115AH HW 5

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1 Problem 1

(a): False. The dimension is $n+1$ since a basis of this space is $\{1, x, ..., x^n\}$.
(b): True. By the dimension theorem, every basis of the same V.S has the same cardinality (so this applies in the finite case).
(c): True. By the Steinitz-Exchange Lemma, the cardinality of any independent set must be less than or equal to that of any spanning set.
(d): True. This is what the Trillium Theorem says. If we have n independent vectors, then they must be spanning and vice versa.
(e): True. If a subspace has dimension 0, then its basis must be the empty set. We showed that if a subspace has dimension n, then it must be equal to V itself.
(f): False. The rank-nullity theorem says that if $T : V \to W$, then T's rank and nullity sum to $dim(V)$, not $dim(W)$.

2 Problem 2

The dimension of R^3 is 3 (considering the standard basis i,j,k) so any lin. indep set must have at most 3 elements. Hence this set with 4 distinct elements cannot be a basis.

3 Problem 3

(a): Claim: $dim(W_1) = dim(W_2)$ if and only if $v \in Span(\{v_1, ..., v_k\})$.

Forward direction: Clearly, since $\{v_1, ..., v_k\} \subseteq \{v_1, ..., v_k, v\}$, then $Span(\{v_1, ..., v_k\}) \subseteq Span(\{v_1, ..., v_k, v\})$ so $W_1 \subseteq W_2$. Since W_1 is a subspace of v.s. W_2 and has the same dimension as W_2, then it must be equal to W_2 itself (we proved this result in class). Hence, $W_1 = W_2$. Clearly, $v \in Span(\{v_1, ..., v_k, v\})$ since $v = 1 \cdot v$, so equivalently, $v \in W_2$. Since $W_1 = W_2$, then $v \in W_1 = Span(\{v_1, ..., v_k\})$ as desired.
Backward direction: Assume $v \in Span(\{v_1, ..., v_k\})$. We want to show $Span(\{v_1, ..., v_k\}) = Span(\{v_1, ..., v_k, v\})$ or equivalently, $W_1 = W_2$.

Then, we know $\{v_1, ..., v_k\} \subseteq Span(\{v_1, ..., v_k\})$ since span of a set contains itself. Also, $\{v\} \subseteq Span(\{v_1, ..., v_k\})$ follows from our assumption. Therefore,

$$\{v_1, ..., v_k\} \cup \{v\} = \{v_1, ..., v_k, v\} \subseteq Span(\{v_1, ..., v_k\})$$

Hence,

$$Span(\{v_1, ..., v_k, v\}) \subseteq Span(\{v_1, ..., v_k\})$$

The other direction is clear. We know that

$$\{v_1, ..., v_k\} \subseteq \{v_1, ..., v_k, v\}$$

Since span preserves this relation, we have

$$Span(\{v_1, ..., v_k\}) \subseteq Span(\{v_1, ..., v_k, v\})$$

Having showed both containment directions,

$$Span(\{v_1, ..., v_k\}) = Span(\{v_1, ..., v_k, v\})$$

Hence,

$$W_1 = W_2$$

so clearly their dimensions are equal.

4 Problem 4

Let us define $U_1 = \{(v, 0) | v \in V\}$, $U_2 = \{(0, w) | w \in W\}$. On a previous HW, we showed $Z = U_1 \oplus U_2$, with U_1, U_2 being subspaces. This means $Z = U_1 + U_2$ and $U_1 \cap U_2 = \{0\}$ by definition. Using problem 6(b)'s trillium theorem, this implies $dim(Z) = dim(U_1) + dim(U_2)$.
Now, let $\{v_1, ..., v_k\}$ be a basis for V. I claim $\{(v_1, 0), ..., (v_k, 0)\}$ is a basis for U_1.
Spanning: Let $(v, 0) \in U_1$. Now, since $v \in V$, it can be represented as

$$v = a_1v_1 + ... + a_kv_k$$

for some scalars $a_1, ..., a_k \in F$. Consider the sum

$$a_1(v_1, 0) + ... + a_k(v_k, 0) = (a_1v_1, 0) + ... + (a_kv_k, 0)$$
$$= (a_1v_1 + ... + a_kv_k, 0) = (v, 0)$$

Hence, $(v, 0) \in Span(\{(v_1, 0), ..., (v_k, 0)\})$ so

$$U_1 \subseteq Span(\{(v_1, 0), ..., (v_k, 0)\})$$

Evidently, $\{(v_1, 0), ..., (v_k, 0)\} \subseteq U_1$ so $Span(\{(v_1, 0), ..., (v_k, 0)\}) \subseteq U_1$, proving the second direction. Therefore our set is spanning.
Independent: Let $a_1, ..., a_k \in F$ be such that

$$a_1(v_1, 0) + ... + a_k(v_k, 0) = 0$$
$$(a_1v_1 + ... + a_kv_k, 0) = 0$$

Note that the zero element of Z is $(0_V, 0_F)$. Equating these components, it must hold that

$$a_1v_1 + ... + a_kv_k = 0_V$$

Since $\{v_1, ..., v_k\}$ is a basis for V and thus independent, then $a_1 = a_2 = ... = a_k = 0$ as desired.
Since it's both spanning and independent, then $\{(v_1, 0), ..., (v_k, 0)\}$ is a basis for U_1 and it has k distinct elements. Hence, $dim(U_1) = k = dim(V)$. By analogous reasoning, $dim(U_2) = dim(W)$.

Therefore, $dim(Z) = dim(U_1) + dim(U_2)$ implies that

$$dim(Z) = dim(V) + dim(W)$$

5 Problem 5

Claim: the set $\{(x-a), x(x-a), ..., x^{n-1}(x-a)\}$ is a basis for W. **Independent:** Let $b_1, ..., b_n \in R$ s.t.

$$b_1(x-a) + ... + b_n x^{n-1}(x-a) = 0$$

$$b_1 x + b_2 x^2 + ... + b_n x^n - a(b_1 + ... + b_n) = 0$$

All the coefficients must be 0 in the 0 polynomial, so

$$b_1 = ... = b_n = 0$$

must hold as desired.

Now, note that W is a subspace of $P_n(R)$ so $dim(W) \leq dim(P_n(R))$. It follows that $dim(W) = dim(P_n(R))$ if and only if $W = P_n(R)$. However, W is a proper subspace since there exists $f \in P_n(R), f \notin W$. For instance, $f = 1$. Hence, $W \neq P_n(R)$ so $dim(W) \neq dim(P_n(R))$, which is $n+1$. Thus, $dim(W) \leq n$ The set above is linearly independent, so it must have less or equal elements to any spanning set, including the basis. Hence, $n \leq dim(W) \leq n$, showing $dim(W) = n$. Hence, $\{(x-a), x(x-a), ..., x^{n-1}(x-a)\}$ is independent and has as many elements as the dimension, proving that it is spanning. So $\{(x-a), x(x-a), ..., x^{n-1}(x-a)\}$ is a basis and the $dim(W) = n$.

6 Problem 6

(a): Since W_1, W_2 are subspaces, then $W_1 \cap W_2$ must be a subspace as we showed. Since $W_1 \cap W_2 \subseteq W_1$ and W_1 has a finite basis, then $W_1 \cap W_2$ must have finite dimension too (b/c $dim(W_1 \cap W_2) \leq dim(W_1)$).
Let $\{u_1, .., u_k\}$ be a basis of $W_1 \cap W_2$. We can extend it to a basis of W_1 that $dim(W_1) = k + m$ (note, independence implies all the vectors must be distinct so we know there are exactly $k + m$ elements):

$$\{u_1, .., u_k, v_1, ..., v_m\}$$

Similarly, extend to a basis of W_2 so that $dim(W_2) = k + p$:

$$\{u_1, .., u_k, w_1, ..., w_p\}$$

Claim: $B_{12} = \{u_1, .., u_k, v_1, ..., v_m, w_1, ..., w_p\}$ is a basis of $W_1 + W_2$.

Spanning: Since $B_{12} \subseteq V$, then $Span(B_{12}) \subseteq V$. Then, let $v \in W_1 + W_2$. Then, $v = w_1 + w_2$ for some $w_1 \in W_1$, $w_2 \in W_2$. Representing w_1, w_2 with their corresponding bases,

$$w_1 = a_{11}u_1 + ... + a_{1k}u_k + a_{21}v_1 + ... + a_{2m}v_m$$

$$w_2 = b_{11}u_1 + ... + b_{1k}u_k + b_{21}w_1 + ... + b_{2p}v_p$$

$$v = (a_{11} + b_{11})u_1 + ... + (a_{1k} + b_{1k})u_k + a_{21}v_1 + ... + a_{2m}v_m + b_{21}w_1 + ... + b_{2p}v_p$$

Thus $v \in Span(B_{12}$ so $V \subseteq Span(B_{12})$. With both containment directions, this shows $Span(B_{12})$ spans V.

Independent: Let $a_1, .., a_k, b_1, ..., b_m, c_1, ..., c_p \in F$ s.t.

$$a_1u_1 + ... + a_ku_k + b_1v_1 + ... + b_mv_m + c_1w_1 + ... + c_pw_p = 0$$

$$(a_1u_1 + ... + a_ku_k) + (c_1w_1 + ... + c_pw_p) = -(b_1v_1 + ... + b_mv_m)$$

Since $u_1, ..., u_k \in W_1 \cap W_2$, then the LHS is in $W_1 \cap W_2$ due to closure of subspaces under linear combination. In particular, $a_1u_1+...+a_ku_k \in W_2$. Since $w_1, ..., w_p \in W_2$, then $-(c_1w_1+...+c_pw_p) \in W_2$. Thus, by closure of addition, $-(b_1v_1 + ... + b_mv_m) \in W_2$. But also, since $v_1, ..., v_m \in W_1$, it is also in W_1. So we can represent this with our basis of $W_1 \cap W_2$. There exist $d_1, ..., d_k \in F$ s.t.

$$-(b_1v_1 + ... + b_mv_m) = d_1u_1 + ... + d_ku_k$$

$$d_1u_1 + ... + d_ku_k + b_1v_1 + ... + b_mv_m$$

Since $\{u_1, ..., u_k, v_1, ..., v_m\}$ is a basis for W_1, then the vectors must be independent. This implies that $d_1 = ... = d_k = b_1 = ... = b_m = 0$, which means

$$b_1v_1 + ... + b_mv_m = 0$$

By analogous reasoning on $c_1w_1 + ... + c_pw_p \in W_1 \cap W_2$, we can also conclude $c_1 = ... = c_p = 0$ and that

$$c_1w_1 + ... + c_pw_p = 0$$

Plugging this into our original linear combination, we are left with

$$a_1u_1 + ... + a_ku_k = 0$$

Since $\{u_1, ...u_k\}$ forms a basis of $W_1 \cap W_2$, it is independent and thus $a_1 = ... = a_k = 0$.

Therefore, we've shown

$$a_1 = ... = a_k = b_1 = ... = b_m = c_1 = ... = c_p = 0$$

So our set B_{12} is linearly independent (so all the vectors must be distinct otherwise that contradicts independence).

Hence, B_{12} is a basis with $k+p+m$ elements, so $dim(W_1+W_2) = k+p+m = (k+m)+(k+p)-k$, or equivalently,

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

(b): First, we prove a fact that we'll use: $\emptyset$ is a basis for $W_1 \cap W_2$. $Span(\emptyset) = \{0\}$ since 0 is the smallest subspace containing $\emptyset$. Also, the $\emptyset$ is independent since we cannot take any linear combinations (so no non-trivial ones summing to 0).

Assume (1) and (2) hold: By 6(a),

$$dim(V) = dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

Since $W_1 \cap W_2$ $\{0\}$, which has basis $\emptyset$, then $dim(W_1 \cap W_2) = 0$, yielding

$$dim(V) = dim(W_1) + dim(W_2)$$

By definition, V is the int. direct sum of W_1, W_2 (since (1) and (2) hold).

Assume (2) and (3) hold: By 6(a),

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2) =$$

Again, $W_1 \cap W_2$ $\{0\}$ implies $dim(W_1 \cap W_2) = 0$, showing that

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2)$$

By (3), this means $dim(V) = dim(W_1 + W_2)$. But since $W_1 + W_2$ is a subspace of V, then by the result from our class, it must hold that $V = W_1 + W_2$ as desired. By definition, V is the int. direct sum of W_1, W_2 (since (1) and (2) hold).

Assume (1) and (3) hold: By 6(a),

$$dim(V) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2)$$

By (3), we have

$$dim(V) = dim(W_1) + dim(W_2)$$

This implies $dim(W_1 \cap W_2) = 0$, so the basis of $W_1 \cap W_2$ must have 0 elements and is thus the empty set. Having shown the empty set is a basis and thus spans $\{0\}$, then $dim(W_1 \cap W_2) = 0$ implies $W_1 \cap W_2 = \{0\}$. By definition, V is the int. direct sum of W_1, W_2 (since (1) and (2) hold).

7 Problem 7

(a): Let $B_1 = \{w_1, .., w_k\}$ be a basis of W_1. Extend it to a basis $B = \{w_1, .., w_k, v_1, ..., v_{n-k}\}$ of V. Let $B_2 = v_1, ..., v_{n-k}$. Then, let $W_2 = Span(v_1, ..., v_{n-k})$. Clearly, B_2 spans W_2 and since $B_2 \subseteq B$ and B is independent, then B_2 is independent. Thus, B_2 is a basis for W_2.
As B is independent, all of its vectors are distinct. This means $B_1 \cap B_2 = \emptyset$, otherwise we would have some $w_i = v_j$ where $w_i \in B_1$, $w_2 \in B_2$. Moreover, $B_1 \cup B_2 = \{w_1, .., w_k, v_1, ..., v_{n-k}\}$ is a basis for V. Moreover, W_1 and W_2 are clearly two subspaces for V. By 10(b) on homework 4, we can conclude $V = W_1 \oplus W_2$ Thus, for any subspace W1 of V, we can construct a subspace W_2 of V such that $V = W_1 \oplus W_2$.
(b): Example one: Let $W_2 = \{(0, y) | y \in R\}$ be the y-axis. Claim: $R^2 = W_1 \oplus W_2$. First, note that $(0, 0) \in W_1 \cap W_2$. Let $w \in W_1 \cap W_2$. Then, equating its two representations, $(0, y) = (x, 0)$ meaning $y = x = 0$, so $w = (0, 0)$. This shows $W_1 \cap W_2 = \{(0, 0)\}$. As a subspace, $W_1 + W_2 \subseteq V$. Now, each $(x, y) \in R^2$ can be written as $(x, y) = (x, 0) + (0, y)$ with $(x, 0) \in W_1, (0, y) \in W_2$. Hence, $V \subseteq W_1 + W_2$, implying $V = W_1 + W_2$. As expected, $V = W_1 \oplus W_2$.
Example two: Let $W_2' = \{(a, a) | a \in R\}$. Clearly, $\{0\} \subseteq W_1 \cap W_2'$. Let $(x, y) \in W_1 \cap W_2'$. Then, equating the two representations,

$$(0, y) = (a, a)$$

Equating the components, this means $a = 0$ and $a = y = 0$. Hence $(x, y) = (0, 0)$ so $W_1 \cap W_2' \subseteq \{0\}$. Since $W_1 + W_2'$ is a subspace of V, then $W_1 + W_2' \subseteq V$. Let $v = (x, y) \in V$. Then,

$$(x, y) = (0, y - x) + (x, x)$$

where $(0, y - x) \in W_1$ and $(x, x) \in W_2'$. Hence, $V \subseteq W_1 + W_2'$. Thus, $V = W_1 \oplus W_2'$.

We've found two different examples W_2, W_2' as desired.

8 Problem 8

(a): By the rank-nullity theorem, $dim(im(T)) + dim(ker(T)) = dim(V)$. Isolating, we get

$$dim(im(T)) = dim(V) - dim(ker(T)) \leq dim(V) < dim(W)$$

Hence, $dim(im(T)) < dim(W)$ so $im(T) \neq W$. We stated that T is surjective if and only if $im(T) = W$, so evidently, T cannot be surjective.
(b): By rank-nullity,

$$dim(ker(T)) = dim(V) - dim(im(T)) > dim(W) - dim(im(T))$$

Since $im(T) \subseteq W$, then $dim(im(T)) \leq dim(W)$ or $dim(W) - dim(im(T)) \geq 0$. Combining these inequalities, we have the strict inequality

$$dim(ker(T)) = dim(V) - dim(im(T)) > 0$$

We proved that T is injective if and only if the kernel is $\{0\}$. Since $\emptyset$ is a basis for this set, then $dim(ker(T))$ must be exactly 0 for T to be injective. However, we showed $dim(ker(T)) > 0$ so this is not possible.

9 Problem 9

(a): First, we're given $V = im(T)+ker(T)$. Then rank nullity tells us $dim(im(T))+dim(ker(T)) = dim(V)$. Clearly, $im(T), ker(T)$ are both subspaces of V. Using 6(b), we conclude $V = im(T) \oplus ker(T)$
(b): First, we're given $im(T) \cap ker(T) = \{0\}$. Then rank nullity tells us $dim(im(T)) + dim(ker(T)) = dim(V)$. Clearly, $im(T), ker(T)$ are both subspaces of V. Using 6(b), we conclude $V = im(T) \oplus ker(T)$
(c): Claim: $im(T) = \{(x, 0)|x \in R\}$. First, for any $(x, 0)$, $(x, 0) = T(0, x) \in im(T)$, so $\{(x, 0)|x \in R\} \subseteq im(T)$. Then, let $(y, 0) \in im(T)$. Clearly, $(y, 0) \in \{(x, 0)|x \in R\}$ so $im(T) \subseteq \{(x, 0)|x \in R\}$.
Claim: $ker(T) = \{(x, 0)|x \in R\}$. First, for any $(x, 0)$, $T(x, 0) = (0, 0)$ so $(x, 0) \in ker(T)$. Then, let $(x, y) \in ker(T)$. Then, $T(x, y) = (y, 0) = (0, 0)$, so this means $y = 0$. Hence, $(x, y) = (x, 0) \in \{(x, 0)|x \in R\}$.
Thus, $im(T) = ker(T) = \{(x, 0)|x \in R\}$, so $im(T)+ker(T) = \{(x, 0)|x \in R\}$. To show this, assume $(x, y) \in im(T) + ker(T)$. Then, it has the form $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0) \in \{(x, 0)|x \in R\}$. Then, consider $(x, 0) \in \{(x, 0)|x \in R\}$. $(x, 0) = (x, 0) + (0, 0)$ so $(x, 0) \in im(T) + ker(T)$.

$$im(T) + ker(T) = \{(x, 0)|x \in R\} = im(T) \cap ker(T)$$

10 Problem 10

(a): Since $im(T), ker(T)$ are subspaces of V (since $T : V \to V$), then $im(T) + ker(T)$ is also a subspace of V so $im(T) + ker(T) \subseteq V$. Then, let $v \in P(R)$. Since we showed T is onto, there exists $v' \in P(R)$ s.t. $v = T(v') = T(v') + 0$. Clearly, $T(v') \in im(T)$ and $0 \in ker(T)$ since the derivative of any constant function is 0. Thus, $v \in im(T) + ker(T)$ so $V \subseteq im(T) + ker(T)$. With both containments, we have $V = im(T) + ker(T)$. However, any constant function $f(x) = c \in R$

is in $im(T)$ since T is onto. Also $f(x) \cap ker(T)$ since it is a constant function. Hence, $f(x) = 1 \in im(T) \cap ker(T)$ and $1 \neq 0$, then $im(T) \cap ker(T) \neq \{0\}$. Therefore, V is not the direct sum of $im(T)$ and $ker(T)$.
(b): Since we showed T is 1-1, then $ker(T) = \{0\}$. Claim: $im(T) + ker(T) = im(T)$. Let $w \in im(T) + ker(T)$. Then, $w = w_1 + w_2$ for $w_1 \in im(T), w_2 \in ker(T)$. But since the kernel is trivial, $w_2 = 0$ so $w = w_1 \in im(T)$. So $im(T) + ker(T) \subseteq im(T)$. Moreover, $im(T) \subseteq im(T) + ker(T)$ by properties of subspaces. Now consider any constant function $f(x) = c$ in V. In HW 3, we showed that polynomials in $im(T)$ must have no constant term. To show this, let $f(x) \in P(R)$ be arbitrary, so $f(x) = a_n x^n + ... + a_1 x + a_0$. Then,

$$T(f)(x) = \frac{a_n}{n+1} x^{n+1} + ... + \frac{a_1}{2} x^2 + a_0 x$$

Thus, $f(x) = c \notin im(T)$. This means we've found a function $f(x) \in V$ yet $f(x) \notin im(T) + ker(T)$. Therefore, $V \neq im(T) + ker(T)$
This shows that in problem 9, the assumption that V is finite is crucial. Evidently, in these infinite-dimensional vector spaces, the same conclusion may fail to hold.

11 Problem 11

T is linear: Let $c \in R$, $f, g \in P_n(R)$.

$$T((cf + g)(x)) = (cf + g)(a)x^n + (cf + g)'(x)$$
$$= (cf(a) + g(a))x^n + (cf(x) + g(x))'$$
$$= cf(a)x^n + g(a)x^n + cf'(x) + g'(x)$$
$$= c(f(a)x^n + f'(x)) + g(a)x^n + g'(x)$$
$$= cT(f(x)) + T(g(x))$$

Having shown $T(cf + g) = cT(f) + T(g)$, T is linear.
T is 1-1: This is equivalent to proving the kernel is trivial. Since T is linear $T(0) = 0$ so $\{0\} \subseteq ker(T)$. Now, let $f(x) \in ker(T)$ this means

$$T(f(x)) = f(a)x^n + f'(x) = 0$$

Since $deg(f) \leq n$, then $deg(f') \leq n - 1$. Hence, the coefficient of x^n in $f(a)x^n + f'(x)$ must be $f(a)$, so it must hold that $f(a) = 0$. This implies $f'(x) = 0$, so $f(x)$ must be the constant function. However, since $f(a) = 0$, $f(x)$ must be the 0 function specifically. Thus, $ker(T) \subseteq \{0\}$. With both directions, we've shown $ker(T) = \{0\}$ so $dim(ker(T)) = 0$ since $\emptyset$ is a basis. Thus, T is 1-1. **T is onto:** Let $V = P_n(R)$ By the rank-nullity theorem, $dim(ker(T)) + dim(im(T)) = dim(V)$. Substituting $dim(ker(T)) = 0$, we get

$$dim(im(T)) = dim(V)$$

Since $T : V \to V$, then $im(T)$ is a subspace of V so $im(T) \subseteq V$ yet $dim(im(T)) = dim(V)$. This implies $im(T) = V$. Since $im(T)$ is the entire codomain, T is onto.
Since it is linear, 1-1, and onto, T is an isomorphism.

12 Problem 12

Let $T(v) \in T(0)$ for $v \in \{0\}$. Then, $T(v) = T(0) = 0 \in \{0\}$ by properties of linear functions. So $T(\{0\}) \subseteq \{0\}$.
Let $T(v) \in T(V)$ for some $v \in V$. $T(v) \in im(T) \subseteq V$, so $T(v) \in V$. So $T(V) \subseteq V$.
Let $T(v) \in T(im(T))$ for some $v \in im(T)$. $T(v) \in im(T)$ since $v \in V$, so $T(im(T)) \subseteq im(T)$.
Let $T(v) \in T(ker(T))$ for some $v \in ker(T)$. $T(v) = 0 \in ker(T)$. Thus $T(ker(T)) \subseteq ker(T)$.

13 Problem 13

Let $T(w_1) \in T(W_1)$ for some $w_1 \in W_1$. Then, $w_1 = w_1 + 0$ is its unique representation where $w_1 \in W_1, 0 \in W_2$. So $T(w_1) = T(w_1 + 0) = w_1$. Thus, $T(w_1) \in W_1$ so this shows $T(W_1) \subseteq W_1$ so W_1 is T-invariant. More specifically, T_{W_1} is the identity function on W_1.
Let $T(w_2) \in T(W_2)$ for some $w_2 \in W_2$. Then, $w_2 = 0 + w_2$ is its unique representation where $0 \in W_1, w_2 \in W_2$. So $T(w_2) = T(0 + w_2) = 0$. Thus, $T(w_2) \in W_2$ since all subspaces contain 0. This shows $T(W_2) \subseteq W_2$ so W_2 is T-invariant. More specifically, T_{W_2} is 0 function.

14 Problem 14

(a): Let $x \in W$. Then $T(x) \in im(T)$ but since W is T-invariant, $T(x) \in T(W) \subseteq W$. Therefore, $T(x) \in im(T) \cap W$. By definition of internal direct sum, $im(T) \cap W = \{0\}$ so $T(x) = 0$ so $x \in ker(T)$. This shows $W \subseteq ker(T)$
(b): By rank-nullity, $dim(V) = dim(im(T)) + dim(ker(T))$. Applying 6(a) (since V and thus any subspace is finite dimensional), we have

$$dim(V) = dim(im(T)) + dim(W) - dim(im(T) \cap W)$$

Note, $dim(im(T) \cap W) = dim(\{0\}) = 0$ as established before. Thus,

$$dim(V) = dim(im(T)) + dim(W)$$

This implies $dim(W) = dim(ker(T))$ In part (a), we showed $W \subseteq im(T)$ so W is a subspace of $im(T)$ and has the same dimension. This implies $W = ker(T)$.
(c): Let $V = P(R)$ the space of all polynomials with real coefficients. We know V is infinite dimensional. Let $T : V \to V$ be s.t. $T(f) = f'$ be the derivative map. We showed this is linear and onto, so $im(T) = V$. Let $W = \{0\}$. Claim: $V = im(T) \oplus W$. First, let $x \in im(T) \cap W$. Then, since $im(T) \cap W \subseteq W$, then $x \in \{0\}$. Clearly $\{0\} \subseteq im(T) \cap W$ since they're subspaces. Thus, $im(T) \cap W = 0$.
Let $v \in V$. Then, $v = v + 0$, where $v \in im(T)$ since T is onto, and $0 \in W = \{0\}$. So $V \subseteq im(T) + W$. Since they are both subspaces of V, $im(T) + W \subseteq V$, showing $V = im(T) + W$. Hence, $V = im(T) \oplus W$.

However, in this case $ker(T)$ consists of all constant real functions, including $f(x) = 1 \neq 0$. Hence, $f(x) = 1 \in ker(T)$ yet $f(x) = 1 \notin W$. Thus, W is only a proper subset of $ker(T)$.