2. Equations and Unknowns

Problem

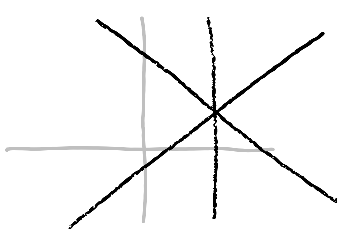
Under what conditions would two simultaneous linear equations have no solution?

Solution

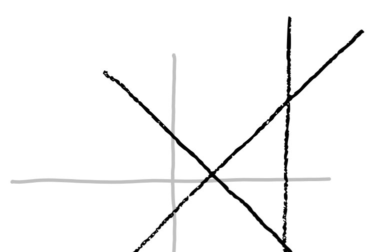
When the lines describing the equations are parallel.

For equations of the form $y = ax + b$, this translates to having the same slope a but different intercepts b.

In general, a system of n linear equations with 2 unknowns has a solution if and only if there exists a point which all n lines representing each equation intersect.



1 solution



No solution

Problem

A line travels through the points (0, 6) and (2, 2). What is the equation of the line?

Solution

The slope is an indication of how far along the y axis the line moves for each additional x.

$$a = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-6}{2-0} = -2$$

Knowing a, we can compute b by plugging in one of the points' coordinates. Both should yield the same result.

$$y = ax + b$$

$6 = -2 \cdot 0 + b$	$2 = -2 \cdot 2 + b$
$b = 6$	$b = 2 + 4 = 6$

Forms for representing a line

One way of representing a linear equation is with a slope-intercept line of the form $y = ax + b$.

The standard form of a linear equation is written as $Ax + By = C$.

The point-slope form of a linear equation is written as $y = m(x-h)+k$ or sometimes $y - y_0 = m(x - x_0)$.

Equation dependencies

A linear combination of a set of equations is any combination of multiples of those equations, added together.

If one of the equations in a set can be rewritten as a linear combination of the others, the set is said to be linearly dependent.

Inversely, if none of the equations in a set can be rewritten as linear combinations of the others, the set is said to be linearly independent.

Solving systems of equations

Problem

Solve for d. Hint: add all of the equations together.

$$a+b+c+d=8$$
$$a+b+c+e=12$$
$$a+b+d+e=16$$
$$a+c+d+e=20$$
$$b+c+d+e=24$$

Solution

Adding all the equations together yields

$$4a+4b+4c+4d+4e=80$$

or

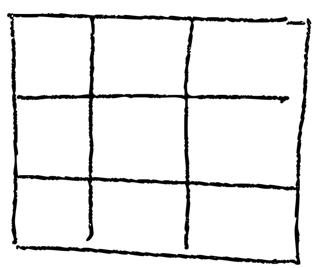
$$a+b+c+d+e=20$$

So we get $d=20-12=8$

Problem

We want to create a 3×3 puzzle. What is the maximum number of different symbols we can use to still make sure the puzzle has a unique solution?

Hint: Determine how many equations there are in total and how many of those can be written as a combination of the others.



Solution

There are 6 equations in total (3 for the rows, 3 for the columns) but 1 of those can always be written in terms of the other 5, so we can use 5 different symbols.

3. Manipulating exponents

Rule

When multiplying exponential terms with the same base, we can add up the exponents.

$$x^a \times x^b = x^{a+b}$$

Example

$$(2\times10^3)\times(3\times10^5) = 2\times3\times10^{3+5} = 6\times10^8$$

Rule

When multiplying exponential terms with the same exponent, we can multiply the bases.

$$x^a \times y^a = (xy)^a$$

Example

$$5^1\times2^2\times5^3\times2^4\times5^5\times2^6 = 5^{1+3+5}\times2^{2+4+6}$$

$$= 5^9\times2^{12}$$

$$= 5^9\times2^9\times2^3$$

$$= 10^9\times2^3$$

$$= 8\times10^9$$

Rule

When dividing exponential terms with the same base, we can subtract the exponents.

$$\frac{x^a}{x^b} = x^{a-b}$$

Example

$$\frac{16x^6}{2x^3} = \frac{16}{2} \times \frac{x^6}{x^3} = 8x^{6-3} = 8x^3$$

Problem

What does this expression evaluate to?

$$\frac{2^{10} \cdot 3^9 \cdot 4^8}{2^7 \cdot 3^6 \cdot 4^5}$$

Solution

$$\frac{2^{10} \cdot 3^9 \cdot 4^8}{2^7 \cdot 3^6 \cdot 4^5} = 2^{10-7} \cdot 3^{9-6} \cdot 4^{8-5}$$

$$= 2^3 \cdot 3^3 \cdot 4^3$$

$$= (2 \times 3 \times 4)^3$$

$$= 24^3$$

Problem

What does the following expression reduce to?

$$\frac{4^8+4^7}{4^5+4^5+4^5+4^5}$$

Solution

$$\frac{4^8+4^7}{4^5+4^5+4^5+4^5} = \frac{4^7\cdot 4^1+4^7}{4^6}$$

$$= \frac{4^7(4+1)}{4^6}$$

$$= 4^1 \times 5$$

$$= 20$$

Negative exponents

An increment by 1 of the exponent in an exponential term represents a multiplication by the base. Inversely, a decrement of 1 represents a division by the base.

$$x^{a+1} = x \cdot x^a$$

$$x^{a-1} = \frac{x^a}{x}$$

Fractional exponents

Fractional exponents are another way to express radicals.

$$x^{\frac{a}{b}} = \left(\sqrt[b]{x}\right)^a$$

Problem

What is x in $3^x \times 9^2 = 27^4$?

Solution

$$3^x \times 9^2 = 3^4 \times 9^4$$
$$= 3^4 \times 9^2 \times 9^2$$
$$= 3^4 \times 3^2 \times 3^2 \times 9^2$$
$$= 3^8 \times 9^2$$

$$x = 8$$

Exponent towers

Exponent towers are expressions in which a base is raised to an exponent, which itself is raised to one or more exponents.

$$x^{a^{b^{c}}}$$

The correct way to evaluate a tower with no parentheses is from the top down. In the example above this means first evaluating $b^c = y$ then $a^y = z$ and finally x^z.

Exponent rules summary

Product rules

$$x^a \times x^b = x^{a+b}$$

$$x^a \times y^a = (xy)^a$$

Quotient rules

$$x^a \div x^b = x^{a-b}$$

$$x^a \div y^a = (x \div y)^a$$

Power rules

$$(x^a)^b = x^{ab}$$

$$x^{a/b} = (\sqrt[b]{x})^a$$

Negative exponents

$$x^{-a} = \frac{1}{x^a}$$

Zero exponents

$$x^0 = 1$$

4. Algebra in Motion

Problem

Two runners start on opposite ends of a circular track and begin running in opposite directions (towards each other). The distance around the track is 400m. If the first runner runs at a speed of 4 m/s and the second runner at a speed of 6 m/s, how many times will they pass each other in 2 minutes?

Solution

They are running towards each other at a rate of $4+6=10$ m/s.
Since they start at opposite ends of the track, they are initially 200m apart and it will take them $200 \div 10 = 20$ s to first pass each other. They subsequently pass each other every $400 \div 10 = 40$s.

In 120s, they will pass each other 3 times.

Problem

Three runners start at the same spot on a circular track that is 360m around. The speeds of the runners are 8m/s, 9m/s and 12m/s, respectively. How long will it be until all 3 runners are all together at the same spot along the track?

Hint: Try working from the perspective of the slowest runner.

Solution

The 1st runner is going $9-8=1$ m/s slower than the 2nd runner, so they will find themselves in the same spot every $360\text{m} \div 1\frac{\text{m}}{\text{s}} = 360\text{s}$.

The 1st runner is going $12-8=4$ m/s slower than the 3rd runner, so they will find themselves in the same spot every $360\text{m} \div 4\frac{\text{m}}{\text{s}} = 90\text{s}$.

Since 360 is a multiple of 90, all 3 runners will find themselves in the same spot every 360 seconds.

Problem

What is F.'s average speed on a journey from A to B if he drives half the way (120 km) at 40 km/h and the other half at 60 km/h?

Solution

The first half of the journey takes him

$$t_1 = \frac{120\,\text{km}}{40\,\text{km/h}} = 3\,\text{h}$$

The 2nd half takes him

$$t_2 = \frac{120\,\text{km}}{60\,\text{km/h}} = 2\,\text{h}$$

His average speed is

$$v = \frac{120\,\text{km} + 120\,\text{km}}{3\,\text{h} + 2\,\text{h}} = 48\,\text{km/h}$$

Problem

A boat starts traveling in a straight path down the Nile river, then travels in a straight path back upstream to where it started.

If no current exists, the entire trip takes 1h.

If there is a positive, constant downstream current for the duration of the trip, will it take more, less or exactly 1 hour to complete the trip?

Solution

Let d be the distance traveled before the boat turns around, b the speed of the boat and c the speed of the current.

When there is no current, the time spent travelling will be

$$t_1 = \frac{2d}{b}$$

When there is a positive current, the time spent travelling will be

$$t_2 = \frac{d}{b+c} + \frac{d}{b-c}$$

$$= \frac{d(b-c)+d(b+c)}{b^2-c^2}$$

$$= \frac{2db}{b^2-c^2}$$

We want to compare t_1 and t_2, so we could subtract them a see if we end up with a positive or with a negative number.

$$t_2 - t_1 = \frac{2db}{b^2-c^2} - \frac{2d}{b}$$

$- 2db^2$
$+ 2dc^2$

$$= \frac{2db^2 - 2d(b^2-c^2)}{b(b^2-c^2)}$$

$$= \frac{2dc^2}{b(b^2-c^2)}$$

$t_2 - t_1 > 0$ hence $t_2 > t_1$