Asymptotic bounds for the combinatorial diameter of random polytopes

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joint work with Gilles Bonnet, Daniel Dadush, Uri Grupel, Galyna Livshyts

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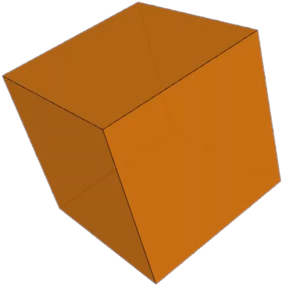
A question about geometry

Given a polyhedron, how many edges do we need to traverse to go from any vertex to any other?



- $P \subset \mathbb{R}^n$
- m facets

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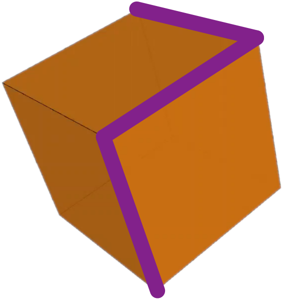

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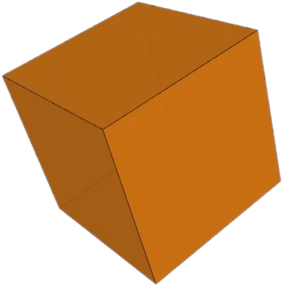

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Given $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$ and a vertex of the polyhedron

$$\{x \in \mathbb{R}^n \mid Ax \leq b\},$$

how many steps do you need to find a vertex maximizing $c^\mathsf{T} x$?

- n variables
- m constraints

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Best constructions:
KW67, unbounded polyhedra with diameter $\geq 1.25(m - n)$.
S12, bounded polyhedra with diameter $\geq 1.05(m - n)$.

Best available diameter bounds

B69, L70, B74:

$$\text{Diameter(P)} \leq 2^{n-2} m.$$

KK92, T14, S19:

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DF94, BdSEHN14, DH16, NSS22:
If $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ where $A \in \mathbb{Z}^{n \times m}$ and every absolute square subdeterminant of A is at most Δ then

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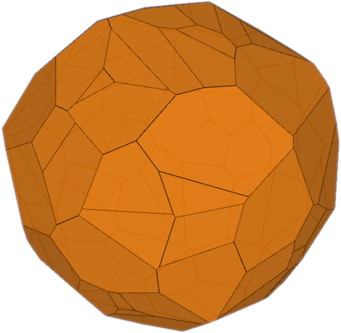
Random polytopes are "well-conditioned on average". Do they have small diameter?

Tantalizing evidence from the simplex method

Let

$$P = \{x \in \mathbb{R}^n : \langle a, x \rangle \leq 1 \; \forall a \in A\}$$

where $A \subset \mathbb{S}^{n-1}$, $|A| = m$ is sampled iid.



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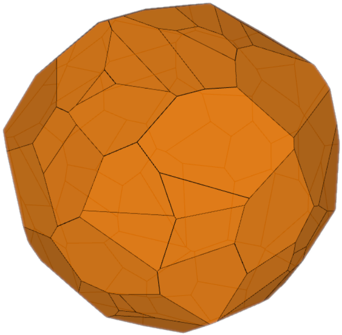

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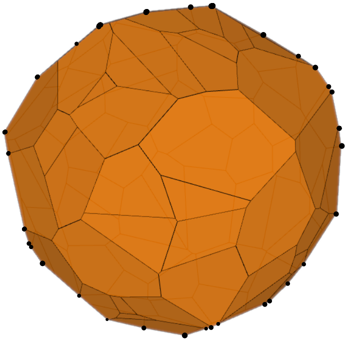

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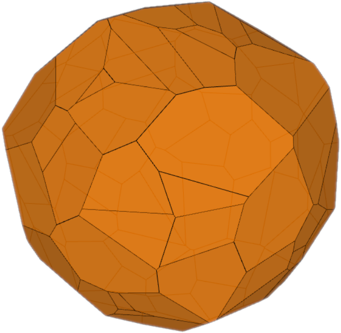

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$\implies$ two vertices optimized by randomly chosen objective vectors are at distance $O(n^2 m^{\frac{1}{n-1}})$ in expectation.

Our results

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where $A \subset \mathbb{S}^{n-1}$ follows a Poisson point process with $\mathbb{E}[|A|] = m > 2^{\Omega(n)}$.

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Then with probability $1 - O(1/m)$ we have

$$\Omega(nm^{\frac{1}{n-1}}) \leq \text{Diameter}(P) \leq O(n^2 m^{\frac{1}{n-1}} + 17^n)$$
$$\Omega(m^{\frac{1}{n-1}}) \leq \text{Diameter}(\text{Conv}(A)) \leq O(nm^{\frac{1}{n-1}} + 17^n).$$

A Weak but Simple Lower bound

A set $B \subset \mathbb{S}^{n-1}$ is called ε-dense if for every $x \in \mathbb{S}^{n-1}$ there exists $b \in B$ with $\|x - b\| \leq \varepsilon$.

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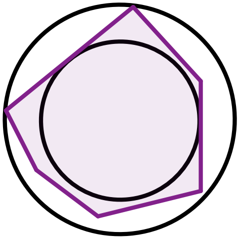

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$\implies$ With high probability Diameter(P) $\geq \Omega(m/\log m)^{\frac{1}{n-1}}$.

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- This happens with high probability $\implies$ shadow size concentrates around its mean.
- Many 2D planes simultaneously have small shadow sizes.
- Can be leveraged to upper bound the diameter (!).

Future directions

- Close the gap between upper and lower bound.
- Other distributions, such as Gaussian?
- What if $m \leq \mathrm{poly}(n)$?