1.5 Centre and commutator subspaces

Reading Seminar, Linckelmann Chapter 1

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In particular, $Z(kG)$ is free as a k-module with rank equal to the number of conjugacy classes in G.

Proposition 1.5.2

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Let G, H be finite groups. There is a unique algebra isomorphism

$$Z(k(G \times H)) \cong Z(kG) \otimes_k Z(kH)$$

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sending the conjugacy class sum of $(x, y) \in G \times H$ in $k(G \times H)$ to $C_x \otimes C_y$, where C_x, C_y are the conjugacy class sums of x and y in kG and kH, respectively.

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Proof

The given map is an algebra homomorphism, induced by the algebra isomorphism in 1.1.4, restricted to the centre of $k(G \times H)$. The explicit description of the centre of a finite group algebra in terms of conjugacy class sums 1.5.1 implies that this induces an isomorphism as stated.

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$kG = Z(kG) \oplus$ ~~~

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Proof

By assumption $Z(B)$ is a direct summand of B, as k-modules, so we can identify $A \otimes_k Z(B)$ as a direct summand of $A \otimes_k B$. Similarly, we identify $Z(A) \otimes_k Z(B)$ with its image in $A \otimes_k B$.

$$B = Z(B) \oplus C$$

$$A \otimes B = A \otimes Z(B) \oplus A \otimes C$$

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Now, naturally $A \otimes_k Z(B)$ centralises $1 \otimes B$ and $Z(A) \otimes_k Z(B)$ is contained in $Z(A \otimes_k B)$.

$$(a \otimes z)(1 \otimes b) = (1 \otimes b)(a \otimes z)$$

$$\underline{a} \otimes \underline{zb} = \underline{a} \otimes \underline{bz}$$

$$A \otimes Z(B) \subseteq \mathrm{Cent}(1 \otimes B)$$

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By assumption $Z(B)$ is a direct summand of B, as k-modules, so we can identify $A \otimes_k Z(B)$ as a direct summand of $A \otimes_k B$. Similarly, we identify $Z(A) \otimes_k Z(B)$ with its image in $A \otimes_k B$.

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$A = \bigoplus k a_i \quad A \otimes B = \bigoplus k a_i \otimes B$

(contd.)

Let $z \in A \otimes_k B$.

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(contd.)

Let $z \in A \otimes_k B$. By the above, z can be written in the form

$$z = \sum_{i \in I} a_i \otimes b_i = \sum (a_i \otimes 1) b_i$$

with uniquely determined elements $b_i \in B$ of which only finitely many are nonzero.

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Suppose that $z \in Z(A \otimes_k B)$. By the first statement, we have $z \in A \otimes_k Z(B)$. Thus z can be written in the form

$$z = \sum_{j \in J} c_j \otimes z_j = \sum c_j (1 \otimes z_j)$$

with uniquely determined elements $c_j \in A$ of which at most finitely many are nonzero.

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$\alpha(\ \ )x\ldots$

$x \in G$ $x^{-1} \in G$ is NOT the inverse of x in $k_\alpha G$.

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  $$z[a, b] = z(ab - ba) = (za)b - b(za) = [za, b].$$

  Therefore $[A, A]$ is a $Z(A)$-submodule of A, and hence $A/[A, A]$ is a $Z(A)$-module.

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Let G be a finite group.

(i) The commutator space $[kG, kG]$ is spanned, as a k-module, by the set of commutators $[x, y] = xy - yx$, with $x, y \in G$. [not the same as G']

(ii) If $x, x' \in G$ are conjugate, then $x - x' \in [kG, kG]$.

$x' = gxg^{-1}$

$x - x' = x - gxg^{-1} = (xg^{-1})g - g(xg^{-1}) = [xg^{-1}, g]$

Proposition 1.5.4

$\sum \alpha_x x$ st $\sum_{y \in C} \alpha_y = 0 \quad \forall C$

Commutator subspaces of group algebras admit the following description.

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(iii) An element $\sum_{x \in G} \alpha_x x$ of kG belongs to the commutator space $\boxed{[kG, kG]}$ if and only if $\sum_{x \in C} \alpha_x = 0$ for any conjugacy class C in G.

$(\Rightarrow)$ $z \in [kG, kG]$; $\underbrace{xy - yx}$ ✓ $\qquad yx = y(xy)y^{-1}$

$[kG, kG] = k\text{-span}([x, y])$

$$\underbrace{\sum_{x\in G}\alpha_x x \in KG}\ \text{st}\ \sum_{x\in C}\alpha_x = 0$$

$$\sum_{x\in G}\alpha_x x = \sum_{C}\sum_{x\in C}\alpha_x x$$

$$= \underbrace{\sum_{C}\underbrace{\sum_{x\in C}\alpha_x\underbrace{(x-x_C)}}}_{\in\ [KG,KG]}\ ,\quad x_C\in C$$

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(iv) Let $\mathcal{K}$ be the set of conjugacy classes, and for $C \in \mathcal{K}$, let $x_C \in C$. The k-module $\sum_{C \in \mathcal{K}} k\,x_C$ is a complement of $[kG, kG]$ in kG.

$$kG = [kG, kG] \oplus \left[\bigoplus_{C} k x_C\right]$$

Remarks B.

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Since the centre $Z(kG)$ has as canonical k-basis the set of conjugacy class sums, the k-dual $Z(kG)^* = \mathrm{Hom}_k(Z(kG), k)$ therefore also has a canonical basis indexed by the set $\mathcal{K}$ of conjugacy classes of G, namely the dual basis $\{\sigma_C : C \in \mathcal{K}\}$, where σ_C sends C to 1 and $C' \neq C$ to 0, for $C, C' \in \mathcal{K}$.

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(i) We have an isomorphism of k-modules $kG/[kG, kG] \cong Z(kG)^*$ sending $x + [kG, kG]$ to the unique element $\sigma \in Z(kG)^*$ which maps the conjugacy class sum of x^{-1} to 1 and any other conjugacy class sum to zero.

$$x + [kG, kG] \longmapsto \sigma \begin{cases} C_{x^{-1}} \longmapsto 1 \\ \text{o/w} \longmapsto 0 \end{cases}$$

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where we regard ϕ as a map $kG \to k$ via the canonical surjection $kG \to kG/[kG, kG]$.

Proposition 1.5.5, 1.5.6

Proof

It follows from 1.5.4 that $kG/[kG, kG]$ is free of rank equal to the number of conjugacy classes of G,

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It follows from 1.5.4 that $kG/[kG, kG]$ is free of rank equal to the number of conjugacy classes of G, and that the image in $kG/[kG, kG]$ of a set $\mathcal{R}$ of representatives of the conjugacy classes is a basis that is independent of the choice of $\mathcal{R}$ because conjugate group elements have the same image in $kG/[kG, kG]$.

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One can prove (ii) by observing that this is the dual map of the isomorphism in (i).

Proposition 1.5.6

Let A, B be k-algebras. We have

$$[A \otimes_k B, A \otimes_k B] = A \otimes_k [B, B] + [A, A] \otimes_k B$$

as k-submodules of $A \otimes_k B$.

So, $[k(G \times H), k(G \times H)]$
$= kG \otimes_k [kH, kH] + [kG, kG] \otimes_k kH$

Proposition 1.5.7

Given a positive integer n, the trace of a square matrix $M = (m_{ij})_{1 \leqslant i,j \leqslant n}$ in $M_n(k)$ is the sum of its diagonal elements $\mathrm{tr}(M) = \sum_{1 \leqslant i \leqslant n} m_{ii}$.

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Taking traces defines a k-linear map $\operatorname{tr} : M_n(k) \to k$. This map is surjective as it maps a matrix with exactly one diagonal entry 1 (and all other entries 0) to 1. An elementary verification shows that the trace is symmetric; that is, $\operatorname{tr}(MN) = \operatorname{tr}(NM)$ for any two matrices $M, N \in M_n(k)$.

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Proposition 1.5.7

Proof

Since $\mathrm{tr}(MN) = \mathrm{tr}(NM)$ it follows that for any $M, N \in M_n(k)$ we have $[M_n(k), M_n(k)] \subseteq \ker(\mathrm{tr})$.

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For $2 \leqslant j \leqslant n$ denote by D_j the matrix with entry $(1, 1)$ equal to 1, entry (j, j) equal to -1 and zero everywhere else; that is, $D_j = E_{11} - E_{jj}$. Again $\mathrm{tr}(D_j) = 0$ and $Dj = [E_{1j}, E_{j1}] \in [M_n(k), M_n(k)]$.

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The set of matrices E_{ij} with $i \neq j$ and D_j with $2 \leqslant j \leqslant n$ is linearly independent, contained in $\ker(\mathrm{tr})$ and has $n^2 - 1$ elements.

Proposition 1.5.7

$M_n(k) = \ker(tr) \oplus k E_{11}$

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This set is a k-basis of $\ker(tr)$ because together with E_{11} it is a k-basis of $M_n(k)$. This shows that $\ker(tr) \subseteq [M_n(k), M_n(k)]$, whence the first equality as stated.

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The last statement is trivial if $n = 1$. Assume that $n \geqslant 2$. It suffices to show that the matrices E_{ij} are in the k-span of $GL_n(k)$.

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If $i \neq j$, then $\mathrm{Id} + E_{ij}$ is invertible, hence $E_{ij} = (\mathrm{Id} + E_{ij}) - \mathrm{Id}$ is in the span of $GL_n(k)$.

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Let J be the matrix with antidiagonal entries 1 and 0 elsewhere. Then J and $J + E_{11}$ are invertible, hence $E_{11} = (J + E_{11}) - J$ is in the span of $GL_n(k)$, completing the proof.

$J =$ $\qquad (-1)^{n+1}$

A k-algebra homomorphism $\alpha : A \to B$ need not send $Z(A)$ to $Z(B)$.

$\alpha : kC_2 \hookrightarrow kS_3$, $e \mapsto e$, $(12) \mapsto (12)$

$\alpha|_{Z(kC_2)}$: $ke + k(12)$ $\qquad$ $ke +$

$Z(kC_2) \not\dashrightarrow$ $\qquad$ $k(12)+(13)+(23)$

$\qquad\qquad\qquad$ $k(123)+(132)$

$\qquad\qquad\qquad$ $Z(kS_3)$

Remarks C., Definition 1.5.8

A k-algebra homomorphism $\alpha : A \to B$ need not send $Z(A)$ to $Z(B)$. But α sends any commutator $[a, b]$ in $[A, A]$ to the commutator $[\alpha(a), \alpha(b)]$ in $[B, B]$ and hence induces a k-linear map $A/[A, A] \to B/[B, B]$. Thus the quotient $A/[A, A]$ has better functoriality properties than the centre $Z(A)$.

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If A, B are group algebras then by taking k-duals one gets from 1.5.5 at least a k-linear (but not necessarily multiplicative) map $Z(B) \longrightarrow Z(A)$, so taking centres becomes contravariantly functorial.

Definition 1.5.8

Let A be a k-algebra, and let M be an (A, A)-bimodule.

Remarks C., Definition 1.5.8

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That is, taking A-fixed points on bimodules, viewed as a functor from the category of (A, A)-bimodules to the category of k- modules, is left exact but not right exact.

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More precisely, for $\phi \in \mathrm{End}_k(U)$ and $a, b \in A$ the k-endomorphism $a \cdot \phi \cdot b$ of U is defined by $(a \cdot \phi \cdot b)(u) = a\phi(bu)$ for all $u \in U$.

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We have

$$(\mathrm{End}_k(U))^A = \mathrm{End}_A(U);$$

indeed, the equality $a \cdot \phi = \phi \cdot a$ is equivalent to $a\phi(u) = \phi(au)$ for all $u \in U$.

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Corollary 1.5.12 implies that idempotents in $Z(A)$ correspond bijectively to projections of A onto bimodule summands of A.

References

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