115AH HW 3

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P.S: I remembered to link the pages this time!!

1 Problem 1

Let $f_1, f_2 \in P(\mathbb{R})$, $c \in \mathbb{R}$.

$$T(cf_1 + f_2)(x) = \int_0^x (cf_1 + f_2)(t)\, dt$$

$$T(cf_1 + f_2)(x) = \int_0^x (cf_1)(t) + f_2(t)\, dt$$

By properties of integrals (separates over sums and constants), this equals

$$c \int_0^x f_1(t)\, dt + \int_0^x f_2(t)\, dt = c \cdot T(f_1)(x) + T(f_2)(x)$$

$$= (c \cdot T(f_1) + T(f_2))(x)$$

Since this holds for all $x \in \mathbb{R}$, then

$$T(cf_1 + f_2) = c \cdot T(f_1) + T(f_2)$$

Thus T is linear.

Let $f_1, f_2 \in P(\mathbb{R})$ s.t. $T(f_1) = T(f_2)$. This means

$$\int_0^x f_1(t)\, dt = \int_0^x f_2(t)\, dt$$

Differentiating both sides w.r.t x, using the fundamental thm of calc, we have

$$f_1(x) = f_2(x)$$

for all x. Hence, $f_1 = f_2$ so T is injective.

However, it is not onto. Let $f(x) \in P(\mathbb{R})$ be arbitrary, so $f(x) = a_n x^n + ... + a_1 x + a_0$. Then,

$$T(f)(x) = \frac{a_n}{n+1} x^{n+1} + ... + \frac{a_1}{2} x^2 + a_0 x$$

Thus, the image of T is $P(\mathbb{R})$ with no constant term. To show this, let $c(x) \in P(\mathbb{R})$ be arbitrary $n+1$ $(n > 0)$ where

$$c(x) = c_{n+1}x^{n+1} + ... + c_1 x$$

If we set $a_n = c_{n+1}(n+1)...a_1 = 2c_2, a_0 = c_1$, then by the above formula, we have

$$T(f)(x) = \frac{a_n}{n+1}x^{n+1} + ... + \frac{a_1}{2}x^2 + a_0 x = c_{n+1}x^{n+1} + ... + c_1 x$$

Consider $g(x) = x + 1$. Then, there is no input that maps to $g(x)$ as it has a constant term. Thus T is not onto.

2 Problem 2

Let $f_1, f_2 \in P(\mathbb{R})$, $c \in \mathbb{R}$.

$$T(cf_1 + f_2) = (cf_1 + f_2)'$$

by derivative rules, this equals

$$cf_1' + f_2' = cT(f_1) + T(f_2)$$

Thus T is linear.

Let $g = a_n x^n + ... + a_1 x + a_0 \in P(\mathbb{R})$. Evidently, $f(x) = \frac{a_n}{n+1}x^{n+1} + ... + \frac{a_1}{2}x^2 + a_0 x$ satisfies $T(f) = f' = g$. Hence, T is onto. However, it is not injective/one-to-one. For any $f \in P(\mathbb{R})$, consider $f + c$ for any constant c. Then,

$$T(f) = f' = T(f+c) = f' + c' = f'$$

Since $f \neq f + c$, yet they map to the same output, T is not injective.

Using results from calculus, if a function f has 0 derivative everywhere i.e. $f'(x) = 0$, then it must be the constant function. The converse also holds: any constant function has 0 derivative. Thus, ker(T) is all constant functions i.e. $f(x) = c$ s.t. $c \in \mathbb{R}$.

3 Problem 3

(a:) False, since a subspace is required to be non-empty.
(b:) False, since W is a subspace of $\mathbb{R}^3$ whereas $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$. However, we could say W and $\mathbb{R}^2$ are isomorphic.
(c:) True. Note that if we have the empty set, by convention, we take any linear combo to be 0. Then, for any non-empty set $\{x_i | i \in I\} \in V$, where i ranges over index set I, then $\sum_{i \in I} 0x_i = 0$ since $0x_=0$ for any $x \in V$. Thus, taking all coefs to be 0, 0 is a linear combo of any set of vectors.
(d:) False. Span is defined as the intersection of all subspaces containing the

given set. Since all subspaces contain 0, then then the Span of the empty set is $\{0\}$, not the empty set.
(e:) True. Given Span(S) = V, the linear combo definition tells us that V is the set of all linear combinations of vectors in S. In other words, for any $v \in V$, there is a representation $v = a_1 x_1 + ... + a_k x_k$ where $x_1 ... x_k \in V$. Special case: if S is the empty set and $V = \{0\}$, this still holds true since by convention, 0 is a linear combination of the empty set.
(f:) False. This is false when S is a set of vectors that is linearly dependent. Counterexample: taking $S = \{(1,0),(0,1),(1,1)\}$, clearly S spans $\mathbb{R}^2$ since any $(x,y) = x(1,0) + y(0,1) + 0(1,1)$. However, $(1,1)$ has two representations as either $1(1,1)$ or $1(1,0) + 1(0,1)$.

4 Problem 4

Let $x \in$ Span(S_1). By definition of span, $x \in W_i$ for all subspaces W_i that contain S_1.

Let U_i be an arbitrary subspace s.t. $S_2 \subseteq U_i$. Since $S_1 \subseteq S_2$, then by transitivity, $S_1 \subseteq U_i$. Thus, since $x \in$ Span(S_1), then $x \in U_i$ since it is a subspace containing S_1. Since this works for all subspaces U_i containing S_2, this shows that

$$x \in \{\cap U_i | S_2 \subseteq U_i\}$$

which is the span of S_2. Thus Span$(S_1) \subseteq$ Span(S_2).
If $V =$Span(S_1), then $V \subseteq$ Span(S_2). However, as we showed in class, Span(S_2) is itself a subspace of V and thus a subset of V by defn, so Span$(S_2) \subseteq V$. Thus, $V =$ Span(S_2) since containment works both ways.

5 Problem 5

(a:) We must show the span of these 3 vectors is $\mathbb{R}^3$. Evidently, the span must be contained in $\mathbb{R}^3$ since it is a subspace. To show the other direction, let $x \in \mathbb{R}^3 = (x_1, x_2, x_3)$. We must show there exists $a_1, a_2 a_3$ s.t. $a_1(1,1,0) + a_2(1,0,1) + a_3(0,1,1) = (x_1, x_2, x_3)$. This equation is equivalent to the system

$$a_1 + a_2 = x_1$$

$$a_1 + a_3 = x_2$$

$$a_2 + a_3 = x_3$$

Solving, we get

$$a_1 = \frac{x_1 + x_2 + x_3}{2} - x_3$$

$$a_2 = \frac{x_1 + x_2 + x_3}{2} - x_2$$

$$a_3 = \frac{x_1 + x_2 + x_3}{2} - x_1$$

Hence, any (x_1, x_2, x_3) can be expressed as a linear combination of the three vectors v_1, v_2, v_3, taking the coefficients above. Since containment works both ways, $\mathbb{R}^3 = Span(v_1, v_2, v_3)$.

(b:) The condition must be that $1 + 1 \neq 0$. Since our construction requires dividing by this number (i.e. multiplying by its multiplicative inverse), F^3 can only be spanned by the vectors if $1 + 1 \neq 0$.

To prove this, assume FTSOC $1 + 1 = 0$. Then, if we attempt to write $(x_1, x_2, x_3) = (0, 0, 1)$ as a linear combination, adding up the equations, we must have

$$a_1 + a_2 + a_2 + a_3 + a_1 + a_3 = x_1 + x_2 + x_3 = 1$$

However, on the LHS, we have

$$(a_1+a_1)+(a_2+a_2)+(a_3+a_3) = a_1(1+1)+a_2(1+1)+a_3(1+1) = a_1 0+a_2 0+a_3 0 = 0$$

This is a contradiction since the LHS $= 0$ but the RHS must equal 1.

Therefore, if $1 + 1 = 0$, then $(0, 0, 1) \in F^3$ cannot be represented as a linear combo of these 3 vectors so F^3 cannot be spanned. Thus, $1+1 \neq 0$ is a necessary condition.

6 Problem 6

Forward direction: assume W is a subspace of V. Let $s \in$Span(W). From the defn of span, s is in the intersection of all subspaces that contain W. Since W is one such subspace, then $s \in W$. Thus Span$(W) \subseteq W$.
We stated in class that $W \subseteq$ Span(W) for any $W \subseteq V$. Hence, $W =$ span(W).
Backwards direction: Assume $W =$span(W). In class, we showed that the span is always a subspace (as it is an intersection of subspaces). It immediately follows that W is a subspace.

7 Problem 7

(a): Note that $(S_1 \cap S_2) \subseteq S_1$ since for all $x \in (S_1 \cap S_2)$, $x \in S_1$. Invoking problem 4, we have Span$(S_1 \cap S_2) \subseteq$ Span(S_1). By analogous reasoning, Span$(S_1 \cap S_2) \subseteq$ Span(S_2). Hence, Span$(S_1 \cap S_2) \subseteq$ Span$(S_1) \cap$ Span(S_2).
(b): Let $V = \mathbb{R}^2$. Let $S_1 = \{(1, 0)\}$, $S_2 = \{(0, 1)\}$. Since $S_1 \cap S_2 = \emptyset$, Span$(S_1 \cap S_2) = \{(0, 0)\}$.

Then, Span(S_1) is all scalar multiples of $(1, 0)$, which is the line $y = 0$. Similarly, Span(S_2) is the line $x = 0$. Thus, the intersection is also $(0, 0)$.
(c): Let $S_1 = \{(1, 0), (0, 1)\}$, $S_2 = \{(1, 0), (0, -1)\}$. Then, $S_1 \cap S_2 = \{(1, 0)\}$, so Span$(S_1 \cap S_2)$, all scalar multiples of this, is the line $y = 0$.

However, Span$(S_1) =$ span$(S_2) = \mathbb{R}^2$ since any two non-parallel vectors span the $\mathbb{R}^2$ space. In this case, Span(S_1) $\cap$Span$(S_2) = \mathbb{R}^2$. Hence, Span$(S_1 \cap S_2)$ is a proper subset.

8 Problem 8

First direction: From the facts our class stated about span, $Span(S_1 \cup S_2)$ is the smallest subspace containing $S_1 \cup S_2$. Let $x \in S_1$. $x = x + 0$ s.t. $x \in Span(S_1)$ (this is since $S_1 \subseteq Span(S_1)$) and $0 \in Span(S_2)$ since all spans are subspaces and subspaces contain 0.
Thus, $x \in Span(S_1) + Span(S_2)$ and so $S_1 \subseteq Span(S_1) + Span(S_2)$. By the same logic, $S_2 \subseteq Span(S_1) + Span(S_2)$. Therefore, $(S_1 \cup S_2) \subseteq Span(S_1) + Span(S_2)$. In Problem 9, we demonstrate the sum of subspaces remains a subspace, so $Span(S_1) + Span(S_2)$ is a subspace containing $(S_1 \cup S_2)$. Therefore,

$$Span(S_1 \cup S_2) \subseteq Span(S_1) + Span(S_2)$$

Second direction: Now, we show the other containment direction. In 9(c), we show that the sum of subspaces is the smallest subspace that contains each individual subspace. We can thus conclude $Span(S_1) + Span(S_2)$ is the smallest subsp X s.t. $Span(S_1) \cup Span(S_2) \subseteq X$. Evidently, $S_1 \subseteq S_1 \cup S_2$, so by problem 4,

$$Span(S_1) \subseteq Span(S_1 \cup S_2)$$

Similarly,

$$Span(S_2) \subseteq Span(S_1 \cup S_2)$$

Therefore, $Span(S_1 \cup S_2)$ is a subspace containing

$$Span(S_1) \cup Span(S_2)$$

But since we stated $Span(S_1) + Span(S_2)$ is the smallest subspace with this property, then

$$Span(S_1) + Span(S_2) \subseteq Span(S_1 \cup S_2)$$

Therefore,

$$Span(S_1) + Span(S_2) = Span(S_1 \cup S_2)$$

9 Problem 9

(a): First, note that since 0 is in each subspace, then $0 = 0 + ... + 0 \in W_1 + W_2 + ... + W_k$
Closure under addition: Let $w, v \in W_1 + W_2 + ... + W_k$. Then, there exist $w_1, v_1 \in W_1$, $w_2, v_2 \in W_2$, ... $w_k, v_k \in W_k$ s.t.

$$w = w_1 + ... + w_k$$

$$v = v_1 + ... + v_k$$

Adding them together,

$$w + v = (w_1 + v_1) + ... + (w_k + v_k)$$

$w_1 + v_1 \in W_1$ due to closure, and same for the other subspaces. Hence, $w + v \in W_1 + ... + W_k$.
Closure under scalar mult: Let $w \in W_1 + W_2 + ... + W_k$ as above. Let $c \in F$.

$$cw = cw_1 + cw_2 + ... + cw_k$$

by distribution of scalar addition. Then, $(cw_1) \in W_1, ..., (cw_k) \in W_k$ so $cw \in W_1 + ... + W_k$.
(b): Let $w \in W_i$ for $1 \leq i \leq k$. Then, $w = w_1 + ... + w_k$, where $w_i = w$ and the remaining $w_1 = w_2 ... w_k = 0$ in their respective subspaces (since 0 is in every subspace).

Thus, $w \in W_1 + ... + W_k$ so

$$W_i \subseteq W_1 + ... + W_k$$

(c): Let $w \in W_1 + ... + W_k$. This means there exist $w_1, ... w_k$ in the respective subspaces s.t.

$$w = w_1 + ... + w_k$$

Since $w_1 \in W_1$ and $W_1 \subseteq X$, then $w_1 \in X$. Applying the same argument, each $w_1, ... w_k \in X$. Since X is a subspace, by closure of addition, then $w \in X$. Hence, $W_1 + W_2 + ... + W_k \subseteq X$.

10 Problem 10

First, we show $V = W_1 + W_2$. Evidently $W_1 + W_2 \subseteq V$ given closure of addition in vector spaces. Note that using problem 8, span$(\{(1,0,0),(0,0,1)\})$ + span$(\{(1,1,1)\})$ is equivalent to span$(\{(1,0,0),(0,0,1),(1,1,1)\})$.

Now, let $x = (x_1, x_2, x_3) \in V$. Then, we must show there are scalars a_1, a_2, a_3 s.t.

$$x = a_1(1,0,0) + a_2(0,0,1) + a_3(1,1,1)$$

This system is equivalent to

$$x_1 = a_1 + a_3$$
$$x_2 = a_3$$
$$x_3 = a_2 + a_3$$

Subtracting eqn 2 from eqn 1 and eqn 3, this system is equivalent to

$$(a_1, a_2, a_3) = (x_1 - x_2, x_3 - x_2, x_2)$$

Hence $x \in$ span$(\{(1,0,0),(0,0,1),(1,1,1)\})$ or equivalent, $x \in W_1 + W_2$. Thus, $V = W_1 + V_2$.
Now, we WTS $W_1 \cap W_2 = \{0\}$. $\{0\} \subseteq W_1 \cap W_2$ is immediate from properties of subspaces. Now, let $w \in W_1 \cap W_2$. Since $w \in W_1$, $w = (x, 0, z)$ for some $x, z \in \mathbb{R}$. Since $w \in W_2$, $w = (a, a, a)$ for some $a \in \mathbb{R}$. Equating the two component-wise, the second entry implies $a = 0$ and thus $w = (0,0,0)$. Hence, $W_1 \cap W_2 \subseteq \{0\}$. This implies $W_1 \cap W_2 = \{0\}$ as desired. Together, these facts show that V is internal direct sum of W_1 and W_2.

11 Problem 11

(a:) We WTS W_1 is a subspace. $0 = 0X + 0X^3 + ... + 0X^n$ for any odd n, so $0 \in W_1$. Closure of addition: let $p(X), q(X) \in W_1$ with $p(X) = a_1X + a_3X^3 + ... + a_nX^n$, $q(X) = b_1X + b_3X^3 + ... + b_mX^m$, n, m odd. WLOG let $n \geq m$

$$p(X) + q(X) = (a_1 + b_1)X + (a_3 + b_3)X^3 + ... + (a_n + b_n)X^n$$

Note, we consider a_i or b_i to be 0 if the term is not in the original polynomial. Clearly, each $(a_i + b_i) \in F$ by closure of addition and $p(X) + q(X)$ only has odd polynomial exponents. Thus, $p(X) + q(X) \in W_1$.
Now, let $c \in F$.

$$c \cdot p(X) = (ca_1)X + (ca_3)X^3 + ... + (ca_n)X^n$$

clearly, each $ca_i \in F$ by closure of multiplication and $c \cdot p(X)$ only has odd polynomial exponents. Thus, $c \cdot p(X) \in W_1$.

Proving that W_2 is a subspace is analogous (but the reasoning is repeated below for completeness).
$0 = 0 + 0X^2 + ... + 0X^n$ for any even n, so $0 \in W_2$. Closure of addition: let $p(X), q(X) \in W_2$ with $p(X) = a_0 + a_2X^2 + ... + a_nX^n$, $q(X) = b_0 + b_2X^2 + ... + b_mX^m$, n, m even. WLOG let $n \geq m$

$$p(X) + q(X) = (a_0 + b_0) + (a_2 + b_2)X^2 + ... + (a_n + b_n)X^n$$

Again, we consider a_i or b_i to be 0 if the term is not in the original polynomial.
Now, let $c \in F$.

$$c \cdot p(X) = (ca_0) + (ca_2)X^2 + ... + (ca_n)X^n$$

clearly, each $ca_i \in F$ by closure of multiplication and $c \cdot p(X)$ only has even polynomial exponents. Thus, $c \cdot p(X) \in W_1$.

(b:) First, we WTS $P(F) = W_1 + W_2$. Evidently, since W_1, W_2 are subspaces of $P(F)$, then $W_1 + W_2 \subseteq P(F)$. Now, let $f(x) = a_0 + a_1x + ... + a_nx^n \in P(F)$ have degree n. Then,

$$f(x) = (a_0 + a_2x^2 ... + a_{2k}x^{2k}) + (a_1x + a_3x^3 ... + a_{2m+1}x^{2m+1})$$

where $2k$ is the greatest even $\leq n$ and $2m + 1$ is the greatest odd $\leq n$. Clearly, this expresses $f(x)$ as a sum of a polynomial in W_1 and one in W_2. So $P(F) \subseteq W_1 + W_2$. Now, we must show $W_1 \cap W_2 = \{0\}$. First, $\{0\} \subseteq W_1 \cap W_2$ given subspaces contain 0. Now, assume $w \in W_1 \cap W_2$. Since $w \in W_1$, we represent it as $a_1x + a_3x^3 + ... + a_{2m+1}x^{2m+1}$. But since $w \in W_2$ also, we write it as $a_0 + a_2x^2 ... + a_{2k}x^{2k}$. Since these expressions must equal, then each coefficient must be 0. This is because the w's representation in W_1 implies that all w's even coefficients must be 0, and vice versa from the W_2 representation. Hence, $w = 0$ so $W_1 \cap W_2 \subseteq \{0\}$. Therefore $W_1 \cap W_2 = \{0\}$. Together, these facts show that $P(F)$ is internal direct sum of W_1 and W_2.

12 Problem 12

First, we WTS $V = W_1 + W_2$. Let $f(x) \in V$. Claim: $f(x) = o(x) + e(x)$ where both functions are in V s.t.

$$o(x) = \frac{f(x) - f(-x)}{2}$$

$$e(x) = \frac{f(x) + f(-x)}{2}$$

Note, $o(x)$ is odd since for all $x \in K$

$$o(-x) = \frac{f(-x) - f(x)}{2} = -o(x)$$

Furthermore, $e(x)$ is even since for all $x \in K$

$$e(-x) = \frac{f(-x) + f(x)}{2} = e(x)$$

It's also clear that $o(x) + e(x) = f(x)$.

Thus, $f(x) \in W_1 + W_2$ so $V \subseteq W_1 + W_2$. Clearly, $W_1 + W_2 \subseteq V$ since they are subspaces of V, which is closed under addition. So $V = W_1 + W_2$.

Now, we WTS $W_1 \cap W_2 = \{0\}$. First, $\{0\} \subseteq W_1 \cap W_2$. We can take $f_1(x) \in W_1$ and $f_2(x) \in W_2$ s.t. both are 0 everywhere. $f_1(x) = 0 = -0 = -f_1(-x)$, so f_1 is odd and $f_2(x) = 0 = f_2(-x)$ so f_2 is even. Now, assume $f(x) \in W_1 \cap W_2$. Then, since $f(x)$ is odd, $-f(x) = f(-x)$ for all $x \in K$. Since $f(x)$ is even, $f(x) = f(-x)$ for all x. By transitivity, $f(x) = -f(x)$ for all x. In the real numbers, this is only possible if $f(x) = 0$ everywhere. This shows $W_1 \cap W_2 \subseteq \{0\}$. Therefore $W_1 \cap W_2 = \{0\}$.

Together, these facts show that V is internal direct sum of W_1 and W_2.

13 Problem 13

Forward direction: Assume V is internal direct sum of W_1 and W_2. Thus, for all $x \in V$, there exists $w_1 \in W_1$, $w_2 \in W_2$, $x = w_1 + w_2$. Now, we want to show this is unique. Assume there exist $w_1' \in W_1$, $w_2' \in W_2$ s.t. $x = w_1' + w_2'$. Then,

$$x = w_1 + w_2 = w_1' + w_2'$$

$$w_1 - w_1' = w_2' - w_2$$

By closure of W_1 subspace, $w_1 - w_1' \in W_1$. By closure of W_2 subspace, the RHS $w_2' - w_2 \in W_2$ so the LHS $w_1 - w_1' \in W_2$ too. This means $w_1 - w_1' \in W_1 \cap W_2$. By defn of internal direct sum, $W_1 \cap W_2 = \{0\}$ so this means $w_1 - w_1' = 0$, implying $w_1 = w_1'$. By analogous reasoning, $w_2 = w_2'$. This means the two representations for x are actually equivalent, so w_1, w_2 are unique as desired.

Backwards direction: Assume each $x \in V$ has such a unique representation $x = w_1 + w_2$. This means $V \subseteq W_1 + W_2$. Clearly, $W_1 + W_2 \subseteq V$ since they're subspaces and due to closure of addition in V.S. V. Hence $W_1 + W_2 = V$. Now, we must show $W_1 \cap W_2 = \{0\}$. Again, the $\{0\} \subseteq$ direction is trivial. Now, suppose $w \in W_1 \cap W_2$. In particular, since $w \in W_2$, then $(-1)w = -w \in W_2$ by closure of scalar mult in subspaces. Thus, $0 = w + (-w)$ is a valid representation in $W_1 + W_2$. However, $0 = 0 + 0$ is another representation. Since we assumed uniqueness of representations, this means $0 = w, 0 = -w$. Hence, $W_1 \cap W_2 \subseteq \{0\}$. This means $W_1 \cap W_2 = \{0\}$.
Together, these facts show that V is internal direct sum of W_1 and W_2.

14 Problem 14

Let $T : Z \to V$ s.t. $T(w_1, w_2) = w_1 + w_2$. The domain is Z and the codomain is V given closure of addition in V.S. In HW 2, we showed Z is indeed a V.S. so T is a function b/w vector spaces.

T is linear: Let $c \in F$, $z_1 = (w_1, w_2), z_2 = (w_1', w_2') \in Z$. First, we have

$$T(cz_1 + z_2) = T((cw_1, cw_2) + (w_1', w_2')) = T(cw_1 + w_1', cw_2 + w_2')$$

using the component-wise defn of scalar multiplication and addition in Z. This equals

$$cw_1 + w_1' + cw_2 + w_2' = c(w_1 + w_2) + (w_1' + w_2') = cT(z_1) + T(z_2)$$

Having shown $T(cz_1 + z_2) = cT(z_1) + T(z_2)$, T is thus linear.
T is onto: Let $v \in V$. Since $V = W_1 + W_2$, there exist $w_1 \in W_1, w_2 \in W_2$ s.t. $v = w_1 + w_2$. this means for $z = (w_1, w_2) \in Z$, $T(z) = v$ thus T is surjective.
T is 1-1: By problem 13, we know that for all $v \in V$, there exist UNIQUE $w_1 \in W_1, w_2 \in W_2$ s.t. $v = w_1 + w_2$. In particular, let $T(z_1) = T(z_2)$, or equivalently, $w_1 + w_2 = w_1' + w_2'$. Since the representation of any element in V must be unique, this implies $w_1 = w_1'$ and $w_2 = w_2'$. Hence, $T(z_1) = T(z_2)$ implies $z_1 = z_2$, proving injectivity.

With these properties, $T(w_1, w_2) = w_1 + w_2$ is indeed an isomorphism.

15 Problem 15

First, we show $Z = U_1 + U_2$. Let $z = (v, w) \in Z$. Then, $z = u_1 + u_2$, where $u_1 = (v, 0) \in U_1$, $u_2 = (0, w) \in U_2$. This means $z \in U_1 + U_2$, implying $Z \subseteq U_1 + U_2$. Moreover, since U_1, U_2 are subspaces of Z, then evidently $U_1 + U_2 \subseteq Z$. Therefore, $Z = U_1 + U_2$.

Now, we show $U_1 \cap U_2 = \{0\}$. Clearly, $\{0\} \subseteq U_1 \cap U_2$ by properties of subspaces. Now, suppose $u \in U_1 \cap U_2$. Since $u \in U_1$, then $u = (v, 0)$ for some

$v \in V$. Since $u \in U_2$, then $u = (0, w)$ for some $w \in W$. Equating these representations component-wise, this implies $v = 0_V, w = 0_W$. Plugging in, this means $u = (0_V, 0_W) = 0_Z$. Hence, $U_1 \cap U_2 \subseteq \{0\}$. With both directions proved, we conclude $U_1 \cap U_2 = \{0\}$.

Together, these facts show that Z is internal direct sum of U_1 and U_2.