Plan : * Record Session? ✓

* Topics important for Final //

* Accompanying example problems //

* General questions //

[In no particular order]

① Deducing truth value of "compound logical statements"

ⓐ $(P \vee Q) \Rightarrow R \equiv (P \Rightarrow R) \wedge (Q \Rightarrow R)$ 1/2 works

ⓑ $(P \wedge (\sim Q)) \wedge (P \wedge Q)$ 1 works

Strategy ~ 2 methods

1. Write down the truth tables

2. Use usual "definitional" logical equiv to simplify expression to just involve $\wedge$, $\vee$ & $\sim$

$(P \Rightarrow Q) \equiv \sim P \vee Q$

$(A \Rightarrow B \equiv \sim A \vee B)$

ⓐ $(P \vee Q) \Rightarrow R \equiv \sim(P \vee Q) \vee R$ ("def" of $\Rightarrow$)

$\equiv (\sim P \wedge \sim Q) \vee R$ (dist of $\sim$)

$\left(\begin{array}{l} a\cdot(b+c) \\ = ab + ac \end{array}\right)$

$$\equiv R \vee (\sim P \wedge \sim Q) \quad (\vee \text{ is commutative})$$

$$\equiv (R \vee \sim P) \wedge (R \vee \sim Q) \quad (\vee \text{ distributes over } \wedge)$$

$\left(\begin{array}{l} A \wedge (B \vee C) \\ (A \wedge B) \vee (A \wedge C) \end{array}\right)$

$$\equiv (\sim P \vee R) \wedge (\sim Q \vee R) \quad (\vee \text{ is commutative})$$

$$\equiv (P \Rightarrow R) \wedge (Q \Rightarrow R) \quad (\text{"df" of } \Rightarrow)$$

2. Negating "english" statements

<u>Strategy</u>

* Translate the given statement into logical stat w/ quantifiers & logical symbols

$\sim(\exists P)$
NOT $\forall \sim P$
but $\forall P$

* Apply negation rules (recall this) [if quantifier negate only that]

* Translate back into english

* Replace negative (e.g. not even) with positives (e.g. odd) whenver possible (e.g. "not a 3rd power / cube (x^3)", no need to replace w/ positive)

Q:

(a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.
(b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

(a) $\exists P, \; Q \wedge R$

P — integers a, b

Q — $ab < 0$

R — $a+b > 0$

Negate

$\forall P, \; \sim Q \vee \sim R$

Translate back:

for all integer a, b, $ab \nless 0$ or $a+b \ngtr 0$

Use positives

for all integers a, b, $ab \geq 0$ or $a+b \leq 0$ $\blacksquare$

(b) $\forall P, \; Q \Rightarrow R$

$\equiv \; \forall P, \; \sim Q \vee R$

P — real numbers x, y

Q — $x \neq y$

R — $x^2+y^2 > 0$

Negate

$\exists P, \; \sim(\sim Q) \wedge \sim R$

$\equiv \; \exists P, \; Q \wedge \sim R$

Translate

there exist real number x, y st $x \neq y$ and $x^2+y^2 \ngtr 0$

Positive

there exist real number x, y st $x \neq y$ and $x^2+y^2 \leq 0$ $\blacksquare$

Example : At least two of my library books are overdue.

Open sentence with domain

$L = \{\text{all library books}\}$

$P(\ell)$: $\underbrace{\ell \geq 2}$, $\underbrace{\text{overdue}}$

negate $\underbrace{\ell < 2}$, not overdue

$\ell \leq 1$, not overdue

at most one library book is not overdue.

3. Induction!

Strategy — Just use strong induction

* divisibility question $\longrightarrow$ (a) $7 \mid 3^{4n+1} - 5^{2n-1}$
* Recursive definition

to general form

↳ (b) $a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3)$ ✪

$a_1 = 1,\ a_2 = 4,\ a_3 = 9,\ a_4 = 16$ ✪

Prove $a_n = n^2$ ☒

Proof (b) : Base Step will be ☆

Induction Step

i for all $1 \le i \le k$

$\underline{1, \dots, k} \Rightarrow k+1$, use ☆

i.e. to prove $a_{k+1} = (k+1)^2$

↳ ☆, apply formula to (they are squares) terms appearing in that

Induction Step

→ Induction Hypothesis

$a_i = i^2$ for all $1 \le i \le k$

i.e. $\left(a_1 = 1^2, a_2 = 2^2, \dots, a_{k-1} = (k-1)^2, a_k = k^2 \right)$

Consider a_{k+1}

☆

$$a_{k+1} = a_{(k+1)-1} - a_{(k+1)-2} + a_{(k+1)-3} - 2(2(k+1)-3)$$

$$= a_k - a_{k-1} + a_{k-2} - 2(2k-1)$$

$$= k^2 - (k-1)^2 + (k-2)^2 - 2(2k-1)$$

$$\vdots$$

$$= (k+1)^2$$

4. Modular Arithmetic. ($\equiv$ mod n)

ⓐ $n \in \mathbb{Z}$, what is the value of ~~how do we investigate~~ $n^3 - n^2 + 1$, mod 4, mod 8, mod 9 ?

$n^3 - n^2 + 1 \equiv ?$ mod 4

$\equiv ?$ mod 8

$\equiv 1$ mod 9

$n = 2k$

$n = 2k+1$

$n = 3k$

$n = 3k+1$

$n = 3k+2$

Strategy : Recall, and can use these properties

$$\left.\begin{array}{l} a \equiv b \bmod n \\ c \equiv d \bmod n \end{array}\right\} \Rightarrow a + c \equiv b + d \bmod n$$

$$\Rightarrow ac \equiv bd \bmod n$$

$n = 3k+1$, compute n^3 & n^2

$n^2 = (3k+1)^2 = 9k^2 + 6k + 1 \equiv 6k + 1 \bmod 9$

$n^3 = (3k+1)^3 = 27k^3 + 27k^2 + 6k + 1 \equiv 6k+1 \bmod 9$

What is $n^3 - n^2 + 1 \bmod 9 \equiv (6k+1) - (6k+1) + 1 \bmod 9 \equiv 1 \bmod 9$.

(b) Is it true that $m^p \equiv 1 \bmod p$ for every $m \in \mathbb{Z}$?
$\quad m = p$

NO! GIVE COUNTEREXAMPLE

give a very specific m st that this isn't true.

$m^3 \equiv 1 \bmod 3$ for $m \in \mathbb{Z}$

$m =$ number st $\text{number}^3 \not\equiv 1 \bmod 3$.

$m = 3$ b/c $3^3 = \underline{\underline{27}} \equiv 0 \bmod 3$

5. Equivalence Relation

Strategy: if a relation is defined as a property, think through the property.

For equiv class: Definitions & unpack.

(a) Relation on lines on $\mathbb{R}^2$

$l_1 \, R \, l_2$ iff $\underline{\underline{l_1 \parallel l_2}}$

Prove this is an equiv relation (assume so, what's the equiv class?)

$$[\ell] = \{\ell_1 \mid \ell_1 R \ell\}$$

$$= \{\ell_1 \mid \ell_1 \parallel \ell\}$$

make things concrete, $\ell : y = mx + b$ slope y-intercept

what does parallel mean — same slope

$y = mx + e$ is $\parallel^{\text{le}}$ to

$$[y = mx + b] = \{\ell_1 \mid \ell_1 \parallel (y = mx + b)\}$$

$$= \{\ell_1 \mid \text{slope}(\ell_1) = m\}$$

$$= \{y = mx + e \mid e \in \mathbb{R}\}$$

e.g.

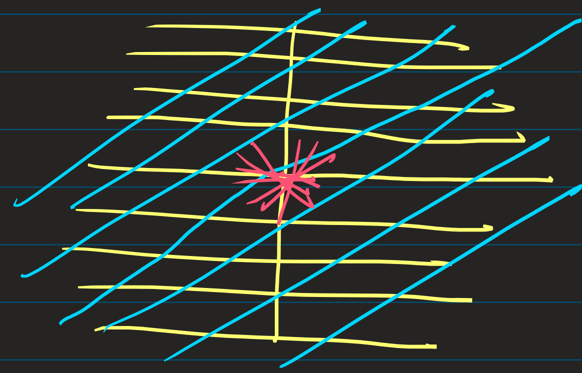
$$[x\text{-axis}] = [y = 0] = \{y = 0 \cdot x + e \mid e \in \mathbb{R}\}$$

$$= \{y = e \mid e \in \mathbb{R}\}$$

In fact,

$$[y = mx] = \{y = mx + e \mid e \in \mathbb{R}\} \quad (*)$$

Partition of $\mathbb{R}^2$ under this equivalence relation.



$y = x$

$$[y=x] = \{y = 1\cdot x + c \mid c \in \mathbb{R}\}$$

$$= \{y = x + c \mid c \in \mathbb{R}\}$$

Take any line: $y = mx+b$, then it will be in equiv class of $y = mx$ (a line passing through the origin)

AND $y = m_1 x \not{R}\ y = m_2 x$ are NOT parallel iff $m_1 \neq m_2$

Set of equiv classes $= \bigcup_{m\in\mathbb{R}} [y=mx] \cup [y\text{-axis}]$

$y = \cancel{mx}+b \in [y = mx]$

(b) $A = \{ a+b\sqrt{2} \mid a, b \in \mathbb{Q} \;\&\; a+b\sqrt{2} \neq 0 \}$

$x R y$ iff $\underline{\frac{x}{y} \in \mathbb{Q}}$

Proof:

(1) Let's prove R is an equiv relation

* Reflexivity

(To show xRx, to show $\frac{x}{x} \in \mathbb{Q}$, to show $1 \in \mathbb{Q}$)

Since $\frac{x}{x} = 1 \in \mathbb{Q}$, therefore xRx

* Symmetric

(To show if xRy, then yRx

" if $\frac{x}{y} \in \mathbb{Q}$, then $\frac{y}{x} \in \mathbb{Q}$

" if something $\in \mathbb{Q}$, then $\frac{1}{\text{something}} \in \mathbb{Q}$)

If $\frac{x}{y} \in \mathbb{Q}$, then $\frac{y}{x} = \frac{1}{x/y} \in \mathbb{Q}$

* Transitive

(To show if xRy, yRz, then xRz

" if $\frac{x}{y}, \frac{y}{z} \in \mathbb{Q}$, then $\frac{x}{z} \in \mathbb{Q}$

(" if smthng$_1$, smthng$_2$ $\in \mathbb{Q}$, then smthng$_1$ × smthng$_2$ $\in \mathbb{Q}$)

So if $\frac{x}{y}, \frac{y}{z} \in \mathbb{Q}$, then $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} \in \mathbb{Q}$.

Equivalence Classes

$$[a+b\sqrt{2}] = \{x \in A \mid x \, R \, a+b\sqrt{2}\}$$

def^n of R

$$= \left\{x \in A \mid \frac{x}{a+b\sqrt{2}} \in \mathbb{Q}\right\}$$

def^n

$$= \left\{x \in A \mid \frac{x}{a+b\sqrt{2}} = k, \quad k \in \mathbb{Q}\right\}$$

algebra

$$= \{x \in A \mid x = k(a+b\sqrt{2}), \quad k \in \mathbb{Q}\}$$

rewriting

$$= \{k(a+b\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\}$$

Figure out when 2 equivalence classes are not equal //

i.e. figure under what conditions 2 elements are NOT related i.e. equiv, classes are distinct

6. Countability

Schröder - Bernstein

There exists a bijective function between A & B

iff ther exist injective function $f: A \to B$ and $g: B \to A$

Punchline To show $|A| = |B|$

you have to produce inj functions

$$f: A \to B$$

$$g: B \to A$$

(a) Prove $|(0,1)| = |[0,1]|$ ✓

Let's find inj $(0,1) \to [0,1]$

$[0,1] \to (0,1)$

Since $(0,1) \subseteq [0,1]$ there's alway a very nice fuction

$f: (0,1) \to [0,1]$, $x \mapsto x$ (inclusion function)

$g: [0,1] \longrightarrow (0,1)$

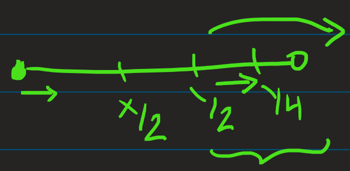
We want to take an $x \in [0,1]$

and manipulate it to get an element in $(0,1)$.

$$0 \le x \le 1$$

Halve-it $\quad 0 \le \frac{x}{2} \le \frac{1}{2} < 1 \quad$ atleast in $[0,1)$

$$0 < 0 + \frac{1}{4} \le \frac{x}{2} + \frac{1}{4} \le \frac{1}{2} + \frac{1}{4} = \frac{3}{4} < 1$$



$$g: [0,1] \longrightarrow (0,1)$$

$$x \longmapsto \frac{x}{2} + \frac{1}{4}$$

$$g(x) = g(y) \Rightarrow \frac{x}{2} + \frac{1}{4} = \frac{y}{2} + \frac{1}{4}$$

$$\Rightarrow \frac{x}{2} = \frac{y}{2}$$

$\Rightarrow x = y$, so g is inj!

7. Functions

9.42

(a) If two functions $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.

! (b) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is onto, then $g \circ f : A \to C$ is onto.

! (c) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is one-to-one, then $g \circ f : A \to C$ is one-to-one.

1+2 (d) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.

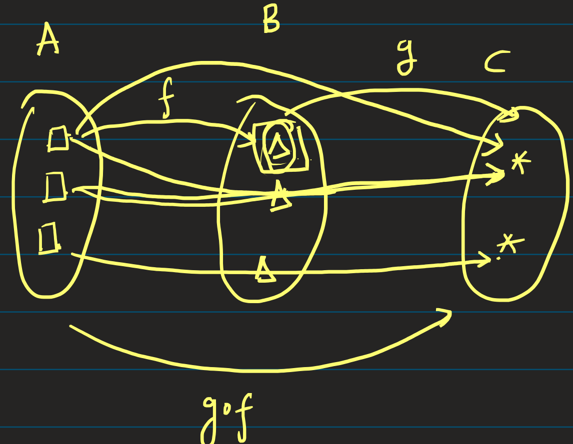
(e) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.

(d) $f : A \to B$ not onto

$g : B \to C$

$g \circ f : A \to C$ is onto.

Example :



$A = \{a, b, c\}$, $B = \{1, 2, 3\}$, $C = \{p, q\}$

$f : A \to B$	$g : B \to C$
$f(a) = 2$	$g(2) = p$
$f(b) = 2$	$g(3) = q$
$f(c) = 3$	$g(1) = p$

$g \circ f : A \to C$

$g \circ f(a) = g(2) = p$

$g \circ f(b) = g(2) = p$

$g \circ f(c) = g(3) = q$