Problem Set #2
OSM
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Exercise 1

- $$< x, y >= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2)$$
$$4 < x, y >= \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$$
where
$$\|\mathbf{x}+\mathbf{y}\|^2 =< x+y, x+y >= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + < x, y > + < y, x >$$
and
$$\|\mathbf{x}-\mathbf{y}\|^2 =< x-y, x-y >= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - < x, y > - < y, x >$$
$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + < x, y > + < y, x > -(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - < x, y > - < y, x >)$$
$$=< x, y > + < y, x > + < x, y > + < y, x >= 4 < x, y >$$

- $$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \frac{1}{2}(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2)$$
$$2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 = (\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2)$$
where
$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 =< x+y, x+y > + < x-y, x-y >$$
$$= \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + < x, y > + < y, x > + \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - < x, y > - < y, x >$$
$$= 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

Exercise 2

Using proof from 1,
$$Re < x, y >= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2)$$
and
$$Re < x, iy >= \frac{1}{4}(\|\mathbf{x}+\mathbf{iy}\|^2 - \|\mathbf{x}-\mathbf{iy}\|^2)$$
Because
$$< x, y >= Re(< x, y >) + iIm(< x, y >)$$
$$< x, y >= \frac{1}{4}(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}+\mathbf{iy}\|^2 - i\|\mathbf{x}-\mathbf{iy}\|^2)$$

Exercise 3

- $$cos(\theta) = \frac{<x, y>}{\|\mathbf{x}\|\|\mathbf{y}\|}$$
  $$<x, x^5> = \int_0^1 x * x^5 dx = \int_0^1 x^6 dx = \frac{x^7}{7}|_0^1 = \frac{1}{7}$$
  $$<x, x> = \int_0^1 x * x dx = \int_0^1 x^2 dx = \frac{x^3}{3}|_0^1 = \frac{1}{3}$$
  $$<x^5, x^5> = \int_0^1 x^5 * x^5 dx = \int_0^1 x^{10} dx = \frac{x^{11}}{11}|_0^1 = \frac{1}{11}$$
  $$cos(\theta) = \frac{\frac{1}{7}}{\sqrt{\frac{1}{3}} * \sqrt{\frac{1}{11}}}$$
  $$\theta = 34.8°$$

- $$<x^2, x^4> = \int_0^1 x^2 * x^4 dx = \int_0^1 x^6 dx = \frac{x^7}{7}|_0^1 = \frac{1}{7}$$
  $$<x^2, x^2> = \int_0^1 x^2 * x^2 dx = \int_0^1 x^4 dx = \frac{x^5}{5}|_0^1 = \frac{1}{5}$$
  $$<x^4, x^4> = \int_0^1 x^4 * x^4 dx = \int_0^1 x^8 dx = \frac{x^9}{9}|_0^1 = \frac{1}{9}$$
  $$cos(\theta) = \frac{\frac{1}{7}}{\sqrt{\frac{1}{5}} * \sqrt{\frac{1}{9}}}$$
  $$\theta = 16.6°$$

Exercise 8

- Sets are orthonormal if the inner product is equal to 0 and norms are equal to 1.
  $$<cos(t), sin(t)> = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(t)dt = \frac{1}{\pi} * \frac{sin^2(t)}{2}|_{-\pi}^{\pi} = 0$$
  $$<cos(t), sin(2t)> = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)sin(2t)dt = \frac{1}{\pi} * -\frac{sin(2t)sin(t) + 2cos(2t)cos(t)}{3}|_{-\pi}^{\pi} = 0$$
  $$<cos(t), cos(2t)> = \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)cos(2t)dt = \frac{1}{\pi} * -\frac{cos(2t)sin(t) - 2sin(2t)cos(t)}{3}|_{-\pi}^{\pi} = 0$$
  $$<sin(t), cos(2t)> = \frac{1}{\pi}\int_{-\pi}^{\pi} sin(t)cos(2t)dt = \frac{1}{\pi} * \frac{cos(2t)cos(t) + 2sin(2t)sin(t)}{3}|_{-\pi}^{\pi} = 0$$

$$< sin(t), sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} sin(t)sin(2t)dt = \frac{1}{\pi} * \frac{sin(2t)cos(t) - 2cos(2t)sin(t)}{3}|_{-\pi}^{\pi} = 0$$

$$< cos(2t), sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(2t)sin(2t)dt = \frac{1}{\pi} * -\frac{cos^2(2x)}{4}|_{-\pi}^{\pi} = 0$$

$$< cos(t), cos(t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(t)cos(t)dt = \frac{1}{\pi} * (\frac{t}{2} + \frac{sin(2t)}{4}|_{-\pi}^{\pi}) = 1$$

$$< sin(t), sin(t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} sin(t)sin(t)dt = \frac{1}{\pi} * (\frac{t}{2} - \frac{sin(2t)}{4}|_{-\pi}^{\pi}) = 1$$

$$< cos(2t), cos(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} cos(2t)cos(2t)dt = \frac{1}{\pi} * (\frac{t}{2} + \frac{sin(4t)}{8}|_{-\pi}^{\pi}) = 1$$

$$< sin(2t), sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} sin(2t)sin(2t)dt = \frac{1}{\pi} * (\frac{t}{2} - \frac{sin(4t)}{8}|_{-\pi}^{\pi}) = 1$$

- $< t, t >= \frac{1}{\pi}\int_{-\pi}^{\pi} t * tdt = \frac{1}{\pi} * \int_{-\pi}^{\pi} t^2dt = \frac{1}{\pi} * \frac{t^3}{3}|_{-\pi}^{\pi}) = \frac{2\pi^2}{3}$
- $$proj_X(cos(3t)) =< X, cos(3t) > \frac{X}{\|\mathbf{X}\|^2}$$
  $$< X, cos(3t) >= \frac{1}{\pi}\int_{-\pi}^{\pi} X * cos(3t)dt = \frac{1}{\pi} * \frac{Xsin(3t)}{3}|_{-\pi}^{\pi} = 0$$
  $$proj_X(cos(3t)) = 0$$
- $$proj_X(t) =< X, t > \frac{X}{\|\mathbf{X}\|^2}$$
  $$< X, t >= \frac{1}{\pi}\int_{-\pi}^{\pi} X * tdt = \frac{1}{\pi} * \frac{Xt^2}{2}|_{-\pi}^{\pi} = 0$$
  $$proj_X(t) = 0$$

Exercise 9

$$R_\theta = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

$$Let\ v\ = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$Let\ w\ = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$T(v) = \begin{bmatrix} cos(\theta)v_1 - sin(\theta)v_2 \\ sin(\theta)v_1 + cos(\theta)v_2 \end{bmatrix}$$

$$T(w) = \begin{bmatrix} cos(\theta)w_1 - sin(\theta)w_2 \\ sin(\theta)w_1 + cos(\theta)w_2 \end{bmatrix}$$

$< T(v), T(w) >= (cos(\theta)v_1 - sin(\theta)v_2)(cos(\theta)w_1 - sin(\theta)w_2) + (sin(\theta)v_1 + cos(\theta)v_2)(sin(\theta)w_1 + cos(\theta)w_2)$

$= cos^2(\theta)(v_1w_1+v_2w_2)+sin(\theta)cos(\theta)(-v_1w_2-v_2w_1+v_1w_2+v_2w_1)+sin^2(\theta)(v_2w_2+v_1w_1)$

$$= (cos^2(\theta) + sin^2(\theta))(v_1w_1 + v_2w_2)$$

$$= v_1w_1 + v_2w_2 =< v, w >$$

Exercise 10

- Since Q is orthornormal, $< a, b >=< Qa, Qb >$.

  $$(Qa)^H Qb = Q^H a^H Qb = a^H b$$

  if and only if $Q^H Q = I$. This also works the other way around, since $< Qa, Qb >=< a, b >$.

- $\|(\mathbf{Qx})^\mathbf{2}\| =< Qx, Qx >=< x, x >= \|\mathbf{x^2}\|$ which means that $\|\mathbf{Qx}\| = \|\mathbf{x}\|$

- Because $Q^H Q = I$, $Q^H = Q^{-1}$,

  $$(Q^H)^H = Q \; and \; (Q^H)^H Q^H = Q^H Q = Q^{-1} Q = I$$

  , which means that Q^{-1} is also an orthonormal matrix.

- Because Q is orthonormal, $(Q^H Q) = I$, which means $(Q^H Q)_{ij} = q_i^H q_j$.

- No.

  $$Let \; Q \; = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

  $$|det(Q)| = 1 \; but \; Q^H Q \neq I$$

- Let Q $= Q_1 Q_2$. $Q_1^H Q_1 = I, \; Q_2^H Q_2 = I$

  $$Q = (Q_1 Q_2)^H (Q_1 Q_2) = Q_1^H Q_2^H Q_1 Q_2 = Q_2^H Q_2 = I$$

  which means that Q is an orthonormal matrix.

Exercise 11

When we apply the Gram-Schmidt orthonormalization process to a collection of linearly dependent vectors, $q_k = 0$.

Exercise 16

- Let $A \in \mathbb{M}_{mxn}(\mathbb{F})$ where $rank(A) = n \leq m$

$$\exists Q \in \mathbb{M}_{mxm} \text{ and upper triangular } R \in \mathbb{M}_{mxn} \text{ where } A = QR$$

$$-Q(-Q)^H = QQ^H = I, \text{ and } (-Q)^H(-Q) = I$$
$$-R \text{ is upper triangular, which means } A = QR = (-Q)(-R)$$

- If A is invertible, it has 2 QR decompositions, QR and $\tilde{Q}\tilde{R}$ where R and $\tilde{R}$ are positive.

$$\tilde{R}^{-1}R = Q^H\tilde{Q},\ \tilde{R}^{-1}R = I$$
$$R = \tilde{R}, Q = \tilde{Q}$$

Exercise 17

$$A^HAx = A^Hb$$
$$(\hat{Q}\hat{R})^H\hat{Q}\hat{R}x = (\hat{Q}\hat{R})^Hb$$
$$\hat{R}^H\hat{Q}^H\hat{Q}\hat{R}x = \hat{R}^H\hat{Q}^Hb$$
$$R^{-1}(\hat{R}^H\hat{Q}^H\hat{Q}\hat{R}x = \hat{R}^H\hat{Q}^Hb)$$
$$\hat{R}x = \hat{Q}^Hb$$

Exercise 23

$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y||$$
$$||x|| - ||y|| \leq ||x - y||$$
$$||y|| = || - y|| = ||x - y + x|| \leq ||x - y|| + ||x||$$
$$||y|| - ||x|| \leq ||x - y||$$

Exercise 24

- $$\textit{Positivity: because } |f(t)| \geq 0, \int_a^b f(t)dt \geq 0$$

$$\textit{Scale preservation: let } a \in \mathbb{R}. ||af(t)||_{L^1} = \int_a^b |af(t)|dt = |a| \int_a^b f(t)dt$$

Triangle inequality: let $g \in C[a,b]$*. because* $|f(t)+g(t)| \leq |f(t)|+|g(t)|$, $\int_a^b |f(t)+g(t)|dt \leq \int_a^b f(t)dt + \int_a^b g(t)dt$

- $$Positivity:\ because\ |f(t)^2| \geq 0, (\int_a^b f(t)^2 dt)^{\frac{1}{2}} \geq 0$$
  $$Scale\ preservation:\ let\ a \in \mathbb{R}. ||af(t)||_{L^2} = \int_a^b (|af(t)|^2 dt)^{\frac{1}{2}} = a \int_a^b f(t)^2 dt)^{\frac{1}{2}}$$
  Triangle inequality: let $g \in C[a,b]$. *because* $|f(t)+g(t)| \leq |f(t)|+|g(t)|$, $(\int_a^b |f(t)+g(t)|^2 dt)^{\frac{1}{2}} \leq (\int_a^b f(t)^2 dt)^{\frac{1}{2}} + (\int_a^b g(t)^2 dt)^{\frac{1}{2}}$

- $$Positivity:\ because\ |f(x)| \geq 0,\ sup_{x\in[a,b]}|f(x)| \geq 0$$
  $$Scale\ preservation:\ let\ a \in \mathbb{R}. ||af||_{L^\infty} = sup_{x\in[a,b]}|a||f(x)| = a sup_{x\in[a,b]}|f(x)|$$
  Triangle inequality: let $g \in C[a,b]$. *because* $|f(t)+g(t)| \leq |f(t)|+|g(t)|$, $sup_{x\in[a,b]}|f(x)+g(x)| \leq sup_{x\in[a,b]}|f(x)| + sup_{x\in[a,b]}|g(x)|$

Exercise 26

Skipped (1)

Exercise 28

- $$\frac{1}{\sqrt{n}||x||_2} \leq \frac{1}{||x||_1} \leq \frac{1}{||x||_2}$$
  $$||Ax||_2 \leq ||Ax||_1 \leq \sqrt{n}||Ax||_2$$
  $$\sqrt{n}||A||_2 \geq ||A||_1$$
  $$\frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq \sqrt{n}||A||_2$$

- $$\frac{1}{\sqrt{n}||x||_\infty} \leq \frac{1}{||x||_2} \leq \frac{1}{||x||_\infty}$$
  $$||Ax||_\infty \leq ||Ax||_2 \leq \sqrt{n}||Ax||_\infty$$
  $$\sqrt{n}||A||_\infty \geq ||A||_2$$
  $$\frac{1}{\sqrt{n}}||A||_\infty \leq ||A||_2 \leq \sqrt{n}||A||_\infty$$

Exercise 29

$$||Q|| = sup_{x\neq 0}\frac{||Q_x||_2}{||x||_2} = sup_{x\neq 0}\frac{||x||_2}{||x||_2} = 1$$
$$Since\ ||Ax|| \leq ||A||||x||,\ ||Rx|| = sup_{A\neq 0}\frac{||Ax||_2}{||A||} \leq sup_{A\neq 0}\frac{||Ax||_2||x||_2}{||Ax||_2}$$
$$Since\ ||Ax||_2 = ||x||_2 = ||Rx||\ and\ ||A|| = 1,$$
$$||A|| \leq \frac{||Ax||}{||x||}$$

Exercise 30

$$\textit{Positivity: because } ||A||_S \geq 0, ||SAS^{-1}|| \geq 0 \textit{ so } ||A|| \geq 0$$

$$\textit{Scale preservation: let } a \in \mathbb{R}. ||aA||_S = ||aSAS^{-1}|| = a||SAS^{-1}|| = a||A||_S$$

$$\textit{Triangle inequality: let } B \in \mathbb{M}_n(\mathbb{F}). ||A+B||_S = ||S(A+B)S^{-1}|| = ||SAS^{-1} + SBS^{-1}|| \leq ||SAS^{-1}|| + ||SBS^{-1}|| = ||A||_S + ||B||_S$$

Exercise 37

$$Let\ p \in V,\ \ p = ax^2 + bx + c =< a, b, c >$$

$$Find\ q\ inV\ \ that\ satisfies\ L[p] = p'(1) = p'q = 2a + b$$

$$q = (2, 1, 0)$$

Exercise 38

$$Let\ p \in V,\ \ p = ax^2 + bx + c,\ \ p = (a, b, c)^T,\ \ p' = D(p) = (0, 2a, b)^T$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$Hermitian = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Exercise 39

- $$< (S+T)*w, v >_v =< w, (S+T)v >_w$$
  $$< w, Sv + Tv >_w =< w, Sv >_w + < w, Tv >_w$$
  $$< S*w, v >_v + < T*w, v >_v =< S*w + T*w, v >_v$$
  $$(S+T)* = S * + T*$$
  $$< (aT)*w, v >_v =< w, (aT)v >_w$$
  $$< w, aTv >_w = a < w, Tv >$$
  $$a < T*w, v > =< \overline{a}T*w, v >$$
  $$(aT)* = \overline{a}T$$

- $$< w, Sv >_w =< S*w, v >_v = \overline{< v, S*w >_v} = \overline{< S**v, w >_w} =< w, S**v >_w$$
  $$S = S**$$

- $$< (ST)*v', v >_v = < v', (ST)v >_v = < v', S(Tv) >_v$$
$$< S*v', Tv >_v = < T*S*v', v >_v$$
$$(ST)* = T*S*$$

- $$T*(T*)^{-1} = (TT^{-1})* = I* = I$$

Exercise 40

- $$Let\ B,\ C \in \mathbb{M}_n(\mathbb{F})$$
$$< B, AC > = < tr(B^H AC) > = tr(A^H B)^H C) = < A^H B, C >$$
$$A* = A^H$$

- $$Let\ A_1, A_2, A_3 \in \mathbb{M}_n(\mathbb{F})$$
$$< A_2, A_3, A_1 > = tr(A_2^H A3A1) = tr(A_1 A_2^H A_3) = tr((A_2 A_1^H)^H A_3) = < A_2, A_1^H, A_3 >$$
$$A_1^H = A_1^* \rightarrow < A_2, A_3, A_1 > = < A_2 A1^*, A_3 >$$

- $$Let\ A, B, C \in \mathbb{M}_n(\mathbb{F})$$
$$< B, AC - CA > = < B, AC > - < B, CA >$$
$$< B, CA > = < BA*, C >$$
$$< B, AC > = tr(B^H AC) = tr((A^H B)^H C) = < A^H B, C > = < A*B, C >$$
$$(T_A)^* = T_{A^*}$$

Exercise 44

$$Let\ x \in \mathbb{F}\ s.t.\ Ax = b,\ then\ \forall y \in N(A^H),$$
$$< y, b > = < y, Ax > = < A^H y, x > = < 0, x > = 0$$
$$If\ \ < y, b > \neq 0,\ then\ b \notin N(A^H) = R(A)$$
$$No\ x \in \mathbb{F}, Ax = b$$

Exercise 45

$$Let\ A \in Skew_n(\mathbb{R}),\ \ B \in Sym_n(\mathbb{R})$$
$$< A, B >=< -A, B >= - < A, B >= tr(AB)$$
$$< A, B >=< B, A >= tr(B^T A) = tr(BA) = tr(AB)$$
$$tr(AB) = -tr(AB) = 0,\ < A, B >= 0$$
$$Let\ B \in Sym_n(\mathbb{R})^{\perp}$$
$$< B + B^T, B >= tr((B + B^T)B) = tr(BB + B^T B) = tr(BB) + tr(B^T B) = 0$$
$$< B^T, B >=< -B, B >$$
$$B^T = -B$$
$$Sym_n(\mathbb{R})^{\perp} = Skew_n(\mathbb{R})$$

Exercise 46

- $$A^H Ax = A^H(Ax) = 0$$
  $$Ax \in N(A^H)\ \ \rightarrow\ \ Ax \in R(A)$$

- $$Let\ x \in N(A),\ \ Ax = 0$$
  $$A^H Ax = 0,\ \ x \in N(A^H A)$$
  $$Let\ x \in N(A^H A),\ \ A^H Ax = 0$$
  $$< Ax, Ax >= x^H A^H Ax = 0$$
  $$||Ax|| = 0\ \ \rightarrow\ \ Ax = 0$$

- $$n = rank(A) + dimN(A),\ \ n = rank(A^H A) + dimN(A^H A)$$
  $$rank(A) = rank(A^H A)$$

- $$n = rank(A) = rank(A^H A)\ \ \rightarrow\ \ A^H A \in \mathbb{M}_n\ \ is\ nonsingular$$

Exercise 47

- $$P^2 = A(A^H A)^{-1} A^H A(A^H A)^{-1} A^H = A(A^H A)^{-1} A^H$$

- $$P^H = (A(A^H A)^{-1} A^H)^H = A((A^H A)^{-1})^H A^H = A((A^H A)^H)^{-1} A^H = A(A^H A)^{-1} A^H = P$$

- $$rank(P) = rank(A(A^H A)^{-1} A^H)$$
  $$tr(A(A^H A)^{-1} A^H) = tr(A^H A(A^H A)^{-1}) = tr(I)$$

  I has dimensions nxn, which means rank(I) = n $\rightarrow$ rank(P) = n

Exercise 48

Skipped (2)

Exercise 50

$$y^2 = \frac{1}{s} + \frac{rx^2}{s} \ where \ A^H Ax = A^H b \ where$$

$$A = \begin{bmatrix} 1 & x_1^2 \\ 1 & x_2^2 \\ \vdots & \vdots \\ 1 & x_n^2 \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{1}{s} \\ \frac{-r}{s} \end{bmatrix}$$

$$b = \begin{bmatrix} y_1^2 \\ y_2^2 \\ \vdots \\ y_n^2 \end{bmatrix}$$

$$A^H b = \begin{bmatrix} \sum_i y_i^2 \\ \sum_i x_i^2 y_i^2 \end{bmatrix}$$