

Open Problems about the Simplex Method

Sophie Huiberts
CNRS, LIMOS

Linear programming

maximize $c^T x$

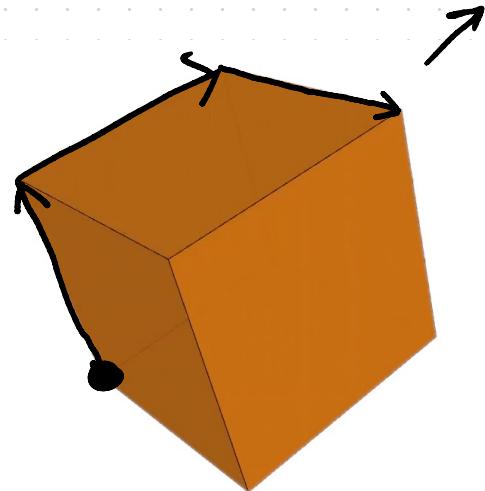
subject to $Ax \leq b$

we get $A \in \mathbb{R}^{n \times d}$

$b \in \mathbb{R}^n$

$c \in \mathbb{R}^d$

We compute $x \in \mathbb{R}^d$



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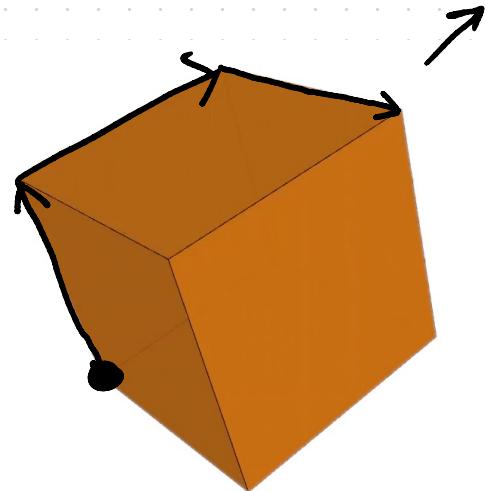
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how many pivot steps?

Linear programming

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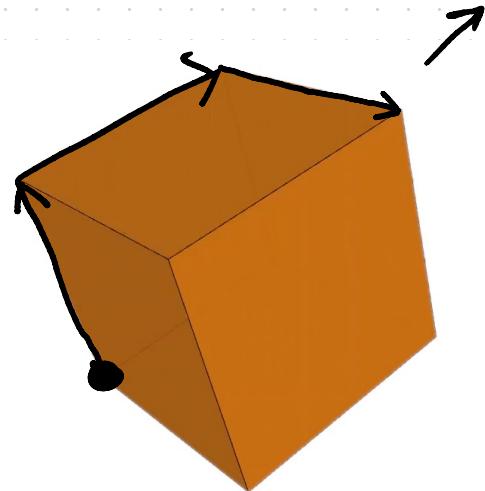
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how many pivot steps?
(assuming non-degeneracy)

In practice

The simplex method takes $2(n+d)$ pivot steps to solve an LP.

Worst-case complexity

Theorem The simplex method
has exponential worst-case
complexity*

*terms and conditions apply

Simplex method is
good in practice
bad in theory

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this is a question for science

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lattice polytopes

Is our theory any good?

At a conference

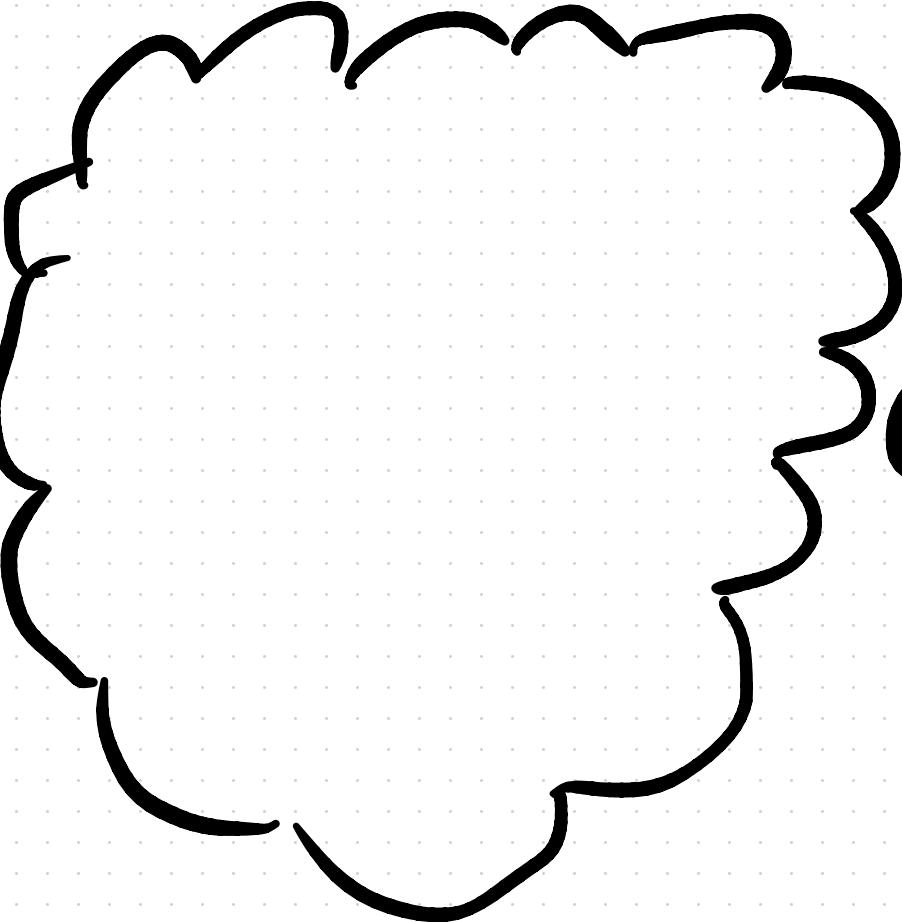


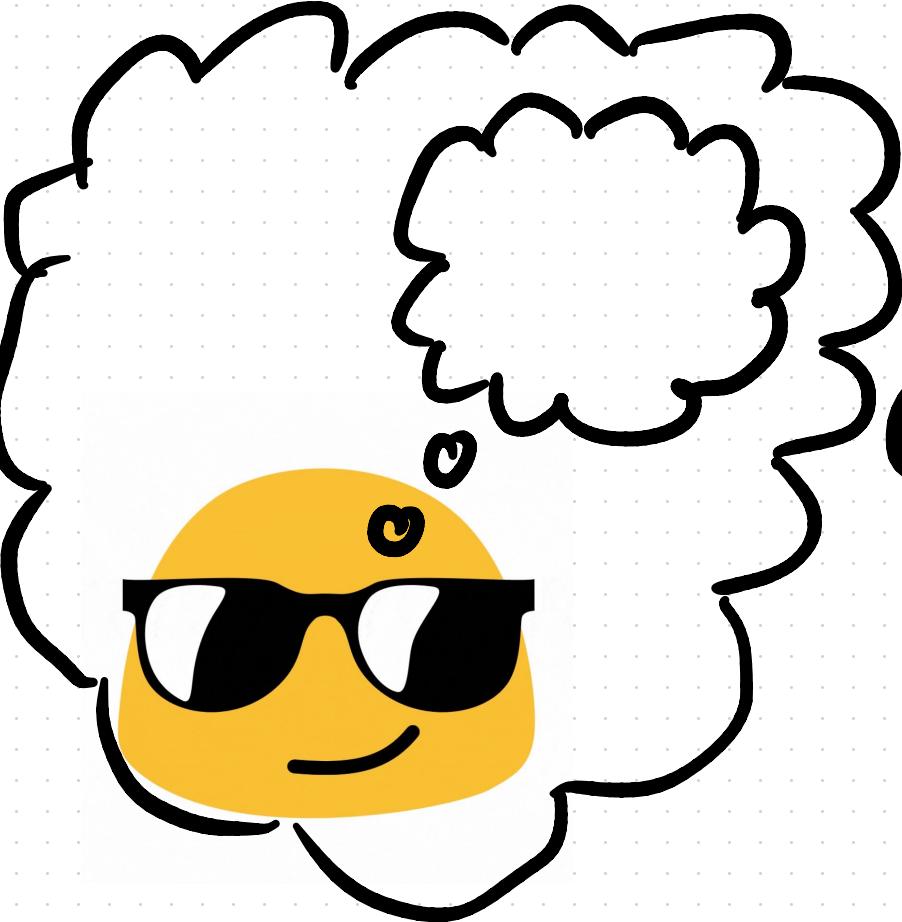
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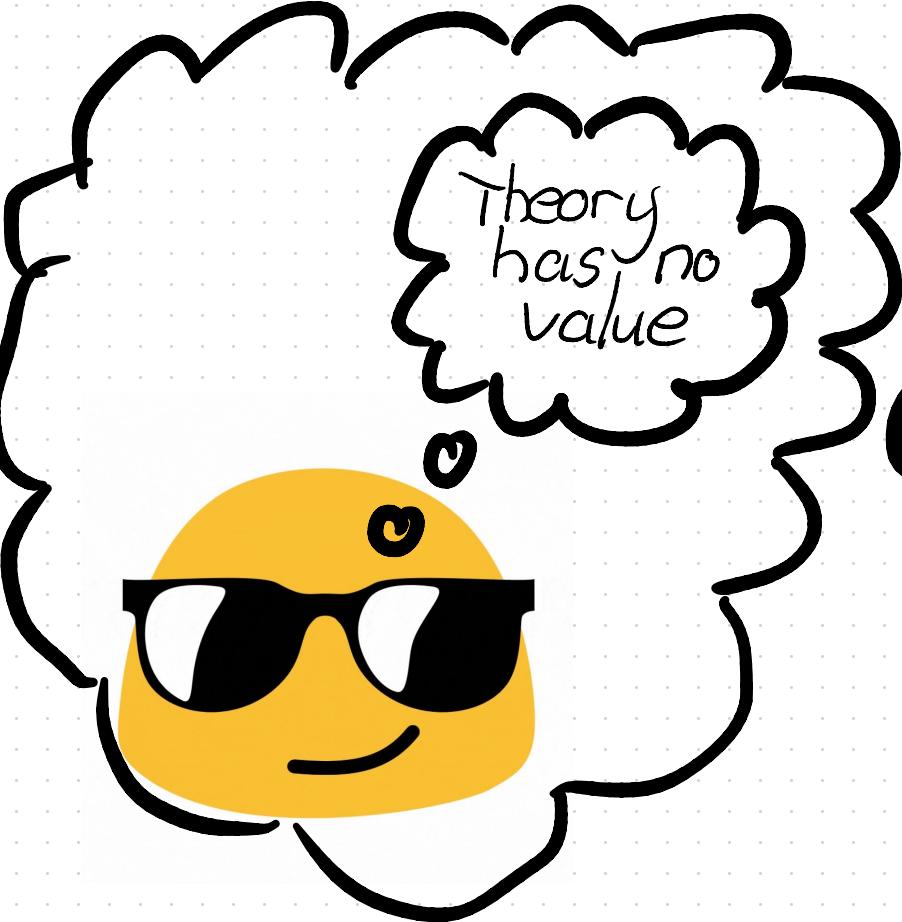


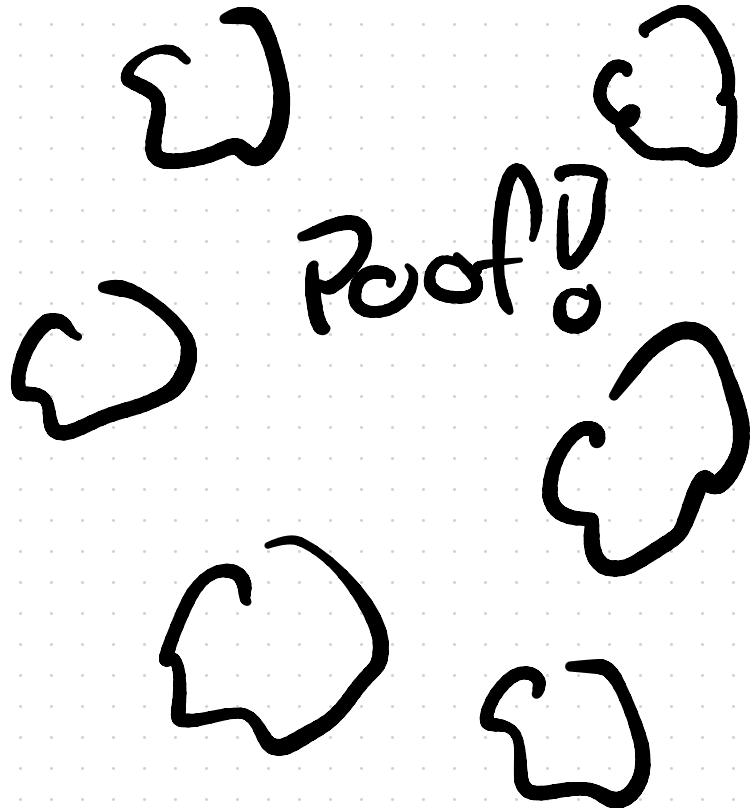
At a conference











Theory
has no
value? . . .



Does theory have any bearing on reality?

Two types of theories

→ Hardness measures

→ Distributional assumptions

↳ Ex: smoothed analysis

Smoothed analysis

Let $\bar{A} \in \mathbb{R}^{n \times d}$ have rows of norm ≤ 1 .

$\bar{b} \in [-1, 1]^n$, $c \in \mathbb{R}^d$

Let \hat{A}, \hat{b} have iid $N(0, \sigma^2)$ entries.

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Theorem

$$\max_{\bar{A}, \bar{b}, c} \mathbb{E}_{\hat{A}, \hat{b}} \left[\begin{array}{l} \text{time to solve} \\ \text{maximize } c^T x \\ \text{s.t. } (\bar{A} + \hat{A})x \leq \bar{b} + \hat{b} \end{array} \right] \leq \text{poly}(n, d, \sigma^{-1})$$

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↑
much effort spent to
get stronger bounds

Why smoothed analysis?

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bad inputs are “brittle” against
measurement/numerical errors

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interpolates between worst case
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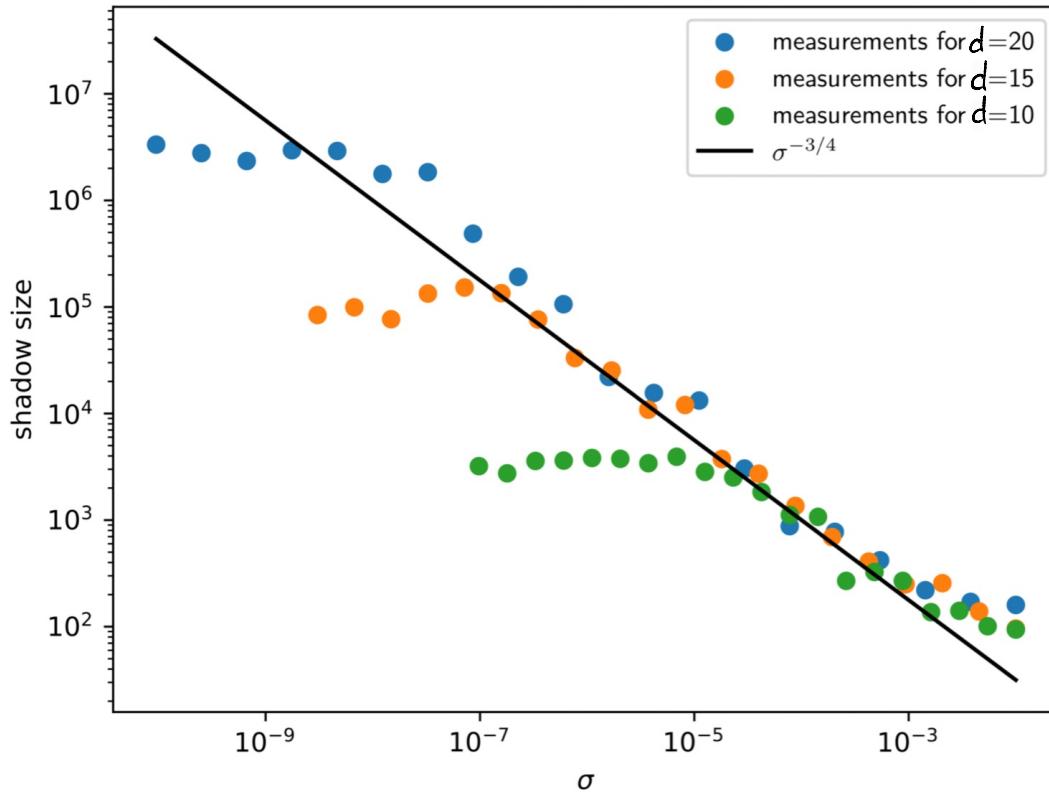
interpolates between worst case
and average case analysis.

algorithm is fast on average in
every large enough neighborhood

Can any of the resulting
insights be tested
experimentally?

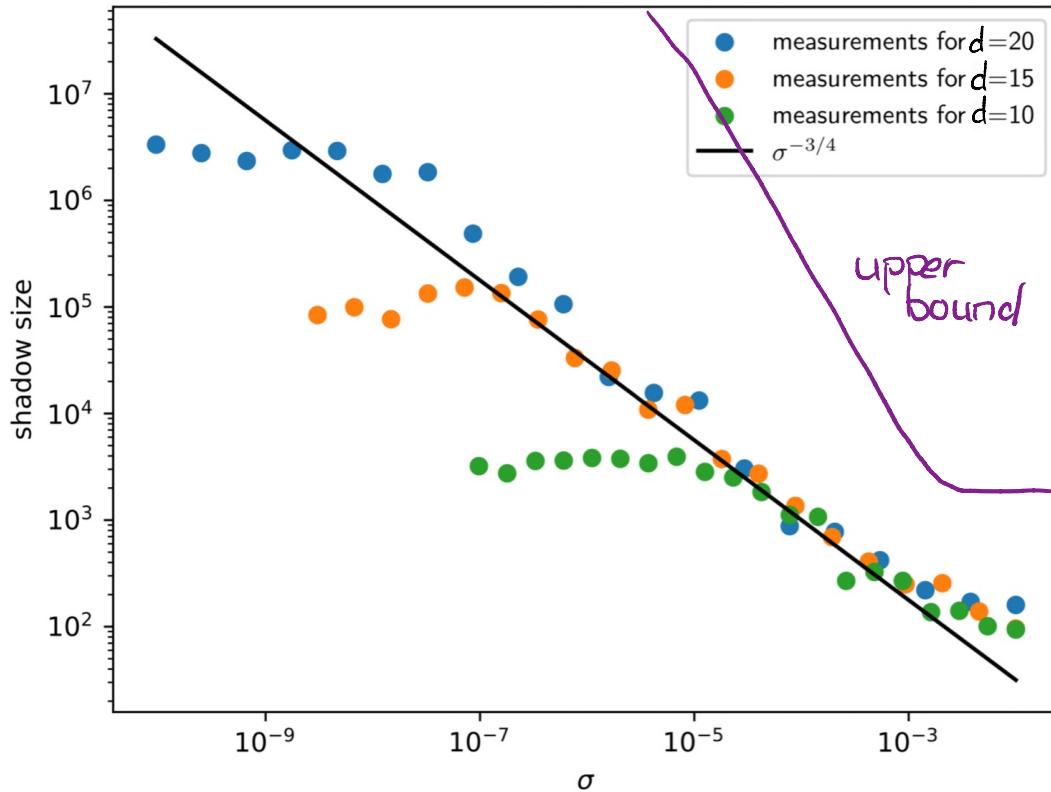
Synthetic data

Measured shadow sizes



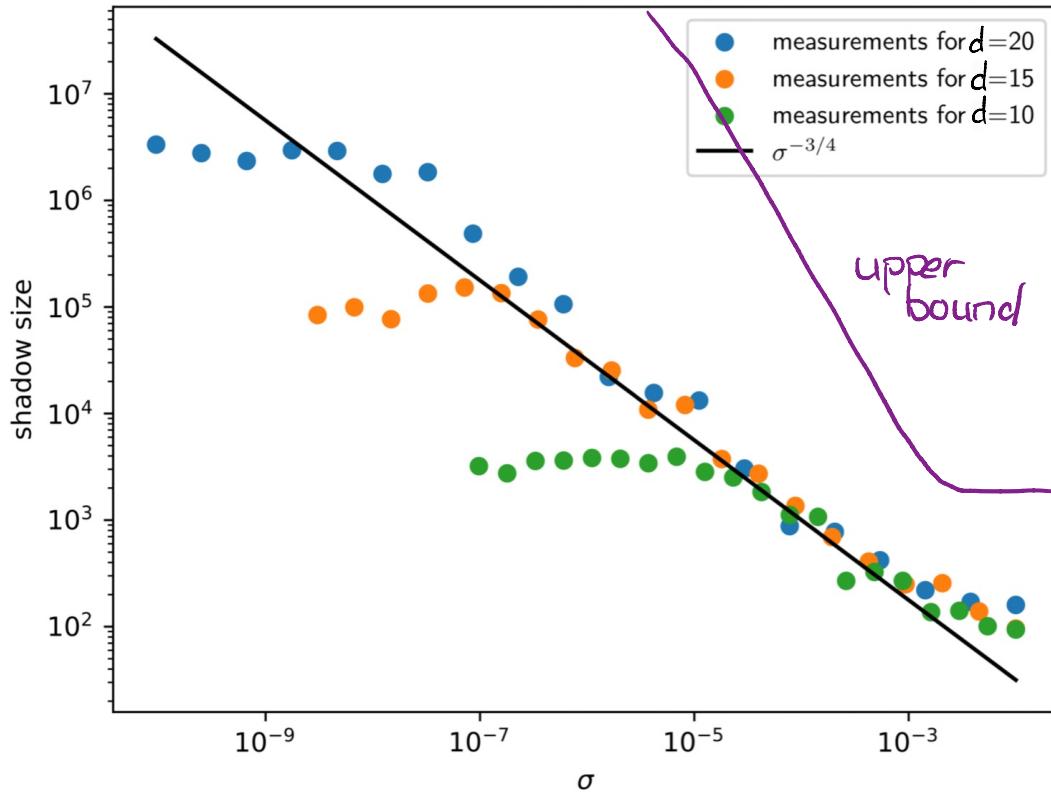
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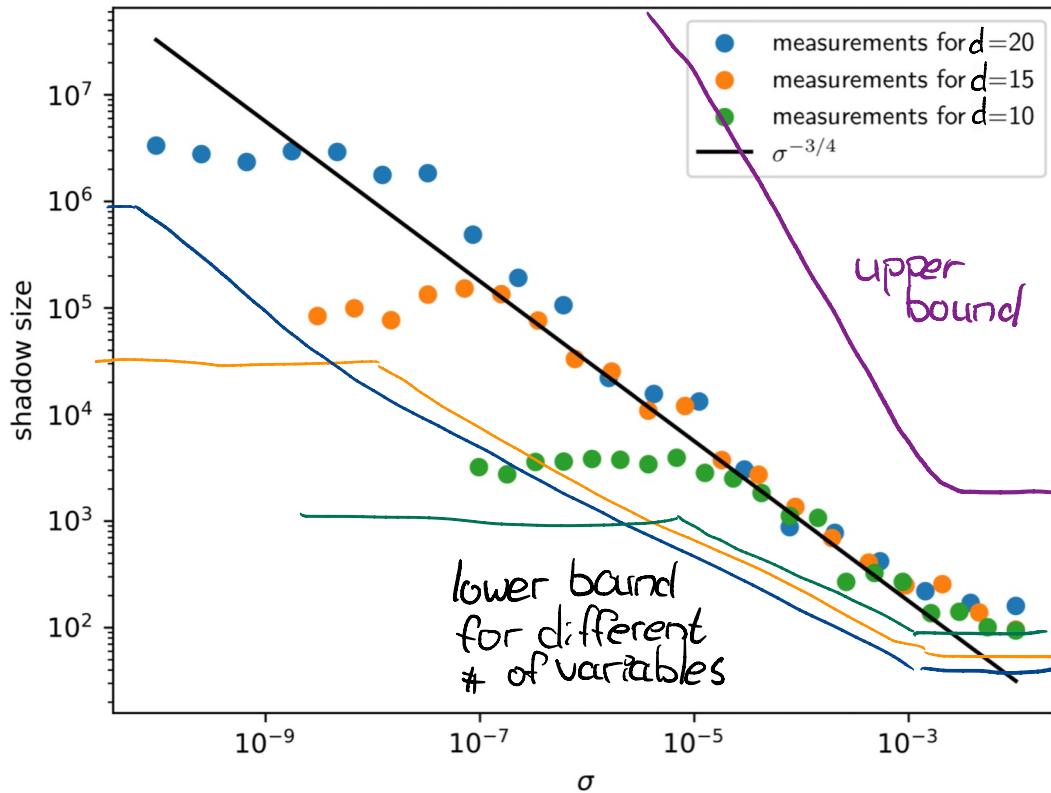
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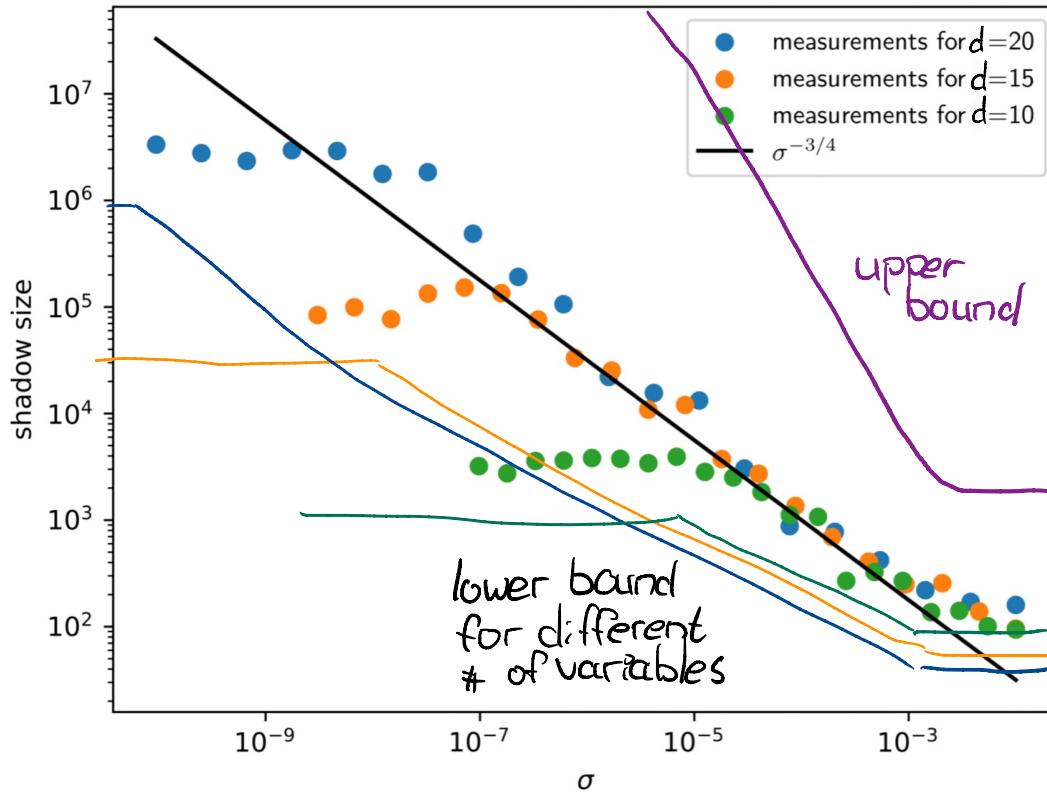
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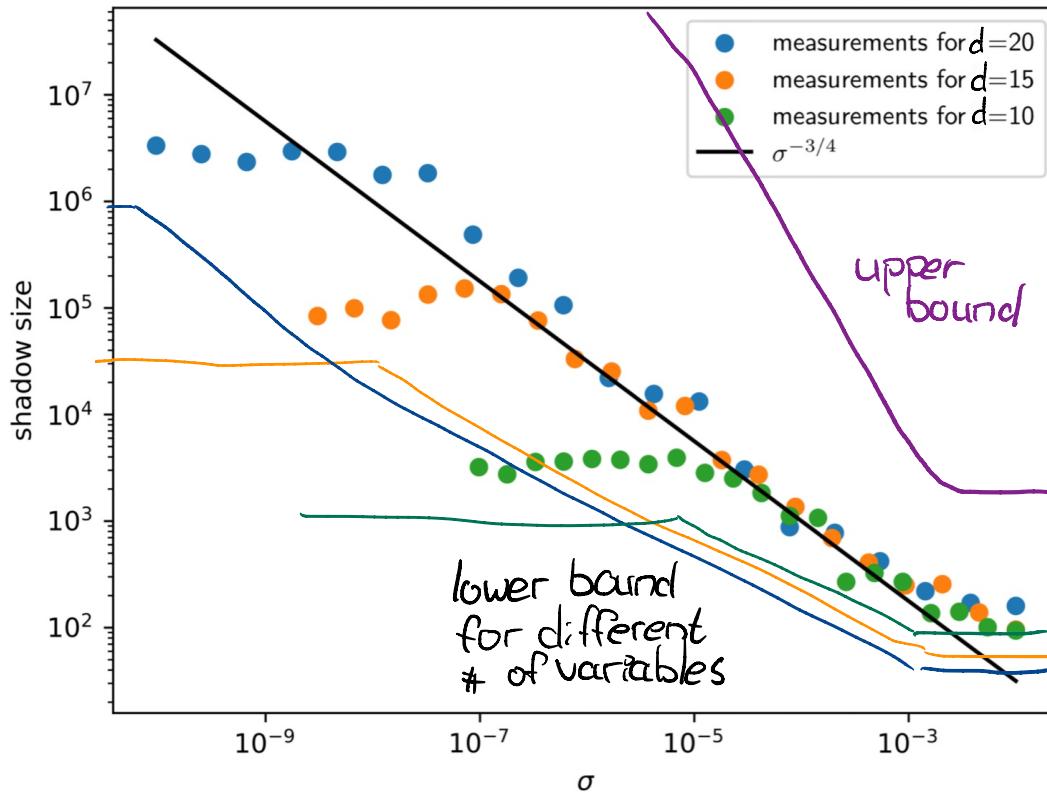
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CAN tell if theorem is tight

Synthetic data

Measured shadow sizes



CAN tell if theorem is tight

CAN NOT tell if theorem is useful

Do there exist
Gaussian distributed
linear programs
in real life?

The first linear program

Given 77 ingredients,

find the cheapest diet

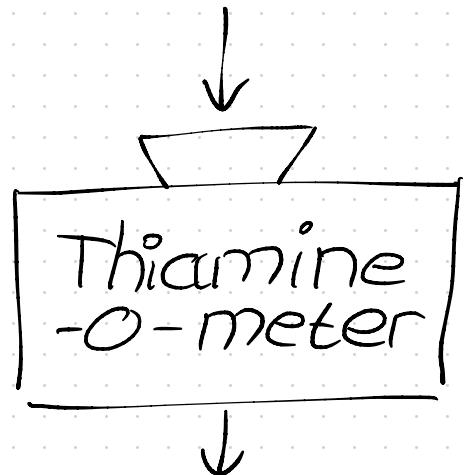
that meets all 9 nutrient needs

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noisy
measurement

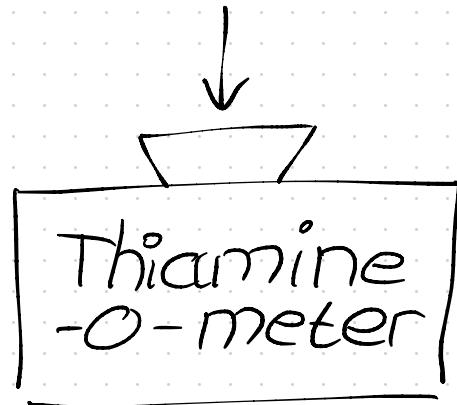
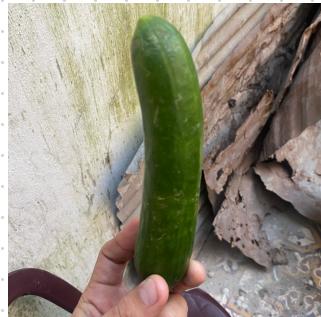
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is this normal to want
and possible to achieve?



noisy
measurement

George Stigler (1911 - 1991)

a quintessential conservative

opposed to rent-control
& price controls

ON THINKING ABOUT GEORGE STIGLER*

C. R. McCann, Jr. and Mark Perlman

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ON THINKING ABOUT GEORGE STIGLER*

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opposed to minimum wage

THE ECONOMICS OF MINIMUM WAGE LEGISLATION

By GEORGE J. STIGLER*

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One final point: We seek to abolish poverty in good part because it leads to undernourishment. In this connection, dietary appraisals show that in any income class, no matter how low, a portion of the families secure adequate diets, and in any income class, as high as the studies go, a

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THE COST OF SUBSISTENCE

GEORGE J. STIGLER
University of Minnesota

These low-cost diets of the professional dieticians thus cost about two or three times as much as a minimum cost diet.

Why do these conventional diets cost so much? The answer is

1945
introducing
diet problem

A calculation of

"minimal resource need"

is an argument for

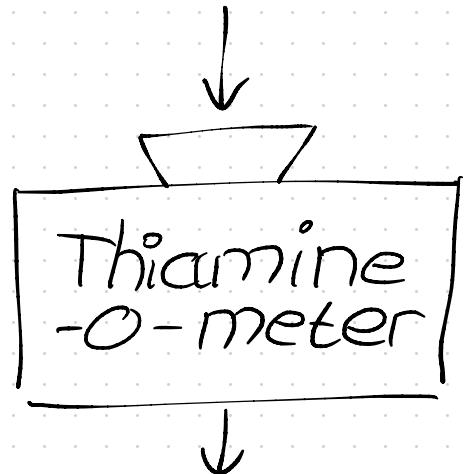
reducing resource availability

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noisy
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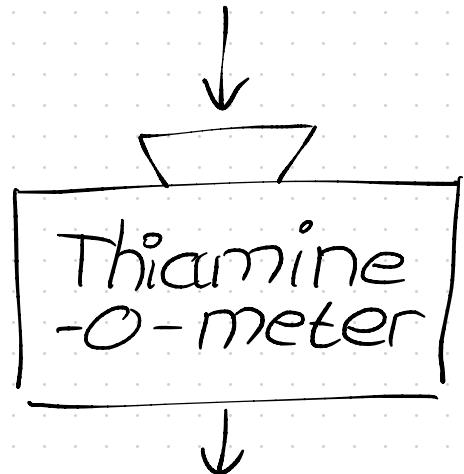
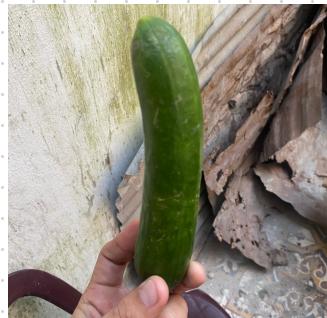
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Is this legal?



↓
noisy
measurement



**INTERNATIONAL COVENANT
ON ECONOMIC, SOCIAL
AND CULTURAL RIGHTS**

1967 human
rights treaty



INTERNATIONAL COVENANT
ON ECONOMIC, SOCIAL
AND CULTURAL RIGHTS



OFFICE OF THE HIGH COMMISSIONER
FOR HUMAN RIGHTS



CESCR General Comment No. 12: The Right to Adequate Food (Art. 11)

1967 human
rights treaty

1999 explanation
of these rights



INTERNATIONAL COVENANT ON ECONOMIC, SOCIAL AND CULTURAL RIGHTS



OFFICE OF THE HIGH COMMISSIONER
FOR HUMAN RIGHTS



CESCR General Comment No. 12: The Right to Adequate Food (Art. 11)

The right to adequate food shall therefore not be interpreted in a narrow or restrictive sense which equates it with a minimum package of calories, proteins and other specific nutrients.

1967 human
rights treaty

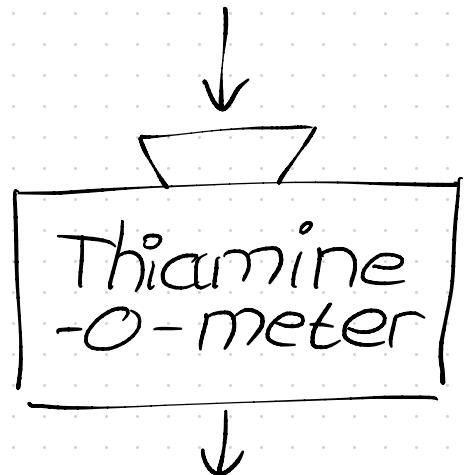
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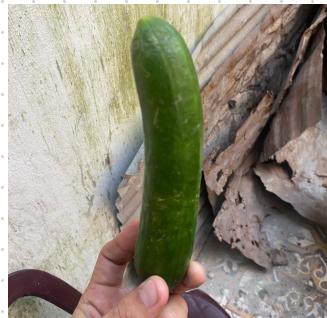
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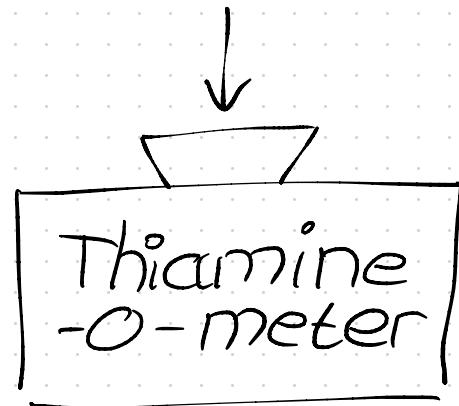
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What about more recent history?



↓
noisy
measurement

SLIDE 1

Ministry of Defense
Coordination of Government Activities in the Territories

Food Consumption in the Gaza Strip – Red Lines

1 January 2008

SLIDE 2

Goals of Analysis

- As part of the policy formulated by the Security Cabinet on September 19, 2007, Israel will limit the entry of goods into the Gaza Strip.
- In order to allow for a basic fabric of life in the Gaza Strip, the deputy defense minister approved allowing 106 trucks carrying basic humanitarian products into the Gaza Strip, mostly food (all products are specified in the appendices). In addition, food in seed form was approved for entry via the aggregate conveyor belt located near the Karni crossing.
- This research examines the main food component.
- The goal of the analysis – to identify the point of intervention for prevention of malnutrition in the Gaza Strip.
- The basis for the analysis is a model formulated by the Ministry of Health (at this point, according to average Israeli consumption) and a model formulated by the Palestinian Ministry of Economy.
- The Ministry of Health is conducting work for calculating the minimal subsistence basket based on the Arab sector in Israel. The “minimum basket” allows nutrition that is sufficient for subsistence without the development of malnutrition.

List of humanitarian products whose transfer into the Gaza Strip is permitted - May 30, 2010

<u>Basic humanitarian food products</u>	<u>Comments</u>
Flour, semolina, wheat and yeast	
Oil, pasta, rice, salt, sugar and saccharine	
Frozen meat and chicken products and fish	
Sausage, canned meat and fish	
Milk products, margarine, milk powder and baby food	
Legumes: broad beans, soy (grains, powder) garbanzo beans, lentils, peas, beans and lupine	
Grains: barley, corn (kernels/ground), oatmeal, sorghum	
Fruit: apples, pears, bananas, loquat, moist dates and avocado, apricots, plums, green almonds, kiwi, mango, pomegranate	Kerem Shalom has a daily quota of 22 agricultural trucks including fruit, vegetables and agricultural inputs
Fresh vegetables: carrots, garlic, pumpkin and onion, green leaves (coriander, dill, parsley, etc.)	Kerem Shalom has a daily quota of 22 agricultural trucks including fruit, vegetables and agricultural inputs
Frozen vegetables	
Processed garbanzo beans without additives or tahina	
Tea and coffee	
Halva, jam	
Basic canned goods	
Eggs for consumption	
Containers and bottles of mineral water	
Spices: black pepper, soup powder, za'atar, sesame, cinnamon, anise, chamomile, sage	
Permitted for donations and international organizations only	
Vitamin enriched biscuits and bottled water	
Tomato paste	

<u>Basic humanitarian products that are not food</u>	<u>Comments</u>
<u>Agricultural inputs</u>	Kerem Shalom has a daily quota of 22 agricultural trucks including fruit, vegetables and agricultural inputs
Breeding eggs, egg cartons and empty containers, sacks and feed bags	
Animal feed (mixture, hay), medications	During the period when the transfer of

list of permitted foods

translation by Gisha

SLIDE 6

General Daily Food Consumption in the Gaza Strip per Ministry of Health Scale (in tons)

Age/Type of food	Male/Female			Female			Male					Total for general population (minus 6- 12 month age bracket)	Food additive for -12 month age bracket)	Total quantity required for general population
	2-3	4-6	7-10	11-24	24-50	51+	11-14	15-18	19-24	24-50	51+			
Grains	11.94	37.15	40.43	63.94	53.52	14.65	25.66	25.71	25.71	68.33	15.23	382.28	3.98	386.26
Vegetables	12.62	37.00	40.52	60.03	50.25	14.64	24.64	24.68	24.68	65.61	14.85	369.53	4.21	373.74
Fruit	16.99	58.80	67.42	102.65	85.92	23.84	43.69	45.02	45.02	119.68	25.80	636.86	6.33	643.19
Milk	39.49	70.18	68.53	140.88	78.61	26.18	51.40	26.34	46.34	82.13	22.13	672.22	13.16	685.38
Meat	14.09	39.83	62.61	60.93	51.00	15.57	24.08	23.38	23.38	62.15	14.95	371.98	4.70	376.67
Oil	0.00	1.18	1.81	0.00	3.39	0.56	0.82	2.22	1.85	7.87	1.06	20.75	0.00	20.75
Sugar	4.35	5.04	5.58	4.95	5.27	2.01	2.87	5.18	4.07	12.78	1.85	53.95	1.45	55.40

- The figures are in tons per calendar day (consumption over seven days per week, unlike supply which is calculated based on five days per week).
- The portion of consumption is measured by the Health Ministry in Israel and provides for 2,000-2,500 calories per adult and 1,550 calories per child.
- The quantities in this table are average consumption according to Israeli standards and are not minimal subsistence portions.
- The Ministry of Health has been requested to calculate the minimal subsistence basket according to the Arab sector in Israel. The “minimal basket” allows for nutrition that is sufficient for subsistence without the development of malnutrition.

SLIDE 13

Additives in Wheat

Number	Added Vitamin/Mineral	Quantity	
1	Thiamine (Vitamin B1)	4.4	Milligram per Kilogram
2	Vitamin B2	2.6	Milligram per Kilogram
3	Niacin	35	Milligram per Kilogram
4	Folic Acid	0.4	Milligram per Kilogram
5	Iron	25	Milligram per Kilogram
6	Folato	1	Milligram per Kilogram
7	Vitamin B6	2.5	Milligram per Kilogram
8	Zinc	15	Milligram per Kilogram
9	Vitamin A	1	Milligram per Kilogram
10	Vitamin B3	0.02	Milligram per Kilogram

SLIDE 14

Summary and Conclusions

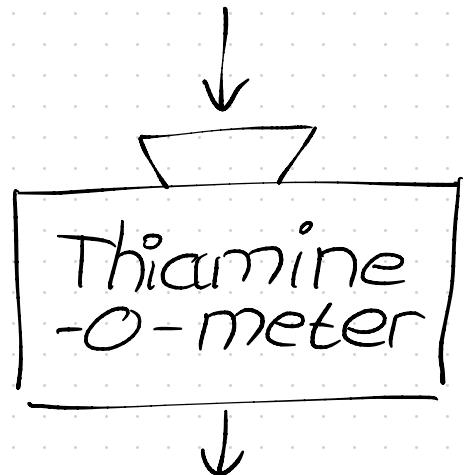
- According to the model supplied by the Israeli Ministry of Health, there is a need for a daily supply of 104 food trucks (5 days a week).
- The model takes into account an exaggerated consumption of milk (3 times the known consumption in the Gaza Strip). Thus, on decreasing the milk component, the working assumption of 106 trucks (+ Karni conveyor belt) which includes about 90 truckloads of basic food, certainly meets nutritional needs in the Gaza Strip.
- The Ministry of Health Model assumes lower consumption of flour than what is known to be in effect.
- The Ministry of Health model is based on the average Israeli consumption, rather than a minimalist basket according to consumption habits in the Arab sector (the Ministry of Health is currently analyzing this).
- Following receipt of the new basket, it will be possible to define a red line as a warning sign.
- The Ministry of Health estimates that the new basket will be 20% lower than the current basket.

The first linear program

Given 77 ingredients,

find the cheapest diet

that meets all 9 nutrient needs



noisy
measurement

The first linear program

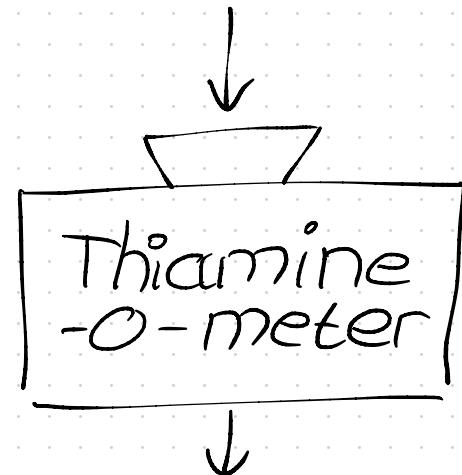
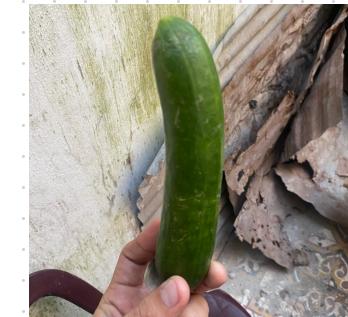
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Type of Operations	No. of repetitions
Multiplication	15,315
Division	1,234
Addition of two numbers	14,561
Addition of 77 numbers	190
Addition of 9 numbers	85

1948 ↪



noisy
measurement

Mathematical Tables Project 1938 - 1948

450 computers employed



Mathematical Tables Project 1938 - 1948



450 computers employed

To perform these computations with desk machines required 5 computers for 21 days, with 4 hours per day supervision by a mathematician.

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- Handbook of Mathematical Functions 1964
- LP's solved by hand until 1952

Linear Programming and Extensions

George B. Dantzig

Dantzig's
famous
book

STIGLER'S NUTRITION MODEL: AN EXAMPLE OF FORMULATION AND SOLUTION

One of the first applications of the simplex algorithm was to the determination of an adequate diet that was of least cost.¹ In the fall of 1947, J. Laderman of the Mathematical Tables Project of the National Bureau of Standards undertook, as a test of the newly proposed simplex method, the first large-scale computation in this field. It was a system with nine equations in seventy-seven unknowns. Using hand-operated desk calculators, approximately 120 man-days were required to obtain a solution.

The particular problem solved was one which had been studied earlier by G. J. Stigler [1945-1], who had proposed a solution based on the substitution of certain foods by others which gave more nutrition per dollar. He then examined a "handful" of the possible 510 ways to combine the selected foods. He did not claim the solution to be the cheapest, but gave good

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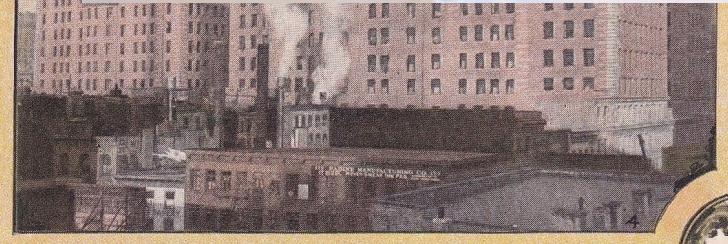
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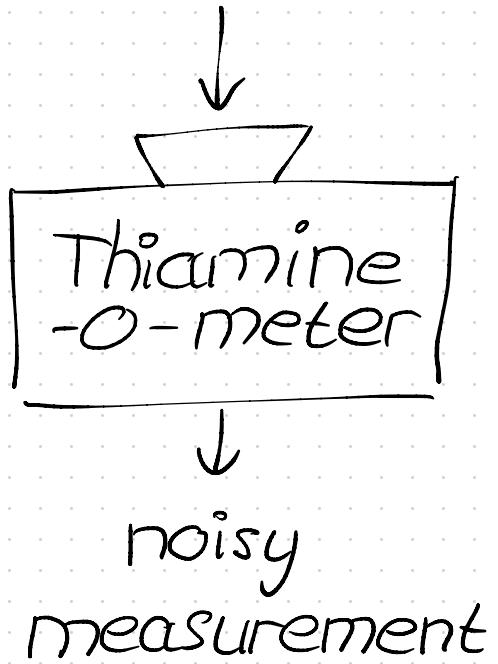
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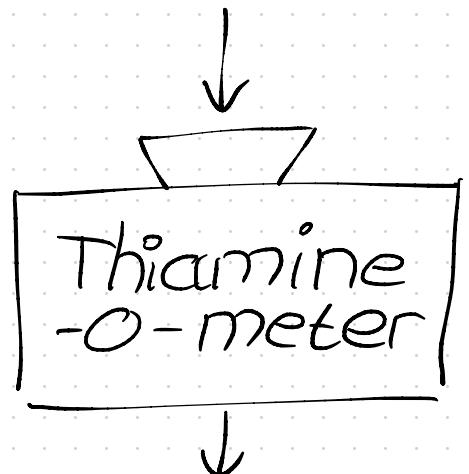
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38. Mathematical Tables Project computers with adding machines



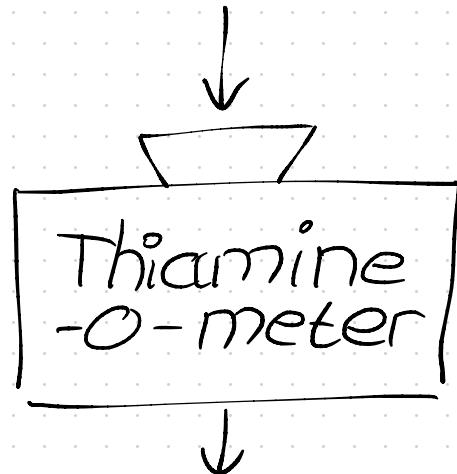
Could smoothed analysis
have predicted that
the first computation
be fast?



noisy
measurement

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Probably not.

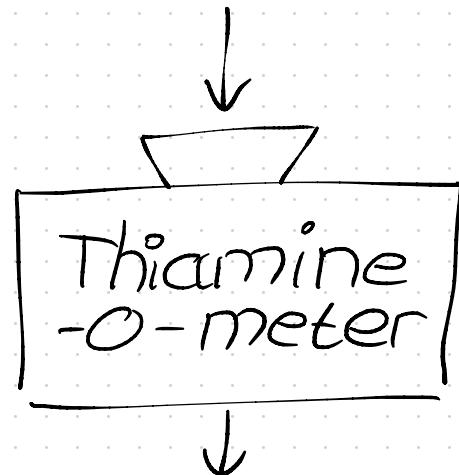


noisy
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1. different pivot rule

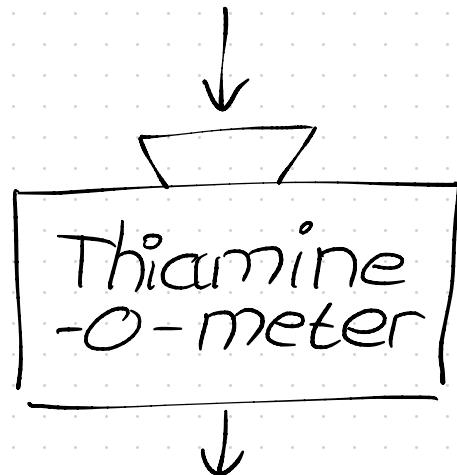


noisy
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1. different pivot rule
2. different phase 1.

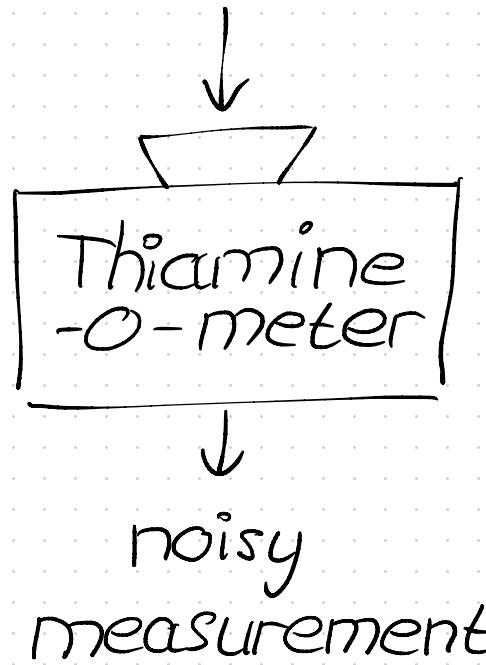


noisy
measurement

Could smoothed analysis
have predicted that
the first computation
be fast?

Probably not.

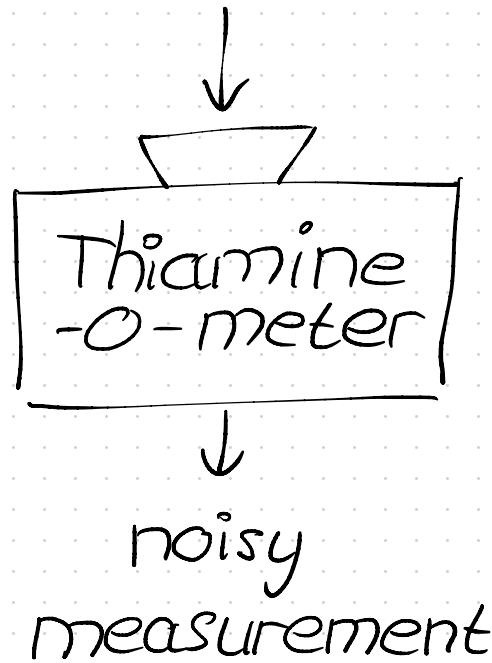
1. different pivot rule
2. different phase 1
3. non-negativity constraints



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1. different pivot rule
2. different phase 1
3. non-negativity constraints
4. multiplicative error $\geq 15\%$,
but need additive error



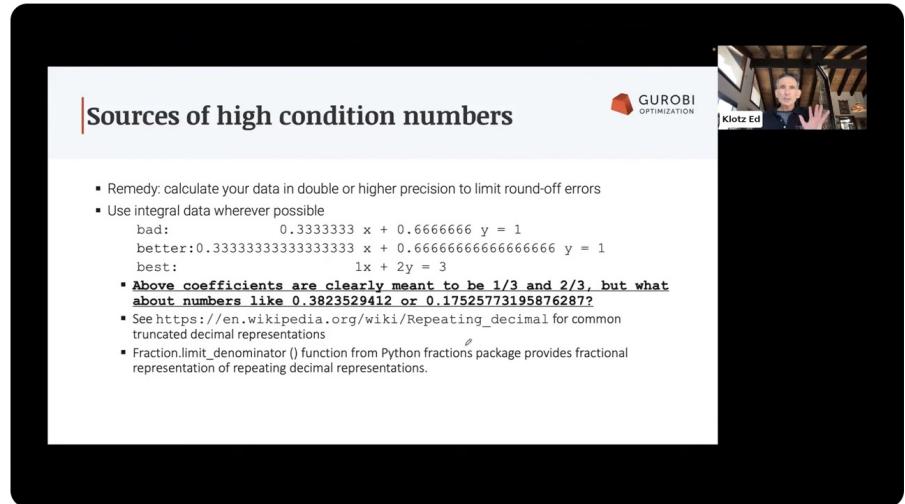
A more fundamental issue?

Smoothed analysis is based on the notion
"more noise is better"

A more fundamental issue?

Smoothed analysis is based on the notion
"more noise is better"

Practitioners say
"less noise is better"



Sources of high condition numbers

GUROBI
OPTIMIZATION

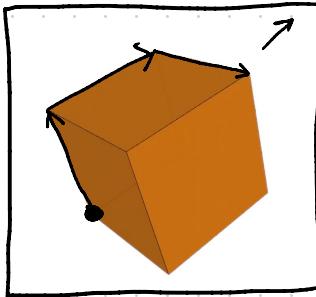
Klotz Ed

- Remedy: calculate your data in double or higher precision to limit round-off errors
- Use integral data wherever possible
 - bad: $0.3333333 x + 0.6666666 y = 1$
 - better: $0.333333333333333 x + 0.666666666666666 y = 1$
 - best: $1x + 2y = 3$
- Above coefficients are clearly meant to be 1/3 and 2/3, but what about numbers like 0.3823529412 or 0.175257731958762877?
- See https://en.wikipedia.org/wiki/Repeating_decimal for common truncated decimal representations
- Fraction.limit_denominator () function from Python fractions package provides fractional representation of repeating decimal representations.

Are we studying the simplex method?

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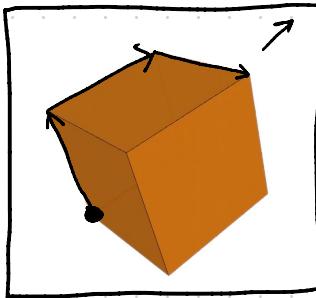
This geometry



doesn't exist.

Are we studying the simplex method?

This geometry

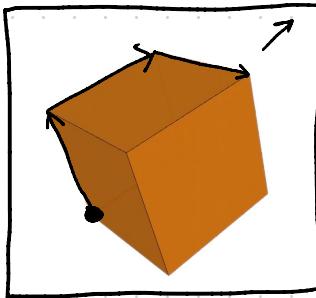


doesn't exist.

What does exist : { linear algebra

Are we studying the simplex method?

This geometry



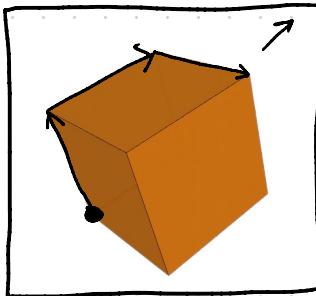
doesn't exist.

What does exist :

{ linear algebra
bound shifting

Are we studying the simplex method?

This geometry



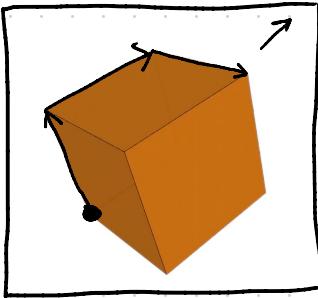
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What does exist :

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doesn't exist.

What does exist:

- { linear algebra
- bound shifting
- bound perturbations
- Harris ratio test



Summary:

Our theory are untested
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Many directions for future work