Asymptotic bounds for the combinatorial diameter of random polytopes

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joint work with Gilles Bonnet, Daniel Dadush, Uri Grupel, Galyna Livshyts

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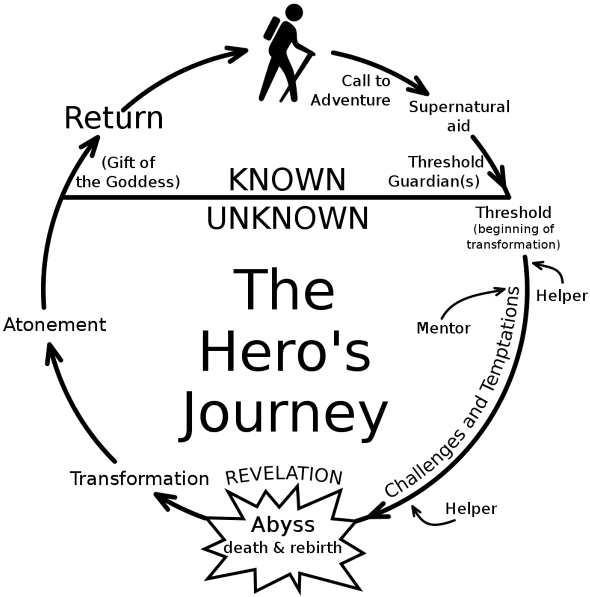
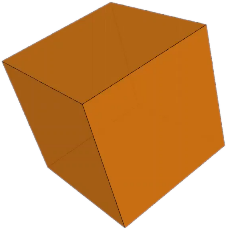
Outline



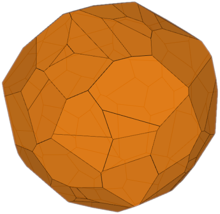
Call to Adventure
Supernatural aid
Threshold Guardian(s)
Threshold
(beginning of transformation)
Helper
Mentor
Challenges and Temptations
Helper
REVELATION
Abyss
death & rebirth
Transformation
Atonement
Return
(Gift of the Goddess)
KNOWN
UNKNOWN
The Hero's Journey

Dramatis Personae

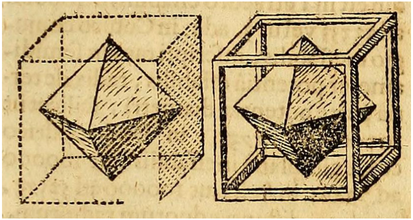
Diameter



Random polytopes

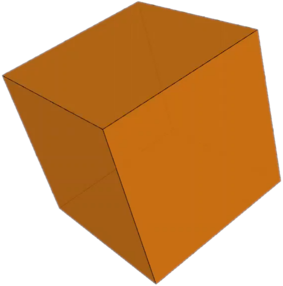


Polar duality



Diameter: a Question About Geometry

Given a polyhedron, how many edges do we need to traverse to go from any vertex to any other?



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- m facets

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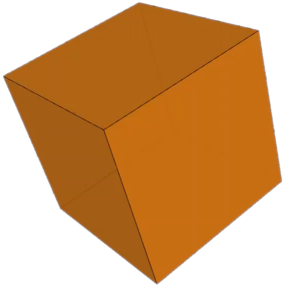

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Given a vertex of the set of points x satisfying

$$Ax \leq b,$$

how many steps do you need to find a vertex maximizing $c^{\mathsf{T}}x$?

- n variables
- m constraints

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- polytopes in $\mathbb{R}^3$
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- fractional stable set polytopes
- polytopes with vertices in $\{0, 1\}^n$
- and many more

But false in general.

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B69, L70, B74:

$$\text{Diameter(P)} \leq 2^{n-2} m.$$

KK92, T14, S19:

$$\text{Diameter(P)} \leq (m - n)^{\log_2 O(n/\log n)}.$$

Polar Duality

Given a convex set $P \subset \mathbb{R}^n$, define

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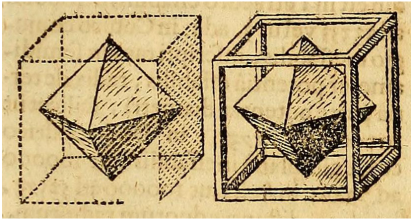
If that P is a bounded polytope and contains the origin in its interior, then:

- P° is a bounded polytope and contains the origin in its interior
- $(P^\circ)^\circ = P$
- if P is simple then P° is simplicial
- vertices of P correspond to facets of P°



Random Models

Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \ \forall a \in A\}, \qquad Q = \mathrm{conv}(A),$$

where $A \subset \mathbb{S}^{n-1}$, $|A| \sim \text{Poison}(m)$ is sampled iid.

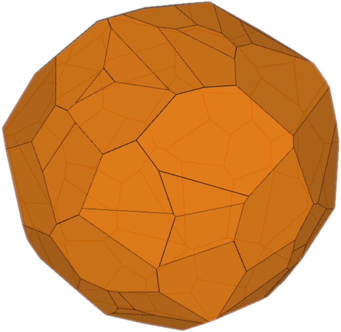


Figure: $P \subset \mathbb{R}^3$ might look like this

Call to Adventure

DF94, BdSEHN14, DH16, NSS22:

If $S = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Z}^{n \times m}$ and every absolute square subdeterminant of A is at most Δ then

$$\text{Diameter(S)} \leq O(n^3 \Delta^2 \log(\Delta)).$$

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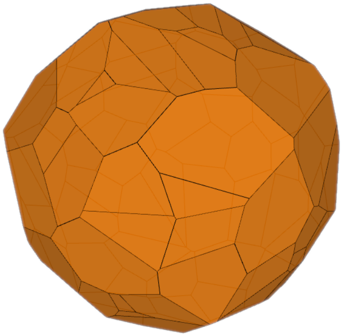

Borgwardt 1977,1982,1987,1999:
consider the projection π onto a fixed two-dimensional subspace $W \subset \mathbb{R}^n$. In expectation, the random polygon $\pi(P)$ has $\Theta(n^2 m^{\frac{1}{n-1}})$ vertices.

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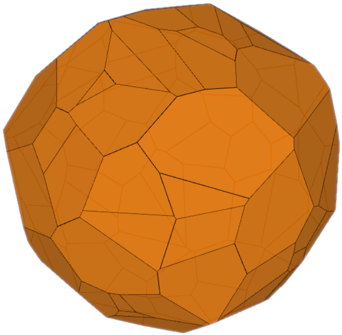

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In particular, any two vertices that project onto the boundary of $\pi(P)$ are connected by a path along the boundary, so of length $O(n^2 m^{\frac{1}{n-1}})$.

Quest: Prove the Following Results

When $A \subset \mathbb{S}^{n-1}$ follows a Poisson point process with $\mathbb{E}[|A|] = m > 2^{\Omega(n)}$, then for

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \ \ \forall a \in A\}, \qquad Q = \mathrm{conv}(A),$$

with probability $1 - O(1/m)$ we have

$$\Omega(nm^{\frac{1}{n-1}}) \leq \mathrm{diam}(P) \leq O(n^2 m^{\frac{1}{n-1}} + 17^n)$$
$$\Omega(m^{\frac{1}{n-1}}) \leq \mathrm{diam}(Q) \leq O(nm^{\frac{1}{n-1}} + 17^n).$$

Mentor: Relating the Diameters

Let $P \subset \mathbb{R}^n$ be a bounded simple polytope containing the origin in its interior. Let $P^\circ = \{x \in \mathbb{R}^n : y^\mathsf{T} x \leq 1 \ \ \forall y \in P\}$.

Lemma: We have $\mathrm{diam}(P) \geq (n-1)\mathrm{diam}(P^\circ) - 2$.

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Proof: consider two vertices of P°, which correspond to two facets F, F' of P. Pick a path $v_1, v_2, \ldots, v_k \in P$ of vertices, with length $k - 1 \leq \mathrm{diam}(P)$ and $v_1 \in F, v_k \in F'$.

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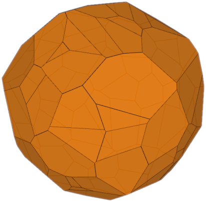
Construct a sequence $F_1, \ldots, F_p$ of facets of P, each consecutive pair having non-empty intersection, such that $v_1 \in F_1, v_k \in F_p$ and $v_2, \ldots, v_{k-1} \in F_1 \cup \cdots \cup F_p$. We can get $p \leq (k-1)/(n-1)$.



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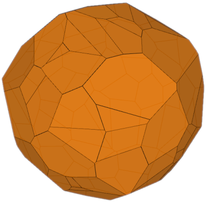

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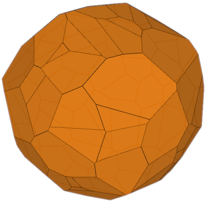

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Hence the vertices of P° corresponding to $F, F_1, \ldots, F_p, F'$ form a path of length at most

$$p + 2 \leq (k-1)/(n-1) + 2 \leq \mathrm{diam}(P)/(n-1) + 2.$$

Revelation: A Simple Lower Bound

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With high probability Diameter(Q) $\geq 1/2\varepsilon = \Omega(m/\log m)^{\frac{1}{n-1}}$.

Transformation: Sketch of the Upper Bound

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- Taking a union bound over many different projections π, this can be extended to a diameter bound.

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- Do the rest of the proof.

Return

- Close the gap between upper and lower bound.
- Other distributions, such as Gaussian?
- What if $m \leq \mathrm{poly}(n)$?