Algebras in Tensor Triangular Categories

Seperability, Descent and Finite Étale Extensions

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18th October, 2021

Some Motivation

Derived Category of Quasi-Coherent Sheaves

Let $j : U \hookrightarrow V$ be a Zariski open Subscheme of a scheme V. It is a well known (well, to some!) theorem that the derived pullback functor $j^* : D^{qcoh}(V) \longrightarrow D^{qcoh}(U)$ is a smashing localization.

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- the localization functor $D^{qcoh}(V) \to D^{qcoh}(V)/D_Z^{qcoh}(V) \cong D^{qcoh}(U)$ has a right adjoint, which is automatically fully faithful.
- Since the right adjoint is fully faithful we can really view $D^{qcoh}(U)$ as being a "piece" of $D^{qcoh}(V)$

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- In many other cases we have "inclusion" maps that induce maps of tensor triangular categories. Are these induced maps also localizations?
- For example: If $H \hookrightarrow G$ is subgroup of a (finite) group G, is the restriction of scalars functor $\mathrm{Stab}(kG) \longrightarrow \mathrm{Stab}(kH)$ (or $D(kG) \longrightarrow D(kH)$) a localization?

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- Multiplication by the unit does nothing: $\mathbb{1} \otimes a \cong a \cong a \otimes \mathbb{1}$

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- The associative isomorphisms above are really a given choice of natural isomorphisms
- There are two distinct maps in the unital isomorphisms (one for tensoring on the left and one for on the right)
- We have not made the claim yet that $a \otimes b \cong b \otimes a$ yet.

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Definition and Some Examples

Def: A **Symmetric Monoidal Category** is a monoidal category with a choice of braiding $\tau : a \otimes b \cong b \otimes a$ that plays nice with the associativity isomorphisms and the two choices of unital isomorphisms.

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- $(Ab, \otimes_{\mathbb{Z}}, \mathbb{Z})$
- More generally for R a commutative ring we have $(R - Mod, \otimes_R, R)$
- Let G be a finite group. Then $(kG - Mod, \otimes_k, k)$
- More generally H be a Hopf Algebra over a field k. Then $(H - Mod, \otimes_k, k)$

Symmetric Monoidal Functors and their Adjoints

(Symmetric) Monoidal Functors

A (lax) monoidal functor $F : (\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}})$ is a functor equipped with a morphism

$$\varphi_0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$$

and a natural transformation

$$\varphi_{a,b} : F(a) \otimes_{\mathcal{D}} F(b) \to F(a \otimes_{\mathcal{C}} b)$$

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FACT: G is a lax monoidal functor. Note that we therefore have the following two maps:

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That is, there is a sort of "multiplication map" for $\mathsf{G}(\mathbb{1}_{\mathcal{D}})$. Let us formalize that.

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We say the ring A is **commutative** if the multiplication map commutes with the braiding: that is if $\mu \circ \tau = \mu$

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- For G finite Group, ring objects in $(kG - \mathrm{Mod}, \otimes_k, k)$ are k-algebras with actions of G as algebra automorphisms.
- Recall a right adjoint G of any strong monoidal functor F is a lax monoidal functor. Then we saw that in fact $G(\mathbb{1})$ is a ring object.

Modules over Ring objects

Given me a ring and I'll give you a Module

Given a ring object we can talk about modules over the ring.
Def: Let A be a (commutative) ring object in a symetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. A left A-module M is an object of $\mathcal{C}$ equipped with a map $\rho : A \otimes M \to M$ such that the following two diagrams commute:

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Remark: These axioms are just souped up versions of the usual two axioms that a.(b.m)=(ab).m and 1.m=m we are familiar with for Modules.

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Note we then have an "extension of scalars functor"

$$F_A := A \otimes - : \mathcal{C} \to A - \mathrm{Mod}_{\mathcal{C}}$$

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Remark: We typically call the essential image of F the category of "Free Modules" and denote it by $A - \mathrm{Free}_{\mathcal{C}}$. The adjunction above then of course restricts to:

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, shift right, "F_A"'] \arrow[dr, shift left, "F_A"'] & \\
A - \mathrm{Free}_{\mathcal{C}} \arrow[ur, shift right, "U_A"'] \arrow[rr, hook] & & A - \mathrm{Mod}_{\mathcal{C}} \arrow[ul, shift left, "U_A"']
\end{tikzcd}$$

Realization of Ring Objects

Modules from an Adjunction

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be an adjoint pair between (Symmetric) Monoidal Categories. Recall that if F is strong Monoidal, then G is lax monoidal, turning $A := G(\mathbb{1})$ into a ring object; so we can consider the category of A-Modules in $\mathcal{C}$.

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Theorem:

There exist unique functors L and K making the following diagram commute:

$$\begin{tikzcd} & \mathcal{C} \arrow[dl, shift right, "F_A"'] \arrow[d, shift right, "F"'] \arrow[dr, shift left, "F_A"] & \\ A - \mathrm{Free}_{\mathcal{C}} \arrow[ur, shift right, "U_A"'] \arrow[r, dashed, "L"'] & \mathcal{D} \arrow[u, shift right, "G"'] \arrow[r, dashed, "K"'] & A - \mathrm{Mod}_{\mathcal{C}} \arrow[ul, shift left, "U_A"] \end{tikzcd}$$

Bring on the Triangulation

Tensor Triangulated Categories

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3. $(SH, \wedge, S)$. The "stable homotopy" category with smash product
4. $(SH(G), \wedge, S)$. The G-equivarient stable homotopy category for a finite group G.

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4. $(SH(G), \wedge, S)$. The G-equivarient stable homotopy category for a finite group G.
5. $(DM^{(\acute{e}t)}(S, R), \otimes, R)$. The derived category of $(\acute{e}tale)$ motives over base scheme S with coefficients in a commutative ring R.

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Some leading questions/examples

Question: Let R be a commutative ring and let A be an R-algebra: Is $A[0]$ a ring object in $D(R)$?

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Answer: Sadly, not always. This is because $(A \otimes^L A)[0] \neq A[0] \otimes^L A[0]$ unless A is flat. So a flat R-Algebra A remains a ring object in $D(R)$.

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Some leading questions/examples

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Separable Rings

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Theorem:

Let A be a separable ring in a tt category $\mathcal{T}$. Then the category $A - \mathrm{Mod}_{\mathcal{T}}$ is canonically triangulated such that the extension of scalars functor

$$F_A : \mathcal{T} \longrightarrow A - \mathrm{Mod}_{\mathcal{T}}$$

is a tt functor.

Examples: Take 1

Smashing Localizations

Before diving into some concrete examples, let us note a particular case of this theory. Let $\mathcal{T}$ be a tt category and consider a smashing localization $\mathrm{L} : \mathcal{T} \to \mathcal{T}$

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Back to OG example

Remark: The case of the open immersion of schemes $U \hookrightarrow V$ can thus be stated as follows:
Letting $A = j_*(\mathcal{O}_U)$ we have that A-Mod $\cong j_*j^*$-Local objects $\cong D^{qcoh}(U)$.
That is, we can view $D^{qcoh}(U)$ as being a sort of Module category **inside** $D^{qcoh}(V)$

Finite Étale Extensions

The Main Construction

Let $\mathcal{C} \underset{F}{\overset{G}{\rightleftarrows}} \mathcal{D}$ be an adjunction of tt categories.

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$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, shift right, "F_A"'] \arrow[d, shift right, "F"'] \arrow[dr, shift left, "F_A"] & \\
A - \mathrm{Free}_{\mathcal{C}} \arrow[ur, shift right, "U_A"'] \arrow[r, dashed, "L"'] & \mathcal{D} \arrow[u, shift right, "G"'] \arrow[r, dashed, "K"'] & A - \mathrm{Mod}_{\mathcal{C}} \arrow[ul, shift left, "U_A"]
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Main Definition:

Let F and A be as above. We say F is a **finite étale extension** if A is a (compact) separable ring object such that

- the functor $\mathcal{D} \xrightarrow{K} A - \mathrm{Mod}_{\mathcal{C}}$ is an tt equivalence of tt categories
- under which the functor F becomes isomorphic to the extension of scalars functor F_A and
- G becomes isomorphic to the forgetful functor U_A

Étale Algebras Galore

The Reason for the Name

Recall that if A is a flat R-Algebra then $A[0]$ remains a ring object in $D(R)$.
Def: An **étale R-Algebra S** is a separable, flat R-algebra of finite presentation.

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Thrm 1: Let S be an étale R-algebra and consider the extension of scalars functor:

$$D(R) \xrightarrow{F := S[0] \otimes^{L}_{R} -} D(S)$$

Then F is a finite étale extension. That is the category $D(S)$ is canonically equivalent to the category of S-Modules inside D(R).

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Thrm 2

Thrm 2: Let $f : V \to X$ be a seperated étale morphism of quasi-compact, quasi-seperated schemes. Then the functor

$$f^* : D^{qcoh}(X) \to D^{qcoh}(V)$$

is a finite étale extension. That is, we have an equivalence of categories
$D^{qcoh}(V) \cong Rf_*(\mathbb{1}) - \mathrm{Mod}_{D^{qcoh}(X)}$

Étale Extensions in Equivarient contexts

Modular Representation Theory

Let G be a finite group and consider the tt category $\mathrm{Stab}(kG)$. Let $H \leqslant G$ be a subgroup and recall that we get the following adjunction:

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Thrm: Let $A^G_H := \mathrm{Ind}^G_H(\mathbb{1}) \cong k(G/H)$. The Restriction to a subgroup functor is a finite étale extension. That is, the category Stab(kH) is canonically isomorphic to the category of A-Modules in Stab(kG) under which the restriction functor is isomorphic to the extension of scalars functor F_A.

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Remark: One can then phrase questions about extending representations of H to G in terms of "descent" of the ring A^G_H. I will not mention much more about this, but will leave some references for you to look at. The big takeaway is that this ring A^G_H satisfies descent iff [G : H] is invertible in k.

Étale Extensions in Equivarient contexts

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Let G be a compact Lie Group (ex; a finite group) and consider the tt category $SH(G)$. Let $H \leqslant G$ be a closed subgroup- we get the following adjunction:

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- $$\begin{tikzcd} & \mathrm{SH}(G) \arrow[d, "\mathrm{Res}^G_H" description] & \\ & \mathrm{SH}(H) \arrow[u, bend left=60, "\mathrm{Ind}^G_H := G_+ \wedge_H -"'] \arrow[u, bend right=60, "\mathrm{CoInd}^G_H := F_H(G_+, -)"] & \end{tikzcd}$$

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$$\begin{tikzcd}[row sep=huge]
\mathrm{SH}(G) \arrow[d, "\mathrm{Res}_H^G" description] \\
\mathrm{SH}(H) \arrow[u, shift left=3, "{\mathrm{Ind}_H^G := G_+ \wedge_H -}"] \arrow[u, shift right=3, "{\mathrm{CoInd}_H^G := F_H(G_+, -)}"']
\end{tikzcd}$$

Note that when $[G : H] < \infty$ there is an isomorphism of functors between $\mathrm{Ind}_H^G \cong \mathrm{CoInd}_H^G$.

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Note that when $[G : H] < \infty$ there is an isomorphism of functors between $\mathrm{Ind}_H^G \cong \mathrm{CoInd}_H^G$.

Theorem:

Let $H \leqslant G$ be a closed subgroup of finite index. Let $A := F_H(G_+, \mathbb{1}_{SH(H)}) \cong G_+ \wedge_H \mathbb{1}_{SH(H)} \cong \sum^\infty (G/H)_+$ Then restriction to H is a finite étale extension; that is the category of A-Modules in SH(G) is equivalent to SH(H).

Further Directions/Questions

Some Topics to Read if Interested

There are many directions one can take with this:

- Read about what the extension of scalars functor does on Spectra
- Classify all separable algebras in a given tt category
- Read about descent for separable algebras
- See how far you can push the analogy of a ring: going up theorem, "residue fields", Galois extensions, etc
- Reading about the behavior of finite étale morphisms on the "big" categories