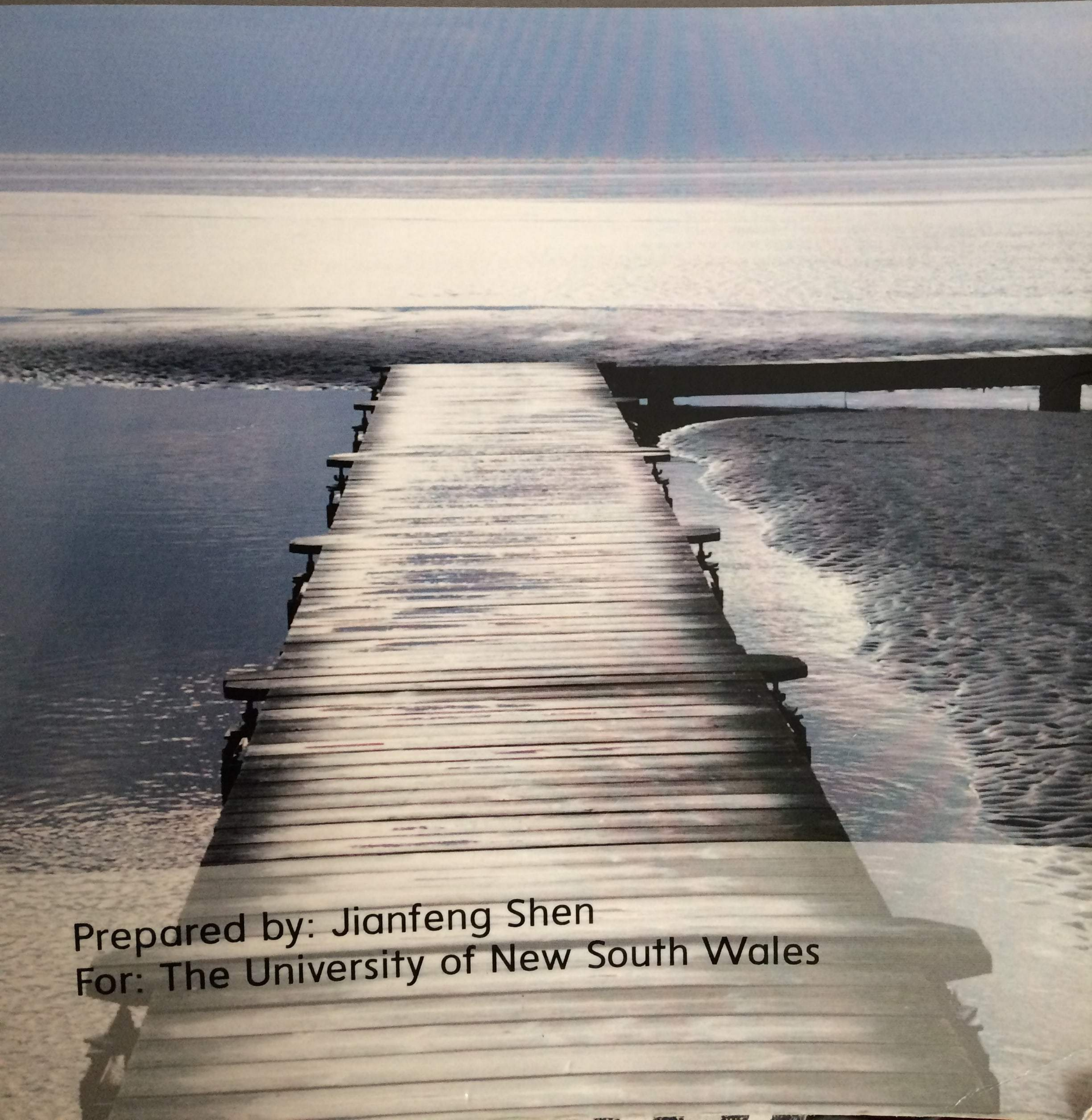
FINS2624

Portfolio Management



Prepared by: Jianfeng Shen
For: The University of New South Wales

Contents

1. Capital Allocation to Risky Assets 2
2. Optimal Risky Portfolios 39
3. Index Models 90
4. The Capital Asset Pricing Model 125
5. Arbitrage Pricing Theory and Multifactor Models of Risk and Return 158
6. The Efficient Market Hypothesis 183
7. Behavioral Finance and Technical Analysis 222
8. Bond Prices and Yields 249
9. The Term Structure of Interest Rates 291
10. Managing Bond Portfolios 319
11. Options Markets: Introduction 362
12. Option Valuation 406
13. Portfolio Performance Evaluation 455

PART II

CHAPTER SIX

Capital Allocation to Risky Assets

THE PROCESS OF constructing an overall portfolio requires you to: (1) select the composition of the risky portfolio and (2) decide how much to invest in it, directing the remaining investment budget to a risk-free investment. The second step is called *capital allocation to risky assets.*

Clearly, to decide on your capital allocation you need to know the risky portfolio and evaluate its properties. Can the construction of that risky portfolio be delegated to an expert? An affirmative answer is necessary for a viable investments management industry. A negative answer would require every investor to learn and implement portfolio management for himself.

To understand the existence of a portfolio management industry in the face of personal investor preferences, we need insight into the nature of risk aversion. We characterize a personal utility function that provides a score for the attractiveness of candidate overall portfolios on the basis of expected return and risk. By choosing the portfolio with the highest score, investors maximize their satisfaction with their choice of investments; that is, they achieve the optimal allocation of capital to risky assets.

The utility model also reveals the appropriate objective function for the construction of an optimal *risky* portfolio and thus explains how an industry can serve investors with highly diverse preferences without the need to know each of them personally.

6.1 Risk and Risk Aversion

In Chapter 5 we introduced the concepts of the holding-period return (HPR) and the excess rate of return over the risk-free rate. We also discussed estimation of the **risk premium** (the *expected* excess return) and the standard deviation of the excess return, which we use as the measure of portfolio risk. We demonstrated these concepts with a scenario analysis of a specific risky portfolio (Spreadsheet 5.1). To emphasize that bearing risk typically must be accompanied by a reward in the form of a risk premium, we first differentiate between speculation and gambling.

Risk, Speculation, and Gambling

One definition of *speculation* is "the assumption of considerable investment risk to obtain commensurate gain." However, this definition is useless without specifying what is meant by "considerable risk" and "commensurate gain."

By "considerable risk" we mean that the risk is sufficient to affect the decision. An individual might reject an investment that has a positive risk premium because the potential gain is insufficient to make up for the risk involved. By "commensurate gain" we mean a positive risk premium, that is, an expected profit greater than the risk-free alternative.

To gamble is "to bet or wager on an uncertain outcome." The central difference between gambling and speculation is the lack of "commensurate gain." Economically speaking, a gamble is the assumption of risk for enjoyment of the risk itself, whereas speculation is undertaken *in spite* of the risk involved because one perceives a favorable risk–return trade-off. To turn a gamble into a speculative venture requires an adequate risk premium to compensate risk-averse investors for the risks they bear. Hence, *risk aversion and speculation are consistent.* Notice that a risky investment with a risk premium of zero, sometimes called a **fair game,** amounts to a gamble. A risk-averse investor will reject it.

In some cases a gamble may *appear* to be speculation. Suppose two investors disagree sharply about the future exchange rate of the U.S. dollar against the British pound. They may choose to bet on the outcome: Paul will pay Mary $100 if the value of £1 exceeds $1.60 one year from now, whereas Mary will pay Paul if the pound is worth less than $1.60. There are only two relevant outcomes: (1) the pound will exceed $1.60, or (2) it will fall below $1.60. If both Paul and Mary agree on the probabilities of the two possible outcomes, and if neither party anticipates a loss, it must be that they assign $p = .5$ to each outcome. In that case the expected profit to both is zero and each has entered one side of a gambling prospect.

What is more likely, however, is that Paul and Mary assign different probabilities to the outcome. Mary assigns it $p > .5$, whereas Paul's assessment is $p < .5$. They perceive, subjectively, two different prospects. Economists call this case of differing beliefs "heterogeneous expectations." In such cases investors on each side of a financial position see themselves as speculating rather than gambling.

Both Paul and Mary should be asking, Why is the other willing to invest in the side of a risky prospect that I believe offers a negative expected profit? The ideal way to resolve heterogeneous beliefs is for Paul and Mary to "merge their information," that is, for each party to verify that he or she possesses all relevant information and processes the information properly. Of course, the acquisition of information and the extensive communication that is required to eliminate all heterogeneity in expectations is costly, and thus up to a point heterogeneous expectations cannot be taken as irrational. If, however, Paul and Mary enter such contracts frequently, they would recognize the information problem in one of two ways: Either they will realize that they are creating gambles when each wins half of the bets, or the consistent loser will admit that he or she has been betting on the basis of inferior forecasts.

CONCEPT CHECK 6.1

Assume that dollar-denominated T-bills in the United States and pound-denominated bills in the United Kingdom offer equal yields to maturity. Both are short-term assets, and both are free of default risk. Neither offers investors a risk premium. However, a U.S. investor who holds U.K. bills is subject to exchange rate risk, because the pounds earned on the U.K. bills eventually will be exchanged for dollars at the future exchange rate. Is the U.S. investor engaging in speculation or gambling?

Risk Aversion and Utility Values

The history of rates of return on various asset classes, as well as elaborate empirical studies, leave no doubt that risky assets command a risk premium in the marketplace. This implies that most investors are risk averse.

Investors who are **risk averse** reject investment portfolios that are fair games or worse. Risk-averse investors consider only risk-free or speculative prospects with positive risk premiums. Loosely speaking, a risk-averse investor "penalizes" the expected rate of return of a risky portfolio by a certain percentage (or penalizes the expected profit by a dollar amount) to account for the risk involved. The greater the risk, the larger the penalty. We believe that most investors would accept this view from simple introspection, but we discuss the question more fully in Appendixes A through C of this chapter.

To illustrate the issues we confront when choosing among portfolios with varying degrees of risk, suppose the risk-free rate is 5% and that an investor considers three alternative risky portfolios as shown in Table 6.1. The risk premiums and degrees of risk (standard deviation, SD) represent the properties of low-risk bonds (*L*), high-risk bonds (*M*), and large stocks (*H*). Accordingly, these portfolios offer progressively higher risk premiums to compensate for greater risk. How might investors choose among them?

Intuitively, a portfolio is more attractive when its expected return is higher and its risk is lower. But when risk increases along with return, the most attractive portfolio is not obvious. How can investors quantify the rate at which they are willing to trade off return against risk?

We will assume that each investor can assign a welfare, or **utility**, score to competing portfolios on the basis of the expected return and risk of those portfolios. Higher utility values are assigned to portfolios with more attractive risk–return profiles. Portfolios receive higher utility scores for higher expected returns and lower scores for higher volatility. Many particular "scoring" systems are legitimate. One reasonable function that has been employed by both financial theorists and the CFA Institute assigns a portfolio with expected return $E(r)$ and variance of returns σ^2 the following utility score:

$$U = E(r) - \tfrac{1}{2}A\sigma^2 \tag{6.1}$$

where U is the utility value and A is an index of the investor's risk aversion. The factor of ½ is just a scaling convention. To use Equation 6.1, rates of return must be expressed as decimals rather than percentages. Notice that the portfolio in question here is the all-wealth investment. Hence, assuming normality, standard deviation is the appropriate measure of risk.

Equation 6.1 is consistent with the notion that utility is enhanced by high expected returns and diminished by high risk. Notice that risk-free portfolios receive a utility score equal to their (known) rate of return, because they receive no penalty for risk. The extent to which the variance of risky portfolios lowers utility depends on A, the

Table 6.1

Available risky portfolios (Risk-free rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

investor's degree of risk aversion. More risk-averse investors (who have the larger values of A) penalize risky investments more severely. Investors choosing among competing investment portfolios will select the one providing the highest utility level. The box on page 174 discusses some techniques that financial advisers use to gauge the risk aversion of their clients.

Example 6.1 Evaluating Investments by Using Utility Scores

Consider three investors with different degrees of risk aversion: $A_1 = 2$, $A_2 = 3.5$, and $A_3 = 5$, all of whom are evaluating the three portfolios in Table 6.1. Because the risk-free rate is assumed to be 5%, Equation 6.1 implies that all three investors would assign a utility score of .05 to the risk-free alternative. Table 6.2 presents the utility scores that would be assigned by each investor to each portfolio. The portfolio with the highest utility score for each investor appears in bold. Notice that the high-risk portfolio, *H*, would be chosen only by the investor with the lowest degree of risk aversion, $A_1 = 2$, while the low-risk portfolio, *L*, would be passed over even by the most risk-averse of our three investors. All three portfolios beat the risk-free alternative for the investors with levels of risk aversion given in the table.

We can interpret the utility score of *risky* portfolios as a **certainty equivalent rate** of return. The certainty equivalent rate is the rate that a risk-free investment would need to offer to provide the same utility score as the risky portfolio. In other words, it is the rate that, if earned with certainty, would provide a utility score equivalent to that of the portfolio in question. The certainty equivalent rate of return is a natural way to compare the utility values of competing portfolios.

A portfolio can be desirable only if its certainty equivalent return exceeds that of the risk-free alternative. A sufficiently risk-averse investor may assign any risky portfolio, even one with a positive risk premium, a certainty equivalent rate of return that is below the risk-free rate, which will cause the investor to reject the risky portfolio. At the same time, a less risk-averse investor may assign the same portfolio a certainty equivalent rate that exceeds the risk-free rate and thus will prefer the portfolio to the risk-free alternative. If the risk premium is zero or negative to begin with, any downward adjustment to utility only makes the portfolio look worse. Its certainty equivalent rate will be below that of the risk-free alternative for all risk-averse investors.

Investor Risk Aversion (A)	Utility Score of Portfolio *L* [$E(r) = .07$; $\sigma = .05$]	Utility Score of Portfolio *M* [$E(r) = .09$; $\sigma = .10$]	Utility Score of Portfolio *H* [$E(r) = .13$; $\sigma = .20$]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	**$.13 - ½ \times 2 \times .2^2 = .09$**
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	**$.09 - ½ \times 3.5 \times .1^2 = .0725$**	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	**$.09 - ½ \times 5 \times .1^2 = .0650$**	$.13 - ½ \times 5 \times .2^2 = .03$

Table 6.2

Utility scores of alternative portfolios for investors with varying degrees of risk aversion

CONCEPT CHECK 6.2

A portfolio has an expected rate of return of 20% and standard deviation of 30%. T-bills offer a safe rate of return of 7%. Would an investor with risk-aversion parameter $A = 4$ prefer to invest in T-bills or the risky portfolio? What if $A = 2$?

In contrast to risk-averse investors, **risk-neutral** investors (with $A = 0$) judge risky prospects solely by their expected rates of return. The level of risk is irrelevant to the risk-neutral investor, meaning that there is no penalty for risk. For this investor a portfolio's certainty equivalent rate is simply its expected rate of return.

A **risk lover** (for whom $A < 0$) is happy to engage in fair games and gambles; this investor adjusts the expected return *upward* to take into account the "fun" of confronting the prospect's risk. Risk lovers will always take a fair game because their upward adjustment of utility for risk gives the fair game a certainty equivalent that exceeds the alternative of the risk-free investment.

We can depict the individual's trade-off between risk and return by plotting the characteristics of portfolios that would be equally attractive on a graph with axes measuring the expected value and standard deviation of portfolio returns. Figure 6.1 plots the characteristics of one portfolio denoted *P*.

Portfolio *P*, which has expected return $E(r_P)$ and standard deviation σ_P, is preferred by risk-averse investors to any portfolio in quadrant IV because its expected return is equal to or greater than any portfolio in that quadrant and its standard deviation is equal to or smaller than any portfolio in that quadrant. Conversely, any portfolio in quadrant I dominates portfolio *P* because its expected return is equal to or greater than *P*'s and its standard deviation is equal to or smaller than *P*'s.

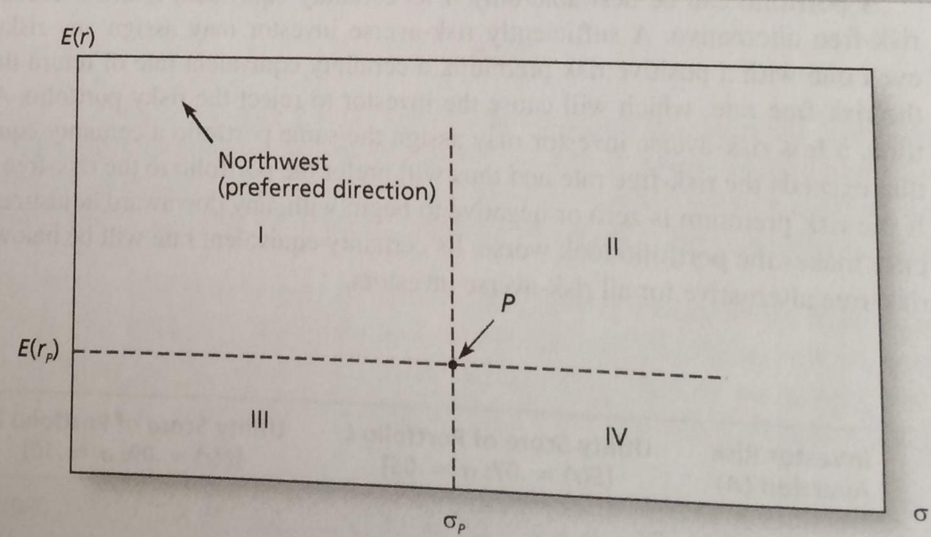


Figure 6.1 The trade-off between risk and return of a potential investment portfolio, *P*

This is the mean-standard deviation, or equivalently, **mean-variance (M-V) criterion**. It can be stated as follows: portfolio *A* dominates *B* if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

and at least one inequality is strict (to rule out indifference).

In the expected return–standard deviation plane in Figure 6.1, the preferred direction is northwest, because in this direction we simultaneously increase the expected return *and* decrease the variance of the rate of return. Any portfolio that lies northwest of *P* is superior to it.

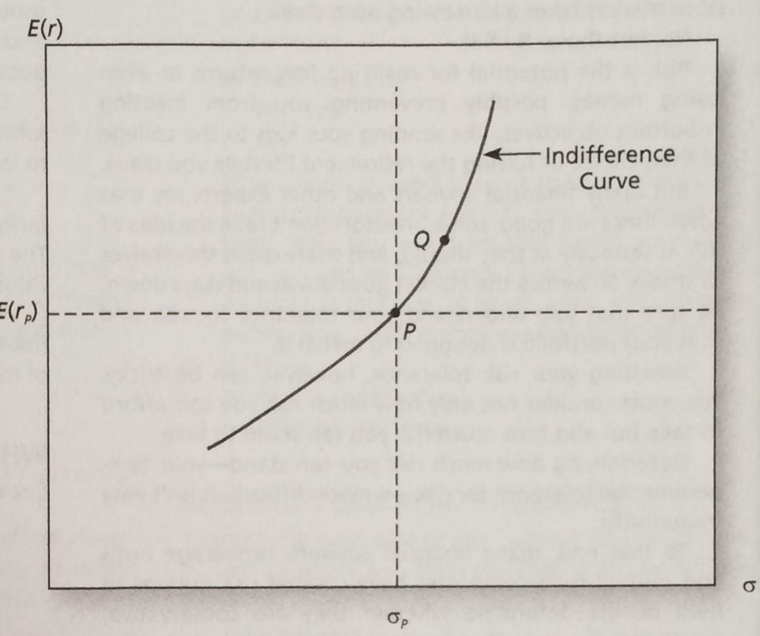


Figure 6.2 The indifference curve

What can be said about portfolios in quadrants II and III? Their desirability, compared with *P*, depends on the exact nature of the investor's risk aversion. Suppose an investor identifies all portfolios that are equally attractive as portfolio *P*. Starting at *P*, an increase in standard deviation lowers utility; it must be compensated for by an increase in expected return. Thus point *Q* in Figure 6.2 is equally desirable to this investor as *P*. Investors will be equally attracted to portfolios with high risk and high expected returns compared with other portfolios with lower risk but lower expected returns. These equally preferred portfolios will lie in the mean–standard deviation plane on a curve called the **indifference curve,** which connects all portfolio points with the same utility value (Figure 6.2).

To determine some of the points that appear on the indifference curve, examine the utility values of several possible portfolios for an investor with $A = 4$, presented in Table 6.3. Note that each portfolio offers

CONCEPT CHECK **6.3**

a. How will the indifference curve of a less risk-averse investor compare to the indifference curve drawn in Figure 6.2?

b. Draw both indifference curves passing through point *P*.

Table 6.3

Utility values of possible portfolios for investor with risk aversion, $A = 4$

Expected Return, $E(r)$	Standard Deviation, σ	Utility $= E(r) - \frac{1}{2} A\sigma^2$
.10	.200	$.10 - .5 \times 4 \times .04 = .02$
.15	.255	$.15 - .5 \times 4 \times .065 = .02$
.20	.300	$.20 - .5 \times 4 \times .09 = .02$
.25	.339	$.25 - .5 \times 4 \times .115 = .02$

WORDS FROM THE STREET

Time for Investing's Four-Letter Word

What four-letter word should pop into mind when the stock market takes a harrowing nose dive?

No, not those. R-I-S-K.

Risk is the potential for realizing low returns or even losing money, possibly preventing you from meeting important objectives, like sending your kids to the college of their choice or having the retirement lifestyle you crave.

But many financial advisers and other experts say that when times are good, some investors don't take the idea of risk as seriously as they should, and overexpose themselves to stocks. So before the market goes down and stays down, be sure that you understand your tolerance for risk and that your portfolio is designed to match it.

Assessing your risk tolerance, however, can be tricky. You must consider not only how much risk you can *afford* to take but also how much risk you can *stand* to take.

Determining how much risk you can stand—your temperamental tolerance for risk—is more difficult. It isn't easy to quantify.

To that end, many financial advisers, brokerage firms and mutual-fund companies have created risk quizzes to help people determine whether they are conservative, moderate or aggressive investors. Some firms that offer such quizzes include Merrill Lynch, T. Rowe Price Associates Inc., Baltimore, Zurich Group Inc.'s Scudder Kemper Investments Inc., New York, and Vanguard Group in Malvern, Pa.

Typically, risk questionnaires include seven to 10 questions about a person's investing experience, financial security and tendency to make risky or conservative choices.

The benefit of the questionnaires is that they are an objective resource people can use to get at least a rough idea of their risk tolerance. "It's impossible for someone to assess their risk tolerance alone," says Mr. Bernstein. "I may say I don't like risk, yet will take more risk than the average person."

Many experts warn, however, that the questionnaires should be used simply as a first step to assessing risk tolerance. "They are not precise," says Ron Meier, a certified public accountant.

The second step, many experts agree, is to ask yourself some difficult questions, such as: How much you can stand to lose over the long term?

"Most people can stand to lose a heck of a lot temporarily," says Mr. Schatsky, a financial adviser in New York. The real acid test, he says, is how much of your portfolio's value you can stand to lose over months or years.

As it turns out, most people rank as middle-of-the-road risk-takers, say several advisers. "Only about 10% to 15% of my clients are aggressive," says Mr. Roge.

WHAT'S YOUR RISK TOLERANCE?

Circle the letter that corresponds to your answer

1. Just 60 days after you put money into an investment, its price falls 20%. Assuming none of the fundamentals have changed, what would you do?
   a. Sell to avoid further worry and try something else
   b. Do nothing and wait for the investment to come back
   c. Buy more. It was a good investment before; now it's a cheap investment, too
2. Now look at the previous question another way. Your investment fell 20%, but it's part of a portfolio being used to meet investment goals with three different time horizons.

2A. What would you do if the goal were five years away?
   a. Sell
   b. Do nothing
   c. Buy more

identical utility, because the portfolios with higher expected return also have higher risk (standard deviation).

Estimating Risk Aversion

How can we estimate the levels of risk aversion of individual investors? A number of methods may be used. The questionnaire in the nearby box is of the simplest variety and, indeed, can distinguish only between high (conservative), medium (moderate), or low (aggressive) levels of the coefficient of risk aversion. More complex questionnaires, allowing subjects to pinpoint specific levels of risk aversion coefficients, ask would-be investors to choose from various sets of hypothetical lotteries.

Access to investment accounts of active investors would provide observations of how portfolio composition changes over time. Coupling this information with estimates of the risk–return combinations of these positions would in principle allow us to calculate investors' implied risk aversion coefficients.

Finally, researchers track behavior of groups of individuals to obtain average degrees of risk aversion. These studies range from observed purchase of insurance policies and durables warranties to labor supply and aggregate consumption behavior.

WORDS FROM THE STREET

2B. What would you do if the goal were 15 years away?

a. Sell
b. Do nothing
c. Buy more

2C. What would you do if the goal were 30 years away?

a. Sell
b. Do nothing
c. Buy more

3. The price of your retirement investment jumps 25% a month after you buy it. Again, the fundamentals haven't changed. After you finish gloating, what do you do?

a. Sell it and lock in your gains
b. Stay put and hope for more gain
c. Buy more; it could go higher

4. You're investing for retirement, which is 15 years away. Which would you rather do?

a. Invest in a money-market fund or guaranteed investment contract, giving up the possibility of major gains, but virtually assuring the safety of your principal
b. Invest in a 50-50 mix of bond funds and stock funds, in hopes of getting some growth, but also giving yourself some protection in the form of steady income
c. Invest in aggressive growth mutual funds whose value will probably fluctuate significantly during the year, but have the potential for impressive gains over five or 10 years

5. You just won a big prize! But which one? It's up to you.

a. $2,000 in cash
b. A 50% chance to win $5,000
c. A 20% chance to win $15,000

6. A good investment opportunity just came along. But you have to borrow money to get in. Would you take out a loan?

a. Definitely not
b. Perhaps
c. Yes

7. Your company is selling stock to its employees. In three years, management plans to take the company public. Until then, you won't be able to sell your shares and you will get no dividends. But your investment could multiply as much as 10 times when the company goes public. How much money would you invest?

a. None
b. Two months' salary
c. Four months' salary

SCORING YOUR RISK TOLERANCE

To score the quiz, add up the number of answers you gave in each category a–c, then multiply as shown to find your score

(a) answers ______ × 1 = ______ points
(b) answers ______ × 2 = ______ points
(c) answers ______ × 3 = ______ points

YOUR SCORE ______ points

If you scored . . . You may be a:

9–14 points	Conservative investor
5–21 points	Moderate investor
22–27 points	Aggressive investor

Source: Reprinted with permission from *The Wall Street Journal*.

6.2 Capital Allocation across Risky and Risk-Free Portfolios

History shows us that long-term bonds have been riskier investments than Treasury bills and that stocks have been riskier still. On the other hand, the riskier investments have offered higher average returns. Investors, of course, do not make all-or-nothing choices from these investment classes. They can and do construct their portfolios using securities from all asset classes. Some of the portfolio may be in risk-free Treasury bills, some in high-risk stocks.

The most straightforward way to control the risk of the portfolio is through the fraction of the portfolio invested in Treasury bills and other safe money market securities versus risky assets. A capital allocation decision implies an asset allocation choice among broad investment classes, rather than among the specific securities within each asset class. Most investment professionals consider asset allocation the most important part of portfolio construction. Consider this statement by John Bogle, made when he was chairman of the Vanguard Group of Investment Companies:

> The most fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? . . . That decision [has been shown to account] for an astonishing 94% of the differences in total returns achieved by institutionally managed pension funds. . . . There is no reason to believe that the same relationship does not also hold true for individual investors.[1]

Therefore, we start our discussion of the risk–return trade-off available to investors by examining the most basic asset allocation choice: the choice of how much of the portfolio to place in risk-free money market securities versus other risky asset classes.

We denote the investor's portfolio of risky assets as *P* and the risk-free asset as *F*. We assume for the sake of illustration that the risky component of the investor's overall portfolio comprises two mutual funds, one invested in stocks and the other invested in long-term bonds. For now, we take the composition of the risky portfolio as given and focus only on the allocation between it and risk-free securities. In the next chapter, we turn to asset allocation and security selection across risky assets.

When we shift wealth from the risky portfolio to the risk-free asset, we do not change the relative proportions of the various risky assets within the risky portfolio. Rather, we reduce the relative weight of the risky portfolio as a whole in favor of risk-free assets.

For example, assume that the total market value of an initial portfolio is $300,000, of which $90,000 is invested in the Ready Asset money market fund, a risk-free asset for practical purposes. The remaining $210,000 is invested in risky securities—$113,400 in equities (*E*) and $96,600 in long-term bonds (*B*). The equities and bond holdings comprise "the" risky portfolio, 54% in *E* and 46% in *B*:

$$E: \quad w_E = \frac{113{,}400}{210{,}000} = .54$$

$$B: \quad w_B = \frac{96{,}600}{210{,}000} = .46$$

The weight of the risky portfolio, *P*, in the **complete portfolio,** including risk-free *and* risky investments, is denoted by *y*:

$$y = \frac{210{,}000}{300{,}000} = .7 \text{ (risky assets)}$$

$$1 - y = \frac{90{,}000}{300{,}000} = .3 \text{ (risk-free assets)}$$

The weights of each asset class in the complete portfolio are as follows:

$$E: \quad \frac{\$113{,}400}{\$300{,}000} = .378$$

$$B: \quad \frac{\$96{,}600}{\$300{,}000} = .322$$

$$\text{Risky portfolio} = E + B = .700$$

The risky portfolio makes up 70% of the complete portfolio.

[1] John C. Bogle, *Bogle on Mutual Funds* (Burr Ridge, IL: Irwin Professional Publishing, 1994), p. 235.

Example 6.2 The Risky Portfolio

Suppose that the owner of this portfolio wishes to decrease risk by reducing the allocation to the risky portfolio from $y = .7$ to $y = .56$. The risky portfolio would then total only $.56 \times \$300{,}000 = \$168{,}000$, requiring the sale of \$42,000 of the original \$210,000 of risky holdings, with the proceeds used to purchase more shares in Ready Asset (the money market fund). Total holdings in the risk-free asset will increase to $\$300{,}000 \times (1 - .56) = \$132{,}000$, the original holdings plus the new contribution to the money market fund:

$$\$90{,}000 + \$42{,}000 = \$132{,}000$$

The key point, however, is that we leave the proportions of each asset in the risky portfolio unchanged. Because the weights of E and B in the risky portfolio are .54 and .46, respectively, we sell $.54 \times \$42{,}000 = \$22{,}680$ of E and $.46 \times \$42{,}000 = \$19{,}320$ of B. After the sale, the proportions of each asset in the risky portfolio are in fact unchanged:

$$E: \quad w_E = \frac{113{,}400 - 22{,}680}{210{,}000 - 42{,}000} = .54$$

$$B: \quad w_B = \frac{96{,}600 - 19{,}320}{210{,}000 - 42{,}000} = .46$$

Rather than thinking of our risky holdings as E and B separately, we may view our holdings as if they were in a single fund that holds equities and bonds in fixed proportions. In this sense we may treat the risky fund as a single risky asset, that asset being a particular bundle of securities. As we shift in and out of safe assets, we simply alter our holdings of that bundle of securities commensurately.

With this simplification, we turn to the desirability of reducing risk by changing the risky/risk-free asset mix, that is, reducing risk by decreasing the proportion y. As long as we do not alter the weights of each security within the risky portfolio, the probability distribution of the rate of return on the risky portfolio remains unchanged by the asset reallocation. What will change is the probability distribution of the rate of return on the *complete* portfolio that consists of the risky asset and the risk-free asset.

CONCEPT CHECK 6.4

What will be the dollar value of your position in equities (E), and its proportion in your overall portfolio, if you decide to hold 50% of your investment budget in Ready Asset?

6.3 The Risk-Free Asset

By virtue of its power to tax and control the money supply, only the government can issue default-free bonds. Even the default-free guarantee by itself is not sufficient to make the bonds risk-free in real terms. The only risk-free asset in real terms would be a perfectly price-indexed bond. Moreover, a default-free perfectly indexed bond offers a guaranteed real rate to an investor only if the maturity of the bond is identical to the investor's desired holding period. Even indexed bonds are subject to interest rate risk, because real interest rates change unpredictably through time. When future real rates are uncertain, so is the future price of indexed bonds.

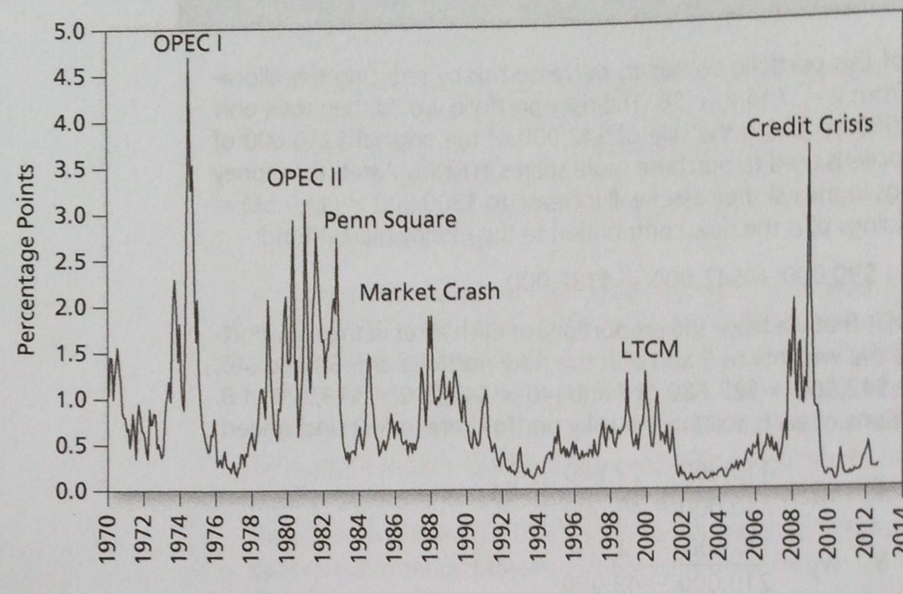


Figure 6.3 Spread between 3-month CD and T-bill rates

Nevertheless, it is common practice to view Treasury bills as "the" **risk-free asset.** Their short-term nature makes their values insensitive to interest rate fluctuations. Indeed, an investor can lock in a short-term nominal return by buying a bill and holding it to maturity. Moreover, inflation uncertainty over the course of a few weeks, or even months, is negligible compared with the uncertainty of stock market returns.

In practice, most investors use a broad range of money market instruments as a risk-free asset. All the money market instruments are virtually free of interest rate risk because of their short maturities and are fairly safe in terms of default or credit risk.

Most money market funds hold, for the most part, three types of securities—Treasury bills, bank certificates of deposit (CDs), and commercial paper (CP)—differing slightly in their default risk. The yields to maturity on CDs and CP for an identical maturity, for example, are always somewhat higher than those of T-bills. The recent history of this yield spread for 90-day CDs is shown in Figure 6.3.

Money market funds have changed their relative holdings of these securities over time but, by and large, T-bills make up only about 15% of their portfolios.[2] Nevertheless, the risk of such blue-chip short-term investments as CDs and CP is minuscule compared with that of most other assets such as long-term corporate bonds, common stocks, or real estate. Hence we treat money market funds as the most easily accessible risk-free asset for most investors.

6.4 Portfolios of One Risky Asset and a Risk-Free Asset

In this section we examine the feasible risk–return combinations available to investors when the choice of the risky portfolio has already been made. This is the "technical" part of capital allocation. In the next section we address the "personal" part of the problem—the individual's choice of the best risk–return combination from the feasible set.

Suppose the investor has already decided on the composition of the risky portfolio, *P*. Now the concern is with capital allocation, that is, the proportion of the investment budget, *y*, to be allocated to *P*. The remaining proportion, $1 - y$, is to be invested in the risk-free asset, *F*.

[2]See **http://www.icifactbook.org/**, Section 4 of Data Tables.

Denote the risky rate of return of P by r_P, its expected rate of return by $E(r_P)$, and its standard deviation by σ_P. The rate of return on the risk-free asset is denoted as r_f. In the numerical example we assume that $E(r_P) = 15\%$, $\sigma_P = 22\%$, and the risk-free rate is $r_f = 7\%$. Thus the risk premium on the risky asset is $E(r_P) - r_f = 8\%$.

With a proportion, y, in the risky portfolio, and $1 - y$ in the risk-free asset, the rate of return on the *complete* portfolio, denoted C, is r_C where

$$r_C = yr_P + (1 - y)r_f \tag{6.2}$$

Taking the expectation of this portfolio's rate of return,

$$\begin{aligned} E(r_C) &= yE(r_P) + (1 - y)r_f \\ &= r_f + y[E(r_P) - r_f] = 7 + y(15 - 7) \end{aligned} \tag{6.3}$$

This result is easily interpreted. The base rate of return for any portfolio is the risk-free rate. In addition, the portfolio is *expected* to earn a proportion, y, of the risk premium of the risky portfolio, $E(r_P) - r_f$. Investors are assumed risk averse and unwilling to take a risky position without a positive risk premium.

With a proportion y in a risky asset, the standard deviation of the complete portfolio is the standard deviation of the risky asset multiplied by the weight, y, of the risky asset in that portfolio.[3] Because the standard deviation of the risky portfolio is $\sigma_P = 22\%$,

$$\sigma_C = y\sigma_P = 22y \tag{6.4}$$

which makes sense because the standard deviation of the portfolio is proportional to both the standard deviation of the risky asset and the proportion invested in it. In sum, the expected return of the complete portfolio is $E(r_C) = r_f + y[E(r_P) - r_f] = 7 + 8y$ and the standard deviation is $\sigma_C = 22y$.

The next step is to plot the portfolio characteristics (with various choices for y) in the expected return–standard deviation plane in Figure 6.4. The risk-free asset, F, appears on the vertical axis because its standard deviation is zero. The risky asset, P, is plotted with a standard deviation of 22%, and expected return of 15%. If an investor chooses to invest solely in the risky asset, then $y = 1.0$, and the complete portfolio is P. If the chosen position is $y = 0$, then $1 - y = 1.0$, and the complete portfolio is the risk-free portfolio F.

What about the more interesting mid-range portfolios where y lies between 0 and 1? These portfolios will graph on the straight line connecting points F and P. The slope of that line is $[E(r_P) - r_f]/\sigma_P$ (rise/run), in this case, 8/22.

The conclusion is straightforward. Increasing the fraction of the overall portfolio invested in the risky asset increases expected return at a rate of 8%, according to Equation 6.3. It also increases portfolio

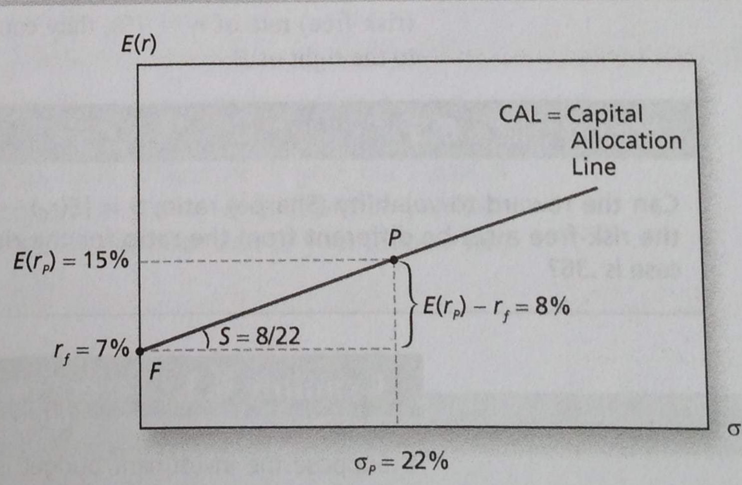


Figure 6.4 The investment opportunity set with a risky asset and a risk-free asset in the expected return–standard deviation plane

[3]This is an application of a basic rule from statistics: If you multiply a random variable by a constant, the standard deviation is multiplied by the same constant. In our application, the random variable is the rate of return on the risky asset, and the constant is the fraction of that asset in the complete portfolio. We will elaborate on the rules for portfolio return and risk in the following chapter.

standard deviation at the rate of 22%, according to Equation 6.4. The extra return per extra risk is thus 8/22 = .36.

To derive the exact equation for the straight line between F and P, we rearrange Equation 6.4 to find that $y = \sigma_C/\sigma_P$, and we substitute for y in Equation 6.3 to describe the expected return–standard deviation trade-off:

$$
\begin{aligned}
E(r_C) &= r_f + y[E(r_P) - r_f] \\
&= r_f + \frac{\sigma_C}{\sigma_P}[E(r_P) - r_f] = 7 + \frac{8}{22}\sigma_C
\end{aligned} \tag{6.5}
$$

Thus the expected return of the complete portfolio as a function of its standard deviation is a straight line, with intercept r_f and slope

$$
S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{8}{22} \tag{6.6}
$$

Figure 6.4 graphs the *investment opportunity set,* which is the set of feasible expected return and standard deviation pairs of all portfolios resulting from different values of y. The graph is a straight line originating at r_f and going through the point labeled P.

This straight line is called the **capital allocation line** (CAL). It depicts all the risk–return combinations available to investors. The slope of the CAL, denoted S, equals the increase in the expected return of the complete portfolio per unit of additional standard deviation—in other words, incremental return per incremental risk. For this reason, the slope is called the **reward-to-volatility ratio.** It also is called the Sharpe ratio (see Chapter 5).

A portfolio equally divided between the risky asset and the risk-free asset, that is, where $y = .5$, will have an expected rate of return of $E(r_C) = 7 + .5 \times 8 = 11\%$, implying a risk premium of 4%, and a standard deviation of $\sigma_C = .5 \times 22 = 11\%$. It will plot on the line FP midway between F and P. The reward-to-volatility ratio is $S = 4/11 = .36$, precisely the same as that of portfolio P.

What about points on the CAL to the right of portfolio P? If investors can borrow at the (risk-free) rate of $r_f = 7\%$, they can construct portfolios that may be plotted on the CAL to the right of P.

CONCEPT CHECK 6.5

Can the reward-to-volatility (Sharpe) ratio, $S = [E(r_C) - r_f]/\sigma_C$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $[E(r_P) - r_f]/\sigma_P$, which in this case is .36?

Example 6.3 Leverage

Suppose the investment budget is \$300,000 and our investor borrows an additional \$120,000, investing the total available funds in the risky asset. This is a *levered* position in the risky asset, financed in part by borrowing. In that case

$$
y = \frac{420{,}000}{300{,}000} = 1.4
$$

and $1 - y = 1 - 1.4 = -.4$, reflecting a short (borrowing) position in the risk-free asset. Rather than lending at a 7% interest rate, the investor borrows at 7%. The distribution of the portfolio rate of return still exhibits the same reward-to-volatility ratio:

$$
E(r_C) = 7\% + (1.4 \times 8\%) = 18.2\%
$$

$$\sigma_C = 1.4 \times 22\% = 30.8\%$$

$$S = \frac{E(r_C) - r_f}{\sigma_C} = \frac{18.2 - 7}{30.8} = .36$$

As one might expect, the levered portfolio has a higher standard deviation than does an unlevered position in the risky asset.

Clearly, nongovernment investors cannot borrow at the risk-free rate. The risk of a borrower's default induces lenders to demand higher interest rates on loans. Therefore, the nongovernment investor's borrowing cost will exceed the lending rate of $r_f = 7\%$. Suppose the borrowing rate is $r_f^B = 9\%$. Then in the borrowing range, the reward-to-volatility ratio, the slope of the CAL, will be $[E(r_P) - r_f^B]/\sigma_P = 6/22 = .27$. The CAL will therefore be "kinked" at point P, as shown in Figure 6.5. To the left of P the investor is lending at 7%, and the slope of the CAL is .36. To the right of P, where $y > 1$, the investor is borrowing at 9% to finance extra investments in the risky asset, and the slope is .27.

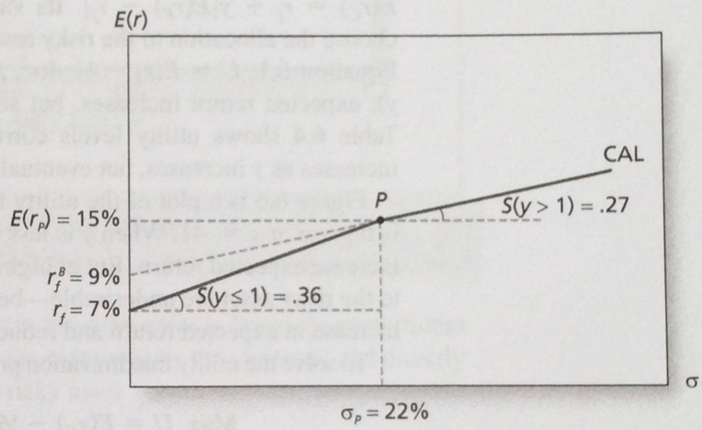


Figure 6.5 The opportunity set with differential borrowing and lending rates

In practice, borrowing to invest in the risky portfolio is easy and straightforward if you have a margin account with a broker. All you have to do is tell your broker that you want to buy "on margin." Margin purchases may not exceed 50% of the purchase value. Therefore, if your net worth in the account is $300,000, the broker is allowed to lend you up to $300,000 to purchase additional stock.[4] You would then have $600,000 on the asset side of your account and $300,000 on the liability side, resulting in $y = 2.0$.

CONCEPT CHECK 6.6

Suppose that there is an upward shift in the expected rate of return on the risky asset, from 15% to 17%. If all other parameters remain unchanged, what will be the slope of the CAL for $y \leq 1$ and $y > 1$?

[4]Margin purchases require the investor to maintain the securities in a margin account with the broker. If the value of the securities falls below a "maintenance margin," a "margin call" is sent out, requiring a deposit to bring the net worth of the account up to the appropriate level. If the margin call is not met, regulations mandate that some or all of the securities be sold by the broker and the proceeds used to reestablish the required margin. See Chapter 3, Section 3.6, for further discussion.

6.5 Risk Tolerance and Asset Allocation

We have shown how to develop the CAL, the graph of all feasible risk–return combinations available for capital allocation. The investor confronting the CAL now must choose one optimal portfolio, *C*, from the set of feasible choices. This choice entails a trade-off between risk and return. Individual differences in risk aversion lead to different capital allocation choices even when facing an identical opportunity set (that is, a risk-free rate and a reward-to-volatility ratio). In particular, more risk-averse investors will choose to hold less of the risky asset and more of the risk-free asset.

The expected return on the complete portfolio is given by Equation 6.3: $E(r_C) = r_f + y[E(r_P) - r_f]$. Its variance is, from Equation 6.4, $\sigma_C^2 = y^2\sigma_P^2$. Investors choose the allocation to the risky asset, *y*, that maximizes their utility function as given by Equation 6.1: $U = E(r) - ½A\sigma^2$. As the allocation to the risky asset increases (higher *y*), expected return increases, but so does volatility, so utility can increase or decrease. Table 6.4 shows utility levels corresponding to different values of *y*. Initially, utility increases as *y* increases, but eventually it declines.

Figure 6.6 is a plot of the utility function from Table 6.4. The graph shows that utility is highest at $y = .41$. When *y* is less than .41, investors are willing to assume more risk to increase expected return. But at higher levels of *y*, risk is higher, and additional allocations to the risky asset are undesirable—beyond this point, further increases in risk dominate the increase in expected return and reduce utility.

To solve the utility maximization problem more generally, we write the problem as follows:

$$\underset{y}{\text{Max}}\ U = E(r_C) - ½A\sigma_C^2 = r_f + y[E(r_P) - r_f] - ½Ay^2\sigma_P^2$$

Students of calculus will recognize that the maximization problem is solved by setting the derivative of this expression to zero. Doing so and solving for *y* yields the optimal position for risk-averse investors in the risky asset, y^*, as follows:[5]

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2} \tag{6.7}$$

Table 6.4

Utility levels for various positions in risky assets (*y*) for an investor with risk aversion $A = 4$

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - ½A\sigma^2$
0	.070	0	.0700
0.1	.078	.022	.0770
0.2	.086	.044	.0821
0.3	.094	.066	.0853
0.4	.102	.088	.0865
0.5	.110	.110	.0858
0.6	.118	.132	.0832
0.7	.126	.154	.0786
0.8	.134	.176	.0720
0.9	.142	.198	.0636
1.0	.150	.220	.0532

[5] The derivative with respect to *y* equals $E(r_P) - r_f - yA\sigma_P^2$. Setting this expression equal to zero and solving for *y* yields Equation 6.7.

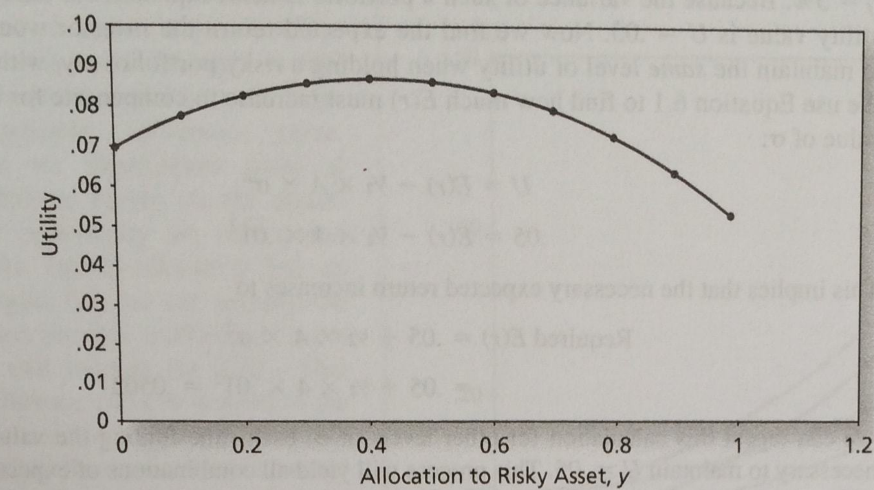


Figure 6.6 Utility as a function of allocation to the risky asset, y

This solution shows that the optimal position in the risky asset is *inversely* proportional to the level of risk aversion and the level of risk (as measured by the variance) and directly proportional to the risk premium offered by the risky asset.

Example 6.4 Capital Allocation

Using our numerical example [$r_f = 7\%$, $E(r_P) = 15\%$, and $\sigma_P = 22\%$], and expressing all returns as decimals, the optimal solution for an investor with a coefficient of risk aversion $A = 4$ is

$$y^* = \frac{.15 - .07}{4 \times .22^2} = .41$$

In other words, this particular investor will invest 41% of the investment budget in the risky asset and 59% in the risk-free asset. As we saw in Figure 6.6, this is the value of y at which utility is maximized.

With 41% invested in the risky portfolio, the expected return and standard deviation of the complete portfolio are

$$E(r_C) = 7 + [.41 \times (15 - 7)] = 10.28\%$$

$$\sigma_C = .41 \times 22 = 9.02\%$$

The risk premium of the complete portfolio is $E(r_C) - r_f = 3.28\%$, which is obtained by taking on a portfolio with a standard deviation of 9.02%. Notice that 3.28/9.02 = .36, which is the reward-to-volatility (Sharpe) ratio of any complete portfolio given the parameters of this example.

A graphical way of presenting this decision problem is to use indifference curve analysis. To illustrate how to build an indifference curve, consider an investor with risk aversion $A = 4$ who currently holds all her wealth in a risk-free portfolio yielding

$r_f = 5\%$. Because the variance of such a portfolio is zero, Equation 6.1 tells us that its utility value is $U = .05$. Now we find the expected return the investor would require to maintain the *same* level of utility when holding a risky portfolio, say, with $\sigma = 1\%$. We use Equation 6.1 to find how much $E(r)$ must increase to compensate for the higher value of σ:

$$U = E(r) - \tfrac{1}{2} \times A \times \sigma^2$$

$$.05 = E(r) - \tfrac{1}{2} \times 4 \times .01^2$$

This implies that the necessary expected return increases to

$$\begin{aligned} \text{Required } E(r) &= .05 + \tfrac{1}{2} \times A \times \sigma^2 \\ &= .05 + \tfrac{1}{2} \times 4 \times .01^2 = .0502 \end{aligned} \tag{6.8}$$

We can repeat this calculation for other levels of σ, each time finding the value of $E(r)$ necessary to maintain $U = .05$. This process will yield all combinations of expected return and volatility with utility level of .05; plotting these combinations gives us the indifference curve.

We can readily generate an investor's indifference curves using a spreadsheet. Table 6.5 contains risk–return combinations with utility values of .05 and .09 for two investors, one with $A = 2$ and the other with $A = 4$. The plot of these indifference curves appears in Figure 6.7. Notice that the intercepts of the indifference curves are at .05 and .09, exactly the level of utility corresponding to the two curves.

Any investor would prefer a portfolio on the higher indifference curve with a higher certainty equivalent (utility). Portfolios on higher indifference curves offer a higher expected return for any given level of risk. For example, both indifference curves for $A = 2$ have the same shape, but for any level of volatility, a portfolio on the curve with utility of .09 offers an expected return 4% greater than the corresponding portfolio on the lower curve, for which $U = .05$.

Figure 6.7 demonstrates that more risk-averse investors have steeper indifference curves than less risk-averse investors. Steeper curves mean that investors require a greater increase in expected return to compensate for an increase in portfolio risk.

Table 6.5

Spreadsheet calculations of indifference curves (Entries in columns 2–4 are expected returns necessary to provide specified utility value.)

	A = 2		A = 4	
σ	$U = .05$	$U = .09$	$U = .05$	$U = .09$
0	.0500	.0900	.050	.090
.05	.0525	.0925	.055	.095
.10	.0600	.1000	.070	.110
.15	.0725	.1125	.095	.135
.20	.0900	.1300	.130	.170
.25	.1125	.1525	.175	.215
.30	.1400	.1800	.230	.270
.35	.1725	.2125	.295	.335
.40	.2100	.2500	.370	.410
.45	.2525	.2925	.455	.495
.50	.3000	.3400	.550	.590

Higher indifference curves correspond to higher levels of utility. The investor thus attempts to find the complete portfolio on the highest possible indifference curve. When we superimpose plots of indifference curves on the investment opportunity set represented by the capital allocation line as in Figure 6.8, we can identify the *highest possible* indifference curve that still touches the CAL. That indifference curve is tangent to the CAL, and the tangency point corresponds to the standard deviation and expected return of the optimal complete portfolio.

To illustrate, Table 6.6 provides calculations for four indifference curves (with utility levels of .07, .078, .08653, and .094) for an investor with $A = 4$. Columns (2)–(5) use Equation 6.8 to calculate the expected return that must be paired

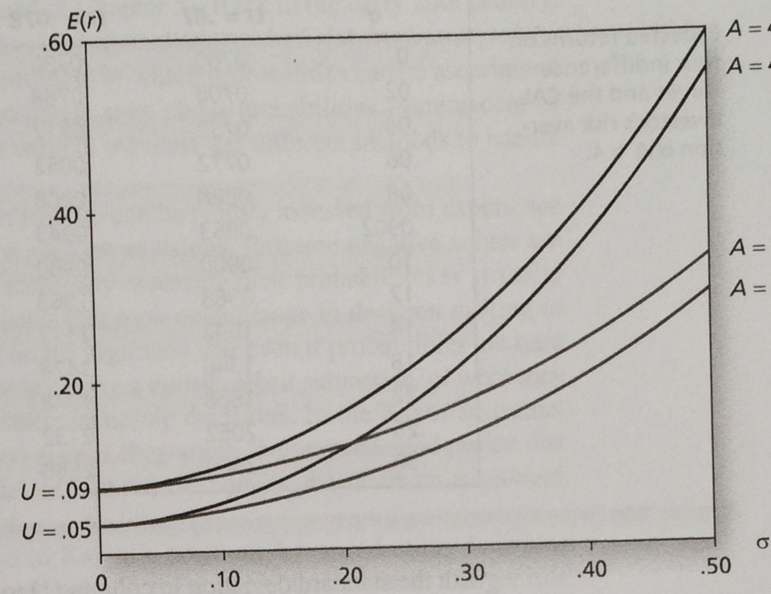


Figure 6.7 Indifference curves for $U = .05$ and $U = .09$ with $A = 2$ and $A = 4$

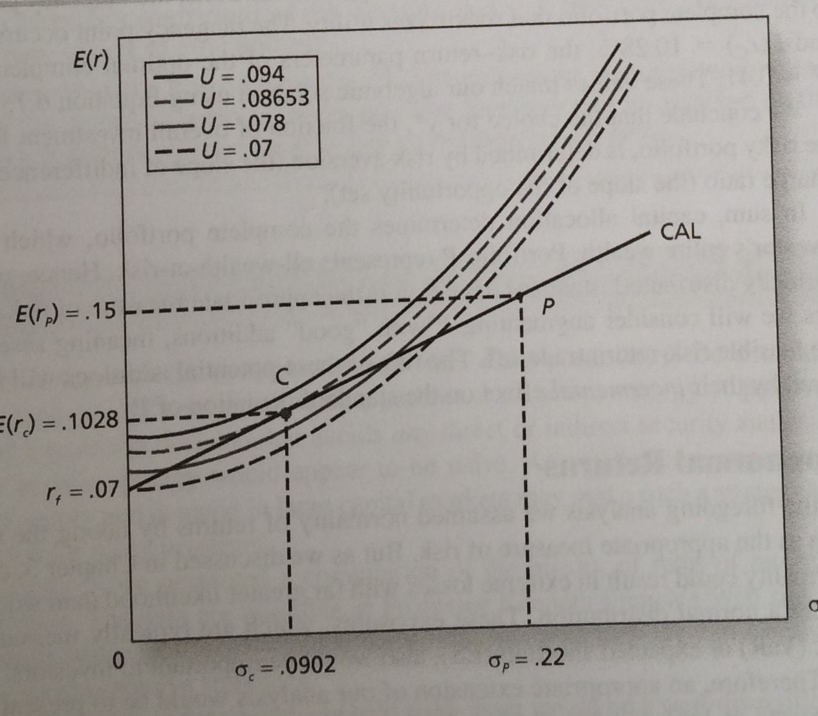


Figure 6.8 Finding the optimal complete portfolio by using indifference curves