Stoch Project 3: ML Estimate

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Contents

Part 1: Derivation of ML Estimators for Exponential and Rayleigh Distributions

Part 1 of this project (pages attached) is the derivation of the maximum likelihood (ML) estimators for the exponential and Rayleigh distributions. First, the likelihood is found. Then the sum of the natural log of the likelihood for each distribution is computed. The derivative of the log likelihood with respect to the parameter is computed and set equal to zero. The last step is to solve for the parameter of the distribution, which is lambda in our case. For the exponential distribution, we get that lambda is equal to the reciprocal of the mean of the distribution. For the Rayleigh distribution, we get that lambda is equal to the square root of the sum of the observations squared divided by 2 times the number of observations.

Part 2: Implemention of ML Estimators

Part 2 of this project is the implementation of the estimators for the parameter of the exponential and Rayleigh distributions. We arbitrarily set lambda equal to 0.6. This is the number we will try to estimate. For each number of observations from 1 to 100, two separate vectors of exponential and Rayleigh distributed numbers are generated. The estimate for lambda is computed based on 10000 trials of each vector. The bias, variance, and mean squared error are computed for each number of observations. Bias is equal to the mean of the estimated lambda minus the actual lambda. The variance is equal to expectation of the square of the difference between the estimator and the mean of the estimator.

Figure 1 shows the plot of the mean squared error versus number of observations for the exponential (blue) and Rayleigh (red) distributions. The mean squared error peaks at 1 observation and then quickly approaches zero. The mean squared error is inversely proportional to number of observations. By 100 observations, the mean squared error is essentially zero.

Figure 2 shows the bias of the exponential (left) and Rayleigh (right) distributions versus number of observations. The bias describes how far off the estimate of lambda is from the true lambda.

The bias of both distributions approaches zero. For the exponential distribution, the bias starts out positive and quickly drops to zero. This means that on average, the estimated lambda for the exponential distribution is greater than the actual lambda. For the Rayleigh distribution, the bias starts out negative and increases towards zero. This means that on average, the estimated lambda for the Rayleigh distribution is smaller than the actual lambda. We can see in this plot that the bias for the Rayleigh distribution is noisy. The value of the bias locally increases and decreases around zero as the number of observations increases.

The ML estimate tries to minimize bias and variance, rather than minimizing the mean squared error.

Figure 3 shows the variance of the exponential (blue) and Rayleigh (red) distributions versus number of observations. The variance describes how close together the estimates of lambda are.

The variances of both distributions get close to zero. The variance of the exponential distribution approaches 10^-2 and the variance of the Rayleigh distribution approaches 10^-3.

As the number of observations increases, the mean squared error, bias, and variance for both the exponential and

Rayleigh distributions get close to zero.

```
clc; close all; clear all;

lambda = 0.6; % choose actual value of parameter
trials = 10000;
observations = 100;

MSE_exp = zeros(1,observations);
bias_exp = zeros(1,observations);
variance_exp = zeros(1,observations);

MSE_ray = zeros(1,observations);
bias_ray = zeros(1,observations);
variance_ray = zeros(1,observations);

for n = 1:observations
    lambda_hat_exp = [];
    lambda_hat_ray = [];

    for i = 1:trials
        x = exprnd(1/lambda, 1, n);
        lambda_hat_exp(1,i) = n/sum(x);

        y = raylrnd(lambda, 1, n);
        lambda_hat_ray(1,i) = sqrt(sum(y.^2)./(2*n));
    end

    % The following formulas were obtained from the Wikipedia page "Estimator"

    % Calculate MSE, Bias, Variance for Exponential Distribution
    MSE_exp(n) = mean((lambda_hat_exp - lambda).^2);
    bias_exp(n) = mean(lambda_hat_exp) - lambda;
    variance_exp(n) = mean((lambda_hat_exp - mean(lambda_hat_exp)).^2); %

    % Calculate MSE, Bias, Variance for Rayleigh Distribution
    MSE_ray(n) = mean((lambda_hat_ray - lambda).^2);
    bias_ray(n) = mean(lambda_hat_ray) - lambda;
    variance_ray(n) = mean((lambda_hat_ray - mean(lambda_hat_exp)).^2);
end

% Plot MSE versus Number of Observations for both Exp and Ray
figure
semilogy(1:observations, MSE_exp)
xlabel('Number of Observations')
ylabel('MSE')

hold on
semilogy(1:observations, MSE_ray)

title('Figure 1: MSE versus Number of Observations')
legend('Exponential Distribution', 'Rayleigh Distribution')

% Plot Bias versus Number of Observations for both Exp and Ray
figure
subplot(1,2,1)
```

```
plot(1:observations,bias_exp)
xlabel('Number of Observations')
ylabel('Bias')
title('Exponential Distribution')

subplot(1,2,2)
plot(1:observations,bias_ray)
xlabel('Number of Observations')
ylabel('Distribution Bias')
title('Rayleigh Distribution')
sgtitle('Figure 2: Bias versus Number of Observations')

% Plot Variance versus Number of Observations for both Exp and Ray
figure
semilogy(1:observations,variance_exp)
hold on

semilogy(1:observations,variance_ray)
xlabel('Observations')
ylabel('Variance')
title('Figure 3: Variance versus Number of Observations')
legend('Exponential Distribution', 'Rayleigh Distribution')
```

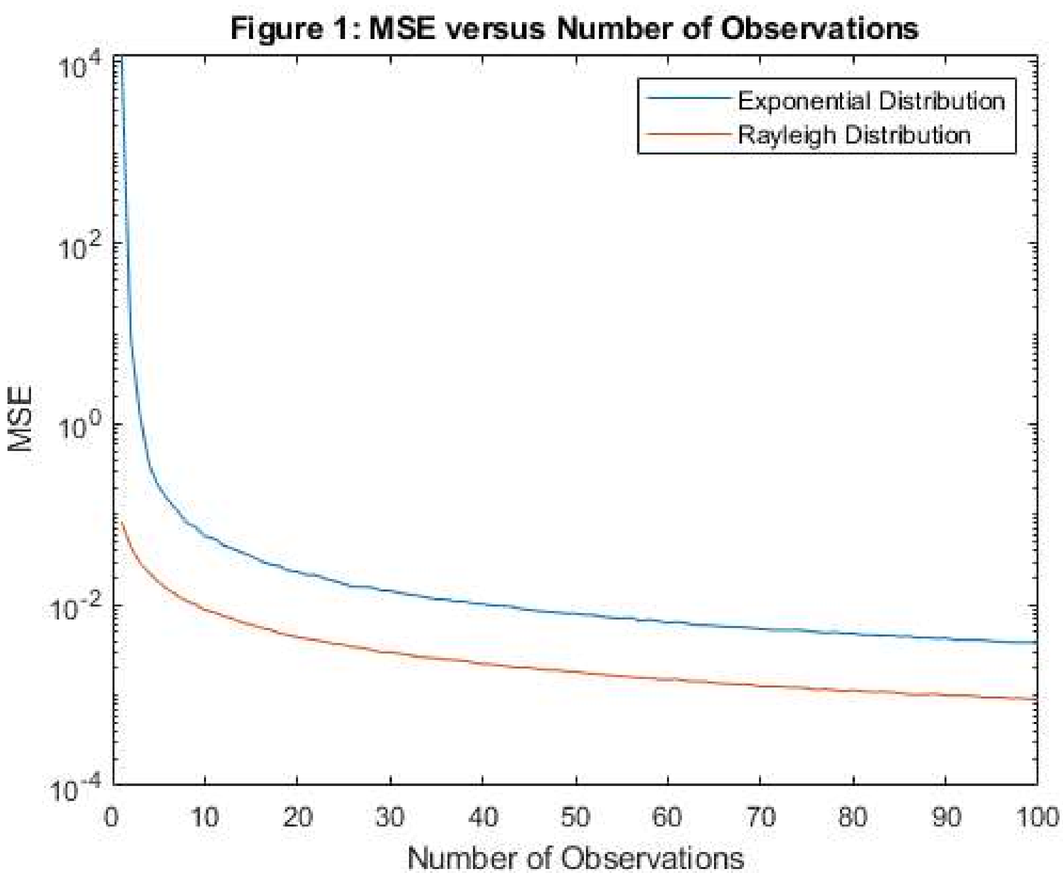


Figure 2: Bias versus Number of Observations

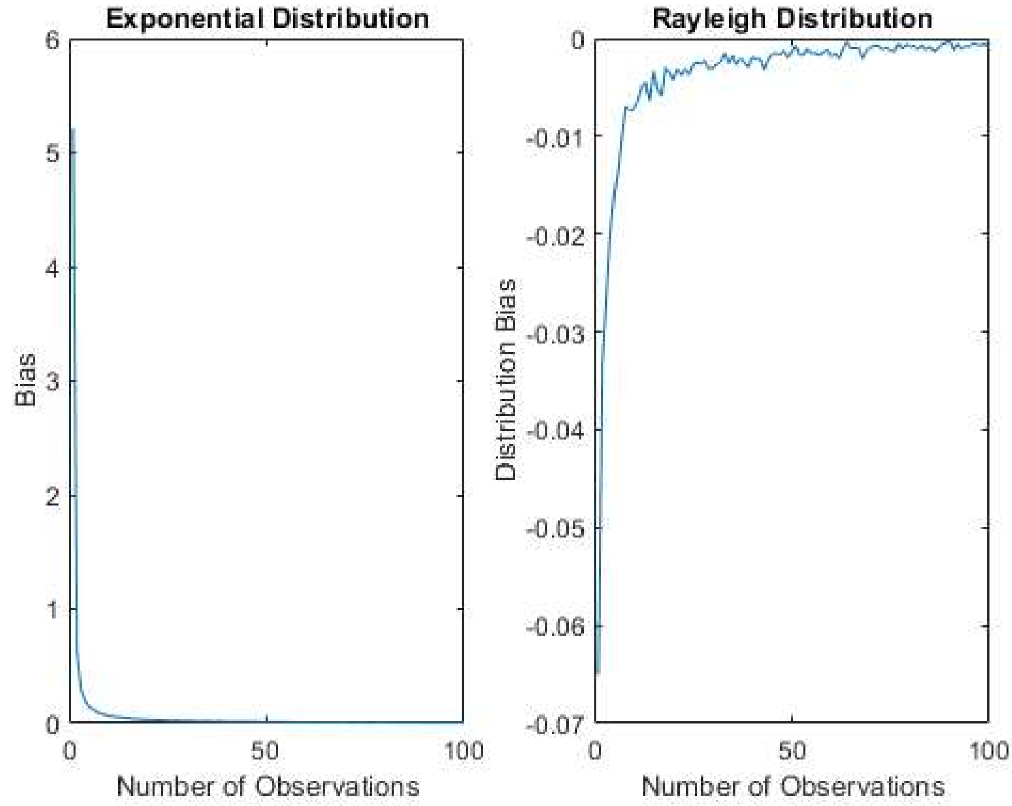
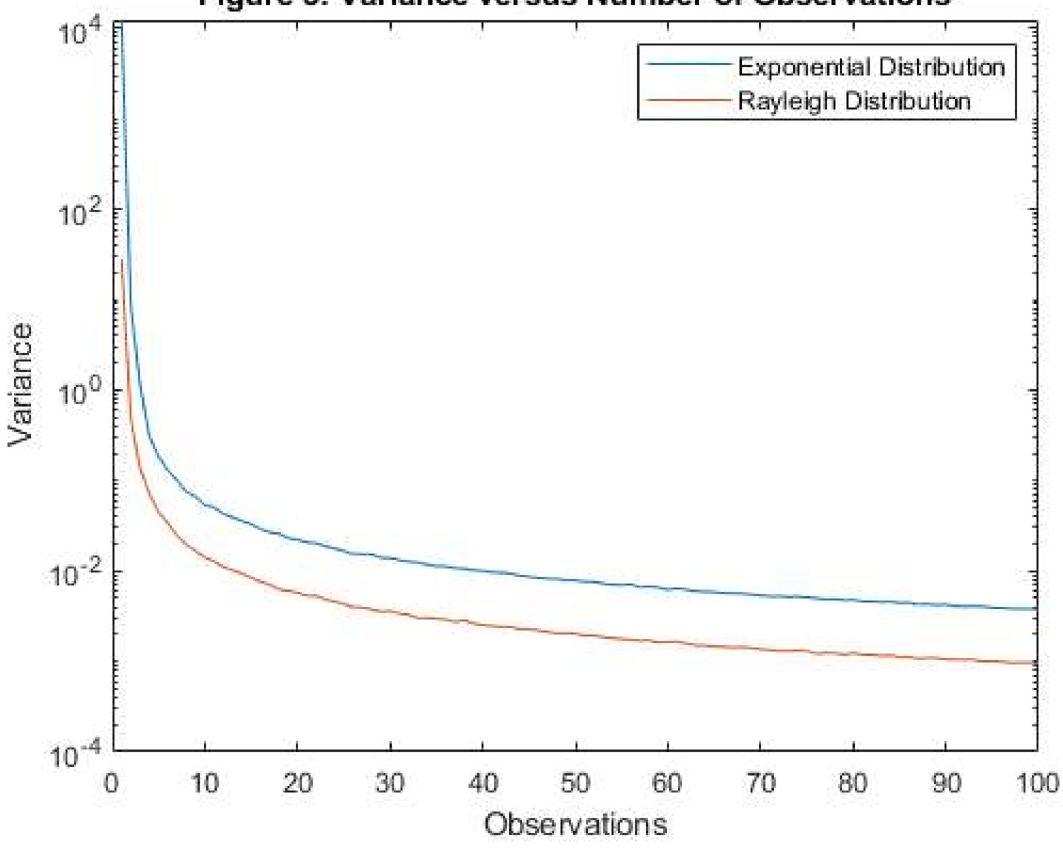


Figure 3: Variance versus Number of Observations



Part 3: Computing Max-Likelihood Estimate of Parameter to Predict the Distribution

A vector of data which was drawn from either an exponential or Rayleigh distribution was given. The log likelihood and estimates for parameter lambda were used to predict which distribution the data was drawn from. First, we used the ML estimator for both the exponential and Rayleigh distributions to estimate the lambda from the data. The estimate of lambda for the exponential distribution was 7.7948 and the estimate of lambda for the Rayleigh distribution was 0.1016. Given these lambda estimates, the log likelihood of the data was calculated for both distributions. The data is more likely to be drawn from the distribution that produced a higher log likelihood. The log likelihood for the exponential distribution is 1.0535e+03 and the log likelihood for the Rayleigh distribution is 1.3655e+03. The data is more likely from a Rayleigh distribution than a exponential distribution.

```
load('data (2).mat', 'data')

% Exponential
lambda_hat_exp = length(data)/sum(data); % Estimate Lambda
likeli_exp = length(data)*log(lambda_hat_exp) - lambda_hat_exp * sum(data) % Log Likelihood

% Rayleigh
lambda_hat_ray = sqrt(sum(data.^2)/(2*length(data))); % Estimate Lambda
likeli_ray = sum(log(data))-(1/(2*lambda_hat_ray.^2))*sum(data.^2)-2*length(data)*log(lambda_
hat_ray) % Log Likelihood
```

```
likeli_exp =

   1.0535e+03


likeli_ray =

   1.3655e+03
```

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Exponential

pdf $\lambda e^{-\lambda x}$

pdf of jth observations $f_x(x_j|\lambda) = \lambda e^{-\lambda x_j}$

Likelihood $L(x_n;\lambda) = f_x(x_j|\lambda)$

Log Likelihood $\ln L(x_1 \ldots x_n;\lambda) = \sum_{j=1}^{n} \ln f_x(x_j|\lambda)$

$$= \sum_{j=1}^{n} \ln(\lambda e^{-\lambda x_j})$$

$$= n \ln \lambda + \sum_{j=1}^{n} (-\lambda x_j)$$

$$= n \ln \lambda - \lambda \sum_{j=1}^{n} x_j$$

Differentiate wrt λ and set equal to 0

$$\frac{n}{\lambda} - \sum_{j=1}^{n} x_j = 0$$

$$\lambda = \frac{n}{\sum_{j=1}^{n} x_j}$$

$$\boxed{\hat{\lambda} = \frac{n}{\sum_{j=1}^{n} x_j}}$$

Rayleigh

pdf $\frac{x}{\lambda^2} e^{-x^2/2\lambda^2}$

pdf of jth observations $f_X(x_j|\lambda) = \frac{x_j}{\lambda^2} e^{-x_j^2/2\lambda^2}$

Likelihood $\mathcal{L}(x_n;\lambda) = f_X(x_j|\lambda)$

Log likelihood $\ln \mathcal{L}(x_1 \ldots x_n;\lambda) = \sum_{j=1}^{n} \ln f(x_j|\lambda)$

$$= \sum_{j=1}^{n} \ln\left(\frac{x_j}{\lambda^2} e^{-x_j^2/2\lambda^2}\right)$$

$$= \sum_{j=1}^{n}\left(\ln\left(\frac{x_j}{\lambda^2}\right) + \ln\left(e^{-x_j^2/2\lambda^2}\right)\right)$$

$$= \sum_{j=1}^{n} \ln x_j - \sum_{j=1}^{n} \ln \lambda^2 + \sum_{j=1}^{n} -x_j^2/2\lambda^2$$

$$= -2n\ln\lambda + \sum_{j=1}^{n} \ln x_j - \frac{1}{2\lambda^2}\sum_{j=1}^{n} x_j^2$$

Differentiate wrt λ and set equal to 0

$$\frac{-2n}{\lambda} + 0 + \frac{1}{\lambda^3}\sum_{j=1}^{n} x_j^2 = 0$$

$$\frac{1}{\lambda}\left(-2n + \frac{1}{\lambda^2}\sum_{j=1}^{n} x_j^2\right) = 0$$

$$-2n + \frac{1}{\lambda^2}\sum_{j=1}^{n} x_j^2 = 0$$

$$\lambda^2 = \frac{\sum_{j=1}^{n} x_j^2}{2n}$$

$$\lambda = \sqrt{\frac{\sum_{j=1}^{n} x_j^2}{2n}}$$

$$\boxed{\hat{\lambda} = \sqrt{\frac{\sum_{j=1}^{n} x_j^2}{2n}}}$$