Tensor-Triangular Classification, Support Theory and the Application on Commutative Noetherian Rings

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Abstact

In this talk, we will discuss a kind of classification up to the tensor-triangular structure in a tensor-triangulated category via the Balmer spectrum. To be more precise, we want to understand the lattice of thick tensor-ideals, a special kind of triangulated subcategories. Then we will talk about the support theory for compactly generated tensor triangulated categories.

We will see the application on algebraic geometry: the derived category of modules over a commutative noetherian ring and the famous result that the Balmer spectrum of the derived category of perfect complexes is homeomorphic to the spectrum of the ring. Finally, we will discuss the derived category of pseudo-coherent complexes over a commutative noetherian ring.

Definition

A **tensor triangulated category** is a triple $(\mathcal{K}, \otimes, 1)$ where $\mathcal{K}$ is a triangulated category, $\otimes : \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a symmetric monoidal tensor product which is exact in each variable, and 1 denotes the unit of the monoidal structure.

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This (as well as the definition of a triangulated category) is an axiomization of the common properties shared by different categories in algebraic topology, homological algebra, algebraic geometry, ... But there is no need to get to the bottom of the axioms now.
The examples include but are not limited to SH, $\mathrm{st}(\mathrm{Mod}(kG))$ and one of our protagonists today: $D(R)$.

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- there is no hope to classify topological spaces up to stable homotopy equivalence,
- there is no hope to classify kG-modules,
- there is no hope to classify complexes of sheaves up to weak equivalence, etc.

Even if we compromise ourselves to small objects, it is still too complicated.

Therefore, we need to seek for alternative ways to understand these categories. Let's assume for now that $\mathcal{K}$ is essentially small.

Definition

A **thick subcategory** of $\mathcal{K}$ is a full triangulated subcategory $\mathcal{J}$ which is closed under direct summands, i.e. if X and Y are objects in $\mathcal{K}$ and $X \oplus Y \in \mathcal{J}$, then X and Y are in $\mathcal{J}$.

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Definition

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More explicitly, a full subcategory is thick if it is closed under suspension, cones and tensoring with arbitrary objects in $\mathcal{K}$.

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Question

When two objects X and Y can be obtained from one another by direct sums, summands, cones suspension and tensors?
That is, when will X and Y generate the same thick tensor-ideals?

Therefore, the **tensor triangular classification**, or simply **tt-classification**, of an essentially small tensor triangulated category is a classification of thick tensor ideals.

History

Let R be a commutative noetherian ring. Denote by $\mathrm{D}(R)$ the derived category of arbitrary R-modules and $D_{\mathrm{perf}}(R)$ the full subcategory of perfect complexes of $\mathrm{D}(R)$. Then $\mathrm{D}_{\mathrm{perf}}(R)$ is an essentially small tensor triangulated category.

History

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Theorem (Hopkins-Neeman, 1987,1992)

Let R be a commutative **noetherian** ring. Then there is an isomorphism

$$\begin{aligned} &\{\text{thick subcategories of } \mathrm{D}_{\mathrm{perf}}(R)\} \\ \cong &\{\text{Thomason subsets of } \mathrm{Spec}(R)\} \end{aligned}$$

of lattices.

A subset of a topological space is a Thomason subset if it is a union of complements of quasi-compact open sets.

It was not until Robert W. Thomason's paper in 1997, the tensor-ideals were noticed.

Theorem (Thomason, 1997)

Let X be a (quasi-compact, quasi-separated) scheme and $\mathrm{D}_{\mathrm{perf}}(X)$ be the full subcategory of perfect complexes of the derived category of X. Then there is an isomorphism of lattices

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Theorem (Thomason, 1997)

Let X be a (quasi-compact, quasi-separated) scheme and $\mathrm{D_{perf}}(X)$ be the full subcategory of perfect complexes of the derived category of X. Then there is an isomorphism of lattices

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As a consequence, by taking affine $X = \mathrm{Spec}R$, Hopkins-Neeman's theorem can be generalized to arbitrary commutative ring. In fact, Thomason proved that when R is noetherian then every thick subcategory of $\mathrm{D_{perf}}(R)$ is a thick tensor ideal.

Balmer Spectrum

Definition

Let $\mathcal{K}$ be an essentially small tensor triangulated category. A proper thick tensor ideal $\mathcal{P}$ of $\mathcal{K}$ is **prime** if $X \otimes Y \in \mathcal{P}$ implies either $X \in \mathcal{P}$ or $Y \in \mathcal{P}$.

Denote by $\mathrm{Spc}(\mathcal{K})$ the set of all prime thick tensor ideals of $\mathcal{K}$, called the **tensor triangular spectrum of $\mathcal{K}$**.

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Denote by $\mathrm{Spc}(\mathcal{K})$ the set of all prime thick tensor ideals of $\mathcal{K}$, called the **tensor triangular spectrum of** $\mathcal{K}$.

Define the Zariski topology on $\mathrm{Spc}(\mathcal{K})$ by letting the following subsets be closed:

$$Z(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | S \cap \mathcal{P} = \emptyset\},$$

where S is any set of objects in $\mathcal{K}$.

Balmer Spectrum

For an object $X \in \mathcal{K}$, the **tensor triangular support** of X is

$$\mathrm{supp}_{\mathcal{K}}(\mathrm{X}) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{K}) | X \notin \mathcal{P}\}.$$

For a subset S of objects, define

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Proposition

$\mathrm{Spc}(\mathcal{K})$, equipped with Zariski topology, is a spectral space.

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- $\mathrm{supp}_{\mathcal{K}}(\mathrm{X} \otimes \mathrm{Y}) = \mathrm{supp}_{\mathcal{K}}(\mathrm{X}) \cap \mathrm{supp}_{\mathcal{K}}(\mathrm{Y})$.

The Classification Theorem

Denote by $\mathrm{Thick}^{\otimes}(\mathcal{K})$ the collection of thick tensor-ideals of $\mathcal{K}$ and $\mathrm{Thom}(\mathrm{Spc}(\mathcal{K}))$ the collection of Thomason subsets of $\mathrm{Spc}(\mathcal{K})$.

Theorem (Paul Balmer, 2005)

There is a pair of mutually inverse maps (of posets):

$$\mathrm{supp}_{\mathcal{K}} : \mathrm{Thick}^{\otimes}(\mathcal{K}) \longrightarrow \mathrm{Thom}(\mathrm{Spc}(\mathcal{K})),$$

and

$$\tau : \mathrm{Thom}(\mathrm{Spc}(\mathcal{K})) \longrightarrow \mathrm{Thick}^{\otimes}(\mathcal{K}),$$

where $\tau(S) = \{X \in \mathcal{K} | \mathrm{supp}_{\mathcal{K}}(\mathrm{X}) \subseteq S\}$.

The Classification Theorem

Remark

In general, the classification theorem establishes the one-one correspondence between the collection of radical thick tensor-ideals of $\mathcal{K}$ and the Thomason subsets of $\mathrm{Spc}(\mathcal{K})$. But in the case $\mathcal{K}$ is rigid where we are most interested in, every thick tensor ideal is radical.

Definition

A **support data** on $\mathcal{K}$ is a pair $(\mathbf{X}, \sigma)$ where $\mathbf{X}$ is a topological space and σ is an assignment which associates to any object $a \in \mathcal{K}$ a closed subset $\sigma(a) \subseteq \mathbf{X}$ subject to the following conditions:

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- $\sigma(X \otimes Y) = \sigma(X) \cap \sigma(Y)$.

Universal Property

Theorem(Paul Balmer, 2005)

$(\mathrm{Spc}(\mathcal{K}), \mathrm{supp}_{\mathcal{K}})$ is a support data on $\mathcal{K}$ and for any support data $(\mathbf{X}, \sigma)$ on $\mathcal{K}$, there exists a unique continuous map $f : \mathbf{X} \to \mathrm{Spc}(\mathcal{K})$ such that $\sigma(X) = f^{-1}(\mathrm{supp}_{\mathcal{K}}(X))$ for all X. Explicitly,

$$f(x) = \{X \in \mathcal{K} | x \notin \sigma(X)\}.$$

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$$f(x) = \{X \in \mathcal{K} | x \notin \sigma(X)\}.$$

Further, if the support data $(\mathbf{X}, \sigma)$ is a classifying support data, then f is a homeomorphism.

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Remark

In fact, $\mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(X))$ is not merely a topological space, it is a locally ringed space. Specially, when $X = \mathrm{Spec}(R)$ for some commutative ring, $\mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R)$ is an affine scheme.

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- by Thomason's Theorem and the fact that $\mathrm{D}_{\mathrm{perf}}(X)$ is rigid, $(X, \mathrm{supp}^{\mathrm{h}})$ is a classifying support data.
- by the universal property of classifying support data, there is a homeomorphism $f : X \to \mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(X))$,

$$f(x) = \{a \in \mathrm{D}_{\mathrm{perf}}(X) | a_x \cong 0 \text{ in } \mathrm{D}_{\mathrm{perf}}(\mathcal{O}_{X,x})\}.$$

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Support Theory

In tt-classification, the property of the tt-category of being essentially small is necessary. Can we generalize the result to the big categories?

Definition

A (not necessarily small) triangulated category $\mathcal{T}$ is **compactly generated** if there is a set G of compact objects of $\mathcal{T}$ such that an object $X \in \mathcal{T}$ is zero if and only if $\mathrm{Hom}_{\mathcal{T}}(g, \Sigma^i X) = 0$ for all $g \in G$ and $i \in \mathbb{Z}$.

Denote by $\mathcal{K} = \mathcal{T}^c$ the full subcategory of compact objects of $\mathcal{T}$. Then $\mathcal{K}$ is an essentially small triangulated category. If $\mathcal{K}$ is rigid, then we say $\mathcal{T}$ is rigidly-compactly generated.

Support Theory

For a rigidly compactly generated tt-category $\mathcal{T}$, Paul Balmer and Giordano Favi in 2011 defined a notion of support $(\mathrm{Spc}(\mathcal{K}), \mathrm{Supp}_{\mathrm{BF}})$ on $\mathcal{T}$.

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Restricting the Balmer-Favi support on $\mathrm{D}(R)$ to $\mathrm{D}_{\mathrm{pseu}}(R)$, by the universal property, there exists a unique map

$$f : \mathrm{Spc}(\mathrm{D}_{\mathrm{perf}}(R)) \cong \mathrm{Spec}(R) \to \mathrm{Spc}(\mathrm{D}_{\mathrm{pseu}}(R)),$$

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such that $f^{-1}(\mathrm{supp}(\mathrm{X})) = \mathrm{Supp}_{\mathrm{BF}})(X)$, for all $X \in \mathrm{D}_{\mathrm{pseu}}(R)$.$f$ is continuous if and only if $\mathrm{Supp}_{\mathrm{BF}})(X)$ is closed for all X.

The End