An Introduction to Supercuspidal Representations

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Review of p-adic numbers

Before diving into representation theory, let's recall some properties of the p-adic numbers $\mathbb{Q}_p$.

p-adic absolute value of $\mathbb{Q}$

Fix a prime p. We can uniquely write a rational number in the form $p^k\left(\frac{u}{v}\right)$, where u, v are coprime to p.
We define the p-adic absolute value as

$$\left|p^k \cdot \left(\frac{u}{v}\right)\right|_p = p^{-k}.$$

This absolute value is non-archimedean – for all $x, y \in \mathbb{Q}$,
$|x+y|_p \leq \max\{|x|_p, |y|_p\}$.
$\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$.

p-adic numbers as a Laurent series

We can represent a p-adic number as a Laurent series in p where we allow finitely negative exponents.
$\mathbb{Q}_p = \{a_{-n} \cdot p^{-n} + a_{n-1} \cdot p^{n-1} + \cdots + a_0 + a_1 \cdot p + a_2 \cdot p^2 + \ldots : a_i \in \mathbb{N} \text{ and } 0 \leq a_i < p\}$.

Review of p-adic numbers

$\mathbb{Z}_p$ – the p-adic integers

Define $\mathbb{Z}_p \subset \mathbb{Q}_p$ as the set

$$\mathbb{Z}_p = \{a_0 + a_1 \cdot p + a_2 \cdot p^2 + \dots \;:\; a_i \in \mathbb{N} \text{ and } 0 \leq a_i < p\} = \{a \in \mathbb{Q}_p \;:\; |a|_p \leq 1\}.$$

$\mathbb{Z}_p$ is the ring of integers of $\mathbb{Q}_p$.

Some key properties of $\mathbb{Z}_p$

- $\mathbb{Z}_p$ is a local ring with unique maximal ideal $p\mathbb{Z}_p := \{a \in \mathbb{Q}_p \;:\; |a|_p < 1\}$.
- The quotient $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$ is the **residue field** of $\mathbb{Q}_p$.
- Every element of $\mathbb{Q}_p$ can be written as $u \cdot p^n$, where $u \in \mathbb{Z}_p^\times$; that is, $u \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$.
- $p^n\mathbb{Z}_p$ is a compact group for all $n \in \mathbb{Z}$.

non-Archimedean local fields

Remark

$\mathbb{Q}_p$ is a non-archimedean local field of characteristic 0. We call finite extensions F of $\mathbb{Q}_p$ ***p*-adic fields**.

$\mathbb{F}_q((T))$

There is another class of non-archimedean local fields, the field extensions of the field of formal Laurent series over a finite field $\mathbb{F}_q((T))$, where q is a power of p.

$$\mathbb{F}_q((T)) = \{a_{-n} \cdot T^{-n} + a_{n-1} \cdot T^{n-1} + \cdots + a_0 + a_1 \cdot T + a_2 \cdot T^2 + \ldots \, : \, a_i \in \mathbb{F}_q\}$$

This is a field of characteristic p, as are all of its finite extensions.

Notation/Definitions

If F is a local non-archimedean field, I will denote by $\mathcal{O}_F$ its ring of integers, with maximal ideal $\mathfrak{p}_F$. Let ϖ be a generator of $\mathfrak{p}_F$ – we call ϖ a uniformizer of F. Finally, let $f = \mathcal{O}_F/\mathfrak{p}_F$ be the residue field of F.

Reductive Groups - Basic Facts and Definitions

- A **reductive group** is a connected linear algebraic group with no nontrivial normal unipotent subgroups.

$$\text{E.g., } \mathbf{SL}_n, \mathbf{GL}_n, \mathbf{PGL}_n, \mathbf{SU}_n, \mathbf{E}_8.$$

- A **Borel subgroup B** is a maximal connected solvable algebraic subgroup of a reductive group. All Borel subgroups of $\mathbf{G}(F)$ are conjugate under $\mathbf{G}(F)$ for any field F.
- Any subgroup $\mathbf{P} \subseteq \mathbf{G}$ containing a Borel subgroup is called a **parabolic subgroup**.
- A **torus T** is a subgroup of $\mathbf{G}$ isomorphic to $\mathbf{GL}_1^r$ for some $r \in \mathbb{N}$. $\mathbf{T}(F)$ is then isomorphic to $(F^\times)^r$, for F algebraically closed.

Levi Decomposition

Definition: Levi decomposition

For any parabolic subgroup $\mathbf{P}$, we have $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$. Here $\mathbf{U}$ a maximal normal unipotent subgroup of $\mathbf{P}$, and $\mathbf{L}$, a reductive group, is called the **Levi factor**.

$\mathbf{SL}_2$ example

Let K be any field, and let $G = \mathbf{SL}_2(K)$. The standard maximal torus of G is given by $\mathbf{T}(K) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in K^\times \right\} \cong K^\times$.

Choosing a positive root, we have **root subgroups**

$$\mathbf{U}_\alpha(K) = \left\langle \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in K \right\rangle, \; \mathbf{U}_{-\alpha}(K) = \left\langle \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : c \in K \right\rangle.$$

Our standard Borel subgroup is

$$B = \mathbf{B}(K) = \langle \mathbf{T}(K), \; \mathbf{U}_\alpha(K) \rangle = \mathbf{T}(K) \ltimes \mathbf{U}_\alpha(K).$$

We find that B and G are the only parabolic subgroups of G.

Levi Decomposition – A slightly more interesting example

Let $G = \mathbf{GL}_3(K)$ for a field K. $\mathbf{GL}_3$ is a reductive group with three positive roots, α, β, and $\alpha+\beta$. We have

$$\mathbf{T}(K) = \left\langle \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} : a, b, c \in K^\times \right\rangle = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

$$\mathbf{U}_\alpha(K) = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{U}_\beta(K) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \mathbf{U}_{\alpha+\beta}(K) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The negative root subgroups are given by the transposes of their corresponding positive root subgroups.

$\mathrm{GL}_3(K)$ continued

We have the following parabolic subgroups and Levi decompositions:

$$G = G \ltimes \mathrm{Id},$$

$$P_\alpha = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \ltimes \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix},$$

$$P_\beta = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \ltimes \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \ltimes \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

p-adic groups

If $\mathbf{G}$ is a reductive group and F is a p-adic field, we call $\mathbf{G}(F)$ a ***p*-adic group**.

$\mathbf{G}(F)$ is a totally disconnect, locally compact group.

Its **maximal compact** open subgroups roughly look like conjugates of $\mathbf{G}(\mathcal{O}_F)$.

Example

E.g., if $G = GL_n(\mathbb{Q}_p)$, $GL_n(\mathbb{Z}_p)$ is the only maximal compact subgroup up to conjugation.

$SL_n(\mathbb{Q}_p)$ has two maximal compact subgroups up to conjugation,

$$SL_2(\mathbb{Z}_p) \text{ and } \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

p-adic representation theory – Smooth representations

Let G be a totally disconnected locally compact group.

Definition

A **representation** of G is a pair (π, V) such that

- V is a complex vector space
- $\pi : G \to \mathrm{Aut}_{\mathbb{C}}(V)(= GL(V))$ is a group homomorphism.

Definition

A vector $v \in V$ is called **smooth** if

$$stab_G(v) := \{g \in G : \pi(g)v = v \text{ for all } g \in G\}$$

is open.

If all vectors $v \in V$ are smooth, we call (π, V) a **smooth representation**.

Representation Theory

If $K \leq G$ is an open compact subgroup, we denote the K-invariants by

$$V^K := \{v \in V : \pi(k)v = v \text{ for any } k \in K\}.$$

Definition

A smooth representation (π, V) of G is called **admissible** if $\dim_{\mathbb{C}} V^K < \infty$ for each open compact subgroup K of G.

Parabolic induction and the Jacquet module

Let (π, V) be a smooth representation of G and $P = LU$ a Levi decomposition of a parabolic subgroup. Let (σ, W) be a continuous representation of L.

Definition - Parabolically induced representations

Denote by

$$Ind_P^G(\sigma) = \{f : G \to W : f(ulg) = \delta_P^{1/2}(l)\sigma(l)f(g) \text{ and } f \in L^2(G/P)\},$$

where $l \in L, u \in U, g \in G$ and δ_P is the modular character of P.

Definition - Jacquet module

Let $V(U) = span_{\mathbb{C}}\{\pi(u)v - v : u \in U, v \in V\}$.

Define $V_U := V/V(U)$, and consider the quotient action of L on V_U:

$$\pi_U(l)(v + V(U)) = \pi(l)v + V(U)$$

The L-representation (π_U, V_U) is called the **Jacquet module** of (π, V) with respect to $P = LU$.

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Normalized Jacquet module

Let (σ, W) be a smooth representation of L.

Remark

By Frobenius reciprocity,

$$Hom_G(\pi, Ind_P^G(\sigma)) \cong Hom_L(\pi_U, \delta_P^{1/2}\sigma).$$

Definition/Remark

To make the Jacquet functor left adjoint to parabolic induction, denote by $(r_L^G(\pi), r_L^G(V))$ the representation $\delta_P^{-1/2}\pi_U$ on V_U. Call this representation the **normalized Jacquet functor** with respect to $P = LU$. Thus, we have

$$Hom_G(\pi, Ind_P^G(\sigma)) \cong Hom_L(r_L^G(\pi), \sigma).$$

Supercuspidal Representations

Let (π, V) be an admissible representation of G.

Definition

We say that (π, V) is **supercuspidal** if for any proper parabolic subgroup $P = LU$ the Jacquet module $V_U = 0$.

Remark

If π is is irreducible, π is supercuspidal if and only if π is not equivalent to a subrepresentation of $Ind_P^G(\sigma)$ for any proper parabolic subgroup $P = LU$, and for any smooth representation σ of L.

Thus, we can think of irreducible supercuspidal representations of G as those native to G; all other irreducible admissible representations come from representations of some proper Levi factor L.

In fact, if (π, V) is a smooth irreducible representation of G, then π appears as a subrepresentation of $Ind_P^G(\sigma)$ for some parabolic $P = LU$ and supercuspidal L-representation σ.

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Cuspidal Representations in the finite setting

Remark

There are some subtleties between what it means to be a supercuspidal representation and what it means to be a cuspidal representation.

Roughly, supercuspidal representations only occur with groups over nonarchimedean fields. In the finite setting, we call the representations with trivial Jacquet modules for every proper parabolic subgroup **cuspidal representations**.

If F is a p-adic field and f is its residue field, we'll see that the cuspidal representations of $\mathbf{G}(f)$ will prove to be important in the construction of supercuspidal representations of $\mathbf{G}(F)$.

Cuspidal representations of $SL_2(\mathbb{F}_3)$

Courtesy of groupprops.subwiki.org, here is the character table of $SL_2(\mathbb{F}_3)$.

In the table below, we denote by ω a primitive cube root of unity.

Representation/conjugacy class representative and size	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **(size 1)**	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **(size 1)**	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **(size 6)**	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ **(size 4)**	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ **(size 4)**	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ **(size 4)**	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ **(size 4)**
trivial	1	1	1	1	1	1	1
nontrivial one-dimensional	1	1	1	ω	ω^2	ω^2	ω
nontrivial one-dimensional	1	1	1	ω^2	ω	ω	ω^2
irreducible two-dimensional (quaternionic, rational character)	2	-2	0	-1	-1	1	1
irreducible two-dimensional	2	-2	0	$-\omega$	$-\omega^2$	ω^2	ω
irreducible two-dimensional	2	-2	0	$-\omega^2$	$-\omega$	ω	ω^2
three-dimensional	3	3	-1	0	0	0	0

Figure: $SL_2(\mathbb{F}_3)$ character table

Remark

Since $SL_2(\mathbb{F}_3)$ only has one nontrivial parabolic subgroup, the Borel subgroup of upper-triangular matrices, we only have to check the dimension of the Jacquet module for $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$.

Representation/conjugacy class representative and size	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (size 1)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (size 1)	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (size 6)	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (size 4)	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ (size 4)	$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ (size 4)	$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$ (size 4)
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irreducible two-dimensional (quaternionic, rational character)	2	-2	0	-1	-1	1	1

Example

One can easily see that each of the non-trivial one-dimensional characters (representations) are cuspidal representations:

Let (π, V) be the first of these non-trivial one-dimensional representations. Since π is one-dimensional, $V \cong \mathbb{C}$.

$V(U) = \text{span}_{\mathbb{C}}\{\pi(u)v - v : u \in \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}, v \in V\}$.

Let $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and let $v \neq 0$. Then $\text{span}_{\mathbb{C}}\{\pi(u)v - v = (\omega - 1)v \neq 0\} = \mathbb{C}$.

Thus, $V_U = V/V(U) = 0$, showing (π, V) is a cuspidal representation of $SL_2(\mathbb{F}_3)$.

*Note: One finds that the irreducible representation whose character is the "two dimensional quaternionic, rational character" is also a supercuspidal representation.

Bruhat-Tits Building – loose definition

Bruhat-Tits building

For any $G = \mathbf{G}(F)$, Bruhat and Tits (1972) define a cell complex called the building $\mathcal{B}(G)$. In this complex each vertex $x \in \mathcal{B}(G)$ corresponds to a maximal compact subgroup of G.

$\mathcal{B}(G)$ is the union of subcomplexes, called apartments, which G acts on transitively. To each split maximal torus T of G there is an associated apartment.

Apartments and Semisimple rank

Let n be the difference in rank between a split maximal torus T of G and maximal torus C of the center $Z(G)$. This is called the **semisimple rank** of G. Each apartment of $\mathcal{B}(G)$ will be isomorphic to $\mathbb{R}^n$.

For example, $SL_2(\mathbb{Q}_p)$ has $n = rk(T) - rk(C) = 1 - 0 = 1$.

For $GL_2(\mathbb{Q}_p)$, one has $n = rk(T) - rk(C) = 2 - 1 = 1$.

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Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$ – Lattices in $\mathbb{Q}_p$

A **lattice** in $\mathbb{Q}_p^2$ is a subgroup $\Lambda \subset \mathbb{Q}_p^2$ which is isomorphic to $\mathbb{Z}_p^2$. Two lattices Λ, Λ' are called **homothetic** if there exists $s \in \mathbb{Q}_p^\times$ such that $s \cdot \Lambda = \Lambda'$.

For example, $p^{-1}\mathbb{Z}_p \oplus p\mathbb{Z}_p$ is homothetic to $p\mathbb{Z}_p \oplus p^3\mathbb{Z}_p$.

We call two homothety classes of lattices $[\Lambda], [\Lambda']$ **adjacent** if for some representatives Λ and Λ', we have $\Lambda \supsetneq \Lambda' \supsetneq p\Lambda$.

For example, we see that

$$\mathbb{Z}_p \oplus \mathbb{Z}_p \supsetneq \mathbb{Z}_p \oplus p\mathbb{Z}_p \supsetneq p\mathbb{Z}_p \oplus p\mathbb{Z}_p.$$

So, $[\mathbb{Z}_p \oplus \mathbb{Z}_p]$ and $[\mathbb{Z}_p \oplus p\mathbb{Z}_p]$ are adjacent.

Remark

One can show that for any lattice class $[\Lambda]$ of $\mathbb{Q}_p^2$ that there are $p + 1$ adjacent lattice classes.

Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$

Theorem

The vertices in $\mathcal{B}(SL_2(\mathbb{Q}_p))$ correspond to homothety classes of lattices $[\Lambda]$. Adjacent lattice classes correspond to adjacent vertices – vertices joined by an edge.

By the remark on the last slide – and some more work – one finds that $\mathcal{B}(SL_2(\mathbb{Q}_p))$ is a $(p+1)$-regular tree.

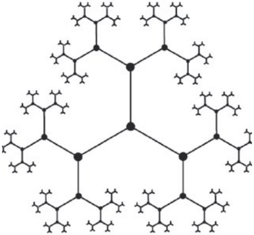

Figure: The Bruhat-Tits building of $SL_2(\mathbb{Q}_2)$

Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$

To every point $x \in \mathcal{B}(G)$ we have a compact subgroup G_x.

In the case of $G = SL_2(\mathbb{Q}_p)$, let's take x and y to be the vertices corresponding to the lattice classes $[\mathbb{Z}_p \oplus \mathbb{Z}_p]$ and $[\mathbb{Z}_p \oplus p\mathbb{Z}_p]$, respectively. Let e be the edge joining x and y.

The groups G_x and G_y are the stabilizers in G of their corresponding lattice classes.

One works out that

$$G_x = SL_2(\mathbb{Z}_p) \text{ and } G_y = \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z}_p) \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_p & p^{-1}\mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

The edge e is stabilized if and only if x and y are stabilized. So

$$G_e = G_x \cap G_y = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$

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Moy-Prasad Filtration

Let $G = \mathbf{G}(F)$.

For all $x \in \mathcal{B}(G)$, Moy and Prasad (1994) developed a filtration of the groups G_x.

$$G_x := G_{x,0} \rhd G_{x,r_1} \rhd G_{x,r_2} \rhd G_{x,r_3} \ldots$$

for positive real numbers $r_i > 0$, with $r_{i+1} > r_i$ for all i.

We'll have it so that if $r_i < r < r_{i+1}$ that $G_{x,r} = G_{x,r_{i+1}}$.

Notation

We define the group $G_{x,r_i+} := \bigcup_{s > r_i} G_{x,s} = G_{x,r_{i+1}}$.

Remark

The quotient $G_{x,0}/G_{x,0+} = G_{x,0}/G_{x,r_1}$ is the f-points of a reductive group.

Subsequent quotients of $G_{x,r_i}/G_{x,r_i+}$ are f vector spaces.

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Moy-Prasad Filtration - Vertices

In the case of $G = SL_2(\mathbb{Q}_p)$, taking x as before (y follows a similar pattern), we have filtration jumps

$$G_{x,0} = \left\langle \mathbf{T}(\mathbb{Z}_p), \mathbf{U}_\alpha(\mathbb{Z}_p), \mathbf{U}_{-\alpha}(\mathbb{Z}_p) \right\rangle = SL_2(\mathbb{Z}_p),$$

$$G_{x,1} = G_{x,0+} = \left\langle \mathbf{T}(1+p\mathbb{Z}_p), \mathbf{U}_\alpha(p\mathbb{Z}_p), \mathbf{U}_{-\alpha}(p\mathbb{Z}_p) \right\rangle = \begin{pmatrix} 1+p\mathbb{Z}_p & p\mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix},$$

$$G_{x,2} = G_{x,1+} = \left\langle \mathbf{T}(1+p^2\mathbb{Z}_p), \mathbf{U}_\alpha(p^2\mathbb{Z}_p), \mathbf{U}_{-\alpha}(p^2\mathbb{Z}_p) \right\rangle = \begin{pmatrix} 1+p^2\mathbb{Z}_p & p^2\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1+p^2\mathbb{Z}_p \end{pmatrix}.$$

Remark: Quotients

$$G_{x,0}/G_{x,1} = \left\langle \mathbf{T}(\mathbb{F}_p), \mathbf{U}_\alpha(\mathbb{F}_p), \mathbf{U}_{-\alpha}(\mathbb{F}_p) \right\rangle \cong SL_2(\mathbb{F}_p).$$

$G_{x,1}/G_{x,2} \cong \mathbb{F}_p^3$ and $G_{x,r}/G_{x,r} \cong \mathbb{F}_p^3$ for all $r \geq 1$.

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In the case of $G = SL_2(\mathbb{Q}_p)$, taking x as before (y follows a similar pattern), we have filtration jumps

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Remark: Quotients

$$G_{x,0}/G_{x,1} = \left\langle \mathbf{T}(\mathbb{F}_p), \mathbf{U}_\alpha(\mathbb{F}_p), \mathbf{U}_{-\alpha}(\mathbb{F}_p) \right\rangle \cong SL_2(\mathbb{F}_p).$$

$$G_{x,1}/G_{x,2} \cong \mathbb{F}_p^3 \text{ and } G_{x,r}/G_{x,r} \cong \mathbb{F}_p^3 \text{ for all } r \geq 1.$$

Moy-Prasad Filtration – Edges

Let e be given as the midpoint of the edge joining x and y. In this case, we have jumps at half-integers.

$$G_{e,0} := G_e = \left\langle \mathbf{T}(\mathbb{Z}_p), \mathbf{U}_\alpha(\mathbb{Z}_p), \mathbf{U}_{-\alpha}(p\mathbb{Z}_p) \right\rangle = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix},$$

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Remark: Quotients

$$G_{e,0}/G_{e,\frac{1}{2}} = \begin{pmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{pmatrix} \cong GL_1(\mathbb{F}_q) \cong \mathbb{F}_p^\times.$$

$$G_{e,\frac{1}{2}}/G_{e,1} = \begin{pmatrix} 0 & \mathbb{F}_p \\ \mathbb{F}_p & 0 \end{pmatrix} \cong \mathbb{F}_p^2.$$

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Depth of a representation

Theorem, Moy-Prasad (1994)

If (π, V) is any smooth representation of G, then there exists a nonnegative rational number $r = \varrho(\pi)$ with that property that r is the minimal number such that $V^{G_{x,r+}} \neq 0$ for some $x \in \mathcal{B}(G)$.

Definition

We define the **depth** of an admissible irreducible representation π of G to be the number $r = \varrho(\pi)$.

Remark

Moy and Prasad (1996) showed that if (π, V) is a depth-zero irreducible supercuspidal representation of G, then $V^{G_{x,0+}}$ can only be non-zero when x is a vertex. They also showed that all depth-zero supercuspidal representations arise from the process of inflating a cuspidal representation from $\mathbf{G}(f)$ to $\mathbf{G}(\mathcal{O}_F)$, and then inducing to $\mathbf{G}(F)$.*

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Compact Induction and Inflation

As I mentioned compact induction and inflation in the previous slide, we will need a rough understanding of these functors in order to construct depth-zero supercuspidal representations.

Compact Induction

Let (σ, W) be a representation of H, a closed subgroup of G.

$cInd_H^G(\sigma)$ is the space of functions $\phi : G \to W$ such that ϕ has support on a finite number of left H cosets, satisfying $\phi(gh^{-1}) = \sigma(h)\phi(g)$ for all $g \in G$, $h \in H$.

Inflation

Let $N \trianglelefteq G$ be a normal subgroup.

$Inf_{G/N}^G$ is a functor which takes a representation of the quotient group G/N and gives back a G-representation π with $N \leq ker(\pi)$.

Depth-zero supercuspidal representation example

Let's consider $G = SL_2(\mathbb{Q}_p)$.

Taking $x \in \mathcal{B}(G)$ to be the vertex with stabilizer

$$G_x = G_{x,0} = \langle \mathbf{T}(\mathbb{Z}_p^\times), \mathbf{U}_\alpha(\mathbb{Z}_p), \mathbf{U}_{-\alpha}(\mathbb{Z}_p) \rangle = SL_2(\mathbb{Z}_p).$$

We saw

$$G_{x,0+} = G_{x,1} = \langle \mathbf{T}(1 + p\mathbb{Z}_p), \mathbf{U}_\alpha(p\mathbb{Z}_p), \mathbf{U}_{-\alpha}(p\mathbb{Z}_p) \rangle, \text{ and}$$

$$G_{x,0}/G_{x,0+} \cong SL_2(\mathbb{F}_p).$$

Depth-zero supercuspidal

We take a cuspidal representation σ of $SL_2(\mathbb{F}_p)$. We can then inflate this representation to $G_{x,0}$. Then, compactly inducing this representation to G will give us a supercuspidal representation π of G.

$$\mathrm{cInd}_{G_{x,0}}^{G}(\mathrm{Inf}_{G_{x,0}/G_{x,0+}}^{G_{x,0}}(\sigma)) = \mathrm{cInd}_{SL_2(\mathbb{Z}_p)}^{SL_2(\mathbb{Q}_p)}(\mathrm{Inf}_{SL_2(\mathbb{F}_p)}^{SL_2(\mathbb{Z}_p)}(\sigma)) = \pi$$

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Minimal positive depth supercuspidal representations

Gross and Reeder (2010) developed a method of constructing **simple supercuspidal representations** – supercuspidal representations of minimal positive depth. These are representations applied to "barycenters of alcoves" (open facets of apartments) on the Bruhat-Tits building.

Reeder and Yu (2014) later generalized these minimal positive depth supercuspidal representations to more than the just the barycenters of alcoves; they named them **epipelagic supercuspidal representations**. The name plays on the theme of depth introduced by Moy and Prasad – the epipelagic zone is the uppermost layer in the ocean in which photosynthesis can occur. For $SL_2(\mathbb{Q}_p)$ these two types of representations will be the same.

Remark

In the $SL_2(\mathbb{Q}_p)$ case, the barycenter of an alcove refers to the midpoint of an edge. Indeed, we found in this case that vertices have integral jumps, while the center of an edge jumps every half integer. In this way, the simple supercuspidals of $SL_2(\mathbb{Q}_p)$ will be depth-$\frac{1}{2}$ representations.

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Simple supercuspidal – $SL_2(\mathbb{Q}_p)$

Moy-Prasad filtration at the center of an edge

Recall that for the edge e joining x and y from before that we have filtration subgroups

$$G_{e,0} = G_e = G_x \cap G_y = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}.$$

$$G_{e,0+} = G_{e,\frac{1}{2}} = \begin{pmatrix} 1 + p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}$$

$$G_{e,\frac{1}{2}+} = G_{e,1} = \begin{pmatrix} 1 + p\mathbb{Z}_p & p\mathbb{Z}_p \\ p^2\mathbb{Z}_p & 1 + p\mathbb{Z}_p \end{pmatrix}$$

And we have quotients

$$G_{e,0}/G_{e,\frac{1}{2}} \cong \begin{pmatrix} \mathbb{F}_p^\times & 0 \\ 0 & \mathbb{F}_p^\times \end{pmatrix} \cong \mathbb{F}_p^\times,$$

$$G_{e,\frac{1}{2}}/G_{e,1} = \begin{pmatrix} 0 & \mathbb{F}_p \\ \mathbb{F}_p & 0 \end{pmatrix} \cong \mathbb{F}_p \oplus \mathbb{F}_p.$$

Simple supercuspidal – $G = SL_2(\mathbb{Q}_p)$

Let χ be an additive character of $G_{e,\frac{1}{2}}/G_{e,1} \cong \mathbb{F}_p \oplus \mathbb{F}_p$ which is nontrivial on each component.

Such a character has the form

$$\chi_{a,b}(u,v) = e^{\frac{2\pi i(au+bv)}{p}},$$

for $0 < a, b \leq p-1$.

Now, we inflate this character to $G_{e,\frac{1}{2}}$, getting ready for compact induction.

Warning

Assuming $p \neq 2$, we have $Z(G) = \pm 1 \not\subset G_{e,\frac{1}{2}} = \begin{pmatrix} 1+p\mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & 1+p\mathbb{Z}_p \end{pmatrix}$. So if we apply compact induction, we will get back two representations of G – two distinct supercuspidal representations.

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So we first induce to $Z(G) \cdot G_{e,\frac{1}{2}} = K$:

$$Ind^{K}_{G_{e,\frac{1}{2}}}(\mathrm{Inf}^{G_{e,\frac{1}{2}}}_{\mathbb{F}_q^2}(\chi_{a,b})) = \sigma_1 \oplus \sigma_2.$$

σ_1 and σ_2 are distinguished by their central character (where they send ± 1). So we pick one of the two. Now,

$$cInd^G_K(\sigma_1) = \pi$$

is a depth-$\frac{1}{2}$ supercuspidal representation.

Remark

Note that our two constructions of supercuspidal representations involved compact induction.

Folklore Conjecture

Every supercuspidal irreducible representation of G can be expressed as $cInd^G_K(\sigma)$, for some compact-mod-center open subgroup $K \subset G$, and some irreducible representation σ of K.

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cInd of p-adic reductive groups

For which reductive groups do we know this compact induction conjecture holds?

Let F be a nonarchimedean local field with residue field of characteristic p.

Groups	Restrictions	Author(s)	Publication Year
$GL_n(F)$	None	Bushnell-Kutzko	1993
$SL_n(F)$	None	Bushnell-Kutzko	1994
Tame groups	p large	Kim	2007
$GL_n(D)$	None	Stevens-Sécherre	2008
Classical groups	No D, $p \neq 2$	Stevens	2008
Semisimple rank-one groups	None	Weissman	2019
Tame groups	$p \nmid \#W$	Fintzen	2021

Here "Tame groups" are reductive groups that split over a tamely ramified extension of F, D is a finite-dimensional division algebra over F, and W is the Weyl group.

Table stolen from one of Marty Weissman's talks

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