

Finding Joy With Combinatorial Games

Sam K. Miller

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Introduction

(Though long personal anecdotes are not the norm for mathematical seminars, I was told I could talk about "whatever I want" for an hour, and I thought this introduction may give others food for thought, and perhaps inspiration for why *they* choose to study mathematics.)

In my former professor Francis Su's retiring-presidential-address-turned-blog-post-turned-book "Mathematics For Human Flourishing," he posits the following question to us: "Why do mathematics?" It is strange - why are so many people from differing cultures and backgrounds, which hold different values and beliefs, united to do math, to study this abstract field which demands so much of us? Especially in today's world, where there are so many people who actively degrade math, who wish for it to be removed from standard curricula, thinking it to be a waste of time and taxes? He believe this question is vital for any mathematician or math enthusiast. In his words:

"Why do mathematics? This is a simple question, but worth considerable reflection. Because how you answer will strongly determine who you think should be doing mathematics, and how you will teach it.

Why is Christopher [an incarcerated student in contact with Francis] sitting in a prison cell studying calculus, even though he won't be using it as a free man for another 25 years? Why was Simone [Weil, philosopher and brother of Andre Weil,] so captivated by transcendent mathematical truths? Why should anyone persist in doing math or seeing herself as a mathematical person when others are telling her in subtle and not so subtle ways she doesn't belong?"

I too believe this is a question worth asking. I fell madly in love with math from a young age, attending math circles, camps, and contests whenever I could through middle and high school. The thought that I could prove, without a doubt, that some property of the numbers would always be satisfied, gave me power. But through my time as an undergrad, I started to lose that special feeling math gave me, instead feeling like I was "going through the motions." I hadn't burned out persay (though the first half of my undergrad could be viewed as residual burnout from high school, where I strove to overachieve in all the ways I possibly could without any regard for sleep or depression), but I was moving forward without any

purpose. I graduated a math major, but rather than continuing my studies, decided to become a software engineer and data scientist for the time being. After all, the grass is always greener, and one may lead a lucrative life developing software at large tech companies. It wasn't until I began that job until the feeling of loss began to set in, and I began to question what about mathematics it was that gave me this feeling of vigor that coding lacked? After all, it's tough to appreciate what you have until it's gone.

Francis's belief is that the practice, not just the pure study, of mathematics allows people to *flourish*, to live a virtuous life by satisfying basic human desires. These desires, in short, are "**play**", "**beauty**", "**truth**", "**justice**", and "**love**." I too ascribe to this belief. Fast forward to the present day, we are students in graduate school, working rigorously and tirelessly into the night to learn abstract mathematics at the forefront of research so we too may leave our mark in the tome of discovered (or created?) mathematical knowledge. We find **beauty** in learning new powerful results, and **truth** in practicing rigorous mathematical thinking day after day. This pursuit can be tiring, draining, sometimes leaving us (or at least me) with imposter syndrome. However, at UCSC, we find **love** in this community, as we support and look out for each other, creating a sense of belonging, giving all of us the opportunity to flourish. And we are fortunate to find **justice** in a community of graduates who are willing to fight for it, for inclusion and equity within the mathematics community. And for our graduate student body, we fight for a living wage, for workers rights, and for justice for those of us who were wronged by the administration and law enforcement.

However, I have not yet touched upon the final of these desires, **play**. Sometimes, I feel that in graduate school, my time for mathematical play is diminished or even forgotten, and this can again lead to my own questioning, self-doubt. We are loaded up with so many exciting new things to learn and study, but courses move so fast and the weeks fly by, and it becomes difficult for me to stop and smell the flowers, to kick back and relax and appreciate just what we've been learning with a clear, unburdened mind. Again, quoting Francis,

Mathematics makes the mind its playground. We play with patterns, and within the structure of certain axioms, we exercise freedom in exploring their consequences, joyful at any truths we find. [...]

And mathematical play builds virtues that enable us to flourish in every area of our lives. For instance, math play builds **hopefulness**: when you sit with a puzzle long enough you are exercising hope that you will eventually solve it. Math play builds **community** — when you share in the delight of working on a problem with another human being. And math play builds **perseverance** — just as weekly soccer practices build up the muscles that make us stronger for the next game, weekly math investigations make us more fit for the next problem whatever that is, even if we don't solve the current problem.

Play is part of human flourishing. *You cannot flourish without play.*

To rediscover my own personal play, I looked back to where I found play in the first place. And as I thought, I realized that from the very start, play, and the freedom to explore, is what first drew me to mathematics! Math competitions of course count, but also simple puzzles

and games, playful problems. And I've seen the echoes of this feeling, in my Representation Theory or Modular Forms courses, where I once again felt inspired to explore. However earlier on, I predominantly found play in combinatorics. I witnessed beautiful combinatorial arguments in math camps through high school (one of which was hosted at UCSC!), but even before that, perhaps my first exposure to this world was through a box set of DVD lectures by Arthur T. Benjamin named "The Joy of Mathematics." It is quite a happy coincidence that some years later, I would study under him and publish my first research paper together with him!

So in the spirit of mathematical play, and in the spirit of those DVD lectures which lit the spark inside of me, today I will share with you a potpourri of combinatorial games and counting arguments, most of which I originally learned from Arthur, and some original results I proved prior to entering UCSC. Unlike most seminars which I find can require intense focus and a good deal of prerequisite knowledge to follow, my hope is that this talk can remain light and playful, while still eliciting that mathematical beauty and truth that we all search for. These are the types of problems that satisfy my desire of play in mathematics, and I hope I can spread that feeling of play within you as well.

1 The Challenging Knight's Tour

Within the community of Chess, there exists a problem known as *The Knight's Tour*. The most simple variant of the puzzle is as follows: we are given an empty chessboard, and a knight. Recall that a knight moves in an "L"-shape, that is, it moves 2 squares vertically then 1 square horizontally, or vice versa; this rule allows for at most 8 possible moves. The most simple objective of the knight's tour is to place the knight on the board, then perform a series of moves with the knight so that it reaches each square on the board exactly once. We call this an open knight's tour. There are more difficult variants which may be performed:

- The **Closed** Knight's Tour: Perform an open knight's tour, with the addition that the knight ends on the same square that it started on.
- The **Challenging** Knight's Tour: First, choose two designated squares, then place the knight on one of those squares. Perform an open knight's tour which ends on the other designated square.

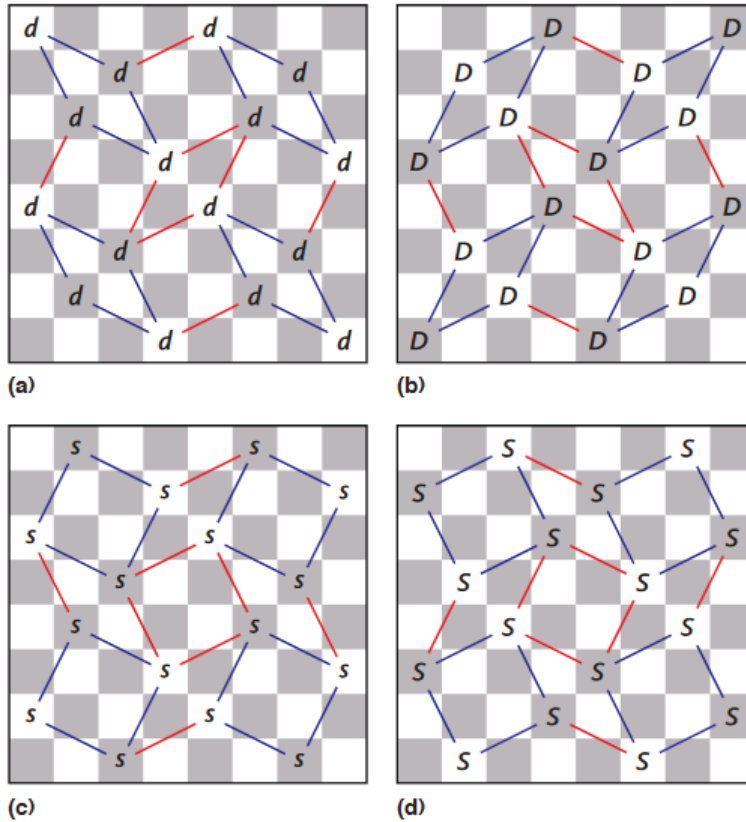
We will shift our attention to the Challenging Knight's Tour, and pose the question: for what combination of squares can we perform a knight's tour? First, note that on a chessboard, when a knight moves, the square it occupies changes color. Since a knight's tour takes 63 moves, a parity argument informs us that the start and end squares must be on opposite colors. We then ask: is the Challenging Knight's Tour always possible for any two chosen squares of opposite color? (Poll for guesses) The solution had been answered via brute-force algorithms in the past, however, in 2016, Arthur Benjamin and I demonstrated with a constructive proof that the tour is indeed always possible.

1.1 Reframing the problem

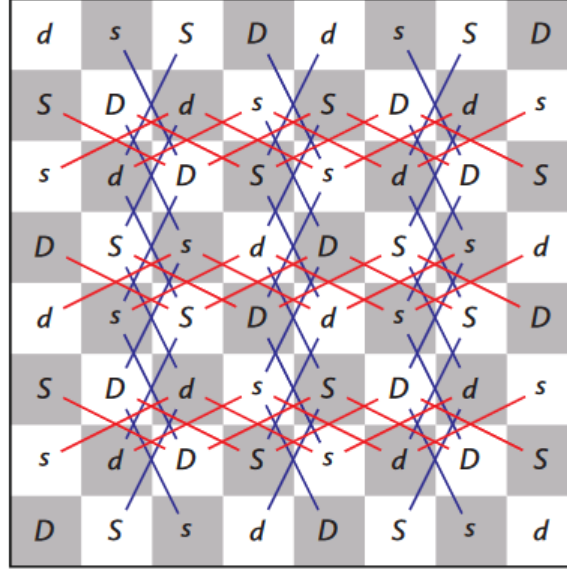
To solve this, we may construct a chessboard graph G with 64 vertices, each corresponding to a square on the chessboard, and with an edge existing between two vertices if and only if the squares are adjacent via knight-move. Additionally, we assign a color to each vertex, corresponding to the square's color on the board. Therefore adjacent vertices must be of opposite color. The problem of solving the challenging knight's tour is then equivalent to proving the following statement:

Theorem: (Benjamin, Miller) For any two opposite-colored vertices v_1, v_2 on G , there exists a Hamiltonian path which starts at v_1 and ends at v_2 .

We will not cover the full proof here, but cover its main strategy. A key observation one may make is that G is decomposable into four isomorphic subgraphs, which are isomorphic to $G_{4,4}$, a 4×4 grid of vertices. We shall call these induced subgraphs the 4 *systems*. On the chessboard, these subgraphs' appearances allow us to further classify them into "diamond" and "square" systems. We denote these systems d, D, s, S .



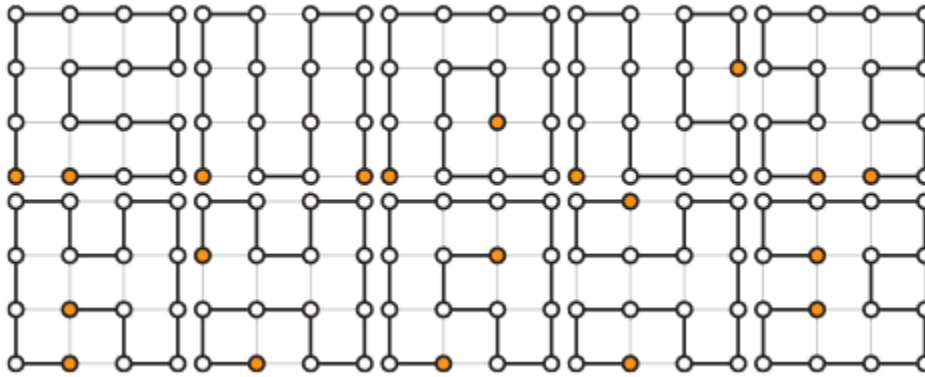
Moreover, if we consider all edges of G not belonging to one of the systems, we see that they form two shoelace patterns between systems of opposite type.



In particular, any vertex belonging to the 4×4 square in the middle of the chessboard is adjacent to vertices belonging to both systems of the opposite type. This suggests a strategy: complete the knight's tour by traversing a system, choosing wisely to end on a square which grants us access to both systems of opposite type, moving to the next system, and repeating until we are done. However, it is not immediately clear that in any situation, we can perform a traversal of $G_{4,4}$ where we can choose the end square.

Lemma: $G_{4,4}$ is traversable, that is, for any pair of distinguished vertices of opposite color in $G_{4,4}$, a Hamiltonian path exists which begins at one vertex and ends at the other.

Proof:

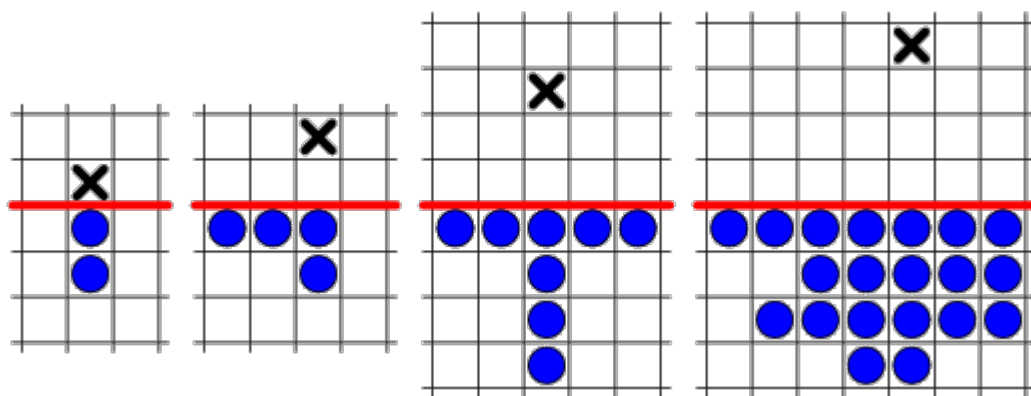


□

Therefore, we may always traverse a system, choosing wisely to end in the center of the board, so we may have access to both adjacent systems. Using this strategy, we immediately have as a result that if the start and end squares given of the challenging knight's tour belong to systems of opposite types, we may perform the tour. Additional care must be taken in the cases where the start and end squares are in either the same system, or different systems of the same type, and this part of the proof is omitted for brevity.

2 Conway's Soldiers

We now shift our attention to a different puzzle, another which contains jumps, but of a slightly different nature. This puzzle is known as Conway's Soldiers, and was devised by John Conway around 1961. We play on an endless grid/chessboard, which contains a horizontal line separating the board into two areas, the friendly territory and the enemy territory. The objective is to place pieces in your friendly territory, then make a sequence of moves that advances the pieces as far into enemy territory as possible. The pieces move similar to they do in peg solitaire - pieces move vertically or horizontally, are only allowed to hop over each other, and any pieces that is hopped over is removed from play.



To get 1 row into enemy territory, it's trivial to see we require 2 checkers minimum. (Let the audience play around and guess the next few). With some work, we may show that the minimum required for 2 rows is 4, for 3 rows is 8, and for 4 rows is 20. Now, would anyone like to guess how many checkers are required to make it 5 rows in? (You may have heard of the term "Combinatorial Explosion" used to describe Ramsey Theorey-esque problems before, perhaps this is an instance of that!)

Theorem: (Conway) It is **impossible** to reach level 5 with a finite amount of checkers!

Proof: Surprisingly, a rather elementary, and quite beautiful proof exists which proves this. Suppose to the contrary that there exists a square p on level 5 we are able to reach. Label that square 0, then for every square on row 4 or below, label that square by the number of steps it would take to move to p moving vertically or horizontally. Now, for the starting configuration which reaches square p , give each checker weight g^k , where k is the label of the square, and $g = (\sqrt{5} - 1)/2$. We say the *weight of the configuration* is the sum of the weights of all the checkers.

	0	
	2	1 2
	3	2 3
...	4	3 4 ...
	5	4 5
	<hr/>	
	6	5 6
	7	6 7
	8	7 8
	...	

As an example, under this labelling, the row 2 configuration would have weight $(g^7 + g^6) + (g^6 + g^5) = g^5 + g^4 = g^3$.

Observe that $g^2 + g = 1$, or more generally, $g^n + g^{n-1} = g^{n-2}$. From this property, we observe that for any jump performed, the weight of the resulting configuration never increases.

- If a checker jumps to a square with lesser labelling, we replace two checkers of weights g^k, g^{k-1} with one checker of weight g^{k-2} , and since $g^k + g^{k-1} = g^{k-2}$, the weight of the new configuration is the same.
- If the checker jumps to a square with greater labelling, we replace two checkers of weight g^k, g^{k+1} with a checker of weight g^{k+2} , and since $g^k + g^{k+1} > g^{k+2}$ (as in general, $g^i > g^{i+1}$ for positive i), the weight of the resulting configuration increases.

Since the weight of the end configuration is at minimum $g^0 = 1$, our starting configuration must also have weight at least 1.

We compute the maximum possible weight of configuration, by bounding checker weights in each row. In row 1, we see that the maximum weight is bounded above by

$$(g^5 + g^6 + g^7 + \dots) + (g^6 + g^7 + \dots) = g^5/(1 - g) + g^6/(1 - g).$$

For $k \geq 2$, $g^k/(1 - g) = g^{k-2}$ (this follows directly from the recursive property of g). Therefore,

$$g^5/(1 - g) + g^6/(1 - g) = g^3 + g^4 = g^2.$$

This bound is strict, as we only may place a finite amount of pieces. Similarly, in the second row, we find the weight is bounded by g^3 , and in general, the n -th row is bounded by g^{n+1} . Therefore, the entire friendly area is strictly bounded by

$$g^2 + g^3 + \dots = g^2/(1 - g) = g^0 = 1.$$

This is a contradiction, since we required a weight of exactly 1 to reach the 5th row in enemy territory! Thus, no configuration exists. \square

Surprisingly, however, Simon Tatham and Gareth Taylor have shown that it is possible to reach the 5th row if we are allowed to place an *infinite* number of checkers. Additionally, it has been shown that in the n -dimensional variant of the game, it is always possible to reach the $(3n - 2)$ th row, and the same argument we used may be applied to show that the $(3n - 1)$ th row is not reachable with a finite number of checkers.

3 Optimal Leapfrogging

To be shown...

4 A Combinatorial Characterization of Determinants

That's right, the determinant of a matrix counts something! To explain what it counts and prove why it counts what it does, I first need to introduce a method of counting which we call D.I.E., or Description-Involution-Exception.

4.1 D.I.E.

To illustrate the method of D.I.E., we will first prove a simple identity. For $0 \leq m \leq n$, we wish to find a closed form of

$$\sum_{k=0}^m \binom{n}{k} (-1)^k.$$

As an aside, there does not exist a "nice" closed form of the non-alternating sum $\sum_{k=0}^m \binom{n}{k}$. Combinatorially, this sum counts the number of subsets of $\mathcal{N} = \{1, \dots, n\}$ of even cardinality, minus the number of subsets of \mathcal{N} of odd cardinality. Observe that there exists a bijection between subsets of even cardinality and of odd cardinality, given by "toggling the 1":

$$\pi : M \subseteq \mathcal{N} \mapsto \begin{cases} M \oplus \{1\} & \text{if } 1 \notin M \\ M \setminus \{1\} & \text{if } 1 \in M \end{cases}$$

As another aside, this bijection provides one of many proofs that $\sum_{k=0}^n \binom{n}{2k} = 2^{n-1}$. Perhaps in a further talk, I will discuss sums of the form $\sum \binom{n}{a+rk}$ for $r \in \mathbb{N}, 0 \leq a \leq k$.

Therefore, for every mutual even-odd subset pair (pairs of subsets which are identical, except for one subset additionally containing 1), their terms in the sum cancel. Thus the only subsets of \mathcal{N} which are not cancelled are those for which "toggling the one" results in a subset not considered by the sum. These subsets are precisely those of cardinality m and which do not contain 1. There are $\binom{n-1}{m}$ of these, all with parity $(-1)^m$. Thus,

$$\sum_{k=0}^m \binom{n}{k} (-1)^k = (-1)^m \binom{n-1}{m}. \quad \square$$

Corollary: $\sum_{k=0}^n \binom{n}{k} (-1)^k = 0$

In this proof, we utilized the technique of D.I.E.

- **Description:** Given an alternating sum $\sum (-1)^k a_k$, we consider what the non-alternating sum $\sum a_k$ counts.
- **Involution:** We find an *involution*, a bijection between most even and odd k -terms.
- **Exception:** We count the number of exceptions to the involution.

4.2 The Problem of the DeterminedAnts

Recall that given a $n \times n$ matrix $A = a_{i,j}$, the *determinant* of A is defined as

$$\det A = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^n a_{i,\sigma(i)}.$$

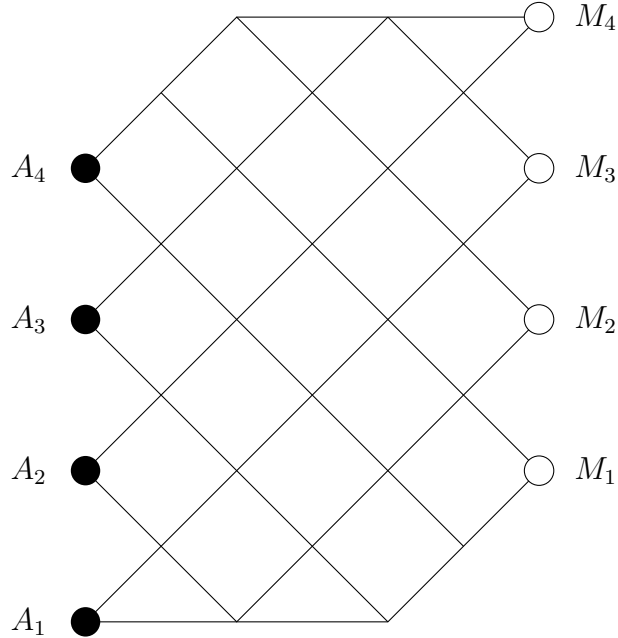
This is an alternating sum, perfect for applying D.I.E.! This means we also need to consider the non-alternating version of the sum.

Definition: Define the *permanent* of A to be

$$\text{per } A = \sum_{\sigma \in S_n} (\sigma) \prod_{i=1}^n a_{i,\sigma(i)}$$

We now introduce **The Problem of the DeterminedAnts!** Consider a directed graph \mathcal{D} , with n designated starting nodes A_i , the ants, and n ending nodes M_j , the morsels. The objective of the "puzzle" is to lead each ant to a morsel so that no two ants end at the same morsel. We require of the digraph that for $i \neq j$, any paths from A_i to M_j intersect paths from A_j to M_i , and that the graph is acyclic.

As an example, we give the following digraph, where ants are only allowed to move to the right:



Definition: The *Ant Matrix* of \mathcal{D} , A , stores in $a_{i,j}$ the number of ways ant A_i can reach morsel M_j , i.e. the number of paths from A_i to M_j .

For example, the ant matrix of the above graph \mathcal{D} is:

$$A = \begin{pmatrix} 14 & 6 & 1 & 0 \\ 20 & 15 & 6 & 1 \\ 15 & 20 & 15 & 6 \\ 6 & 15 & 20 & 14 \end{pmatrix}$$

Now, how many ways are there to solve the puzzle? (pause to see if anyone sees it)

Theorem: The *permanent* of A counts the number of pairs of simultaneous paths from a unique ant to a unique morsel.

Proof: Suppose \mathcal{D} has n ants and morsels. Then, each $\sigma \in S_n$ is in 1 to 1 correspondence to a unique pairing of ants to morsels:

$$\pi : \sigma \mapsto \{(1, \sigma(1)), (2, \sigma(2)), \dots, (n, \sigma(n))\}.$$

Given a pairing of ants to morsels $\pi(\sigma)$, the number of pairings of paths of the form $(1, \sigma(1)), \dots, (n, \sigma(n))$ is given by:

$$a_{1, \sigma(1)} \cdots a_{n, \sigma(n)}.$$

Therefore, the number of ways to guide the ants is given by:

$$\sum_{\sigma \in S_n} a_{1, \sigma(1)} \cdots a_{n, \sigma(n)} = \text{per } A. \quad \square$$

Now suppose we change the rules of the puzzle to additionally require that no two ants' paths touch. Now, how many ways there are to solve the puzzle? (pause again)

Theorem: (Gessel, Viennot) The *determinant* of A counts the number of simultaneous paths from a unique ant to a unique morsel, for which no two paths touch.

Proof: This result follows from D.I.E. We have a characterisation of the non-alternating sum per A , which counts all pairings of paths, so det A counts the pairings of paths corresponding to even permutations, minus the pairings of paths corresponding to odd permutations. We wish to find an involution between even and odd pairings.

For a valid pairing of paths, consider the smallest path (ordered by ant number) which intersects another path, and take its first touching. Create a new valid pairing of paths by tail swapping the two paths, that is, if $A_i \rightarrow M_j$ and $A_k \rightarrow M_l$, the new pairing has $A_i \rightarrow M_l$ and $A_k \rightarrow M_j$. This new path has a corresponding permutation with opposite sign! Moreover, note that performing the process again yields the starting path, since the tail swap does not change the first point of intersection of the path.

Therefore, we have found a sign changing involution! Moreover, it is easy to classify the exceptions to the involution - it is precisely the pairings of paths which have no intersections.

These all correspond to $1 \in S_n$, an even permutation, by how we restricted the graph. Thus, we conclude $\det A$ counts the number of valid pairings of paths from ants to morsels which do not touch. \square

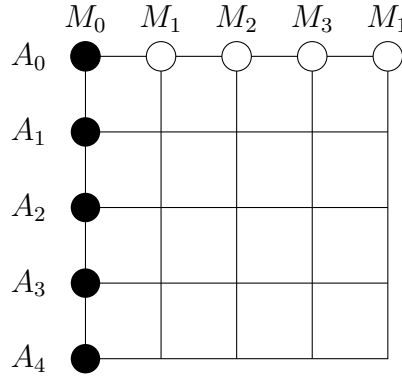
This is a gameified corollary of the Lindström–Gessel–Viennot lemma, proven in 1985 by Gessel and Viennot, which is a more general statement concerning the situation when there exist nonintersecting pairings of paths which do not correspond to $1 \in S_n$. In this case, the determinant counts the even nonintersecting pairings minus the odd nonintersecting pairings. This lemma shows up in the proof of equivalence of two definitions of Schur polynomials, symmetric polynomials which show up in the study of Young tableaux.

As a corollary, we may determine the determinant of Pascal's matrix P_n - generated in the same way as Pascal's triangle. More precisely, $p_{i,j} = \binom{i+j}{j}$ (with the indexing starting at 0). For example,

$$P_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}.$$

Theorem: For $n \geq 1$, $\det P_n = 1$.

Proof: Consider the grid graph $G_{n,n}$ (pictured here is $G_{4,4}$)



Let us additionally suppose $G_{n,n}$ is a digraph, with directions only moving up and right, and set up the ants and morsels for the problem of the DeterminedAnts like in the diagram. Observe that there are precisely $\binom{i+j}{j}$ paths for ant A_i to reach morsel M_j , as the ant must make $i + j$ moves, j of which must move to the right. Therefore, P_n is the ant matrix. However, it is also clear that there is exactly 1 nonintersecting pairing of paths, so we conclude $\det P_n = 1$. \square

5 Knock Em Down

To be shown...