* Relations

* Properties

* Equivalence Classes

(8.4) $A=\{a,b,c\}$ & $B=\{1,2,3,4\}$

$R_1=\{(a,2),(a,3),(b,1),(b,3),(b,4)\}$

Is $(1,b)\in R_2$ ✗
$(3,b)\in R_2$ ✗

$R_2=\{(1,b),(1,c),(2,a),(2,b),(3,c),(4,a),(4,c)\}$

R on A, where xRy if for some $n\in B$
xR_1n & nR_1y

"x is related to why, if there's an $n\in B$ bridging them"

Ans Ask 9 questions

$(R\subseteq A\times A)$

- Is $(a,a)\in R$? $n\in B$
  $(a,n)\in R_1$ & $(n,a)\in R_2$

  $(b,a)\in R$?

  $(c,a)\in R$?

- Is $(a,b)\in R$?
  (a,m) & (m,b) $\boxed{m\in B}$

  $(b,b)\in R$? NO!
  $(b,\ell),(\ell,b)$ ✗
  $\in R_1$ $\in R_2$

  $(c,b)\in R$?

  $(a,c)\in R$?

  $(b,c)\in R$?
  $(b,k),(k,c)$.
  $\in R_1$ $\in R_2$

  $(c,c)\in R$?

Example

$A = \mathbb{Z}, \quad B = \mathbb{Z}_{>0} = \mathbb{N}$

$R_1 = \{ (x, n) \mid n = |x| \} \subseteq \mathbb{Z} \times \mathbb{N}$

$R_2 = \{ (n, x) \mid x = -n \} \subseteq \mathbb{N} \times \mathbb{Z}$

R on $\mathbb{Z}$ similarly as before i.e.

$x R y$ if for some $n \in \mathbb{N}$ $x R_1 n$ & $n R_2 y$

Question 1: Is $(-1, 3) \in R$?

2: Is $(-1, -1) \in R$?

Answer : (1) Suppose it was i.e. $-1 R 3$

$-1 R_1 n$ and $\underline{n R_2 3}$ for some $n \in \mathbb{N}$

$\hookrightarrow (-1, n) \in R_1 \rightsquigarrow n = |-1| = 1 \rightsquigarrow X$

$\hookrightarrow (n, 3) \in R_2 \rightsquigarrow -n = 3, \; n = -3 \rightsquigarrow X$

Contradiction! Hence $(-1, 3) \notin \underline{\underline{R}}$.

(2) $-1 R -1 \rightsquigarrow -1 R_1 m$ & $m R_2 -1$

$\hookrightarrow m = |-1|$ & $-m = -1$

$m = 1$

i.e. for $m = 1$, $-1 \, R_1 \, 1$ & $1 \, R_2 \, -1$

Hence $(-1, -1) \in \underline{\underline{R}}$.

Properties — reflexive
— symmetric
— transitive

- Relation R on lines in $\mathbb{R}^3$

$l_1 \, R \, l_2$ if they're NOT parallel &
do NOT intersect
(skew lines)

* Reflexive? NO!

If reflexive ⇝ $l \, R \, l$ ⇝ l NOT parallel to l
and doesn't intersect l

BUT l is parallel to itself
l intersect itself at infinitely many points.

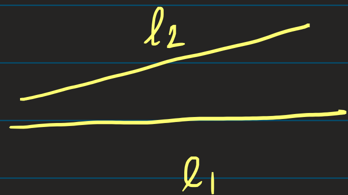
* Symmetric? YES

$$\text{Saying } \ell_1 \overset{\checkmark}{R} \ell_2 \rightsquigarrow \ell_1 \,\&\, \ell_2 \text{ are skew lines}$$

$$\rightsquigarrow \ell_2 \,\&\, \ell_1 \text{ are skew lines}$$

$$\rightsquigarrow \ell_2 R \ell_1 \checkmark$$

* Transitive?



$\underline{\ell_2} R \ell_1 \qquad \ell_1 R \underline{\ell_2}$

BUT $\ell_2 \not{R} \ell_2$

If a relation is symmetric but NOT reflexive, it cannot be transitive $(aRb,\ bRa$ but! $a \not{R} a)$

8.10 $\quad \underline{|A| = 4} \quad$ what is $\max |R|$ st $\underline{R} \cap \underline{R}^{-1} = \phi$

$A = \{a, b, c, d\}$

$R \subseteq A \times A$ //

$$A \times A = \{(a,a), (b,b), (c,c), (d,d)\} \cup \{(a,b), (b,a)\}$$

$$\cup \{(a,c), (c,a)\} \cup \{(a,d), (d,a)\}$$

$\cup \{(b,c), (c,b)\} \cup \{(b,d), (d,b)\}$

$\cup \{(c,d), (d,c)\}$ ✓

So $|R| \le 6$ hence $\max |R| = \underline{\underline{6}}$

Equivalence Classes

8.32

$A = \{a+b\sqrt{2} \mid a,b \in \mathbb{Q}$ & $a+b\sqrt{2} \neq 0\}$ ✓

$x \underline{R} y$ if $\boxed{\frac{x}{y} \in \mathbb{Q}}$.

* Reflexive

$\frac{x}{x} = 1 \in \mathbb{Q}$

* Symmetric $\frac{x}{y} \in \mathbb{Q} \Rightarrow \frac{y}{x} = \frac{1}{x/y} \in \mathbb{Q}$

* Transitive $\frac{x}{y}, \frac{y}{z} \in \mathbb{Q}$; $\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} \in \mathbb{Q}$

Equivalence Class

$r \in A$, $[r]$

$$[a+b\sqrt{2}] \overset{\text{def}}{=} \{x \in A \mid x \, R \, \underline{\underline{a+b\sqrt{2}}}\}$$

$$\Big\}\ \text{def}^{n}$$

$$= \left\{x \in A \;\middle|\; \boxed{\frac{x}{a+b\sqrt{2}}} \in \mathbb{Q}\setminus\{0\}\right\}$$

$$\Big\}\ \text{def}$$

$$= \left\{x \in A \;\middle|\; \frac{x}{a+b\sqrt{2}} = k, \quad k \in \mathbb{Q}\setminus\{0\}\right\}$$

$$\Big\}\ \text{algebra}$$

$$= \{x \in A \mid x = k(a+b\sqrt{2}), \quad k \in \mathbb{Q}\setminus\{0\}\}$$

$$\Big\}\ \text{rewriting}$$

$$= \{k(a+b\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\} \quad //$$

I $[a+b\sqrt{2}]$ when $a, b \neq 0$

II $[b\sqrt{2}]$ where $b \neq 0$ $(a=0)$

III $[a]$ wher $a \neq 0$ $(b=0)$

Claim: II is just $[\sqrt{2}]$

III is just $[1]$

I encapsulates many distinct classes

$$[1+\sqrt{2}] = \{k(1+\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\}$$

$$[2] = \{2k \mid k \in \mathbb{Q}\setminus\{0\}\} = \mathbb{Q}\setminus\{0\} = [1]$$

$$[3\sqrt{2}] = \{3\sqrt{2}\cdot k \mid k \in \mathbb{Q}\setminus\{0\}\} = [\sqrt{2}]$$

$$\underbrace{[a]}_{=[1]},\ \underbrace{[b\sqrt{2}]}_{=[\sqrt{2}]},\ \underbrace{[a+b\sqrt{2}],\ [c+d\sqrt{2}]}_{a,b,c,d\neq 0}$$

when is $[a+b\sqrt{2}] = [c+d\sqrt{2}]$

$\iff c+d\sqrt{2} \in [a+b\sqrt{2}]$

$\iff c+d\sqrt{2} = k(a+b\sqrt{2})$ for some $k \in \mathbb{Q}\setminus\{0\}$

$\iff c+d\sqrt{2} = ka + kb\sqrt{2}$

$\iff c = ka\ ,\quad d = kb$

$\iff \frac{a}{c} = \frac{b}{d}\ \left(= \frac{1}{k}\right)$

Hence $[a+b\sqrt{2}] \neq [c+d\sqrt{2}]$ when

$a,b,c,d \neq 0$

iff $\frac{a}{c} \neq \frac{b}{d}$

$$[1+\sqrt{2}] = \{k(1+\sqrt{2}) \mid k \in \mathbb{Q}\setminus\{0\}\}$$

$$\shortparallel$$

$$[2+4\sqrt{2}] = \{\ell(2+4\sqrt{2}) \mid \ell \in \mathbb{Q}\setminus\{0\}\}$$

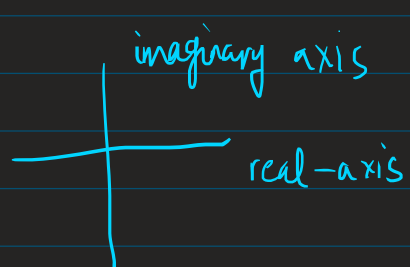
We will show $[1+\sqrt{2}] \subseteq [2+4\sqrt{2}]$

and $[2+4\sqrt{2}] \subseteq [1+\sqrt{2}]$

$x \in [2+4\sqrt{2}]$,
$$\begin{aligned} x &= \ell(2+4\sqrt{2}) \\ &= 2\ell(1+\sqrt{2}) \\ &\in [1+\sqrt{2}] \end{aligned}$$

$y \in [1+\sqrt{2}]$,
$$\begin{aligned} y &= k(1+\sqrt{2}) \\ &= \frac{k}{2}(2+4\sqrt{2}) \\ &\in [2+4\sqrt{2}] \end{aligned}$$

$\mathbb{C}$



$a+bi$

$a\in\mathbb{R},\ b\in\mathbb{R}$

A

$\mathbb{Q}\sqrt{2}$

$\sqrt{2}$-axis

$[a]$, $[b\sqrt{2}]$ → y-axis

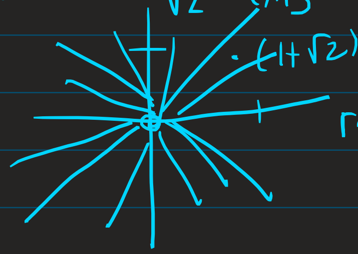
x-axis

$a+b\sqrt{2}$

$a \in \mathbb{Q}$

$b \in \mathbb{Q}$

$\mathbb{Q}$ ←



rational axis

$k(a+b\sqrt{2})$

$\frac{a}{c} + \frac{b}{d}$

$\boxed{y = mx}$