HW6 Problem 6

non-trivial

G then there exists a group H & a normal subgroup $N \trianglelefteq H$ st $\underbrace{G}_{\text{im}\,\phi} \cong H/N$ $\longrightarrow \ker\phi$

$(H \neq G,\ N \neq \{e\})$; $\quad G/\{e\} \cong G$

FLI applied to the $id: G \longrightarrow G$

$H = G \times G = \{(g_1, g_2) \mid g_1, g_2 \in G\}$

w/ coordinate-wise multiplication.

i.e. WANT a SURJECTIVE hmph

$$\pi: G \times G \longrightarrow G \qquad \text{(canonical projection)}$$
$$(g_1, g_2) \longmapsto g_1$$

$$\pi((g_1, g_2)(g_3, g_4)) = \pi(g_1 g_3, g_2 g_4)$$
$$= g_1 g_3$$
$$= \pi(g_1, g_2)\, \pi(g_3 g_4)$$

What is im π ?

π is surjective

$$\pi : G \times G \longrightarrow G$$
$$(g, h) \longmapsto g$$

For any $g \in G$ (what element in $G \times H$ should we pick such that π of that is g?)

$(g, e) \in G \times G$

$\pi(g, e) = g$

Next thing to check then ker π is not trivial

$\ker \pi \leq G \times G$

To show $\ker \pi$ is a bigger group than just $\{(e, e)\}$.

i.e. give an element $(g, h) \in G \times G$ w/ $(g, h) \neq (e, e)$ st

$$\pi(g,h)=e$$

i.e. $(g,h)\in\ker\pi$

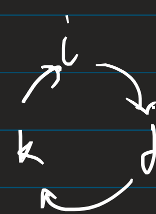
$\pi(e,g)$ for some $g\neq e$

$=e$

$$(G\times G)\Big/\underbrace{\ker\pi}_{\cong\, G} \cong \operatorname{im}\pi = G$$

Problem 5 (Remark).

$$\underline{\underline{Q_8}} = \langle i,j \mid i^2=j^2,\ i^4=1,\ \overbrace{ij}^{k}=-ji\rangle$$



Quaternion group of order 8

There are ONLY $\underline{\underline{5}}$ groups of order 8 (up to isomorphism)

* $\mathbb{Z}/8\mathbb{Z}$
* $\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
* $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$
* $D_8 \cong \underbrace{\mathbb{Z}/4\mathbb{Z}}_{\langle r\rangle} \overset{\text{semidirect}}{\rtimes} \underbrace{\mathbb{Z}/2\mathbb{Z}}_{\langle s\rangle}$ $\boxed{srs=r^3}$
* $\underline{\underline{Q_8}}$

$$D_{2n} = \mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

$\langle r \rangle$ $\quad$ $\langle s \rangle$

$srs = r^{n-1}$

List of all groups of order upto 10 upto isomorphism

$\{e\}$ order 1

$\mathbb{Z}/p\mathbb{Z} = C_p$ order $p = 2, 3, 5, 7$

$e \neq x \in G$, $|G| = p$. a prime

$o(x) \mid |G| = p \Rightarrow o(x) = \cancel{1}$ or p

Hence $o(x) = p \Rightarrow |\langle x \rangle| = p$

$\langle x \rangle \leq G \Rightarrow G = \langle x \rangle$ $\boxtimes$

<u>Order 4</u>

* $\mathbb{Z}/4\mathbb{Z} = C_p$
* $V_4 / K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Order 6

* $C_6 \cong \mathbb{Z}/6\mathbb{Z}$
* S_3

Order 8 — Just gave

Order 9 — $\mathbb{Z}/9\mathbb{Z}$

$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$

p^2 — $\mathbb{Z}/p^2\mathbb{Z}$

p^2 — $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

Order 10

* $\mathbb{Z}/10\mathbb{Z}$
* D_{10} (symmetries of a regular pentagon)

$2p$ — $\mathbb{Z}/2p\mathbb{Z}$

$2p$ — D_{2p}

Why care about ideals?

— You can construct fields using them!

$\mathbb{R}[x]$ ⇝ ring of polynomials w/ coefficients in $\mathbb{R}$.

$x^2+1 \in \mathbb{R}[x]$

$$(x^2+1) = \{ f(x^2+1) \mid f \in \mathbb{R}[x] \}$$

$$\phi : \mathbb{R}[x] \longrightarrow \mathbb{C} \qquad \underline{\underline{\text{surjective}}}$$

$$x \longmapsto i$$

$$(a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0)$$

$$\longmapsto a_n i^n + a_{n-1} i^{n-1} + \cdots + a_1 i + a_0$$

$$(3x+2) \longmapsto 3i+2$$

$$x^2+4x+1 \longmapsto i^2+4i+1 = 4i$$

$$a+bx \longmapsto a+bi$$

$$\mathbb{R}[x] / \underbrace{\ker\phi}_{(x^2+1)} \cong \mathbb{C}$$

(It's like in the quotient we made $x^2+1=0$

$\underline{x}$, $\boxed{x^2+1=0}$)

$\mathbb{C} := \mathbb{R}[x]/(x^2+1)$

$\cancel{V}$ I $\qquad$ $v_1 + v_2 \in \cancel{V}\ I$ (adding two "vectors" makes sense)

$\alpha \in \cancel{\mathbb{R}}\ R$ $\qquad$ $\alpha \cdot v \in \cancel{V}\ I$ (scalar multiplication makes sense)

I an ideal is like a "subspace" of a ring.

(2-dim vector space) $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ (abelian group)

(vector subspaces) $\qquad$ (subgroups)

(other structure) $\qquad R \qquad$ (ring)

(sub-other structure) $\qquad$ (subring)

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HW7 Pb4

$|A| = 6$ $\longrightarrow$ commutative ring

A is an abelian group of order 6. (wrt addition)

$0 \neq x \in A$, $o(x) \,\big|\, |A| = 6$

$\underbrace{o(x) = 2 \text{ or } 3}$

you'll find that this isn't an integral domain.

Pb 5

A, B rings. $\longrightarrow$ groups wrt $+$

A ring hmph — group hmph + another condition.

$f : A \longrightarrow B$ (ring)

* $\boxed{f(x+y) = f(x) + f(y)}$ (group)

* $f(xy) = f(x)\, f(y)$