(Tensor) Triangulated Categories

What are they good for- part 2?

David Rubinstein

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An analogue of Ring Theory

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   3.1 We call such "rings" Tensor-Triangulated Categories
   3.2 With this product structure in hand (to be defined soon), we can make progress on some promised Unification we hinted at last week. Let us recall those leading questions now.

Some Leading Questions

Unification

1. Since so many examples of these Triangulated Categories exist ranging from algebra/geometry (examples 2,3,4) to topology (1,5,6) to analysis (7) the vague hope is that by studying "Triangulated Categories" writ large, one can learn things about all these topics in one fell swoop.
2. For example- Can we relate the ideas of line bundles in Algebraic-Geometry and Endotrivial Modules in Representation Theory? Moreover, can we relate Spec(R) for a commutative ring with Support Varieties $\mathcal{V}_G := \mathrm{Proj}(H^\bullet(G,k))$
3. There is a famous nilpotence theorem of Hopkins and Smith in Algebraic Topology- are there analogous nilpotent theorems in other contexts? In Alg Topology this nilpotence theorem provides a stratification for our category, can we expect the same for other nilpotence theorems?

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   1.1 We have the following natural isomorphisms:

$$l_a : 1 \otimes a \cong a$$
$$r_a : a \otimes 1 \cong a$$
$$\tau_{a,b} : a \otimes b \cong b \otimes a$$

   along with associativity axioms in such a way that everything behaves nice with one another.

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   along with associativity axioms in such a way that everything behaves nice with one another.
2. We think of $\otimes$ as a commutative product and 1 as the unit. Indeed, in the literature, you will often see the above referred to as a "Symmetric Monoidal Category" (although this can refer to any tensor category, not necessarily triangulated), or an "Axiomatic Stable Homotopy Theory"

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2. Then for each $k \in \mathcal{K}$ we denote the duel of k as $k^\vee := \mathrm{hom}(k, \mathbb{1})$. There is a natural map $k^\vee \otimes l \to \mathrm{hom}(k, l)$ given from the counit of the adjunction.

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2. Then for each $k \in \mathcal{K}$ we denote the duel of k as $k^{\vee} := \mathrm{hom}(k, \mathbb{1})$. There is a natural map $k^{\vee} \otimes l \to \mathrm{hom}(k, l)$ given from the counit of the adjunction.
3. We say that $\mathcal{K}$ is rigid, if this natural map $k^{\vee} \otimes l \xrightarrow{\sim} \mathrm{hom}(k, l)$ is an isomorphism for all k,l. (sometimes this condition is called "Strongly Duelizable")

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4. Rigidity is an assumption that just simplifies our life a lot- for example, the tensor-hom adjunctions imply the k is a direct summand of $k \otimes k \otimes k^{\vee}$ and that $k^{\vee}$ is a direct summand of $k \otimes k^{\vee} \otimes k^{\vee}$. We will note in a few slides why this is useful.

Examples of tt Categories

Bring on the Examples

The following are examples of tt categories. We shall write them all as $(\mathcal{K}, \otimes, 1)$

1. Let k be a field of char p dividing the order of the group G. Then $(\mathrm{stab}(kG), \otimes_k, k)$ is a tt category, where the tensor is the usual tensor with diagonal action of g
$g(M \otimes_k N) = gM \otimes_k gN$

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4. $(\mathcal{SH}^{\text{finite}}, \wedge, \mathbb{S}^0)$ or more generally the G-Equivariant Stable Homotopy Category for G a Group $(\mathcal{SH}^{\text{finite}}(G), \wedge, \mathbb{S}^0)$
5. Variations of the theme for points 2 and 3 for X a quasi compact, quasi separated scheme.

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3. Let $(\mathcal{T}, \otimes, 1)$ be another closed, rigid, tt category. Then a tt functor $F : \mathcal{K} \to \mathcal{T}$ is a triangulated functor that is also "strong monoidal" (that is $F(a \otimes b) \cong F(a) \otimes F(b)$- with a whole bunch of compatibility axioms as well)

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Rmk: The assumption that $\mathcal{K}$ is rigid, and the remark about $k, k^\vee$ being summands of respective tensors implies that a tensor ideal $\mathcal{I}$ is automatically closed under taking duals, and is radical ($k^{\otimes n} \in \mathcal{I} \implies k \in \mathcal{I}$). This second part will be very important in a few slides.

Verdier Localization revisited

Example of tt functor

We saw last talk that given a thick subgcategory $\mathcal{C} \subseteq \mathcal{K}$ that we could form the quotient category $\mathcal{K}/\mathcal{C}$ in such a way that it was triangulated, and where the universal quotient functor $\mathrm{q} : \mathcal{K} \to \mathcal{K}/\mathcal{C}$ is a triangulated functor. We want to extend this construction to the tt world.

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1. Now let $\mathcal{I} \subseteq \mathcal{K}$ be a tt ideal. Then we can form the triangulated category $\mathcal{K}/\mathcal{I}$ with universal functor $\mathrm{q} : \mathcal{K} \to \mathcal{K}/\mathcal{I}$ just because $\mathcal{I}$ is thick.

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2. The point is, that now we have the following, two hopefully unsurprising, facts:
    2.1 $\mathcal{K}/\mathcal{I}$ is a tt category.
    2.2 The universal functor $\mathrm{q} : \mathcal{K} \to \mathcal{K}/\mathcal{I}$ is a tensor functor.

TT Geometry-take 1

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2. In other words we want to know when we can reach an object Y from X using all the structure at hand. That is we want to know if Y is in the tensor ideal generated by X!
3. Therefore, our main task is to classify thick tensor ideals of $\mathcal{K}$! How to do this.....?

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2. The support of an object k is the collection of all prime ideals k is NOT in, supp(k)=$\{\mathcal{P} \in Spc(\mathcal{K}) : k \notin \mathcal{P}\}$
   2.1 Rmk: Recall the kernal of the universal functor $q : \mathcal{K} \to \mathcal{K}/\mathcal{P}$ is precisely P. So saying $k \notin \mathcal{P}$ amounts to saying $k \neq 0$ in the tt category $\mathcal{K}/\mathcal{P}$

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4. We shall see that classifying the tt ideals of $\mathcal{K}$ more or less amounts to classifying nice subsets of $\mathrm{Spc}(\mathcal{K})$. Let us first see some basic properties of the Spectrum.

More on Balmer Spectrum

It should hopefully come as no surprise that the Balmer spectrum satisfies a universal property. To state what the universal property is, we need a few results first: Let $\mathcal{K}, \mathrm{Spc}(\mathcal{K}), \mathrm{supp}(k)$ be as above. Then,

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We say a "Support Data" on a tt category is a pair (X, σ) where X is a topological space and $\sigma(k)$ is a closed subset in X for each $k \in \mathcal{K}$ satisfying conditions 1-5 above.

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$$\text{Thrm} : (Spc(\mathcal{K}), supp) \text{ is the terminal support data on } \mathcal{K}$$

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TT Geometry- Take 3

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TT Geometry- Take 3

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3. Moreover, $\mathrm{Spc}(\mathcal{K})$ is a "Spectral" topological space: that is, it is T_0; it is quasi-compact; the quasi-compact open subsets of $\mathrm{Spc}(\mathcal{K})$ are closed under finite intersections and form an open basis of $\mathrm{Spc}(\mathcal{K})$; and every non-empty irreducible closed subset of $\mathrm{Spc}(\mathcal{K})$ has a generic point.

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4. The assignment $\mathcal{K} \to \mathrm{Spc}(\mathcal{K})$ is a contravarient functor. Given a tt functor $\mathcal{K} \xrightarrow{F} \mathcal{T}$ we get a continuous (spectral) map $\mathrm{Spc}(\mathcal{T}) \xrightarrow{\mathrm{Spc}(F)} \mathrm{Spc}(\mathcal{K})$

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3. Now let $\mathcal{P} \in \mathrm{Spc}(K)$, and let $V = \bigcup_{x \in \mathcal{P}} \mathrm{supp}(x)$. Then V is a Thomason subset of $\mathrm{Spc}(\mathcal{K})$. Moreover,

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4. **The two assignments above give an order preserving bijection between**

$$\mathrm{Thom}(\mathrm{Spc}(\mathcal{K})) \leftrightarrow \mathrm{Thick}^{\otimes}(\mathcal{K})$$

the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$ and the set of tt ideals in $\mathcal{K}$.

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Applications

Before we provide some of the more classical examples of classifications, let us give some consequences of this notion of Spc.

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Now, recall that we noticed that $\mathrm{Spc}(\mathcal{K})$ is a "spectral space." It is a theorem that every spectral space is isomorphic to Spec(R) for some commutative ring R. In any tt-category, one can take the endomorphism ring of the unit, and get a commutative ring. So we might ask,

1. Is $\mathrm{Spc}(\mathcal{K}) \cong \mathrm{Spec}(\mathrm{End}_{\mathcal{K}}(1))$?
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3. Thrm: There exist two natural continuous maps
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   Moreover, these maps are very often surjective. We call these maps the comparison maps, and we will make use of them shortly.

Derived Category of a (nice enough) Scheme

Reproducing a Scheme

1. Thrm (Thomason, Neeman): Let X be a quasi-compact, quasi seperated scheme, and let $\mathcal{K} = D^{\mathrm{perf}}(X)$ be the derived category of perfect complexes. Then the Spectrum of $\mathcal{K}$ is isomorphic to the underlying scheme itself $|X|$ via a homeomorphism $|X| \xrightarrow{\sim} \mathrm{Spc}(D^{\mathrm{perf}}(X))$ given by:

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4. There are generalizations of these to (nice enough) stacks and singularity categories a la Stevenson. They are beyond my current knowledge however.

Stable Module Category

Support Varieties

Let G be a finite group and k be a field of charactersistic p dividing the order of G. Then consider $\mathcal{K}$ =stab(kG) the category of finite dimensional kG-modules, and recall that we can identify stab(kG) as the Verdier quotient stab(kG) $\cong D^b(kG)/D^{perf}(kG)$ (see my last talk)

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3. Rmk: The two above homeomorphisms hold for G a finite group scheme as well.

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1. We will consider the "p-local Stable Homotopy category". This consists of objects in SH^c such that $\pi_*(X) \otimes \mathbb{Z}_{(p)} \cong \pi_*(X)$. Denote this subcategory by SH^c_p. This is the Bausfield Localization $q : SH^c \to SH^c_p$ with respect to the Homology theory $\pi_\bullet(-) \otimes \mathbb{Z}_(p)$

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2. For each integer $n \geq 1$ and prime p, there is a Homology theory, called Morava k-theory, denoted as $K_{p,n} : SH^c_p \to \mathbb{F}_p[v_n, v_n^{-1}] - Mod$.
   Let us denote $\mathcal{C}_{p,n} := q^{-1}(ker(K_{p,n}))$ and $\mathcal{C}_{0,1}$ to be the kernal of the rationalization functor $\pi_\bullet(-) \otimes \mathbb{Q} \cong H_\bullet(-, \mathbb{Q}) : SH^c \to SH^c_{\mathbb{Q}} \cong D^b(\mathbb{Q})$

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3. Recall the "comparison map" defined some slides ago: $\rho : Spc(\mathcal{K}) \to Spec(End_{\mathcal{K}}(1))$. In this case, the unit is $1 = \mathbb{S}^0$ and $\mathrm{End}_{\mathcal{K}}(\mathbb{S}^0) = \mathbb{Z}$

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4. Then it turns out the Spectrum of SH^c is given by pulling back this comparison map $\rho_{\mathcal{K}}$. The picture is as follows:

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$$
\begin{tikzcd}[column sep=small]
 & \mathcal{C}_{2,\infty} & \mathcal{C}_{3,\infty} & \cdots & \mathcal{C}_{p,\infty} & \cdots \\
\operatorname{Spec}(\mathrm{SH}^c) = \arrow[dddddd, "\rho_{\mathrm{SH}^c}"'] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & \\
 & \mathcal{C}_{2,n+1} \arrow[d, dash] & \mathcal{C}_{3,n+1} \arrow[d, dash] & \cdots & \mathcal{C}_{p,n+1} \arrow[d, dash] & \cdots \\
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 & \mathcal{C}_{2,2} \arrow[drr, dash] & \mathcal{C}_{3,2} \arrow[dr, dash] & \cdots & \mathcal{C}_{p,2} \arrow[dl, dash] & \cdots \\
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\operatorname{Spec}(\mathbb{Z}) = & 2\mathbb{Z} \arrow[drr, dash] & 3\mathbb{Z} \arrow[dr, dash] & \cdots & p\mathbb{Z} \arrow[dl, dash] & \cdots \\
 & & & (0) & &
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In the above picture, a line indicates that the higher prime is in the closure of the lower one. We have more precisely:

1. The pre-image of the dense point $(0) \in Spec(\mathbb{Z})$ is the single point $\mathcal{C}_{0,1}$

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3. $\mathcal{C}_{0,1}$ is the unique dense point in $\mathrm{Spc}(\mathrm{SH}^c)$. For each prime p and integer $1 \leq n \neq \infty$ we have the closure $\overline{\{\mathcal{C}_{p,n}\}} = \{\mathcal{C}_{p,m} : n \leq m \leq \infty\}$. The closed points of $\mathrm{Spc}(\mathrm{SH}^c)$ are precisely the $\mathcal{C}_{p,\infty}$ for all p.

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4. The support of an object x is:
   - 4.1 supp(x)= $\emptyset$ when $x \cong 0$
   - 4.2 supp(x)= Spc(SH^c) when $x \notin \mathcal{C}_{0,1}$
   - 4.3 supp(x)= a finite union of "collumns" when $x \in \mathcal{C}_{0,1}$. More concretely, supp(x)= finite unions of $\overline{\mathcal{C}_{p,m_p}}$ where $\mathcal{C}_{p,m_p} := \{\mathcal{C}_{p,n} : m_p \leq n \leq \infty\}$ and where m_p is the "type" of p.

The Original classification

In the above picture, a line indicates that the higher prime is in the closure of the lower one. We have more precisely:

1. The pre-image of the dense point $(0) \in Spec(\mathbb{Z})$ is the single point $\mathcal{C}_{0,1}$
2. For each prime p, the pre-image of $p\mathbb{Z}$ consists of the column $\mathcal{C}_{p,n}$ and where $\mathcal{C}_{p,\infty} = \ker(q_p : SH^c \to SH^c_p)$.
3. $\mathcal{C}_{0,1}$ is the unique dense point in $Spc(SH^c)$. For each prime p and integer $1 \le n \ne \infty$ we have the closure $\overline{\{\mathcal{C}_{p,n}\}} = \{\mathcal{C}_{p,m} : n \le m \le \infty\}$. The closed points of $Spc(SH^c)$ are precisely the $\mathcal{C}_{p,\infty}$ for all p.
4. The support of an object x is:
   - 4.1 supp(x)= $\emptyset$ when $x \cong 0$
   - 4.2 supp(x)= Spc(SH^c) when $x \notin \mathcal{C}_{0,1}$
   - 4.3 supp(x)= a finite union of "columns" when $x \in \mathcal{C}_{0,1}$. More concretely, supp(x)= finite unions of $\overline{\mathcal{C}_{p,m_p}}$ where $\mathcal{C}_{p,m_p} := \{\mathcal{C}_{p,n} : m_p \le n \le \infty\}$ and where m_p is the "type" of p.
5. The Thomason Subsets of Spc(SH^c) are
   - 5.1 the empty set and the whole space itself
   - 5.2 Arbitrary unions of columns $\overline{\mathcal{C}_{p,m_p}}$
6. A great way to think about each column is that it expresses a "chromatic refinement" between the representing spectra $H\mathbb{Q}$ and $H\mathbb{F}_p$ (ie, between the primes (0) and $p\mathbb{Z}$).

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1. The examples above, while providing a great conceptual framework that unifies seemingly disjoint work, are more repackaging of old theorems rather than brave new work.
2. In many of the cases in fact, computing $\mathrm{Spc}(\mathcal{K})$ is done by ALREADY knowing the tt ideals of $\mathcal{K}$. But we would like to do the reverse: Given a tt category $\mathcal{K}$ compute $\mathrm{Spc}(\mathcal{K})$ from first principles, and then from that, DEDUCE the tt-ideals of $\mathcal{K}$.

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3. Recent work by Balmer-Sanders (2017) does just that, for the case of $\mathrm{SH}^c(G)$ for G a finite group. They describe $\mathrm{Spc}(\mathrm{SH}^c(G))$ as a set for all finite groups G, and get close to completely describing the topology. Let us say a little about $\mathrm{SH}^c(G)$, and then we can give the statement of the theorem, and present an absolutely beautiful picture after the fact.

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4. Recall the comparison map we have used a few times now $\rho_{\mathcal{K}} : \mathrm{Spc}(\mathcal{K}) \to \mathrm{Spc}(\mathrm{End}_{\mathcal{K}}(1))$. In this case, the endomorphism ring is the Burnside Ring of G, $\mathrm{End}_{\mathcal{K}}(1) \cong A(G)$.

SH(G)

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5. Remember we remarked that the computation of $\mathrm{Spc}(SH^c)$ provided a "refinement" between the primes at (0) and $p\mathbb{Z}$- we get a similar refinement of the spectrum of A(G) in this case as well.

SH(G)- Take 2

$\mathrm{Spc}(\mathrm{SH}^c)$ as a set.

1. Thrm: All G-equivariant primes are obtained by pulling back non-equivariant primes via geometric fixed point functors with respect to the various subgroups $H \leqslant G$. Moreover, there is no redundancy, in the sense that the primes $\mathcal{P}(H, p, n) = \mathcal{P}(H', p', n')$ iff H is conjugate to H' and the chromatic primes $\mathcal{C}_{p,n} = \mathcal{C}_{p',n'}$ coincide in SH^c (where $\mathcal{P}(H, p, n) := (\Phi^H)^{-1}(\mathcal{C}_{p,n})$ are the pulled back primes in SH^c by the "geometric fixed point functor").
   If $K \trianglelefteq H$ has nonzero index then $\mathcal{P}(K, p, n+1) \subseteq \mathcal{P}(H, p, n)$ for every $n \geq 1$. There is no inclusion $\mathcal{P}(K, q, n) \subseteq \mathcal{P}(H, p, m)$ unless the Chromatic primes are included, $\mathcal{C}_{q,n} \subseteq \mathcal{C}_{p,m}$ and K is conjugate to a q-subnormal subgroup of H.

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2. This completely describes the topology for groups of square free order. For example, the following picture is the spectrum of $SH(C_p)$.

$\mathrm{Spc}(\mathrm{SH}(C_p))$

SH but on Steroids