# Stratification of Tensor Triangular Categories

## Applications to Motivic Categories

David Rubinstein

University of California, Santa Cruz

# Outline of Presentation

## The Mathematical Landscape is Large

Categories $\mathcal{C}$ are pervasive in all fields of mathematics

- Algebra
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- Geometry
  - $\mathcal{C} = \mathsf{SmMan}$;
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Unfortunately, we are confronted with "wild classification problems"

- Can't classify all finite dim representations of group G in positive characteristic case;
- Can't classify finite CW complexes up to homotopy equivalence;
- No more hope for classifying all complexes of sheaves on an algebraic variety V;
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- Technically after a classification the thick tensor ideals

## Some Historical Examples

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  - The pair $(\mathrm{Spc}(\mathcal{K}), \mathrm{supp})$ is the universal space with well-behaved notion of support

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For mild assumptions on $\mathcal{K}$ there is a bijection

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- $Spc(D^{perf}(R)) \cong Spec(R)$
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- The large, non-compact objects, however, are very interesting!
  - Objects representing cohomology theories are not compact in SH
  - There are major open questions about the structure of the larger objects (e.g. The telescope conjecture)

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- In recent work Barthel, Heard and Sanders (BHS, 2021) developed a support theory for noetherian large tt-categories $\mathcal{T}$
    - There is no "$\mathrm{Spc}(\mathcal{T})$" but can consider $\mathrm{Spc}(\mathcal{T}^c)$
    - The support for arbitrary objects will be a subset of $\mathrm{Spc}(\mathcal{T}^c)$

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- D(X) for X noetherian scheme
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# Motives and their Derived Categories

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- Prove some longstanding fundamental conjectures in algebraic geometry (e.g. The Milnor conjecture and the Bloch-Kato conjecture).

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- Idea: study first a "piece" of the category; the Tate motives
- The localizing subcategory generated by the Tate twists is the (large) category of Tate motives, denoted by $DTM(\mathbb{F}, R)$.

## Étale Motives

There is a similar story to tell with so called étale motives.

- There is a (large) derived category of étale motives, denoted $\mathrm{DM}^{\text{ét}}(\mathbb{F}, R)$, again with an étale motive functor:

# Motives and their Derived Categories

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These two constructions are very similar: there is an "étale sheafification" functor

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## Étale Motives

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Moreover,

- Whenever $\mathbb{Q} \subset R$, $\alpha_{ét}$ is an equivalence of categories.

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## The Balmer spectrum of $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$

### Theorem (Gallauer 2019)

(1) *The Balmer spectrum of* $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c$ *is the following picture:*

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*In this picture the specialization relations are pointing upwards.*

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(2) $\mathrm{Spc}(\mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z})^c) \cong \mathrm{Spec}(\mathbb{Z})$

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(3) *The étale sheafification map induces a map* $\mathrm{Spec}(\mathbb{Z}) \xrightarrow{\mathrm{Spc}(\alpha_{\acute{e}t})} \mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$ *which is a homeomorphism onto the subspace* $\{\mathfrak{m}_o, e_p\}$

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## Remarks on These Computations

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{tikzcd}[arrows=-]
m_2 \arrow[d] & m_3 \arrow[d] & \cdots & m_p \arrow[d] & \cdots \\
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$(\mathfrak{m}_p)$: kernal of map $\gamma_p^* : \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \to \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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$(e_p)$: kernals of the composite

$$\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \xrightarrow{\gamma^*} \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c \xrightarrow{\alpha_{\text{ét}}} \mathrm{DTM}^{\text{ét}}(\overline{\mathbb{Q}}, \mathbb{Z}/p\mathbb{Z})^c$$

Again, this kernal coincides with those motives whose mod-p étale cohomology vanishes.

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Again, this kernal coincides with those motives whose mod-p étale cohomology vanishes.

$(\mathfrak{m}_0)$: kernal of the rationalization map $\gamma^* : \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c \to \mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Q})^c$, which coincides with those motives whose rational motivic cohomology vanishes.

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

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$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{array}{ccccc} \mathfrak{m}_2 & \mathfrak{m}_3 & \cdots & \mathfrak{m}_p & \cdots \\ | & | & & | & \\ \mathfrak{e}_2 & \mathfrak{e}_3 & \cdots & \mathfrak{e}_p & \cdots \\ & \diagdown & \diagdown \quad \diagup & & \\ & & \mathfrak{m}_0 & & \end{array}$$

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- Morally there are only 3 flavors of primes in $\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c)$
- Reduce the problem to each "vertical slice" in the spectrum and just consider the 3 primes in each slice

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## How To Establish Minimality at the Primes

$$\mathrm{Spc}(\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})^c) = \begin{tikzcd}[arrows=dash] \mathfrak{m}_2 \arrow[d] & \mathfrak{m}_3 \arrow[d] & \cdots & \mathfrak{m}_p \arrow[d] & \cdots \\ \mathfrak{e}_2 \arrow[dr] & \mathfrak{e}_3 \arrow[d] & \cdots & \mathfrak{e}_p \arrow[dll] & \cdots \\ & \mathfrak{m}_0 & & & \end{tikzcd}$$

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$(\mathfrak{m}_0)$: Reduces to showing minimality in $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Q})$.

# My Problem: Stratification for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$

## Final Comments and Summery

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- In summary, we want to get a classification for the localizing tensor ideals for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$.

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- In summary, we want to get a classification for the localizing tensor ideals for $\mathrm{DTM}(\overline{\mathbb{Q}}, \mathbb{Z})$.
- Using the results of Sanders, et al we are tasked with checking a certain minimality condition at every prime.
- In this case, we can first take vertical slices of the spectrum, and then check minimality at local categories for mod p and rational coefficients