

1.18 Complexes and homotopy

Reading Seminar, Linckelmann Chapter 1

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2nd September 2021

Chain Homotopy

Let \mathcal{C} be an additive category and let $(X, \delta), (Y, \varepsilon)$ be complexes over \mathcal{C} .

$$X: \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

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homotopic to 0

Or equivalently, if

$$f_n - f'_n = \varepsilon_{n+1} \circ h_n + h_{n-1} \circ \delta_n$$

for any $n \in \mathbb{Z}$.

We also say h is a *homotopy from f to f'* .

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- For cochain complexes, we define analogously a *cochain homotopy* to be a graded morphism of degree -1 satisfying the analogous property.

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of chain maps, for any two complex X, Y over \mathcal{C} .

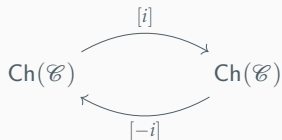
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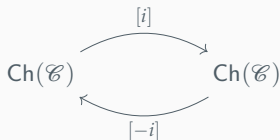
We denote by $K^+(\mathcal{C})$, $K^-(\mathcal{C})$, $K^b(\mathcal{C})$ the full subcategories of $K(\mathcal{C})$ consisting of bounded above, bounded below, bounded complexes over \mathcal{C} , respectively.

Recall that $\text{Ch}(\mathcal{C})$ admits the *shift automorphism*, for any integer i



Homotopy Category

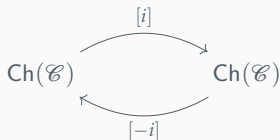
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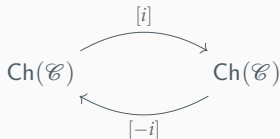
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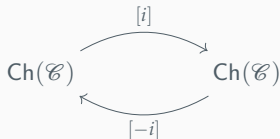
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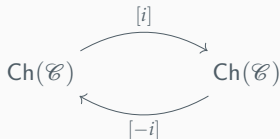


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This automorphism preserves any of the subcategories $\text{K}^+(\mathcal{C})$, $\text{K}^-(\mathcal{C})$, $\text{K}^b(\mathcal{C})$.

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$$H_*(f)([z]) = [f(z)]$$

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$$H_n(h \circ \delta + \varepsilon \circ h)([z]) = \underbrace{[(h \circ \delta + \varepsilon \circ h)_n(z)]}_{\text{im } \delta_{n+1}} = \underbrace{[(h_{n-1} \circ \delta_n + \varepsilon_{n+1} \circ h_n)(z)]}_{\text{im } \delta_{n+1}} = \underbrace{[\varepsilon_{n+1}(h_n(z))]}_{\text{im } \delta_{n+1}}$$

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2/2

Note that $\varepsilon_{n+1}(h_n(z)) \in \text{im } \varepsilon_{n+1}$ and $H_n(h \circ \delta + \varepsilon \circ h)([z]) \in H_n(Y) = \ker \varepsilon_n / \text{im } \varepsilon_{n+1}$,

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$$(f-f') - 0 = f-f' = \boxed{}$$

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$$0 = H_* (f-f') = H_*(f) - H_*(f')$$

If $f \sim f'$, then by (i), $f - f'$ induces the zero map in homology, and thus $H_*(f) = H_*(f')$; so we have proved (ii).

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$$g \circ f \sim \text{id}_X \Rightarrow H_*(g \circ f) = H_*(\text{id}_X) = \text{id}_{H_*(X)}$$

Suppose f has a homotopy inverse g . Then, by (ii), we have

$$\text{id}_{H_*(X)} = H_*(g \circ f) = \underbrace{H_*(g) \circ H_*(f)} \quad \text{and} \quad \text{id}_{H_*(Y)} = H_*(f \circ g) = H_*(f) \circ H_*(g)$$

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Suppose f has a homotopy inverse g . Then, by (ii), we have

$$\text{id}_{H_*(X)} = H_*(g \circ f) = H_*(g) \circ H_*(f) \quad \text{and} \quad \text{id}_{H_*(Y)} = H_*(f \circ g) = H_*(f) \circ H_*(g)$$

So, $H_*(f) : H_*(X) \rightarrow H_*(Y)$ is an isomorphism, and hence f is a quasi-isomorphism; we have proved (iii).

Homology induced by Homotopy

Proof of Prop. 1.18.3 (contd.)

2/2

Note that $\varepsilon_{n+1}(h_n(z)) \in \text{im } \varepsilon_{n+1}$ and $H_n(h \circ \delta + \varepsilon \circ h)([z]) \in H_n(Y) = \ker \varepsilon_n / \text{im } \varepsilon_{n+1}$, and therefore

$$H_n(h \circ \delta + \varepsilon \circ h)([z]) = [\varepsilon_{n+1}(h_n(z))] = [0].$$

Hence $(h \circ \delta + \varepsilon \circ h)$ induces the zero map in homology, whence (i).

If $f \sim f'$, then by (i), $f - f'$ induces the zero map in homology, and thus $H_*(f) = H_*(f')$; so we have proved (ii).

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So, $H_*(f) : H_*(X) \rightarrow H_*(Y)$ is an isomorphism, and hence f is a quasi-isomorphism; we have proved (iii).

If $X \simeq 0$, then X is quasi-isomorphic to 0 by (iii), which is equivalent to $H_*(X) = 0$, giving us (iv). ■

(Co-)Homology as Homotopy Classes

For an algebra A , X a complex of A -modules, V an A -module, and n an integer.

$$X: \quad \cdots \longrightarrow X_{n+1} \xrightarrow{\delta_{n+1}} X_n \xrightarrow{\delta_n} X_{n-1} \xrightarrow{\delta_{n-1}} \cdots$$

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We denote by $V[n]$ the complex that is equal to V in degree n and zero in all other degrees, with the zero differential.

$$V[n] : \quad \cdots \longrightarrow 0 \xrightarrow{0_{n+1}} V \xrightarrow{0_n} 0 \xrightarrow{0_{n-1}} \cdots$$

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We regard $\mathrm{Hom}_A(X, V)$ as a cochain complex, obtained from applying the contravariant functor $\mathrm{Hom}_A(-, V)$ to X .

$$\cdots \longleftarrow \mathrm{Hom}_A(X_{n+1}, V) \xleftarrow{-\circ \delta_{n+1}} \mathrm{Hom}_A(X_n, V) \xleftarrow{-\circ \delta_n} \cdots$$

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and $\text{Hom}_A(V, X)$ as a chain complex, obtained from applying the covariant functor $\text{Hom}_A(V, -)$ to X .

$$\cdots \longrightarrow \text{Hom}(V, X_{n+1}) \xrightarrow{\delta_{n+1} \circ -} \text{Hom}(V, X_n) \xrightarrow{\delta_n \circ -} \cdots$$

(Handwritten annotations: a red checkmark above the first term, a red arrow pointing from the first term to the second, and a red squiggle above the second term)

Proposition 1.18.4

Let A be a k -algebra, V an A -module, and (X, δ) a complex of A -modules. Let n be an integer.

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Proof of Prop. 1.18.4

An element in degree n of $\operatorname{Hom}_A(V, X)$ is given by an A -linear map $\zeta : V \rightarrow X_n$,

(Co-)Homology as Homotopy Classes

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Proof of Prop. 1.18.4

An element in degree n of $\operatorname{Hom}_A(V, X)$ is given by an A -linear map $\zeta : V \rightarrow X_n$, and this belongs to the kernel of the differential at degree n if and only if $\delta_n \circ \zeta = 0$.

(Co-)Homology as Homotopy Classes

Proof of Prop. 1.18.4 (contd.)

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This is equivalent to asserting that ζ defines a chain map $V[n] \rightarrow X$. (ζ_n)_n $\zeta_n = \zeta$ $\zeta_m = 0$ $m \neq n$

$$\begin{array}{ccccccc}
 V[n] : & \cdots & \longrightarrow & 0 & \xrightarrow{0_{n+1}} & V & \xrightarrow{0_n} & 0 & \xrightarrow{0_{n-1}} & \cdots \\
 & & & \downarrow 0 & \text{///} & \downarrow \zeta & \text{///} & \downarrow 0 & & \\
 X : & \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots
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 \end{array}$$

$[\zeta] = [0]$
 in homology
 $\zeta \in \text{im}(\delta_{n+1} \circ -)$

Now, ζ belongs to the image of the differential if and only if $\zeta = \delta_{n+1} \circ \eta$ for some A -linear map $\eta : V \rightarrow X_{n+1}$

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2/4

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This is equivalent to asserting that ζ , as a chain map, is homotopic to zero; and so we have proved the first isomorphism in (i).

Proof of Prop. 1.18.4 (contd.)

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The second isomorphism in (i) follows from the first isomorphism we just proved and the fact that $\text{Hom}_A(A, -)$ is isomorphic to the identity functor on $\text{Mod}(A)$.

Proof of Prop. 1.18.4 (contd.)

3/4

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Proof of Prop. 1.18.4 (contd.)

4/4

Now, ζ belongs to the image of the differential at degree n of $\text{Hom}_A(X, V)$ if and only if $\zeta = \eta \circ \delta_n$ for some A -linear map $\eta : X_{n-1} \rightarrow V$

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This is equivalent to asserting that ζ , as a chain map, is homotopic to zero; and so we have proved the first isomorphism in (ii).

Proof of Prop. 1.18.4 (contd.)

4/4

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Proposition 1.18.5

Let \mathcal{C} be an abelian category, let P be a complex of projective objects in \mathcal{C} , I a complex of injective objects in \mathcal{C} and let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of complexes over \mathcal{C} .

Projectives, Injectives and the Homotopy Category

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(i) Suppose that X is acyclic and that one of P, Y is bounded below. The map

$$g \circ - : \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(P, Y) \rightarrow \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(P, Z)$$

is surjective and induces an isomorphism

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(ii) Suppose that Z is acyclic and that one of Y, I is bounded below. The map

$$- \circ f : \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(Y, I) \rightarrow \operatorname{Hom}_{\operatorname{Ch}(\mathcal{C})}(X, I)$$

is surjective and induces an isomorphism

$$\operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(Y, I) \cong \operatorname{Hom}_{\mathbf{K}(\mathcal{C})}(X, I).$$

Proof of Prop. 1.18.5

Denote by $\delta, \varepsilon, \zeta, \pi$ the differentials of X, Y, Z, P , respectively.

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Projectives, Injectives and the Homotopy Category

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Let n be an integer. Suppose we have already constructed morphisms $\underline{p_i} : \underline{P_i} \rightarrow Y_i$ satisfying

$$\underline{g_i \circ p_i = q_i} \quad \text{and} \quad \underline{\varepsilon_i \circ p_i = p_{i-1} \circ \pi_i} \quad \checkmark$$

for $i < n$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5

Denote by $\delta, \varepsilon, \zeta, \pi$ the differentials of X, Y, Z, P , respectively.

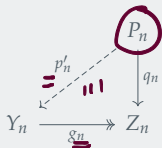
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for $i < \underline{n}$.



Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \rightarrow Y_n$ such that $\underbrace{g_n \circ p'_n}_{= q_n} = q_n$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5

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Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \rightarrow Y_i$ satisfying

$$g_i \circ p_i = q_i \quad \text{and} \quad \varepsilon_i \circ p_i = p_{i-1} \circ \pi_i$$

for $i < n$.

$$\begin{array}{ccc} & P_n & \\ p'_n \swarrow & \downarrow q_n & \\ Y_n & \xrightarrow{g_n} & Z_n \end{array}$$

Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \rightarrow Y_n$ such that $g_n \circ p'_n = q_n$.

That is, p'_n satisfies the first of the two conditions above,

Projectives, Injectives and the Homotopy Category

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Since one of P, Y is bounded below, we have $q_i = 0$ for all sufficiently small integers i (if Y is bounded below, then so is Z by the surjectivity of g .) So take $p_i = 0$ for i sufficiently small.

Let n be an integer. Suppose we have already constructed morphisms $p_i : P_i \rightarrow Y_i$ satisfying

$$g_i \circ p_i = q_i \quad \text{and} \quad \varepsilon_i \circ p_i = p_{i-1} \circ \pi_i$$

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Since g_n is an epimorphism and P_n is projective, there is a morphism $p'_n : P_n \rightarrow Y_n$ such that $g_n \circ p'_n = q_n$.

That is, p'_n satisfies the first of the two conditions above, but we may have to adjust p'_n to make sure that it is compatible with the differentials as in the second condition.

Note that

$$\underbrace{g_{n-1} \circ (s_n \circ p'_n - p_{n-1} \circ \pi_n)}_{\text{because } q \text{ is a chain map.}} = \underbrace{\zeta_n \circ g_n}_{\text{}} \circ p'_n - \underbrace{g_{n-1} \circ p_{n-1}}_{\text{}} \circ \pi_n = \zeta_n \circ \dot{q}_n - \underbrace{q_{n-1} \circ \pi_n}_{\text{}} = 0$$

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Therefore we have

$$\operatorname{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \operatorname{im} f_{n-1}$$

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

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Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

because q is a chain map. Therefore we have

$$\text{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \text{im } f_{n-1}$$

$$\begin{array}{ccc}
 & P_n & \\
 \swarrow \sigma & \downarrow (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) & \\
 X_{n-1} & \xrightarrow{f_{n-1}} & \text{im } f_{n-1}
 \end{array}$$

(Handwritten red marks: a checkmark above σ , a double slash $//$ on the dashed arrow, and an equals sign $=$ next to the vertical arrow.)

Consequently, there is a morphism $\sigma : P_n \rightarrow X_{n-1}$ such that $\underbrace{f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n}_{\text{Handwritten red underline and checkmark}}$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

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$$\begin{array}{ccc} & P_n & \\ \swarrow \sigma & \downarrow (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) & \\ X_{n-1} & \xrightarrow{f_{n-1}} & \text{im } f_{n-1} \end{array}$$

Consequently, there is a morphism $\sigma : P_n \rightarrow X_{n-1}$ such that $f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n$.

Moreover, we have

$$\begin{aligned} \underline{f_{n-2} \circ \delta_{n-1} \circ \sigma} &= \varepsilon_{n-1} \circ \underline{f_{n-1} \circ \sigma} \\ &= \varepsilon_{n-1} \circ \underline{\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n} = \underline{-p_{n-2} \circ \pi_{n-1} \circ \pi_n} = \underline{0} \end{aligned}$$

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

2/6

Note that

$$g_{n-1} \circ (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = \zeta_n \circ g_n \circ p'_n - g_{n-1} \circ p_{n-1} \circ \pi_n = \zeta_n \circ q_n - q_{n-1} \circ \pi_n = 0$$

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$$\text{im}(\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) \subseteq \ker g_{n-1} = \text{im } f_{n-1}$$

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Consequently, there is a morphism $\sigma : P_n \rightarrow X_{n-1}$ such that $\underline{f_{n-1} \circ \sigma = \varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n}$.

$$f \circ g = 0 \Rightarrow g = 0$$

Moreover, we have

$$\begin{aligned} f_{n-2} \circ \delta_{n-1} \circ \sigma &= \varepsilon_{n-1} \circ f_{n-1} \circ \sigma \\ &= \varepsilon_{n-1} \circ \varepsilon_n \circ p'_n - \varepsilon_{n-1} \circ p_{n-1} \circ \pi_n = -p_{n-2} \circ \pi_{n-1} \circ \pi_n = 0 \end{aligned}$$

and hence $\underline{\delta_{n-1} \circ \sigma} = 0$, as f_{n-2} is a monomorphism.

Proof of Prop. 1.18.5 (contd.)

3/6

Thus we have $\underline{\operatorname{im} \sigma} \subseteq \underline{\ker \delta_{n-1}} = \underline{\operatorname{im} \delta_n}$, where the last equality holds as X is acyclic.

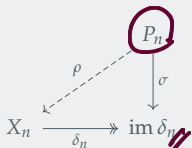
$$0 = H_n(X) = \ker \delta_{n-1} / \operatorname{im} \delta_n$$

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

3/6

Thus we have $\text{im } \sigma \subseteq \ker \delta_{n-1} = \text{im } \delta_n$, where the last equality holds as X is acyclic.



Thus there is a morphism $\rho : P_n \rightarrow X_n$ such that $\sigma = \delta_n \circ \rho$.

Proof of Prop. 1.18.5 (contd.)

3/6

Thus we have $\text{im } \sigma \subseteq \ker \delta_{n-1} = \text{im } \delta_n$, where the last equality holds as X is acyclic.

$$\begin{array}{ccc} & P_n & \\ \swarrow \rho & \downarrow \sigma & \\ X_n & \xrightarrow{\delta_n} & \text{im } \delta_n \end{array}$$

Thus there is a morphism $\rho : P_n \rightarrow X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = \underline{p'_n - f_n \circ \rho}$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

3/6

Thus we have $\text{im } \sigma \subseteq \ker \delta_{n-1} = \text{im } \delta_n$, where the last equality holds as X is acyclic.

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Thus there is a morphism $\rho : P_n \rightarrow X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = \underline{p'_n - f_n \circ \rho}$. We still have $\underline{g_n \circ p_n} = \underline{g_n \circ p'_n - \overset{\text{yes}}{g_n \circ f_n \circ \rho}} = g_n \circ p'_n = q_n$,

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

3/6

Thus we have $\text{im } \sigma \subseteq \ker \delta_{n-1} = \text{im } \delta_n$, where the last equality holds as X is acyclic.

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Thus there is a morphism $\rho : P_n \rightarrow X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = p'_n - f_n \circ \rho$. We still have $g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n$,

and we now also have the compatibility with the differentials

$$\begin{aligned} \varepsilon_n \circ p_n &= \varepsilon_n \circ p'_n - \varepsilon_n \circ f_n \circ \rho \\ &= \varepsilon_n \circ p'_n - \underbrace{f_{n-1} \circ \delta_n}_{\text{circled}} \circ \rho \\ &= \varepsilon_n \circ p'_n - \underbrace{f_{n-1} \circ \sigma} \\ &= \cancel{\varepsilon_n \circ p_n} - (\cancel{\varepsilon_n \circ p'_n} - p_{n-1} \circ \pi_n) = \underline{\underline{p_{n-1} \circ \pi_n}} \end{aligned}$$

as required.

Thus we have $\text{im } \sigma \subseteq \ker \delta_{n-1} = \text{im } \delta_n$, where the last equality holds as X is acyclic.

$$\begin{array}{ccc} & P_n & \\ \rho \swarrow & \downarrow \sigma & \\ X_n & \xrightarrow{\delta_n} & \text{im } \delta_n \end{array}$$

Thus there is a morphism $\rho : P_n \rightarrow X_n$ such that $\sigma = \delta_n \circ \rho$.

Set $p_n = p'_n - f_n \circ \rho$. We still have $g_n \circ p_n = g_n \circ p'_n - g_n \circ f_n \circ \rho = g_n \circ p'_n = q_n$,

and we now also have the compatibility with the differentials

$$\begin{aligned} \varepsilon_n \circ p_n &= \varepsilon_n \circ p'_n - \varepsilon_n \circ f_n \circ \rho \\ &= \varepsilon_n \circ p'_n - f_{n-1} \circ \delta_n \circ \rho \\ &= \varepsilon_n \circ p'_n - f_{n-1} \circ \sigma \\ &= \varepsilon_n \circ p'_n - (\varepsilon_n \circ p'_n - p_{n-1} \circ \pi_n) = p_{n-1} \circ \pi_n \end{aligned}$$

as required. This shows the surjectivity of the map given by composition with g .

Proof of Prop. 1.18.5 (contd.)

4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

Proof of Prop. 1.18.5 (contd.)

4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

$$i = j \circ f$$

If $p \sim 0$, there is a homotopy $h : P \rightarrow Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Proof of Prop. 1.18.5 (contd.)

4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \rightarrow Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

Composing with g yields

$$q = g \circ p = \underline{g \circ \varepsilon} \circ h + g \circ h \circ \pi = \zeta \circ \boxed{g \circ h} + \boxed{g \circ h} \circ \pi,$$

Proof of Prop. 1.18.5 (contd.)

4/6

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$$q = g \circ p = g \circ \varepsilon \circ h + g \circ h \circ \pi = \zeta \circ g \circ h + g \circ h \circ \pi,$$

thus $q \sim 0$ via the homotopy $g \circ h : P \rightarrow Z$.

Proof of Prop. 1.18.5 (contd.)

4/6

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Composing with g yields

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thus $q \sim 0$ via the homotopy $g \circ h : P \rightarrow Z$.

Suppose $q \sim 0$, then there is a homotopy $t : P \rightarrow Z$ such that $q = \zeta \circ t + t \circ \pi$.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

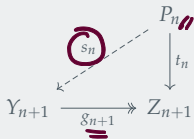
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Since g_{n+1} is an epimorphism, $t_n : P_n \rightarrow Z_{n+1}$ then lifts to a morphism $s_n : P_n \rightarrow Y_{n+1}$; that is, the homotopy $t : P \rightarrow Z$ lifts to some homotopy $s : P \rightarrow Y$. //

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

4/6

We need to show that $p \sim 0$ if and only if $q \sim 0$.

If $p \sim 0$, there is a homotopy $h : P \rightarrow Y$ such that $p = \varepsilon \circ h + h \circ \pi$.

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Suppose $q \sim 0$, then there is a homotopy $t : P \rightarrow Z$ such that $q = \zeta \circ t + t \circ \pi$.

$$\begin{array}{ccc} & P_n & \\ \swarrow s_n & \downarrow t_n & \\ Y_{n+1} & \xrightarrow{g_{n+1}} & Z_{n+1} \end{array}$$

Since g_{n+1} is an epimorphism, $t_n : P_n \rightarrow Z_{n+1}$ then lifts to a morphism $s_n : P_n \rightarrow Y_{n+1}$; that is, the homotopy $t : P \rightarrow Z$ lifts to some homotopy $s : P \rightarrow Y$.

So, $p' := \underbrace{\varepsilon \circ s + s \circ \pi}_{\text{homotopy}} : P \rightarrow Y$ is a chain map such that $p' \sim 0$ via s , trivially.

Projectives, Injectives and the Homotopy Category

Proof of Prop. 1.18.5 (contd.)

4/6

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Proof of Prop. 1.18.5 (contd.)

4/6

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So, $p' := \varepsilon \circ s + s \circ \pi : P \rightarrow Y$ is a chain map such that $p' \sim 0$ via s , trivially. Furthermore, one immediately checks that $g \circ p' = q$, but p' need not be equal to p .

It suffices to show that $p - p' \sim 0$. Since $g \circ (p - p') = 0$, we may assume that $q = 0$.

Proof of Prop. 1.18.5 (contd.)

5/6

Then $g \circ p = q = 0$, hence $\operatorname{im} p \subseteq \ker g = \operatorname{im} f$.

Proof of Prop. 1.18.5 (contd.)

5/6

Then $g \circ p = q = 0$, hence $\text{im } p \subseteq \ker g = \text{im } f$.

$$\begin{array}{ccc} & P & \\ \swarrow u & \downarrow p & \\ X & \xrightarrow{f} & \text{im } f \end{array}$$

This implies that there is a chain map $u : P \rightarrow X$ such that $f \circ u = p$.

Proof of Prop. 1.18.5 (contd.)

5/6

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It suffices to show that $\underline{u} \sim 0$. This is again done inductively.

Proof of Prop. 1.18.5 (contd.)

5/6

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This implies that there is a chain map $u : P \rightarrow X$ such that $f \circ u = p$.

It suffices to show that $u \sim 0$. This is again done inductively.

Given an integer n , suppose that we have morphisms $h_i : P_i \rightarrow X_{i+1}$ satisfying

$$u_i = \delta_{i+1} \circ h_i + h_{i-1} \circ \pi_i$$

for any $i < n$.

Proof of Prop. 1.18.5 (contd.)

5/6

Then $g \circ p = q = 0$, hence $\text{im } p \subseteq \ker g = \text{im } f$.

$$\begin{array}{ccc} & P & \\ & \downarrow p & \\ X & \xrightarrow{f} & \text{im } f \end{array}$$

(Note: A dashed arrow labeled u points from P to X in the original image.)

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for any $i < n$.

Using this equality for $i = n - 1$, we get

$$\delta_n \circ (u_n - h_{n-1} \circ \pi_n) = \delta_n \circ u_n - \delta_n \circ h_{n-1} \circ \pi_n$$

=

$$= \delta_n \circ u_n - (u_{n-1} - h_{n-2} \circ \pi_{n-1}) \circ \pi_n = \delta_n \circ u_n - u_{n-1} \circ \pi_n = 0,$$

as u is a chain map.

Then $g \circ p = q = 0$, hence $\text{im } p \subseteq \ker g = \text{im } f$.

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as u is a chain map. Therefore

$$\text{im}(u_n - h_{n-1} \circ \pi_n) \subseteq \ker \delta_n = \text{im } \delta_{n+1}.$$

(X is acyclic)

Proof of Prop. 1.18.5 (contd.)

6/6

$$\begin{array}{ccc} & P_n & \\ \swarrow h_n & \downarrow (u_n - h_{n-1} \circ \pi_n) & \\ X_{n+1} & \xrightarrow[\delta_{n+1}]{\quad} \text{im } \delta_{n+1} & \end{array}$$

Hence, as P_n is projective, there is a map $h_n : P_n \rightarrow X_{n+1}$ such that

$$\delta_{n+1} \circ h_n = u_n - h_{n-1} \circ \pi_n$$

Proof of Prop. 1.18.5 (contd.)

6/6

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thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

Proof of Prop. 1.18.5 (contd.)

6/6

$$\begin{array}{ccc} & P_n & \\ \swarrow h_n & \downarrow (u_n - h_{n-1} \circ \pi_n) & \\ X_{n+1} & \xrightarrow[\delta_{n+1}]{\quad} \text{im } \delta_{n+1} & \end{array}$$

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thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

So, we have $u \sim 0$. This completes the proof of (i).

Proof of Prop. 1.18.5 (contd.)

6/6

$$\begin{array}{ccc} & P_n & \\ \swarrow h_n & \downarrow (u_n - h_{n-1} \circ \pi_n) & \\ X_{n+1} & \xrightarrow[\delta_{n+1}]{\quad} \text{im } \delta_{n+1} & \end{array}$$

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thus $u_n = \delta_{n+1} \circ h_n + h_{n-1} \circ \pi_n$.

So, we have $u \sim 0$. This completes the proof of (i).

The proof of (ii) is obtained by dualising the arguments. ■

Corollary 1.18.6

Let \mathcal{C} be an abelian category, let P be a complex of projective objects in \mathcal{C} , I a complex of injective objects in \mathcal{C} and let X be an acyclic complex of objects in \mathcal{C} .

Corollary 1.18.6

Let \mathcal{C} be an abelian category, let P be a complex of projective objects in \mathcal{C} , I a complex of injective objects in \mathcal{C} and let X be an acyclic complex of objects in \mathcal{C} .

- (i) If one of P , X is bounded below, then $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(P, X) = \{0\}$.

Corollary 1.18.6

Let \mathcal{C} be an abelian category, let P be a complex of projective objects in \mathcal{C} , I a complex of injective objects in \mathcal{C} and let X be an acyclic complex of objects in \mathcal{C} .

- (i) If one of P , X is bounded below, then $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(P, X) = \{0\}$.
- (ii) If one of X , I is bounded below, then $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(X, I) = \{0\}$.

Corollary 1.18.6

Let \mathcal{C} be an abelian category, let P be a complex of projective objects in \mathcal{C} , I a complex of injective objects in \mathcal{C} and let X be an acyclic complex of objects in \mathcal{C} .

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- (ii) If one of X , I is bounded below, then $\mathrm{Hom}_{\mathbf{K}(\mathcal{C})}(X, I) = \{0\}$.

Proof of Cor. 1.18.6

Apply the above result to the short exact sequences $0 \rightarrow X \rightarrow X \rightarrow 0 \rightarrow 0$ and $0 \rightarrow 0 \rightarrow X \rightarrow X \rightarrow 0$. ■

- An abelian category \mathcal{C} is said to have *enough projective objects* if for any object X in \mathcal{C} there exists a projective object P in \mathcal{C} and an epimorphism $\pi : P \rightarrow X$.

Projectives, Injectives and the Homotopy Category

- An abelian category \mathcal{C} is said to have *enough projective objects* if for any object X in \mathcal{C} there exists a projective object P in \mathcal{C} and an epimorphism $\pi : P \rightarrow X$.
- Dually, \mathcal{C} is said to have *enough injective objects* if for any object X in \mathcal{C} there exists an injective object I and a monomorphism $\iota : X \rightarrow I$.

Projectives, Injectives and the Homotopy Category

- An abelian category \mathcal{C} is said to have *enough projective objects* if for any object X in \mathcal{C} there exists a projective object P in \mathcal{C} and an epimorphism $\pi : P \rightarrow X$.
- Dually, \mathcal{C} is said to have *enough injective objects* if for any object X in \mathcal{C} there exists an injective object I and a monomorphism $\iota : X \rightarrow I$.
- If A is a k -algebra, then the category of A -modules $\text{Mod}(A)$ has enough projective and injective objects.

$$\bigoplus_{m \in M} A \longrightarrow M$$

$$n \cdot : A \rightarrow A \times \mathbb{Q}, \mathbb{Q}/\mathbb{Z}$$

Projectives, Injectives and the Homotopy Category

- An abelian category \mathcal{C} is said to have *enough projective objects* if for any object X in \mathcal{C} there exists a projective object P in \mathcal{C} and an epimorphism $\pi : P \rightarrow X$.
- Dually, \mathcal{C} is said to have *enough injective objects* if for any object X in \mathcal{C} there exists an injective object I and a monomorphism $\iota : X \rightarrow I$.
- If A is a k -algebra, then the category of A -modules $\text{Mod}(A)$ has enough projective and injective objects.
- If A is Noetherian, then the category of finitely generated A -modules $\text{mod}(A)$ has enough projective objects, but need not have enough injective objects.

Corollary 1.18.7

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Projectives, Injectives and the Homotopy Category

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Let P be the complex that is zero in any degree other than n and that in degree n is a projective object P_n in \mathcal{C} such that there is an epimorphism $\pi : P_n \rightarrow \ker \delta_n$; this is possible since \mathcal{C} has enough projective objects.

Proof of Cor. 1.18.7 (contd.)

2/2

Then π defines a chain map from P to X

$$\begin{array}{ccccccccc} P : & \cdots & \longrightarrow & 0 & \xrightarrow{0_{n+1}} & P_n & \xrightarrow{0_n} & 0 & \xrightarrow{0_{n-1}} & \cdots \\ & & & \downarrow 0 & & \downarrow \pi & & \downarrow 0 & & \\ X : & \cdots & \longrightarrow & X_{n+1} & \xrightarrow{\delta_{n+1}} & X_n & \xrightarrow{\delta_n} & X_{n-1} & \xrightarrow{\delta_{n-1}} & \cdots \end{array}$$

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To see this, suppose that was the case, then there is an $\eta : P_n \rightarrow X_{n+1}$ such that $\pi = \delta_{n+1} \circ \eta$.

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Projectives, Injectives and the Homotopy Category

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By dualising the above proof, one shows (ii). ■

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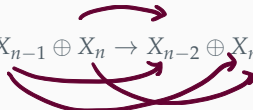
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relative version $C(f)$

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This does not necessarily imply that f is split as a chain map because the family $(s_n)_{n \in \mathbb{Z}}$ is not required to be a chain map.

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These chain maps yield a degreewise split short exact sequence of complexes

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If X is bounded below, bounded above or bounded, so is $C(X)$.

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Let \mathcal{C} be an additive category and let X be a complex over \mathcal{C} . We have $C(X) \simeq 0$.

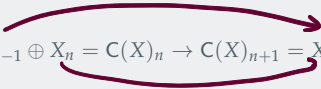
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Proposition 1.18.11

Let A be a k -algebra, Y, Z complexes of A -modules, and let $g : Y \rightarrow Z$ be a chain map. The following are equivalent.

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By 1.18.10, the cone $C(Z)$ is contractible.

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Proof of Prop. 1.18.11

By 1.18.10, the cone $C(Z)$ is contractible. Therefore $g : Y \rightarrow Z$ is a quasi-isomorphism if and only if $g \oplus p_Z : Y \oplus C(Z)[-1] \rightarrow Z$ is one; furthermore the latter map is surjective.

$\underbrace{g \oplus p_Z}_{H_\star(g)} \checkmark$

Proof of Prop. 1.18.11 (contd.)

2/2

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Since $\text{Mod}(A)$ has enough injective objects, a variation of the above arguments shows the equivalence between (i) and (iii). ■

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Further Results on Homotopy

Proposition 1.18.12

(has an interpretation in terms of *relative projectivity*) Let \mathcal{C} be an additive category and X a complex over \mathcal{C} . Denote by $\mathcal{F} : \text{Ch}(\mathcal{C}) \rightarrow \text{Gr}(\mathcal{C})$ the forgetful functor sending a complex (X, δ) to the underlying graded object X . The following are equivalent.

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Suppose that (i) holds.

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Proof of Prop. 1.18.12 (contd.)

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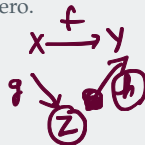
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Further Results on Homotopy

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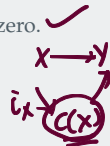
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Proof of Prop. 1.18.13 (contd.)

2/3

Assume (iii); that is, there exists a chain map $f' : C(X) \rightarrow Y$ such that $f = f' \circ i_X$.

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2/3

Assume (iii); that is, there exists a chain map $f' : C(X) \rightarrow Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \rightarrow Z$,

Proof of Prop. 1.18.13 (contd.)

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$$\begin{array}{ccc} X & \xrightarrow{\iota} & Z \\ & \searrow i_X & \swarrow i_X \circ \rho \\ & C(X) & \end{array}$$

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Further Results on Homotopy

Proof of Prop. 1.18.13 (contd.)

2/3

Assume (iii); that is, there exists a chain map $f' : C(X) \rightarrow Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \rightarrow Z$, so we can find a graded morphism $\rho : Z \rightarrow X$ such that $\rho \circ \iota = \text{id}_X$ in $\text{Gr}(\mathcal{C})$. Therefore, we have the following commutative diagram

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That is, the following diagram commutes

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Further Results on Homotopy

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$$g \circ \iota = f' \circ g' \circ \iota = f' \circ i_X = f$$

That is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \iota & \swarrow g \\ & Z & \end{array}$$

Hence, (iv) holds.

Further Results on Homotopy

Proof of Prop. 1.18.13 (contd.)

2/3

Assume (iii); that is, there exists a chain map $f' : C(X) \rightarrow Y$ such that $f = f' \circ i_X$. Consider a degreewise split monomorphism $\iota : X \rightarrow Z$, so we can find a graded morphism $\rho : Z \rightarrow X$ such that $\rho \circ \iota = \text{id}_X$ in $\text{Gr}(\mathcal{C})$. Therefore, we have the following commutative diagram

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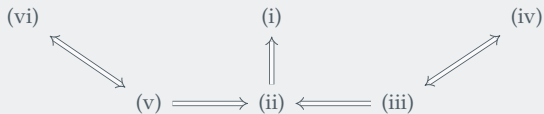
$$g \circ \iota = f' \circ g' \circ \iota = f' \circ i_X = f$$

That is, the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \iota & \swarrow g \\ & Z & \end{array}$$

Hence, (iv) holds. Similarly, (v) implies (vi).

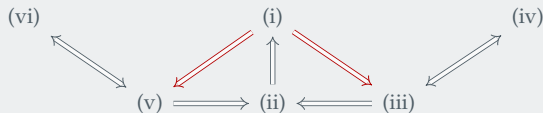
Our progress



Proof of Prop. 1.18.13 (contd.)

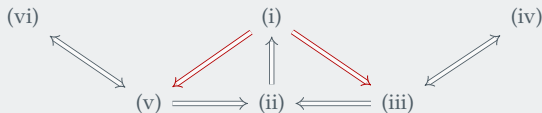
3/3

Our progress



Suppose (i) holds, we show then that both (iii) and (v) hold.

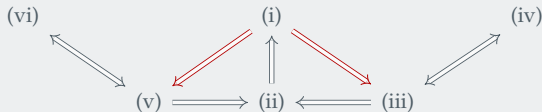
Our progress



Suppose (i) holds, we show then that both (iii) and (v) hold.

Denote by δ, ε the differentials of X, Y , respectively, and let $h : X \rightarrow Y$ be a homotopy satisfying $f = h \circ \delta + \varepsilon \circ h$.

Our progress



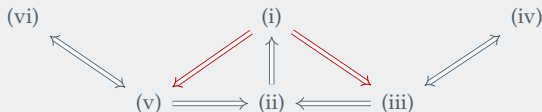
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We define a graded morphisms $r : C(X) \rightarrow Y$ and $s : X \rightarrow C(Y)[-1]$ by setting, for any integer n ,

$$r_n = (h_{n-1} \quad f_n) : X_{n-1} \oplus X_n \rightarrow Y_n \quad \text{and} \quad s_n = \begin{pmatrix} f_n \\ h_n \end{pmatrix} : X_n \rightarrow Y_n \oplus Y_{n+1}$$

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One readily checks that r and s are chain maps satisfying $f = r \circ i_X = p_X \circ s$. ■

Homotopy and Split Complexes

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Proposition 1.18.15

Let A be a k -algebra and X a complex of A -modules.

- (i) The complex X is split if and only if $X \cong Y \oplus H_*(X)$ for some contractible complex Y , where $H_*(X)$ is considered as a complex with zero differential.

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Proof of Prop. 1.18.15 (contd.)

2/3

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Recall that $\operatorname{im}(\delta) \subseteq \ker(\delta)$, one then shows that $\ker(\delta) = \operatorname{im}(\delta) \oplus H$, as graded A -modules,

Homotopy and Split Complexes

Proof of Prop. 1.18.15 (contd.)

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The graded submodule $Y = \operatorname{im}(s \circ \delta) \oplus \operatorname{im}(\delta)$ is a subcomplex of X

Proof of Prop. 1.18.15 (contd.)

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Since $H \subseteq \ker(\delta)$, the graded submodule H of X is in fact a subcomplex of X with zero differential. By construction, $H \cong H_*(X)$.

The graded submodule $Y = \operatorname{im}(s \circ \delta) \oplus \operatorname{im}(\delta)$ is a subcomplex of X because δ maps $\operatorname{im}(\delta)$ to zero and $\operatorname{im}(s \circ \delta)$ to $\operatorname{im}(\delta \circ s \circ \delta) = \operatorname{im}(\delta)$.

Proof of Prop. 1.18.15 (contd.)

3/3

We need to show that Y is contractible.

Homotopy and Split Complexes

Proof of Prop. 1.18.15 (contd.)

3/3

We need to show that Y is contractible. On $Y = \text{im}(s \circ \delta) \oplus \text{im}(\delta)$ we define the homotopy h such that h is zero on the summand $\text{im}(s \circ \delta)$ and equal to s on the summand $\text{im}(\delta)$.

$$h = \begin{pmatrix} 0 & s \end{pmatrix}$$

$$\text{id}_Y = \delta \circ h + h \circ \delta$$
$$\delta \text{id}_Y$$

Proof of Prop. 1.18.15 (contd.)

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For the converse, we have $X \cong Y \oplus H_*(X)$ for a contractible complex (Y, ε) ; this complex is split acyclic. Indeed, Y is acyclic by 1.18.3 (iii),

Proof of Prop. 1.18.15 (contd.)

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For the converse, we have $X \cong Y \oplus H_*(X)$ for a contractible complex (Y, ε) ; this complex is split acyclic. Indeed, Y is acyclic by 1.18.3 (iii), and if h is a homotopy on Y such that $\varepsilon \circ h + h \circ \varepsilon = \operatorname{id}_Y$,

Proof of Prop. 1.18.15 (contd.)

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Now, since $X \cong Y \oplus H_*(X)$, we can find chain maps $f : X \xrightarrow{\sim} Y \oplus H_*(X) : g$ such that $g \circ f = \operatorname{id}_X$ and $f \circ g = \operatorname{id}_{Y \oplus H_*(X)}$.

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Proof of Prop. 1.18.15 (contd.)

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The second statement follows. ■

Corollary 1.18.16

Let A be a k algebra, and let X, Y be split complexes of A -modules. A chain map $f : X \rightarrow Y$ is a quasi-isomorphism if and only if f is a homotopy equivalence.

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Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes.

Corollary 1.18.16

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Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes. Therefore $X \simeq H_*(X)$ and $Y \simeq H_*(Y)$.

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Let A be a k algebra, and let X, Y be split complexes of A -modules. A chain map $f : X \rightarrow Y$ is a quasi-isomorphism if and only if f is a homotopy equivalence.

Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes. Therefore $X \simeq H_*(X)$ and $Y \simeq H_*(Y)$. It follows that if f is a quasi-isomorphism, then f is a homotopy equivalence.

Homotopy and Split Complexes

Corollary 1.18.16

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Proof of Cor. 1.18.16

By 1.18.15 we have $X \cong C \oplus H_*(X)$ and $Y \cong D \oplus H_*(Y)$, where C and D are some contractible complexes. Therefore $X \simeq H_*(X)$ and $Y \simeq H_*(Y)$. It follows that if f is a quasi-isomorphism, then f is a homotopy equivalence. The converse is clear by 1.18.3 (iii). ■

Homotopy and Split Complexes

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We have $H_m(X) = X_m / \text{im}(\delta_{m+1})$, which is projective by the assumptions.

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Since X_m is projective, so is $\text{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \rightarrow \text{im}(\delta_{m+1})$ is split surjective.

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Since X_m is projective, so is $\text{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \rightarrow \text{im}(\delta_{m+1})$ is split surjective. Thus $X_{m+1} \cong \text{im}(\delta_{m+1}) \oplus \ker(\delta_{m+1})$.

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Since X_m is projective, so is $\text{im}(\delta_{m+1})$, and hence the map $\delta_{m+1} : X_{m+1} \rightarrow \text{im}(\delta_{m+1})$ is split surjective. Thus $X_{m+1} \cong \text{im}(\delta_{m+1}) \oplus \ker(\delta_{m+1})$. Hence $\ker(\delta_{m+1})$ is projective as well.

Proof of Prop. 1.18.17 (contd.)

2/2

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- a complex X' that coincides with X in all degrees, except in degree $m+1$, where $X'_{m+1} = \ker \delta_{m+1}$ and $X'_m = \{0\}$.

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All terms and homology of X' are still projective, and hence X' is split by induction. X is therefore split by 1.18.15 (i) and the result follows. ■

Homotopy and Split Complexes

Proposition 1.18.18

Let \mathcal{C} be an abelian category and let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a degreewise split short exact sequence of chain complexes over \mathcal{C} .

- (i) The chain map f is a homotopy equivalence if and only if $Z \simeq 0$. In that case, f is a split monomorphism.
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Proof of Prop. 1.18.18

Suppose that f is a homotopy equivalence, and let $f' : Y \rightarrow X$ be a homotopy inverse of f . Then $f' \circ f \sim \text{id}_X$, or equivalently $\text{id}_X - f' \circ f \sim 0$.

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Suppose that f is a homotopy equivalence, and let $f' : Y \rightarrow X$ be a homotopy inverse of f . Then $f' \circ f \sim \text{id}_X$, or equivalently $\text{id}_X - f' \circ f \sim 0$. By 1.18.13, $\text{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism.

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Suppose that f is a homotopy equivalence, and let $f' : Y \rightarrow X$ be a homotopy inverse of f . Then $f' \circ f \sim \text{id}_X$, or equivalently $\text{id}_X - f' \circ f \sim 0$. By 1.18.13, $\text{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \rightarrow X$ be a chain map such that $\text{id}_X - f' \circ f = t \circ f$.

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Homotopy and Split Complexes

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Suppose that f is a homotopy equivalence, and let $f' : Y \rightarrow X$ be a homotopy inverse of f . Then $f' \circ f \sim \text{id}_X$, or equivalently $\text{id}_X - f' \circ f \sim 0$. By 1.18.13, $\text{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \rightarrow X$ be a chain map such that $\text{id}_X - f' \circ f = t \circ f$. Then $\text{id}_X = (f' + t) \circ f$, hence f is split as a chain map with retraction $f' + t$.

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Suppose that f is a homotopy equivalence, and let $f' : Y \rightarrow X$ be a homotopy inverse of f . Then $f' \circ f \sim \text{id}_X$, or equivalently $\text{id}_X - f' \circ f \sim 0$. By 1.18.13, $\text{id}_X - f' \circ f$ factors through f since f is a degreewise split monomorphism. Let $t : Y \rightarrow X$ be a chain map such that $\text{id}_X - f' \circ f = t \circ f$. Then $\text{id}_X = (f' + t) \circ f$, hence f is split as a chain map with retraction $f' + t$. Hence, f and g induce an isomorphism of the given exact sequence with

$$0 \longrightarrow X \xrightarrow{i} X \oplus Z \xrightarrow{p} Z \longrightarrow 0$$

Proof of Prop. 1.18.18 (contd.)

2/2

Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q .

Proof of Prop. 1.18.18 (contd.)

2/2

Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q . Let π and ι be the canonical retract and section of i and p respectively, we then have $q \sim \pi$.

Proof of Prop. 1.18.18 (contd.)

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Here i and p are the canonical maps, and i remains a homotopy equivalence, say with homotopy inverse q . Let π and ι be the canonical retract and section of i and p respectively, we then have $q \sim \pi$. Now

$$\mathrm{id}_Z = p \circ \iota = p \circ \mathrm{id}_{X \oplus Z} \circ \iota \sim p \circ (i \circ q) \circ \iota \sim p \circ i \circ (\pi \circ \iota) = p \circ i \circ 0 = 0$$

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Corollary 1.18.19

Let X, Y be chain complexes over an abelian category \mathcal{C} .

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In that case, if the complexes X, Y are both bounded above (resp. bounded below, bounded), then the complexes P, Q can be chosen to be bounded above (resp. bounded below, bounded), too.

Homotopy and Split Complexes

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Suppose that $f : X \rightarrow Y$ is a homotopy equivalence.

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Suppose that $f : X \rightarrow Y$ is a homotopy equivalence. Set $P = C(Y)[-1]$ and $p = p_Y$.

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Suppose that $f : X \rightarrow Y$ is a homotopy equivalence. Set $P = C(Y)[-1]$ and $p = p_Y$. Note that p is a degreewise split epimorphism.

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Suppose that $f : X \rightarrow Y$ is a homotopy equivalence. Set $P = C(Y)[-1]$ and $p = p_Y$. Note that p is a degreewise split epimorphism. Thus the chain map $(f, p) : X \oplus P \rightarrow Y$ is a degreewise split epimorphism.

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Suppose that $f : X \rightarrow Y$ is a homotopy equivalence. Set $P = C(Y)[-1]$ and $p = p_Y$. Note that p is a degreewise split epimorphism. Thus the chain map $(f, p) : X \oplus P \rightarrow Y$ is a degreewise split epimorphism. Since $P \simeq 0$, the chain map (f, p) is still a homotopy equivalence. By 1.18.18, the complex $Q = \ker(f, p)$ satisfies $Q \simeq 0$ and $X \oplus P \cong Y \oplus Q$ in $\text{Ch}(\mathcal{C})$. The converse is trivial. The last statement follows from the fact that if Y is bounded above (resp. bounded below, bounded), so is $C(Y)$. ■

Corollary 1.18.20

Let \mathcal{A} be an abelian category.

- (i) Let P, Q be bounded below chain complexes consisting of projective objects in \mathcal{A} , and let $f : P \rightarrow Q$ be a chain map. Then f is a quasi-isomorphism if and only if f is a homotopy equivalence.
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Suppose that f is a quasi-isomorphism. Since the cone $C(P)$ is contractible by 1.18.10, we may replace Q by $Q \oplus C(P)$ and f by $\begin{pmatrix} f \\ i_X \end{pmatrix}$.

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Let A be a k -algebra, X a complex of right A -modules, and Y, Z complexes of left A -modules.

Final Remarks

Let A be a k -algebra, X a complex of right A -modules, and Y, Z complexes of left A -modules.

If $h : Y \rightarrow Z$ is a homotopy, then $\text{id}_X \otimes h$ is a homotopy from $X \otimes_A Y$ to $X \otimes_A Z$.

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Similar statements hold for functors of the form $\text{Hom}_A(Y, -)$ and $\text{Hom}_A(-, Z)$.

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Fin.

- [1] Linckelmann, Markus (2018). *The Block Theory of Finite Group Algebras*. Cambridge University Press.