P Power sets cardinality, cartesian products

intersection

$\to$ indexed

$|A| = ?$ $A = \{1, 2, \dots, n\}$

$|P(A)| = 2^{|A|}$

$P(A) \to ?$ ✓ $P(A) = \{B \mid B \subseteq A\}$ set-builder notation.

$|P(A)| = 2^{|A|}$ // $\to$ a surprising fact?

Binomial theorem?

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

Remember? Or have you seen this?

$$\to \frac{n!}{r!\,(n-r)!}$$

$n! = 1 \cdot 2 \cdots n$

ways we choose r things from n things without repitition

$P(A) \sim B \subseteq A$ $\to$ $n = |A|$; $0 \leq |B| \leq n$ agreed?

0 elements in the set $\sim$ 1 $\phi \to \binom{n}{0} = \frac{n!}{n!(n-0)!} = 1$

1 elements in the set $\sim \binom{n}{1}$ //

A =

proof

2 elements in the set $\sim \binom{n}{2}$

⋮

r n element $\sim \binom{n}{r}$

⋮

n element $\sim \binom{n}{n}$

$$|P(A)| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{r} + \cdots + \binom{n}{n} = (1+1)^n$$

$$= 2^n$$ //

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$$

$n=1$

LHS. $1+x$

$$\text{RHS} = \sum_{r=0}^{1} \binom{n}{r} x^r = \binom{1}{0} x^0 + \binom{1}{1} x$$

$$= 1 + x \checkmark$$

assume works for n

$$(1+x)^{n+1} = (1+x)^n (1+x) = \left(\sum_{r=0}^{n} \binom{n}{r} x^r\right)(1+x)$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r + x \cdot \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r + \sum_{r=0}^{r} \binom{n}{r} x^{r+1}$$

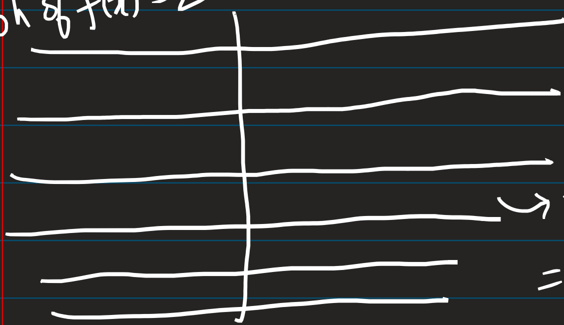
$=$ PAUSE (reindex the sum, take sums of $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r}$)

$$= (1+x)^{n+1}$$

Cartesian Products//..

$\underline{\underline{A \times B}} \rightsquigarrow \mathbb{R}^2 = \underline{\underline{\mathbb{R}}} \times \underline{\underline{\mathbb{R}}} = \{(x,y) \mid x,y \in \mathbb{R}\}$

graph of $f(x)=2$



$\mathbb{R}\times\{2\}$

$\mathbb{R}\times\{1\}$

$\hookrightarrow$ x-axis $= \mathbb{R}\times\{0\}$

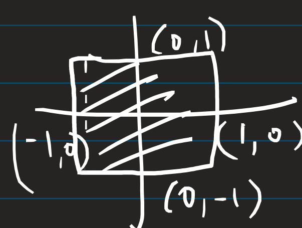
$\underbrace{\mathbb{R}\times\mathbb{Z}}$, $\mathbb{R}\times\mathbb{Q} = \{(x,p) \mid x\in\mathbb{R}, p\in\mathbb{Q}\}$

$\rightarrow \{(x,n) \mid x\in\mathbb{R}, n\in\mathbb{Z}\} = \bigcup_{n\in\mathbb{Z}} \mathbb{R}\times\{n\}$

$\hookrightarrow$ times (no cross product)

$\mathbb{R}^2 =$  $\mathbb{R} = \bigcup_{y\in\mathbb{R}} \mathbb{R}\times\{y\}$

Intersections.



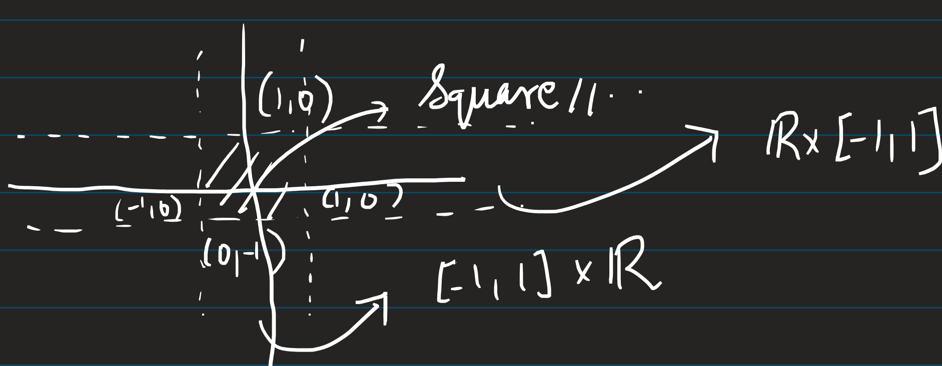
square of side length $\underline{\underline{2}}$

Describe it as $\underline{\underline{\text{a set}}}$

$S = [\overset{-1}{\cancel{0}},1] \times [\overset{-1}{\cancel{0}},1]$

$= \{(x,y) \mid -1 \le x,y \le 1\}$

$S = \{(x,y) \mid \max\{|x|,|y|\} \le 1\}$



$S = \underline{\mathbb{R}\times[-1,1]} \cap \underline{[-1,1]\times\mathbb{R}} = \underbrace{[-1,1]\times[-1,1]}$

ϕ, $\{\phi\}$ $\phi \subseteq A$ $A = \{1, 2\}$ $C = \{\phi, 1, 2\}$

$|\phi| = 0$ $\phi \in A$ $B = \{3, 4\}$ $D = \{\phi, 3, 4\}$

$|\{\phi\}| = 1$ $\underbrace{A \cap B = \phi}$ $C \cap D = \{\phi\}$ have an element in common.

$\rightarrow$ they have NO element in common

A, $P(A) = \{B \mid B \subseteq A\} \ni \phi$ $A = \{1, 2, 3\}$

"shadow of ϕ"

$\phi \subseteq A$ ✓

assume $\phi \not\subseteq A$

ϕ has an element $\alpha \alpha \alpha$

$[B \not\subseteq S$

B has an element that is not in S$]$

$A \times B$, $C \subseteq A$, $D \subseteq B$ is $C \times D \subseteq A \times B$?

$1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$ ✓ Yes

$C \times D \subseteq A \times B$; $(x, y) \in C \times D$ and show $(x, y) \in A \times B$

$x \in C \subseteq A$ $\Rightarrow$ $x \in A$ $\Rightarrow$ $(x, y) \in A \times B$

$y \in D \subseteq B$ $y \in B$

Are all subsets of $A \times B$ of the form $C \times D$ where $C \subseteq A$, $B \subseteq D$?

It is no, the answer

$A\times B$, $\quad |P(A\times B)| = 2^{|A\times B|} = 2^{|A|\cdot|B|}$

subset of $A\times B$ is $2^{|A|\cdot|B|}$ //

$|\{C\times D \mid C\subseteq A,\ D\subseteq B\}| = (\#\text{ possibilities of } C)$

$\cdot\ (\#\text{ possibilities of } D)$ //

$= |P(A)|\cdot|P(B)| = 2^{|A|}\cdot 2^{|B|} = 2^{|A|+|B|}$ //

$\mathbb{R}^2$, $\quad \Delta = \{(x,x)\mid x\in\mathbb{R}\} \leftarrow$

$y=x$

$\neq$

$U\times V$, $\quad U,V\subseteq \mathbb{R}$

Suppose $\Delta = U\times V$

① $U=V$ //

② $U=\mathbb{R}$

$\Delta=\mathbb{R}^2$ not true!

① $x\in U$, $y\in V$, then $(x,y)\in U\times V=\Delta$

$\Rightarrow x=y\in V \Rightarrow U\subseteq V$

$V\subseteq U$; $\quad U=V$

$\Delta = \underset{\sim}{U} \times \underset{\sim}{U}$

if $|U| > 1$, then $x, y \in U$ such that $x \neq y$

$(x, y) \in U \times U$ but $(x, y) \notin \Delta$.

so $|U| = 1$ but $U \times U = \{(x, x) \mid x \in U\}$

BUT $|\Delta| > 1$, contradiction //