① Relations

② Properties

③ Equiv. Classes

(Q.4) $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$

$R_1 = \{(a,2), (a,3), (b,1), (b,3), (c,4)\} \subseteq A \times B$

$R_2 = \{(1,b), (1,c), (2,a), (2,b), (3,c), (4,a), (4,c)\} \subseteq B \times A$

(a,b) (c,a) (c,c)

$R \subseteq A \times A$

$x R y$ if for some $n \in B$ $x R_1 n$ & $n R_2 y$

What is $R \subseteq A \times A$

if for some $n \in B$
$(a,n) \in R_1$ & $(n,a) \in R_2$
$n = 2$ ✓

So ① Is $(a,a) \in R$? ✓ Is $(c,a) \in R$?

$(a,n) \in R_1$

Is $(a,b) \in R$? $(n,b) \in R_2$ ✓ Is $(c,b) \in R$?

Is $(a,c) \in R$? ✓ Is $(c,c) \in R$?

Is $(b,a) \in R$?

Is $(b,b) \in R$?

Is $(b,c) \in R$?

(eg) $A = \mathbb{Z}$, $B = \mathbb{N}$

$R_1 = \{(x,n) \mid |x| = n\} \subseteq A \times B$

$\in (-2,2), (2,2)$

$R_2 = \{(n,x) \mid x = -n\} \quad \subseteq B \times A$

$\in (1,-1), (2,-2), \ldots$

R on $\mathbb{Z} \times \mathbb{Z}$ similarly as before

i.e. xRy if for some $n \in \mathbb{N}$ xR_1n & nR_2y

Question: $(-1, 3) \in R$?

Answer Suppose it was, then by defn there exists a

$n \in \mathbb{N}$ st $\quad \underbrace{(-1, n) \in R_1}$ & $(n,3) \in R_2$

Since $(-1,n) \in R_1$, therefore $n = |-1| = 1$

Since $(n,3) \in R_2$, therefore $3 = -n$, i.e. $n = -3$

Contradiction!

Hence, $(-1, 3) \notin R$.

Question: Is $(-1,-1) \in R$

Answer: If it was then for some $m \in \mathbb{N}$ we would have

$(-1, m) \in R_1$ & $(m,-1) \in R_2$

$\hookrightarrow m = |-1|$ $\quad \hookrightarrow -1 = -m$

$= 1$

since $(-1,1) \in R_1$

so $m = 1$ works i.e. $(1,-1) \in R_2$

$\therefore (-1,-1) \in R$ $\boxtimes$

Determine Relation Properties

* Relation R on lines in $\mathbb{R}^2$: $l_1 \, R \, l_2$ if

$l_1 \parallel l_2$ (parallel to each other)

* Reflexive: Yes! a line is $\parallel^{le}$ to itself!

* Symmetric: Yes! $l_1 \parallel l_2 \Leftrightarrow l_2 \parallel l_1$

* Transitive: Yes!

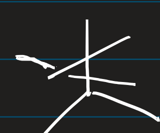
$l_1 \parallel l_2$ & $l_2 \parallel l_3$

so $l_1 \parallel l_3$

* Relation R on lines in $\mathbb{R}^3$: $l_1 \, R \, l_2$ if

l_1 don't intersect l_2 and is not parallel

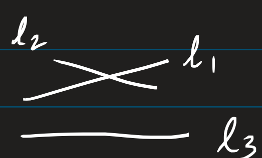
either to l_2



* NOT reflexive

* Symmetric!

* Transitive? NO!



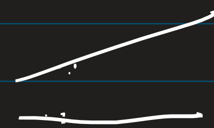
$l_1 \, R \, l_3$ $l_3 \, R \, l_2$

BUT $l_2 \not R \, l_1$

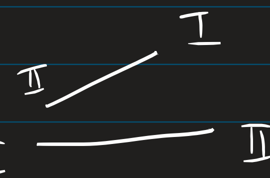
* $l_1 R l_2$ $\quad$ $l_2 R l_1$

$\not\Downarrow$

$l_1 \not R \, l_1$



* Symmetric: If $l_1 R l_2$ i.e.



l_1 & l_2 NOT intersecting
NOT parallel

l_2 & l_1 NOT intersecting
NOT parallel

$l_2 R l_1$

* Reflexive: Is l_1 NOT intersecting itself ✗

NOT parallel to itself ✗

↳ same slope as itself

If true, then yes, reflexive

lines have at least 1 pt in common

i.e. a line intersects itself at INFINITELY many points

Equivalence Classes

$A = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}$ & $a+b\sqrt{2} \neq 0\}$

xRy if $\frac{x}{y} \in \mathbb{Q}$; what are the equivalence classes

* $\frac{x}{x} = 1 \in \mathbb{Q}$ * $\frac{x}{y} \in \mathbb{Q}$, $\frac{y}{x} \in \mathbb{Q}$

* $\frac{x}{y} \in \mathbb{Q}$, $\frac{y}{z} \in \mathbb{Q}$ $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} \in \mathbb{Q}$

What are its equivalence classes?

$$[0] \underset{\text{def}}{=} \{x \mid x - 0 \equiv 0 \bmod n\} = \{x \mid x \equiv 0 \bmod n\}$$
$$= \{x \mid x = nk,\ k \in \mathbb{Z}\}$$
$$= \{nk \mid k \in \mathbb{Z}\}$$

$$[a+b\sqrt{2}] = \{x \in A \mid xR(a+b\sqrt{2})\}$$

def^n of rel^n

$$= \left\{x \in A \;\middle|\; \frac{x}{a+b\sqrt{2}} \in \mathbb{Q}\right\}$$

def^n of set belonging

$$= \left\{x \in A \;\middle|\; \frac{x}{a+b\sqrt{2}} = k,\quad k \in \mathbb{Q} \setminus \{0\}\right\}$$

some

What is the def^n of an eq class?
$[a] = \{b \mid aRb\}$

$$\overset{\text{algebra}}{=} \{x \in A \mid x = k(a+b\sqrt{2}),\ k \in \mathbb{Q}\setminus\{0\}\}$$

rewriting set

$$= \{k(a+b\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\}$$

$$[1+\sqrt{2}] = \{k(1+\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\}$$

$$= \{k + k\sqrt{2} \mid k \in \mathbb{Q}\setminus\{0\}\}$$

$$m = 2 \cdot \frac{m}{2}$$

$$[2] = \{2k \mid k \in \mathbb{Q}\setminus\{0\}\} = \mathbb{Q}\setminus\{0\}$$

$$[3] = \{3k \mid k \in \mathbb{Q}\setminus\{0\}\} = \mathbb{Q}\setminus\{0\}$$

When is $[a+b\sqrt{2}] = [c+d\sqrt{2}]$

when $a+b\sqrt{2} \in [c+d\sqrt{2}]$

i.e. $a+b\sqrt{2} = k(c+d\sqrt{2})$ if $k \in \mathbb{Q}\setminus\{0\}$

$a = ck$ & $b = dk$. for some $k \in \mathbb{Q}\setminus\{0\}$

$\frac{a}{c} \in \mathbb{Q}$ $\quad$ $\frac{b}{d} \in \mathbb{Q}$

Therefore $[a+b\sqrt{2}] \neq [c+d\sqrt{2}]$ iff $\boxed{\frac{a}{c} \neq \frac{b}{d}}$

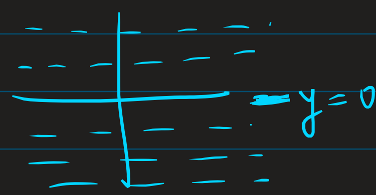
eg $\quad l_1 \parallel l_2 \quad$ question $\quad [x\text{-axis}]$

$$[x\text{-axis}] = \{\, l \mid l \parallel x\text{-axis} \,\}$$

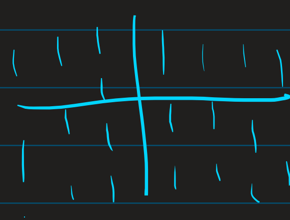
$$= \{\, l \mid \text{slope of } l \text{ is } 0 \,\}$$

$$= \{\, y = mx + b \mid m = 0 \,\}$$

$$= \{\, y = b \,\}$$



$$= \{\, y = b \mid b \in \mathbb{R} \,\}$$



$$[y\text{-axis}] = \{\, x = c \mid c \in \mathbb{R} \,\}$$

$$[y = x] = \{\, y = x + d \mid d \in \mathbb{R} \,\}$$