Fundamental Groups

Topological, Étale, Tannakian

UCSC Graduate Colloquium

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Table of contents

Galois Theory

Quick Review: Field Extensions

Algebraic Extensions

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An element of an algebraic extension $L|k$ is *separable* over k if its minimal polynomial is separable; the extension itself is called *separable* if every element of L is separable over k.

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Algebraic Closure

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From now on by "a separable closure of k" we shall mean its separable closure in some chosen algebraic closure.

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The set of elements of L that are fixed by the action of $\mathrm{Aut}(L|k)$, denoted $L^{\mathrm{Aut}(L|k)}$, forms a field extension of k. So $k \to L$ factors as

$$\begin{tikzcd} L^{\mathrm{Aut}(L|k)} \arrow[rr] & & L \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

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Galois Correspondence for Finite Extensions

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Let $L|k$ be a finite Galois extension with Galois group G. The maps

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The extension $M|k$ is Galois if and only if $H := \mathrm{Gal}(L|M)$ is a normal subgroup of G; in this case we have $\mathrm{Gal}(M|k) \cong G/H$.

Galois Group of Infinite Galois Extensions

Let $K|k$ be a Galois extension of fields. The Galois groups of *finite* Galois subextensions of $K|k$ together with the homomorphisms $\phi_{ML} : \mathrm{Gal}(M|k) \to \mathrm{Gal}(L|k)$ form an inverse system whose inverse limit is isomorphic to $\mathrm{Gal}(K|k)$.

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Profinite Groups

Profinite groups are endowed with a natural topology. They are compact and totally disconnected. Moreover, the open subgroups are precisely the closed subgroups of finite index.

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Galois Correspondence for Infinite Extensions (Krull)

Let L be a subextension of the Galois extension $K|k$. Then $\mathrm{Gal}(K|L)$ is a closed subgroup of $G = \mathrm{Gal}(K|k)$.

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In particular, these all apply to $k_s|k$ and the absolute Galois group of k.

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We start with a base field k, of which we fix separable and algebraic closures $k_s \subseteq \bar{k}$. Let $\mathrm{Gal}(k) := \mathrm{Gal}(k_s|k)$.

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So we may consider the finite set $\mathrm{Hom}_k(L, k_s)$ which is endowed by a natural left action of $\mathrm{Gal}(k)$ given by $(g, \phi) \mapsto g \circ \phi$ for $g \in \mathrm{Gal}(k)$, $\phi \in \mathrm{Hom}_k(L, k_s)$.

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Here Galois extensions give rise to $\mathrm{Gal}(k)$-sets isomorphic to some finite quotient of $\mathrm{Gal}(k)$.

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A finite dimensional k-algebra A is étale (over k) if it is isomorphic to a finite direct product of separable extensions of k.

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Topological Fundamental Group

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Galois Covers

Covers

We define the full subcategory $\mathsf{Cov}(X)$ of the category Top/X where $p : Y \to X$ are subject to the condition: each point of X has an open neighbourhood V for which $p^{-1}(V) \cong V \times I$, where I is discrete.

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Even (properly discontinuous) action

Let G be a group acting continuously from the left on a topological space Y. The action of G is even if each point $y \in Y$ has some open neighbourhood U such that the open sets gU are pairwise disjoint for all $g \in G$.

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If G is a group acting evenly on a connected space Y, the projection $p_G : Y \to G\backslash Y$ turns Y into a cover of $G\backslash Y$.

Galois Covers

Henceforth we fix a base space X which will be assumed locally connected.

Automorphism Group

Given a cover $p : Y \to X$, its automorphisms are to be automorphisms of Y as a space over X, i.e. topological automorphisms compatible with the projection p.

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Note that for each point $x \in X$ the group $\mathrm{Aut}(Y|X)$ maps the fibre $p^{-1}(x)$ onto itself, so $p^{-1}(x)$ is equipped with a natural action of $\mathrm{Aut}(Y|X)$.

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Conversely, if G is a group acting evenly on a connected space Y, the automorphism group of the cover $p_G : Y \to G \backslash Y$ is precisely G.

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Intermediate Covers

For a Galois cover $p : Y \to X$, a connected cover $q : Z \to X$ is an intermediate cover if the following diagram commutes for some $f : Y \to Z$

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "p"'] & Z \arrow[d, "q"] \\ & X \end{tikzcd}$$

Galois Covers

Galois Correspondence

Let $p : Y \to X$ be a Galois cover with $G = \mathrm{Aut}(Y|X)$. The maps

$$\begin{tikzcd} \{\text{Intermediate covers as before}\} \arrow[d, shift left, "Z \mapsto \mathrm{Aut}(Y|Z)"] \\ \{\text{Subgroups of } G\} \arrow[u, shift left, "H\backslash Y \mapsfrom H"] \end{tikzcd}$$

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The cover $q : Z \to X$ is Galois if and only if $H := \mathrm{Aut}(Y|Z)$ is a normal subgroup of G; in this case we have $\mathrm{Gal}(Z|X) \cong G/H$.

Topological Fundamental Group

(Topological) Fundamental Group

For a topological group X, the *(topological) fundamental group of X with base point x* $\pi_1(X, x)$ is the group of homotopy classes of loops based at $x \in X$, where the group operation is given by concatenation of loops: for loops $\gamma_1, \gamma_2 : [0, 1] \to X$,

$$(\gamma_1 \bullet \gamma_2)(x) = \begin{cases} \gamma_2(2x) & 0 \leqslant x \leqslant 1/2 \\ \gamma_1(2x - 1) & 1/2 \leqslant x \leqslant 1 \end{cases}$$

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If X is *path-connected*, i.e. any two points x and y may be joined by a path ρ, then $\pi_1(X, x)$ is non-canonically isomorphic to $\pi_1(X, y)$ via $\gamma \mapsto \rho \bullet \gamma \bullet \rho^{-1}$.

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Given a cover $p : Y \to X$, the fibre $p^{-1}(x)$ over a point $x \in X$ carries a natural action by the group $\pi_1(X, x)$.

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Fibre Functor

Fix a pointed space (X, x). We define a functor

$$\mathrm{Fib}_x : \mathsf{Cov}(X) \longrightarrow \left\{ \begin{matrix} \text{category of sets equipped} \\ \text{with a left } \pi_1(X,x)\text{-action} \end{matrix} \right\}, \quad (p : Y \to X) \longmapsto p^{-1}(x)$$

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Call a cover $Y \to X$ *finite* if it has finite fibres; for connected X these have the same cardinality, called the *degree* of the cover.

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Let X as above, let x and y be two base points and consider the universal covers $\widetilde{X}_x$ and $\widetilde{X}_y$ that represent Fib_x and Fib_y. Then there's a bijection between homotopy classes of paths joining y to x, sometimes denoted as $\pi_1(X; y, x)$, and $\mathrm{Isom}_X(\widetilde{X}_x, \widetilde{X}_y)$.

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Furthermore, the monodRomy action is translated as follows: for any automorphism $\phi \in \mathrm{Aut}(\mathrm{Fib}_x)$, and a cover $Y \to X$, there's by definition a morphism $\mathrm{Fib}_x(Y) \to \mathrm{Fib}_x(Y)$ induced by ϕ which then gives a natural left action of $\mathrm{Aut}(\mathrm{Fib}_x)$ on $\mathrm{Fib}_x(Y)$.

Étale Fundamental Group

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Étale Covers

Schemes

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ having an open covering $\{U_i\}_{i \in I}$ such that for all i the locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ are isomorphic to affine schemes $(\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$.

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The underlying topological space of the fibre X_p is homeomorphic to the subspace $\phi^{-1}(p)$ of the underlying space of X.

Finite Étale Covers

A morphism $\phi : X \to S$ is *flat* if, for every $x \in X$, the induced map of stalks $\mathcal{O}_{S,\phi(x)} \to \mathcal{O}_{X,x}$ makes $\mathcal{O}_{X,x}$ a flat $\mathcal{O}_{S,\phi(x)}$-module.

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A morphism $\phi : X \to S$ is called *finite* if it's affine, i.e. for any affine open $\operatorname{Spec} A$ of S, $\phi^{-1}(\operatorname{Spec} A)$ is an affine open of X, say $\operatorname{Spec} B$, and the corresponding map of rings $\phi^\sharp : A \to B$ makes B a finitely generated module over A.

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A surjective finite étale morphism is called a *finite étale cover*.

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The fibres of ϕ are spectra of finite étale algebras if and only if its geometric fibres are of the form $\mathrm{Spec}(\Omega \times \cdots \times \Omega)$, i.e. they are finite disjoint unions of points defined over Ω.

Fibre Functor

For a scheme S denote by $\mathsf{F\acute{E}t}_S$ the full subcategory of Sch/S whose objects are finite étale covers of S.

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We have thus defined a set-valued functor

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Étale Fundamental Group

Given a pointed scheme $(S, \bar{s})$, we define the *étale fundamental group* $\pi_1(S, \bar{s}) := \mathrm{Aut}(\mathrm{Fib}_{\bar{s}})$.

Étale Fundamental Group

Fibre Functor Revisited

Note that there's a natural left action of $\mathrm{Aut}(\mathrm{Fib}_{\overline{s}})$ on $\mathrm{Fib}_{\overline{s}}(X)$ for each finite étale cover X of S.

Étale Fundamental Group

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induces an equivalence of categories. Here connected covers correspond to sets with transitive $\pi_1(S, \overline{s})$-action, and Galois covers to finite quotients of $\pi_1(S, \overline{s})$.

Revisiting Galois Theory

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Although in the above example the fibre functor is identified with the functor $\mathrm{Hom}(\operatorname{Spec} k_s, -)$, this does not mean that the fibre functor is representable, for k_s is not a finite étale k-algebra. However, it is the union of its finite Galois subextensions which are.

Pro-representable Functors

Let $\mathcal{C}$ be a category, and F a set-valued functor on $\mathcal{C}$. We say that F is *pro-representable* if there exists an inverse system $P = (P_\alpha, \phi_{\alpha\beta})$ of objects of $\mathcal{C}$ indexed by a directed partially ordered set Γ and a natural isomorphism

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Homotopy Exact Sequence

Let X be a quasi-compact and geometrically integral scheme over a field k; that is, $X_{\bar{k}} := X \times_k \bar{k}$ is integral, for an algebraic closure $\bar{k}$ of k.

Étale Fundamental Group

Paths

Let S be a connected scheme. Given two geometric points $\bar{s} : \mathrm{Spec}(\Omega) \to S$ and $\bar{s}' : \mathrm{Spec}(\Omega') \to S$,

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$$1 \longrightarrow \pi_1(X_{\bar{k}}, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow \mathrm{Gal}(k_s|k) \longrightarrow 1$$

induced by the maps $X_{\bar{k}} \to X$ and $X \to \operatorname{Spec} k$ is exact.

Under Base Change

Let $k \subseteq K$ be an extension of algebraically closed fields, and let X be a proper integral scheme over k. Denote $X_K \coloneqq X \times_k K$. The map $\pi_1(X_K, \overline{x}_K) \to \pi_1(X, \overline{x})$ induced by the projection $X_K \to X$ is an isomorphism for every geometric point $\overline{x}$ of X.

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Comparison with Topological Fundamental Group

Let X be a connected scheme of finite type over $\mathbf{C}$. The analytification functor $(Y \to X) \mapsto (Y^{\mathrm{an}} \to X^{\mathrm{an}})$ induces an equivalence of the category of finite étale covers of X with that of finite topological covers of X^{an}.

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$$\pi_1^{\mathrm{top}}\widehat{(X^{\mathrm{an}}, \overline{x})} \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \overline{x})$$

Tannakian Fundamental Group

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Representations of Affine Group Schemes

Affine Group Schemes

Let k be a field. An *affine group scheme* G over k is a functor from $\mathsf{Alg}_k \to \mathsf{Grp}$ that, when viewed as a set-valued functor, is representable by some k-algebra A.

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Coalgebras & Comodules

A *coalgebra* over k is a k-vector space equipped with a comultiplication $\Delta : A \to A \otimes_k A$ and a counit map $\iota : A \to k$ subject to the coassociativity and counit axioms. Here, Δ and ι are only assumed to be maps of k-vector spaces.

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Let A be a coalgebra over a field k. A *right A-comodule* is a k-vector space M together with a k-linear map $\rho : M \to M \otimes_k A$ such that certain diagrams commute.

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Representations of Affine Group Schemes (contd.)

If moreover M is finite dimensional over k and we fix a k-basis of M, giving a representation of G becomes equivalent to giving a morphism of group schemes $G \to \mathbf{GL}_n$, i.e. a morphism of group-valued functors.

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Given a k-coalgebra A, the k-linear dual $A^* := \mathrm{Hom}_k(A, k)$ is a k-algebra with unit that is not necessarily commutative.

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Given a finite dimensional k-coalgebra A, the contravariant functor $V \mapsto V^*$ induces an anti-isomorphism between the category of finitely generated right A-comodules and that of finitely generated left A^*-modules.

Tensor Categories

A *tensor category (with a unit)*, i.e. a *symmetric monoidal category*, is a category $\mathscr{C}$ together with

- a functor $- \otimes - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$; ("tensor product")
- a natural isomorphism $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$; ("associator")
- an object $\mathbb{1} \in \mathscr{C}$; ("unit")
- a natural isomorphism $\lambda_X : \mathbb{1} \otimes X \to X$; ("left unitor")
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A *tensor functor* between two tensor categories $\mathscr{C}$ and $\mathscr{D}$ is a functor $F : \mathscr{C} \to \mathscr{D}$ together with an isomorphism $\phi_0 : \mathbb{1}_{\mathscr{D}} \to F(\mathbb{1}_{\mathscr{C}})$ and a natural isomorphism $\phi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ such that certain diagrams commute.

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We call a tensor category *rigid* if for each object X, there's a *dual* X^*, that is there exist morphisms $\varepsilon_X : X \otimes X^* \to \mathbb{1}$ and $\delta_X : \mathbb{1} \to X^* \otimes X$ so that certain diagrams commute.

Revisiting Comodf_A

Let A be a coalgebra over a field k, and ω the forgetful functor from Comodf_A to Vecf_k.

Assume that there is a tensor category structure on Comodf_A for which ω becomes a tensor functor, where Vecf_k carries its usual tensor structure.

Tensor Categories

Revisiting Comodf_A

Let A be a coalgebra over a field k, and ω the forgetful functor from Comodf_A to Vecf_k.

Assume that there is a tensor category structure on Comodf_A for which ω becomes a tensor functor, where Vecf_k carries its usual tensor structure.

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- If moreover the tensor category structure on Comodf_A is rigid, then A has the structure of a Hopf algebra.
- Assume moreover the tensor category structure on Comodf_A is commutative, and ω respects the commutativity constraints. Then A is a commutative Hopf algebra, and Comodf_A becomes equivalent to the category Rep_G of finite dimensional representations of the associated affine group scheme G.

More on Rep_G

Observe that given a commutative k-algebra R, the forgetful functor ω on Rep_G induces a tensor functor $\omega \otimes R : V \mapsto V \otimes_k R$ with values in the tensor category of R-modules.

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The rule $\phi \mapsto \phi^*$ induces a bijection between group scheme homomorphisms $G \to H$ and tensor functors $F : \mathsf{Rep}_H \to \mathsf{Rep}_G$ satisfying $\omega_G \circ F = \omega_H$.

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$$\nabla : \mathcal{V} \to \Omega_{X/K} \otimes \mathcal{V}$$

(that is, curvature $\nabla_1 \circ \nabla = 0$, where $\nabla_1 : \Omega_{X/K} \otimes \mathcal{V} \to \Omega^2_{X/K} \otimes \mathcal{V}$)

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by subbundles stable under ∇ such that each $\mathcal{V}_i/\mathcal{V}_{i+1}$ is a trivial bundle with connection.

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$$F_b^{\mathrm{dR}} : \mathsf{Un}(X) \to \mathsf{Vecf}_K, \ (\mathcal{V}, \nabla) \mapsto \mathcal{V}_b$$

that associates to each vector bundle its stalk at b.

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Here, we also have the notion of (torsor of) paths

$$\pi_1^{\mathrm{dR}}(X; b, x) \coloneqq \mathbf{Isom}^{\otimes}(F_b^{\mathrm{dR}}, F_x^{\mathrm{dR}})$$

Fin.

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