115AH HW 4

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1 Problem 1

(a) False. Linearly dept means at least ONE vector is redundant, not necessarily all the vectors. For instance, let $S = \{(0,1),(1,0),(0,-1)\}$. Thus, S is dependent since $(0,1) = -1(0,-1)$ is redundant. However, $(1,0)$ cannot be written as a combo of $(0,1)$ and $(0,-1)$ since in such a linear combo, the first component will always be 0, whereas $(1,0)$ requires 1 to be in the first component.
(b) False. Consider the set $S = \{(0,1),(1,0),(0,-1)\}$. The subset $\{(0,1),(1,0)\}$ is independent (since it is the standard basis). To modify this statement, we can say SUPERSETS of linearly dependent sets are independent. (i.e. if $S_1 \subseteq S_2$ and S_1 is dependent, then S_2 is dependent).
(c) True. If all linear combos that total to 0, with vectors in S_2, must be the trivial combination, then the same must hold for S_1 if $S_1 \subseteq S_2$.
(d) False. This is one of the conditions for linearity, and it does not necessarily imply the other condition $T(cx) = cT(x), \forall x \in V$. (Constructing a specific counter-example is difficult, but I believe there's one such in the complex numbers).
(e) False. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto $y = 0$. Projections are linear transformations. Consider $S = \{(0,1),(1,0)\}$, which is independent. But $T(S) = \{(0,0),(1,0)\}$ which is dependent since it contains 0.
(f) False. Let $x_1 = (0,1), x_2 = (0,2)$. Let $y_1 = (0,1)$ and $y_2 = (0,3)$. There exists no linear T. Since this would mean $T(0,2) = 2T(0,1) = (0,2)$ yet we require $T(0,2) = (0,3)$. **(g)** False. Let $V = \mathbb{R}^2$. It's known that any two non-parallel vectors form a basis here, so for instance $B_1 = \{(0,1),(1,0)\}$ and $B_2 = \{(0,1),(-1,0)\}$ are both valid bases.

2 Problem 2

Forward: Let $\{u,v\}$ be distinct vectors that are dependent. Then, by problem 4 (equivalent formulation of dependence), there exist $x \in \{u,v\}$ s.t. $x \in Span(\{u,v\} \backslash \{x\})$. WLOG let this redundant vector be u. Then, u can be written as a linear combo of distinct vectors in $\{v\}$, meaning $u = kv$, for some $k \in F$. This validates the conclusion.
Backwards: WLOG assume u is a scalar multiple of v, meaning $u = kv$, for some $k \in F$. By definition, $u \in Span(\{v\})$. Hence, u is redundant in the set $\{u,v\}$. By problem 4, then $\{u,v\}$ is dependent.

3 Problem 3

If three vectors span a plane, they are dependent. For instance, $\{(1,1,0),(1,0,0),(0,1,0)\}$ is dependent since $1(1,1,0) - 1(1,0,0) - 1(0,1,0) = 0$ is a non-trivial combo. However, none of the vectors are direct scalar multiples of each other.

4 Problem 4

Forward: Let S be dependent. Then there exists a non-trivial combo equal to 0. In particular, there are distinct vectors $x_1, x_2, ..., x_k \in S$, $a_1, ..., a_k \in F$ NOT ALL 0 s.t.

$$a_1x_1 + ... + a_kx_k = 0$$

Let us take any $a_j, 1 \leq j \leq k$ s.t. $a_i \neq 0$. Rearranging the relation, we get

$$x_j = -\frac{1}{a_j}\sum_{i=1, i\neq j}^{k} a_ix_i = \sum_{i=1, i\neq j}^{k} -\frac{a_i}{a_j}x_i$$

Hence, $x_j \in S$ and $x_j \in Span(S \backslash x_j)$ as desired.
Backward: By assumption, there exists $x \in S$ s.t. $x \in Span(\{S\}\backslash x)$. This means there exists distinct vectors $x_1, ...x_k \in \{S\}\backslash x\}$ where

$$x = \sum_{i=1}^{k} a_ix_k$$

Re-arranging, we get

$$x - \sum_{i=1}^{k} a_ix_k = 0$$

Since $x_1...x_k$ are distinct and all $\neq x$, the set $\{x, x_1, ..., x_k\}$ has distinct vectors. However, this linear combination of distinct vectors in S equals 0 yet is not trivial (as x has coefficient 1). As $\{x, x_1, ..., x_k\} \subseteq S$, this proves S is dependent.

5 Problem 5

Forward: Let S be independent. Then, let $S_1 \subseteq S$ be a finite subset. Let $x_1, ..., x_k \in S_1$ be distinct vectors and let $a_1, ..., a_k \in F$ be s.t.

$$a_1x_1 + ... + a_kx_k = 0$$

$S_1 \subseteq S$, so thus $x_1, ..., x_k \in S$. Since S is independent, then it must hold that $a_1 = a_2... = a_k = 0$. This shows S_1 is also independent. Since S_1 was arbitrary, this shows that every finite subset of S is linearly independent.
Backwards: Assume every finite subset of S is linearly independent. Let $x_1, ..., x_k \in S$ be distinct vectors where

$$a_1x_1 + ... + a_kx_k = 0$$

Let $S_1 = \{x_1, ..., x_k\}$. Clearly, $S_1 \subseteq S$ is finite, and thus independent by assumption. Then, we must have $a_1 = a_2... = a_k = 0$. This tell us for any linear combo of vectors in S that equals 0, the coefficients must all be 0. Thus, S is dependent.

6 Problem 6

Consider the linear combination of vectors in $S = \{v_1, .., v_k\}$ (the set defined in the problem). Note, if $i \neq j$, then $v_i \neq v_j$. Otherwise, assuming $v_i = v_j$, then $w_i = T(v_i) = T(v_j) = w_j$ but this would contradict the w's being distinct. Thus, the vectors in S are distinct. Consider such a linear combination:

$$a_1 v_1 + ... + a_k v_k = 0$$

Applying T to both sides,

$$T(a_1 v_1 + ... + a_k v_k) = T(0)$$

Applying linearity,

$$a_1 T(v_1) + ... + a_k T(v_k) = 0$$

Substituting in the w's, we get

$$a_1 w_1 + ... + a_k w_k = 0$$

Since the set $\{w_1, ..., w_k\}$ is independent, then $a_1 = a_2 ... a_k = 0$, which in turn shows that $S = \{v_1, .., v_k\}$ is independent.

7 Problem 7

Forward: Let T be 1-1. Let L be a linearly indep subset of V. Let $l_1, ..., l_k \in T(L)$ be distinct. Suppose

$$a_1 l_1 + ... + a_k l_k = 0$$

By definition, $l_1 = T(x_1)$ for some $x_1 \in L$ etc etc $l_k = T(x_k), x_k \in L$. Note, since the l's are distinct, the x's must be too (using similar reasoning in problem 6). Hence,

$$a_1 T(x_1) + ... + a_k T(x_k) = 0$$

By linearity,

$$T(a_1 x_1 + ... + a_k x_k) = 0$$

Since T is 1-1, the kernel must be trivial (as we showed in class). This implies

$$\sum_{i=1}^{k} a_i x_i = 0$$

Since L is independent by assumption and the x_i's are distinct, this implies $a_1 = a_2 = ... = a_k = 0$. Hence, $T(L)$ is independent for all choices of independent subsets L.

Backward: Suppose FTSOC T is not 1-1. Then, this implies the kernel is not trivial. There exists $x_0 \in V$ s.t. $x_0 \neq 0$ yet $T(x_0) = 0$. Consider the set $L = \{x_0\}$. L is independent (b/c if $ax_0 = 0$, then $a = 0$ or $x_0 = 0$ but here we assumed $x \neq 0$ so thus $a = 0$). Hence $T(L) = \{T(x_0)\} = \{0\}$ is dependent. Thus, this shows that there exists some lin. indpt set L where $T(L)$ is not independent, completing our proof by contrapositive.

8 Problem 8

Since T is an isomorphism, it is 1-1. Since B is a basis, it is an indep subset of V. Thus, Problem 7 tells us $T(B)$ is independent.
Since T is an isomorphism, it is onto. Let $w \in W$. Then, there exists $v \in V$ s.t. $T(v) = w$. Since B is a basis of V, this implies $Span(B) = V$. Then, there exists $x_1, ...x_k \in B$, $a_1, ...a_k \in F$ s.t.

$$v = a_1x_1 + ... + a_kx_k$$

Applying linearity,

$$T(v) = a_1T(x_1) + ... + a_kT(x_k) = w$$

Thus, w is a linear combo of $T(x_1), ..., T(x_k) \in T(B)$, so $w \in Span(T(B))$ so $W \subseteq Span(T(B))$. Clearly, $Span(T(B))$ lies in the vector space W i.e. $Span(T(B)) \subseteq W$. Hence, $Span(T(B)) = W$. Since $T(B)$ is both independent and spanning, it is a basis.

9 Problem 9

Let $E_{ij} \in M_{k\times n}(F)$ be the matrix with all entries 0's, except the entry in (i,j) is 1. Claim:

$$E = \{E_{ij} | 1 \leq i \leq k, 1 \leq j \leq n\}$$

is a basis for $M_{k\times n}(F)$.
Independent: Let $a_{ij} \in F$ for $1 \leq i \leq k, 1 \leq j \leq n$. Let us have the following linear combo:

$$\sum_{1\leq i\leq k, 1\leq j\leq n} a_{ij}E_{ij} = 0$$

Evaluating entry-wise, the $k \times n$ LHS matrix equals

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,n} \end{bmatrix}$$

Since addition and scalar multiplication are entry-wise, it's evident that the $0 \in M_{k\times n}(F)$ is the matrix with all 0 entries. If the matrix above equals 0, then each entry must be 0. So all $a_{ij} = 0$ as desired, and E is independent.
Spanning: Let $b \in M_{k\times n}(F)$ be an arbitrary matrix with entries b_{ij} as above. Then, we WTS $b \in Span(E)$. Using the logic above, if we take coefficients $b_{ij} \in F$ for each $E_{ij} \in E$, then

$$\sum_{1\leq i\leq k, 1\leq j\leq n} b_{ij}E_{ij} = \begin{bmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{k,1} & \cdots & b_{k,n} \end{bmatrix}$$

Hence, the matrix b is indeed a linear combination of our set. So $M_{k\times n}(F) \subseteq Span(E)$. Also, since $Span(E) \subseteq M_{k\times n}(F)$ since $E \subseteq$ the vector space $M_{k\times n}(F)$. Therefore, $Span(E) = M_{k\times n}(F)$ As both an independent and spanning set,

$$E = \{E_{ij} | 1 \leq i \leq k, 1 \leq j \leq n\}$$

forms a basis of $M_{k\times n}(F)$.

10 Problem 10

(a)

Showing $Span(B_1 \cup B_2) = V$**:** Let $v \in V$. Since $V = W_1 + W_2$, there exist $w_1 \in W_1, w_2 \in W_2$ s.t. $v = w_1 + w_2$. Since B_1 is a basis for W_1, it spans W. There exist $x_1, x_2, .., x_k \in B_1$, $a_1, .., a_k \in F$ s.t.

$$w_1 = a_1x_1 + ... + a_kx_k$$

Similarly, for some $y_1, y_2, .., y_k \in B_2$, $b_1, .., b_k \in F$, we have

$$w_2 = b_1y_1 + ... + b_ky_k$$

Thus, for all $v \in V$, we have

$$v = (a_1x_1 + ... + a_kx_k) + (b_1y_1 + ... + b_ky_k)$$

so $v \in Span(B_1 \cup B_2)$ so $V \subseteq Span(B_1 \cup B_2)$. Moreover, $B_1 \subseteq W_1 \subseteq V$ and similarly $B_2 \subseteq V$ so $B_1 \cup B_2 \subseteq V$, which means $Span(B_1 \cup B_2) \subseteq V$ since V is a subspace containing $B_1 \cup B_2$. Hence, $B_1 \cup B_2$ spans V.

Showing $B_1 \cup B_2$ **is independent:** Let $x_1, .., x_k \in B_1 \cup B_2$ be distinct and $a_1, ..., a_k \in F$ s.t.

$$a_1x_1 + ... + a_kx_k = 0$$

Since each x_i belongs to B_1 or B_2, let there be m of them in B_1, and the remaining $k-m$ lie in B_2. Re-label the terms as $a_{1,1}x_{1,1}, a_{1,2}x_{1,2}, ..., a_{1,m}x_{1,m}$ for the B_1 terms and $a_{2,1}x_{2,1}, a_{2,2}x_{2,2}, ..., a_{2,k-m}x_{2,k-m}$. Since W_1, W_2 are subspaces, they are closed under linear combination so

$$a_{1,1}x_{1,1} + ... + a_{1,m}x_{1,m} \in W_1$$

$$a_{2,1}x_{2,1} + ... + a_{2,k-m}x_{2,k-m} \in W_2$$

Substituting into the linear combo $a_1x_1 + ... + a_kx_k = 0$ above, they sum to 0. By properties of internal direct sum, $0 \in V$ must have a unique representation, implying

$$a_{1,1}x_{1,1} + ... + a_{1,m}x_{1,m} = 0$$

Since the vectors lie in B_1, which is linearly independent, then

$$a_{1,1} = a_{1,2} = ... = a_{1,m} = 0$$

By similar logic,

$$a_{2,1}x_{2,1} + ... + a_{2,k-m}x_{2,k-m} = 0$$

Hence

$$a_{2,1} = a_{2,2} = ... = a_{2,k-m} = 0$$

Hence, all the coefs of the original linear combo must be 0, so $B_1 \cup B_2$ is independent. Therefore, $B_1 \cup B_2$ is a basis.

Now,

$$B_1 \cap B_2 \subseteq B_1 \subseteq W_1$$

Similarly,

$$B_1 \cap B_2 \subseteq W_2$$

Therefore,

$$B_1 \cap B_2 \subseteq W_1 \cap W_2 = \{0\}$$

Thus, either $B_1 \cap B_2 = \{0\}$ or $B_1 \cap B_2 = \emptyset$. However, $0 \notin B_1$ and $0 \notin B_2$ since bases must be linearly independent (and any set that contains 0 has non-trivial linear combos equal to 0). So the first scenario is not possible. Thus, $B_1 \cap B_2 = \emptyset$ must hold.

(b) First, we show $V = W_1 + W_2$. First, $W_1 + W_2 \subseteq V$ is obvious since both are subspaces of V. Now, let $v \in V$. Since $B_1 \cup B_2$ is a basis (and thus spanning), there are $x_1, .., x_k \in B_1 \cup B_2$ and $a_1, ..., a_k \in F$ where

$$v = a_1x_1 + ... + a_kx_k$$

Using the same notation in pt (a), re-label the terms as $a_{1,1}x_{1,1}, a_{1,2}x_{1,2}, ..., a_{1,m}x_{1,m}$ for the B_1 terms and $a_{2,1}x_{2,1}, a_{2,2}x_{2,2}, ..., a_{2,k-m}x_{2,k-m}$. Special case: if $m = 0$ then $v \in W_2$ and evidently $v \in W_1 + W_2$, so this case is trivial. The case where $k = m$ works the same way, implying $v \in W_1$ and thus $v \in W_1 + W_2$. Now, assume $m, k - m > 0$ (i.e. we have at least one vector from each basis). Hence,

$$v = (a_{1,1}x_{1,1} + ... + a_{1,m}x_{1,m}) + (a_{2,1}x_{2,1} + ... + a_{2,k-m}x_{2,k-m})$$

Clearly, the first term is in W_1 since it uses B_1 and the second term is in W_2. Hence, $v \in W_1 + W_2$ so $V \subseteq W_1 + W_2$. This shows $V = W_1 + W_2$

Now, consider $w \in W_1 \cap W_2$. Since $w \in W_1$, then $w \in Span(B_1)$. Then, there are vectors $x_1, ...x_k \in B_1$, $a_1, ..., a_k \in F$ s.t.

$$w = a_1x_1 + ... + a_kx_k$$

Similarly, since $w \in W_2$, we can write it as

$$w = b_1y_1 + ... + b_my_m$$

Equating the two and moving them to one side:

$$a_1x_1 + ... + a_kx_k - (b_1y_1 + ... + b_my_m) = 0$$

First, we know $x_i \neq y_j$ since $x_i \in B_1, y_j \in B_2$ but $B_1 \cap B_2 = \emptyset$. Moreover, within the x_i's, WLOG we can assume the vectors are all distinct (since if they're not, we can collapse them into one term until the vectors are all distinct). Same with the y_j's. Hence all these vectors are distinct. Since $B_1 \cup B_2$ is a basis of V, then it must be independent. All the x_i and y_j are in $B_1 \cup B_2$, so this implies all the coefficients must be 0. Therefore, $a_1 = a_2 = ...a_k = 0$, so

$$w = 0x_1 + ... + 0x_k = 0$$

Hence, $W_1 \cap W_2 \subseteq \{0\}$. Obviously, $\{0\} \subseteq W_1 \cap W_2$ as subspaces. This proves that $W_1 \cap W_2 = \{0\}$. Therefore, having shown both $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$, the conclusion follows.

11 Problem 11

Assume FTSOC that $Span(B\backslash\{x\}) = V$, for some $x \in B$. Then, $x \in V = Span(B\backslash\{x\})$. By Problem 4, this means B must be linearly dependent, contradicting the fact that it is a basis (and thus indpt). Thus, if B is a basis, then $Span(B\backslash\{x\}) \neq V$ so B is a minimal spanning set.

12 Problem 12

Let $S \subseteq V$ be a minimal spanning set. It suffices to show that S is indpt. By problem 4 (taking the contrapositive), it suffices to show that for all $x \in S$, $x \notin Span(S\backslash\{x\})$ (i.e. no redundant vectors).

Assume FTSOC that there exists $x \in S$ s.t. $x \in Span(S\backslash\{x\})$. Clearly, $Span(S\backslash\{x\}) \subseteq Span(S)$ since $S\backslash\{x\} \subseteq S$. Furthermore, we know $S\backslash\{x\} \subseteq Span(S\backslash\{x\})$ since the span of a set contains itself. By assumption, $x \in Span(S\backslash\{x\})$ or equivalently, $\{x\} \subseteq Span(S\backslash\{x\})$. This implies

$$(S\backslash\{x\}) \cup \{x\} = S \subseteq Span(S\backslash\{x\})$$

Since $Span(S\backslash\{x\})$ is a subspace containing S, then $Span(S)$ must be a subset of it by defn of span. Thus, $Span(S) \subseteq Span(S\backslash\{x\})$. Having shown both containment directions, this implies $Span(S) = Span(S\backslash\{x\}) = V$. This contradicts S being a minimal spanning set, so our assumption that there exists a redundant vector $x \in S$ must be false.

Thus, S must have no redundant vectors, implying that is independent (again, by problem 4's contrapostive).

Since S is minimally spanning and thus independent, S must be a basis.

13 Problem 13

(a): Let

$$W_1 = \{(0, y) | y \in \mathbb{R}\}$$

$$W_2 = \{(x, 0) | x \in \mathbb{R}\}$$

Clearly, $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$ since for any $v = (x, y)$,

$$v = (0, y) + (x, 0)$$

Taking the projection onto W_1 (the y-axis), we get

$$T(v) = (0, y)$$

(b): Let

$$W_1 = \{(0, y) | y \in \mathbb{R}\}$$

$$W_2 = \{(a, a) | x \in \mathbb{R}\}$$

Clearly, $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$ since for any $v = (x, y)$,

$$v = (0, y) + (x, 0)$$

Taking the projection onto W_1 (the y-axis), we get

$$T(v) = (0, y)$$

(b):

14 Problem 14

(a): Let $v_1, v_2 \in V, c \in F$. Let v_1 have the unique rep $v_1 = w_1 + w_2$, let $v_2 = w_1' + w_2'$. Now,

$$T(cv_1 + v_2) = T(cw_1 + w_1' + cw_2 + w_2')$$

By definition of T, this equals the component in W_1, discarding the component in W_2. Hence, we get

$$T((cw_1 + w_1') + (cw_2 + w_2')) = cw_1 + w_1' = cT(v_1) + T(v_2)$$

Thus T is linear.

(b): Since $W_1 = V$, it must be that $W_2 = \{0\}$. This is because if there were a non-zero $w_2 \in W_2$, then automatically, $w_2 \in W_1 \cap W_2$ but this would contradict defn of internal direct sum $(W_1 \cap W_2) = \{0\}$.

Hence, for any $v \in V$, it is represented uniquely as $v + 0$, where $v \in W_1$, $0 \in W_2$. Hence, T is the identity transformation where $T(v) = v$

(c): Since $W_1 = \{0\}$, it must be that $W_2 = V$. This is because $V = W_1 + W_2$. For any $v = w_1 + w_2$, then w_1 must be 0, so w_2 must be v. Hence, for all $v \in V$, $v \in W_2$, so $V \subseteq W_2$. Clearly, $W_2 \subseteq V$ so $W_2 = V$.

Hence, for any $v = 0 + v$, where $0 \in W_1$, $v \in W_2$, $T(v) = 0$ so T is the 0 transformation.