Simple Formula for Nuclear Charge Radius *

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Abstract

A new formula for the nuclear charge radius is proposed, dependent on the mass number A and neutron excess $N-Z$ in the nucleus. It is simple and it reproduces all the experimentally available mean square radii and their isotopic shifts of even–even nuclei much better than other frequently used relations.

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1 Introduction

The aim of the present investigation is to find a phenomenological macroscopic formula for the nuclear radius. A charge distribution inside a nucleus depends not only on it's mass number A and the ground state deformation caused by shell effects, but also on the neutron excess $N-Z$. The equilibrium deformation of a nucleus can be established by the minimalization of the potential energy obtained by one of the microscopic methods. But to know the range (R) of nuclear charge in any (ϑ,φ) direction, it is necessary to describe properly the radius (R_{00}) of the corresponding spherical nucleus which has been deformed keeping its volume constant.

The simplest known liquid drop formula for the nuclear radius

$$R_{00} = r_0 \cdot A^{1/3} \tag{1}$$

with $r_0 = 1.2 fm$ doesn't lead to the experimentally known mean square radii $\langle r^2 \rangle$ (MSR) [1] or their isotopic shifts [2]. A popular liquid droplet model [3] gives also rather unsatisfactory results for MSR.

We are going to find the best dependence of R_{00} on mass number (A) and neutron excess ($N-Z$). A corresponding formula which reproduces the experimentally measured isotopic shifts of MRS was already proposed in refs. [4, 5]. That formula was obtained without the fit to the absolute values of the model independent of MSR data [1]. Now we would like to propose a new simple expression for the proton radius (R_{00}) which reproduces not only the isotopic shifts but also the magnitude of MSR for all even–even nuclei with proton numbers $Z \geq 8$.

2 Description of the calculation

The MSR values of even–even nuclei are calculated macroscopically with a uniform charge distribution

$$\rho_0 = \begin{cases} \frac{Ze}{V} & \text{for} \quad r \leq R \\ 0 & \text{for} \quad r > R \end{cases} \tag{2}$$

where V is the volume of a nucleus conserved during deformation

$$V = \frac{4}{3}\pi R_{00}^3 = \int_0^{2\pi} d\,\varphi \int_{-1}^{1} d\,\cos\vartheta \int_0^{R(\{\beta_\lambda\};\vartheta,\varphi)} r^2\,d\,r \ . \tag{3}$$

R is the radius dependent on deformation, calculated with the equilibrium shapes $\{\beta_\lambda\}$ taken from ref. [6].

The MSR values are given by the integral

$$\langle r^2 \rangle = \rho_0 \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \int_0^{R(\{\beta_\lambda\};\vartheta,\varphi)} r^4 dr \tag{4}$$

and they are evaluated for axially symmetric nuclei with the radius

$$R(\{\beta_\lambda\},\vartheta) = R_0(\{\beta_\lambda\}) \left[1 + \sum_{\lambda=1}^{6} \beta_\lambda P_\lambda(\cos\vartheta)\right] \ , \tag{5}$$

where P_λ are the Legender polynomials and

$$R_0(\{\beta_\lambda\}) = R_{00} \left\{ \frac{1}{2} \int_{-1}^{1} \left[1 + \sum_{\lambda=1}^{6} \beta_\lambda P_\lambda(\cos\vartheta) \right] d\cos\vartheta \right\}^{-1/3} \quad . \tag{6}$$

The last relation originates from the volume conservation condition when a nucleus deforms and R_{00} is the radius of the corresponding spherical nucleus. For practical use the formula (4) for small quadrupole deformations could be estimated approximatly by:

$$\begin{aligned} \langle r^2 \rangle &= \frac{Q_{00}}{Z} \left\{ 1 + \frac{8}{\sqrt{5\pi}} \beta^2 + \cdots \right\} \\ &= \frac{Q_{00}}{Z} \left\{ 1 + \frac{8}{5\sqrt{5\pi}} \left(\frac{Q_{20}}{Q_{00}} \right)^2 + \cdots \right\} , \end{aligned} \tag{7}$$

where $Q_{00} = \frac{3}{5} e\, Z\, R_{00}^2$ is the electric monopole moment of the spherical nucleus and Q_{20} is the electric quadrupole moment of the deformed nucleus.

The charge radius R_{00} in formula (6) is chosen in the following form (see also [4-5])

$$R_{00} = r_0 A^{1/3} \left(1 + \frac{\kappa}{A} - \alpha \frac{N-Z}{A} \right) \tag{8}$$

The set of r_0, κ and α parameters are obtained by the fit of $\langle r^2 \rangle^{exp}$ [1] and its isotopic shifts $\delta < r^2 >_{exp}^{A,A'}$ [2] for even–even nuclei with the proton numbers $Z \geq 8$. We have assumed that the sum of the mean square errors:

$$\Sigma^2_{\langle r^2 \rangle} = \sum_{\substack{i=1 \\ Z \geq Z_{min}}}^{86} \left(\langle r^2 \rangle_i - \langle r^2 \rangle_i^{exp} \right) \tag{9}$$

should be minimal. Simultaneously we have kept the minimal sum of square errors of MSR isotopic shifts for all experimentally measured sets of even–even isotopes

$$\delta \langle r^2 \rangle^{A,A'} = \langle r^2 \rangle^{A} - \langle r^2 \rangle^{A'} \tag{10}$$

i.e.

$$\Sigma^2_{\delta\langle r^2 \rangle} = \sum_{\substack{i=1 \\ Z \geq Z_{min}}}^{220} \left(\delta \langle r^2 \rangle_i^{A,A'} - \delta \langle r^2 \rangle_{i,exp}^{A,A'} \right) \quad . \tag{11}$$

The results obtained with our simple formula for the charge radius (8) will be compared in the next section with the liquid droplet estimates of MSR [3].

3 Results

After an analysis of 22 isotopic shifts of MSR for even–even nuclei we have proposed in Ref. [5] a formula for R_{00} containing the isotopic term,

$$R_{00} = r_0 A^{1/3} \left(1 - \alpha \frac{N-Z}{A}\right) \tag{12}$$

with the parameters $\alpha = 0.2$ and $r_0 = 1.25 fm$, but without the $\sim 1/A$ term, i.e. $\kappa = 0$. This formula was good enough to reproduce all the MSR isotopic shifts of even–even elements above $Z \geq 38$ and the average magnitude of MSR for the gold isotopes. Unfortunately it was rather impossible to get a reasonable agreement of MSR values of lighter elements $8 \leq Z < 38$ without introducing an additional κ/A term. Fitting R_{00} described by formula (6) to the MSR [1] and $\delta\langle r^2\rangle^{A,A'}$ [2] data for all the $Z \geq 8$ elements we have recorded the following set of parameters:

$$r_0 = 1.240 fm, \;\; \kappa = 1.646, \;\; \alpha = 0.191$$

. In every case the sum of square errors of MSR is more than six times lower than in the liquid droplet model [3]. It is interesting, that the α parameter is almost stable around $\alpha \sim 0.2$, independently on the number of fitted data and inclusion of the $\sim 1/A$ term. The values r_0 and α are almost the same for the case of $Z > 38$ elements without the κ/A in R_{00}.

We have also tested the different powers of A in the new term κ/A^n ($n = \frac{1}{3}, \frac{2}{3}, \cdots 2$). We have achives the best agreement for $n = 1$, but for $n = \frac{1}{3}$ and $\frac{2}{3}$ the quality of the fitment was comparable with that for $n = 1$.

The results of our investigation are illustrated in the figures 1-5. In the Figs. 1-3 the results of the MSR values obtained in the three cases of R_{00} are illustrated.

In Fig. 1 the MSR values obtained with the traditional liquid drop formula (1) are plotted. They are obviously to small even for the spherical case (dashed line) in comparison with the experimental data [1] marked by stars. The difference between the macroscopic results including the effect of the deformations of nuclei taken from Ref. [6] and the experimental data $\langle r^2\rangle_{exp}$ are represented by the length of vertical lines shown under each experimentally known element ("bars"). The differences are rather large, especially for the light elements.

In Fig. 2 the MSR results were obtained using our previous formula [5], but with the better set of parameters ($r_0 = 1.256 fm$, $\alpha = 0.202$), fitting, both $< r^2 >$ and $\delta\langle r^2\rangle^{A,A'}$ values. The fitment is much better than that in Fig. 1, both for heavier and lighter elements, even in the spherical case. When including deformation, the differences (bars on the figure) between macroscopic and experimental value $\langle r^2\rangle_{th} - \langle r^2\rangle_{exp}$ almost disappear, but the lighter elements are still not well reproduced. It was necessary to introduce an additional term in the formula for the charge radius in order to increase the model estimates of MSR for the lightest nuclei.

In Fig. 3 the MSR results for the formula (8) with the best set of parameters fitted to the $\langle r^2\rangle_{exp}$ and $\delta\langle r^2\rangle^{A,A'}_{exp}$ data are shown. The agreement is good. In spite of a few cases the differences between the macroscopic and experimental MSR values have almost disappeared when the deformations of the nuclides were taken into account [6].

One can compare these results with the liquid droplet calculation according to ref. [3] illustrated in Fig. 4, where the theoretical results are much smaller than the experimental data. Also the results for MSR obtained with the newest version of the droplet model [6] are only a little better than those of [3] in Fig. 4. It is obvious that a new analysis of liquid droplet parameters in order to reproduce MSR values is needed. We hope that our investigation will be helpful.

The importance of introducing $(N-Z)$ dependence in R_{00} is illustrated in Figs. 5-6, where the MSR isotopic shifts $\delta\langle r^2\rangle^{A,A'}_{exp}$ of all the experimentally known even–even isotopes [2] are compared with our estimates.

In Fig. 5 the differences between the macroscopic results calculated with equilibrium deformations of [7] and experimental data [1]

$$\Delta\langle r^2\rangle^{A,A'} \equiv \delta\langle r^2\rangle^{A,A'}_{th} - \delta\langle r^2\rangle^{A,A'}_{exp}$$

are plotted in the case of the traditional liquid drop formula (1). The differences are large. They reach even $1\,fm^2$ for some nuclei. When including the R_{00} depending on isospin $\frac{N-Z}{A}$ (Fig. 6), these differences decrease significantly. One has to add that the same quality of the fit of $\delta\langle r^2\rangle^{A,A'}$ as with formula (8) was already achieved for the two parametric formula (12), but the magnitude of MSR values for light nuclei are not well reproduced.

4 Conclusions

We can draw the following conclusions from our investigations:

- The traditional liquid drop formula, depending on mass number A only, is not able to give a good mean square nuclear charge radius $\langle r^2\rangle$ or its isotopic shifts.

- In order to reproduce the data of $\langle r^2\rangle$ and $\delta\langle r^2\rangle$ of the even–even nuclei one should use the three parametric formulas for the spherical nuclear charge radius

$$R_{00} = 1.240\,A^{1/3}\left(1 + \frac{1.646}{A} + 0.191\frac{N-Z}{A}\right) fm$$

- For the nuclei with $Z \geq 38$ one can omit the $\sim 1/A$ term and use the formula

$$R_{00} = 1.256\,A^{1/3}\left(1 - 0.202\frac{N-Z}{A}\right) fm$$

- The liquid droplet model gives systematically larger MSR estimates of the nuclear charge.

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6 Figures Captions

Fig. 1. The mean square charge radii in fm^2 of even even nuclei obtained without deformation (dashed line) with the traditional formula liquid drop compared with experimental data [1] (crosses). The differences between the macroscopic results evaluated including equilibrium deformations [6] and the experimental data [1] are illustrated by the bars around the ordinates line. The macroscopic MSR values are to small.

Fig. 2. The same as in Fig. 1 but with the isospin dependence in the R_{00} formula (12). The differences for light nuclei are still of the order $\sim 1fm^2$.

Fig. 3. The same as in Fig. 1 but with our new formula for R_{00} (8). The experimental data is almost ideally reproduced.

Fig. 4. The same as in Fig. 1 for the liquid droplet macroscopic results with parameters of Ref. [3]. The theoretical values are to small up to $2fm^2$.

Fig. 5. The differences between the liquid drop and the experimental isotopic shifts of the MSR of the charge in fm^2. The discrepancy between these macroscopic results and the experimental data exceed $1fm^2$.

Fig. 6. The same as in Fig. 5 for the three parametric R_{00} formula (7). The deviations between our theoretical estimates and experimental data do not exceed $0.5fm^2$.