Cohomology on Sites

UCSC Graduate Colloquium

Deewang Bhamidipati

10th January 2022

Sheaves

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Topology Categorified

Category of Open Sets

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- Given two open sets U, V, say

$$\mathrm{Hom}_{\mathsf{Open}(X)}(U, V) = \begin{cases} \{U \hookrightarrow V\} & \text{if } U \subseteq V \\ \varnothing & \text{otherwise} \end{cases}$$

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Note that X is the final object of $\mathsf{Open}(X)$ and finite (fibered) products exist and are given by finite intersections. We reiterate, the following is a Cartesian square in $\mathsf{Open}(X)$,

$$\begin{tikzcd} U \cap V \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, hook] \\ V \arrow[r, hook] & X \end{tikzcd}$$

that is, $U \times_X V = U \cap V$ in $\mathsf{Open}(X)$.

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- Finally, whenever $U \hookrightarrow V \hookrightarrow W$, the following diagram commutes in Ab

$$\begin{tikzcd}
\mathscr{F}(W) \arrow[rr, "\mathrm{res}_{W,U}"] \arrow[dr, "\mathrm{res}_{W,V}"'] & & \mathscr{F}(U) \\
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One can suitably replace the target category to obtain *presheaf of sets, groups, rings, modules, algebras* etc.

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- Consider a topological space (resp. smooth manifold) X, then associating to each open set U the ring of $\mathbb{R}$-valued continuous (resp. smooth) functions on U gives us a presheaf (of rings).

$$\mathcal{F}(U) = \{ f : U \to \mathbb{R} \mid f \text{ cts} \mid \text{smooth} \}$$

$$f \longmapsto f|_U$$

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- If X is a smooth (resp. complex) manifold, associating to each open set U the abelian group of smooth (resp. holomorphic) differential forms on U gives us a presheaf (of abelian groups).

$U \mapsto \Omega^p(U)$

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In all these examples, the restriction maps are just that, honest restriction of functions.

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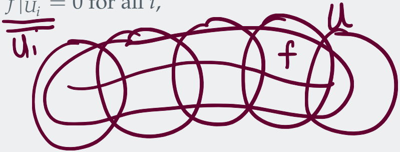

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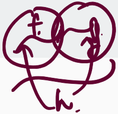
(S2) *Gluability axiom*. Suppose $\{U_i\}_{i\in I}$ is an open cover of an open set U, and $f_i \in \mathscr{F}(U_i)$ are such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ for all i, j,



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$(\ldots)$

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(…) Equivalently, these axioms can be succinctly stated as saying the following sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\rho := (\mathrm{res}_{U,U_i})} \prod_i \mathscr{F}(U_i) \underset{\rho_j := (\mathrm{res}_{U_j,\, U_i\cap U_j})_{i,j}}{\overset{\rho_i := (\mathrm{res}_{U_i,\, U_i\cap U_j})_{i,j}}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

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is exact for a given covering of $\{U_i\}_{i \in I}$ of U.

(S1) corresponds to exactness at $\mathscr{F}(U)$, that is the injectivity of ρ. While (S2) corresponds to exactness at $\prod_i \mathscr{F}(U_i)$. This can be interpreted as saying $\mathrm{im}(\rho) = \mathrm{eq}(\rho_i, \rho_j)$, the equaliser;

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$f \longmapsto f_i|_{U_i \cap U_j} - f_j|_{U_i \cap U_j} = 0 \quad \forall i,j$

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$\operatorname{Spec} A = D(1) \longmapsto A_1 = A$

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Therefore, a morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ is a natural transformation of functors. That is, for each open set U, we have a group homomorphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ such that whenever $U \hookrightarrow V$ the following diagram commutes

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\phi(V)"] \arrow[d, "\mathrm{res}^{\mathscr{F}}_{V,U}"'] & \mathscr{G}(V) \arrow[d, "\mathrm{res}^{\mathscr{G}}_{V,U}"] \\ \mathscr{F}(U) \arrow[r, "\phi(U)"'] & \mathscr{G}(U) \end{tikzcd}$$

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Example

For any abelian group A, the constant sheaf $\underline{A}$ is the sheafification of the constant presheaf $\underline{A}_{\mathrm{pre}}$.

Sheaf Cohomology

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The takeaway being that $\psi(X)$ is not necessarily surjective.

The standard example that illustrates this is the *exponential exact sequence*.

Global Sections Functor

Exponential Exact Sequence

Let $X = \mathbb{C}^*$ (the punctured complex plane) with the standard topology, and define $\mathscr{O}_X$ to be the sheaf of holomorphic functions. Also let $\mathscr{O}_X^*$ to be the sheaf of invertible (nowhere zero) holomorphic functions (this is a sheaf of abelian groups under multiplication).

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But

$$\exp : \Gamma(X, \mathscr{O}_X) \to \Gamma(X, \mathscr{O}_X^*) = \Gamma(X, \mathscr{O}_X)^*$$

is not surjective since there does not exist a global complex logarithm.

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we obtain a long exact sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X, \mathscr{F}) \arrow[r] & \Gamma(X, \mathscr{G}) \arrow[r] & \Gamma(X, \mathscr{H}) \arrow[dll, bend left] \\ & H^1(X, \mathscr{F}) \arrow[r] & H^1(X, \mathscr{G}) \arrow[r] & H^1(X, \mathscr{H}) \arrow[dll, bend left] \\ & H^2(X, \mathscr{F}) \arrow[r] & \cdots & \end{tikzcd}$$

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While c associates to each (isomorphism class of) line bundle $\mathscr{L}$ its first Chern class $c_1(\mathscr{L})$.

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Let $\mathscr{C}$ be a category, a *Grothendieck topology* on $\mathscr{C}$ is an assignment, for each object U of $\mathscr{C}$, of a set $\mathbf{cov}(U)$ of families of morphisms $\{U_i \to U\}_{i \in I}$ (called *coverings*) such that

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- if $\{U_i \to U\}_i$ is in $\mathbf{cov}(U)$ and $\{V_{ij} \to U_i\}_j$ is in $\mathbf{cov}(U_i)$, then $\{V_{ij} \to U_i \to U\}_{i,j}$, obtained by composition, is in $\mathbf{cov}(U)$.

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So, given a site $\mathscr{T}$, we have the underlying category denoted as $\mathsf{Cat}(\mathscr{T})$ and a collection of covers for each object which we denote $\mathsf{Cov}(\mathscr{T})$.

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Note that if $\mathsf{Cat}(\mathscr{T})$ has a terminal object $\bigstar$, then $\mathscr{T} = \bigstar_{\mathscr{T}}$.

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- *(takes covers to covers)* if $\{U_i \to U\}_i$ is a covering of U, then $\{f^*(U_i) \to f^*(U)\}_i$ is a covering of $f^*(U)$.

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Remark

To be precise, what we've described here is really what is called a *Grothendieck pretopology*; this notion suffices for our purposes. So, as is commonly done, we will abuse terminology and keep using the term Grothendieck topology.

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It's this example that motivates the notion of a continuous function of sites.

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- A morphism $f : X \to Y$ is *locally of finite presentation*, if for any affine open set $\operatorname{Spec} A \subseteq Y$ and affine open set $\operatorname{Spec} B \subseteq f^{-1}(\operatorname{Spec} A)$ the corresponding ring map $A \to B$ makes B an A-algebra of finite presentation. That is, there exists a surjective ring map $A[x_1, \ldots, x_n] \twoheadrightarrow B$ such that the kernel is a finitely generated ideal of $A[x_1, \ldots, x_n]$.

- A morphism $f : X \to Y$ that is locally of finite presentation is *unramified*, if for each point $y \in Y$, the fiber $f^{-1}(y) = X \times_Y \operatorname{Spec} \kappa(y) \cong \coprod_i \operatorname{Spec} k_i$, where k_i's are finite separable extensions of the residue field at y, $\kappa(y)$.

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Sheaves and Sheaf Cohomology on Sites

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It's worth pointing out that if we're given a category $\mathscr{C}$ which can be equipped with different Grothendieck topologies then the category of presheaves $[\mathscr{C}^{\mathrm{op}}, \mathsf{Ab}]$ has multiple subcategories of sheaves with respect each of these Grothendieck topologies.

Sheaves on Sites

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Then h_G is a sheaf on X_E which we also denote as G.

Sheaves on Sites

Representable Sheaves

Let $E \in \{\text{Zar}, \text{ét}, \text{fppf}\}$ and consider the site X_E for a scheme X. For any commutative X-group scheme G, we can consider the functor of points

$$h_G : (\mathsf{Sch}/X)^{\text{op}} \to \mathsf{Ab},\ T \mapsto \text{Hom}_S(T, G).$$

Then h_G is a sheaf on X_E which we also denote as G.

Subcategory of the Category of Sheaves

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This way, we can view Sch/X as a subcategory of $\mathsf{Sh}(X_E)$.

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As before, the functor $i : \mathsf{Sh}(\mathscr{T}) \hookrightarrow \mathsf{PSh}(\mathscr{T})$ has a left adjoint

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The category $\mathsf{Sh}(\mathscr{T})$ has enough injectives, this is a consequence of it being a Grothendieck category (an AB5 category with a family of generators).

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we obtain a long exact sequence of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & \Gamma_E(X, \mathscr{F}) \arrow[r] & \Gamma_E(X, \mathscr{G}) \arrow[r] & \Gamma_E(X, \mathscr{H}) \arrow[dll, out=-30, in=150] & \\ & H^1_E(X, \mathscr{F}) \arrow[r] & H^1_E(X, \mathscr{G}) \arrow[r] & H^1_E(X, \mathscr{H}) \arrow[r] & H^2_E(X, \mathscr{F}) \arrow[r] & \cdots \end{tikzcd}$$

Some Examples

Hilbert's Theorem 90

Let X be any scheme, then

$$H^1_{\text{fppf}}(X, \mathbb{G}_m) \cong H^1_{\text{ét}}(X, \mathbb{G}_m) \cong H^1_{\text{Zar}}(X, \mathscr{O}_X^*) \cong \text{Pic}(X)$$

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(this sequence is exact in $\mathsf{Sh}(X_{\text{ét}})$ provided n is coprime to the characteristic of X.) $(\ldots)$

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$$\mathcal{O}_X \xrightarrow{\phi} \mathscr{L}^{\otimes n} \xrightarrow{(\psi^{-1})^{\otimes n}} \mathcal{O}_X^{\otimes n} \xrightarrow{\text{can}} \mathcal{O}_X$$

is multiplication by a^{-1}.

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where the latter is the i^{th} Galois cohomology group (that is, group cohomology of G_k) with coefficients in $M_{\mathscr{F}}$.

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The above group was denoted by Grothendieck as $\operatorname{Br}'(X)$.

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The classical Hilbert's Theorem 90 tells us that the latter cohomology group is trivial, while the one stated above tells us that the former cohomology group is $\operatorname{Pic}(\operatorname{Spec} k)$. This is trivial, since the only line bundles on $\operatorname{Spec} k$ are the trivial ones.

Brauer Groups

Since we have $H^2(G_k, k_{\text{sep}}^\times) \cong \operatorname{Br}(k)$, the Brauer group of k which can be defined as the group of Morita equivalent central simple algebras over k, we call, for any scheme X, the group $H^2_{\text{ét}}(X, \mathbb{G}_m)$ the *cohomological Brauer group of* X.

The above group was denoted by Grothendieck as $\operatorname{Br}'(X)$. He reserved $\operatorname{Br}(X)$ for the group of Morita equivalent *Azumaya algebras* (a generalisation of central simple algebras).

Some Examples

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Fin.

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