115AH HW 6

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1 Problem 1

(a):
Forward: Let g be a left inverse of f. For all $x \in X$, then $g(f(x)) = x$. Now, suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in X$. Applying g to either side, $g(f(x_1)) = g(f(x_2))$. By definition of left inverse, this reduces to $x_1 = x_2$ so f is injective.
Backwards: Let f be injective. Then, we define a function $g : Y \to X$ as follows. For $y \in Y$ where $y \in im(f)$, there exists some $x \in X$ such that $f(x) = y$, and x must be unique due to injectivity. Then, set $g(y) = x$. Now, fix some $x_1 \in X$. For $y \in Y$ s.t. $y \notin im(f)$, set $g(y) = x_1$.

We check that this satisfies the definition of left inverse. Let $x_1 \in X$, and let $f(x_1) = y_1$. We have $g(f(x_1)) = g(y_1)$. Again, by injectivity, x_1 is the unique element in X mapping to y_1, so $g(y_1) = x_1$. This tells us for all $x_1 \in X$,

$$g(f(x_1)) = x_1$$

so g is indeed a left inverse of f.
(b):
Forward: Let $y \in Y$. Let g be a right inverse, so by definition, for all $y \in Y$, $f(g(y)) = y$ where $g(y) \in X$. Therefore, f is surjective since every element in the codomain Y gets mapped to by at least one element in X.
Backwards: Let f be surjective. We define a function $g : Y \to X$ as follows. Let $y \in Y$. Surjectivity implies that the preimage set of y, $\{x \in X | f(x) = y\}$ is nonempty. Let $P = \{f^{-1}(y) | y \in Y\}$, where $f^{-1}(y)$ is the set of all preimages. Note that P is a collection of non-empty subsets of X.
For all $y \in Y$, we specify $g(y) \in f^{-1}(y)$ by saying $g(y) = x'$ where x' can be chosen as any element in y's pre-image set. Here, g picks one element from each nonempty set in P, which is a valid construction by the axiom choice. We check that this satisfies the definition of right inverse. Let $y \in Y$. Since $g(y) \in f^{-1}(y)$, by definition of preimage, $f(g(y)) = y$ as desired.
(c):
Forward: by definition, a two sided inverse is both a left and right inverse. f is both injective and surjecive by parts (a) and (b) and thus bijective.
Backwards: Let f be bijective. We define a function $g : Y \to X$ as follows. Let $y \in Y$. Surjectivity implies that the preimage set of y, $\{x \in X | f(x) = y\}$ is nonempty as there exists at least one $x \in X$ that maps to y. By injectivity, if $y = f(x_1) = f(x_2)$, then $x_1 = x_2$ so the pre-image of y must have exactly one element x_y. Let $g(y) = x_y \in f^{-1}(y)$ for all $y \in Y$.
We check g satisfies the defn of left inverse. Let $x \in X$, and let $f(x) = y$. We have $g(f(x)) = g(y)$. Again, by injectivity, x is the unique element in X mapping to y, so $g(y) = x$. So for all $x \in X$,

$g(f(x)) = x$ so g is a left inverse of f.
We check g satisfies the defn of right inverse. Let $y \in Y$. Since $g(y) \in f^{-1}(y)$, by definition of preimage, $f(g(y)) = y$ as desired for all $y \in Y$.
Therefore, if f is bijective, it has a two sided inverse g as we just constructed.
(d): For some invertible function f, let $g, g' : Y \to X$ both be inverses of f. Consider the (valid) composition $g \circ f \circ g'$. Since function composition is associative, this equals

$$g \circ f \circ g' = g \circ (f \circ g') = g \circ 1_Y = g$$

this follows since g' is a right inverse, then any function composed with the identity function (on either side) is just itself by definition.
On the other hand,

$$g \circ f \circ g' = (g \circ f) \circ g' = 1_X \circ g' = g'$$

this follows since g is a left inverse, then any function composed with the identity function (on either side) is just itself by definition. Therefore, this shows $g = g'$ so if f has an inverse, it is unique.

2 Problem 2

(a): If T has a right inverse, then it is surjective. Knowing $dim(V) = dim(W)$, it follows that T is 1-1 by the corollary from class. By definition of a right inverse, for all $w \in W$, then $T(S(w)) = w$. Let $v \in V$, and let $T(v) = w_1 \in W$. Then,

$$T(S(T(v))) = T(S(w_1)) = w_1 = T(v)$$

Since T is injective and $S(T(v)), v \in V$ map to the same $w_1 \in W$, this implies $S(T(v)) = v$ for all $v \in V$. Therefore, S also satisfies the definition of left inverse so it is a two-sided inverse, so T is invertible with $T^{-1} = S$
(b): If T has a left inverse, it is injective. Knowing $dim(V) = dim(W)$, it follows that T is surjective by the corollary from class. By definition of of left inverse, we have

$$S(T(v)) = v, \forall v \in V$$

Let $w \in W$. Since T is surjective, there exists $v_1 \in V$ s.t. $T(v_1) = w$. Then,

$$T(S(w)) = T(S(T(v_1))) = T(v_1) = w$$

$$T(S(w)) = w$$

Therefore, S also satisfies the definition of right inverse so it is a two-sided inverse. T is invertible with $T^{-1} = S$

3 Problem 3

(a): Let $V = \mathbb{R}^3, W = \mathbb{R}^2$. Let $T : V \to W$ be defined as $T(x, y, z) = (x, y)$. Clearly, T is surjective. As a subspace, $im(T) \subseteq W$. Then, let $(x, y) \in W$. Then, $(x, y, 0) \in V$ and $T(x, y, 0) = (x, y)$ so $(x, y) \in im(T)$. Therefore $im(T) = W$. By problem 1, this implies T has a right inverse.

Note $dim(V) = 3 > dim(W) = 2$, so by HW 5 problem 8, T cannot be injective. By problem 1, this is equivalent to saying T does not have a left inverse.
(b): Let $S : W \to V$ be defined as $S(x, y) = (x, y, 0)$. Let $S_1 : W \to V$ be defined as $S_1(x, y) = (x, y, 1)$. We check they both satisfy the right inverse definition.

$$\forall (x, y) \in W, T(S(x, y)) = T(x, y, 0) = (x, y)$$

Similarly,

$$\forall (x, y) \in W, T(S_1(x, y)) = T(x, y, 1) = (x, y)$$

Therefore S, S_1 are distinct right inverses of f.
(c): Let $V = W = P(\mathbb{R})$. Let T be the integration function. Specifically, for $f \in V$,

$$T(f) = \int_0^x f(t)dt$$

We previously showed this map is injective (and thus has a left inverse), but is not surjective (so no right inverse).
(d): Let $S : P(\mathbb{R}) \to P(\mathbb{R})$ be the derivative map, so $S(f) = f'$. We check that

$$\forall f \in P(\mathbb{R}), S(T(f))(x) = S(\int_0^x f(t)dt) = f(x)$$

by the fundamental theorem of calculus. As $S(T(f)) = f$, S is a left inverse.
Let $S_1 : P(\mathbb{R}) \to P(\mathbb{R})$ be a different function defined as $S_1(f) = f' + f(0)$. First, for the constant function $f_1(x) = 1 \in P(\mathbb{R})$, $S(f_1) = 0$ whereas $S_1(f_1) = 0 + 1 = 1$ so these are indeed two distinct functions. We check that

$$\forall f \in P(\mathbb{R}), S_1(T(f))(x) = S_1(\int_0^x f(t)dt) = f(x) + \int_0^0 f(t)dt = f(x) + 0 = f(x)$$

This shows $S_1(T(f)) = f$, so S_1 is a left inverse. Therefore S, S_1 are distinct right inverses of f.

4 Problem 4(a)

Let V, W be finite dimensional vector spaces where $dim(V) = n$, $dim(W) = p$. Let $\mathcal{B}, \mathcal{C}$ be ordered bases for V, W respectively. By the matrix isomorphism theorem, $\Phi_{\mathcal{B},\mathcal{C}} : \mathcal{L}(V, W) \to M_{p\times n}(F)$, where $\Phi_{\mathcal{B},\mathcal{C}}(T) = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ is an isomorphism and thus invertible. Then, let $\Phi^{-1}_{\mathcal{B},\mathcal{C}}(A) = T$ where T is a linear function. Equivalently, $A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$

Assume for the sake of contradiction A has a left inverse B s.t. $BA = I_n$. Then, $B \in M_{n\times p}(F)$. $\Phi_{\mathcal{C},\mathcal{B}} : \mathcal{L}(W, V) \to M_{n\times p}(F)$ where $\Phi_{\mathcal{C},\mathcal{B}}(S) = {}_{\mathcal{B}}[S]_{\mathcal{C}}, S \in \mathcal{L}(W, V)$ is an isomorphism and thus invertible. we let $S = \Phi^{-1}_{\mathcal{C},\mathcal{B}}(B)$. Equivalently, $B = {}_{\mathcal{B}}[S]_{\mathcal{C}}$
Considering the composition $S \circ T : V \to V$ with $T : V \to W$, $S : W \to V$, we use the matrix multiplication theorem to get

$${}_{\mathcal{B}}[ST]_{\mathcal{B}} = {}_{\mathcal{B}}[S]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[T]_{\mathcal{B}} = BA = I_n$$

By the matrix times vector theorem, then for $x \in V$, we have

$$_{\mathcal{B}}[(ST)(x)] = {_{\mathcal{B}}}[ST]_{\mathcal{B}} \cdot {_{\mathcal{B}}}[x] = I_n \cdot {_{\mathcal{B}}}[x] = {_{\mathcal{B}}}[x]$$

Since $\phi_{\mathcal{B}} : V \to F^n$ is an isomorphism, it has an inverse . Therefore

$$\phi_{\mathcal{B}}^{-1}({_{\mathcal{B}}}[(ST)(x)]) = \phi_{\mathcal{B}}^{-1}({_{\mathcal{B}}}[x])$$

$$(ST)(x) = x$$

$$S(T(x)) = x, \forall x \in V$$

Therefore, S is a left inverse of T, implying T is injective. But since $dim(W) = p < dim(V) = n$, then T cannot be injective as we proved on HW 5, leading to a contradiction. This shows that A cannot have a left inverse if $p < n$

5 Problem 4(b)

As before, let $dim(V) = n$, $dim(W) = p$. Assume for a contradiction that A has a right inverse $B \in M_{n\times p}(F)$. Let $\Phi_{\mathcal{B},\mathcal{C}} : \mathcal{L}(V, W) \to M_{p\times n}(F)$, $\Phi_{\mathcal{C},\mathcal{B}} : \mathcal{L}(W, V) \to M_{n\times p}(F)$ be defined as above. Since they are isomorphisms and thus invertible, we write

$$\Phi_{\mathcal{B},\mathcal{C}}^{-1}(A) = T \iff A = {_{\mathcal{C}}}[T]_{\mathcal{B}}$$

$$\Phi_{\mathcal{C},\mathcal{B}}^{-1}(B) = S \iff B = {_{\mathcal{B}}}[S]_{\mathcal{C}}$$

For some $T \in \mathcal{L}(V, W)$, $S \in \mathcal{L}(W, V)$. Now, consider the composition $TS : W \to W$. By the matrix times vector theorem,

$$_{\mathcal{C}}[TS]_{\mathcal{C}} = {_{\mathcal{C}}}[T]_{\mathcal{B}} \cdot {_{\mathcal{B}}}[S]_{\mathcal{C}} = AB = I_p$$

By the matrix vector theorem, since $TS : W \to W$, we have

$$_{\mathcal{C}}[(TS)(x)] = {_{\mathcal{C}}}[ST]_{\mathcal{C}} \cdot {_{\mathcal{C}}}[x] = I_p \cdot {_{\mathcal{C}}}[x] = {_{\mathcal{C}}}[x]$$

This shows $\phi_{\mathcal{C}}((TS)(x)) = \phi_{\mathcal{C}}(x), \forall x \in W$. Since $\phi_{\mathcal{C}} : W \to F_p$ is an isomorphism, it is invertible so we have

$$\phi_{\mathcal{C}}^{-1}(\phi_{\mathcal{C}}((TS)(x))) = \phi_{\mathcal{C}}^{-1}(\phi_{\mathcal{C}}(x))$$

$$(TS)(x) = x$$

$$T(S(x)) = x, \forall x \in W$$

Therefore, S is a right inverse of T, implying T is surjective. However, since $dim(V) = n < dim(W) = p$, T cannot be surjective as proved on HW 5, leading to a contradiction. This proves A cannot have a right inverse if $n < p$.

6 Problem 4(c)

See the setup above with matrix A corresponding to T. Let $B, B' \in M_{n\times n}(F)$ where $B, B' : W \to V$ both be two sided inverses of A. Let $\Phi_{\mathcal{C},\mathcal{B}} : \mathcal{L}(W, V) \to M_{n\times p}(F)$ be defined as above. For some $S, S' \in \mathcal{L}(W, V)$

$$\Phi_{\mathcal{C},\mathcal{B}}^{-1}(B) = S, \Phi_{\mathcal{C},\mathcal{B}}(B') = S' \iff B = {}_{\mathcal{B}}[S]_{\mathcal{C}}, B' = {}_{\mathcal{B}}[S']_{\mathcal{C}}$$

Since $STS' : W \to W$, applying the matrix multiplication theorem (twice since two compositions). Since function composition is associative, we write $STS' = S \circ (TS')$ where $TS' : W \to W$

$${}_{\mathcal{B}}[STS']_{\mathcal{C}} = {}_{\mathcal{B}}[S]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[TS']_{\mathcal{C}} = {}_{\mathcal{B}}[S]_{\mathcal{C}} \cdot ({}_{\mathcal{C}}[T]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[S']_{\mathcal{C}}) = B(AB')$$

Since B' is a right inverse, then $AB' = I_n$ so this product becomes

$$B(AB') = BI_n = B$$

Once again, we consider this composition but with $STS' = (ST) \circ S'$ where $ST : V \to V$

$${}_{\mathcal{B}}[STS']_{\mathcal{C}} = {}_{\mathcal{B}}[ST]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[S']_{\mathcal{C}} = ({}_{\mathcal{B}}[S]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[T]_{\mathcal{B}}) \cdot {}_{\mathcal{B}}[S']_{\mathcal{C}} = (BA)B'$$

Since B is a left inverse, then $BA = I_n$ so this product becomes

$$(BA)B' = I_nB' = B'$$

Thus, this shows that $B = B'$ as desired, so A^{-1} is unique. (Note, this problem could have been shorter if we assume matrix multiplication is associative. We didn't technically prove that, so instead I used that function composition is associative which we know for sure).

7 Problem 5(a)

Let V, W be finite dimensional vector spaces with $dim(V) = n = dim(W)$ with ordered bases $\mathcal{B}, \mathcal{C}$ respectively. By the matrix isomorphism theorem, $\Phi_{\mathcal{B},\mathcal{C}} : \mathcal{L}(V, W) \to M_{n\times n}(F)$, where $\Phi_{\mathcal{B},\mathcal{C}}(T) = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ is an isomorphism and thus invertible. The same applies to $\Phi_{\mathcal{C},\mathcal{B}} : \mathcal{L}(W, V) \to M_{n\times p}(F)$. Let us denote

$$\Phi_{\mathcal{B},\mathcal{C}}^{-1}(A) = T \in \mathcal{L}(V, W) \iff A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$$

Let $B \in M_{n\times n}(F)$ be the left inverse of A. We denote

$$\Phi_{\mathcal{C},\mathcal{B}}^{-1}(B) = S \in \mathcal{L}(W, V) \iff B = {}_{\mathcal{B}}[S]_{\mathcal{C}}$$

By the matrix multiplication theorem for $ST : V \to V$,

$${}_{\mathcal{B}}[ST]_{\mathcal{B}} = {}_{\mathcal{B}}[S]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[T]_{\mathcal{B}} = BA = I_n$$

From here, we can apply same logic as 4(a) (with the matrix vector theorem, so some steps are omitted) to conclude

$${}_{\mathcal{B}}[(ST)(x)] = {}_{\mathcal{B}}[x]$$

Again, since $\phi_{\mathcal{B}}$ is an isomorphism, then we apply the inverse to get

$$(ST)(x) = x$$

$$S(T(x)) = x, \forall x \in V$$

This shows S is a left inverse of T. Since V, W have the same dimension, then by problem 2, T is invertible with $T^{-1} = S$, meaning S is also a right inverse of T. For all $w \in W$, then $(TS)(w) = w$ where $TS : W \to W$. Now, let us write out the ordered basis $\mathcal{C} = \{v_1, ..., v_n\}$. $(TS)(v_1) = v_1, ..., (TS)(v_n) = v_n$ The j-th column of the matrix ${}_{\mathcal{C}}[TS]_{\mathcal{C}}$, by definition, ${}_{\mathcal{C}}[(TS)(v_j)] = {}_{\mathcal{C}}[v_j]$. Since $v_j = 0v_1 + ... + 0v_{j-1} + 1v_j + 0v_{j+1} + ...$, then

$$_{\mathcal{C}}[v_j] = \begin{bmatrix} \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

where the jth row is 1 and the remaining are 0. Considering the jth column of ${}_{\mathcal{C}}[TS]_{\mathcal{C}}$ for $1 \leq j \leq n$, we see that all entries $(j, j) = 1$ while the others are 0. Thus

$$_{\mathcal{C}}[TS]_{\mathcal{C}} = I_n$$

Finally, the matrix multiplication theorem tells us

$$_{\mathcal{C}}[TS]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}\, {}_{\mathcal{B}}[S]_{\mathcal{C}} = AB$$

Thus, $AB = I_n$ so B is a right inverse of A, and thus the unique two-sided inverse of A. So $A^{-1} = B$.

8 Problem 5(b)

(Same setup as above with A corresponding to $T : V \to W$, B to $S : W \to V$, except this time we assume B is a right inverse of A)
By the matrix multiplication theorem for $TS : W \to W$,

$$_{\mathcal{C}}[TS]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[S]_{\mathcal{C}} = AB = I_p$$

From here, we can apply same logic as 4(b) (with the matrix vector theorem, so some steps are omitted) to conclude

$$_{\mathcal{C}}[(TS)(x)] = {}_{\mathcal{C}}[x]$$

Applying $\phi_{\mathcal{C}}^{-1}$ yields

$$(TS)(x) = x$$

$$(T(S(x)) = x, \forall x \in W$$

This shows S is a left inverse of T. Since V, W have the same dimension, then by problem 2, T is invertible with $T^{-1} = S$, meaning S is also a left inverse of T. For all $x \in V$, then $(ST)(x) = x$ where $ST : V \to V$. By analogous reasoning as 5(a), this shows $(ST)(v_j) = v_j$ for each $v_j \in \mathcal{B}$, so taking the B-coordinate, we've shown that the jth column of ${}_{\mathcal{B}}[ST]_{\mathcal{B}}$ is just the column vector with all 0 entries except the jth row. This shows

$$_{\mathcal{B}}[ST]_{\mathcal{B}} = I_n$$

We also know by the matrix multiplication theorem that

$$ {}_{\mathcal{B}}[ST]_{\mathcal{B}} = {}_{\mathcal{B}}[S]_{\mathcal{C}}\, {}_{\mathcal{C}}[T]_{\mathcal{B}} = BA = I_n $$

Therefore, B is a left inverse of A and thus the unique two sided inverse $A^{-1} = B$.

9 Problem 6

(a): Let $A \in M_{2\times 3}(\mathbb{R})$ be

$$ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} $$

Then, I claim $B \in M_{3\times 2}(\mathbb{R})$ with

$$ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} $$

is a right inverse. Indeed, we have

$$ AB = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + & 1\cdot 0 + 0\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + 1\cdot 0 + 0\cdot 0 & 0\cdot 0 + 1\cdot 1 + 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 $$

(b): Again, let $A \in M_{2\times 3}(\mathbb{R})$ be

$$ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} $$

We showed that B above was a right inverse. Now, let $B' \in M_{3\times 2}(\mathbb{R})$ with

$$ B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix} $$

We multiply

$$ AB' = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 + 0\cdot 2 + & 1\cdot 0 + 0\cdot 1 + 0\cdot 2 \\ 0\cdot 1 + 1\cdot 0 + 0\cdot 2 & 0\cdot 0 + 1\cdot 1 + 0\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 $$

Thus, B, B' are distinct right inverses of A.
c): Let $A \in M_{3\times 2}(\mathbb{R})$ be

$$ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} $$

Then, let $B \in M_{2\times 3}(\mathbb{R})$ be

$$ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} $$

In part (a), we multiplied to show that (except here A, B are swapped)

$$ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = I_2 $$

This shows $BA = I_2$, so B is a left inverse of A.
d): Again, let A be as in part (c), and we showed that B in part(c) was a left inverse. Let $B' \in M_{2\times 3}(\mathbb{R})$ with

$$B' = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

We multiply

$$B'A = \begin{bmatrix} 1\cdot 1 + 0\cdot 0 + 2\cdot 0 & 1\cdot 0 + 0\cdot 1 + 2\cdot 0 \\ 0\cdot 1 + 1\cdot 0 + 2\cdot 0 & 0\cdot 0 + 1\cdot 1 + 2\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Thus, B, B' (as defined in part (c), (d)) are distinct left inverses of A.

10 Problem 7

(a) Let V be finite dimensional vector space with $dim(V) = n$ with ordered basis $\mathcal{B}$. Using matrix isomorphism theorem setup, we have

$$\Phi^{-1}_{\mathcal{B},\mathcal{B}}(B) = T \in \mathcal{L}(V,V) \iff B = {}_{\mathcal{B}}[T]_{\mathcal{B}}$$

$$\Phi^{-1}_{\mathcal{B},\mathcal{B}}(A) = S \in \mathcal{L}(V,V) \iff A = {}_{\mathcal{B}}[S]_{\mathcal{B}}$$

Since $AB \in M_{n\times n}(F)$ is invertible, it has an inverse $C \in M_{n\times n}(F)$ s.t. $C(AB) = I_n$. We denote

$$\Phi^{-1}_{\mathcal{B},\mathcal{B}}(C) = U \in \mathcal{L}(V,W) \iff C = {}_{\mathcal{B}}[U]_{\mathcal{B}}$$

B **is invertible:** Consider the composition $UST : V \to V$. By the matrix multiplication theorem (considering $UST = U \circ (ST)$ where $ST : V \to V$ we have

$${}_{\mathcal{B}}[UST]_{\mathcal{B}} = {}_{\mathcal{B}}[U]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[ST]_{\mathcal{B}} = {}_{\mathcal{B}}[U]_{\mathcal{B}} \cdot ({}_{\mathcal{B}}[S]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}}) = C(AB)$$

Regarding $UST = (US) \circ T$ where $US : V \to V$ (since function comp is assoc), we use analogous reasoning to conclude

$${}_{\mathcal{B}}[UST]_{\mathcal{B}} = {}_{\mathcal{B}}[US]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}} = (CA)B$$

Thus, $(CA)B = C(AB) = I_n$. Since $C, A \in M_{n\times n}(F)$, $CA \in M_{n\times n}(F)$ and CA is a left inverse of B. By problem 5, it is actually a two sided inverse of B, so $B^{-1} = CA$. B is invertible as desired.

A **is invertible:** We essentially do the same thing, but with C as a right inverse of AB, considering the composition $STU : V \to V$. By the matrix multiplication theorem ($STU = (ST) \circ U$ where $ST : V \to V$ we have

$${}_{\mathcal{B}}[STU]_{\mathcal{B}} = {}_{\mathcal{B}}[ST]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[U]_{\mathcal{B}} = ({}_{\mathcal{B}}[S]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}})\, {}_{\mathcal{B}}[U]_{\mathcal{B}} = (AB)C$$

Regarding $STU = S \circ (TU)$ where $TU : V \to V$ (since function comp is assoc), we use analogous reasoning to conclude

$${}_{\mathcal{B}}[STU]_{\mathcal{B}} = {}_{\mathcal{B}}[S]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[TU]_{\mathcal{B}} = A(BC)$$

Using C as a right inverse, we have $A(BC) = (AB)C = I_n$. Since $C, B \in M_{n\times n}(F)$, $BC \in M_{n\times n}(F)$ and BC is a right inverse of A. By problem 5, then BC is a two sided inverse so A is invertible

with $A^{-1} = BC$.
(b) The example from problem 6(a) works here. Let $A \in M_{2\times 3}(\mathbb{R})$ be

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Let $B \in M_{3\times 2}(\mathbb{R})$ be

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

I already showed that $AB = I_2$, so AB is invertible since the I_2 is its own inverse ($I_2 \cdot I_2 = I_2$, which proves it is a two-sided inverse).

11 Problem 8

(a): Let $T(x) \in im(T)$ where $x \in V$ and $x = w_1 + w_2$. By definition of projection, $T(x) = w_1 \in W_1$ so $im(T) \subseteq W_1$. Let $w_1 \in W_1$. Clearly, $w_1 \in V$ so $T(w_1) = T(w_1 + 0) = w_1$ so $w_1 \in im(T)$. With both containments, $im(T) = W_1$.

Let $x \in ker(T)$ where $x = w_1 + w_2$. Again by definition, $T(x) = w_1$. Since $x \in ker(T)$, then $T(x) = 0$ so $w_1 = 0$. This means $x = 0 + w_2 = w_2 \in W_2$. Thus $ker(T) \subseteq W_2$. Let $w_2 \in W_2$. Clearly, $w_2 \in V$ so $T(w_2) = T(0+w_2) = 0$, so $w_2 \in ker(T)$. With both containments, $ker(T) = W_2$
(b): Consider $x \in V$ where $x = w_1 + w_2$. $T(x) = w_1$ by definition. In part (a) we justified that $T(w_1) = w_1$ for any $w_1 \in W_1$ is. This means $T^2(x) = T(T(x)) = T(w_1) = w_1$. This shows $T(x) = T^2(x)$ for all $x \in V$ thus $T = T^2$.

12 Problem 9

(a): Since $im(T), ker(T)$ are subspaces of V, then $im(T) + ker(T)$ is also a subspace of V as shown in previous HWs. Thus $im(T) + ker(T) \subseteq V$. Let $x \in V$. Consider $x - T(x) \in V$.

$$T(x - T(x)) = T(x) - T(T(x)) = 0$$

where we used linearity and the hypothesis $T = T^2$. Thus $x - T(x) \in ker(T)$. Clearly, $T(x) \in im(T)$ by definition. Then, for any $x \in V$, we've shown $x = T(x) + (x - T(x))$ where $T(x) \in im(T)$ and $x - T(x) \in ker(T)$ so $x \in im(T) + ker(T)$. Therefore $V \subseteq im(T) + ker(T)$. With both containments, we have $V = im(T) + ker(T)$.

Clearly, $\{0\} \subseteq im(T) \cap ker(T)$ since $T(0) = 0$. Let $x \in im(T) \cap ker(T)$. Then there exists $v \in V$ s.t. $T(v) = x$. This implies $T(T(v) = T(x)$. Since $T^2 = T$, then the $T(T(v)) = T(v)$ and since $x \in ker(T)$, then $T(x) = 0$. This tells us $T(v) = 0$ but also, $T(v) = x$, so this means $x = 0$. Hence, $im(T) \cap ker(T) \subseteq \{0\}$
Therefore $V = im(T) \oplus ker(T)$. **(b):** We know $V = W_1 \oplus W_2$ where $W_1 = im(T), W_2 = ker(T)$. For each $x \in V$, let x be represented as $w_1 + w_2$, where we know $w_1 = T(x), w_2 = x - T(x)$. We have $T(x) = T(w_1 + w_2) = w_1$ clearly since $w_1 = T(x)$. Thus T here satisfies the definition of the projection function onto $W_1 = im(T)$ along $W_2 = ker(T)$

13 Problem 10

(a): By definition, the ith column of this matrix is the $\mathcal{C}$-coordinates of $T(v_i)$ where v_i is the ith element of the ordered basis $\mathcal{B}$. Calculations shown below:

$$T(X^3) = 3X^2 = 0 \cdot 1 + 0 \cdot X + 3 \cdot X^2$$

$$T(X^2) = 2X = 0 \cdot 1 + 2 \cdot X + 0 \cdot X^2$$

$$T(X) = 1 = 1 \cdot 1 + 0 \cdot X + 0 \cdot X^2$$

$$T(1) = 1 = 0 \cdot 1 + 0 \cdot X + 0 \cdot X^2$$

Compiling these coordinates in order, we get

$$_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix}$$

(b): We write f in terms of the ordered basis:

$$f = 2X^3 + (-5)X^2 + 0X + 4 \cdot 1$$

Thus, the B-coordinates are

$$[f]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -5 \\ 0 \\ 4 \end{bmatrix}$$

By the matrix times vector theorem, we have

$$_{\mathcal{C}}[T(f)] = {}_{\mathcal{C}}[T]_{\mathcal{B}} \cdot [f]_{\mathcal{B}}$$

Substituting in, this becomes

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -5 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -10 \\ 6 \end{bmatrix}$$

By definition of $\mathcal{C}$-coordinates, this means

$$T(f) = 0 \cdot 1 + (-10)X + 6X^2 = -10X + 6X^2$$

(c): By definition, the ith column of this matrix is the $\mathcal{D}$-coordinates of $S(v_i)$ where v_i is the ith element of the ordered basis $\mathcal{C}$. Let $\mathcal{D} = \{(1,0),(0,1)\}$ be the standard basis of $\mathbb{R}^2$. Calculations shown below:

$$S(1) = (1,1) = 1(1,0) + 1(0,1)$$

$$S(X) = (-1,1) = -1(1,0) + 1(0,1)$$

$$S(X^2) = (1,1) = 1(1,0) + 1(0,1)$$

Compiling these coordinates in order, we get

$$_{\mathcal{D}}[S]_{\mathcal{C}} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Consider the composition $ST : P_3(\mathbb{R}) \to \mathbb{R}^2$, where $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$, $S : P_2(\mathbb{R}) \to \mathbb{R}^2$ are linear functions. By the matrix multiplication theorem,

$$_{\mathcal{D}}[ST]_{\mathcal{B}} = {}_{\mathcal{D}}[S]_{\mathcal{C}} \cdot {}_{\mathcal{C}}[T]_{\mathcal{B}}$$

Substituting in, this equals

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

14 Problem 11

Let $B_1 = \{w_1, ..., w_k\}$ be a basis of W. Let $\mathcal{B}_\infty = [w_1, ..., w_k]$ be an ordered basis. Since B_1 is independent, by the basis extension theorem, there exists a basis B for V such that $B_1 \subseteq B$. Let

$$B = \{w_1, ..., w_k, v_1, ...v_m\}$$

where $k + m = n$. Let $\mathcal{B} = [w_1, ..., w_k, v_1, ...v_m]$ be an ordered basis. For the linear operator $T : V \to V$, we compute $_{\mathcal{B}}[T]_{\mathcal{B}}$. By definition, for $1 \leq i \leq k$, the ith column of the matrix is $_{\mathcal{B}}[T(w_i)]$ For each $w_i \in W$, since W is T-invariant, then $T(w_i) \in W$. We can express $T(w_i)$ uniquely as a linear combination of the basis for W. In other words, there exist unique scalars $a_1, ..., a_k \in F$ s.t.

$$T(w_i) = a_1 w_1 + ... + a_k w_k = a_1 w_1 + ... + a_k w_k + 0v_1 + ... + 0v_m$$

By definition of $_{\mathcal{B}}[T]_{\mathcal{B}}$ (I now denote $A = {}_{\mathcal{B}}[T]_{\mathcal{B}}$ for notation purposes), we have

$$T(w_i) = A_{1,i} w_1 + ... + A_{k,i} w_k + A_{(k+1),i} v_1 + ... + A_{ni} v_m$$

We've expressed $T(w_i)$ as two linear combinations of the basis $\mathcal{B}$. The representation is unique, so we equate the coefficients

$$A_{1,i} = a_1, ..., A_{k,i} = a_k$$

$$A_{k+1,i} = ... = A_{n,i} = 0$$

This shows that for each i where $1 \leq i \leq k$ (i.e. the first k columns), entries in the bottom $n - k$ rows must be 0. Thus, we indeed get a 0 matrix in the bottom left of $_{\mathcal{B}}[T]_{\mathcal{B}}$ that is $(n - k) \times k$.

15 Problem 12

Let $B_1 = \{v_1, ..., v_k\}, B_2 = \{w_1, ..., w_m\}$ be bases of W_1, W_2 respectively. Since $V = W_1 \oplus W_2$, then $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2$ is a basis for V so $dim(V) = k + m$. Let $\mathcal{B} = [v_1, ..., v_k, w_1, ..., w_m]$ be an ordered basis for V. I claim that $_{\mathcal{B}}[T]_{\mathcal{B}}$ is a diagonal matrix.

Case 1: cols 1 through k For i s.t. $1 \leq i \leq k$, $v_i \in W_1$ so $v_i = v_i + 0$ so $T(v_i) = 1v_i$. Hence, the representation of $T(v_i)$ as a linear combo of $\mathcal{B}$ is that all coefficients are 0 except for that of the $1 \cdot v_i$ term. By definition of ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ (denote $A = {}_{\mathcal{B}}[T]_{\mathcal{B}}$ for notation),

$$T(v_i) = A_{1,i}v_1 + ... + A_{k,i}v_k + A_{(k+1),i}w_1 + ... + A_{(k+m),i}w_m$$

We've expressed $T(v_i)$ as two linear combinations of the basis $\mathcal{B}$. The representation is unique, so we equate the coefficients

$$A_{i,i} = 1$$

$$A_{j,i} = 0, \forall j \neq i$$

This shows that for each i where $1 \leq i \leq k$ (i.e. the first k columns), the (i,i)th entry in ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ must be 1 while the other (off-diagonal entries) must be 0.
Case 2: cols k+1 through k+m Since each $w_i \in W_2$, then $T(w_i) = T(0 + w_i) = 0$. Thus,

$$T(w_i) = 0v_1 + ... + 0v_k + 0w_1 + ... + 0w_m$$

For $1 \leq i \leq m$, the $k+i$th element of $\mathcal{B}$ is w_i. By definition of ${}_{\mathcal{B}}[T]_{\mathcal{B}} = A$ considering the $k+i$th column, we get

$$T(w_i) = A_{1,k+i}v_1 + ... + A_{k+m,k+i}w_m$$

We've expressed $T(v_i)$ as two linear combinations of the basis $\mathcal{B}$. The representation is unique, so we equate the coefficients

$$A_{j,k+i} = 0, \forall 1 \leq j \leq k+m$$

This shows that for each column from $k+1, ..., k+m$, its column vector entries are all 0.
Therefore, these two cases show that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is a diagonal $(k+m) \times (k+m)$ matrix with entries $(1,1), ..., (k,k)$ equal to 1, then the remaining entries equal to 0 (including the remaining diagonal $(k+1), ..., (k+m)$).

16 Problem 13

Let $B_k = \{v_1, ..., v_k\}$ be a basis for $ker(T) \subseteq V$. Then, B_k is independent so by basis extension theorem, there exists a basis $B = \{v_1, ..., v_k, u_1, ..., u_m\}$ for V where $B_k \subseteq B$. Let $\mathcal{B} = [v_1, ..., v_k, u_1, ..., u_m]$ be an ordered basis for V. Now, consider the set $\{T(u_1), ..., T(u_m)\} \in W$. Claim: this set is linearly independent. Let $a_1, ..., a_m \in F$ such that

$$a_1T(u_1) + ... + a_mT(u_m) = 0$$

$$T(a_1u_1 + ... + a_mu_m) = 0$$

Therefore,

$$a_1u_1 + ... + a_mu_m \in Ker(T) = Span(\{v_1, ..., v_k\})$$

There exist $b_1, ..., b_k \in F$ s.t.

$$a_1u_1 + ... + a_mu_m = b_1v_1 + ... + b_kv_k$$

Equivalently,

$$a_1u_1 + ... + a_mu_m - b_1v_1 ... - b_kv_k = 0$$

Since $\{v_1, ..., v_k, u_1, ..., u_m\}$ is a basis, it must be independent, implying

$$a_1 = ... = a_m = 0$$

as desired, proving $\{T(u_1), ..., T(u_m)\} \in W$ is independent. Let us extend this to a basis C of W with $C = \{T(u_1), ..., T(u_m), w_1, ..., w_k\}$ There are k additional vectors since $dim(V) = dim(W) = k + m$, and $\{T(u_1), ..., T(u_m)\}$ is a set of m linearly independent and thus distinct vectors. Let $\mathcal{C} = [w_1, ..., w_k, T(u_1), ..., T(u_m)]$ be an ordered basis of W.

Claim: ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ is a diagonal matrix.
Case 1: cols 1 through k For i s.t. $1 \leq i \leq k$, $v_i \in ker(T)$ so

$$T(v_i) = 0 = 0w_1 + ... + 0w_k + 0T(u_1) + ... + 0T(u_m)$$

By definition of ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ (denote $A = {}_{\mathcal{C}}[T]_{\mathcal{B}}$ for notation),

$$T(v_i) = A_{1,i}w_1 + ... + A_{k,i}w_k + A_{k+1,i}T(u_1) + ... + A_{k+m,i}T(u_m)$$

We've expressed $T(v_i)$ as two linear combinations of the basis $\mathcal{C}$. The representation is unique, so we equate the coefficients

$$A_{j,i} = 0, \forall j, 1 \leq j \leq k + m$$

This shows that for each i where $1 \leq i \leq k$, all entries in that column must be 0. Thus all entries in the first k columns must be 0.
Case 2: cols k+1 through k+m For $1 \leq i \leq m$, the $k + i$th element of $\mathcal{B}$ is u_i, where

$$T(u_i) = 0w_1 + ... + 0w_k + ... + 0T(u_{i-1}) + 1T(u_i) + 0T(u_{i+1})....$$

In this unique representation, all coefficients besides that of the term $1T(u_i)$ are 0.
By definition of ${}_{\mathcal{C}}[T]_{\mathcal{B}} = A$ considering the $k + i$th column, we get

$$T(v_i) = A_{1,k+i}w_1 + ... + A_{k,k+i}w_k + A_{k+1,k+i}T(u_1) + ... + A_{k+m,k+i}T(u_m)$$

Given $T(u_i)$'s unique representation in terms of the $\mathcal{C}$ basis, we equate coefficients

$$A_{k+i,k+i} = 1$$

$$A_{j,k+i} = 0, \forall j \neq k + i$$

This shows that for each column $k + i$, for $1 \leq i \leq m$, the $k + i$th entry in the column vector is 1 but the remaining entries are 0. Therefore, these two cases show that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is a diagonal $(k+m) \times (k+m)$ matrix with entries $(1, 1), ..., (k, k)$ equal to 0, then $(k+1, k+1), ..., (k+m, k+m)$ entries equal to 1, the the remaining entries (off-diagonal) equal to 0.

Todo: piecewise for 1(a). make sure 4-6 are right for notation make sure bases problems are right