1.18 Complexes and homotopy

Reading Seminar, Linckelmann Chapter 1

Deewang Bhamidipati

2nd September 2021

Chain Homotopy

Let $\mathscr{C}$ be an additive category and let (X, δ), (Y, ε) be complexes over $\mathscr{C}$.

$$X: \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

$$Y: \quad \cdots \longrightarrow Y_{n+1} \xrightarrow[\varepsilon_{n+1}]{} Y_n \xrightarrow[\varepsilon_n]{} Y_{n-1} \xrightarrow[\varepsilon_{n-1}]{} \cdots$$

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$$f - f' = \varepsilon \circ h + h \circ \delta.$$

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Or equivalently, if

$$f_n - f'_n = \varepsilon_{n+1} \circ h_n + h_{n-1} \circ \delta_n$$

for any $n \in \mathbb{Z}$.

We also say *h is a homotopy from f to f'*.

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- For cochain complexes, we define analogously a *cochain homotopy* to be a graded morphism of degree -1 satisfying the analogous property.

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We denote by $\mathsf{K}^{+}(\mathscr{C})$, $\mathsf{K}^{-}(\mathscr{C}), \mathsf{K}^{b}(\mathscr{C})$ the full subcategories of $\mathsf{K}(\mathscr{C})$ consisting of bounded above, bounded below, bounded complexes over $\mathscr{C}$, respectively.

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Therefore $[i]$ of $(\mathscr{C})$ descends to an automorphism, still denoted $[i]$, of the homotopy category $\mathsf{K}(\mathscr{C})$.

This automorphism preserves any of the subcategories $\mathsf{K}^+(\mathscr{C})$, $\mathsf{K}^-(\mathscr{C}), \mathsf{K}^b(\mathscr{C})$.

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$H_*(f)([z]) = [f(z)]$

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$$H_n(h \circ \delta + \varepsilon \circ h)([z]) = [\underline{(h \circ \delta + \varepsilon \circ h)_n(z)}] = [(\underline{h_{n-1} \circ \delta_n} + \varepsilon_{n+1} \circ h_n)(z)] = \underline{[\varepsilon_{n+1}(h_n(z))]}$$

Proof of Prop. 1.18.3 (contd.) 2/2

Note that $\varepsilon_{n+1}(h_n(z)) \in \operatorname{im}\varepsilon_{n+1}$ and $H_n(h \circ \delta + \varepsilon \circ h)([z]) \in H_n(Y) = \ker\varepsilon_n / \operatorname{im}\varepsilon_{n+1}$,

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$g \circ f \sim id_X \Rightarrow H_*(g \circ f) = H_*(id_X)$
$= id_{H_*(X)}$

Suppose f has a homotopy inverse g. Then, by (ii), we have

$$\mathrm{id}_{H_*(X)} = H_*(g \circ f) = H_*(g) \circ H_*(f) \quad \text{and} \quad \mathrm{id}_{H_*(Y)} = H_*(f \circ g) = H_*(f) \circ H_*(g)$$

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$$\mathrm{id}_{H_*(X)} = H_*(g \circ f) = H_*(g) \circ H_*(f) \quad \text{and} \quad \mathrm{id}_{H_*(Y)} = H_*(f \circ g) = H_*(f) \circ H_*(g)$$

So, $H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism, and hence f is a quasi-isomorphism; we have proved (iii).

Proof of Prop. 1.18.3 (contd.) 2/2

Note that $\varepsilon_{n+1}(h_n(z)) \in \operatorname{im}\varepsilon_{n+1}$ and $H_n(h \circ \delta + \varepsilon \circ h)([z]) \in H_n(Y) = \ker\varepsilon_n / \operatorname{im}\varepsilon_{n+1}$, and therefore

$$H_n(h \circ \delta + \varepsilon \circ h)([z]) = [\varepsilon_{n+1}(h_n(z))] = [0].$$

Hence $(h \circ \delta + \varepsilon \circ h)$ induces the zero map in homology, whence (i).

If $f \sim f'$, then by (i), $f - f'$ induces the zero map in homology, and thus $H_*(f) = H_*(f')$; so we have proved (ii).

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So, $H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism, and hence f is a quasi-isomorphism; we have proved (iii).

If $X \simeq 0$, then X is quasi-isomorphic to 0 by (iii), which is equivalent to $H_*(X) = 0$, giving us (iv). ∎

(Co-)Homology as Homotopy Classes

For an algebra A, X a complex of A-modules, V an A-module, and n an integer.

$$X : \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

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We denote by $V[n]$ the complex that is equal to V in degree n and zero in all other degrees, with the zero differential.

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We regard $\mathrm{Hom}_A(X, V)$ as a cochain complex, obtained from applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to X.

$$\cdots \longleftarrow \mathrm{Hom}_A(X_{n+1}, V) \xleftarrow{-\circ\,\delta_{n+1}} \mathrm{Hom}_A(X_n, V) \xleftarrow{-\circ\,\delta_n} \cdots$$

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and $\mathrm{Hom}_A(V, X)$ as a chain complex, obtained from applying the covariant functor $\mathrm{Hom}_A(V, -)$ to X.

h

$$\cdots \longrightarrow \mathrm{Hom}(V, X_{n+1}) \xrightarrow{\delta_{n+1}\,\circ -} \mathrm{Hom}(V, X_n) \xrightarrow{\delta_n\,\circ -} \cdots$$

(Co-)Homology as Homotopy Classes

Proposition 1.18.4

Let A be a k-algebra, V an A-module, and (X, δ) a complex of A-modules. Let n be an integer.

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Proof of Prop. 1.18.4

An element in degree n of $\mathrm{Hom}_A(V, X)$ is given by an A-linear map $\zeta : V \to X_n$, and this belongs to the kernel of the differential at degree n if and only if $\delta_n \circ \zeta = 0$.

(Co-)Homology as Homotopy Classes

Proof of Prop. 1.18.4 (contd.) 2/4

This is equivalent to asserting that ζ defines a chain map $V[n] \to X$. $(\zeta_n)_n$ $\zeta_n = \zeta$ $\zeta_m = 0$ $m \neq n$

$$\begin{tikzcd}
V[n]: & \cdots \arrow[r] & 0 \arrow[r, "0_{n+1}"] \arrow[d, "0"] & V \arrow[r, "0_n"] \arrow[d, "\zeta"] & 0 \arrow[r, "0_{n-1}"] \arrow[d, "0"] & \cdots \\
X: & \cdots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"'] & X_n \arrow[r, "\delta_n"'] & X_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots
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Proof of Prop. 1.18.4 (contd.) 2/4

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\end{tikzcd}$$

$[\zeta] = [0]$ in homology

$\zeta \in \operatorname{im}(\delta_{n+1} \circ -)$

Now, ζ belongs to the image of the differential if and only if $\underline{\zeta = \delta_{n+1} \circ \eta}$ for some A-linear map $\eta : V \to X_{n+1}$

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Proof of Prop. 1.18.4 (contd.) 2/4

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\end{tikzcd}$$

This is equivalent to asserting that ζ, as a chain map, is homotopic to zero; and so we have proved the first isomorphism in (i).

Proof of Prop. 1.18.4 (contd.) 3/4

The second isomorphism in (i) follows from the first isomorphism we just proved and the fact that $\mathrm{Hom}_A(A, -)$ is isomorphic to the identity functor on $\mathsf{Mod}(A)$.

Proof of Prop. 1.18.4 (contd.) 3/4

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Similarly as before, an element in degree n of the cochain complex $\mathrm{Hom}_A(X, V)$ is given by an A-linear map $\zeta : X_n \to V$, and this belongs to the kernel of the differential at degree n if and only if $\zeta \circ \delta_{n+1} = 0$.

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$$\begin{tikzcd}
X: & \cdots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "0"] & X_n \arrow[r, "\delta_n"] \arrow[d, "\zeta"'] & X_{n-1} \arrow[r, "\delta_{n-1}"] \arrow[d, "0"] & \cdots \\
V[n]: & \cdots \arrow[r] & 0 \arrow[r, "0_{n+1}"'] & V \arrow[r, "0_n"'] & 0 \arrow[r, "0_{n-1}"'] & \cdots
\end{tikzcd}$$

Proof of Prop. 1.18.4 (contd.)

Now, ζ belongs to the image of the differential at degree n of $\mathrm{Hom}_A(X, V)$ if and only if $\zeta = \eta \circ \delta_n$ for some A-linear map $\eta : X_{n-1} \to V$

Proof of Prop. 1.18.4 (contd.) 4/4

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This is equivalent to asserting that ζ, as a chain map, is homotopic to zero; and so we have proved the first isomorphism in (ii).

Proof of Prop. 1.18.4 (contd.) 4/4

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This is equivalent to asserting that ζ, as a chain map, is homotopic to zero; and so we have proved the first isomorphism in (ii).

Naturality in X is readily checked. ∎

Proposition 1.18.5

Let $\mathscr{C}$ be an abelian category, let P be a complex of projective objects in $\mathscr{C}$, I a complex of injective objects in $\mathscr{C}$ and let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of complexes over $\mathscr{C}$.

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(i) Suppose that X is acyclic and that one of P, Y is bounded below. The map

$$g \circ - : \mathrm{Hom}_{\mathsf{Ch}(\mathscr{C})}(P, Y) \to \mathrm{Hom}_{\mathsf{Ch}(\mathscr{C})}(P, Z)$$

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(ii) Suppose that Z is acyclic and that one of Y, I is bounded below. The map

$$- \circ f : \mathrm{Hom}_{\mathsf{Ch}(\mathscr{C})}(Y, I) \to \mathrm{Hom}_{\mathsf{Ch}(\mathscr{C})}(X, I)$$

is surjective and induces an isomorphism

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Proof of Prop. 1.18.5

Denote by δ, ε, ζ, π the differentials of X, Y, Z, P, respectively.

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Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \to Y_i$ satisfying

$$\underline{g_i \circ p_i = q_i} \quad \text{and} \quad \underline{\varepsilon_i \circ p_i = p_{i-1} \circ \pi_i}$$

for $i < n$.

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$$\begin{tikzcd} & P_n \arrow[dl, dashed, "p'_n"'] \arrow[d, "q_n"] \\ Y_n \arrow[r, twoheadrightarrow, "g_n"'] & Z_n \end{tikzcd}$$

Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \to Y_n$ such that $\underbrace{g_n \circ p'_n = q_n}$.

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Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \to Y_n$ such that $g_n \circ p'_n = q_n$.

That is, p'_n satisfies the first of the two conditions above,

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Since one of P, Y is bounded below, we have $q_i = 0$ for all sufficiently small integers i (if Y is bounded below, then so is Z by the surjectivity of g.) So take $p_i = 0$ for i sufficiently small.

Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \to Y_i$ satisfying

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Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \to Y_n$ such that $g_n \circ p'_n = q_n$.

That is, p'_n satisfies the first of the two conditions above, but we may have to adjust p'_n to make sure that it is compatible with the differentials as in the second condition.

Proof of Prop. 1.18.5 (contd.)

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map.

Proof of Prop. 1.18.5 (contd.) 2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Therefore we have

$$\operatorname{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \operatorname{im} f_{n-1}$$

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Proof of Prop. 1.18.5 (contd.) 2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Therefore we have

$$\operatorname{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \operatorname{im} f_{n-1}$$

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\sigma"'] \arrow[d, "(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n)"] \\ X_{n-1} \arrow[r, two heads, "f_{n-1}"'] & \operatorname{im} f_{n-1} \end{tikzcd}$$

Consequently, there is a morphism $\sigma : P_n \to X_{n-1}$ such that $f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n$.

Proof of Prop. 1.18.5 (contd.) 2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

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Consequently, there is a morphism $\sigma : P_n \to X_{n-1}$ such that $f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n$.

Moreover, we have

$$\begin{aligned} f_{n-2} \circ \delta_{n-1} \circ \sigma &= \varepsilon_{n-1} \circ f_{n-1} \circ \sigma \\ &= \boxed{\varepsilon_{n-1} \circ \varepsilon_n} \circ p'_n - \varepsilon_{n-1} \circ p_{n-1} \circ \pi_n = -p_{n-2} \circ \pi_{n-1} \circ \pi_n = 0 \end{aligned}$$

Proof of Prop. 1.18.5 (contd.) 2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Therefore we have

$$\mathrm{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \mathrm{im}\, f_{n-1}$$

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\sigma"'] \arrow[d, "(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n)"] \\ X_{n-1} \arrow[r, two heads, "f_{n-1}"'] & \mathrm{im}\, f_{n-1} \end{tikzcd}$$

Consequently, there is a morphism $\sigma : P_n \to X_{n-1}$ such that $f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n$.

$f \circ g = 0 \Rightarrow g = 0$

Moreover, we have

$$\begin{aligned} f_{n-2} \circ \delta_{n-1} \circ \sigma &= \varepsilon_{n-1} \circ f_{n-1} \circ \sigma \\ &= \varepsilon_{n-1} \circ \varepsilon_n \circ p'_n - \varepsilon_{n-1} \circ p_{n-1} \circ \pi_n = -p_{n-2} \circ \pi_{n-1} \circ \pi_n = 0 \end{aligned}$$

and hence $\delta_{n-1} \circ \sigma = 0$, as f_{n-2} is a monomorphism.

Proof of Prop. 1.18.5 (contd.) 3/6

Thus we have $\operatorname{im}\sigma \subseteq \ker\delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

$0 = H_n(X) = \ker\delta_{n-1}/\operatorname{im}\delta_n$

Proof of Prop. 1.18.5 (contd.) 3/6

Thus we have $\operatorname{im}\sigma \subseteq \ker\delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\rho"'] \arrow[d, "\sigma"] \\ X_n \arrow[r, two heads, "\delta_n"'] & \operatorname{im}\delta_n \end{tikzcd}$$

Thus there is a morphism $\rho : P_n \to X_n$ such that $\sigma = \delta_n \circ \rho$.

Proof of Prop. 1.18.5 (contd.)

Thus we have $\operatorname{im}\sigma \subseteq \ker\delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\rho"'] \arrow[d, "\sigma"] \\ X_n \arrow[r, two heads, "\delta_n"'] & \operatorname{im}\delta_n \end{tikzcd}$$

Thus there is a morphism $\rho : P_n \to X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = \underline{p'_n - f_n \circ \rho}$.

Proof of Prop. 1.18.5 (contd.) 3/6

Thus we have $\operatorname{im}\sigma \subseteq \ker\delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

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Thus there is a morphism $\rho : P_n \to X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = p'_n - f_n \circ \rho$. We still have $g_n \circ p_n = g_n \circ p'_n - \overset{\text{ses}}{g_n \circ f_n} \circ \rho = g_n \circ p'_n = q_n$,

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Proof of Prop. 1.18.5 (contd.)

Thus we have $\operatorname{im}\sigma \subseteq \ker \delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

$$\begin{tikzcd}
 & P_n \arrow[dl, dashed, "\rho"'] \arrow[d, "\sigma"] \\
X_n \arrow[r, two heads, "\delta_n"'] & \operatorname{im}\delta_n
\end{tikzcd}$$

Thus there is a morphism $\rho : P_n \to X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = p'_n - f_n \circ \rho$. We still have $g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n$,

and we now also have the compatibility with the differentials

$$\begin{aligned}
\varepsilon_n \circ p_n &= \varepsilon_n \circ p'_n - \varepsilon_n \circ f_n \circ \rho \\
&= \varepsilon_n \circ p'_n - f_{n-1} \circ \delta_n \circ \rho \\
&= \varepsilon_n \circ p'_n - f_{n-1} \circ \sigma \\
&= \cancel{\varepsilon_n \circ p'_n} - (\cancel{\varepsilon_n \circ p'_n} - p_{n-1} \circ \pi_n) = p_{n-1} \circ \pi_n
\end{aligned}$$

as required.

Proof of Prop. 1.18.5 (contd.)

Thus we have $\operatorname{im}\sigma \subseteq \ker\delta_{n-1} = \operatorname{im}\delta_n$, where the last equality holds as X is acyclic.

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "\rho"'] \arrow[d, "\sigma"] \\ X_n \arrow[r, two heads, "\delta_n"'] & \operatorname{im}\delta_n \end{tikzcd}$$

Thus there is a morphism $\rho : P_n \to X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = p'_n - f_n \circ \rho$. We still have $g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n$,

and we now also have the compatibility with the differentials

$$\begin{aligned} \varepsilon_n \circ p_n &= \varepsilon_n \circ p'_n - \varepsilon_n \circ f_n \circ \rho \\ &= \varepsilon_n \circ p'_n - f_{n-1} \circ \delta_n \circ \rho \\ &= \varepsilon_n \circ p'_n - f_{n-1} \circ \sigma \\ &= \varepsilon_n \circ p'_n - (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = p_{n-1} \circ \pi_n \end{aligned}$$

as required. This shows the surjectivity of the map given by composition with g.

Proof of Prop. 1.18.5 (contd.) 4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

Proof of Prop. 1.18.5 (contd.)

We need to show that $p \sim 0$ if and only if $q \sim 0$. $q = g \circ p$

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Proof of Prop. 1.18.5 (contd.) 4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

$$q = g \circ p = \underline{g \circ \varepsilon} \circ h + g \circ h \circ \pi = \underline{\zeta \circ} \boxed{g \circ h} + \boxed{g \circ h} \circ \pi,$$

Proof of Prop. 1.18.5 (contd.)

We need to show that $p \sim 0$ if and only if $q \sim 0$.

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Composing with g yields

$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

thus $q \sim 0$ via the homotopy $g \circ h : P \to Z$.

Proof of Prop. 1.18.5 (contd.)

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

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$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

thus $q \sim 0$ via the homotopy $g \circ h : P \to Z$.

Suppose $q \sim 0$, then there is a homotopy $t : P \to Z$ such that $q = \zeta \circ t + t \circ \pi$.

Proof of Prop. 1.18.5 (contd.) 4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

thus $q \sim 0$ via the homotopy $g \circ h : P \to Z$.

Suppose $q \sim 0$, then there is a homotopy $t : P \to Z$ such that $q = \zeta \circ t + t \circ \pi$.

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "s_n"'] \arrow[d, "t_n"] \\ Y_{n+1} \arrow[r, two heads, "g_{n+1}"'] & Z_{n+1} \end{tikzcd}$$

Since g_{n+1} is an epimorphism, $t_n : P_n \to Z_{n+1}$ then lifts to a morphism $s_n : P_n \to Y_{n+1}$; that is, the homotopy $t : P \to Z$ lifts to some homotopy $s : P \to Y$.

Proof of Prop. 1.18.5 (contd.) 4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

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So, $p' := \varepsilon \circ s + s \circ \pi : P \to Y$ is a chain map such that $p' \sim 0$ via s, trivially.

Proof of Prop. 1.18.5 (contd.)

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

thus $q \sim 0$ via the homotopy $g \circ h : P \to Z$.

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So, $p' := \varepsilon \circ s + s \circ \pi : P \to Y$ is a chain map such that $p' \sim 0$ via s, trivially. Furthermore, one immediately checks that $g \circ p' = q$, but p' need not be equal to p.

Proof of Prop. 1.18.5 (contd.) 4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \to Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

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Since g_{n+1} is an epimorphism, $t_n : P_n \to Z_{n+1}$ then lifts to a morphism $s_n : P_n \to Y_{n+1}$; that is, the homotopy $t : P \to Z$ lifts to some homotopy $s : P \to Y$.

So, $p' := \varepsilon \circ s + s \circ \pi : P \to Y$ is a chain map such that $p' \sim 0$ via s, trivially. Furthermore, one immediately checks that $g \circ p' = q$, but p' need not be equal to p.

It suffices to show that $p - p' \sim 0$. Since $g \circ (p - p') = 0$, we may assume that $q = 0$.

Proof of Prop. 1.18.5 (contd.)

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

Proof of Prop. 1.18.5 (contd.) 5/6

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[dl, dashed, "u"'] \\ X \arrow[r, two heads, "f"'] & \operatorname{im} f \end{tikzcd}$$

This implies that there is a chain map $u : P \to X$ such that $f \circ u = p$.

Proof of Prop. 1.18.5 (contd.)

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

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This implies that there is a chain map $u : P \to X$ such that $f \circ u = p$.

It suffices to show that $u \sim 0$. This is again done inductively.

Proof of Prop. 1.18.5 (contd.)

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Given an integer n, suppose that we have morphisms $h_i : P_i \to X_{i+1}$ satisfying

$$u_i = \delta_{i+1} \circ h_i + h_{i-1} \circ \pi_i$$

for any $i < n$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.) 5/6

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

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$$u_i = \delta_{i+1} \circ h_i + h_{i-1} \circ \pi_i$$

for any $i < n$.

Using this equality for $i = n - 1$, we get

$$\begin{aligned} \delta_n \circ (u_n - h_{n-1} \circ \pi_n) &= \delta_n \circ u_n - \delta_n \circ h_{n-1} \circ \pi_n \\ &= \delta_n \circ u_n - (u_{n-1} - h_{n-2} \circ \pi_{n-1}) \circ \pi_n = \delta_n \circ u_n - u_{n-1} \circ \pi_n = 0, \end{aligned}$$

as u is a chain map.

Proof of Prop. 1.18.5 (contd.) 5/6

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

$$\begin{tikzcd} & P \arrow[dl, dashed, "u"'] \arrow[d, "p"] \\ X \arrow[r, two heads, "f"'] & \operatorname{im} f \end{tikzcd}$$

This implies that there is a chain map $u : P \to X$ such that $f \circ u = p$.

It suffices to show that $u \sim 0$. This is again done inductively.

Given an integer n, suppose that we have morphisms $h_i : P_i \to X_{i+1}$ satisfying

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$$\begin{aligned} \delta_n \circ (u_n - h_{n-1} \circ \pi_n) &= \delta_n \circ u_n - \delta_n \circ h_{n-1} \circ \pi_n \\ &= \delta_n \circ u_n - (u_{n-1} - h_{n-2} \circ \pi_{n-1}) \circ \pi_n = \delta_n \circ u_n - u_{n-1} \circ \pi_n = 0, \end{aligned}$$

as u is a chain map. Therefore

$$\operatorname{im}(u_n - h_{n-1} \circ \pi_n) \subseteq \ker \delta_n = \operatorname{im} \delta_{n+1}.$$ (X is acyclic)

Proof of Prop. 1.18.5 (contd.) 6/6

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "h_n"'] \arrow[d, "(u_n - h_{n-1}\circ \pi_n)"] \\ X_{n+1} \arrow[r, two heads, "\delta_{n+1}"'] & \operatorname{im}\delta_{n+1} \end{tikzcd}$$

Hence, as P_n is projective, there is a map $h_n : P_n \to X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

Proof of Prop. 1.18.5 (contd.) 6/6

$$
\begin{tikzcd}
 & P_n \arrow[dl, dashed, "h_n"'] \arrow[d, "(u_n - h_{n-1}\circ\pi_n)"] \\
X_{n+1} \arrow[r, two heads, "\delta_{n+1}"'] & \operatorname{im}\delta_{n+1}
\end{tikzcd}
$$

Hence, as P_n is projective, there is a map $h_n : P_n \to X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

Proof of Prop. 1.18.5 (contd.) 6/6

$$\begin{tikzcd} & & P_n \arrow[dll, dashed, "h_n"'] \arrow[d, "(u_n - h_{n-1} \circ \pi_n)"] \\ X_{n+1} \arrow[rr, two heads, "\delta_{n+1}"'] & & \operatorname{im} \delta_{n+1} \end{tikzcd}$$

Hence, as P_n is projective, there is a map $h_n : P_n \to X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

So, we have $u \sim 0$. This completes the proof of (i).

Proof of Prop. 1.18.5 (contd.) 6/6

$$\begin{tikzcd} & P_n \arrow[dl, dashed, "h_n"'] \arrow[d, "(u_n - h_{n-1}\circ\pi_n)"] \\ X_{n+1} \arrow[r, two heads, "\delta_{n+1}"'] & \operatorname{im}\delta_{n+1} \end{tikzcd}$$

Hence, as P_n is projective, there is a map $h_n : P_n \to X_{n+1}$ such that

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thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

So, we have $u \sim 0$. This completes the proof of (i).

The proof of (ii) is obtained by dualising the arguments. ∎

Corollary 1.18.6

Let $\mathscr{C}$ be an abelian category, let P be a complex of projective objects in $\mathscr{C}$, I a complex of injective objects in $\mathscr{C}$ and let X be an acyclic complex of objects in $\mathscr{C}$.

Corollary 1.18.6

Let $\mathscr{C}$ be an abelian category, let P be a complex of projective objects in $\mathscr{C}$, I a complex of injective objects in $\mathscr{C}$ and let X be an acyclic complex of objects in $\mathscr{C}$.

(i) If one of P, X is bounded below, then $\mathrm{Hom}_{\mathsf{K}(\mathscr{C})}(P, X) = \{0\}$.

Corollary 1.18.6

Let $\mathscr{C}$ be an abelian category, let P be a complex of projective objects in $\mathscr{C}$, I a complex of injective objects in $\mathscr{C}$ and let X be an acyclic complex of objects in $\mathscr{C}$.

(i) If one of P, X is bounded below, then $\mathrm{Hom}_{\mathsf{K}(\mathscr{C})}(P, X) = \{0\}$.

(ii) If one of X, I is bounded below, then $\mathrm{Hom}_{\mathsf{K}(\mathscr{C})}(X, I) = \{0\}$.

Corollary 1.18.6

Let $\mathscr{C}$ be an abelian category, let P be a complex of projective objects in $\mathscr{C}$, I a complex of injective objects in $\mathscr{C}$ and let X be an acyclic complex of objects in $\mathscr{C}$.

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(ii) If one of X, I is bounded below, then $\mathrm{Hom}_{\mathsf{K}(\mathscr{C})}(X, I) = \{0\}$.

Proof of Cor. 1.18.6

Apply the above result to the short exact sequences $0 \longrightarrow X \longrightarrow X \longrightarrow 0 \longrightarrow 0$ and $0 \longrightarrow 0 \longrightarrow X \longrightarrow X \longrightarrow 0$. ∎

Projectives, Injectives and the Homotopy Category

- An abelian category $\mathscr{C}$ is said to have *enough projective objects* if for any object X in $\mathscr{C}$ there exists a projective object P in $\mathscr{C}$ and an epimorphism $\pi : P \to X$.

Projectives, Injectives and the Homotopy Category

- An abelian category $\mathscr{C}$ is said to have *enough projective objects* if for any object X in $\mathscr{C}$ there exists a projective object P in $\mathscr{C}$ and an epimorphism $\pi : P \to X$.
- Dually, $\mathscr{C}$ is said to have *enough injective objects* if for any object X in $\mathscr{C}$ there exists an injective object I and a monomorphism $\iota : X \to I$.

Projectives, Injectives and the Homotopy Category

- An abelian category $\mathscr{C}$ is said to have *enough projective objects* if for any object X in $\mathscr{C}$ there exists a projective object P in $\mathscr{C}$ and an epimorphism $\pi : P \to X$.
- Dually, $\mathscr{C}$ is said to have *enough injective objects* if for any object X in $\mathscr{C}$ there exists an injective object I and a monomorphism $\iota : X \to I$.
- If A is a k-algebra, then the category of A-modules $\mathrm{Mod}(A)$ has enough projective and injective objects.

$\bigoplus_{m \in M} A \twoheadrightarrow M$

$n \cdot : A \to A$

$\mathbb{Q}, \mathbb{Q}/\mathbb{Z}$

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- If A is a k-algebra, then the category of A-modules $\mathsf{Mod}(A)$ has enough projective and injective objects.
- If A is Noetherian, then the category of finitely generated A-modules $\mathsf{mod}(A)$ has enough projective objects, but need not have enough injective objects.

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Let P be the complex that is zero in any degree other than n and that in degree n is a projective object P_n in $\mathscr{C}$ such that there is an epimorphism $\pi : P_n \to \ker \delta_n$; this is possible since $\mathscr{C}$ has enough projective objects.

Proof of Cor. 1.18.7 (contd.) 2/2

Then π defines a chain map from P to X

$$\begin{tikzcd}
P: & \cdots \arrow[r] & 0 \arrow[r, "0_{n+1}"] \arrow[d, "0"] & P_n \arrow[r, "0_n"] \arrow[d, "\pi"] & 0 \arrow[r, "0_{n-1}"] \arrow[d, "0"] & \cdots \\
X: & \cdots \arrow[r] & X_{n+1} \arrow[r, "\delta_{n+1}"'] & X_n \arrow[r, "\delta_n"'] & X_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots
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To see this, suppose that was the case, then there is an $\eta : P_n \to X_{n+1}$ such that $\pi = \delta_{n+1} \circ \eta$.

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By dualising the above proof, one shows (ii). ∎

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relative version $C(f)$

$X \xrightarrow{f} Y \to C_f \to X[1]$

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This does not necessarily imply that f is split as a chain map because the family $(s_n)_{n \in Z}$ is not required be a chain map.

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If X is bounded below, bounded above or bounded, so is $\mathsf{C}(X)$.

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Proof of Prop. 1.18.10

Denote by δ and Δ the differentials of X and $\mathsf{C}(X)$, respectively, and define a homotopy h on $\mathsf{C}(X)$ by

$$h_n = \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} : X_{n-1} \oplus X_n = \mathsf{C}(X)_n \to \mathsf{C}(X)_{n+1} = X_n \oplus X_{n+1}$$

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Further Results on Homotopy

Proposition 1.18.11

Let A be a k-algebra, Y, Z complexes of A-modules, and let $g : Y \to Z$ be a chain map. The following are equivalent.

(i) The chain map $g : Y \to Z$ is a quasi-isomorphism.

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Proof of Prop. 1.18.11

By 1.18.10, the cone $\mathsf{C}(Z)$ is contractible. Therefore $g : Y \to Z$ is a quasi-isomorphism is and only if $\underbrace{g \oplus p_Z}_{H_*(g)} : Y \oplus \mathsf{C}(Z)[-1] \to Z$ is one; furthermore the latter map is surjective.

Further Results on Homotopy

Proof of Prop. 1.18.11 (contd.) 2/2

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Since $\mathsf{Mod}(A)$ has enough injective objects, a variation of the above arguments shows the equivalence between (i) and (iii). ∎

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(ii) For any chain map $f : Y \to Z$ of complexes over $\mathscr{C}$ and any chain map $u : X \to Z$ such that there is a graded morphism $v : X \to Y$ satisfying $u = f \circ v$ as graded morphisms, there is a chain map $w : X \to Y$ satisfying $u = f \circ w$.

Further Results on Homotopy

Proposition 1.18.12

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Proof of Prop. 1.18.12

Suppose that (i) holds.

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Suppose that (i) holds. Let h be a homotopy on X such that $h \circ \delta + \delta \circ h = \mathrm{id}_X$, where δ is the differential of X. Let $f : Y \to Z$, $u : X \to Z$ be chain maps and $v : X \to Y$ be a graded morphism such that $u = f \circ v$ as graded morphisms. Denote by ε, ζ the differentials of Y, Z, respectively.

Proof of Prop. 1.18.12 (contd.) 2/2

Set $w = v \circ h \circ \delta + \varepsilon \circ v \circ h$; one readily checks w is a chain map.

Proof of Prop. 1.18.12 (contd.) 2/2

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f \circ w &= f \circ v \circ h \circ \delta + f \circ \varepsilon \circ v \circ h \\
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Moreover, since $\mathsf{C}(X) \simeq 0$ there is a homotopy h on $\mathsf{C}(X)$ such that $\Delta \circ h + h \circ \Delta = \mathrm{id}_{\mathsf{C}(X)}$.

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For $w \circ h \circ i_X$, we note

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and hence $X \simeq 0$ via the homotopy $w \circ h \circ i_X$ on X. Thus (i) holds.

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and hence $X \simeq 0$ via the homotopy $w \circ h \circ i_X$ on X. Thus (i) holds. A similar argument, using the fact that p_X is a split epimorphism in each degree, shows that (ii) implies (i). ∎

Proposition 1.18.13

Let $\mathscr{C}$ be an additive category and let $f : X \to Y$ be a chain map of complexes over $\mathscr{C}$. The following are equivalent.

(i) $f \sim 0$.

Further Results on Homotopy

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Let $\mathscr{C}$ be an additive category and let $f : X \to Y$ be a chain map of complexes over $\mathscr{C}$. The following are equivalent.

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$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \\ & Z \arrow[ur, "h"'] & \end{tikzcd}$$

Proof of Prop. 1.18.13

Clearly (ii) implies (i).

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$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_X"'] & Y \\ C(X) \arrow[ur] & \end{tikzcd}$$

Proof of Prop. 1.18.13

Clearly (ii) implies (i). Similarly, since $\mathsf{C}(X) \simeq 0$, both (iii) and (v) imply (ii).

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Furthermore, (iv) implies (iii), because i_X is a degreewise split monomorphism;

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Furthermore, (iv) implies (iii), because i_X is a degreewise split monomorphism; and (vi) implies (v), because p_X is a degreewise split epimorphism.

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$.

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$,

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$.

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$. Therefore, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "i_X"'] & & Z \arrow[dl, "i_X \circ \rho"] \\ & \mathsf{C}(X) & \end{tikzcd}$$

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$. Therefore, we have the following commutative diagram

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Since $\mathsf{C}(X) \simeq 0$, by 1.18.12 (iii), there exists a *chain map* $g' : Z \to \mathsf{C}(X)$ such that $g' \circ \iota = i_X$.

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$. Therefore, we have the following commutative diagram

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Since $\mathsf{C}(X) \simeq 0$, by 1.18.12 (iii), there exists a *chain map* $g' : Z \to \mathsf{C}(X)$ such that $g' \circ \iota = i_X$. Consider the chain map $g = f' \circ g' : Z \to Y$;

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$. Therefore, we have the following commutative diagram

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Since $\mathsf{C}(X) \simeq 0$, by 1.18.12 (iii), there exists a *chain map* $g' : Z \to \mathsf{C}(X)$ such that $g' \circ \iota = i_X$. Consider the chain map $g = f' \circ g' : Z \to Y$; note that

$$g \circ \iota = f' \circ g' \circ \iota = f' \circ i_X = f$$

Proof of Prop. 1.18.13 (contd.) 2/3

Assume (iii); that is, there exists a chain map $f' : \mathsf{C}(X) \to Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \to Z$, so we can find a graded morphism $\rho : Z \to X$ such that $\rho \circ \iota = \mathrm{id}_X$ in $\mathsf{Gr}(\mathscr{C})$. Therefore, we have the following commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\iota"] \arrow[dr, "i_X"'] & & Z \arrow[dl, "i_X \circ \rho"] \\ & \mathsf{C}(X) & \end{tikzcd}$$

Since $\mathsf{C}(X) \simeq 0$, by 1.18.12 (iii), there exists a *chain map* $g' : Z \to \mathsf{C}(X)$ such that $g' \circ \iota = i_X$. Consider the chain map $g = f' \circ g' : Z \to Y$; note that

$$g \circ \iota = f' \circ g' \circ \iota = f' \circ i_X = f$$

That is, the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\iota"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

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Hence, (iv) holds.

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Hence, (iv) holds. Similarly, (v) implies (vi).

Proof of Prop. 1.18.13 (contd.) 3/3

Our progress

$$
\begin{tikzcd}
\text{(vi)} \arrow[dr, Leftrightarrow] & & \text{(i)} & & \text{(iv)} \arrow[dl, Leftrightarrow] \\
& \text{(v)} \arrow[r, Rightarrow] & \text{(ii)} \arrow[u, Rightarrow] & \text{(iii)} \arrow[l, Rightarrow] &
\end{tikzcd}
$$

Further Results on Homotopy

Proof of Prop. 1.18.13 (contd.) 3/3

Our progress

$$
\begin{tikzcd}
\text{(vi)} \arrow[dr, Leftrightarrow] & & \text{(i)} \arrow[dl, Rightarrow, red] \arrow[dr, Rightarrow, red] & & \text{(iv)} \arrow[dl, Leftrightarrow] \\
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Suppose (i) holds, we show then that both (iii) and (v) hold.

Further Results on Homotopy

Proof of Prop. 1.18.13 (contd.) 3/3

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Suppose (i) holds, we show then that both (iii) and (v) hold.

Denote by δ, ε the differentials of X, Y, respectively, and let $h : X \to Y$ be a homotopy satisfying $f = h \circ \delta + \varepsilon \circ h$.

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We define a graded morphisms $r : \mathsf{C}(X) \to Y$ and $s : X \to \mathsf{C}(Y)[-1]$ by setting, for any integer n,

$$r_n = \begin{pmatrix} h_{n-1} & f_n \end{pmatrix} : X_{n-1} \oplus X_n \to Y_n \quad \text{and} \quad s_n = \begin{pmatrix} f_n \\ h_n \end{pmatrix} : X_n \to Y_n \oplus Y_{n+1}$$

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\begin{tikzcd}
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One readily checks that r and s are chain maps satisfying $f = r \circ i_X = p_X \circ s$. ∎

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Proposition 1.18.15

Let A be a k-algebra and X a complex of A-modules.

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Proof of Prop. 1.18.15 (contd.) 2/3

Then $\delta \circ s$ and $s \circ \delta$ are graded idempotent endomorphisms of degree 0 of X.

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Recall that $\mathrm{im}(\delta) \subseteq \ker(\delta)$, one then shows that $\ker(\delta) = \mathrm{im}(\delta) \oplus H$, as graded A-modules,

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The graded submodule $Y = \mathrm{im}(s \circ \delta) \oplus \mathrm{im}(\delta)$ is a subcomplex of X because δ maps $\mathrm{im}(\delta)$ to zero and $\mathrm{im}(s \circ \delta)$ to $\mathrm{im}(\delta \circ s \circ \delta) = \mathrm{im}(\delta)$.

Proof of Prop. 1.18.15 (contd.) 3/3

We need to show that Y is contractible.

Proof of Prop. 1.18.15 (contd.) 3/3

We need to show that Y is contractible. On $Y = \mathrm{im}(s \circ \delta) \oplus \mathrm{im}(\delta)$ we define the homotopy h such that h is zero on the summand $\mathrm{im}(s \circ \delta)$ and equal to s on the summand $\mathrm{im}(\delta)$.

$h = (0 \quad s)$

$\mathrm{id}_Y = \underline{\delta} \circ h + h \circ \delta$

$\delta|_Y$

Proof of Prop. 1.18.15 (contd.) 3/3

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Now, since $X \cong Y \oplus H_*(X)$, we can find chain maps $f : X \leftrightarrows Y \oplus H_*(X) : g$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_{Y \oplus H_*(X)}$.

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The second statement follows. ∎

Corollary 1.18.16

Let A be a k algebra, and let X, Y be split complexes of A-modules. A chain map $f : X \to Y$ is a quasi-isomorphism if and only if f is a homotopy equivalence.

Homotopy and Split Complexes

Corollary 1.18.16

Let A be a k algebra, and let X, Y be split complexes of A-modules. A chain map $f : X \to Y$ is a quasi-isomorphism if and only if f is a homotopy equivalence.

Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes.

Homotopy and Split Complexes

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Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes. Therefore $X \simeq H_*(X)$ and $Y \simeq H_*(Y)$.

Homotopy and Split Complexes

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Homotopy and Split Complexes

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Homotopy and Split Complexes

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Let A be a k-algebra, and let X be a bounded complex of projective A-modules such that $H_i(X)$ is projective for all integers i. Then X is split.

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We have $H_m(X) = X_m / \operatorname{im}(\delta_{m+1})$, which is projective by the assumptions.

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Homotopy and Split Complexes

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Since X_m is projective, so is $\operatorname{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \to \operatorname{im}(\delta_{m+1})$ is split surjective.

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Since X_m is projective, so is $\operatorname{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \to \operatorname{im}(\delta_{m+1})$ is split surjective. Thus $X_{m+1} \cong \operatorname{im}(\delta_{m+1}) \oplus \ker(\delta_{m+1})$.

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Since X_m is projective, so is $\operatorname{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \to \operatorname{im}(\delta_{m+1})$ is split surjective. Thus $X_{m+1} \cong \operatorname{im}(\delta_{m+1}) \oplus \ker(\delta_{m+1})$. Hence $\ker(\delta_{m+1})$ is projective as well.

Proof of Prop. 1.18.17 (contd.) 2/2

It follows that the complex X is the direct sum of the following complexes

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- a contractible complex, where the two terms are in degree $m+1$ and m, and

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- a complex X' that coincides with X in all degrees, except in degree $m+1$, where $X'_{m+1} = \ker \delta_{m+1}$ and $X'_m = \{0\}$.

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All terms and homology of X' are still projective, and hence X' is split by induction.

Homotopy and Split Complexes

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All terms and homology of X' are still projective, and hence X' is split by induction. X is therefore split by 1.18.15 (i) and the result follows. ∎

Proposition 1.18.18

Let $\mathscr{C}$ be an abelian category and let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a degreewise split short exact sequence of chain complexes over $\mathscr{C}$.

(i) The chain map f is a homotopy equivalence if and only if $Z \simeq 0$. In that case, f is a split monomorphism.

(ii) The chain map g is a homotopy equivalence if and only if $X \simeq 0$. In that case, g is a split epimorphism.

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Proof of Prop. 1.18.18

Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f.

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Proof of Prop. 1.18.18

Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f. Then $f' \circ f \sim \mathrm{id}_X$, or equivalently $\mathrm{id}_X - f' \circ f \sim 0$.

Homotopy and Split Complexes

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Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f. Then $f' \circ f \sim \mathrm{id}_X$, or equivalently $\mathrm{id}_X - f' \circ f \sim 0$. By 1.18.13, $\mathrm{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism.

Homotopy and Split Complexes

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Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f. Then $f' \circ f \sim \mathrm{id}_X$, or equivalently $\mathrm{id}_X - f' \circ f \sim 0$. By 1.18.13, $\mathrm{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \to X$ be a chain map such that $\mathrm{id}_X - f' \circ f = t \circ f$.

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Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f. Then $f' \circ f \sim \mathrm{id}_X$, or equivalently $\mathrm{id}_X - f' \circ f \sim 0$. By 1.18.13, $\mathrm{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \to X$ be a chain map such that $\mathrm{id}_X - f' \circ f = t \circ f$. Then $\mathrm{id}_X = (f' + t) \circ f$,

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(ii) The chain map g is a homotopy equivalence if and only if $X \simeq 0$. In that case, g is a split epimorphism.

Proof of Prop. 1.18.18

Suppose that f is a homotopy equivalence, and let $f' : Y \to X$ be a homotopy inverse of f. Then $f' \circ f \sim \mathrm{id}_X$, or equivalently $\mathrm{id}_X - f' \circ f \sim 0$. By 1.18.13, $\mathrm{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \to X$ be a chain map such that $\mathrm{id}_X - f' \circ f = t \circ f$. Then $\mathrm{id}_X = (f' + t) \circ f$, hence f is split as a chain map with retraction $f' + t$.

Homotopy and Split Complexes

Proposition 1.18.18

Let $\mathscr{C}$ be an abelian category and let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a degreewise split short exact sequence of chain complexes over $\mathscr{C}$.

(i) The chain map f is a homotopy equivalence if and only if $Z \simeq 0$. In that case, f is a split monomorphism.

(ii) The chain map g is a homotopy equivalence if and only if $X \simeq 0$. In that case, g is a split epimorphism.

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$$0 \longrightarrow X \xrightarrow{i} X \oplus Z \xrightarrow{p} Z \longrightarrow 0$$

Homotopy and Split Complexes

Proof of Prop. 1.18.18 (contd.) 2/2

Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q.

Proof of Prop. 1.18.18 (contd.) 2/2

Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q. Let π and ι be the canonical retract and section of i and p respectively, we then have $q \sim \pi$.

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Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q. Let π and ι be the canonical retract and section of i and p respectively, we then have $q \sim \pi$. Now

$$\mathrm{id}_Z = p \circ \iota = p \circ \mathrm{id}_{X \oplus Z} \circ \iota \sim p \circ (i \circ q) \circ \iota \sim p \circ i \circ (\pi \circ \iota) = p \circ i \circ 0 = 0$$

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Conversely, suppose that $Z \simeq 0$.

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Conversely, suppose that $Z \simeq 0$. Since g is degreewise split surjective, it follows from 1.18.12 (ii) that g is split surjective as a chain map.

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Hence $Z \simeq 0$.

Conversely, suppose that $Z \simeq 0$. Since g is degreewise split surjective, it follows from 1.18.12 (ii) that g is split surjective as a chain map. Thus f and g induce an isomorphism $Y \simeq X \oplus Z$,

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Corollary 1.18.19

Let X, Y be chain complexes over an abelian category $\mathscr{C}$.

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In that case, if the complexes X, Y are both bounded above (resp. bounded below, bounded), then the complexes P, Q can be chosen to be bounded above (resp. bounded below, bounded), too.

Homotopy and Split Complexes

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Suppose that $f : X \to Y$ is a homotopy equivalence. Set $P = \mathsf{C}(Y)[-1]$ and $p = p_Y$. Note that p is a degreewise split epimorphism. Thus the chain map $(f, p) : X \oplus P \to Y$ is a degreewise split epimorphism. Since $P \simeq 0$, the chain map (f, p) is still a homotopy equivalence. By 1.18.18, the complex $Q = \ker(f, p)$ satisfies $Q \simeq 0$ and $X \oplus P \cong Y \oplus Q$ in $\mathsf{Ch}(\mathscr{C})$. The converse is trivial. The last statement follows from the fact that if Y is bounded above (resp. bounded below, bounded), so is $\mathsf{C}(Y)$. ■

Homotopy and Split Complexes

Corollary 1.18.20

Let $\mathscr{A}$ be an abelian category.

(i) Let P, Q be bounded below chain complexes consisting of projective objects in $\mathscr{A}$, and let $f : P \to Q$ be a chain map. Then f is a quasi-isomorphism if and only if f is a homotopy equivalence.

(ii) Let I, J be bounded below cochain complexes consisting of injective objects in $\mathscr{A}$, and let $g : I \to J$ be a cochain map. Then g is a quasi-isomorphism if and only if g is a homotopy equivalence.

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Suppose that f is a quasi-isomorphism. Since the cone $\mathsf{C}(P)$ is contractible by 1.18.10, we may replace Q by $Q \oplus \mathsf{C}(P)$ and f by $\begin{pmatrix} f \\ i_X \end{pmatrix}$.

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Homotopy and Split Complexes

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Final Remarks

Let A be a k-algebra, X a complex of right A-modules, and Y, Z complexes of left A-modules.

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If $h : Y \to Z$ is a homotopy, then $\mathrm{id}_X \otimes h$ is a homotopy from $X \otimes_A Y$ to $X \otimes_A Z$. In particular, if $Y \simeq Z$, then $X \otimes_A Y \simeq X \otimes_A Z$.

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Similar statements hold for functors of the form $\mathrm{Hom}_A(Y, -)$ and $\mathrm{Hom}_A(-, Z)$.

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Fin.

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