Problem 5

S_5 $\quad \overbrace{(13)(12)}(45)$

Step I. any perm can be written as a product of disjoint cycles

$$\sigma = \underbrace{(13)(12)}(45) = (123)(45)$$

Step II. Disjoint cycles commute

$$(123)(45) = (45)(123)$$

Step III. Figure out inverse

$$\left((123)(45)\right)^{-1} = (45)^{-1}(123)^{-1}$$

$$= (45)(132)$$

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$$\left[\begin{aligned} &(a_1 a_2 \cdots a_r)^{-1} \\ &= (a_1 a_r a_{r-1} \cdots a_2) \end{aligned} \right]$$

Step IV Suppose σ & σ^{-1} were conjugates

then $\exists \rho \in S_5$ st

$$\sigma^{-1} = \rho \sigma \rho^{-1}$$

$$(132)(45) = \rho\,(123)(45)\,\rho^{-1}$$
$$= \rho\,(123)e\,(45)\,\rho^{-1}$$
$$= \rho\,(123)\,\rho^{-1}\rho\,(45)\,\rho^{-1}$$
$$= [\rho(123)\rho^{-1}][\rho(45)\rho^{-1}]$$

Step V. Recall, for some $\tau \in S_n$

$$\tau\,(a_1\,a_2 \cdots a_r)\,\tau^{-1} = (\tau(a_1)\,\tau(a_2) \cdots \tau(a_r))$$

equivalently (multiplying by τ on the right)

$$\tau\,(a_1\,a_2 \cdots a_r) = (\tau(a_1)\,\tau(a_2) \cdots \tau(a_r))\,\tau$$

$ghg^{-1} = h'$

$g h g^{-1} g = h' g$

$= gh$

$$x \notin \{a_1, \ldots, a_r\}$$

$$\tau\,\underbrace{(a_1\,a_2 \cdots a_r)}\,(x) = \tau(x) \qquad \text{(LHS)}$$

$(123)(4) = 4$

$$\parallel$$

$$\underbrace{(\tau(a_1) \cdots \tau(a_r))}\,\underbrace{\tau(x)} = \tau(x) \qquad \text{(RHS)}$$

$f(\underbrace{g(x)}) = f(x)$

$$\tau\,(a_1\,a_2 \cdots a_r)\,(a_i) = \tau(a_{i+1}) \qquad \text{(LHS)}$$

$$\parallel$$

$$(\tau(a_1) \cdots \tau(a_r))\,\tau\,(a_i) = \tau(a_{i+1}) \qquad \text{(RHS)}$$

$$(132)(45) = \rho(123)\rho^{-1} \cdot \rho(45)\rho^{-1}$$

$$= (\rho(1)\ \rho(2)\ \rho(3))\ (\rho(4)\ \rho(5))$$

$\rho(1) = 1$

$\rho(2) = 3$

$\rho(3) = 2$

$\rho(4) = 4$

$\rho(5) = 5$

$$\rho = (2\ 3)$$

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$$

$$= (a_1^1\, a_2^1 \cdots a_{n_1}^1)\ (a_1^2\, a_2^2 \cdots a_{n_2}^2) \cdots (a_1^k\, a_2^k \cdots a_{n_k}^k)$$

$$\sigma^{-1} = \rho \sigma \rho^{-1} = \rho \sigma_1 \overset{e}{\swarrow} \sigma_2 \overset{\rho \cdots}{\swarrow} \cdots \sigma_k \rho^{-1}$$

$$= (\rho \sigma_1 \rho^{-1})(\rho \sigma_2 \rho^{-1}) \cdots (\rho \sigma_k \rho^{-1})$$

Problem #2.

$$|A_n| = \frac{n!}{2} = \frac{1}{2}|S_n|$$

$A_n = \{\sigma \in S_n \mid sgn(\sigma) = 1\}$ i.e. elements that are a product of even-many transpositions

$O_n = \{ \sigma \in S_n \mid \operatorname{sgn}(\sigma) = -1 \} \subseteq S_n$ (not a subgroup)

$$S_n = A_n \amalg O_n$$

$$\# S_n = \# A_n + \# O_n$$

To prove $\# A_n = \# O_n$, so let's produce a bijection

$$f : A_n \longrightarrow O_n$$

$$\tau \longmapsto \tau(1\,2)$$

$$\operatorname{sgn}(\tau(12)) = \operatorname{sgn}(\tau)\operatorname{sgn}(1\,2)$$

$$= 1(-1) = -1$$

in O_n

$$g : O_n \longrightarrow A_n$$

$$\rho \longmapsto \rho(12)$$

$$\operatorname{sgn}(\rho(12)) = \operatorname{sgn}(\rho)\operatorname{sgn}(1\,2)$$

$$= (-1)(-1) = 1$$

in A_n

$$f \circ g(\rho) = f(\rho(12)) = \rho(12)(12) = \rho = \mathrm{id}_{O_n}(\rho)$$

$$g \circ f(\tau) = g(\tau(12)) = \tau(12)(12) = \tau = \mathrm{id}_{A_n}(\tau)$$

Therefore $f: A_n \longrightarrow O_n$ is a bij

$\# A_n = \# O_n$

$\# S_n = 2 \cdot \# A_n \Rightarrow \# A_n = \frac{1}{2} \# S_n$.

Quotient groups $N \trianglelefteq G$

$\hookrightarrow$ (self-conjugate groups)

$gNg^{-1} = N \quad \forall g \in G$

$gN = Ng$

Claim: $A_n \trianglelefteq S_n$

for any $\sigma \in S_n$, $\sigma A_n \sigma^{-1} \underset{(\supseteq)}{\overset{\subseteq}{=}} A_n$

$\{\sigma \rho \sigma^{-1} \mid \underbrace{\rho \in A_n}\} = A_n$

$\sigma \rho \sigma^{-1} \in \sigma A_n \sigma^{-1}$

To show $\operatorname{sgn}(\sigma \rho \sigma^{-1}) = 1$, then $\sigma \rho \sigma^{-1} \in A_n$

$\operatorname{sgn}(\sigma \rho \sigma^{-1}) = \operatorname{sgn}(\sigma) \operatorname{sgn}(\rho) \operatorname{sgn}(\sigma)^{-1}$

$= \operatorname{sgn}(\sigma)^2 \cdot 1 = 1$

$\sigma \rho \sigma^{-1} \in A_n$ for any $\sigma \in S_n$

Hence $\sigma A_n \sigma^{-1} \subseteq A_n$

To show $A_n \subseteq \sigma A_n \sigma^{-1}$

$\rho \in A_n$, so $\sigma^{-1} \rho \sigma \in A_n$ (for any $\sigma \in S_n$)

Hence $\sigma^{-1} \rho \sigma = \alpha \in A_n$

multiply by σ on the left & σ^{-1} on the right

$$\rho = \underbrace{\sigma \sigma^{-1}} \rho \underbrace{\sigma \sigma^{-1}} = \sigma \alpha \sigma^{-1} \in \sigma A_n \sigma^{-1}$$

$$A_n \subseteq \sigma A_n \sigma^{-1}$$

$$\# \, S_n / A_n = \#S_n / \#A_n = 2$$

$$S_n/A_n = \{ A_n, \tau A_n \}$$

$$= \{ \underline{\underline{A_n}}, \underline{\underline{(1\,2)}} A_n \}$$

odd

even

$$\mathbb{Z}/2\mathbb{Z} = \{ 2\mathbb{Z}, 1 + 2\mathbb{Z} \}$$

odd

even

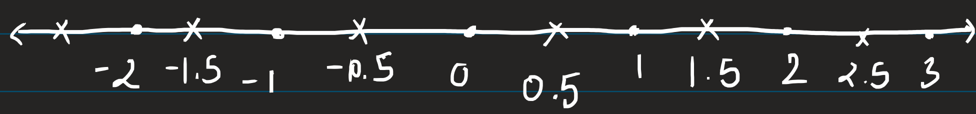
$(\mathbb{Z},+) \trianglelefteq (\mathbb{R},+)$

$\mathbb{Z}/3\mathbb{Z}$

$2, 5;\ 1, 7$

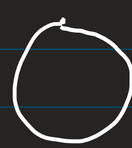
$\mathbb{R}/\mathbb{Z} = \{\mathbb{Z},$

$\mathbb{R}$



$\mathbb{R}/\mathbb{Z}$, step I

Final step



$\mathbb{R}/\mathbb{Z} \cong S^1$

$\mathbb{Z}/n\mathbb{Z}$

$p = \underline{r} + nq$

$x \in \mathbb{R}$, $\quad n.a_1a_2a_3\ldots$

$= n + \underline{0.a_1a_2}\ldots$

$\mathbb{R}/\mathbb{Z}$

$$\phi : \mathbb{R} \longrightarrow S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$$

$$= \{ re^{i\theta} \in \mathbb{C} \mid r = 1 \}$$

$$= \{ e^{i\theta} \mid \theta \}$$

$$x \longmapsto e^{2\pi i x}$$

$$\frac{\theta}{2\pi} \longmapsto e^{i\theta} \in S^1$$

$$x+y \longmapsto e^{2\pi i (x+y)} = e^{2\pi i x + 2\pi i y} = e^{2\pi i x} \cdot e^{2\pi i y}$$

$\ker\phi \trianglelefteq \mathbb{R}$, $\quad \mathbb{R}/\ker\phi \cong \mathrm{im}\,\phi$ ✓

$$\mathbb{R}/\mathbb{Z} \cong S^1$$

Alternate proof of $|A_n| = \frac{n!}{2}$.

$$\mathrm{sgn} : S_n \longrightarrow \{\pm 1\}$$

$$= S_n/A_n \cong \{\pm 1\} \longrightarrow \ker \mathrm{sgn} = A_n$$

$$\#(S_n/A_n) = \#S_n/\#A_n = \#\{\pm 1\} = 2$$

$$\#A_n = \frac{1}{2}\#S_n$$