Realizing Finite Groups as Automizers

Sylvia Bayard

UCSC Graduate Colloquium

April 25, 2022

Automorphism Groups

- For an object X in a category C, the automorphism group of X in C, denoted $\mathrm{Aut}_C(X)$ is the group of all C-isomorphisms $X \to X$ under composition. For a group G, we write $\mathrm{Aut}(G) := \mathrm{Aut}_{\mathbf{Grp}}(G)$, the group of all group automorphisms of G.
- The inner automorphism group of G is the normal subgroup of automorphisms that G induces on itself by conjugation:

$$\mathrm{Inn}(G) = \{x \mapsto gxg^{-1} : g \in G\} \trianglelefteq \mathrm{Aut}(G)$$

- In general, Inn(G) is isomorphic to the quotient group $G/Z(G)$.

Which groups can be automorphism groups?

- It is not true in general that a finite group A is isomorphic to $\mathrm{Aut}(G)$ for some finite group G.

Proposition

Let A be a finite cyclic group of odd, non-trivial order. Then there exists no finite group G such that $\mathrm{Inn}(G) \cong A$. Moreover, there exists no finite group G such that $\mathrm{Aut}(G) \cong A$.

Automizers of Subgroups

- For a subgroup H of G, we define the group $\mathrm{Aut}_G(H)$ to be the group of automorphisms on H induced by conjugation by elements of G.

$$\mathrm{Aut}_G(H) := \{\varphi \in \mathrm{Aut}(H) : \varphi = c_g|_H \text{ for some } g \in G\}.$$

This group is called the automizer of H in G, and we have

$$\mathrm{Aut}_G(H) \cong N_G(H)/C_G(H).$$

Proposition

Let A be a finite group. Then there exists a group G with subgroup U such that $\mathrm{Aut}_G(U) \cong A$. Moreover, we can choose U to be any homocyclic group of rank at least $|A|$.

Wreath Products

- Let G and H be finite groups, and let $\alpha : G \to \mathrm{Sym}(n)$ be a homomorphism. We define the wreath product $H \wr_\alpha G$ of H and G to be the semidirect product $H^n \rtimes G$ where G acts on H^n by permuting the direct product according to α.
- When the permutation action of G is obvious, we omit α and just write $H \wr G$.
- We can now prove the above theorem. Let A be any finite group, and let $\alpha : A \to \mathrm{Sym}(|A|)$ be the left regular representation. Then for any natural $n > 1$, construct $G := C_n \wr_\alpha A = U \rtimes A$, where $U = C_n^{|A|}$. Then we can see immediately that $\mathrm{Aut}_G(U) \cong N_G(U)/C_G(U) = G/U \cong A$.

- We all know that D_8 is the group of reflection and rotation symmetries of the square. Suppose we have n squares which we may independently rotate and reflect. In addition, we may perform any permutation on the squares. Then the resulting group is $D_8 \wr \mathrm{Sym}(n)$.
- Consider the matrices in $GL_n(\mathbb{C})$ with precisely one non-zero element in each row and column. We can think of these as weighted permutation matrices. Now consider the group G of such matrices whose non-zero entries are all on the unit circle. Then $G \cong S^1 \wr \mathrm{Sym}(\mathrm{n})$. This is a Lie group with $n!$ connected components.
- Let p be a prime, and let W_n be the Sylow p-subgroup of $\mathrm{Sym}(p^n)$. Then $W_1 \cong C_p$, and in general $W_{n+1} \cong W_n \wr C_p$.

Onto the Research Proper

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Definition

Let H be a subgroup of G. The normal closure of H in G, $\langle H^G \rangle$, is the smallest normal subgroup of G which also contains H. In general,

$$\left\langle H^G \right\rangle = \left\langle ghg^{-1} : h \in H, g \in G \right\rangle .$$

That is, it is the subgroup generated by G-conjugates of H.

We show in our paper that one can construct a finite perfect group G with a homocyclic subgroup U such that $\mathrm{Aut}_G(U)$ is isomorphic to an arbitrary finite A, and $\left\langle U^G \right\rangle = G$.

The Park Embedding

To achieve this, we will use a 2016 result of Dr. Sejong Park.

Theorem (Park 2016.)

Let $\mathcal{F}$ be a fusion system on a finite p-group S. Then there is a finite group G having S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.

More specifically, we use the observation of [War19] that this applies to "fusion" over any finite group, not just p-groups. To understand this, we need to know what a fusion system is in the first place.

Fusion Systems

Definition

A fusion system over a group S is a category $\mathcal{F}$, where the objects $\mathrm{Ob}(\mathcal{F})$ of $\mathcal{F}$ are the set of subgroups of S and for all $P, Q \leq S$:

- $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ consists of injective homomorphisms from P to Q, including all such morphisms induced by S-conjugation. Notice this includes all inclusion functions as conjugation by the identity element.
- Each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomophism from P to $\varphi(P)$ composed with an inclusion.

This is identical to Puig's definition found in [AKO11], but with the requirement that S is a p-group is omitted. One quick example of a fusion system for a p-group $S \leq G$ is $\mathcal{F}_S(G)$, which is induced by G-conjugation. Note that not all morphisms need to extend to a morphism of the entirety of S.

Fusion System Background

- Fusion systems arise in the study of a finite group's Sylow p-subgroups, the subgroups of maximal prime-power order for a given prime p.
- Sylow p-subgroups are an essential tool for understanding finite groups, as Sylow's Theorems offer strong conditions on the properties of these subgroups and how they fit into the group.
- By understanding the possible fusion systems of p-groups, we can better understand what fusion a group can induce on its Sylow p-subgroups. In general, the goal is to deduce global properties of a group through the "local" properties at a prime. In our case, however, the main goal is simply to ensure certain conjugation in our embedding to maximise the normal closure.

Realizing fusion systems inside finite groups

Theorem ([War19], cf. [Par16])

Let $\mathcal{F}$ be a fusion system over a finite group S. Then there exists a finite group G having S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.

- This work is immediately relevant to our broader goal, since for any $U \in S$, $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}_{\mathcal{F}_S(G)}(U) = \mathrm{Aut}_G(U)$.

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- A core component to this construction is an S-S-biset, which is a set X having a left and a right S-action such that for all $x \in X$, $u, v \in S$, $(ux)v = u(xv)$.
- We can treat any S-S-biset as a left $S \times S$-set by the action $(u, v)x = uxv^{-1}$. Throughout the rest of this talk, we will use "S-S-biset" in accordance with the work we cite, but we will actually work in the language of left $S \times S$-sets.

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Left Semi-characteristic Biset

Definition ([Par16])

Let $\mathcal{F}$ be a fusion system over a finite group S, and let X be an S-S biset. For any $Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, define the "twisted diagonal subgroup" $\Delta(Q, \varphi) := \{(q, \varphi(q)) : q \in Q\}$. Then X is called a "left semi-characteristic S-S-biset" if:

- X is $\mathcal{F}$-generated: Every orbit of X is isomorphic to the $S \times S$-set $(S \times S)/\Delta(Q, \varphi)$ for some $Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$.
- X is left $\mathcal{F}$-stable: For every $Q \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, S)$, ${}_QX \cong {}_{\varphi}X$ are isomorphic as Q-sets, where ${}_QX$ is the action of $Q \times S$ on X and ${}_{\varphi}X$ is the action of $Q \times S$ through $\varphi \times \mathrm{id}_S$.

Let X be such a biset, and define for the remainder of the talk the group $G := \mathrm{Aut}({}_1X)$ of bijections $X \to X$ which preserve the action of $1 \times S$. [Par16] shows that this is precisely the group G from the previous theorem; where $\mathcal{F}_S(G) = \mathcal{F}$.

Burnside Rings and the Mark Homomorphism

- For a finite group G, let the Burnside ring of G, denoted $B(G)$ be the $\mathbb{Z}$-module generated by isomorphism classes of finite G-sets, where addition is the disjoint union. $B(G)$ can be given a ring structure by the Cartesian product of sets. Now for a conjugation class $[H]$ of subgroups of G and an $X \in B(G)$, let $\varphi_{[H]}(X) := |X^H|$. This extends to an injective ring homomorphism $\varphi : B(G) \to \mathbb{Z}^{\mathrm{Conj}(G)}$.
- From this, we may conclude that two finite G-sets X and Y are isomorphic if and only if for all $H \leq G$, $|X^H| = |Y^H|$. Thus Burnside rings are incredibly useful for constructing G-sets with certain properties by carefully adding up fixed points for particular subgroups.
- To construct a left semi-characteristic biset, we will use the Burnside ring with rational coefficients $\mathbb{Q} \otimes_{\mathbb{Z}} B(G)$.

Constructing a Left Semi-characteristic Biset [BLO03]

Let $\mathcal{F}' = \mathcal{F} \times \mathcal{F}_S(S)$ be the product fusion system on $S \times S$. Observe that the set of subgroups of the form $\Delta(P, \varphi)$ with $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ is closed under $\mathcal{F}'$-conjugacy and taking subgroups. Our goal is to build an S-S biset X which has constant fixed point set size on $\mathcal{F}'$-conjugacy classes of twisted diagonal subgroups. Given that X is $\mathcal{F}$-generated, the above is sufficient to show it is $\mathcal{F}$-stable.

Let

$$X_0 = \sum_{[\alpha] \in \mathrm{Out}_{\mathcal{F}}(S)} (S \times S)/\Delta(S, \alpha),$$

where the sum denotes disjoint union. This is a $\mathcal{F}$-generated virtual S-S biset with nonnegative rational coefficients having the property that $|X_0^{\Delta(S,\beta)}| = |N_{S\times S}(\Delta(S, \mathrm{id}))/\Delta(S, \mathrm{id})| = |Z(S)|$ for all $\beta \in \mathrm{Aut}_{\mathcal{F}}(S)$. Thus, the fixed point sizes in X_0 are constant on $\mathcal{F}'$-conjugacy classes of twisted diagonal subgroups $\Delta(S, \beta)$ with $\beta \in \mathrm{Aut}_{\mathcal{F}}(S)$.

Let $\mathcal{H}$ be a set of subgroups of $S \times S$ of the form $\Delta(P, \varphi)$ with $\varphi\colon P \to S$ in $\mathcal{F}$ such that $\mathcal{H}$ is closed under $\mathcal{F}'$-conjugacy and taking subgroups. Assume given inductively an $\mathcal{F}$-generated virtual S-S biset X_0 with nonnegative rational coefficients such that fixed point sizes on X_0 are constant on $\mathcal{F}'$-conjugacy classes of twisted diagonal subgroups which are not in $\mathcal{H}$. Let $\mathcal{P}$ be an $\mathcal{F}'$-conjugacy class in $\mathcal{H}$ whose members are maximal under inclusion among the subgroups in $\mathcal{H}$, and let $\Delta(P, \varphi) \in \mathcal{P}$ be a subgroup for which the fixed point set $X_0^{\Delta(P,\varphi)}$ has largest size (among the elements of $\mathcal{P}$). Define

$$X_1 = X_0 + \sum_{\Delta} \frac{|X_0^{\Delta(P,\varphi)}| - |X_0^{\Delta}|}{|N_{S \times S}(\Delta)/\Delta|}(S \times S)/\Delta$$

where the sum runs over a set of representatives Δ for the subgroups in $\mathcal{P}$ up to $S \times S$-conjugacy.

$$X_1 = X_0 + \sum_{\Delta} \frac{|X_0^{\Delta(P,\varphi)}| - |X_0^{\Delta}|}{|N_{S\times S}(\Delta)/\Delta|}(S \times S)/\Delta$$

Thus, X_1 is an $\mathcal{F}$-generated virtual S-S biset with nonnegative rational coefficients. A subgroup $D \notin \mathcal{H} - \mathcal{P}$ has a fixed point on $(S \times S)/\Delta$ if and only if D is $S \times S$-conjugate to some Δ, and in this case the number of such fixed points is $|N_{S\times S}(\Delta)/\Delta|$. So by construction, $|X_1^D| = |X_0^D|$ for each subgroup $D \leq S \times S$ which is not in $\mathcal{H}$, and $|X_1^D| = |X_0^{\Delta(P,\varphi)}|$ for each $D \in \mathcal{P}$. In particular, fixed point sizes on X_1 are constant on $\mathcal{F}'$-conjugacy classes of twisted diagonal subgroups which are not in $\mathcal{H} - \mathcal{P}$.

Repeat inductively until we have a virtual S-S-biset with non-negative rational coefficients $X_{\mathbb{Q}}$. Then we may choose a natural number m such that $X := mX_{\mathbb{Q}}$ is a virtual S-S-biset with non-negative integer coefficients; thus, X is an S-S-biset satisfying our fixed point condition on twisted diagonal subgroups. [Par16] goes on to show that X is a left semi-characteristic biset for $\mathcal{F}$.

Theorem (Park 2016)

Let $\mathcal{F}$ be a fusion system on a finite group S, and let X be a semi-characteristic biset for $\mathcal{F}$. Now let $G := \mathrm{Aut}({}_1X)$ be the group of bijections of X which preserve the right action (action of $1 \times S$). Then $G \cong S \wr \mathrm{Sym}(n)$ for some natural number n. Further, we may identify S with its left multiplication on X so that $S \leq G$. Then $\mathcal{F} = \mathcal{F}_S(G)$.

This is what we use to realize finite groups as automizers in our paper.

Our core result

Theorem ([BL22])

For each finite group A, there exist a finite perfect group G and a homocyclic abelian subgroup U such that $\langle U^G \rangle = G$ and $\mathrm{Aut}_G(U) \cong A$.

Assume $A \neq 1$. Let e be the exponent of A. Consider the homocyclic group $U = C_e^{|A|} \times C_e^{|A|}$ with regular action of A on each $C_e^{|A|}$ factor, and let $S := U$ be the semidirect product with respect to this action. Thus, $\mathrm{Aut}_S(U) \cong A$. Let $\mathcal{V}$ be the collection of all rank 2 homocyclic subgroups of S of order e^2, and define

$$\mathcal{F} = \langle \mathrm{Aut}(V) \mid V \in \mathcal{V} \rangle_S.$$

Since the domain of all morphisms added by V is at most e^2, and $|U| \geq e^4$, $\mathrm{Aut}_{\mathcal{F}}(U) = \mathrm{Aut}_S\, U \cong A$. In the paper, we show there is a subfamily $\tilde{\mathcal{V}}$ of $\mathcal{V}$ such that $S = \left\langle \tilde{\mathcal{V}} \right\rangle$ and $\cap \tilde{\mathcal{V}} = \{1\}$. When we build $S \leq \Gamma := S \wr \mathrm{Sym}(n) = B \rtimes \mathrm{Sym}(n)$ as per [Par16], the above conditions are sufficient to show that $S \leq \Gamma'$, $S \cap B = \{1\}$ and $n > 2|A| \geq 4$, where S^n here is the base subgroup of Γ.

Now we take $G := \Gamma'$. As mentioned above, $S \leq G$. Let H be the alternating subgroup inside $\mathrm{Sym}(n) \in \Gamma = S \wr \mathrm{Sym}(n)$. We show in the paper that $G = H[H, B]$, and that G is a perfect group. Notice that this implies that $\langle H^G \rangle = G$. Thus if $\langle U^G \rangle \cap N \neq 1$, $H \leq \langle U^G \rangle$, and thus $\langle U^G \rangle = G$. Suppose the contrary for contradiction. Since $U \cap B = \{1\}$, $\langle U^G \rangle$ projects onto a nontrivial normal subgroup of H. Simplicity of H implies this image is H itself, so $|H|$ divides $|\langle U^G \rangle|$. Since, as we mentioned earlier, $n > 2|A| \geq 4$, H is a simple group. Now we may pick a prime p with $|A| < p < 2|A|$ so that p divides $|H|$ but not $|S|$, so p also does not divide B. Then $|H|^2$ does not divide $|G|$. But since $\langle U^G \rangle \cap H = \{1\}$, $\langle U^G \rangle H$ is a subgroup of G with order divisible by $|H|^2$. We have reached a contradiction, so $\langle U^G \rangle = G$.

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Thank you!

Questions?