115AH HW 1

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1 Problem 1

(a) There exists an $x \in A$ s.t. for all $b \in B$, $b \leq x$
(b) For all $x \in A$, there exists $b \in B$ s.t. $b \leq x$
(c) There exists $x, y \in \mathbb{R}^2$ s.t. $f(x) = f(y)$ and $x \neq y$
(d) There exists $b \in \mathbb{R}^2$ s.t. for all $x \in R$, $f(x) \neq b$
(e) There exists $x, y \in \mathbb{R}^2$ and $\epsilon \in P$ s.t. for all $\delta \in P$, $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$
(f) There exists $\epsilon \in \mathbb{R}$ s.t. for all $\delta \in P$, there exists $x, y \in$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$

2 Problem 2

The negation is such: There exists an integer $n > 0$ s.t. for all real $x > 0$, $x \geq 1/n$

The original sentence is true since given any positive integer, any x chosen in the range $(0, 1/n)$, such as $1/2n$ satisfies the condition.

3 Problem 3

(a) We can find a value M that bounds the magnitude of all $x \in S$ i.e. $-M \leq x \leq M$.

The negation is: For all M, there exists an $x \in S$ s.t. $|x| > M$. This means that there is no finite bound M that works for the entire set S. (since given any number M, we can always find an $x \in S$ outside the bounds.

(b) Given any $x \in S$, there is a bound M for that specific x (satisfying $|x| \leq M$).

The negation is: There exists a $x \in S$ s.t. for all numbers M, $|x| > M$

(a) implies (b). If (a) is true, this gives us a single value $M = M_0$ that bounds all the elements in S. Therefore, given any $x \in S$, we can just choose M_0 for its bound, which satisfies (b).

4 Problem 4

We WTS that a has additive inverse $-1 \cdot a$. By axiom $(F4)$, we must verify that $a + (-1 \cdot a) = 0$

$$a + (-1 \cdot a)$$

By multiplicative identity (F3) and commutativity of multiplication (F1):

$$a \cdot 1 + a \cdot (-1)$$

By distribution (F5) and additive inverse (F4) , this is

$$a \cdot (1 + (-1)) = a \cdot 0 = 0$$

Note: in class, we showed $a \cdot 0 = 0$ for all $a \in F$. Thus, $-1 \cdot a$ is the unique add. inv of a.

5 Problem 5

(a)
i: We WTS that $(a \cdot b)$ has add inv $(-a) \cdot b$
By $(F4)$, it suffices to verify $a \cdot b + (-a) \cdot b = 0$

$$a \cdot b + (-a) \cdot b = b \cdot a + b \cdot (-a)$$

by commutativity of multiplication (F1) By distribution (F5), this equals

$$b \cdot (a + (-a))$$

$$b \cdot 0 = 0$$

since any field element times 0 is 0. So the unique additive inverse of $(a \cdot b)$ is $(-a) \cdot b$.

$$(-a) \cdot b = -(a \cdot b)$$

ii: We WTS that $(a \cdot b)$ has add inv $a \cdot (-b)$ Similar to part i, we compute

$$a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 = 0$$

Having we've verified $(F4)$,

$$-(a \cdot b) = a \cdot (-b)$$

(b) By part (a) i,

$$(-a)(-b) = -(a \cdot (-b))$$

Substituting part (a) ii,

$$-(a \cdot (-b)) = -(-(a \cdot b))$$

By double additive inverses (as shown in class)

$$-(-(a \cdot b)) = a \cdot b$$

as desired, so $(-a)(-b) = ab$

6 Problem 6

No, X is not a field. Addition is not closed. For instance

$$1/3 + 1/3 = 2/3 \notin X$$

so $(F0)$ is violated. Moreover, the additive identity 0 is not in X so thus $(F3)$ is violated.

7 Problem 7

We must verify all the field axioms for E. **(F0)** Closure of addition/mult is given.
(F1) Commutativity of E follows because for any $a, b \in E$, $a, b \in F$ as E is a subset of F, and since F is a field, $a + b = b + a$ and $ab = ba$
(F2) Similar to F1, additive/multiplicative associativity in E is true since for any $a, b, c \in E$, $a, b, c \in F$ and the desired properties follow. **(F3)** Additive identity of E. We're given that 0_F, denoting the add. id of F, is in E. For any $a \in E$, we know $a \in F$. Thus, $0_F + a = a$. This means 0_F is also the additive identity element in E. Multiplicative identity of E. We're given that 1_F, denoting the mult. id of F, is in E. For any $a \in E$, we know $a \in F$. Thus, $1_F \cdot a = a$. This means 1_F is also the multiplicative identity element in E.

(F4) Additive inverse in E. We're given that $-a_F$, denoting the add. inv of a in F, is in E. For any $a \in E$, we know $a \in F$. Thus, $a + (-a_F) = 0_F$. Earlier, we proved $0_F = 0_E$ so

$$a + (-a_F) = 0_E$$

Therefore, for every $a \in E$, $-a_F$ is also $-a$ in E.

Multiplicative inverse of E. We're given that for all $a \in E$ s.t. $a \neq 0_F$ (equivalently, $a \neq 0_E$), a_F^{-1}, denoting the mult. inv of a in F, is in E. For any $a \in E$, we know $a \in F$. Thus, $a \cdot a_F^{-1} = 1_F = 1_E$. Therefore, for every $a \in E$ s.t. $a \neq 0$, a_F^{-1} is also a^{-1} in E.

(F5) Lastly, the distributive property is true since for any $a, b, c \in E$, $a, b, c \in F$ and the desired properties follow from F's field properties.

Thus, E is a field.

8 Problem 8

We can verify the properties in the previous problem.
First, we show addition is closed. Let $x = a + bi, y = c + di$ be in $\mathbb{Q}(i)$. Then, $x + y = (a + b) + (c + d)i$. Since $\mathbb{Q}$ is a field (and is thus closed under addition), then $a, b \in \mathbb{Q} \to a + b \in \mathbb{Q}$ and $c, d \in \mathbb{Q} \to c + d \in \mathbb{Q}$ so therefore $x + y \in \mathbb{Q}(i)$ as its real and imaginary parts are rational.

Similarly, we show multiplication is closed. Let $x = a + bi, y = c + di$ be in $\mathbb{Q}(i)$. Then, $xy = (ac - bd) + (ad + bc)i$. Since $\mathbb{Q}$ is a field, $a, b, c, d \in \mathbb{Q}$ means $ac, bd \in \mathbb{Q}$ due to closure of multiplication. By closure of addition, then $ac + (-bd) \in \mathbb{Q}$. By analogous reasoning, $ad + bc \in Q$. Therefore, $xy \in \mathbb{Q}(i)$ as its real and imaginary parts are rational.

The additive identity of $\mathbb{C}$, $0 = 0 + 0i \in \mathbb{Q}(i)$. The multiplicative identity of $\mathbb{C}$, $1 = 1 + 0i \in \mathbb{Q}(i)$ as desired.

Consider an arbitrary $x = a + bi \in \mathbb{Q}(i)$. Then, $-x = -a - bi$, the additive inverse, is in $\mathbb{Q}(i)$ because since $a, b \in \mathbb{Q}$, then $-a, -b \in \mathbb{Q}$.
Similarly, for $x = a + bi \in \mathbb{Q}(i)$ s.t. $x \neq 0$, then $1/x = \frac{a}{a^2+b^2} + \frac{-b}{a^2+b^2}i$. Note that since a, b cannot both be 0, the denominator is always non-zero. Since a, b are rational, then so are $\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}$ so

$$x^{-1} \in \mathbb{Q}(i)$$

To verify that x^{-1} is indeed the inverse, we show $x(x^{-1}) = 1$:

$$(a + bi) \cdot (\frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}i) = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

Therefore, for all nonzero $x \in \mathbb{Q}(i)$, $x^{-1} \in \mathbb{Q}(i)$.
Having verified the assumptions in the previous problem, this shows $\mathbb{Q}(i)$ is a field, as a subfield of $\mathbb{C}$.

9 Problem 9

Let $a \in F$, WTS that $a = 0$. Suppose 0 has a multiplicative inverse. By defn of multiplicative identity, we have

$$a = 1 \cdot a = (0 \cdot b) \cdot a = 0 \cdot (b \cdot a)$$

(using associativity above). Having showed that any element in F times 0 is 0, then

$$a = 0 \cdot (b \cdot a) = (b \cdot a) \cdot 0 = 0$$

note, we used commutativity of multiplication
Thus, if 0 were to have a multiplicative inverse, every element in F is forced to be 0.

10 Problem 10

Suppose $e \in V$ is also an additive identity. Then, consider $0 + e$. By (VS3), $0 + e = 0$ since e is an additive identity. Using additive commutativity (VS1), this equals $e + 0$. By (VS3), $e + 0 = e$ since 0 is an additive identity.

Therefore, $e = 0$, and the additive identity 0 is unique.

11 Problem 11

Let $x \in V$, and suppose there exist $y_1 \in V$, $y_2 \in V$ s.t. $x + y_1 = 0$, $x + y_2 = 0$. Consider the sum

$$y_1 + x + y_2$$

By additive associativity (VS2),

$$y_1 + x + y_2 = y_1 + (x + y_2) = (y_1 + x) + y_2$$

In the first expression, since $x + y_2 = 0$, this equals

$$y_1 + 0$$

By additive identity (VS3),

$$y_1 + 0 = y_1$$

In the second expression, since $x + y_1 = 0$, by additive commutativity, we have $y_1 + x = 0$. Thus, the sum becomes

$$0 + y_2 = y_2 + 0 = y_2$$

using (VS3).

Since $y_1 + x + y_2 = y_1 = y_2$, the additive inverse of any $x \in V$ is unique.

12 Problem 12

(a): By $(F3)$, $0 + 0 = 0$. Therefore, for $0 \in F$ and any $x \in V$,

$$0x = (0 + 0)x$$

By (VS8), scalar multiplication distributes over scalar addition so

$$(0 + 0)x = 0x + 0x$$

This tells us

$$0x = 0x + 0x$$

Since $0x \in V$ by closure of scalar multiplication, then $-(0x) \in V$ exists by (VS4), satisfying the additive inverse property $0x + -(0x) = 0$

Adding $-(0x)$ on both sides,

$$0x + -(0x) = 0 = (0x + 0x) + -(0x)$$

By additive associativity (VS2), the RHS equals

$$0x + (0x + -(0x)) = 0x + 0 = 0x$$

using the additive inverse, then additive identity properties (VS4), (VS3). Therefore, substituting into the above equation,

$$0 = 0x$$

as desired.

(b): By (VS3), $0 + 0 = 0$ where $0 \in V$. Therefore, for $0 \in V$ and any $a \in F$,

$$a0 = a(0 + 0)$$

By (VS7), scalar multiplication distributes over vector addition so

$$a(0 + 0) = a0 + a0$$

This tells us

$$a0 = a0 + a0$$

Since $a0 \in V$ by closure of scalar multiplication, then $-(a0) \in V$ exists by (VS4), satisfying the additive inverse property $a0 + -(a0) = 0$

Adding $-(a0)$ to both sides,

$$a0 + -(a0) = 0 = (a0 + a0) + -(a0)$$

By additive associativity (VS2), the RHS can be computed as

$$a0 + (a0 + -(a0)) = a0 + 0 = a0$$

using the additive inverse, then additive identity properties (VS4), (VS3). Therefore, the above equation simpilfies to

$$0 = a0$$

as desired.

13 Problem 13

This is equivalent to showing if $ax = 0$ holds, then if $a \neq 0$, then $x = 0$. Since $a \neq 0$, by (F4), a^{-1} exists. We multiply the original equation

$$ax = 0$$

$$a^{-1}(ax) = a^{-1}0$$

On the LHS, by associativity of scalar multiplication (VS6), $a^{-1}(ax) = (aa^{-1})x$

$$(aa^{-1})x = 1x = x$$

by multiplicative inverse (F4) and then (VS5), which says $1 \in F$ satisfies $1x = x$ for all $x \in V$.
On the RHS, using problem 12(b),

$$a^{-1}0 = 0$$

Setting LHS = RHS yields

$$x = 0$$

Hence, $a \neq 0$ implies $x = 0$, which is equivalent to saying $a = 0$ or $x = 0$.

14 Problem 14

(a): Since $a \neq 0$, by (F4), a^{-1} exists. We multiply the original equation on both sides:

$$a^{-1}(ax) = a^{-1}(ay)$$

Using the same argument as problem 13,

$$a^{-1}(ax) = (aa^{-1})x = 1x = x$$

$$a^{-1}(ay) = (aa^{-1})x = 1x = y$$

Thus the equation simpifies to

$$x = y$$

as desired.

(b): Since $bx \in V$ by closure of scalar multiplication, then $-(bx) \in V$ exists by (VS4), satisfying the additive inverse property $bx + -(bx) = 0$

We add $-(bx)$ to both sides:

$$ax + -(bx) = bx + -(bx)$$

$$ax + -(bx) = 0$$

Now, we WTS that $-(bx) = (-b)x$. Consider $bx + (-b)x$. By (VS8), this equals

$$(b + -b)x$$

By defn of add. inverse in a field, $b + (-b) = 0$. Hence,

$$(b + -b)x = 0x$$

by 12(a), $0x = 0$ for $0 \in F$ and any $x \in V$. This shows that

$$bx + (-b)x = 0$$

Hence, by (F4) $-(bx) = (-b)x$. We plug into the previous equation

$$ax + -(bx) = 0$$

$$ax + (-b)x = 0$$

By (VS8), this implies

$$(a + (-b))x = 0$$

Now, since $a + (-b) \in F$, $x \in V$, problem 13 implies that either $a + (-b) = 0$ or $x = 0$. As we're given $x \neq 0$, then $a + (-b) = 0$ must hold. Using associativity, commutativity, and additive inverse/identity properties in a field,

$$a + (-b) = 0$$

$$(a + (-b)) + b = 0 + b$$

$$a + ((-b) + b) = b$$

$$a + (b + (-b)) = b$$

$$a + 0 = a = b$$

We've shown that if $ax = bx$ and $x \neq 0$, then $(a + (-b))x = 0$ which implies $a + (-b) = 0$, from which $a = b$ follows.