Fibonacci Numbers

Note on proof writing.

Geometric Series

Product of limits

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① There exists a $p \neq 2$ st $p \mid n$, then n can be written as a sum of 2 or more consecutive numbers.

$n = 2^r s$, s is odd $\to$ $s = 2t+1$

$2^r - t$, $2^r - t + 1$

$2-1$ $2+1$

$n = \underbrace{2^r (2t+1)}$

$2 (2 \cdot 1 + 1) = 6 = 1+2+3$ $2^r - t$

2 $+2$

2^r 4 t

$(2 \cdot 4 + 1) = 9 = 4+5$

$(2t+1)$ $t + (t+1)$

$\underbrace{2^r > t}$ or $\underbrace{2^r \leq t}$

$\Updownarrow$

$t - 2^r \geq 0$

$2^r \geq t+1$

$2^r - t - 1 \geq 0$

$(t - 2^r) + (t - 2^r + 1) + \cdots + = 2^r (2t+1)$

$(2^r - t - 1)(2^r - t)$ 2^r

(66) $A = \{0, 1\}$, $B = \{1, 2\}$ $\qquad A = \{0,1\}$, $B = \{2\}$.

$A = 1$, $B = 0$.

$P(A \cup B)$ $\qquad \boxed{A = 2, \; B = 1}$

$$2^{|A \cup B|} = 2^{|A| + |B|} = 2 = 8$$

$$P(A) \cup P(B) = 2^A + 2^B \qquad 2 + 1 \text{ or.}$$

$$= 5$$

Prove: $\quad a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

Induction — $P(n)$ statement

$$P(n): a + ar + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Base Step: $P(1)$ is true.

assume $P(1)$ is true

$n = 1$ $\qquad \boxed{ar^{1-1} = \dfrac{a(1-r^1)}{1-r}}$ $\qquad$ WRONG!

$$a = a$$

$\square$

$P(1)$: $A = B$

$ar^{1-1} = a$

LHS: A separately

so LHS = RHS

RHS: B separately $\quad \dfrac{a(1-r)}{1-r} = a$

Fibonacci Numbers

↪ Naming is an issue, documented by Indians many ages ago.

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha = \phi = \frac{1+\sqrt{5}}{2}$$

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$\alpha + \beta = 1$

$\alpha\beta = -1$

$\alpha^2 = 1+\alpha$

$\beta^2 = 1+\beta$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha > 1 \qquad \frac{1+\sqrt{5}}{2}$$

$$1.5 = \frac{1}{2} + \frac{\sqrt{4}}{2} < \frac{1}{2} + \frac{\sqrt{5}}{2}$$

Therefore $\boxed{\frac{1}{\alpha} < 1}$

Geometric progression sum

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

Suppose $a = 1$

$$\boxed{1 + r + \cdots + r^{n-1} = \frac{1-r^n}{1-r}}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } 0<x<1$$

$0 < x < 1$, then $\lim_{n\to\infty} x^n = 0$

$$1+r+\cdots+r^{n-1} = \frac{1-r^n}{1-r}$$

suppose $0<r<1$, then $\sum r^n = \lim_{n\to\infty} \frac{1-r^n}{1-r}$ $\to 0$

$$= \frac{1}{1-r}$$

If $0<r<1$, then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

What if we had $\sum_{n=0}^{\infty} ar^n = a\sum_{n=0}^{\infty} r^n$

$$= \frac{a}{1-r}$$

Now, what about $\sum_{n=0}^{\infty} r^{2n} = \sum_{n=0}^{\infty} (r^2)^n$

$(r<1)$

$(r^2<1)$

$$= \frac{1}{1-r^2}$$

$$\sum_{n=0}^{\infty} \Box^n = \frac{1}{1-\Box} \quad \text{if } 0<\Box<1$$

Finally, what can you say about

$$\sum_{n=0}^{\infty} r^{2n+1} = \sum_{n=0}^{\infty} r^{2n} \cdot r$$

$$= r \sum_{n=0}^{\infty} r^{2n}$$

$$= r \cdot \frac{1}{1-r^2}$$

$$= \frac{r}{1-r^2}$$

$r = \frac{1}{\alpha}$; $\alpha = \frac{1+\sqrt{5}}{2}$

$$\frac{1}{\alpha} = \frac{2}{1+\sqrt{5}} = \frac{2}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}}$$

$\alpha\beta = -1$

$\underbrace{\frac{1}{\alpha}} = \underbrace{-\beta}$

$$= \frac{2(1-\sqrt{5})}{1^2 - (\sqrt{5})^2}$$

$$= \frac{2(\sqrt{5}-1)}{5-1} = \frac{\sqrt{5}-1}{2}$$

$$= -\beta$$

$\frac{1}{\alpha} < 1$, so!

$$\sum_{n=0}^{\infty} \Big(\underbrace{\frac{1}{\alpha}}_{r}\Big)^n = \underbrace{\frac{1}{1-\frac{1}{\alpha}}}_{\frac{1}{1-r}} = \frac{1}{1+\beta}$$

$$= \frac{1}{\beta^2}$$

$\beta^2 = 1+\beta$; $\alpha\beta = -1$

$\alpha^2\beta^2 = 1$

$$= \alpha^2$$

* If $0 < \square < 1$, then $\sum_{n=0}^{\infty} a\square^n = \frac{1 \cdot a}{1-1\square}$ //

* $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ ||

* $\alpha + \beta = 1$, $\alpha\beta = -1$ $\rightsquigarrow$ α & β are roots of $x^2 - x - 1$ ||

* $\alpha^2 = \alpha + 1$ & $\beta^2 = \beta + 1$

$$\sum \frac{1}{\alpha^{2n}} \quad \& \quad \sum \frac{1}{\alpha^{2n+1}}$$

Suppose you have sequences a_n & b_n

$$\left[\begin{array}{l} \lim_{n\to\infty} a_n = a \\ \lim_{n\to\infty} b_n = b \end{array}\right. \qquad \begin{array}{l} \lim_{n\to\infty} 1 + \frac{1}{n} = 1 \\ \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \end{array}$$

Do you know what $\underbrace{\lim_{n\to\infty} a_n b_n}$ is?

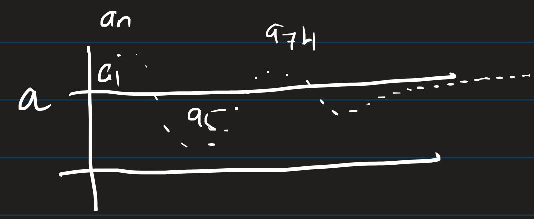
$$\lim_{n\to\infty} a_n b_n = ab$$

We know $\lim_{n\to\infty} 1 + \frac{1}{n} = \lim_{n\to\infty} \frac{n+1}{n} = 1$ //

So we also know $\lim_{n\to\infty} 1 + \frac{k}{n} = 1$

$$= \lim_{n\to\infty} \frac{n+k}{n} \quad //$$

$a_1, a_2, \dots, a_k, a_{k+1}, \dots$



If $\lim_{n\to\infty} a_n = a$

then $\boxed{\lim_{n\to\infty} a_{n+k}} = a$

$a_k, a_{k+1}, a_{k+2}, \dots$

$\frac{F_{n+1}}{F_n}$

$a_n = \frac{n+1}{n} = \frac{b_{n+1}}{b_n}$ where $b_n = n$

$\lim_{n\to\infty} a_n = 1$

$$\lim_{n\to\infty} \boxed{\frac{n+k}{n}} = \lim_{n\to\infty} \frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{n+k-1}{n+k-2}\cdot\frac{n+k}{n+k-1}$$

eg $\frac{5}{2} = \underbrace{\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}}$

$$= \lim_{n\to\infty}\frac{n+1}{n}\cdot\lim_{n\to\infty}\frac{n+2}{n+1} \cdots \lim_{n\to\infty}\frac{n+k-1}{n+k-2}\cdot\lim_{n\to\infty}\frac{n+k}{n+k-1}$$

$$= \underbrace{\lim_{n\to\infty} a_n}_{\alpha}\cdot\underbrace{\lim_{n\to\infty} a_{n+1}}_{\alpha}\cdots\underbrace{\lim_{n\to\infty} a_{n+k-1}}_{\alpha}\underbrace{\lim_{n\to\infty} a_{n+k}}_{\alpha}$$

$= 1$

What would be $\lim_{n\to\infty}\frac{F_{n+k}}{F_n}$ knowing $\lim_{n\to\infty}\frac{F_{n+1}}{F_n} = \alpha$

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \lim_{n\to\infty} \frac{F_{n+1}}{F_n} \cdot \frac{F_{n+2}}{F_{n+1}} \cdots \frac{F_{\square}}{F_{\square}}$$

$n = n+k-1$

$$\boxed{\lim_{\substack{n\to\infty \\ n+k-1\to\infty}} \frac{F_{n+k}}{F_{n+k-1}} = \alpha}$$

$$\lim_{n\to\infty} \frac{F_{n+k}}{F_n} = \lim_{n\to\infty} \left(\frac{F_{n+1}}{F_n} \cdot \frac{F_{n+2}}{F_{n+1}} \cdots \frac{F_{n+k}}{F_{n+k-1}} \right)$$

$= \alpha \cdot \alpha \cdots \alpha$ ✓ (How many?)

$= \alpha^{k}$