1. Group Algebras

Fix k to be a unital commutative ring & G a group.

(I) kG

Def^n The group algebra kG is the free k-algebra with G as a basis and with multiplication induced by the group multiplication.

$$kG = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \lambda_g \neq 0 \text{ for finitely many terms, } g \in G \right\}$$

sum is defined coefficient-wise

multiplication :

$$\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{h \in G} \mu_h h\right) = \sum_{g,h\in G} \lambda_g \mu_h gh$$

$$= \sum_{z \in G} \left(\sum_{gh = z} \lambda_g \mu_h \right) z$$

We view G as sitting in kG, $g \in G$ is viewed as the element $1 \cdot g \in kG$.
With that, the $1_{kG} = 1 \cdot e_G$, e_G is the group identity.

$$\begin{aligned} k &\longrightarrow kG \\ x &\longmapsto x \cdot e_G \end{aligned}$$

so kG is a k-algebra, since $x \cdot e_G \in Z(kG)$.

II Idempotents

* An idempotent in an algebra A, is a non-zero element i st $i^2 = i$
* i, j as idempotents are called orthogonal if $ij = 0 = ji$. If this happens, $i+j$ is also an idempotent.
* If i is an idempotent st $i \neq 1$, then $1-i$ is also an idempotent. And i and $1-i$ are orthogonal.

* An idempotent i is called primitive if it CANNOT be written as a sum of two orthogonal idempotents.
* a (primitive) decomposition of an idempotent e in A is a finite set I of pairwise orthogonal, idempotents in A s.t. $e = \sum_{i \in I} i$.
(primitive)

Prop^n Let $H \leq G$, H is finite, then if $|H|$ is invertible in k, then

$$e_H = \frac{1}{|H|} \sum_{y \in H} y$$

is an idempotent in kG. Further, if $H \trianglelefteq G$, then $e_H \in Z(kG)$.

Proof To show $e_H^2 = e_H$.

$$\left(\sum_{y \in H} y\right)^2 = \sum_{y \in H} y \cdot \sum_{x \in H} x = \sum_{y \in H} \sum_{x \in H} x = |H| \cdot \sum_{x \in H} x$$

Divide by $|H|^2$, giving us $e_H^2 = e_H$.

Assume $H \trianglelefteq G$,

Using the fact that $Z(kG) = \{x \in kG \mid gxg^{-1} = x, \forall g \in G\}$, we note

$$ge_Hg^{-1} = \frac{1}{|H|}\sum_{y \in H} gyg^{-1} = \frac{1}{|H|}\sum_{x \in g^{-1}H} x = \sum_{x \in H} x = e_H$$

Hence $e_H \in Z(kG)$ ▨

(III) Resuming kG

Prop^n G, H be groups and let $\phi : G \to H$ be a group homomorphism. Then there exists a unique k-algebra morphism $\alpha : kG \to kH$ st

$\alpha(x) = \phi(x)$, for every $x \in G$.

Proof: Extend ϕ k-linearly to get α. α is multiplicative since ϕ is a grp hmph. ▨

So, we have a functor $k(-): \mathrm{Grp} \longrightarrow {}_k\mathrm{Algebras}$.

* Related functor: $(-)^\times : {}_k\mathrm{Algebras} \longrightarrow \mathrm{Grp}$

$$A \longmapsto A^\times$$

$$(f: A \longrightarrow B) \longmapsto (f|_{A^\times} : A^\times \longrightarrow B^\times)$$

Fact: $k(-) \dashv (-)^\times$

* A k-algebra A is called a division ring if $A^\times = A \setminus \{0\}$.
Commutative division rings are fields.

* Every module over a division ring is free, proof is similar to showing modules over a field are free.

IV Tensor Products

* Fact: The tensor product of two k-algebras A and B: $A \otimes_k B$ is also a k-algebra.

Thm: Let G, H be groups. Then there's a unique algebra isomorphism

$$k(G \times H) \cong kG \otimes_k kH$$

sending $(x, y) \in G \times H$ to $x \otimes y$.

Proof

$$\phi : G \times H \longrightarrow kG \otimes_k kH$$
$$(x, y) \longmapsto x \otimes y$$

extend to a k-algebra map $\alpha : k(G \times H) \longrightarrow kG \otimes_k kH$

$$(x, y) \longmapsto x \otimes y$$

mapping basis to basis, so an isomorphism.

Comment: the direct product of group algebras is NOT a group algebra

Take k to be a field of char 2, and consider $kC_2 \times kC_2$

If this was a group algebra, then either $kC_2 \times kC_2 \cong kC_4$ or $\cong kV_4$. (1.11.5)

But we can directly see that kC_4 & kV_4 don't have non-trivial idempotents

but $kC_2 \times kC_2$ does have non-trivial idempotents: $(1,0)$ and $(0,1)$.

V Noetherian-ness & Artinian-ness

Prop^n If $|G| < \infty$ and k is Noetherian (resp. Artinian), then kG and $Z(kG)$ is Noetherian (resp. Artinian)

Proof FACT from commutative algebra: f.g.-module over a Noetherian (resp. Artinian) ring is also Noetherian (resp. Artinian). kG is f.g. over k.

[FACT: $0 \to M' \to M \to M'' \to 0$

M is N (resp. A) iff M' & M'' is N. (resp A.)]

Therefore kG is N. (resp A.) as a k-module, hence an asceding chain of k-submodules of kG becomes stationary.

Consider an ascending chain of left ideals I_i of kG. But every left ideal I_i is a k-submodule, hence the chain becomes stationary since kG is a Noetherian k-module. Similarly, kG is right-Noetherian as a ring, hence Noetherian. Similarly for Artinian-ness.

$Z(kG)$ is also a k-submodule of kG, and so Noetherian (resp. Aritinian) as a module. Similar reasoning as above gives us that $Z(kG)$ is a Noetherian (resp. Artinian) as a ring. $\boxtimes$

VI Representations

Def^n A representation of G over k is a pair (V, ρ) where V is a k-module and $\rho : G \longrightarrow GL(V) = Aut_k(V)$, a group homomorphism.

* Morphisms b/w (V,ρ) and (W,σ) is a linear map $\phi : V \to W$ st $\forall g \in G$ $\phi \circ \rho(g) = \sigma(g) \circ \phi$.

* If V is a free module w/ finite rank, say n, then $GL(V) \cong GL_n(k)$ by choosing a basis of V. Then $\rho : G \longrightarrow GL_n(k)$ is our group homomorphism.

* (V, ρ), a representation and define a kG-module structure on V.
For any $v \in V$, $g \cdot v = \rho(g)(v)$

Conversely, given a kG-module V, we define a $\rho : G \longrightarrow GL(V)$
$$g \longmapsto (v \mapsto g \cdot v)$$

* ${}_{kG}\mathrm{Mod}$ is an abelian category and has a tensor product

U, V be two kG-modules, and $U \otimes_k V$ is a k-module.

Then define the G-action as $g \cdot (u \otimes v) := gu \otimes gv \quad \forall g \in G, u \in U, v \in V$.

VII Permutation representation & modules.

Let M be a G-set.

Def^n Let kM be defined as the kG-module, which as a k-module is free w/ basis M. Concretely, take $\sum_{g \in G} \lambda_g g \in kG$ and $\sum_{m \in M} \mu_m m$, then

$$\left(\sum_{g \in G} \lambda_g g\right) \cdot \left(\sum_{m \in M} \mu_m m\right) = \sum_{g \in G} \sum_{m \in M} \lambda_g \mu_m \, g \cdot m$$

* A kG- module V is called a permutation module if $V \cong kM$ for a G-set M, or equivalently: V is a free k-module and it has a k-basis that is G-stable.

* If G acts transitively on a k-basis of V, V is a transitive permutation module

eg

* The trivial kG-module k is a permutation module since $k \cong k\{*\}$.

* The regular left kG-module is the permutation module obtained by the regular action of G on itself by left-multiplication.

* For any $n \in \mathbb{Z}_+$, then S_n naturally acts on $\Omega = \{1, \dots, n\}$ and the associated kG-module $k\Omega$ is called the natural permutation module of S_n.

Prop^n: Let U and V be two permutation modules, so are $U \oplus V$ and $U \otimes_k V$.

Proof. Let M be a G-stable k-basis of U, and N a G-stable k-basis of V. Then $M \sqcup N$ will be a stable k-basis of $U \oplus V$, making this a permutation module.

Consider the set $\{ m \otimes n \mid (m,n) \in M \times N \}$, this is a G-stable k-basis of $U \otimes_k V$.

Prop^n: Let G be finite, and M is a transitive G-set. Let $m \in M$ and let $H = \{ x \in G \mid xm = m \}$ stabilizer of m in G. Then the map that sends $x \in G$ to $xm \in M$ induces an isomorphism of G-sets $G/H \cong M$ and induces an isomorphism of kG-modules.

Proof: Uses facts we know.

VIII Indecomposable Modules

Let A be a k-algebra.

* U, an A-module, is indecomp. if it CANNOT be written as $U \cong V \oplus V'$ for some non-zero A-modules V and V'.

* $\pi \in \mathrm{End}_A(U)$ is an idempotent, then $U = \pi(U) \oplus (1_U - \pi)(U)$. This says that U is indecomp $\Leftrightarrow$ 1_U is the only idempotent in $\mathrm{End}_A(U)$.

* An A-module S is called simple if $S \neq 0$ has no non-zero proper submodules.

* A simple module is indecomp. but not vice-versa, in general.

* A transitive permutation KG-module need not be indecomposable

* Direct summand of a permutation module need not be a permutation module.

Prop^n Let G be finite and M a G-set. Then the permutation module kM has a trivial submodule, namely $k\left(\sum_{m\in M} m\right)$. In particular, if $|M| \geq 2$, then kM is not simple.

Proof: Action of G permutes elements in M, so fixes $\sum_{m\in M} m$. Hence $k\left(\sum_{m\in M} m\right)$ is a kG-submodule of kM. ∎