Smoothed analysis of the Simplex method

Sophie Huiberts

Joint work with Yin Tat Lee and Xinzhi Zhang

Once upon a time...

maximize $c^T x$

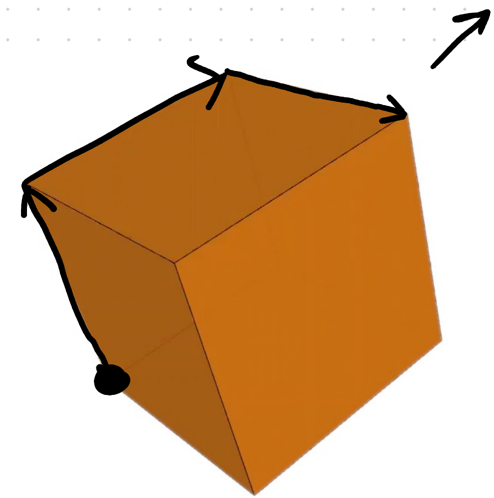
subject to $Ax \leq b$

we get $A \in \mathbb{R}^{n \times d}$

$b \in \mathbb{R}^n$

$c \in \mathbb{R}^d$

we compute $x \in \mathbb{R}^d$



Different simplex methods

- most negative reduced cost
- greatest improvement
- steepest edge

- approximate steepest edge

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Different simplex methods

- most negative reduced cost
- greatest improvement
- steepest edge
- approximate steepest edge
- whatever they do in real software
- shadow vertex rule (nice in theory)

LP History

75th anniversary celebration

The first large LP

Mathematical Tables Project

- 450 computers employed
- 1938 – 1948

Grier '05

The first large LP

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You will recall that 77 foods and 9 nutrient elements were involved in this problem. The number of operations by type are as follows:

Type of Operations	No. of repetitions
Multiplication	15,315
Division	1,234
Addition of two numbers	14,561
Addition of 77 numbers	190
Addition of 9 numbers	85

To perform these computations with desk machines required 5 computers for 21 days, with 4 hours per day supervision by a mathematician.

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- Handbook of Mathematical Functions (1964)

Grier '05

Linear Programming and Extensions

George B. Dantzig

STIGLER'S NUTRITION MODEL: AN EXAMPLE OF FORMULATION AND SOLUTION

One of the first applications of the simplex algorithm was to the determination of an adequate diet that was of least cost.[1] In the fall of 1947, J. Laderman of the Mathematical Tables Project of the National Bureau of Standards undertook, as a test of the newly proposed simplex method, the first large-scale computation in this field. It was a system with nine equations in seventy-seven unknowns. Using hand-operated desk calculators, approximately 120 man-days were required to obtain a solution.

The particular problem solved was one which had been studied earlier by G. J. Stigler [1945-1], who had proposed a solution based on the substitution of certain foods by others which gave more nutrition per dollar. He then examined a "handful" of the possible 510 ways to combine the selected foods. He did not claim the solution to be the cheapest but gave good

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Historical Takeaways

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- human computers played an important role in early comb. opt. history.

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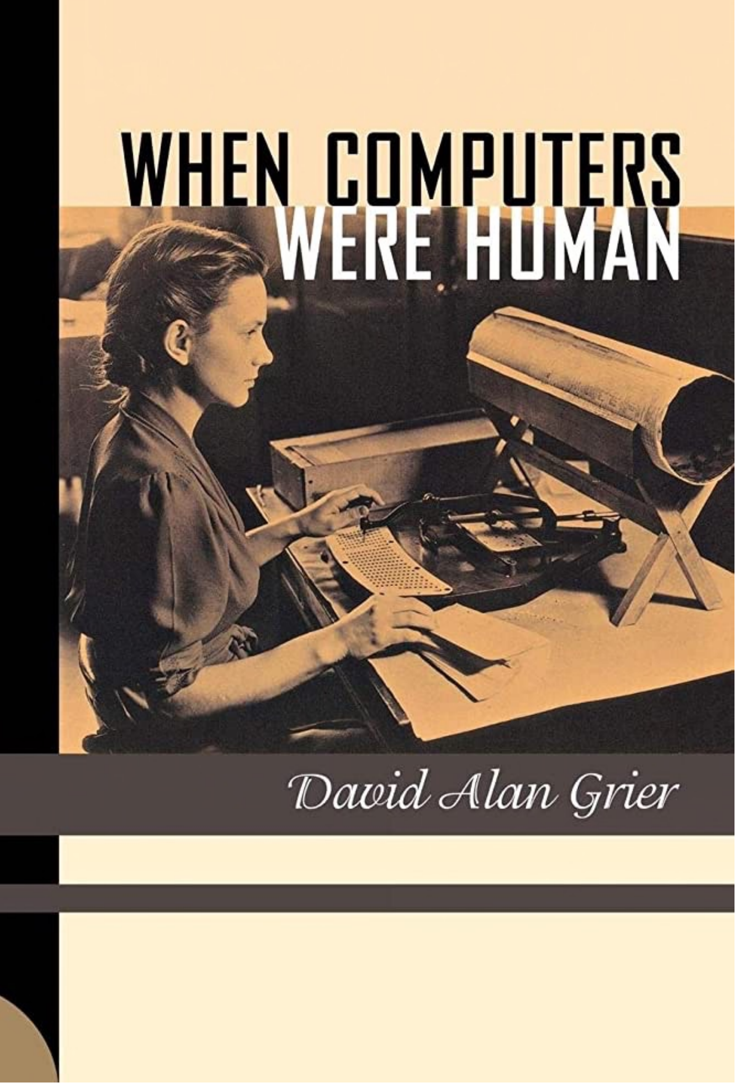
Historical Takeaways

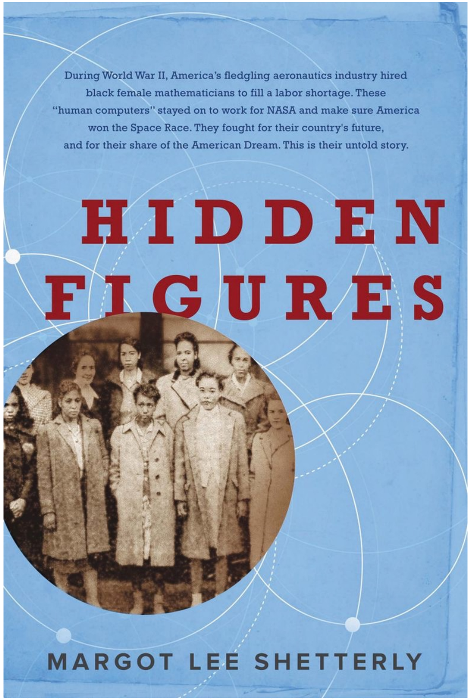
Professional Ambition • 243

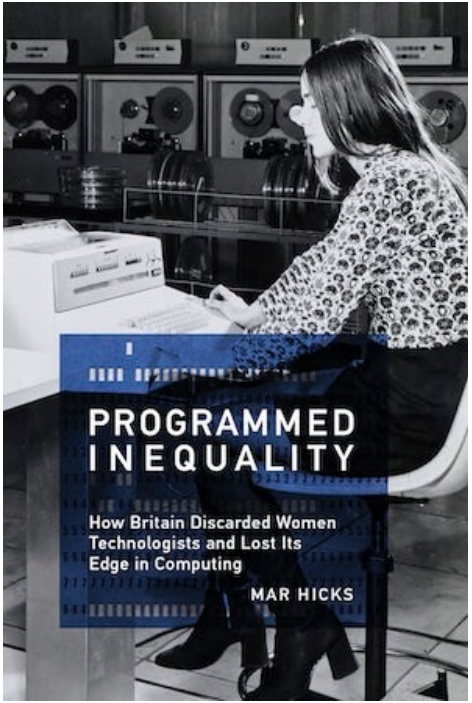
38. Mathematical Tables Project computers with adding machines

- human computers played an important role in early comb. opt. history.
- their contributions were made invisible by contemporary white men
- their demographics were exactly those underrepresented in our field, then & now



Consider including this history in your lectures





Every day...

The simplex method visits $\sim 2(n+d)$ vertices before reaching an optimal one

Only few documented cases where $> 10(n+d)$ were needed

But one day

Theorem There exist A, b, c, x^0 with $n = 2d$ such that the simplex method visits 2^d vertices

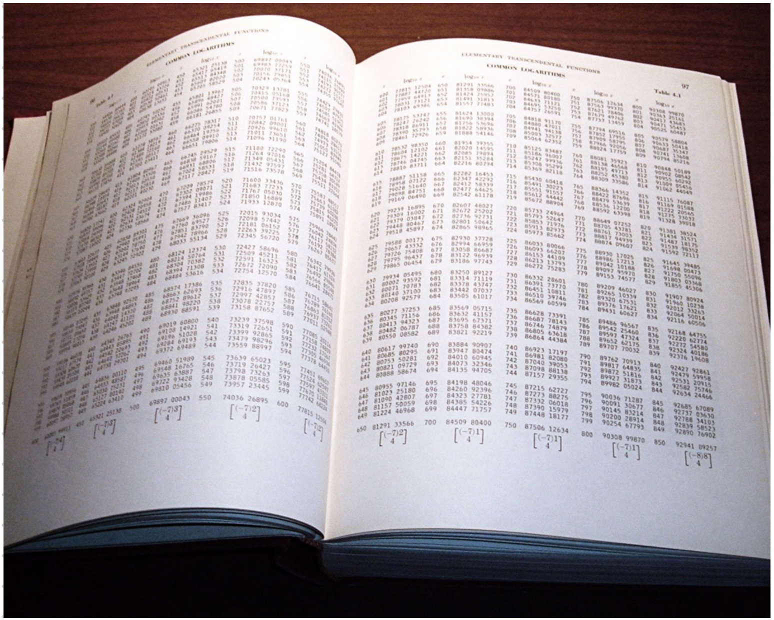
Klee Minty '72
Amenta Ziegler '98

Because of that

Theorem if the rows of A are iid from a rotationally symmetric distribution, and $b = 1$ then the simplex method visits $O(d^2 n^{\frac{1}{d-1}})$ vertices in expectation.

Borgwardt '87

Yes, but


ELEMENTARY TRANSCENDENTAL FUNCTIONS
COMMON LOGARITHMS
Table 4.1

Smoothed complexity

Let $\bar{A} \in \mathbb{R}^{n \times d}$ have rows of norm at most 1,

$\bar{b} \in [-1, 1]^n$, $c \in \mathbb{R}^d$

Let $\hat{A}$, $\hat{b}$ have iid $N(0, \sigma^2)$ entries.

The smoothed complexity is $\max_{\bar{A}, \bar{b}, c} \mathbb{E}_{\hat{A}, \hat{b}} [T(\bar{A} + \hat{A}, \bar{b} + \hat{b}, c)]$

where $T(A, b, c)$ is the time to solve $\max c^T x$ s.t. $Ax \leq b$

Spielman Teng '04

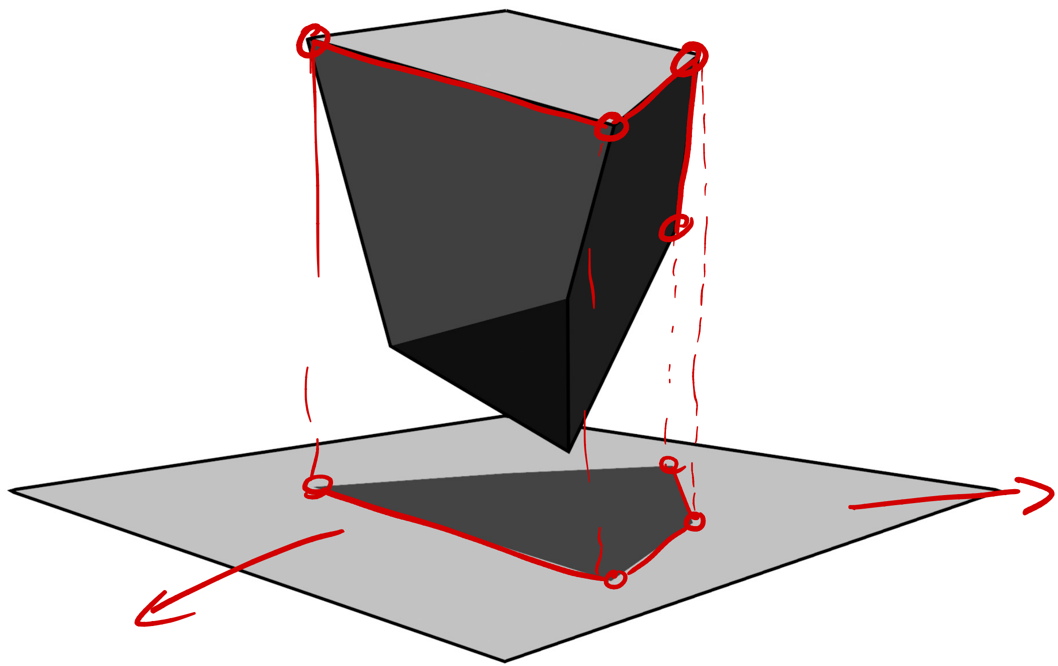
Why smoothed analysis

- in any large enough neighborhood, simplex is fast on most instances

- if worst-case analysis is adversarial, then smoothed analysis weakens this adversary.

- independent measurement/numerical errors do not conspire against your algorithm

Shadow vertex simplex method



Shadow simplex method

Theorem To bound the running time of the simplex method, it suffices to consider projections of polyhedra

$$\pi_W\left(\left\{ x : \quad Ax \le b \qquad \right\}\right)$$

where W is the worst case 2d subspace, and count the number of vertices.

Borgwardt '87
Spielman Teng '04
Vershynin '09
Dadush Huiberts '17

Clever auxilliary LP's

Theorem it suffices to consider projections of the form

$$\Pi_W\left(\{x : (\bar{A} + \hat{A})x \le 1\}\right)$$

where W is a fixed 2d subspace independent of $\hat{A}$.

Borgwardt '87
Spielman Teng '04
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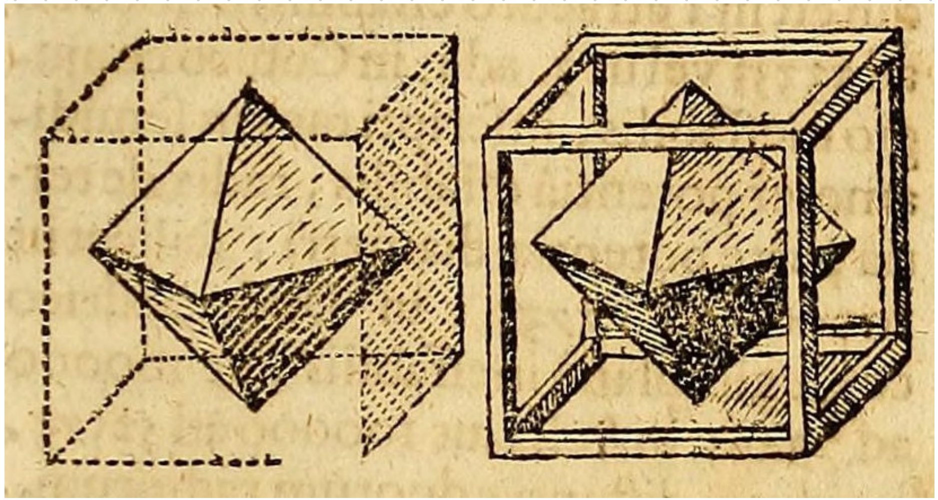
Polar duality

$S \subseteq \mathbb{R}^d \qquad S^\circ = \{x \in \mathbb{R}^d : y^T x \leq 1 \ \forall y \in S\}$



convex body $S \ni 0$ $\qquad$ $S^\circ \ni 0$ is convex body

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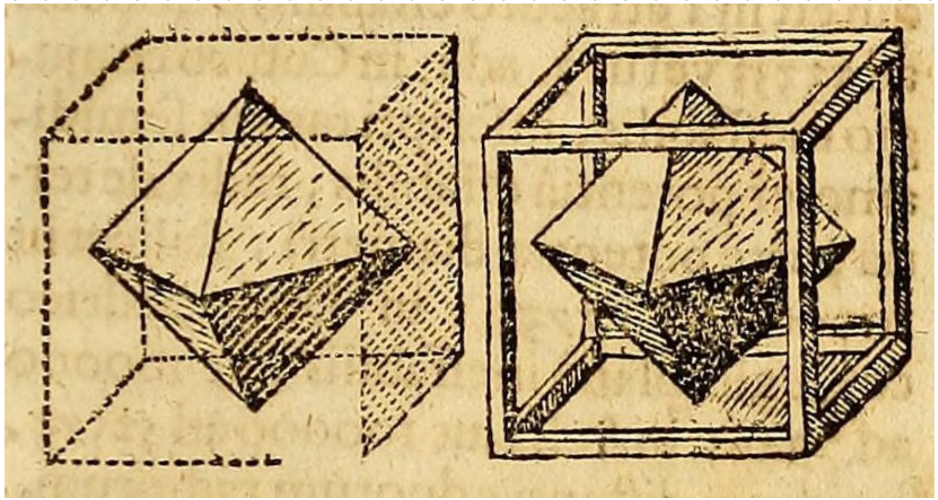

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k-dimensional faces $\qquad$ $d-k$ - dimensional faces

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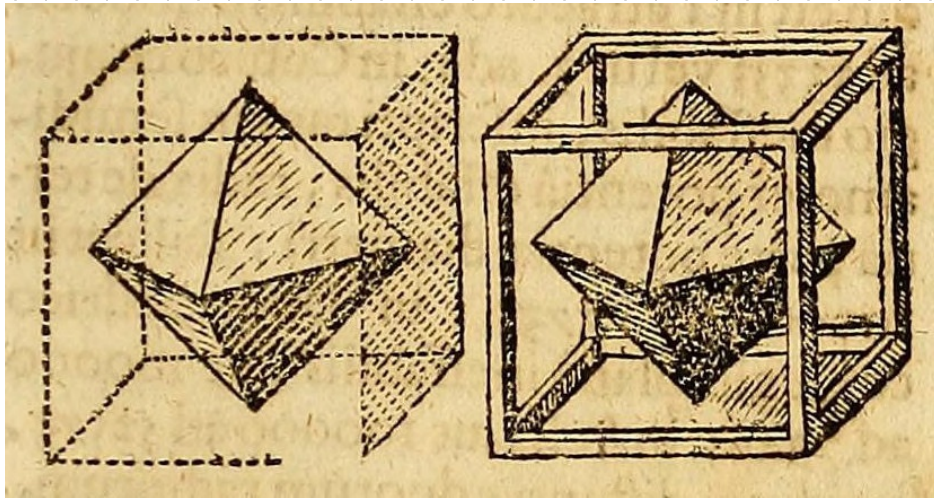

convex body $S \ni 0$	$S^\circ \ni 0$ is convex body
k-dimensional faces	$d-k$-dimensional faces
$S + V$	$S^\circ \cap V^\circ$

Clever auxilliary LP's (polar edition)

Theorem it suffices to consider intersections of the form

$$W \cap \operatorname{conv}(\bar{a}_1 + \hat{a}_1, \ldots, \bar{a}_n + \hat{a}_n)$$

where W is a fixed 2d subspace independent of $\hat{A}$.

Borgwardt '87
Spielman Teng '04
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Key quantity to analyze (summary)

"Theorem" the smoothed complexity of the simplex method is

$$\max_{\substack{\bar{a}_1,\dots,\bar{a}_n \in \mathbb{B}_2^d \\ W \subseteq \mathbb{R}^d}} \mathbb{E}_{\hat{a}_1,\dots,\hat{a}_n} \left[\#\text{ of vertices of } W \cap \operatorname{conv}(\bar{a}_1 + \hat{a}_1, \dots, \bar{a}_n + \hat{a}_n) \right]$$

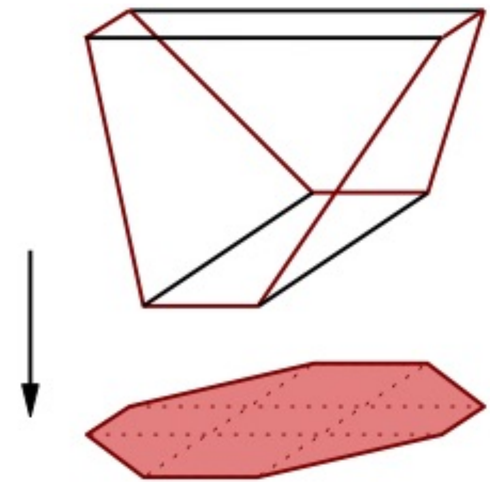
Results

	Expected Number of Vertices
Spielman, Teng '04	$O(d^3 n \sigma^{-6})$
Deshpande, Spielman '05	$O(dn^2 \sigma^{-2} \log n)$
Vershynin '09	$O(d^3 \sigma^{-4} \log^7 n)$
Dadush, Huiberts '18	$O(d^2 \sigma^{-2} \log^{1/2} n)$
Huiberts, Lee, Zhang '22	$O(d^{13/4} \sigma^{-3/2} \log^{7/4} n)$
Borgwardt '87	$\Omega(d^{3/2} \sqrt{\log n})$
Huiberts, Lee, Zhang '22	$\Omega(\min(2^d, \frac{1}{\sqrt{\sigma d \sqrt{\log n}}}))$

Lower bound

easy construction from
ongoing work with
Eleonore Bach

Extended formulations



Theorem (Ben-Tal, Nemirovski '01) For every k, there exists $P \subseteq \mathbb{R}^{k+4}$ with $2k+4$ facets such that

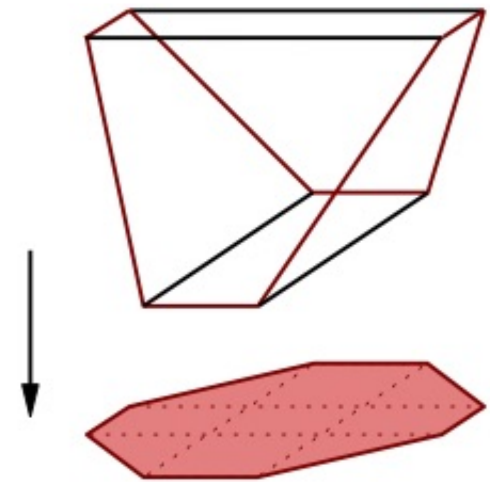
$$\pi_W(P)$$

is a regular 2^k-gon

By John's Theorem '48, we may assume

$$\mathbb{B}_2^{k+4} \subseteq P \subseteq (k+4)\,\mathbb{B}_2^{k+4}$$

Extended formulations



Theorem (Ben-Tal, Nemirovski '01)
For every k, there exists $P^\circ \subseteq \mathbb{R}^{k+4}$ with $2k+4$ vertices such that

$$P^\circ \cap W$$

is a regular 2^k-gon

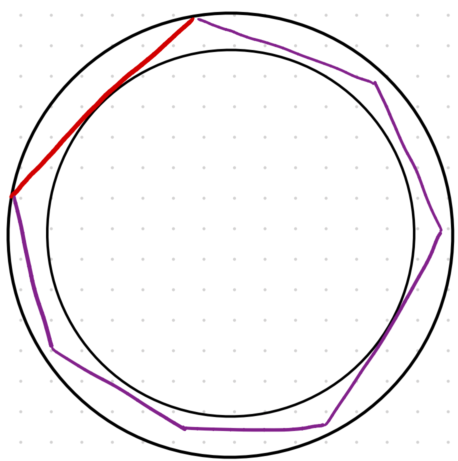
By John's Theorem '48, we may assume

$$\frac{1}{k+4}\mathbb{B}_2^{k+4} \subseteq P^\circ \subseteq \mathbb{B}_2^{k+4}$$

Edge counting

Lemma 1 if $T \subseteq \mathbb{R}^2$ is a polygon and $\alpha \mathbb{B}_2^2 \subseteq T \subseteq \beta \mathbb{B}_2^2$
then T has $\Omega\left(\sqrt{\frac{\alpha}{\beta - \alpha}}\right)$ edges

proof by picture



$$\# \text{edges} \geq \frac{\text{Perimeter}}{\text{longest edge length}}$$

Round intersection stays round

Lemma 2 if $r > 2\varepsilon > 0$ and

$a_1, \dots, a_n, \tilde{a}_1, \dots, \tilde{a}_n \in \mathbb{R}^d$ satisfy

i) $r\mathbb{B}_2^d \subseteq \mathrm{conv}(a_1, \dots, a_n)$

ii) $\|a_i - \tilde{a}_i\|_2 \leq \varepsilon$ for all $i = 1, \dots, n$

then

$$\left(1 - \frac{2\varepsilon}{r}\right)\mathrm{conv}(a_1, \dots, a_n) \subseteq \mathrm{conv}(\tilde{a}_1, \dots, \tilde{a}_n) \subseteq \left(1 + \frac{\varepsilon}{r}\right)\mathrm{conv}(a_1, \dots, a_n)$$

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Lemma 2 if $r > 2\varepsilon > 0$ and

constant — linear in noise size

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"Perturbing does not affect roundness too much"

Proof of lower bound

i) Get $\mathrm{conv}(a_1, \ldots, a_n)$ from B-T, N '01

with $\frac{1}{30}(1-4^{-k})\mathbb{B}_2^d \cap W \subseteq \mathrm{conv}(a_1, \ldots, a_n) \cap W \subseteq \frac{1}{30}\mathbb{B}_2^d \cap W$

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iii) Lemma 1 gives

$$(1-2\varepsilon d)\,\mathrm{conv}(a_1, \dots, a_n) \subseteq \mathrm{conv}(\tilde{a}_1, \dots, \tilde{a}_n) \subseteq (1+\varepsilon d)\,\mathrm{conv}(a_1, \dots, a_n)$$

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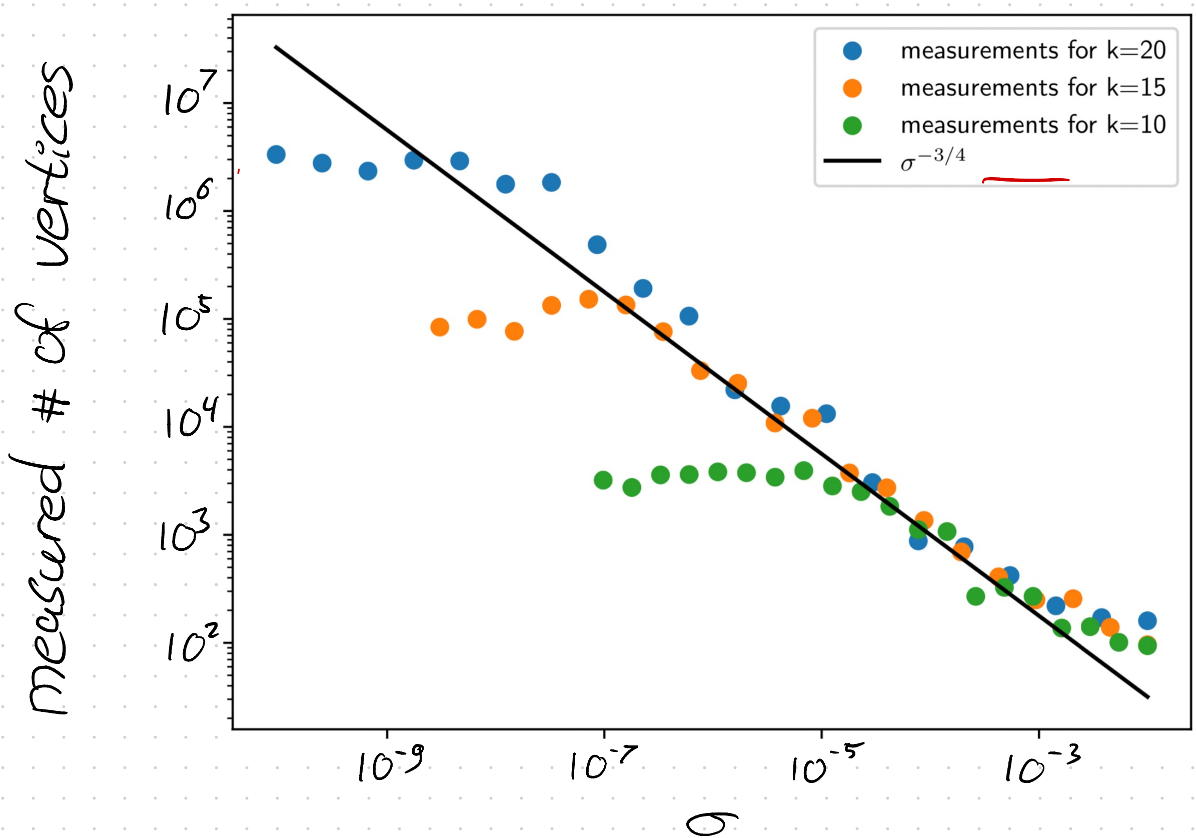
iv) Lemma 2 + i) + iii) gives the result

Our analysis is not tight wrt σ



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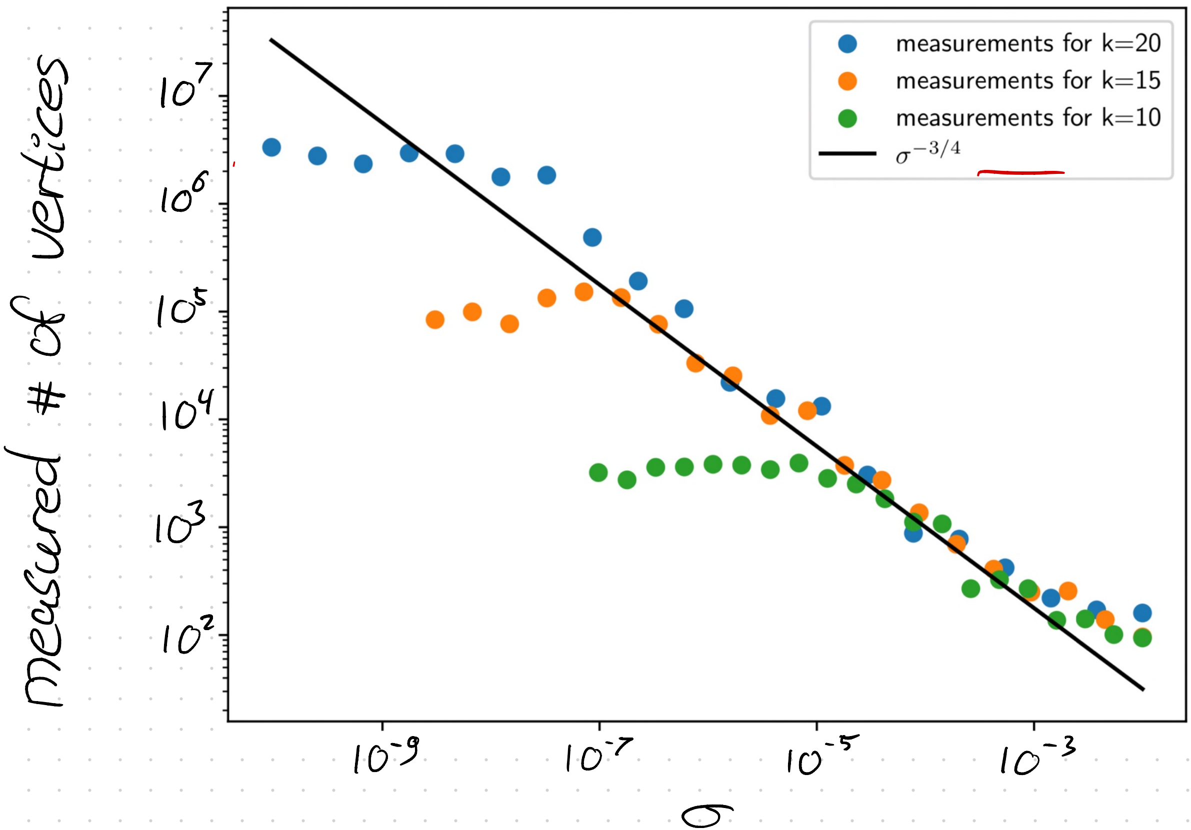

Proven:

$O(\sigma^{-3/2})$

$\Omega(\sigma^{-1/2})$

measured:

$\sigma^{-3/4}$?

Upper bound is
a similar story

- expected edge lengths
- expected exterior angles

Open problems

- Tighter bounds are better
- Sparse noise would add a sense of realism
- Noise inspired by in-software perturbations

Upper bound

by analogy

Upper bound analogy

Let $\bar{a}_1, \ldots, \bar{a}_n \in \mathbb{B}_2^2$ be fixed,

$\hat{a}_1, \ldots, \hat{a}_n \sim N(0, \sigma^2)$ iid.

How many vertices does

$\mathrm{conv}(\bar{a}_1 + \hat{a}_1, \ldots, \bar{a}_n + \hat{a}_n)$ have?

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DGGT '16	$O(\sqrt{\log n} + \sigma^{-1}\sqrt{\log n})$
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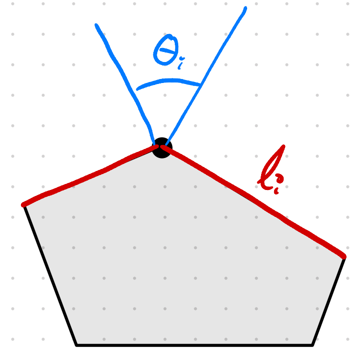
Theorem $\max\limits_{\bar{a}_1, \ldots, \bar{a}_n} \mathbb{E}_{\hat{a}_1, \ldots, \hat{a}_n}[\# \text{ vertices}] = O\left(\sqrt{\log n} + \frac{\sqrt[4]{\log n}}{\sqrt{\sigma}}\right)$

Upper bound sketch: two potentials

$$\mathbb{E}[\#\text{vertices of } \operatorname{conv}(\bar{a}_1+\hat{a}_1,\dots,\bar{a}_n+\hat{a}_n)]$$

$$= \sum_{i=1}^{n} \Pr[\bar{a}_i+\hat{a}_i \text{ is a vertex}]$$



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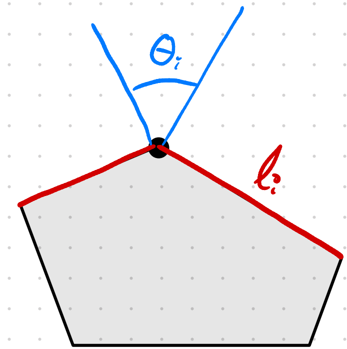

Define ℓ_i is the sum length of edges touching $\bar{a}_i + \hat{a}_i$

θ_i is the exterior angle at $\bar{a}_i + \hat{a}_i$

if $\bar{a}_i + \hat{a}_i$ is a vertex. Otherwise $\ell_i = \theta_i = 0$.

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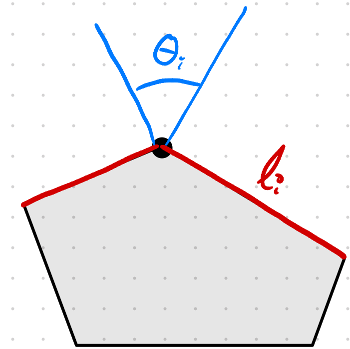

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Note:
$$\sum_{i=1}^{n}\mathbb{E}[\ell_i]=2\cdot\mathbb{E}[\text{perimeter of } \mathrm{conv}(\bar{a}_1+\hat{a}_1,\ldots,\bar{a}_n+\hat{a}_n)]$$
$$\sum_{i=1}^{n}\mathbb{E}[\theta_i]=2\pi$$

Upper bound sketch: potentials vs probability

$$\mathbb{E}[\ell_i] = \mathbb{E}[\ell_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}] \Pr[\bar{a}_i + \hat{a}_i \text{ is a vertex}]$$

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$$\Longrightarrow$$

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Similarly,

$$\Pr[\bar{a}_i + \hat{a}_i \text{ is a vertex}] = \frac{\mathbb{E}[\theta_i]}{\mathbb{E}[\theta_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}]}$$

Upper bound sketch: potentials vs probability

$$\mathbb{E}[\ell_i] = \mathbb{E}[\ell_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}] \Pr[\bar{a}_i + \hat{a}_i \text{ is a vertex}]$$

$$\Longrightarrow$$

$$\Pr[\bar{a}_i + \hat{a}_i \text{ is a vertex}] = \frac{\mathbb{E}[\ell_i]}{\mathbb{E}[\ell_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}]}$$

Similarly,

$$\Pr[\bar{a}_i + \hat{a}_i \text{ is a vertex}] = \frac{\mathbb{E}[\theta_i]}{\mathbb{E}[\theta_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}]}$$

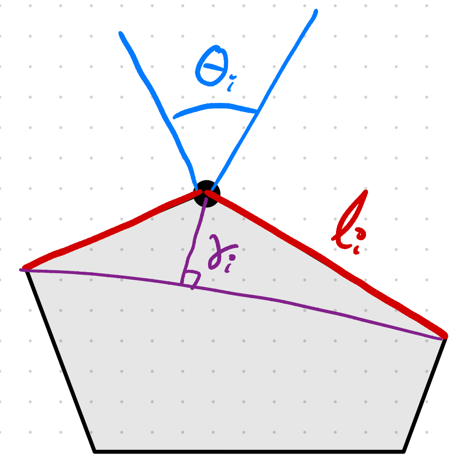
Key lemma if $\mathbb{E}[\ell_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}] \leq t$

then $\mathbb{E}[\theta_i \mid \bar{a}_i + \hat{a}_i \text{ is a vertex}] \geq \dfrac{\sigma}{t\sqrt{\log n}}$

Upper bound sketch: Planar geometry

Define γ_i as the distance from

$\bar{a}_i + \hat{a}_i$ to $\operatorname{conv}(\bar{a}_j + \hat{a}_j : j \neq i)$.



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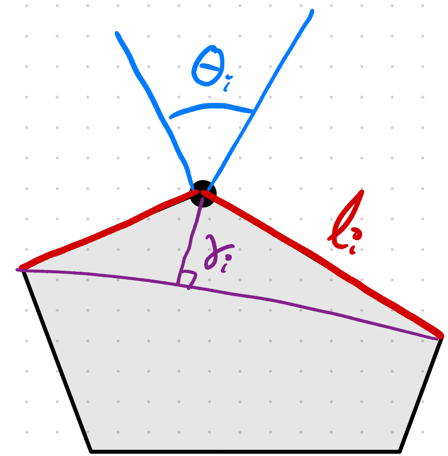

Get $\theta_i \geq \dfrac{\gamma_i}{\ell_i}$

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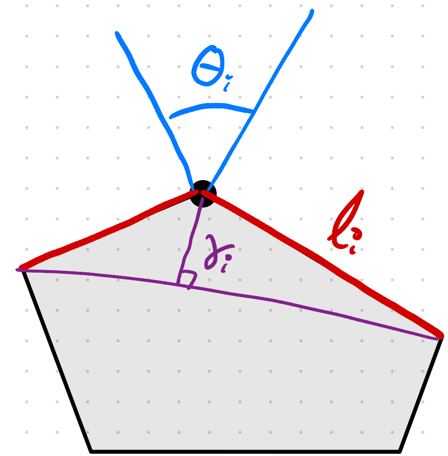

Get $\theta_i \geq \dfrac{\gamma_i}{\ell_i}$

Prove that $\Pr\left[\gamma_i \geq \dfrac{\sigma}{\sqrt{\log n}}\right] \geq \dfrac{2}{3}$.

Upper bound sketch:

$$\mathbb{E}[\#\text{ vertices of } \operatorname{conv}(\overline{a_1}+\hat{a}_1, \ldots, \overline{a_n}+\hat{a}_n)]$$

$$= \sum_{i=1}^{n} \Pr[\overline{a_i} + \hat{a}_i \text{ is a vertex}]$$

Upper bound sketch:

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$$\leq \sum_{i=1}^{n} \frac{1}{t} \mathbb{E}[\ell_i] + \frac{t\sqrt{\log n}}{\sigma} \mathbb{E}[\theta_i]$$

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$$= \frac{2}{t}\cdot\mathbb{E}[\text{perimeter of } \operatorname{conv}(\overline{a}_1+\hat{a}_1, \dots, \overline{a}_n+\hat{a}_n)] + \frac{2\pi t\sqrt{\log n}}{\sigma}$$

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$$= O\left(\sqrt{\log n} + \frac{\sqrt[4]{\log n}}{\sqrt{\sigma}}\right)$$