The Curious Case of Integer Linear Programming

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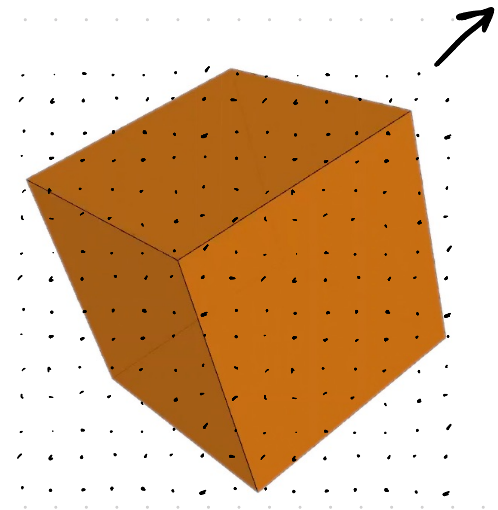
Integer Linear Program

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax \le b \\ & x \in \mathbb{Z}^d \end{aligned}$$



we get $A \in \mathbb{R}^{n \times d}$

$b \in \mathbb{R}^n$

$c \in \mathbb{R}^d$

we compute $x \in \mathbb{Z}^d$

Pros	Cons
Versatile Practical	NP Hard

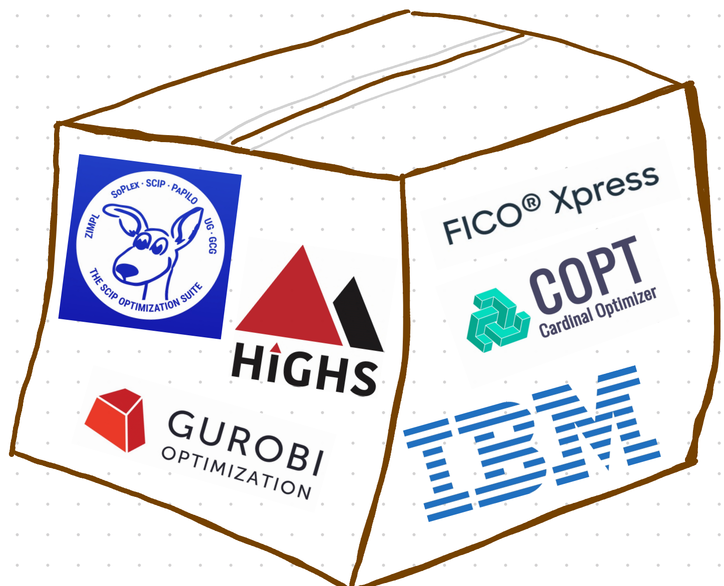
Solving Integer Linear Programs is NP Hard

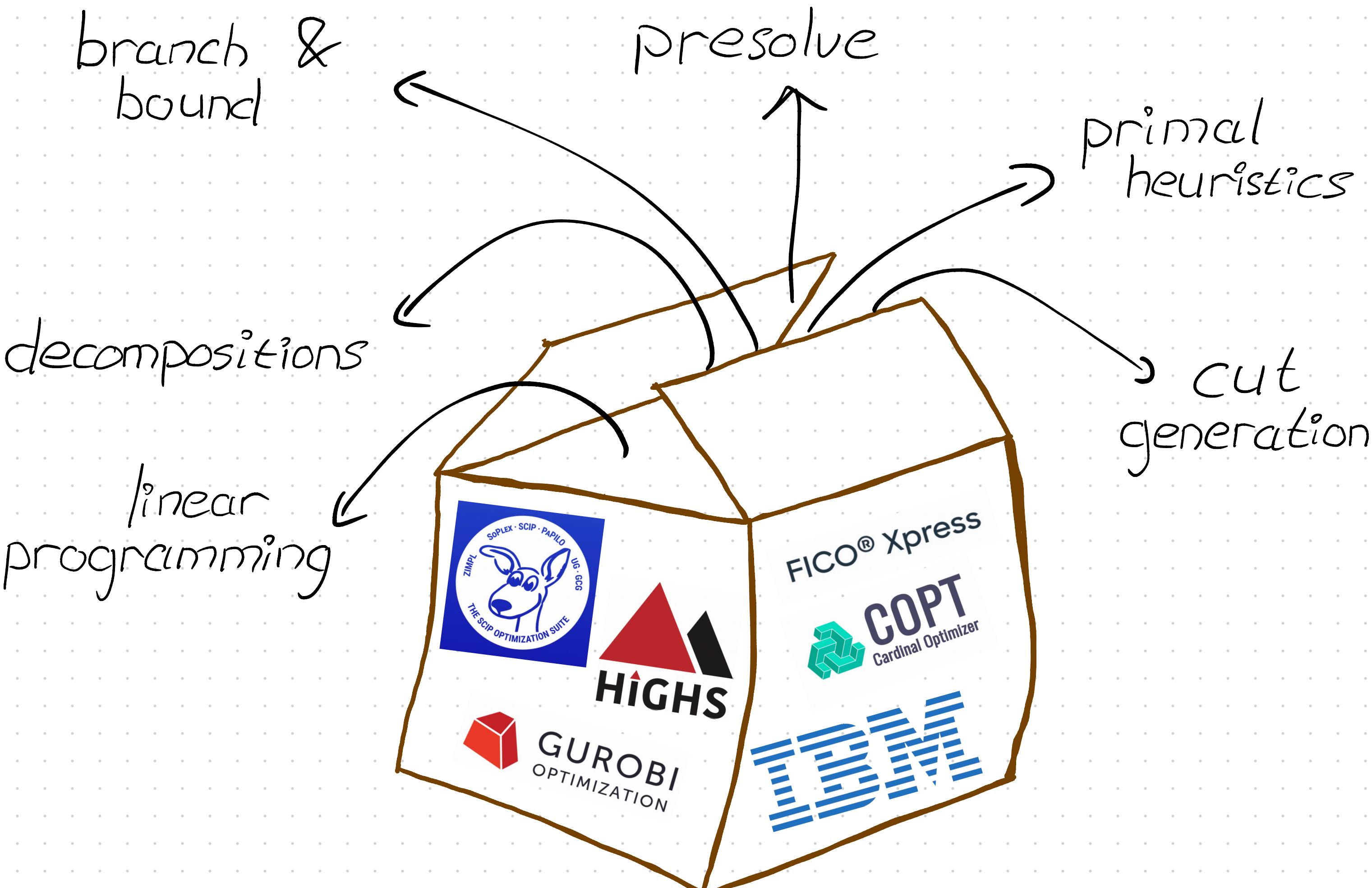
Widely believed that no polynomial time algorithm can exist.

All known algorithms take exponential time in the worst case.

Efficient Software Libraries Do Exist.

ILP's with tens of thousands of variables are solved regularly.

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presolve
primal heuristics
decompositions
cut generation
linear programming
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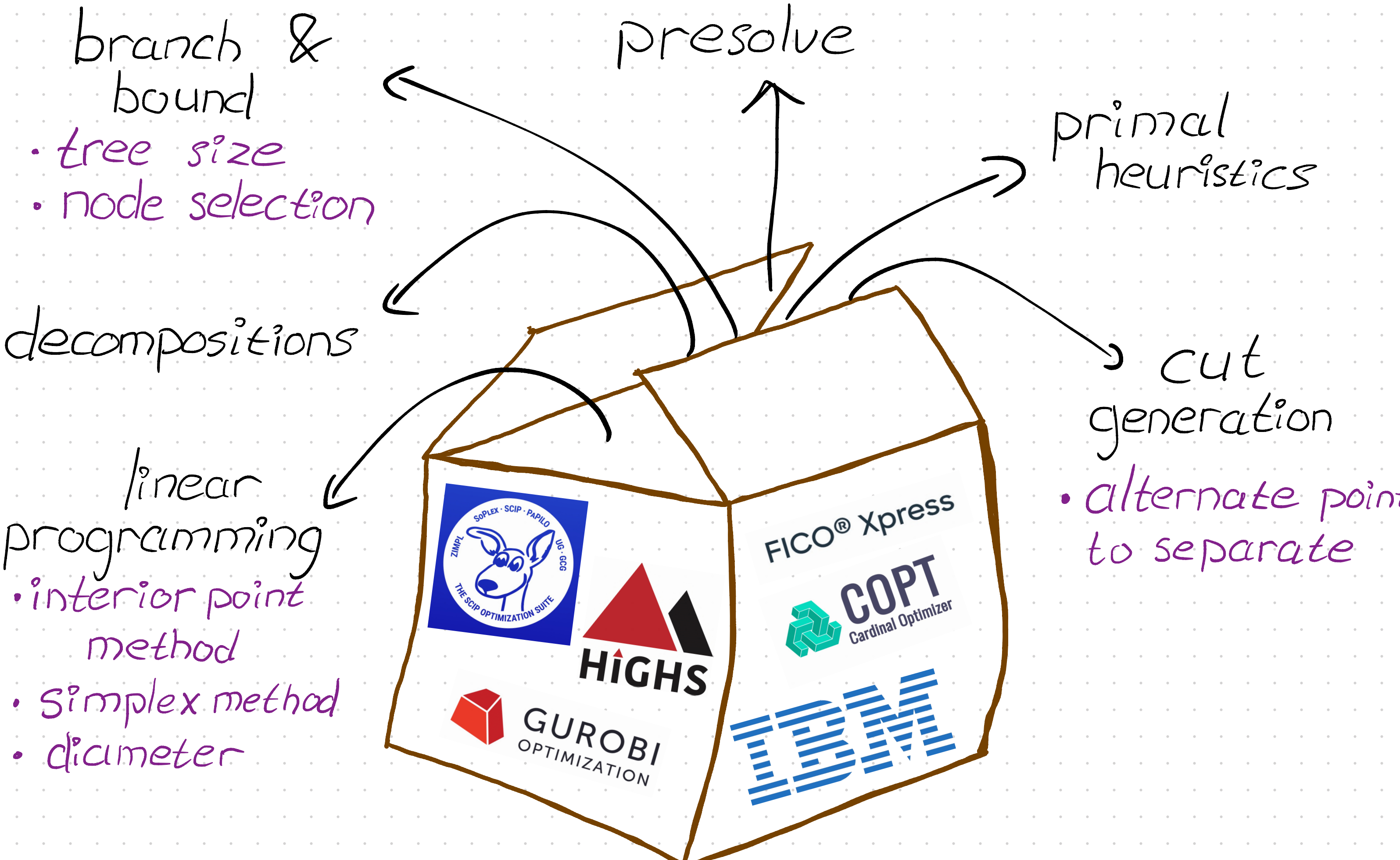
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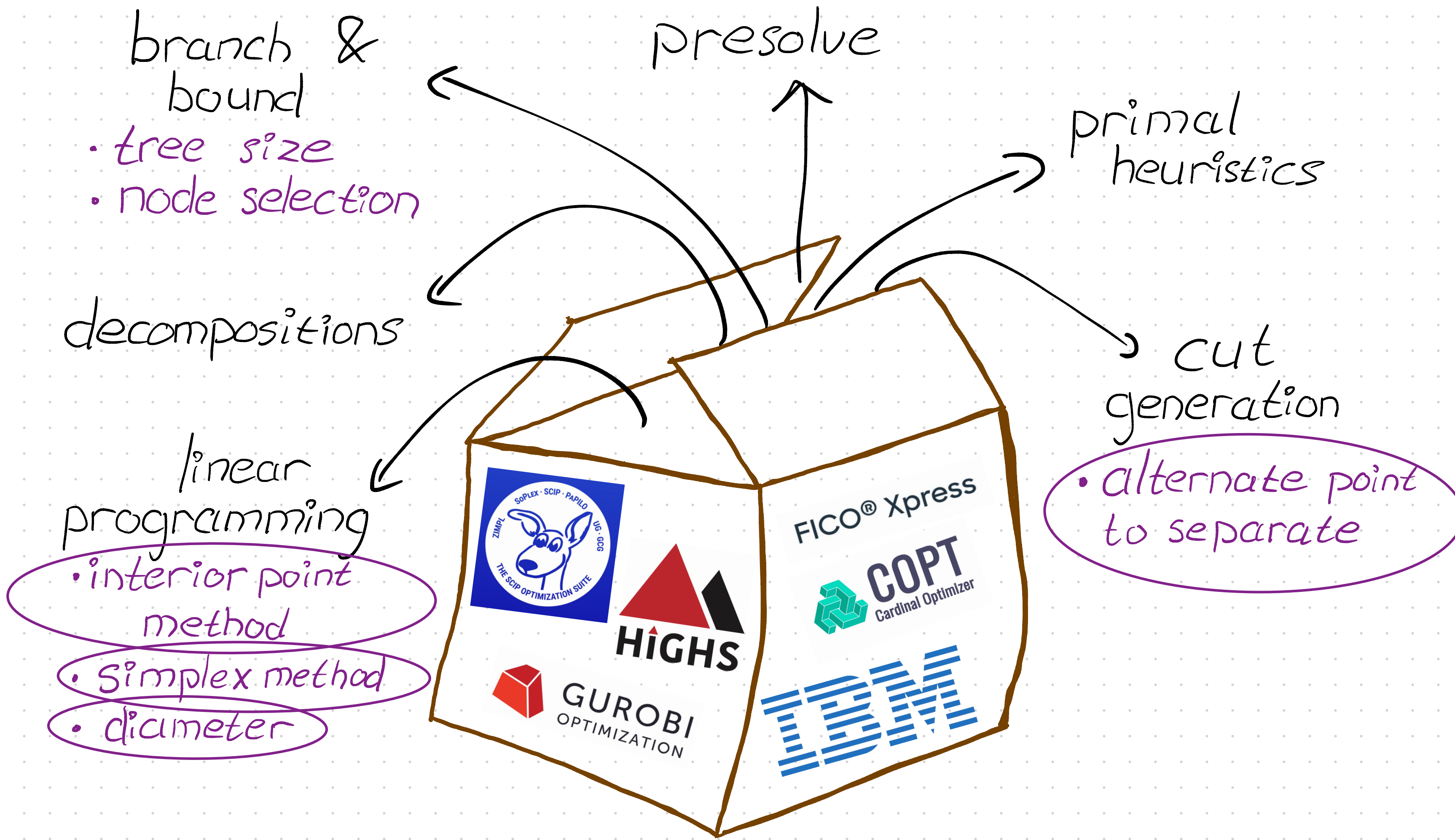
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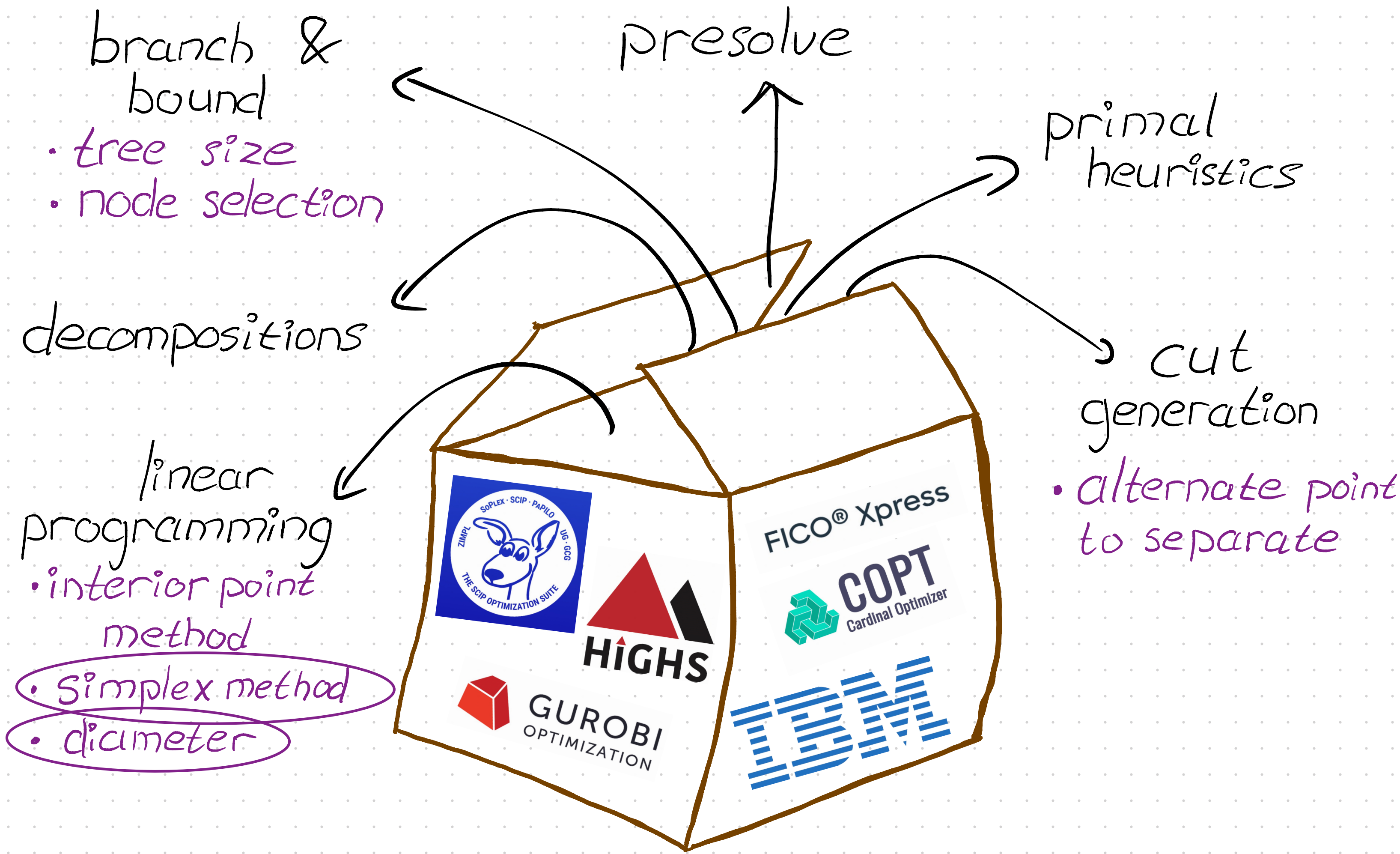
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Case Study: the Simplex Method

$$\text{maximize } c^T x$$
$$\text{subject to } Ax \leq b$$
$$x \in \mathbb{R}^d$$

we get $A \in \mathbb{R}^{n \times d}$

$b \in \mathbb{R}^n$

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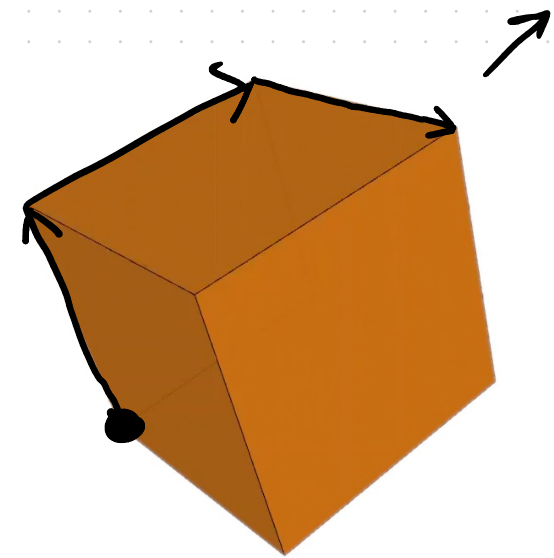

The First Linear Program

table listing foods, each with nutrient info & price

Compute a diet:
- eat a non-negative amount of each food
- meet all nutrient needs
- as cheap as possible

In Practice

The simplex method visits $\sim 2(n+d)$ vertices before reaching an optimal one

Only a few documented cases where $> 10(n+d)$ iterations were performed

In Theory

Theorem There exist input A, b, c, x^0, $n = 2d$, |
such that the simplex method
visits 2^d vertices

Klee Minty '72
Many, many others '72 - '23

Because of that

Borgwardt '87

Theorem There is a simplex method which, if the rows of A are iid uniform from the sphere and $b=1$, visits $\Theta(d^2 n^{\frac{1}{d-1}})$ vertices in expectation

Because of that

Borgwardt '87

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Theorem If $n \gg 2^{d^3}$ then this concentrates around the mean.

Bonnet
Dadush
Grupel
Huiberts
Livshyts
'22

Extension to diameter

Theorem If the rows of A are iid uniform from the sphere and $n \gg 2^{d^2}$, then whp the graph of $\{x : Ax \leq 1\}$ has

$$\Omega(d\, n^{\frac{1}{d-1}}) < \text{diameter} < O(d^2\, n^{\frac{1}{d-1}})$$

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How Realistic is Average Case?

Smoothed complexity

Let $\bar{A} \in \mathbb{R}^{n \times d}$ have rows of norm at most 1,

$\bar{b} \in [-1, 1]^n$, $c \in \mathbb{R}^d$

Let $\hat{A}$, $\hat{b}$ have iid $N(0, \sigma^2)$ entries.

The smoothed complexity is $\max_{\bar{A}, \bar{b}, c} \mathbb{E}_{\hat{A}, \hat{b}} \left[T(\bar{A} + \hat{A}, \bar{b} + \hat{b}, c) \right]$

where $T(A, b, c)$ is the time to solve $\max c^T x$ s.t. $Ax \leq b$

Spielman Teng '04

Why smoothed analysis

- independent measurement/numerical errors do not conspire against your algorithm
- interpolates between worst and average case
- shows simplex is fast on average in every large enough neiborhood

Results

	Expected Number of Pivots
Spielman, Teng '01	$O(n^{86} d^{55} \sigma^{-30})$
Vershynin '09	$O(d^3 \log^3 n\sigma^{-4})$
Dadush, Huiberts '18	$O(d^2 \sqrt{\log n}\sigma^{-2})$
Huiberts, Lee, Zhang '23	$O(d^{13/4} \log^{7/4} n\sigma^{-3/2})$
Borgwardt '87	$\Omega(d^{3/2} \sqrt{\log n})$
Huiberts, Lee, Zhang '23	$\Omega(\min(2^d, \frac{1}{\sqrt{\sigma d \sqrt{\log n}}}))$

Many Questions Remain