An Introduction to Stability and Banach Algebras

Ryan Pugh

February 11, 2021

Toeplitz and Hankel Matrices

Toeplitz Matrices

A Toeplitz matrix is a matrix (usually infinite but sometimes finite) that is constant along the diagonals:

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \dots \\ a_1 & a_0 & a_{-1} & a_{-2} & \dots \\ a_2 & a_1 & a_0 & a_{-1} & \dots \\ a_3 & a_2 & a_1 & a_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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One question we can ask about them is: When are these matrices bounded ($=$ continuous)?

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This function $a \in L^\infty$ is called the *symbol* for the matrix and we refer to the matrix by $T(a)$.

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3. In what context do they arise?

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Similar to Toeplitz matrices, we have the following fact:

Theorem

The matrix A (resp. $\tilde{A}$) generates a bounded operator on ℓ^2 if and only if there exists a function $b \in L^\infty$ such that $b_n = a_n$ (resp. $b_{-n} = a_{-n}$) for all $n \geq 1$.

Connecting the Two

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Worth noting is that if we further require a and b to be continuous, then we can actually conclude that the Hankel matrices generated by them are *compact*. This is useful for several reasons.

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can tell us things about the Fredholmness of $T(a)$ or $T(b)$ if we know things about a and b. In particular, if $a \in C$ is invertible with continuous inverse, then

$$T(a)T(a^{-1}) = Id + K_1$$

and

$$T(a^{-1})T(a) = Id + K_2$$

so we can conclude right away $T(a)$ and $T(a^{-1})$ are Fredholm.

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Wiener-Hopf Factorization: writing a symbol a in a Banach algebra A as $a = a_- \chi_n a_+$ where
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Now iterate the previous theorem twice:

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Nothing special about χ_n in this equality, it just becomes relevant in WHF.There's lots of theory in WHF, this isn't even scratching the surface !

Playground for Problems: Approximation Methods

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Some Things to Consider

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1. Does the sequence $\{P_n A P_n\}$ approximate A *well*? (and what does "well" mean?)
2. Is the sequence stable?

Appropriate Approximations

We say that a sequence $\{A_n\}$ is an *appropriate approximation* for A if

1. There exists an $n_0 \in \mathbb{N}$ such that A_n is invertible for all $n \geq n_0$ and
2. For all $y \in \ell^2$, the (unique) solutions to $A_n x^{(n)} = P_n y$ converge in ℓ^2 to a solution $x \in \ell^2$ of the equation $Ax = y$.

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We have the following:

Theorem

Let $A \in B(\ell^2)$ and let $\{A_n\}$ be an approximating sequence for A. Then $\{A_n\}$ is an appropriate approximating sequence for A if and only if A is invertible, the matrices A_n are all invertible for sufficiently large n, and $A_n^{-1} := A_n^{-1} P_n$ converges strongly to A^{-1}.

Stability

Closely related to this idea of appropriate approximations is the notion of stability. We call a sequence of matrices $\{A_n\}$ stable if A_n is invertible for all sufficiently large n (say $n \geq n_0$) and $sup_{n \geq n_0} ||A_n^{-1}|| < \infty$.

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Theorem

Let $A \in B(\ell^2)$ and let $\{A_n\}$ be an approximating sequence for A. Then $\{A_n\}$ is an appropriate approximating sequence for A if and only if A is invertible and $\{A_n\}$ is a stable sequence.

Methods for Determining Stability

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- The more algebraic approach follows this strategy: Can we build a Banach algebra G so that $\{A_n\}$ being stable is equivalent to something being invertible in G?
- We'll revisit this notion soon without getting too technical, but for now let's have a bit of fun.

Some Tools for the Toolbox

Banach Algebras

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3. $\mathbb{R}$ and $\mathbb{C}$ (fan favorite?)
4. The Wiener algebra $W = \{f \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{f}_n| < \infty\}$. Can you guess what the norm is?

Another Example

Define the set $\mathcal{F}$ to be the set of all sequences $\{A_n\}$ of operators (matrices) $A_n \in B(Im(P_n))(\cong \mathbb{C}^{nxn})$ for which

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This set is a Banach algebra when equipped with the above norm and the following operations:

$$\{A_n\}+\{B_n\} := \{A_n+B_n\}, \lambda\{A_n\} := \{\lambda A_n\}, \{A_n\}\{B_n\} := \{A_nB_n\}$$

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Denote by $\mathcal{N}$ the subset of $\mathcal{F}$ consisting of all $\{C_n\} \in \mathcal{F}$ such that $||C_n|| \to 0$ as $n \to \infty$. This is a closed 2-sided ideal of $\mathcal{F}$ and hence $\mathcal{F}/\mathcal{N}$ is a Banach algebra.

Our First Pass at Algebraization of Stability

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For finite Toeplitz matrices, we have

$$T_n(a)T_n(b) = T_n(ab) - P_n H(a) H(\tilde{b}) P_n - W_n H(\tilde{a}) H(b) W_n$$

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Banach Algebras: Ideals

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- Maximal ideals are closely related to *multiplicative linear functionals*, which are nonzero linear functionals $f : A \to \mathbb{C}$ satisfying $f(ab) = f(a)f(b)$.
- Thanks to Gelfand-Mazur, we know there is a one-to-one correspondence between multiplicative linear functionals and maximal ideals of a Banach algebra (via associating with each functional its kernel).

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Do you think we can equip it with a topology?

Maximal Ideal Space: Giving it a Topology

For each element $a \in A$ we can assign it a function $\hat{a} : M \to \mathbb{C}$ defined by $\hat{a}(f) = f(a)$. This function is called the *Gelfand transform* of a. The map $\gamma : A \to C(M)$ is called the *Gelfand map*.

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Define the set $\hat{A} := \{\hat{a} : a \in A\}$. The *Gelfand topology* is the coarsest (weakest) topology on M such that all functions $\hat{a} \in \hat{A}$ are continuous. If you prefer, we can think of M as a subspace of A^* and then the Gelfand topology is nothing but the subspace topology where A^* is equipped with the weak-* topology.

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Can show that M is a compact Hausdorff space.

Why is M compact?

(Outline)

1. First show it is closed using the definition of the open sets in M: For $\phi \in M$, the open ball of radius $\epsilon > 0$ centered at ϕ is

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3. Then sit and fondly remember Banach-Alaoglu (the closed unit ball is compact in the weak$^*-$ topology)
4. Finally, since it is a closed subset of a compact set, it is itself compact

Relating to Invertibility

Theorem

Let A be a commutative Banach algebra with unit and let M be the maximal ideal space of A. An element $a \in A$ is invertible if and only if $\hat{a}(m) \neq 0$ for all $m \in M$.

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In English: The Gelfand map is a Banach algebra homomorphism of A into $C(M)$ which preserves spectra. This is seen by showing that the range of $\hat{a}$ is equal to the spectrum of a.

Last Theorem in Action: The Wiener Algebra

Recall: The Wiener algebra is $W = \{a \in C(\mathbb{T}) : \sum_{n\in\mathbb{Z}} |\hat{a}_n| < \infty\}$

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The last theorem then gives a proof of Wiener's Theorem, since $\hat{a} : M\ (\cong \mathbb{T}) \to \mathbb{C}$ is exactly the mapping $\hat{a}(\phi_\tau) = \phi_\tau(a) := a(\tau)$

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In the case where A is not commutative, we have the following theorem:

Theorem

Let A be a Banach algebra with identity element e and let Z be a closed subalgebra of the center of A which contains e. Denote the maximal ideal space of Z by Ω and for each maximal ideal $\omega \in \Omega$ let J_ω be the smallest closed two-sided ideal of A which contains the set ω. Then an element $a \in A$ is invertible in A if and only if the coset $a + J_\omega$ is invertible in A/J_ω for every $\omega \in \Omega$.

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- If the center is trivial, we get that a is invertible if and only if a is invertible (spicy, I know).

Example: Operators of Local Type

- The set $J := \{A \in B(\ell^2) : AT(c) - T(c)A \in K(\ell^2)\, \forall c \in C\}$ are called *operators of local type*. It contains the compact operators and it contains $T(a)$ with $a \in L^\infty$.

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Can therefore use Allen-Douglass with $A = J^\pi$ and $Z = \{T(c) + K(\ell^2) : c \in C\}$ to study Fredholmness of operators of local type.

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4. The quotient of any C^*-algebra with a self-adjoint, closed, 2-sided ideal will also be a C^*-algebra

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This is more useful than it may seem...

C^*-algebra Homomorphisms

- A C^*-algebra homomorphism f is a homomorphism in the algebraic sense between two C^*-algebras with the added property that $f(a^*) = f(a)^*$

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- It can be shown that the image $f(A)$ is a subalgebra of the codomain
- We say that such a homomorphism $f : A \to B$ between unital C^*-algebras *preserves spectra* if $spf(a) = spa$ for all $a \in A$ and we call it an isometry if $||f(a)|| = ||a||$ for all $a \in A$

C^*-algebra Homomorphisms

The following fact is very useful:

Theorem

Let A and B be unital C^*-algebras and $f : A \to B$ be a C^*-algebra homomorphism.

1. *If f preserves spectra, then f also preserves norms.*
2. *If f is injective, then f preserves spectra.*

Back to Stability

Remember that when discussing stability from an algebraic perspective, we want to see if we can reduce stability of a sequence to invertibility in an algebra. The fact that injective *-homomorphisms preserve spectra is huge in this regard !

Any questions?