Problem Set #3

OSM
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Exercise 2

- $$Let D = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

  All eigenvalues of D are 0 because D is upper triangular. Therefore, the algebraic multiplicity is 3. Because D has 1 eigenvector for eigenvalue of 0 and the eigenspace of 0 is span(1), the geometric multiplicity is 1.

Exercise 4

- $$Let A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  So det(A) = ad - bc and if $A^H = A$, then $a = \overline{a}, b = \overline{c}, d = \overline{d}$, which means a and d are real. Because $bc = \overline{c}c = ||c||^2$ is also real.

  $$p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - (a+d)\lambda + ad - ||c||^2$$

  $$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + ||c||^2}}{2}$$

  Because $(a-d)^2 + ||c||^2 \geq 0, \lambda$ is real.

- $$Let A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

  Because $a = -\overline{a}, b = -\overline{c}, d = -\overline{d}$, a and d are imaginary. Because $bc = -\overline{c}c = -||c||^2$, a and d are also negative.

  $$p(\lambda) = \lambda^2 - tr(A)\lambda + det(A) = \lambda^2 - (a+d)\lambda + ad - ||c||^2$$

  $$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + ||c||^2}}{2}$$

  Because $(a-d)^2 + ||c||^2 < 0, \lambda$ is imaginary.

Exercise 6

If matrix A is upper triangular, $det(A) = \Pi_{i=1}^{n} a_{ii}$ and its eigenvalues are such that $det(\lambda I - A) = 0$.

$$det(\lambda I - A) = \Pi_{i=1}^{n} a_{ii} = 0$$

This implies that the eigenvalues are given by the diagonal elements of matrix A.

Exercise 8

- Because this set is orthonormal given the inner product $\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$, each element is independent and forms a basis for V.

- $Dsin(x) = cos(x), Dcos(x) = -sin(x), Dcos(2x) = -2sin(2x), Dsin(2x) = 2cos(2x)$, so

$$D = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

- $span(sin(x), cos(x))\ and\ span(sin(2x), cos(2x))$

Exercise 13

$det(\lambda I - A) = \lambda^2 - 1.4\lambda + 0.4$ which means the roots and eigenvalues are 1 and 0.4, and

$$\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix}$$

has a solution of $[2\ \ 1]^T$. The eigenvector for $\lambda = 0.4$ has a null space

$$\begin{bmatrix} 0.4 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}$$

with a solution of $[1\ \ 1]^T$, which means

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

Exercise 15

Let $(\lambda_i)_{i=1}^n$ be the eigenvalues of matrix $A \in M_n(\mathbb{F})$ and $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. Because A can be diagonalized as PBP^{-1}

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n = a_0PP^{-1} + a_1PBP^{-1} + a_2PB^2P^{-1} + \cdots + a_nPB^nP^{-1} = Pf(B$$

Because each term in f(B) is a diagonal matrix, the eigenvalues are $(f(\lambda_i))_{i=1}^n$

Exercise 16

- $A^n = PC^nP^{-1}$

$$C^n = \begin{bmatrix} 1^n & 0 \\ 0 & 0.4^n \end{bmatrix}$$

$$A^k = \frac{1}{3}\begin{bmatrix} 2 + 0.4^k & 2 - 2 * 0.4^k \\ 1 - 0.4^k & 1 + 2 * 0.4^k \end{bmatrix}$$

$$B = \frac{1}{3}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

$$A^k - B = \frac{1}{3}\begin{bmatrix} 0.4^k & -2*0.4^k \\ -0.4^k & 2*0.4^k \end{bmatrix}$$

where each term converges with respect to the 1-norm.

- $-0.4^k + 2*0.4^k \to 0 \; as \; k \to \infty$ and the ∞ norm is the largest row sum, so

$$||A^k - B||_F = \sqrt{tr(\begin{bmatrix} 2*0.4^{2k} & -4*0.4^{2k} \\ -4*0.4^{2k} & 8*0.4^{2k} \end{bmatrix})}$$

$$= \sqrt{10*0.4^{2k}} \to 0 \; as \; k \to \infty$$

Because $||A^k - B||_F \to 0$, the answer doesn't depend on the norm.

- $f(A) = 3I + 5A + A^3 \; where \; f(1) = 3+5+1 = 9 \; and \; f(0.4) = 3+5*0.4+0.4^3 = 5.06$

Exercise 18

Let λ be an eigenvalue of $A \in M_n(\mathbb{F})$. Because A and A^T have the same characteristic polynomial, λ is an eigenvalue of A^T, which means $A^T x = \lambda x \to (A^T x)^T = (\lambda x)^T$. Therefore, $x^T A = \lambda x^T$

Exercise 20

If A is Hermitian and orthonormally similar to B,

$$B = PAP^H = PA^H P^H = (PAP^H)^H = B^H$$

Exercise 24

If A is Hermitian,

$$< x, Ax >= x^H Ax = x^H A^H x =< Ax, x >= \overline{< x, Ax >}$$

. Because the numerator is real, the Rayleigh quotient takes real values. If A is skew Hermitian,

$$< x, Ax >= x^H Ax = -x^H A^H x = - < Ax, x >= -\overline{< x, Ax >}$$

Because the numerator is imaginary, the Rayleigh quotient takes imaginary values.

Exercise 25

- *Because* $< x_j, x_j >= x_j^H x_j = 1$, *and* $< x_i, x_j >= x_i^H x_j = 0$, $(x_1 x_1^H + \cdots + x_n x_n^H) x_j = x_j x_j^H x_j = I x_j$, *so* $I = x_1 x_1^H + \cdots + x_n x_n^H$.

- *Because* $(\lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H)x_j = \lambda x_j x_j^H x_j = \lambda x_j = Ax_j,\ so\ A = \lambda_1 x_1 x_1^H + \cdots + \lambda_n x_n x_n^H$

Exercise 27

Let e_i be the ith vector of the standard basis. $Because\ < x, Ax >= x^H Ax > 0\ for\ all\ x \neq 0, 0 < e_i^H Ae_i = a_{ii}$ must be real and positive.

Exercise 31

- $Let\ A = USigmaV^H, y = V^H x$

$$||A||_2 = sup_{x\neq 0}\frac{||Ax||_2}{||x||_2} = sup_{x\neq 0}\frac{||U\Sigma V^H x||_2}{||x||_2} = sup_{x\neq 0}\frac{||\Sigma V^H x||_2}{||x||_2} = sup_{y\neq 0}\frac{||\Sigma y||_2}{||Vy||_2} =$$

$$sup_{y\neq 0}\frac{||\Sigma y||_2}{||y||_2} = sup_{y=1}||\Sigma y||_2 = \sigma_1$$

- $Let\ A = U\Sigma V^H\ so\ A^{-1} = V\Sigma^{-1}U^H$. Because $\frac{1}{\sigma_1}, \ldots, \frac{1}{\sigma_n}$ are the diagonal entries of Σ^{-1}, $\frac{1}{\sigma_n}$ is the largest singular value of A^{-1} and $||A^{-1}||_2 = \frac{1}{\sigma_n}$

- $Let\ A = U\Sigma V^H\ so\ A^T = V\Sigma^T U^T\ and\ A^H = V\Sigma^H U^H.$

$$||A||_2^2 = ||A^T||_2^2 = ||A^H||_2^2 = \sigma_1^2$$
$$A^H A = V\Sigma^H U^H U\Sigma V^H = V\Sigma^H \Sigma V^H = V\Sigma^2 V^H$$
$$||A^H A||_2 = \sigma_1^2 = ||A||_2^2$$

- $$||UAV||_2^2 = ||(UAV)^H UAV||_2 = ||V^H A^H AV||_2 = ||A^H AVV^H||_2 = ||A^H A||_2 = ||A||_2^2$$

Exercise 32

- $$||UAV||_F = \sqrt{tr(V^H A^H U^H UAV)} = \sqrt{tr(V^H A^H AV)} = \sqrt{tr(A^H AVV^H)} = \sqrt{tr(A^H A)} = ||A||_F$$

- $$||A||_F = ||U\Sigma V^H||_F = ||\Sigma||_F = \sqrt{tr(\Sigma^H \Sigma)} = (\sum_{i=1}^{r}\sigma_i^2)^{\frac{1}{2}} = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2)^{\frac{1}{2}}$$

Exercise 33

Because $||A||_2 = \sigma_1$ with σ_1 being the largest singular value of A,

$$sup_{||x||_2=1,||y||_2=1}|y^H Ax \leq sup_{||x||_2=1,||y||_2=1}||y||_2||\Sigma x||_2$$
$$= sup_{||x||_2=1}||\Sigma x||_2 \leq \sigma_1$$
$$sup_{||x||_2=1,||y||_2=1}|y^H Ax| \geq |y^H Ax| = \sigma_1$$

Exercise 36

$$Let\ A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$A^H A = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

where det(A) = -2 and the singular values are 1 and 2 with eigenvalues $\pm\sqrt{2}$.

Exercise 38

- $$AA^\dagger A = U_1 V_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = U_1 \Sigma_1 \Sigma_1^{-1} \Sigma_1 V_1^H = U_1 \Sigma_1 V_1^H = A$$

- $$A^\dagger A A^\dagger = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} \Sigma_1 \Sigma_1^{-1} U_1^H = V_1 \Sigma_1^{-1} U_1^H = A^\dagger$$

- $$(AA^\dagger)^H = (U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H)^H = U_1 \Sigma_1 V_1^H V_1 \Sigma_1^{-1} U_1^H = AA^\dagger$$

- $$(A^\dagger A)^H = (V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H)^H = V_1 \Sigma_1^{-1} U_1^H U_1 \Sigma_1 V_1^H = A^\dagger A$$

- $$(AA^\dagger A)A^\dagger = A^\dagger A \rightarrow (AA^\dagger)(AA^\dagger) = A^\dagger A$$

  Let $U_1 = [u_1, \ldots, u_n]$ where U_1 is an orthonormal basis for $\mathbb{R}(A)$.

  $$AA^\dagger = U_1 U_1^H x = U_1[u_1^H x, \ldots, u_n^H x] = \sum_{i=1}^{n} u_i^H x u_i = \sum_{i=1}^{n} < u_i, x > u_i = proj_{\mathbb{R}(A)} x$$

- Let $V_1 = [v_1, \ldots, v_n]$ where V_1 is an orthonormal basis for $\mathbb{R}(A^H)$.

  $$A^\dagger A x = V_1 V_1^H x = V_1[v_1^H x, \ldots, v_n^H x] = \sum_{i=1}^{n} v_i^H x v_i = \sum_{u=1}^{n} < v_i, x > v_i = proj_{\mathbb{R}(A^H)} x$$