Stable Simplicies of p-adic Representations in Bruhat-Tits Buildings

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p-adic representations and stable lattices

p-adic representations

Let G be a group and p a prime number. A p-**adic representation** of G is a group homomorphism

$$\rho\colon G \longrightarrow \mathrm{GL}_K(V),$$

from G to the group of K-linear automorphisms of a finite-dimensional vector space V over a local field K with residue characteristic p.

$\shortparallel$

e.g. $\mathbb{Q}_p$

Example

Let D_8 be the dihedral group of order 8:

$$D_8 = \left\langle r, s \;\middle|\; r^4 = s^2 = (rs)^2 = 1 \right\rangle.$$

Then the following gives an irreducible p-adic representation in $V = K^2$ (with K be any local field with residue characteristic p)

unless char $K = 2$

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Notations:

K a non-Archimedean local field; e.g. $\mathbb{Q}_p$

val the valuation on K;

K° the ring of integers $\{x \in K \mid \mathrm{val}(x) \geqslant 0\}$;

$K^{\circ\circ}$ the maximal ideal $\{x \in K \mid \mathrm{val}(x) > 0\}$;

ϖ a uniformizer, namely $K^{\circ\circ} = \varpi K^\circ$;

κ the residue field $K^\circ / K^{\circ\circ}$.

Stable lattices

$V = Ke_1 + Ke_2$

$L = K^\circ e_1 + K^\circ e_2$

For V a finite-dimensional vector space V over K, a **lattice** in V is a finitely generated K°-submodule L of V spanning the entire V.

Given a p-adic representation $\rho\colon G \to \mathrm{GL}_K(V)$, a **stable lattice** in V is a lattice which is stable under the action of G.

Given a stable lattice L, one can obtain a representative over the residue filed κ by reduction:

$$L \longrightarrow L \otimes_{K^\circ} \kappa.$$

$\rightsquigarrow \bar{\rho} : G \longrightarrow GL_\kappa(L \otimes_{K^\circ} \kappa)$

Example

Let D_8 be the dihedral group of order 8 with generators r and s as before. Let ρ be the representation

$\mathcal{K} = \mathbb{Q}_p$

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$\mathbb{Z}_p e_1 + \mathbb{Z}_p e_2$

Then the lattice $K^\circ e_1 + K^\circ e_2$ (where (e_1, e_2) is the standard basis of K^2) is a stable lattice. Its reduction is irreducible if $p \neq 2$, but if $p = 2$, it has a stable subspace $\kappa(\overline{e_1} + \overline{e_2})$.

$\rightsquigarrow \mathbb{F}_p \overline{e_1} + \mathbb{F}_p \overline{e_2}$

Stable lattices

Two lattices L and L' are **homothetic** if there is some $x \in K^\times$ such that

$$L' = xL.$$

Therefore it is reasonable to consider the set

$$S(\rho)^0 = \{\text{stable lattices of } \rho\}/_{\text{homothety}}$$

Its cardinality $h(\rho) = |S(\rho)^0|$ is called the **class number** of ρ.

Stable lattices

However, it is difficult to compute $h(\rho)$ in general.

Question

How $h(\rho)$ behaves along a totally ramified extensions E/K?

Difficulty: unlike the ideal class group of K, the homothety class set $S(\rho)^0$ is merely a set, not a group!

Known results

$\rho : G \to GL_K(V)$

K local field

ρ has stable lattice if and only if it is **precompact** (the image of ρ has compact closure).

$\mathrm{Im}(\rho) \subset GL_K^1(V) := \{ g \in GL_K(V) : \mathrm{val}(\det(g)) = 0 \}$

When ρ is precompact, it is said to has **regular reduction** if for some stable lattice L, the Jordan-Hölder constituents of the reduction of $L \otimes_{K^\circ} \kappa$ are pairwise inequivalent. Otherwise, ρ has **irregular reduction**.

$L \otimes_{K^\circ} k \supset W_1 \supset W_2 \supset \cdots$

$\parallel$

W_0

s.t. W_i / W_{i+1} irr

Example

Let D_8 be the dihedral group of order 8 with generators r and s as before. Let ρ be the representation

$$r \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad s \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then ρ has regular reduction if $p \neq 2$ and irregular reduction if $p = 2$.

$p=2$ case: $\quad \mathbb{F}_2 \bar{e}_1 + \mathbb{F}_2 \bar{e}_2 = L \otimes_{\mathcal{O}_K} k =: \bar{V}$

$W = \mathbb{F}_2 (\bar{e}_1 + \bar{e}_2) \subsetneq \bar{V} \qquad \bar{V}/W \not\cong W$ as D_8-modules.

Known results

Junecue Suh, "*Stable lattices in p-adic representations I. Regular reduction and Schur algebra*", Journal of Algebra, vol.575, 2021, pp.192-219.

Theorem 1.2

If ρ has regular reduction, the class number of the base change $\rho \otimes_K E$ in totally ramified extensions E/K is a polynomial of $[E : K]$.

Junecue Suh, "*Stable lattices in p-adic representations II. Irregularity and entropy*", Journal of Algebra, vol.591, 2022, pp.379-409.

Theorem 1.1.

If ρ has irregular reduction,the growth of the class number along a tower of finite totally ramified extensions is at least the exponential of the degree.

Stable simplices

ultrametric

A **norm** on V is a map $\alpha \colon V \to \mathbb{R} \cup \{\infty\}$ such that for any $x, y \in V$ and any $t \in K$,

(a) $\alpha(tx) = \mathrm{val}(t) + \alpha(x)$;

(b) $\alpha(x + y) \geqslant \inf\{\alpha(x), \alpha(y)\}$;

(c) $\alpha(x) = \infty$ if and only if $x = 0$.

The set of norms on V is denoted by $\mathcal{N}(V)$.

If α is a norm, then so is $\alpha + c$ for any $c \in \mathbb{R}$. Such a norm is said to be **homothetic** to α. The set of homothety classes of norms on V is denoted by $\mathscr{X}(V)$.

k-simplex in $\mathscr{X}(V)$ with a metric structure

The **Bruhat-Tits building of** $\mathrm{GL}(V)$ is the set $\mathscr{X}(V)$ equipped with a natural simplicial structure.

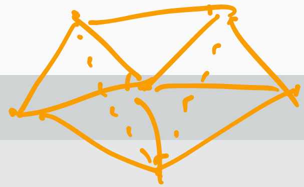
Any lattice L in V defines a norm: "distance between x and L".

$$x \in K^\times \longmapsto \sup\{\mathrm{val}(t) \mid t \in K^\times, x \in tL\}.$$

Two homothetic lattices defines two homothetic norms. Moreover, the points in $\mathscr{X}(V)$ arising from this way are precisely the **vertices** (0-simplices) in the simplicial structure of $\mathscr{X}(V)$.

$L' = xL$

$d_{L'} = d_L + \mathrm{val}(x)$



Theorem

Let $S(\rho)$ denote the set of fixed points:

$$S(\rho) = \{x \in \mathscr{X}(V) \mid g.x = x \text{ for all } g \in G\}.$$

Then $x, y \in S(\rho) \Rightarrow [x\, y] \in S(\rho)$

geodisc

$x \in S(\rho) \Rightarrow \forall$ simplex F s.t. $F \ni x$ implies $F \in S(\rho)$

(i) $S(\rho)$ *is a convex and simplicial subset.*

(ii) Its set of vertices (0-simplices) is $S(\rho)^0$.

(iii) $S(\rho)$ *is compact if and only if* $h(\rho)$ *is finite if and only if* ρ *is irreducible.*

(iv) Maximal simplices in $S(\rho)$ *has dimension* $r(\rho) - 1$ *where* $r(\rho)$ *is the length of any composition series of the reduction* $L \otimes_{K^\circ} \kappa$ *of any stable lattice* L

$L \otimes_{K^\circ} \kappa = W_0 \supset W_1 \supset \cdots \supset W_r$

arise from a $(r-1)$-simplex $[L_0, L_1, \dots, L_r]$

The simplicial set of fixed points

$\alpha \longmapsto \alpha \otimes E : \quad v' \in V \otimes E \mapsto \sup \{ \inf_k \{ \alpha(v_k) + val(t_k) \} \}$

$v' = \sum_k \underset{V}{v_k} \otimes \underset{E}{t_k}$!

The main idea is to compare the image of $S(\rho)$ under the natural embedding

$$\mathscr{X}(V) \hookrightarrow \mathscr{X}(V \otimes_K E)$$

and the simplicial subset $S(\rho \otimes_K E)$.

$\rho \otimes_K E : G \to GL_E(V \otimes_K E)$

In the regular reduction, they coincide. In the irregular case, the set $S(\rho \otimes_K E)$ will grow along the tower of finite totally ramified extensions. To control its growth, one needs to know how the simplicial ball grows.

Generalization

$\rho \rightsquigarrow \rho \oplus \rho^*$ on $V \oplus V^*$

$\rho \oplus \rho^* : G \to Sp(V \oplus V^*)$

It is often the case that a p-adic representation $\rho\colon G \to \mathrm{GL}_K(V)$ actually lands in a nice subgroup of $\mathrm{GL}_K(V)$.

Example

The vector space has a non-degenerate symplectic form and the action of G respects this form. Then ρ lands in $\mathrm{Sp}(V)$.

In general, we can consider nice subgroups of $\mathrm{GL}_K(V)$ arising as groups of K-rational points of **reductive groups**.

Any reductive group admits a **Bruhat-Tits building**, which is a metric space with a polysimplicial structure on it. The points in it has concrete interpretation for particular reductive groups.

Example

(a) The Bruhat-Tits building of $\mathrm{GL}(V)$ consists of homothety classes of norms.

(b) The Bruhat-Tits building of $\mathrm{Sp}(V)$ consists of self-dual norms.

Stable simplices

Theorem

Suppose the representation ρ factors through (the group of K-rational points of) a reductive group whose Bruhat-Tits building is $\mathscr{B}$.

Let $S(\rho)$ denote the set of fixed points:

$$S(\rho) = \{x \in \mathscr{B} \mid g.x = x \text{ for all } g \in G\}.$$

Then

(i) $S(\rho)$ is a convex and simplicial subset.

(ii) $S(\rho)$ is compact if and only if ρ is has no non-trivial subrepresentation of the same type.

$\rho: G \to Sp(V)$

V has no non-trivial G-stable symplectic subspace.

e.g. $\rho: G \xrightarrow{\text{irr}} GL(V) \to \rho \oplus \rho^*$ is not irr but $S(\rho \oplus \rho^*)$ is compact.

Stable simplices

It is reasonable to study the subset $S(\rho)$ in general. In particular, the number $h(\rho)$ of vertices in $S(\rho)$.

Conjecture (work in-progress)

If ρ has "regular" reduction, then $h(\rho)$ grows as a polynomial along totally ramified extensions. If ρ has "irregular" reduction, then $h(\rho)$ grows at least exponentially along totally ramified extensions.

Remark: The meaning of "regular" / "irregular" reduction in this conjecture is different from the $\mathrm{GL}(V)$ case.

In $Sp(V)$ case. regular means the following: L stable lattice

symplectic form take integers value on L and induce non-deg form on $L \otimes_{K^\circ} k$.

$G \to Sp(L \otimes_{K^\circ} k)$ $\quad$ $W_0 \supset W_1 \supset \cdots \supset W_r$ totally isotropic stable.

Thanks!

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