2. Complex numbers

The rule $\sqrt{a \times b} = \sqrt{a} \cdot \sqrt{b}$ is generally only valid when a and b are <u>positive</u> real numbers, so rather than saying $\sqrt{-121} = \sqrt{-1}\sqrt{121} = i\sqrt{121}$, the square root of a negative number is <u>defined</u> to be i times the square root of its absolute value.

Hence, $\sqrt{x^2} = x$ rather than $\pm x$

Imaginary powers

The value of i raised to increasing integer powers follows a pattern:

$i^1 = i \qquad i^2 = -1 \qquad i^3 = -i \qquad i^4 = 1 \qquad i^5 = i \qquad \ldots$

Knowing this pattern means the value of i^n, where n is a positive integer, can be quickly determined.

$$
\begin{aligned}
i^{81} &= (i^{80}) \cdot (i^1) \\
&= (i^4)^{20} \cdot (i^1) \\
&= 1^{20} \cdot i \\
&= i
\end{aligned}
$$

Note that $1/i = -i$, as demonstrated by:

$i^2 = -1 \qquad i = -1/i \qquad -i = 1/i$

Real and imaginary parts

The complex conjugate of a complex number z, written $\bar{z}$, is the number with the same real part while the imaginary part is negated. So if $z = a + bi$ then $\bar{z} = a - bi$.

The triangle inequality

The magnitude of a complex number z, denoted $|z|$, is the distance from the origin to the point in the complex plane representing the complex number.

Note that for any two complex numbers z and w, we have $|z+w| \leqslant |z| + |w|$. This is known as the triangle inequality. Furthermore, replacing w with $-w$ also yields $|z-w| \leqslant |z| + |w|$.

$|z+w| = |z| + |w|$ may only be true when z and w lie on the same line from the origin.

Problem

What is the minimum value of

$$|z-i|+|z-3+2i| ?$$

Solution

Substituting the values in our problem yields

$$|\underbrace{(z-i)}_{a}-\underbrace{(z-3+2i)}_{b}| \leq |z-i|+|z-3+2i|$$

Note that we used the modified version of the triangle inequality $|a-b| \leq |a|+|b|$ to make sure the two z's cancel, giving us a constant lower bound independent of z's value.

$$|-i-(-3+2i)| = |3-3i| \leq |z-i|+|z-3+2i|$$

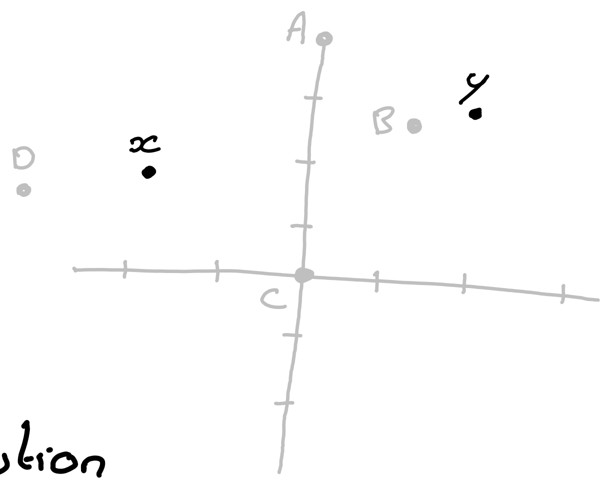
Using $|a+ib|=\sqrt{a^2+b^2}$, we find the lower bound

$$|3-3i| = \sqrt{3^2+(-3)^2} = \sqrt{18} = 3\sqrt{2}$$

Problem

x and y are two complex numbers shown on the complex plane below.

If z is a complex number such that $|z-x|+|z-y|=|x-y|$, which of the following points could represent z?



Solution

B is the correct answer. We must think of the magnitudes as representing the distance between two points.

$|z-x|+|z-y|=|x-y|$ means that the sum of the distances from z to x and from z to y is equal to the distance from x to y. As shown in the proof of the triangle inequality, this can only occur when z lies on the line segment between x and y.