Topology

Given a cat $\mathcal{C}$, a topology T on $\mathcal{C}$ is a collection of morphisms

$\{U_i \longrightarrow U\}_i$ called coverings st

(T1) $\phi : U' \xrightarrow{\sim} U$, then $\{U' \xrightarrow{\phi} U\} \in T$

(T2) $\{U_i \longrightarrow U\} \in T$ & $V \longrightarrow U$ in $\mathcal{C}$, then

$$\{U_i \times_U V \longrightarrow V\} \in T$$

$$\begin{tikzcd} U_i \times_U V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\ U_i \arrow[r] & U \end{tikzcd}$$

(usual sense

$$\begin{tikzcd} f^{-1}(U_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d, "f"] \\ U_i \arrow[r, hook] & U \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r] & U \end{tikzcd}$$
)

(T3) $\{U_i \longrightarrow U\}_i \in T$ & $\{V_{ij} \longrightarrow U_i\}_j \in T$

$$\{V_{ij} \longrightarrow U\}_{i,j} \in T$$

We call $(\mathcal{C}, T)$ a site

Note : SGA calls it a pretopology (sieves for topology)
↳ the def^n given

* Grothendieck pre-top "generates" a Grothendieck top
(acts a "basis of topology")

* But an actual basis of topology is not a Grothendieck pre-topology

$$X \rightsquigarrow \underline{\underline{\mathcal{O}(X)}}$$

Morphisms of Topologies

$$F : (\mathcal{C}, T) \longrightarrow (\mathcal{D}, J)$$

$$F : \mathcal{C} \longrightarrow \mathcal{D} \quad \text{functor}$$

* $\{U_i \longrightarrow U\} \in T$, then $\{F(U_i) \longrightarrow F(U)\} \in J$

* $\{U_i \longrightarrow U\} \in T \qquad V \longrightarrow U$ in $\mathcal{C}$

$$F(U_i \times_U V) \xrightarrow{\sim} F(U_i) \times_{F(U)} F(V)$$

needs to be an isomorphism $\forall i$.

(Pre) sheaves on Sites

Given a site $(\mathcal{X}, T)$, a $\mathcal{C}$-valued presheaf on $(\mathcal{X}, T)$ is a

$$F : \mathcal{X}^{op} \longrightarrow \mathcal{C}$$

Morphism of presheaves $\rightsquigarrow$ natural transformation of functors.

Sheaves (fix $\mathcal{X}$)

A presheaf F on T is a sheaf if for every of abelian groups
$\{U_i \longrightarrow U\}_i \in T$

$$0 \longrightarrow F(U) \longrightarrow \prod_i F(U_i) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

exact

$Shv_{\mathcal{X}}(T)$ is a full subcategory of $PSh_{\mathcal{X}}(T)$.

Representable Presheaves : Fix $Z \in \mathcal{C}$, then

$U \longmapsto Hom(U, Z)$ is representable presheaf.

Epimorphisms

$U \xrightarrow{f} V$ in $\mathcal{C}$ is an epimorphism

$$\mathrm{Hom}(V,Z) \longrightarrow \mathrm{Hom}(U,Z) \text{ is injective}$$
$$g \longmapsto g\circ f$$

$$g\circ f = h\circ f \Rightarrow g=h \qquad \text{(right-cancellation)}$$

Effective Epimorphism

$U \longrightarrow V$ is an effective epi if

$$\mathrm{Hom}(V,Z) \longrightarrow \mathrm{Hom}(U,Z) \rightrightarrows \mathrm{Hom}(U\times_V U, Z)$$

is exact for all $Z\in\mathcal{C}$

Universal Effective Epimorphism

$U \longrightarrow V$ is a u.e.e if for any $V' \longrightarrow V$ in $\mathcal{C}$

$U\times_V V' \longrightarrow V'$ is an effective epi.

(Taking $V'=V$ tells you that $U\longrightarrow V$ is effective epi, in particular)

$$\begin{tikzcd} U \arrow[r, "id"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner"] & U \arrow[d, "f"] \\ V \arrow[r, "id"'] & V \end{tikzcd}$$

(Universal, Effective) Epi for families

$\{U_i \to U\}_i$ is

* epic if

$$\mathrm{Hom}(U, Z) \longrightarrow \prod_i \mathrm{Hom}(U_i, Z) \text{ is injective}$$

* effective epic if

$$\mathrm{Hom}(U, Z) \longrightarrow \prod_i \mathrm{Hom}(U_i, Z) \rightrightarrows \prod_{i,j} \mathrm{Hom}(U_i \times_U U_j, Z)$$

is exact

* u.e.e. if

for any $V \longrightarrow U$ in $\mathcal{C}$

$\{U_i \times_U V \longrightarrow V\}_i$ is a family of

effective epi's.

Canonical Topology

Given a category $\mathcal{X}$, let the canonical topology T_{can}

be the collection of all u.e.c's in $\mathcal{X}$.

Presheaves on the can topology

$\mathrm{Hom}_Z : U \longmapsto \mathrm{Hom}(U, Z)$ is a sheaf!

$$(\mathcal{X}, T_{can})^{op} \longrightarrow \mathcal{S}ets$$

for any $Z \in \mathcal{C}$

Let T be any other topology on $\mathcal{X}$ st all representable presheaves are sheaves,

$$\mathrm{Hom}_Z : U \longmapsto \mathrm{Hom}(U, Z) \text{ are sheaves}$$

so $\{U_i \to U\}_i$ is a family of u.e.c.

$$\mathrm{id} : \mathcal{X} \longrightarrow \mathcal{X}$$

$$T \longrightarrow T_{can}$$

so T_{can} is the finest top where ALL rep presheaves are sheaves!

The example of G-sets

Let G be any group, $G\mathcal{S}et$ the category of G-sets. Give it the canonical topology T_G

$\{U_i \xrightarrow{\phi_i} U\}_i$ is a family of u.e.e.'s iff $U = \bigcup_i \phi_i(U_i)$

Representable presheaf is a sheaf

$$Hom_Z : U \longmapsto Hom(U, Z) \text{ for any } Z \in \mathcal{C}$$

Proposition

$$_G Set \longrightarrow Shv(T_G)$$

$$Hom_{-} : \quad Z \longmapsto Hom(-, Z)$$

$$F(G) \longleftarrow F \qquad : F(-)$$

$$Z \longmapsto Hom(-, Z) \longrightarrow Hom(G, Z) \cong Z \quad \checkmark$$

$$\boxed{\begin{array}{l} s \in F(G) \quad g \cdot s = F(R_g)(s) \in F(G) \\ R_g : G \to G \qquad R_{g_1 g_2} = R_{g_2} \circ R_{g_1} \\ F(R_G) : F(G) \longrightarrow F(G) \end{array}}$$

$$\phi \longmapsto \phi(e)$$

$$\phi(g) = g\,\phi(e)$$

$$F \longmapsto F(G) \longmapsto Hom(-, F(G))$$

$U \in {}_G Set$ $\qquad \{G \xrightarrow{\phi_u} U\}_{u \in U}$ is a family of u.e.e

$$\phi_u : G \longrightarrow U \qquad \phi_u(G) = Gu$$
$$g \longmapsto g \cdot u$$

$$U = \bigcup_{u \in U} Gu$$

$$\begin{tikzcd} F(\mathcal{U}) \arrow[r, "F(\phi_u)"] & \prod_{u \in \mathcal{U}} F(G) \arrow[r, shift left, "F(\phi_u)"] \arrow[r, shift right, "F(\phi_v)"'] & \prod_{u,v} F(G \times_{\mathcal{U}} G) \end{tikzcd}$$

injective

$$\prod_{u \in \mathcal{U}} F(G) \cong \prod_{u \in \mathcal{U}} \mathrm{Hom}(G, F(G))$$

$$\cong \mathrm{Hom}\left(\coprod_{u \in \mathcal{U}} G, F(G)\right)$$

$$\cong \mathrm{Hom}(\mathcal{U}, F(G))$$

*

$$\begin{tikzcd} G \times_{\mathcal{U}} G \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & G \arrow[d, "\phi_w"] \\ G \arrow[r, "\phi_v"'] & \mathcal{U} \end{tikzcd}$$

$$G \times_{\mathcal{U}} G = \{(g_1, g_2) \in G \times G \mid \phi_w(g) = \phi_v(g)\}$$

Suppose v & w are not in the same orbit, and assume $G \times_{\mathcal{U}} G \neq \phi$

$(g_1, g_2) \in G \times_{\mathcal{U}} G$ but

$$g_1 \cdot v = \phi_v(g_1) = \phi_w(g_2) = g_2 \cdot w \Rightarrow v = g_1^{-1} g_2 \cdot w \quad \text{contradiction!}$$

If v, w, in the same orbit, then

$$\begin{tikzcd} G \arrow[r, "R_{g^{-1}}"] \arrow[d, "id"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\phi_{g \cdot v}"] \\ G \arrow[r, "\phi_v"'] & \mathcal{U} \end{tikzcd}$$

$$F(U) \xrightarrow{F(\phi_u)} \underbrace{\prod_{u \in \mathcal{U}} F(G)} \underset{F(R_{g^{-1}})}{\overset{id_{F(G)}}{\rightrightarrows}} \underbrace{\prod_{u,v}}_{\text{same orbit}} F(G)$$

injective

surjective

probably equal

Hence $F(U) \cong Hom(U, F(G))$

$$F \cong Hom(-, F(G))$$

* $G \longrightarrow H$

$${}_H Set \longrightarrow {}_G Set$$

$$T_H \longrightarrow T_G$$

Aside

$$\mathbb{Z}_p = \varprojlim_a \mathbb{Z}/p^a\mathbb{Z}$$

$$\hat{\mathbb{Z}} = \varprojlim_{\mathbb{N}} \mathbb{Z}/n\mathbb{Z} \qquad (\mathbb{N}, |)$$

$$\mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \qquad \gcd(m,n) = 1$$

only 0-map

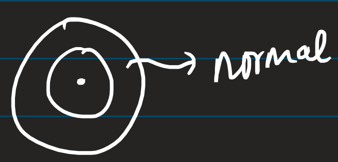
$(\{\text{open normal subgroups}\}, \subseteq) \supseteq$ subposet

$$N \subseteq M$$

$$G/M \longrightarrow G/N$$

$$\hat{G} = \varprojlim_{N} G/N \qquad \tilde{G} = \varprojlim_{\text{subpo}} G/N$$

Top G



$\varprojlim G/(a+N$