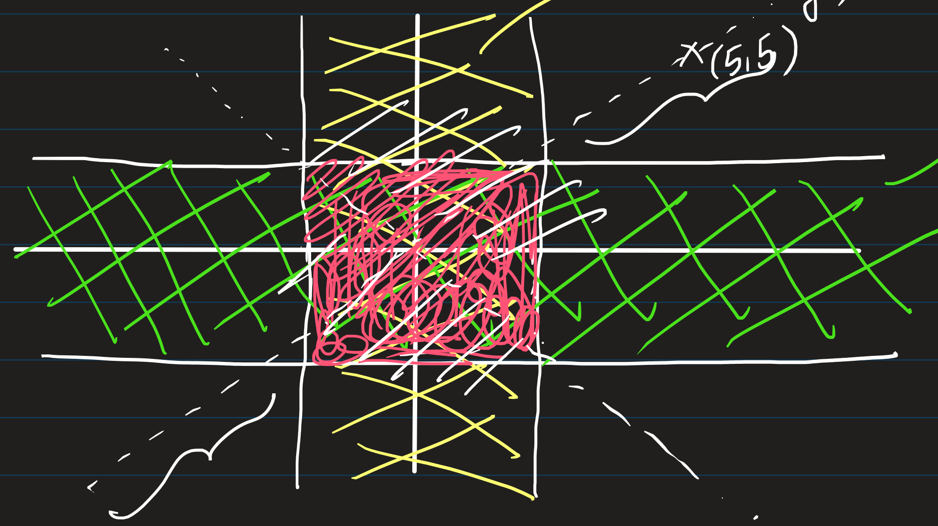
3 [2]

$R = \{(x,y) \in \mathbb{R}^2 \mid |x| \le 1,\ |y| \le 1\} = \{(x,y) \mid |x| \le 1\} \cap \{(x,y) \mid |y| \le 1\}$




R is not reflexive

$5 \not R 5$

$(5,5) \in \mathbb{R}^2$ but $|5| > 1$ //

so $(5,5) \notin R$

$\mathbb{Q}$ are countable (picture is fine!) //

$\mathbb{Z}$ are countable ~ can write down

+ve — even numbers

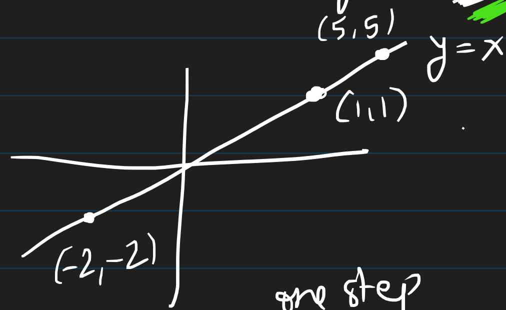
−ve — odd numbers //

$f: \mathbb{Z} \to \mathbb{N}$

$$\boxed{\begin{array}{l} n \longmapsto 2n \\ -n \longmapsto 2n-1 \end{array}} \Bigg\} \star$$

* Is the set $\{(x,x) \mid x\in\mathbb{R}\} \subseteq R$

$(5,5)$ What it means to reflexive. $\{(\pi,\pi)\mid$




one step

$$\{(x,y)\mid y=x\} = \{(x,x)\mid x\in\mathbb{R}\}$$

$|y|\le 1$

$(x,y)\in R$ is $(y,x)\in R$

$|x|\le 1$

$\underbrace{|x|\le 1 \,\&\, |y|\le 1}$

Transitive

want $(x,z)\in R$

$(x,y), (y,z)\in R$

what does it mean $|x|\le 1, |z|\le 1$

$\cancel{|x|\le 1}, |y|\le 1, \cancel{|z|\le 1}$

$R\subseteq \mathbb{R}^4$

* 9.42 (a), (b), (c)

* Mid, 3[3]

* 10.22

* Mid, 5 (eq classes)

* 10.8

9.42

ⓐ Let $f: A \to B$, $g: B \to C$ be bijective

Then $g \circ f: A \to C$ is bijective

Proof: Method 1 — prove $g \circ f$ is injective & surjective

Inject To show if $g(f(x)) = g(f(y))$

for some $x, y \in A$, then $x = y$.

By injective (since $g$ is bijective) of $g$

$f(x) = f(y)$

By injective of $f$, $\boxed{x = y}$.

Surjective

$g \circ f: A \to C$ is surj

Consider $c \in C$ to find a $a \in A$ st

$g(f(a)) = c$.

$c \in C$, since $g$ is surjective $\exists b \in B$ st $g(b) = c$

for $b \in B$, by surj of $f$, $\exists a \in A$ st $f(a) = b$

Hence $g(f(a)) = c$

Method 2. bijective function has an inverse function

If a function has an inverse, it's bijective

$h$ an $h^{-1}$ is st

$$h \circ h^{-1} = id_{dom(h^{-1})}$$

$$h^{-1} \circ h = id_{dom(h)}$$

$f$ & $g$ are bij, consider the inverse $f^{-1}$ & $g^{-1}$

We'll produce an inverse for $g \circ f$ to conclude $g \circ f$ is bijective

Claim: $f^{-1} \circ g^{-1}$ is the inverse of $g \circ f$

To show

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = \text{identity}$$

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$$

and $(f^{-1} \circ g^{-1}) \circ (g \circ f) = \text{identity}$

$$(AB)^T = B^T A^T$$

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1}$$

$$= g \circ id \circ g^{-1} = g \circ g^{-1} = id$$

$$(f^{-1}\circ g^{-1})\circ(g\circ f) = f^{-1}\circ(g\circ g^{-1})\circ f$$
$$= f^{-1}\circ id\circ f$$
$$= f^{-1}\circ f = id.$$

$$\boxed{\begin{aligned} &f\circ(g\circ h) \\ &= (f\circ g)\circ h \end{aligned}}$$

$$\boxed{(g\circ f)^{-1} = f^{-1}\circ g^{-1}}$$

Midterm

3[3]

$$R = \{(x,y)\in\mathbb{R} \mid |x-y| \le 1\}$$

Reflexive: Is $\underbrace{(x,x)\in R}$ for any $x\in\mathbb{R}$

i.e. $(x,x)$ has to satisfy the property that defines R

Is $|x-x|\le 1$?

Yes! since $|x-x| = 0$ $(0\le 1)$

Symmtric(ity): If $(x,y)\in R$, is $(y,x)\in R$

i.e. if $|x-y|\le 1$, is $|y-x|\le 1$

Yes, since $\underbrace{|y-x| = |x-y|}\le 1$

Transitive: NOT true, If $|x-y|\le 1$, $|y-z|\le 1$, then $|x-z|\le 1$ //

Because the $\Delta^{le}$ inequality

$$|x-z| = |x-y+y-z| = |x-y+(y-z)| \leq |x-y| + |y-z| \leq 2$$

$$\underbrace{x=1, \quad y=0}_{|x-y|=1\leq 1} \quad \underbrace{, \quad z=-1}_{|y-z|=|0-(-1)|=1 \leq 1}$$

BUT. $|x-z| = |1-(-1)| = 2 > 1$

Counterexample//..

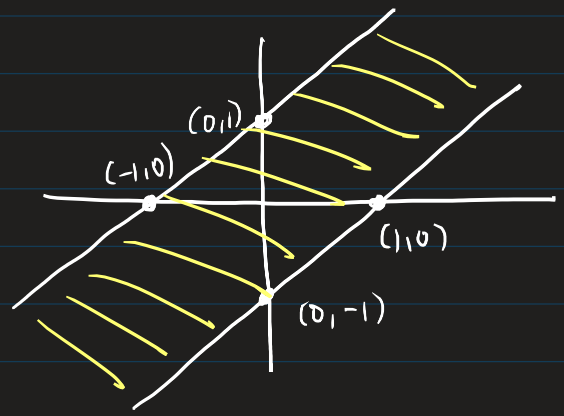
$$|x-y| \leq 1$$

$$-1 \leq x-y \leq 1$$

$$-1-x \leq -y \leq 1-x$$

$$1+x \geq y \geq x-1$$

$$\underbrace{1-x}_{y=1-x} \leq y \leq \underbrace{1+x}_{\to \text{line } y=1+x}$$



Midterm 5[2]

$R$ on $\mathbb{Z}$ is defined as $aRb$ if $|a-2| = |b-2|$

Let's find equivalence classes

Suppose $|x| = |y|$

what can $x$ be in terms of $y$?

$x = y$ or $x = -y$

$x = \pm y$

$[a] = \{ b \in \mathbb{Z} \mid aRb \}$ def$^n$ of equiv class

$= \{ b \in \mathbb{Z} \mid |a-2| = |b-2| \}$ def$^n$ of $R$

$= \{ b \in \mathbb{Z} \mid (a-2) = \pm (b-2) \}$ unpacking abs value

$= \{ b \in \mathbb{Z} \mid b-2 = a-2$ or $b-2 = -(a-2) \}$

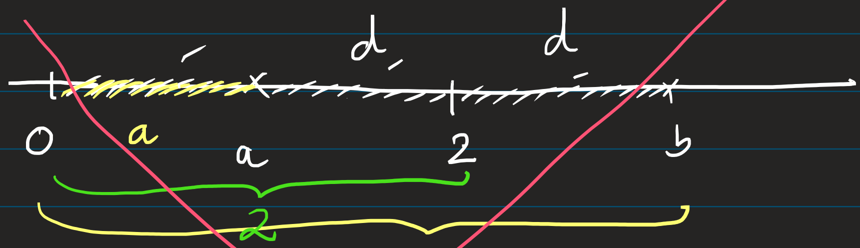
$b-2 = 2-a \Rightarrow b = 4-a$

$= \{ b \in \mathbb{Z} \mid b = a$ or $b = 4-a \}$

$= \{ a, 4-a \}$

$[1] = \{1, 4-1\}$ $\qquad\qquad$ $[2] = \{2, 4-2\} = \{2\}$

$[3] = \{3, 4-3\}$




$$\left.\begin{aligned} b &= a+2d \\ 2 &= a+d \end{aligned}\right\}$$

$$b = a+d+d = 2+d$$

so $\boxed{d = 2-b}$

10-22

Find a $f : (-2,2) \longrightarrow \mathbb{R}$ st $f$ is bijective

From the book we have a bij $g : (-1,1) \longrightarrow \mathbb{R}$ ✓

Let's construct a bij function $h : (-2,2) \longrightarrow (-1,1)$

$g \circ h$ will be bij

This the $f$ we're looking for

$x \longmapsto e^x > 0$ $\qquad$ $-0.5$ $\qquad$ $h(x) = h(y)$

$h : (-2,2) \longrightarrow (-1,1)$ $\qquad$ $\frac{x}{2} = \frac{y}{2}$

$x \longmapsto x/2$ $\qquad$ $x = y$ ✓

$2y \longmapsto y$ surj ///

why is it bij b/c we can construct an inverse fn

$$k: (-1,1) \longrightarrow (-2,2)$$
$$y \longmapsto 2y$$

$$h \circ k\,(y) = h(k(y)) = h(2y) = \frac{2y}{2} = y$$

$$k \circ h(x) = k(h(x)) = k(x/2) = 2(x/2) = x$$

$k = h^{-1}$, so $h$ is bij.

$h$ is surj if for a $y \in (-1,1)$ we can find a $x \in (-2,2)$ st $h(x) = y$

let $x = 2y$

$$\cancel{g(x) = \frac{x}{|x|-1}}$$

10.8

$$\boxed{\cancel{g: \mathbb{Z} \to \mathbb{N},}}$$

$$f: \mathbb{N} \to \mathbb{Z}$$

$$f(n) = \frac{1 + (-1)^n(2n+1)}{4} \; ; \; f(n) = \begin{cases} \frac{n+1}{2} & n \text{ is even} \\ -\frac{n}{2} & n \text{ is odd} \end{cases}$$

show it's inj & surj

Let $f(n)=f(m)$

$$\frac{1+(-1)^n(2n+1)}{\cancel{4}} = \frac{1+(-1)^m(2m+1)}{\cancel{4}}$$

$$(-1)^n(2n+1) = (-1)^m(2m+1)$$

$$(-1)^{n-m}(2n+1) = 2m+1$$

Case I: $n-m$ is $\underline{\text{even}}$ $\quad 2n+1 = 2m+1 \Rightarrow n=m$

Case II: $\boxed{n-m \text{ is odd}}$ $\quad$ so $-(2n+1) = 2m+1$

$$-2n-1 = 2m+1$$

$$2(m+n)+2=0$$

$$m+n=-1$$

$m,n \in \mathbb{N}$ can't happen

$$m = \frac{1+(-1)^n(2n+1)}{4}, \qquad m = \frac{1+(2n+1)}{4} = \frac{n+1}{2}, \quad n = 2m-1$$

$n$ is even, solve for $n$ ~ $f(\quad)=n$

$n$ is odd solve for $n$ ~ $f(\quad)=n$

$$f(2m-1)=m$$

$m \in \mathbb{Z}$ there exist an $\underline{n \in \mathbb{N}}$ st $m=f(n)$