Knots, Links, and a Little Magic

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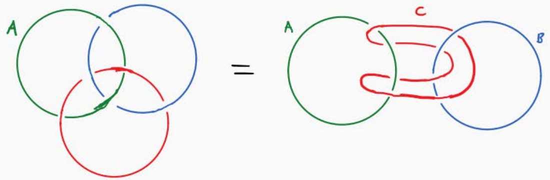
UCSC Graduate Student

Spring 2022

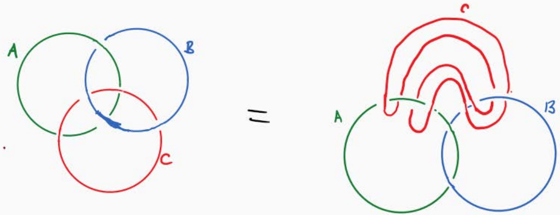
Department of Mathematics
UCSC Undergraduate Colloquium by AWM

Let's Begin with Some Magic!

Case 1:



Case 2:



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Tools we will need

- Some Topology
- Some (Abstract) Algebra
- Knots and Links
- Knot and Link Complements
- The Knot Group
- The Wirtinger Presentation

Definitions

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- Two continuous functions from one topological space to another are called **homotopic** if one can be "continuously deformed" into the other. This is a weaker notion than being homeomorphic.

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- A **(topological) embedding** is an injective continuous map

$$f : X \to Y$$

that yields a homeomorphism between X and $f(X)$

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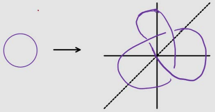
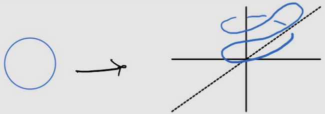
What is the difference?

The Trefoil:



A Trivial Knot:



Knot Diagrams

In $X = \mathbb{R}^3$ to draw knots, we use **regular projections** to some plane in X with corresponding *over-under crossings*.

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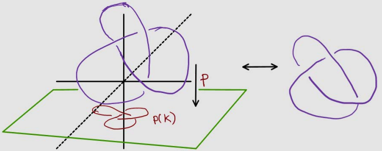
Example



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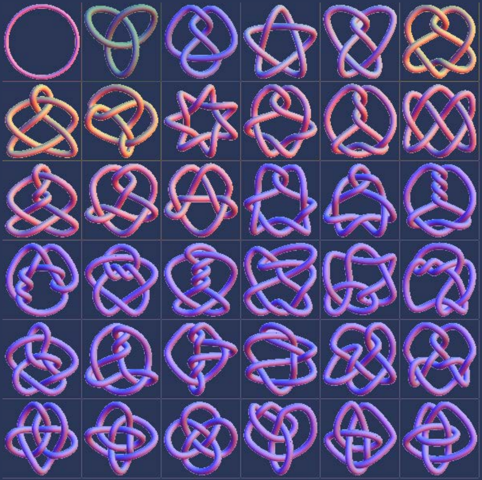

What is a link?

A subset L of a 3-dimensional space X is a **link** if it is an embedding of $\mathcal{S}^0 \cup \ldots \cup \mathcal{S}^0 \cup \mathcal{S}^1 \cup \ldots \cup \mathcal{S}^1$.

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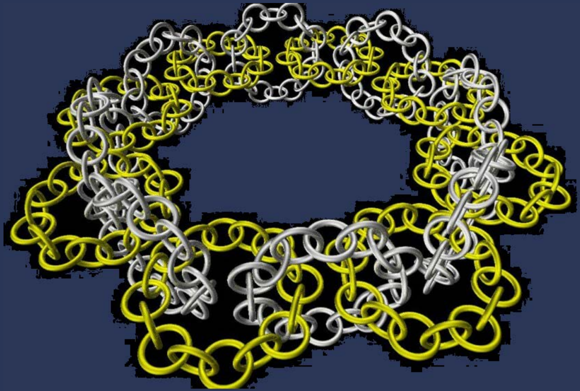
Remark: The number of components need not be finite.

Example: Antione's Necklace



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We are often more interested in **knot types modulo ambient isotopies**.

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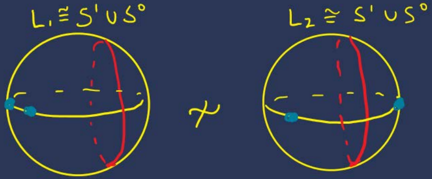

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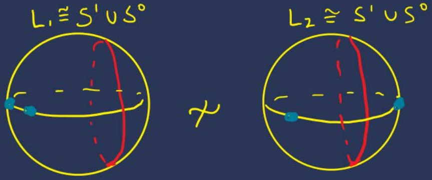

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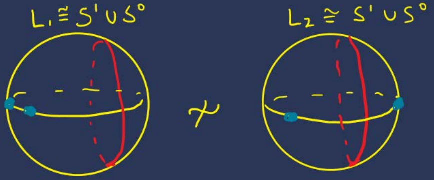

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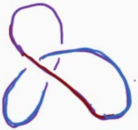

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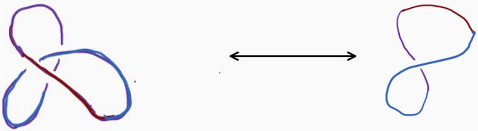

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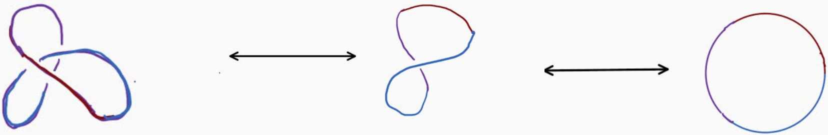

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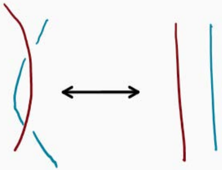

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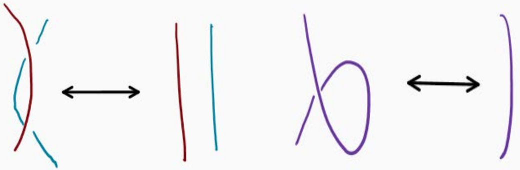

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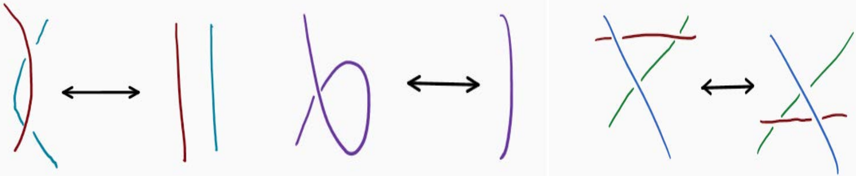

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More Background

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- The loops on a space modulo homotopy equivalence give us a lot of information about the space itself.
- The group of all loops on a space X modulo homotopy equivalence is called the **fundamental group of** X, and is denoted $\pi_1(X)$.

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- A **group presentation** is a way to specify a group with a set of *generators*, $\mathcal{S}$, and a set of *relations*, $\mathcal{R}$, denoted $G = \langle \mathcal{S} \mid \mathcal{R} \rangle$.

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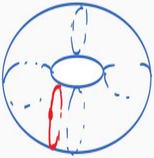

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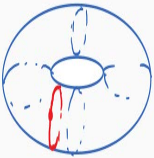

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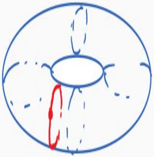

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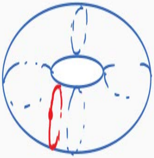

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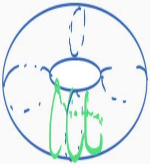

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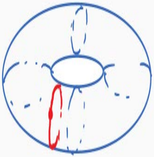

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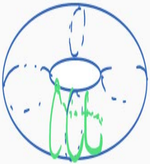

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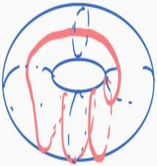

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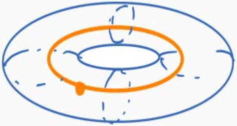

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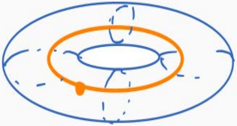

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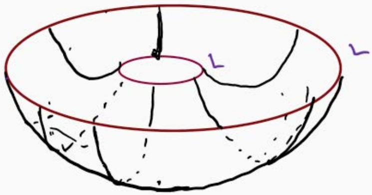

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Therefore, there are only two knot types on $\mathbb{T}^2$: the inessential, and everything else!

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- ◇ Any two meridians are ambient isotopic.
- ◇ There are infinitely many ambient isotopies of longitudes.

The Solid Torus

Let $V = \mathcal{S}^1 \times D^1 =$ a solid torus.
We still consider a meridian M and a longitude L, simple closed curves on ∂V.

- Any two meridians are ambient isotopic.
- There are infinitely many ambient isotopies of longitudes.
- If we embedded V into $\mathcal{S}^3$ and take the **complement** $X = \mathcal{S}^3 \setminus V$, $\pi_1(X)$ can tell us a lot about the embedding.

The Solid Torus

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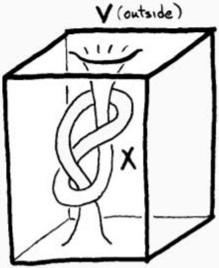
If X is not a solid torus, it is sometimes called a **cube-with-knotted-hole**



The Knot Group

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A thickened knot is an embedding of a solid torus. So it is enlightening to consider the fundamental group of **knot complements**, $X = \mathbb{R}^3 \setminus K$ (or $\mathcal{S}^3 \setminus K$).

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We call $\pi_1(X)$ the **knot group** of K.

Remark: The knot complement is a complete topological invariant for knots, but not for links. Though we still utilize this for many examples.

The Wirtinger Presentation

An algorithm to compute the knot group:

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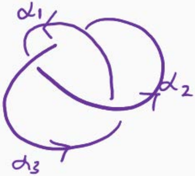
- Choose an orientation for K.

The Wirtinger Presentation

An algorithm to compute the knot group:

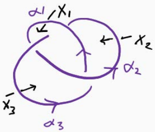
◇ Choose an orientation for K.

◇ The knot diagram consists of n arcs, separated by n over-under crossings. Label each arc $\alpha_1, \alpha_2, ..., \alpha_n$ so that α_i connects to α_{i+1} and α_n to α_1.



The Wirtinger Presentation

- At each α_i, draw an arrow labeled x_i passing under the arc using a *right-hand rule*. These arrows represent loops in the knot complement.



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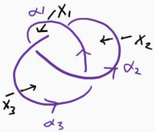
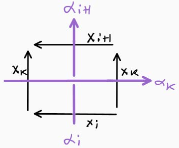


Case 1:

- At each over-under crossing, observe the relations described by the x_i's.



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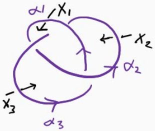

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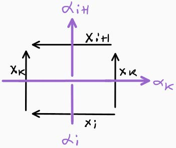

r_i : $x_k x_{i+1} = x_i x_k$

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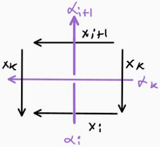

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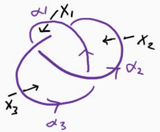

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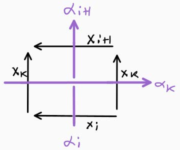

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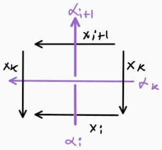

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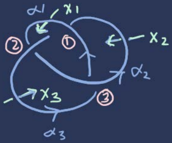
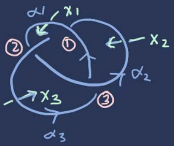
This yeilds $\pi_1(X) = \langle x_1, ..., x_n | r_1, ..., r_n \rangle$.



Example: The Trefoil



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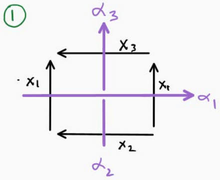

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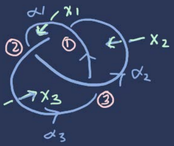

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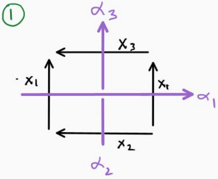

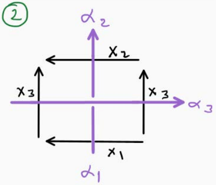

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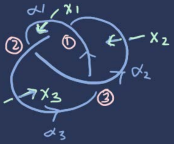

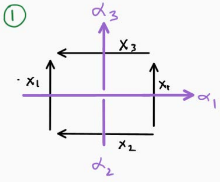

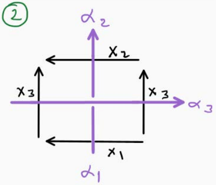

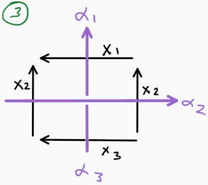

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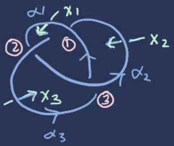

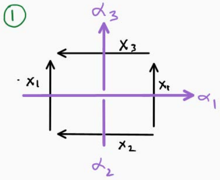

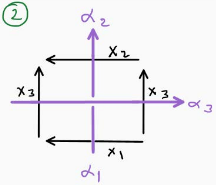

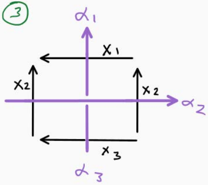

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$$\begin{aligned} \implies \pi_1(X) &= \langle x_1, x_2, x_3 | x_1 x_3 = x_2 x_1 = x_3 x_2 \rangle \\ &= \langle x_1, x_2 | x_1 x_2 x_1 = x_2 x_1 x_2 \rangle \end{aligned}$$

The Magic Trick Revealed

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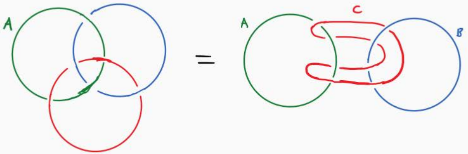

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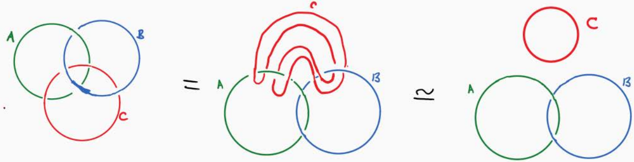
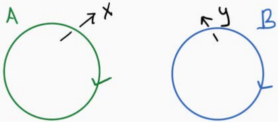


The Magic Trick Revealed

Let $L_1 = A \cup B$ and $X_1 = \mathbb{R}^3 \setminus L_1$.



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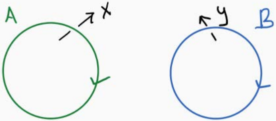

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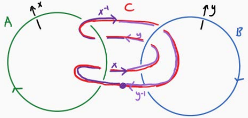
Now consider what $C \in \pi_1(X_1)$:



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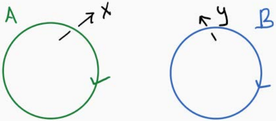

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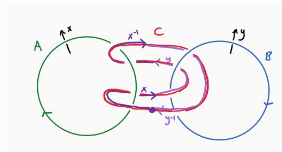

By observation $C = xyx^{-1}y^{-1} = [x, y] \in \pi_1(X_1)$, the commutator.

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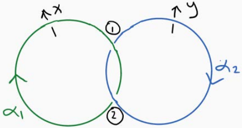
Now let $L_2 = A \cup B$ with A and B linked and $X_2 = \mathbb{R}^3 \setminus L_2$.



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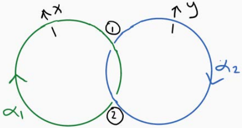
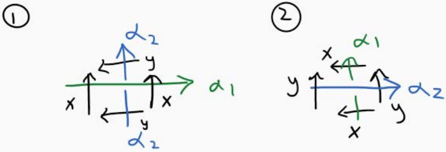


Consider the relation defined at the over-under crossings.



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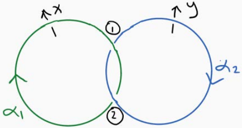

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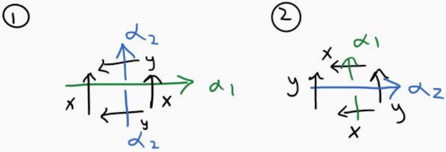

This yields $\pi_1(X_2) = \langle x, y | xy = yx \rangle$.

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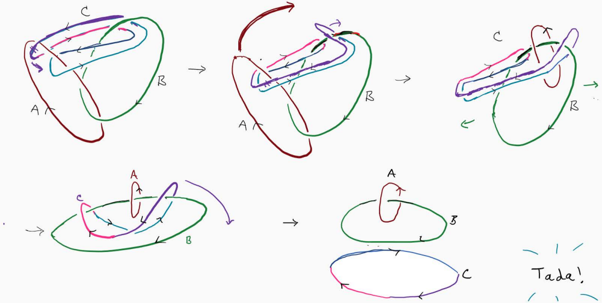

Bibliography

References

[1] D. Rolfsen, *Knots and Links*. AMS Chelsea Publishing, Rhode Island, 1990, Reprinted with corrections: 2003, pp. 1–65.

Thank you!