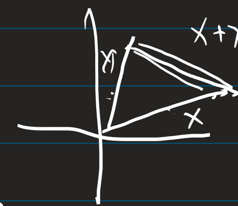
Midterm, Assignment 4, Written assignment, Quiz

31 56 24

$\hookrightarrow \equiv$ is $=$ (kinda!)

$\hookrightarrow$ AM $\geq$ GM



$|x| - |y| \leq |x+y|$ ✓

$|x| = |x+y-y|$

$= |x+y+(-y)|$

$\leq |x+y| + |-y|$ Δ^{le} ineq

$= |x+y| + |y|$

$|x| - |y| \leq |x+y|$ substract $-|y|$

(reverse Δ^{le} inequality)

$\left[\ \underline{\underline{|x+y| \leq |x| + |y|}}\ \right]$ ✪

$\hookrightarrow$ seen this?

$|x+y| \leq |x| + |y|$

Case I: $x, y \geq 0$

$|x+y| = x+y$

$||$

$|x| + |y| = x+y$ ✓

Case II $x, y \leq 0$

$|x+y| = -(x+y)$ ✓

$||$

$|x| + |y| = -x-y$

Case III. $x > 0,\ y \leq 0$

$|x+y|$ $\qquad |x|,\ |y|$

$\underbrace{+ve + -ve}_{+ve}$ $\qquad x = |x|,\ -y = |y|$

$(4, -3)$

$x+y \leq \underbrace{x-y}_{+ve \text{ number}}$ $\rightarrow 4+3$

$4-3$

$2, -3.$

$-(x+y) \leq x-y$ ✓

56 $(A-B)\cup(A-C) = A-(B\cap C)$

1 set containments

(1) $(A-B)\cup(A-C) \subseteq A-(B\cap C)$

(2) $A-(B\cap C) \subseteq (A-B)\cup(A-C)$

$x \in (A-B)\cup(A-C)$

$x\in A-B$ or $x\in A-C$

$(x\in A$ and $x\notin B)$ or $(x\in A$ and $x\notin C)$

$(x\in A)$ and $(x\in \bar{B}$ or $x\in\bar{C})$

$x\in A$ and $x\in \bar{B}\cup\bar{C}$ $=$

$x\in A$ and $x\in \overline{B\cap C}$

$x\in A-(B\cap C)$

set operations

$(A-B)\cup(A-C)$

$=(A\cap\bar{B})\cup(A\cap\bar{C})$

$=A\cap(\bar{B}\cup\bar{C})$

$a\cdot(b+c) = a\cdot b + a\cdot c$

$= A\cap\overline{(B\cap C)}$

$= A-(B\cap C)$

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arbitrary $x \in A$

$A = \{x\in\mathbb{R} \mid x>0\} = \{x^2 \mid x\in\mathbb{R}\} = B$

To show by set equality $A\subseteq B$ and $B\subseteq A$

Is $B \subseteq A$ an obvious thing? suppose $x \in B \therefore x = y^2 > 0$
so $x \in A$

i.e. any number that's a square is an element of A?

$A \subseteq B$ pick a positive number $x \in A$

show $x = y^2$ for some y

so $x \in B$

so what y works? $y = \sqrt{x}$

$\boxed{A = B}$

Chapter 3

24 $n \in \mathbb{Z}$

$2n^2 + n$ is odd iff $\cos\left(\frac{n\pi}{2}\right)$ is even

Proof. $2n^2+n$ is odd $\Rightarrow$ $\cos\left(\frac{n\pi}{2}\right)$ is even

AND

$\cos\left(\frac{n\pi}{2}\right)$ is even $\Rightarrow$ $2n^2+n$ is odd

PAUSE

(24) x & y are even

$x^2 \equiv y^2 \mod 16$ iff

(a) $x, y \equiv 0 \mod 4$ (or) (b) $x, y \equiv 2 \mod 4$.

$\boxed{(\Rightarrow)\quad x^2 \equiv y^2 \mod 16 \Rightarrow (a) \text{ or } (b)}$

$(\Leftarrow)$ (a) or (b) $\Rightarrow x^2 \equiv y^2 \mod 16$ ✓

$R \Rightarrow (P \vee Q)$

Proof: Suppose $\boxed{x^2 \equiv y^2 \mod 16}$, to show (a) or (b)

$\boxed{P \vee Q}$

~~(a)~~ to show (b) holds

$\equiv \sim(\sim P) \vee Q$

$\equiv \neg P \Rightarrow Q$

n — ~~even~~ / odd

$\longrightarrow$ $x^2 \equiv y^2 \mod 16$ and $x \not\equiv 0 \mod 4$ or $y \not\equiv 0 \mod 4$

To show $x \equiv 2 \mod 4$ and $y \equiv 2 \mod 4$.

Case I.

Suppose $x \not\equiv 0 \mod 4$ so x is even $\boxed{x \equiv 2 \mod 4}$

and $\boxed{x^2 \equiv y^2 \mod 16}$

$x^2 = y^2 + 16l$

$x = \underbrace{2 + 4k}$

$(2+4k)^2 - 16l = y^2$, so $4 + 16k^2 + 16k - 16l = y^2$

$4(1+4k^2-4k-4l)=y^2$

i.e. $y^2 \equiv 0 \bmod 4$ then $y \equiv 2 \bmod 4$ (or) $y \equiv 0 \bmod 4$.

Proof. [If $x^2 \equiv y^2 \bmod 16$, then (a) or (b)]

Contrapositive: if $\boxed{\text{(a) and (b) are not true}}$

then $x^2 \not\equiv y^2 \bmod 16$

(a) is not true ~ $\underbrace{x \not\equiv 0 \bmod 4}$ or $\underbrace{y \not\equiv 0 \bmod 4}$

(b) is not true ~ $x \not\equiv 2 \bmod 4$ or $y \not\equiv 2 \bmod 4$

Case I: $x \not\equiv 0 \bmod 4$, $\boxed{x \equiv 2 \bmod 4}$

$y \not\equiv 2 \bmod 4$, $\boxed{y \equiv 0 \bmod 4}$

x y

0 2 0 2

(a) x — 2
or
y — 2

and

(b) x — 0
or
y — 0

$x = 2+4k$

$y = 4l$

$x^2 = 4+16k^2+16k$

$y^2 = 16l^2$

$x^2-y^2 = 16(\_\_\_)$

(A or B) and ($\neg$A or $\neg$B)

($\cancel{\text{A and}\neg\text{A}}$) or (A and $\neg$B) or (B and $\neg$A) or ($\cancel{\text{B and}\neg\text{B}}$)

$x^2 = 16k^2 + 16k + 4$

$y^2 = 16l^2$

$$x^2 - y^2 = \underbrace{16(k^2 - l^2) + 16k + 4}$$

$\equiv 4 \mod 16$

$\not\equiv 0 \mod 16$

Quiz $\Rightarrow$ congruences ✓

inequalities ✓

(AM $\geq$ GM) 2 numbers — arithmetic mean $\geq$ geometric mean

a, b $\qquad \frac{a+b}{2} \geq \sqrt{ab}$

To show

$$\frac{a^2+b^2+2ab}{2} = \left(\frac{a+b}{4}\right)^2 \geq (\sqrt{ab})^2 = ab$$

$$\frac{a^2+b^2+2ab}{4} - ab = \frac{a^2+b^2}{2} \geq 0$$

$\frac{(a-b)^2}{2}$

SCRATCH WORK

Proof: ① $a = x^2$, and $b = y^2$ okay

$x = \sqrt{a}$ $\qquad y = \sqrt{b}$

② assume AM $\geq$ GM ✓ doesn't lead to a contradiction

Proof. $\left(\frac{a-b}{2}\right)^2 > 0$

$$\frac{a^2+b^2-2ab}{4} > 0$$

$$\frac{a^2+b^2+2ab-4ab}{4} > 0$$

$$\frac{a^2+b^2+2ab}{4} - ab > 0$$

$$\left(\frac{a+b}{2}\right)^2 > \left(\sqrt{ab}\right)^2$$

$x^2 > y^2$

$\Downarrow$?

$x > y$

Why can we conclude

$$\frac{a+b}{2} > \sqrt{ab}$$

$$\frac{a+b}{2} > \sqrt{ab}$$

SCRATCH

$\iff a+b > 2\sqrt{ab}$

$\iff \underbrace{a+b-2\sqrt{ab}}_{} > 0$

Get everything to one side

$\underbrace{\qquad}_{} > 0$

$x = \sqrt{a}$

$y = \sqrt{b}$

$\iff \underbrace{a+b-2\sqrt{a}\cdot\sqrt{b}}_{x^2+y^2-2xy}$

$\iff (\sqrt{a})^2+(\sqrt{b})^2 - 2\cdot\sqrt{a}\cdot\sqrt{b} = \left(\sqrt{a}-\sqrt{b}\right)^2$

Proof: $\left(\sqrt{a}-\sqrt{b}\right)^2 \geq 0$ as a square

$$a+b-2\sqrt{ab} \geq 0$$

$$a+b \geq 2\sqrt{ab}$$

$$\frac{a+b}{2} \geq \sqrt{ab}.$$

2, 3

GM $\geq$ HM

$$\sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}}.$$

AM $\geq$ GM $\geq$ HM.

$$\frac{x_1+\cdots+x_n}{n}$$

$$\sqrt[n]{x_1\cdots x_n}$$

$$\frac{n}{\frac{1}{x_1}+\cdots+\frac{1}{x_n}}$$
