* HW questions ~ 9.48, 9.42 (b), 9.56
* For Quiz
* Remarks
* Midterm – 4.3

Midterm 4 — (2) & (3)

(2) $2^2+4^2+\cdots+(2n)^2 \sim =$

$$\left[\cancel{1+2^2+\cdots+(2n)^2 \sim \frac{(2n)(2n+1)(4n+1)}{6}}\right]$$

$$2^2+4^2+\cdots+(2n)^2 = 2^2\left(1+2^2+\cdots+n^2\right)$$

$$= 4\cdot\frac{n(n+1)(2n+1)}{6}$$

(3) $1^2+3^2+\cdots+(2n-1)^2 \sim$

$$\left[\cancel{1+2^2+\cdots+(2n-1)^2 \sim \frac{(2n-1)(2n)(4n-1)}{6}}\right]$$

$$\boxed{1^2+2^2+\cdots+(2n)^2} = \left(1^2+3^2+\cdots+(2n-1)^2\right) + \left(2^2+4^2+\cdots+(2n)^2\right)$$

$$\frac{2n(2n+1)(4n+1)}{6} \qquad\qquad 4\,\frac{n(n+1)(2n+1)}{6}$$

subtract ~ $\dfrac{4n^3-n}{3}$

(1) $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

$n=2$

$n=k$

$n=(2n)$

$n=4n$

$1^2+2^2+\cdots+(2n)^2 = \dfrac{(2n)((2n)+1)(2(2n)+1)}{6}$

$\dfrac{2n(n+1)(2n+1)}{6} \neq \dfrac{2n(2n+1)(4n+1)}{6}$

eg. $1^2+2^2 = 5$

$1^2+2^2+3^2+4^2 = 5+9+16 = 30$

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* U, $A, B \subseteq U$

$A^c = \bar{A} = U \setminus A = U - A = \{x \in U \mid x \notin A\}$

Question: if $\bar{A} = \bar{B}$

is $A = B$? Yes

$+1+1$

$+1+1$

$+1+1$

$+1+1$

What is $\bar{\bar{A}} = A$?

Right!

Oh! I need to show $A \subseteq B$ and $B \subseteq A$.

* Injective & Surjective

→ Injectivity $\quad f : A \to B$ is injective

if $f(x) = f(y) \Rightarrow x = y$ [Good def^n for problems]

Contrapositive

$$x \neq y \Rightarrow f(x) \neq f(y).$$

"distinct elements get mapped to distinct elements"

→ Surjectivity $\quad f : A \to B$ is surjective

if $f(A) = B$

Recall $f(A) = \{ f(x) \mid x \in A \} \subseteq B$

f is surjective iff $\quad B \subseteq f(A)$

$$\boxed{b \in B, \text{ then } \quad b \in f(A) \quad \text{i.e. } \exists x \in A \text{ st } \; b = f(a)}$$

* Let A and B be finite sets

Question: How many functions are there from A to B?

$$B^A = \{ f : A \to B \mid f \text{ a function} \}$$

$$|B^A| = |B|^{|A|}$$

$\#$ possible bijection $\rightsquigarrow$ $|A| = |B|$

$\rightsquigarrow$ $|A|!$

$f : A \to B$

$a \in A$, $f(a)$ = what is this in B.

$|A| = n$, $A = \{a_1, \dots, a_n\}$

$$f(a_1) \quad f(a_2) \quad \dots \quad f(a_n)$$

$$m = |B| \quad m \quad \dots \quad m$$

$$\underbrace{m \cdot m \cdots m}_{n = |A| - \text{times}} = m^n = |B|^{|A|}$$

$f : A \to B$, $|A| = |B| = n$

$A = \{a_1, \dots, a_n\}$

$f(a_1) = \lambda = f(a_2)$

$$f(a_1) \quad f(a_2) \quad \cdots \quad f(a_n)$$

$$n \quad n-1 \quad n-2 \quad \cdots \quad 1$$

$$n(n-1)(n-2)\cdots 1 = n!$$

Remark

* $|A| = n$

Then the set of all bijections b/w (between) A and A

i.e. $\{f : A \to A \mid f \text{ bijective}\}$ is denoted S_n or $Sym(n)$

and called the Symmetric Group.

* $|B^A| = |B|^{|A|}$

In the special case $B = \{0,1\}$ i.e. $|B| = 2$

$$B^A = 2^A = \{f : A \to \{0,1\} \mid f \text{ a function}\}$$

$$|B^A| = |2^A| = 2^{|A|}$$

$\mathcal{P}(A) = \{S \mid S \subseteq A\}$ is st $|\mathcal{P}(A)| = 2^{|A|}$

Question: Is 2^A & $\mathcal{P}(A)$ bijective? Yes!

A, B $\quad |A| = |B| = 3$ $\quad A = \{a_1, a_2, a_3\}$, $\quad B = \{b_1, b_2, b_3\}$ $\quad f(a_i) = b_i$

$$I : \mathcal{P}(A) \longrightarrow 2^A$$
$$S \longmapsto f_S$$

so f_S is a function $f_S : A \longrightarrow \{0,1\}$

$$f_S(a) = \begin{cases} 1 & a \in S \\ 0 & a \notin S \end{cases}$$

"indicator function of S"

Can create $J : 2^A \longrightarrow \mathcal{P}(A)$ st $\boxed{\begin{aligned} I \circ J &= id_{2^A} \\ J \circ I &= id_{\mathcal{P}(A)} \end{aligned}}$

$$f \longmapsto \{x \in A \mid f(x) = 1\}$$
$$= f^{-1}(1)$$

I & J are inverses

Key word relating ↑ and Chapter 10 — Continuum Hypothesis

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Question: $f: A \longrightarrow \underline{\underline{B}}$ is a function that's not surjective

Can we "edit" the target of f such that f is surjective?

$$f(A) = \{ f(x) \mid x \in A \}$$

$$f : A \longrightarrow \underset{\substack{\shortparallel \\ T}}{f(A)}$$ is actually surjective

We want $T = f(A)$ (but it is! because we made it so!)

Question: $f: \underline{\underline{A}} \longrightarrow B$ is a function that's not injective

Can we "edit" the source of f such that f is injective ?

Hard question!

We will conflate or not differentiate b/w elements of A that have the same value under f

Fix: Introduce an equivalence relation on A, R

$$xRy \quad \text{iff} \quad f(x) = f(y)$$

fact: R is an equivalence relation

Consider the set of equivalence classes with respect to R

$A/R = \{ [x] \mid x \in A \}$, called the quotient of A by R.

read as "A mod / modulo R"

$$\tilde{f} : A/R \longrightarrow B$$
$$\tilde{f}([x]) := f(x)$$

Claim: $\tilde{f}$ is injective

Consider $[x], [y] \in A/R$ st

$$\tilde{f}([x]) = \tilde{f}([y])$$
$$\parallel \qquad\qquad \parallel$$
$$f(x) = f(y)$$

$$x \, R \, y$$

$$[x] = [y]$$

Hence $\tilde{f}$ is injective.

So together $\quad f : A \to B$ and R as above

$\tilde{f} : A/R \to B$ is injective

$\tilde{f} : A/R \to \tilde{f}(A/R) = \{ \tilde{f}([x]) \mid x \in A \}$

is surjective $= \{ f(x) \mid x \in A \} = f(A)$

A/R is bijective to $f(A)$.

$*$ $$f : \mathbb{R} \longrightarrow \mathbb{R}$$
$$x \longmapsto x^2$$

f is neither injective, nor surjective

$2, -2 \mapsto 4$ a negative no. is not a square

Fix: R is the eq

$$xRy \Leftrightarrow f(x) = f(y) \Leftrightarrow x^2 = y^2 \Leftrightarrow x = \pm y$$

i.e. $[x] = \{x, -x\} = [-x] = [|x|]$

$$\mathbb{R}/R = \{[x] \mid x \in \mathbb{R}\} = \{[|x|] \mid x \in \mathbb{R}\} \longleftrightarrow [0, \infty)$$

$\cancel{[-2]} = [2]$ $\longrightarrow$ 2

$$f(\mathbb{R}) = \{f(x) \mid x \in \mathbb{R}\} = \{x^2 \mid x \in \mathbb{R}\}$$
$$= [0, \infty)$$

$$\tilde{f} : [0, \infty) \longrightarrow [0, \infty)$$
$$x \longmapsto x^2$$

$$\tilde{f}^{-1} : x \longmapsto \sqrt{x}$$