Derivations of strongly rational vertex operator algebras

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Classical derivations

$D = \frac{d}{dx}$ $D = f(x)\frac{d}{dx}$.

Given an algebraic structure $(A, *)$,
(For instance, a commutative algebra $(\mathbb{C}[x], \cdot)$, a Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$, or an associative algebra $(M_n(\mathbb{C}), \cdot)$)
a derivation on A is a linear map $D : A \to A$ satisfying the "Leibniz rule":

$$D(a * b) = (Da) * b + a * (Db),$$

for all $a, b \in A$.

Derivation of semisimple Lie algebras

Definition

A finite dimensional Lie algebra $\mathfrak{g}$ over an algebraic closed field k is callled semisimple if its maximal solvable ideal $\mathrm{rad}(\mathfrak{g}) = 0$.

$\mathfrak{sl}(2,k) = ke \oplus kh \oplus kf$, with relations

$[h,e] = 2e$, $[h,f] = -2f$, $[e,f] = h$.

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Theorem (H.Weyl)

A finite dimensional Lie algebra $\mathfrak{g}$ is semisimple if and only if every finite dimensional $\mathfrak{g}$ module is a direct sum of irreducible $\mathfrak{g}$-modules.

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A derivation D of $\mathfrak{g}$ is a linear map $D : \mathfrak{g} \to \mathfrak{g}$ s.t.

$$D[x, y] = [Dx, y] + [x, Dy],$$

for all $x, y \in \mathfrak{g}$.

Derivation of semisimple Lie algebras

$$adx[y,z] = [adx(y), z] + [y, adx(z)].$$

Example

Fix $x \in \mathfrak{g}$, the linear map $adx : \mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$ is a derivation of $\mathfrak{g}$. Such derivations are called inner derivations.

Derivation of semisimple Lie algebras

Example

Fix $x \in \mathfrak{g}$, the linear map $adx : \mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$ is a derivation of $\mathfrak{g}$. Such derivations are called inner derivations.

Theorem (K.Yosida, 1938)

If a finite dimenional Lie algebra $\mathfrak{g}$ is semisimple, then all derivations of $\mathfrak{g}$ are inner derivations.

$\forall D : \mathfrak{g} \to \mathfrak{g}$, s.t. $D[x, y] = [Dx, y] + [x, Dy]$,
$\exists a \in \mathfrak{g}$ s.t. $D = ad a$.

Derivation of semisimple associative algebras

$M_n(\mathbb{C})$

A vector space A, with $\cdot : A \times A \to A$

Definition

An associative algebra A over a ring R is called semisimple if its left regular module ${}_AA$ is a direct sum of irreducible left A-modules.

Ex: (1) $\mathbb{F}[G]$ with G finite, char $\mathbb{F} = 0$ is semisimple.

(2) $M_n(\mathbb{D})$. $\mathbb{D}$ is a division ring.

Derivation of semisimple associative algebras

Definition

An associative algebra A over a ring R is called semisimple if its left regular module ${}_A A$ is a direct sum of irreducible left A-modules.

Let A be an associative algebra, and let M be a bimodule over A. A derivation of A with coefficient in M is a linear map $D : A \to M$ s.t.

$$D(a \cdot b) = a.(Db) + (Da).b$$

${}_A A_A$

$(a.x).b = a.(x.b)$

for all $a, b \in A$.

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Example

For $u \in M$, the linear map $D_u(a) := u.a - a.u$ is a derivation of A with coefficients in M. D_u is called an inner derivation.

Derivation of semisimple associative algebras

In the Hochschild cochain complex

$$0 \to M = C^0(A, M) \xrightarrow{d} C^1(A, M) \xrightarrow{d} C^2(A, M) \to \dots,$$

one has $C^n(A,M) = \{ f : \underbrace{A \times A \cdots \times A}_{n} \to M \mid f \text{ is multilinear} \}.$

$$Z^1(A, M) = \mathrm{Der}(A, M)$$

$$B^1(A, M) = \mathrm{Inn}(A, M).$$

For $f \in C^n(A,M)$,

$$(df)(a_1, a_2, \dots, a_{n+1}) = a_1 f(a_2, \dots, a_{n+1}) + \sum_{i=1}^{n} f(a_1, \dots, a_i a_{i+1}, \dots, a_{n+1})$$

$$+ (-1)^{n+1} f(a_1, \dots, a_n) . a_{n+1}$$

Derivation of semisimple associative algebras

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Theorem (Hochschild)

Let A be an associative algebra over a perfect field $\mathbb{F}$. Then TFAE:

1. *A is finite dimensional and semisimple.*

Derivation of semisimple associative algebras

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Theorem (Hochschild)

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2. *$H^1(A, M) = 0$. i.e. The derivations of A with coefficients in M are all inner.*
3. *$H^n(A, M) = 0$ for all $n \geq 1$.*

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$V \rightsquigarrow A(V)$ — Zhu's Alg.

Thm: $A(V)$ is Noetherian

(1984) Borcherds.

The Heisenberg VOA

A vertex operator algebra(VOA) is a graded vector space $V = \bigoplus_{n=0}^{\infty} V_n$ with $\dim V_n < \infty$ for all $n \geq 0$, and a linear map

$M(1,0) \simeq \mathbb{C}[x_1, x_2, x_3, \ldots]$

the state-field correspondence.

$$Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$$

$$a \mapsto Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

$\omega = \frac{1}{2} x_1^2, \ \mathbb{1} = 1.$

Jacobi Identity

($Y(a, z)$ is called the vertex operator of a) and two special elements $\mathbf{1} \in V_0$, $\omega \in V_2$ (called the Virasoro element) satisfying certain axioms.

The vacuum elt

$$V = \bigoplus_{n=0}^{\infty} V_n$$

- The graded subspace V_n is called the nth-level of V, and for $k \in \mathbb{Z}$, $\boxed{a_k b}$ is called the kth-product of a and b.

$$Y(a,z) = \sum_{k \in \mathbb{Z}} \underset{\substack{\uparrow \\ \mathrm{End}(V)}}{a_k} \cdot z^{-n-1}, \quad a_k b \qquad a_0 \approx [a, -]$$

Definition of Vertex Operator Algebras

- The graded subspace V_n is called the nth-level of V, and for $k \in \mathbb{Z}$, $a_k b$ is called the kth-product of a and b. $\mathbf{1} \in V_0.$
- For $a \in V_m$, m is called the weight of a. So in particular, $\mathbf{1}$ has weight 0 and ω has weight 2. $\omega \in V_2$

Definition of Vertex Operator Algebras

- The graded subspace V_n is called the nth-level of V, and for $k \in \mathbb{Z}$, $a_k b$ is called the kth-product of a and b.
- For $a \in V_m$, m is called the weight of a. So in particular, $\mathbf{1}$ has weight 0 and ω has weight 2.
- The first level $V_1 \subset V$ is a Lie algebra with respect to the bracket:

$$[a, b] := a_0 b,$$

for all $a, b \in V_1$.

Thm (Dong & Mason 2004) If V is strongly rational, then V_1 is a reductive Lie alg

Strongly rational VOAs

Let V be a VOA. A module over V is a graded vector space $M = \bigoplus_{n=0}^{\infty} M(n)$, together with a linear map

$A \times M \to M$

$$Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]]$$
$$a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}$$

satisfying certain axioms that are similar to the VOA axioms.

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satisfying certain axioms that are similar to the VOA axioms.

Zhu (1990)

Definition

A VOA V is called $\boxed{\text{rational}}$ if the category of V module is semisimple. i.e. every V-module is a direct sum of irreducible V-modules.

Strongly rational VOAs

Definition

A VOA V is called strongly rational, if V is of CFT-type: $V_0 = \mathbb{F}\mathbf{1}$, simple, $V \cong V'$, $\omega_2 V_1 = 0$, rational and C_2-cofinite i.e. $\dim V/span\{a_{-2}b : a, b \in V\} < \infty$.

Strongly rational VOAs

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1. The Virasoro VOA $L(c, 0)$ with $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ and $p, q \in \{2, 3, ...\}$ is strongly rational.
2. The affine VOA $L_{\hat{\mathfrak{g}}}(k, 0)$ is strongly rational.
3. The lattice VOA V_L is strongly rational.
4. The Heisenberg VOA $M_{\hat{\mathfrak{h}}}(k, 0)$ is NOT strongly rational.

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Derivation of VOAs

Let V be a VOA. A derivation D of V is a linear map $D : V \to V$ satisfying $D\mathbf{1} = 0$, $D\omega = 0$, and the "Leibniz rule":

$$D(Y(a,z)b) = Y(Da,z)b + Y(a,z)Db$$

for all $a, b \in V$. The space of derivations of V is denoted by $\mathrm{Der}(V)$.

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$Y(u,z) = \sum u_n z^{-n-1}$

u_0

for all $a, b \in V$. The space of derivations of V is denoted by $\mathrm{Der}(V)$.

Example

Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA. For any $u \in V_1$, $\boxed{u_0 : V \to V}$ is a derivation of V. It is called an inner derivation. The space of inner derivations is denoted by $\boxed{\mathrm{Inn}(V).}$

Structure of the derivation algebra

Der(A) is a Lie alg with
$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$.

Theorem (Dong & Griess 2002)

Let V be a strongly rational VOA, and assume that $V_0 + \dots + V_N$ generates V. Then the derivation Lie algebra $\mathrm{Der}(V)$ *is a direct sum of ideals* $\mathrm{Inn}(V)$ *and* $\mathrm{Inn}(V)^{\perp} = \{d \in \mathrm{Der}(V) : tr|_{V_N} D \circ u_0 = 0, \forall u \in V_1\}$.

$\mathrm{Der}(V) = \mathrm{Inn}(V) \oplus \mathrm{Inn}(V)^{\perp}$.

Structure of the derivation algebra

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Conjecture(Dong): If V is strongly rational, then $\mathrm{Der}(V) = \mathrm{Inn}(V)$.

Examples of the derivation algebra

$D\omega = 0$

- If $V = L(c,0)$ is a Virasoro VOA, then $L(c,0) = \langle \omega \rangle$

$$\mathrm{Der}(L(c,0)) = 0 = \mathrm{Inn}(L(c,0)).$$

Examples of the derivation algebra

- If $V = L(c,0)$ is a Virasoro VOA, then

$$\mathrm{Der}(L(c,0)) = 0 = \mathrm{Inn}(L(c,0)).$$

Frenkel & Zhu [1992]

$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K.$

- If $V = L_{\hat{\mathfrak{g}}}(k,0)$ is an affine VOA, then

$\mathfrak{g}$ is f.d. s.s. Lie alg.

$$\mathrm{Der}(L_{\hat{\mathfrak{g}}}(k,0)) \cong \mathrm{Der}(\mathfrak{g}) = \boxed{\mathrm{Inn}(\mathfrak{g})} = \boxed{\mathrm{Inn}(L_{\hat{\mathfrak{g}}}(k,0)).}$$

$\mathrm{Der}(L_{\hat{\mathfrak{g}}}(k,0)) \xrightarrow{\sim} \mathrm{Der}(\mathfrak{g})$

$D \longmapsto D|_{V_1 = \mathfrak{g}}$

Examples of the derivation algebra

L - A positive definite even lattice. e.g. E_8. D_{16}

FLM (1988).

- If $V = V_L$ is a lattice VOA associate to a positive definite even lattice L of rank 1 or 2, then $\boxed{\mathrm{Der}(V_L) = \mathrm{Inn}(V_L)}$.

Examples of the derivation algebra

- If $V = V_L$ is a lattice VOA associate to a positive definite even lattice L of rank 1 or 2, then $\mathrm{Der}(V_L) = \mathrm{Inn}(V_L)$.
- If $V = M_{\hat{\mathfrak{h}}}(k, 0)$ is a Heisenberg VOA associate to a vector space $\mathfrak{h}$ of dimension n, then

$$\mathrm{Der}(M_{\hat{\mathfrak{h}}}(k, 0)) \cong \underbrace{\{A \in M_n(\mathbb{C}) : A^t = -A\}}_{= O(n, \mathbb{C})},$$

while $\mathrm{Inn}(M_{\hat{\mathfrak{h}}}(k, 0)) = 0$.

Explorations of the derivation conjecture

Let V be a strongly rational VOA, we can prove the derivation conjecture under certain conditions.

Explorations of the derivation conjecture

$V_{\mathbb{R}}/\mathbb{R}$ is a sub VOA of V
s.t. $\mathbb{C}\otimes_{\mathbb{R}} V_{\mathbb{R}} = V$.

Let V be a strongly rational VOA, we can prove the derivation conjecture under certain conditions.

Theorem

If V possesses a real VOA form $V_{\mathbb{R}}$ s.t. $\mathrm{Der}(V) = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Der}(V_{\mathbb{R}})$, then $\mathrm{Der}(V) = \mathrm{Inn}(V)$.

$V_{\mathbb{Z}}$

Thank you for your attention!

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