115AH HW 7

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1 Problem 1

(a): Let $x, y \in V$, $c \in F$.

$$q(cx + y) = (cx + y) + W$$

By definition of addition of equivalence classes, this equals

$$(cx + W) + (y + W)$$

By definition of scalar multiplication of equiv classes, this becomes

$$c(x + W) + (y + W) = cq(x) + q(y)$$

Therefore q is linear since $q(cx + y) = cq(x) + q(y), \forall x, y, \in V$.
Surjective: Consider an arbitrary equiv class in $\bar{x} \in V/W$ for some $x \in V$. We showed the coset $\bar{x} = x + W$. By definition, $q(x) = x + W$ so q is surjective since every element in V/W gets mapped to.
(b) Preliminary claim: for any $w \in W$, $W = w + W$. For $w_1 \in W$, we have $w_1 = w + (w_1 - w)$, where $w_1 - w \in W$ due to closure. Thus $w_1 \in w + W$, so $W \subseteq w + W$. On the flip side, suppose $w_1 \in w + W$, so $w_1 = w + w_2$ for some $w_2 \in W$. Due to closure, $w + w_2 \in W$ so $w + W \subseteq W$ too.

Let $x \in ker(q)$. This means $q(x) = x + W = 0 + W$. By the claim above, $0 \in W$ so $q(x) = \bar{x} = W$. Due to reflexivity of equiv relations, x must be in its own equiv class, so $x \in W$, implying $ker(q) \subseteq W$.
Let $w \in W$. Then $q(w) = w + W = W = 0 + W$ (again, by above claim). This is the 0 vector in V/W, hence $w \in ker(q)$. We've shown $ker(q) \subseteq W$ and $W \subseteq ker(q)$, so $W = ker(q)$.
(c) By rank-nullity (since q linear, V finite dimensional),

$$dim(im(q)) + dim(ker(q)) = dim(V)$$

Since q is surjective, $im(q) = V/W$, and by part (b), $ker(q) = W$. Substituting,

$$dim(V/W) + dim(W) = dim(V)$$

2 Problem 2

Since $y \in im(T)$, there is at least one $x \in V$ s.t. $T(x) = y$ (i.e. S is not empty). Fix a particular $x_0 \in S$ (the choice can be arbitrary). Claim: $S = x_0 + ker(T)$. For the first containment, suppose $s \in S$.

$$T(s - x_0) = T(s) - T(x_0) = y - y = 0$$

meaning $s - x_0 = k$, for some $k \in ker(T)$. Thus, $s = x_0 + k$ so $s \in x_0 + ker(T) \implies S \subseteq x_0 + ker(T)$. Now, suppose $s_1 \in x_0 + ker(T)$. Then $s_1 = x_0 + k$ for some $k \in ker(T)$.

$$T(s_1) = T(x_0) + T(k) = y + 0 = y$$

By definition, this means $s_1 \in S$. Therefore, $x_0 + ker(T) \subseteq S$. With both containments, $S = x_0 + ker(T)$.

3 Problem 3

(a):
Forwards direction: Suppose $\bar{T}$ is well defined. Let $w \in W$. Let $x \in V$, so that $x + W \in V/W$. Let $y = x + w \iff y - x = w \in W$. Since y is equivalent to x, $y \in \bar{x} \iff \bar{y} = \bar{x}$. Since $\bar{T}$ is well defined, $\bar{T}(x + W) = \bar{T}(y + W)$, which by defn implies

$$T(x) = T(y) = T(x + w)$$

This implies

$$T(x + w) - T(x) = T(w) = 0$$

so $w \in ker(T)$, allowing us to conclude $W \subseteq ker(T)$
Backwards direction: Suppose $W \subseteq ker(T)$. Suppose $x, y \in V$ s.t. $x + W = y + W$ i.e. their equivalence classes are equal. This implies $y \in \bar{x}$, so $y \in x + W \implies y = x + w$ for some $w \in W$.

$$\bar{T}(y + W) = T(y) = T(x + w)$$

$$= T(x) + T(w) = T(x) = \bar{T}(x + W)$$

Since $\bar{x} = \bar{y} \implies \bar{T}(\bar{x}) = \bar{T}(\bar{y})$, we've proved $\bar{T}$ is well-defined.
(b): Let $c \in F$, $\bar{x}, \bar{y} \in V/W$ for some $x, y \in V$. By definition of scalar mult on V/W, we have

$$c\bar{x} = c(x + W) = (cx) + W$$

Then, by definition of addition on V/W,

$$c\bar{x} + \bar{y} = (cx + W) + (y + W) = (cx + y) + W$$

Plugging into $\bar{T}$,

$$\bar{T}(c\bar{x} + \bar{y}) = \bar{T}((cx + y) + W) = T(cx + y)$$

Using linearity of T,

$$T(cx + y) = cT(x) + T(y)$$

By definition, $\bar{T}(\bar{x}) = T(x), \bar{T}(\bar{y}) = T(y)$. Plugging in,

$$cT(x) + T(y) = c\bar{T}(\bar{x}) + \bar{T}(\bar{y})$$

Thus, we've shown for any $c \in F$, $\bar{x}, \bar{y} \in V/W$,

$$\bar{T}(c\bar{x} + \bar{y}) = c\bar{T}(\bar{x}) + \bar{T}(\bar{y})$$

thus $\bar{T} : V/W \to Z$ is linear.

4 Problem 4

(a): From problem 3, we know $\bar{T}$ is linear.
Injective: Suppose $\bar{T}(\bar{x}) = 0$ for some $\bar{x} \in V/ker(T)$. By definition, $T(x) = 0$ so $x \in ker(T)$. In problem 2(b), I showed for any $w \in W$, $W = w + W$ where W is a subspace of V. Applying this, $\bar{x} = x + ker(T)$ and $x \in ker(T)$ implies $x + ker(T) = ker(T) = 0 + ker(T)$, so $\bar{x}$ is the 0 element of V/W. This shows $ker(\bar{T}) \subseteq \{0_{V/ker(T)}\}$. To show the other direction, take some $\bar{x} \in \{0_{V/ker(T)}\}$. This is a set with one element, so $\bar{x} = 0_{V/ker(T)} = 0 + ker(T)$. We have

$$\bar{T}(\bar{x}) = \bar{T}(0 + ker(T)) = T(0) = 0$$

This shows the other containment, that $\{0_{V/ker(T)}\} \subseteq ker(\bar{T})$. Thus, $ker(\bar{T}) = \{0_{V/ker(T)}\}$ so the kernel is trivial, implying $\bar{T}$ is 1-1.
Surjective: Let $z \in Z$. Since T is surjective, $T(x) = z$ for some $x \in V$. Thus,

$$\bar{T}(x + ker(T)) = T(x) = z, x + ker(T) \in V/ker(T)$$

Hence $\bar{T}$ is surjective since each $z \in Z$ gets mapped to by some element in $V/ker(T)$.
(b): To show a diagram commutes, we must show for all possible paths that start and end at the same node, those function compositions are equal. Here, the only such directed path goes from $V \to Z$. It suffices to show $T : V \to Z$ is the same function as $\bar{T} \circ q : V \to Z$. Let $x \in V$.

$$(\bar{T} \circ q)(x) = \bar{T}(q(x)) = \bar{T}(x + ker(T)) = T(x)$$

This immediately follows from the definitions (and well-defined-ness) of both functions as we showed. As

$$(\bar{T} \circ q)(x) = T(x), \forall x \in V$$

, the functions are equal so the diagram commutes.

5 Problem 5

Consider the composition $T = 1_W \circ T \circ 1_V$. We have $1_V : V \to V$, where we use ordered bases $\mathcal{B}'$ input, $\mathcal{B}$ output. Then, $T : V \to W$ we use $\mathcal{B}$ input, $\mathcal{C}$ output. Then, $1_W : W \to W$, we use $\mathcal{C}$ input, $\mathcal{C}'$ output. From the matrix multiplication theorem,

$${}_{\mathcal{C}'}[T]_{\mathcal{B}'} = {}_{\mathcal{C}'}[1_W]_{\mathcal{C}}\, {}_{\mathcal{C}}[T]_{\mathcal{B}}\, {}_{\mathcal{B}}[1_V]_{\mathcal{B}'}$$

As defined in the problem, $P = {}_{\mathcal{B}'}[1_V]_{\mathcal{B}}$. By the change of coordinate theorem, $P^{-1} = {}_{\mathcal{B}}[1_V]_{\mathcal{B}'}$ and we know $Q = {}_{\mathcal{C}'}[1_W]_{\mathcal{C}}$ Plugging into the above,

$${}_{\mathcal{C}'}[T]_{\mathcal{B}'} = Q\, {}_{\mathcal{C}}[T]_{\mathcal{B}} P^{-1}$$

6 Problem 6

(a): Having defined $\{w_1, ...w_n\}$, a set with n vectors, it suffices to prove they are independent by the Trillium Theorem for finite bases. Let $a_1, ..., a_n \in F$ be s.t.

$$\sum_{j=1}^{n} a_j w_j = 0$$

Writing out w_j, we have

$$\sum_{j=1}^{n} a_j \sum_{i=1}^{n} P_{ij} v_i = 0$$

$$\sum_{j=1}^{n}\sum_{i=1}^{n}(P_{ij}a_j)v_i = \sum_{i=1}^{n}\sum_{j=1}^{n}(P_{ij}a_j)v_i = 0$$

Now, the v_i is a constant w.r.t. to the j index, so we can factor it out

$$\sum_{i=1}^{n}(\sum_{j=1}^{n}(P_{ij}a_j))v_i = 0$$

Now, this becomes a linear combo of the v_i's with the inner sum acting as scalars. Since the v_i's are independent, this means

$$\sum_{j=1}^{n} P_{ij}a_j = 0, \forall 1 \leq i \leq n$$

Define $\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$. Since the LHS is the dot product between the ith row of matrix P and $\vec{a}$, by defn of matrix vector multiplication, $\sum_{j=1}^{n} P_{ij}a_j = 0$ is the ith component of $P\vec{a}$.

$$P\vec{a} = \vec{0}$$

Since P is invertible, we take

$$P^{-1}(P\vec{a}) = P^{-1}\vec{0}$$

$$(P^{-1}P)\vec{a} = I_n\vec{a} = \vec{a} = \vec{0}$$

Therefore, all components $a_1 = ... = a_n = 0$, so $w_1, ..., w_n$ are independent (and thus distinct). We therefore have n independent vectors, so $\{w_1, ..., w_n\}$ forms a basis of V. Thus, $\mathcal{B}' = [w_1, ..., w_n]$ is an ordered basis of V.

(b) For each w_j for $1 \leq j \leq n$, we have

$$w_j = \sum_{i=1}^{n} P_{ij} v_i$$

Equivalently, the jth column of P forms the ${}_{\mathcal{B}}[w_j] = {}_{\mathcal{B}}[1_V(w_j)]$ so by definition $P = {}_{\mathcal{B}}[1_V]_{\mathcal{B}'}$ with $\mathcal{B}' = [w_1, ..., w_n]$ an ordered basis of V.

7 Problem 7

Backwards direction: assume C is similar to ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ for some ordered basis $\mathcal{B}$. There exists an invertible matrix S s.t.

$$C = S\,{}_{\mathcal{B}}[T]_{\mathcal{B}}S^{-1}$$

We substitute $P = S^{-1}$ so equivalently,

$$C = P^{-1}\,{}_{\mathcal{B}}[T]_{\mathcal{B}}P$$

Let each w_j for $1 \leq j \leq n$ be defined as:

$$w_j = \sum_{i=1}^{n} P_{ij}v_i$$

By part (a), since P is invertible (since $P = S^{-1}$ so $P^{-1} = S$), $\{w_1, ..., w_n\}$ is a basis of V. We denote $\mathcal{B}' = [w_1, ..., w_n]$ as an ordered basis.

Since jth column of P forms the $\mathcal{B}$-coordinates of $w_j = 1_V(w_j)$, then by definition

$$P = {}_{\mathcal{B}}[1_V]_{\mathcal{B}'}$$

Substituting in,

$$C = {}_{\mathcal{B}}[1_V]^{-1}_{\mathcal{B}'} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[1_V]_{\mathcal{B}'}$$

Since ${}_{\mathcal{B}}[1_V]^{-1}_{\mathcal{B}'} = {}_{\mathcal{B}'}[1_V]_{\mathcal{B}}$, by the change of basis theorem we have

$$C = {}_{\mathcal{B}'}[1_V]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[T]_{\mathcal{B}} \cdot {}_{\mathcal{B}}[1_V]_{\mathcal{B}'} = {}_{\mathcal{B}'}[T]_{\mathcal{B}'}$$

with $\mathcal{B}' = [w_1, ..., w_n]$ as defined above.

8 Problem 8

(a): Let $x \in im(T^{k+1})$. Then $T^{k+1}(v) = x$ for some $v \in V$. Since function composition is associative, we can write $(T^k \circ T)(v) = T^k(T(v)) = x$. As $T : V \to V$,then $T(v) \in V$ therefore $x \in im(T^k)$. This means $im(T^{k+1}) \subseteq im(T^k)$.

Let $x \in ker(T^k)$ so $T^k(x) = 0$. Applying T on both sides, $T(T^k(x)) = T(0) = 0$. Writing $T^{k+1} = T \circ T^k$, then

$$T^{k+1}(x) = 0 \implies x \in ker(T^{k+1})$$

(b): By part (a), since $im(T^{k+1})$ is a subspace contained in $im(T^k)$, then $dim(im(T^{k+1})) \leq dim(im(T^k))$. Assume FTSOC that equality is never achieved, so inequality is strict:

$$dim(im(T^{k+1})) < dim(im(T^k))$$

Since V is a finite dimensional vector space, then its subspaces must have finite dimension too. Let $dim(im(T)) = n$ where n is finite. Since inequality is strict, we have $dim(im(T^2)) < n \implies dim(im(T^2)) \leq n - 1$. Then,

$$dim(im(T^3)) < dim(im(T^2)) = n - 1 \implies dim(im(T^3)) \leq n - 2$$

Continuing in this manner, we reach

$$dim(im(T^{n+1})) \leq n - n = 0 \implies dim(im(T^{n+2})) < 0$$

Since dimension must be non-negative, this is a contradiction. Hence, there must be some $k > 0$ s.t. nullity$(im(T^k))$ = nullity$(im(T^{k+1}))$.
(c): For this k, we have $dim(im(T^{k+1})) = dim(im(T^k))$. Since $im(T^{k+1})$ is a subspace contained in $im(T^k)$, then by the corollary to the Trillium theorem, $im(T^{k+1}) = im(T^k)$. By rank-nullity theorem, since $T^k, T^{k+1} : V \to V$, we have

$$rank(T^{k+1}) + nullity(T^{k+1}) = dim(V) = rank(T^k) + nullity(T^k)$$

Since $rank(T^{k+1}) = rank(T^k)$ for this choice of k, then this equal implies $nullity(T^{k+1}) = nullity(T^k)$. Above, we showed $ker(T^k)$ is a subspace contained in $ker(T^{k+1})$ (both finite dimensional since they're subspaces of V). Since they have the same dimension, by the corollary to trillium theorem, they must be equal: $ker(T^k) = ker(T^{k+1})$.
(d): We proceed by induction to show $im(T^k) \cap ker(T^j) = \{0\}$ for $j \geq 1$.
Base case: $j = 1$. Let $x \in im(T^k) \cap ker(T)$. Then there exists $v \in V$ s.t. $x = T^k(v)$ and also $T(x) = 0$. This implies $T^{k+1}(v) = 0$, so $v \in ker(T^{k+1})$. For this particular k, we have $ker(T^{k+1}) = ker(T^k)$. Thus, $v \in ker(T^k) \implies x = T^k(v) = 0$. This shows $im(T^k) \cap ker(T) \subseteq \{0\}$. Clearly, $\{0\} \subseteq im(T^k) \cap ker(T)$ since every subspace of V contains 0. This validates the base case, that $im(T^k) \cap ker(T) = \{0\}$.
Induction step: Now, our induction hypothesis is that $im(T^k) \cap ker(T^j) = \{0\}$ for some $j \geq 1$. We seek to prove this holds for $j+1$. Suppose $x \in im(T^k) \cap ker(T^{j+1})$. This means $x = T^k(v)$ for some $v \in V$ and $T^{j+1}(x) = 0$. First, $x = T^k(v)$ implies

$$T(x) = T(T^k(v)) = T^{k+1}(v) = T^k(T(v))$$

Since $T(v) \in V$, this means $T(x) \in im(T^k)$. Similarly, from the kernel condition, we derive

$$T^j(T(x)) = 0$$

so $T(x) \in ker(T^j)$. Thus, $T(x) \in im(T^k) \cap ker(T^j)$. By our induction hypothesis, this means $T(x) \in \{0\}$ so $T(x) = 0$. From this, we have $x \in im(T^k) \cap ker(T)$. Then from the base case, we know $im(T^k) \cap ker(T) = \{0\}$ so $x \in \{0\}$. This proves $im(T^k) \cap ker(T^{j+1}) \subseteq \{0\}$. The other direction is obvious since 0 is contained in any subspace. Therefore, we've completed the inductive step to show $im(T^k) \cap ker(T^{j+1}) = \{0\}$ if the same holds for j.

We've just proved $im(T^k) \cap ker(T^j) = \{0\}$ for $j \geq 1$. Taking $j = k$, we get

$$im(T^k) \cap ker(T^k) = \{0\}$$

From rank-nullity, we know

$$dim(im(T^k)) + dim(ker(T^k)) = dim(V)$$

From 6(b), the trillium theorem for finite-dimensional direct sums, this implies

$$V = im(T^k) \oplus ker(T^k)$$

for this value of k we derived.

9 Problem 9

We choose the ordered basis $\mathcal{B} = [1, X, X^2]$ for $P_2(\mathbb{R})$ and compute ${}_{\mathcal{B}}[T]_{\mathcal{B}}$.

$$T(1) = X^2 = 0(1) + 0(X) + 1(X^2)$$

$$T(X) = 1 + X^2 = 1(1) + 0(X) + 1(X^2)$$

$$T(X^2) = 2X + X^2 = 0(1) + 2(X) + 1(X^2)$$

This gives us ${}_{\mathcal{B}}[T(1)], {}_{\mathcal{B}}[T(X)], {}_{\mathcal{B}}[T(X^2)]$, which form the columns of the matrix

$${}_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

By definition, $det(T) = det({}_{\mathcal{B}}[T]_{\mathcal{B}})$

$$det \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix} = 1 \cdot det \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = 2$$

Where we used cofactor expansion on the first column. Thus $det(T) = 2$.

10 Problem 10

Justification for the approach: In class, we showed ${}_{\mathcal{B}}[T]_{\mathcal{B}}$, for some basis $\mathcal{B} = [v_1, ..., v_n]$, is a diagonal matrix with entries $\lambda_1, ..., \lambda_n \in F$ if and only if $\mathcal{B}$ is an eigenbasis, so we must have $T(v_1) = \lambda_1, ..., T(v_n) = \lambda_n$. Consider any ordered basis B'. Its coordinate function is an isomorphism, so this condition is equivalent to

$${}_{\mathcal{B}'}[T(v_i)] = \lambda_i \, {}_{\mathcal{B}'}[v_i] \iff {}_{\mathcal{B}'}[T]_{\mathcal{B}'} \cdot {}_{\mathcal{B}'}[v_i] = \lambda_i \cdot {}_{\mathcal{B}'}[v_i]$$

for $1 \leq i \leq n$. So it suffices to choose a $\mathcal{B}'$, find the eigenvectors of ${}_{\mathcal{B}'}[T]_{\mathcal{B}'}$, then set these to the $\mathcal{B}'$-coordinates of the v_i's, which will be eigenvectors that form a basis (provided the lambdas are distinct).

In class, we showed λ is an eigenvalue if and only if $det(T - \lambda I) = 0$. This is equivalent to $det({}_{\mathcal{B}'}[T]_{\mathcal{B}'} - \lambda I_n) = 0$ any ordered basis B'.

(a): Let $B' = [1, x]$, so

$$T(1) = 1 + 2x$$

$$T(x) = -6 - 6x$$

With these coordinates, we have

$${}_{\mathcal{B}'}[T]_{\mathcal{B}'} = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$$

$$det({}_{\mathcal{B}'}[T]_{\mathcal{B}'} - \lambda I) = det \begin{bmatrix} 1-\lambda & -6 \\ 2 & -6-\lambda \end{bmatrix} = (1-\lambda)(-6-\lambda) + 12 = \lambda^2 + 5\lambda + 6$$

The roots are $\lambda = -2, -3$.

For $\lambda = -2$, all eigenvectors lie in the $ker \begin{bmatrix} 1-(-2) & -6 \\ 2 & -6-(-2) \end{bmatrix} = ker \begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}$ Row reducing the matrix, we get the solution set is

$$k \begin{bmatrix} 2 \\ 1 \end{bmatrix}, k \in \mathbb{R}$$

so we set $v_1 \in V$ s.t.

$$_{\mathcal{B}'}[v_1] = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -3$, all eigenvectors lie in the $ker \begin{bmatrix} 1-(-3) & -6 \\ 2 & -6-(-3) \end{bmatrix} = ker \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$ Row reducing the matrix and taking one solution, we set $v_2 \in V$ s.t.

$$_{\mathcal{B}'}[v_2] = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Converting the coordinates above, we have

$$v_1 = 2 + x, v_2 = 3 + 2x$$

are an eigenbasis (since their eigenvals are distinct). For $\mathcal{B} = [v_1, v_2]$,

$$_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$$

(b): Let $B' = [1, x, x^2]$, so

$$T(1) = 1(1) + 1(x) + 0(x^2)$$
$$T(x) = x + 2x + 3 = 3(1) + 3(x) + 0(x^2)$$
$$T(x^2) = x(2x) + 4x + 9 = 9(1) + 4(x) + 2(x^2)$$

With these coordinates, we have

$$_{\mathcal{B}'}[T]_{\mathcal{B}'} = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$det(_{\mathcal{B}'}[T]_{\mathcal{B}'} - \lambda I) = det \begin{bmatrix} 1-\lambda & 3 & 9 \\ 1 & 3-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Expanding along row 3, this becomes

$$(2-\lambda) \cdot det \begin{bmatrix} 1-\lambda & 3 \\ 1 & 3-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2 - 4\lambda)$$

The roots are $\lambda = 0, 2, 4$.

For $\lambda = 0$, this corresponds to $ker \begin{bmatrix} 1 & 3 & 9 \\ 1 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}$. Row reducing we get

$$ker \begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & -5 \\ 0 & 0 & 2 \end{bmatrix} = ker \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Taking a solution, we set $v_1 \in V$ s.t.

$$_{\mathcal{B}'}[v_1] = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 2, \lambda = 4$ we get

$$\mathcal{B}'[v_2] = \begin{bmatrix} -3 \\ -13 \\ 4 \end{bmatrix}, \mathcal{B}'[v_3] = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Expanding, we get the eigenbasis $\mathcal{B} = [v_1, v_2, v_3]$ where $v_1 = -3+x, v_2 = -3-13x+4x^2, v_3 = 1+x$ thus our matrix

$$_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(c): Let $B' = [E_{11}, E_{12}, E_{21}, E_{22}]$ (the standard basis).

$$T(E_{11}) = 0 \cdot E_{11} + 0 \cdot E_{12} + 1 \cdot E_{21} + 0 \cdot E_{22}$$

$$T(E_{12}) = 0 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 1 \cdot E_{22}$$

$$T(E_{21}) = 1 \cdot E_{11} + 0 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22}$$

$$T(E_{22}) = 0 \cdot E_{11} + 1 \cdot E_{12} + 0 \cdot E_{21} + 0 \cdot E_{22}$$

With these coordinates, we form the columns of the matrix

$$_{\mathcal{B}'}[T]_{\mathcal{B}'} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$_{\mathcal{B}'}[T]_{\mathcal{B}'} - I\lambda = \begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & -\lambda \end{bmatrix}$$

Taking the determinant yields:

$$-\lambda \cdot det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} + 1 \cdot det \begin{bmatrix} 0 & -\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -\lambda \end{bmatrix} = -\lambda(-\lambda \cdot \lambda^2 + \lambda) + (-1 \cdot (\lambda^2 - 1))$$

$$\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = (\lambda - 1)^2(\lambda + 1)^2$$

Taking $\lambda = -1$, we seek

$$ker \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Taking a basis for this space, we set

$$_{\mathcal{B}'}[v_1] = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, {}_{\mathcal{B}'}[v_2] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Taking $\lambda = 1$, we seek

$$ker \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

So we set

$$_{\mathcal{B}'}[v_3] = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, {}_{\mathcal{B}'}[v_4] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Expanding out, we get $\mathcal{B} = [v_1, v_2, v_3, v_4]$ s.t.

$$v_1 = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Finally, we have

$$_{\mathcal{B}}[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11 Problem 11

(a):

Forward direction (by contrapositive): Suppose 0 is an eigenvalue. Then, there exists $v \neq 0$ s.t. $T(v) = 0v = 0$, meaning the kernel is not trivial, so T is not 1-1, T is not invertible.

Backwards direction (by contrapositive): Suppose T is not invertible. It is either not injective or not surjective. But since T is a linear operator on a finite dimensional VS, it must be neither injective nor surjective (these properties are if and only if in this case). In particular, as T is not injective, the kernel is non-trivial so there exists $v \neq 0$ s.t. $T(v) = 0v = 0$, so by definition 0 is an eigenvalue.

(b):

Forward direction: Note that since T is invertible, by part (a) any eigenvalue λ must be non-zero so λ^{-1} exists. Suppose λ is an eigenvalue of T. Then there exists $v \neq 0$ s.t.

$$T(v) = \lambda v$$

Applying T^{-1} (which also must be linear) yields

$$T^{-1}(T(v)) = T^{-1}(\lambda v)$$

$$v = \lambda \cdot T^{-1}(v)$$
$$\lambda^{-1} \cdot v = \lambda^{-1}(\lambda \cdot T^{-1}(v)) = (\lambda^{-1}\lambda)T^{-1}(v) = T^{-1}(v)$$

This shows λ^{-1} is an eigenvalue of T^{-1} again since v was non-zero.
Backwards direction: Suppose λ^{-1} is an eigenvalue of T^{-1}. By definition there exists a vector $v \neq 0$ s.t.

$$T^{-1}(v) = \lambda^{-1}v$$

Using simliar reasoning as above,

$$T(T^{-1}(v)) = v = \lambda^{-1}T(v) \iff \lambda v = T(v)$$

This shows λ is an eigenvalue of T since v was non-zero.
(c): The forward direction's reasoning still holds true: T invertible means T injective, so the kernel is trivial so 0 CANNOT have non-zero eigenvectors so 0 is not an eigenvalue.
BUT, the backwards direction is false, since it relies on the fact that V is finite dimensional. For a counterexample, consider $V = P(\mathbb{R})$. $T : V \to V$ defined as

$$T(f) = \int_0^X f(t)\,dt$$

Previously, we showed T is injective (again, this means kernel is trivial so 0 is NOT an eigenvalue). However, we also showed T fails to be surjective, thus T is not invertible.
So T is not invertible, but 0 is not an eigenvalue. Thus, the backwards direction may fail if V is infinite dimensional.