Perfect Isometries and Broué's Abelian Defect Group Conjecture

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G

$$\begin{tikzcd}
& G \arrow[d, dash] & \\
& N_G(P) & \\
P \arrow[ur, dash, "\lhd"] \arrow[dr, dash] & & C_G(P) \arrow[ul, dash, "\lhd"'] \arrow[dl, dash] \\
& Z(P) &
\end{tikzcd}$$

$\mathbb{Q}(S_{\|G\|})$	I	x	y
χ_1	1	1	1
χ_2	1	1	-1
χ_3	2	-1	0

$$\begin{tikzcd}
 & G \arrow[d, no head] & \\
 & N_G(P) \arrow[dl, no head, "\triangleleft"'] \arrow[dr, no head, "\triangleleft"] & \\
P \arrow[dr, no head] & & C_G(P) \arrow[dl, no head] \\
 & Z(P) &
\end{tikzcd}$$

	1	x	y
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φ_2	2	-1

$$
\begin{tikzcd}
& G \arrow[d, dash] & \\
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P \arrow[dr, dash] & & C_G(P) \arrow[dl, dash] \\
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\end{tikzcd}
$$

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\end{tikzcd}
$$

$P_R(G)$

$R_F(G)$

$R_K(G)$

φ

	I	x	y
χ_1	1	1	1
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G

$P_R(G)$ $\qquad$ $R_F(G)$

$R_K(G)$

$\mathcal{F}_p(G)$

$H^2(G,A)$

$$\begin{array}{c} (P,1) \\ | \\ (1,e) \end{array}$$

$$\begin{tikzcd}
 & G \arrow[d, dash] & \\
 & N_G(P) & \\
P \arrow[ur, "\triangleleft"] \arrow[dr, dash] & & C_G(P) \arrow[ul, "\triangleleft"'] \arrow[dl, dash] \\
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φ_2	2	-1

$$\begin{tikzcd} & G \arrow[d, dash] & \\ & N_G(P) & \\ P \arrow[ur, "\trianglelefteq"] & & C_G(P) \arrow[ul, "\trianglelefteq"'] \\ & Z(P) \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

$\mathcal{F}_p(G)$

$$\begin{tikzcd} (P,1) \arrow[d, dash] \\ (1,e) \end{tikzcd}$$

$H^2(G,A)$

$$\begin{tikzcd} P_R(G) \arrow[rr] \arrow[dr] & & R_F(G) \\ & R_K(G) \arrow[ur] & \end{tikzcd}$$

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G

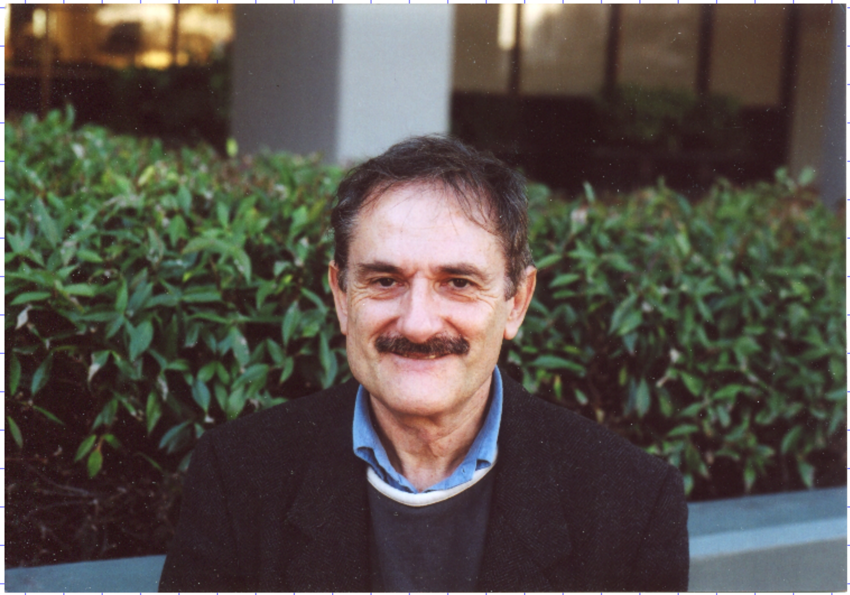
H

G

	g_1	g_2	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	1	1	1	-1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

H

	h_1	h_2	h_3	h_4	h_5
ψ_1	1	1	1	1	1
ψ_2	1	1	-1	1	-1
ψ_3	1	1	1	-1	-1
ψ_4	1	1	-1	-1	1
ψ_5	2	-2	0	0	0



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Student of C. Chevalley
and J.-P. Serre

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[2] Michel Broué, Rickard equivalences and block theory. Groups '93 Galway/St. Andrews Vol. 1 (1993), 58-79, London Math Soc. Lecture Note Series, 211

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$$\mathrm{char}(K) \nmid |G| \quad \xRightarrow{\text{Maschke}} \quad \text{the group algebra } KG \text{ is semisimple}$$

(e.g. $K = \mathbb{Q}$ or $\mathbb{C}$)

$\mathrm{char}(F) = p \mid |G|$?

$$\mathrm{char}(F) = p \mid |G|$$

<u>Defn</u>: G = finite group, p = prime number.
A p-modular system is a triple (K, R, F)
where

R = complete discrete valuation ring of characteristic 0 $\qquad K \hookrightarrow R \twoheadrightarrow F$

K = the field of fractions of R

F = the residue field of R, characteristic p.

The p-modular system is large enough for G if R contains a primitive $|G|^{th}$ root of unity.

G = finite group, (K, R, F) = p-modular system large enough.

Recall: over K, have

$$\mathrm{Irr}_K(G) := \{\text{characters of irreducible } KG\text{-modules}\}$$

$$R_K(G) := \mathbb{Z}\,\mathrm{Irr}_K(G)$$

The character χ_V of $V \in {}_{KG}\mathrm{mod}$ is defined: if $g \in G$,

$$\chi_V(g) = \mathrm{trace}\begin{pmatrix} V \longrightarrow V \\ v \longmapsto gv \end{pmatrix}.$$

G = finite group, (K, R, F) = p-modular system large enough.

$W \in {}_{FG}\text{mod}$. $\quad g \in G_{p'} := \{g \in G : p \nmid |g|\}$.

- The eigenvalues of $W \to W$, $w \mapsto gw$, are "p'-roots of unity" in F. These lift uniquely to p'-roots of unity in R. The sum of the R-lifts of the eigenvalues is denoted $\varphi_W(g)$, and $\varphi_W : G_{p'} \to R$ is the Brauer character of W.
- $\mathrm{IBr}(G) := \{$Brauer characters of simple FG-modules$\}$
- $R_F(G) := \mathbb{Z}\,\mathrm{IBr}(G)$.

G = finite group, (K, R, F) = p-modular system large enough.

Can also consider

$P_R(G) :=$ subgroup of $R_K(G)$ spanned by the characters of projective indecomposable RG-modules.

G = finite group, (K, R, F) = p-modular system large enough.

Brauer's CDE Triangle

$$\begin{tikzcd}
RG^{proj} \arrow[r] & P_R(G) \arrow[rr, "c"] \arrow[dr, hook] & & R_F(G) & FG^{mod} \arrow[l] \\
& & R_K(G) \arrow[ur, "d"'] & &
\end{tikzcd}$$

d (decomposition)

$\uparrow$ Grothendieck gp

KG^{mod}

Blocks

G, (K, R, F) as before.

$$RG = B_1 \oplus B_2 \oplus \cdots \oplus B_t$$

unique!

indecomposable ideals called the **blocks** of RG

$$\mathcal{B}l(RG) := \{B_i\}_{i=1}^{t}$$

Blocks

G, (K, R, F) as before.

$$RG = B_1 \oplus B_2 \oplus \cdots \oplus B_t$$

$RGe_i =: B_i$ $\uparrow$ $\downarrow$ e_i = identity of B_i

$$1 = e_1 + e_2 + \cdots + e_t$$

pairwise orthogonal primitive idempotents of $Z(RG)$.

$\mathrm{bli}(RG) := \{e_i\}_{i=1}^{t}$

$M \in_{RG} \mathrm{mod}$. $\quad \mathcal{B}\ell(RG) = \{B_i\}_{i=1}^{t}$ $\quad bl_i(RG) = \{e_i\}_{i=1}^{t}$

$$M = e_1 M \oplus e_2 M \oplus \cdots \oplus e_t M$$

$$e_i M \in {}_{B_i}\mathrm{mod}$$

Say M belongs to B_i if M is annihilated by the $B_j \neq B_i$.

- $\mathrm{Irr}(G) = \dot{\bigcup}_{B \in \mathcal{B}\ell(RG)} \mathrm{Irr}(B)$ $\quad \chi = \chi_V \in \mathrm{Irr}(B) \iff$ V is irred. KG-module belonging to B

- $\mathrm{IBr}(G) = \dot{\bigcup}_{B \in \mathcal{B}\ell(RG)} \mathrm{IBr}(B)$

- $R_K(G) = \bigoplus_{B \in \mathcal{B}\ell(RG)} R_K(B)$ and $R_F(G) = \bigoplus_{B \in \mathcal{B}\ell(RG)} R_F(B)$

- For each $B \in \mathcal{B}\ell(RG)$,

$$\begin{tikzcd} P_R(B) \arrow[rr, "c_B"] \arrow[dr, bend right] & \arrow[d, phantom, "\equiv"] & R_F(B) \\ & R_K(B) \arrow[ur, "d_B"'] & \end{tikzcd}$$

ex) Character table of A_4

	1	2	3A	3B
χ_1	1	1	1	1
χ_2	1	1	ζ	ζ^2
χ_3	1	1	ζ^2	ζ
χ_4	3	-1	0	0

ζ = primitive 3^{rd} root of unity

$p=2$: 1 is the unique block idempotent, so $\mathcal{Bl}(RA_4) = \{B_1\}$ and $\mathrm{Irr}(B_1) = \{\chi_1, \chi_2, \chi_3, \chi_4\}$

$p=3$: there are 2 block idempotents:

$$e_{B_1} = \frac{1}{4}(1 + (12)(34) + (13)(24) + (14)(23))$$

$$e_{B_2} = \frac{1}{4}(3 - (12)(34) - (13)(24) - (14)(23))$$

$$\mathcal{Bl}(RA_4) = \{B_1, B_2\}$$

- $\mathrm{Irr}(B_1) = \{\chi_1, \chi_2, \chi_3\}$
- $\mathrm{Irr}(B_2) = \{\chi_4\}$.

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Toward the definition of "perfect isometry."

Let G, H be finite groups, (K, R, F) = p-modular system large enough for both G and H.

We are interested in correspondences

$$\mathrm{Irr}_K(A) \xrightarrow{\sim} \mathrm{Irr}_K(B) \qquad \begin{array}{l} A \in \mathrm{Bl}(RH) \\ B \in \mathrm{Bl}(RG) \end{array}$$

such a bijection induces

$$R_K(A) = \mathbb{Z}\,\mathrm{Irr}_K(A) \xrightarrow{\sim} \mathbb{Z}\,\mathrm{Irr}_K(B) = R_K(B)$$

an iso that maps basis elt to basis elt.

Defn: an iso. $I : R_K(A) \to R_K(B)$ is an isometry if for all $\psi \in \mathrm{Irr}(A)$, $I(\psi) = \pm\chi$ for some $\chi \in \mathrm{Irr}(B)$

$A \in \mathcal{B}\ell(RH)$, $B \in \mathcal{B}\ell(RG)$.

Fact: there is a bijection

Grothendieck gp of $_B\text{mod}_A$

$$R_K(B,A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(R_K(A), R_K(B))$$

$$\mu \longmapsto I_\mu$$

where $I_\mu(\nu)(g) = \frac{1}{|H|} \sum_{h \in H} \mu(g,h)\nu(h)$, $\forall \nu \in R_K(A), g \in G$.

(Note: $R_K(B,A) \subseteq R_K(G \times H)$)

(Broué)

<u>Defn</u>: Let G, H be finite groups, let (K, R, F) be a p-modular system large enough, and let $A \in Bl(RH)$, $B \in Bl(RG)$. An isometry $I: R_K(A) \to R_K(B)$ is a perfect isometry if the unique $\mu \in R_K(B, A)$ such that $I = I_\mu$ satisfies both:

(1) For all $g \in G$, $h \in H$,

$$\frac{\mu(g,h)}{|C_G(g)|} \in R \quad \text{and} \quad \frac{\mu(g,h)}{|C_H(h)|} \in R, \quad \text{and}$$

(2) If $\mu(g,h) \neq 0$ then $g \in G_{p'}$ if and only if $h \in H_{p'}$.

$A \in \mathcal{Bl}(RH)$, $B \in \mathcal{Bl}(RG)$, and $I: R_K(A) \to R_K(B)$ a perfect isometry. Then:

① I induces an R-algebra iso. $Z(A) \xrightarrow{\sim} Z(B)$.

$A \in \mathcal{Bl}(RH)$, $B \in \mathcal{Bl}(RG)$, and $I : R_K(A) \to R_K(B)$ a perfect isometry. Then:

1. I induces an R-algebra iso. $Z(A) \xrightarrow{\sim} Z(B)$.
2. I maps $P_R(A)$ onto $P_R(B)$.

$A \in \mathcal{B}\ell(RH)$, $B \in \mathcal{B}\ell(RG)$, and $I : R_K(A) \to R_K(B)$ a perfect isometry. Then:

1. I induces an R-algebra iso. $Z(A) \xrightarrow{\sim} Z(B)$.
2. I maps $P_R(A)$ onto $P_R(B)$.
3. I maps $\ker(d_A)$ onto $\ker(d_B)$. In particular, there is a unique map $\bar{I} : R_F(A) \to R_F(B)$ such that $d_B \circ I = \bar{I} \circ d_A$.

$$\begin{tikzcd}
P_R(A) \arrow[rr, "c_A"] \arrow[dr, hook] \arrow[rrr, bend left] & & R_F(A) \arrow[rrr, bend left] & P_R(B) \arrow[rr, "c_B"] \arrow[dr, hook] & & R_F(B) \\
& R_K(A) \arrow[ur, "d_A"'] \arrow[rrr, bend right] & & & R_K(B) \arrow[ur, "d_B"'] &
\end{tikzcd}$$

$A \in \mathcal{B}\ell(RH)$, $B \in \mathcal{B}\ell(RG)$, and $I : R_K(A) \to R_K(B)$ a perfect isometry. Then:

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4. The Cartan matrices C_A and C_B have the same determinant and elementary divisors (with the same multiplicities).

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4. The Cartan matrices C_A and C_B have the same determinant and elementary divisors (with the same multiplicities).
5. If $\psi \in Irr(A)$, $\chi \in Irr(B)$ such that $I(\psi) = \pm\chi$ then $\text{height}(\psi) = \text{height}(\chi)$.

Broué's Abelian Defect Group Conjecture (version #1):

Let $B \in \mathcal{B}\ell(RG)$, let D be a "defect group" of B, and let $A \in \mathcal{B}\ell(RN_G(D))$ be the "Brauer correspondent" of B. If D is abelian then there exists a <u>perfect isometry</u> between A and B.

In particular, A and B have isomorphic centers, isomorphic CDE triangles, etc.

ex) Let $G = A_5$ and $p = 2$. There is a block $B \in Bl(RA_5)$ with $\mathrm{Irr}(B) = \{\chi_1, \chi_3, \chi_4, \chi_5\}$

	1	2	3	5A	5B
• χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
• χ_3	5	1	-1	0	0
• χ_4	3	-1	0	α	β
• χ_5	3	-1	0	β	α

$\alpha = \frac{1}{2}(1+\sqrt{5})$

$\beta = \frac{1}{2}(1-\sqrt{5})$

$D \in \mathrm{Syl}_2(A_5)$, D abelian.

$N_{A_5}(D) \cong A_4$. $A = RA_4$

	1	2	3A	3B
• ψ_1	1	1	1	1
• ψ_2	1	1	ζ	ζ^2
• ψ_3	1	1	ζ^2	ζ
• ψ_4	3	-1	0	0

ζ = primitive 3^{rd} root of unity

Can check!

$I: \psi_1 \mapsto \chi_1$
$\psi_2 \mapsto -\chi_5$
$\psi_3 \mapsto -\chi_4$
$\psi_4 \mapsto -\chi_3$

is a perfect isometry $R_K(A) \to R_K(B)$.

Note: I preserves char. values on the 2-elements.

Let A, B be as before. An isotypy between A and B is a family of "compatible" perfect isometries between corresponding blocks of centralizers of p-subgroups.

Broué's Abelian Defect Group Conjecture (version #2):

Let $B \in \mathcal{Bl}(RG)$, let D be a "defect group" of B, and let $A \in \mathcal{Bl}(RN_G(D))$ be the "Brauer correspondent" of B. If D is abelian then there exists an <u>isotypy</u> between A and B.

version #2 $\Rightarrow$ version #1

Recall: $\cdot\, D^b(A)$ is the bounded derived category of ${}_A\mathrm{mod}$.

$\cdot$ A and B are derived equivalent if the triangulated categories $D^b(A)$, $D^b(B)$ are equivalent.

Broué's Abelian Defect Group Conjecture (version #3):

Let $B \in Bl(RG)$, let D be a "defect group" of B, and let $A \in Bl(RN_G(D))$ be the "Brauer correspondent" of B. If D is abelian then A and B are derived equivalent.

Let A, B be as in the conjecture.

If A, B are derived equiv. $\Rightarrow$ there is a two-sided tilting complex X such that

$$\cdot \otimes_A X : D^b(A) \to D^b(B)$$

is an equivalence.

Can show: X induces an equivalence

$$D^b(K \otimes_R A) \cong D^b(K \otimes_R B).$$

The Grothendieck gp of $D^b(K \otimes_R A)$ is iso. to $R_K(A)$, so X induces isomorphism $R_K(A) \to R_K(B)$, and this isomorphism turns out to be a perfect isometry.

(So version #3 $\Rightarrow$ version # 1)