How many vertices are there in a simplicial ball of radius r (in a Brihat-Tits Building)?

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$$\mathrm{SV}(r) \sim c(n) \cdot r^{\epsilon(n)} q^{\pi(n) r},$$

where $c(n)$ is a rational function[1] of q, and $\epsilon(n)$ and $\pi(n)$ are in the following tabluar.

X_n	$\epsilon(n)$	$\pi(n)$
A_n (n is odd)	0	$(\frac{n+1}{2})^2$
A_n (n is even)	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n ($n = 3$)	0	5
B_n ($n \geqslant 4$)	0	$\frac{n^2}{2}$
C_n ($n \geqslant 2$)	0	$\frac{n(n+1)}{2}$
D_n ($n = 4$)	2	6
D_n ($n \geqslant 5$)	1	$\frac{n(n-1)}{2}$

[1] not really

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Bruhat-Tits buildings are buildings associated with reductive groups over local fields.

What is a (Bruhat-Tits) building?

Definition

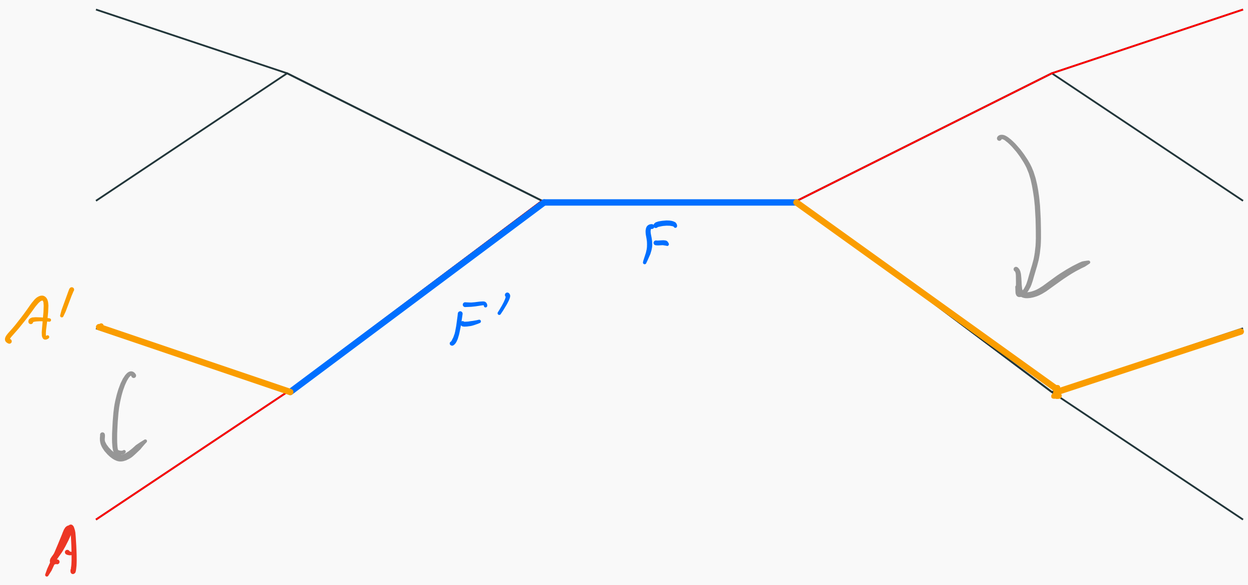
A ***(Euclidean) building*** is a set $\mathscr{B}$ equipped with a polysimplicial complex $\mathfrak{F}$ of ***facets*** and a family $\mathcal{A}$ of ***apartments***, such that:

EB0. each apartment is isomorphic to an abstract one $\mathscr{A}$;

EB1. any two facets F, F' are contained in the same apartment;

EB2. any two apartment A, A' containing F, F' are isomorphic through an isomorphism fixing F and F' pointwise.

What is a (Bruhat-Tits) building?



Definition

A ***(Euclidean) apartment*** is a Euclidean affine space $\mathbb{A}$ equipped with a reflection group W. The reflection group W divides the space $\mathbb{A}$ into ***facets***, forming a polysimplicial complex.

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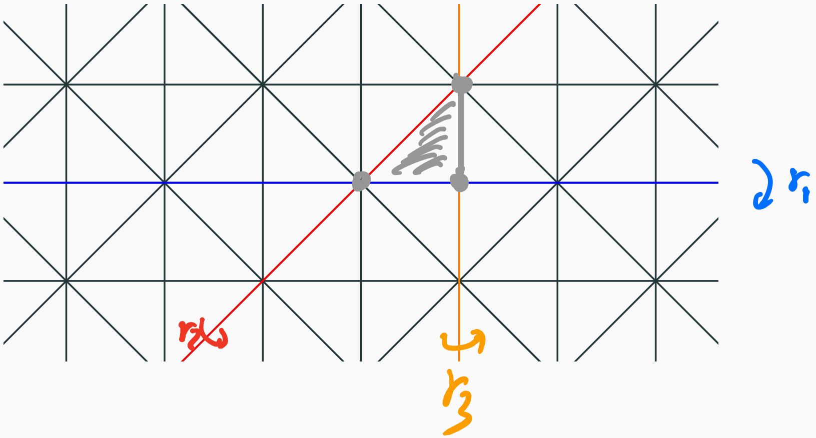


vertex = facet of lowest dimension.

G: a reductive group over a local field K

What is a (Bruhat-Tits) building?

$$G \xrightsquigarrow{\text{Bruhat-Tits}} \mathscr{B}(G)$$

a building

What is a (Bruhat-Tits) building?

$$\begin{tikzcd} \mathbf{G} \arrow[r, rightsquigarrow, "\text{Bruhat-Tits}"] & \mathscr{B}(\mathbf{G}) \arrow[loop above, "G = \mathbf{G}(K)"] \end{tikzcd}$$

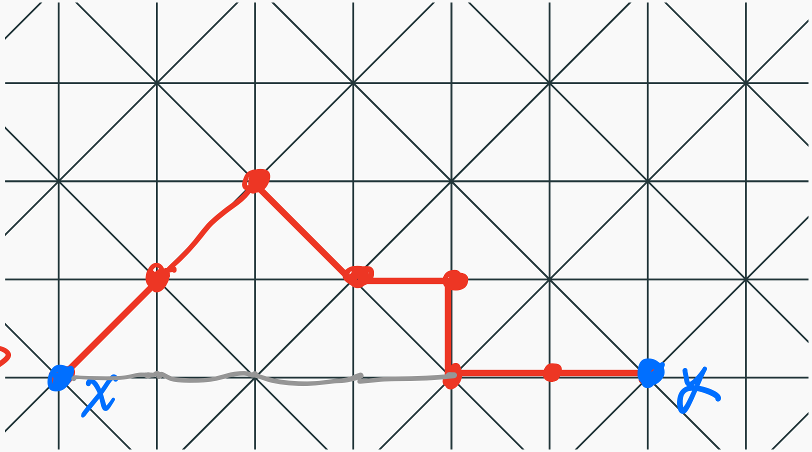
What are simplicial volumes?

What is a simplicial ball?

path = sequence of adjacent vertices;

simplicial distance = minimal length of pathes;

length = 7



$d_S(x, y) = \min\{\text{length of path from } x \text{ to } y\}.$

$= 6$.

Definition

The ***simplicial ball with center*** x ***and radius*** r is the set of all vertices with simplicial distance no larger than r from x:

$$B_s(x, r) := \left\{ y \in \mathscr{B}^0 \;\middle|\; d_s(x, y) \leqslant r \right\}.$$

Definition

The ***simplicial ball with center*** x ***and radius*** r is the set of all vertices with simplicial distance no larger than r from x:

$$B_S(x, r) := \{y \in \mathscr{B}^0 \mid d_S(x, y) \leqslant r\}.$$

We may focus on $x = o$, the origin of an apartment $\mathscr{A}$ and denote $B_S(o, r)$ by $B_S(r)$ for short. Special vertex

Its cardinality is called the ***simplicial volume*** $\mathrm{SV}(r)$.

Fact: Any vertex is adjacent to a special one.

Notations

Before moving on, let's fix the following notations:

- K is a non-Archimedean local field;
- val is the valuation on K;
- K° is the ring of integers $\{x \in K \mid \mathsf{val}(x) \geqslant 0\}$;
- $K^{\circ\circ}$ is the maximal ideal $\{x \in K \mid \mathsf{val}(x) > 0\}$;
- ϖ is a uniformizer, namely $K^{\circ\circ} = \varpi K^\circ$;
- κ is the residue field $K^\circ / K^{\circ\circ}$, with cardinality q. $< \infty$

Example: $\mathscr{B}(\mathrm{GL}(V))$

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Definition

A ***norm*** on V is a map $\alpha : V \longrightarrow \mathbb{R} \cup \{\infty\}$ such that for any $x, y \in V$ and any $t \in K$,

exp $\|\cdot\|_\alpha$.

(a) $\alpha(tx) = \mathrm{val}(t) + \alpha(x)$;
(b) $\alpha(x+y) \geqslant \inf\{\alpha(x), \alpha(y)\}$;
(c) $\alpha(x) = \infty$ if and only if $x = 0$.

Two norms are ***homothetic*** if they are different by a constant.

$\alpha(x) - \alpha'(x) = c$

constant

Let V be a vector space over K of dimension n. Then the Bruhat-Tits building $\mathscr{B}(\mathrm{GL}(V))$ can be interpreted as follows:

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a k-simplex is a ***lattice chain*** of rank $k+1$.

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> a k-simplex is a ***lattice chain*** of rank $k+1$.

Definition

A ***lattice*** in V is a finitely generated K°-submodule spanning V.

A ***lattice chain*** is a decreasing sequence of lattices which is invariant (as a set) under homotheties. The number of homothety classes of lattices in a lattice chain is called its ***rank***.

$$\cdots \supsetneq \varpi^{-1} L_r \supsetneq \underbrace{L_0 \supsetneq L_1 \supsetneq \cdots \supsetneq L_r}_{\text{rank} = r+1} \supsetneq \varpi L_0 \supsetneq \cdots$$

Let V be a vector space over K of dimension n. Then the Bruhat-Tits building $\mathscr{B}(\mathrm{GL}(V))$ can be interpreted as follows:

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- a k-simplex is a ***lattice chain*** of rank $k+1$.

Proposition

Any norm gives rise to a lattice chain. Conversely, any lattice chain gives rise to a k-simplex of norms.

Proposition 2.1.1

Let x and y be two vertices in $\mathscr{B}(\mathrm{GL}(V))$ and $r \geqslant 0$ an integer. Then $\boxed{d_s(x, y) \leqslant r}$ if and only if there exist lattices L and L' such that

$$\underline{x = [L]}, \quad \underline{y = [L']}, \quad \text{and } L \supseteq L' \supseteq \varpi^r L.$$

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Fix a lattice L and choose o to be the point $[L]$.

Q: How many lattices L' are there (up to homotheties) satisfying

$$L \supseteq L' \supseteq \varpi^r L.$$

Theorem 2.1.2 in Junecue Suh, "*Stable lattices in p-adic representations II. Irregularity and entropy*", Journal of Algebra, vol.591, 2022, pp.379-409.

$\asymp r^{\varepsilon} q^{\kappa r}$

no inf about coefficient.

The computation strategy

The strategy is to employ a ***strongly transitive*** and ***type-preserving*** automorphism group G. Then we have

$$\mathrm{SV}(r) = \sum_{x} [P_o : P_{o,x}] ,$$

where

- P_o is the stabilizer of o in G,
- $P_{o,x}$ is the stabilizer of x in P_o, and
- the summation is taking over the intersection of $B_s(r)$ with a fundamental domain of the action of P_o.

The computation strategy

The group G comes from the Bruhat-Tits theory on reductive groups over local fields.

$$\begin{tikzcd} \mathrm{G} \arrow[rr, squiggly, "\text{Bruhat-Tits}"] & & \mathscr{B}(\mathrm{G}) \arrow[loop above, "G = \mathrm{G}(K)"] \end{tikzcd}$$

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vertex has dim 0

We may focus on ***reduced*** Bruhat-Tits building and assume G is a ***simply-connected (semi-)simple*** group.

$$\begin{tikzcd} \text{reductive} \arrow[r, rightsquigarrow] \arrow[dr] & \text{building} \arrow[r] & \text{reduced building} \\ & \text{root datum} \arrow[u] \arrow[r] & \text{root system} \arrow[u] \end{tikzcd}$$

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We may focus on ***reduced*** Bruhat-Tits building and assume G is a ***simply-connected (semi-)simple*** group. Then $G = \mathrm{G}(K)$ is a strongly transitive and type-preserving automorphism group of $\mathscr{B}(\mathrm{G})$.

Parahoric reduction

of o

Under our assumption, the stabilizer P_o is the ***parahoric subgroup*** attached to the point o. By Bruhat-Tits theory, there is a smooth model $\mathfrak{G}_o$ of G such that $\mathfrak{G}_o(K^\circ) = P_o$.

K°-grp scheme

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Let $\overline{P_o}$ denote the reductive quotient of $\mathfrak{G}_o(\kappa)$. Then we have

$$P_o^+ \hookrightarrow P_o \twoheadrightarrow \overline{P_o}.$$

a reductive group.
o is special
⇒ has root sys Ξ
same as B.

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$$P_{o,x}^{ur} \longrightarrow \overline{P_{o,x}}^{ur}$$

Note that $P_{o,x} = P_o \cap P_x$. Let $\overline{P_{o,x}}$ denote its image under above reduction. Then we have

$$[P_o : P_{o,x}] = \underbrace{\left[\overline{P_o} : \overline{P_{o,x}}\right]}_{\text{over } k} \cdot \underbrace{\left[P_o^+ : P_o^+ \cap P_{o,x}\right]}.$$

Poincare polynomial

$\overline{P_o}$ is a reductive group over the finite field κ and $\overline{P_{o,x}}$ is a parabolic subgroup of it with type $I_{o,x} := \{a \in \Delta \mid a(x) = 0\}$.

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Generalized Bruhat Decomposition

If G is a reductive group and P is a parabolic subgroup of it, then the quotient G/P is a disjoint union of affine spaces, called ***Schubert cells*** and hence

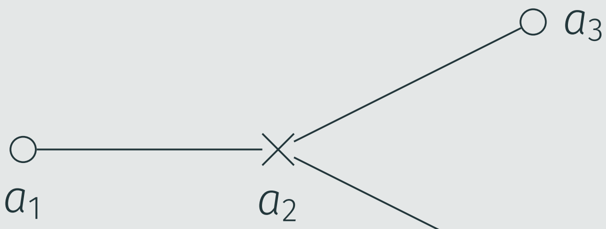
$$|G/P| = \sum q^{\ell(w)},$$

which can be presented by a Poincaré polynomial $\mathcal{P}_{\Phi;I}$ (where $\mathbf{\Phi}$ is the root system of G and I is the type of P).

Example

The Poincaré polynomial $\mathcal{P}_{D_4;I}$ of the following

$\frac{\mathcal{P}_{D_4}}{\mathcal{P}_{A_1}^3}$



is $(z^2+1)^2(z^2+z+1)(z^3+1)$.

Concave functions

$$\left[P_0^+ : P_0^+ \cap P_x \right]$$

$P_0 \to \overline{P_0}$ $\qquad$ $P_{0,x} \to \overline{P_{0,x}}$

A ***concave function*** is a function on $\widetilde{\Phi} := \Phi \cup \{0\}$ such that

$$\forall (a_i)_{i\in I} \subseteq \widetilde{\Phi}, \sum_{i\in I} a_i \in \widetilde{\Phi} \implies \sum_{i\in I} f(a_i) \geqslant f(\sum_{i\in I} a_i).$$

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Any concave function f defines a smooth model $\mathfrak{G}_f$ of G such that $\mathfrak{G}_f(K^\circ)$ is a bounded subgroup of G and there is a Bruhat decomposition for schemes:

root subgrps

$$\prod_{a\in\Phi_f^+} \mathfrak{U}_{a,f(a)} \cdot \mathfrak{T}_{f(0)} \cdot \prod_{a\in\Phi_f^-} \mathfrak{U}_{a,f(a)} \longrightarrow \mathfrak{G}_f$$

tori

If $f(0) > 0$, it induces an isomorphism on special fibers.

$K^0/\varpi^N K^0$

The index $\left[P_o^+ : P_o^+ \cap P_{o,x}\right]$ can be computed using concave functions.

$x \in \mathcal{B}$ $\quad x \in \mathcal{A} \ni o$

The index $[P_o^+ : P_o^+ \cap P_{o,x}]$ can be computed using concave functions.

Indeed, there are concave functions

$$f_{0+} : \underline{a \mapsto 0+}, \qquad \text{and} \qquad f_{0+,x} : \underline{a \mapsto \max\{0+, -a(x)\}},$$

such that

$$\mathfrak{G}_{f_{0+}}(K^\circ) = P_o^+ \qquad \text{and} \qquad \mathfrak{G}_{f_{0+,x}}(K^\circ) = P_o^+ \cap P_{o,x}.$$

$$f_{0+}(0) = f_{0+,x}(0) = 0+ > 0.$$

Using the Schematic Bruhat decomposition, we have

$$
\begin{aligned}
[P_o^+ : P_o^+ \cap P_{o,x}] &= \prod_{a\in\widetilde{\Phi}} [U_{f_{o+},a} : U_{f_{o+,x},a}] \\
&= \prod_{a\in\Phi} \left[U_{a,o+} : U_{a,\max\{0+,-a(x)\}}\right] \\
&= \prod_{a\in\Phi} q^{\lceil \max\{0+,-a(x)\}\rceil - \lceil 0+\rceil} \\
&= \prod_{a(x)<0} q^{\lceil -a(x)\rceil -1} = \prod_{a(x)>0} q^{\lceil a(x)\rceil -1}.
\end{aligned}
$$

$\{U_{a,k}\}_{k\in\mathbb{R}}$

We have

$$\mathrm{SV}(r) = \sum_{x} \frac{\mathscr{P}_{\Phi;I_{o,x}}(q)}{q^{\deg(\mathscr{P}_{\Phi;I_{o,x}})}} \prod_{a(x)>0} q^{\lceil a(x) \rceil}$$

where x is taking over the intersection of $B_s(r)$ with a fundamental domain of the action of P_o.

Fundamental domain

Using the ***(vectorial) Bruhat decomposition***

$$G = BNB',$$

one can show that for any Weyl chamber ${}^vC$, the convex cone $\overline{o + {}^vC}$ is a fundamental domain for P_o.

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one can show that for any Weyl chamber ${}^{v}C$, the convex cone $\overline{o + {}^{v}C}$ is a fundamental domain for P_o.

So the summation is taking over $B_S(r) \cap \overline{o + {}^{v}C}$.

$$B_S(r, {}^{v}C, l) := \left\{ x \in B_S(r) \cap \overline{o + {}^{v}C} \;\middle|\; l_{o,x} = l \right\}.$$

The simplicial volume formula

Theorem

Let $\mathscr{B}$ be a Bruhat-Tits building of a split reductive group. Then the simplicial volume in it can be computed by the formula

$$\mathrm{SV}(r) = \sum_{I\subseteq\Delta} \frac{\mathscr{P}_{\Phi;I}(q)}{q^{\deg\left(\mathscr{P}_{\Phi;I}\right)}} \boxed{\sum_{x\in B_S(r,{}^{\vee}C,I)} \prod_{a(x)>0} q^{\lceil a(x)\rceil}}.$$

simple roots

Explicit description of $B_S(r, {}^vC, l)$

Explicit description of $B_S(r, {}^{v}C, I)$

Lemma

If $\mathscr{B}$ is of split classical type, then a vertex x in the apartment is within simplicial distance r from the origin if and only if $\lceil a_0(x) \rceil \leqslant r$.

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highest root.

1. Any edge intersects a wall with a vertex. So $d(0, x) \geqslant a_0(x)$.
2. If x, y are special and $a_i(x) = a_i(y)$ except $i = i_0$, then there is a path of length $|a_{i_0}(x) - a_{i_0}(y)|h_{i_0}$ between them.
3. If $x =$ is not special, then there is a path from a special vertex x_0 to x with expected length.

Nontrivial

If $\mathscr{B}$ is of type A_n and $I = \Delta \setminus \{\ell_1, \cdots, \ell_t\}$:

$$B_S(r, {}^\vee C, I) = \left\{ o + c_1\omega_{\ell_1} + \cdots + c_t\omega_{\ell_t} \,\middle|\, c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t \leqslant r \right\},$$

where ω_i are the fundamental coweights. $\alpha_i(\omega_j) = \delta_{ij}$

If $\mathscr{B}$ is of type A_n and $I = \Delta \setminus \{\ell_1, \cdots, \ell_t\}$:

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where ω_i are the fundamental coweights.

If $\mathscr{B}$ is of type C_n and $I = \Delta \setminus \{\ell_1, \cdots, \ell_t\}$:

$$B_S(r, {}^V C, I) = \left\{o + c_1\omega'_{\ell_1} + \cdots + c_t\omega'_{\ell_t} \,\middle|\, c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t \leqslant r\right\},$$

where $\omega'_i = h_i^{-1}\omega_i$ with $a_0 = \sum_i h_i a_i$.

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where $\omega'_i = h_i^{-1} \omega_i$ with $a_0 = \sum_i h_i a_i$.

If $\mathscr{B}$ is of type B_n or D_n, then the description is complicated.

If $\mathscr{B}$ is of type B_n, we consider

$$\partial B_S(r, {}^\vee C, I) := B_S(r, {}^\vee C, I) - B_S(r-1, {}^\vee C, I).$$

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$$\partial B_S(r, {}^V C, I) := B_S(r, {}^V C, I) - B_S(r-1, {}^V C, I).$$

Then $\partial B_S(r, {}^V C, I)$ is contained in the supset $\partial(r, I)^0$ if $\ell_1 > 1$ and in $\partial(r, I)^0 \cup \partial(r, I)^1$ if $\ell_1 = 1$. Here

$$\partial(r, I)^0 := \left\{ x = o + c_1 \omega'_{\ell_1} + \cdots + c_t \omega'_{\ell_t} \;\middle|\; c_i \in \mathbb{Z}_{>0}, c_1 + \cdots + c_t = r \right\},$$
$$\partial(r, I)^1 := \partial(r, I)^0 - \tfrac{1}{2}\omega_1.$$

Next, for any $J \subseteq \Delta$, we define $\mathcal{D}(\Delta)_J$ to be the set

$$o + \mathcal{P}^{\vee}(B_n) - \sum_{j \in J} \tfrac{1}{2}\omega_j.$$

Then we define $\partial(r, I)_J$ to be the intersection of $\mathcal{D}(\Delta)_J$ with the supset.

The complement of $\partial B_S(r, {}^{\vee}C, I)$ in the supset is

$$\partial(r, I)_{\{1\}} \sqcup \partial(r, I)_{\{1,2\}} \sqcup \cdots \sqcup \partial(r, I)_{\{n-1,n\}}.$$

Asymptotic behavior

Theorem

Let $\mathscr{B}$ be a Bruhat-Tits building of split type A_n, B_n, C_n, or D_n. Then the simplicial volume SV(r) *in it has the following asymptotic dominant relation as $r \to \infty$:*

$$\mathrm{SV}(r) \asymp r^{\epsilon(n)} q^{\pi(n) r},$$

where $\epsilon(n)$ and $\pi(n)$ are in the following tabluar.

$$C_1 r^{\varepsilon} q^{\pi r} \leq SV(r) \leq C_2 r^{\varepsilon} q^{\pi r}$$

$$r \to \infty$$

Theorem

X_n	$\epsilon(n)$	$\pi(n)$
A_n (n is odd)	0	$(\frac{n+1}{2})^2$
A_n (n is even)	1	$\frac{n}{2}(\frac{n}{2}+1)$
B_n $(n = 3)$	0	5
B_n $(n \geqslant 4)$	0	$\frac{n^2}{2}$
C_n $(n \geqslant 2)$	0	$\frac{n(n+1)}{2}$
D_n $(n = 4)$	2	6
D_n $(n \geqslant 5)$	1	$\frac{n(n-1)}{2}$

Theorem

If $\mathscr{B}$ is not of the type B_n $(n \geqslant 4)$ or D_n $(n \geqslant 5)$, then we have

$$\mathrm{SV}(r) \sim c(n) \cdot r^{\epsilon(n)} q^{\pi(n) r},$$

where $c(n)$ is a rational function of q.

If $\mathscr{B}$ is of the type B_n $(n \geqslant 4)$ or D_n $(n \geqslant 5)$, then we have

$$\mathrm{SV}(2r) \sim c_0(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$

$$\mathrm{SV}(2r+1) \sim c_1(n) \cdot r^{\epsilon(n)} q^{2\pi(n) r},$$

where $c_0(n)$ and $c_1(n)$ are rational functions of q.

Remark

1. *Buildings of Exceptional type are not considered since the simplicial distance lemma fails in these cases.*
2. *The results extends to buildings of not necessarily irreducible ones by decomposition.*
3. *The results apply to buildings of split classical types. Note that a non-split reductive group can have a building of split type.*

$$\mathcal{B}(GL_n(L)) \cong \mathcal{B}(GL_n(K))^{[L:K]}$$

L/K

Exponential polynomials

Definition

A *q-number* is an element of $\mathbb{Q}(q^{1/2})$ having no poles at $q > 1$. The ring of q-numbers is denoted by $\mathbb{Q}(q^{1/2})_{q>1}$. A *$q$-function* $f(z)$ is a function from $\mathbb{Z}$ to $\mathbb{Q}(q^{1/2})_{q>1}$.

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A q-***exponential polynomial*** is a finite sum of the form

$$\underline{f(x) = \sum_{\nu} f_{\nu}(x) q^{\nu x},}$$

where $\nu \in \frac{1}{2}\mathbb{Z}$, and each c_{ν} is a polynomial of x with q-number coefficients.

The ***order*** $\mathrm{ord}(f)$ of f is the largest ν such that $f_\nu \neq 0$. The ***degree*** of f is the degree of the polynomial $f_{\mathrm{ord}(f)}$.

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For any polynomial f_ν, we write it as

$$f_\nu(z) = \sum_{n=0}^{\deg(f_\nu)} c_{\nu,n} \binom{z}{n},$$

where $c_{\nu,n}$ are q-numbers and

$$\binom{z}{n} = \frac{1}{n!} z(z-1)\cdots(z-n+1).$$

The ***leading coefficient*** $\mathrm{lead}(f)$ of f is defined to be $c_{\mathrm{ord}(f),\deg(c_\nu)}$.

A q-exponential polynomial can be viewed as a q-function. Then the key asymptotic property of such a q-function $f(z)$ is

$$f(z) \sim \mathrm{lead}(f)\binom{z}{\deg(f)} q^{\mathrm{ord}(f)z}.$$

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$$\boxed{f(z) \sim \mathrm{lead}(f)\binom{z}{\deg(f)} q^{\mathrm{ord}(f) z}.}$$

A parity q-exponential polynomial f is said to be ***rational*** if all the exponential coefficients ν are integers and all the coefficient $c_{\nu,n}$ take values in $\mathbb{Q}(q)_{q>1}$.

The $\mathbb{Q}(q)_{q>1}$-algebra of (rational) q-exponential polynomials is closed under differences and anti-differences.

Lemma

Given a multi-summation of q-exponentials

$$\mathsf{S}(z) = \sum_{\mathbf{c} \in \mathbb{Z}^{\mathbf{i}}_{>0} \,:\, \mathbf{1} \cdot \mathbf{c} = z} q^{\boldsymbol{\mu} \cdot \mathbf{c}},$$

and let $\mathbf{i}_{\max} = \{i \in \mathbf{i} \mid \mu_i = \mu_{\max} := \max \boldsymbol{\mu}\}$. *Then* $\mathsf{S}(z)$ *is a q-exponential polynomial and we have*

$$\mathsf{S}(z) \sim \prod_{i \notin \mathbf{i}_{\max}} (q^{\mu_{\max} - \mu_i} - 1)^{-1} \cdot \binom{z}{|\mathbf{i}_{\max}| - 1} q^{\mu_{\max} z}.$$

Moreover, if $\boldsymbol{\mu}$ *takes integral values, then* $\mathsf{S}(z)$ *is rational.*

Asymptotic study for A_n

If n is odd, then

$$\mathrm{SV}(r) \sim \left(\sum_{\substack{I \subseteq \Delta \\ \frac{n+1}{2} \notin I}} \frac{q^{\left(\frac{n+1}{2}\right)^2 - \deg\left(\mathscr{P}_{A_n;I}\right)} \mathscr{P}_{A_n;I}(q)}{\prod_{\substack{1 \leqslant i \leqslant t+1 \\ i \neq i_0}} \left(q^{\left(\ell_i - \frac{n+1}{2}\right)^2} - 1 \right)} \right) \cdot q^{\left(\frac{n+1}{2}\right)^2 r}.$$

If n is even, then

$$\mathrm{SV}(r) \sim \left(\sum_{\substack{I \subseteq \Delta \\ \frac{n}{2}, \frac{n}{2}+1 \notin I}} \frac{q^{\frac{n}{2}\left(\frac{n}{2}+1\right) - \deg\left(\mathscr{P}_{A_n;I}\right)} \mathscr{P}_{A_n;I}(q)}{\prod_{\substack{1 \leqslant i \leqslant t+1 \\ i \neq i_0, i_0+1}} \left(q^{\left(\ell_i - \frac{n}{2}\right)\left(\ell_i - \frac{n}{2} - 1\right)} - 1 \right)} \right) \cdot r q^{\frac{n}{2}\left(\frac{n}{2}+1\right) r}.$$

However, for root systems other than A_n, we have to deal with the ceil functions appearing in

$$\sum_{x \in B_S(r, {}^V C, I)} \prod_{a(x)>0} q^{\lceil a(x) \rceil}.$$

In general, the exponents contain ceil functions and may not be integral. But by considering summations related to supsets and subsets, we can show that

$$\boxed{\mathrm{SV}(r) \asymp \binom{r}{\epsilon(n)} q^{\pi(n) r}.}$$

Parity exponential polynomials

allow coefficients to be

parity functions instead of constant.

Definition

A ***parity*** q***-exponential polynomial*** is a finite sum of the form

$$f(x) = \sum_{\nu} f_\nu(x) q^{\nu x},$$

where $\nu \in \frac{1}{2}\mathbb{Z}$, and each c_ν is a polynomial of x whose coefficients are parity functions taking q-number values.

The ***order***, ***degree***, and ***leading coefficient*** of f are defined similarly as a q-exponential polynomial.

A parity q-exponential polynomial can be viewed as a q-function. Then the key asymptotic property of such a q-function $f(z)$ is

$$f(z) \sim \mathsf{lead}(f)\binom{z}{\deg(f)}q^{\mathrm{ord}(f)z}.$$

However, since the leading coefficient $\mathsf{lead}(f)$ depends on the parity of z, the asymptotic growth $f(z)$ along even integers and odd integers can be different.

A parity q-exponential polynomial can be viewed as a q-function. Then the key asymptotic property of such a q-function $f(z)$ is

$$f(z) \sim \operatorname{lead}(f)\binom{z}{\deg(f)} q^{\operatorname{ord}(f)z}.$$

A parity q-exponential polynomial f is said to be ***rational*** if all the exponential coefficients ν are integers and all the coefficient $c_{\nu,n}$ take values in $\mathbb{Q}(q)_{q>1}$.

The $\mathbb{Q}(q)_{q>1}$-algebra of (rational) parity q-exponential polynomials is closed under differences and anti-differences.

Lemma

Given a multi-summation as below

$$S(z) = \sum_{\mathbf{c} \in \mathbb{Z}_{>0}^{\mathbf{i}}\,:\, \mathbf{1}\cdot\mathbf{c}=z} q^{\boldsymbol{\mu}\cdot\mathbf{c}+\epsilon(\mathbf{c})},$$

where $\epsilon(\mathbf{c})$ *is a multivariable parity function. Let* $\mathbf{i}_{\max} = \{i \in \mathbf{i} \mid \mu_i = \mu_{\max} := \max \boldsymbol{\mu}\}$. *Then* $S(z)$ *is a parity q-exponential polynomial and we have*

$$S(z) \sim \boxed{\frac{1}{2^{|\mathbf{i}_{\max}|-1}} \prod_{i \notin \mathbf{i}_{\max}} \left(q^{2(\mu_{\max}-\mu_i)} - 1\right)^{-1} \cdot E(z)} \cdot \binom{z}{|\mathbf{i}_{\max}|-1} q^{\mu_{\max} z},$$

order

deg

leading coefficient

Lemma

where $E(z)$ is a parity rational q-function and can be obtained as follows. Fix an $i^* \in \mathbf{i}_{\max}$ and let $\mathbf{i}_{i^*} = \mathbf{i} \setminus \{i^*\}$. Then

$$E(z) = \sum_{\mathbf{s} \in \mathbb{F}_2^{\mathbf{i}_{i^*}}} q^{\epsilon(\tau_{i^*}\mathbf{s}(z+\mathbf{1}\cdot\mathbf{s}))+\boldsymbol{\mu}\star_{i^*}\mathbf{s}},$$

where $\boldsymbol{\mu} \star_{i^*} \mathbf{s} = \sum_{i \in \mathbf{i}} (\mu_{i^*} - \mu_i)\epsilon_0(s_i)$. Moreover, if $\boldsymbol{\mu}$ and ϵ take integral values, then $\mathsf{S}(z)$ is ratioanl.

Asymptotic study for C_n

$$\mathrm{SV}(r) \sim \left(\sum_{\substack{I \subseteq \Delta \\ n \notin I}} \frac{\mathscr{P}_{C_{n};I}(q)}{q^{\mathbf{deg}\left(\mathscr{P}_{C_{n};I}\right)}} \frac{E_I}{\prod_{i=1}^{t-1} \left(q^{(n+1-\ell_i)(n-\ell_i)} - 1\right)} \frac{q^{\frac{n(n+1)}{2}}}{q^{\frac{n(n+1)}{2}} - 1} \right) \cdot q^{\frac{n(n+1)}{2} r},$$

where

$$E_I(r) = \sum_{\mathbf{s} \in \mathbb{F}_2^{t-1}} q^{\epsilon_I(s_1, \cdots, s_{t-1}) + \sum_{i=1}^{t-1} \frac{1}{2}(n+1-\ell_i)(n-\ell_i)\epsilon_0(s_i)},$$

$$\epsilon_I(c_1, \cdots, c_{t-1}) = \sum_{1 \leqslant i < j \leqslant t} (\ell_i - \ell_{i-1})(\ell_j - \ell_{j-1}) \epsilon_0(c_i + \cdots + c_{j-1}).$$

Asymptotic study for B_n and D_n

For B_3:

$$\mathrm{SV}(r) \sim \frac{cq^5}{q^5-1} \cdot q^{5r},$$

where

$$c = \frac{\mathscr{P}_{C_3;\{a_2,a_3\}}(q)}{q^5} + \frac{\mathscr{P}_{C_3;\{a_3\}}(q)}{q^8} \cdot \frac{1+q^4}{q^2-1} + \frac{\mathscr{P}_{C_3;\{a_2\}}(q)}{q^8} \cdot \frac{1+q+q^2}{q-1} + \frac{\mathscr{P}_{C_3;\emptyset}(q)}{q^9} \cdot \frac{1+q+2q^2+q^4}{(q-1)(q^2-1)}.$$

For B_n ($n \geqslant 4$):

$$\mathrm{SV}(2r) \sim \left(\sum_{n \notin I} \frac{\mathscr{P}_{C_{n;I}}(q)}{q^{\mathbf{deg}(\mathscr{P}_{C_{n;I}})}} \frac{E_{I,0} q^{n^2} + E_{I,1}}{q^{n^2} - 1} C_I \right) \cdot q^{n^2 r},$$

$$\mathrm{SV}(2r+1) \sim \left(\sum_{n \notin I} \frac{\mathscr{P}_{C_{n;I}}(q)}{q^{\mathbf{deg}(\mathscr{P}_{C_{n;I}})}} \frac{E_{I,0} q^{n^2} + E_{I,1} q^{n^2}}{q^{n^2} - 1} C_I \right) \cdot q^{n^2 r}.$$

For D_4:

$$\mathrm{SV}(r) \sim \frac{cq^6}{q^6-1} \cdot \binom{r}{2} q^{6r},$$

where

$$c = \frac{\mathcal{P}_{D_4;\{a_2\}}(q)}{q^{11}} \cdot (1+q^4) + \frac{\mathcal{P}_{D_4;\emptyset}(q)}{q^{12}} \cdot \frac{q^4}{1-q^4}\left(1+2q^4+q^3\right).$$

For D_n ($n \geqslant 5$):

$$\mathrm{SV}(2r) \sim \left(\sum_{n \notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \frac{E_{I,0} q^{n(n-1)} + E_{I,1}}{q^{n(n-1)} - 1} C_I \right) \cdot r q^{n(n-1)r},$$

$$\mathrm{SV}(2r+1) \sim \left(\sum_{n \notin I} \frac{\mathscr{P}_{D_n;I}(q)}{q^{\deg(\mathscr{P}_{D_n;I})}} \frac{E_{I,0} q^{n(n-1)} + E_{I,1} q^{n(n-1)}}{q^{n(n-1)} - 1} C_I \right) \cdot r q^{n(n-1)r}.$$

Thanks!