Problem Set #1

OSM
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Exercise 1.3

- Not an algebra
- Algebra
- Algebra and σ-algebra

Exercise 1.7

{∅, X} is the smallest possible σ-algebra because by definition, a σ-algebra must have an empty set, which is included in this, and its complement must also be in the set, which is also included in {∅, X}. On the other hand, the power set is the largest possible σ-algebra because it lists basically all the combinations of X.

Exercise 1.10

- Since each S_α is a σ-algebra, that means that $\emptyset \in S_\alpha$ for all α, which implies that $\emptyset \in \cap_\alpha S_\alpha$
- If $A \in S_\alpha$ for all α, then $A^c \in S_\alpha$ for all α because each S_α is a σ-algebra, which implies that $A^c \in \cap S_\alpha$
- If $A \in S_\alpha$ for all α, then $\cup A \in S_\alpha$ for all α because each S_α is a σ-algebra, which implies that $\cup A \in \cap_\alpha S_\alpha$
- Therefore, if $\{S_\alpha\}$ is a σ-algebra, then $\cap_\alpha S_\alpha$ is also a σ-algebra.

Exercise 1.17

- Because μ is a nonnegative function and

$$\mu(\cup_{i=1}^{\infty})A_i = \sum_{i=1}^{\infty} \mu A_i$$

then

$$B = A \cup (B \cap A^c) \; because \; A \subset B$$

then

$$\mu(A) + \mu(B \cap A^c) = \mu(B)$$

then

$$\mu(A) \leq \mu(B)$$

then

$$A, B \in S, A \subset B \; then \; \mu(A) \leq \mu(B)$$

- Let A_m, $A_n \in$ A.

  Let D = $A_m \cap A_n^c$, E = $A_n \cap A_m^c$, and F = $A_m \cap A_n$.

  Because

  $$\mu(\cup_{i=1}^{\infty})A_i = \sum_{i=1}^{\infty} \mu A_i$$

  then $\mu(A_n \cup A_m) = \mu(D \cup E \cup F)$, and because D, E, and F are disjoint, then

  $$\mu(A_n \cup A_m) = \mu(D) + \mu(E) + \mu(F)$$

  . Solving for the left side of the equation:

  $$\mu(A_n) + \mu(A_m) = \mu(D \cup F) + \mu(E \cup F)$$

  $$\mu(A_n) + \mu(A_m) = \mu(D) + \mu(F) + \mu(E) + \mu(F)$$

  which means

  $$\mu(D) + \mu(E) + \mu(F) \leq \mu(D) + \mu(F) + \mu(E) + \mu(F)$$

  which means if

  $$\{A_i\}_{i=1}^{\infty} \subset A \text{ },then\ \mu(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$$

Exercise 1.18

- Because $B \in S$ then $(A \cap B) \in S$. Because $\lambda(A) = \mu(A \cap B)$ and $B \in S$, then $\lambda(A) = \mu(A \cap B)$ is also a measure (X, S).

Exercise 1.20

- Let $B_1 = A_1$ and $B_i = A_i \cap A_{i-1}^c$ for $i \leq 2$. Also, $A = \cup_{n \in N} B_n$ and $A_n = \cup_{n=1}^{n} B_n$. This means that $\lim_{n\to\infty}(A_1 \cap A_n^c) = A_1 \cap A^c$

  Using the proof from i),

  $$\mu(A_1) - \mu(A_n) = \lim_{n\to\infty} \mu(A_1 \cap A_n^c) = \mu(A_1 \cap A^c) = \mu(A_1) = \mu(A_n)$$

  which means

  $$\lim_{n\to\infty} \mu(A_n) = \mu(\cap_{i=1}^{\infty} A_i)$$

Exercise 2.10

Since both E $\in$ X and B $\in$ X, we know that both are in X. Therefore, there are three options: E = B, $E \cap B = \emptyset$ or $E \cap B \neq \emptyset$.

If E = B, then $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$ where $\mu^*(B \cap E) = \mu^*(B)$ and $\mu^*(B \cap E^c) = 0$, which means $\mu^*(B) = \mu^*(B)$, which means $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

If $E \cap B = \emptyset$, then $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$ where $\mu^*(B \cap E) = 0$ and $\mu^*(B \cap E^c) = \mu^*(B)$, which means $\mu^*(B) = \mu^*(B)$, which means $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

If $E \cap B \neq \emptyset$, then $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$ where $\mu^*(B \cap E) + \mu^*(B \cap E^c) = \mu^*(B)$, which means $\mu^*(B) = \mu^*(B)$, which means $\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$.

This means that

$$\mu^*(B) = \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

Exercise 2.14

Let O be the set of open sets and let A be defined as $O \subset A$. Because $B(\mathbb{R}) = \sigma(O) \subset \sigma(A) \subset \{S_\alpha\}$

Exercise 3.1

Let $\{x_i\}_{i=1}^{\infty}$ be elements of X. For any $\epsilon > 0$, $A_i = (x_i - \frac{\epsilon}{2^i}, x_i + \frac{\epsilon}{2^i})$, then $X \subset \cup_{i=1}^{\infty} A_i$ and $M(\cup A_i) \leq \sum_{i=1}^{\infty}(2^{1-i}\epsilon) = 2\epsilon$, which means $M(X) = 0$.

Exercise 4.15

Because $\{s : 0 \leq s \leq f\}$ and $\{s : 0 \leq s \leq g\}$ where s is simple, measurable, $\int_E f d\mu = \int_E g d\mu$

Note:

Sorry I just don't understand much of this math at all, even after working with other people in the program and watching multiple YouTube videos on measure theory. I especially don't understand how to prove things and what certain equations mean - I've never taken a proof-based math class. I genuinely feel bad for not turning in a complete problem set, but I really don't understand this and feel like this math is beyond my level - is there any advice you could give me in terms of how to catch up to this level?