
Proof: Case 1: $a \geq 1$. we want to prove than $a^{1/n} \geq 1$ by contrapositive.
Let $\epsilon > 0$, there is $N_1 > a$, and there is N_2 such that for all $n > N_2$,

$$|n^{\frac{1}{n}} - 1| < \epsilon$$

Take $N = \max\{N_1, N_2\}$, for $n > N$, since $1 \leq a^{1/n} < n^{1/n}$,

$$|a^{\frac{1}{n}} - 1| < |n^{\frac{1}{n}} - 1| < \epsilon$$

□

Def: A sequence (s_n) goes to infinity, denoted by

$$\lim s_n = \infty$$

There is some N such that for all $n > N$

$$s_n > M$$

(no absolute value since it is infinity)
Similarly, $\lim s_n = -\infty$ if for all M, there is N such that for all $n > N$,

$$s_n < M$$

Note: $-\infty$ and ∞ are not real numbers. None of the theorems we proved above applies. For example,

$$s_n = \frac{1}{n}, t_n = n$$

$$\lim s_n = 0, \lim t_n = \infty$$

$$\lim s_n t_n = \lim 1 = 1$$

if $r_n = n^2, \lim r_n = \infty$

$$\lim s_n r_n = \lim n = \infty$$

Proof writing practice: prove that $\lim_{n\to\infty}(\sqrt{n}-7)=\infty$

Proof. Let $M \in \mathbb{R}$,

$$N = (M+7)^2$$

let $n > N$, then

$$\begin{aligned} n &> (M+7)^2 \\ \sqrt{n} &> M+7 \\ \sqrt{n}-7 &> M \end{aligned}$$

□

Theorem: Let (s_n) and (r_n) are sequences. If $\lim s_n = \infty$ and $\boldsymbol{\lim r_n = r > 0}$ with $r \in \mathbb{R}$, then

$$\lim s_n r_n = \infty$$

Proof: Let $M \in \mathbb{R}$, we want to show that there is N such that $s_n r_n > M$ for $n > N$. Let $0 < m < r$, since $r_n \to r$, there is some N_1 such that for all $n > N_1$,

$$m < r_n$$

Since $\lim s_n = \infty$, there is N_2 such that for all $n > N_2$,

$$\frac{M}{m} < s_n$$

Then take $N = \max\{N_1, N_2\}$ for all $n > N$.

$$s_n r_n > s_n \cdot m > \frac{M}{m} \cdot m > M$$

□

Exercise: Find similar theorems for all combinations of $\lim s_n = \pm\infty$ and $\lim r_n$ is positive or negative.

Theorem: Let (s_n) be a sequence with $s_n > 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} s_n = \infty \text{ iff. } \lim_{n\to\infty} \frac{1}{s_n} = 0$$

Exercise: Find a similar theorem when $s_n < 0$ for all n.

Theorem: Let (s_n) be a sequence with $s_n \neq 0$ for all $n \in \mathbb{N}$, then (s_n) is unbounded iff $\lim \frac{1}{s_n} = 0$

20 July 23

Theorem: Let (s_n) be a sequence with $s_n > 0$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} s_n = \infty \text{ iff. } \lim_{n\to\infty} \frac{1}{s_n} = 0$$

Proof: $\Rightarrow$ Assume $\lim_{n\to\infty} s_n = \infty$, we want to show $\lim_{n\to\infty} \frac{1}{s_n} = 0$
Let $\epsilon > 0$, take $M = \frac{1}{\epsilon}$
Since $\lim_{n\to\infty} s_n = \infty$, we have that there is N s.t. for all $n > N$,

$$s_n > M = \frac{1}{\epsilon} > 0$$

so that there is

$$|\frac{1}{s_n} - 0| < \epsilon$$

$\Leftarrow$ Assume $\lim_{n\to\infty} \frac{1}{s_n} = 0$, we want to show $\lim_{n\to\infty} s_n = \infty$
Let $M > 0$, take $\epsilon = \frac{1}{M}$
Since $\lim \frac{1}{s_n} = 0$, There is some N s.t. for all $n > N$,

$$0 < |\frac{1}{s_n} - 0| = \frac{1}{s_n} < \epsilon = \frac{1}{M}$$

Thus,

$$0 < M < s_n$$

□

20.1 Chapter 17

Functions: A function is something that takes an input and spits out an output
(i) The set of inputs of a function f is called the domain of f, denoted dom(f)
(ii) Often, a function is given by a rule or assignment. That is, for every $x \in$ dom(f), we specify the value of the output $f(x)$.
Example:

1. A sequence is a function $s : \mathbb{N} \to \mathbb{R}$
2. $f : \mathbb{R} \to \mathbb{R}$ $f(x) = \frac{1}{x}$ is not a function sicne x need to be $x \neq 0$
3. $g : \mathbb{R} \to \mathbb{R}$

   $$g(x) = \begin{cases} \dfrac{1}{x} \text{ if } x \neq 0 \\ 1 \text{ if } x = 0 \end{cases}$$

4. $f : [-2, 2] \to \mathbb{R}$ $f(x) = \sqrt{4 - x^2}$

Definition: Let f be a function with $\text{dom}(f) \subseteq \mathbb{R}$, we say f is continuous at $x \in \text{dom}(f)$ if for every sequence (x_n) if elements in $\text{dom}(f)$ that converges to x, we have that

$$\lim f(x_n) = f(x)$$

Theorem: Let f be a function, f is continuous at $x \in \text{dom}(f)$ **iff**
For all $\epsilon > 0$, there is $\delta > 0$ s.t. for all $y \in \text{dom}(f)$, if

$$|x - y| < \delta$$

then

$$|f(x) - f(y)| < \epsilon$$

Proof: $\Rightarrow$ Assume that f is continuous at x. Suppose towards a contradiction that:
There is some $\epsilon > 0$ s.t. for every $\delta > 0$, there is $y \in \text{dom}(f)$ such that

$$|x - y| < \delta \text{ and } |f(x) - f(y_n)| \geq \epsilon$$

Fix $\epsilon > 0$, for every $n \in \mathbb{N}$ there is $y \in \text{dom}(f)$ s.t.

$$|x - y_n| < \frac{1}{n} \text{ and } |f(x) - f(y)| \geq \epsilon$$

·First, note that $\lim y_n = x$ becasue let $\epsilon^{'} > 0$ want to find N s.t. for all $n > N$,

$$|y_n - x| = |x - y_n| < \epsilon^{'}$$

·Take $N = \frac{1}{\epsilon^{'}}$ then for $n > N$ we have $\frac{1}{n} < \frac{1}{N} < \epsilon^{'}$. so

$$|x - y_n| < \frac{1}{n} < \epsilon^{'}$$

$y_n \to x$ by definition of continuity,

$$\lim f(y_n) = f(x)$$

There is some N s.t. for all $n > N$,

$$\epsilon \leq |f(y_n) - f(x)| < \epsilon$$

$\epsilon < \epsilon$, a contradiction
$\Leftarrow$ Proof in July 24 $\square$

21 July 24

Proof: Continue:

$\Leftarrow$ Suppose that for all $\epsilon > 0$, there is $\delta > 0$ s.t. for all $x \in \text{dom}(f)$ if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Let (a_n) such that $\lim a_n = a$, want to show that $\lim f(a_n) = f(a)$

Let $\epsilon > 0$, take N such that for all $n > N$,

$$|a_n - a| < \delta$$

since $\lim a_n = a$.

By assumption for all $n > N$,

$$|f(a_n) - f(a)| < \epsilon$$

$\square$

Example: Let $f(x) = 2x^2 + 1, f : \mathbb{R} \to \mathbb{R}$ prove that f is continuous.

Proof: (1) Using definition 1: want to show that $f(a_n) = 2a_n^2+1$ converges to $f(a) = 2a+1$

$$\lim 2(a_n)^2 + 1 = 2\lim(a_n)^2 + \lim 1 = 2(\lim a_n)^2 + 1 = 2a^2 + 1 = 2f(a) + 1$$

(2) Using definition 2:

Let $\epsilon > 0$, take $\delta = \min\{\frac{\epsilon}{4}, \frac{\epsilon}{8|a|}\}$

$$\begin{aligned} |f(x) - f(a)| &= |2x^2 + 1 - (2a^2 + 1)| \\ &= 2|x - a| \cdot |x + a| \\ &\leq 2|x - a|(|x - a| + |2a|) \\ &< 2(\sqrt{\frac{\epsilon}{4}})^2 + 4(\frac{\epsilon}{8|a|})|a| = \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ |f(x) - f(a)| &< \epsilon \end{aligned}$$

$\square$

Proof Writing practice:

(1) Draw a graph of the function $f(x) = x^2 \sin(\frac{1}{x})$

(2) f continuous at 0? and proof

Theorem: Let f be a function. Let $a \in \text{dom}(f), r \in \mathbb{R}$.

If f is continuous at a, then rf is continuous at a and $|f|$ is continuous at a. where

$$(rf)(x) = r \cdot f(x)$$

$$|f|(x) = |f(x)|$$

Proof: (i) Using definition 1, Let (a_n) such that $\lim a_n = a$, want to show that

$$\lim rf(a_n) = rf(a)$$

$$\lim rf(a_n) = r \cdot \lim f(a_n) = r \cdot f(a)$$

(ii) $|f|$ is constant at a usign Def 2.
Let $\epsilon > 0$, take δ such that if $|x - a| < \delta$, then

$$|f(x) - f(a)| < \epsilon$$

$$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)| < \epsilon$$

$\square$

Exercise:

$$\big||f(a)| - |f(b)|\big| \leq |f(a) - f(b)|$$

Let f, g be functions:

1. $(f + g)(x) = f(x) + g(x)$

$$\mathrm{dom}(f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

2. $(f \cdot g)(x) = f(x)g(x)$

$$\mathrm{dom}(f + g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

3. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$

$$\mathrm{dom}(\tfrac{f}{g}) = \mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) | g(x) \neq 0\}$$

4. $(f \circ g)(x) = f(g(x))$

$$\mathrm{dom} = \{x \in \mathrm{dom}(f) | f(x) \in \mathrm{dom}(g)\}$$

5. $\max\{f, g\}(x) = \max\{f(x), g(x)\}$

6. $\min\{f, g\}(x) = \min\{f(x), g(x)\}$

$$\mathrm{dom}(\max / \min) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$$

22 July 29

Theorem: If f, g are continuous, then

$$f+g, fg, \frac{f}{g}, g\circ f, \max\{f,g\}, \min\{f,g\} \text{ are continuous.}$$

Example:
(1): $f(x) = 1$ for all $x \in \mathbb{R}$ f is continuous
Assume that (x_n) is a sequence such that $\lim x_n = a$, want to prove that $\lim f(x_n) = f(a)$

$$f(x_n) = 1 \text{ for all n and } f(a) = 1$$

so $\lim f(x_n) = a$
(2): $g(x) = x$ for all $x \in \mathbb{R}$
g is continuous
take (x_n) such that $x_n \to a$, for every $n, g(x_n) = x_n, g(a) = a$

$$\lim g(x_n) = \lim x_n = a = g(a)$$

(3): $\frac{f}{g}(x) = \frac{1}{x}$
Continuous, since 0 is not in the domain of x

Let $a \in \operatorname{dom}(f) \cap \operatorname{dom}(g)$
(1): $f+g$ is constant at a
take (x_n) s.t. $\lim x_n = a$

$$\begin{aligned}\lim(f+g)(x_n) &= \lim f(x_n) + \lim g(x_n)\\ &= \lim f(a) + \lim g(a)\\ &= f(a) + g(a) \text{ (because f and g are continuous)}\\ &= (f+g)(a)\end{aligned}$$

(2/3) are similar to exercise (2)/(3)
(4) Let $a \in \operatorname{dom}(g\circ f)$
Let $x_n \to a$ where $x_n \in \operatorname{dom}(g\circ f)$ for all n. So, $x_n \in \operatorname{dom}(f)$

$$\lim f(x_n) = f(a) \text{ becasue f is constant}$$

$f(x_n) \in \operatorname{dom}(g)$ for every n

$$f(x_n) \to f(a) \in \operatorname{dom}(g)$$

Since g is continuous,

$$g(f(x_n)) \to g(f(a))$$

So

$$\lim(g\circ f)(x_n) = (g\circ f)(a)$$

Lemma 22.1. *For* $a, b \in \mathbb{R}, \max\{a,b\} = \frac{1}{2}(a+b) + \frac{1}{2}|a-b|$

Proof: Case 1: $a \geq b$

$$\max\{a,b\} = a, \text{ and } a - b \geq 0$$

$$\frac{1}{2}(a+b) + \frac{1}{2}|a-b| = \frac{1}{2}(a+b) + \frac{1}{2}(a-b) = a$$

Case 2: Similar to case 1 $\square$

Note: $\text{dom}(\max\{f,g\}) = \text{dom}(f) \cap \text{dom}(g) = \text{dom}(\min\{f,g\})$

(5): $\max\{f,g\}(x) = \frac{1}{2}(f(x)+g(x)) + \frac{1}{2}|f(x)-g(x)|$

This is continuous because we are only adding and multiplying and taking absolute values of continuous funcitons and all of those remain continuous

Example: A polynomial is a function of the form

Proof-Writing Activity: Find a formula for $\min\{a,b\}$ similar to the one for max and prove that if f, g are continuous functions, then $\min\{f,g\}$ is continuous.

$$\min\{f,g\} = \frac{1}{2}\big(f(x)+g(x)\big) - \frac{1}{2}|f(x)-g(x)|$$

Proof: Similar to the explainaiton in (5)

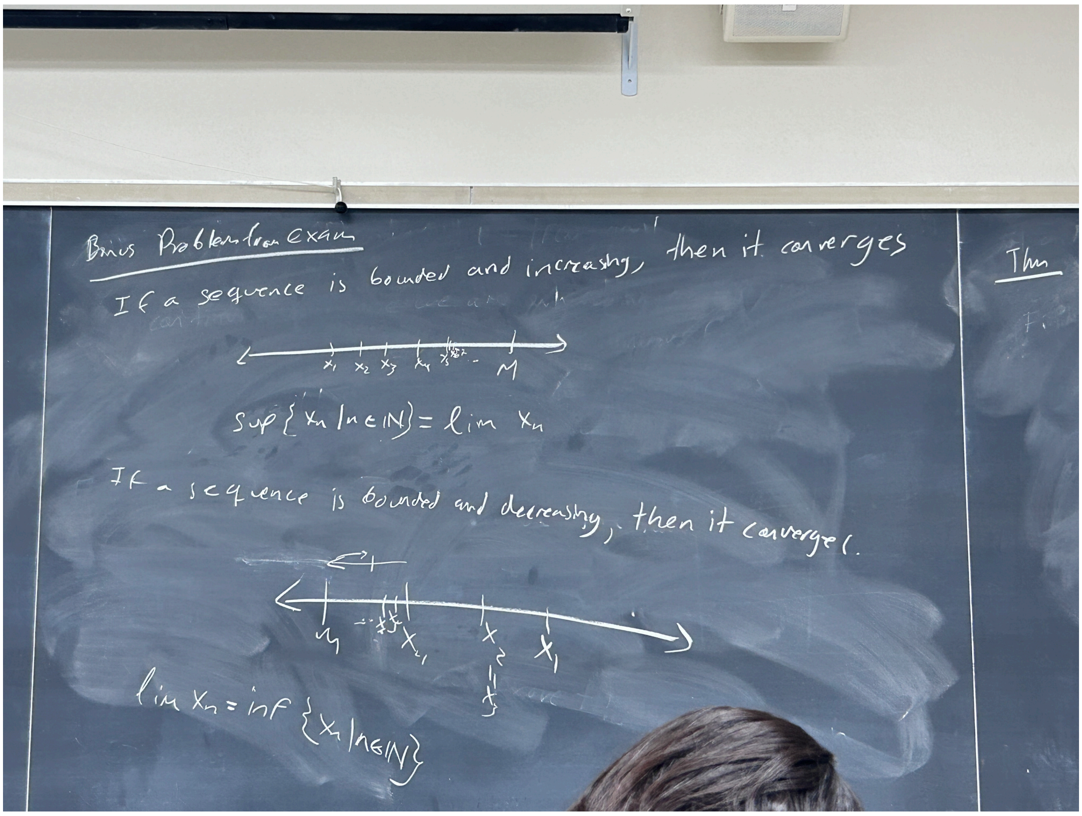
Case 1: $f(x) < g(x)$

$$\begin{aligned} \min\{f(x),g(x)\} &= f(x), \text{ and } f(x)-g(x) < 0 \\ \frac{1}{2}\big(f(x)+g(x)\big) - \frac{1}{2}|f(x)-g(x)| &= \frac{1}{2}f(x) + \frac{1}{2}g(x) + \frac{1}{2}f(x) - \frac{1}{2}g(x) \\ &= f(x) \end{aligned}$$

$\square$

Def: A sequence (x_n) is monotone if either

$$\begin{cases} \text{(1) For all } n < m, x_n \leq x_m \text{ (increasing)} \\ \text{(2) For all } n < m, x_m \leq x_n \text{ (decreasing)} \end{cases}$$



Theorem: Any bounded monotone sequence converges

Theorem: (Bolzano-Weierstros) Any bounded seuqence has a monotone subsequence

23 July 30

<u>Definition</u>: Let $(s_n)_{n\in\mathbb{N}}$ be a seuqence:
A sequence $(t_k)_{k\in\mathbb{N}}$ is a subsequence of (s_n)
(1): For all k, there is $n \in \mathbb{N}$ such that

$$t_k = s_n$$

(2): If $k_1 < k_2$, and $t_{k_1} = s_{n_1}, t_{k_2} = s_{n_2}$, then

$$n_1 < n_2$$

<u>Notation</u>: If $(s_n)_{n\in\mathbb{N}}$ is a subsequence, then $(s_{n_k})_{k\in\mathbb{N}}$ is a subsequence where

$$k \to 8k + 1$$

<u>Example1</u>: $s_n = (-1)^n$ does not converge
$s_{n_k} = (-1)^{2k}$, $(s_{n_k}) = (1, 1, 1, \dots)$ converge
Note: A subsequence might converge even if the original doesn't
<u>Exercise</u>: If (s_n) is a sequence and $\lim s_n = s$, then for every subsequences, (s_{n_k}) we have

$$\lim s_{n_k} = s$$

Theorem: (Bolzano-Weierstrass):
Every bounded sequence has a monotone subsequence
<u>Cor</u>: Every bounded sequence has a convergetn subsequence
<u>Def</u>: We say that s_n is dominant if for all $m > n$, we have that

$$s_m \leq s_n$$

Proof: Take $n_1 = 1$,
Assume we have efined n_k and so fare (s_{nk}) is increasing,
s_{n_k} is not a dominant term in (s_n). So there is $n_k < m$ such that

$$s_{n_k} \leq s_m$$

Call $n_{k+1} = m$,
then (s_{n_k}) is an increasing subsequence, thus monotone

Case 1: There are only finitely many dominant terms
There is $N \in \mathbb{N}$ such that for all $n > N$, s_n is not domonant.
Let $n = N + 1$,
Define $n_{k+1} = m$ to be the least $m > n_k$ such that $s_{n_k} \leq s_m$

Case 2: There are infinitely many dominant terms
Let s_{n_k} be the k^{th} dominant term,

Claim: (s_{n_k}) is decreasing: s_{n_j} is dominant, for all $m > n$, we have

$$s_m \le s_{n_j}$$

In particular, if $k \ge j$, then $n_k > n_j$

$$s_{n_k} \le s_{n_j}$$

Let s_{n_k} be the k^{th} dominant term □

Theorem: Every bounded monotone sequence converges
Theorem: Every sequence has a monotone subsequence
Theorem: If a sequence if bounded, then every subsequence is bounded
Cor: If (s_n) is any bounded sequence, it has a convergent subsequence
Theorem: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded. That is, there is $M \in \mathbb{R}$ such that

$$|f(x)| \le M \text{ for all } x \in [a, b]$$

In other words, $A = \{f(x)|x \in \text{dom}(f)\}$ is bounded
Moreover, it achieves it's max amd min. That is, there are $x_1, x_2 \in [a, b]$ such that for all $x \in [a, b]$,

$$f(x) \le f(x_1) \text{ and } f(x_2) \le f(x)$$

Proof: Suppose towards a contradiction that f is not bounded.
For every $n \in \mathbb{N}$ there is $x_n \in [a, b]$ such that

$$n < |f(x_n)|$$

So (x_n) is a sequence in $\text{dom}(f) = [a, b]$. Since $a \le x \le b$, (x_n) is bouded
So (x_n) has a convergend subsequence, call it (x_{n_k}),

$$\lim x_{n_k} = s$$

□

Lemma 23.1. *(1): If* $r_n < y$ *for all* n *and* $\lim r_n = r$*, then* $r \le y$
(2): If $y < r_n$ *for all* n *and* $\lim r_n = r$*, then* $y \le r$*.*

ways to proof:

$$\inf\{r_n|n \in \mathbb{N}\} \le \lim r_n \le \sup\{r_n|n \in \mathbb{N}\}$$

$$\begin{aligned}
&a \le \inf y_{n_k} \le \lim y_{n_k} \le \sup y_{n_k} \le b \\
&y \in [a, b] \\
&\lim y_{n_k} = y, \text{ by continuity,} \\
&\infty = \lim y_{n_k} = f(y) \text{ a contradiction}
\end{aligned}$$

24 July 31

24.1 Chapter 18

Theorem: If $f : [a, b]$ is continuous, then f is bounded and it attains its max and min. That is, there are $x_0, x_1 \in [a, b]$ such that for all $x \in [a, b]$,

$$f(x_0) \leq f(x) \leq f(x_1)$$

Proof: Let $M = \sup\{f(x)|x \in [a, b]\}$
Since f is bounded, $M \in \mathbb{R}$,
For every $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound of $\{f(x)|x \in [a, b]\}$
ther eis $f(s_n)$ such that $M - \frac{1}{n} \leq f(s_n) \leq M$
which means $|M - f(s_n)| < \frac{1}{n}$
$\lim f(s_n) = M$
the problem is that $\lim s_n$ may not exist(converge)
Since each $s_n \in [a, b]$, (s_n) is boudned
Let (s_{n_k}) be a convergent sequence, $\lim s_{n_k} = s$,
then $\lim f(s_n) = \lim f(s_{n_k}) = f(s)$ so $M = f(s)$ □
For the min, take $m = \inf\{f(x)|x \in [a, b]\}$

Theorem: **(Intermediate Value Theorem (IVT))**
Let $f : [a, b] \to \mathbb{R}$ continuous.
If $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$
then there is $c \in [a, b]$ such that

$$f(c) = y$$

Proof: Case 1: $f(a) \leq f(b)$
Let y : s.t. $f(a) < y < f(b)$
Want to find $c \in [a, b]$ s.t. $f(c) = y$
Let $S := \{x \in [a, b]|f(x) < y\}$
Let $c = \sup S$, For every $n \in \mathbb{N}$, $c - \frac{1}{n}$ is not an upper bound of S
So, there is $c_n \in S$ such that

$$c - \frac{1}{n} \leq c_n \leq c$$

So $|c - c_n| < \frac{1}{n}$
$\lim c_n = c$
Since $f(c_n) < y$ for every n,
$\lim f(c_n) = f(call) \leq y$
then we want to show $y \leq f(c)$
Let $s_n = \min\{c + \frac{1}{n}, b\}$ (to make sure $c + \frac{1}{n}$ is in the domain)

So $s_n > c$ for all n,

$$\lim s_n = c$$

$f(s_n) \nless y$ since $s_n \in S$ and $c = \sup S$
So $y \leq f(s_n)$ for every $n \in \mathbb{N}$

$$y \leq \lim f(s_n) = f(c)$$

Case 2: identical proof $\square$

· if input are in interval, the output would also be interval.

Theorem: Brover's Fixed Point: (Proof writing practice)

Let $f : [0,1] \to [0,1]$ be continuous, prove that there is $c \in [0,1]$ such that

$$f(c) = c$$

Since f is continuous and $y = x$ is continuous, then $f - x$ is also continuous.

25 August 1

Define: Let $S \subseteq \mathbb{R}$ and $a \in \mathbb{R}\{-\infty, \infty\}, L \in \mathbb{R}$,
if f is a functin defined on $S(S \subseteq \text{dom}(f))$, we say that $\lim_{x \to a^s} f(x) = L$, read as "the limit as x approaches a through s of $f(x)$ is L" if for every sequence (x_n) in S with $x_n \to a$, we have $\lim f(x_n) = L$
Example:
(1) If $s = \text{dom}(f)$, then f is continuous at a iff

$$\lim_{x \to a^S} f(x) = f(a)$$

(2) $\lim_{x \to a} f(x) = L$ means that $S = (b, c)$ such that $b < a < c$
(3) $f : \mathbb{R} \to \mathbb{R}, f(x) = 0$ for all $x \in \mathbb{R}$
$S = \{-1, 1\}$

$$\lim_{x \to 0^S} f(x) = r \text{for any } r \in \mathbb{R}$$

Since there is no x converge to 0 (proof by contradiction through negation of definition that there is a sequence (x_n) in S such that $x_n \to 0$ and $\lim f(x_n) \neq r$)
Note: If a is not the limit of any sequence in S, then $\lim_{x \to a^S} f(x)$ makes no sense.
(4)

$$\lim_{x \to a^-} f(x) \text{ means that } S = (b, a) \text{ with } b < a$$

$$\lim_{x \to a^+} f(x) \text{ means that } S = (a, c) \text{ with } c > a$$

Exercise: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$
Corollary: from textbook

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(5):

$$\lim_{x \to \infty} f(x) = L \text{ means that } a = \infty \text{ and } S = (c, \infty) \text{ for some } c \in \mathbb{R}$$

$$\lim_{x \to -\infty} f(x) = L \text{ means that } a = -\infty \text{ and } S = (-\infty, b) \text{ for some } b \in \mathbb{R}$$

(6): $\lim_{x \to 3^-} x^2 + 1 = 10$, let $S = (0, 3)$, want to show that for every sequence (x_n) in S such that $x_n \to 3$, then $f(x) \to 10$
(7): $\lim_{x \to 0^-} \frac{1}{x} = -\infty$. Let (x_n) be a sequence such that $x_n < 0$ for all n and $\lim x_n = 0$
We want to show $\lim f(x_n) = -\infty$,

$$f(x_n) = \frac{1}{x_n}, \lim f(x_n) = \lim \frac{1}{x_n} = 0$$

Note: $\left\{\begin{array}{l} s_n > 0 \text{ for all n} : \lim s_n = \infty \text{ iff } \lim \dfrac{1}{s_n} = 0 \\ s_n = 0 \text{ for all n} : (s_n) \text{ is unbounded iff } \lim \dfrac{1}{s_n} = 0 \\ s_n < 0 \text{ for all n} : \lim s_n = -\infty \text{ iff } \lim \dfrac{1}{s_n} = 0 \end{array}\right\}$

Theorem: Let f_1 and f_2 functions, L_1 and $L_2 \in \mathbb{R}$,

$$\lim_{x \to a^S} f_1(x) = L_1 \text{ and } \lim_{x \to a^S} f_2(x) = L_2$$

Then

$$\lim_{x \to a^S} (f_1 + f_2)(x) = L_1 + L_2$$

$$\lim_{x \to a^S} (f_1 \cdot f_2)(x) = L_1 \cdot L_2$$

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26.1 Limits of Functins

Def: Let $S \subseteq \mathbb{R}, a \in \mathbb{R} \cup \{-\infty.\infty\}, f$ a function with $S \subseteq \text{dom}(f), L \in \mathbb{R}$
Then

$$\lim_{x \to a^S} f(x) = L$$

If for any sequence (s_n) in S that converges to a, we have that

$$\lim f(s_n) = L$$

- For the "usual" two-sided limit, S can be any open interval containing a. This includes $\mathbb{R} = (-\infty, \infty)$. In this case,

$$\lim_{x \to a^S} f(x) = \lim_{x \to a} f(x)$$

- For the one-sided limits, S can be any open interval where a is an end point.

$$S_1 = (b, a) \text{ with } b < a, S_2 = (a, c) \text{ for some } a < c$$

$$\lim_{x \to a^{S_1}} f(x) = \lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^{S_2}} f(x) = \lim_{x \to a^+} f(x)$$

Theorem: Let f_1, f_2 be funcitons, and $L_1, L_2 \in \mathbb{R}$ such that

$$\lim_{x \to a^S} f_1(x) = L_1, \lim_{x \to a^S} f_2(x) = L_2$$

Then,

$$\cdot \lim_{x \to a^S} f_1 + f_2(x) = L_1 + L_2$$
$$\cdot \lim_{x \to a^S} f_1 f_2(x) = L_1 \cdot L_2$$
$$\cdot \lim_{x \to a^S} \frac{f_1}{f_2}(x) = \frac{L_1}{L_2}$$
$$\cdot \text{ as long as } L_2 \neq 0 \text{ and } f_2(x) \neq 0 \text{ for } x \in S$$

Proof: (a) Let (s_n) be a sequence in S such that $\lim s_n = a$,
want to show: $\lim f_1 + f_2(s_n) = L_1 + L_2$

$$\begin{aligned} \lim f_1 + f_2(s_n) &= \lim f_1(s_n) + f_2(s_n) \\ &= \lim f_1(s_n) + \lim f_2(s_n) \\ &\text{(Using Therom about sequences)} \\ &= L_1 + L_2 \end{aligned}$$

□

Continuity: f is continuous at a if

$$\lim_{x\to a} f(x) = f(\lim_{x\to a} x) = f(a)$$

Theorem: $\lim_{x\to a^S} f(x) = L$ iff
for all $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in S$ if $|x-a| < \delta$, then $|f(x) - L| < \epsilon$
Proof: $\Rightarrow$, By contradiction
Assume $\lim_{x\to a^S} f(x) = L$ and there is $\epsilon > 0$ such that for all $\delta > 0$, we can find $x \in S$ such that

$$|x-a| < \delta \text{ and } |f(x) - L| \geq \epsilon$$

For each $n \in \mathbb{N}$ there is $x_n \in S$ such that $|x_n - a| < \frac{1}{n}$ but

$$|f(x_n) - L| \geq \epsilon$$

Note that $\lim x_n = a$
Let $\epsilon > 0$, let $N = \frac{1}{\epsilon}$ for all $n > N$, $\frac{1}{n} < \epsilon$ so

$$|x_n - a| < \frac{1}{n} < \frac{1}{N} = \epsilon$$

Since $\lim_{x\to a^S} f(x) = L$, we have that

$$\lim f(x_n) = L$$

This means that for any $\epsilon > 0$, there is N such that for all $n > N$, $|f(x_n) - L| < \epsilon$,

$$\epsilon \leq |f(x_n) - L| < \epsilon$$

A contradiction
$\Leftarrow$ Let (s_n) be a sequence in S such that $\lim s_n = a$, we want to show that $\lim f(s_n) = L$
Let $\epsilon > 0$, we need to find N such that for all $n > N$,

$$|f(s_n) - L| < \epsilon$$

Since $\lim s_n = a$, there is some n such that for all $n > N$,

$$|s_n - a| < \delta$$

So

$$|f(s_n) - L| < \epsilon$$

□

Note: For limits of functions, a does not need to be in $\text{dom}(f)$, $\lim_{x\to a} f(x)$ still makes sense
For example, under limits of functions, $f(x) = x^2 \sin(\frac{1}{x})$ is continuous instead of uncontinuous since in this case, $\lim_{x\to 0} f(x) = 0$

26.2 Differentiation

Def Let f be a function, $a \in \text{dom}(f)$,

$$f^{'}(a) \coloneqq \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ if it exists}$$

Equivalently, taking $x = a + h$,

$$f^{'}(a) \coloneqq \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$\cdot f^{'}$ is a function that takes a and outputs the slope of the tangent to the graph of $f(x)$ at a, if it is well-defined

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$$\text{dom}(f^{'}) \subseteq \text{dom}(f)$$

For example, $f(x) = |x|$ is differentiable at almost everywhere **except 0**
$\text{dom}(f^{'}) = \mathbb{R}\backslash\{0\} \nsubseteq \text{dom}(f) = \mathbb{R}$
Example: $f(x) = x^2$, then $f^{'}(x) = 2x$
Let $a \in \mathbb{R}$,

$$f^{'}(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = \lim_{x\to a} x + a = 2a$$

So $f^{'}(x) = 2x$

Theorem: let f be a function and $a \in \text{dom}(f)$,
If f is a differentiable at a, then f is continuous at a.
Proof: we know that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists, want to show that $\lim_{x\to a} f(x) = f(a)$
If $x \neq a$,

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a}(x-a) \cdot \frac{f(x)-f(a)}{x-a} + f(a) \\
&= \lim_{x\to a}(x-a) \cdot \lim_{x\to a} \frac{f(x)-f(a)}{x-a} + \lim_{x\to a} f(a) \\
&\text{(Knowing these 3 limits are all } \in \mathbb{R}) \\
&= 0 \cdot f(a) + f(a) \\
&= f(a)
\end{aligned}$$

□

Example (Weierstrass Monster Function):
There is a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere but is not differentiable at any point.

Theorem: Let f, g be functions differentiable at a. Let $c \in \mathbb{R}$.

$$(cf)^{'}(x) = c \cdot f^{'}(x)$$

$$(f + g)^{'}(x) = f^{'}(x) + g^{'}(x)$$

$$\text{Product rule } (fg)^{'}(x) = f^{'}(x)g(x) + f(x)g^{'}(x)$$

$$\text{Quotient rule } \frac{f}{g}^{'}(x) = \frac{f^{'}(x)g(x) + f(x)g^{'}(x)}{(g(x))^2} \text{ as long as } g(x) \neq 0$$

Proof:

$$(cf)^{'}(x) = \lim_{x\to a} \frac{c \cdot f(x) - c \cdot f(a)}{x - a} = c \cdot \frac{f(x) - f(a)}{x - a} = c \cdot f^{'}(a)$$

$\square$

Proof:

$$\begin{aligned}(f+g)^{'}(a) &= \lim_{x\to a}\frac{(f(x)+g(x))-(f(a)+g(a))}{x-a}\\ &= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\\ &= f^{'}(a)+g^{'}(a)\end{aligned}$$

$\square$

Proof:

$$\begin{aligned}(fg)^{'}(a) &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}\\ &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}\\ &= \lim_{x\to a}\frac{f(x)g(x)-f(a)g(x)}{x-a}+\lim_{x\to a}\frac{f(a)g(x)-f(a)g(a)}{x-a}\\ &= \lim_{x\to a}g(x)\cdot\frac{f(x)-f(a)}{x-a}+\lim_{x\to a}f(a)\cdot\frac{g(x)-g(a)}{x-a}\\ &= g(a)\cdot f^{'}(a)+f(a)\cdot g^{'}(a)\end{aligned}$$

$\square$

Proof:

$$\begin{aligned}\frac{f}{g}^{'}(x) &= \lim_{x\to a}\frac{\frac{f(x)}{g(x)}-\frac{f(a)}{g(a)}}{x-a}\\ &= \lim_{x\to a}\frac{\frac{f(x)g(a)-f(a)g(x)}{g(x)g(a)}}{x-a}\\ &= \frac{\lim_{x\to a}\frac{f(x)g(a)-f(a)g(x)}{x-a}}{\lim_{x\to a}\frac{g(x)g(a)}{x-a}}\\ &= \frac{\lim_{x\to a}\frac{f(x)g(a)-f(a)g(a)+f(a)g(a)-f(a)g(x)}{x-a}}{\lim_{x\to a}\frac{g(x)g(a)}{x-a}}\\ &= \frac{\lim_{x\to a}\frac{[f(x)g(a)-f(a)g(a)]-[f(a)g(x)-f(a)g(a)]}{x-a}}{\lim_{x\to a}\frac{g(x)g(a)}{x-a}}\\ &= \frac{g(a)\cdot f^{'}(a)-f(a)\cdot g^{'}(a)}{g(a)^2}\end{aligned}$$

$\square$

Theorem: If f is differentiable at a and g is differentiable at $f(a)$, then

$$(g\circ f)^{'}(a)=[g(f(a))]^{'}=g^{'}(f(a))\cdot f^{'}(a)$$

Proof:

$$\begin{aligned}
(g \circ f)^{'}(a) &= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \\
&= \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} \cdot \frac{f(x) - f(a)}{f(x) - f(a)} \\
&= \lim_{f(x) \to f(a)} \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= g^{'}(f(a)) \cdot f^{'}(a)
\end{aligned}$$

Note: $f(x)$ may equal to $f(a)$ at many points, so problem may appear, so the prrof above is incomplete now. □

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Theorem: (Mean Value Theorem)
If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a.b)$ such that

$$f^{'}(c) = \frac{f(b) - f(a)}{b - a}$$