Fibonacci Numbers ✓ Product of limits

Geometric Series ✓ Note on Proof writing ✓

Assignment 6 / HW 6

Prove for $r \neq 1$

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

<u>Proof</u>: $P(n)$: $a + ar + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

<u>Base Step</u> <u>$P(1)$</u>

$\hookrightarrow$ $\boxed{ar^{1-1} = \dfrac{a(1-r^1)}{1-r}}$ $\not\propto$ needs to be proved

$\hookrightarrow$ $a = a$ ∎

<u>In this situation</u> $P(n)$: $A = B$

$\underline{\underline{P(n)}}$
- $P(1)$ you want to prove the equality
- LHS — A Separately $= \cdots$ Ans
- RHS — B separately $= \cdots$ Ans

LHS = RHS

LHS: $ar^{1-1} = a$; RHS: $\dfrac{a(1-r^1)}{1-r} = a$

* Geometric Series

$$\underline{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}} \quad \text{when } |x| < 1$$

$0 < r < 1$

$M - \frac{1}{r}\left(\frac{r}{1-r}\right)$

$M - (r-1) + \frac{1}{1-r}$

Q1

What is $\sum_{n=k}^{\infty} r^n$?

I

$$\sum_{n=1}^{\infty} r^n \stackrel{?}{=} \sum_{n=0}^{\infty} r^{n+1}$$

r^{n+2}

$r^n \to r^{n+k}$

$\boxed{\frac{r^k}{1-r}}$

$$= \sum_{n=0}^{\infty} r^n \cdot r$$

$$= r \cdot \sum_{n=0}^{\infty} r^n$$

$$= \frac{r}{1-r}$$

II

$2 \qquad (1+r) - (1+r)$

$$\sum_{n=1}^{\infty} r^n + 1 - 1$$

$$\sum_{n=1}^{\infty} r^n + r^0 - 1$$

$$= \sum_{n=0}^{\infty} r^n - 1$$

$$= \frac{1}{1-r} - 1$$

$$= \frac{1-(1-r)}{1-r}$$

$$= \frac{r}{1-r}$$

Q2

What is $\sum_{n=0}^{\infty} r^{2n}$?

since $0 < r < 1$

$0 < r^2 < 1$

$$\sum_{n=0}^{\infty} r^{2n} = \sum_{n=0}^{\infty} (r^2)^n$$

$$= \frac{1}{1-r^2}$$

$$\sum_{n=0}^{\infty} \square^n = \frac{1}{1-\square}$$

Q2A What is $\sum_{n=1}^{\infty} r^{2n}$

$$\sum_{n=1}^{\infty} r^{2n} = r^2 + r^4 + r^6 + \cdots$$

$$= r^2(1 + r^2 + r^4 + \cdots)$$

$$= r^2 \cdot \sum_{n=0}^{\infty} r^{2n}$$

$$\frac{r^2}{1-r^2}$$

$$= \sum_{n=0}^{\infty} r^{2n+2}$$

$$= \sum_{n=0}^{\infty} r^{2(n+1)}$$

Q3 What is $\sum_{n=0}^{\infty} r^{2n+1}$

why $\frac{r}{1-r^2}$?

$$\sum_{n=0}^{\infty} r^{2n+1} = \sum_{n=0}^{\infty} r^{2n} \cdot r$$

$$= r \underbrace{\sum_{n=0}^{\infty} r^{2n}}$$

$$= r \cdot \frac{1}{1-r^2} = \frac{r}{1-r^2}$$

JMJ

Q3A

What is $\sum_{n=1}^{\infty} r^{2n+1}$?

$\hookrightarrow \frac{r^3}{1-r^2}$ — J

I

$$\sum_{n=1}^{\infty} r^{2n+1} = r \underbrace{\sum_{n=1}^{\infty} r^{2n}}$$

$$= r \cdot \frac{r^2}{1-r^2}$$

$$= \frac{r^3}{1-r^2}$$

II

$$\sum_{n=1}^{\infty} r^{2n+1} = \sum_{n=0}^{\infty} r^{2(n+1)+1} = \sum_{n=0}^{\infty} r^{2n+3}$$

$$= r^2 \cdot \sum_{n=0}^{\infty} r^{2n+1}$$

$$= r^2 \cdot r/1-r^2 = r^3/1-r^2$$

* $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ ✪

* $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$ ✪

* $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ ✪

$\alpha, \beta \in \mathbb{R}$ st $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$

$$\left.\begin{array}{l} * \ \alpha+\beta = 1 \\ * \ \alpha\beta = -1 \end{array}\right\}$$

* $\underline{\alpha^2} = \alpha + 1$
* $\underline{\beta^2} = \beta + 1$

α & β are zeros of the polynomial $\underline{x^2} - x - 1 = 0$

$(x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$

$\underline{\alpha} = \frac{1+\sqrt{5}}{2} > \frac{1+\sqrt{4}}{2} = \frac{3}{2} = 1.5 > 1$

so $\underbrace{\frac{1}{\alpha}} < 1$; since $\alpha\beta = -1 \Rightarrow \underbrace{-\beta = \frac{1}{\alpha}}$

We can talk about $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha}\right)^n$!

$n \to 2n$, $n \to 2n+1$

$$\underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{\alpha}\right)^n}_{1} = \frac{1}{1-\frac{1}{\alpha}} = \frac{1}{1+\beta} = \frac{1}{\beta^2} = \alpha^2$$

$[\alpha\beta = -1$, square them! $\alpha^2\beta^2 = 1]$

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* $\underbrace{\lim_{n\to\infty} a_n}$ $= a$

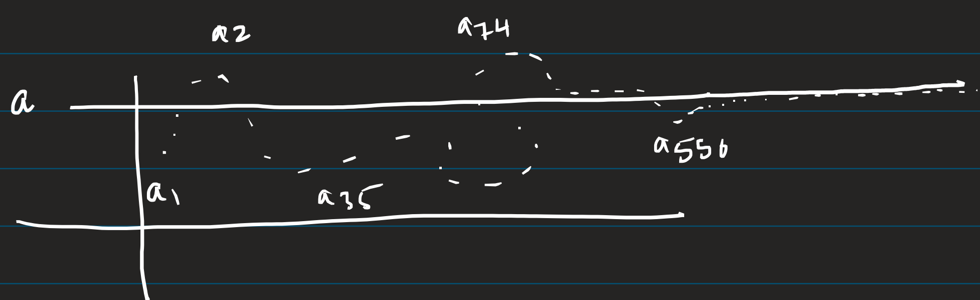
$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha \quad ☆$$

$$R_n = \frac{F_{n+1}}{F_n}$$

$$\lim_{n\to\infty} a_{n+2} = a$$

$$\lim_{n\to\infty} a_{n+35} = a$$

$$\boxed{\lim_{n\to\infty} R_n = \alpha}$$



$$\underbrace{\lim_{n\to\infty} a_{n+k} = a} \quad \text{for any } k \in \mathbb{Z}_{>0}$$

$$\lim_{n\to\infty} R_n = \alpha \; ; \; \lim_{n\to\infty} R_{n+4} = \alpha$$

$$= \lim_{n\to\infty} F_{\underline{n+4}+1} / F_{n+4} = \alpha$$

$$\lim_{n\to\infty} \frac{F_{n+k+1}}{F_{n+k}} = \alpha \qquad \text{for any } k \in \mathbb{Z}_{>0}$$

$*$ $\lim_{n\to\infty} a_n = a \qquad \lim_{n\to\infty} b_n = b$

What is $\lim_{n\to\infty} a_n b_n$?

$$\underbrace{\lim_{n\to\infty} a_n b_n} = ab = \underbrace{\left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)}$$

$*$ $\underbrace{a_n = \frac{n+1}{n} = 1 + \frac{1}{n}}$, $\underbrace{\lim_{n\to\infty} a_n = 1}$, so $\lim_{n\to\infty} a_{n+k} = 1$

$\lim_{n\to\infty} \frac{n+k}{n} = 1$ // for any k

$\left(= 1 + \frac{k}{n}\right)$

$$\lim_{n\to\infty} \underbrace{\frac{n+k}{n}}_{=} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdot \frac{n+3}{n+2} \cdots \frac{n+k-1}{n+k-2} \cdot \frac{n+k}{n+k-1}$$

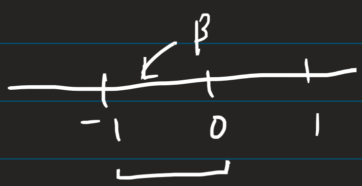
$$= \lim_{n\to\infty} \underbrace{a_n a_{n+1} a_{n+2} \cdots a_{n+k-2}\, a_{n+k-1}}$$

$$= \boxed{\underbrace{\lim_{n\to\infty} a_n} \cdot \underbrace{\lim_{n\to\infty} a_{n+1}} \cdots \underbrace{\lim_{n\to\infty} a_{n+k-1}}}$$

$$= \underbrace{1 \cdot 1 \cdots 1}_{k\text{-times}} \ // \qquad = 1$$

* $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$

Recall we said $-\beta = \frac{1}{\alpha} < 1$



so $\beta > -1$

$-\beta < 1$

$\lim_{n\to\infty} x^n = 0$ $\quad |x| < 1$

$$\lim_{n\to\infty} (-\beta)^n = 0$$

$$\lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \beta^n = 0$$

so $\lim_{n\to\infty} \beta^n = 0$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}$$

$$= \frac{\lim \alpha^{n+1} - \cancel{\lim \beta^{n+1}} \to 0}{\lim \alpha^n - \cancel{\lim \beta^n} \to 0}$$

$$= \lim \frac{\alpha^{n+1}}{\alpha^n} = \alpha$$

$$* \quad \underbrace{\lim_{n\to\infty} \frac{F_{n+1}}{F_n}}_{\substack{\parallel \\ \ell}} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n}$$

$$= \lim_{n\to\infty} 1 + \frac{F_{n-1}}{F_n}$$

$$= 1 + \underbrace{\lim_{n\to\infty} 1 / \frac{F_n}{F_{n-1}}} = 1/\ell$$

If $S_n = \frac{F_n}{F_{n-1}}$, then $S_{n+1} = \frac{F_{n+1}}{F_n}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n+1} = \ell$$

$$\ell = 1 + \frac{1}{\ell} \Rightarrow \ell^2 - 1 - \ell = 0$$

so $\ell = \alpha$ or $\cancel{\beta}$