Dice Probability

Jonathan Stokes

February 10, 2024

ADD: How is this related to the birthday problem????!!!!

What is the probability of rolling a 1 and a 2 when rolling 3 dice? First lets define the following events.

$$A\text{: 1 does appear in a roll of 3 dice.} \tag{1}$$

$$B\text{: 2 does appear in a roll of 3 dice.} \tag{2}$$

To find the probability of events A and B we need to find the probability of $\neg A$ and $\neg B$ which is the probability of rolling any of the other 5 dice faces of the 3 independent dice. Therefore

$$P(\neg A) = P(\neg B) = \left(\frac{5}{6}\right)^3 \tag{3}$$

and by analogy the probability of not rolling a 1 and a 2 is

$$P(\neg A \cap \neg B) = \left(\frac{4}{6}\right)^3 \tag{4}$$

It follows from eq. (3) that the probability of rolling either a 1 or a 2 is

$$P(A) = P(B) = 1 - P(A) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216} \tag{5}$$

To answer our initial question above we need to find $P(A \cap B)$.

Given eq. (4) the conditional probability of not rolling a 1 given not rolling a 2 is

$$P(\neg A|\neg B) = \frac{P(\neg A \cap \neg B)}{P(\neg B)} = \frac{\left(\frac{4}{6}\right)^3}{\left(\frac{5}{6}\right)^3} = \left(\frac{4}{5}\right)^3 \tag{6}$$

and given the law of total probability

$$P(A) = P(A|B)P(B) + P(A|\neg B)P(\neg B) \tag{7}$$

Therefore from eq. (6) and since $P(A|\neg B) + P(\neg A|\neg B) = 1$,

$$P(A|B) = \frac{P(A) - P(A|\neg B)P(\neg B)}{P(B)} = \frac{\left(1 - \left(\frac{5}{6}\right)^3\right) - \left(1 - \left(\frac{4}{6}\right)^3\right)\left(\frac{5}{6}\right)^3}{1 - \left(\frac{5}{6}\right)^3} = \frac{30}{91} \tag{8}$$

and it follows that

$$P(A \cap B) = P(A|B)P(B) = \left(\frac{30}{91}\right)\left(\frac{91}{216}\right) = \frac{5}{36} \approx 0.138 \tag{9}$$

Now does this generalize for $n \geq 2$ dice rolled? Lets first define the following events

$$A_n\text{: 1 does appear in a roll of } n \text{ dice.} \tag{10}$$

$$B_n\text{: 2 does appear in a roll of } n \text{ dice.} \tag{11}$$

$$P(A_n \cap B_n) = \left(\frac{\left(1 - \left(\frac{5}{6}\right)^n\right) - \left(1 - \left(\frac{4}{6}\right)^n\right)\left(\frac{5}{6}\right)^n}{1 - \left(\frac{5}{6}\right)^n} \right) \left(1 - \left(\frac{5}{6}\right)^n\right) \tag{12}$$

and as $n \to \infty$, $P(A_n \cap B_n) \to 1$ as expected.

Below seems overly complicated and probably wrong. Perhaps try making the recursion more explicit. Show how m=3 is build from m=2? Also check that m=3 solution is correct in simulations.

Now say we want to know the probability of rolling the sequence of dice 1 to m where $m \in \{2, 3, 4, 5\}$. Given $n > m$ (DOES THIS WORK FOR n=m????, what happens if m=6???????) lets define the events

$$A_{n,i}\text{: } i \text{ does appear in a roll of } n \text{ dice.} \tag{13}$$

for $i \in \{1 \dots m\}$.

$$P(\neg A_{n,i}) = \left(\frac{5}{6}\right)^n \tag{14}$$

$$P\left(\bigcap_{i=1}^{m} \neg A_{n,i}\right) = \left(\frac{6-m}{6}\right)^n \tag{15}$$

Now expanding the conditional probability **(that is not the right term)** we know that.

$$P\left(\bigcap_{i=1}^{m} A_{n,i}\right) = P\left(A_{n,1} | \bigcap_{i=2}^{m} A_{n,i}\right) P\left(\bigcap_{i=2}^{m} A_{n,i}\right) \tag{16}$$

$$= \prod_{j-1}^{m} P\left(A_{n,j} | \bigcap_{i=j+1}^{m} A_{n,j}\right) \tag{17}$$

and we also know from above that $P\left(A_{n,j} | \bigcap_{i=j+1}^{m} A_{n,j}\right)$ has the form

$$P\left(A_{n,j} | \bigcap_{i=j+1}^{m} A_{n,j}\right) = \frac{P(A_{n,j}) - P(A_{n,j} | \bigcap_{i=j+1}^{m} \neg A_{n,i}) \left(1 - P\left(\bigcap_{i=j+1}^{m} \neg A_{n,j}\right)\right)}{\left(1 - P\left(\bigcap_{i=j+1}^{m} A_{n,j}\right)\right)} \tag{18}$$

Now from eq. (15) we can see that

$$\left(1 - P\left(\bigcap_{i=j+1}^{m} \neg A_{n,j}\right)\right) = \left(\frac{6-m+i}{6}\right)^n \tag{19}$$

and from eq. (14)

$$P(A_{n,j}) = 1 - \left(\frac{5}{6}\right)^n \tag{20}$$

also

$$1 - P\left(\bigcap_{i=j+1}^{m} A_{n,j}\right) = 1 - P\left(\bigcap_{i=1}^{c=m-j} A_{n,i}\right) \tag{21}$$

setting up a recursion by substitution into eq. (16) with $c = 2$ as the base case which we solved for above. This leaves

$$P(A_{n,j}|\bigcap_{i=j+1}^{m}\neg A_{n,i}) = 1 - P(\neg A_{n,j}|\bigcap_{i=j+1}^{m}\neg A_{n,i}) \tag{22}$$

$$= 1 - \frac{P(\bigcap_{i=j}^{m}\neg A_{n,i})}{P(\neg A_{n,j})} \tag{23}$$

$$= 1 - \frac{\left(\frac{6-m+j}{6}\right)^n}{\left(\frac{5}{6}\right)^n} \tag{24}$$

$$= 1 - \left(\frac{6-m+j}{5}\right)^n \tag{25}$$

substitute all this back into eq. (18) and the result of that into eq. (16).