Physcis 20.3 Numerical Solutions of Ordinary Differential Equations

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1 The Motion of a Mass on a Spring

Figure 1 shows a few cycles of oscillation when the explicit Euler method is implemented. This graph was made with initial condition set to $h = 0.004, N = 10000, x_0 = 2, y_0 = 2, t_0 = 0$.

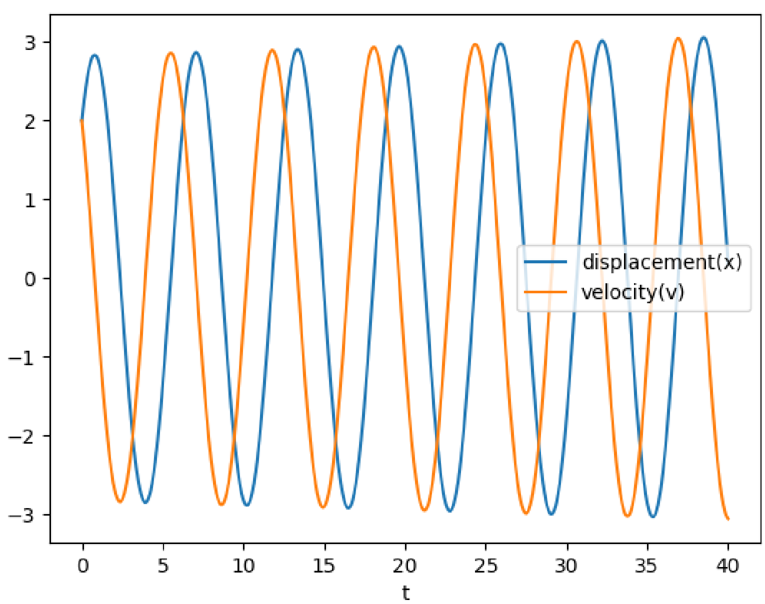


Figure 1: Motion of a mass on a spring implementing the explicit Euler method

2 Analytic Solution

2.1 Derivation of the Analytic Solution

Rearranging $F = ma = -kx$ we have that $a = -\frac{k}{m}x$. Let $a = \frac{d^2x}{dt^2}$ and $\frac{k}{m} = \omega^2$. We then have $a = \frac{d^2x}{dt^2} = \omega^2 x$. Using the Ansatz principle we guess that the solution to this equation is in its most general form

$$x(t) = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t$$

. This can be rewritten in terms of two new constants B and C.

$$x(t) = B\cos\omega t + C\sin\omega t$$

. By finding the derivative of the above equation we get that

$$v(t) = -B\omega\sin\omega t + C\omega\cos\omega t$$

To find the constants C and D, substitute $t = 0$ into both of the equations.

$$v_0 = v(0) = -\omega\sin 0 + C\omega\cos 0 = \omega C$$

So, $C = \frac{v_0}{\omega}$

$$x_0 = x(0) = B\cos 0 + C\sin 0 = B$$

So, $B = x_0$ Therefore we have the general solution

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

$$v(t) = -x_0\omega\sin\omega t + v_0\cos\omega t$$

But because $\omega = 1$ in this case we have that the final analytical solution to the mass of the spring is

$$x(t) = x_0\cos t + v_0\sin t \tag{1}$$

$$v(t) = -x_0\sin t + v_0\cos t \tag{2}$$

2.2 Global Errors

Figure 2 shows that as the global error increases exponentially with time. This graph was made with initial conditions set to $h = 0.005, N = 50000, x_0 = 2, y_0 = 2, t_0 = 0$. The increasing error could have also been seen at a lower N, however, a larger N was chosen tot demonstrate the exponential increase.

3 Truncation Error

Figure 3 shows that the truncation error is proportional to h for small values of h. We can see that as h increases, so does the truncation error. I chose $h_0 = 0.001$ and then calculated the error for $h_0, h_0/2, h_0/4, h_0/8, h_1/16$ to produce this plot.

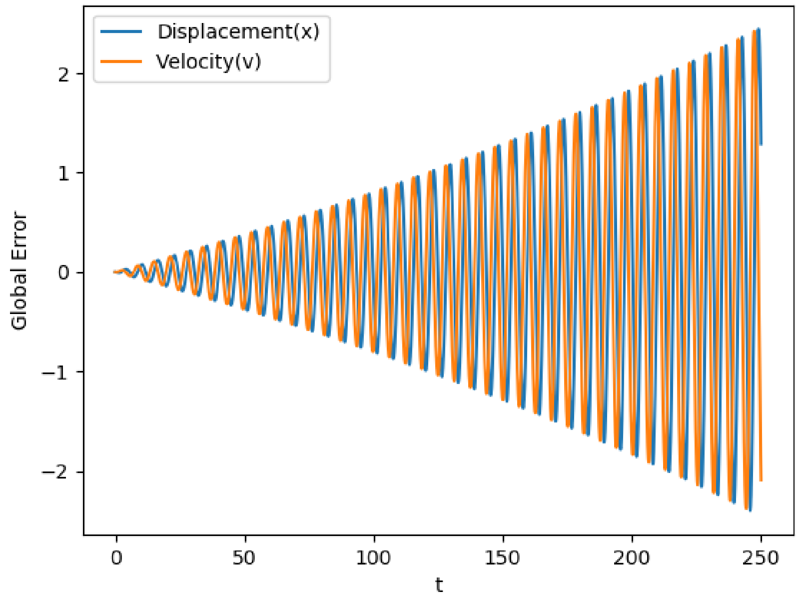


Figure 2: Global errors between the analytic solution and the explicit Euler method

4 Numerical Evolution of Total Energy

One can calculate the normalized total energy of a physical system by $E = x^2 + v^2$. Figure 4 shows how this energy evolves over a long period of time. The trend we can see is that the energy increases exponentially with time. We can see that this is similar to the global error which we also saw to be increasing over time, see figure 2. For this physical system we would actually expect the energy to remain constant as energy should be conserved. However, it is evident that due to the exponentially increasing error, E also increases exponentially. The parameters used for the plot of figure 4 are $h = 0.004, N = 100000, x_0 = 2, y_0 = 2, t_0 = 0$.

5 The Implicit Euler Equation

5.1 Expressions for the Implicit Euler Method

The implicit Euler method requires the unknown values of x_{i+1} to update the known values of x_i. This can be encoded in the linear system

$$\begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix} \cdot \begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} = \begin{pmatrix} x_i \\ v_i \end{pmatrix} \tag{3}$$

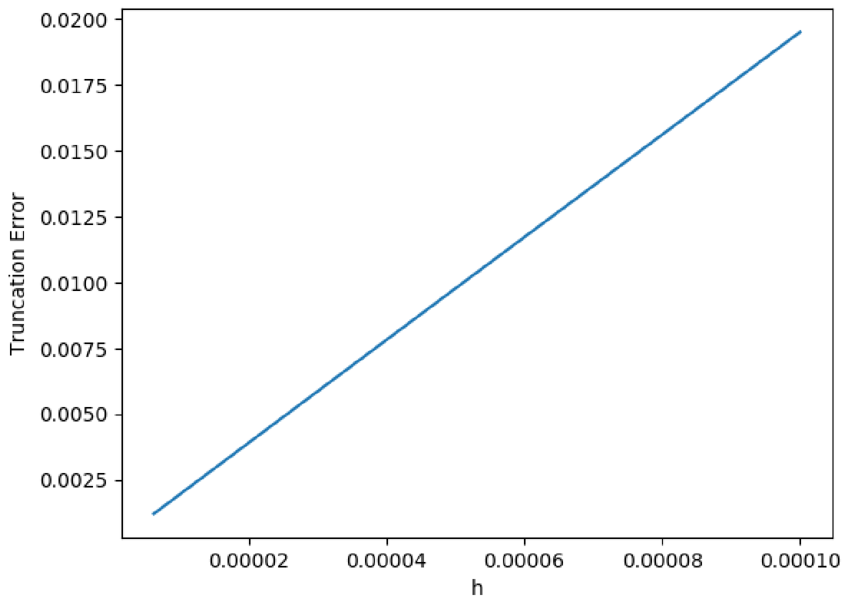


Figure 3: Maximum value of $x_{analytic} - x$ versus h for several different h

However, we want to find an expression for the unknown x_{i+1} in terms of the known value x_i. To do this we must first find the inverse of the first matrix. This gives

$$\begin{pmatrix} 1 & -h \\ h & 1 \end{pmatrix}^{-1} = \frac{1}{1+h^2} \cdot \begin{pmatrix} 1 & h \\ -h & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{1+h^2} & \frac{h}{1+h^2} \\ \frac{-h}{1+h^2} & \frac{1}{1+h^2} \end{pmatrix} \tag{4}$$

We can then use this inverse matrix to find the values of x_{i+1} and v_{i+1} in terms of x_i and v_i. We get the linear system

$$\begin{pmatrix} \frac{1}{1+h^2} & \frac{h}{1+h^2} \\ \frac{-h}{1+h^2} & \frac{1}{1+h^2} \end{pmatrix} \cdot \begin{pmatrix} x_i \\ v_i \end{pmatrix} = \begin{pmatrix} x_{i+1} \\ v_{i+1} \end{pmatrix} \tag{5}$$

Using equation 5 we can get the implicit Euler method in terms of x_i and v_i

$$x_{i+1} = \frac{x_i}{1+h^2} + \frac{v_i}{1+h^2}, v_{i+1} = \frac{v_i}{1+h^2} - \frac{x_i}{1+h^2} \tag{6}$$

If we plot these expressions much we get a graph like in figure 5. This plot looks extremely similar to the plot generated when implementing the explicit Euler equations. We can compare it to figure 1 because the parameters used are exactly the same.

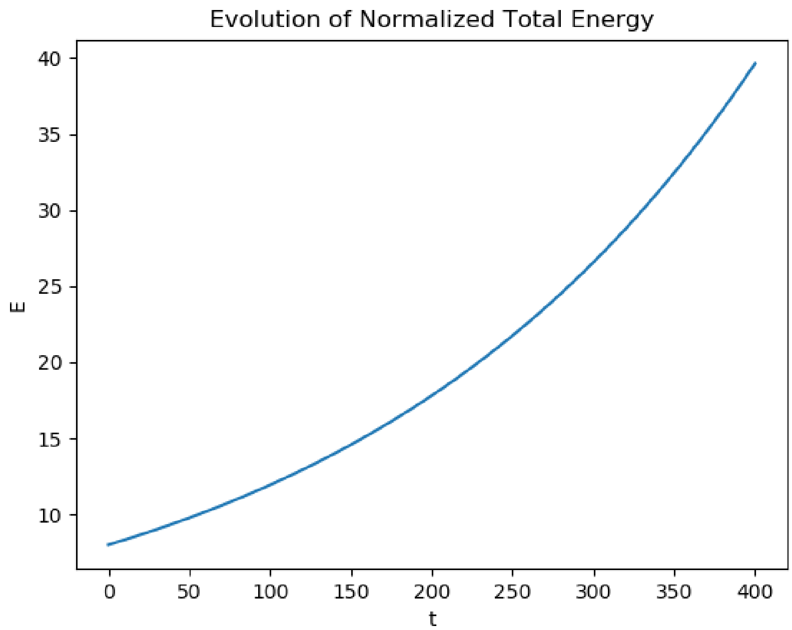


Figure 4: Long term evolution of the total energy with time, explicit Euler method

5.2 Global Errors

If we plot the global errors generated using the implicit functions we get a graph like in fig 6. This graph shows an evolution of total energy similar to figure 2 for the explicit Euler equation. However, the error does not exponentially increase like in figure 2, instead it first increases and then tails off in a logarithmic fashion. Again, the graphs are comparable because I used the same parameters as for the explicit method.

5.3 Evolution of Total Energy

Unlike in figure 4 for the explicit Euler equation, where the energy grew exponentially, the energy in figure 7 decreases exponentially. Again, the energy is not constant. Thus, the implicit Euler method also does not model the physical system such that energy is conserved. We saw that the error instead of increasing exponentially only had a logarithmic increase. It is because of this that we see a decrease in energy and the errors become greater for the implicit Euler method.

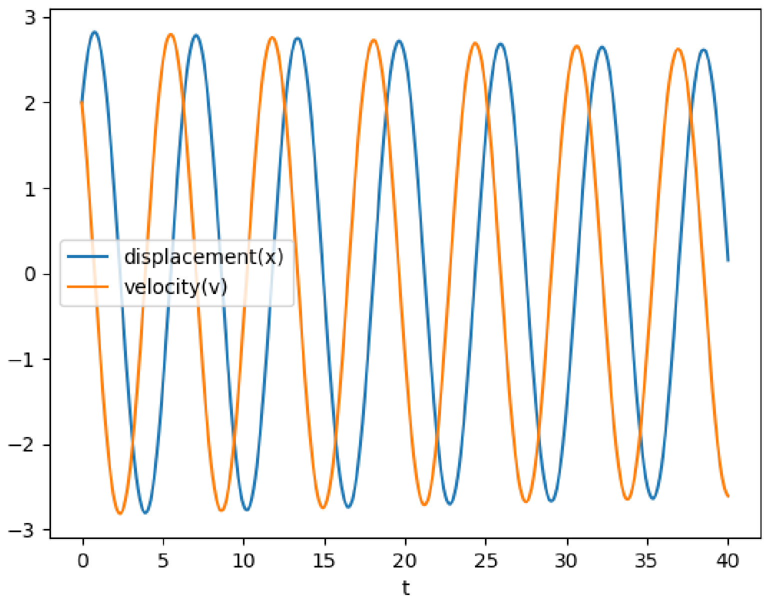


Figure 5: Implementing the implicit Euler equation

6 Phase-space Trajectories

Figure 8 shows very clearly that the errors in the explicit method deviate positively and the errors in the implicit method deviate negatively. We can see this because in the explicit Euler phase-space the deviation from the analytic solutions are towards the outside of the circle. This means that there are always positive errors calculated by this solution. On the other hand, for the implicit Euler method, the deviations are inside the orange circle and therefore, the errors were always negative.

7 Symplectic Euler Method

7.1 Phase-space trajectory

When the phase space for the symplectic method is plotted along side the analytic for the value of $h = 0.004$ that was also used in figure 8, we can see no difference between the analytic solutions and the solutions generated by the symplectic method. However, if the value of h is decreased to $h = 0.04$ we have figure 9(b). The circle of the symplectic method slightly deviates from the analytic solutions, however, it corrects itself and returns back. Thus, it never

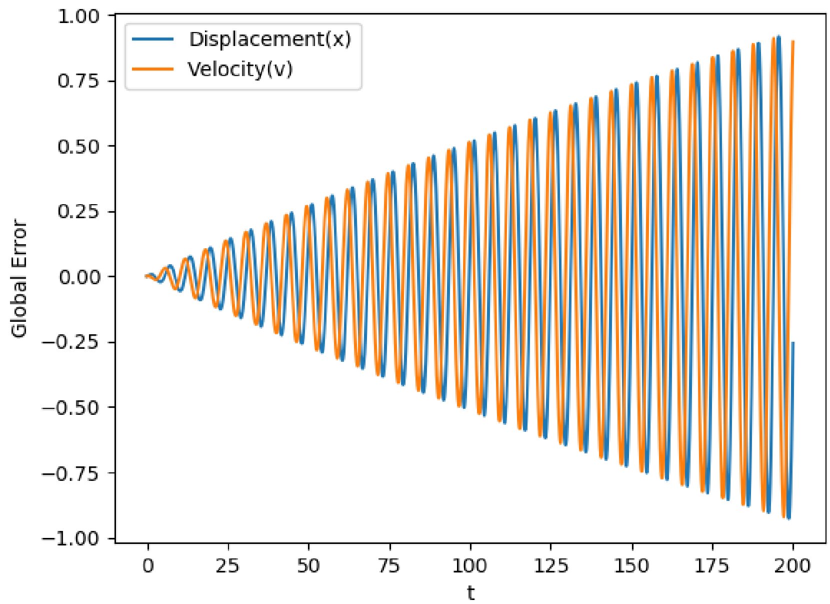


Figure 6: Global errors between the analytic solution and the implicit Euler method

diverges in one particular direction from the orange circle like in figure 8.

7.2 Evolution of total energy

Figure 10 shows the long term evolution of the total energy of the symplectic Euler method compared to the analytical total energy. The analytic energy holds a constant value as expected because energy is conserved in the physical system. We can see that the energy of the symplectic grows, however when it reaches a certain height it returns to the total energy for the analytic solution. This suggests that the error of the symplectic method is in a sense, self-correcting. Although it does not fully conserve energy, it returns back to a given point. And therefore it never diverges significantly like in the explicit or implicit Euler methods. We can also relate figure 10 to 9(b). The way the energy error is corrected back is similar to the way that the symplectic trajectory deviated yet returned to the trajectory of the analytic solution. This suggests that the error created by the symplectic method is both positive and negative and thus self-correcting.

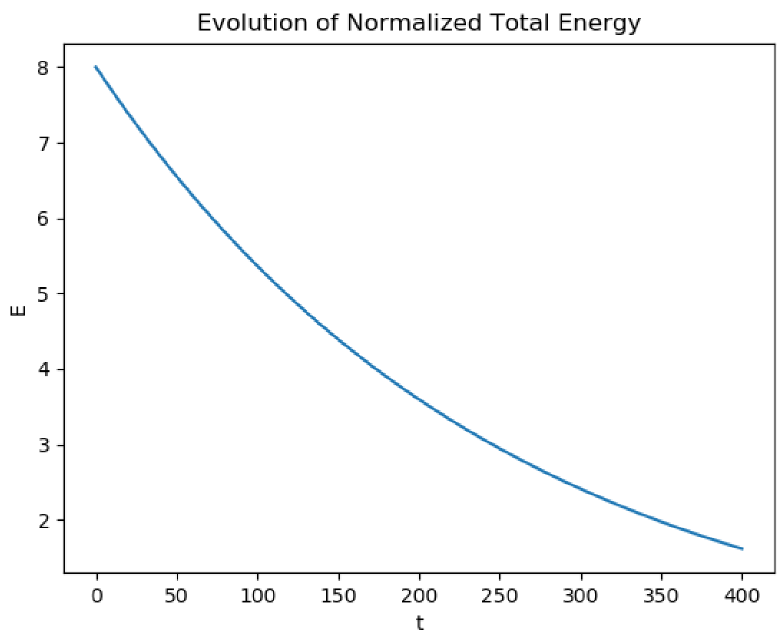
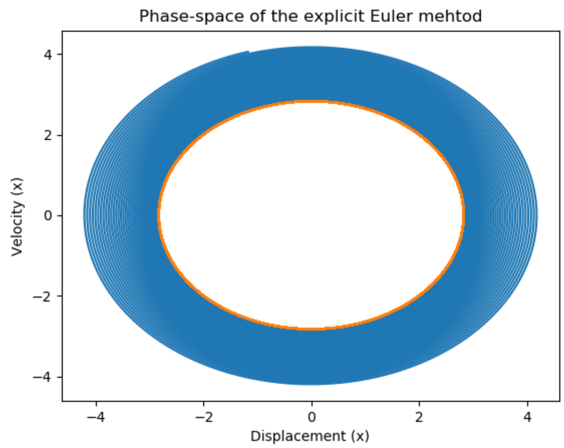
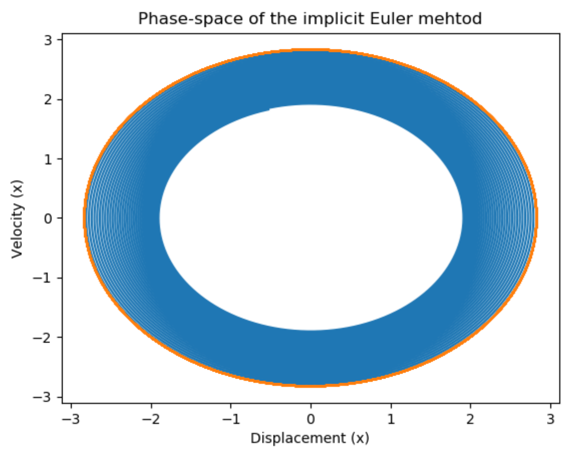


Figure 7: Long term evolution of the total energy with time, implicit Euler method

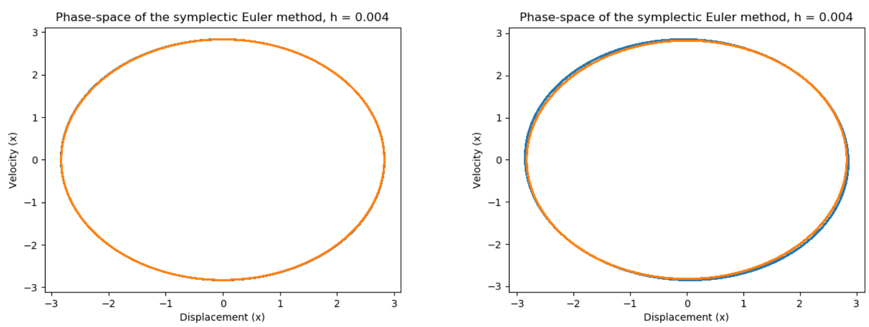


(a) Phase-space trajectory produces by the explicit Euler method



(b) Phase-space trajectory produces by the implicit Euler method

Figure 8: Phase-space trajectories produced by the implicit and explicit Euler methods. The orange circle in both represents the trajectory produced by the analytic solutions



(a) Phase-space trajectory produced by the symplectic method at $h = 0.004$

(b) Phase-space trajectory produced by the symplectic method at $h = 0.04$

Figure 9: Phase-space trajectories produced by the symplectic Euler method at different values of h. The orange circle in both represents the trajectory produced by the analytic solutions

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Figure 10: Long term evolution of the total energy with time, symplectic Euler method