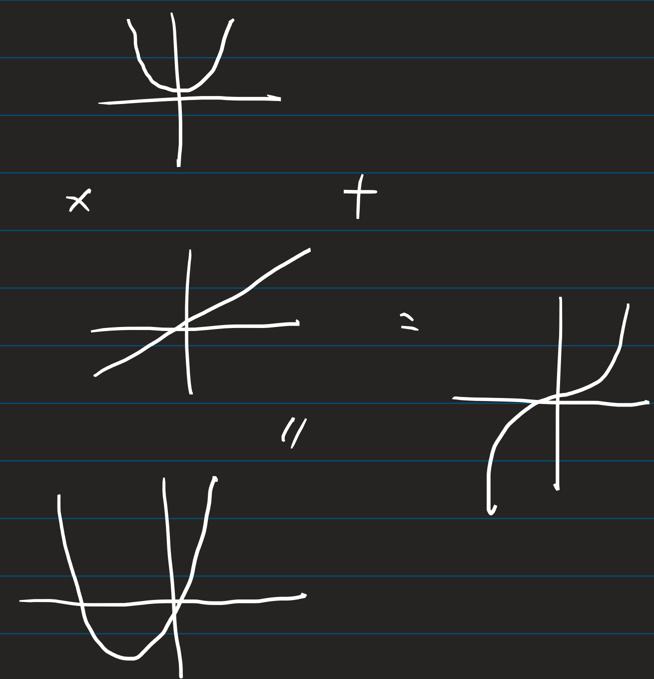
04/06

Groups, Homomorphisms. & HW

$f: \mathbb{R} \longrightarrow \mathbb{R} \quad \rightsquigarrow \quad \Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$

$f(x) = x^2$

$g(x) = 2x$



$(f+g)(x) = f(x) + g(x)$

$= x^2 + 2x$

$= x(x+2)$

$(f \cdot g)(x) = f(x) \cdot g(x) = 2x^3$

$+ \qquad \longrightarrow$

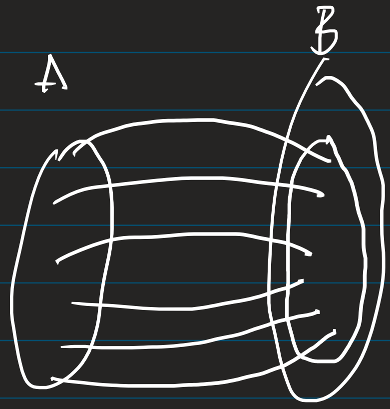
$- \qquad \longleftarrow$

$G = \{f : \mathbb{R} \longrightarrow \mathbb{R}\}$ as a group w/ $+$

$\mathbb{Z} \longrightarrow G$ //

Construct a group homomorphism $\phi : \mathbb{Z} \to G$ st ϕ is inj.

$\mathbb{Z} \cong \phi(\mathbb{Z}) \le G$, $\mathbb{Z}$ should be bijctive to its $\phi(\mathbb{Z})$



$^{-1}$

$$f : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$$
$$x \longmapsto x$$

$$f : \mathbb{R}_{>0} \longrightarrow f(\mathbb{R}_{>0})$$
$$x \longmapsto x$$

$$\phi : \mathbb{Z}, + \longrightarrow G$$
$$n \longmapsto f_n \in G$$

$f_n(x) =$ n / x^n / nx

$\mathbb{R} \longrightarrow \mathbb{R}$

I. Is ϕ a grp hmph?

II. Is it inj?

1. $\phi(n)(x) = n$ [$\phi(n) = f_n$ where $f_n(x) = n$]

$\phi(n+m)(x) = n+m = \phi(n)(x) + \phi(m)(x)$

i.e. $\phi(n+m) = \phi(n) + \phi(m)$

2. $\phi(n)(x) = x^n$

$\phi(n+m)(x) = x^{n+m} = x^n \cdot x^m$ $\qquad \phi : \mathbb{Z},+ \to (G,\times)$

(uh oh! we wanted $x^n + x^m = \phi(n)(x) + \phi(m)(x)$)

$\phi(1+1)(3) = 3^2 = 9$

$\phi(1)(3) = 3^1 = 3$

$+ \quad = 6$ $\quad \not\times$

$\phi(1)(3) = 3^1 = 3$

3. $\phi(n)(x) = nx$

$\phi(n+m)(x) = (n+m)x = nx + mx = \phi(n)(x) + \phi(m)(x)$

Grp hmph!

$\phi(n) = \phi(m)$, is $n = m$?

1. $\phi(n) = \phi(m)$

$f_n = f_m$ i.e. $f_n(x) = f_m(x) \quad \forall x \in \mathbb{R}$

$\begin{array}{ccc} \shortparallel & & \shortparallel \\ n & = & m \end{array}$ ✓

3. $\phi(n) = \phi(m)$

$f_n = f_m$ i.e. $f_n(x) = f_m(x) \quad \forall x \in \mathbb{R}$

$\overset{\shortparallel}{n x} = m x \qquad \forall x \in \mathbb{R}$

Then let $x = 1$, $n(1) = m(1)$

$$\overset{\shortparallel}{n} = \overset{\shortparallel}{m}$$

Can treat $\underset{(w/+)}{\mathbb{Z}}$ as "a subgroup" of $\underset{(w/+)}{G}$

<u>Exercise</u> Can you repeat the argument and show

$\underset{(w/+)}{\mathbb{R}}$ is "a subgroup" of $\underset{(w/+)}{G}$

$\phi(a)(x)$ — a, ax

Matrices! w/ addition

$$\boxed{\begin{gathered} GL_n(\mathbb{R}) = \{ A \in M_{n\times n}(\mathbb{R}) \mid \det A \neq 0 \} \quad \text{"general linear group"} \\ A^{-1} \cdot A = I \qquad \text{non-abelian group.} \\ AB \neq BA \end{gathered}}$$

$$\det : GL_n(\mathbb{R}) \longrightarrow \mathbb{R} \setminus \{0\} = (\mathbb{R}^{\times}, \cdot)$$

$$\det(AB) = \det A \ \det B$$

$a \neq 0$

$$\det(aI) = \det\begin{pmatrix} a & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix} = a^n$$

$$\det\begin{pmatrix} a & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = a$$

$$\det\begin{pmatrix} a & & \\ & 1 & \\ & & \ddots \end{pmatrix} = a = \det\begin{pmatrix} 1 & & \\ & a & \\ & & 1 & \\ & & & \ddots \end{pmatrix}$$

"kernel" measures failure of injectivity

$$SL_n(\mathbb{R}) = \{ A \in M_{n \times n}(\mathbb{R}) \mid \det A = 1 \}$$

"special linear group"

* $G = \{ f : \mathbb{R} \to \mathbb{R} \}$ w/ add^n is a BIG group

* $GL_n(\mathbb{R})$ is an example of a non-abelian group
(matrices wrt mult^n)

* det is a grp hmph
* Can think of different groups as subgroups of other groups via inj group hmph.