AM-GM inequality

2-numbers a, b

3-numbers in last HW p, q, r

AM (arithmetic mean) of a, b is

$\frac{a+b}{2}$

of p, q, r

$\frac{p+q+r}{3}$

GM (geometric mean) of a, b is

$\sqrt{ab}$

of p, q, r is

$\sqrt[3]{pqr}$

$\frac{a+b}{2} \geq \sqrt{ab}$; $a, b > 0$

$\frac{p+q+r}{3} \geq \sqrt[3]{pqr}$; $p, q, r \in \mathbb{R}$

$a = c^2$, $b = d^2$

$(c = \sqrt{a}$, $d = \sqrt{b})$

To prove that $\dfrac{c^2+d^2}{2} \geq cd$

SCRATCH

get everything on left
w/ 0 on right

$$c^2+d^2 \geq 2cd$$

$$\underbrace{c^2+d^2-2cd}_{} \geq 0$$

looks like the formula for $(c-d)^2$
≥ 0

Proof. Let $c=\sqrt{a}$ & $d=\sqrt{b}$, note

$$(c-d)^2 \geq 0$$

$$\Rightarrow \quad c^2+d^2-2cd \geq 0$$

$$\Rightarrow \quad c^2+d^2 \geq 2cd$$

$$\Rightarrow \quad \frac{c^2+d^2}{2} \geq cd$$

$$\Rightarrow \quad \boxed{\frac{a+b}{2} \geq \sqrt{ab}}$$

∎

Problem is given in words $(\Rightarrow)$ if ~, then ~

Sentence

[if $\underbrace{\text{m is odd}}_{(P}$, $\underset{\wedge}{\text{and}}$ $\underbrace{\text{n is even}}_{Q)}$, $\underset{\Rightarrow}{\text{then}}$ $\underbrace{\text{m+n is odd}}_{R}$,

$$(P \wedge Q) \Rightarrow R$$

Negate this

$$\neg(A \Rightarrow B)$$

$$(P \wedge Q) \wedge (\neg R)$$

$$\equiv A \wedge \neg B$$

[m is odd and n is even, and m+n is even.

Contrapositive

$$(P \Rightarrow Q) \equiv \underbrace{(\neg Q)}_{} \Rightarrow \underbrace{(\neg P)}_{}$$

if $\neg Q$, then $\neg P$

$(P \Rightarrow Q)$ by contrapos'n

$$\boxed{\neg P \Rightarrow \neg Q}$$

Contradiction

$$(P \Rightarrow Q)$$

$\boxed{(P \wedge \neg Q)}$ then show things wrong

Parity.

x, y have the same parity only if

$x \equiv y \mod 2$

$\hookrightarrow$ $x - y \equiv 0 \mod 2$

$\hookrightarrow$ $2 \mid x - y$ i.e. $x - y$ is even.

$\hookrightarrow$ x & y are even and odd together.

NOT having the same parity

$\hookrightarrow$ $x - y \not\equiv 0 \mod 2$, $x - y \equiv 1 \mod 2$

$\hookrightarrow$ $2 \nmid x - y$

$\hookrightarrow$ one of x & y is even and the other is odd.

x and y are odd

$x = 2k+1$, $k \in \mathbb{Z}$

$y = 2l+1$, $l \in \mathbb{Z}$

(A) Contradiction

↳ $\sqrt{2}$ is irrational

↳ infinitely many primes

↳ $\sqrt{3}$ is irrational

($\sqrt{p}$ is irrational, where p is prime).

① infinitely many primes,

Assume falsity of statement, and show things go wrong.

Assume there are finitely many primes

$P = \{p_1, p_2, \ldots, p_n\}$ is the set of all primes

$N = p_1 p_2 \cdots p_n + 1$ //

N is a number, ANY number has a prime divisor

p is a prime divisor of N ie. $p \mid N$ and p is prime

So $p \in P$, then $p \mid p_1 \cdots p_n$ //

$$p_1 \cdots p_n = p(p_1 \cdots p_n)$$

$p \mid N$, $p_1 \cdots p_n \equiv 0 \mod p$

$N \equiv 0 \mod p$

$N - p_1 \cdots p_n \equiv 0 \mod p$

$1 \equiv 0 \mod p$

i.e. $p \mid 1$ NOT possible.
(ABSURD!)

since p is prime ($p > 1$)

So our assumption was false, can't be finitely many primes.

p	2	
$q = p+1$	3	
$r = pq+1$	5	
$s = pqr+1$	31	
$\vdots$	$\vdots$	∞

$\sqrt{3}$ is irrational

ASSUME NOT i.e. let $\sqrt{3} \in \mathbb{Q}$

$\sqrt{3} = \frac{a}{b}$ i.e. $\gcd(a,b) = 1$

$\cancel{6}$ $4 \mid 36 = 6^2$ $\boxed{4 \nmid 6}$

$\sqrt{3} b = a$

$a^2 = 3b^2$ which means $\boxed{3 \mid a^2 \Rightarrow \boxed{3 \mid a}}$

Then $\underline{3 \mid a}$

a — $3k$, $3k+1$, $3k+2$ NOT divisible by 3

$a = 3p, \quad p \in \mathbb{Z}$

$a^2 = 9p^2$

$3b^2 = 9p^2$

$b^2 = 3p^2$

$3 \mid b^2 \Rightarrow 3 \mid b$

BUT! we assumed a & b had NO common factor!

How can it have 3 as a common factor?

~~So~~ ABSURD!

$\sqrt{3}$ is not rational

QUIZ

* Closure properties

$S \subseteq \mathbb{R}$, We say S is "$\underbrace{\text{closed under add}^n}$ / $\underbrace{\text{multipl}^n}$"

if $a, b \in S$

$a + b \in S$ $\qquad$ $a \cdot b \in S$

$\mathbb{Z}$ are closed under add^n & multiplication"

sum of integers is an integer
product of " " " "

* $S = \{x \in \mathbb{R} \mid |x| < 1\}$, then let's show S is closed under multiplication.

$\rightarrow$ $x, y \in S$, to show $xy \in S$

$E = \{2k \mid k \in \mathbb{Z}\}$, $A = \{x \in \mathbb{Z} \mid x^2 = 3\}$

property

they can be written as

they have to satisfy the property

To show $|xy| < 1$ $\quad (x \in S$ & $y \in S$; $|x| < 1, |y| < 1)$

$\boxed{HW}$ $\quad |xy| = |x| \cdot |y| < 1 \cdot 1 = 1$

so $xy \in S$

* $T = \{x \in \mathbb{R} \mid |x| < 2\}$

Is T closed under multiplication?

$x = 1.5$

$y = 1.5$

$1.9999\ldots$, $1.5 < 2$

so $x, y \in T$

COUNTEREXAMPLES

BUT! $x \cdot y = (1.5)^2 = 2.25 > 2$

so $xy \notin T$.

$2 < \_\_$

$2 < x^2$

$x > \sqrt{2}$ $= 1.4141\ldots$

* True or not! $\forall a \in \mathbb{R}, \exists f:$

For every $a \in \mathbb{R}$, you can find a unique function $f: \mathbb{R} \to \mathbb{R}$ st $f(a) = 0$

What's f?

$\exists f(a) = 0$

$f: \mathbb{R} \to \mathbb{R}$

$x \mapsto 0$

$\mapsto x^0 - 1$

$f(a) = 0$

For every $a \in \mathbb{R}$, there exists $f: \mathbb{R} \to \mathbb{R}$ st $f(a) = 0$ & $f(b) \neq 0$ when $b \neq a$

$f(x) = x - a$

$f(x) = a - a = 0$

$f(b) = 0$

$f(c) = 0$

$\boxed{f(x) = x - a}$

$f(a) = a - a = 0$

$f(b) = b - a \neq 0 \Leftrightarrow b \neq a$

* set w/ 4 distinct integer

$S = \{a_1, a_2, a_3, a_4\}$

(not equal to each other)

Then there are 2 numbers whose difference is divisible by 3.

TRUE! 3k , 3k+1 , 3k+2

a_1 — $3k$

a_2 — $3l+1$

a_3 — $3m+2$

a_4: $3b$, $3b+1$, $3b+2$

$$S = \{a_1, a_2, a_3, a_4\}$$

$$a_1 \equiv \begin{cases} 0 \mod 3 \\ 1 \mod 3 \\ 2 \mod 3 \end{cases}$$

$a_2 \equiv$ ~~$\cancel{\;}$~~ $1 \mod 3$

$a_2 \equiv 1 \mod 3$

$a_1 \equiv 1 \mod 3$

$a_2 - a_1 \equiv 1 - 1 \mod 3$

$\equiv 0 \mod 3$

$a_1 \equiv 0 \mod 3$

$a_2 \equiv 1 \mod 3$

$a_3 \equiv 2 \mod 3$

$a_1 - a_4$

$$a_4 \equiv \begin{cases} 0 \mod 3 \checkmark \\ 1 \mod 3 \\ 2 \mod 3 \end{cases}$$

$a_4 - a_2$

$a_4 - a_3$