Open Problems about the Simplex Method

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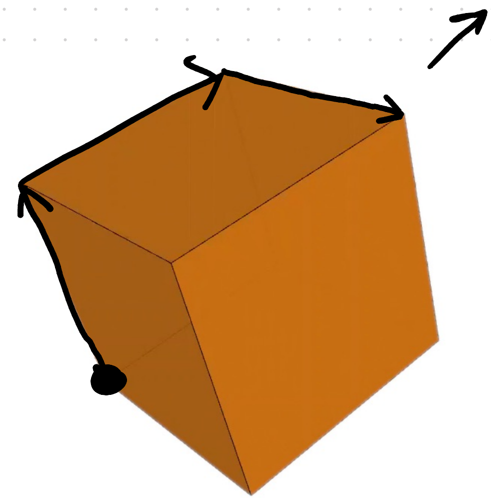
Linear programming

maximize $c^T x$
subject to $Ax \leq b$

we get $A \in \mathbb{R}^{n \times d}$
$b \in \mathbb{R}^n$
$c \in \mathbb{R}^d$

we compute $x \in \mathbb{R}^d$



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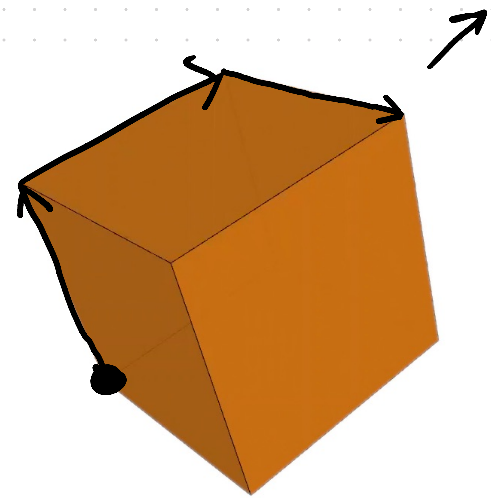

how many pivot steps?

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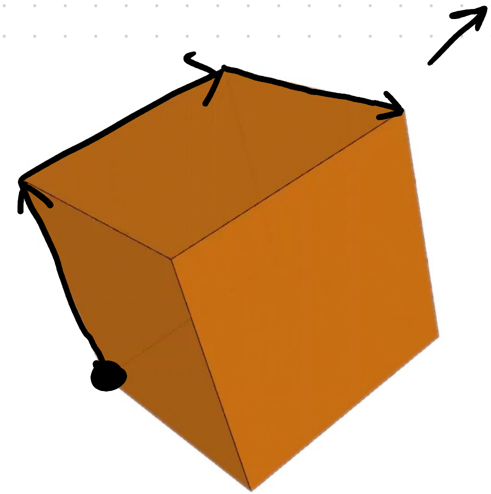

how many pivot steps?
(assuming non-degeneracy)

In practice

The simplex method takes $2(n+d)$ pivot steps to solve an LP.

Worst-case complexity

Theorem The simplex method has exponential worst-case complexity*

*terms and conditions apply

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good in practice
bad in theory

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Average-case analysis

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Lattice polytopes

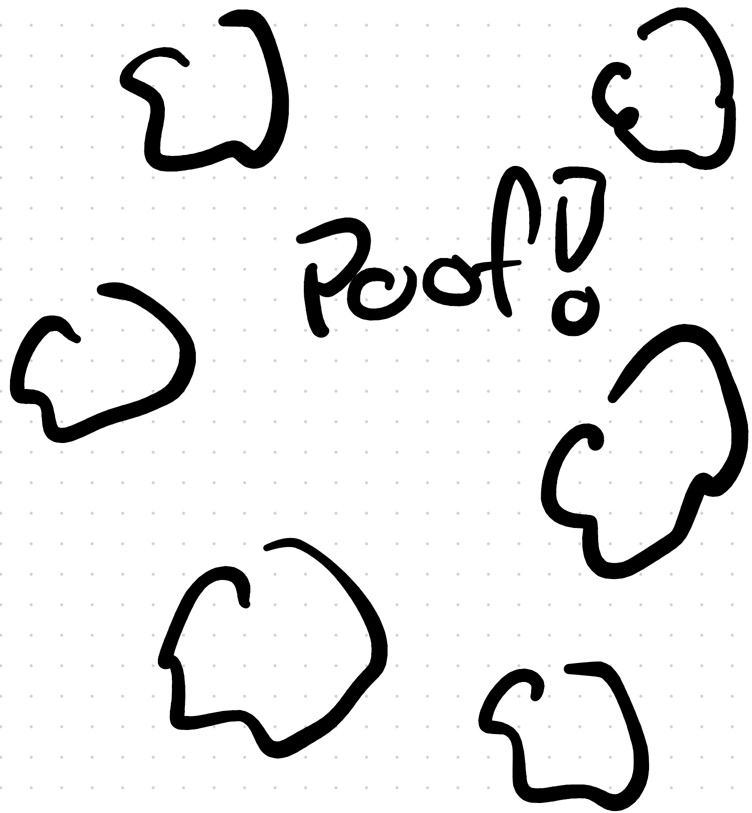
Is our theory any good?

At a conference

At a conference

At a conference


Theory has no value


Poof!


Theory has no value?

Does theory have any bearing on reality?

Instancewise assumptions

slack ratios

Distributional assumptions

smoothed analysis

Slack ratios

Suppose V is the vertex set of feasible region

$$\begin{aligned} &\text{maximize } c^T x \\ &\text{subject to } Ax \le b \end{aligned}$$

Let
$$\kappa = \frac{\min\{(b-Ax)_i : x \in V,\ i=1,\dots,n,\ (Ax)_i < b_i\}}{\max\{\|b-Ax\|_\infty : x \in V\}}$$

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After T pivot steps the optimality gap is $\propto e^{-\frac{\kappa T}{d}}$

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Optimal solution after $\frac{nd}{\kappa}\log(n/\kappa)$ steps

This $e^{-\frac{kT}{d}}$ was Dantzig's intuition for developping the simplex method.

never published?

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Independently rediscovered by Kitahara & Mizuno.

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Kuno, Sano & Tsuruda proved computing κ is NP-Hard

A posteriori κ: run simplex, observe slacks

Name	$m \times n$	pivots	est δ	est γ	est bound
afiro	27×59	16	2.615	440.4	3.260×10^{6}
kb2	43×84	61	2.478×10^{-2}	72.15	1.781×10^{8}
sc50b	50×98	49	7.226×10^{-1}	324.9	3.185×10^{7}
blend	74×157	92	1.497×10^{-2}	38.27	5.208×10^{8}
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scagr7	129×269	140	3.304	6552	1.236×10^{9}
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bore3d	233×548	200	2.091×10^{-4}	301.2	5.209×10^{12}
sctap1	300×780	328	1.154×10^{-8}	2.000	1.444×10^{15}
agg2	516×818	218	3.047×10^{-2}	1.676×10^{6}	8.060×10^{14}
scagr25	471×971	742	3.007×10^{-2}	1.118×10^{4}	4.964×10^{12}
standmps	467×1542	279	1.998×10^{-2}	101.1	7.718×10^{10}

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CAN conclude bound is nontrivial

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CAN conclude bound is nontrivial

CAN NOT conclude much else

Kuno
Sano
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Open Questions:

Is κ also NP-Hard to approximate?

can a MIP solver do it anyway?

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Does a good κ correlate with faster solving?

does presolve improve κ?

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Are these the right questions to ask?

Smooted analysis

Let $\bar{A} \in \mathbb{R}^{n \times d}$ have rows of norm ≤ 1.

$\bar{b} \in [-1, 1]^n$, $c \in \mathbb{R}^d$

Let $\hat{A}$, $\hat{b}$ have iid $N(0, \sigma^2)$ entries

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Then

$$\max_{\bar{A}, \bar{b}, c} \mathbb{E}_{\hat{A}, \hat{b}} \left[\begin{array}{c} \text{time to solve} \\ \text{maximize } c^T x \\ \text{s.t } (\bar{A} + \hat{A})x \leq \bar{b} + \hat{b} \end{array} \right] \leq \text{poly}(n, d, \sigma^{-1})$$

Why smoothed analysis?

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independent measurement/numerical errors do not conspire against your algorithm.

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interpolates between worst case and average case analysis.

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interpolates between worst case and
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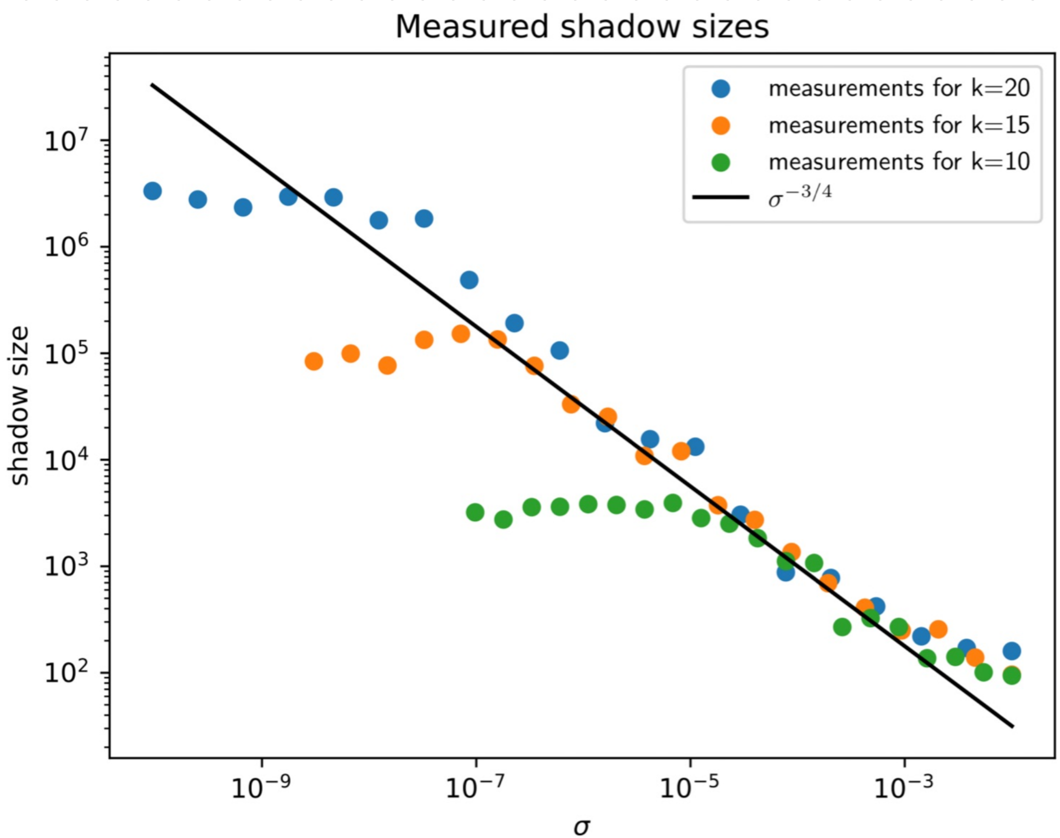
shows algorithm is fast on average
in every large enough neighborhood

Can any of the resulting insights be tested experimentally?

Synthetic data



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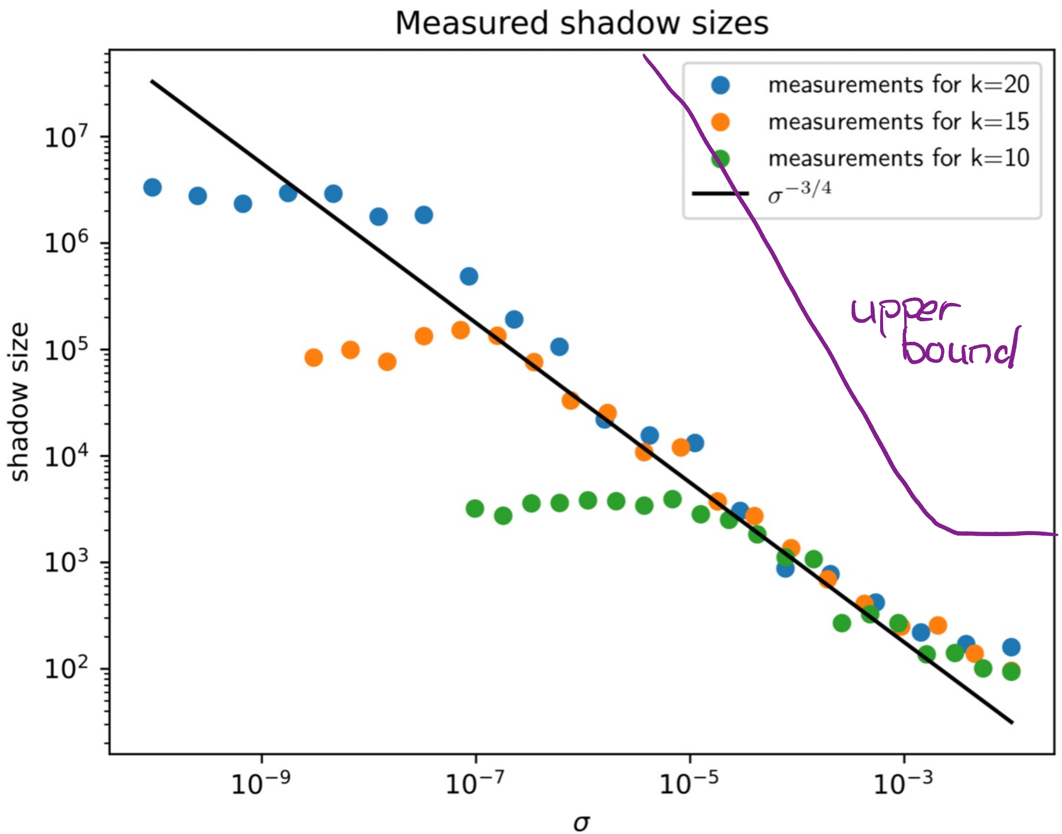

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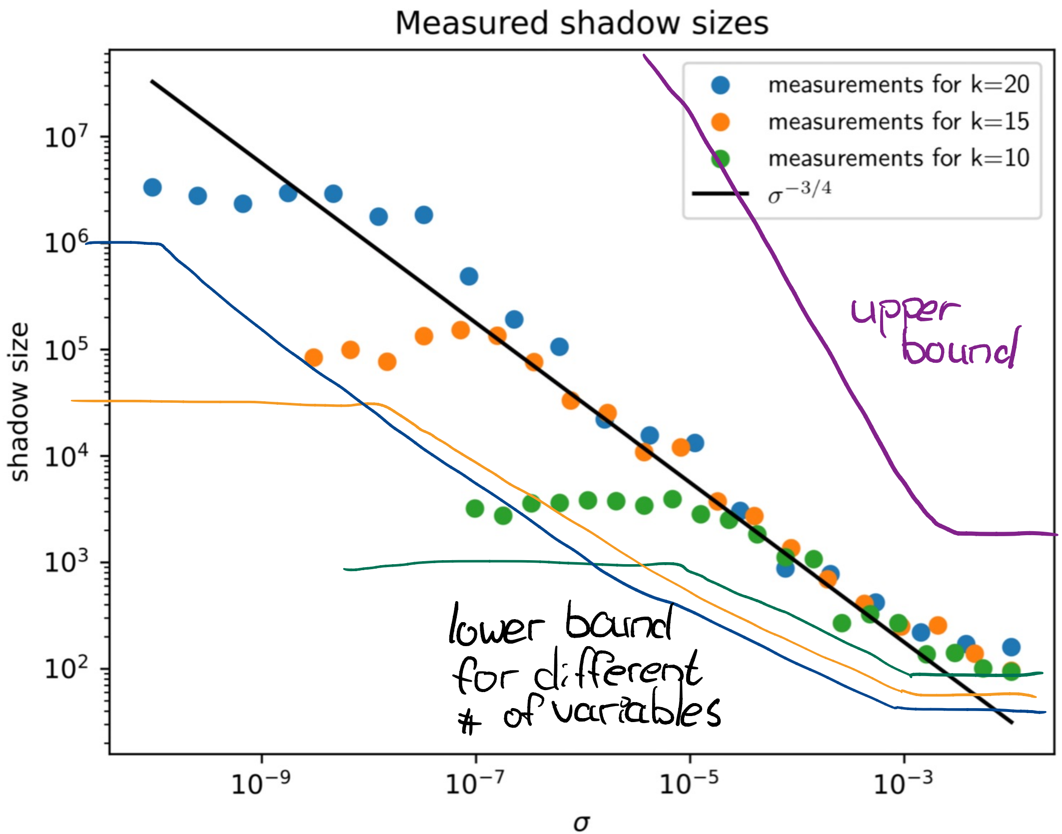

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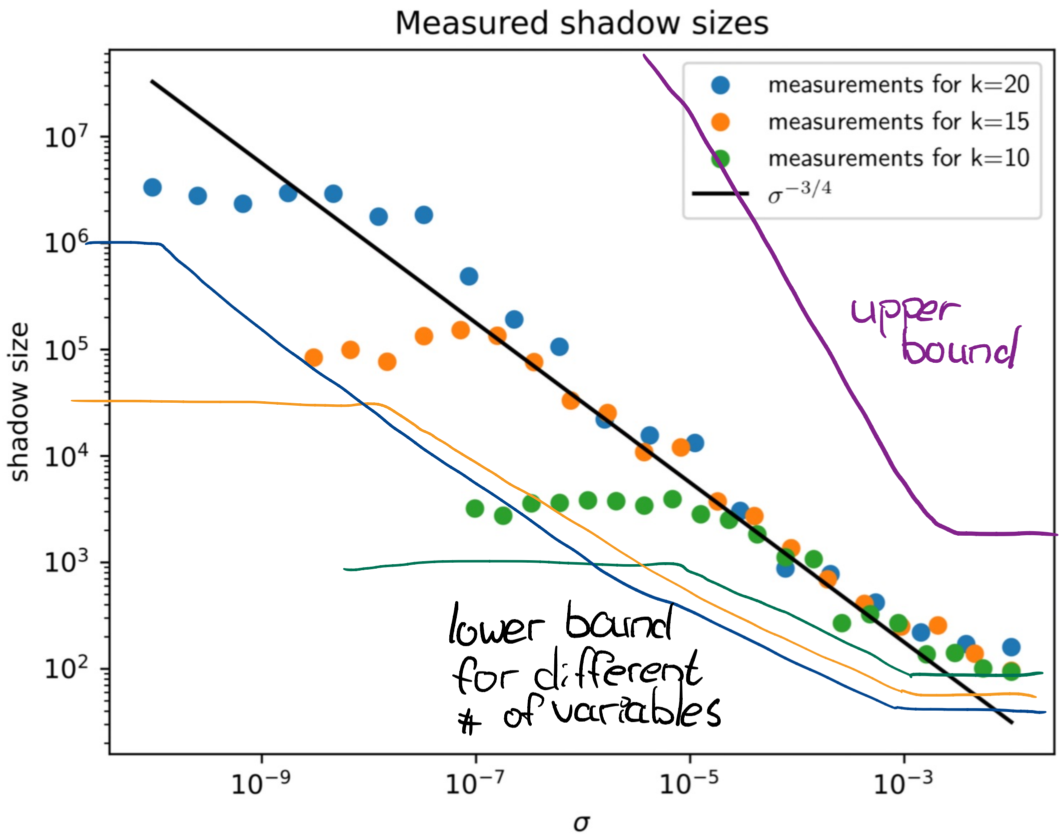

CAN tell if theorem is tight

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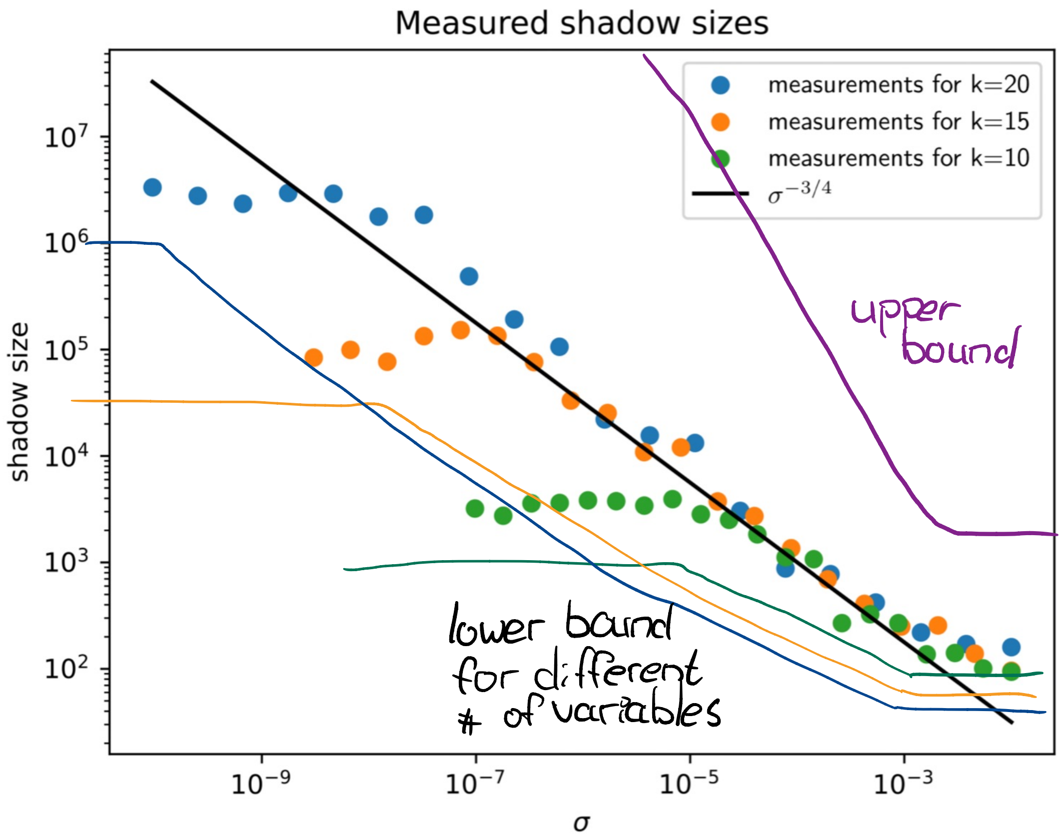

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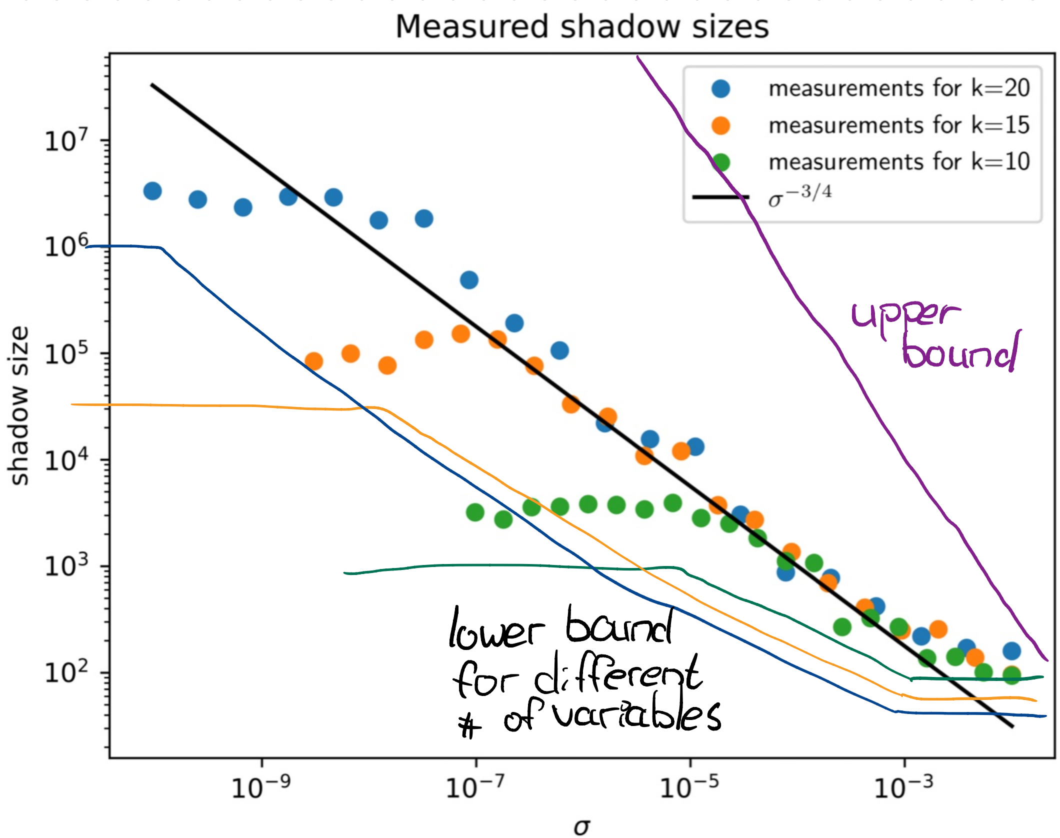

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The first linear program (1948)

Given 77 ingredients,
find the cheapest diet
that meets all 9 nutrient needs

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Side Question:

is this normal to want
and possible to achieve?

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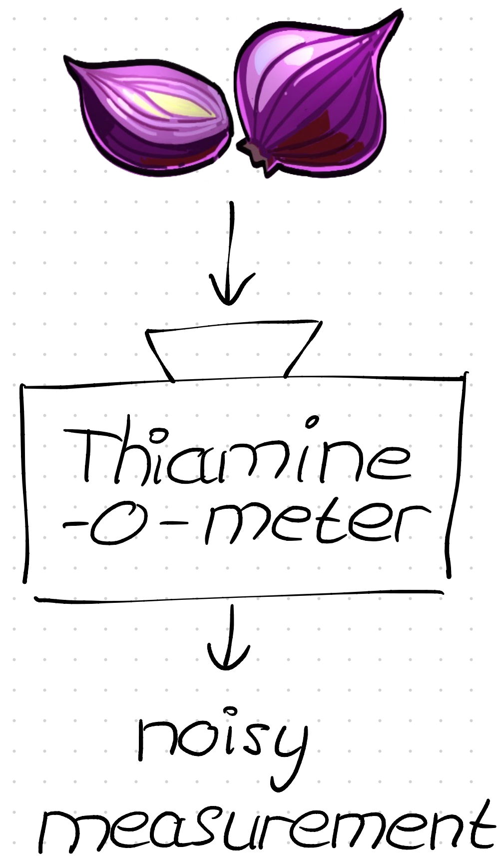

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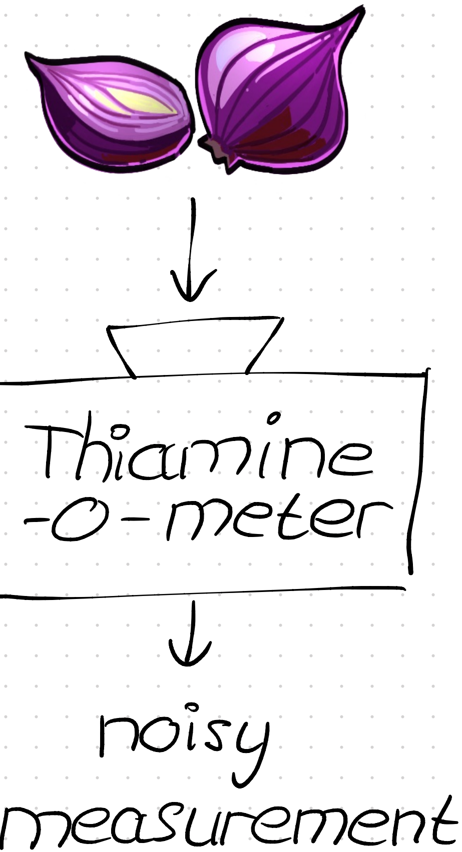
Type of Operations	No. of repetitions
Multiplication	15,315
Division	1,234
Addition of two numbers	14,561
Addition of 77 numbers	190
Addition of 9 numbers	85



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Thiamine
-o-meter

noisy
measurement

Open question: how many
pivot steps is that?

Intermission

History lesson

Mathematical Tables Project 1938 - 1948

450 computers employed

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To perform these computations with desk machines required 5 computers for 21 days, with 4 hours per day supervision by a mathematician.

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Linear Programming and Extensions

George B. Dantzig

Dantzig's famous book

STIGLER'S NUTRITION MODEL: AN EXAMPLE OF FORMULATION AND SOLUTION

One of the first applications of the simplex algorithm was to the determination of an adequate diet that was of least cost.[1] In the fall of 1947, J. Laderman of the Mathematical Tables Project of the National Bureau of Standards undertook, as a test of the newly proposed simplex method, the first large-scale computation in this field. It was a system with nine equations in seventy-seven unknowns. Using hand-operated desk calculators, approximately 120 man-days were required to obtain a solution.

The particular problem solved was one which had been studied earlier by G. J. Stigler [1945-1], who had proposed a solution based on the substitution of certain foods by others which gave more nutrition per dollar. He then examined a "handful" of the possible 510 ways to combine the selected foods. He did not claim the solution to be the cheapest but gave good

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38. Mathematical Tables Project computers with adding machines

Historical takeaways

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Human computers played an important role in early comb. opt. history

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Their contributions were made invisible by contemporary white men

Historical takeaways

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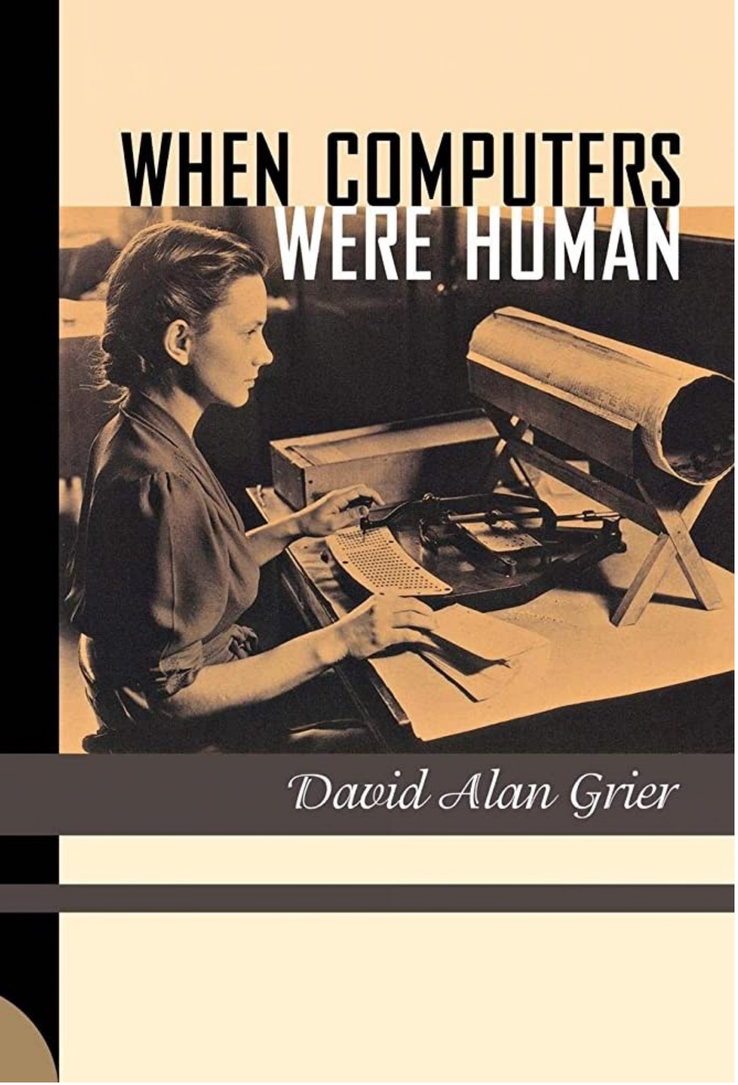
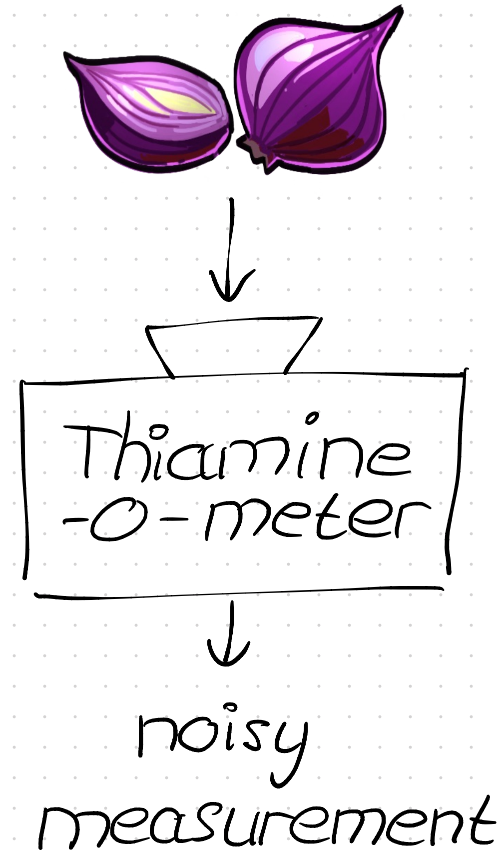
Their demographics are exactly those underrepresented in our field today



Consider including this history in your lectures

today's source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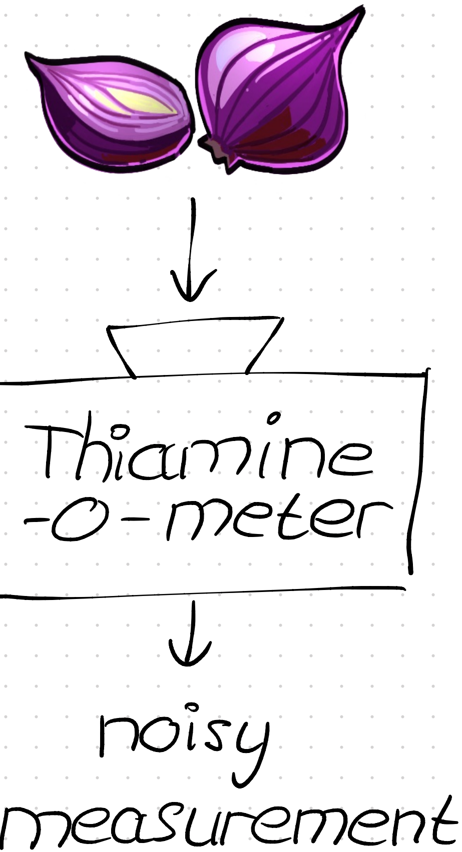
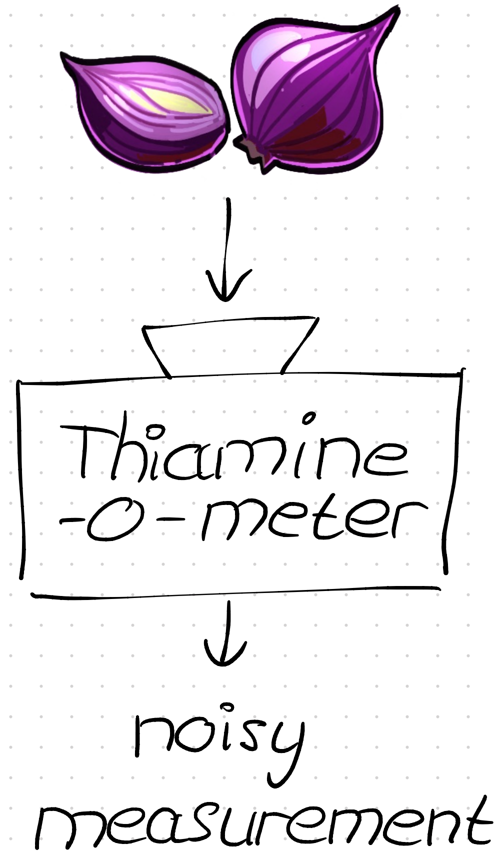
Could smoothed analysis have *predicted* that the first computation be fast?



Could smoothed analysis have predicted that the first computation be fast?

Probably not.



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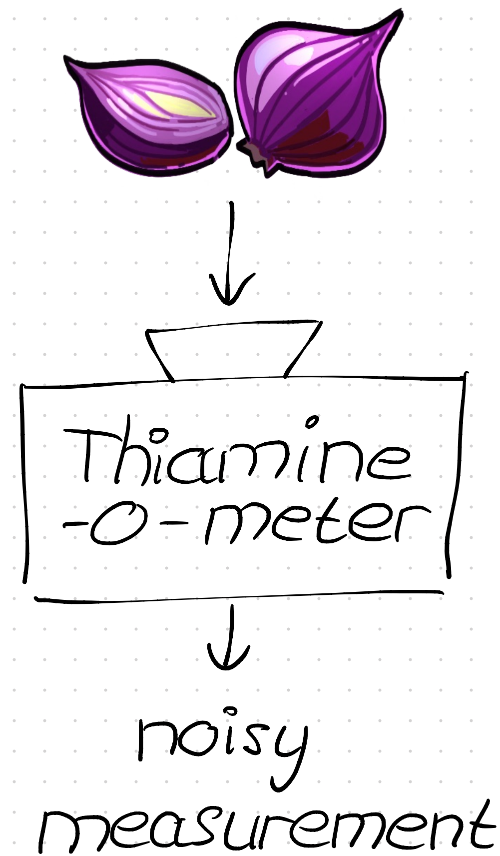
1. different pivot rule



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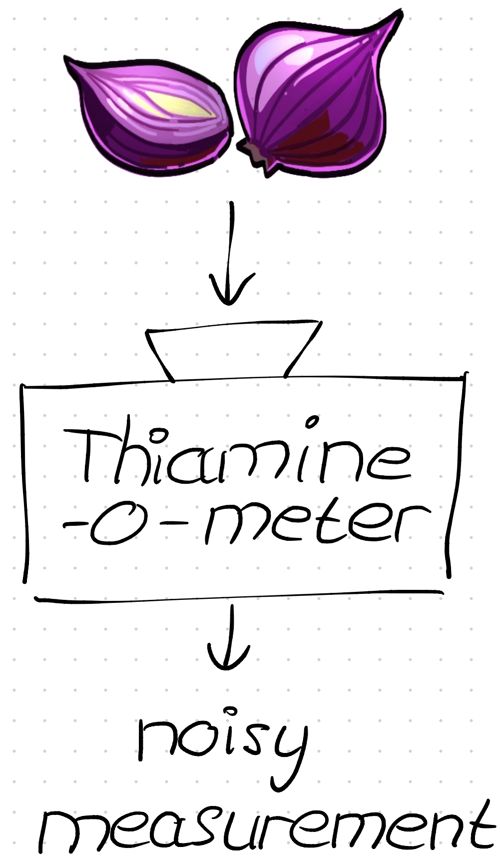
2. different phase 1.



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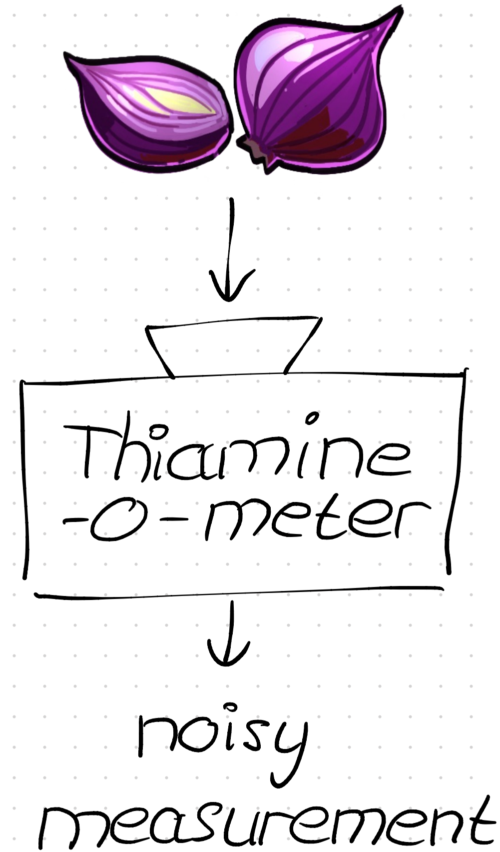

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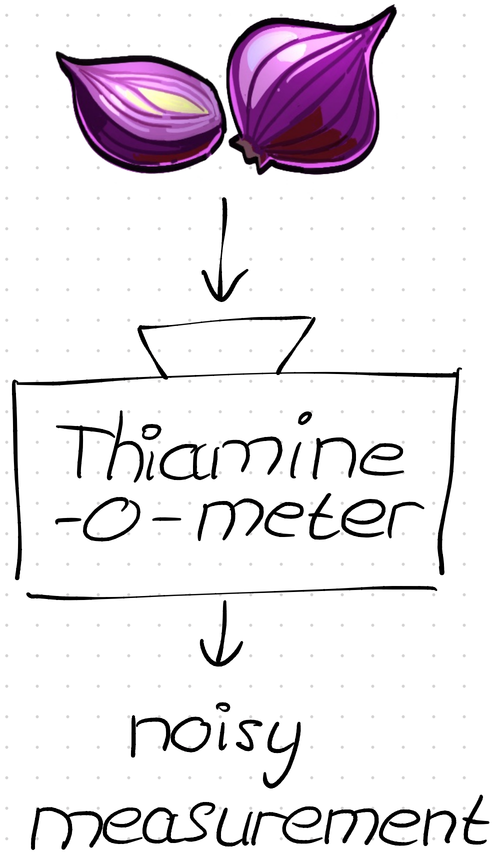
1. different pivot rule
2. different phase 1
3. non-negativity constraints
4. multiplicative error ≥ 15%, but need additive error



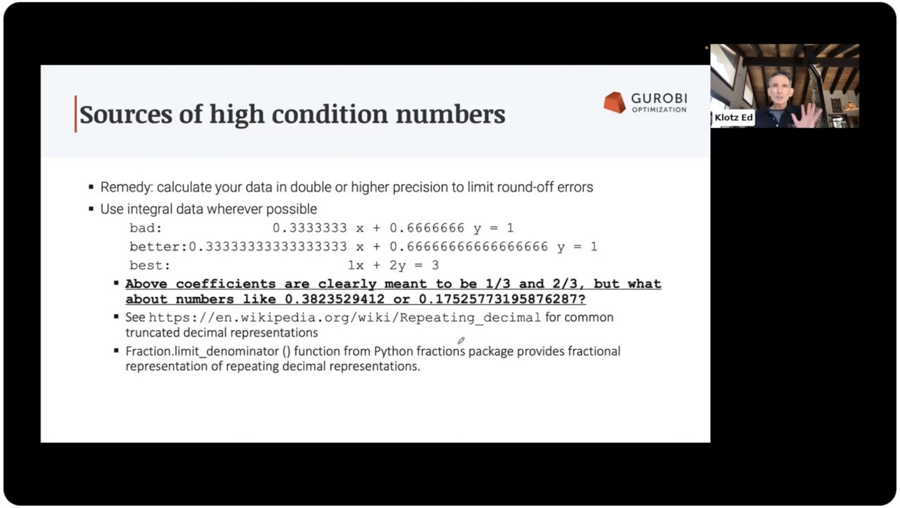
A more fundamental issue?

Smoothed analysis is based on the notion "more noise is better"

A more fundamental issue?

Smoothed analysis is based on the notion
"more noise is better"

Practitioners say
"less noise is better"



Pinpointing Numerical Issues: Tool for Diagnosis & Explanation of Ill Conditioning in LPs & MILPs

Are we studying the simplex method?

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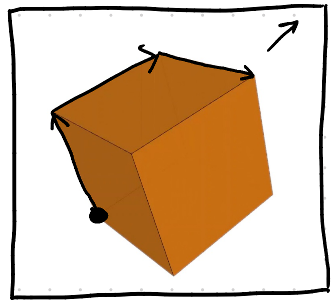
This geometry  doesn't exist.

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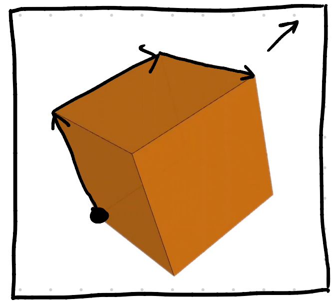
This geometry  doesn't exist.

What does exist:
- linear algebra
- bound shifting
- bound perturbations
- Harris ratio test

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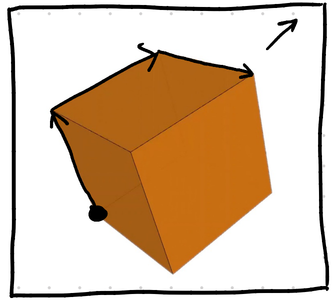
This geometry  doesn't exist.

Paula Harris

What does exist: $\left\{ \begin{array}{l} \text{linear algebra} \\ \text{bound shifting} \\ \text{bound perturbations} \\ \text{Harris ratio test} \end{array} \right.$

Summary:

Our theory are untested
untestable?

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We have little/no data

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We disagree with practitioners

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Are we looking at the wrong algorithm?