Question

(a) Consider a group G, for each $g \in G$ define a map

$$
\begin{aligned}
c_g : G &\longrightarrow G \\
h &\longmapsto g h g^{-1}
\end{aligned}
$$

$c_g(h) = g h g^{-1}$

$c_g(g) = g g g^{-1} = g$

Call $\mathrm{Inn}(G) := \{ c_g \mid g \in G \}$

Prove $\mathrm{Inn}(G) \leq \mathrm{Aut}(G)$ i.e. show

grp under composition

identity $- \mathrm{id}_G$

✓ (i) $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ [c_g is a $\underbrace{\text{bijective}}_{\text{create an inverse}}$ homomorphism]

✓ (ii) $\mathrm{Inn}(G) \neq \varnothing$

✓ (iii) $c_g \circ c_h^{-1} \in \mathrm{Inn}(G) \quad \forall g, h \in G$

Remark: $\mathrm{Inn}(G)$ is called the group of inner automorphisms.

(b) Consider the function

$$
\begin{aligned}
\phi : G &\longrightarrow \mathrm{Aut}(G) \\
g &\longmapsto c_g
\end{aligned}
$$

Show

(i) ϕ is a group homomorphism $\quad \phi(gh) = C_{gh} = C_g \circ C_h = \phi(g)\circ\phi(h)$

(ii) Compute $\ker \phi$ (do you see $\operatorname{im}\phi = \operatorname{Inn}(G)$)

(iii) Use the first isomorphism theorem to write $\operatorname{Inn}(G)$ as a quotient of G. $\quad G/\ker\phi \cong \operatorname{im}\phi$

Remark: Turns out $\operatorname{Inn}(G) \trianglelefteq \operatorname{Aut}(G)^{*}$, and we call

$$\operatorname{Out}(G) := \operatorname{Aut}(G)/\operatorname{Inn}(G)$$

the group of outer automorphisms of G.

* This is because, taking any $\psi \in \operatorname{Aut}(G)$ we have

$$\underbrace{\psi \circ C_g \circ \psi^{-1}}_{} = C_{\psi(g)} \in \operatorname{Inn}(G)$$

(a) (i) $\quad C_g(xy) = gxyg^{-1} = gxeyg^{-1} = gxg^{-1}gyg^{-1}$

$$= C_g(x)\, C_g(y)$$

Claim: $C_g^{-1} = C_{g^{-1}}$

$$C_g \circ C_{g^{-1}} = \mathrm{id} = C_{g^{-1}} \circ C_g$$

$$C_{g^{-1}} \circ C_g(x) = C_{g^{-1}}(gxg^{-1}) = \underbrace{g^{-1}(g}_{}\,x\,\underbrace{g^{-1})(g^{-1})^{-1}}_{} = x$$

$$= \mathrm{id}(x)$$

$\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$

(ii) Claim: $id \in \mathrm{Inn}(G)$ ✓

Claim: $id = C_e$

$$C_e(x) = exe^{-1} = x = id(x)$$

Hence $\mathrm{Inn}(G) \neq \emptyset$

(iii) $C_g \circ C_{\underline{\underline{h}}}^{-1} = C_g \circ C_{h^{-1}}$

Claim: $C_g \circ C_{h^{-1}} = C_{gh^{-1}} \in \mathrm{Inn}(G)$

$$\begin{aligned} C_g \circ C_{h^{-1}}(x) &= C_g\left(h^{-1} x (h^{-1})^{-1}\right) \\ &= g(h^{-1}xh)g^{-1} \\ &= gh^{-1}x(gh^{-1})^{-1} \\ &= C_{gh^{-1}}(x) \end{aligned}$$

(b) $\phi(gh) = \phi(g) \circ \phi(h)$

i.e. $C_{gh} = C_g \circ C_h$

$C_{gh}(x) = ghxh^{-1}g^{-1} = g(hxh^{-1})g^{-1}$

$= C_g(hxh^{-1})$

$= C_g(C_h(x))$

$= C_g \circ C_h(x)$

(ii) Claim: $\ker\phi = Z(G) = \{g \in G \mid gx = xg, \ \forall x \in G\}$

* $\ker\phi \subseteq Z(G)$ & • $Z(G) \subseteq \ker\phi$

* $g \in \ker\phi \Rightarrow \phi(g) = id$, i.e. $C_g = id$

Let $x \in G$ arbitrary

$gxg^{-1} = C_g(x) = id(x) = x \Rightarrow gx = xg$

$\Rightarrow g \in Z(G)$.

• $g \in Z(G)$ and show $C_g = id$ (i.e. $g \in \ker\phi$)

$C_g(x) = \underbrace{gx}g^{-1} = \underbrace{xg}g^{-1} = x = id(x)$

Hence $g \in \ker\phi$

Therefore $\ker\phi = Z(G)$

(iii) So by FI, we have

$$G/Z(G) \cong \mathrm{Inn}(G) \longrightarrow \mathrm{im}\,\phi$$

$\ker\phi$

Recall (classwork 1)

$$S^1 = \{z \in \mathbb{C}^\times \mid |z| = 1\} \leq \mathbb{C}^\times$$

(group under multiplication)

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$p_2 : S^1 \longrightarrow S^1$$
$$z \longmapsto z^2,$$

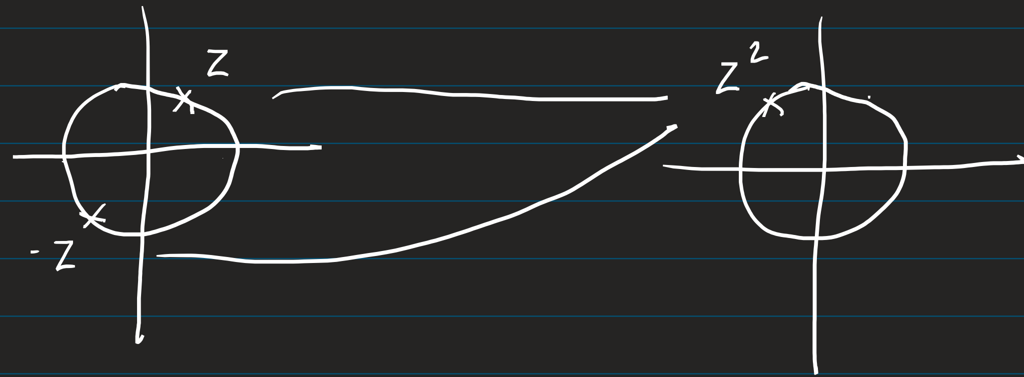
$$(zw)^2 = z^2 w^2$$

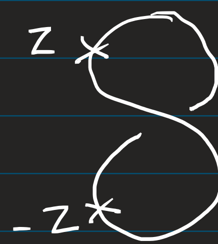
surjective group hmph.

$$\ker p_2 = \{z \in S^1 \mid z^2 = 1\} = \{\pm 1\}$$

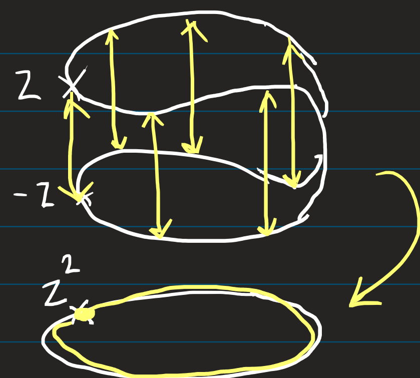
$$p_2(z) = p_2(-z)$$

$$S^1/\{\pm 1\} \cong S^1$$





$$\begin{array}{c} S^1 \\ \downarrow \\ S^1 \end{array}$$



$$p_n : S^1 \twoheadrightarrow S^1$$

$$z \longmapsto z^n$$

$$\ker p_n = \mu_n = \{n^{th} \text{ roots of unity}\}$$

$$S^1/\mu_n \cong S^1$$