Demystifying the p-adics

an invitation to the non-Archimedean world

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- Historically, and perhaps surprisingly, p-adic theory yielded a proof of the first of the Weil Conjectures, the question on rationality of the concerned zeta function, by Dwork.
- Their weirdness just makes them very fun!

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and other miscellaneous things...

The p-adic Norm: a non-standard absolute value

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Let's fix a prime number p, and consider any rational number $r \in \mathbb{Q}$, then we can write

$$r = p^k \frac{m}{n}$$

for some $k \in \mathbb{Z}$, such that $p \nmid m, n$.

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Define

$$|\cdot|_p : \mathbb{Q} \to \mathbb{R}_+$$

as

$$|r|_p = \left| p^k \frac{m}{n} \right|_p \coloneqq p^{-k} \quad \text{and} \quad |0|_p \coloneqq 0$$

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$$|r|_p = \left| p^k \frac{m}{n} \right|_p := p^{-k} \quad \text{and} \quad |0|_p := 0$$

The p-adic norm satisfies the *strong triangle inequality*

$$|r + s|_p \leq \max\{|r|_p, |s|_p\}, \quad \text{for any } r, s \in \mathbb{Q}$$

This inequality implies the regular triangle inequality

We call such a norm *non-Archimedean*.

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The natural numbers are **bounded** with respect to $|\cdot|_p$

$$|n|_p = |1 + \cdots + 1|_p \leq \max\{|1|_p, \ldots, |1|_p\} = |1|_p = 1$$

for any $n \in \mathbb{N}$.

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$$d_p(r, s) \coloneqq |r - s|_p$$

A metric induced by a non-Archimedean norm is called an *ultrametric*.

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We define, for any $r, s \in \mathbb{Q}$

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Therefore the strong triangle inequality

$$d_p(r, s) \leq \max\{d_p(r, t), d_p(t, s)\}$$

is also called the *ultrametric inequality*.

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$$a_0 = 1, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$$

is a Cauchy sequence of rationals such that $a_n^2 \to 2$.

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Yes, $(\mathbb{Q}_p, |\cdot|_p)$ is the completion of the normed space $(\mathbb{Q}, |\cdot|_p)$.

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- For $n = 3$, any solution mod 7^3 is necessarily also a solution mod 7^2. So, let $x_3 \coloneqq x_2 + 7^2 y = 3 + 1 \cdot 7 + 7^2 y$; solving the equation for y, you get $y \equiv 2 \bmod 7$. Therefore, we let $x_3 = 3 + 1 \cdot 7 + 2 \cdot 7^2 = 108$.

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Iteratively, we have define a sequence of rationals (x_n), where

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This is a Cauchy sequence with respect to the 7-adic norm, and it clearly doesn't converges in $\mathbb{Q}$. Because if it did, that would be a rational number $x \in \mathbb{Q}$ such that $x^2 \equiv 2 \bmod 7^n$, for every $n \in \mathbb{N}$.

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$$\mathbb{Z}_p^\times \coloneqq \{x \in \mathbb{Q}_p \mid |x|_p = 1\}$$

Ostrowski's theorem classification of completions of $\mathbb{Q}$

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So, the only completions of $\mathbb{Q}$ are $\mathbb{Q}$ (with respect to the trivial norm), $\mathbb{R}$ and $\mathbb{Q}_p$, for any prime p.

Constructing $\mathbb{Q}_p$: construction II, 1

The completion construction is a bit opaque. We construct $\mathbb{Q}_p$ more algebraically.

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Think Taylor series/holomorphic functions. Let the coefficients of the “Taylor expansion" of any $x \in \mathbb{Z}_p$ notationally be $a_n(x)$.

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We can now define the p-adic metric via this p-adic norm.

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Think Laurent series/meromorphic functions

$$\mathbb{Q}_p := \mathrm{Frac}(\mathbb{Z}_p) = \left\{ \sum_{n \geq -N} a_n p^n \;\middle|\; 0 \leq a_n \leq p-1, \ \forall n \right\}$$

$|\cdot|_p$ lifts, and $\mathbb{Q}$ is dense in $\mathbb{Q}_p$.

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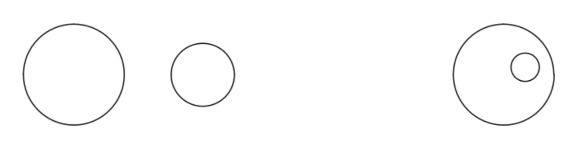

Metric topology of $\mathbb{Q}_p$: topological nature

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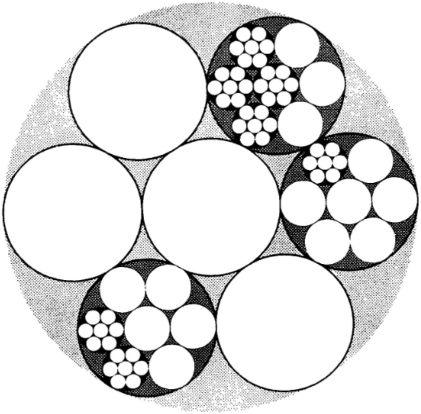
Let's use this visualize $\mathbb{Z}_p$, for $p = 7$ and $p = 3$!



$\mathbb{Z}_7$

Artist's conception of the 3-adic unit disk.

Drawing by A.T. Fomenko of Moscow State University, Moscow, U.S.S.R.

Metric topology of $\mathbb{Q}_p$: Euclidean Models of $\mathbb{Z}_p$

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You can construct Cantor-like sets $\mathcal{C}^p$, where numbers can be written in base $2p-1$; starting with the unit interval, and continuing recursively, divide into $2p-1$ pieces and delete the second piece. Taking the intersection of the resulting pieces gives you $\mathcal{C}^p$.The following map

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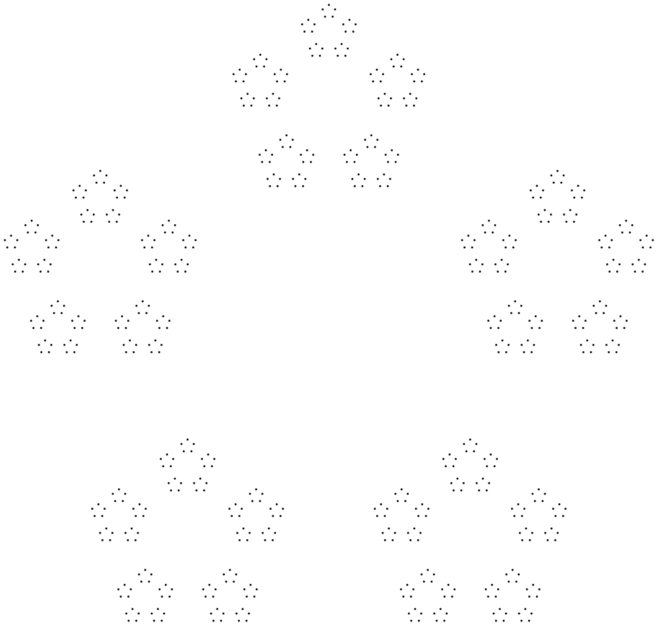
gives a homeomorphism between $\mathbb{Z}_p$ and the cantor set $\mathcal{C}^p$.

two-dimensional

Will skip the technical details, not that hard! Here's one model of $\mathbb{Z}_5$.



Functions in $\mathbb{Q}_p$: series & power series

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Series are simpler to study with respect to $|\cdot|_p$

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convergence of series

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A p-adic series

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Logarithms

The series

$$-\log_p(1-x) := \sum_{n=1}^{\infty} \frac{x^n}{n}$$

converges in the region $|x|_p < 1$

Functions in $\mathbb{Q}_p$: continuous functions $C(\mathbb{Z}_p, \mathbb{Q}_p)$

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While there exists no nice description of $C([0,1], \mathbb{R})$, there does exist one for $C(\mathbb{Z}_p, \mathbb{Q}_p)$!

Theorem (Mahler, 1961)

Functions in $\mathbb{Q}_p$: Mahler's Theorem

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$$f(x) = \sum_{k=0}^{\infty} a_n(f) \binom{x}{n}$$

Hensel's Lemma: statement(s)

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Let $f(x) \in \mathbb{Z}_p[x]$, and suppose there exists a $a \in \mathbb{Z}_p$ such that

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Let $f(x) \in \mathbb{Z}_p[x]$, and suppose there exists a $a_1 \in \mathbb{Z}_p$ such that $|f(a_1)|_p < 1$ and $|f'(a_1)|_p = 1$. Then, for each $n \geq 1$

$$a_{n+1} \coloneqq a_n - \frac{f(a_n)}{f'(a_n)}$$

defines a convergent sequence whose limit $\alpha \in \mathbb{Z}_p$ is the unique p-adic integer such that $|\alpha - a_1| < 1$ and $f(\alpha) = 0$. Then there exists a unique $\alpha \in \mathbb{Z}_p$ such that $\alpha \equiv a \bmod p$ and $f(\alpha) = 0$.

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Polynomial Form of Hensel’s Lemma (Lifting factors mod p)

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Let $f(x) \in \mathbb{Z}_p[x]$, and suppose there exist $G(x), H(x) \in \mathbb{F}_p[x]$ such that

$$f(x) \equiv G(x)H(x) \bmod p$$

with $(G, H) = 1$, and $G(x)$ monic. Then there exist polynomials $g(x), h(x) \in \mathbb{Z}_p[x]$ such that $g(x) \equiv G(x) \bmod p$, $h(x) \equiv H(x) \bmod p$ with $g(x)$ monic and

$$f(x) = g(x)h(x)$$

Hensel's Lemma: applications

roots of unity

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For any prime p and any positive integer m not divisible by p, there exists a primitive m^{th} root of unity in $\mathbb{Q}_p$ if and only if m divides $p-1$.

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Recall $\mathbb{R}^\times / \mathbb{R}^{\times 2} \cong \mathbb{Z}/2\mathbb{Z}$.

Local and Global: towards the principal

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an example of "local to global"

A number $x \in \mathbb{Q}$ is a square if and only if it is a square in every $\mathbb{Q}_p$, $p \leq \infty$.

Local and Global: the principal, in practice

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Local-Global Principal

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The existence or non-existence of solutions in $\mathbb{Q}$ (global solutions) of a diophantine equation can be detected by studying, for each $p \leq \infty$, the solutions of the equation in $\mathbb{Q}_p$ (local solutions).

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Theorem (Hasse-Minkowski)

Let

$$F(x_1, x_2, \ldots, x_n) = \sum_{i,j} c_{ij} x_i x_j \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$$

be a quadratic form (that is, a homogeneous polynomial of degree 2 in n variables). The equation

$$F(x_1, x_2, \ldots, x_n) = 0$$

has non-trivial solutions in Q if and only if it has non-trivial solutions in $\mathbb{Q}_p$ for each $p \leq \infty$.

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But, you can still hope for local information telling you something about the global picture.

Beyond $\mathbb{Q}_p$: a quick download

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- You can make sense of ramification index and residue degree, and so talk about ramified and unramified extensions.

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- The unit closed balls are the analogues of rings of integers, and they're local rings, similar to $\mathbb{Z}_p$.
- You can make sense of ramification index and residue degree, and so talk about ramified and unramified extensions.
- What do they look like? Set theoretically, with each extension, you allow for fractional powers in the Laurent series expansion. The closure allows any fractional power.

Beyond $\mathbb{Q}_p$: a quick download

some quick facts

- Unexpectedly, $\mathbb{Q}_p$ is not algebraically closed.
- But unlike $\mathbb{R}$, it's not a finite extension away from being so.
- So there's a whole host of finite extensions of $\mathbb{Q}_p$ that can be studied, they're called local number fields.
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Beyond $\mathbb{Q}_p$: irreducibility criterions and extensions

Gauss's Lemma in $\mathbb{Q}_p$

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Suppose that $f(x) \in \mathbb{Z}_p[x]$ factors in $\mathbb{Q}_p[x]$, so that $f(x) = g(x)h(x)$, with $g(x), h(x) \in \mathbb{Q}_p[x]$ and non-constant.

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Let $f(x) \in \mathbb{Z}_p[x]$ be a monic polynomial whose reduction mod p is irreducible in $\mathbb{F}_p[x]$. Then $f(x)$ is irreducible over $\mathbb{Q}_p$.

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For each integer $n \geq 1$ there is an extension of $\mathbb{Q}_p$ which has degree exactly n and which "comes from" the unique extension of degree n of the finite field $\mathbb{F}_p$.

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Beyond $\mathbb{Q}_p$: extension of $|\cdot|_p$

a finite extension of $\mathbb{Q}_p$ is a complete ultrametric space

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Let $K/\mathbb{Q}_p$ be a finite extension of degree n. The function $|\cdot|_p^K : K \longrightarrow \mathbb{R}_+$ defined by

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Problem: $\overline{\mathbb{Q}}_p$ isn't complete! (Contrast with $\overline{\mathbb{R}} = \mathbb{C}$.)

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As you will hope from my experiences with classic algebraic number theory, $n = ef$.

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Unramified Extensions

For each f there is exactly one unramified extension of degree f. It can be obtained by adjoining to $\mathbb{Q}_p$ a primitive $(p^f - 1)$-st root of unity.

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The result: $\mathbb{C}_p$ and $\mathbb{C}$ are $\mathbb{Q}$-isomorphic.

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The result: $\mathbb{C}_p$ and $\mathbb{C}$ are $\mathbb{Q}$-isomorphic. So essentially, $\mathbb{C}_p$ is just $\mathbb{C}$ with a very weird topology.

$\mathbb{C}_p$: has rich algebra but not big enough for all of analysis

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For the purposes of functional analysis, we want to work over *spherically complete* normed fields, this a normed field where every nested sequence of closed balls has a non-empty intersection.

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$\mathbb{C}_p$: has rich algebra but not big enough for all of analysis

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Fin.

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