R be a ring (commutative & has $1 \in R$)

* R is called a local ring if it has only one maximal ideal. $\mathfrak{m}$

$k := R/\mathfrak{m}$; the residue field of R.

eg $\mathbb{C}[[x]]$ or $\mathbb{C}[\![x]\!]$ Ring of formal power series

$$\mathbb{C}[\![x]\!] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in \mathbb{C} \right\}$$

add^n: $$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

identity 0 power series ; $0 = \sum_{i=0}^{\infty} 0 x^i$

mult^n: $$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right)$$

$$= \sum_{k=0}^{\infty} c_k x^k$$

$$c_k = \sum_{i=0}^{k} a_i b_{k-i} = \left(a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0\right)$$

aside: $(a_3 x^3 + \underbrace{a_2 x^2} + \underbrace{a_1 x} + \underbrace{a_0})(b_3 x^3 + \underbrace{b_2 x^2} + \underbrace{b_1 x} + \underbrace{b_0})$

$$= (a_2 b_0 + a_1 b_1 + a_0 b_2)\, x^2 + (a_3 b_0 + a_0 b_3)\, x^3 + \cdots$$

$$1 = 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$$

$\mathbb{C}[x]$ subring of $\mathbb{C}[[x]]$

$$\underbrace{a_n x^n + \cdots + a_1 x + a_0} = a_0 + a_1 x + \cdots + a_n x^n + 0 \cdot x^{n-1} + \cdots$$

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1. Claim: $(x) = \{x \cdot p(x) \mid p(x) \in \mathbb{C}[[x]]\}$ is the ONLY maximal ideal.

$$ev_0 : \mathbb{C}[[x]] \longrightarrow \mathbb{C}$$
$$\sum_{i=0}^{\infty} a_i x^i \longmapsto a_0$$

Surjective: $z \in \mathbb{C}$, then $p(x) = z + 0 \cdot x + 0 \cdot x^2 + \cdots$

Then $ev_0(p(x)) = z$

kernel: $\ker ev_0 = (x)$

(i) $(x) \subseteq \ker ev_0$

$x \cdot p(x) \in (x)$

$$ev_0(x \cdot p(x)) = ev_0\left(x \cdot (a_0 + a_1 x + a_2 x^2 + \cdots)\right)$$

$$= ev_0\left(a_0 x + a_1 x^2 + a_2 x^3 + \cdots\right)$$

$$= 0 \qquad \Rightarrow \qquad x \cdot p(x) \in \ker ev_0$$

Hence $(x) \subseteq \ker ev_0$

(ii) $\ker ev_0 \subseteq (x)$

$q(x) \in \ker ev_0$

$$0 = ev_0(q(x)) = ev_0(\underbrace{a_0 + a_1 x + a_2 x^2 + \cdots}) = a_0$$

$$q(x) = a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= x(a_1 + a_2 x + a_3 x^2 + \cdots) \in (x)$$

Hence $\ker ev_0 \subseteq (x)$

Therefore $\ker ev_0 = (x)$

Hence by FIT

$$\mathbb{C}[[x]]/(x) \cong \mathbb{C}, \text{ a field}$$

$m = (x)$ is maximal!

② If $p(x) \notin \mathfrak{m}$, then $p(x) \in \mathbb{C}[[x]]^\times$

$\exists\, q(x) \in \mathbb{C}[[x]]$ st $p(x) \cdot q(x) = 1$ ★

More precisely, $\mathbb{C}[[x]] \setminus \mathfrak{m} = \mathbb{C}[[x]]^\times$

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$q(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$1 = p(x) \cdot q(x) = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_0)x^2 + (a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0)x^3 + (a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0)x^4 + \cdots$$

$$\to 1 + 0 \cdot x + 0 \cdot x^2 + \cdots$$

$1 = a_0 b_0 \Rightarrow b_0 = 1/a_0$

$0 = a_0 b_1 + a_1 b_0 \Rightarrow b_1 = (-a_1 b_0)\, b_0$

$0 = a_0 b_2 + a_1 b_1 + a_2 b_0 \Rightarrow b_2 = -(a_1 b_1 + a_2 b_0)\, b_0$

$0 = a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \Rightarrow b_3 = -(a_3 b_0 + a_2 b_1 + a_1 b_2)\, b_0$

$0 = a_0 b_4 + a_1 b_3 + a_2 b_2 + a_3 b_1 + a_4 b_0 = \vdots$

$p(x) \notin (x)$ i.e. $p(x) \neq x \cdot (f(x)) \Rightarrow \boxed{a_0 \neq 0}$

If $a_0 = 0$; $p(x) = a_1 x + a_2 x^2 + \cdots = x(a_1 + a_2 x + \cdots)$
$\in (x)$

$a_0 b_0 = 1 \Rightarrow b_0 = 1/a_0$

$k > 0$ $\quad \underbrace{\sum_{i=0}^{k} a_i b_{k-i} = 0} \Rightarrow b_k = -b_0 \sum_{i=1}^{k} a_i b_{k-i}$

$0 = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0$

$\Rightarrow a_0 b_k = -(a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$

$\Rightarrow b_k = -\frac{1}{a_0}(a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$

$= -b_0 (a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0)$

$= -b_0 \sum_{i=1}^{k} a_i b_{k-i}$

Classwork 4

Q. Consider $p(x) = x^2 - 2x + 1$. This is not in $m = (x)$

What is $q(x) = 1/p(x)$?

Answer : $x^2 - 2x + 1 = (x-1)^2 = (1-x)^2$

so, if we can find what $r(x) = \frac{1}{1-x}$

$$q(x) = r(x)^2$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

$$r(x) = c_0 + c_1 x + c_2 x^2 + \cdots$$

$$(r(x))(1-x) = 1$$

$$1 = (c_0 + c_1 x + c_2 x^2 + \cdots) - (c_0 x + c_1 x^2 + c_2 x^3 + \cdots)$$

$$= c_0 + (c_1 - c_0)x + (c_2 - c_1)x^2 + (c_3 - c_2)x^3 + \cdots$$

$c_0 = 1$; $\quad 1 = c_0 = c_1 = c_2 = c_3 = \cdots$

$$q(x) = (1 + x + x^2 + x^3 + \cdots)(1 + x + x^2 + x^3 + \cdots)$$

$$= 1 + (1+1)x + (1+1+1)x^2 + (1+1+1+1)x^3 + \cdots$$

$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots = \sum_{i=0}^{\infty} (i+1)x^i$$

$p(x) \notin (x) = m$, then $p(x) \in \mathbb{C}[[x]]^\times$

$$\mathbb{C}[[x]] \setminus m = \mathbb{C}[[x]]^\times$$

Let n be another maximal ideal st $n \neq m$

$\exists\, r(x) \in n$ st $\underline{r(x) \notin m}$, then $r(x) \in \mathbb{C}[[x]]^\times$
$s(x)$

n a maximal ideal contains a unit ✗

$r(x) \in n$

$1 = s(x) \cdot r(x) \in n$

$1 \in n$

every element of the ring $\mathbb{C}[[x]]$ is in n

$a(x) \in \mathbb{C}[[x]]$

$a(x) \cdot 1 \in n \Rightarrow \mathbb{C}[[x]] \subseteq n$

$n = \mathbb{C}[[x]]$

I, $(I,+) \leq (R,+)$

& $x \in I$, $r \in R$ then $rx \in I$

So $n = m$. m is the <u>unique</u> maximal ideal.