Classwork
Section 01B

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MATH 111T — Spring 2021

Classwork 1 (Week 3)

Problem 1. We have the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \cdot)$, the latter has as a subgroup $(\mathbb{R}^\times_{>0}, \cdot)$. Give a homomorphism

$$\varphi : (\mathbb{R}, +) \to (\mathbb{R}^\times_{>0}, \cdot)$$

such that (a) φ is an isomorphism. (b) φ is not an isomorphism.

Problem 2. (a) For $(\mathbb{R}^\times, \cdot)$ as above, is $(\mathbb{R}^\times_{<0}, \cdot)$ a subgroup? (b) Consider the group $(\mathbb{C}^\times, \cdot)$, show $S \coloneqq \{z \in \mathbb{C} \, : \, |z| = 1\}$ is a subgroup.

Extra Problem. Consider the group $G = \{f : \mathbb{R} \to \mathbb{R} \, : \, f \text{ is bijective}\}$. This is a group under composition. Prove that the following set

$$H = \{f_r : \mathbb{R} \to \mathbb{R}, \; f_r(x) = x^r \, : \, r = p/q, \; p, q \text{ are odd}\}$$

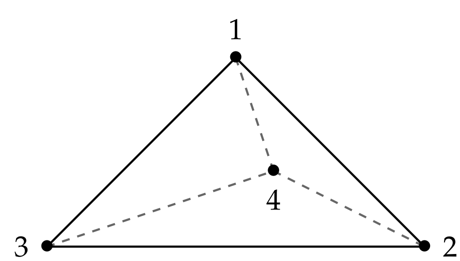
is a subset of G, and also a subgroup of G.

Why did we require p, q to be odd?

Classwork 2 (Week 5)

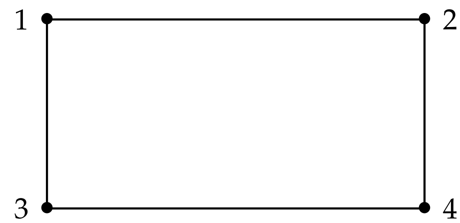
Problem 1. Consider the group $G = \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, find an n and subgroup $H \leqslant S_n$ such that $G \cong H$.

Problem 2. Consider a regular tetrahedron, T



Carefully write down the cycles that correspond to the rotations of T. Which subgroup of S_4 can you identify it with?

Extra Problem. Consider the following rectangle R



Carefully write down the cycles that correspond to the symmetries of R. Which subgroup of S_4 can you identify it with?

(Warning: Rotating the rectangle by 90 degrees is not a symmetry of R as the resultant rectangle has a distinctly different shape than R.)

Classwork 3 (Week 7)

Problem 1. Consider a group G.

(a) For each $g \in G$ define a map $c_g : G \to G,\ h \mapsto ghg^{-1}$. Call $\mathrm{Inn}(G) \coloneqq \{c_g \ : \ g \in G\}$. Prove that $\mathrm{Inn}(G) \leqslant \mathrm{Aut}(G)$, that is, show

  (i) $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$.

  (ii) $\mathrm{Inn}(G) \neq \varnothing$.

  (iii) $c_g \circ c_h^{-1} \in \mathrm{Inn}(G)$, for any $g, h \in G$.

(b) Consider the function $\varphi : G \to \mathrm{Aut}(G),\ g \mapsto c_g$. Show that

  (i) φ is a group homomorphism.

  (ii) Compute $\ker \varphi$; note that $\mathrm{im}\, \varphi = \mathrm{Inn}(G)$.

  (iii) Use the first isomorphism theorem to write $\mathrm{im}\, \varphi = \mathrm{Inn}(G)$ as a quotient of G.

Classwork 4 (Week 9)

Consider the ring of formal power series

$$\mathbb{C}[\![x]\!] = \left\{ \sum_{i=0}^{\infty} a_i x^i \ : \ a_i \in \mathbb{C} \right\}$$

where addition is defined as

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) + \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

and multiplication as

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right) = \sum_{k=0}^{\infty} c_k x^k, \qquad c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

Consider the ideal $\mathfrak{m} \coloneqq (x) = \{xf(x) \,:\, f(x) \in \mathbb{C}[\![x]\!]\}$, we have that $\mathfrak{m}$ is the unique maximal ideal since $\mathbb{C}[\![x]\!] \setminus \mathfrak{m} = \mathbb{C}[\![x]\!]^{\times}$.

We unpack this a bit; one notes that $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots \notin \mathfrak{m} = (x)$ if and only if $a_0 \neq 0$. The inverse of $p(x)$, say $q(x)$ is the power series $b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$ such that

$$b_0 = 1/a_0, \qquad b_k = -b_0 \left(\sum_{i=1}^{k} a_i b_{k-i}\right), \quad k > 0$$

Problem 1. Consider the element $p(x) = x^2 - 2x + 1 \in \mathbb{C}[\![x]\!]$, clearly $p(x) \notin \mathfrak{m}$. Find the first five terms of the power series $1/p(x)$.

Extra Problem. Consider the element $p(x) = x + 1 \in \mathbb{C}[\![x]\!]$, clearly $p(x) \notin \mathfrak{m}$. Find the first five terms of the power series $1/p(x)$.