$R \subseteq \mathbb{N} \times \mathbb{N}$

aRb if a^2+b^2 is even

* Reflexive: is aRa?

  Note $a^2+a^2 = 2a^2$ is even, $\therefore$ aRa !

  for $a \in \mathbb{N}$.

* Symmetric: if aRb, then bRa?

  Given a^2+b^2 is even, is b^2+a^2 even?

  Yes, since $b^2+a^2 = a^2+b^2$.

* Transitive: if $\underbrace{aRb}$ & $\underbrace{bRc}$, then aRc?

  Given a^2+b^2 and b^2+c^2 is even, is a^2+c^2 even?

  $a^2+b^2 = 2k$ & $b^2+c^2 = 2l$

  $a^2 + 2b^2 + c^2 = 2(k+l)$

  Then $a^2+c^2 = 2(k+l-b^2)$, hence even!

Equivalence classes

Let $a \in \mathbb{N}$

$$[a] = \{ b \in \mathbb{N} \mid aRb \} \overset{\text{def}^n}{=} \{ b \in \mathbb{N} \mid a^2+b^2 \text{ is even} \}$$

//

Question is : when is $[a]=[b]$, for some $a, b \in \mathbb{N}$

if $a \in [b]$ or $b \in [a]$

WLOG, assume $a \in [b]$

i.e. a^2+b^2 is even //

Hence $[a] \neq [b]$ when a^2+b^2 is NOT even //

— i.e. a & b have different parity //

i.e. a — odd / even, b — even / odd

$[1]$, $[2]$ ▨

3 → $[1]$, 6 → $[2]$

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(32) $A = \{ a+b\sqrt{2} \mid a+b\sqrt{2} \neq 0,\ a, b \in \mathbb{Q} \}$

$\boxed{xRy \quad \text{then} \quad \frac{x}{y} \in \mathbb{Q}}$ $\boxed{(x,y) \in R \iff \frac{x}{y} \in \mathbb{Q} //}$

$R \subseteq A \times A$

$(x,x) \in R$ //, i.e. $\frac{x}{x} \in \mathbb{Q}$ //

* Reflexivity : $x \in A \ (\mathbb{Q})$, $\frac{x}{x} = 1 \in \mathbb{Q}$

* Symmetry : $\boxed{\frac{x}{y} \in \mathbb{Q}}$ i.e. $\frac{x}{y} = \frac{p}{q}$ where $p, q \in \mathbb{Z}$

$$\frac{1}{x/y} = \frac{y}{x} = \frac{q}{p} \in \mathbb{Q} \qquad \text{hence symmetric!} \qquad r \in \mathbb{Q},\ \frac{1}{r} \in \mathbb{Q}$$

* Transitivity : Suppose $\underbrace{\frac{x}{y}, \frac{y}{z} \in \mathbb{Q}}$. Is $\underbrace{\frac{x}{z} \in \mathbb{Q}}$?

$\underbrace{(x,y), (y,z) \in R}$. Is $\underbrace{(x,z) \in R}$

$r, s \in \mathbb{Q}$ is $r \cdot s \in \mathbb{Q}$

$$\frac{x}{y}, \frac{y}{z} \in \mathbb{Q} \quad \text{then} \quad \underbrace{\frac{x}{y} \cdot \frac{y}{z}}_{\frac{x}{z}} \in \mathbb{Q}$$

$$\frac{x}{z} \in \mathbb{Q}$$

Equivalence Classes

$a+b\sqrt{2} \in A$

$$[a+b\sqrt{2}] \overset{\text{def}^n}{=} \{x \in A \mid x \, R \, a+b\sqrt{2}\}$$

def^n

$$= \left\{x \in A \,\middle|\, \frac{x}{a+b\sqrt{2}} \in \mathbb{Q}\right\}$$

def^n

$$= \left\{x \in A \,\middle|\, \frac{x}{a+b\sqrt{2}} = k \in \mathbb{Q}\right\}$$

algebra

$$= \{x \in A \mid x = k(a+b\sqrt{2}), \quad k \in \mathbb{Q}\}$$

rewriting

$$= \{k(a+b\sqrt{2}) \mid k \in \mathbb{Q} \setminus \{0\}\} \quad \checkmark$$

$\textcircled{I}$ $[a+b\sqrt{2}] \neq [c+d\sqrt{2}]$ // Investigation!

$c, d \neq 0$

$\underline{a+b\sqrt{2}} \in \underline{[c+d\sqrt{2}]}$ ✓

$a+b\sqrt{2} = \underline{k(c+d\sqrt{2})}$ $\quad k \in \mathbb{Q}\setminus\{0\}$

$\underline{a+b\sqrt{2} = kc + kd\sqrt{2}}$

$\therefore \underline{a = kc \ \& \ b = kd}$

$\therefore \dfrac{a}{c} \neq \dfrac{b}{d} \ (= k)$ //

[We assumed c & d $\neq 0$
What if $c=0$ or $d=0$]

II $c=0$

$[a+b\sqrt{2}] = [d\sqrt{2}]$ $\quad k \cdot a$ $\quad$ Investigation!

$a+b\sqrt{2} = kd\sqrt{2}$ $\quad \frac{1}{k} \cdot a$

$\underline{a=0}$ and $\underline{b=kd}$

$[a+b\sqrt{2}] = [kd\sqrt{2}] = [d\sqrt{2}] = [\sqrt{2}]$

$[kd\sqrt{2}] = [\sqrt{2}]$

$[kd\sqrt{2}] \subseteq [\sqrt{2}]$ ✓ and $[\sqrt{2}] \subseteq [kd\sqrt{2}]$ ✓

Let $x \in [kd\sqrt{2}]$ i.e. $x = r(kd\sqrt{2}) = \underline{(rkd)}\sqrt{2}$

$x \in [\sqrt{2}]$

$y \in [\sqrt{2}]$, $y = s\sqrt{2} = \underline{\dfrac{s}{kd}} \cdot \underline{(kd\sqrt{2})} \in [kd\sqrt{2}]$

III $d=0$

$[a+b\sqrt{2}] = [d]$

$a+b\sqrt{2} = kd$

$b=0$ & $a=kd$

$[a+b\sqrt{2}] = [kd] = [d] = [1]$

I ~~~ $[1] = \{k\cdot 1 \mid k \in \mathbb{Q}\setminus\{0\}\} = \mathbb{Q}\setminus\{0\}$

II ~~~ $[\sqrt{2}] = \{k\sqrt{2} \mid k \in \mathbb{Q}\setminus\{0\}\}$

Others: $[c+d\sqrt{2}] \neq [a+b\sqrt{2}]$ when $a,b,c,d \neq 0$ $0\}$

when $\frac{a}{c} \neq \frac{b}{d}$

$[1+\sqrt{2}] \neq [2+3\sqrt{2}]$ b/c $\frac{1}{2} \neq \frac{1}{3}$

$[n+(n+1)\sqrt{2}] \neq [1+\sqrt{2}]$ $\forall n \in \mathbb{N}$

since $\frac{1}{n} \neq \frac{1}{n+1}$

lines in $\mathbb{R}^2$: $\underbrace{l_1 R l_2}$ if $\underbrace{l_1 \parallel^{le} l_2}_{\text{slope of } l_1 \,=\, \text{slope of } l_2}$ ..

(1) Reflexive: ✓

(2) Symmetric ✓

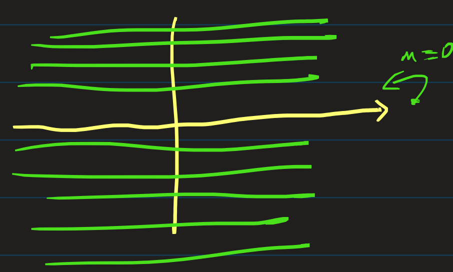
(3) Transitive: ✓

$$[x\text{-axis}] = \{ l \mid l \parallel x\text{-axis} \}$$

$$= \{ l \mid \text{slope of } l \text{ is } 0 \}$$

$$= \{ y = mx + b \mid m = 0 \}$$

$$= \{ y = b \mid b \in \mathbb{R} \}$$



$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \dots$$

$\frac{1}{2} := [(1,2)]$

$$[(1,2)] = \{ (k, 2k) \mid k \in \mathbb{Z} \} \ni (4,8)$$

$(2,4) \in$ $\qquad \ni (3,6)$

$\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ $\qquad (a,b) \, R \, (c,d)$

$ad = bc$

$\left[\begin{array}{c} \frac{a}{b} = \frac{c}{d} \\ ad = bc \end{array} \right]$

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z} \setminus \{0\}) / \sim$$