115AH HW 2

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April 2025

1 Problem 1

(a) : f, g both have domain X, codomain $\mathbb{R}$.

$$f(0) = 2 * 0 + 1 = 1$$

$$g(0) = 1 + 4 * 0 - 2 * 0^2 = 1$$

$$f(1) = 2 * 1 + 1 = 3$$

$$g(1) = 1 + 4 * 1 - 2 * 1^2 = 3$$

Since $\forall x_0 \in X = \{0, 1\}, f(x_0) = g(x_0)$, then $f = g$.

(b) : Since $f + g$ is defined as $(f + g)(x) = f(x) + g(x)$, then $(f + g)$ has domain X. Moreover, since $\mathbb{R}$ is closed under addition, then if $f(x), g(x) \in \mathbb{R}$, then $f(x) + g(x) \in \mathbb{R}$, so $f + g$ has codomain $\mathbb{R}$. Hence, $f + g$ and h have the same domain and codomain.

$$(f + g)(0) = f(0) + g(0) = 2$$

$$h(0) = 5^0 + 1 = 2$$

$$(f + g)(1) = f(1) + g(1) = 6$$

$$h(1) = 6$$

Since $\forall x_0 \in X = \{0, 1\}, (f + g)(x_0) = h(x_0)$, then $f + g = h$.

2 Problem 2

The additive identity is $0_v = (0, 1)$ since for all $x = (a_1, a_2) \in V$, $x + 0_v = x$. However, this means $(VS4)$, additive inverses, are violated. Consider $(1, 0)$. If it had an additive inverse, there exists (b_1, b_2) s.t.

$$(1, 0) + (b_1, b_2) = 0_v = (0, 1)$$

This requires

$$(1 + b_1, 0 * b_2) = (0, 1)$$

However, on LHS, the second coordinate is always 0 so it cannot be 1. Therefore, $(1, 0)$ does not have an additive inverse.

Moreover, (VS8), saying scalar multiplication distributes over scalar addition, is violated. Consider $c, d \in \mathbb{R}$, $x = (a_1, a_2) \in V$.

$$(c + d)x = ((c + d)a_1, a_2)$$

$$cx + dx = (ca_1, a_2) + (da_1, a_2) = (ca_1 + da_1, a_2^2)$$

Thus, for $x = (a_1, a_2) \in V$ s.t. $a_2 \neq a_2^2$, then

$$(c + d)x \neq cx + dx$$

Thus, V is not a vector space.

3 Problem 3

Here, all the additive properties hold since addition is defined as normal. Closure of addition and scalar mult also holds. However, (VS8) saying scalar multiplication distributes over scalar addition, is violated. Consider $c, d \in \mathbb{R}$, $x = (a_1, a_2) \in V$.

$$(c + d)x = ((c + d)a_1, \frac{a_2}{c + d})$$

$$cx + dx = (ca_1, a_2/c) + (da_1, a_2/d) = (ca_1 + da_1, a_2(\frac{1}{c} + \frac{1}{d}))$$

For arbitrary scalars $c, d \in \mathbb{R}$, $\frac{1}{c+d}$ may not equal $\frac{1}{c} + \frac{1}{d}$, in which case

$$(c + d)x \neq cx + dx$$

Thus, V is not a vector space.

4 Problem 4

(VS0): Addition and scalar multiplication are closed by the givens (since they evaluate to 0_v.

(VS1): Addition commutes since for all $x, y \in V$, $x + y = y + x$. This is because x, y are forced to be 0_v, thus $0_v + 0_v = 0_v + 0_v$

(VS2): Addition associates since for all $x, y, z \in V$,

$$(x + y) + z = (0_v + 0_v) + 0_v = 0_v + 0_v = 0_v = y + x$$

$$x + (y + z) = 0_v + (0_v + 0_v) = 0_v + 0_v = 0_v$$

So $(x + y) + z = x + (y + z)$ in V.

(VS3): 0_v is the additive identity since for all $x \in V$, we know $x = 0$ so

$$x + 0_v = 0_v + 0_v = 0_v = x$$

$$x + 0_v = x$$

(VS4): For all $x \in V$, 0 is the additive inverse: We know $x = 0_v$, and

$$0_v + 0_v = 0_v \to x + 0_v = 0_v$$

(VS5): For all $x \in V$, $1x = 1 * 0_v = 0_v = x$ by the given.
(VS6): For all $x \in V$, $a, b \in F$, we have

$$(ab)x = (ab)0_v = 0_v$$

by the given (since $(ab) \in F$ due to closure of multiplication in fields).

$$a(bx) = a(b0_v) = a0_v = 0_v$$

Thus, $(ab)x = a(bx)$ holds in V.
(VS7): For all $a \in F$, $x, y \in V$,

$$a(x + y) = a(0_v + 0_v) = a0_v = 0_v$$

$$ax + ay = a0_v + a0_v = 0_v + 0_v = 0_v$$

Thus, $a(x + y) = ax + ay$ holds in V.
(VS8): For all $a, b \in F$, $x \in V$,

$$(a + b)x = (a + b)0_v = 0_v$$

by the given, since $(a + b) \in F$ by additive closure in fields.

$$ax + bx = a0_v + b0_v = 0_v + 0_v = 0_v$$

Thus, the $(a + b)x = ax + bx$ holds in V. With all the axioms verified, $V = \{0\}$ over F is indeed a vector space.

5 Problem 5

(VS0): Let $z_1 = (v_1, w_1), z_2 = (v_2, w_2) \in Z$. Then

$$z_1 + z_2 = (v_1 + v_2, w_1 + w_2)$$

Since $v_1 + v_2 \in V, w_1 + w_2 \in W$ by closure of addition in vector spaces, then $z_1 + z_2 \in Z$.

Let $c \in F, z_1 \in Z$.

$$cz_1 = (cv_1, cw_1)$$

Since $cv_1 \in V, cw_1 \in W$ due to closure of scalar mult in vector spaces, then $cz_1 \in Z$
(VS1): Let $z_1, z_2 \in Z$.

$$z_1 + z_2 = (v_1 + v_2, w_1 + w_2)$$

$$z_2 + z_1 = (v_2 + v_1, w_2 + w_1)$$

Since $v_1 + v_2 = v_2 + v_1$ and $w_1 + w_2 = w_2 + w_1$ due to commutativity of addition in vector spaces, then $z_1 + z_2 = z_2 + z_1$

(VS2): Let $z_1, z_2, z_3 \in Z$.

$$(z_1 + z_2) + z_3 = (v_1 + v_2, w_1 + w_2) + (v_3, w_3) = ((v_1 + v_2) + v_3, (w_1 + w_2) + w_3)$$

By associativity of addition in V, W, the above equals

$$(v_1 + (v_2 + v_3), w_1 + (w_2 + w_3)) = (v_1, w_1) + (v_2 + v_3, w_2 + w_3) = z_1 + (z_2 + z_3)$$

so associativity of addition holds in Z.

(VS3): The additive identity is $(0_v, 0_w) \in Z$ because for any $z = (v, w) \in Z$

$$z + (0_v, 0_w) = (v + 0_v, w + 0_w) = (v, w) = z$$

using the properties of $0_v, 0_w$

(VS4): For any $z = (v, w) \in Z$, consider $y = (-v, -w) \in Z$.

$$z + y = (v + (-v), w + (-w)) = (0_v, 0_w) = 0_z$$

Therefore, $y = (-v, -w) \in Z$ is the unique additive inverse of z.

(VS5): For any $z = (v, w) \in Z$,

$$1z = (1v, 1w) = (v, w) \rightarrow 1z = z$$

Note, $1v = v, 1w = w$ since (VS5) holds in vector spaces V, W.

(VS6): Let $a, b \in F$, $z \in Z$.

$$(ab)z = ((ab)v, (ab)w)$$

Since (VS6) holds in V, W, then we re-arrange the above to

$$(a(bv), a(bw)) = a(bv, bw) = a(bz)$$

Thus, scalar multiplication associates in Z.

(VS7): Let $a \in F$, $z_1, z_2 \in V$.

$$a(z_1 + z_2) = a(v_1 + v_2, w_1 + w_2) = (a(v_1 + v_2), a(w_1 + w_2)) = (av_1 + av_2, aw_1 + aw_2)$$

Note, we applied (VS7) in V, W.

$$= (av_1, aw_1) + (av_2, aw_2) = az_1 + az_2$$

Therefore, scalar multiplication distributes over vector addition in Z.

(VS8): Let $a, b \in F$, $z \in V$.

$$(a + b)z = ((a + b)v, (a + b)w)$$

Using (VS8) in V, W over each component, this equals

$$(av + bv, aw + bw) = (av, aw) + (bv, vw) = az + bz$$

Thus, scalar multiplication distributes over scalar addition in Z. Having verified the axioms, Z is a vector space.

6 Problem 6

(a): Yes, $\mathbb{C}^n$ is a vector space over $\mathbb{R}$. We know $\mathbb{R}$ is a subfield of $\mathbb{C}$. Thus every scalar in $\mathbb{R}$ is also in $\mathbb{C}$ and the vector space axioms follow from $\mathbb{C}^n$ being a V.S over $\mathbb{C}$.

(b): No, $\mathbb{R}^n$ is NOT a vector space over $\mathbb{C}$ because (VS0), closure of scalar multiplication is violated. For instance, if $n = 2$, based on how scalar multiplication is defined,

$$(1 + i) \cdot (1, 1) = (1 + i, 1 + i) \notin \mathbb{R}^n$$

(c): In class, we stated F^n is a V.S over F so we will use that fact here. Thus, $(VS0)$ on closure of addition holds since the vectors are the same, only the scalar field has changed. Closure of scalar mult holds since for all $x \in F^n, a \in E$, this implies $a \in F$. Since F^n is a V.S. over F, we know $ax \in F^n$
(VS1, VS2) Commutativity and associativity of vector addition follow since these properties hold when we take F^n over F.
VS3: the additive identity is the n-tuple $(0_F, 0_F ... 0_F) \in F^n$
VS4: the additive inverse of any $(a_1, a_2 ..., a_n) \in F$ is $(-a_1, -a_2 ..., -a_n) \in F$
VS5: Note that in HW 1, we proved that if E is proper a subfield of F, it must have the same identity element as F, so $1_E = 1_F$. Thus, knowing F^n is a V.S over F, we know $1_F x = x$ for all $x \in F^n$. Substituting in, we have $1_E x = x$ as desired. **VS6**: Scalar multiplication is associative since for all $a, b \in E$, $x \in F^n$, we know $a, b \in F$ so the property is satisfied by F^n being a V.S. over F.
VS7, VS8: By similar reasoning, the distributive properties follow since E is a subfield of F and we know it holds in that case.

7 Problem 7

(a) By definition of additive identity, $0_v + 0_v = 0_v$. By linearity,

$$T(0_v) = T(0_v + 0_v) = T(0_v) + T(0_v)$$

Now, we add Since $T(0_v) \in W$, by it has additive inverse $-T(0_v) \in W$. Adding to both sides,

$$T(0_v) + -T(0_v) = (T(0_v) + T(0_v)) + -T(0_v)$$

$$0_w = T(0_v) + (T(0_v) + -T(0_v))$$

by associativity. This implies

$$0_w = T(0_v) + 0_w = T(0_v)$$

as desired.
(b)

Forward direction: Suppose $T : V \to W$ is linear. Let $x, y \in V, c \in F$. Using linearity of addition then linearity of scalar mult,

$$T(cx + y) = T(cx) + T(y) = cT(x) + T(y)$$

Backward direction: Suppose for all $x, y \in V, c \in F$, $T(cx + y) = cT(x) + T(y)$ Consider $y = 0_v$. Then, for all $x \in V$,

$$T(cx) = T(cx + 0_v) = cT(x) + T(0_v)$$

By part (a), $T(0_v) = 0_w$ so this becomes

$$= cT(x) + 0_w = cT(x)$$

Hence, linearity of scalar multiplication holds.

Then, consider $c = 1$ and $x, y \in V$ arbitrary. We know

$$T(1x + y) = 1T(x) + T(y)$$

By the VS axioms in V, $1x = x$ and similarly, by the VS axioms in W, $1T(x) = T(x)$. Substituting into the above, this implies

$$T(x + y) = T(x) + T(y)$$

so linearity of addition holds too. Therefore T is linear.

(c) Let T be linear.

$$T(x - y) = T(x + -y) = T(x) + T(-y)$$

We WTS $T(-y) = -T(y)$. Consider the sum

$$T(y) + T(-y)$$

By linearity of addition, this equals

$$T(y + -y) = T(0_v) = 0_w$$

by part (a). Therefore, $T(-y)$ is the unique add. inv of $T(y)$ so $T(-y) = -T(y)$. Substituting back above,

$$T(x) + T(-y) = T(x) + -T(y) = T(x) - T(y)$$

Therefore, if T linear, then $T(x - y) = T(x) - T(y)$ holds for all $x, y \in V$. **(d)** Forward direction: Suppose $T : V \to W$ is linear. By linearity of addition,

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} T(a_i x_i)$$

By linearity of scalar mult, this becomes

$$\sum_{i=1}^{n} a_i T(x_i)$$

as desired.
Backwards direction: Suppose for all $x_1 ... x_n \in V$, $a_1, ..., a_n \in F$, the following holds

$$T(\sum_{i=1}^{n} a_i x_i) = \sum_{i=1}^{n} a_i T(x_i)$$

Let $a_1 = a_2 = 1$, and the remaining scalars be 0. Then,

$$\sum_{i=1}^{n} a_i x_i = 1x_1 + 1x_2 + 0x_3 ... + 0x_n$$

Since $0x = 0$ for all $x \in V$ and $1x = 1$ for all $x \in V$, the sum becomes

$$x_1 + x_2$$

Thus, the LHS is $T(x_1 + x_2)$. On the RHS,

$$\sum_{i=1}^{n} a_i T(x_i) = 1T(x_1) + 1T(x_2) + 0T(x_3) + ... + 0T(x_n)$$

Since $0w = 0$ for all $w \in W$ and $1w = 1$ for all $w \in W$, the RHS is

$$T(x_1) + T(x_2)$$

Since LHS = RHS, this tells us for any $x_1, x_2 \in V$,

$$T(x_1 + x_2) = T(x_1) + T(x_2)$$

Now, we let $a_1 \in F$ be arbitrary, and set $a_2 = ... = a_n = 0$. Again, let $x_1, x_2 ..., x_n \in V$ be arbitrary. Subsituting in, the LHS is

$$T(a_1 x_1 + 0x_2 + ... + 0x_n) = T(a_1 x_1)$$

Then, the RHS is

$$a_1 T(x_1) + 0T(x_2) + ... + 0T(x_2) = a_1 T(x_1)$$

Since LHS = RHS, this says that for any $x_1 \in V$, $a_1 \in F$,

$$T(a_1 x_1) = a_1 T(x_1)$$

so linearity of scalar multiplication is satisfied. Thus T is linear.

8 Problem 8

Since $4x^2+x+3 = 3(x^2+1)+(x^2+x)$, then by linearity of addition then scalar multiplication,

$$T(4x^2+x+3) = T(3(x^2+1)) + T(x^2+x) = 3T(x^2+1) + T(x^2+x)$$

Substituting the known values,

$$T(4x^2+x+3) = 3(2x+3) + x^2 - 5 = x^2 + 6x + 4$$

9 Problem 9

Assume for the sake of contradiction that T is linear and satisfies both these relations did exist. Then,

$$T(6x^2+9) = 3T(2x^2+3) = 3(x+1) = 3x+3$$

However, we have assume $T(6x^2+9) = 2x+3$. Since $3x+3 \neq 2x+3$, then this is a contradiction so a linear transformation T cannot satisfy both conditions at once.

10 Problem 10

(a): First, we show $a(ST) = (aS)T$. Let $x \in V$. The LHS function evaluated on x is

$$(a(S \circ T))(x) = a \cdot (S \circ T)(x) = a \cdot (S(T(x))$$

The RHS function equals

$$((aS) \circ T)(x) = (aS)(T(x)) = a \cdot S(T(x))$$

The conclusion follows.
Then, we show $a(ST) = S(aT)$. Let $x \in V$. LHS function evaluated on x is

$$(S \circ (aT))(x) = S(aT(x)) = a \cdot S(T(x))$$

Where we used linearity of S in the last step.

Thus, all three expressions equal $a \cdot S(T(x))$.

(b) Let $x \in V$. The LHS function evaluated on x is

$$(S \circ (T_1 + T_2))(x) = S((T_1 + T_2)(x)) = S(T_1(x) + T_2(x))$$

By linearity of S, this equals

$$S(T_1(x)) + S(T_2(x)) = (S \circ T_1)(x) + (S \circ T_2)(x) = (S \circ T_1 + S \circ T_2)(x)$$

Hence, this shows that

$$S(T_1 + T_2) = ST_1 + ST_2$$

as desired.

(c) Let $x \in V$. The LHS function evaluated on x is

$$((S_1+S_2)\circ T)(x) = (S_1+S_2)(T(x)) = S_1(T(x))+S_2(T(x)) = (S_1\circ T+S_2\circ T)(x)$$

the second step follows by defn of adding functions. The conclusion follows.

11 Problem 11

(a) First, the zero vector $(0_F, ..., 0_F) \in W_1$ since $0_F + ... + 0_F = 0_F$ (since we repeatedly apply the additive identity).
Closure under addition: let $a = (a_1, ..., a_n), b = (b_1, ..., b_n) \in W_1$. We know

$$a_1 + ... + a_n = 0 = b_1 + ... + b_n$$

Now, consider $a + b = (a_1 + b_1, ..., a_n + b_n)$. Adding the coordinates,

$$(a_1 + b_1) + ... + (a_n + b_n)$$

We use commutativity and associativity to re-arrange this sum

$$(a_1 + ... + a_n) + (b_1 + ... + b_n) = 0_F + 0_F = 0_F$$

Hence $a + b \in W_1$.
Closure under salar multiplication: let $a = (a_1, ..., a_n) \in W$, so $a_1 + ... + a_n = 0$. For all $c \in F$, $ca = (ca_1, ..., ca_n)$. Summing the coordinates, we get

$$ca_1 + ... + ca_n = c(a_1 + ... + a_n) = c0_F = 0_F$$

hence $ca \in W_1$. Having all 3 properties verified, W_1 is a subspace.

(b) The zero vector $(0_F, ..., 0_F) \notin W_2$ since $0_F + ... + 0_F = 0_F \neq 1_F$, violating this condition. It is not closed under scalar multiplication either. For all $x \in W_2$, note that $0_F \cdot x$ is the zero vector, which we just showed is not in W_2.

12 Problem 12

No, this is not a subspace as it is not closed under addition. For instance, if we take these two elements

$$x^n, (-1)x^n + x^{n-1}$$

for some $n \geq 1$, adding the polynomials yields

$$(1 + -1)x^n + x^{n-1} = x^{n-1}$$

which has degree $n - 1$ and is non-zero, thus not belonging to W.

13 Problem 13

Let us denote the zero function $f_0 : X \to F$ s.t. $f_0(x) = 0_F$ for all $x \in X$. Note that for arbitrary $f : X \to F$, $f + f_0 = f$, since

$$(f + f_0)(x) = f(x) + 0_F = f(x), \forall x \in X$$

Hence f_0 is the additive identity (the 0 vector in this vector space). Clearly, $f_0 \in W$ since $f_0(x_0) = 0$.
Closure under addition: let $f_1, f_2 \in W$.

$$(f_1 + f_2)(x_0) = f_1(x_0) + f_2(x_0) = 0_F + 0_F = 0_F$$

so $f_1 + f_2 \in W$. Closure under scalar mult: let $f_1 \in W$ and $c \in F$. Then,

$$(cf)(x_0) = c \cdot f(x_0) = c \cdot 0_F = 0_F$$

so $cf \in W$. Thus, with these 3 properties, W is a subspace.

14 Problem 14

(a): Again, let us denote the zero function $f_0 : K \to F$ s.t. $f_0(x) = 0_F$ for all $x \in K$. For all $x \in K$, $f_0(x) = 0 = -0 = -f_0(-x)$ so $f_0 \in W_1$.
Closure under addition: let $f_1, f_2 \in W_1$. For all $x \in K$,

$$(f_1 + f_2)(x) = -f_1(-x) + -f_2(-x) = -(f_1(-x) + f_2(-x)) = -(f_1 + f_2)(-x)$$

Hence $f_1 + f_2 \in W_1$.
Closure under scalar mult: let $f \in W_1$ and $c \in F$. For all $x \in K$,

$$(cf)(x) = c \cdot f(x) = c(-f(-x)) = -(cf(-x)) = -(cf)(-x)$$

Thus $cf \in W_1$, so W_1 is a subspace.

(b): f_0, the zero vector in this space, as defined above, satisfies $f_0(x) = 0 = f_0(-x), \forall x \in X$ so $f_0 \in W_2$.
Closure under addition: let $f_1, f_2 \in W_1$. For all $x \in K$,

$$(f_1 + f_2)(x) = f_1(x) + f_2(x) = f_1(-x) + f_2(-x) = (f_1 + f_2)(-x)$$

Hence $f_1 + f_2 \in W_1$.
Closure under scalar mult: let $f \in W$ and $c \in F$. For all $x \in K$,

$$(cf)(x) = c \cdot f(x) = c \cdot f(-x) = (cf)(-x)$$

As $cf \in W_2$, W_2 is a subspace.