Introduction to Statistics

Part 2: Inferential statistics and modelling

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Beyond the sample

Course content

Part 2: Friday 1st November, 2024

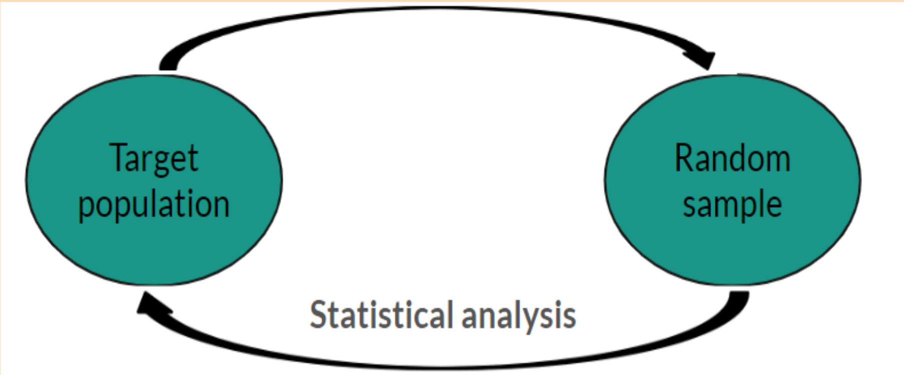
- Inferential statistics
  - Central limit theorem
  - p-values and confidence intervals: how to interpret and communicate results
  - Where do these values come from?

Course content

Part 2: Friday 1st November, 2024

- Statistical modelling
  - What are models and why are they useful to data analytics?
  - How to choose an appropriate model based on the research question
  - Model outputs and their interpretations

What are inferential statistics?



What are inferential statistics?

Inferential statistics make inferences about target population based on a **random, representative** sample.

Combine sample estimates with **sample size** and level of **precision**

Most common inferential statistics: **p-values** and **confidence intervals**

Measures of precision

Precision of an estimate quantified by **standard error (SE)**

Based on sample size and sample variability

Different formula for each type of estimate (e.g. mean, percentage, difference between means)

$$SE(\bar{x}) = \frac{SD}{\sqrt{n}}$$

Measures of precision

Larger SE → **less precise** Smaller SE → **more precise**

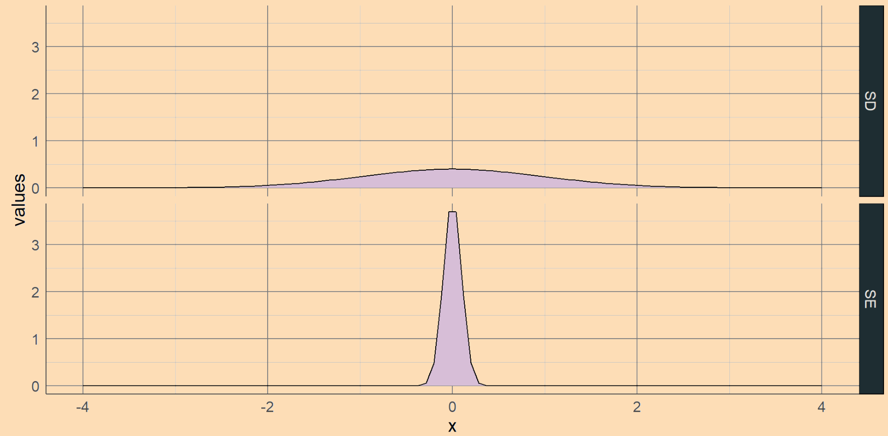
$$SE(\bar{x}) = \frac{SD}{\sqrt{n}}$$

For every parameter of interest:

- Larger sample, higher precision → **lower standard error**
- More variability, lower precision → **higher standard error**

Inferential statistics work based on the **central limit theorem**

Central limit theorem



Confidence intervals

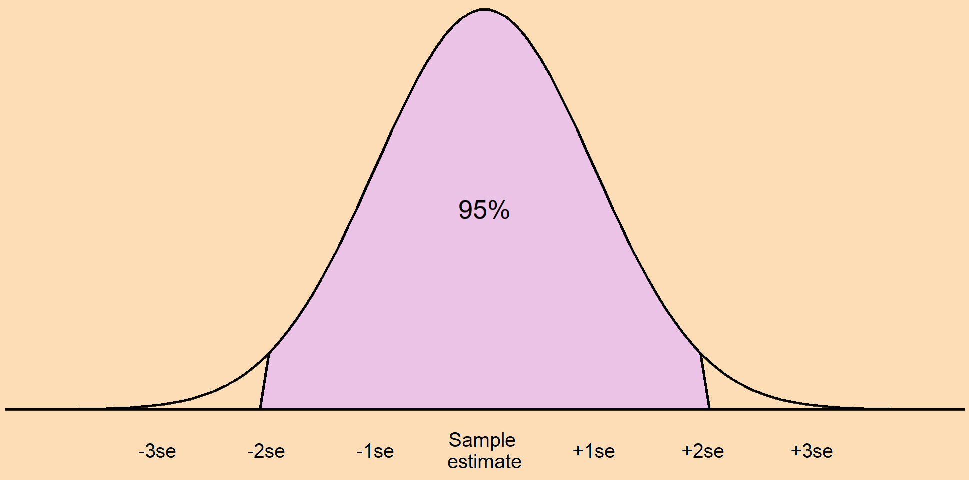
- A range of values the true population parameter is compatible with
- Based on sample estimate, precision, and confidence level

Confidence levels	Number of SEs
80%	1.282
90%	1.645
95%	1.960
99%	2.576
99.9%	3.291

Confidence intervals

- A range of values the true population parameter is compatible with
- Based on sample estimate, precision, and confidence level
- Based on central limit theorem, can capture ranges we would expect a percentage of parameter estimates to lie:

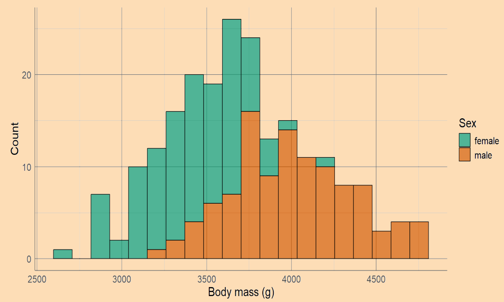
$$\bar{x} \pm 1.96 \times SE(\bar{x})$$



Confidence interval example

Let's compare the body mass of our penguins between sexes.

First, we want to check the distribution of these samples:



Confidence interval example

Both groups appear to be normally distributed, so we can compare the **means**.

Mean body mass of male penguins: **4010.28g**

Mean body mass of female penguins: **3419.16g**

Difference in the means (of the sample): 4010.28g - 3419.16g

= 591.12g

Confidence interval example

Difference in the means (of the sample): 591.12g

Standard error of the mean difference**= 4.23g**

95% confidence interval: $591.12 \pm 1.96 \times 4.23$

= 582.83g, 599.42g

But what does that mean??

Confidence interval example

95% confidence interval: 582.83g, 599.42g

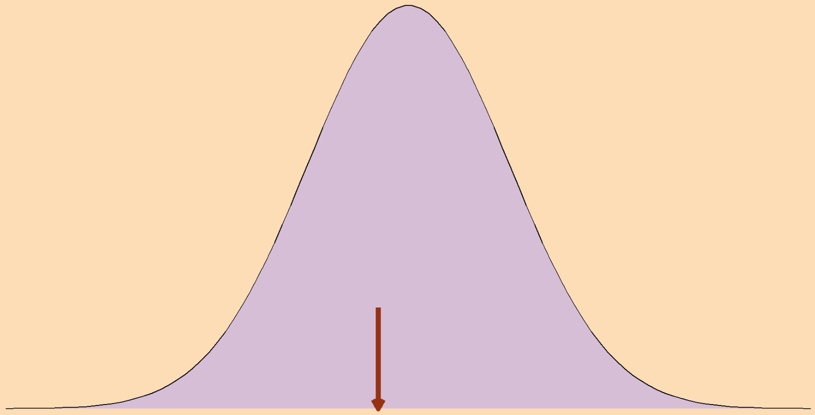
We are **95% confident** that male penguins were between 582.83g and 599.42g heavier than female penguins on average.

Note that this confidence interval only contains positive values.

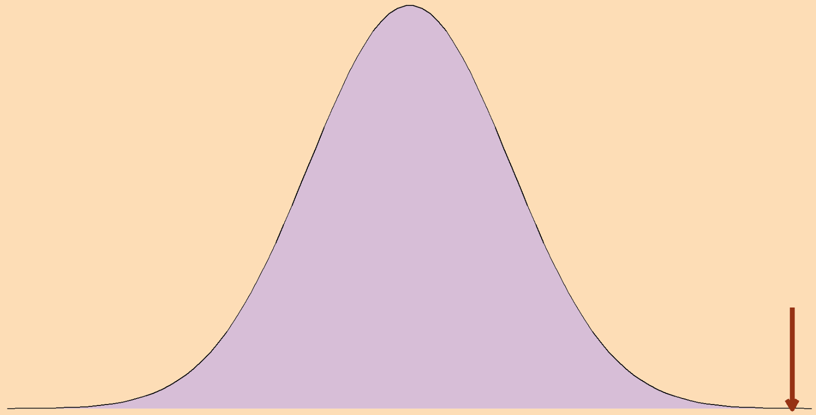
p-values

- Probability of obtaining a result as extreme or more extreme as the sample if the null hypothesis is true
- Null hypothesis (H0): no difference/association

p-values



p-values



p-values

- Probability of obtaining a result as extreme or more extreme as the sample if the null hypothesis is true
- Null hypothesis (H0): no difference/association
- Low p-value: less evidence to support the null hypothesis
  - Very low p-value is known as **statistically significant**

Statistical significance

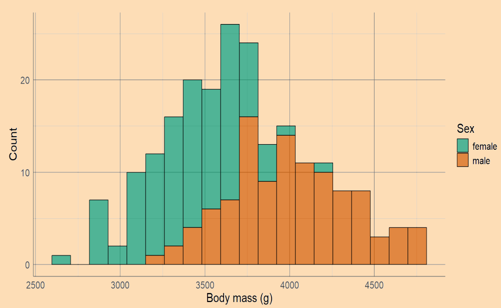
Often significance is defined by arbitrary cut-off (usually 0.05)

Be careful with these arbitrary definitions, it is not how probability behaves!

$p < 0.05$ is significant **at the 5% level**

We never **accept** or **reject** a null hypothesis

p-values example



p-values example

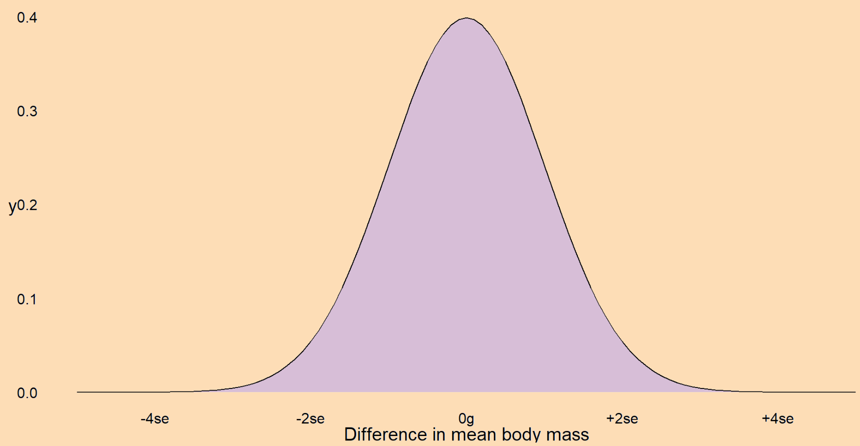
As we are comparing groups, our null hypothesis is that there is **no difference** in the target population.

Sample mean difference: **591.12g**

Standard error of the difference: **4.23g**

p-values assume that the null hypothesis is true

p-value example



p-value example

The observed sample mean difference is (591.12 - 0 ÷ 4.23) = **139.68 standard errors** away from the null hypothesis.

This is so far that we can't even see it on our histogram!

The probability of this happening **if the null were true** is VERY VERY small (**$p < 0.00000000001$**).

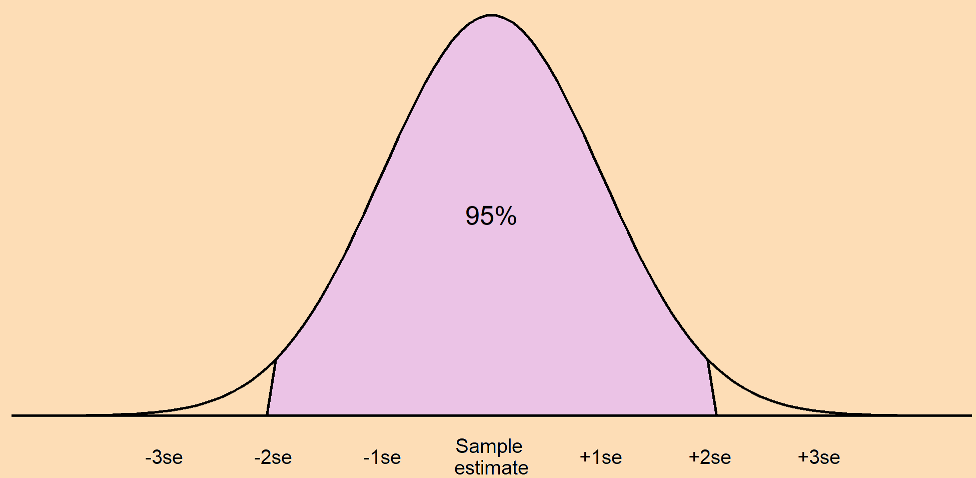
In this case, we would say this difference is **highly significant**

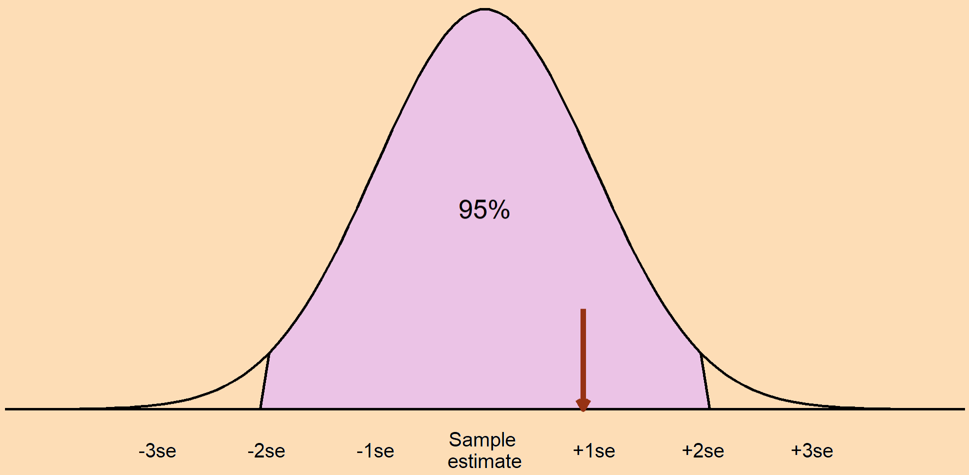
Relationship between p-values and confidence intervals

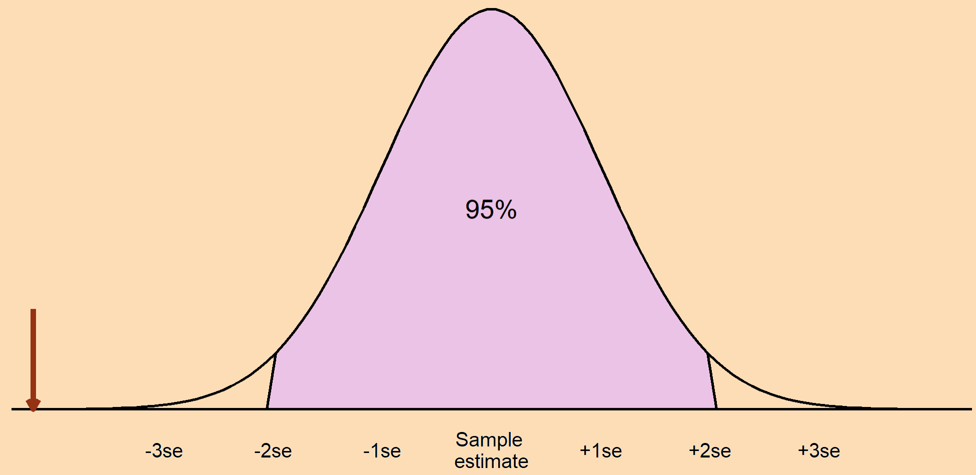
Confidence intervals and p-values are based on the same information and so agree with one another

If a p-value is **above 0.05**, the sample estimate is **less than 1.96 SEs away**. This means it will be **within the 95% confidence interval**

If the null hypothesis is **outside the 99% confidence interval**, it is **over 2.576 SEs away** from the sample estimate so **$p < 0.01$**







Exercise 3:

Inferential statistics

Table 1: Proportion of men who committed a proven reoffence in a one-year period (reoffending rate) after support from Only Connect compared with a matched comparison group

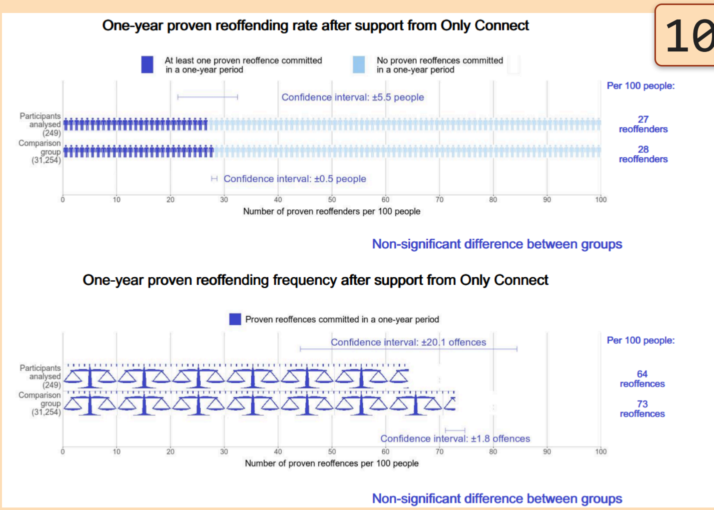
Number in treatment group	Number in comparison group	Treatment group rate (%)	Comparison group rate (%)	Estimated difference (% points)	Significant difference?	p-value
249	31,254	27	28	-7 to 4	No	0.66

Table 2: Number of proven reoffences committed in a one-year period (reoffending frequency - offences per person) by men who received support from Only Connect compared with a matched comparison group

Number in treatment group	Number in comparison group	Treatment group frequency	Comparison group frequency	Estimated difference	Significant difference?	p-value
249	31,254	0.64	0.73	-0.29 to 0.12	No	0.40

Table 3: Average time (days) to first proven reoffence in a one-year period for men who received support from Only Connect, compared with a matched comparison group (reoffenders only)

Number in treatment group	Number in comparison group	Treatment group time (days)	Comparison group time (days)	Estimated difference	Significant difference?	p-value
67	7,503	184	166	-6 to 43	No	0.14



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Statistical modelling

Inferential statistics

Now we know how to **interpret** and **communicate** inferential statistics…how do we calculate them?

Almost all parameters that we can estimate from a sample can be presented with inferential statistics.

- Mean
- Proportion/percentage
- Correlation coefficients
- Difference in means
- Difference in proportions
- **Model coefficients**

Statistical models

Models aim to explain complex process in a simple way.

Statistical models explain these processes using a **mathematical equation:**

$g(\mathrm{Y}) = \alpha + \beta_1 X_1 + \ldots + \beta_n X_n$

Model equations generally consist of

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Model equations generally consist of **outcome(s)**,

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Model equations generally consist of outcome(s), **predictor(s)**

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Statistical models explain these processes using a **mathematical equation:**

$g(Y) = \alpha + \beta_1 X_1 + \ldots + \beta_n X_n$

Model equations generally consist of outcome(s), predictor(s) and **coefficients**

Statistical vs machine learning models

Machine learning models (MLM) are very powerful when making **predictions**.

However, the way they get these predictions is often shrouded in mystery

Models are not **interpretable**

Statistical vs machine learning models

Statistical models are based on **probabilistic assumptions**

Makes them stricter but **interpretable**

Statistical models are better at **explaining** and **understanding** processes

Regression models

Common statistical model are **regression models**.

Also known as **linear models, generalised linear models or GLMs**

Choice of regression type depends on the **type** of outcome variable(s).

All aim to fit a linear equation to a **transformation** of the outcome ($g(Y)$)

Regression models

Outcome type	Regression	Transformation
Continuous	Linear	Identity
Count/rate	Poisson	Log
Binary	Logistic	Logit
Ordinal	Ordinal logistic	Logit
Nominal	Multinomial	Logit

Linear regression

The simplest statistical modelling approach is a linear model.

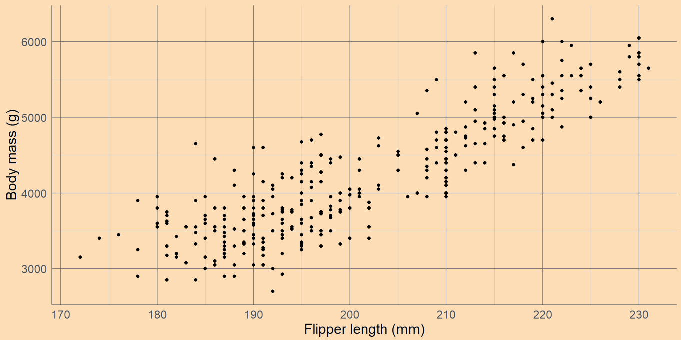
This is because coefficient estimates are related to the outcome itself:

$Y = \alpha + \beta_1 X_1 + \ldots$

When X_1 is the only predictor and a continuous variable, linear regression fits a straight line to the data

Linear regression

Let’s fit a model to explore the relationship between penguin’s body mass and flipper length.



Linear regression

Let’s fit a model to explore the relationship between penguin’s body mass and flipper length.

Here, the outcome is **body mass** and the predictor is **flipper length**:

Body mass = α + $\beta\times$ flipper length

- α = intercept
- Predicted outcome where predictors = 0
- β = slope
- Expected change in outcome for a unit increase in predictor

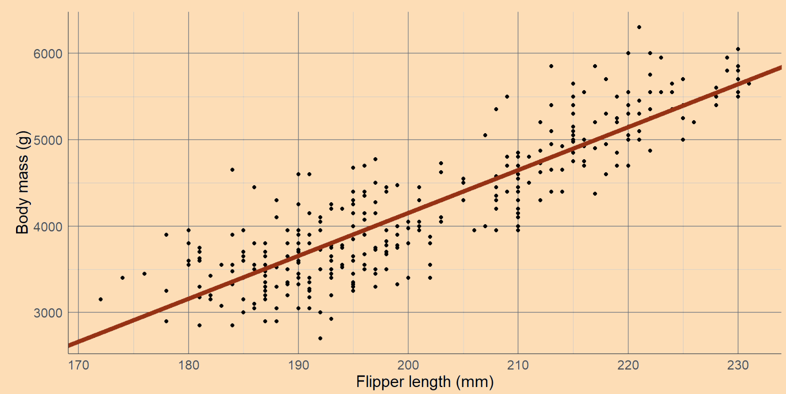
Linear regression

Body mass = -5780.83 + 49.69 $\times$ flipper length

	Coefficient estimates	95% confidence interval	p-value
Intercept	-5,780.83	[-6382.36, -5179.3]	<0.001
Flipper length	49.69	[46.7, 52.67]	<0.001

Linear regression

Body mass = -5780.83 + 49.69 $\times$ flipper length



Linear regression

Body mass = -5780.83 + 49.69 $\times$ flipper length

- Intercept = -5780.83

Penguins with flipper length of 0mm had a predicted body mass of **-5780.83g**

- Slope = 49.69

For every unit increase in flipper length (1mm), body mass was expected to increase by **49.69g**

Linear regression

Confidence intervals give a range of values the coefficients are **compatible with.**

In this sample, the average increase in body mass for every 1mm increase in flipper length was **49.69g**

But at the **population level**, we are 95% confident that this increase could lie between **46.7g** and **52.67g**

Linear regression

p-values test the null hypothesis of **no association**: $\beta = 0$

These p-values are both too small to be printed in their entirety, therefore the coefficients are **statistically significant**

For intercept, this has no real use.

For the slope, we have shown a **significant association** between flipper length and body mass

Multiple regression

One of the benefits of using a regression is that we can take account of **confounders**

Confounders = background variables that are related to both the outcome and predictor variable(s)

Confounders can **create** false associations or **hide** true associations if not properly accounted for

Multiple regression

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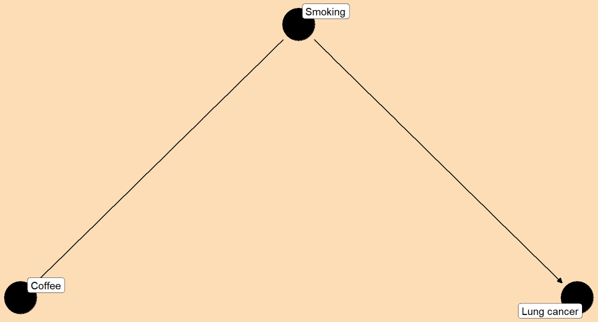


Multiple regression

Let's extend the previous model to account for the sex of penguins:

Multiple regression

Let's extend the previous model to account for the sex of penguins:

Body mass = α + $\beta_1 \times$ flipper length + $\beta_2 \times$ male

Categorical variables are added as **dummy variables**

male = 1 if sex = male

male = 0 if sex = female

Multiple regression

Body mass = -5410.3 + 46.98 × flipper length + 347.85 × male

	Coefficient estimates	95% confidence interval	p-value
Intercept	-5,410.30	[-5972.52, -4848.09]	<0.001
Flipper length	46.98	[44.15, 49.82]	<0.001
Male	347.85	[268.49, 427.21]	<0.001

Multiple regression

Body mass = -5410.3 + 46.98 × flipper length + 347.85 × male

Coefficients now represent expected change in outcome for unit increase in predictor **after adjusting for other predictors**

There is a **significant positive** association between flipper length and body mass after adjusting for differences between sexes

Model evaluation

There are going to be many potential models to answer our research question…how do we choose the best one??

- Consider the **intention** of the model
- Use **common sense** and prior knowledge
- Aim to find the most **parsimonious**

Model evaluation

Model evaluation can involve comparisons of model fitting statistics.

R-squared value: proportion of the outcome explained by the model

Adjusted R-squared penalises the R-squared value based on the number of predictors included in the model

Model evaluation

Adjusted R-squared value for flipper-only model: **0.76**

Adjusted R-squared value for flipper + sex model: **0.8**

Adding sex still increased the adjusted R-squared value, indicating its addition was **worthwhile**

Model evaluation

Prediction metrics are another family of useful model evaluation tools

They compare the observed outcome with the fitted model predictions.

- RMSE: root mean squared error $\sqrt{\frac{1}{n}\sum(y_i - \hat{y}_i)^2}$
- MAE: mean absolute error $\frac{1}{n}\sum|y_i - \hat{y}_i|$

Useful as they provide a measure of fit **in context**

Model evaluation

Model	RMSE	MAE
Flipper only	393.12	313.00
Flipper + sex	354.28	283.25

Adding sex into the model **reduced** performance metrics. This means it **improved** prediction.

If both prediction errors are large (in context of the problem), consider trying to improve them in some way

Model diagnostics

Linear regression is a **parametric** method: it has assumptions that must be checked

- **L**inearity: can present the outcome as a linear combination of predictors
- **I**ndependent predictors: no multicollinearity present
- **N**ormally distributed residuals
- **E**qual variance of residuals AKA homoskedasticity

Model diagnostics

Predictors must be **independent** of one another.

Correlation can be accounted for to some degree, and dependency can exist between > 2 variables

Variance inflation factor (VIF): Measure of multicollinearity

$VIF_i = \frac{1}{1-R_i^2}$ for each predictor

Model diagnostics

VIF = 10 $\rightarrow R^2 = 0.9$: 90% of the variation in that predictor is explained by other predictors

Multicollinearity leads to unstable coefficient estimates and invalid inferential statistics

When there is evidence of multicollinearity (VIF > 5-ish), **remove the offending variable(s)!**

Model diagnostics

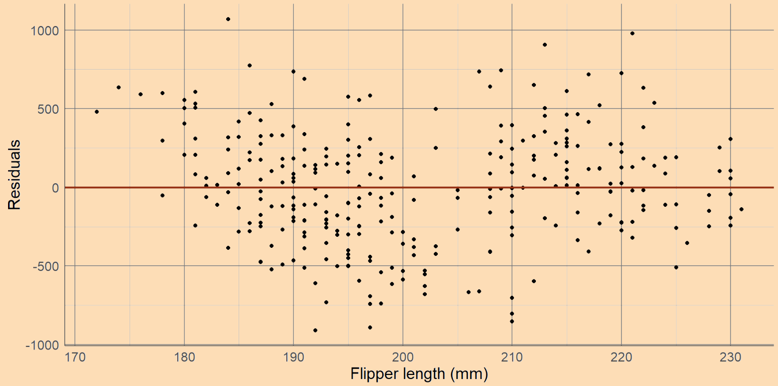
Residuals = model error terms: observed outcome - predicted outcome

Used to check final three assumptions:

- Linearity: plot residuals against each predictor
- Normal distribution: plot a histogram of residuals
- Equal variance: plot residuals against the predicted outcome

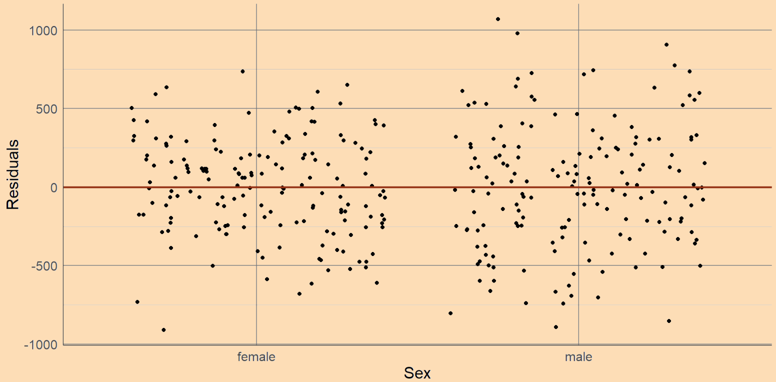
Model diagnostics

Linearity: check residual vs. predictor plots for patterns



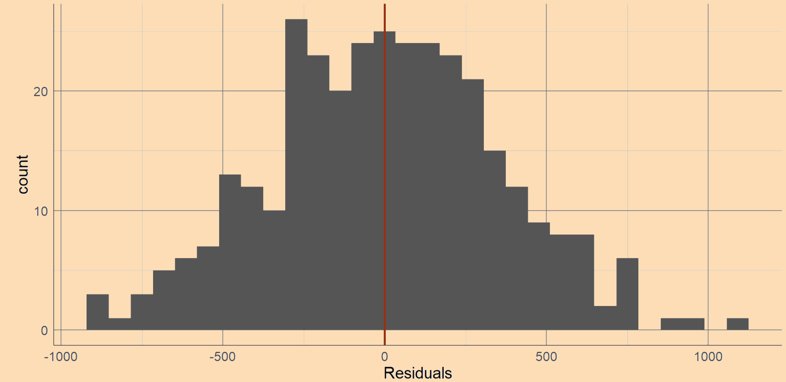
Model diagnostics

Linearity: check residual vs. predictor plots for patterns



Model diagnostics

Normally distribution residuals



Model diagnostics

Homoskedasticity: check residual vs. prediction plots for patterns

Generalised linear models

When the outcome variable is not continuous, another regression model must be chosen.

Estimated coefficients relate to a **transformed** version of the outcome.

For example, a fitted poisson model (of counts) looks like this:

$$log(Y) = \alpha + \beta_1 X_1 + \beta_2 X_2 + \ldots$$

Coefficients (α, β_1, β_2) relate to the **log** of the outcome.

Generalised linear models

To interpret coefficients, they must be **back-transformed** to relate to the original outcome.

For example, in the poisson case, we apply the **exponential function** (the opposite of the log) to the equation

This makes interpretation of other regression models slightly more difficult **but not impossible!**

Generalised linear models

Statistical models are based on probabilistic assumptions.

These assumptions must be true for results for be valid

Different regression types have different assumptions but all share these two:

1. Observations must be **independent** of one another

2. It must be possible to represent the relationship between the (transformed) outcome and predictors using a **linear equation**

Beyond GLMs

When either of these assumptions are not valid, we must consider **other models**

Mixed models (or GLMM, multilevel models, random effect models) account for dependency structures in data:

- Spatial data
- Temporal data
- Data on clusters (e.g. households)

Beyond GLMs

When either of these assumptions are not valid, we must consider **other models**

Additive models (GAMs) are useful when modelling non-linear relationships

Allow smooth functions of predictors, $s(X)$, to be entered into a model:

g(Y) = $\alpha + \beta_1 X_1 + \cdots + s(X_j)$

Final thoughts

Final thoughts

- Statistics is a **huge** topic
- Do not underestimate planning stage: research questions, biases and exploratory work
- Complex analysis **can not** overcome bad data
- Do not make inferential statements about sample estimates and do not make **causal** statements unless performing causal analysis

Final thoughts

- Choose analysis methods based on the **research question** rather than the available data
- Models should be built to address this question and using common sense/background knowledge **not based on p-values**
- If a method requires assumptions to be met, check these **before** communicating results

Final thoughts

- Many **free** statistical software packages available
- **R** is a favourite of statisticians (me included!) and has a huge online community to help learn and TONS of free resources
- **Python** is a favourite of data scientists and has a rapidly growing community
- **Excel** will do basic stats but is limited and prone to issues!

Thank you for listening!