So what is a Shimura Curve?

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Abstract

Shimura varieties are one of the objects needed to realize the ambitious Langlands' correspondence. They are infamous for being rather technical. Historically, their one dimensional avatar, called Shimura curves, were studied first, in the context of class field theory. In this talk, we will build up to a working definition of what a Shimura curve is, using classical modular curves as our Ariadne's Thread.

<u>Motivation</u> : Class field theory

Construction of <u>class fields</u> of totally real number fields

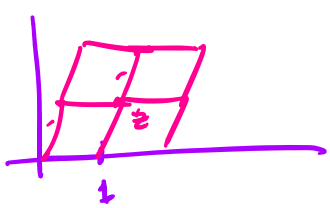
H. Pasten Shimura curve & ABC conj

I <u>Review of Modular Curves</u>

$\Gamma(1) := SL_2(\mathbb{Z})$ acts on $\mathcal{H} = SL_2(\mathbb{R})/SO(2)$ via Möbius transformations ; i.e. given $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma(1)$,

$$z \longmapsto \gamma \cdot z = \frac{az+b}{cz+d}$$

For $z \in \mathcal{H}$, we have $\Lambda_z = \langle 1, z \rangle_{\mathbb{Z}}$



$1, z$ = fundamental periods

elliptic curve $E_z = \mathbb{C}/\Lambda_z$

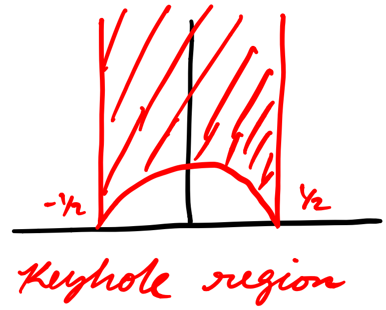
$E_z \cong E_{z'} \iff z$ and z' lie in the same $\Gamma(1)$-orbit.

$$\{\alpha \in \mathbb{C} \mid \alpha \cdot \Lambda_z = \Lambda_z\} \cong \mathrm{End}(E_z) \cong \begin{cases} \mathbb{Z} \\ \mathcal{O} \subset \mathbb{Q}(\tau)/\mathbb{Q} \quad \text{CM} \end{cases}$$

imag deg 2 field

Prop 1 $\{E/\mathbb{C}\}/\sim \;\xleftrightarrow{\;\sim\;}\; \Gamma(1)\backslash\mathcal{H}$

$$\Gamma(1)\backslash\mathcal{H} \cong \Omega$$



Notation Γ = congruence subgp of $SL_2(\mathbb{Z})$

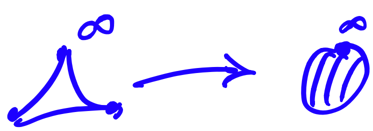
$Y(\Gamma) := \Gamma\backslash\mathcal{H}$ $\qquad\qquad \mathcal{H}^* := \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$

$X(\Gamma) := \Gamma\backslash\mathcal{H}^* = Y(\Gamma) \cup \Gamma\backslash(\mathbb{Q}\cup\{\infty\})$

$X(1) = X(\Gamma(1)) = SL_2(\mathbb{Z})\backslash\mathcal{H} \cup \{\infty\}$

Prop 2 (i) $X(1)$ is the coarse moduli sp of $E/\mathbb{C}$

(ii) $X(1) \cong \mathbb{P}^1_{\mathbb{C}}$ via the j-invt



congruence subgps	classical Modular Curves	Coarse moduli spaces of cpx ell. curves with TORSION DATA
$\Gamma(1) = SL_2(\mathbb{Z})$	$Y(1) \cong \mathbb{P}^1$	$\{E/\mathbb{C}\}$
$\uparrow$	$\Uparrow$	
$\Gamma_0(N) \equiv \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \bmod N$	$Y_0(N)$	$\{E/\mathbb{C}$ w/ a $\mathbb{Z}/N\mathbb{Z}$ subgp$\}$
$\uparrow$	$\Uparrow$	
$\Gamma_1(N) \equiv \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \bmod N$	$Y_1(N)$	$\{E/\mathbb{C}$ w/ a pt of order $N\}$
$\uparrow$	$\Uparrow$	
$\Gamma(N) \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \bmod N$	$Y(N)$	$\{E/\mathbb{C}$ w/ basis of N-torsion pts with fixed Weil pairing$\}$

Rmk 2 Function field of $X_0(N)$ is $\mathbb{Q}(j(\tau), j(N\tau))$ for some $\tau \in \mathbb{C}$

$\Rightarrow \exists\, p(x,y) \in \mathbb{Q}[x,y]$ s.t. $p(j(\tau), j(n\tau)) = 0$

In fact $p(x,y) \in \mathbb{Z}[x,y]$

Used to show E w/ CM by $\mathcal{O}_\tau \Rightarrow j(E)$ is an alg integer

Recall: Hilbert class field H/K = max'l unramified abelian extn of field K

Fact $Gal(H/K) \cong Cl(K)$ ideal class gp

$\Rightarrow [H:K] = h_K$

Thm 1 i) $j(E)$ is an algebraic integer

ii) $\deg(j(E)) = h(\tau) = \# Cl(\mathbb{Q}(\tau))$

iii) H_K obtained by adjoining $j(E)$ to K.

iv) Max'l ab extn of $\mathbb{Q}(\tau)$ obtained by adjoining $j(E)$ and x-coords of torsion pts of E/K

v) $Gal(H/K)$ action of $j(E)$ described explicitly.

II Groups & Algebras

Dfn 1 Groups G_1 and G_2 are commensureable if $\exists$ sgp's of finite index $H_1 \subset G_1$, $H_2 \subset G_2$ s.t. $H_1 \cong H_2$.

Dfn 2 Let $G(\mathbb{Q}) \subseteq GL_n(\mathbb{Q})$ be an algebraic gp over $\mathbb{Q}$. sgp $\Gamma_0 \subseteq G(\mathbb{Q})$ is arithmetic if Γ_0 cms w/ $G(\mathbb{Z})$.

Dfn 3 sgp $\Gamma \subseteq SL_2(\mathbb{R})$ is cocompact if $\exists$ $\mathbb{Q}$-alg gp G and gp epi $\varphi : G(\mathbb{R}) \twoheadrightarrow SL_2(\mathbb{R})$ w/ cpt $Ker(\varphi)$

$$\underset{\text{arithmetic s.gp}}{G(\mathbb{Q}) \supseteq \Gamma_0} \longmapsto \Gamma = \varphi(\Gamma_0)$$

If we set $G = SL_2(\cdot)$

then congruent s.gp $\Gamma(1) = SL_2(\mathbb{Z})$

are arithmetic s.gps

Problem Want cocpt arithmetic sgps

Solution Pick different $\mathbb{Q}$-alg str

s.t. $G(\mathbb{R}) = SL_2(\mathbb{R})$

candidate Quaternion alg over $\mathbb{Q}$.

Dfn 3 Given F char $F \neq 2$. Quaternion alg.

$\left(\frac{a,b}{F}\right)$ is an F-alg w/ a basis $\{1, i, j, ij\}$

s.t. $i^2 = a$ e.g. $\left(\frac{-1,-1}{\mathbb{R}}\right)$

$j^2 = b$

$ij = -ji$

Prop 3 Let B be a quarternion $\mathbb{Q}$-alg. Then $B_p := B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ has 2 possible structures:-

i) Ramified B_p is a division alg.

ii) split $B_p \cong M_2(\mathbb{Q}_p)$

$$\mathbb{Q}_\infty = \mathbb{R}$$

III Shimura Curves

Let B = quarternion $\mathbb{Q}$-alg split at ∞ $= B_\infty$

M_2 = 2×2 matrices

$$SL_2(\mathbb{Z}) \subset M_2(\mathbb{Z}) \subset M_2(\mathbb{Q})$$

$$\underbrace{\mathcal{O}_1^\times}_{\text{norm 1 units}} \subset \underbrace{\mathcal{O}}_{\text{max'l order}} \subset B$$

where $\mathcal{O}_1^\times(\mathbb{R}) = SL_2(\mathbb{R})$

For $\alpha \in SL_2(\mathbb{R})$ w/ char eqn $\lambda^2 - tr(\alpha)\lambda + 1 = 0$

α is parabolic when $|tr(\alpha)| = 2 \Rightarrow \lambda \in \{\pm 1\}$

Fact $\mathcal{O}_1^\times$ has no parabolic elts

$\Rightarrow \mathcal{O}_1^\times \backslash \mathcal{H}$ is cocpt.

Dfn 4 $X(1) := \mathcal{O}_1^\times \backslash \mathcal{H}$ is called a Shimura curve

Thm 2 $X(1)$ is a coarse moduli sp of pair

(A, ι) $\quad A =$ ab surface

$\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$

Totally real number field F s.t. $[F:\mathbb{Q}] = n$

B = quaternion alg over F split at exactly 1 $\mathbb{R}$ place

$\mathcal{O}_1^\times$ is a cocpt arithmetic s.sp of $SL_2(\mathbb{R})$

$\cap|$

$\mathcal{O}$ max'l order

$\cap|$

B

$X(1) = \mathcal{O}_1^\times \backslash \mathcal{H}$ Shimura curve

= moduli sp of ab var's of dim $2n$ equipped w/ $\mathcal{O}$-actions

Thm 3 (Shimura) Let M be a totally imaginary extn of F, isomorphic to a quadratic subfield of B over F.

Let C_F = max'l ab. extn of F unramified at all the finite primes

$$\Big[\varphi : X(1) \longrightarrow \underset{\sim}{V}\Big] = \text{canonical model of } X(1)$$

complete alg curve $/C_F$

Then H_M is obtained from $C_F \cdot M$ by adjoining $\varphi(z)$, where z = regular fixed pt of M on $\mathcal{H}$

right analogue of $j(E)$

Action of $\mathrm{Gal}(H/M)$ can be described explicitly using Shimura reciprocity law.

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http://www.math.columbia.edu/~chaoli/docs/ShimuraCurves.html