Table 6.6

Expected returns on four indifference curves and the CAL. Investor's risk aversion is $A = 4$.

σ	$U = .07$	$U = .078$	$U = .08653$	$U = .094$	CAL
0	.0700	.0780	.0865	.0940	.0700
.02	.0708	.0788	.0873	.0948	.0773
.04	.0732	.0812	.0897	.0972	.0845
.06	.0772	.0852	.0937	.1012	.0918
.08	.0828	.0908	.0993	.1068	.0991
.0902	.0863	.0943	.1028	.1103	.1028
.10	.0900	.0980	.1065	.1140	.1064
.12	.0988	.1068	.1153	.1228	.1136
.14	.1092	.1172	.1257	.1332	.1209
.18	.1348	.1428	.1513	.1588	.1355
.22	.1668	.1748	.1833	.1908	.1500
.26	.2052	.2132	.2217	.2292	.1645
.30	.2500	.2580	.2665	.2740	.1791

with the standard deviation in column (1) to provide the utility value corresponding to each curve. Column (6) uses Equation 6.5 to calculate $E(r_C)$ on the CAL for the standard deviation σ_C in column (1):

$$E(r_C) = r_f + [E(r_P) - r_f]\frac{\sigma_C}{\sigma_P} = 7 + [15 - 7]\frac{\sigma_C}{22}$$

Figure 6.8 graphs the four indifference curves and the CAL. The graph reveals that the indifference curve with $U = .08653$ is tangent to the CAL; the tangency point corresponds to the complete portfolio that maximizes utility. The tangency point occurs at $\sigma_C = 9.02\%$ and $E(r_C) = 10.28\%$, the risk–return parameters of the optimal complete portfolio with $y^* = 0.41$. These values match our algebraic solution using Equation 6.7.

We conclude that the choice for y^*, the fraction of overall investment funds to place in the risky portfolio, is determined by risk aversion (the slope of indifference curves) and the Sharpe ratio (the slope of the opportunity set).

In sum, capital allocation determines the complete portfolio, which constitutes the investor's entire wealth. Portfolio P represents all-wealth-at-risk. Hence, when returns are normally distributed, standard deviation is the appropriate measure of risk. In future chapters we will consider augmenting P with "good" additions, meaning assets that improve the feasible risk-return trade-off. The risk of these potential additions will have to be measured by their *incremental* effect on the standard deviation of P.

Nonnormal Returns

In the foregoing analysis we assumed normality of returns by taking the standard deviation as the appropriate measure of risk. But as we discussed in Chapter 5, departures from normality could result in extreme losses with far greater likelihood than would be plausible under a normal distribution. These exposures, which are typically measured by value at risk (VaR) or expected shortfall (ES), also would be important to investors.

Therefore, an appropriate extension of our analysis would be to present investors with forecasts of VaR and ES. Taking the capital allocation from the normal-based analysis as a benchmark, investors facing fat-tailed distributions might consider reducing their allocation to the risky portfolio in favor of an increase in the allocation to the risk-free vehicle.

There are signs of advances in dealing with extreme values (in addition to new techniques to handle transaction data mentioned in Chapter 5). Back in the early 20th century, Frank Knight, one of the great economists of the time, distinguished *risk* from *uncertainty*, the difference being that risk is a known problem in which probabilities can be ascertained while uncertainty is characterized by ignorance even about probabilities (reminiscent of the black swan problem). Hence, Knight argued, we must use different methods to handle uncertainty and risk.

Probabilities of moderate outcomes in finance can be readily assessed from experience because of the high relative frequency of such observations. Extreme negative values are blissfully rare, but for that very reason, accurately assessing their probabilities is virtually impossible. However, the Bayesian statistics that took center stage in decision making in later periods rejected Knight's approach on the argument that even if probabilities are hard to estimate objectively, investors nevertheless have a notion, albeit subjective, of what they may be and must use those beliefs to make economic decisions. In the Bayesian framework, these so-called priors must be used even if they apply to unprecedented events that characterize uncertainty. Accordingly, in this school of thought, the distinction between risk and uncertainty is deemed irrelevant.

Economists today are coming around to Knight's position. Advanced utility functions attempt to distinguish risk from uncertainty and give these uncertain outcomes a larger role in the choice of portfolios. These approaches have yet to enter everyday practice, but as they are developed, practical measures are certain to follow.

CONCEPT CHECK 6.7

a. If an investor's coefficient of risk aversion is $A = 3$, how does the optimal asset mix change? What are the new values of $E(r_C)$ and σ_C?

b. Suppose that the borrowing rate, $r_f^B = 9\%$ is greater than the lending rate, $r_f = 7\%$. Show graphically how the optimal portfolio choice of some investors will be affected by the higher borrowing rate. Which investors will *not* be affected by the borrowing rate?

6.6 Passive Strategies: The Capital Market Line

The CAL is derived with the risk-free and "the" risky portfolio, *P*. Determination of the assets to include in *P* may result from a passive or an active strategy. A **passive strategy** describes a portfolio decision that avoids *any* direct or indirect security analysis.[6] At first blush, a passive strategy would appear to be naive. As will become apparent, however, forces of supply and demand in large capital markets may make such a strategy the reasonable choice for many investors.

In Chapter 5, we presented a compilation of the history of rates of return on different portfolios. The data are available at Professor Kenneth French's Web site, **mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html**. We can use these data to examine various passive strategies.

A natural candidate for a passively held risky asset would be a well-diversified portfolio of common stocks such as "All U.S." described in Chapter 5. Because a passive strategy

[6] By "indirect security analysis" we mean the delegation of that responsibility to an intermediary such as a professional money manager.

requires that we devote no resources to acquiring information on any individual stock or group of stocks, we must follow a "neutral" diversification strategy. One way is to select a diversified portfolio of stocks that mirrors the value of the corporate sector of the U.S. economy. This results in a portfolio in which, for example, the proportion invested in Microsoft stock will be the ratio of Microsoft's total market value to the market value of all listed stocks.

The most popular value-weighted index of U.S. stocks is the Standard & Poor's Composite Index of 500 large capitalization U.S. corporations (the S&P 500). Table 6.7 summarizes the performance of the S&P 500 portfolio over the 87-year period 1926–2012, as well as for four subperiods. Table 6.7 shows the average return for the portfolio, the return on rolling over 1-month T-bills for the same period, as well as the resultant average excess return and its standard deviation. The Sharpe ratio was .40 for the overall period, 1926–2012. In other words, stock market investors enjoyed a .40% average excess return over the T-bill rate for every 1% of standard deviation. The large standard deviation of the excess return (20.48%) is one reason we observe a wide range of average excess returns and Sharpe ratios across subperiods (varying from .21 to .74). Using the statistical distribution of the difference between the Sharpe ratios of two portfolios, we can estimate the probability of observing a deviation of the Sharpe ratio for a particular subperiod from that of the overall period, assuming the latter is the true value. The last column of Table 6.7 shows that the probabilities of finding such widely different Sharpe ratios over the subperiods are actually quite substantial.

We call the capital allocation line provided by 1-month T-bills and a broad index of common stocks the **capital market line** (CML). A passive strategy generates an investment opportunity set that is represented by the CML.

How reasonable is it for an investor to pursue a passive strategy? We cannot answer such a question without comparing the strategy to the costs and benefits accruing to an active portfolio strategy. Some thoughts are relevant at this point, however.

First, the alternative active strategy is not free. Whether you choose to invest the time and cost to acquire the information needed to generate an optimal active portfolio of risky assets, or whether you delegate the task to a professional who will charge a fee, constitution of an active portfolio is more expensive than a passive one. The passive portfolio requires negligible cost to purchase T-bills and management fees to either an exchange-traded fund

	Average Annual Returns			S&P 500 Portfolio		
Period	S&P 500 Portfolio	1-Month T-Bills	Risk Premium	Standard Deviation	Sharpe Ratio (Reward-to-Volatility)	Probability*
1926–2012	11.67	3.58	8.10	20.48	0.40	—
1989–2012	11.10	3.52	7.59	18.22	0.42	0.94
1968–1988	10.91	7.48	3.44	16.71	0.21	0.50
1947–1967	15.35	2.28	13.08	17.66	0.74	0.24
1926–1946	9.40	1.04	8.36	27.95	0.30	0.71

Table 6.7

Average annual return on large stocks and 1-month T-bills; standard deviation and Sharpe ratio of large stocks over time

*The probability that the estimate of the Sharpe ratio over 1926–2012 equals the true value and that we observe the reported, or an even more different Sharpe ratio for the subperiod.

WORDS FROM THE STREET

Investors Sour on Pro Stock Pickers

Investors are jumping out of mutual funds managed by professional stock pickers and shifting massive amounts of money into lower-cost funds that echo the broader market.

Through November 2012, investors pulled $119.3 billion from so-called actively managed U.S. stock funds according to the latest data from research firm Morningstar Inc. At the same time, they poured $30.4 billion into U.S. stock exchange-traded funds.

The move reflects the fact that many money managers of stock funds, which charge fees but also dangle the prospect of higher returns, have underperformed the benchmark stock indexes. As a result, more investors are choosing simply to invest in funds tracking the indexes, which carry lower fees and are perceived as having less risk.

The mission of stock pickers in a managed mutual fund is to outperform the overall market by actively trading individual stocks or bonds, with fund managers receiving higher fees for their effort. In an ETF (or indexed mutual fund), managers balance the share makeup of the fund so it accurately reflects the performance of its underlying index, charging lower fees.

Morningstar says that when investors have put money in stock funds, they have chosen low-cost index funds and ETFs. Some index ETFs cost less than 0.1% of assets a year, while many actively managed stock funds charge 1% a year or more.

While the trend has put increasing pressure lately on stock pickers, it is shifting the fortunes of some of the biggest players in the $14 trillion mutual-fund industry.

Fidelity Investments and American Funds, among the largest in the category, saw redemptions or weak investor interest compared with competitors, according to an analysis of mutual-fund flows done for *The Wall Street Journal* by research firm Strategic Insight, a unit of New York-based Asset International.

At the other end of the spectrum, Vanguard, the world's largest provider of index mutual funds, pulled in a net $141 billion last year through December, according to the company.

Many investors say they are looking for a way to invest cheaply, with less risk.

Source: Adapted from Kirsten Grind, *The Wall Street Journal*, January 3, 2013. Reprinted with permission.

or a mutual fund company that operates a market index fund. Vanguard, for example, operates the Index 500 Portfolio that mimics the S&P 500 index fund. It purchases shares of the firms constituting the S&P 500 in proportion to the market values of the outstanding equity of each firm, and therefore essentially replicates the S&P 500 index. The fund thus duplicates the performance of this market index. It has one of the lowest operating expenses (as a percentage of assets) of all mutual stock funds precisely because it requires minimal managerial effort.

A second reason to pursue a passive strategy is the free-rider benefit. If there are many active, knowledgeable investors who quickly bid up prices of undervalued assets and force down prices of overvalued assets (by selling), we have to conclude that at any time most assets will be fairly priced. Therefore, a well-diversified portfolio of common stock will be a reasonably fair buy, and the passive strategy may not be inferior to that of the average active investor. (We will elaborate on this argument and provide a more comprehensive analysis of the relative success of passive strategies in later chapters.) The nearby box points out that passive index funds have actually outperformed most actively managed funds in the past decades and that investors are responding to the lower costs and better performance of index funds by directing their investments into these products.

To summarize, a passive strategy involves investment in two passive portfolios: virtually risk-free short-term T-bills (or, alternatively, a money market fund) and a fund of common stocks that mimics a broad market index. The capital allocation line representing such a strategy is called the *capital market line*. Historically, based on 1926 to 2012 data, the passive risky portfolio offered an average risk premium of 8.1% and a standard deviation of 20.48%, resulting in a reward-to-volatility ratio of .40.

Passive investors allocate their investment budgets among instruments according to their degree of risk aversion. We can use our analysis to deduce a typical investor's risk-aversion parameter. From Table 1.1 in Chapter 1, we estimate that approximately 65.6%

of net worth is invested in a broad array of risky assets.[7] We assume this portfolio has the same reward–risk characteristics that the S&P 500 has exhibited since 1926, as documented in Table 6.7. Substituting these values in Equation 6.7, we obtain

$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2} = \frac{.081}{A \times .2048^2} = .656$$

which implies a coefficient of risk aversion of

$$A = \frac{.081}{.656 \times .2048^2} = 2.94$$

Of course, this calculation is highly speculative. We have assumed that the average investor holds the naive view that historical average rates of return and standard deviations are the best estimates of expected rates of return and risk, looking to the future. To the extent that the average investor takes advantage of contemporary information in addition to simple historical data, our estimate of $A = 2.94$ would be an unjustified inference. Nevertheless, a broad range of studies, taking into account the full range of available assets, places the degree of risk aversion for the representative investor in the range of 2.0 to 4.0.[8]

CONCEPT CHECK 6.8

Suppose that expectations about the S&P 500 index and the T-bill rate are the same as they were in 2012, but you find that a greater proportion is invested in T-bills today than in 2012. What can you conclude about the change in risk tolerance over the years since 2012?

[7] We include in the risky portfolio real assets, half of pension reserves, corporate and noncorporate equity, and half of mutual fund shares. This portfolio sums to $50.05 trillion, which is 65.6% of household net worth. (See Table 1.1.)

[8] See, for example, I. Friend and M. Blume, "The Demand for Risky Assets," *American Economic Review* 64 (1974); or S. J. Grossman and R. J. Shiller, "The Determinants of the Variability of Stock Market Prices," *American Economic Review* 71 (1981).

SUMMARY

1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero risk premium. It will not be undertaken by a risk-averse investor.
3. Investors' preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if it is received with certainty, would yield the same utility as the risky portfolio.

5. Shifting funds from the risky portfolio to the risk-free asset is the simplest way to reduce risk. Other methods involve diversification of the risky portfolio and hedging. We take up these methods in later chapters.

6. T-bills provide a perfectly risk-free asset in nominal terms only. Nevertheless, the standard deviation of real rates on short-term T-bills is small compared to that of other assets such as long-term bonds and common stocks, so for the purpose of our analysis we consider T-bills as the risk-free asset. Money market funds hold, in addition to T-bills, short-term relatively safe obligations such as CP and CDs. These entail some default risk, but again, the additional risk is small relative to most other risky assets. For convenience, we often refer to money market funds as risk-free assets.

7. An investor's risky portfolio (the risky asset) can be characterized by its reward-to-volatility ratio, $S = [E(r_P) - r_f]/\sigma_P$. This ratio is also the slope of the CAL, the line that, when graphed, goes from the risk-free asset through the risky asset. All combinations of the risky asset and the risk-free asset lie on this line. Other things equal, an investor would prefer a steeper-sloping CAL, because that means higher expected return for any level of risk. If the borrowing rate is greater than the lending rate, the CAL will be "kinked" at the point of the risky asset.

8. The investor's degree of risk aversion is characterized by the slope of his or her indifference curve. Indifference curves show, at any level of expected return and risk, the required risk premium for taking on one additional percentage point of standard deviation. More risk-averse investors have steeper indifference curves; that is, they require a greater risk premium for taking on more risk.

9. The optimal position, y^*, in the risky asset, is proportional to the risk premium and inversely proportional to the variance and degree of risk aversion:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

Graphically, this portfolio represents the point at which the indifference curve is tangent to the CAL.

10. A passive investment strategy disregards security analysis, targeting instead the risk-free asset and a broad portfolio of risky assets such as the S&P 500 stock portfolio. If in 2012 investors took the mean historical return and standard deviation of the S&P 500 as proxies for its expected return and standard deviation, then the values of outstanding assets would imply a degree of risk aversion of about $A = 2.94$ for the average investor. This is in line with other studies, which estimate typical risk aversion in the range of 2.0 through 4.0.

Related Web sites for this chapter are available at www.mhhe.com/bkm

KEY TERMS

risk premium
fair game
risk averse
utility
certainty equivalent rate
risk neutral
risk lover
mean-variance (M-V) criterion
indifference curve
complete portfolio
risk-free asset
capital allocation line
reward-to-volatility ratio
passive strategy
capital market line

KEY EQUATIONS

Utility score: $U = E(r) - \frac{1}{2}A\sigma^2$

Optimal allocation to risky portfolio: $y^* = \dfrac{E(r_P) - r_f}{A\sigma_P^2}$

PROBLEM SETS

Basic

1. Which of the following choices best completes the following statement? Explain. An investor with a higher degree of risk aversion, compared to one with a lower degree, will prefer investment portfolios
   a. with higher risk premiums.
   b. that are riskier (with higher standard deviations).
   c. with lower Sharpe ratios.
   d. with higher Sharpe ratios.
   e. None of the above is true.
2. Which of the following statements are true? Explain.
   a. A lower allocation to the risky portfolio reduces the Sharpe (reward-to-volatility) ratio.
   b. The higher the borrowing rate, the lower the Sharpe ratios of levered portfolios.
   c. With a fixed risk-free rate, doubling the expected return and standard deviation of the risky portfolio will double the Sharpe ratio.
   d. Holding constant the risk premium of the risky portfolio, a higher risk-free rate will increase the Sharpe ratio of investments with a positive allocation to the risky asset.
3. What do you think would happen to the expected return on stocks if investors perceived higher volatility in the equity market? Relate your answer to Equation 6.7.

Intermediate

4. Consider a risky portfolio. The end-of-year cash flow derived from the portfolio will be either \$70,000 or \$200,000 with equal probabilities of .5. The alternative risk-free investment in T-bills pays 6% per year.
   a. If you require a risk premium of 8%, how much will you be willing to pay for the portfolio?
   b. Suppose that the portfolio can be purchased for the amount you found in (*a*). What will be the expected rate of return on the portfolio?
   c. Now suppose that you require a risk premium of 12%. What is the price that you will be willing to pay?
   d. Comparing your answers to (*a*) and (*c*), what do you conclude about the relationship between the required risk premium on a portfolio and the price at which the portfolio will sell?
5. Consider a portfolio that offers an expected rate of return of 12% and a standard deviation of 18%. T-bills offer a risk-free 7% rate of return. What is the maximum level of risk aversion for which the risky portfolio is still preferred to bills?
6. Draw the indifference curve in the expected return–standard deviation plane corresponding to a utility level of .05 for an investor with a risk aversion coefficient of 3. (*Hint:* Choose several possible standard deviations, ranging from 0 to .25, and find the expected rates of return providing a utility level of .05. Then plot the expected return–standard deviation points so derived.)
7. Now draw the indifference curve corresponding to a utility level of .05 for an investor with risk aversion coefficient $A = 4$. Comparing your answer to Problem 6, what do you conclude?
8. Draw an indifference curve for a risk-neutral investor providing utility level .05.
9. What must be true about the sign of the risk aversion coefficient, *A*, for a risk lover? Draw the indifference curve for a utility level of .05 for a risk lover.

For Problems 10 through 12: Consider historical data showing that the average annual rate of return on the S&P 500 portfolio over the past 85 years has averaged roughly 8% more than the Treasury bill return and that the S&P 500 standard deviation has been about 20% per year. Assume these values are representative of investors' expectations for future performance and that the current T-bill rate is 5%.

10. Calculate the expected return and variance of portfolios invested in T-bills and the S&P 500 index with weights as follows:

W_{bills}	W_{index}
0	1.0
0.2	0.8
0.4	0.6
0.6	0.4
0.8	0.2
1.0	0

11. Calculate the utility levels of each portfolio of Problem 10 for an investor with $A = 2$. What do you conclude?
12. Repeat Problem 11 for an investor with $A = 3$. What do you conclude?

Usetheseinputs for Problems 13 through 19: You manage a risky portfolio with expected rate of return of 18% and standard deviation of 28%. The T-bill rate is 8%.

13. Your client chooses to invest 70% of a portfolio in your fund and 30% in a T-bill money market fund. What is the expected value and standard deviation of the rate of return on his portfolio?
14. Suppose that your risky portfolio includes the following investments in the given proportions:

Stock A	25%
Stock B	32%
Stock C	43%

What are the investment proportions of your client's overall portfolio, including the position in T-bills?

15. What is the reward-to-volatility ratio (S) of your risky portfolio? Your client's?
16. Draw the CAL of your portfolio on an expected return–standard deviation diagram. What is the slope of the CAL? Show the position of your client on your fund's CAL.
17. Suppose that your client decides to invest in your portfolio a proportion y of the total investment budget so that the overall portfolio will have an expected rate of return of 16%.
    a. What is the proportion y?
    b. What are your client's investment proportions in your three stocks and the T-bill fund?
    c. What is the standard deviation of the rate of return on your client's portfolio?
18. Suppose that your client prefers to invest in your fund a proportion y that maximizes the expected return on the complete portfolio subject to the constraint that the complete portfolio's standard deviation will not exceed 18%.
    a. What is the investment proportion, y?
    b. What is the expected rate of return on the complete portfolio?
19. Your client's degree of risk aversion is $A = 3.5$.
    a. What proportion, y, of the total investment should be invested in your fund?
    b. What is the expected value and standard deviation of the rate of return on your client's optimized portfolio?
20. Look at the data in Table 6.7 on the average risk premium of the S&P 500 over T-bills, and the standard deviation of that risk premium. Suppose that the S&P 500 is your risky portfolio.
    a. If your risk-aversion coefficient is $A = 4$ and you believe that the entire 1926–2012 period is representative of future expected performance, what fraction of your portfolio should be allocated to T-bills and what fraction to equity?
    b. What if you believe that the 1968–1988 period is representative?
    c. What do you conclude upon comparing your answers to (a) and (b)?

21. Consider the following information about a risky portfolio that you manage, and a risk-free asset: $E(r_P) = 11\%$, $\sigma_P = 15\%$, $r_f = 5\%$.

    a. Your client wants to invest a proportion of her total investment budget in your risky fund to provide an expected rate of return on her overall or complete portfolio equal to 8%. What proportion should she invest in the risky portfolio, *P*, and what proportion in the risk-free asset?

    b. What will be the standard deviation of the rate of return on her portfolio?

    c. Another client wants the highest return possible subject to the constraint that you limit his standard deviation to be no more than 12%. Which client is more risk averse?

KAPLAN SCHWESER

22. Investment Management Inc. (IMI) uses the capital market line to make asset allocation recommendations. IMI derives the following forecasts:

    - Expected return on the market portfolio: 12%.
    - Standard deviation on the market portfolio: 20%.
    - Risk-free rate: 5%.

    Samuel Johnson seeks IMI's advice for a portfolio asset allocation. Johnson informs IMI that he wants the standard deviation of the portfolio to equal half of the standard deviation for the market portfolio. Using the capital market line, what expected return can IMI provide subject to Johnson's risk constraint?

For Problems 23 through 26: Suppose that the borrowing rate that your client faces is 9%. Assume that the S&P 500 index has an expected return of 13% and standard deviation of 25%, that $r_f = 5\%$, and that your fund has the parameters given in Problem 21.

23. Draw a diagram of your client's CML, accounting for the higher borrowing rate. Superimpose on it two sets of indifference curves, one for a client who will choose to borrow, and one who will invest in both the index fund and a money market fund.

24. What is the range of risk aversion for which a client will neither borrow nor lend, that is, for which $y = 1$?

25. Solve Problems 23 and 24 for a client who uses your fund rather than an index fund.

26. What is the largest percentage fee that a client who currently is lending ($y < 1$) will be willing to pay to invest in your fund? What about a client who is borrowing ($y > 1$)?

Challenge

For Challenge Problems 27, 28, and 29: You estimate that a passive portfolio, that is, one invested in a risky portfolio that mimics the S&P 500 stock index, yields an expected rate of return of 13% with a standard deviation of 25%. You manage an active portfolio with expected return 18% and standard deviation 28%. The risk-free rate is 8%.

27. Draw the CML and your funds' CAL on an expected return–standard deviation diagram.

    a. What is the slope of the CML?

    b. Characterize in one short paragraph the advantage of your fund over the passive fund.

28. Your client ponders whether to switch the 70% that is invested in your fund to the passive portfolio.

    a. Explain to your client the disadvantage of the switch.

    b. Show him the maximum fee you could charge (as a percentage of the investment in your fund, deducted at the end of the year) that would leave him at least as well off investing in your fund as in the passive one. (*Hint:* The fee will lower the slope of his CAL by reducing the expected return net of the fee.)

29. Consider again the client in Problem 19 with $A = 3.5$.

    a. If he chose to invest in the passive portfolio, what proportion, *y*, would he select?

    b. Is the fee (percentage of the investment in your fund, deducted at the end of the year) that you can charge to make the client indifferent between your fund and the passive strategy affected by his capital allocation decision (i.e., his choice of *y*)?

Use the following data in answering CFA Problems 1–3:

CFA® PROBLEMS

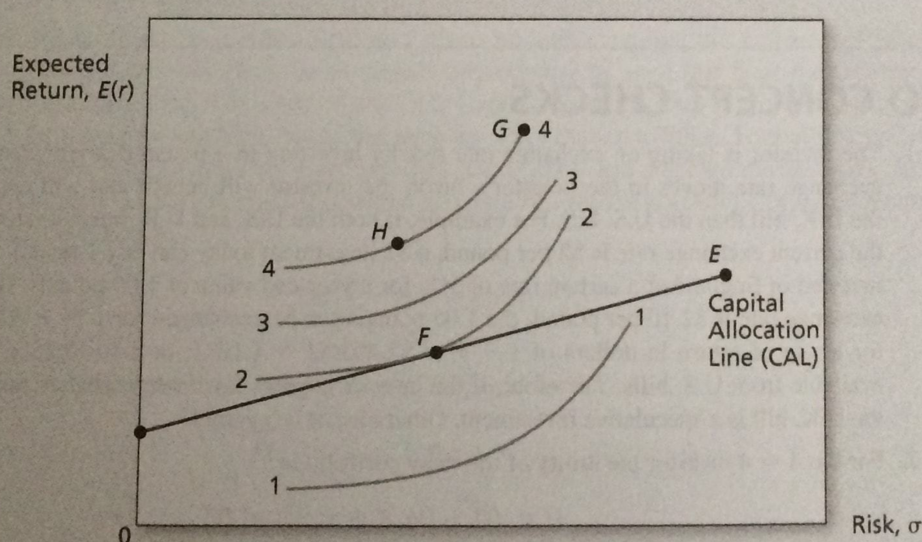
Utility Formula Data

Investment	Expected Return, $E(r)$	Standard Deviation, σ
1	.12	.30
2	.15	.50
3	.21	.16
4	.24	.21

$U = E(r) - \frac{1}{2}A\sigma^2$, where $A = 4$

1. On the basis of the utility formula above, which investment would you select if you were risk averse with $A = 4$?
2. On the basis of the utility formula above, which investment would you select if you were risk neutral?
3. The variable (A) in the utility formula represents the:
   a. investor's return requirement.
   b. investor's aversion to risk.
   c. certainty equivalent rate of the portfolio.
   d. preference for one unit of return per four units of risk.

Use the following graph to answer CFA Problems 4 and 5.



4. Which indifference curve represents the greatest level of utility that can be achieved by the investor?
5. Which point designates the optimal portfolio of risky assets?
6. Given \$100,000 to invest, what is the expected risk premium in dollars of investing in equities versus risk-free T-bills on the basis of the following table?

Action	Probability	Expected Return
Invest in equities	.6	\$50,000
	.4	−\$30,000
Invest in risk-free T-bills	1.0	\$ 5,000

7. The change from a straight to a kinked capital allocation line is a result of the:
   a. Reward-to-volatility ratio increasing.
   b. Borrowing rate exceeding the lending rate.
   c. Investor's risk tolerance decreasing.
   d. Increase in the portfolio proportion of the risk-free asset.
8. You manage an equity fund with an expected risk premium of 10% and an expected standard deviation of 14%. The rate on Treasury bills is 6%. Your client chooses to invest $60,000 of her portfolio in your equity fund and $40,000 in a T-bill money market fund. What is the expected return and standard deviation of return on your client's portfolio?
9. What is the reward-to-volatility ratio for the *equity fund* in CFA Problem 8?

E-INVESTMENTS EXERCISES

There is a difference between an investor's *willingness* to take risk and his or her *ability* to take risk. Take the quizzes offered at the Web sites below and compare the results. If they are significantly different, which one would you use to determine an investment strategy?

http://mutualfunds.about.com/library/personalitytests/blrisktolerance.htm
http://mutualfunds.about.com/library/personalitytests/blriskcapacity.htm

SOLUTIONS TO CONCEPT CHECKS

1. The investor is taking on exchange rate risk by investing in a pound-denominated asset. If the exchange rate moves in the investor's favor, the investor will benefit and will earn more from the U.K. bill than the U.S. bill. For example, if both the U.S. and U.K. interest rates are 5%, and the current exchange rate is $2 per pound, a $2 investment today can buy 1 pound, which can be invested in England at a certain rate of 5%, for a year-end value of 1.05 pounds. If the year-end exchange rate is $2.10 per pound, the 1.05 pounds can be exchanged for 1.05 × $2.10 = $2.205 for a rate of return in dollars of $1 + r =$ $2.205/$2 = 1.1025, or $r =$ 10.25%, more than is available from U.S. bills. Therefore, if the investor expects favorable exchange rate movements, the U.K. bill is a speculative investment. Otherwise, it is a gamble.
2. For the $A = 4$ investor the utility of the risky portfolio is

$$U = .02 - (\tfrac{1}{2} \times 4 \times .3^2) = .02$$

while the utility of bills is

$$U = .07 - (\tfrac{1}{2} \times 4 \times 0) = .07$$

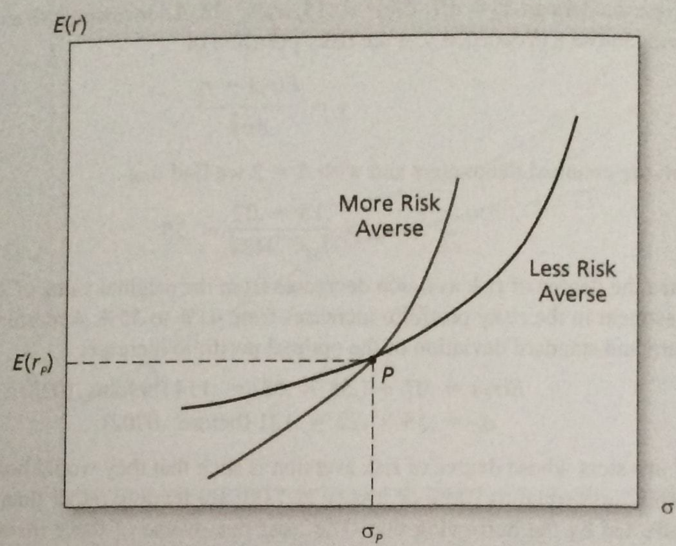
The investor will prefer bills to the risky portfolio. (Of course, a mixture of bills and the portfolio might be even better, but that is not a choice here.)

Even for the $A = 2$ investor, the utility of the risky portfolio is

$$U = .20 - (\tfrac{1}{2} \times 2 \times .3^2) = .11$$

while the utility of bills is again .07. The less risk-averse investor prefers the risky portfolio.

3. The less risk-averse investor has a shallower indifference curve. An increase in risk requires less increase in expected return to restore utility to the original level.



4. Holding 50% of your invested capital in Ready Assets means that your investment proportion in the risky portfolio is reduced from 70% to 50%.

   Your risky portfolio is constructed to invest 54% in E and 46% in B. Thus the proportion of E in your overall portfolio is $.5 \times 54\% = 27\%$, and the dollar value of your position in E is $\$300{,}000 \times .27 = \$81{,}000$.

5. In the expected return–standard deviation plane all portfolios that are constructed from the same risky and risk-free funds (with various proportions) lie on a line from the risk-free rate through the risky fund. The slope of the CAL (capital allocation line) is the same everywhere; hence the reward-to-volatility ratio is the same for all of these portfolios. Formally, if you invest a proportion, y, in a risky fund with expected return $E(r_P)$ and standard deviation σ_P, and the remainder, $1 - y$, in a risk-free asset with a sure rate r_f, then the portfolio's expected return and standard deviation are

$$E(r_C) = r_f + y[E(r_P) - r_f]$$
$$\sigma_C = y\sigma_P$$

   and therefore the reward-to-volatility ratio of this portfolio is

$$S_C = \frac{E(r_C) - r_f}{\sigma_C} = \frac{y[E(r_P) - r_f]}{y\sigma_P} = \frac{E(r_P) - r_f}{\sigma_P}$$

   which is independent of the proportion y.

6. The lending and borrowing rates are unchanged at $r_f = 7\%$, $r_f^B = 9\%$. The standard deviation of the risky portfolio is still 22%, but its expected rate of return shifts from 15% to 17%.

   The slope of the two-part CAL is

$$\frac{E(r_P) - r_f}{\sigma_P} \text{ for the lending range}$$

$$\frac{E(r_P) - r_f^B}{\sigma_P} \text{ for the borrowing range}$$

   Thus in both cases the slope increases: from 8/22 to 10/22 for the lending range, and from 6/22 to 8/22 for the borrowing range.

7. *a.* The parameters are $r_f = .07$, $E(r_P) = .15$, $\sigma_P = .22$. An investor with a degree of risk aversion A will choose a proportion y in the risky portfolio of

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

With the assumed parameters and with $A = 3$ we find that

$$y = \frac{.15 - .07}{3 \times .0484} = .55$$

When the degree of risk aversion decreases from the original value of 4 to the new value of 3, investment in the risky portfolio increases from 41% to 55%. Accordingly, both the expected return and standard deviation of the optimal portfolio increase:

$$E(r_C) = .07 + (.55 \times .08) = .114 \text{ (before: .1028)}$$
$$\sigma_C = .55 \times .22 = .121 \text{ (before: .0902)}$$

b. All investors whose degree of risk aversion is such that they would hold the risky portfolio in a proportion equal to 100% or less ($y \leq 1.00$) are lending rather than borrowing, and so are unaffected by the borrowing rate. The least risk-averse of these investors hold 100% in the risky portfolio ($y = 1$). We can solve for the degree of risk aversion of these "cut off" investors from the parameters of the investment opportunities:

$$y = 1 = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{.08}{.0484\,A}$$

which implies

$$A = \frac{.08}{.0484} = 1.65$$

Any investor who is more risk tolerant (that is, $A < 1.65$) would borrow *if* the borrowing rate were 7%. For borrowers,

$$y = \frac{E(r_P) - r_f^B}{A\sigma_P^2}$$

Suppose, for example, an investor has an A of 1.1. When $r_f = r_f^B = 7\%$, this investor chooses to invest in the risky portfolio:

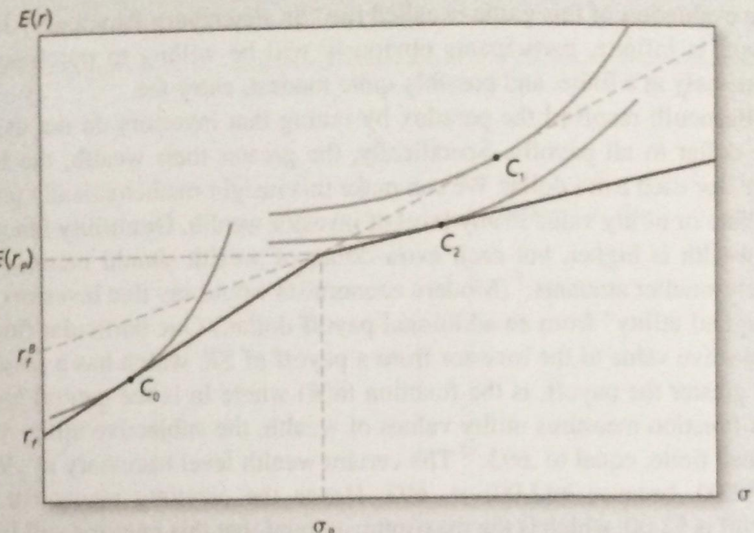
$$y = \frac{.08}{1.1 \times .0484} = 1.50$$

which means that the investor will borrow an amount equal to 50% of her own investment capital. Raise the borrowing rate, in this case to $r_f^B = 9\%$, and the investor will invest less in the risky asset. In that case:

$$y = \frac{.06}{1.1 \times .0484} = 1.13$$

and "only" 13% of her investment capital will be borrowed. Graphically, the line from r_f to the risky portfolio shows the CAL for lenders. The dashed part *would* be relevant if the borrowing rate equaled the lending rate. When the borrowing rate exceeds the lending rate, the CAL is kinked at the point corresponding to the risky portfolio.

The following figure shows indifference curves of two investors. The steeper indifference curve portrays the more risk-averse investor, who chooses portfolio C_0, which involves lending. This investor's choice is unaffected by the borrowing rate. The more risk-tolerant investor is portrayed by the shallower-sloped indifference curves. If the lending rate equaled the borrowing rate, this investor would choose portfolio C_1 on the dashed part of the CAL. When the borrowing rate goes up, this investor chooses portfolio C_2 (in the borrowing range of the kinked CAL), which involves less borrowing than before. This investor is hurt by the increase in the borrowing rate.



8. If all the investment parameters remain unchanged, the only reason for an investor to decrease the investment proportion in the risky asset is an increase in the degree of risk aversion. If you think that this is unlikely, then you have to reconsider your faith in your assumptions. Perhaps the S&P 500 is not a good proxy for the optimal risky portfolio. Perhaps investors expect a higher real rate on T-bills.

APPENDIX A: Risk Aversion, Expected Utility, and the St. Petersburg Paradox

We digress in this appendix to examine the rationale behind our contention that investors are risk averse. Recognition of risk aversion as central in investment decisions goes back at least to 1738. Daniel Bernoulli, one of a famous Swiss family of distinguished mathematicians, spent the years 1725 through 1733 in St. Petersburg, where he analyzed the following coin-toss game. To enter the game one pays an entry fee. Thereafter, a coin is tossed until the *first* head appears. The number of tails, denoted by n, that appears until the first head is tossed is used to compute the payoff, $\$R$, to the participant, as

$$R(n) = 2^n$$

The probability of no tails before the first head ($n = 0$) is 1/2 and the corresponding payoff is $2^0 = \$1$. The probability of one tail and then heads ($n = 1$) is $1/2 \times 1/2$ with payoff $2^1 = \$2$, the probability of two tails and then heads ($n = 2$) is $1/2 \times 1/2 \times 1/2$, and so forth.

The following table illustrates the probabilities and payoffs for various outcomes:

Tails	Probability	Payoff = \$ R(n)	Probability × Payoff
0	1/2	\$1	\$1/2
1	1/4	\$2	\$1/2
2	1/8	\$4	\$1/2
3	1/16	\$8	\$1/2
⋮	⋮	⋮	⋮
n	$(1/2)^{n+1}$	$\$2^n$	\$1/2

The expected payoff is therefore

$$E(R) = \sum_{n=0}^{\infty} \Pr(n)R(n) = \tfrac{1}{2} + \tfrac{1}{2} + \cdots = \infty$$

The evaluation of this game is called the "St. Petersburg Paradox." Although the expected payoff is infinite, participants obviously will be willing to purchase tickets to play the game only at a finite, and possibly quite modest, entry fee.

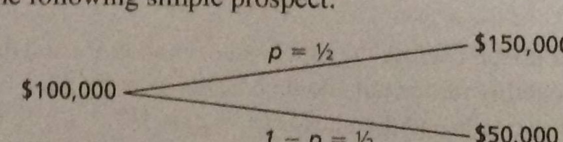
Bernoulli resolved the paradox by noting that investors do not assign the same value per dollar to all payoffs. Specifically, the greater their wealth, the less their "appreciation" for each extra dollar. We can make this insight mathematically precise by assigning a welfare or utility value to any level of investor wealth. Our utility function should increase as wealth is higher, but each extra dollar of wealth should increase utility by progressively smaller amounts.[9] (Modern economists would say that investors exhibit "decreasing marginal utility" from an additional payoff dollar.) One particular function that assigns a subjective value to the investor from a payoff of $\$R$, which has a smaller value per dollar the greater the payoff, is the function $\ln(R)$ where ln is the natural logarithm function. If this function measures utility values of wealth, the subjective utility value of the game is indeed finite, equal to .693.[10] The certain wealth level necessary to yield this utility value is \$2.00, because $\ln(2.00) = .693$. Hence the certainty equivalent value of the risky payoff is \$2.00, which is the maximum amount that this investor will pay to play the game.

Von Neumann and Morgenstern adapted this approach to investment theory in a complete axiomatic system in 1946. Avoiding unnecessary technical detail, we restrict ourselves here to an intuitive exposition of the rationale for risk aversion.

Imagine two individuals who are identical twins, except that one of them is less fortunate than the other. Peter has only \$1,000 to his name while Paul has a net worth of \$200,000. How many hours of work would each twin be willing to offer to earn one extra dollar? It is likely that Peter (the poor twin) has more essential uses for the extra money than does Paul. Therefore, Peter will offer more hours. In other words, Peter derives a greater personal welfare or assigns a greater "utility" value to the 1,001st dollar than Paul does to the 200,001st. Figure 6A.1 depicts graphically the relationship between the wealth and the utility value of wealth that is consistent with this notion of decreasing marginal utility.

Individuals have different rates of decrease in their marginal utility of wealth. What is constant is the *principle* that the per-dollar increment to utility decreases with wealth. Functions that exhibit the property of decreasing per-unit value as the number of units grows are called concave. A simple example is the log function, familiar from high school mathematics. Of course, a log function will not fit all investors, but it is consistent with the risk aversion that we assume for all investors.

Now consider the following simple prospect:



This is a fair game in that the expected profit is zero. Suppose, however, that the curve in Figure 6A.1 represents the investor's utility value of wealth, assuming a log utility function. Figure 6A.2 shows this curve with numerical values marked.

Figure 6A.2 shows that the loss in utility from losing \$50,000 exceeds the gain from winning \$50,000. Consider the gain first. With probability $p = .5$, wealth goes from \$100,000

[9] This utility is similar in spirit to the one that assigns a satisfaction level to portfolios with given risk and return attributes. However, the utility function here refers not to investors' satisfaction with alternative portfolio choices but only to the subjective welfare they derive from different levels of wealth.

[10] If we substitute the "utility" value, $\ln(R)$, for the dollar payoff, R, to obtain an expected utility value of the game (rather than expected dollar value), we have, calling $V(R)$ the expected utility,

$$V(R) = \sum_{n=0}^{\infty} \Pr(n) \ln[R(n)] = \sum_{n=0}^{\infty} (1/2)^{n+1} \ln(2^n) = .693$$

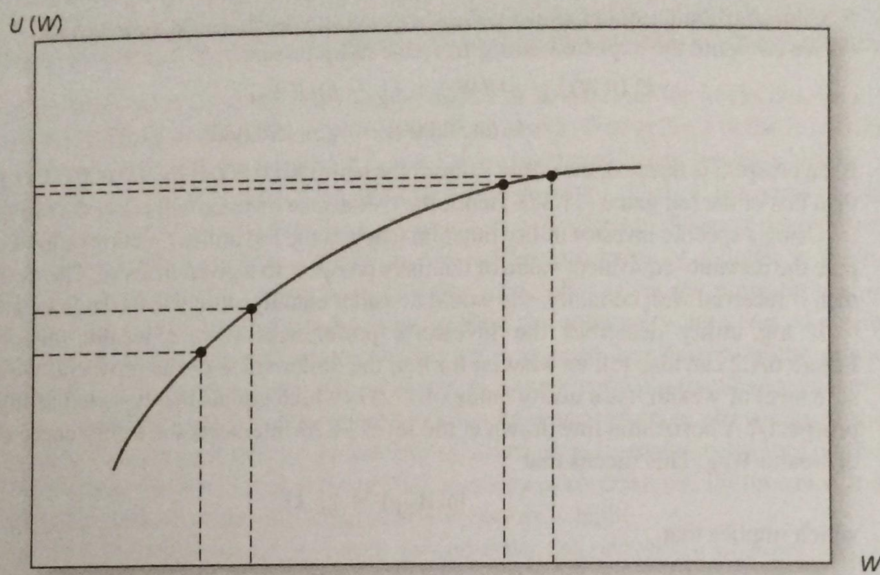


Figure 6A.1 Utility of wealth with a log utility function

to \$150,000. Using the log utility function, utility goes from ln(100,000) = 11.51 to ln(150,000) = 11.92, the distance G on the graph. This gain is G = 11.92 − 11.51 = .41. In expected utility terms, then, the gain is $pG = .5 \times .41 = .21$.

Now consider the possibility of coming up on the short end of the prospect. In that case, wealth goes from \$100,000 to \$50,000. The loss in utility, the distance L on the graph, is L = ln(100,000) − ln(50,000) = 11.51 − 10.82 = .69. Thus the loss in expected utility

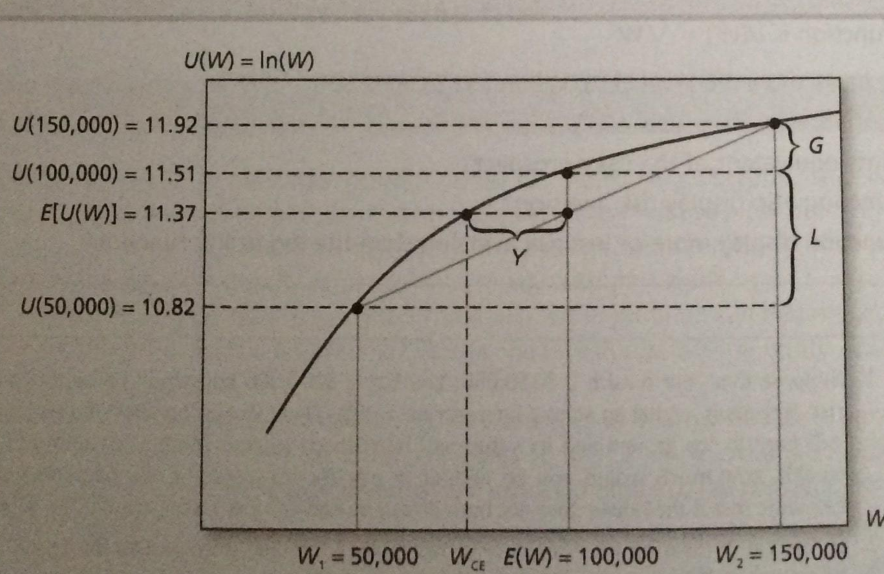


Figure 6A.2 Fair games and expected utility terms is $(1 - p)L = .5 \times .69 = .35$, which exceeds the gain in expected utility from the possibility of winning the game.

We compute the expected utility from the risky prospect:

$$E[U(W)] = pU(W_1) + (1 - p)U(W_2)$$
$$= \tfrac{1}{2}\ln(50{,}000) + \tfrac{1}{2}\ln(150{,}000) = 11.37$$

If the prospect is rejected, the utility value of the (sure) \$100,000 is ln(100,000) = 11.51, greater than that of the fair game (11.37). Hence the risk-averse investor will reject the fair game.

Using a specific investor utility function (such as the log utility function) allows us to compute the certainty equivalent value of the risky prospect to a given investor. This is the amount that, if received with certainty, she would consider equally attractive as the risky prospect.

If log utility describes the investor's preferences toward wealth outcomes, then Figure 6A.2 can also tell us what is, for her, the dollar value of the prospect. We ask, What sure level of wealth has a utility value of 11.37 (which equals the expected utility from the prospect)? A horizontal line drawn at the level 11.37 intersects the utility curve at the level of wealth W_{CE}. This means that

$$\ln(W_{CE}) = 11.37$$

which implies that

$$W_{CE} = e^{11.37} = \$86{,}681.87$$

W_{CE} is therefore the certainty equivalent of the prospect. The distance Y in Figure 6A.2 is the penalty, or the downward adjustment, to the expected profit that is attributable to the risk of the prospect.

$$Y = E(W) - W_{CE} = \$100{,}000 - \$86{,}681.87 = \$13{,}318.13$$

This investor views \$86,681.87 for certain as being equal in utility value as \$100,000 at risk. Therefore, she would be indifferent between the two.

CONCEPT CHECK 6A.1

Suppose the utility function is $U(W) = \sqrt{W}$

a. What is the utility level at wealth levels \$50,000 and \$150,000?

b. What is expected utility if p still equals .5?

c. What is the certainty equivalent of the risky prospect?

d. Does this utility function also display risk aversion?

e. Does this utility function display more or less risk aversion than the log utility function?

PROBLEMS: APPENDIX A

1. Suppose that your wealth is \$250,000. You buy a \$200,000 house and invest the remainder in a risk-free asset paying an annual interest rate of 6%. There is a probability of .001 that your house will burn to the ground and its value will be reduced to zero. With a log utility of end-of-year wealth, how much would you be willing to pay for insurance (at the beginning of the year)? (Assume that if the house does not burn down, its end-of-year value still will be \$200,000.)
2. If the cost of insuring your house is \$1 per \$1,000 of value, what will be the certainty equivalent of your end-of-year wealth if you insure your house at:

   a. ½ its value.

   b. Its full value.

   c. 1½ times its value.

APPENDIX B: Utility Functions and Equilibrium Prices of Insurance Contracts

The utility function of an individual investor allows us to measure the subjective value the individual would place on a dollar at various levels of wealth. Essentially, a dollar in bad times (when wealth is low) is more valuable than a dollar in good times (when wealth is high).

Suppose that all investors hold the risky S&P 500 portfolio. Then, if the portfolio value falls in a worse-than-expected economy, all investors will, albeit to different degrees, experience a "low-wealth" scenario. Therefore, the equilibrium value of a dollar in the low-wealth economy would be higher than the value of a dollar when the portfolio performs better than expected. This observation helps explain the apparently high cost of portfolio insurance that we encountered when considering long-term investments in the previous chapter. It also helps explain why an investment in a stock portfolio (and hence in individual stocks) has a risk premium that appears to be so high and results in probability of shortfall that is so low. Despite the low probability of shortfall risk, stocks still do not dominate the lower-return risk-free bond, because if an investment shortfall should transpire, it will coincide with states in which the value of dollar returns is high.

Does revealed behavior of investors demonstrate risk aversion? Looking at prices and past rates of return in financial markets, we can answer with a resounding yes. With remarkable consistency, riskier bonds are sold at lower prices than are safer ones with otherwise similar characteristics. Riskier stocks also have provided higher average rates of return over long periods of time than less risky assets such as T-bills. For example, over the 1926 to 2012 period, the average rate of return on the S&P 500 portfolio exceeded the T-bill return by around 8% per year.

It is abundantly clear from financial data that the average, or representative, investor exhibits substantial risk aversion. For readers who recognize that financial assets are priced to compensate for risk by providing a risk premium and at the same time feel the urge for some gambling, we have a constructive recommendation: Direct your gambling impulse to investment in financial markets. As Von Neumann once said, "The stock market is a casino with the odds in your favor." A small risk-seeking investment may provide all the excitement you want with a positive expected return to boot!

APPENDIX C: The Kelly Criterion

To take a step upwards from the gamble of the St. Petersburg Paradox, consider a sequence of identical one-period investment prospects, each with two possible payoffs (with rates of return expressed as decimals): a positive excess return, b, with probability p, and a negative excess return, $-a$ $(a > 0)$, with probability $q = 1 - p$. J.L. Kelly[11] considered this a basic form of a capital allocation problem and determined the optimal investment in such a sequence of bets for an investor with a log utility function (described in Appendix A).

Investing a fraction y in the prospect and the remainder in the risk-free asset provides a total rate of return of $1 + r + by$ with probability p, or $1 + r - ay$ with probability q. Because Kelly employs a log utility function, the expected utility of the prospect, per dollar of initial wealth, is:

$$E[U(y)] = p \ln(1 + r + yb) + q \ln(1 + r - ay) \tag{6.C.1}$$

[11] J.L. Kelly Jr., "A New Interpretation of Information Rate," *Bell System Technical Journal* 35 (1956), 917–56.

The investment that maximizes the expected utility has become known as the Kelly criterion (or Kelly formula). The criterion states that the fraction of total wealth invested in the risky prospect is independent of wealth and is given by:

$$y = (1 + r)\left(\frac{p}{a} - \frac{q}{b}\right) \tag{6.C.2}$$

This will be the investor's asset allocation in each period.

The Kelly formula calls for investing more in the prospect when p and b are large and less when q and a are large. Risk aversion stands out since, when the gains and losses are equal, i.e., when $a = b$, $y = (1 + r)(p - q)/a$, the larger the win/loss spread (corresponding to larger values of a and b), the smaller the fraction invested. A higher interest rate also increases risk taking (an income effect).

Kelly's rule is based on the log utility function. One can show that investors who have such a utility function will, in each period, attempt to maximize the geometric mean of the portfolio return. So the Kelly formula also is a rule to maximize geometric mean, and it has several interesting properties: (1) It never risks ruin, since the fraction of wealth in the risky asset in Equation 6C.2 never exceeds $1/a$. (2) The probability that it will outperform any other strategy goes to 1 as the investment horizon goes to infinity. (3) It is myopic, meaning the optimal strategy is the same regardless of the investment horizon. (4) If you have a specified wealth goal (e.g., \$1 million), the strategy has the shortest expected time to that goal. Considerable literature has been devoted to the Kelly criterion.[12]

SOLUTION TO CONCEPT CHECK

A.1. *a.* $U(W) = \sqrt{W}$

$U(50{,}000) = \sqrt{50{,}000} = 223.61$

$U(150{,}000) = 387.30$

b. $E(U) = (.5 \times 223.61) + (.5 \times 387.30) = 305.45$

c. We must find W_{CE} that has utility level 305.45. Therefore

$$\sqrt{W_{CE}} = 305.45$$
$$W_{CE} = 305.45^2 = \$93{,}301$$

d. Yes. The certainty equivalent of the risky venture is less than the expected outcome of \$100,000.

e. The certainty equivalent of the risky venture to this investor is greater than it was for the log utility investor considered in the text. Hence this utility function displays less risk aversion.

[12] See, for example, L.C. MacLean, E.O. Thorp, W.T. Ziemba, Eds., *The Kelly Capital Growth Criterion: Theory and Practice* (World Scientific Handbook in Financial Economic Series), Singapore: World Scientific Publishing Co., 2010.