A flight over Gromov's Non-squeezing Theorem

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January 2022

Summary

- Introduction
- Examples
- Problem
- Nonsqueezing theorem
- Capacities

Introduction

- A symplectic form in a smooth manifold M is a closed non degenerate 2-form ω.
- A pair (M, ω) where M is a smooth manifold, and ω is a symplectic form on M is called a symplectic manifold.
- We will always consider the $\mathbb{R}^{2n}$ with coordinates $(p_1, ...p_n, q_1, ..., q_2)$, and symplectic form $\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$.

Consequences of the Definition

- Any symplectic manifold is even dimension
- Any symplectic manifold is orientable.
- If ω is a symplectic form on a manifold M, then ω^n is a volume form.
- Any symplectomorphism preserve volume, i.e, if $f : (M, \omega) \to (N, \eta)$ is a symplectomorphism (diffeomorphims s.t $f^*\eta = \omega$) then $f^*\eta^n = \omega^n$
- If a closed manifold M admits a symplectic form, then $H_2(M, \mathbb{R})$ is non trivial

Examples and Non Examples

- Examples

- We will always consider the $\mathbb{R}^{2n}$ with coordinates $(p_1, ...p_n, q_1, ..., q_2)$ and symplectic form $\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i$.
- Any orientable surface ($\mathbb{S}^2, \mathbb{T}^2$, Genus g surface...).
- $\mathbb{CP}^n$.
- $(B^{2n}(r), \omega_0)$, where $B^{2n}(r) = \{(p, q) \in \mathbb{R}^{2n} / |p|^2 + |q|^2 < r^2\}$.
- $(Z^{2n}(r), \omega_0)$, where $Z^{2n}(r) = \{(p, q) \in \mathbb{R}^{2n} / p_1^2 + q_1^2 < r^2\} = B^2(r) \times \mathbb{R}^{2n-2}$.

- Non Examples

- $\mathbb{RP}^n, \mathbb{S}^n (n > 2)$.

Question

We have seem that $Symp(M, \omega) \subset Diff_v(M, \omega^n)$. How different are these two groups? Is there any other symplectic invariant besides volume? For example consider the map $\phi : B^{2n}(r) \to Z^{2n}(R)$ given by

$$\phi(p, q) = (\epsilon p_1, \frac{1}{\epsilon} p_2, ..., \epsilon q_1, \frac{1}{\epsilon} q_2 ...).$$

This is a volume preserving embedding for ϵ small enough, but not a symplectic embedding (only when $\epsilon = 1$).

Theorem - Moser

Let M be a compact, connected and oriented manifold. If α and β are two volume forms such that their volume agree, i.e.,

$$\int_M \alpha = \int_M \beta.$$

then there is a diffeomorphism f on M such that $f^*\beta = \alpha$.

this theorem is saying that volume is the only invariant of volume preserving diffeomorphisms.

In $\mathbb{R}^2$ the symplectic form is equal to the volume form, so then volume is the only invariant

Nonsqueezing theorem

Theorem

[Gromov's Nonsqueezing theorem, 1985] If there is a symplectic embedding

$$B^{2n}(r) \hookrightarrow Z^{2n}(R).$$

then

$$r \leq R.$$

Capacity

(Symplectic Capacity) Consider the class of all symplectic manifolds of fixed dimension $2n$. A symplectic capacity is a map which associate to every symplectic manifold (M, w), a non negative number or ∞, satisfying the following properties.

- Monotonicity: $c(M, w) \leq c(N, \tau)$.
- If there exists a symplectic embedding $\phi : (M, \omega) \to (N, \tau)$.
- Conformality: $c(M, \alpha\omega) = |\alpha| c(M, \omega)$.
- Nontriviality: $c(B^{2n}(1)) > 0$ and $c(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$.

Properties

For $U \subset (\mathbb{R}^{2n}, \omega_0)$ open, and $\lambda \neq 0$,

$$c(\lambda U) = \lambda^2 c(U)$$

.

In particular if $c(B(1)) = \pi = c(Z(1))$, then $c(B(r)) = r^2\pi = c(Z(r))$.

Proof's main idea

At first, it is not so clear that there exist a capacity in $\mathbb{R}^{2n}$, but it is clear that the existence of a capacity such that

$$c(B^{2n}(1)) = c(Z^{2n}(1)) = \pi,$$

implies the Gromov's nonesqueezing. Gromov's proof involves J-holomorphic curves.

Capacities

Any smooth function $H : M \to \mathbb{R}$ is associated with a vector field X_H by the relation

$$\omega(X_H, .) = -dH$$

A T-periodic solution for the Hamiltonian equation is a solution $x(t)$

$$x^{'}(t) = X_H(x(t))$$

such that $x(T) = x(0)$. In particular singularities of the Hamiltonian are periodic solutions.

Consider $\eta(M, \omega)$ to be the set of smooth functions that satisfies

- there is a compact $K \subset M$ (depending on H) such that $K \subset (M - \partial M)$, and
$$H(M - K) = m(H) \ (a\ constant).$$
- There is an open set $U \subset M$ on which
$$H(U) \equiv 0.$$
- $0 \leq H \leq m(H)$, for all $x \in M$.

the constant $m(H) = max(H) - min(H)$ is called oscillation of H
A function in $\eta(M, \omega)$ will be called admissible if all the periodic solutions are either constant $x(t) = x(0)$ for all t or have period $T > 1$
denoting the set of admissible functions by $\eta_a(M, \omega)$, we define

$$c_0 = sup\{m(H) \ / \ H \in \eta_a(M, \omega)\}.$$

Theorem

c_0 is a symplectic capacity such that $c(B(1)) = \pi = c(Z(1))$.

Darboux theorem

Darboux Theorem

Let (M, ω) be a symplectic manifold, and p any point in M. Then there exists coordinates $(U, x_1, ..., x_n, y_1, ..., y_n)$ such that

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

In particular, for every point $p \in M^{2n}$, there exists an symplectic embedding $(B^{2n})(r), \omega_0) \hookrightarrow (M^{2n}, \omega)$. It is natural to define the following: (Gromov width)

$$D(M, \omega) = sup\{\pi r^2 \ / \ there\ exists\ an\ embedding\ (B^{2n})(r), \omega_0) \hookrightarrow (M^{2n}, \omega)\}.$$

Gromov width

We define the Gromov width by

$$D(M,\omega) = sup\{\pi r^2 \; / \; \textit{there exists an embedding } (B^{2n}(r),\omega_0) \hookrightarrow (M^{2n},\omega)\}.$$

Gromov width

The Gromov width $D(M,\omega)$ is a symplectic capacity.

Clearly the Gromov width is well defined (can be ∞) and is a Symplectic invariant.

Thank you.

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