No Periodic Geodesics in Jet Space

Alejandro Bravo Doddoli

Math Department, UCSC

April 4, 2022

Out Line

- we will answer a question asked by Enrico Le Donne:

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.
  - Bijection between subRiemannian geodesics and (F, I).

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.
  - Bijection between subRiemannian geodesics and (F, I).
  - Classification of subRiemannian geodesics.

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.
  - Bijection between subRiemannian geodesics and (F, I).
  - Classification of subRiemannian geodesics.
  - Proof of theorem.

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.
  - Bijection between subRiemannian geodesics and (F, I).
  - Classification of subRiemannian geodesics.
  - Proof of theorem.
- Cut points in J^k.

Out Line

- we will answer a question asked by Enrico Le Donne: Does the space of k-jets, denoted here by J^k, have periodic geodesics?

Theorem

J^k does not have periodic geodesics.

- Out line
  - J^k as subRiemannian manifold.
  - Bijection between subRiemannian geodesics and (F, I).
  - Classification of subRiemannian geodesics.
  - Proof of theorem.
- Cut points in J^k.
- Action-Angles for the Hamiltonian geodesic flow on J^k.

Space of k-jets

- The k-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is its k-th order Taylor expansion at x_0.

Space of k-jets

- The k-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is its k-th order Taylor expansion at x_0. We will encode this k-jets as a $k+2$-tuple of real numbers as follows:

$$\begin{aligned}(j^k f)(x_0) &= (x_0, f^k(x_0), f^{k-1}(x_0), \ldots, f^1(x_0), f(x_0)) \\ &= (x, u_k, \ldots, u_1, u_0).\end{aligned}$$

Space of k-jets

- The k-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is its k-th order Taylor expansion at x_0. We will encode this k-jets as a $k+2$-tuple of real numbers as follows:

$$
\begin{aligned}
(j^k f)(x_0) &= (x_0, f^k(x_0), f^{k-1}(x_0), \dots, f^1(x_0), f(x_0)) \\
&= (x, u_k, \dots, u_1, u_0).
\end{aligned}
$$

- J^k is endowed with a natural rank 2 distribution $\mathcal{D} \subset T\mathcal{J}^k$ characterized by the k Pfaffian equations

Space of k-jets

- The k-jet of a smooth function $f : \mathbb{R} \to \mathbb{R}$ at the point $x_0 \in \mathbb{R}$ is its k-th order Taylor expansion at x_0. We will encode this k-jets as a $k+2$-tuple of real numbers as follows:

$$(j^k f)(x_0) = (x_0, f^k(x_0), f^{k-1}(x_0), \dots, f^1(x_0), f(x_0)) \\ = (x, u_k, \dots, u_1, u_0).$$

- J^k is endowed with a natural rank 2 distribution $\mathcal{D} \subset T\mathcal{J}^k$ characterized by the k Pfaffian equations

$$\theta^1 = u_1 dx - du_0, \;\; \theta^2 = u_2 dx - du_1, \; \dots, \; \theta^k = u_k dx - du_{k-1}.$$

We define $\mathcal{D} := \ker \{\theta^1, \dots, \theta^k\}$.

Space of k-jets

- The **alternate** coordinates θ_j for J^k with an index swapping and sign change:

$$\theta_i = \sum_{i=0}^{i} (-1)^i \frac{x^{k-i}}{(k-i)!} u_{k-i}.$$

Space of k-jets

- The **alternate** coordinates θ_j for J^k with an index swapping and sign change:

$$\theta_i = \sum_{i=0}^{i} (-1)^i \frac{x^{k-i}}{(k-i)!} u_{k-i}.$$

- D is globally framed by two vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}. \tag{1}$$

Space of k-jets

- The **alternate** coordinates θ_j for J^k with an index swapping and sign change:

$$\theta_i = \sum_{i=0}^{i} (-1)^i \frac{x^{k-i}}{(k-i)!} u_{k-i}.$$

- D is globally framed by two vector fields

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}. \tag{1}$$

- A subRiemannian structure on J^k is defined by declaring these two vector fields to be orthonormal .

Bijection between geodesic and (F, I)

- Let $F(x)$ be a polynomial of degree k or less.

Bijection between geodesic and (F, I)

- Let $F(x)$ be a polynomial of degree k or less.
- I is a maximum closed interval, called Hill interval, for x such that $F^2(x) < 1$, if x is in the interior of I.

Bijection between geodesic and (F, I)

- Let $F(x)$ be a polynomial of degree k or less.
- I is a maximum closed interval, called Hill interval, for x such that $F^2(x) < 1$, if x is in the interior of I.
- Here maximum means that if I' is a closed interval such that $I \subseteq I'$ and $F^2(x) < 1$ if x is in the interior of I', then $I = I'$.

Bijection between geodesic and (F, I)

- Let $F(x)$ be a polynomial of degree k or less.
- I is a maximum closed interval, called Hill interval, for x such that $F^2(x) < 1$, if x is in the interior of I.
- Here maximum means that if I' is a closed interval such that $I \subseteq I'$ and $F^2(x) < 1$ if x is in the interior of I', then $I = I'$.
- Constant polynomials corresponds to geodesics whose projection to (x, θ_0) are lines and the abnormal geodesics when $F(x) = \pm 1$.

Bijection between geodesic and (F, I)

- Let $F(x)$ be a polynomial of degree k or less.
- I is a maximum closed interval, called Hill interval, for x such that $F^2(x) < 1$, if x is in the interior of I.
- Here maximum means that if I' is a closed interval such that $I \subseteq I'$ and $F^2(x) < 1$ if x is in the interior of I', then $I = I'$.
- Constant polynomials corresponds to geodesics whose projection to (x, θ_0) are lines and the abnormal geodesics when $F(x) = \pm 1$.
- The Hill interval I is compact, if and only if, $F(x)$ is not a constant polynomial; in this case, if $I = [x_0, x_1]$, then $F^2(x_1) = F^2(x_0) = 1$.

Bijection between geodesic and (F, I)

- We will consider a Hamiltonian system with variable (p_x, x) and the potential function $\frac{1}{2}F^2(x)$.

Bijection between geodesic and (F, I)

- We will consider a Hamiltonian system with variable (p_x, x) and the potential function $\frac{1}{2}F^2(x)$.

$$H_F(p_x, \dot{x}) = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x),$$

Bijection between geodesic and (F, I)

- We will consider a Hamiltonian system with variable (p_x, x) and the potential function $\frac{1}{2}F^2(x)$.

$$H_F(p_x, \dot{x}) = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x),$$

- The Hamiltonian equations are given by

$$\dot{x} = p_x, \quad \dot{p}_x = -F'(x)F(x).$$

Bijection between geodesic and (F, I)

- We will consider a Hamiltonian system with variable (p_x, x) and the potential function $\frac{1}{2}F^2(x)$.

$$H_F(p_x, \dot{x}) = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x),$$

- The Hamiltonian equations are given by

$$\dot{x} = p_x, \quad \dot{p}_x = -F'(x)F(x).$$

- We chose $H_F = \frac{1}{2}$, then if $I = [x_0, x_1]$ we have $\dot{x} = 0$ if and only if $x = x_0, x_1$. So the dynamic takes place in the Hill interval.

Bijection between geodesic and (F, I)

- We will consider a Hamiltonian system with variable (p_x, x) and the potential function $\frac{1}{2}F^2(x)$.

$$H_F(p_x, \dot{x}) = \frac{1}{2}p_x^2 + \frac{1}{2}F^2(x),$$

- The Hamiltonian equations are given by

$$\dot{x} = p_x, \quad \dot{p}_x = -F'(x)F(x).$$

- We chose $H_F = \frac{1}{2}$, then if $I = [x_0, x_1]$ we have $\dot{x} = 0$ if and only if $x = x_0, x_1$. So the dynamic takes place in the Hill interval.
- Moreover, x_0 is equilibrium point if and only if $F'(x_0) = 0$.

Bijection between geodesic and (F, I)

- Given the solution $x(t)$, we define a plane curve $c(t) = (x(t), \theta_0(t))$ by the ODE

$$\dot{\theta}_0 = F(x(t)).$$

Bijection between geodesic and (F, I)

- Given the solution $x(t)$, we define a plane curve $c(t) = (x(t), \theta_0(t))$ by the ODE

$$\dot{\theta}_0 = F(x(t)).$$

- Using the Hamilton equation we have

$$\dot{x}(t) = \pm\sqrt{1 - F^2(x(t))}. \tag{2}$$

Bijection between geodesic and (F, I)

- Given the solution $x(t)$, we define a plane curve $c(t) = (x(t), \theta_0(t))$ by the ODE

$$\dot{\theta}_0 = F(x(t)).$$

- Using the Hamilton equation we have

$$\dot{x}(t) = \pm\sqrt{1 - F^2(x(t))}. \tag{2}$$

- By construction $|\dot{c}(t)| = 1$.

Bijection between geodesic and (F, I)

- Given the solution $x(t)$, we define a plane curve $c(t) = (x(t), \theta_0(t))$ by the ODE

$$\dot{\theta}_0 = F(x(t)).$$

- Using the Hamilton equation we have

$$\dot{x}(t) = \pm\sqrt{1 - F^2(x(t))}. \tag{2}$$

- By construction $|\dot{c}(t)| = 1$.
- Also, $\kappa(t) = F'(x(t))$.

Bijection between geodesic and (F, I)

- We lift the curve $(x(t), \theta_0(t))$ to a curve $\gamma(t)$ on J^k by solving

Bijection between geodesic and (F, I)

- We lift the curve $(x(t), \theta_0(t))$ to a curve $\gamma(t)$ on J^k by solving

$$\dot{\theta}_i(t) = \dot{x}^i(t) F(x(t)), \quad i \in 1, \cdots, k.$$

$\gamma(t)$ is unique upto a constant of integration.

Bijection between geodesic and (F, I)

- We lift the curve $(x(t), \theta_0(t))$ to a curve $\gamma(t)$ on J^k by solving

$$\dot{\theta}_i(t) = \dot{x}^i(t)F(x(t)), \quad i \in 1, \cdots, k.$$

$\gamma(t)$ is unique upto a constant of integration.

Theorem (Monroy-Anzaldo)

The above prescription yields a geodesic in J^k parameterized by arclength. Conversely, any arc-length parameterized geodesic in J^k can be achieved by this prescription applied to some polynomial $F(x)$ of degree k or less.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.
    - γ is x-periodic, $x(t+L) = x(t)$, if $F'(x_0) \neq 0$ and $F'(x_1) \neq 0$.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.
    - γ is x-periodic, $x(t+L) = x(t)$, if $F'(x_0) \neq 0$ and $F'(x_1) \neq 0$.
    - γ is homoclinic if $F'(x_0) = 0$ and $F'(x_1) \neq 0$, or, $F'(x_0) \neq 0$ and $F'(x_1) = 0$.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.
  - γ is x-periodic, $x(t+L) = x(t)$, if $F'(x_0) \neq 0$ and $F'(x_1) \neq 0$.
  - γ is homoclinic if $F'(x_0) = 0$ and $F'(x_1) \neq 0$, or, $F'(x_0) \neq 0$ and $F'(x_1) = 0$.
  - γ is heteroclinic if $F'(x_0) = 0$ and $F'(x_1) = 0$.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.
    - γ is x-periodic, $x(t+L) = x(t)$, if $F'(x_0) \neq 0$ and $F'(x_1) \neq 0$.
    - γ is homoclinic if $F'(x_0) = 0$ and $F'(x_1) \neq 0$, or, $F'(x_0) \neq 0$ and $F'(x_1) = 0$.
    - γ is heteroclinic if $F'(x_0) = 0$ and $F'(x_1) = 0$.
- We just need to see that x-periodic curves are not periodic.

Classification of geodesics

- γ is a line-geodesics if $F(x)$ is a constant polynomial.
- γ is a non line-geodesic if $F(x)$ is non constant polynomial. Let $[x_0, x_1]$ be the Hill interval.
- According to classical mechanical theory of one-degree of freedom, there are three possibilities for the qualitative behavior of the coordinate x.
    - γ is x-periodic, $x(t+L) = x(t)$, if $F'(x_0) \neq 0$ and $F'(x_1) \neq 0$.
    - γ is homoclinic if $F'(x_0) = 0$ and $F'(x_1) \neq 0$, or, $F'(x_0) \neq 0$ and $F'(x_1) = 0$.
    - γ is heteroclinic if $F'(x_0) = 0$ and $F'(x_1) = 0$.
- We just need to see that x-periodic curves are not periodic. In this case we have $1 - F^2(x) = (x - x_0)(x - x_1)q(x)$, where $q(x) \neq 0$ if $x \in [x_0, x_1]$.

Classification of geodesics

Example (Heisenberg group)

If $k = 1$ then $F(x)$ is at most degree 1 polynomial, then geodesics are circles or lines.

Classification of geodesics

Example (Heisenberg group)

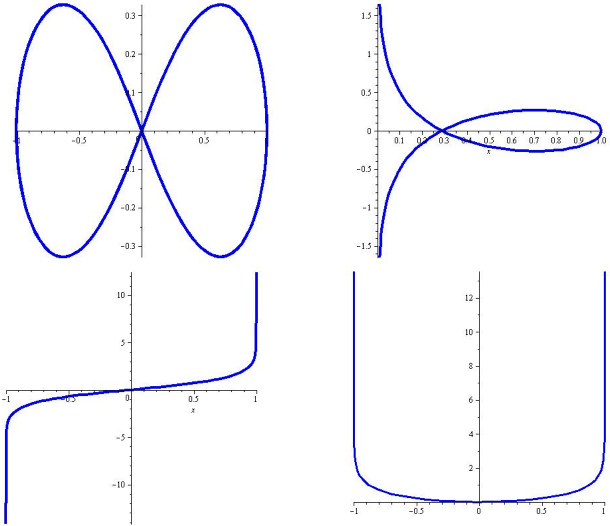
If $k = 1$ then $F(x)$ is at most degree 1 polynomial, then geodesics are circles or lines.

Example (Engel group- Euler-Elastica)

If $F(x) = bx^2 + a$ with $b < 0$.

- One Hill-interval, if $-1 < a < 1$.
- Two Hill-interval, if $a = 1$, Euler-Soliton, $F(0) = 1$ and $F'(0) = 0$.
- Two Hill-interval, if $1 < a$.

Classification of geodesics



Proof

Proposition (Periods)

Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ *in* J^k *be an* x*-periodic geodesic corresponding to the pair* (F, I)*. Then the* x*-period is*

$$L(F, I) = 2\int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

Proof

Proposition (Periods)

Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ *in* J^k *be an* x*-periodic geodesic corresponding to the pair* (F, I). *Then the* x*-period is*

$$L(F, I) = 2\int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

and is twice the time it takes for $x(t)$ *to cross its Hill interval exactly once.*

Proof

Proposition (Periods)

Let $\gamma(t) = (x(t), \theta_0(t), \cdots, \theta_k(t))$ in J^k be an x-periodic geodesic corresponding to the pair (F, I). Then the x-period is

$$L(F, I) = 2 \int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

and is twice the time it takes for $x(t)$ to cross its Hill interval exactly once. After one period, the changes $\Delta\theta_i := \theta_i(t+L) - \theta_i(t)$ for $i = 0, 1, \ldots, k$ undergone by θ_i are given by

$$\Delta\theta_i(F, I) = \frac{2}{i!} \int_I \frac{x^i F(x) dx}{\sqrt{1 - F^2(x)}}.$$

Proof

- Then a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.

Proof

- Then a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.
- Because that period L is finite, we can define an inner product in the space of polynomials of degree k or less in the following way

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1 - F^2(x)}}. \tag{3}$$

Proof

- Then a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.
- Because that period L is finite, we can define an inner product in the space of polynomials of degree k or less in the following way

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1 - F^2(x)}}. \tag{3}$$

- This inner product is not degenerated.

Proof

- Then a x-periodic geodesic $\gamma(t)$ corresponding to the pair (F, I) is periodic if and only if $\Delta\theta_i(F, I) = 0$ for all i.
- Because that period L is finite, we can define an inner product in the space of polynomials of degree k or less in the following way

$$< P_1(x), P_2(x) >_F := \int_I \frac{P_1(x)P_2(x)dx}{\sqrt{1 - F^2(x)}}. \tag{3}$$

- This inner product is not degenerated.
- We have $\Delta\theta_i(F, I) =< x^i, F(x) >_F$.

Proof

Proof.

- We will proceed by contradiction.

Proof.

- We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant.

Proof

Proof.

- We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant.
- Then $\Delta\theta_i(F, I) = 0$ for all i in $0, \cdots, k$.

Proof

Proof.

- We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant.
- Then $\Delta\theta_i(F, I) = 0$ for all i in $0, \cdots, k$.
- The condition $\Delta\theta_i(F, I) = 0$ for all i is equivalent to $F(x)$ being perpendicular to x^i for all $i \in 0, 1, \cdots, k$.

Proof

Proof.

- We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant.
- Then $\Delta\theta_i(F, I) = 0$ for all i in $0, \cdots, k$.
- The condition $\Delta\theta_i(F, I) = 0$ for all i is equivalent to $F(x)$ being perpendicular to x^i for all $i \in 0, 1, \cdots, k$.
- But $\{x^i\}$ is a base for the space of polynomials of degree k or less, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero.

Proof

Proof.

- We will proceed by contradiction. Let us assume $\gamma(t)$ is a periodic geodesic on J^k corresponding to the pair (F, I), where $F(x)$ is not constant.
- Then $\Delta\theta_i(F, I) = 0$ for all i in $0, \cdots, k$.
- The condition $\Delta\theta_i(F, I) = 0$ for all i is equivalent to $F(x)$ being perpendicular to x^i for all $i \in 0, 1, \cdots, k$.
- But $\{x^i\}$ is a base for the space of polynomials of degree k or less, then $F(x)$ is perpendicular to any vector, so $F(x)$ is zero.
- This is a contradiction to the assumption that $F(x)$ is not a constant polynomial.

$\square$

Upper bound for the cut time

Definition

Let $\gamma : \mathbb{R} \to M$ be a geodesic in a length space (e.g. a subRiemannian manifold) parameterized by arclength.

- The cut time of γ is

$$t_{cut}(\gamma) := \sup\{0 < t : \gamma|_{[0,t]} \text{ is lenght-minimizing}\}.$$

Upper bound for the cut time

Definition

Let $\gamma : \mathbb{R} \to M$ be a geodesic in a length space (e.g. a subRiemannian manifold) parameterized by arclength.

- The cut time of γ is

$$t_{cut}(\gamma) := \sup\{0 < t : \gamma|_{[0,t]} \text{ is lenght-minimizing}\}.$$

Proposition

Let $\gamma(t)$ be a x-periodic geodesic corresponding to (F, I) on J^k. Then $t_{cut}(\gamma) \leq L(F, I)$.

Upper bound for the cut time

Proof.

- Let $\gamma(t)$ be the geodesic corresponding to $F(x)$ and $[x_0, x_1]$.

Upper bound for the cut time

Proof.

- Let $\gamma(t)$ be the geodesic corresponding to $F(x)$ and $[x_0, x_1]$.
- Let $x(0) \in (x_0, x_1)$. Then there are exactly two geodesics passing through $\gamma(0)$ and associated to $F(x)$.

Upper bound for the cut time

Proof.

- Let $\gamma(t)$ be the geodesic corresponding to $F(x)$ and $[x_0, x_1]$.
- Let $x(0) \in (x_0, x_1)$. Then there are exactly two geodesics passing through $\gamma(0)$ and associated to $F(x)$.
- Namely, the given one $\gamma(t)$ and $\tilde{\gamma}(t)$ characterized by $\dot{\tilde{x}}(0) = -\dot{x}(0)$.

Upper bound for the cut time

Proof.

- Let $\gamma(t)$ be the geodesic corresponding to $F(x)$ and $[x_0, x_1]$.
- Let $x(0) \in (x_0, x_1)$. Then there are exactly two geodesics passing through $\gamma(0)$ and associated to $F(x)$.
- Namely, the given one $\gamma(t)$ and $\tilde{\gamma}(t)$ characterized by $\dot{\tilde{x}}(0) = -\dot{x}(0)$.
- Then $\tilde{x}(t) = x(-t)$ for all t. By x-periodicity we have $x(L) = \tilde{x}(L)$, (here $L(F, I) = L$).

Upper bound for the cut time

Proof.

- Let $\gamma(t)$ be the geodesic corresponding to $F(x)$ and $[x_0, x_1]$.
- Let $x(0) \in (x_0, x_1)$. Then there are exactly two geodesics passing through $\gamma(0)$ and associated to $F(x)$.
- Namely, the given one $\gamma(t)$ and $\tilde{\gamma}(t)$ characterized by $\dot{\tilde{x}}(0) = -\dot{x}(0)$.
- Then $\tilde{x}(t) = x(-t)$ for all t. By x-periodicity we have $x(L) = \tilde{x}(L)$, (here $L(F, I) = L$).
- The periods Proposition tells us that γ and $\tilde{\gamma}$ have the same θ_i, periods, $\Delta\theta_i$. Thus

$$\gamma(L) = \gamma(0) + (0, \Delta\theta_0, \cdots, \Delta\theta_k) = \tilde{\gamma}(L).$$

$\square$

Upper bound for the cut time

Proof.

- The geodesics curves are distinct, showing that $t_{cut}(\gamma) \leq L$.

Upper bound for the cut time

Proof.

- The geodesics curves are distinct, showing that $t_{cut}(\gamma) \leq L$.
- In case $x(0) = x_0, x_1$, we can prove that $\gamma(L)$ is conjugate to $\gamma(0)$ along γ, so again $t_{cut}(c) \leq L$.

$\square$

Upper bound for the cut time

Proof.

- The geodesics curves are distinct, showing that $t_{cut}(\gamma) \leq L$.
- In case $x(0) = x_0, x_1$, we can prove that $\gamma(L)$ is conjugate to $\gamma(0)$ along γ, so again $t_{cut}(c) \leq L$.

$\square$

Conjecture

- *We say that the arc-length parameterized geodesic $\gamma : \mathbb{R} \to X$ is equi-optimal if its cut times are independent of initial point on the geodesic.*

Upper bound for the cut time

Proof.

- The geodesics curves are distinct, showing that $t_{cut}(\gamma) \leq L$.
- In case $x(0) = x_0, x_1$, we can prove that $\gamma(L)$ is conjugate to $\gamma(0)$ along γ, so again $t_{cut}(c) \leq L$.

$\square$

Conjecture

- *We say that the arc-length parameterized geodesic $\gamma : \mathbb{R} \to X$ is equi-optimal if its cut times are independent of initial point on the geodesic. We say that a length space is equi-optimal if all the geodesics are equi-optimal.*

Upper bound for the cut time

Proof.

- The geodesics curves are distinct, showing that $t_{cut}(\gamma) \leq L$.
- In case $x(0) = x_0, x_1$, we can prove that $\gamma(L)$ is conjugate to $\gamma(0)$ along γ, so again $t_{cut}(c) \leq L$.

$\square$

Conjecture

- *We say that the arc-length parameterized geodesic $\gamma : \mathbb{R} \to X$ is equi-optimal if its cut times are independent of initial point on the geodesic. We say that a length space is equi-optimal if all the geodesics are equi-optimal.*
- *J^k is equi-optimal.*

Action-Angles coordinates

- Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates on T^*J^k, or in short way as (p, q).

Action-Angles coordinates

- Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates on T^*J^k, or in short way as (p, q).
- Let $P_1, P_2 : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields X_1, X_2, in terms of the coordinates (p, q) are given by

$$P_1(p,q) := p_x, \qquad P_2(p,q) := \sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}.$$

Action-Angles coordinates

- Let $(p_x, p_{\theta_0}, \cdots, p_{\theta_k}, x, \theta_0, \cdots, \theta_k)$ be the traditional coordinates on T^*J^k, or in short way as (p, q).
- Let $P_1, P_2 : T^*J^k \to \mathbb{R}$ be the momentum functions of the vector fields X_1, X_2, in terms of the coordinates (p, q) are given by

$$P_1(p,q) := p_x, \qquad P_2(p,q) := \sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}.$$

- Then the Hamiltonian governing the geodesic on J^k is

$$H(p,q) := \frac{1}{2}(P_1^2 + P_2^2) = \frac{1}{2}p_x^2 + \frac{1}{2}\Big(\sum_{i=0}^{k} p_{\theta_i} \frac{x^i}{i!}\Big).$$

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.
- Then $\mathcal{I} = \{h = 1/2, p_{\theta_0}, \cdots, p_{\theta_k}\}$ is a set of $(k+2)$ constants of motion,

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.
- Then $\mathcal{I} = \{h = 1/2, p_{\theta_0}, \cdots, p_{\theta_k}\}$ is a set of $(k+2)$ constants of motion, then the Hamiltonian system is integrable.

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.
- Then $\mathcal{I} = \{h = 1/2, p_{\theta_0}, \cdots, p_{\theta_k}\}$ is a set of $(k+2)$ constants of motion, then the Hamiltonian system is integrable.
- if $\gamma(t)$ is a geodesic corresponding to the pair (F, I), where $F(x) = \sum_{i=0}^{k} a_i x^i$,

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.
- Then $\mathcal{I} = \{h = 1/2, p_{\theta_0}, \cdots, p_{\theta_k}\}$ is a set of $(k+2)$ constants of motion, then the Hamiltonian system is integrable.
- if $\gamma(t)$ is a geodesic corresponding to the pair (F, I), where $F(x) = \sum_{i=0}^{k} a_i x^i$, then $p_{\theta_i} = i! a_i$ and

$$\frac{1}{2} = H|_{\mathcal{I}}(p, q) = \frac{1}{2} p_x^2 + \frac{1}{2} F^2(x) =: H_F(p_x, x). \tag{4}$$

Action-Angles coordinates

- Using the Hamilton equations we have $\dot{p}_{\theta_i} = \partial H / \partial \theta_i = 0$.
- Notice that H does not depend on t, so $h := H$ and p_{θ_i} are constants of motion.
- Then $\mathcal{I} = \{h = 1/2, p_{\theta_0}, \cdots, p_{\theta_k}\}$ is a set of $(k+2)$ constants of motion, then the Hamiltonian system is integrable.
- if $\gamma(t)$ is a geodesic corresponding to the pair (F, I), where $F(x) = \sum_{i=0}^{k} a_i x^i$, then $p_{\theta_i} = i! a_i$ and

$$\frac{1}{2} = H|_{\mathcal{I}}(p, q) = \frac{1}{2} p_x^2 + \frac{1}{2} F^2(x) =: H_F(p_x, x). \tag{4}$$

- Here $H_F(p_x, x)$ is a Hamiltonian function in the phase plane (x, p_x) whose dynamic takes place on the Hill interval $I = [x_0, x_1]$.

Action-Angles coordinates

- the condition $h = 1/2$ implies that its solution $(x(t), p_x(t))$ lies in a simple closed curve given by

$$\alpha_{(F,I)} := \{(p_x, x) : \frac{1}{2} = H_F(p_x, x) \text{ and } x_0 \leq x \leq x_1\}.$$

Action-Angles coordinates

- the condition $h = 1/2$ implies that its solution $(x(t), p_x(t))$ lies in a simple closed curve given by

$$\alpha_{(F,I)} := \{(p_x, x) : \frac{1}{2} = H_F(p_x, x) \text{ and } x_0 \leq x \leq x_1\}.$$

- Note that $\alpha_{(F,I)}$ is smooth if only if x_0 and x_1 are regular points of $F(x)$,

Action-Angles coordinates

- the condition $h = 1/2$ implies that its solution $(x(t), p_x(t))$ lies in a simple closed curve given by

$$\alpha_{(F,I)} := \{(p_x, x) : \frac{1}{2} = H_F(p_x, x) \text{ and } x_0 \leq x \leq x_1\}.$$

- Note that $\alpha_{(F,I)}$ is smooth if only if x_0 and x_1 are regular points of $F(x)$, in other words, $\alpha_{(F,I)}$ is smooth if and only if the corresponding geodesic $\gamma(t)$ is x-periodic.

Action-Angles coordinates

- the condition $h = 1/2$ implies that its solution $(x(t), p_x(t))$ lies in a simple closed curve given by

$$\alpha_{(F,I)} := \{(p_x, x) : \frac{1}{2} = H_F(p_x, x) \text{ and } x_0 \leq x \leq x_1\}.$$

- Note that $\alpha_{(F,I)}$ is smooth if only if x_0 and x_1 are regular points of $F(x)$, in other words, $\alpha_{(F,I)}$ is smooth if and only if the corresponding geodesic $\gamma(t)$ is x-periodic.
- Then the the Arnold-Liouville manifold $M_{\mathcal{I}}$ is given by

$$M_{\mathcal{I}} := \{(p, q) \in T^*J^k : \frac{1}{2} = H_F(p_x, x), \;\; p_{\theta_i} = i!a_i\}.$$

Action-Angles coordinates

- In the case $\gamma(t)$ is x-periodic, $M_{\mathcal{I}}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha_{(F,I)}$.

Action-Angles coordinates

- In the case $\gamma(t)$ is x-periodic, $M_{\mathcal{I}}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha_{(F,I)}$.
- In order to construct the canonical transformation $\Phi(p, q) = (\mathcal{I}, \phi)$, we will look for its generating function $S(\mathcal{I}, q)$, of the second type,

Action-Angles coordinates

- In the case $\gamma(t)$ is x-periodic, $M_{\mathcal{I}}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha_{(F,I)}$.
- In order to construct the canonical transformation $\Phi(p,q) = (\mathcal{I}, \phi)$, we will look for its generating function $S(\mathcal{I}, q)$, of the second type, that satisfying the three following conditions

$$p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mathcal{I}}, \quad H(\frac{\partial S}{\partial q}, q) = h = \frac{1}{2}.$$

Action-Angles coordinates

- In the case $\gamma(t)$ is x-periodic, $M_{\mathcal{I}}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha_{(F,I)}$.
- In order to construct the canonical transformation $\Phi(p,q) = (\mathcal{I}, \phi)$, we will look for its generating function $S(\mathcal{I}, q)$, of the second type, that satisfying the three following conditions

$$p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mathcal{I}}, \quad H(\frac{\partial S}{\partial q}, q) = h = \frac{1}{2}.$$

For more detail on the definition of $S(\mathcal{I}, q)$, see Section 50, Mathematical methods of classical mechanics.

Action-Angles coordinates

- In the case $\gamma(t)$ is x-periodic, $M_{\mathcal{I}}$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^{k+1}$, where $\mathbb{S}^1$ is the simple closed and smooth curve $\alpha_{(F,I)}$.
- In order to construct the canonical transformation $\Phi(p,q) = (\mathcal{I}, \phi)$, we will look for its generating function $S(\mathcal{I}, q)$, of the second type, that satisfying the three following conditions

$$p = \frac{\partial S}{\partial q}, \quad \phi = \frac{\partial S}{\partial \mathcal{I}}, \quad H(\frac{\partial S}{\partial q}, q) = h = \frac{1}{2}.$$

  For more detail on the definition of $S(\mathcal{I}, q)$, see Section 50, Mathematical methods of classical mechanics.
- To find $S(\mathcal{I}, q)$, we will solve the subRiemannian Hamilton-Jacobi equation associated to the subRiemannian geodesic flow.

Action-Angles coordinates

- That is,

$$\frac{1}{2} = h = \frac{1}{2}(\frac{\partial S}{\partial x})^2 + \frac{1}{2}(\sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial S}{\partial \theta_i})^2$$

Action-Angles coordinates

- That is,

$$\frac{1}{2} = h = \frac{1}{2}(\frac{\partial S}{\partial x})^2 + \frac{1}{2}(\sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial S}{\partial \theta_i})^2$$

- Take the ansatz

$$S(\mathcal{I}, q) := f(x) + \sum_{i=0}^{k} i! a_i \theta_i,$$

as a solution.

Action-Angles coordinates

- That is,

$$\frac{1}{2} = h = \frac{1}{2}(\frac{\partial S}{\partial x})^2 + \frac{1}{2}(\sum_{i=0}^{k} \frac{x^i}{i!}\frac{\partial S}{\partial \theta_i})^2$$

- Take the ansatz

$$S(\mathcal{I}, q) := f(x) + \sum_{i=0}^{k} i! a_i \theta_i,$$

as a solution.

- The equation H-J equation becomes equation $h = H_F$, then the generating function is given by

$$S(\mathcal{I}, q) = \pm \int_{x_0}^{x} \sqrt{2h - F^2(x)}dx + \sum_{i=0}^{n} i! a_i \theta_i. \tag{5}$$

Action-Angles coordinates

- We can see that conditions 1 and 3 are satisfied: $p = \partial S / \partial q$ and $H(\partial S/\partial q, q) = H(p(\mathcal{I}, q), q) = h$.

Action-Angles coordinates

- We can see that conditions 1 and 3 are satisfied: $p = \partial S/\partial q$ and $H(\partial S/\partial q, q) = H(p(\mathcal{I}, q), q) = h$.
- To find the new coordinates, use the condition 2

$$\frac{\partial S}{\partial h}|_{\mathcal{I}} = \int_{x(t_0)}^{x} \frac{dx}{\sqrt{1 - F^2(x)}} = \phi_t$$

$$\frac{\partial S}{\partial a_i}|_{\mathcal{I}} = -\int_{x(0)}^{x} \frac{x^i F(x) dx}{\sqrt{1 - F^2(x)}} + i!\theta_i = \phi_i.$$

Action-Angles coordinates

- We can see that conditions 1 and 3 are satisfied: $p = \partial S/\partial q$ and $H(\partial S/\partial q, q) = H(p(\mathcal{I}, q), q) = h$.
- To find the new coordinates, use the condition 2

$$\frac{\partial S}{\partial h}|_{\mathcal{I}} = \int_{x(t_0)}^{x} \frac{dx}{\sqrt{1 - F^2(x)}} = \phi_t$$

$$\frac{\partial S}{\partial a_i}|_{\mathcal{I}} = -\int_{x(0)}^{x} \frac{x^i F(x) dx}{\sqrt{1 - F^2(x)}} + i!\theta_i = \phi_i.$$

- Using the Hamilton equations for the new coordinates $(\mathcal{I}, \phi)$, $\partial H/\partial h = 1$ and $\partial H/\partial a_i = 0$, we have $\phi_t = t$ and $\phi_i = const$.

Action-Angles coordinates

- Using the initial conditions $x(t_0) = x_0$, we see that the $\phi_i(t_0) = i!\theta_i(t_0)$ and we can rewrite θ_i as follow

$$\theta_i = \frac{1}{i!}\int_{x_0}^{x} \frac{x^i F(x)dx}{\sqrt{1 - F^2(x)}} + \theta_i(t_0).$$

Action-Angles coordinates

- Using the initial conditions $x(t_0) = x_0$, we see that the $\phi_i(t_0) = i!\theta_i(t_0)$ and we can rewrite θ_i as follow

$$\theta_i = \frac{1}{i!}\int_{x_0}^{x} \frac{x^i F(x)dx}{\sqrt{1 - F^2(x)}} + \theta_i(t_0).$$

- If $\gamma(t)$ is a geodesic parameterized by arc-length corresponding to the pair (F, I), by construction the S solution is a calibration function for $\gamma(t)$, that is,

$$1 = ||\dot{\gamma}|| = dS(\dot{\gamma}(t)).$$

Action-Angles coordinates

- Let us assume $\gamma(t)$ is x-periodic corresponding to (F, I) with period L.

Action-Angles coordinates

- Let us assume $\gamma(t)$ is x-periodic corresponding to (F, I) with period L.
- We consider the change in $S(\mathcal{I}, q)$ after $\gamma(t)$ travel form t to $t+L$, in other words,

$$\int_t^{t+L} dS(\dot{\gamma}(t)) = \int_t^{t+L} (\sqrt{2h - F^2(x)}\dot{x} + \sum_{i=0}^{n} i! a_i \dot{\theta}_i) dt$$

$$hL = 2\int_I \sqrt{2h - F^2(x)} dx + \sum_{i=0}^{n} i! a_i \Delta\theta_i(F, I).$$

Action-Angles coordinates

- Let us assume $\gamma(t)$ is x-periodic corresponding to (F, I) with period L.
- We consider the change in $S(\mathcal{I}, q)$ after $\gamma(t)$ travel form t to $t + L$, in other words,

$$\int_t^{t+L} dS(\dot{\gamma}(t)) = \int_t^{t+L} (\sqrt{2h - F^2(x)}\dot{x} + \sum_{i=0}^{n} i! a_i \dot{\theta}_i) dt$$

$$hL = 2\int_I \sqrt{2h - F^2(x)} dx + \sum_{i=0}^{n} i! a_i \Delta\theta_i(F, I).$$

- $2\int_I \sqrt{2h - F^2(x)} dx$ is the area enclosed by the loop $\alpha_{(F,I)}$.

Action-Angles coordinates

- Therefore, the function S is a multiple-value function on $M_{\mathcal{I}}$,

Action-Angles coordinates

- Therefore, the function S is a multiple-value function on $M_{\mathcal{I}}$, this term has no effect on the derivative $\partial S/\partial q$, but it leads to the multiple-valuedness of $\phi = \partial S/\partial \mathcal{I}$.

Action-Angles coordinates

- Therefore, the function S is a multiple-value function on $M_{\mathcal{I}}$, this term has no effect on the derivative $\partial S/\partial q$, but it leads to the multiple-valuedness of $\phi = \partial S/\partial \mathcal{I}$.
- To see how this affect in ϕ_t and ϕ_i, we derive the above equation with respect to h and a_i, using $h = 1/2$, to find

$$L(F, I) = 2\int_I \frac{dx}{\sqrt{1 - F^2(x)}},$$

$$0 = -2\int_I \frac{x^i F(x) dx}{\sqrt{1 - F^2(x)}} + i!\Delta\theta_i(F, I).$$